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Structure of Hecke algebras of modular forms modulo p

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Generalizing the recent results of Bellaïche and Khare for the level-1 case, we study the structure of the local components of the shallow Hecke algebras (i.e., Hecke algebras without U_p and U_ℓ for all primes ℓ dividing the level N) acting on the space of modular forms modulo p for $\Gamma_0(N)$ and $\Gamma_1(N)$. We relate them to pseudodeformation rings and prove that in many cases, the local components are regular complete local algebras of dimension 2.

1. Introduction

The p-adic Hecke algebra acting on modular forms of level N of all weights, which is generated by the Hecke operators away from Np, has been well studied in the past due to its connection with p-adic families of modular forms and deformation rings of Galois representations (see [Emerton 2011] for the precise definition and more details). One can similarly define a mod p Hecke algebra acting on modular forms modulo p of level N (in the sense of Serre and Swinnerton-Dyer) of all weights. The main aim of this paper is to study the structure of these Hecke algebras acting on modular forms modulo p and their relation with suitable deformation rings in characteristic p. These objects were previously studied by Jochnowitz [1982], Khare [1998], Nicolas and Serre [2012a; 2012b], and Bellaïche and Khare [2015]. In this article, we generalize, with minor changes, the results Bellaïche and Khare [2015] proved for $p \geq 5$ and $N = 1$.

Before proceeding further, we fix some notation. In all of this paper, we fix a prime number $p > 3$ and a positive integer $N \geq 1$ not divisible by p. We shall denote by K a finite extension of $\mathbb{Q}_p$, by $\mathcal{O}$ the ring of integers of K, by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, by π the generator of $\mathfrak{p}$ and by $\mathbb{F}$ the finite residue field of $\mathcal{O}$. We call $G_{\mathbb{Q},Np}$ the Galois group of a maximal algebraic extension of $\mathbb{Q}$ unramified outside $\{\ell \text{ s.t. } \ell \mid Np\} \cup \{\infty\}$ over $\mathbb{Q}$. For a prime q not dividing Np, we denote by $\mathrm{Frob}_q \in G_{\mathbb{Q},Np}$ a Frobenius element at q. We denote by c a complex conjugation in $G_{\mathbb{Q},Np}$. We write $G_{\mathbb{Q}_\ell}$ for $\mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ for every prime

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ℓ dividing Np. There are natural maps $i_\ell : G_{\mathbb{Q}_\ell} \to G_{\mathbb{Q},Np}$, well defined up to conjugacy. For every prime ℓ dividing Np, we write I_ℓ for the inertia subgroup of $G_{\mathbb{Q}_\ell}$. For a representation ρ of $G_{\mathbb{Q},Np}$, we shall denote by $\rho|_{G_{\mathbb{Q}_\ell}}$ the composition of i_ℓ with ρ: this is a representation of $G_{\mathbb{Q}_\ell}$, well defined up to an isomorphism. We denote by $\omega_p : G_{\mathbb{Q},Np} \to \mathbb{F}^*$ the cyclotomic character modulo p.

We will now define the Hecke algebra that we want to study and the space of modular forms on which it acts. For the rest of the introduction, Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(M)$ means either $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. Following [Bellaïche and Khare 2015] (henceforth abbreviated as [BK]), we shall denote by $S_k^\Gamma(\mathcal{O})$ the module of cuspidal modular forms of weight k for Γ with Fourier coefficients in $\mathcal{O}$. We see it as a submodule of $\mathcal{O}[[q]]$ by the q-expansion. We denote by $S_{\le k}^\Gamma(\mathcal{O})$ the submodule $\sum_{i=0}^{i=k} S_i^\Gamma(\mathcal{O})$ of $\mathcal{O}[[q]]$. Note that this sum is direct (see [BK, Section 1.2]). We denote by $S_{\le k}^\Gamma(\mathbb{F})$ the image of $S_{\le k}^\Gamma(\mathcal{O})$ under the reduction map $\mathcal{O}[[q]] \to \mathbb{F}[[q]]$, which reduces each coefficient of a power series in $\mathcal{O}[[q]]$ modulo $\mathfrak{p}$. Thus, $S_{\le k}^\Gamma(\mathbb{F}) = \sum_{i=0}^{i=k} S_i^\Gamma(\mathbb{F})$, where $S_i^\Gamma(\mathbb{F})$ is the space of cuspidal modular forms of weight i for Γ over $\mathbb{F}$ in the sense of Serre and Swinnerton-Dyer, i.e., it is the image of $S_i^\Gamma(\mathcal{O})$ under the reduction map considered above. The map $S_{\le k}^\Gamma(\mathcal{O})/\mathfrak{p}S_{\le k}^\Gamma(\mathcal{O}) \to S_{\le k}^\Gamma(\mathbb{F})$ is surjective but not an isomorphism in general. Let

$$S^\Gamma(\mathcal{O}) = \bigcup_{k=0}^{\infty} S_{\le k}^\Gamma(\mathcal{O}) \quad \text{and} \quad S^\Gamma(\mathbb{F}) = \bigcup_{k=0}^{\infty} S_{\le k}^\Gamma(\mathbb{F}).$$

All the modules considered above have a natural action of the Hecke operators T_n for $(n, Np) = 1$. We denote by $\mathbb{T}_k^\Gamma$ the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_{\le k}^\Gamma(\mathcal{O}))$ generated by the T_n's with $(n, Np) = 1$. We denote by A_k^Γ the $\mathbb{F}$-subalgebra of $\mathrm{End}_{\mathbb{F}}(S_{\le k}^\Gamma(\mathbb{F}))$ generated by the T_n's with $(n, Np) = 1$. From the relations between the Hecke operators, we see that $\mathbb{T}_k^\Gamma$ or A_k^Γ is generated by the Hecke operators T_q and S_q for primes q not dividing Np (see [BK, Section 1.2] for more details). Here, S_q is the operator acting on forms of weight k as the multiplication by $\langle q\rangle q^{k-2}$, where $\langle q\rangle$ is the diamond operator corresponding to q.

We have a natural morphism of $\mathbb{F}$-algebras $\mathbb{T}_k^\Gamma/\mathfrak{p}\mathbb{T}_k^\Gamma \to A_k^\Gamma$, which is surjective, but in general not an isomorphism. We set

$$\mathbb{T}^\Gamma = \varprojlim \mathbb{T}_k^\Gamma \quad \text{and} \quad A^\Gamma = \varprojlim A_k^\Gamma.$$

Thus, the Hecke algebras $\mathbb{T}^\Gamma$ and A^Γ act on $S^\Gamma(\mathcal{O})$ and $S^\Gamma(\mathbb{F})$, respectively. We obtain a surjective map $\mathbb{T}^\Gamma/\mathfrak{p}\mathbb{T}^\Gamma \to A^\Gamma$ from the surjective maps considered above. We call A^Γ the Hecke algebra modulo p of level Γ and this is the central object of our study.

The rings $\mathbb{T}^\Gamma$ and A^Γ are complete and semilocal. Actually, if $\mathbb{F}$ is large enough, then, by the existence of Galois representations attached to eigenforms and by the Deligne–Serre lifting lemma, the maximal ideals, and hence the local components of both $\mathbb{T}^\Gamma$ and A^Γ, are in bijection with the set of isomorphism classes of Γ-modular Galois representations $\bar{\rho}: G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ (see [BK, Section 1.2] for more details). Here and below, Γ-*modular* means that $\bar{\rho}$ is the semisimplified reduction of a stable lattice for the Galois representation $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(K)$ attached by Deligne to an eigenform in $S^\Gamma(\mathcal{O})$. Observe that, since $\bar{\rho}$ is odd, if $\bar{\rho}$ is irreducible, then it is absolutely irreducible. We define $\mathbb{T}^\Gamma_{\bar{\rho}}$ and $A^\Gamma_{\bar{\rho}}$ to be the local components of $\mathbb{T}^\Gamma$ and A^Γ corresponding to a Γ-modular representation $\bar{\rho}$. These rings are complete local rings. The surjective map $\mathbb{T}^\Gamma/\mathfrak{p}\mathbb{T}^\Gamma \to A^\Gamma$ sends $\mathbb{T}^\Gamma_{\bar{\rho}}/\mathfrak{p}\mathbb{T}^\Gamma_{\bar{\rho}}$ onto $A^\Gamma_{\bar{\rho}}$. As A^Γ (resp. $\mathbb{T}^\Gamma$) is semilocal and an inverse limit of artinian rings, A^Γ (resp. $\mathbb{T}^\Gamma$) splits into the product of its local components. Thus, it is enough to study the structure of each local component to understand the structure of A^Γ, once we determine the number of its local components.

The advantage of working with the local components is that one can relate them to suitable deformation rings. By gluing pseudorepresentations attached to modular eigenforms of level Γ lifting the system of eigenvalues corresponding to a Γ-modular representation $\bar{\rho}$, one gets a pseudorepresentation of $G_{\mathbb{Q},Np}$ taking values in $A^\Gamma_{\bar{\rho}}$ and deforming $(\operatorname{tr}\bar{\rho}, \det\bar{\rho})$. Let $\tilde{R}^0_{\bar{\rho}}$ be the universal deformation ring with constant determinant of the pseudorepresentation $(\operatorname{tr}\bar{\rho}, \det\bar{\rho})$ in the category of local, profinite $\mathbb{F}$-algebras with residue field $\mathbb{F}$. The pseudorepresentation obtained above induces a local, surjective morphism $\tilde{R}^0_{\bar{\rho}} \to A^\Gamma_{\bar{\rho}}$ when either $\Gamma = \Gamma_0(N)$ or $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$. Otherwise, it induces a local, surjective map $\tilde{R}^0_{\bar{\rho}} \to (A^{\Gamma_1(N)}_{\bar{\rho}})^{\mathrm{red}}$. See Section 2 for more details.

We define a Γ-modular representation $\bar{\rho}$ to be *unobstructed* if the tangent space of $\tilde{R}^0_{\bar{\rho}}$ has dimension 2. If $\bar{\rho}$ is irreducible and $p \nmid \phi(N)$, then $\bar{\rho}$ is unobstructed in our sense if and only if it is unobstructed in the sense of Mazur [1989, Section 1.6]. But our notion is weaker than Mazur's notion if $p \mid \phi(N)$. See Section 10, where the notion of unobstructedness is studied in detail.

We prove the following results concerning the Hecke algebra $A^\Gamma_{\bar{\rho}}$ and its relation with the deformation ring $\tilde{R}^0_{\bar{\rho}}$:

Theorem 1. *Let $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$. If $\bar{\rho}$ is a Γ-modular representation, then both $\tilde{R}^0_{\bar{\rho}}$ and $A^\Gamma_{\bar{\rho}}$ have Krull dimension at least* 2.

Theorem 2. *Suppose either $\Gamma = \Gamma_0(N)$ or $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$. If $\bar{\rho}$ is a Γ-modular representation which is unobstructed, then the morphism $\tilde{R}^0_{\bar{\rho}} \to A^\Gamma_{\bar{\rho}}$ is an isomorphism, and $A^\Gamma_{\bar{\rho}}$ is isomorphic to a power series ring in two variables $\mathbb{F}[[x, y]]$. If $p \mid \phi(N)$ and $\bar{\rho}$ is an unobstructed $\Gamma_1(N)$-modular representation, then the morphism $\tilde{R}^0_{\bar{\rho}} \to (A^{\Gamma_1(N)}_{\bar{\rho}})^{\mathrm{red}}$ is an isomorphism, and $(A^{\Gamma_1(N)}_{\bar{\rho}})^{\mathrm{red}}$ is isomorphic to a power series ring in two variables $\mathbb{F}[[x, y]]$.*

In the last section, we shall give some conditions under which a $\Gamma_1(N)$-modular representation $\bar{\rho}$ is unobstructed and the corresponding local component $A^{\Gamma_1(N)}_{\bar{\rho}}$

is not reduced: see Proposition 27 and Proposition 28. We shall also give some examples of nonreduced Hecke algebras: see the remarks and discussion after Proposition 28.

Remark. (1) Our approach to prove Theorem 1 and Theorem 2 does not depend on whether $\bar{\rho}$ is irreducible or not. This is mainly because the results that we use from deformation theory of Galois representations are available for both reducible and irreducible $\bar{\rho}$'s. So, the proof does not become simpler when $\bar{\rho}$ is irreducible.

(2) Note that from Theorem 2, it follows that if $\bar{\rho}$ is a $\Gamma_0(N)$-modular representation which is unobstructed and if $p \nmid \phi(N)$, then the natural restriction morphism $A_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_0(N)}$ is an isomorphism. In fact, if $p \nmid \phi(N)$, then this happens for any $\Gamma_0(N)$-modular representation $\bar{\rho}$. Indeed, fixing such a $\bar{\rho}$, we see that the corresponding system of eigenvalues for the diamond operators is trivial. As $p \nmid \phi(N)$, by Hensel's lemma, any system of eigenvalues lifting $\bar{\rho}$ is trivial for the diamond operators. Hence, the diamond operators act trivially on $S^{\Gamma_1(N)}(\mathcal{O})_{\bar{\rho}}$. Therefore, $S^{\Gamma_1(N)}(\mathcal{O})_{\bar{\rho}} = S^{\Gamma_0(N)}(\mathcal{O})_{\bar{\rho}}$. So, $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma_0(N)}$ are isomorphic. Thus, $A_{\bar{\rho}}^{\Gamma_1(N)}$ and $A_{\bar{\rho}}^{\Gamma_0(N)}$ are isomorphic for all the $\Gamma_0(N)$-modular representations $\bar{\rho}$ if p does not divide $\phi(N)$. However, this argument breaks down if p divides $\phi(N)$.

Theorem 3. *Assume that $\bar{\rho}$ is a $\Gamma_1(N)$-modular representation coming from a newform of level N and is absolutely irreducible after restriction to the Galois group of $\mathbb{Q}(\zeta_p)$. If $\ell \mid N$, $p \mid \ell^2 - 1$ and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is unramified, then assume $\ell^2 \mid N$. If $\ell \mid N$, $p \mid \ell - 1$, and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible, ramified and not a sum of two ramified characters, then assume that the highest power of ℓ dividing N is greater than the highest power of ℓ dividing the Artin conductor of $\bar{\rho}$. If $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible, assume, in addition, that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is not isomorphic to $\chi \otimes \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ nor to $\chi \otimes \left(\begin{smallmatrix} 1 & * \\ 0 & \omega_p \end{smallmatrix}\right)$, where χ is any character $G_{\mathbb{Q}_p} \to \mathbb{F}^*$. Then $A_{\bar{\rho}}^{\Gamma_1(N)}$ has Krull dimension 2. Moreover, $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$.*

Similarly, if $\bar{\rho}$ is a $\Gamma_0(N)$-modular representation satisfying the hypotheses as above, then $A_{\bar{\rho}}^{\Gamma_0(N)}$ has Krull dimension 2. Moreover, if p does not divide $\phi(N)$, then $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_0(N)})^{\mathrm{red}}$.

These results are generalizations of Theorem III, Theorem I and Theorem II of [BK], respectively. Note that, it is possible to have mod p deformation rings with constant determinant which are nonreduced, but we do not know of any such examples.

Remark. If $p \nmid N$, then we know that the module of mod p modular forms for $\Gamma_1(N)$ is the same as the module of mod p modular forms for $\Gamma_1(Np^e)$ for $e \geq 1$ (see the remark after Corollary I.3.6 on page 23 of [Gouvêa 1988] for more details).

Hence, the corresponding mod p Hecke algebras are also the same. So, even though we are assuming that $p \nmid N$, all the theorems that we prove for mod p Hecke algebras for $\Gamma_1(N)$ above will still be true without this assumption on N.

Note that, Theorem 2 easily follows from Theorem 1 and the definition of unobstructed.

The idea of the proof of Theorem 1 is similar to the idea used in proving Theorem III in [BK]. So, we try to find a relation between the characteristic-0 Hecke algebra $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ with the characteristic-p Hecke algebra $A^{\Gamma}_{\bar{\rho}}$. However, there are some difficulties, such as $S^{\Gamma}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$ not being isomorphic to $S^{\Gamma}(\mathbb{F})$, in comparing them directly. To overcome this problem, like Bellaïche and Khare [2015], we also work with the divided congruence modules of Katz. Using the results of Katz and the methods of [BK], we get the relation between characteristic-0 and characteristic-p *full* Hecke algebras, i.e., the relation between the characteristic-0 and characteristic-p Hecke algebras generated by the Hecke operators T_q, qS_q for primes q not dividing Np, U_ℓ for primes ℓ dividing N and U_p.

Now, we need to analyze how the addition of the U_ℓ and U_p operators changes our Hecke algebras in characteristic 0 and p. We can control the change caused by the U_p operator in a similar way to that done in the level-1 case in [BK]. This allows us to get a relation between Hecke algebras in characteristic 0 and p generated by Hecke operators away from p. Note that, the proof for the $N = 1$ case finishes at this step. However, for $N > 1$, we still need to study the effect of adding the extra operators U_ℓ for primes ℓ dividing N to our original Hecke algebras. This differentiates the case of $N > 1$ from $N = 1$. In this direction, we prove that in characteristic 0, if $\bar{\rho}$ is new, i.e., if $\bar{\rho}$ is not $\Gamma(N')$-modular for any proper divisor N' of N, then the operator U_ℓ acting on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ is integral over $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ for every prime ℓ dividing N. This gives us the finiteness of the U_ℓ's over the mod p Hecke algebra automatically for a new $\bar{\rho}$ and, along with the Gouvêa–Mazur infinite fern argument, leads us to Theorem 1 in those cases. Finally, we prove that if $\bar{\rho}$ is also a $\Gamma(M)$-modular representation for some M dividing N, then the natural map $A^{\Gamma}_{\bar{\rho}} \to A^{\Gamma(M)}_{\bar{\rho}}$ is surjective using the pseudorepresentation attached to it and the Chebotarev density theorem. Theorem 1 in the case where $\bar{\rho}$ is new, along with the surjectivity established above, leads us to Theorem 1 for all the local components of the mod p Hecke algebra. We would like to point out that, in contrast with [BK], our method does not give the precise kernel of the map $\mathbb{T}^{\Gamma}_{\bar{\rho}} \to A^{\Gamma}_{\bar{\rho}}$ in all the cases while proving Theorem 1. However, in many cases, we can find the kernel up to some nilpotence.

To prove Theorem 3, we need to use a result of Böckle and flatness of $\mathbb{T}^{\Gamma_1(N)}_{\bar{\rho}}$ over the Iwasawa algebra $\mathcal{O}[\![T]\!]$ which is proved in the same way as in [BK]. This, along with Theorem 1, would imply the first part of Theorem 3. We prove, using techniques and results similar to the ones sketched in the previous paragraph, that the map $\mathbb{T}_{\bar\rho}^{\Gamma_1(N)}/(\mathfrak{p}, T) \to A_{\bar\rho}^{\Gamma_1(N)}$ has nilpotent kernel if $\bar\rho$ satisfies the hypotheses of Theorem 3. We use it, along with the results of Böckle, to conclude the second part of Theorem 3 for the $\Gamma_1(N)$ case. The theorem for the $\Gamma_0(N)$ case then follows easily from the theorem for the $\Gamma_1(N)$ case.

Finally, we would like to remark on a possible generalization of these results to the case of Hilbert modular forms. A well behaved theory for mod p Hilbert modular forms with arbitrary weights, which includes a lot of tools and facts for mod p modular forms that we use, is available (see the works of Andreatta and Goren [2005] for more details). Moreover, an analogue of the Gouvêa–Mazur infinite fern argument is also true for certain local components of the p-adic Hecke algebra acting on the space of Hilbert modular forms due to work of Chenevier [2011, Theorem 5.9]. However, contrary to the case of modular forms, the theory of divided congruence modules of Katz, and properties of Hecke operators acting on them, is not known for Hilbert modular forms. We expect results similar to what we have proved above to hold for Hilbert modular forms once we know the theory of divided congruence modules of Hilbert modular forms and the infinite fern argument for all the local components. The results will depend on the space of mod p Hilbert modular forms we consider. For instance, we expect the lower bound on the Krull dimension of the mod p Hecke algebra for Hilbert modular forms of parallel weights to be 2, while the corresponding lower bound for the mod p Hecke algebra for Hilbert modular forms of arbitrary weights to be $2n$, where n is the degree of extension of the totally real field over $\mathbb{Q}$.

2. Deformation rings and Hecke algebras

The goal of this section is to relate mod p Hecke algebras with appropriate deformation rings.

Using [Bellaïche 2012b, Step 1 of the proof of Theorem 1], which is essentially an argument of gluing pseudorepresentations attached to modular eigenforms of a fixed level and all weights, we get the following lemma (see also [BK, Proposition 2]):

Lemma 4. *Let Γ be either $\Gamma_0(N)$ or $\Gamma_1(N)$. For a Γ-modular representation $\bar\rho$, there exists a unique continuous pseudorepresentation*

$$(\tau^\Gamma, \delta^\Gamma) : G_{\mathbb{Q},Np} \to \mathbb{T}_{\bar\rho}^\Gamma$$

such that $\tau^\Gamma(c) = 0$, $\tau^\Gamma(\mathrm{Frob}_q) = T_q$ and $\delta^\Gamma(\mathrm{Frob}_q) = qS_q$ for all the primes q not dividing Np. We have

$$\tau^\Gamma \pmod{m_{\mathbb{T}_{\bar\rho}^\Gamma}} = \operatorname{tr}\bar\rho \quad \textit{and} \quad \delta^\Gamma \pmod{m_{\mathbb{T}_{\bar\rho}^\Gamma}} = \det\bar\rho.$$

By composing $(\tau^\Gamma, \delta^\Gamma)$ with the natural morphism $\mathbb{T}_{\bar\rho}^\Gamma \to A_{\bar\rho}^\Gamma$, we get a pseudorepresentation $(\tilde\tau^\Gamma, \tilde\delta^\Gamma) : G_{\mathbb{Q},Np} \to A_{\bar\rho}^\Gamma$ lifting $(\operatorname{tr}\bar\rho, \det\bar\rho)$.

A deformation (t, d) of a pseudorepresentation $(\bar{t}, \bar{d})$ is called a deformation with constant determinant if $d = \bar{d}$.

Lemma 5. *If* $\Gamma = \Gamma_0(N)$ *or if* $p \nmid \phi(N)$ *and* $\Gamma = \Gamma_1(N)$, *then for a* Γ*-modular representation* $\bar{\rho}$, $(\tilde{\tau}^\Gamma, \tilde{\delta}^\Gamma)$ *is a deformation of* $(\operatorname{tr}\bar{\rho}, \det\bar{\rho})$ *with constant determinant.*

Proof. Let q be a prime not dividing Np. Note that, in the $\Gamma_0(N)$ case, S_q acts as multiplication by q^{k-2} on a weight k modular form with Fourier coefficients in $\mathcal{O}$. If two modular forms of level N are congruent modulo $\mathfrak{p}$, then their weights are congruent modulo $p-1$. So, S_q acts like a constant on $S^{\Gamma_0(N)}(\mathbb{F})_{\bar{\rho}}$ for every $\bar{\rho}$. Since $\tilde{\delta}^\Gamma(\mathrm{Frob}_q) = qS_q$ and the set of Frob_q for primes q not dividing Np is dense in $G_{\mathbb{Q},Np}$, we get that $\tilde{\delta}^\Gamma$ is constant and we are done in the $\Gamma_0(N)$ case.

Applying the same reasoning to the $\Gamma_1(N)$ case, we get that, for every prime q not dividing Np, S_q acts like $c_q\langle q\rangle$ on $S^{\Gamma_1(N)}(\mathbb{F})_{\bar{\rho}}$, where c_q is an invertible constant. Since $S_q \in A_{\bar{\rho}}^{\Gamma_1(N)}$, it follows that $\langle q\rangle \in A_{\bar{\rho}}^{\Gamma_1(N)}$ for every prime q not dividing Np. If p does not divide $\phi(N)$, then the order of every diamond operator $\langle q\rangle$ is coprime to p. We have chosen $\mathbb{F}$ to be large enough so that it contains all the mod p system of eigenvalues. Thus, by Hensel's lemma, $\langle q\rangle$, and hence S_q, will be constant in $A_{\bar{\rho}}^{\Gamma_1(N)}$ for every $\bar{\rho}$ and every prime q not dividing Np. Therefore, $\tilde{\delta}^\Gamma$ is constant in the $\Gamma_1(N)$ case if p does not divide $\phi(N)$, by the same argument as in the $\Gamma_0(N)$ case. □

If $p \mid \phi(N)$, then the determinant $\tilde{\delta}^\Gamma$ may not be constant. See Section 10 for more details. If the order of $\langle q\rangle$ is p^e, then $(\langle q\rangle - 1)^{p^e} = 0$ in $A_{\bar{\rho}}^{\Gamma_1(N)}$. Therefore, in $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$, we have $\langle q\rangle = 1$ for all such $\langle q\rangle$. Thus, from the proof of the lemma above, it follows that if $p \mid \phi(N)$, then the determinant

$$(\tilde{\delta}^{\Gamma_1(N)})^{\mathrm{red}} : G_{\mathbb{Q},Np} \to ((A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}})^*$$

is constant.

Let $R_{\bar{\rho}}$ be the universal deformation ring of the pseudorepresentation $(\operatorname{tr}\bar{\rho}, \det\bar{\rho})$ in the category of local profinite $\mathcal{O}$-algebras with residue field $\mathbb{F}$, $\tilde{R}_{\bar{\rho}}$ be the corresponding universal deformation ring mod $\mathfrak{p}$ and $\tilde{R}^0_{\bar{\rho}}$ be the corresponding universal deformation ring mod $\mathfrak{p}$ with constant determinant (see [Chenevier 2014; BK, Section 1.4] for more details regarding the existence and properties of these rings). For a Γ-modular representation $\bar{\rho}$, the pseudorepresentation $(\tau^\Gamma, \delta^\Gamma)$ defines a local morphism $R_{\bar{\rho}} \to \mathbb{T}^\Gamma_{\bar{\rho}}$ which is the identity, modulo their maximal ideals. Similarly, we get a local morphism $\tilde{R}_{\bar{\rho}} \to A^\Gamma_{\bar{\rho}}$. From the previous paragraphs, we see that the morphism $\tilde{R}_{\bar{\rho}} \to A^\Gamma_{\bar{\rho}}$ factors through $\tilde{R}^0_{\bar{\rho}}$ if $\Gamma = \Gamma_0(N)$ or if $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$. However, this is not true in general. But, from the above discussion, we see that for a $\Gamma_1(N)$-modular representation $\bar{\rho}$, the map $\tilde{R}_{\bar{\rho}} \to (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ factors through $\tilde{R}^0_{\bar{\rho}}$. All the morphisms considered above are surjective. Indeed, for a prime q not dividing Np, the images of the trace and the determinant, coming from the universal pseudorepresentation of Frob_q, under the morphisms above are T_q and qS_q, respectively.

3. Relation between the full Hecke algebras in characteristic 0 and p

The goal of this section is to obtain a relation, similar to [BK, Proposition 16], between the full Hecke algebras, but for both the cases, $\Gamma_1(N)$ and $\Gamma_0(N)$. We follow the approach of [BK]. We briefly recall, without proofs, some important results of the theory of divided congruences of Katz needed for our purpose. Then, we state some results about the comparisons of various Hecke algebras; these are similar to the results given in Sections 3–6 of [BK], and since their proofs are more or less the same, we do not give them in full detail here. Instead, we mostly refer readers to the proofs of corresponding results in [BK] and provide some additional details when required. In this section, we follow the notation of [BK] for the divided congruence modules and the Hecke algebras along with an additional index representing the level. For instance, $D^{\Gamma_1(N)}(\mathcal{O})$ will represent the divided congruence module of cuspidal forms for $\Gamma_1(N)$ over $\mathcal{O}$. These modules and the Hecke algebras acting on them are defined in exactly the same way as their level-1 counterparts in [BK] after making appropriate level changes. The main references for this section are [Katz 1975; Hida 1986; BK]. Throughout this section, Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$.

3.1. ***The divided congruence modules of Katz.*** In this and the following subsection, we quickly list all the results that are needed from the theory of divided congruence modules. These results are the level-N counterparts of the results that appear in [BK].

We now define the divided congruence modules of Katz. Let $S^{\Gamma}_{\leq k}(K)$ be the subspace of $K[\![q]\!]$ given by $\sum_{i=0}^{i=k} S^{\Gamma}_i(K)$, where $S^{\Gamma}_i(K)$ is the space of cusp forms of weight i and level Γ with Fourier coefficients in K, which is identified as a subspace of $K[\![q]\!]$ via q-expansions. Let $D^{\Gamma}_{\leq k}(\mathcal{O})$ be the $\mathcal{O}$-submodule of $\mathcal{O}[\![q]\!]$ given by the intersection of $S^{\Gamma}_{\leq k}(K)$ with $\mathcal{O}[\![q]\!]$. These modules are called *divided congruence modules* because they capture congruences between cusp forms of different weights (see Remark 3 of Section 2 of [BK] for more details). We define $D^{\Gamma}_{\leq k}(\mathbb{F})$ as the image of $D^{\Gamma}_{\leq k}(\mathcal{O})$ under the reduction map $\mathcal{O}[\![q]\!] \to \mathbb{F}[\![q]\!]$. Let

$$D^{\Gamma}(\mathcal{O}) = \bigcup_{k=0}^{\infty} D^{\Gamma}_{\leq k}(\mathcal{O}) \quad \text{and} \quad D^{\Gamma}(\mathbb{F}) = \bigcup_{k=0}^{\infty} D^{\Gamma}_{\leq k}(\mathbb{F}).$$

We call $D^{\Gamma}(\mathcal{O})$ the divided congruence module of cuspidal forms of level Γ and $D^{\Gamma}(\mathbb{F})$ the divided congruence module of cuspidal forms modulo p of level Γ. See Section 2 of [BK] for more details.

Lemma 6. *The natural map $D^{\Gamma}_{\leq k}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \to D^{\Gamma}_{\leq k}(\mathbb{F})$ is an isomorphism.*

Proof. If $f \in D^{\Gamma}_{\leq k}(\mathcal{O})$ lies in the kernel of the natural surjective map

$$D^{\Gamma}_{\leq k}(\mathcal{O}) \to D^{\Gamma}_{\leq k}(\mathbb{F}),$$

then f/π lies in both $S^{\Gamma}_{\le k}(K)$ and $\mathcal{O}[\![q]\!]$. Hence, it lies in $D^{\Gamma}_{\le k}(\mathcal{O})$ which implies the lemma. See the proof of Lemma 5 of [BK]. $\square$

We now recall, without proof, two theorems of Katz which relate the divided congruence module of cuspidal forms mod p of level Γ with the space of mod p cuspforms of level Γ:

Proposition 7 [Katz 1975, Corollary 1.7]. (1) *There exists a unique action of $\mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$ on $D^{\Gamma_1(N)}(\mathcal{O})$, denoted by $((x, y), f) \mapsto (x, y).f$, such that for $(x, y) \in \mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$, and $f \in S_k^{\Gamma_1(N)}(\mathcal{O}) \subset D^{\Gamma_1(N)}(\mathcal{O})$,*

$$(x, y).f = x^k \langle y\rangle f.$$

(2) *There exists a unique action of $\mathbb{Z}_p^*$ on $D^{\Gamma_0(N)}(\mathcal{O})$ denoted by $(x, f) \mapsto x.f$ such that for $x \in \mathbb{Z}_p^*$, and $f \in S_k^{\Gamma_0(N)}(\mathcal{O}) \subset D^{\Gamma_0(N)}(\mathcal{O})$,*

$$x.f = x^k f.$$

Theorem 8 [Katz 1975, Section 4]. *The space $S^{\Gamma}(\mathbb{F})$ is the space of invariants of $1 + p\mathbb{Z}_p$ acting on $D^{\Gamma}(\mathbb{F})$.*

See also [Hida 1986, Theorem 1.1].

The two results of Katz recalled in this subsection are proved only for $p > 3$. We do not know whether they also hold for $p = 2, 3$.

3.2. ***Hecke operators on the divided congruence modules.*** On $D^{\Gamma}(\mathcal{O})$ we can define the Hecke operators T_q and S_q, for primes q not dividing Np, and U_ℓ, for primes ℓ dividing N. Their action on the q-expansions is given in the same way as it is given on the q-expansions of the classical modular forms. See the proof of Corollary-and-Definition 7 of [BK] and [Hida 1986, page 243] for more details.

Now we introduce *partially full* Hecke algebras which are generated by the U_ℓ's for every prime ℓ dividing N along with the T_n's for $(n, Np) = 1$. We denote these Hecke algebras by supplementing the index with pf. Thus, $\mathbb{T}_k^{\Gamma,\mathrm{pf}}$ is the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S^{\Gamma}_{\le k}(\mathcal{O}))$ generated by the T_n's with $(n, Np) = 1$ and U_ℓ's with prime ℓ dividing N, while $A_k^{\Gamma,\mathrm{pf}}$ is the $\mathbb{F}$-subalgebra of $\mathrm{End}_{\mathbb{F}}(S^{\Gamma}_{\le k}(\mathbb{F}))$ generated by the T_n's with $(n, Np) = 1$ and U_ℓ's with prime ℓ dividing N. We can consider the projective limits

$$\mathbb{T}^{\Gamma,\mathrm{pf}} = \varprojlim \mathbb{T}_k^{\Gamma,\mathrm{pf}}, \quad A^{\Gamma,\mathrm{pf}} = \varprojlim A_k^{\Gamma,\mathrm{pf}}.$$

Lemma 9. (1) *The subalgebra of $\mathrm{End}_{\mathcal{O}}(D^{\Gamma}_{\le k}(\mathcal{O}))$ generated by the Hecke operators T_q, S_q for primes q not dividing Np and U_ℓ for primes ℓ dividing N is naturally isomorphic to $\mathbb{T}_k^{\Gamma,\mathrm{pf}}$.*

(2) *The subalgebra of $\mathrm{End}_{\mathcal{O}}(D^{\Gamma}_{\le k}(\mathcal{O}))$ generated by the Hecke operators T_q, S_q for primes q not dividing Np is naturally isomorphic to $\mathbb{T}_k^{\Gamma}$.*

Proof. Both parts of the lemma follow from the observations that $S^{\Gamma}_{\le k}(\mathcal{O})$ is a cotorsion submodule of $D^{\Gamma}_{\le k}(\mathcal{O})$ and that the action of Hecke operators on $D^{\Gamma}_{\le k}(\mathcal{O})$ extends their action on $S^{\Gamma}_{\le k}(\mathcal{O})$. Indeed, as $S^{\Gamma}_{\le k}(\mathcal{O})$ is a cotorsion submodule of $D^{\Gamma}_{\le k}(\mathcal{O})$, if $f \in D^{\Gamma}_{\le k}(\mathcal{O})$, then $\pi^n f \in S^{\Gamma}_{\le k}(\mathcal{O})$ for some n which implies that a Hecke operator vanishing on $S^{\Gamma}_{\le k}(\mathcal{O})$ also vanishes on $D^{\Gamma}_{\le k}(\mathcal{O})$. See the proof of Lemma 8 of [BK] for more details. □

Lemma 10. (1) *The homomorphism* $\phi : \mathbb{Z}_p^* \to \mathrm{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$, *defined by* $\phi(x)f = (x,1).f$ *for* $f \in D^{\Gamma_1(N)}(\mathcal{O})$, *takes values in the subalgebra* $\mathbb{T}^{\Gamma_1(N)}$ *and hence, in* $\mathbb{T}^{\Gamma_1(N),\mathrm{pf}}$.

(2) *The homomorphism* $\phi : \mathbb{Z}_p^* \to \mathrm{End}_{\mathcal{O}}(D^{\Gamma_0(N)}(\mathcal{O}))$, *defined by* $\phi(x)f = x.f$ *for* $f \in D^{\Gamma_0(N)}(\mathcal{O})$, *takes values in the subalgebra* $\mathbb{T}^{\Gamma_0(N)}$ *and hence, in* $\mathbb{T}^{\Gamma_0(N),\mathrm{pf}}$.

Proof. The proof of part (1) is almost the same as the proof of Lemma 9 of [BK]. We only need to change the last step of the proof slightly, so we sketch it briefly here. Following the same proof, we see that $\mathbb{T}^{\Gamma_1(N)}$ is a closed subset of $\mathrm{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$ under the weak topology, and the map $\phi : \mathbb{Z}_p^* \to \mathrm{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$ is continuous for the weak topology. For a prime q which does not divide Np and is $1 \pmod N$, one has $\phi(q) = q^2 S_q \in \mathbb{T}^{\Gamma_1(N)}$, since $\langle q \rangle$ is the trivial operator as q is $1 \pmod N$. If $x \in \mathbb{Z}_p^*$, there exists, by the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions, a sequence of primes q_n, (different from primes dividing Np) that are $1 \pmod N$, that converges to x p-adically. Hence, $\phi(q_n)$ converges to $\phi(x)$ in $\mathrm{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$. Therefore, $\phi(x) \in \mathbb{T}^{\Gamma_1(N)}$. The proof of part (2) is the same as the proof of Lemma 9 of [BK]. □

Let Λ be the Iwasawa algebra $\mathcal{O}[[1+p\mathbb{Z}_p]]$. By choosing a topological generator of $1+p\mathbb{Z}_p$, say $1+p$, one gets an isomorphism $\Lambda \simeq \mathcal{O}[[T]]$. Under this isomorphism, the maximal ideal m_Λ of Λ gets mapped to (π, T). We get a morphism $\psi : \Lambda \to \mathbb{T}^{\Gamma}$ of $\mathcal{O}$-algebras from the group homomorphism $\phi : 1+p\mathbb{Z}_p \to (\mathbb{T}^{\Gamma})^*$. Using the morphism ψ, we can consider $\mathbb{T}^{\Gamma}$ as a Λ-algebra.

3.3. *Divided congruence modules of level* $\Gamma_0(Np)$ *and* $\Gamma_1(Np)$.

In this subsection, we consider the divided congruence modules of cuspidal forms for levels $\Gamma_0(Np)$ and $\Gamma_1(Np)$. They are defined in the same way as in the level-N case after just changing the level. For the rest of this section, Γ still means $\Gamma_0(N)$ or $\Gamma_1(N)$ and $\Gamma(p)$ means either $\Gamma_0(Np)$ or $\Gamma_1(Np)$ accordingly.

Proposition 11. *The closures of* $D^{\Gamma}(\mathcal{O})$ *and* $D^{\Gamma(p)}(\mathcal{O})$ *in* $\mathcal{O}[[q]]$ *provided with the topology of uniform convergence, are equal.*

Proof. See [Gouvêa 1988, Proposition I.3.9] and Section 1 of [Hida 1986]. □

Corollary 12. *There is an isomorphism preserving* q*-expansions of*

$$D^{\Gamma(p)}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \simeq D^{\Gamma}(\mathbb{F}).$$

Corollary 13. *The algebras $\mathbb{T}^{\Gamma(p),\mathrm{pf}}$ and $\mathbb{T}^{\Gamma,\mathrm{pf}}$ are naturally isomorphic, as are also the algebras $\mathbb{T}^{\Gamma(p)}$ and $\mathbb{T}^{\Gamma}$.*

Proof. The natural restriction maps between the Hecke algebras of level Np and N are isomorphisms because their action is continuous and the modules on which they act have the same closure. See the proof of Corollary 13 of [BK] for more details. □

3.4. ***Full Hecke algebras.*** In this subsection, we consider *full* Hecke algebras, i.e., the Hecke algebras generated by all the operators T_q, S_q for primes q with $(q, Np) = 1$, U_ℓ for primes ℓ dividing N and U_p. So, $\mathbb{T}_k^{\Gamma(p),\mathrm{full}}$, $DA_k^{\Gamma,\mathrm{full}}$ and $A_k^{\Gamma,\mathrm{full}}$ are the Hecke algebras generated by the Hecke operators T_q, S_q for primes q with $(q, Np) = 1$, U_ℓ for primes ℓ dividing N and U_p acting on $D_{\leq k}^{\Gamma(p)}(\mathcal{O})$, $D_{\leq k}^{\Gamma}(\mathbb{F})$ and $S_{\leq k}^{\Gamma}(\mathbb{F})$, respectively. We denote by $\mathbb{T}^{\Gamma(p),\mathrm{full}}$, $DA^{\Gamma,\mathrm{full}}$ and $A^{\Gamma,\mathrm{full}}$ the full Hecke algebras acting on $D^{\Gamma(p)}(\mathcal{O})$, $D^{\Gamma}(\mathbb{F})$ and $S^{\Gamma}(\mathbb{F})$, respectively, which are obtained by taking the inverse limits over the weights k of appropriate Hecke algebras, as before.

Proposition 14 (perfect duality). *The pairings*

$$\mathbb{T}_k^{\Gamma(p),\mathrm{full}} \times D_{\leq k}^{\Gamma(p)}(\mathcal{O}) \to \mathcal{O}, \quad DA_k^{\Gamma,\mathrm{full}} \times D_{\leq k}^{\Gamma}(\mathbb{F}) \to \mathbb{F}, \quad A_k^{\Gamma,\mathrm{full}} \times S_{\leq k}^{\Gamma}(\mathbb{F}) \to \mathbb{F},$$

given by $(t, f) \mapsto a_1(tf)$, are perfect.

Proof. This is well known but we recall the proof here. Suppose $f \in D_{\leq k}^{\Gamma(p)}(\mathcal{O})$ is such that $a_1(tf) = 0$ for all $t \in \mathbb{T}_k^{\Gamma(p),\mathrm{full}}$. This means $a_1(T_n f) = a_n(f) = 0$ for all n coprime to Np, $a_1(U_\ell f) = a_\ell(f) = 0$ for all primes ℓ dividing N and $a_1(U_p f) = a_p(f) = 0$. Thus, we have $a_n(f) = 0$ for all n which implies $f = 0$. Now, suppose $t \in \mathbb{T}_k^{\Gamma(p),\mathrm{full}}$ is such that $a_1(tf) = 0$ for all $f \in D_{\leq k}^{\Gamma(p)}(\mathcal{O})$. This means $a_1(s(tf)) = a_1(t(sf)) = 0$ for all $s \in \mathbb{T}_k^{\Gamma(p),\mathrm{full}}$. Thus, from the previous part, we get that $tf = 0$ for all $f \in D_{\leq k}^{\Gamma(p)}(\mathcal{O})$ which means $t = 0$. The proof for other cases goes in the exact same way. □

Corollary 15. *The map $\mathbb{T}_k^{\Gamma(p),\mathrm{full}} \to DA_k^{\Gamma,\mathrm{full}}$ induces an isomorphism*

$$\mathbb{T}_k^{\Gamma(p),\mathrm{full}} \otimes_{\mathcal{O}} \mathbb{F} \xrightarrow{\sim} DA_k^{\Gamma,\mathrm{full}}.$$

Hence, we get an isomorphism $\mathbb{T}^{\Gamma(p),\mathrm{full}} \otimes_{\mathcal{O}} \mathbb{F} \simeq DA^{\Gamma,\mathrm{full}}$.

Proof. By the perfect duality above and Lemma 6, we see that the rank of the torsion-free $\mathcal{O}$-module $\mathbb{T}_k^{\Gamma(p),\mathrm{full}}$ and the dimension of $DA_k^{\Gamma,\mathrm{full}}$ as an $\mathbb{F}$-vector space are the same, which implies the result. See the proof of Corollary 15 of [BK] for more details. □

The composition $\Lambda \to \mathbb{T}^{\Gamma,\mathrm{pf}} \to \mathbb{T}^{\Gamma(p),\mathrm{full}}$ defines a Λ-algebra structure on $\mathbb{T}^{\Gamma(p),\mathrm{full}}$.

Proposition 16. $\mathbb{T}^{\Gamma(p),\mathrm{full}}/m_\Lambda \mathbb{T}^{\Gamma(p),\mathrm{full}} \simeq A^{\Gamma,\mathrm{full}}$.

Proof. The proposition follows from Theorem 8 (a theorem of Katz that we recalled in Section 3.1), the perfect duality and its corollary above. See the proof of Proposition 16 of [BK] for more details. □

4. Relation between the components of Partial and Full Hecke algebras

Throughout this section Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(p)$ means either $\Gamma_1(Np)$ or $\Gamma_0(Np)$ accordingly. Recall that we have a direct product decomposition $\mathbb{T}^{\Gamma} = \prod \mathbb{T}^{\Gamma}_{\bar{\rho}}$ and a direct sum decomposition $S^{\Gamma}(\mathcal{O}) = \bigoplus S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$, where the product and sum are taken over all the Γ-modular representations. Note that, $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ is the intersection of the subspace of $S^{\Gamma}(\overline{\mathbb{Q}_p})$ generated by the eigenforms lifting the system of eigenvalues corresponding to $\bar{\rho}$ with $S^{\Gamma}(\mathcal{O})$. The decompositions above are such that $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ is also the largest quotient of $\mathbb{T}^{\Gamma}$ which acts faithfully on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$. Let $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ be the largest quotient of $\mathbb{T}^{\Gamma,\mathrm{pf}}$ which acts faithfully on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$. Define $A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ in a similar way.

Note that, since $D^{\Gamma}(\mathcal{O})$ contains $S^{\Gamma}(\mathcal{O})$ as a cotorsion submodule, we have a direct sum decomposition $D^{\Gamma}(\mathcal{O}) = \bigoplus D^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ similar to that of $S^{\Gamma}(\mathcal{O})$. Moreover, $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ and $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ are the largest of quotients of $\mathbb{T}^{\Gamma,\mathrm{pf}}$ and $\mathbb{T}^{\Gamma}$ respectively, acting faithfully on $D^{\Gamma}(\mathcal{O})_{\bar{\rho}}$.

By a result of Serre and Tate (see [Jochnowitz 1982, Lemma 4.4]), the subalgebras of $\mathbb{T}^{\Gamma(p),\mathrm{full}}$ and $A^{\Gamma,\mathrm{full}}$ generated by the Hecke operators T_q, S_q for primes q not dividing Np and U_p are semilocal and the local components of both of them are in bijection with the set of $\mathbb{F}$-valued systems of eigenvalues of the Hecke operators T_q, S_q for primes q not dividing Np and U_p appearing in $S^{\Gamma}(\mathbb{F})$. Hence, they are in bijection with the pairs $(\bar{\rho}, \lambda)$, where $\bar{\rho} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ is a Γ-modular representation attached to some eigenform $f \in S^{\Gamma}(\mathbb{F})$ and λ is the eigenvalue of U_p on f (see [Jochnowitz 1982]).

So, we get a direct sum decomposition $D^{\Gamma(p)}(\mathcal{O}) = \bigoplus D^{\Gamma(p)}(\mathcal{O})_{\bar{\rho},\lambda}$ similar to that of $S^{\Gamma}(\mathcal{O})$ seen above. Now, let us define $\mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},\lambda}$ to be the largest quotient of $\mathbb{T}^{\Gamma(p),\mathrm{full}}$ acting faithfully on $D^{\Gamma(p)}(\mathcal{O})_{\bar{\rho},\lambda}$ and $A^{\Gamma,\mathrm{full}}_{\bar{\rho},\lambda}$ to be the largest quotient of $A^{\Gamma,\mathrm{full}}$ acting faithfully on $S^{\Gamma}(\mathbb{F})_{\bar{\rho},\lambda}$.

We get, using the Chinese remainder theorem and the definitions above, the product decompositions $\mathbb{T}^{\Gamma(p),\mathrm{full}} = \prod \mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},\lambda}$, and $\mathbb{T}^{\Gamma,\mathrm{pf}} = \prod \mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$. Similarly, we get product decompositions of $A^{\Gamma,\mathrm{full}}$ and $A^{\Gamma,\mathrm{pf}}$. Here, the products are finite products as the pairs $(\bar{\rho}, \lambda)$ are finitely many.

Proposition 17. (1) *For a Γ-modular representation $\bar{\rho}$, one has a natural isomorphism of $A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$-algebras $A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}[\![U_p]\!] \simeq A^{\Gamma,\mathrm{full}}_{\bar{\rho},0}$.*

(2) *For a Γ-modular representation $\bar{\rho}$, one has a natural isomorphism of $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$-algebras $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}[\![U_p]\!] \simeq \mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},0}$.*

Proof. The first part of this proposition is proved by Jochnowitz [1982, Theorem 6.3]. The proof of the second part of the proposition is the same as that of Proposition 17 of [BK], where they adapt the proof of Jochnowitz in characteristic 0, which relies on the interplay between the operators U_p and V. Here, V is the operator which sends $\sum a_n q^n$ to $\sum a_n q^{pn}$. The same argument works here. $\square$

5. Finiteness of $\mathbb{T}_{\bar\rho}^{\Gamma,\mathrm{pf}}$ over $\mathbb{T}_{\bar\rho}^{\Gamma}$ for new $\bar\rho$

Throughout this section, Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(M)$ means either $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. Let us call a Γ-modular representation $\bar\rho$ *new* if it is not $\Gamma(M)$-modular for any proper divisor M of N. Thus, the system of eigenvalues of the Hecke operators corresponding to $\bar\rho$ does not have a nontrivial eigenspace in $S^{\Gamma(M)}(\mathbb{F})$ for any proper divisor M of N. Let $S_k^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p)$ be the $\mathcal{O}$-submodule of $S_k^{\Gamma}(\overline{\mathbb{Q}}_p)$ consisting of new modular forms of weight k and level Γ. Let

$$S_{\le k}^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p)=\sum_{i=0}^{k} S_i^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p) \quad\text{and}\quad S^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p)=\bigcup_{k=0}^{\infty} S_{\le k}^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p).$$

The following lemma follows directly from the discussion above and the description of $S^{\Gamma}(\mathcal{O})_{\bar\rho}$ given in the previous section:

Lemma 18. *If $\bar\rho$ is a new Γ-modular representation, then $S^{\Gamma}(\mathcal{O})_{\bar\rho}$ is an $\mathcal{O}$-submodule of $S^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p)$.*

Now we recall a well-known result regarding the Galois representations attached to level-N newforms:

Lemma 19 [Emerton et al. 2006, Lemma 2.6.1]. *Let f be a classical newform of tame level N over $\overline{\mathbb{Q}}_p$. Let ρ_f be the p-adic Galois representation attached to f and let V_f denote its underlying space. Let ℓ be a prime dividing N and let $a_\ell(f)$ be the U_ℓ eigenvalue of f. Let $(V_f)_{I_\ell}$ be the vector space of I_ℓ coinvariants of V_f. Then the following are equivalent*:

(1) *$a_\ell(f)$ is a nonunit*,

(2) *$a_\ell(f)=0$*,

(3) *$(V_f)_{I_\ell}=0$.*

If these equivalent conditions do not hold, then we have that $(V_f)_{I_\ell}$ is one dimensional and $a_\ell(f)$ is equal to the eigenvalue of Frob_ℓ *acting on this line.*

Proposition 20. *If $\bar\rho$ is a new Γ-modular representation, then $\mathbb{T}_{\bar\rho}^{\Gamma,\mathrm{pf}}$ is finite over $\mathbb{T}_{\bar\rho}^{\Gamma}$. More precisely, U_ℓ (for every $\ell \mid N$) is integral over $\mathbb{T}_{\bar\rho}^{\Gamma}$ of degree at most 2. If one of the following condition holds*:

(1) *$\Gamma=\Gamma_0(N)$ and there does not exist a prime ℓ such that $\ell \parallel N$ and $p \mid \ell+1$.*

(2) $\Gamma = \Gamma_1(N)$ *and if there exists a prime* ℓ *such that* $p \mid \ell^2 - 1$ *and* $\ell \parallel N$, *then* $p \mid \ell + 1$ *and* $\det \bar{\rho}(I_\ell) \neq 1$.

then $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}} = \mathbb{T}^{\Gamma}_{\bar{\rho}}$.

Proof. Let $\Gamma = \Gamma_0(N)$. Since $\bar{\rho}$ is new, we know, from Lemma 18, that

$$S^{\Gamma}(\mathcal{O})_{\bar{\rho}} \subset S^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p).$$

Now, $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ is the largest quotient of $\mathbb{T}^{\Gamma,\mathrm{pf}}$ which acts faithfully on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$. Let ℓ be a prime dividing N. By Theorem 5 of [Atkin and Lehner 1970], if ℓ^2 divides N, then U_ℓ acts like 0 on $S_k^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p)$ and if $\ell \parallel N$, then U_ℓ^2 acts like ℓ^{k-2} on $S_k^{\Gamma,\mathrm{new}}(\overline{\mathbb{Q}}_p)$. Hence, if ℓ^2 divides N, then U_ℓ acts like 0 on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ and if $\ell \parallel N$, then U_ℓ^2 acts like $\ell^{-2}\phi(\ell)$ on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$, where $\phi : \mathbb{Z}_p^* \to \mathbb{T}^{\Gamma}_{\bar{\rho}}$ is the map considered in Lemma 10. Thus, if $\ell^2 \mid N$, then $U_\ell = 0$, and if $\ell \parallel N$, then $U_\ell^2 - \ell^{-2}\phi(\ell) = 0$ in $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$.

Suppose $\ell \parallel N$ and let f be a newform of level $\Gamma_0(N)$ lifting the system of eigenvalues corresponding to $\bar{\rho}$. Let π_ℓ be the ℓ-component of the automorphic representation corresponding to f and ρ_f be the p-adic Galois representation attached to f. As $\ell^2 \nmid N$, π_ℓ is either principal series or special (see [Carayol 1989, Section 1.2]). So, it follows from the local Langlands correspondence, that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is either a direct sum of two characters or a nontrivial extension of a character by its cyclotomic twist (see Sections 3 and 5 of [Weston 2004]). As f is a newform of level Γ, the Artin conductor of ρ_f is N (the level of f). Since $\ell \parallel N$, the exponent of ℓ appearing in the Artin conductor of ρ_f is exactly 1 which means $(\rho_f)^{I_\ell}$, the subspace of ρ_f on which I_ℓ acts trivially, is one dimensional. So, if $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a direct sum of two characters, then one of them is unramified and the other is tamely ramified. Otherwise, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by its cyclotomic twist. As f is a modular form of level $\Gamma_0(N)$, its nebentypus is trivial, which means $\det \rho_f(I_\ell) = 1$. This implies that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is not a direct sum of two characters and hence, $\rho_f|_{G_{\mathbb{Q}_\ell}} \simeq \left(\begin{smallmatrix} \epsilon_p\chi & * \\ 0 & \chi \end{smallmatrix}\right)$, where $*$ is nonzero and ramified, χ is an unramified character and ϵ_p is the p-adic cyclotomic character of $G_{\mathbb{Q}_\ell}$.

As $a_\ell(f)$, the U_ℓ eigenvalue of f, is nonzero, by Lemma 19 above, it is the eigenvalue of Frob_ℓ acting on $(\rho_f)_{I_\ell}$. Thus, $a_\ell(f) = \chi(\mathrm{Frob}_\ell)$. Let x be a lift of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. Note that

$$\mathrm{tr}(\rho_f \circ i_\ell(x)) = \epsilon_p\chi(\mathrm{Frob}_\ell) + \chi(\mathrm{Frob}_\ell) = (\ell+1)\chi(\mathrm{Frob}_\ell) = (\ell+1)a_\ell(f),$$

which means $a_\ell(f) = \mathrm{tr}(\rho_f \circ i_\ell(x))/(\ell+1)$. Suppose $p \nmid \ell+1$, which implies that $\ell+1$ is a unit in $\mathbb{T}^{\Gamma}_{\bar{\rho}}$. Then on every newform f of level Γ lifting $\bar{\rho}$, the action of U_ℓ coincides with the action of $(\tau^{\Gamma} \circ i_\ell)(x)/(\ell+1)$, which lies in $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ as $\ell+1$ is a unit in $\mathbb{T}^{\Gamma}_{\bar{\rho}}$. As $\bar{\rho}$ is new, every eigenform of level Γ lifting $\bar{\rho}$ is a newform. This implies that $U_\ell - (\tau^{\Gamma} \circ i_\ell)(x)/(\ell+1)$ acts like 0 on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ and hence, $U_\ell = (\tau^{\Gamma} \circ i_\ell)(x)/(\ell+1)$ in $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$. Therefore, $U_\ell \in \mathbb{T}^{\Gamma}_{\bar{\rho}}$ if $\ell \parallel N$ and $p \nmid \ell+1$.

Note that, $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ is generated by the Hecke operators U_ℓ over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. So, by combining the discussion of the last two paragraphs, we get that $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}} = \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if there does not exist a prime ℓ such that $\ell \,\|\, N$ and $p \mid \ell+1$. Otherwise, we have $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if either $\ell^2 \mid N$ or $p \nmid \ell+1$ and $U_\ell^2 - \ell^{-2}\phi(\ell) = 0$ if $\ell \,\|\, N$. Therefore, we conclude that $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ is a finite extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma}$.

Let $\Gamma = \Gamma_1(N)$. There is a continuous pseudorepresentation

$$(\tau^\Gamma, \delta^\Gamma) : G_{\mathbb{Q},Np} \to \mathbb{T}_{\bar{\rho}}^{\Gamma}$$

such that $\tau^\Gamma(\mathrm{Frob}_q) = T_q$, $\delta^\Gamma(\mathrm{Frob}_q) = qS_q$ for primes q not dividing Np. Hence, we get a continuous pseudorepresentation $(t, d) : G_{\mathbb{Q},Np} \to \mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ since $\mathbb{T}_{\bar{\rho}}^{\Gamma} \subset \mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$. For every prime ℓ dividing N, let us choose an element g_ℓ of $G_{\mathbb{Q}_\ell}$ which gets mapped to Frob_ℓ under the quotient map $G_{\mathbb{Q}_\ell} \to G_{\mathbb{Q}_\ell}/I_\ell$. We have already fixed a natural map $i_\ell : G_{\mathbb{Q}_\ell} \to G_{\mathbb{Q},Np}$. Hence, we get a pseudorepresentation

$$(t \circ i_\ell, d \circ i_\ell) : G_{\mathbb{Q}_\ell} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}.$$

Now, consider the characteristic polynomial $Q_\ell(x)$ of g_ℓ which is defined by

$$Q_\ell(x) = x^2 - (t \circ i_\ell)(g_\ell)x + (d \circ i_\ell)(g_\ell).$$

Let f be a newform of level N lifting the system of eigenvalues corresponding to $\bar{\rho}$. Denote by ρ_f the p-adic Galois representation attached to f and by $a_\ell(f)$ its U_ℓ eigenvalue. By Lemma 19, if $a_\ell(f) = 0$, then U_ℓ kills f. If $a_\ell(f) \neq 0$, then $a_\ell(f)$ is the root of the characteristic polynomial $P_f(x)$ of $\rho_f \circ i_\ell(g_\ell)$. But $Q_\ell(U_\ell)f = P_f(a_\ell(f))f = 0$. Thus, in this case, $Q_\ell(U_\ell)$ kills f. We will now determine if it is possible to have two newforms f and g of level N lifting $\bar{\rho}$ such that $a_\ell(f) \neq 0$ but $a_\ell(g) = 0$ or equivalently (by Lemma 19), $(\rho_f)_{I_\ell}$ is one dimensional but $(\rho_g)_{I_\ell} = 0$.

Let f be a newform of level Γ lifting $\bar{\rho}$ as above and π_ℓ be the ℓ-component of the automorphic representation corresponding to f. So, π_ℓ is one of the following: principal series, special and supercuspidal (see [Weston 2004, Section 3]). As f is a newform, the Artin conductor of ρ_f is N (the level of f). Suppose $\ell \mid N$ but $\ell^2 \nmid N$. Then, from the analysis carried out in the $\Gamma_0(N)$ case above, we see that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is either a sum of an unramified character and a tamely ramified character or a nontrivial extension of an unramified character by its cyclotomic twist. Hence, the space $(\rho_f)_{I_\ell}$ of I_ℓ coinvariants is also one dimensional. Therefore, if $\ell \,\|\, N$, then for every newform f of level Γ lifting $\bar{\rho}$, $(\rho_f)_{I_\ell}$ is one dimensional and hence, $a_\ell(f) \neq 0$.

Now suppose $\ell^2 \mid N$, and moreover, the space of I_ℓ coinvariants of ρ_f is one dimensional. So, the exponent of ℓ appearing in the Artin conductor of ρ_f is at least 2. This means that π_ℓ is not special, as otherwise $(\rho_f)_{I_\ell}$ being one dimensional will imply that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by its cyclotomic twist, and hence, the exponent of ℓ appearing in the Artin conductor of ρ_f is 1. The nontriviality of $(\rho_f)_{I_\ell}$ also implies that π_ℓ is not extraordinary supercuspidal (see the proof of [Weston 2004, Proposition 3.2] for more details). Suppose π_ℓ is supercuspidal but not extraordinary. Then, by the local Langlands correspondence,

$$\rho_f|_{G_{\mathbb{Q}_\ell}} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_\ell}} \chi,$$

where K is a quadratic extension of $\mathbb{Q}_\ell$, G_K is the absolute Galois group of K, χ is a character of G_K taking values in $\overline{\mathbb{Q}}_p$ and moreover, ρ_f is irreducible (see Section 3 of [Weston 2004]). But as $(\rho_f)_{I_\ell}$ is one dimensional, we get that χ is an unramified character of G_K. However, since the maximal unramified extension of K is an abelian extension of $\mathbb{Q}_\ell$, this implies that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a sum of two characters, contradicting the hypothesis that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irreducible. Hence, π_ℓ is not supercuspidal. Therefore, π_ℓ is principal series which means that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a sum of two characters χ_1 and χ_2. Moreover, $(\rho_f)_{I_\ell} \neq 0$ implies that one of them is unramified, while $\ell^2 \mid N$ implies that the other is wildly ramified. Without loss of generality, suppose χ_1 is wildly ramified and χ_2 is unramified.

Thus, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}} = \overline{\chi_1} \oplus \overline{\chi_2}$, where $\overline{\chi_1}$ and $\overline{\chi_2}$ are the reductions of χ_1 and χ_2 in characteristic p, respectively. Note that, $\overline{\chi_2}$ is unramified while $\overline{\chi_1}$ is wildly ramified as $\ell \neq p$. Let g be another newform of level Γ lifting $\bar{\rho}$ and π'_ℓ be the ℓ-component of the automorphic representation corresponding to g. If π'_ℓ is special, then $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of a character by the cyclotomic twist of itself. This would imply that both $\overline{\chi_1}$ and $\overline{\chi_2}$ are either unramified or ramified, which is not the case. Hence, π'_ℓ is not special. If π'_ℓ is extraordinary supercuspidal, then $\bar{\rho}|_{I_\ell}$ is irreducible as $p \geq 5$ (see proof of [Weston 2004, Proposition 3.2]). So, π'_ℓ is not extraordinary supercuspidal. If π'_ℓ is supercuspidal but not extraordinary, then $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is induced from a character of the absolute Galois group of a quadratic extension of $\mathbb{Q}_\ell$. Moreover, the subspace of ρ_g fixed by I_ℓ is trivial. But the subspace of $\bar{\rho}$ fixed by I_ℓ is one dimensional. Thus, the exponent of ℓ in the Artin conductor of ρ_g is greater than the exponent of ℓ in the Artin conductor of $\bar{\rho}$ (see [Carayol 1989, Section 1.1]). But [Carayol 1989, Proposition 2], along with the assumption that π'_ℓ is supercuspidal, implies that $\bar{\rho}$ is unramified at ℓ which gives us a contradiction. Therefore, π'_ℓ is not supercuspidal.

This means that π'_ℓ is principal series and hence, $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is a sum of two characters, say χ'_1 and χ'_2. Without loss of generality, suppose χ'_1 is a lift of $\overline{\chi_1}$ and χ'_2 is a lift of $\overline{\chi_2}$. As $\overline{\chi_1}$ is wildly ramified, the Artin conductor of $\overline{\chi_1}$ is same as the Artin conductor of χ'_1 and the Artin conductor of χ_1 (see [Carayol 1989, Section 1.2]). As the exponent of ℓ in the Artin conductor of ρ_f is the sum of the exponents of ℓ in the Artin conductors of χ_1 and χ_2 and its exponent in the Artin conductor of ρ_g is the sum of its exponents in the Artin conductors of χ'_1 and χ'_2. As both f

and g are newforms of level N, the exponents of ℓ in the Artin conductors of ρ_f and ρ_g are same. This implies that the Artin conductors of χ_2 and χ_2' are same, which means that χ_2' is also unramified. In particular, we see that $(\rho_g)_{I_\ell}$ is one dimensional. As a consequence, we see that if f is a newform of level Γ lifting $\bar{\rho}$ and $(\rho_f)_{I_\ell}$ is one dimensional, then, for every newform g of level Γ lifting $\bar{\rho}$, $(\rho_g)_{I_\ell}$ is one dimensional.

Let us continue with the assumption that there is a newform f_0 of level Γ lifting $\bar{\rho}$ such that $(\rho_{f_0})_{I_\ell}$ is one dimensional. Observe that in this case, from the discussion so far, we get that for every newform g of level Γ lifting $\bar{\rho}$, $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is reducible and moreover, at least one character appearing in the semisimplification of $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is unramified. Thus, $(\bar{\rho}|_{G_{\mathbb{Q}_\ell}})^{\mathrm{ss}} = \alpha \oplus \beta$, where $(\bar{\rho}|_{G_{\mathbb{Q}_\ell}})^{\mathrm{ss}}$ is the semisimplification of $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ and α and β are characters of $G_{\mathbb{Q}_\ell}$ such that at least one of them is unramified. Note that, both α and β are defined over $\mathbb{F}$. Indeed, both $\alpha|_{I_\ell}$ and $\beta|_{I_\ell}$ take values in $\mathbb{F}$ as one of them is unramified and $\det(\bar{\rho})$ is defined over $\mathbb{F}$. So it follows from the previous discussion that the image of I_ℓ under $\bar{\rho}$ in $\mathrm{GL}_2(\mathbb{F})$ is abelian and hence, is upper-triangular under a suitable basis. Therefore, since I_ℓ is normal in $G_{\mathbb{Q}_\ell}$, it follows that if $\bar{\rho}$ is ramified at ℓ, then under the same basis, the image of $G_{\mathbb{Q}_\ell}$ under $\bar{\rho}$ in $\mathrm{GL}_2(\mathbb{F})$ is also upper-triangular, which implies that both α and β are defined over $\mathbb{F}$. If $\bar{\rho}$ is unramified at ℓ, then $\bar{\rho}$ is reducible as it is new. As $\bar{\rho}$ is semisimple, it follows that both α and β are defined over $\mathbb{F}$.

Suppose $\alpha \neq \beta$. Then, by [Bellaïche and Chenevier 2009, Theorem 1.4.4, Chapter 1], $\mathbb{T}^{\Gamma}_{\bar{\rho}}[G_{\mathbb{Q}_\ell}]/\big(\ker(\tau^{\Gamma} \circ i_\ell(G_{\mathbb{Q}_\ell}))\big)$ is a generalized matrix algebra (GMA) of the form

$$\begin{pmatrix} \mathbb{T}^{\Gamma}_{\bar{\rho}} & B \\ C & \mathbb{T}^{\Gamma}_{\bar{\rho}} \end{pmatrix}, \tag{1}$$

where B and C are finitely generated $\mathbb{T}^{\Gamma}_{\bar{\rho}}$-modules contained in the total fraction ring of $\mathbb{T}^{\Gamma}_{\bar{\rho}}$, and the diagonal entries reduce to α and β modulo the maximal ideal of $\mathbb{T}^{\Gamma}_{\bar{\rho}}$. Moreover, $BC \subset \mathbb{T}^{\Gamma}_{\bar{\rho}}$ and it is an ideal of $\mathbb{T}^{\Gamma}_{\bar{\rho}}$. Let us call it I.

For a newform f of level Γ lifting $\bar{\rho}$, let $\phi_f : \mathbb{T}^{\Gamma}_{\bar{\rho}} \to \overline{\mathbb{Q}_p}$ be the map which sends a Hecke operator to its f-eigenvalue and denote its kernel by P_f. From the previous paragraph, we see that the 2 dimensional pseudocharacter $\tau^{\Gamma} \circ i_\ell$ of $G_{\mathbb{Q}_\ell}$ is a sum of two characters modulo P_f and hence, is reducible modulo P_f for every newform f of level Γ lifting $\bar{\rho}$. Therefore, by [Bellaïche and Chenevier 2009, Proposition 1.5.1, Chapter 1], it follows that $I \subset P_f$ for every newform f of level Γ lifting $\bar{\rho}$. This means that if $x \in I$, then $x(f) = 0$ for every newform f of level Γ lifting $\bar{\rho}$. As every eigenform of level Γ lifting $\bar{\rho}$ is a newform, we see that $I = 0$. Thus, the projection on the diagonal entries of the GMA (1) above gives two characters $\tilde{\alpha}$ and $\tilde{\beta}$ of $G_{\mathbb{Q}_\ell}$ taking values in $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ such that, $\tilde{\alpha}$ is a deformation of α, while $\tilde{\beta}$ is a deformation of β.

As at least one of α and β is unramified; without loss of generality, assume that β is unramified. Suppose α is ramified. Then, it follows, from the analysis above, that for a newform f of level Γ lifting $\bar{\rho}$, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a direct sum of an unramified character and a ramified character. As α is ramified, the unramified character appearing in $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a lift of β and hence, it is the image of $\tilde{\beta}$ modulo P_f. This means that the reduction of $\tilde{\beta}$ modulo P_f gives an unramified character for every newform f of level Γ lifting $\bar{\rho}$. Hence, by the reasoning used in the previous paragraph, we see that $\tilde{\beta}$ is an unramified character of $G_{\mathbb{Q}_\ell}$. By Lemma 19, it follows that for every newform f of level Γ lifting $\bar{\rho}$, the U_ℓ eigenvalue of f is the reduction of $\tilde{\beta}(\mathrm{Frob}_\ell)$ modulo P_f. Thus, $U_\ell - \tilde{\beta}(\mathrm{Frob}_\ell)$ annihilates every newform f of level Γ lifting $\bar{\rho}$. Therefore, $U_\ell = \tilde{\beta}(\mathrm{Frob}_\ell)$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ and hence, $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ as $\tilde{\beta}(\mathrm{Frob}_\ell) \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$.

Now, suppose α is also unramified. This means that for a newform f of level Γ lifting $\bar{\rho}$, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is either a nontrivial extension of an unramified character by its cyclotomic twist or a direct sum of an unramified character and a tamely ramified character and in both cases, $\ell \parallel N$ (see [Carayol 1989, Proposition 2]). Moreover, in the second case, $p \mid \ell - 1$. Suppose $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a direct sum of an unramified character and a tamely ramified character. Let ϵ be the nebentypus of f. Note that,

$$\det(\rho_f(I_\ell)) = \epsilon((\mathbb{Z}/\ell\mathbb{Z})^*) \neq 1,$$

but its reduction is 1 in characteristic p. So, by [Carayol 1989, Proposition 3], there exists a newform g of level Γ lifting $\bar{\rho}$ such that $\epsilon'((\mathbb{Z}/\ell\mathbb{Z})^*) = 1$, where ϵ' is the nebentypus of g. Thus, $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by its cyclotomic twist. This means that either α/β or β/α is the cyclotomic character ω_p. But as $p \mid \ell - 1$, $\omega_p(\mathrm{Frob}_\ell) = 1$ and hence, ω_p is the trivial character. However, this means that $\alpha = \beta$, which contradicts our assumption that $\alpha \neq \beta$. Therefore, we get that, if α is unramified, then $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by the cyclotomic twist of itself for every newform f of level Γ lifting $\bar{\rho}$. By [Carayol 1989, Proposition 3] and the discussion above, it follows that $p \nmid \ell - 1$, for otherwise there would exist a newform g lifting $\bar{\rho}$ such that $\det(\rho_g(I_\ell)) \neq 1$, which gives a contradiction.

As α is also unramified, we saw above that either α/β or β/α is ω_p. If both of them are ω_p, then $\omega_p^2 = 1$, which means $p \mid \ell^2 - 1$. As $p \nmid \ell - 1$, we get that $p \mid \ell + 1$ if both of them are ω_p. Suppose $p \nmid \ell + 1$, which implies that exactly one of them is ω_p. Without loss of generality, assume α/β is ω_p. Now in this case, the image of $\tilde{\alpha}$ is the cyclotomic twist of the image of $\tilde{\beta}$ modulo P_f for every newform f of level Γ lifting $\bar{\rho}$. Thus, for every such newform f, $G_{\mathbb{Q}_\ell}$ acts by the image of $\tilde{\beta}$ modulo P_f on $(\rho_f)_{I_\ell}$. Hence, the U_ℓ eigenvalue of f is the reduction of $\tilde{\beta}(\mathrm{Frob}_\ell)$ modulo P_f for every newform f of level Γ lifting $\bar{\rho}$. Therefore, by the reasoning used in the previous case, we see that $U_\ell = \tilde{\beta}(\mathrm{Frob}_\ell)$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ which means that $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$.

Now suppose that there exists a newform f_0 of level Γ lifting $\bar{\rho}$ such that $(\rho_{f_0})_{I_\ell} = 0$, which means $U_\ell f_0 = 0$. Then, by our analysis above, it follows that $\ell^2 \mid N$ and $U_\ell f = 0$ for all newforms f of level Γ lifting $\bar{\rho}$. As $\bar{\rho}$ is new of level Γ, we get, by the reasoning used above, that $U_\ell = 0$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$.

From the discussion so far, we see that for a prime ℓ dividing N, $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if one of the following conditions hold:

(1) $\ell^2 \mid N$,

(2) $\ell \parallel N$ and $p \nmid \ell^2 - 1$,

(3) $\ell \parallel N$, $p \mid \ell + 1$ and $\det \bar{\rho}(I_\ell) \neq 1$.

Otherwise, $Q_\ell(U_\ell)$ kills every newform f of level N lifting $\bar{\rho}$. Since $\bar{\rho}$ is new, every eigenform of level N lifting $\bar{\rho}$ is a newform. This implies that either $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ or $Q_\ell(U_\ell) = 0$ for every prime ℓ dividing N. Since the pseudorepresentation (t, d) takes values in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$, it follows that for every prime ℓ dividing N, $Q_\ell(x)$ is a monic polynomial with coefficients in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. Hence, U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ for every prime ℓ dividing N of degree at most 2. Therefore, $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ is finite over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ and moreover, $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}} = \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if $\ell \parallel N$ and $p \mid \ell^2 - 1$ implies that $p \mid \ell + 1$ and $\det \bar{\rho}(I_\ell) \neq 1$ as $\mathbb{T}_{\bar{\rho}}^{\Gamma,\mathrm{pf}}$ is generated by these U_ℓ's over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. □

Remark. (1) Note that, the proof given above for the $\Gamma_1(N)$ case works for the $\Gamma_0(N)$ case as well. But we give a different proof because it gives us a more precise result in the $\Gamma_0(N)$ case and it is also simpler than the proof in the $\Gamma_1(N)$ case.

(2) Even though we do a detailed analysis in the $\Gamma_1(N)$ case above, it is not really necessary just to prove that U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. Indeed, by the reasoning used in the last paragraph of the proof above, we can easily prove that $U_\ell Q_\ell(U_\ell) = 0$, which proves that U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. For most of our purposes, we only need the result that U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. But we give this detailed analysis to obtain a more precise result which is helpful in getting a more precise version of Theorem 3 in some cases (see the remark after the proof of Theorem 3).

6. The case of $\bar{\rho}$ which is not new

Throughout this section Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(M)$ means either $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. Let $\bar{\rho}$ be a Γ-modular representation which is not new. Thus, the system of $\mathbb{F}$-valued eigenvalues corresponding to $\bar{\rho}$ has a nontrivial eigenspace in $S^{\Gamma(M)}(\mathbb{F})$ for some proper divisor M of N. Let us denote by $M_{\bar{\rho}}$ the smallest divisor of N such that the system of $\mathbb{F}$-valued eigenvalues corresponding to $\bar{\rho}$ has a nontrivial eigenspace in $S^{\Gamma(M_{\bar{\rho}})}(\mathbb{F})$. Note that, $\bar{\rho}$ is then a new $\Gamma(M_{\bar{\rho}})$-modular representation.

Lemma 21. *If $\bar{\rho}$ is a Γ-modular representation which is not new, then the natural map of local algebras $r : A^{\Gamma}_{\bar{\rho}} \to A^{\Gamma(M_{\bar{\rho}})}_{\bar{\rho}}$ obtained by restriction is surjective.*

Proof. First note that $S^{\Gamma(M_{\bar{\rho}})}(\mathbb{F})_{\bar{\rho}} \subset S^{\Gamma}(\mathbb{F})_{\bar{\rho}}$. Hence, using this inclusion, we obtain a natural map $r : A^{\Gamma}_{\bar{\rho}} \to A^{\Gamma(M_{\bar{\rho}})}_{\bar{\rho}}$ by restriction. Its image $\mathrm{Im}(r)$ is complete and hence, closed. It contains all the Hecke operators T_q and qS_q (considered as operators on $S^{\Gamma(M_{\bar{\rho}})}(\mathbb{F})_{\bar{\rho}}$) for all the primes q not dividing Np. There is a continuous pseudorepresentation $(\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}, \tilde{\delta}^{\Gamma(M_{\bar{\rho}})}) : G_{\mathbb{Q},M_{\bar{\rho}}p} \to A^{\Gamma(M_{\bar{\rho}})}_{\bar{\rho}}$ and $\mathrm{Im}(r)$ contains the Hecke operators $T_q = \tilde{\tau}^{\Gamma(M_{\bar{\rho}})}(\mathrm{Frob}_q)$ for primes q not dividing Np. By the Chebotarev density theorem, the set of Frob_q for primes q not dividing Np is dense in $G_{\mathbb{Q},M_{\bar{\rho}}p}$. Since $\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}$ is continuous and $\mathrm{Im}(r)$ is closed, it contains $\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}(g)$ for every $g \in G_{\mathbb{Q},M_{\bar{\rho}}p}$. It also contains $qS_q = \tilde{\delta}^{\Gamma(M_{\bar{\rho}})}(\mathrm{Frob}_q)$ for all the primes q not dividing Np. As $\tilde{\delta}^{\Gamma(M_{\bar{\rho}})}$ is continuous and $\mathrm{Im}(r)$ is closed, by the Chebotarev density theorem, we see that it contains $\tilde{\delta}^{\Gamma(M_{\bar{\rho}})}(g)$ for every $g \in G_{\mathbb{Q},M_{\bar{\rho}}p}$. For every prime q not dividing $M_{\bar{\rho}}$, we have $\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}(\mathrm{Frob}_q) = T_q$ and $\tilde{\delta}^{\Gamma(M_{\bar{\rho}})}(\mathrm{Frob}_q) = qS_q$. Hence, we see that $\mathrm{Im}(r)$ contains T_q, qS_q for all the primes q not dividing $M_{\bar{\rho}}p$. Thus, $\mathrm{Im}(r)$ is a closed subalgebra of $A^{\Gamma(M_{\bar{\rho}})}_{\bar{\rho}}$ containing all of its generators. Hence, $\mathrm{Im}(r) = A^{\Gamma(M_{\bar{\rho}})}_{\bar{\rho}}$. $\square$

7. Proof of Theorem 1

Throughout this section Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$, $\Gamma(p)$ means either $\Gamma_1(Np)$ or $\Gamma_0(Np)$ accordingly and $\Gamma(M)$ means $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. As we noted before, the natural map $\mathbb{T}^{\Gamma,\mathrm{pf}} \to \prod \mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ is an isomorphism and it lifts the natural map $\mathbb{T}^{\Gamma} \to \prod \mathbb{T}^{\Gamma}_{\bar{\rho}}$. The corresponding statement for $A^{\Gamma,\mathrm{pf}}$ and A^{Γ} is also true. The natural surjective map $\mathbb{T}^{\Gamma,\mathrm{pf}} \to A^{\Gamma,\mathrm{pf}}$ sends $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ onto $A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$. The natural surjective map $\mathbb{T}^{\Gamma(p),\mathrm{full}} \to A^{\Gamma,\mathrm{full}}$ takes $\mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},\lambda}$ onto $A^{\Gamma,\mathrm{full}}_{\bar{\rho},\lambda}$.

Thus, from Proposition 16 and the discussion above, it follows that $A^{\Gamma,\mathrm{full}}_{\bar{\rho},0}$ is isomorphic to $\mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},0}/m_{\Lambda}\mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},0}$. By Proposition 17, $\mathbb{T}^{\Gamma(p),\mathrm{full}}_{\bar{\rho},0} = \mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}[\![U_p]\!]$ and $A^{\Gamma,\mathrm{full}}_{\bar{\rho},0} = A^{\Gamma}_{\bar{\rho}}[\![U_p]\!]$. Thus, $A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ is isomorphic to $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}/m_{\Lambda}\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$. As the ideal m_{Λ} is generated by two elements, the *Hauptidealsatz* implies that

$$\dim A^{\Gamma,\mathrm{pf}}_{\bar{\rho}} \geq \dim \mathbb{T}^{\Gamma(p),\mathrm{pf}}_{\bar{\rho}} - 2.$$

Suppose $\bar{\rho}$ is a new Γ-modular representation. Then, we have proved so far that $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ is a finite extension of $\mathbb{T}^{\Gamma}_{\bar{\rho}}$. Observe that $\mathbb{T}^{\Gamma}_{\bar{\rho}}$ is mapped onto $A^{\Gamma}_{\bar{\rho}}$ under the surjective map $\mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}} \to A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$. This follows from the fact that the map between the partially full Hecke algebras is obtained by first reducing the partially full Hecke algebra in characteristic 0 modulo $\mathfrak{p}$ to get an action on $D^{\Gamma}(\mathbb{F})_{\bar{\rho}}$ and then restricting it to the subspace $S^{\Gamma}(\mathbb{F})_{\bar{\rho}}$. Hence, we see that $A^{\Gamma,\mathrm{pf}}_{\bar{\rho}}$ is finite over $A^{\Gamma}_{\bar{\rho}}$.

Therefore, $\dim A^{\Gamma,\mathrm{pf}}_{\bar{\rho}} = \dim A^{\Gamma}_{\bar{\rho}}$ and $\dim \mathbb{T}^{\Gamma,\mathrm{pf}}_{\bar{\rho}} = \dim \mathbb{T}^{\Gamma}_{\bar{\rho}}$. By the Gouvêa–Mazur infinite fern argument, we see that $\dim \mathbb{T}^{\Gamma}_{\bar{\rho}} \geq 4$ (see [Emerton 2011, Corollary 2.28; Gouvêa and Mazur 1998, Theorem 1]). Hence, $\dim A^{\Gamma}_{\bar{\rho}} \geq \dim \mathbb{T}^{\Gamma}_{\bar{\rho}} - 2 \geq 2$.

Now suppose that $\bar{\rho}$ is not a new Γ-modular representation. Thus, as remarked before, there exists a proper divisor $M_{\bar{\rho}}$ of N such that $\bar{\rho}$ is a new $\Gamma(M_{\bar{\rho}})$-modular representation (we can take it to be the smallest divisor of N such that the eigenspace corresponding to the system of eigenvalues corresponding to $\bar{\rho}$ is nonzero in $S^{\Gamma_1(M_{\bar{\rho}})}(\mathbb{F})$). If $M_{\bar{\rho}} = 1$, then we know that $\dim A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})} = 2$ by Theorem III of [BK]. If $M_{\bar{\rho}} > 1$, then since $\bar{\rho}$ is a new $\Gamma(M_{\bar{\rho}})$-modular representation, repeating the argument given in the previous paragraph for $A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ gives $\dim A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})} \geq 2$. (We can do this since, all the results till the previous section are valid for any level not divisible by p. This condition is satisfied by $M_{\bar{\rho}}$ as it divides N which is coprime to p.) Thus, in any case, we see that $\dim A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})} \geq 2$. By Lemma 21, we have a surjective map $A_{\bar{\rho}}^{\Gamma} \to A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$. Hence, $\dim A_{\bar{\rho}}^{\Gamma} \geq 2$ when $\bar{\rho}$ is not a new Γ-modular representation. Thus, Theorem 1 is proved for all the Γ-modular representations $\bar{\rho}$.

8. Proof of Theorem 3

Theorem 22 (Böckle, Diamond–Flach–Guo, Gouvêa–Mazur, Kisin). *Under the hypotheses of Theorem 3, the natural map $R_{\bar{\rho}} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism between local rings of dimension* 4.

Proof. If $\bar{\rho}$ is a $\Gamma_1(N)$-modular representation which also satisfies the hypotheses of Theorem 3, then mimicking the proof of Theorem 18 of [BK] gives the result of the theorem. We can mimic the argument since, under all these assumptions, the infinite fern argument of Gouvêa–Mazur and [Böckle 2001, Theorems 2.8, 3.1 and 3.9; Diamond 1996, Theorem 1.1; Diamond et al. 2004, Theorem 3.6; Kisin 2004, Main Theorem], which are the key ingredients of the proof of Theorem 18 of [BK], hold. Note that the Hecke algebra appearing in [Böckle 2001, Theorem 3.9] is of a higher level N' such that $N \mid N'$ and N and N' have the same prime factors (see the discussion after [Böckle 2001, Theorem 2.7] for more details). But the hypotheses of Theorem 3, along with [Carayol 1989, Proposition 2], ensure that all eigenforms of level N' lifting $\bar{\rho}$ arise from newforms of level dividing N, which means that the natural restriction map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N')} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. ☐

Lemma 23. *Under the hypotheses of Theorem 3, the algebra $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is flat over Λ for the structure of Λ-algebra on $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ defined at the end of Section 3.2.*

Proof. The proof is similar to that of Lemma 19 of [BK], but we need to make a few changes. Let $\bar{\rho}$ be a $\Gamma_1(N)$-modular representation. If the p-primary part of $(\mathbb{Z}/N\mathbb{Z})^*$ is $\prod_{i=1}^{i=n} \mathbb{Z}/p^{e_i}\mathbb{Z}$, then the universal deformation ring of $\det \bar{\rho}$ is

$$R_{\det \bar{\rho}} = \Lambda[Y_1, \ldots, Y_n]/(Y_1^{p^{e_1}} - 1, \ldots, Y_n^{p^{e_n}} - 1). \tag{2}$$

Thus, if p does not divide $\phi(N)$, then $R_{\det \bar{\rho}} = \Lambda$. Using the same arguments as used in the proof of [BK, Lemma 19], we get that $R_{\bar{\rho}}$ is flat over $R_{\det \bar{\rho}}$. Since $R_{\det \bar{\rho}}$

is finite and free over Λ, it is also flat over Λ. So, we get that $R_{\bar{\rho}}$ is flat over Λ. To verify that the map $\Lambda \to R_{\bar{\rho}} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is the same as the map $\Lambda \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ given at the end of Section 3.2, we first show that it is true for all the primes $q \in 1 + p\mathbb{Z}_p$ which are $1 \pmod N$. This is shown in exactly the same way as it is showed for all the primes $q \in 1 + p\mathbb{Z}_p$ in the proof of [BK, Lemma 19]. Since we are choosing the primes which are $1 \pmod N$, the nebentypus is trivial at those primes and hence, we can use the argument used in the proof of [BK, Lemma 19] without modifications. By Dirichlet's theorem on primes in arithmetic progressions, we know that the set of all such primes is dense in $1 + p\mathbb{Z}_p$. Thus, by continuity, we see that the two maps considered above are the same. Under the hypotheses of Theorem 3, the map $R_{\bar{\rho}} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. Hence, we conclude that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is flat over Λ for the Λ-algebra structure defined on $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ at the end of Section 3.2. $\square$

There is a surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_1(N)}$ whose kernel contains $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ by Proposition 16. Hence, we get a surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_1(N)}$. Since $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is flat over Λ by Lemma 23, it follows, from [Eisenbud 1995, Theorem 10.10], that the Krull dimension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is equal to $\dim \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} - 2 = 4 - 2 = 2$. Thus, the Krull dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$ is at most 2. But, by Theorem 1, the dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$ is at least 2. Hence, the dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$ is exactly 2. This concludes the proof of first part of Theorem 3 for the $\Gamma_1(N)$ case.

Let us assume that $\bar{\rho}$ satisfies the hypothesis of Theorem 3 and is also new. Then we have proved that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ and $A_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ is an integral extension of $A_{\bar{\rho}}^{\Gamma_1(N)}$. By Proposition 16, we see that the kernel of the natural map $\mathbb{T}^{\Gamma_1(N),\mathrm{pf}} \to A^{\Gamma_1(N),\mathrm{pf}}$ is $m_\Lambda \mathbb{T}^{\Gamma_1(N),\mathrm{pf}}$. Thus, the kernel of the natural map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_1(N)}$ is contained in every prime ideal of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ containing $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$. Hence, by the going-up theorem, the kernel is contained in every prime ideal of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ containing $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. Therefore, the natural surjective map $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}} \to (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ is an isomorphism. Under the hypothesis of Theorem 3, the surjective map $R_{\bar{\rho}} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. Thus, from the proof of Lemma 23, it follows that $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$ is isomorphic to $(R_{\bar{\rho}}/m_\Lambda R_{\bar{\rho}})^{\mathrm{red}}$ (this follows from the fact that the universal deformation ring of $\det \bar{\rho}$ is given by Equation (2) and hence, its maximal ideal is $\mathrm{Rad}(m_\Lambda)$) and the map $R_{\bar{\rho}} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ considered above induces an isomorphism $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}} \to (\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. As seen before, we have a surjective map $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}} \to (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ and this map factors through $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. Since both the maps $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}} \to (\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ and $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}} \to (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ are isomorphisms, we see that $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ under the map mentioned above.

Now suppose $\bar{\rho}$ satisfies the hypotheses of Theorem 3 but is not new. Note that, while proving the theorem for a new $\bar{\rho}$ of level N, we use the result that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ to prove that the kernel of the surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_1(N)}$ is nilpotent modulo $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$, which in turn implies the isomorphism between $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. This, along with the isomorphism between $R_{\bar{\rho}}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$, gives us the isomorphism between $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$ and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. In the present situation, we know that $R_{\bar{\rho}}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ are isomorphic. So, to prove the theorem in the current case, it suffices to prove that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. To prove this, we will first compare the Artin conductor of $\bar{\rho}$ with the level N and then do a case by case analysis using the technique used in the proof of Proposition 20. Let C be the Artin conductor of $\bar{\rho}$. Recall that we assume $\bar{\rho}$ to be semisimple. As there exists a newform of level N giving rise to $\bar{\rho}$, we see that $N = C \prod r^{v(r)}$, where the product is over finitely many primes r and $v(r) \le 2$ for every r (see [Carayol 1989, Proposition 2] and the discussion before it). Note that the product may include some divisors of C.

Let ℓ be a prime dividing N. So, if f is an eigenform of level N lifting the system of eigenvalues associated to $\bar{\rho}$, then f comes from a newform g of level M such that the highest power of ℓ dividing N/M is at most 2. Thus, $f(z) = \sum_{i=1}^{i=r} g(d_i z)$, where the d_i's are some divisors of N/M. Now we will analyze the action of U_ℓ on f case by case to find a monic polynomial with coefficients in $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ satisfied by U_ℓ.

Case 1. $\ell \mid M$ but $\ell \nmid N/M$: In this case, $\ell \nmid d_i$ for all i and hence, f is new at ℓ. Therefore, from the proof of Proposition 20, we see that $U_\ell Q_\ell(U_\ell)(f) = 0$, where $Q_\ell(x)$ is the characteristic polynomial of a lift of Frob_ℓ in $G_{\mathbb{Q},Np}$, as considered in the proof of Proposition 20.

Case 2. $\ell \mid M$ and $\ell \,\|\, N/M$: In this case, for every i either $\ell \,\|\, d_i$ or $\ell \nmid d_i$ and g is an eigenform for U_ℓ. So, $U_\ell f$ is an eigenform of level N/ℓ, which is also new at ℓ if it is nonzero. So, by the same logic as in the previous case, $U_\ell Q_\ell(U_\ell)(U_\ell f) = 0$.

Case 3. $\ell \mid M$ and $\ell^2 \mid N/M$: In this case, for every i either $\ell \nmid d_i$ or ℓ divides d_i with multiplicity at most 2 and g is an eigenform for U_ℓ. So, $U_\ell^2 f$ is an eigenform of level N/ℓ^2 and it is new at ℓ when it is nonzero. Thus, we get $U_\ell Q_\ell(U_\ell)(U_\ell^2 f) = 0$.

Case 4. $\ell \nmid M$ and $\ell \nmid d_i$ for all i: If $\ell \nmid M$ and $\ell \nmid d$, then it follows directly from the description of the action of T_ℓ and U_ℓ on the q-expansions of modular forms, along with the assumption that g is an eigenform of level M, that U_ℓ stabilizes the subspace of modular forms of level $Md\ell$ generated by $g(dz)$ and $g(\ell dz)$ (see [Bellaïche 2010, Lemma III.7.2] and the discussion around it). Moreover, the characteristic polynomial of U_ℓ over this subspace is $\big(U_\ell^2 - \operatorname{tr}\rho_g(\mathrm{Frob}_\ell) U_\ell + \det \rho_g(\mathrm{Frob}_\ell)\big)$, where ρ_g is the p-adic Galois representation of $G_{\mathbb{Q},Mp}$ attached to g. Thus, in this case, we get that $Q_\ell(U_\ell)(f) = 0$.

Case 5. $\ell \nmid M$ and ℓ divides at least one of the d_i's: Note that, $U_\ell g(d_i z)$ is an eigenform of level N/ℓ if $\ell \,\|\, d_i$ and $U_\ell^2 g(d_i z)$ is an eigenform of level N/ℓ^2 if $\ell^2 \mid d_i$. This means that for all i, one of $g(d_i z)$, $U_\ell g(d_i z)$ or $U_\ell^2 g(d_i z)$ is an eigenform of level coprime to ℓ. So, by case 4, we have $Q_\ell(U_\ell)(U_\ell^2 g(d_i z)) = 0$ for all i as both $Q_\ell(U_\ell)(U_\ell g(d_i z)) = 0$ and $Q_\ell(U_\ell)(g(d_i z)) = 0$ imply $Q_\ell(U_\ell)(U_\ell^2 g(d_i z)) = 0$. Hence, we have $Q_\ell(U_\ell)(U_\ell^2 f) = 0$.

From the previous paragraphs, we conclude that, if f is an eigenform of level N lifting $\bar{\rho}$, then $(U_\ell^3 Q_\ell(U_\ell))f = 0$, which means $U_\ell^3 Q_\ell(U_\ell) = 0$. As $Q_\ell(x)$ is a monic polynomial with coefficients in $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$, U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ for all primes ℓ dividing N. Therefore, $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ and $A_{\bar{\rho}}^{\Gamma_1(N),\mathrm{pf}}$ is an integral extension of $A_{\bar{\rho}}^{\Gamma_1(N)}$. Now using the argument used for the new case above, we get that $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. This concludes the proof of the second part of Theorem 3 for the $\Gamma_1(N)$ case.

Let $\bar{\rho}$ be a $\Gamma_0(N)$-modular representation which satisfies the hypothesis of Theorem 3. Then, by the arguments above, it follows that $A_{\bar{\rho}}^{\Gamma_1(N)}$ has Krull dimension 2 and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ is isomorphic to $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$. We have a natural surjective map $A_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_0(N)}$. By Theorem 1, the Krull dimension of $A_{\bar{\rho}}^{\Gamma_0(N)}$ is at least 2. Hence, we conclude that the Krull dimension of $A_{\bar{\rho}}^{\Gamma_0(N)}$ is exactly 2. If p does not divide $\phi(N)$, then as observed in the introduction, the natural surjective map $A_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_0(N)}$ is an isomorphism. Hence, we get that $(A_{\bar{\rho}}^{\Gamma_0(N)})^{\mathrm{red}}$ is isomorphic to $(\tilde{R}_{\bar{\rho}}^0)^{\mathrm{red}}$. Thus, Theorem 3 is proved in the $\Gamma_0(N)$ case.

Remark. Let $\bar{\rho}$ be a $\Gamma_1(N)$-modular representation satisfying the hypothesis of Theorem 3 and assume $p \nmid \phi(N)$. Moreover, assume that if $\ell \parallel N$ and $p \mid \ell + 1$, then $\det \bar{\rho}(I_\ell) \neq 1$. Hence, from the proof of Proposition 20 and the proof of Theorem 1, we see that the kernel of the map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_1(N)}$ is $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. As $p \nmid \phi(N)$, the kernel of the surjective map $R_{\bar{\rho}} \to \tilde{R}_{\bar{\rho}}^0$ is $m_\Lambda R_{\bar{\rho}}$. As $\bar{\rho}$ satisfies the hypotheses of Theorem 3, we get that $R_{\bar{\rho}} \simeq \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. Hence, by combining all the observations above, we see that the natural surjective map $\tilde{R}_{\bar{\rho}}^0 \to A_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. Thus, for this case, we get a stronger statement than Theorem 3.

9. Proof of Theorem 2

Throughout this section Γ means either $\Gamma_0(N)$ or $\Gamma_1(N)$ with $p \nmid \phi(N)$. Assume that $\bar{\rho}$ is an unobstructed Γ-modular representation. So, by assumption, the cotangent space of $\tilde{R}_{\bar{\rho}}^0$ has dimension 2 and this implies that its Krull dimension is at most 2. We have a surjective morphism $\tilde{R}_{\bar{\rho}}^0 \to A_{\bar{\rho}}^{\Gamma}$. The Krull dimension of $A_{\bar{\rho}}^{\Gamma}$ is at least 2 by Theorem 1. Hence, the Krull dimension of $\tilde{R}_{\bar{\rho}}^0$ is exactly 2 and it is a regular local ring of dimension 2. Therefore, by [Eisenbud 1995, Proposition 10.16], it is isomorphic to $\mathbb{F}[[x, y]]$. Hence, the surjective map $\tilde{R}_{\bar{\rho}}^0 \to A_{\bar{\rho}}^{\Gamma}$ is an isomorphism. This proves Theorem 2 for the cases considered above.

If $\bar{\rho}$ is an unobstructed $\Gamma_1(N)$-modular representation and $p \mid \phi(N)$, then we have a surjective morphism $\tilde{R}_{\bar{\rho}}^0 \to (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. By Theorem 1, the Krull dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$, and hence also of $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$, is at least 2. Therefore, the argument used in the previous paragraph proves Theorem 2 in this case as well. This completes the proof of Theorem 2 in all the cases.

10. Unobstructed modular representations

In this section, we study $\Gamma_1(N)$-modular representations to determine if and when they can be unobstructed. If $\bar{\rho}$ is irreducible, then $\bar{\rho}$ is unobstructed in our sense if and only if $H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho})$ has dimension 2 (see [BK, Section 1.4]). By Tate's global Euler characteristic formula, we see that $\dim H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) \geq 2$ with equality if and only if $H^2(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) = 0$. If p does not divide $\phi(N)$, then this is equivalent to $H^2(G_{\mathbb{Q},Np}, \mathrm{ad}\bar{\rho}) = 0$, as $\mathrm{ad}\bar{\rho} = 1 \oplus \mathrm{ad}^0\bar{\rho}$ and, by the global Euler characteristic formula, $H^2(G_{\mathbb{Q},Np}, 1) = 0$. Thus, in this case, $\bar{\rho}$ is unobstructed in our sense if and only if it is unobstructed in the sense of Mazur [1989, Section 1.6]. The study of such unobstructed representations is carried out in [Weston 2004; 2005] when p does not divide $\phi(N)$.

However, if $\bar{\rho}$ is irreducible and p divides $\phi(N)$, then the global Euler characteristic formula tells us that $H^2(G_{\mathbb{Q},Np}, 1) \neq 0$ and hence, $H^2(G_{\mathbb{Q},Np}, \mathrm{ad}\bar{\rho}) \neq 0$. Thus, we see that $\bar{\rho}$ is always obstructed in the sense of Mazur if $p \mid \phi(N)$ (see [Weston 2004, Theorem 4.5]). However, it is not clear a priori if $\bar{\rho}$ can be unobstructed in our sense when $p \mid \phi(N)$. We devote most of this section to study the unobstructedness of reducible and irreducible $\bar{\rho}$'s when $p \mid \phi(N)$.

Throughout this section, we assume that $p \mid \phi(N)$ unless otherwise stated. Note that, as $p \nmid N$ and $p \mid \phi(N)$, there exists at least one prime divisor ℓ of N such that $p \mid \ell - 1$, i.e., $\ell \equiv 1 \pmod{p}$. Let $\mathcal{C}$ be the category of local profinite $\mathbb{F}$-algebras. Let $D_{\bar{\rho}}$ be the functor from $\mathcal{C}$ to $\mathcal{S}et$ of deformations of the pseudorepresentation $(\mathrm{tr}\,\bar{\rho}, \det\bar{\rho})$ and $D^0_{\bar{\rho}}$ be its subfunctor of deformations with constant determinant.

Proposition 24. *If $p \mid \phi(N)$, then every reducible $\Gamma_1(N)$-modular representation is obstructed.*

Proof. Let $\bar{\rho}$ be a reducible $\Gamma_1(N)$-modular representation. Up to a twist, $\bar{\rho}$ is of the form $1 \oplus \chi$ where χ is an odd character of $G_{\mathbb{Q},Np}$ taking values in $\mathbb{F}$. The character χ is odd because $\bar{\rho}$ is odd. By the main theorem of [Bellaïche 2012a], we have the following exact sequence involving the tangent space of $D_{\bar{\rho}}$:

$$0 \to \mathrm{Tan}(D_1 \oplus D_\chi) \xrightarrow{i} \mathrm{Tan}(D_{\bar{\rho}}) \to H^1(G_{\mathbb{Q},Np}, \chi) \otimes H^1(G_{\mathbb{Q},Np}, \chi^{-1}) \to H^2(G_{\mathbb{Q},Np}, 1)^2.$$

Here, D_χ and D_1 are the deformation functors of χ and 1 as characters of $G_{\mathbb{Q},Np}$, and i is the map which sends a pair of deformations (α, β) of $(\chi, 1)$ on $\mathbb{F}[\epsilon]/(\epsilon^2)$ to the deformation $\alpha + \beta$ of $\mathrm{tr}\,\bar{\rho} = 1 + \chi$.

Let γ be an element of $H^1(G_{\mathbb{Q},Np}, \chi) \otimes H^1(G_{\mathbb{Q},Np}, \chi^{-1})$ which is the image of some deformation (t, d) of $(\mathrm{tr}\,\bar{\rho}, \det\bar{\rho})$ to $\mathbb{F}[\epsilon]/(\epsilon^2)$. Thus, d is a deformation of $\det\bar{\rho} = \chi$ and hence, an element of $\mathrm{Tan}(D_\chi) \subset \mathrm{Tan}(D_1 \oplus D_\chi)$. Thus, subtracting the image of this element under the map i from (t, d) gives us an element of $\mathrm{Tan}(D^0_{\bar\rho})$ whose image in $H^1(G_{\mathbb{Q},Np}, \chi) \otimes H^1(G_{\mathbb{Q},Np}, \chi^{-1})$ is still γ. Therefore, we have an exact sequence:

$$0 \to \mathrm{Tan}((D_1 \oplus D_\chi)^0) \xrightarrow{i} \mathrm{Tan}(D^0_{\bar\rho}) \to H^1(G_{\mathbb{Q},Np}, \chi) \otimes H^1(G_{\mathbb{Q},Np}, \chi^{-1}) \to H^2(G_{\mathbb{Q},Np}, 1)^2.$$

Here, $(D_1 \oplus D_\chi)^0$ is the subfunctor of $D_1 \oplus D_\chi$ parameterizing the deformations (α, β) of $(1, \chi)$ such that $\alpha\beta$ is constant.

Observe that, $\dim H^1(G_{\mathbb{Q},Np}, 1) = n+1$, where n is the number of prime divisors of N which are 1 mod p. Hence, $\dim \mathrm{Tan}((D_1 \oplus D_\chi)^0) = n+1$. Since, i is an injective map, we get $\dim \mathrm{Tan}(D^0_{\bar\rho}) \geq n+1$. Thus, if $n \geq 2$, then $\dim \mathrm{Tan}(D^0_{\bar\rho}) \geq 3$. Hence, $\bar\rho$ is obstructed when $n \geq 2$, i.e., when there are at least two primes dividing N which are 1 mod p.

Let us now assume that $n = 1$. Thus, there is a unique prime ℓ which divides N and which is 1 mod p. In this case, $\dim H^1(G_{\mathbb{Q},Np}, 1) = 2$ and by Tate's global Euler characteristic formula, $\dim H^2(G_{\mathbb{Q},Np}, 1) = 1$. From the exact sequence above, we get that

$$\dim \mathrm{Tan}(D^0_{\bar\rho}) \geq 2 + \dim H^1(G_{\mathbb{Q},Np}, \chi) \dim H^1(G_{\mathbb{Q},Np}, \chi^{-1}) - 2 \dim H^2(G_{\mathbb{Q},Np}, 1).$$

We shall distinguish between two cases:

First case. $\chi|_{G_{\mathbb{Q}_\ell}} = 1$: Note that $\omega_p|_{G_{\mathbb{Q}_\ell}} = 1$, since $\ell \equiv 1 \pmod{p}$.

If $\chi \neq \omega_p$, then, by the Greenberg–Wiles version of Poitou–Tate duality [Washington 1997, Theorem 2], we get that

$$\dim H^1(G_{\mathbb{Q},Np}, \chi) \geq \dim H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) - \dim H^0(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) + \dim H^1(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) - \dim H^0(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}).$$

As $\ell \equiv 1 \pmod{p}$, $\dim H^1(G_{\mathbb{Q}_\ell}, 1) = 2$. By the local Euler characteristic formula,

$$\dim H^1(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) - \dim H^0(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) = 1 + \dim H^2(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) \geq 1.$$

Since, $\chi|_{G_{\mathbb{Q}_\ell}} = 1$, we see that $\dim H^1(G_{\mathbb{Q},Np}, \chi) \geq 2 - 1 + 1 = 2$.

If $\chi = \omega_p$, then, using the Kummer exact sequence, we get that a class in $H^1(G_{\mathbb{Q},Np}, \omega_p)$ is represented by a cocycle of the form $g \mapsto g(\alpha)/\alpha$, with $\alpha \in \overline{\mathbb{Q}}$, $\alpha^p \in \mathbb{Q}$ and $v_q(\alpha^p) = 0$ for all primes $q \nmid Np$ (See [Washington 1997, Section 1] for more details). Hence, $\dim H^1(G_{\mathbb{Q},Np}, \omega_p) = n'$, where n' = number of distinct prime factors of Np. Hence, $\dim H^1(G_{\mathbb{Q},Np}, \omega_p) \geq 2$. Since, $\chi^{-1}|_{G_{\mathbb{Q}_\ell}} = 1$ as well, the same argument gives us $\dim H^1(G_{\mathbb{Q},Np}, \chi^{-1}) \geq 2$.

Combining all the above calculations yields $\dim \mathrm{Tan}(D^0_{\bar\rho}) \geq 2 + (2)(2) - 2(1) = 4$. Hence, if $n = 1$ and $\chi|_{G_{\mathbb{Q}_\ell}} = 1$, then $\bar\rho$ is obstructed.

Second case. $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$: As $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$, $\omega_p \otimes \chi \neq 1$. Thus, $H^0(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = 0$, $H^2(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \chi|_{G_{\mathbb{Q}_\ell}}) = 0$ and hence, $H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = 0$.

Similarly, we get that $H^1(G_{\mathbb{Q}_\ell}, \chi^{-1}|_{G_{\mathbb{Q}_\ell}}) = 0$. Thus, the calculations done in the first case do not imply that $\dim \mathrm{Tan}(D^0_{\bar{\rho}}) > 2$ which is required to prove that $\bar{\rho}$ is obstructed.

Assume that $\bar{\rho}$ is unobstructed. So, the dimension of the tangent space of $\tilde{R}^0_{\bar{\rho}}$ is 2. By Theorem 1, its Krull dimension is at least 2. Hence, $\tilde{R}^0_{\bar{\rho}}$ is isomorphic to $\mathbb{F}[[x, y]]$, the power series ring in two variables. Let $(t^{\mathrm{univ}}, d^{\mathrm{univ}})$ be the universal pseudorepresentation with constant determinant in characteristic p deforming $(\mathrm{tr}\,\bar{\rho}, \det \bar{\rho})$.

By [Bellaïche and Chenevier 2009, Theorem 1.4.4, Chapter 1],

$$\tilde{R}^0_{\bar{\rho}}[G_{\mathbb{Q}_\ell}]/(\ker(t^{\mathrm{univ}} \circ i_\ell(G_{\mathbb{Q}_\ell})) \text{ is a GMA of the form } \begin{pmatrix} \tilde{R}^0_{\bar{\rho}} & B \\ C & \tilde{R}^0_{\bar{\rho}} \end{pmatrix},$$

where B and C are finitely generated $\tilde{R}^0_{\bar{\rho}}$-modules and the diagonal entries reduce to 1 and χ modulo the maximal ideal of $\tilde{R}^0_{\bar{\rho}}$. By [Bellaïche and Chenevier 2009, Theorem 1.5.5, Chapter 1], we get injective maps $(B/m_{\tilde{R}^0_{\bar{\rho}}} B)^* \hookrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\chi, 1)$ and $(C/m_{\tilde{R}^0_{\bar{\rho}}} C)^* \hookrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(1, \chi)$. Since we have

$$H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = H^1(G_{\mathbb{Q}_\ell}, \chi^{-1}|_{G_{\mathbb{Q}_\ell}}) = 0,$$

it implies that $\mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\chi, 1) = \mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(1, \chi) = 0$ and hence, $B = C = 0$. Thus, $t^{\mathrm{univ}} \circ i_\ell(G_{\mathbb{Q}_\ell}) = \kappa_1 + \kappa_2$, where κ_1 and κ_2 are characters of $G_{\mathbb{Q}_\ell}$ taking values in $(\tilde{R}^0_{\bar{\rho}})^*$ and are deformations of 1 and χ, respectively. Therefore, $t^{\mathrm{univ}} \circ i_\ell$ factors through $G^{\mathrm{ab}}_{\mathbb{Q}_\ell}$, the abelianization of $G_{\mathbb{Q}_\ell}$.

By local class field theory, $G^{\mathrm{ab}}_{\mathbb{Q}_\ell} = \mathbb{Z}^*_\ell \times \hat{\mathbb{Z}}$. Let p^e be the highest power of p dividing $\ell - 1$. Let $a \in G^{\mathrm{ab}}_{\mathbb{Q}_\ell}$ be the unique element of order p^e. Since, a has order p^e and $\mathbb{F}[[x, y]]$ does not have any nontrivial element of order p, it follows that $\kappa_1(a) = \kappa_2(a) = 1$. Let b be a lift of a in $G_{\mathbb{Q}_\ell}$. Hence, $t^{\mathrm{univ}} \circ i_\ell(b) = 2$, i.e., it is constant. Hence, $t \circ i_\ell(b) = 2$ for any deformation t of $\mathrm{tr}\,\bar{\rho}$ as a pseudocharacter of $G_{\mathbb{Q},Np}$ with constant determinant in characteristic p. But it is easy to construct an explicit such deformation t with $t \circ i_\ell(b) \neq 2$.

Indeed, to construct such a deformation, first observe that the maximal pro-p subgroup $G^{\mathrm{ab},p}_{\mathbb{Q},Np}$ of $G^{\mathrm{ab}}_{\mathbb{Q},Np}$ is $\mathbb{Z}/p^e\mathbb{Z} \times \mathbb{Z}_p$. Consider a deformation $\tilde{1}$ of 1 as a character of $G_{\mathbb{Q},Np}$ to $R = \mathbb{F}[[x]][y]/(y^{p^e} - 1)$ which maps $G^{\mathrm{ab},p}_{\mathbb{Q},Np} \to 1 + m_R$ in the following way: the topological generator of $\mathbb{Z}_p$ is mapped to $1 + x$ and the generator of $\mathbb{Z}/p^e\mathbb{Z}$ is mapped to y. The character $\chi/\tilde{1}$, which we will denote by $\tilde{\chi}$, is a deformation of χ. Let $\tilde{\rho} = \tilde{1} \oplus \tilde{\chi}$. Thus, $(\mathrm{tr}\,\tilde{\rho}, \det \tilde{\rho})$ is a deformation of $(\mathrm{tr}\,\bar{\rho}, \det \bar{\rho})$ to R with constant determinant. We claim that $\mathrm{tr}(\tilde{\rho} \circ i_\ell(b)) \neq 2$.

To prove the claim, first consider the map $i^{\mathrm{ab}}_\ell : G^{\mathrm{ab}}_{\mathbb{Q}_\ell} \to G^{\mathrm{ab}}_{\mathbb{Q},Np}$ induced from i_ℓ by passing to the abelianizations of both the groups. By class field theory, $G^{\mathrm{ab}}_{\mathbb{Q},Np} = \prod \mathbb{Z}^*_q$, where the product is taken over primes q which divide Np. By the local-global compatibility of class field theory, the $\mathbb{Z}^*_\ell$ component of $G^{\mathrm{ab}}_{\mathbb{Q},Np}$, lies in the image of $\mathbb{Z}^*_\ell \subset G^{\mathrm{ab}}_{\mathbb{Q}_\ell}$ under the map i^{ab}_ℓ. Thus, the unique element of $G^{\mathrm{ab}}_{\mathbb{Q},Np}$

of order p^e is $i_\ell^{\mathrm{ab}}(a)$. Therefore,

$$\mathrm{tr}(\tilde{\rho} \circ i_\ell(b)) = \tilde{1}(i_\ell^{\mathrm{ab}}(a)) + \tilde{\chi}(i_\ell^{\mathrm{ab}}(a)) = y + y^{-1}.$$

If $y + y^{-1} = 2$, then it would imply $(y-1)^2 = 0$. But this relation does not hold in R. Hence, $\mathrm{tr}(\tilde{\rho} \circ i_\ell(b)) \neq 2$ and our claim is proved.

Thus, we get a contradiction to our hypothesis that $\bar{\rho}$ is unobstructed. Therefore, in this case as well, $\bar{\rho}$ is obstructed.

Therefore, combining all the results above, we get that if $\bar{\rho}$ is a reducible $\Gamma_1(N)$-modular representation and if $p \mid \phi(N)$, then $\bar{\rho}$ is obstructed. □

Remark. Now suppose that $\bar{\rho}$ is a reducible $\Gamma_1(N)$-modular representation and p does not divide $\phi(N)$. Up to a twist, $\bar{\rho} = 1 \oplus \chi$. If $p \nmid \phi(N)$, then

$$\dim H^1(G_{\mathbb{Q},Np}, 1) = 1,$$

and Tate's global Euler characteristic formula implies that $H^2(G_{\mathbb{Q},Np}, 1) = 0$. The exact sequence of [Bellaïche 2012a] considered on page 25 implies

$$\dim \mathrm{Tan}(D^0_{\bar{\rho}}) = 1 + \dim H^1(G_{\mathbb{Q},Np}, \chi) \dim H^1(G_{\mathbb{Q},Np}, \chi^{-1}).$$

If $\chi = \omega_p$, then by the Kummer theory argument above, it follows that the number of distinct prime divisors of Np is $\dim H^1(G_{\mathbb{Q},Np}, \omega_p)$ and hence, greater than 1. Hence, if $\chi = \omega_p, {\omega_p}^{-1}$, then $\dim \mathrm{Tan}(D^0_{\bar{\rho}}) > 2$, so in this case $\bar{\rho}$ is obstructed. If $\chi \neq \omega_p, {\omega_p}^{-1}$, then by the Greenberg–Wiles version of Poitou–Tate duality, we get that

$$k + 1 \leq \dim H^1(G_{\mathbb{Q},Np}, \chi) \leq k + 1 + \dim A(\chi^{-1}\omega_p)$$

and

$$k' + 1 \leq \dim H^1(G_{\mathbb{Q},Np}, \chi^{-1}) \leq k' + 1 + \dim A'(\chi\omega_p),$$

where k is the number of prime divisors ℓ of N such that $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, k' is the number of prime divisors ℓ of N such that $\chi^{-1}|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, $A(\chi^{-1}\omega_p)$ is the part of the p-torsion subgroup of the class group of the totally real abelian extension F of $\mathbb{Q}$ fixed by $\mathrm{Ker}(\chi^{-1}\omega_p)$ on which $\mathrm{Gal}(F/\mathbb{Q})$ acts by $\chi^{-1}\omega_p$ and $A'(\chi\omega_p)$ is the part of the p-torsion subgroup of the class group of the totally real abelian extension F' of $\mathbb{Q}$ fixed by $\mathrm{Ker}(\chi\omega_p)$ on which $\mathrm{Gal}(F'/\mathbb{Q})$ acts by $\chi\omega_p$.

Proposition 25. *Suppose $p \mid \phi(N)$. Let $\bar{\rho}$ be an absolutely irreducible $\Gamma_1(N)$-modular representation such that $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible for at least one prime ℓ dividing N which is* 1 *mod p. Then, $\bar{\rho}$ is obstructed.*

Proof. Let $\bar{\rho}$ be an absolutely irreducible $\Gamma_1(N)$-modular representation. By [Weston 2005, Lemma 2.5], we have

$$\dim_{\mathbb{F}} H^2(G_{\mathbb{Q},Np}, \mathrm{ad}\bar{\rho}) = \dim_{\mathbb{F}} Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}\bar{\rho}) + \Sigma_{q \mid Np} \dim_{\mathbb{F}} H^0(G_{\mathbb{Q}_q}, \omega_p \otimes \mathrm{ad}\bar{\rho}).$$

Note that $\mathrm{ad}\bar{\rho} = 1 \oplus \mathrm{ad}^0\bar{\rho}$. By Tate's global Euler characteristic formula, we get that $\dim_{\mathbb{F}} H^2(G_{\mathbb{Q},Np}, 1) = n$, where n = number of prime divisors of N which are 1 mod p. For a prime q, $\dim_{\mathbb{F}} H^0(G_{\mathbb{Q}_q}, \omega_p)$ is 1 if q is 1 mod p and 0, otherwise. By removing the contributions of the trivial representation from both sides of the formula above, we get

$$\dim_{\mathbb{F}} H^2(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) = \dim_{\mathbb{F}} Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}^0\bar{\rho}) + \Sigma_{q \mid Np} \dim_{\mathbb{F}} H^0(G_{\mathbb{Q}_q}, \omega_p \otimes \mathrm{ad}^0\bar{\rho}).$$

Now, let ℓ be a prime dividing N such that $\ell - 1$ is divisible by p. Suppose that $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible. Thus, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is an extension of a character χ_1 by a character χ_2. If $\chi_1 \neq \chi_2$, then, by Tate's local Euler characteristic formula, $\mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\chi_1, \chi_2) = 0$. So, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}} = \chi_1 \oplus \chi_2$ and $\mathrm{ad}^0\bar{\rho}|_{G_{\mathbb{Q}_\ell}} = 1 \oplus \chi_1\chi_2{}^{-1} \oplus \chi_1{}^{-1}\chi_2$. As $p \mid \ell - 1$, $\omega_p(G_{\mathbb{Q}_\ell}) = 1$ and $H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \mathrm{ad}^0\bar{\rho}) = H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0\bar{\rho})$ and in this case, both of them are nonzero. Hence, if $\chi_1 \neq \chi_2$, then, by the formula above, $H^2(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) \neq 0$ and $\bar{\rho}$ is obstructed.

If $\chi_1 = \chi_2 = \chi$, then $\rho|_{G_{\mathbb{Q}_\ell}}$ is either $\chi \oplus \chi$ or a nontrivial extension of χ by itself. If $\rho|_{G_{\mathbb{Q}_\ell}} = \chi \oplus \chi$, then clearly $1 \subset \mathrm{ad}^0\bar{\rho}$. If $\rho|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of χ by itself, then choose a basis of $\bar{\rho}$ such that $\bar{\rho}(G_{\mathbb{Q}_\ell})$ is upper triangular and identify $\mathrm{ad}^0\bar{\rho}$ with the subspace of trace 0 matrices of $M_2(\mathbb{F})$. Then, $G_{\mathbb{Q}_\ell}$ acts trivially on the subspace of $\mathrm{ad}^0\bar{\rho}$ generated by the element $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Thus, $1 \subset \mathrm{ad}^0\bar{\rho}$ in this case also. Therefore, if $\chi_1 = \chi_2$, then

$$H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \mathrm{ad}^0\bar{\rho}) = H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0\bar{\rho}) \neq 0$$

and hence, by the formula above, $H^2(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) \neq 0$, which implies that $\bar{\rho}$ is obstructed. This concludes the proof of Proposition 25. □

We would like to determine the cases when the situation considered as above would arise, i.e., when $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ would be reducible. Note that, if f is an eigenform of level N and ρ_f is the Galois representation attached to it, then we know all the possible descriptions of $\rho_f|_{G_{\mathbb{Q}_\ell}}$ for a prime ℓ dividing N. So, we will now analyze all the possible descriptions of $\rho_f|_{G_{\mathbb{Q}_\ell}}$ for an eigenform f lifting $\bar{\rho}$ to determine which of them will make $\bar{\rho}$ obstructed and when can they arise. This will give us some conditions on $\bar{\rho}$ which will force it to be obstructed.

Let f be an eigenform of level N lifting $\bar{\rho}$, ℓ be a prime dividing N which is 1 mod p, ρ_f be the p-adic Galois representation attached to f and π_ℓ be the ℓ-component of the automorphic representation associated to f. If π_ℓ is either principal series or special, then by the local Langlands correspondence, we see that $\rho_f|_{G_{\mathbb{Q}_\ell}}$, and hence $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$, is reducible (see Sections 3 and 5 of [Weston 2004] for a similar analysis). This implies that $\bar{\rho}$ is obstructed. In particular, if $\ell \mid N$ but $\ell^2 \nmid N$, then we know that π_ℓ is either special or principal series (see [Carayol 1989, Section 1.2]). If $(\bar{\rho})^{I_\ell}$, the subspace of $\bar{\rho}$ on which I_ℓ acts trivially, is nonzero, then clearly $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible. If there exists an eigenform f of level M which lifts $\bar{\rho}$ such that there exists at least one prime divisor ℓ of N which is 1 mod p and which does not divide M, then $\bar{\rho}$ is unramified at ℓ and hence, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible. Therefore, in these cases, $\bar{\rho}$ is obstructed.

On the other hand, if π_ℓ is supercuspidal, then, as $\ell \neq 2$, it follows, from the local Langlands correspondence, that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is induced from a character χ of $G_K = \mathrm{Gal}(\overline{\mathbb{Q}_\ell}/K)$, where K is a quadratic extension of $\mathbb{Q}_\ell$ and moreover, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irreducible (see the proof of Proposition 3.2 of [Weston 2004]). Let χ^σ be the $\mathrm{Gal}(K/\mathbb{Q}_\ell)$-conjugate character of χ. So, $\rho_f|_{G_K} \simeq \chi \oplus \chi^\sigma$. If $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible, then clearly $\bar{\chi} = \overline{\chi^\sigma}$, where $\bar{\chi}$ and $\overline{\chi^\sigma}$ are the reductions of χ and χ^σ in characteristic p, respectively. This means that χ/χ^σ factors through the maximal abelian pro-p quotient of G_K. But it follows, from local class field theory and the assumption that $p \mid \ell - 1$, that the maximal abelian pro-p extension of K is also abelian over $\mathbb{Q}_\ell$. So, if χ/χ^σ factors through the maximal abelian pro-p quotient of G_K, then it implies that $\rho_f|_{G_{\mathbb{Q}_\ell}} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_\ell}} \chi$ is reducible, contradicting the hypothesis that it is irreducible. Thus, we see that if π_ℓ is supercuspidal and $p \mid \ell - 1$, then $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is not reducible. In summary we have:

Corollary 26. *Suppose $p \mid \phi(N)$. Let $\bar{\rho}$ be an absolutely irreducible $\Gamma_1(N)$-modular representation satisfying one of the following conditions*:

(1) *There exists at least one prime ℓ which is* 1 *mod p such that $\ell \mid N$ and $\ell^2 \nmid N$.*

(2) *There exists an eigenform f of level M which lifts $\bar{\rho}$ such that there is at least one prime divisor ℓ of N which is* 1 *mod p and which either does not divide M or divides M with multiplicity* 1.

(3) *The subspace $(\bar{\rho})^{I_\ell}$ is nontrivial.*

Then, $\bar{\rho}$ is obstructed.

The only case which remains to be considered is when $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is irreducible for all the prime divisors ℓ of N which are 1 mod p. Let $\bar{\rho}$ be such a $\Gamma_1(N)$-modular representation. In this case, $H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \mathrm{ad}^0\bar{\rho}) = H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0\bar{\rho}) = 0$ for all such primes ℓ. Note that, if f is any eigenform which lifts $\bar{\rho}$, then the local component π_ℓ of the automorphic representation associated to f is supercuspidal at all such primes. The analysis of contributions coming from p and other prime divisors of N in the formula above, which are not 1 mod p, is done in [Weston 2004, Sections 3–5; 2005, Section 3] to give conditions for vanishing of corresponding H^0's. We can assume that those conditions hold as well so that we can ignore those primes. Note that these conditions neither depend on each other nor put any restrictions on each other. So assuming them together does not yield any immediate contradiction. Now, moreover assume that $\bar{\rho}$ does not come from a weight 1 modular form. We need this hypothesis to use the results of Section 3 of [Diamond et al. 2004]. We will analyze the only remaining group $Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}^0 \bar{\rho})$ in this case following the approach of Weston [2004; 2005].

Let g be an eigenform such that ρ_g is a minimally ramified lift of $\bar{\rho}$ (such a lift does exist; see [Diamond et al. 2004, Section 3.2]). Hence, it is an eigenform of level N and its weight k lies between 2 and $p-1$. Let K be a finite extension of $\mathbb{Q}_p$ generated by the eigenvalues of g, $\mathcal{O}$ be its ring of integers and $\mathfrak{m}$ be its maximal ideal. Denote by A_{ρ_g} the module $(K/\mathcal{O})^3$ on which $G_{\mathbb{Q},Np}$ acts via $\mathrm{ad}^0 \rho_g$. We define $H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g})$ as in [Diamond et al. 2004, Section 2.1] (see also [Weston 2005, Section 2.2]).

By [Diamond et al. 2004, Theorem 2.15], we get that

$$H^1_f(G_{\mathbb{Q}}, \rho_g) = H^1_f(G_{\mathbb{Q}}, \rho_g(1)) = 0,$$

where $\rho_g(1)$ is the Tate twist of ρ_g and $H^1_f(G_{\mathbb{Q}}, \rho_g)$, $H^1_f(G_{\mathbb{Q}}, \rho_g(1))$ are the Bloch–Kato Selmer groups defined as in Section 2.1 of [Diamond et al. 2004]. It follows, from [Weston 2005, Lemma 2.6], that

$$\dim_{\mathbb{F}} Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}^0 \bar{\rho}) \leq \dim_{\mathbb{F}} H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g})[\mathfrak{m}],$$

where $H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g})[\mathfrak{m}]$ is the $\mathfrak{m}$-torsion of $H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g})$. Now, [Diamond et al. 2004, Theorem 3.7] implies that the length of $H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g})$ is $v_{\mathfrak{m}}(\eta_g^{\varnothing})$. Here, $v_{\mathfrak{m}}$ is the $\mathfrak{m}$-adic valuation and $\eta_g^{\varnothing}$ is the congruence ideal of g defined in [Diamond et al. 2004, Section 1.7]. From the proof of [Weston 2005, Proposition 4.2], we see that $v_{\mathfrak{m}}(\eta_g^{\varnothing}) > 0$ if and only if $\mathfrak{m}$ is a congruence prime for g, i.e., if there exists an eigenform h which lifts $\bar{\rho}$ and which is not a Galois conjugate of g.

Let ϵ be the nebentypus of g. Let ψ be a character of $(\mathbb{Z}/N\mathbb{Z})^*$ whose order is a power of p (there exists such a character as $p \mid \phi(N)$). Thus, ϵ and $\psi\epsilon$ have the same reduction modulo $\mathfrak{m}$. Hence, from [Carayol 1989, Proposition 3], it follows that there exists an eigenform h of level N and nebentypus $\psi\epsilon$ which lifts $\bar{\rho}$. If h is a Galois conjugate of g, then $\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ and $\psi\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ will be Galois conjugates of each other. Since, g is a minimal lift of $\bar{\rho}$, the p-part of $\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ is trivial. But the p-part of $\psi\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ is nontrivial as ψ is a character of p-power order. So, they can't be Galois conjugates of each other. Therefore, g and h are not Galois conjugates of each other. Thus, $\mathfrak{m}$ is a congruence prime for g and $H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g}) \neq 0$. However, this does not ensure that $Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}^0 \bar{\rho})$ is nonzero because we do not know whether the injection $Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}^0 \bar{\rho}) \to H^1_{\varnothing}(G_{\mathbb{Q}}, A_{\rho_g})[\mathfrak{m}]$ given in the proof of Lemma 2.6 of [Weston 2005], which we used in the last paragraph, is an isomorphism, and in general it is very difficult to prove such a result (see [Weston 2005, Lemma 2.6, Remark 2.7] and the discussion preceding them for more details). But, based on the calculations above, we do expect that $Ш^1(G_{\mathbb{Q},Np}, \omega_p \otimes \mathrm{ad}^0 \bar{\rho})$ is nonzero.

Let us consider the modular representations coming from a weight-1 modular form. In the proposition below, we do not put the condition $p\,|\,\phi(N)$. For a number field L with ring of integers $\mathcal{O}_L$, we denote by $A(L)[p]$ the p-torsion subgroup of the class group of L and we let

$$U(L) = \ker\left(\frac{\mathcal{O}_L^*}{(\mathcal{O}_L^*)^p} \to \left(\prod_{v\,|\,p} \frac{\mathcal{O}_{L_v}^*}{(\mathcal{O}_{L_v}^*)^p}\right)\right).$$

Proposition 27. *Let $\bar{\rho}$ be a $\Gamma_1(N)$-modular representation coming from a* regular *modular form f of weight 1 which has either RM or CM by F* (see [Bellaïche and Dimitrov 2016] for the definition of regular). *Let H be the extension of $\mathbb{Q}$ which is fixed by* $\mathrm{Ker}(\mathrm{ad}^0\bar{\rho})$. *Moreover, assume the following conditions*:

(1) *If $\ell\,|\,N$ and $p\,|\,\ell+1$, then ℓ does not stay inert in F.*

(2) *If ℓ is a prime divisor of N, then $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is irreducible.*

(3) $A(F)[p] = 0$ *and* $\mathrm{Hom}_G(A(H)[p], \mathrm{ad}^0\bar{\rho}) = 0$.

(4) $\mathrm{Hom}_G(U(F), \mathrm{ad}^0\bar{\rho}) = 0$ *and if F is imaginary, then* $\mathrm{Hom}_G(U(H), \mathrm{ad}^0\bar{\rho}) = 0$.

Then, $\bar{\rho}$ is unobstructed.

Proof. Note that, $\bar{\rho} = \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \delta$, where F is a quadratic extension of $\mathbb{Q}$ which is either real or imaginary and $\delta : G_F \to \mathbb{F}^*$ is a character, as it comes from a weight-1 form which has RM or CM. Let $\mathrm{Gal}(F/\mathbb{Q}) = G' = \{1, \sigma\}$. As f is a regular weight-1 eigenform, $\rho_f|_{G_{\mathbb{Q}_p}} = \chi_1 \oplus \chi_2$, where χ_1 and χ_2 are distinct, unramified characters from $G_{\mathbb{Q}_p} \to (\overline{\mathbb{Q}}_p)^*$. So, F and the fixed field H of $\mathrm{Ker}(\mathrm{ad}^0\bar{\rho})$ are unramified at p. From the hypothesis (2) above, we see that all prime divisors ℓ of N are either inert or ramified in F.

Thus, from above, it follows that $\mathrm{ad}^0\bar{\rho} = \epsilon \oplus \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \chi$, where ϵ is the character of $G_\mathbb{Q}$ of order 2 corresponding to F and $\chi = \delta/\delta^\sigma$. So

$$H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) = H^1(G_{\mathbb{Q},Np}, \epsilon) \oplus H^1(G_{\mathbb{Q},Np}, \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \chi).$$

We will analyze each part of this sum separately. We mostly follow the notation of [Bellaïche and Dimitrov 2016].

First, we will look at the group $H^1(G_{\mathbb{Q},Np}, \epsilon)$ appearing above. From the inflation-restriction sequence, we see that $H^1(G_{\mathbb{Q},Np}, \epsilon) = (\mathrm{Hom}(G_{F,S}, \mathbb{F}) \otimes \epsilon)^{G'}$, where S is the set of the places of F dividing Np. From class field theory and the hypothesis (3) above, we have the following exact sequence of G'-modules:

$$0 \to \mathrm{Hom}(G_{F,S}^{\mathrm{ab}}, \mathbb{F}) \to \mathrm{Hom}\left(\prod_{v\,|\,Np} \mathcal{O}_{F_v}^*, \mathbb{F}\right) \to \mathrm{Hom}(\mathcal{O}_F^*, \mathbb{F}).$$

Now, by hypothesis (1), we have:

$$\mathrm{Hom}\Big(\prod_{v\mid Np}\mathcal{O}_{F_v}^*,\mathbb{F}\Big)=\mathrm{Hom}\Big(\prod_{v\mid p}\mathcal{O}_{F_v}\prod_{\substack{\ell\mid N\\ p\mid \ell-1}}\prod_{v\mid\ell}\mathbb{Z}/(\ell-1)\mathbb{Z},\mathbb{F}\Big)$$

as F is unramified at p. The factors in the second product come from places of F dividing such primes and, since ℓ is either inert or ramified in F, each factor appears only once. Observe that G' acts trivially on each of them. Thus,

$$\Big(\mathrm{Hom}\Big(\prod_{\substack{\ell\mid N\\ p\mid \ell-1}}\prod_{v\mid\ell}\mathbb{Z}/(\ell-1)\mathbb{Z},\mathbb{F}\Big)\otimes\epsilon\Big)^{G'}=0.$$

If F is imaginary, then $\mathcal{O}_F^*$ is a finite group and its order is not divisible by p, as p is unramified in F. Thus, $U(F)=\mathcal{O}_F^*/(\mathcal{O}_F^*)^p=0$. Hence, in this case, it follows that $\big(\mathrm{Hom}\big(\prod_{v\mid p}\mathcal{O}_{F_v},\mathbb{F}\big)\otimes\epsilon\big)^{G'}$ has dimension 1. Therefore, $H^1(G_{\mathbb{Q},Np},\epsilon)$ has dimension 1. If F is real, then the free part of $\mathcal{O}_F^*$ has rank 1 and the torsion part is $\{1,-1\}$. Suppose p is not split in F. Then, it is inert in F and the fundamental unit of F generates the residue field of $\mathcal{O}_{F_p}$ over $\mathbb{F}_p$. Suppose p is split in F. Then $\prod_{v\mid p}\mathcal{O}_{F_v}=\mathbb{Z}_p\times\mathbb{Z}_p$ and the action of G' switches them. Moreover, under the G'-equivariant diagonal embedding $\mathcal{O}_F^*\to\prod_{v\mid p}\mathcal{O}_{F_v}^*=\mathbb{Z}_p^*\times\mathbb{Z}_p^*$, the fundamental unit of F gets mapped to an element of the form $(a,-a)$ as the nontrivial element of G' sends the unit to its inverse. So, it follows, from the discussion above and hypothesis (4), that in both the cases, G' acts trivially on the subspace of elements of $\mathrm{Hom}\big(\prod_{v\mid p}\mathcal{O}_{F_v},\mathbb{F}\big)$ which vanish on $\mathcal{O}_F^*$. An element of $\mathrm{Hom}(G_{F,S}^{\mathrm{ab}},\mathbb{F})$ vanishes on the fundamental unit. Thus, combining this and the previous paragraph, we see that G' acts trivially on $\mathrm{Hom}(G_{F,S}^{\mathrm{ab}},\mathbb{F})$, when F is totally real. It follows that, in this case, $H^1(G_{\mathbb{Q},Np},\epsilon)=\big(\mathrm{Hom}(G_{F,S}^{\mathrm{ab}},\mathbb{F})\otimes\epsilon\big)^{G'}=0$.

We shall now analyze the second factor appearing in the sum above. Let H be the extension of $\mathbb{Q}$ which is fixed by $\mathrm{Ker}(\mathrm{ad}^0\bar{\rho})$. Thus, $G=\mathrm{Gal}(H/\mathbb{Q})$ is a dihedral group D_{2n} which is nonabelian and $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ is an irreducible representation of G. By the arguments used above, we get that

$$H^1(G_{\mathbb{Q},Np},\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi)=(H^1(G_{H,S},\mathbb{F})\otimes\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi)^G.$$

By hypothesis (3), $H^1(G_{H,S},\mathbb{F})$ is the subspace of $\mathrm{Hom}\big(\prod_{v\mid Np}\mathcal{O}_{H_v}^*,\mathbb{F}\big)$ vanishing on $\mathcal{O}_H^*$. Since H is unramified at p, we get

$$\mathrm{Hom}\Big(\prod_{v\mid Np}\mathcal{O}_{H_v}^*,\mathbb{F}\Big)=\mathrm{Hom}\Big(\prod_{v\mid p}\mathcal{O}_{H_v}\prod\mathbb{Z}/p\mathbb{Z},\mathbb{F}\Big),$$

where the last product is taken over all places v of H which divide N and whose residue fields have p-th roots of unity. If ℓ is a such a prime, then the G-submodule

$\mathrm{Hom}\big(\prod_{v\mid\ell}\mathbb{Z}/p\mathbb{Z},\mathbb{F}\big)$ of the module above is isomorphic to $\mathrm{Ind}_{D_\ell}^{G}\alpha$, where D_ℓ is the image of $G_{\mathbb{Q}_\ell}$ under $\mathrm{ad}^0\bar{\rho}$ and α is the character by which it acts on the p-th roots of unity. Note that, by hypothesis (2) above, $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ is an irreducible representation of D_ℓ for every $\ell\mid N$. So, $\big(\mathrm{Hom}\big(\prod\mathbb{Z}/p\mathbb{Z},\mathbb{F}\big)\otimes\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi\big)^G=0$.

Assume F is imaginary. As H is unramified at p and $p\nmid|G|$, we see, using arguments from [Bellaïche and Dimitrov 2016, Section 3.2], that every irreducible representation of G occurs in $\mathrm{Hom}(G_{H,S}^{\mathrm{ab}},\mathbb{F})$ with multiplicity at least 1. The additional multiplicities would only arise from $U(H)$. But, from hypothesis (4) above, we have $\mathrm{Hom}_G\big(U(H),\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi\big)=0$. Therefore, $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ occurs in $\mathrm{Hom}(G_{H,S}^{\mathrm{ab}},\mathbb{F})$ with multiplicity 1. Hence, $\big(\mathrm{Hom}(G_{H,S},\mathbb{F})\otimes\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi\big)^G=H^1\big(G_{\mathbb{Q},Np},\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi\big)$ has dimension 1. If F is real, then $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ is totally odd. Thus, from the arguments in [Bellaïche and Dimitrov 2016, Section 3.2], we see that $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ occurs in the subspace of $\mathrm{Hom}\big(\prod_{v\mid p}\mathcal{O}_{H_v}^*,\mathbb{F}\big)$ vanishing on $\mathcal{O}_H^*$ with multiplicity 2. Note that, we do not need to consider contribution from $U(H)$ as $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ does not occur in $\mathcal{O}_H^*/(\mathcal{O}_H^*)^p$. Therefore, combining this with the previous paragraph, we see that $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi$ occurs in $\mathrm{Hom}(G_{H,S}^{\mathrm{ab}},\mathbb{F})$ with multiplicity 2. Hence, $H^1(G_{\mathbb{Q},Np},\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}\chi)$ has dimension 2.

Combining these results we see that if $\bar{\rho}$ satisfies the conditions of the proposition, then $H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0\bar{\rho})$ has dimension 2. Hence, $\bar{\rho}$ is unobstructed in our sense. □

Remark. (1) The hypotheses (3) and (4) are similar to those used by Mazur [1989, Sections 1.12 and 1.13], where he studies unobstructed representations unramified outside a single prime. Note that, for a number field L, $A(L(\zeta_p))[p]=0$ implies that $U(L)=0$ (see the remark after Proposition 1 of [Boston 1990]).

(2) We know that $G'=\mathrm{Gal}(F/\mathbb{Q})$ acts on $A(F)[p]$ by the character ϵ. So, if $A(F)[p]\neq 0$, then it would contribute to $H^1(G_{\mathbb{Q},Np},\epsilon)$ in addition to what we have calculated above. Hence, in this case, we have $\dim H^1(G_{\mathbb{Q},Np},\epsilon)>1$. Thus, considering the above calculations, we see that $\dim H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0\bar{\rho})>2$. Hence, if $A(F)[p]\neq 0$, then $\bar{\rho}$ will be obstructed.

(3) Suppose f is a weight-1 form which has RM or CM by F, and there exists a prime ℓ such that $\ell\mid N$ and $p\mid\ell+1$. Moreover, assume that ℓ stays inert in F. Thus, using the notations of the proof above, we see that

$$\mathrm{Hom}\Big(\prod_{v\mid Np}\mathcal{O}_{F_v}^*,\mathbb{F}\Big)=\mathrm{Hom}\Big(\prod_{v\mid p}\mathcal{O}_{F_v}\prod_{\substack{\ell\mid N\\ p\mid\ell-1}}\prod_{v\mid\ell}\mathbb{Z}/(\ell-1)\mathbb{Z}\prod\mathbb{Z}/p\mathbb{Z},\mathbb{F}\Big),$$

where the last product is taken over all the prime divisors ℓ of N which stay inert in F and which are -1 mod p. Observe that, they are isomorphic to ϵ as G' representations. The projection of the image of $\mathcal{O}_F^*$ in $\prod_{v\mid Np}\mathcal{O}_{F_v}^*$ onto this product is also trivial as F is unramified at p. Hence, this product also contributes to $H^1(G_{\mathbb{Q},Np}, \epsilon)$, making it bigger. All the other calculations at the places above p in the proof above still remain valid. As a result, we see that $\dim H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0\bar{\rho}) > 2$. Hence, in this case, $\bar{\rho}$ is obstructed.

Note that, the assumptions in the proposition above do not yield any immediate contradiction. Thus, we see that our notion of unobstructedness is weaker than Mazur's notion of unobstructedness.

Proposition 28. *Let $\bar{\rho}$ be a modular representation satisfying the hypotheses of Proposition 27 and suppose $p \mid \phi(N)$. Let the p-primary part of $(\mathbb{Z}/N\mathbb{Z})^*$ be $\mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z}$. Assume moreover that*:

(1) *$\bar{\rho}$ is a new $\Gamma_1(N)$-modular representation.*

(2) *For all primes ℓ dividing N such that $p \mid \ell^2 - 1$, U_ℓ acts like 0 on all newforms of level N which lift $\bar{\rho}$.*

Then, the corresponding local component $A_{\bar{\rho}}$ of the mod p Hecke algebra is isomorphic to $\mathbb{F}[\![x, y]\!][y_1, \ldots, y_n]/(y_1^{p^{e_1}}, \ldots, y_n^{p^{e_n}})$ and thus, is not reduced.

Proof. As $\bar{\rho}$ satisfies the hypotheses of Proposition 27, it satisfies the hypotheses of Theorem 3 as well. Indeed, $\bar{\rho}$ comes from a regular, weight-1 form and is unramified at p. So, $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is of the required form. If ℓ is a prime divisor of N which is 1 mod p, then it splits completely in $\mathbb{Q}(\zeta_p)$. As $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is irreducible, $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is also irreducible. Therefore, from the results of Böckle [2001], Diamond [1996], Diamond et al. [2004], and Kisin [2004], it follows that the surjective map $R_{\bar{\rho}} \to \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. The hypothesis (2) along with the proof of Proposition 20 and the proof of Theorem 1 implies that the kernel of the surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \to A_{\bar{\rho}}^{\Gamma_1(N)}$ is $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. By Proposition 27, we see that $R_{\bar{\rho}} \simeq \mathcal{O}[\![x, y, T]\!][z_1, \ldots, z_n]/(z_1^{p^{e_1}} - 1, \ldots, z_n^{p^{e_n}} - 1)$. Combining all of the above gives us that $A_{\bar{\rho}} \simeq \mathbb{F}[\![x, y]\!][y_1, \ldots, y_n]/(y_1^{p^{e_1}}, \ldots, y_n^{p^{e_n}})$. □

Remark. (1) If the first condition of Proposition 28 is satisfied, then to check the second condition, it is sufficient, by Lemma 19, to check that $\bar{\rho}|_{I_\ell}$ is irreducible.

(2) Suppose $\bar{\rho}$ is a $\Gamma_1(N)$-modular representation which satisfies the two assumptions of Proposition 28 and the assumptions of Theorem 3. Moreover, assume $p \mid \phi(N)$ and let the p-primary part of $(\mathbb{Z}/N\mathbb{Z})^*$ be $\mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z}$. Then, from the proof above, we see that $A_{\bar{\rho}}^{\Gamma_1(N)} \simeq R_{\bar{\rho}}/(\pi, T)$. Now, as $p \neq 2$,

$$R_{\bar{\rho}} \simeq R_{\bar{\rho}}^0[T, y_1, \ldots, y_n]/(y_1^{p^{e_1}} - 1, \ldots, y_n^{p^{e_n}} - 1),$$

where $R_{\bar{\rho}}^0$ is the universal deformation ring of $\bar{\rho}$ with constant determinant (see the proof of [BK, Lemma 19] for more details). So, $A_{\bar{\rho}}^{\Gamma_1(N)}$ is isomorphic to $\tilde{R}_{\bar{\rho}}^0[z_1, \ldots, z_n]/(z_1^{p^{e_1}}, \ldots, z_n^{p^{e_n}})$ and hence, is not reduced. So, the assumptions of Proposition 27 are not necessary to get nonreduced Hecke algebras but are necessary to find the precise structure of the Hecke algebra.

We now give some examples of nonreduced Hecke algebras following the previous remark. Let ℓ be a prime such that $p \mid \ell - 1$ and ℓ is 3 modulo 4. Let $K = \mathbb{Q}(\sqrt{-\ell})$ and let h_K be its class number. If p splits in K, then define n to be the smallest integer such that p is not split in the anticyclotomic extension of degree ℓ^n of K and $\ell^n \nmid h_K$. Otherwise, define it to be the smallest integer such that $\ell^n \nmid h_K$. Let $\chi : G_K \to \mathbb{Z}_\ell \to \mathbb{Z}/\ell^n\mathbb{Z} \to \bar{\mathbb{F}}^*$, where the first map is given by the anticyclotomic $\mathbb{Z}_\ell$ extension of K, and the last map is the inclusion of the ℓ^n-th roots of unity into $\bar{\mathbb{F}}^*$. Now, we see, from a classical theorem of Hecke, that $\bar{\rho} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ on $G_{\mathbb{Q},p\ell}$ is an odd, irreducible representation coming from a weight-1 newform f of level ℓ^{2n+3}.

As $\chi^\sigma = \chi^{-1}$ and χ has odd order, they are distinct characters of G_K and they only coincide on $\ker(\chi)$. Now, ℓ ramifies in K and the prime v' of K lying above ℓ ramifies in the anticyclotomic extension of K of degree ℓ^n as $\ell^n \nmid h_K$. As a consequence, we see that $\bar{\rho}|_{I_\ell}$ and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ are irreducible. As ℓ splits completely in $\mathbb{Q}(\zeta_p)$, it follows that $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is also irreducible. If p is inert in K, then clearly $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is a sum of distinct characters as p is unramified in K. If p is split in K, then for a place v of K above p, χ and χ^σ are distinct characters of G_{K_v}, as χ has odd order and p does not split completely in the anticyclotomic extension of K of degree ℓ^n. As the anticyclotomic extension of K is unramified at p, we see that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is not a direct sum of characters which are cyclotomic twists of each other. Therefore, $\bar{\rho}$ is a new $\Gamma_1(\ell^{2n+3})$-modular representation satisfying the two conditions of Proposition 28 and the conditions of Theorem 3. Hence, from the remark above, it follows that $A_{\bar{\rho}}^{\Gamma_1(\ell^{2n+3})}$ is not reduced.

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Split abelian surfaces over finite fields and reductions of genus-2 curves

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Dedicated to the memory of Professor Tom M. Apostol

For prime powers q, let $\operatorname{split}(q)$ denote the probability that a randomly chosen principally polarized abelian surface over the finite field $\mathbb{F}_q$ is not simple. We show that there are positive constants c_1 and c_2 such that, for all q,

$$c_1(\log q)^{-3}(\log\log q)^{-4} < \operatorname{split}(q)\sqrt{q} < c_2(\log q)^4(\log\log q)^2,$$

and we obtain better estimates under the assumption of the generalized Riemann hypothesis. If A is a principally polarized abelian surface over a number field K, let $\pi_{\text{split}}(A/K, z)$ denote the number of prime ideals $\mathfrak{p}$ of K of norm at most z such that A has good reduction at $\mathfrak{p}$ and $A_{\mathfrak{p}}$ is not simple. We conjecture that, for sufficiently general A, the counting function $\pi_{\text{split}}(A/K, z)$ grows like $\sqrt{z}/\log z$. We indicate why our theorem on the rate of growth of $\operatorname{split}(q)$ gives us reason to hope that our conjecture is true.

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1. Introduction

Let A/K be a principally polarized absolutely simple abelian variety over a number field. Murty and Patankar have conjectured [Murty 2005; Murty and Patankar 2008] that if the absolute endomorphism ring of A is commutative, then the reduction $A_\mathfrak{p}$ is simple for almost all primes $\mathfrak{p}$ of $\mathcal{O}_K$. (See [Achter 2009; Zywina 2014] for work on this conjecture.) Given this, it makes sense to try to quantify the (conjecturally density zero) set of primes of good reduction for which $A_\mathfrak{p}$ is *split*; that is, for which $A_\mathfrak{p}$ is isogenous to a product of abelian varieties of smaller dimension. Specifically, define the counting function

$$\pi_{\mathrm{split}}(A/K, z) = \#\bigl\{\mathfrak{p} : \mathcal{N}(\mathfrak{p}) \le z \text{ and } A_\mathfrak{p} \text{ is split}\bigr\}.$$

Some upper bounds for the rate of growth of this function are available. For instance, a special case of [Achter 2012, Theorem B, p. 42] states that if the image of the ℓ-adic Galois representation attached to the g-dimensional abelian variety A is the full group of symplectic similitudes, then

$$\pi_{\mathrm{split}}(A/K, z) \ll \frac{z(\log\log z)^{1+\frac{1}{3(2g^2+g+1)}}}{(\log z)^{1+\frac{1}{6(2g^2+g+1)}}} \qquad \text{for all } z \ge 3;$$

if one is willing to assume a generalized Riemann hypothesis, one can further show that

$$\pi_{\mathrm{split}}(A/K, z) \ll z^{1-\frac{1}{4g^2+3g+4}}(\log z)^{\frac{2}{4g^2+3g+4}} \qquad \text{for all } z \ge 3. \tag{1}$$

However, there is no reason to believe that even (1) does a very good job of capturing the actual behavior of the function $\pi_{\mathrm{split}}(A/K, z)$. The purpose of the present paper is to explain and support the following hope.

Conjecture 1.1. *Let A/K be a principally polarizable abelian surface with absolute endomorphism ring* $\operatorname{End}_{\bar{K}} A \cong \mathbb{Z}$. *Then there is a constant $C_A > 0$ such that*

$$\pi_{\mathrm{split}}(A/K, z) \sim C_A \frac{\sqrt{z}}{\log z} \quad \textit{as } z \to \infty.$$

This statement bears some resemblance to the Lang–Trotter conjecture [1976], whose enunciation we briefly recall. Let $E/\mathbb{Q}$ be an elliptic curve with $\operatorname{End}_{\overline{\mathbb{Q}}} E \cong \mathbb{Z}$, and fix a nonzero integer a. Let $\pi(E, a, z)$ be the number of primes $p < z$ such that $E_p(\mathbb{F}_p) - (p+1) = a$. Then Lang and Trotter conjecture that $\pi(E, a, z) \sim C_{E,a}\sqrt{z}/\log z$ as $z \to \infty$, for some constant $C_{E,a}$. They also give a conjectural formula for the constant $C_{E,a}$, but we shall ignore such finer information here.

In Section 2, we review a framework under which one might expect such counting functions to grow like $\sqrt{z}/\log z$. Roughly speaking, the philosophy of Section 2 suggests that Conjecture 1.1 should hold if the probability that a randomly chosen principally polarized abelian surface over $\mathbb{F}_q$ is split varies like $q^{-1/2}$. The bulk of our paper is taken up with a proof of a theorem which says that, up to factors of $\log q$, this is indeed the case.

For every positive integer n we let $\mathcal{A}_n$ denote the moduli stack of principally polarized n-dimensional abelian varieties, so that for every field K the objects of $\mathcal{A}_n(K)$ are the K-isomorphism classes of such principally polarized varieties over K. For every n and K we also let $\mathcal{A}_{n,\mathrm{split}}(K)$ denote the subset of $\mathcal{A}_n(K)$ consisting of the principally polarized abelian varieties (A, λ) for which A is not simple over K. This is perhaps an abuse of notation, because there is no geometrically defined substack $\mathcal{A}_{n,\mathrm{split}}$ giving rise to the sets $\mathcal{A}_{n,\mathrm{split}}(K)$; our definition of "split" is sensitive to the field of definition.

Theorem 1.2. *We have*

$$\frac{1}{(\log q)^3(\log\log q)^4} \ll \frac{\#\mathcal{A}_{2,\mathrm{split}}(\mathbb{F}_q)}{q^{5/2}} \ll (\log q)^4(\log\log q)^2 \quad \textit{for all } q.$$

If the generalized Riemann hypothesis is true, *we have*

$$\frac{1}{(\log q)(\log\log q)^6} \ll \frac{\#\mathcal{A}_{2,\mathrm{split}}(\mathbb{F}_q)}{q^{5/2}} \ll (\log q)^2(\log\log q)^4 \quad \textit{for all } q.$$

Since $\mathcal{A}_2$ is irreducible of dimension 3, Theorem 1.2 implies that, up to logarithmic factors, the chance that a randomly chosen principally polarized abelian surface over $\mathbb{F}_q$ is split varies like $q^{-1/2}$.

The paper closes by presenting some numerical data, including evidence in favor of Conjecture 1.1. We also indicate what we believe to be true when one considers varieties that are geometrically split, and not just split over the base field.

After the first-named author gave a preliminary report on this work, including some data obtained using `sage`, William Stein suggested contacting Andrew Sutherland for help with more extensive calculations. Sutherland provided us with the program `smalljac` [Kedlaya and Sutherland 2008], which we ran on our own computers to obtain data on the mod-p reductions of the curve $y^2 = x^5 + x + 6$ over $\mathbb{Q}$; later, Sutherland kindly used his own computers, running a program based on the algorithm in [Harvey and Sutherland 2016], to provide us with reduction data for this curve for all primes up to 2^{30}. It is a pleasure to acknowledge Sutherland's assistance. The data presented in Sections 11.1 and 11.2 was obtained using gp and MAGMA.

As we were writing up the various asymptotic estimates of number-theoretic functions that appear in this paper, the second-named author thought frequently of Professor Tom M. Apostol, in whose undergraduate Caltech course Math 160 he first became familiar with such computations. Not long after we completed this paper, Apostol passed away. We dedicate this work to his memory.

Notation and conventions. If Z is a set of real numbers and f and g are real-valued functions on Z, we use the Vinogradov notation

$$f(z) \ll g(z) \quad \text{for } z \in Z$$

to mean that there is a constant C such that $|f(z)| \le C|g(z)|$ for all $z \in Z$. If Z contains arbitrarily large positive reals, we use

$$f(z) \sim g(z) \quad \text{as } z \to \infty$$

to mean that $f(z)/g(z) \to 1$ as $z \to \infty$, and we write $f(z) \asymp g(z)$ to mean that there are positive constants C_1 and C_2 such that $C_1|g(z)| \le |f(z)| \le C_2|g(z)|$ for all sufficiently large z.

When we are working over a finite field $\mathbb{F}_q$, we will use without further comment the letter p to denote the prime divisor of q. This convention unfortunately conflicts with the standard use in analytic number theory of the letter p as a generic prime, for instance when writing Euler product representations of arithmetic functions. In such situations in this paper (see for example equation (4) in Section 4), we will instead use ℓ to denote a generic prime, and we explicitly allow the possibility that $\ell = p$.

A *curve* over a field K is a smooth, projective, irreducible variety over K of dimension one, and a *Jacobian* is the neutral component of the Picard scheme of such a curve.

2. Conjectures of Lang–Trotter type

Let $\mathcal{M}$ be a moduli space of PEL type [Shimura 1966]. Let K be a number field, let $\Delta \in \mathcal{O}_K$ be nonzero, and let S be the set of primes of K that do not divide Δ. If $\mathfrak{p}$ is a prime of K we let $\mathbb{F}_\mathfrak{p}$ denote its residue field. Equip each finite set $\mathcal{M}(\mathbb{F}_\mathfrak{p})$ with the uniform probability measure, and let $\underline{A}_\mathfrak{p}$ be a random variable on $\mathcal{M}(\mathbb{F}_\mathfrak{p})$. Suppose that for each $\mathfrak{p} \in S$ a subset $T_\mathfrak{p} \subset \mathcal{M}(\mathbb{F}_\mathfrak{p})$ is specified, with indicator function $I_\mathfrak{p}$. Let $\underline{A} = \prod_{\mathfrak{p} \in S} \underline{A}_\mathfrak{p}$, and let

$$\pi(\underline{A}, I, z) = \sum_{\mathfrak{p} \in S : \mathcal{N}(\mathfrak{p}) < z} \frac{\# T_\mathfrak{p}}{\# \mathcal{M}(\mathbb{F}_\mathfrak{p})}$$

be the expected value of $\sum_{\mathfrak{p} \in S : \mathcal{N}(\mathfrak{p}) \le z} I_\mathfrak{p}(\underline{A}_\mathfrak{p})$. If $\# T_\mathfrak{p}/\#\mathcal{M}(\mathbb{F}_\mathfrak{p}) \asymp 1/\mathcal{N}(\mathfrak{p})^m$, then Landau's prime ideal theorem [1903a, p. 670] yields the estimate $\pi(\underline{A}, I, z) \asymp \int_2^z 1/(x^m \log x)\, dx$. In particular, for $m = \frac{1}{2}$ one finds that $\pi(\underline{A}, I, z) \asymp \sqrt{z}/\log z$.

Henceforth, assume $\mathcal{M}$ and $T_\mathfrak{p}$ are chosen so that the above holds with $m = \frac{1}{2}$. Now suppose that $A \in \mathcal{M}(\mathcal{O}_K[1/\Delta])$, and let

$$\pi(A, I, z) = \sum_{\mathfrak{p} \in S : \mathcal{N}(\mathfrak{p}) \le z} I_\mathfrak{p}(A_\mathfrak{p}).$$

If one *assumes* that (A is sufficiently general, and thus) A is well modeled by the random variable $\underline{A}$, then one predicts that

$$\pi(A, I, z) \asymp \frac{\sqrt{z}}{\log z}. \tag{2}$$

(By "sufficiently general" one might mean, for example, that the Mumford–Tate group of A is the same as the group attached to the Shimura variety $\mathcal{M}$; but this will not be pursued here.)

For instance, let $\mathcal{A}_1$ be the moduli stack of elliptic curves, and let a be a nonzero integer. On one hand, since $\mathcal{A}_1$ is irreducible and one-dimensional, we have the estimate $\#\mathcal{A}_1(\mathbb{F}_q) \asymp q$. On the other hand, the number of isomorphism classes of elliptic curves over $\mathbb{F}_q$ with trace of Frobenius a is the Kronecker class number $H(a^2 - 4q)$. Up to (at worst) logarithmic factors, the class number $H(a^2 - 4q)$ grows like $\sqrt{|a^2 - 4q|} \sim 2\sqrt{q}$ (see Lemma 4.4). In this case, the prediction (2) yields the Lang–Trotter conjecture.

We interpret Theorem 1.2 as saying that the number of principally polarized split abelian surfaces over $\mathbb{F}_q$ is approximately $q^{5/2}$. This, combined with the fact that $\dim \mathcal{A}_2 = 3$ and thus $\#\mathcal{A}_3(\mathbb{F}_q) \asymp q^3$, is the inspiration behind Conjecture 1.1.

In spite of the apparent depth and difficulty of the Lang–Trotter conjecture, we are certainly not the first to have attempted to formulate analogous conjectures in related contexts. Murty [1999] poses the problem of counting the primes $\mathfrak{p}$ for which, in a given Galois representation $\rho : \mathrm{Gal}(K) \to \mathrm{GL}_r(\mathcal{O}_\lambda)$, the trace of Frobenius $\mathrm{tr}(\rho(\sigma_\mathfrak{p}))$ is a given number a. The work of Bayer and González [1997] is philosophically more similar to the present paper. Bayer and González consider a modular abelian variety $A/\mathbb{Q}$ and study the number of primes p such that the reduction A_p has p-rank zero. Unfortunately, in most situations, both [Bayer and González 1997, Conjecture 8.2, p. 69] and [Murty 1999, Conjecture 2.15, p. 199] predict a counting function $\pi(z)$ which either grows like $\log\log z$ or is absolutely bounded. In contrast, Conjecture 1.1 has the modest virtue of involving functions that grow visibly over the range of computationally feasible values of z.

3. Split abelian surfaces over finite fields

In this section we articulate the proof of Theorem 1.2, which gives asymptotic upper and lower bounds on the number of principally polarized abelian surfaces over finite fields such that the abelian surface is isogenous to a product of elliptic curves. There are several different types of such surfaces, each of which we analyze separately.

First, there are the abelian surfaces over $\mathbb{F}_q$ that are isogenous to a product $E_1 \times E_2$ of two ordinary elliptic curves, with E_1 and E_2 lying in two different isogeny classes. We call this the *ordinary split nonisotypic* case.

Proposition 3.1. *The number W_q of principally polarized ordinary split nonisotypic abelian surfaces over $\mathbb{F}_q$ satisfies*

$$W_q \ll \begin{cases} q^{5/2}(\log q)^4(\log\log q)^2 & \text{for all } q\text{, unconditionally,} \\ q^{5/2}(\log q)^2(\log\log q)^4 & \text{for all } q\text{, under GRH.} \end{cases}$$

Second, there are the abelian surfaces over $\mathbb{F}_q$ that are isogenous to the square of an ordinary elliptic curve. We call this the *ordinary split isotypic* case.

Proposition 3.2. *The number X_q of principally polarized ordinary split isotypic abelian surfaces over $\mathbb{F}_q$ satisfies*

$$X_q \ll \begin{cases} q^2(\log q)^2|\log\log q| & \text{for all } q\text{, unconditionally,} \\ q^2(\log q)(\log\log q)^2 & \text{for all } q\text{, under GRH.} \end{cases}$$

Third, there are the abelian surfaces over $\mathbb{F}_q$ that are isogenous to the product of two elliptic curves, exactly one of which is supersingular. We call this the *almost ordinary split* case.

Proposition 3.3. *The number Y_q of principally polarized almost ordinary split abelian surfaces over $\mathbb{F}_q$ satisfies*

$$Y_q \ll \begin{cases} q^2(\log q)(\log\log q)^2 & \text{for all } q\text{, unconditionally,} \\ q^2|\log\log q|^3 & \text{for all } q\text{, under GRH.} \end{cases}$$

And fourth, there are the abelian surfaces over $\mathbb{F}_q$ that are isogenous to the product of two supersingular elliptic curves. We call this the *supersingular split* case.

Proposition 3.4. *The number Z_q of principally polarized supersingular split abelian surfaces over $\mathbb{F}_q$ satisfies $Z_q \ll q^2$ for all q.*

To prove the lower bound in Theorem 1.2, we estimate the number of ordinary split nonisotypic surfaces.

Proposition 3.5. *The number W_q of principally polarized ordinary split nonisotypic abelian surfaces over $\mathbb{F}_q$ satisfies*

$$W_q \gg \begin{cases} \dfrac{q^{5/2}}{(\log q)^3(\log\log q)^4} & \text{for all } q\text{, unconditionally,} \\[2ex] \dfrac{q^{5/2}}{(\log q)(\log\log q)^6} & \text{for all } q\text{, under GRH.} \end{cases}$$

It is clear that together these propositions provide a proof of Theorem 1.2. We will prove the propositions in the following sections. We begin with some background information and results on endomorphism rings of elliptic curves over finite fields (Section 4) and a review of "gluing" elliptic curves together (Section 5).

4. Endomorphism rings of elliptic curves over finite fields

In this section we set notation and give some background information on endomorphism rings of elliptic curves over finite fields. With the exception of the concepts of “strata” and of the “relative conductor”, most of the results on endomorphism rings we mention are standard (see [Waterhouse 1969, Chapter 4] and [Schoof 1987], and note that [Schoof 1987, Theorem 4.5, p. 194] corrects a small error in [Waterhouse 1969, Theorem 4.5, p. 541]).

Let E be an elliptic curve over a finite field $\mathbb{F}_q$. The substitution $x \mapsto x^q$ induces an endomorphism $\mathrm{Fr}_E \in \operatorname{End} E$ called the *Frobenius* endomorphism. The characteristic polynomial of Fr_E (acting, say, on the ℓ-adic Tate module of E for some $\ell \neq p$) is of the form $f_E(T) = T^2 - a(E)T + q$ for an integer $a(E)$, the *trace of Frobenius*. Two elliptic curves E and E' are isogenous if and only if $a(E) = a(E')$, and Hasse [1936a; 1936b; 1936c] showed that $|a(E)| \leq 2\sqrt{q}$. We will denote the isogeny class corresponding to a by

$$\mathcal{I}(\mathbb{F}_q, a) = \{E/\mathbb{F}_q : a(E) = a\}.$$

The isogeny class $\mathcal{I}(\mathbb{F}_q, a)$ is called *ordinary* if $\gcd(a, q) = 1$, and *supersingular* otherwise (see [Waterhouse 1969, p. 526 and Chapter 7]). The supersingular curves E are characterized by the property that $E[p](\overline{\mathbb{F}}_q) \cong \{0\}$.

If $E/\mathbb{F}_q$ is a supersingular elliptic curve, then $\operatorname{End}_{\overline{\mathbb{F}}_q} E$ is a maximal order in $\mathbb{Q}_{p,\infty}$, the quaternion algebra over $\mathbb{Q}$ ramified exactly at $\{p, \infty\}$. There are two possibilities for $\operatorname{End} E$ itself. It may be that all of the geometric endomorphisms of E are already defined over $\mathbb{F}_q$, so that $\operatorname{End} E$ is a maximal order in $\mathbb{Q}_{p,\infty}$; this happens when q is a square and $a(E)^2 = 4q$. The other possibility is that $\operatorname{End} E$ is an order in an imaginary quadratic field; in this case, the discriminant of $\operatorname{End} E$ is either $-p$, $-4p$, -3, or -4. (See Table 1 in Section 8 for the exact conditions that determine the various cases.)

Suppose $\mathcal{I}(\mathbb{F}_q, a)$ is an isogeny class with $a^2 \neq 4q$. Then

$$\mathcal{O}_{a,q} := \mathbb{Z}[T]/(T^2 - aT + q)$$

is an order in the imaginary quadratic field

$$K_{a,q} := \mathbb{Q}\big(\sqrt{a^2 - 4q}\big),$$

and is isomorphic to the subring $\mathbb{Z}[\mathrm{Fr}_E]$ of $\operatorname{End} E$ for every $E \in \mathcal{I}(\mathbb{F}_q, a)$. An order $\mathcal{O}$ in $K_{a,q}$ occurs as $\operatorname{End} E$ for some $E \in \mathcal{I}(\mathbb{F}_q, a)$ if and only if $\mathcal{O} \supseteq \mathcal{O}_{a,q}$ and $\mathcal{O}$ is maximal at p (see [Waterhouse 1969, Theorem 4.2, pp. 538–539] or [Schoof 1987, Theorem 4.3, p. 193]). Note that the maximality at p is automatic when $\mathcal{I}(\mathbb{F}_q, a)$ is ordinary, because in that case q is coprime to the discriminant $a^2 - 4q$ of $\mathcal{O}_{a,q}$. If we let $\mathcal{I}(\mathbb{F}_q, a, \mathcal{O})$ denote the set of isomorphism classes of elliptic curves in

$\mathcal{I}(\mathbb{F}_q, a)$ with endomorphism ring $\mathcal{O}$, we can write $\mathcal{I}(\mathbb{F}_q, a)$ as a disjoint union

$$\mathcal{I}(\mathbb{F}_q, a) = \bigsqcup_{\mathcal{O} \supseteq \mathcal{O}_{a,q}} \mathcal{I}(\mathbb{F}_q, a, \mathcal{O}),$$

where $\mathcal{O}$ ranges over all orders of $K_{a,q}$ that contain $\mathcal{O}_{a,q}$ and that are maximal at p. If a is coprime to q, or if $a = 0$ and q is not a square, then each of the sets $\mathcal{I}(\mathbb{F}_q, a, \mathcal{O})$ appearing in the equality above is a torsor for the class group $\mathrm{Cl}(\mathcal{O})$ of the order $\mathcal{O}$. In particular, $\#\mathcal{I}(\mathbb{F}_q, a, \mathcal{O})$ is equal to the class number $h(\mathcal{O})$ of $\mathcal{O}$ (see [Schoof 1987, Theorem 4.5, p. 194]).

We will refer to a nonempty set of the form $\mathcal{I}(\mathbb{F}_q, a, \mathcal{O})$ as a *stratum* of elliptic curves over $\mathbb{F}_q$. Given a stratum $\mathcal{S}$, we will denote the associated trace by $a(\mathcal{S})$ and the associated quadratic order by $\mathcal{O}_{\mathcal{S}}$. If $\mathcal{S}$ and $\mathcal{S}'$ are two strata over $\mathbb{F}_q$, we say that $\mathcal{S}$ and $\mathcal{S}'$ are *isogenous*, and write $\mathcal{S} \sim \mathcal{S}'$, if the elliptic curves in $\mathcal{S}$ are isogenous to those in $\mathcal{S}'$ — that is, if $a(\mathcal{S}) = a(\mathcal{S}')$.

For any imaginary quadratic order $\mathcal{O}$ we let $\Delta(\mathcal{O})$ denote the discriminant of $\mathcal{O}$ and $\Delta^*(\mathcal{O})$ the associated fundamental discriminant — that is, the discriminant of the integral closure of $\mathcal{O}$ in its field of fractions. Then

$$\Delta(\mathcal{O}) = \mathfrak{f}(\mathcal{O})^2 \Delta^*(\mathcal{O}),$$

where $\mathfrak{f}(\mathcal{O})$ is the *conductor* of $\mathcal{O}$. For a trace of Frobenius a with $a^2 \neq 4q$ we will write $\Delta_{a,q}$, $\mathfrak{f}_{a,q}$, and $\Delta^*_{a,q}$ for the corresponding quantities associated to $\mathcal{O}_{a,q}$.

Let $E/\mathbb{F}_q$ be an elliptic curve whose endomorphism ring is a quadratic order. We define the *relative conductor* $\mathfrak{f}_{\mathrm{rel}}(E)$ of E by

$$\mathfrak{f}_{\mathrm{rel}}(E) = \frac{\mathfrak{f}(\mathcal{O}_{a,q})}{\mathfrak{f}(\mathrm{End}\, E)};$$

this quantity is also equal to the index of $\mathcal{O}_{a,q} \cong \mathbb{Z}[\mathrm{Fr}_E]$ in $\mathrm{End}\, E$. If $E/\mathbb{F}_q$ is a supersingular elliptic curve with endomorphism ring equal to an order in a quaternion algebra, we adopt the convention $\mathfrak{f}_{\mathrm{rel}}(E) = 0$. The relative conductor depends only on the stratum of E, so for a stratum $\mathcal{S}$ we may define $\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})$ to be the relative conductor of any curve in $\mathcal{S}$.

Proposition 4.1. *Let $E/\mathbb{F}_q$ be an elliptic curve with* End E *a quadratic order.*

(a) *The relative conductor $\mathfrak{f}_{\mathrm{rel}}(E)$ is the largest integer r such that there exists an integer b with*
$$\frac{\mathrm{Fr}_E - b}{r} \in \mathrm{End}\, E.$$

(b) *The relative conductor $\mathfrak{f}_{\mathrm{rel}}(E)$ is the largest integer r for which* Fr_E *acts as an integer on the group scheme $E[r]$.*

(c) *If E is ordinary, the relative conductor $\mathfrak{f}_{\mathrm{rel}}(E)$ is the largest integer r, coprime to q, for which* Fr_E *acts as an integer on the group $E[r](\overline{\mathbb{F}}_q)$.*

Proof. Let $\mathcal{O}$ be the maximal order containing $\operatorname{End} E$ and let ω be an element of $\mathcal{O}$ such that $\mathcal{O} = \mathbb{Z}[\omega]$. Write $\mathrm{Fr}_E = u + v\omega$ for integers u and v; then $\mathbb{Z}[\mathrm{Fr}_E] = \mathbb{Z} + v\mathcal{O}$, so $v = \mathfrak{f}(\mathbb{Z}[\mathrm{Fr}_E])$.

On one hand, suppose r is an integer for which there is an integer b with $(\mathrm{Fr}_E - b)/r \in \operatorname{End} E$. Then $\operatorname{End} E \supseteq \mathbb{Z} + (v/r)\mathcal{O}$, so r is a divisor of the relative conductor. On the other hand, if s is the relative conductor of E, then $\operatorname{End} E = \mathbb{Z} + (v/s)\mathcal{O} = \mathbb{Z}[(v/s)\omega]$, so $(\mathrm{Fr}_E - u)/s$ is an element of $\operatorname{End} E$. This proves (a).

If Fr_E acts as an integer b on the group scheme $E[r]$, then the endomorphism $\mathrm{Fr}_E - b$ kills $E[r]$. This implies that $\mathrm{Fr}_E - b$ factors through multiplication-by-r, which means that $(\mathrm{Fr}_E - b)/r$ lies in $\operatorname{End} E$. Conversely, if $(\mathrm{Fr}_E - b)/r$ lies in $\operatorname{End} E$, then Fr_E acts on $E[r]$ as the integer b. Thus, (b) follows from (a).

Suppose E is ordinary. The endomorphism Fr_E does not act as an integer on the group scheme $E[p]$, because it acts noninvertibly (consider the local part of $E[p]$), but not as zero (consider the reduced part of $E[p]$). Therefore, the integer defined by (b) will not change if we add the requirement that r be coprime to p. For integers r coprime to p, the group scheme $E[r]$ is determined by the Galois module $E[r](\overline{\mathbb{F}}_q)$. Thus, (c) follows from (b). □

Corollary 4.2. *Let $E/\mathbb{F}_q$ be an elliptic curve with relative conductor r, and let n be a positive integer. The largest divisor d of n such that* Fr_E *acts as an integer on $E[d]$ is equal to* $\gcd(n, r)$.

Proof. When $\operatorname{End} E$ is a quadratic order, this follows immediately from Proposition 4.1. If the endomorphism ring of E is an order in a quaternion algebra, then q is a square and $\mathrm{Fr}_E = \pm\sqrt{q}$; that is, Fr_E *is* an integer, so that $d = n = \gcd(n, 0)$. □

Later in the paper we will need to have bounds on the sizes of the automorphism groups of schemes of the form $E[n]$ for ordinary E and positive integers n. Our bounds will involve the Euler function $\varphi(n)$ as well as the arithmetic function ψ defined by $\psi(n) = n \prod_{\ell \mid n} (1 + 1/\ell)$.

Proposition 4.3. *Let E be an elliptic curve over $\mathbb{F}_q$, let n be a positive integer, and let $g = \gcd(n, \mathfrak{f}_{\mathrm{rel}}(E))$. If E is supersingular, assume that n is coprime to q. Then*

$$\varphi(n) \leq \frac{\#\operatorname{Aut} E[n]}{g^2 \varphi(n)} \leq \psi(n). \tag{3}$$

Proof. Every term in the inequality is multiplicative in n, so it suffices to consider the case where n is a prime power ℓ^e.

Suppose $\ell = p$. In this case, E must be ordinary by assumption. Note that the relative conductor divides the discriminant $a^2 - 4q$, where $a = a(E)$ is coprime to p because E is ordinary. Therefore the relative conductor is coprime to p, so $g = 1$.

The group scheme $E[n]$ is the product of a reduced-local group scheme G_1 and a local-reduced group scheme G_2, each of rank n. The group scheme G_1 is geometrically isomorphic to $\mathbb{Z}/n$, with Frobenius acting as multiplication by an integer (which is congruent to a modulo q). The automorphism group of G_1 is $(\mathbb{Z}/n)^\times$, and has cardinality $\varphi(n)$.

The group scheme G_2 is geometrically isomorphic to $\boldsymbol{\mu}_n$, the group scheme of n-th roots of unity, with Frobenius acting as power-raising by an integer. The automorphism group of G_2 is also $(\mathbb{Z}/n)^\times$, and has cardinality $\varphi(n)$.

Since there are no nontrivial morphisms between G_1 and G_2, the automorphism group of $E[n]$ is the product of the automorphism groups of G_1 and G_2. Thus, when n is a power of p the middle term of (3) is equal to $\varphi(n)$, and the two inequalities of (3) both hold.

Now suppose $\ell \neq p$. In this case, the group scheme $E[n]$ can be understood completely in terms of its geometric points and the action of Frobenius on them. The group $E[n](\overline{\mathbb{F}}_q)$ is isomorphic to $(\mathbb{Z}/n)^2$, and if we fix such an isomorphism the Frobenius endomorphism is given by an element γ of $\mathrm{GL}_2(\mathbb{Z}/n)$ whose trace is a and whose determinant is q. The automorphism group of $E[n]$ is then isomorphic to the subgroup of $\mathrm{GL}_2(\mathbb{Z}/n)$ consisting of those elements that commute with γ; that is, the centralizer $Z(\gamma)$ of γ.

Let r be the largest divisor of n such that Fr_E acts as an integer on $E[r]$; Proposition 4.1 shows that $r = g$. Then there is an integer d (uniquely determined modulo g) and a matrix $\beta \in \mathrm{GL}_2(\mathbb{Z}/n)$ such that

$$g \cdot \beta \in g\,\mathrm{Mat}_2(\mathbb{Z}/n) \cong \mathrm{Mat}_2(\mathbb{Z}/(n/g))$$

is cyclic and such that $\gamma = d \cdot I + g \cdot \beta$. (See [Avni et al. 2009; Williams 2012] for details.)

Given this expression for γ, we can explicitly compute the centralizer $Z(\gamma)$. If $g = n$ then $Z(\gamma) = \mathrm{GL}_2(\mathbb{Z}/n)$, so $Z(\gamma)$ has order $n\psi(n)\varphi(n)^2$. If g is a proper divisor of n then $Z(\gamma)$ is the group of all $\alpha \in \mathrm{GL}_2(\mathbb{Z}/n)$ such that the image of α in $\mathrm{GL}_2(\mathbb{Z}/(n/g)) \subset \mathrm{Mat}_2(\mathbb{Z}/(n/g))$ lies in the $\mathbb{Z}/(n/g)$-span of I and β. The order of this subgroup of $\mathrm{GL}_2(\mathbb{Z}/(n/g))$ is equal to $\varphi(n/g)$ times

$$\begin{cases} \psi(n/g) & \text{if } \beta \bmod \ell \text{ has no eigenvalues in } \mathbb{Z}/\ell, \\ n/g & \text{if } \beta \bmod \ell \text{ has 1 eigenvalue in } \mathbb{Z}/\ell, \\ \varphi(n/g) & \text{if } \beta \bmod \ell \text{ has 2 eigenvalues in } \mathbb{Z}/\ell, \end{cases}$$

so the order of its preimage in $\mathrm{GL}_2(\mathbb{Z}/n)$ is either $g^2\psi(n)\varphi(n)$ or $g^2 n\varphi(n)$ or $g^2\varphi(n)^2$. In every case we find that

$$g^2\varphi(n)^2 \le \#Z(\gamma) \le g^2\psi(n)\varphi(n),$$

which gives (3). (Alternative methods of calculating $Z(\gamma)$ can be found in [Williams 2012].) $\square$

Later in the paper we would like to have estimates for the sizes of isogeny classes and strata; since these sizes are given by class numbers, we close this section by reviewing some bounds on class numbers.

We denote the class number of an imaginary quadratic order $\mathcal{O}$ by $h(\mathcal{O})$; this is the size of the group of equivalence classes of invertible fractional ideals of $\mathcal{O}$. We let $H(\mathcal{O})$ denote the Kronecker class number of $\mathcal{O}$, defined by

$$H(\mathcal{O}) = \sum_{\mathcal{O}' \supseteq \mathcal{O}} h(\mathcal{O}'),$$

where the sum is over all quadratic orders that contain $\mathcal{O}$. If Δ is the discriminant of an imaginary quadratic order $\mathcal{O}$, we write $h(\Delta)$ and $H(\Delta)$ for $h(\mathcal{O})$ and $H(\mathcal{O})$, respectively.

Lemma 4.4. *We have*

$$h(\Delta) \ll \begin{cases} |\Delta|^{1/2} \log|\Delta| & \textit{for fundamental } \Delta < 0, \\ |\Delta|^{1/2} \log|\Delta| \log\log|\Delta| & \textit{for all } \Delta < 0; \end{cases}$$

$$H(\Delta) \ll |\Delta|^{1/2} \log|\Delta| (\log\log|\Delta|)^2 \quad \textit{for all } \Delta < 0.$$

If the generalized Riemann hypothesis is true, we have

$$h(\Delta) \ll \begin{cases} |\Delta|^{1/2} \log\log|\Delta| & \textit{for fundamental } \Delta < 0, \\ |\Delta|^{1/2} (\log\log|\Delta|)^2 & \textit{for all } \Delta < 0; \end{cases}$$

$$H(\Delta) \ll |\Delta|^{1/2} (\log\log|\Delta|)^3 \quad \textit{for all } \Delta < 0.$$

Proof. The unconditional bound on $h(\Delta)$ for fundamental Δ comes from [Cohen 1993, Exercise 5.27, p. 296], and the conditional bound from [Littlewood 1928, Theorem 1, p. 367].

For an arbitrary negative discriminant Δ, write $\Delta = \mathfrak{f}^2 \Delta^*$ for a fundamental discriminant Δ^*, and let χ be the quadratic character modulo Δ^*. Then

$$h(\Delta) = \mathfrak{f}\, h(\Delta^*) \prod_{\ell \mid \mathfrak{f}} \left(1 - \frac{\chi(\ell)}{\ell}\right) \le \mathfrak{f}\, h(\Delta^*) \prod_{\ell \mid \mathfrak{f}} \left(1 + \frac{1}{\ell}\right) \le h(\Delta^*) \sigma(\mathfrak{f}), \tag{4}$$

where σ is the sum-of-divisors function (and we recall that ℓ ranges over all prime divisors of $\mathfrak{f}$). Since $\sigma(n) \ll n \log\log n$ for $n > 2$ by [Hardy and Wright 1968, Theorem 323, p. 266], we find that

$$h(\Delta) \ll \mathfrak{f}\, h(\Delta^*) \log\log|\Delta| \quad \text{for all } \Delta < 0.$$

Combining this with the class number bounds for fundamental discriminants gives us the bounds for arbitrary discriminant.

For Kronecker class numbers, note that

$$H(\Delta) = \sum_{f \mid \mathfrak{f}} h(f^2\Delta^*) = \sum_{f \mid \mathfrak{f}} f\, h(\Delta^*) \prod_{\ell \mid f} \left(1 - \frac{\chi(\ell)}{\ell}\right)$$
$$\le h(\Delta^*) \left(\sum_{f \mid \mathfrak{f}} f\right) \prod_{\ell \mid \mathfrak{f}} \left(1 + \frac{1}{\ell}\right) \le \mathfrak{f}^{-1}\, h(\Delta^*) \sigma(\mathfrak{f})^2,$$

so that

$$H(\Delta) \ll \mathfrak{f}\, h(\Delta^*) (\log\log|\Delta|)^2 \quad \text{for all } \Delta < 0.$$

This leads to the desired bounds on $H(\Delta)$. □

5. Gluing elliptic curves

In this section, we review work of Frey and Kani [1991] that explains how to construct principally polarized abelian surfaces from pairs of elliptic curves provided with some extra structure. First, we discuss isomorphisms of torsion subgroups of elliptic curves.

Let E and F be elliptic curves over a field K and let $n > 0$ be an integer. We let $\operatorname{Isom}(E[n], F[n])$ denote the set of group scheme isomorphisms between the n-torsion subschemes of E and F. The Weil pairing gives us nondegenerate alternating pairings

$$E[n] \times E[n] \to \boldsymbol{\mu}_n \quad \text{and} \quad F[n] \times F[n] \to \boldsymbol{\mu}_n$$

from the n-torsion subschemes of E and of F to the n-torsion of the multiplicative group scheme. Via the Weil pairing, we get a map

$$m : \operatorname{Isom}(E[n], F[n]) \to \operatorname{Aut} \boldsymbol{\mu}_n \cong (\mathbb{Z}/n\mathbb{Z})^\times.$$

For every $i \in (\mathbb{Z}/n\mathbb{Z})^\times$ we let $\operatorname{Isom}^i(E[n], F[n])$ denote the set $m^{-1}(i)$, so that $\operatorname{Isom}^1(E[n], F[n])$ consists of the group scheme isomorphisms that respect the Weil pairing, and $\operatorname{Isom}^{-1}(E[n], F[n])$ consists of the anti-isometries from $E[n]$ to $F[n]$.

If η is an anti-isometry from $E[n]$ to $F[n]$, then the graph G of η is a subgroup scheme of $(E \times F)[n]$ that is maximal isotropic with respect to the product of the Weil pairings. It follows from [Mumford 1974, Corollary to Theorem 2, p. 231] that n times the canonical principal polarization on $E \times F$ descends to a principal polarization λ on the abelian surface $A := (E \times F)/G$. In this situation, we say that the polarized surface (A, λ) is obtained by *gluing* E and F together along their n-torsion subgroups via η.

Frey and Kani [1991] show that every principally polarized abelian surface (A, λ) that is isogenous to a product of two elliptic curves arises in this way; furthermore, if such an A is not isogenous to the square of an elliptic curve, then the E, F, n, and η that give rise to the polarized surface (A,λ) are unique up to isomorphism and up to interchanging the triple (E,F,η) with (F,E,η^{-1}).

Frey and Kani also note that if the polarized surface (A,λ) constructed in this way is the canonically polarized Jacobian of a curve C, then there are minimal degree-n maps $\alpha : C \to E$ and $\beta : C \to F$ such that $\alpha_*\beta^* = 0$; here *minimal* means that α and β do not factor through nontrivial isogenies. Conversely, every pair of minimal degree-n maps $\alpha : C \to E$ and $\beta : C \to F$ such that $\alpha_*\beta^* = 0$ arises in this way.

6. Ordinary split nonisotypic surfaces

In this section we will prove Proposition 3.1. The proof depends on three lemmas, whose proofs we postpone until the end of the section.

Lemma 6.1. *The number W_q of principally polarized ordinary split nonisotypic abelian surfaces over $\mathbb{F}_q$ is at most*

$$\sum_{\mathcal{S}}\sum_{\mathcal{S}'\not\sim\mathcal{S}} h(\mathcal{O}_{\mathcal{S}})h(\mathcal{O}_{\mathcal{S}'})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S}') \sum_{n\,\mid\,(a(\mathcal{S})-a(\mathcal{S}'))} \psi(n),$$

where the first sum is over ordinary strata $\mathcal{S}$, and the second is over ordinary strata $\mathcal{S}'$ not isogenous to $\mathcal{S}$.

Lemma 6.2. *We have*

$$\sum_{d\,\mid\,n}\psi(d) \ll n(\log\log n)^2 \quad \textit{for all } n>1.$$

Lemma 6.3. *We have*

$$\sum_{\text{ordinary } E/\mathbb{F}_q} \mathfrak{f}_{\mathrm{rel}}(E) \ll \begin{cases} q(\log q)^2 & \textit{for all } q\textit{, unconditionally,}\\ q(\log q)|\log\log q| & \textit{for all } q\textit{, under GRH.}\end{cases}$$

Given these lemmas, the proof of Proposition 3.1 is straightforward.

Proof of Proposition 3.1. From Lemmas 6.1 and 6.2 we find that

$$W_q \ll q^{1/2}(\log\log q)^2 \sum_{\mathcal{S}}\sum_{\mathcal{S}'\not\sim\mathcal{S}} h(\mathcal{O}_{\mathcal{S}})h(\mathcal{O}_{\mathcal{S}'})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S}') \quad \text{for all } q.$$

Since

$$\sum_{\mathcal{S}}\sum_{\mathcal{S}'\not\sim\mathcal{S}} h(\mathcal{O}_{\mathcal{S}})h(\mathcal{O}_{\mathcal{S}'})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S}') < \Bigl(\sum_{\mathcal{S}} h(\mathcal{O}_{\mathcal{S}})\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})\Bigr)^2 = \Bigl(\sum_{\text{ordinary } E/\mathbb{F}_q} \mathfrak{f}_{\mathrm{rel}}(E)\Bigr)^2,$$

we have

$$W_q \ll q^{1/2}(\log\log q)^2 \Biggl(\sum_{\text{ordinary } E/\mathbb{F}_q} \mathfrak{f}_{\mathrm{rel}}(E)\Biggr)^2 \quad \text{for all } q.$$

Combining this with Lemma 6.3, we find that we have

$$W_q \ll \begin{cases} q^{5/2}(\log q)^4(\log\log q)^2 & \text{for all } q, \text{ unconditionally,}\\ q^{5/2}(\log q)^2(\log\log q)^4 & \text{for all } q, \text{ under GRH.} \end{cases} \qquad \square$$

Now we turn to Lemmas 6.1, 6.2, and 6.3. The proof of Lemma 6.1 itself requires some notation and a preparatory result.

Fix an elliptic curve $E/\mathbb{F}_q$ and a stratum $\mathcal{S}$ of elliptic curves over $\mathbb{F}_q$. For a positive integer n, let

$$\mathrm{Isom}(E,\mathcal{S},n) = \{(E,E',\eta) : E' \in \mathcal{S}, \eta \in \mathrm{Isom}(E[n],E'[n])\},$$
$$\mathrm{Isom}^{-1}(E,\mathcal{S},n) = \{(E,E',\eta) : E' \in \mathcal{S}, \eta \in \mathrm{Isom}^{-1}(E[n],E'[n])\}.$$

Lemma 6.4. *Suppose that either $\mathcal{S}$ is ordinary, or that $a(\mathcal{S}) = 0$ and q is a nonsquare. If $\mathrm{Isom}^{-1}(E,\mathcal{S},n)$ is nonempty then $\gcd(n,\mathfrak{f}_{\mathrm{rel}}(E)) = \gcd(n,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S}))$, and we have*

$$\#\,\mathrm{Isom}^{-1}(E,\mathcal{S},n) \le 2\psi(n)h(\mathcal{O}_{\mathcal{S}})\gcd(n,\mathfrak{f}_{\mathrm{rel}}(E))\gcd(n,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})).$$

In particular, if $\mathfrak{f}_{\mathrm{rel}}(E) \neq 0$, then

$$\#\,\mathrm{Isom}^{-1}(E,\mathcal{S},n) \le 2\psi(n)h(\mathcal{O}_{\mathcal{S}})\,\mathfrak{f}_{\mathrm{rel}}(E)\,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S}).$$

Proof. Suppose that $\mathrm{Isom}^{-1}(E,\mathcal{S},n)$ is nonempty. Then there is an $E' \in \mathcal{S}$ for which there is an isomorphism $E[n] \cong E'[n]$. Corollary 4.2 then shows that $\gcd(n,\mathfrak{f}_{\mathrm{rel}}(E)) = \gcd(n,\mathfrak{f}_{\mathrm{rel}}(E')) = \gcd(n,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S}))$.

The class group $\mathrm{Cl}(\mathcal{O}_{\mathcal{S}})$ acts on $\mathcal{S}$, and the assumption that either $\mathcal{S}$ is ordinary or that $a(\mathcal{S}) = 0$ and q is a nonsquare implies that $\mathcal{S}$ is a torsor for the class group. Define an action of $\mathrm{Aut}\,E[n] \times \mathrm{Cl}(\mathcal{O}_{\mathcal{S}})$ on the nonempty set $\mathrm{Isom}(E,\mathcal{S},n)$ by setting

$$(\alpha,[\mathfrak{a}])\circ(E,E',\eta) = (E,[\mathfrak{a}]*E',[\mathfrak{a}]\circ\eta\circ\alpha^{-1}).$$

It is clear that $\mathrm{Isom}(E,\mathcal{S},n)$ is a torsor for $\mathrm{Aut}\,E[n] \times \mathrm{Cl}(\mathcal{O}_{\mathcal{S}})$ under this action, so using Proposition 4.3 we find that

$$\#\,\mathrm{Isom}(E,\mathcal{S},n) \le (\#\,\mathrm{Aut}\,E[n])h(\mathcal{O}_{\mathcal{S}}) \le g^2\varphi(n)\psi(n)h(\mathcal{O}_{\mathcal{S}}),$$

where $g = \gcd(n,\mathfrak{f}_{\mathrm{rel}}(E))$. Therefore

$$\#\,\mathrm{Isom}(E,\mathcal{S},n) \le \varphi(n)\psi(n)h(\mathcal{O}_{\mathcal{S}})\gcd(n,\mathfrak{f}_{\mathrm{rel}}(E))\gcd(n,\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})).$$

In the preceding section we defined a map $m : \mathrm{Isom}(E[n],E'[n]) \to \mathrm{Aut}\,\boldsymbol{\mu}_n$ that sends a group scheme isomorphism to the automorphism of $\boldsymbol{\mu}_n$ induced by the Weil pairing. This gives rise to a map from $\mathrm{Isom}(E, \mathcal{S}, n)$ to $\mathrm{Aut}\,\boldsymbol{\mu}_n$, which we continue to denote by m, that sends a triple (E, E', η) to $m(\eta)$. We claim that the image of this map is a coset of a subgroup of $\mathrm{Aut}\,\boldsymbol{\mu}_n$ of index at most 2.

To see this, we use the theory of complex multiplication, the Galois-equivariance of the Weil pairing, and class field theory for the extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ as follows. Let K be the field of fractions of $\mathcal{O}_{\mathcal{S}}$. Given $[\mathfrak{a}] \in \mathrm{Cl}(\mathcal{O}_{\mathcal{S}})$ and $(E, E', \eta) \in \mathrm{Isom}(E, \mathcal{S}, n)$, we have

$$m((1, [\mathfrak{a}]) \circ (E, E', \eta)) = (\mathcal{N}_{K/\mathbb{Q}}(\mathfrak{a}), \mathbb{Q}(\zeta_n)/\mathbb{Q}) \circ m(\eta) \in \mathrm{Aut}\,\boldsymbol{\mu}_n,$$

where $(\,\cdot\,, \mathbb{Q}(\zeta_n)/\mathbb{Q})$ denotes the Artin symbol for the extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. Since the group of norms of idèle classes of K has index $[K:\mathbb{Q}] = 2$ in the group of idèle classes of $\mathbb{Q}$, the image of the map m is a coset of a subgroup of index at most 2.

Therefore, the number of elements in $\mathrm{Isom}^{-1}(E, \mathcal{S}, n)$ is at most $2/\varphi(n)$ times the number of elements in $\mathrm{Isom}(E, \mathcal{S}, n)$, and we obtain the inequality in the lemma. □

Proof of Lemma 6.1. As we noted in Section 5, every principally polarized ordinary split nonisotypic surface over $\mathbb{F}_q$ is obtained in exactly two ways by gluing two ordinary nonisogenous curves E and E' together along their n-torsion. Since we must then have $E[n] \cong E'[n]$, the traces of Frobenius of E and E' must be congruent to one another modulo n; that is, $n \mid (a(E) - a(E'))$. Summing over ordinary E and E', we find that

$$\begin{aligned}
2W_q &= \sum_{E} \sum_{E' \not\sim E} \sum_{n \,|\, (a(E)-a(E'))} \#\,\mathrm{Isom}^{-1}(E[n], E'[n]) \\
&= \sum_{E} \sum_{\mathcal{S}' \not\sim E} \sum_{n \,|\, (a(E)-a(\mathcal{S}'))} \#\,\mathrm{Isom}^{-1}(E, \mathcal{S}', n) \\
&\le \sum_{E} \sum_{\mathcal{S}' \not\sim E} \sum_{n \,|\, (a(E)-a(\mathcal{S}'))} 2\psi(n) h(\mathcal{O}_{\mathcal{S}'})\, \mathfrak{f}_{\mathrm{rel}}(E)\, \mathfrak{f}_{\mathrm{rel}}(\mathcal{S}') \qquad \text{(by Lemma 6.4)} \\
&\le 2 \sum_{E} \sum_{\mathcal{S}' \not\sim E} h(\mathcal{O}_{\mathcal{S}'})\, \mathfrak{f}_{\mathrm{rel}}(E)\, \mathfrak{f}_{\mathrm{rel}}(\mathcal{S}') \sum_{n \,|\, (a(E)-a(\mathcal{S}'))} \psi(n) \\
&= 2 \sum_{\mathcal{S}} \sum_{\mathcal{S}' \not\sim \mathcal{S}} h(\mathcal{O}_{\mathcal{S}}) h(\mathcal{O}_{\mathcal{S}'})\, \mathfrak{f}_{\mathrm{rel}}(\mathcal{S})\, \mathfrak{f}_{\mathrm{rel}}(\mathcal{S}') \sum_{n \,|\, (a(\mathcal{S})-a(\mathcal{S}'))} \psi(n),
\end{aligned}$$

which proves the lemma. □

Proof of Lemma 6.2. Denote the sum on the left by $f(n)$, so that f is a multiplicative function. We calculate that $f(n)/n \le \prod_{\ell \mid n} (1 + 1/\ell)/(1 - 1/\ell)$. Taking this inequality and multiplying by the square of the identity $\varphi(n)/n = \prod_{\ell \mid n} (1 - 1/\ell)$, we find that

$$\frac{f(n)}{n(\log\log n)^2} \left(\frac{\varphi(n) \log\log n}{n} \right)^2 \le \prod_{\ell \mid n} \left(1 - \frac{1}{\ell^2} \right) \le 1.$$

Landau [1903b] showed that $\liminf \varphi(n)(\log\log n)/n = e^{-\gamma}$, where γ is Euler's constant. The lemma follows. $\square$

Our proof of Lemma 6.3 requires an estimate from analytic number theory. Let C be the multiplicative arithmetic function defined on prime powers ℓ^e by $C(\ell^e) = 2(1+1/\ell)$.

Lemma 6.5. *We have*

$$\sum_{n\le x} C(n) \ll x\log x \quad \textit{for all } x > 1.$$

Proof. Let D be the Dirichlet product [Apostol 1976, §2.6] of C with the Möbius function μ, so that

$$C(n) = \sum_{d\,|\,n} D(d).$$

We compute that D is the multiplicative function defined on prime powers ℓ^e by

$$D(\ell^e) = \begin{cases} 1+2/\ell & \text{if } e = 1,\\ 0 & \text{if } e > 1.\end{cases}$$

Then

$$\sum_{n\le x} C(n) = \sum_{n\le x}\sum_{d\,|\,n} D(d) = \sum_{d\le x} D(d)\left\lfloor \frac{x}{d}\right\rfloor \le x\sum_{d\le x}\frac{D(d)}{d},$$

so we need only show that $\sum_{d\le x} D(d)/d \ll \log x$ for $x > 1$.

Note that

$$\sum_{i=0}^{\infty}\frac{D(\ell^i)}{\ell^i} = 1 + \frac{1}{\ell} + \frac{2}{\ell^2},$$

so that

$$\sum_{d\le x}\frac{D(d)}{d} \le \prod_{\ell\le x}\left(1 + \frac{1}{\ell} + \frac{2}{\ell^2}\right).$$

Taking logarithms, we find that

$$\begin{aligned}\log\sum_{d\le x}\frac{D(d)}{d} &\le \sum_{\ell\le x}\log\left(1+\frac{1}{\ell}+\frac{2}{\ell^2}\right)\\ &= \sum_{\ell\le x}\frac{1}{\ell} + c + O\left(\frac{1}{x}\right)\\ &= \log\log x + c' + O\left(\frac{1}{\log x}\right),\end{aligned}$$

where c and c' are constants and where the last equality comes from [Apostol 1976, Theorem 4.12, p. 90]. Exponentiating, we find that $\sum_{d\le x} D(d)/d \ll \log x$ for $x \ge 2$, as desired. $\square$

Proof of Lemma 6.3. First we compute a bound on the sum of the relative conductors of the elliptic curves in a fixed ordinary isogeny class. Let a be an integer, coprime to q, with $a^2 < 4q$. Recall from Section 4 that we write $\Delta_{a,q} := a^2 - 4q = \mathfrak{f}_{a,q}^2\, \Delta_{a,q}^*$, where $\Delta_{a,q}^*$ is a fundamental discriminant. Let $\widetilde{\mathcal{O}}_{a,q}$ be the quadratic order of discriminant $\Delta_{a,q}^*$. As we noted in Section 4, the isogeny class $\mathcal{I}(\mathbb{F}_q, a)$ is the union of strata $\mathcal{S} = \mathcal{I}(\mathbb{F}_q, a, \mathcal{O})$, where the orders $\mathcal{O} \subseteq \widetilde{\mathcal{O}}_{a,q}$ have discriminant $f^2\Delta_{a,q}^*$ for the divisors f of $\mathfrak{f}_{a,q}$. The curves in $\mathcal{S}$ have relative conductor $\mathfrak{f}_{a,q}/f$, and the number of curves in $\mathcal{S}$ is equal to $h(\mathcal{O})$. If we let χ denote the quadratic character modulo $\Delta_{a,q}^*$, then

$$h(\mathcal{O}) = f h(\Delta_{a,q}^*) \prod_{\ell \mid f} \left(1 - \frac{\chi(\ell)}{\ell}\right).$$

Thus,

$$\begin{aligned} \sum_{E \in \mathcal{I}(\mathbb{F}_q, a)} \mathfrak{f}_{\mathrm{rel}}(E) &= \sum_{f \mid \mathfrak{f}_{a,q}} \frac{\mathfrak{f}_{a,q}}{f} f h(\Delta_{a,q}^*) \prod_{\ell \mid f} \left(1 - \frac{\chi(\ell)}{\ell}\right) \\ &= \mathfrak{f}_{a,q}\, h(\Delta_{a,q}^*) \sum_{f \mid \mathfrak{f}_{a,q}} \prod_{\ell \mid f} \left(1 - \frac{\chi(\ell)}{\ell}\right). \end{aligned}$$

Lemma 4.4 tells us that $h(\Delta) \ll |\Delta|^{1/2} \log|\Delta|$ for all fundamental discriminants $\Delta < 0$. Combining this with the fact that $|f_{a,q}^2 \Delta_{a,q}^*| = 4q - a^2 < 4q$ we see that there is a constant c such that for all q and a we have

$$\sum_{E \in \mathcal{I}(\mathbb{F}_q, a)} \mathfrak{f}_{\mathrm{rel}}(E) < c q^{1/2} (\log q) A(\mathfrak{f}_{a,q}),$$

where A is the arithmetic function defined by

$$A(n) = \sum_{d \mid n} \prod_{\ell \mid d} \left(1 + \frac{1}{\ell}\right) = \sum_{d \mid n} \frac{\psi(d)}{d}.$$

Additionally, if the generalized Riemann hypothesis is true we can use Lemma 4.4 to find that there is a constant c' such that for all q and a we have

$$\sum_{E \in \mathcal{I}(\mathbb{F}_q, a)} \mathfrak{f}_{\mathrm{rel}}(E) < c' q^{1/2} |\log\log q| A(\mathfrak{f}_{a,q}).$$

Thus, to prove the lemma it will suffice to show that we have

$$\sum_{\substack{1 \le a \le 2\sqrt{q} \\ \gcd(a,q)=1}} A(\mathfrak{f}_{a,q}) \ll q^{1/2} \log q \quad \text{for all } q. \tag{5}$$

Note that the sum on the left side of (5) is equal to

$$\sum_{\substack{1\le a\le 2\sqrt{q}\\ \gcd(a,q)=1}} \sum_{d \mid \mathfrak{f}_{a,q}} \frac{\psi(d)}{d} = \sum_{1\le d\le 2\sqrt{q}} \frac{\psi(d)}{d} \#\bigl\{a : 1 \le a \le 2\sqrt{q},\ \gcd(a,q)=1,\ d \mid \mathfrak{f}_{a,q}\bigr\}.$$

If $d \mid \mathfrak{f}_{a,q}$ then $a^2 \equiv 4q \mod d^2$, so let us first consider, for a fixed d, estimates for the number of a in the interval $\bigl[1, 2\sqrt{q}\bigr]$ with $a^2 \equiv 4q \mod d^2$.

We have

$$\begin{aligned}\#\bigl\{a : 1 \le a \le 2\sqrt{q},\ a^2 \equiv 4q \mod d^2\bigr\}&\\ &\le \#\bigl\{a : 1 \le a \le d^2\lceil 2\sqrt{q}/d^2\rceil,\ a^2 \equiv 4q \mod d^2\bigr\}\\ &= \lceil 2\sqrt{q}/d^2\rceil \#\bigl\{a : 1 \le a \le d^2,\ a^2 \equiv 4q \mod d^2\bigr\}.\end{aligned}$$

Thus, if we let B_q denote the multiplicative arithmetic function given by

$$B_q(n) = \#\bigl\{a : 1 \le a \le n^2,\ a^2 \equiv 4q \mod n^2\bigr\}$$

then we have

$$\begin{aligned}\sum_{\substack{1\le a\le 2\sqrt{q}\\ \gcd(a,q)=1}} A(\mathfrak{f}_{a,q}) &\le \sum_{\substack{d\le 2\sqrt{q}\\ \gcd(d,q)=1}} \frac{\psi(d)}{d} \#\bigl\{a : 1 \le a \le 2\sqrt{q},\ \gcd(a,q)=1,\ d \mid \mathfrak{f}_{a,q}\bigr\}\\ &\le \sum_{\substack{d\le 2\sqrt{q}\\ \gcd(d,q)=1}} \frac{\psi(d)}{d} \lceil 2\sqrt{q}/d^2\rceil B_q(d)\\ &\le \sum_{\substack{d\le 2\sqrt{q}\\ \gcd(d,q)=1}} \frac{2\sqrt{q}}{d^2}\frac{\psi(d)}{d} B_q(d) + \sum_{\substack{d\le 2\sqrt{q}\\ \gcd(d,q)=1}} \frac{\psi(d)}{d} B_q(d). \qquad (6)\end{aligned}$$

If ℓ is a prime that does not divide q and if $e > 0$ then

$$B_q(\ell^e) \le \begin{cases} 2 & \text{if } \ell \ne 2\\ 8 & \text{if } \ell = 2,\end{cases}$$

so

$$\frac{\psi(d)}{d} B_q(d) \le 4C(d)$$

for all d coprime to q, where C is the function from Lemma 6.5. For every $\epsilon > 0$ we have $C(d) \ll d^\epsilon$ for all d, so

$$\sum_{\substack{d\le 2\sqrt{q}\\ \gcd(d,q)=1}} \frac{1}{d^2}\frac{\psi(d)}{d} B_q(d) \le 4 \sum_{\substack{d\le 2\sqrt{q}\\ \gcd(d,q)=1}} \frac{C(d)}{d^2} \le 4\sum_{d=1}^{\infty} \frac{C(d)}{d^2} < \infty \quad \text{for all } q. \qquad (7)$$

This shows that the first term on the right side of (6) is $\ll \sqrt{q}$ for all q. To bound the second term on the right side of (6), we compute that

$$\sum_{\substack{d \le 2\sqrt{q} \\ \gcd(d,q)=1}} \frac{\psi(d)}{d} B_q(d) \le 4 \sum_{d \le 2\sqrt{q}} C(d) \ll q^{1/2} \log q \quad \text{for all } q, \tag{8}$$

by Lemma 6.5. Combining (6) with (7) and (8) proves (5), and completes the proof of the lemma. □

7. Ordinary split isotypic surfaces

In this section we will prove Proposition 3.2. As in the preceding section, we state several lemmas which lead to a quick proof of the proposition. Lemma 7.1 follows from Lemma 6.4. We postpone the proofs of Lemmas 7.2, 7.3, and 7.4 until the end of the section.

Lemma 7.1. *For every ordinary $E/\mathbb{F}_q$ and positive integer n we have*

$$\sum_{E' \sim E} \#\operatorname{Isom}^{-1}(E, E', n) \le 2\psi(n)\, \mathfrak{f}_{\mathrm{rel}}(E) \sum_{E' \sim E} \mathfrak{f}_{\mathrm{rel}}(E'). \qquad \square$$

Lemma 7.2. *Let $E/\mathbb{F}_q$ be an elliptic curve and let $C/\mathbb{F}_q$ be a smooth genus-2 curve with $\operatorname{Jac} C \sim E^2$. Then there is a finite morphism $C \to E$ of degree at most $\sqrt{2q}$. If E is supersingular with all endomorphisms defined over $\mathbb{F}_q$, then there is a finite morphism $C \to E$ of degree at most $q^{1/4}$.*

Lemma 7.3. *We have*

$$\sum_{n \le x} \psi(n) = \frac{15}{2\pi^2} x^2 + O(x \log x).$$

For every pair of isogenous curves E and E' over $\mathbb{F}_q$, we let $s(E, E')$ denote the degree of the smallest isogeny from E to E'.

Lemma 7.4. *Let $E/\mathbb{F}_q$ be an ordinary elliptic curve with $\mathfrak{f}_{\mathrm{rel}}(E) = 1$, and let $\mathcal{S}$ be a stratum of curves isogenous to E. Then*

$$\sum_{E' \in \mathcal{S}} \frac{1}{s(E, E')^2} < \frac{\zeta(3)}{\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})^2},$$

where ζ is the Riemann zeta function.

Proof of Proposition 3.2. Proposition 3.2 gives an upper bound on the number of principally polarized abelian surfaces isogenous to the square of an ordinary elliptic curve. We would like to instead consider Jacobians. This requires that we first dispose of those principally polarized surfaces that are not Jacobians of smooth curves; according to [González et al. 2005, Theorem 3.1, p. 270], these are the polarized surfaces that are products of elliptic curves with the product polarization, together with the restrictions of scalars of polarized elliptic curves over the quadratic extension of our base field. But the restriction of scalars of an elliptic curve over $\mathbb{F}_{q^2}$ with trace of Frobenius b is an abelian surface over $\mathbb{F}_q$ with Weil polynomial $x^4 - bx^2 + q^2$, and such a surface is never isogenous to the square of an ordinary elliptic curve, because in that case its Weil polynomial would have to be $(x^2 - ax + q)^2$, where a is coprime to q. Therefore, to dispose of the non-Jacobians, we need only consider products of elliptic curves, with the product polarization.

The number of elliptic curves in an ordinary isogeny class with trace of Frobenius equal to a is equal to the Kronecker class number $H(a^2 - 4q)$ of the discriminant $a^2 - 4q$ (see [Schoof 1987, Theorem 4.6, pp. 194–195]). From Lemma 4.4 we know that $H(\Delta) \ll |\Delta|^{1/2} \log|\Delta| (\log\log|\Delta|)^2$ for all negative discriminants Δ. Therefore the number of product surfaces $E \times E'$ with E and E' both in a fixed ordinary isogeny class over $\mathbb{F}_q$ is $\ll q(\log q)^2(\log\log q)^4$; summing over isogeny classes, we find that the number of product surfaces $E \times E'$ with E and E' ordinary and isogenous to one another is $\ll q^{3/2}(\log q)^2(\log\log q)^4$. Thus, the contribution of the non-Jacobians to the ordinary split isotypic polarized surfaces is much less than the bound claimed in Proposition 3.2. (Of course, for present purposes, it suffices to observe that the number of non-Jacobians is bounded by the square of the number of elliptic curves over $\mathbb{F}_q$; but the estimate provided here is closer to the actual truth.)

Fix an integer a with $|a| \le 2\sqrt{q}$ and $\gcd(a, q) = 1$, and let E_a be an elliptic curve over $\mathbb{F}_q$ with $a(E) = a$ and with $\operatorname{End} E_a \cong \mathcal{O}_{a,q}$, so that $\mathfrak{f}_{\mathrm{rel}}(E_a) = 1$. Suppose C is a curve over $\mathbb{F}_q$ whose Jacobian is isogenous to E_a^2. By Lemma 7.2 there is a morphism from C to E_a of degree at most $\sqrt{2q}$. We can write this map as a composition of a minimal map $C \to E$ (see Section 5) with an isogeny $E \to E_a$, and it follows that the degree of the minimal map $C \to E$ is at most $\sqrt{2q}/s(E, E_a)$. If we let N_a denote the number of genus-2 curves with Jacobians isogenous to E_a^2, we find that

$$
\begin{aligned}
N_a &\le \sum_{E \sim E_a} \#\bigl\{C \text{ with minimal maps to } E \text{ of degree at most } \sqrt{2q}/s(E, E_a)\bigr\} \\
&\le \sum_{E \sim E_a} \sum_{n \le \sqrt{2q}/s(E,E_a)} \sum_{E' \sim E} \#\operatorname{Isom}^{-1}(E, E', n) \\
&\le \sum_{E \sim E_a} \sum_{n \le \sqrt{2q}/s(E,E_a)} 2\psi(n)\,\mathfrak{f}_{\mathrm{rel}}(E) \sum_{E' \sim E} \mathfrak{f}_{\mathrm{rel}}(E') \qquad (9) \\
&= 2\Bigl(\sum_{E' \sim E_a} \mathfrak{f}_{\mathrm{rel}}(E')\Bigr) \sum_{E \sim E_a} \mathfrak{f}_{\mathrm{rel}}(E) \sum_{n \le \sqrt{2q}/s(E,E_a)} \psi(n) \\
&\ll \Bigl(\sum_{E' \sim E_a} \mathfrak{f}_{\mathrm{rel}}(E')\Bigr) \sum_{E \sim E_a} \mathfrak{f}_{\mathrm{rel}}(E) \frac{2q}{s(E, E_a)^2} \qquad \text{for all } a \text{ and } q. \quad (10)
\end{aligned}
$$

Here (9) follows from Lemma 7.1 and (10) follows from Lemma 7.3. Now we group the curves E isogenous to E_a by their strata. Recall that we have $a^2-4q=\mathfrak{f}_{a,q}^2\,\Delta^*_{a,q}$, and that the strata of curves isogenous to E_a are indexed by the divisors f of $\mathfrak{f}_{a,q}$. We find that

$$\begin{aligned}
N_a &\ll q\Biggl(\sum_{E'\sim E_a}\mathfrak{f}_{\mathrm{rel}}(E')\Biggr)\sum_{\mathcal{S}\sim E_a}\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})\sum_{E\in\mathcal{S}}\frac{1}{s(E,E_a)^2} && \text{for all } a \text{ and } q\\
&\ll q\Biggl(\sum_{E'\sim E_a}\mathfrak{f}_{\mathrm{rel}}(E')\Biggr)\sum_{f\,\mid\,\mathfrak{f}_{a,q}}\frac{1}{f} && \text{for all } a \text{ and } q \qquad (11)\\
&\ll q\Biggl(\sum_{E'\sim E_a}\mathfrak{f}_{\mathrm{rel}}(E')\Biggr)|\log\log q| && \text{for all } a \text{ and } q, \qquad (12)
\end{aligned}$$

where (11) follows from Lemma 7.4 and (12) follows from the asymptotic upper bound [Hardy and Wright 1968, Theorem 323, p. 266]

$$e^\gamma=\limsup_{n>0}\frac{\sum_{d\,\mid\,n}d}{n\log\log n}=\limsup_{n>0}\frac{\sum_{d\,\mid\,n}d/n}{\log\log n}=\limsup_{n>0}\frac{\sum_{d\,\mid\,n}1/d}{\log\log n}.$$

Recall that X_q is the number of principally polarized ordinary split isotypic abelian surfaces over $\mathbb{F}_q$. Then X_q is the sum over all a coprime to q of the N_a (together with the negligible contribution from those abelian surfaces that are isomorphic, as principally polarized abelian varieties, to products of isogenous elliptic curves), and we find that

$$X_q\ll q|\log\log q|\Biggl(\sum_{\text{ordinary } E/\mathbb{F}_q}\mathfrak{f}_{\mathrm{rel}}(E)\Biggr)\quad\text{for all } q.$$

Proposition 3.2 then follows from Lemma 6.3. □

Proof of Lemma 7.2. Choose a divisor of degree 1 on C, and let L be the additive group of morphisms from C to E that send the given divisor to the identity of E. Let $\boldsymbol{E}$ be the base extension of E from $\mathbb{F}_q$ to the function field F of C. The *Mordell–Weil lattice* of $\boldsymbol{E}$ over F is the group $\boldsymbol{E}(F)/E(\mathbb{F}_q)$ provided with the pairing coming from the canonical height. The natural map $L\to\boldsymbol{E}(F)/E(\mathbb{F}_q)$ is a bijection, and the quadratic form on L obtained from the height pairing on $\boldsymbol{E}(F)$ is twice the degree map (see [Silverman 1994, Theorem III.4.3, pp. 217–218]). Let $a=a(E)$, and let π and $\bar\pi$ be the roots in $\mathbb{C}$ of the characteristic polynomial of Frobenius for E, so that $\pi+\bar\pi=a$. The Birch and Swinnerton-Dyer conjecture for constant elliptic curves over function fields (proved by Milne [1968, Theorem 3, pp. 100–101]) shows that the determinant of the Mordell–Weil lattice is a divisor of

$$\begin{cases}(\pi-\bar\pi)^4=(a^2-4q)^2 & \text{if } \pi\neq\bar\pi,\\ q^2 & \text{if } \pi=\bar\pi;\end{cases}$$

note that $\pi = \bar{\pi}$ if and only if E is supersingular with all of its endomorphisms rational over $\mathbb{F}_q$.

The $\mathbb{Z}$-rank of L is twice the $\mathbb{Z}$-rank of $\operatorname{End} E$. If $\pi \neq \bar{\pi}$, so that $\operatorname{End} E$ is an imaginary quadratic order, then L is a $\mathbb{Z}$-module of rank 4. Applying [Cassels 1978, Theorem 12.2.1, p. 260] we find that there is a nonzero element of L of degree at most

$$\tfrac{1}{2}\gamma_4 |a^2 - 4q|^{1/2},$$

where γ_4 is the Hermite constant for dimension 4. Using the fact that $\gamma_4 = \sqrt{2}$ (see [Hermite 1850]), we obtain the bound in the lemma.

If $\pi = \bar{\pi}$ then $\operatorname{End} E$ is an order in a quaternion algebra and L is a $\mathbb{Z}$-module of rank 8. We find that there is a nonzero element of L with degree at most

$$\tfrac{1}{2}\gamma_8 q^{1/4}.$$

The value of γ_8 was determined by Blichfeldt [1935] to be 2, so there is a map from C to E of degree at most $q^{1/4}$. □

Proof of Lemma 7.3. First we note that

$$\psi(n) = \sum_{d \mid n} |\mu(d)| \frac{n}{d},$$

where μ is the Möbius function. Then, arguing as in the proof of [Apostol 1976, Theorem 3.7, p. 62], we see that

$$\begin{aligned}
\sum_{n \le x} \psi(n) &= \sum_{\substack{d,q \\ dq \le x}} |\mu(d)| q = \sum_{d \le x} |\mu(d)| \sum_{q \le x/d} q \\
&= \sum_{d \le x} |\mu(d)| \left(\frac{1}{2} \left(\frac{x}{d} \right)^2 + O\left(\frac{x}{d} \right) \right) \\
&= \frac{1}{2} x^2 \sum_{d \le x} \frac{|\mu(d)|}{d^2} + O\left(x \sum_{d \le x} \frac{1}{d} \right) \\
&= \frac{1}{2} c x^2 + O(x \log x),
\end{aligned}$$

where

$$c = \sum_{d=1}^{\infty} \frac{|\mu(d)|}{d^2} = \prod_{\ell} \left(1 + \frac{1}{\ell^2} \right) = \prod_{\ell} \frac{(1 - 1/\ell^4)}{(1 - 1/\ell^2)} = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2}.$$ □

Proof of Lemma 7.4. Let $\mathcal{O} = \mathcal{O}_{\mathcal{S}}$ be the order corresponding to the stratum $\mathcal{S}$. We claim that there is an elliptic curve $\widetilde{E}$ in $\mathcal{S}$ and an isogeny $f : E \to \widetilde{E}$ with the property that every isogeny from E to an elliptic curve in $\mathcal{S}$ factors through f.

One way to see this is via the theory of Deligne modules [Deligne 1969; Howe 1995]. If we let π be the Frobenius for E and let K be the quadratic field $\mathbb{Q}(\pi)$, then the Deligne modules of the elements of $\mathcal{S}$ can be viewed as lattices in K with endomorphism rings equal to $\mathcal{O}$, while the Deligne module for E can be viewed as a lattice $\Lambda \subset K$ with $\operatorname{End}\Lambda = \mathbb{Z}[\pi]$. The curve $\widetilde{E}$ is the elliptic curve corresponding to the Deligne module $\Lambda \otimes \mathcal{O}$, and the isogeny f corresponds to the inclusion $\Lambda \subset \Lambda \otimes \mathcal{O}$. In particular, we see that the degree of f is equal to $\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})$.

The isogenies from $\widetilde{E}$ to the other elements of $\mathcal{S}$ correspond to the invertible ideals $\mathfrak{a} \subset \mathcal{O}$, with different ideals giving rise to different isogenies. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be the (distinct) ideals corresponding to the smallest isogenies from $\widetilde{E}$ to the elements of $\mathcal{S}$, where $n = \#\mathcal{S}$. Then

$$\sum_{E' \in \mathcal{S}} \frac{1}{s(E, E')^2} = \sum_{i=1}^{n} \frac{1}{\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})^2 \mathcal{N}(\mathfrak{a}_i)^2} < \frac{1}{\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})^2} \sum_{\text{all } \mathfrak{a}} \frac{1}{\mathcal{N}(\mathfrak{a})^2},$$

where the final sum is over all invertible ideals $\mathfrak{a} \subseteq \mathcal{O}$; that is, the final sum is equal to $\zeta_{\mathcal{O}}(2)$, where $\zeta_{\mathcal{O}}$ is the zeta function for the order $\mathcal{O}$.

Kaneko [1990, Proposition, p. 202] gives an explicit formula for $\zeta_{\mathcal{O}}(s)$ in terms of the zeta function for K and the conductor of $\mathcal{O}$. It is not hard to check that for real $s > 1$ the Euler factor at ℓ for $\zeta_{\mathcal{O}}(s)$ is bounded above by $1/(1 - \ell^{1-2s})$, so that $\zeta_{\mathcal{O}}(s) < \zeta(2s - 1)$, where ζ is the Riemann zeta function. In particular, $\zeta_{\mathcal{O}}(2) < \zeta(3)$, and the lemma follows. □

8. Almost ordinary split surfaces

In this section we prove Proposition 3.3, which gives an upper bound on the number of principally polarized almost ordinary split abelian surfaces. We base the proof on two lemmas, which we prove at the end of the section.

Lemma 8.1. *Let E_0 be a supersingular elliptic curve over a finite field $\mathbb{F}_q$, with q a square, and suppose the Frobenius endomorphism on E_0 is equal to multiplication by s, where $s^2 = q$. Let $\mathcal{S}$ be a stratum of ordinary elliptic curves over $\mathbb{F}_q$, and suppose n is a positive integer such that* $\operatorname{Isom}^{-1}(E_0, \mathcal{S}, n)$ *is nonempty. Then*

(a) *the integer n is coprime to q,*

(b) *the relative conductor $\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})$ is divisible by n, and*

(c) *the trace $a(\mathcal{S})$ satisfies $4a(\mathcal{S}) \equiv 8s \mod n^2$.*

Lemma 8.2. *We have*

$$\sum_{n \le x} \frac{\psi(n)}{n} = \frac{15}{\pi^2} x + O(\log x).$$

Proof of Proposition 3.3. In analogy with Section 6, we will bound the number Y_q of principally polarized almost ordinary split abelian surfaces over $\mathbb{F}_q$ by estimating the number of surfaces obtained by gluing a supersingular E_0 to an ordinary E along their n-torsion subgroups. The methods we use will depend on whether or not E_0 has all of its endomorphisms defined over $\mathbb{F}_q$. Let $Y_{q,1}$ denote the number of principally polarized surfaces we get from E_0 with all of the endomorphisms defined, and let $Y_{q,2}$ denote the number we get from E_0 with not all endomorphisms defined. We will show that $Y_{q,1}$ and $Y_{q,2}$ each satisfy the bound of Proposition 3.3.

First let us bound $Y_{q,1}$; that is, we consider the case where all of the endomorphisms of E_0 are defined over $\mathbb{F}_q$. In this case, q is a square and the characteristic polynomial of E_0 is $(T-2s)^2$, where $s^2 = q$; furthermore, [Schoof 1987, Theorem 4.6, pp. 194–195] tells us that there are

$$\frac{1}{12}\left(p+6-4\left(\frac{-3}{p}\right)-3\left(\frac{-4}{p}\right)\right) \le \frac{\sqrt{q}}{2}$$

such curves for each of the two possible values of s, so at most $\sqrt{q}$ curves in total.

Fix such an E_0 and fix an integer $n > 0$. Suppose $\mathcal{S}$ is an ordinary stratum of elliptic curves over $\mathbb{F}_q$ such that $\operatorname{Isom}^{-1}(E_0, \mathcal{S}, n)$ is nonempty. If n is even let $m = \frac{n}{2}$; otherwise let $m = n$. We see from Lemma 8.1 that the trace $a(\mathcal{S})$ of $\mathcal{S}$ is an integer congruent to $2s$ modulo m^2, but not equal to $2s$. The number of such integers a in the Weil interval is at most $\lfloor 4\sqrt{q}/m^2 \rfloor$.

Given such an integer a, write $a^2 - 4q = \mathfrak{f}_{a,q}^2\, \Delta^*_{a,q}$ for a fundamental discriminant $\Delta^*_{a,q}$. Let χ be the quadratic character modulo $\Delta^*_{a,q}$, and for each divisor d of $\mathfrak{f}_{a,q}/n$ let $\mathcal{S}_d$ be the stratum $\mathcal{S}$ with $a(\mathcal{S}) = a$ and $\mathfrak{f}(\mathcal{S}) = d$. Using Lemma 6.4 we find that

$$\begin{aligned}
\sum_{E \in \mathcal{I}(\mathbb{F}_q, a)} \#\operatorname{Isom}^{-1}(E_0[n], E[n]) &= \sum_{\substack{\mathcal{S}\text{ with}\\ a(\mathcal{S})=a\\ \text{and } n \,\mid\, \mathfrak{f}_{\mathrm{rel}}(\mathcal{S})}} \#\operatorname{Isom}^{-1}(E_0, \mathcal{S}, n)\\
&= \sum_{d \,\mid\, (\mathfrak{f}_{a,q}/n)} \#\operatorname{Isom}^{-1}(E_0, \mathcal{S}_d, n)\\
&\le 2 \sum_{d \,\mid\, (\mathfrak{f}_{a,q}/n)} \psi(n) h(\mathcal{O}_{\mathcal{S}_d}) n^2\\
&= 2\psi(n) n^2 H\left(\frac{a^2-4q}{n^2}\right),
\end{aligned}$$

where $H(x)$ is the Kronecker class number. Thus,

$$\sum_{E \in \mathcal{I}(\mathbb{F}_q, a)} \#\operatorname{Isom}^{-1}(E_0[n], E[n]) \ll \begin{cases} 2\psi(n) n q^{1/2} (\log q)(\log\log q)^2 & \text{for all } a \text{ and } q\text{, unconditionally,}\\ 2\psi(n) n q^{1/2} |\log\log q|^3 & \text{for all } a \text{ and } q\text{, under GRH.}\end{cases}$$

Conditions on q	Conditions on p	$a(\mathcal{S})$	$\Delta(\mathcal{O}_\mathcal{S})$	$\mathfrak{f}_{\mathrm{rel}}(\mathcal{S})$	$\#\mathcal{S}$
q nonsquare	—	0	$-4p$	$\sqrt{q/p}$	$h(-4p)$
	$p \equiv 3 \bmod 4$	0	$-p$	$2\sqrt{q/p}$	$h(-p)$
	$p = 2$	$\pm\sqrt{2q}$	-4	$\sqrt{q/2}$	1
	$p = 3$	$\pm\sqrt{3q}$	-3	$\sqrt{q/3}$	1
q square	—	0	-4	$\sqrt{q}$	$1-(-4/p)$
	—	$\pm\sqrt{q}$	-3	$\sqrt{q}$	$1-(-3/p)$

Table 1. The supersingular strata $\mathcal{S}$ over $\mathbb{F}_q$ with not all endomorphisms defined over $\mathbb{F}_q$. Here q is a power of a prime p.

Summing over the $\lfloor 4\sqrt{q}/m^2 \rfloor$ possible values of a for a given n, and then summing over the possible $n < 4\sqrt{q}$, and then summing over the possible curves E_0, we find that

$$Y_{q,1} \ll q^{3/2}(\log q)(\log\log q)^2 \sum_{n=1}^{4\sqrt{q}} \frac{\psi(n)}{n} \ll q^2(\log q)(\log\log q)^2 \quad \text{for all } q.$$

If the generalized Riemann hypothesis holds, we get the better bound

$$Y_{q,1} \ll q^2 |\log\log q|^3 \quad \text{for all } q.$$

Now we turn to estimating $Y_{q,2}$, the number of principally polarized split surfaces isogenous to a surface of the form $E_0 \times E$, where E is ordinary and E_0 is supersingular with not all endomorphisms defined. Using [Schoof 1987, Theorems 4.2, 4.3, and 4.5, pp. 194–195], we find that the possible strata of such curves E_0 are as listed in Table 1.

Let E_0 be a supersingular curve with not all endomorphisms defined. If we are to glue E_0 to an ordinary elliptic curve E along the n-torsion of the two curves, then n must be coprime to q. In that case, the greatest common divisor of $\mathfrak{f}_{\mathrm{rel}}(E_0)$ and n is either 1 or 2, as we see from Table 1. It follows from Lemma 6.4 that for every ordinary stratum $\mathcal{S}$ we have

$$\#\,\mathrm{Isom}^{-1}(E_0, \mathcal{S}, n) \le 8\psi(n)h(\mathcal{O}_\mathcal{S}),$$

so the total number of curves obtained from gluing E_0 to an ordinary elliptic curve is bounded by

$$8 \sum_{\text{ordinary } \mathcal{S}} h(\mathcal{O}_\mathcal{S}) \sum_{n \mid (a(\mathcal{S})-a(E_0))} \psi(n) \ll q^{1/2}(\log\log q)^2 \sum_{\text{ordinary } \mathcal{S}} h(\mathcal{O}_\mathcal{S}) \quad \text{for all } q$$

by Lemma 6.2. This last sum is simply the number of ordinary elliptic curves over $\mathbb{F}_q$, which (one shows) is at most $2q+4$, so the number of curves obtained as above from a fixed E_0 is $\ll q^{3/2}(\log\log q)^2$ for all q.

If q is a square there are at most 6 possible E_0, and we find that

$$Y_{q,2} \ll q^{3/2}(\log\log q)^2 \quad \text{for all square } q.$$

If q is not a square, then Lemma 4.4 shows that the number of possible E_0 is $\ll q^{1/2}\log q$ for all q unconditionally, and $\ll q^{1/2}|\log\log q|$ for all q under the generalized Riemann hypothesis. This leads to

$$Y_{q,2} \ll \begin{cases} q^2(\log q)(\log\log q)^2 & \text{for all } q\text{, unconditionally,} \\ q^2|\log\log q|^3 & \text{for all } q\text{, under GRH,} \end{cases}$$

and completes the proof of Proposition 3.3. □

Proof of Lemma 8.1. Since $\operatorname{Isom}^{-1}(E_0, \mathcal{S}, n)$ is nonempty, there is an $E \in \mathcal{S}$ with $E_0[n] \cong E[n]$. The p-torsion of E_0 is a local-local group scheme, while $E[p]$ has no local-local part, so n must not be divisible by p. This proves (a).

Lemma 6.4 shows that $\gcd(n, \mathfrak{f}_{\mathrm{rel}}(E_0)) = \gcd(n, \mathfrak{f}_{\mathrm{rel}}(\mathcal{S}))$. Since $\mathfrak{f}_{\mathrm{rel}}(E_0) = 0$, we find that $n \mid \mathfrak{f}_{\mathrm{rel}}(\mathcal{S})$. This proves (b).

Let $a = a(\mathcal{S})$. From (b) we know that $a^2 - 4q \equiv 0 \bmod n^2$, and we also know that $a \equiv a(E_0) = 2s \bmod n$. Since $a - 2s \equiv 0 \bmod n$ we have

$$0 \equiv a^2 - 4as + 4s^2 \equiv 4s^2 - 4as + 4s^2 \equiv 8s^2 - 4as \mod n^2.$$

Since s is coprime to n by (a), we can divide through by s to obtain (c). □

Proof of Lemma 8.2. The proof is quite similar to that of Lemma 7.3. We have

$$\begin{aligned}\sum_{n\le x} \frac{\psi(n)}{n} &= \sum_{\substack{d,q \\ dq\le x}} \frac{|\mu(d)|}{d} = \sum_{d\le x} \frac{|\mu(d)|}{d} \left\lfloor \frac{x}{d} \right\rfloor \\ &= x\sum_{d\le x} \frac{|\mu(d)|}{d^2} + O(\log x) = cx + O(\log x),\end{aligned}$$

where $c = \sum_{d=1}^{\infty} |\mu(d)|/d^2 = 15/\pi^2$. □

9. Supersingular split surfaces

In this section we prove Proposition 3.4, which gives a bound on the number Z_q of principally polarized supersingular split abelian surfaces over $\mathbb{F}_q$.

We must first introduce some terminology and some background results. Let A be an abelian surface over a finite field $\mathbb{F}_q$ of characteristic p, and let $\boldsymbol{\alpha}_p$ denote the (unique) local-local group scheme of rank p over $\mathbb{F}_q$. The *a-number* of A is the dimension of the $\mathbb{F}_q$-vector space $\operatorname{Hom}(\boldsymbol{\alpha}_p, A)$. If A has a-number 2 then A is called *superspecial*; all superspecial surfaces over $\mathbb{F}_q$ are geometrically isomorphic to one another, and they are all geometrically isomorphic to the square of a supersingular elliptic curve. A supersingular surface A has a-number equal to either 1 or 2; if the a-number is 1, then A has a unique local-local subgroup scheme of rank p, and the quotient of A by this subgroup scheme is a superspecial surface.

Let $\mathcal{A}_2^{\mathrm{ss}}$ denote the supersingular locus of the coarse moduli space of principally polarized abelian surfaces. Koblitz [1975, p. 193] shows that the only singularities of $\mathcal{A}_2^{\mathrm{ss}}$ are at the superspecial points, and from [Oort 1974, Proof of Corollary 4.7, p. 117] we know that each irreducible component of $\mathcal{A}_{2,\bar{\mathbb{F}}_p}^{\mathrm{ss}}$ is a curve of genus 0. Also, every component contains a superspecial point. Therefore, the nonsuperspecial locus of $\mathcal{A}_{2,\bar{\mathbb{F}}_p}^{\mathrm{ss}}$ is a disjoint union of components, each of which is isomorphic to an open affine subset of $\boldsymbol{A}^1$.

Moreover, the number of irreducible components of $\mathcal{A}_{2,\bar{\mathbb{F}}_p}^{\mathrm{ss}}$ is equal to the class number $H_2(1, p)$ of the nonprincipal genus of $\mathbb{Q}_{p,\infty}^2$ (see [Katsura and Oort 1987, Theorem 5.7, p. 133]). Hashimoto and Ibukiyama [1981] (see also [Ibukiyama et al. 1986, Remark 2.17, p. 147]) provide a formula for $H_2(1, p)$ which shows both that $H_2(1, p) = \frac{1}{2880}p^2 + O(p)$ and that $H_2(1, p) \le \frac{1}{4}p^2$ for all p.

For convenience, we also state the following lemma.

Lemma 9.1. *Let (A, λ) be a principally polarized abelian surface over $\bar{\mathbb{F}}_q$ that has a model over $\mathbb{F}_q$. Then the number of distinct $\mathbb{F}_q$-rational models of (A, λ) is at most* 1152*.*

Proof. The size of the automorphism group of a principally polarized abelian surface over a finite field is bounded by 1152 (by 72, if the characteristic is greater than 5); for Jacobians, this follows from Igusa's enumeration [1960, §8] of the possible automorphism groups, and for products of polarized elliptic curves and for restrictions of scalars of elliptic curves it is an easy exercise. (We know from [González et al. 2005, Theorem 3.1, p. 270] that every principally polarized abelian surface is of one of these three types.) By [Brock and Granville 2001, Lemma 7.2, pp. 85–86], the number of $\mathbb{F}_q$-rational forms of such a polarized surface is bounded by this same number. □

With these preliminaries out of the way, we may proceed to the proof of Proposition 3.4. The proof splits into cases, depending on whether or not the base field is a prime field. First we consider the case where q ranges over the set of primes p.

We may assume that $p > 3$. In that case, we see from Table 1 that there is only one isogeny class of supersingular elliptic curves, the isogeny class $\mathcal{I}(\mathbb{F}_p, 0)$ of trace-0 curves, which consists of 1 or 2 strata.

Pick a trace-0 elliptic curve $E_0/\mathbb{F}_p$ whose endomorphism ring has discriminant $-4p$. If (A, λ) is a principally polarized abelian surface over $\mathbb{F}_p$ with A

isogenous to E_0^2, then either A is a product of elliptic curves with the product polarization, or A is the restriction of scalars of an elliptic curve over $\mathbb{F}_{p^2}$ with trace $-2p$, or A is the Jacobian of a curve C. (See [González et al. 2005, Theorem 3.1, p. 270].) The number of elliptic curves in $\mathcal{I}(\mathbb{F}_p, 0)$ is $H(-4p)$; using Lemma 4.4 we see that the number of products of such elliptic curves is $\ll p(\log p)^2(\log\log p)^4 \ll p^2$ for all primes p. The number of supersingular elliptic curves over $\mathbb{F}_{p^2}$ with trace $-2p$ is equal to the number of supersingular j-invariants, which is $\frac{1}{12}p + O(1)$; therefore the number of restrictions of scalars of such curves is $\ll p$. Thus, we may focus our attention on the case where (A, λ) is the Jacobian of a curve C.

In this case, we know from Lemma 7.2 that C has a map of degree at most $\sqrt{2p} < p$ to E_0, so C has a *minimal* map of degree at most p to a curve E in $\mathcal{I}(\mathbb{F}_p, 0)$. We see that the polarized variety (A, λ) can be obtained by gluing together two elliptic curves E and E' in $\mathcal{I}(\mathbb{F}_p, 0)$ along their n-torsion, for some $n < p$. It follows that the a-number of A is 2, so A is superspecial. By [Ibukiyama and Katsura 1994, Remark 3, p. 41], the number of principally polarized superspecial abelian surfaces over $\overline{\mathbb{F}}_p$ which admit a model over $\mathbb{F}_p$ is $\ll ph(-p)$, which in turn is $\ll p^{3/2}(\log p)|\log\log p|$ by Lemma 4.4. By Lemma 9.1, we get the same bound for the number of superspecial curves over $\mathbb{F}_p$. This shows that Proposition 3.4 holds as q ranges over the set of primes.

Now we let q range over the set of proper prime powers. Let $q = p^e$ for some prime p and $e > 1$. First we bound the number of principally polarized superspecial split surfaces.

By [Ibukiyama and Katsura 1994, Theorem 2, p. 41], the total number of superspecial curves over $\overline{\mathbb{F}}_q$ is equal to the class number $H_2(1, p) \le \frac{1}{4}p^2$ mentioned above, so by Lemma 9.1 there are at most $\frac{1152}{4}p^2 = 288p^2$ superspecial curves over $\mathbb{F}_q$. Similarly, the number of supersingular j-invariants is $\frac{1}{12}p + O(1)$, so the number of distinct products of polarized supersingular elliptic curves over $\overline{\mathbb{F}}_q$ is also bounded by a constant times p^2; by Lemma 9.1, this shows that the number of principally polarized superspecial split abelian surfaces over $\mathbb{F}_q$ that are not Jacobians is $\ll p^2$. Since $q \ge p^2$, the number of principally polarized superspecial split surfaces is $\ll q$.

We are left with the task of estimating the number of nonsuperspecial supersingular split curves over $\mathbb{F}_q$. To do this, we appeal to a moduli space argument. As noted above, the coarse moduli space of nonsuperspecial supersingular curves is geometrically a union of $\frac{1}{2880}p^2 + O(p)$ components, each one an open subvariety of $\boldsymbol{A}^1$. Thus, the number of $\mathbb{F}_q$-rational points on this moduli space is at most $\frac{1}{2880}p^2q + O(pq)$. By Lemma 9.1, each rational point on the moduli space corresponds to at most 1152 curves over $\mathbb{F}_q$, so there are $\ll p^2q \ll q^2$ principally polarized supersingular split abelian surfaces over $\mathbb{F}_q$. $\square$

10. A lower bound for the number of split surfaces

In this section we prove Proposition 3.5.

Let ℓ be a prime coprime to q. We say that two elliptic curves E and F over $\mathbb{F}_q$ are *of the same symplectic type modulo* ℓ if (in the notation of Section 5) the set $\operatorname{Isom}^1(E[\ell], F[\ell])$ is nonempty, that is, if there is an isomorphism $E[\ell] \to F[\ell]$ of group schemes that respects the Weil pairing. Clearly, if E and F have the same symplectic type modulo ℓ then their traces of Frobenius are congruent modulo ℓ, so for each residue class modulo ℓ, the elliptic curves whose traces lie in that residue class are distributed among some number of symplectic types.

Lemma 10.1. *Let ℓ be an odd prime coprime to q and let $a \in \mathbb{Z}/\ell$.*

(a) *If $a^2 \not\equiv 4q \mod \ell$ then all elliptic curves $E/\mathbb{F}_q$ with $a(E) \equiv a \mod \ell$ are of the same symplectic type.*

(b) *If $a^2 \equiv 4q \mod \ell$, there are at most three symplectic types of elliptic curves with trace congruent to a. If we fix an ℓ-th root of unity $\zeta \in \overline{\mathbb{F}}_q$, these three types are determined as follows*:
   1. *Those E for which Frobenius acts as an integer on $E[\ell]$.*
   2. *Those E for which Frobenius does not act as an integer on $E[\ell]$, and for which the Weil pairing $e(P, \operatorname{Fr}_E(P))$ is of the form ζ^x with $x \in (\mathbb{Z}/\ell)^\times$ a square for all $P \in E[\ell](\overline{\mathbb{F}}_q)$ with $\operatorname{Fr}_E(P) \neq \frac{a}{2}P$.*
   3. *Those E for which Frobenius does not act as an integer on $E[\ell]$, and for which the Weil pairing $e(P, \operatorname{Fr}_E(P))$ is of the form ζ^x with $x \in (\mathbb{Z}/\ell)^\times$ a nonsquare for all $P \in E[\ell](\overline{\mathbb{F}}_q)$ with $\operatorname{Fr}_E(P) \neq \frac{a}{2}P$.*

Corollary 10.2. *For each odd ℓ coprime to q, there are at most $\ell + 4$ symplectic types of elliptic curves modulo ℓ over $\mathbb{F}_q$.* □

Proof of Lemma 10.1. Let E be an elliptic curve over $\mathbb{F}_q$ and let G be the automorphism group of $E[\ell]$. In Section 5 we defined a map $m : G \to \operatorname{Aut} \boldsymbol{\mu}_\ell$. If $a^2 \not\equiv 4q \mod \ell$ then m is surjective, so there is an isometry between $E[\ell]$ and $F[\ell]$ for any two curves E and F of trace a. Likewise, if Frobenius acts as a constant on $E[\ell]$ then m is surjective, so if Frobenius acts as $\frac{a}{2}$ on $E[\ell]$ and $F[\ell]$ then there is an isometry between those two group schemes.

On the other hand, if Frobenius does not act semisimply then the image of m is a coset of a subgroup of index 2, that is, a coset of the subgroup of squares, and is an isometry between $E[\ell]$ and $F[\ell]$ for two such curves E and F if and only if the image of m is the same for both of them. □

Lemma 10.3. *Let ℓ be a prime coprime to q and with $\ell \equiv 1 \mod 4$. If two elliptic curves E and F over $\mathbb{F}_q$ have the same symplectic type modulo ℓ, then there are at least $\ell - 1$ elements of* $\operatorname{Isom}^{-1}(E[\ell], F[\ell])$.

Proof. Since E and F have the same symplectic type modulo ℓ there is an isometry $\eta : E[\ell] \to F[\ell]$. Let b be an integer with $b^2 \equiv -1 \bmod \ell$. Then $b\eta$ is an anti-isometry, so $\mathrm{Isom}^{-1}(E[\ell], F[\ell])$ is nonempty. From Proposition 4.3 we know that $\#\operatorname{Aut} E[\ell] \ge (\ell-1)^2$, so $\#\mathrm{Isom}^1(E[\ell], E[\ell])$ is at least $\ell - 1$, and it follows that there are at least this many elements of $\mathrm{Isom}^{-1}(E[\ell], F[\ell])$. □

Proof of Proposition 3.5. Let c be a constant such that

$$H(\Delta) < c|\Delta|^{1/2}\log|\Delta|(\log\log|\Delta|)^2$$

for all negative discriminants Δ; such a constant exists by Lemma 4.4. We will show that for every prime $\ell \ne p$ with $\ell \equiv 1 \bmod 4$ and with

$$\ell < \frac{q^{1/2}}{1600c^2(\log q)^2(\log\log q)^4} \tag{13}$$

there are more than $\frac{2}{5}q^2$ triples (E_1, E_2, η), where E_1 and E_2 are nonisogenous ordinary elliptic curves over $\mathbb{F}_q$ and $\eta : E_1[\ell] \to E_2[\ell]$ is an anti-isometry. Dirichlet's theorem shows that there are constants $c' \ge 13$, $c'' > 0$ such that when $q \ge c'$ the number of such primes ℓ is at least

$$\frac{c'' q^{1/2}}{(\log q)^3(\log\log q)^4},$$

so for $q \ge c'$ we will have at least

$$\frac{c'' q^{5/2}}{5(\log q)^3(\log\log q)^4}$$

distinct principally polarized abelian surfaces, thus proving the unconditional part of Proposition 3.5.

Let ℓ be a prime as above, let $t \le \ell + 4$ be the number of symplectic types of curves modulo ℓ, and let $S_1, \ldots, S_t$ be the sets of ordinary curves of the t different symplectic types. We would like to count the number of pairs of curves (E_1, E_2) where E_1 and E_2 are not isogenous to one another but are of the same symplectic type. The number of ordered pairs (E_1, E_2) where E_1 and E_2 are of the same type is

$$\sum_{i=1}^{t} (\# S_i)^2.$$

This sum is minimized when the elliptic curves are evenly distributed across the symplectic types. It is easy check that when $q \ge 13$ there are always at least $\frac{5}{3}q$ ordinary elliptic curves over $\mathbb{F}_q$, so we see that

$$\sum_{i=1}^{t} (\# S_i)^2 \ge t\left(\frac{5q}{3t}\right)^2 \ge \frac{25q^2}{9(\ell+4)} \ge \frac{125q^2}{81\ell} > \frac{3q^2}{2\ell}.$$

On the other hand, the number of ordered pairs (E_1, E_2) of ordinary elliptic curves that are isogenous to one another is

$$\sum_{\substack{-2\sqrt{q}<a<2\sqrt{q}\\ \gcd(a,q)=1}} H(a^2-4q)^2.$$

Using Lemma 4.4 and the definition of c, we see that each summand is at most

$$c^2(4q)(\log(4q))^2(\log\log(4q))^4 < 400c^2q(\log q)^2(\log\log q)^4,$$

so the number of such ordered pairs is at most

$$1600c^2q^{3/2}(\log q)^2(\log\log q)^4 \le q^2/\ell.$$

Thus, the number of ordered pairs (E_1, E_2) of nonisogenous curves that have the same symplectic type is at least $\frac{1}{2}(q^2/\ell)$. By Lemma 10.3, this gives us more than $\frac{1}{2}(\ell-1)q^2/\ell > \frac{2}{5}q^2$ triples (E_1, E_2, η) where E_1 and E_2 are nonisogenous ordinary elliptic curves and $\eta : E_1[\ell] \to E_2[\ell]$ is an anti-isometry, as we wanted.

If the generalized Riemann hypothesis holds, we modify our argument as follows. We take c to be a constant such that

$$H(\Delta) < c|\Delta|^{1/2}(\log\log|\Delta|)^3$$

for all negative discriminants Δ, and consider primes $\ell \equiv 1 \bmod 4$ bounded by

$$\ell < \frac{q^{1/2}}{1600c^2(\log\log q)^6}$$

instead of by (13). Again we find that for each such ℓ we have more than $\frac{2}{5}q^2$ triples (E_1, E_2, η), where E_1 and E_2 are nonisogenous ordinary elliptic curves and $\eta : E_1[\ell] \to E_2[\ell]$ is an anti-isometry. Dirichlet's theorem then leads to the desired estimate for W_q. □

11. Numerical data, evidence for Conjecture 1.1, and further directions

In this section we present summaries of some computations that help give some indication of the behavior of several of the quantities that we study and provide bounds for, and we give some evidence that seems to support Conjecture 1.1. We close with some thoughts about possible extensions of our results.

11.1. ***The sum of the relative conductors.*** In Section 6 we proved Proposition 3.1, which gives an upper bound on the number of principally polarized ordinary split nonisotypic abelian surfaces over a finite field $\mathbb{F}_q$. The key to the argument is

Lemma 6.3, which gives an upper bound for the sum of the relative conductors of the ordinary elliptic curves over $\mathbb{F}_q$. The lemma shows that there is a constant c such that for all q this sum is at most $cq(\log q)^2$. However, we suspect that the sum of the relative conductors grows more slowly than this; it is perhaps even $O(q)$.

We computed this sum for all prime powers q less than 10^7. For q in the range $(10^3, 10^4)$, the sum lies between $2.07q$ and $4.27q$; for q in the range $(10^4, 10^5)$, the sum lies between $2.14q$ and $3.95q$; for q in the range $(10^5, 10^6)$, the sum lies between $2.09q$ and $3.82q$; and for q in the range $(10^6, 10^7)$, the sum lies between $2.10q$ and $3.77q$. Note that as q ranges through these successive intervals, the upper bound on $1/q$ times the sum of the relative conductors decreases; this is why we are tempted to suspect that the sum of the relative conductors is $O(q)$.

11.2. ***The probability that a principally polarized abelian surface is split.*** If S is a finite collection of geometric objects having finite automorphism groups, we define the *weighted cardinality* $\#' S$ of S by

$$\#' S = \sum_{s \in S} \frac{1}{\# \operatorname{Aut} s}.$$

It is well known that the weighted cardinality can lead to cleaner formulas than the usual cardinality. For instance, the weighted cardinality of the set of genus-2 curves over $\mathbb{F}_q$ is equal to q^3 [Brock and Granville 2001, Proposition 7.1, p. 87]. A principally polarized abelian surface over a field is either a Jacobian, a product of polarized elliptic curves, or the restriction of scalars of a polarized elliptic curve over a quadratic extension of the base field [González et al. 2005, Theorem 3.1, p. 270]. One can show that the weighted cardinality of the set of products of polarized elliptic curves over $\mathbb{F}_q$ is $\frac{1}{2}q^2$, as is the weighted cardinality of the set of restrictions of scalars. Thus, if we let $\mathcal{A}_2$ denote the moduli stack of principally polarized abelian surfaces, then

$$\#' \mathcal{A}_2(\mathbb{F}_q) = q^3 + q^2.$$

For each prime power q we let

$$c_q = \frac{\sqrt{q} \cdot \#' \mathcal{A}_{2,\mathrm{split}}(\mathbb{F}_q)}{\#' \mathcal{A}_2(\mathbb{F}_q)} = \frac{\sqrt{q} \cdot \#' \mathcal{A}_{2,\mathrm{split}}(\mathbb{F}_q)}{q^3 + q^2}.$$

For all primes $q < 300$ and for $q = 521$ we computed the exact value of c_q by direct enumeration of curves and computation of zeta functions. For $q \in \{1031, 2053, 4099, 16411, 65537\}$ (the smallest primes greater than 2^i for $i = 10, 11, 12, 14, 16$) we computed approximations to c_q by randomly sampling genus-2 curves (with probability inversely proportional to their automorphism groups), and then adjusting the probabilities to account for the non-Jacobians. We computed enough examples

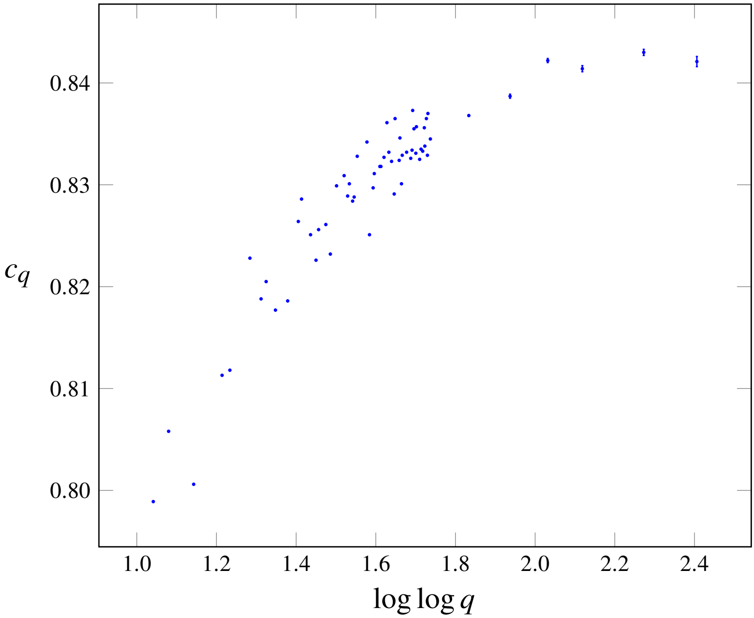

Figure 1. The values of c_q for the primes q with $17 \leq q \leq 293$, together with $q \in \{521, 1031, 2053, 4099, 16411, 65537\}$. The values of c_q for the five largest q were computed experimentally; the error bars indicate one standard deviation.

for each of these q to determine c_q with a standard deviation of less 0.0005. The result of the computations is displayed in Figure 1; the (almost invisible) error bars on the rightmost five data points indicate the standard deviation. Note that the horizontal axis is $\log \log q$; even so, the graph looks sublinear. This encourages us to speculate that perhaps the c_q are bounded away from 0 and ∞.

11.3. ***Reductions of a fixed surface.*** Let A/K be a principally polarizable abelian surface over a number field such that the absolute endomorphism ring $\operatorname{End}_{\overline{K}} A$ is isomorphic to $\mathbb{Z}$, and recall the counting function $\pi_{\mathrm{split}}(A/K, z)$ introduced in Section 1. Conjecture 1.1 states that $\pi_{\mathrm{split}}(A/K, z) \sim C_A \sqrt{z}/\log z$; we tested this against actual data on the splitting behavior of a particular surface A over $\mathbb{Q}$.

Let A be the Jacobian of the curve over $\mathbb{Q}$ with affine model $y^2 = x^5 + x + 6$. Using the methods of [Harvey and Sutherland 2016], Andrew Sutherland computed for us the primes $p < 2^{30}$ for which the mod-p reduction of A is split, thereby giving us the exact value of $\pi_{\mathrm{split}}(A/\mathbb{Q}, z)$ for all $z \leq 2^{30}$. We numerically fit curves of the form $a\sqrt{z}/(\log z)^b$ and of the form $c\sqrt{z}/\log z$ to this function. For curves of the form $a\sqrt{z}/(\log z)^b$, the best-fitting exponent b was $b \approx 1.02269$, reasonably close to our conjectural value of 1. For curves of the form $c\sqrt{z}/\log z$,

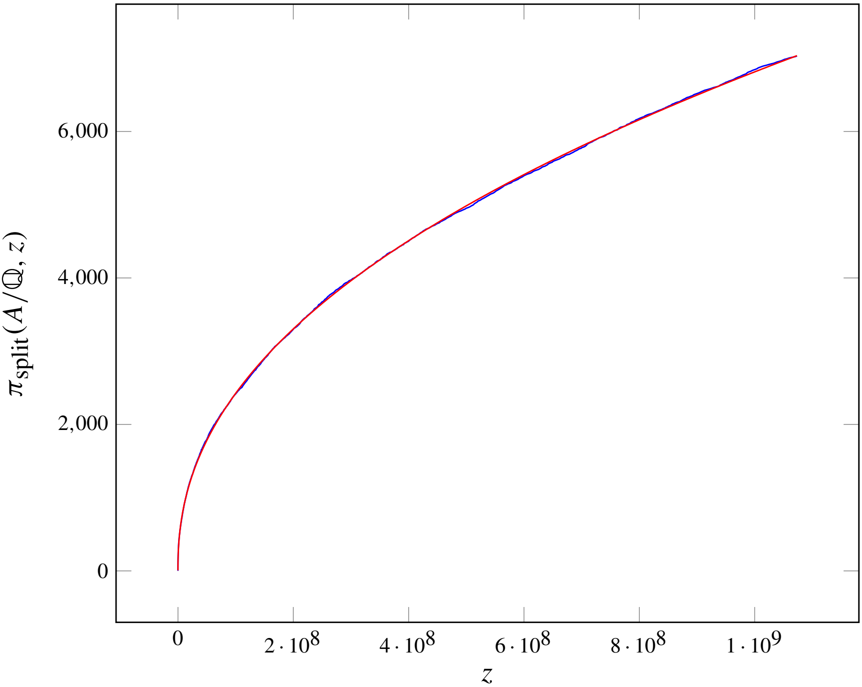

Figure 2. The blue curve plots the function $\pi_{\text{split}}(A/\mathbb{Q}, z)$ for the Jacobian A of the curve $y^2 = x^5 + x + 6$ over $\mathbb{Q}$. The red curve is $c\sqrt{z}/\log z$, with $c \approx 4.4651$.

the best-fitting constant c was $c \approx 4.4651$. In Figure 2 we present the actual data (in blue) alongside the best-fitting function $c\sqrt{z}/\log z$ (in red); the figure shows that the idealized function is in close agreement with the actual function.

11.4. ***Further directions.*** We noted in Section 1 that our definition of $\mathcal{A}_{2,\text{split}}(\mathbb{F}_q)$ was perhaps not as natural as it could be — one could also ask about principally polarized surfaces that split over $\overline{\mathbb{F}}_q$, not just over $\mathbb{F}_q$ itself. We suspect that a result like Theorem 1.2 holds for this more general type of splitting. To prove such a theorem, one would need to estimate the number of principally polarized surfaces in several types of isogeny classes: the simple ordinary isogeny classes that are geometrically split (which are enumerated in [Howe and Zhu 2002, Theorem 6, p. 145]), and the supersingular isogeny classes (which are all geometrically split). There are a number of ways one could try to estimate the number of principally polarized surfaces in these isogeny classes; for instance, the techniques of [Howe 2004] might be of use. We will not speculate further on this here.

Let $\mathcal{A}_{2,\text{geom. split}}(\mathbb{F}_q)$ denote the subset of $\mathcal{A}_2(\mathbb{F}_q)$ consisting of those principally polarized varieties that are not geometrically simple, and for each q let d_q denote

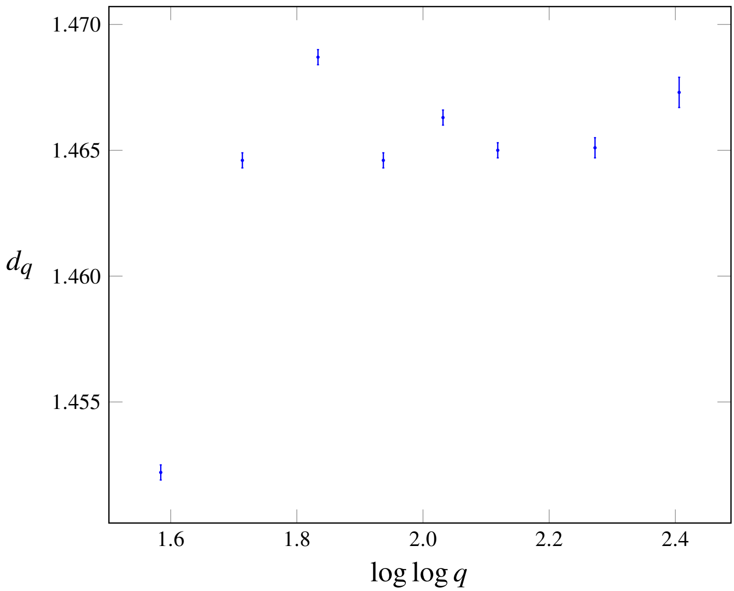

Figure 3. The experimentally computed values of d_q for the primes q in $\{131, 257, 521, 1031, 2053, 4099, 16411, 65537\}$. Error bars indicate one standard deviation.

the ratio

$$d_q = \frac{\sqrt{q}\cdot \#'\mathcal{A}_{2,\text{geom. split}}(\mathbb{F}_q)}{\#'\mathcal{A}_2(\mathbb{F}_q)} = \frac{\sqrt{q}\cdot \#'\mathcal{A}_{2,\text{geom. split}}(\mathbb{F}_q)}{q^3+q^2}.$$

While collecting the data presented in Section 11.2 we also collected data on d_q by random sampling of curves. Figure 3 presents the results for $q \in \{131, 257, 521, 1031, 2053, 4099, 16411, 65537\}$. The figure suggests that perhaps d_q is bounded away from 0 and ∞.

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A tropical approach to nonarchimedean Arakelov geometry

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Chambert-Loir and Ducros have recently introduced a theory of real valued differential forms and currents on Berkovich spaces. In analogy to the theory of forms with logarithmic singularities, we enlarge the space of differential forms by so called δ-forms on the nonarchimedean analytification of an algebraic variety. This extension is based on an intersection theory for tropical cycles with smooth weights. We prove a generalization of the Poincaré–Lelong formula which allows us to represent the first Chern current of a formally metrized line bundle by a δ-form. We introduce the associated Monge–Ampère measure μ as a wedge-power of this first Chern δ-form and we show that μ is equal to the corresponding Chambert-Loir measure. The $*$-product of Green currents is a crucial ingredient in the construction of the arithmetic intersection product. Using the formalism of δ-forms, we obtain a nonarchimedean analogue at least in the case of divisors. We use it to compute nonarchimedean local heights of proper varieties.

MSC2010: primary 14G40; secondary 14G22, 14T05, 32P05.

Keywords: differential forms on Berkovich spaces, Chambert-Loir measures, tropical intersection theory, nonarchimedean Arakelov theory.

0. Introduction

Weil's adelic point of view was to compactify the ring of integers $\mathscr{O}_K$ of a number field K by the archimedean primes. Arakelov's brilliant idea was to add metrics on the "fibre at infinity" of a surface over $\mathscr{O}_K$ which gave a good intersection theory for arithmetic divisors. This Arakelov theory became popular after Faltings used it to prove Mordell's conjecture. A higher dimensional arithmetic intersection theory was developed by Gillet and Soulé. Their theory combines algebraic intersection theory on a regular model $\mathscr{X}$ over $\mathscr{O}_K$ with differential geometry on the associated complex manifold X^{an} of the generic fibre X of $\mathscr{X}$. Roughly speaking, an arithmetic cycle on $\mathscr{X}$ is given by a pair $(\mathscr{Z}, g_Z)$, where $\mathscr{Z}$ is a cycle on $\mathscr{X}$ with generic fibre Z and g_Z is a current on X^{an} satisfying the equation

$$dd^c g_Z = [\omega_Z] - \delta_Z$$

for a smooth differential form ω_Z and the current of integration δ_Z over Z^{an}. The arithmetic intersection product uses the algebraic intersection product for algebraic cycles in the first component and the $*$-product of Green currents in the second component. This arithmetic intersection theory is nowadays called Arakelov theory. It found many nice applications such as Faltings's proof of the Mordell–Lang conjecture for abelian varieties and the proof of Ullmo and Zhang of the Bogomolov conjecture for abelian varieties.

It is an old dream to handle archimedean and nonarchimedean places in a similar way. This means that we are looking for a description in terms of currents for the contributions of the nonarchimedean places to Arakelov theory. Such a nonarchimedean Arakelov theory at finite places was developed by Bloch–Gillet–Soulé relying strongly on the conjectured existence of resolution of singularities for models in mixed characteristics. The use of models also has another disadvantage since they are not suitable to describe canonical metrics as for line bundles on abelian varieties with bad reduction. A more analytic nonarchimedean Arakelov theory was developed by Chinburg and Rumely, and Zhang in the case of curves. A crucial role is played here by the reduction graph of the curve. Without any doubt, the latter should be replaced by the Berkovich analytic space associated to the curve and this was done by Thuillier in his thesis introducing a nonarchimedean potential theory. Chambert-Loir and Ducros [2012] recently introduced differential forms and currents on Berkovich spaces. These provide us with a new tool to give an analytic description of nonarchimedean Arakelov theory in higher dimensions.

We recall the definition of differential forms given in [loc. cit.]. We restrict here to the algebraic case. Let U be an n-dimensional very affine open variety which means that U has a closed embedding into a multiplicative torus $T = \mathbb{G}_m^r$ over a nonarchimedean field K. By definition, such a field K is endowed with a complete nonarchimedean absolute value $|\cdot|$. Let $t_1, \dots, t_r$ be the torus coordinates. Then we have the tropicalization map

$$\mathrm{trop}: T^{\mathrm{an}} \to \mathbb{R}^r, \quad t \mapsto (-\log|t_1|, \dots, -\log|t_r|).$$

By the Bieri–Groves theorem, the tropical variety $\mathrm{Trop}(U) := \mathrm{trop}(U^{\mathrm{an}})$ is a finite union of n-dimensional polyhedra. More precisely, $\mathrm{Trop}(U)$ is an n-dimensional tropical cycle which means that $\mathrm{Trop}(U)$ is a polyhedral complex endowed with canonical weights satisfying a balancing condition. Let $x_1, \dots, x_r$ be the coordinates on $\mathbb{R}^r$. Then Lagerberg's superforms on $\mathbb{R}^r$ are formally given by

$$\alpha = \sum_{\substack{|I|=p,\\ |J|=q}} \alpha_{IJ} d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \cdots \wedge d''x_{j_q},$$

where I (resp. J) consists of $i_1 < \cdots < i_p$ (resp. $j_1 < \cdots < j_q$), $\alpha_{IJ} \in C^\infty(\mathbb{R}^r)$. We have differential operators d' and d'' on the space of superforms given by

$$d'\alpha := \sum_{\substack{|I|=p,\\ |J|=q}} \sum_{i=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_i \wedge d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \cdots \wedge d''x_{j_q}$$

and

$$d''\alpha := \sum_{\substack{|I|=p,\\ |J|=q}} \sum_{j=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_j} d''x_j \wedge d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \cdots \wedge d''x_{j_q}.$$

They are the analogues of the differential operators ∂ and $\bar\partial$ in complex analysis. The space of superforms on $\mathbb{R}^r$ with the usual wedge product is a differential bigraded $\mathbb{R}$-algebra with respect to d' and d''. The space of supercurrents on $\mathbb{R}^r$ is given as the topological dual of the space of superforms.

Every superform α induces a differential form on U^{an} and two superforms α, α' induce the same form if and only if they restrict to the same superform on $\mathrm{Trop}(U)$. In general, a differential form on an n-dimensional variety X is given locally for the Berkovich analytic topology on very affine open subsets by Lagerberg's superforms which agree on common intersections (see [Gubler 2016] for more details). The wedge product and the differential operators can be carried over to X^{an} leading to a sheaf $A^{\cdot,\cdot}$ of differential forms on X^{an}. Integration of superforms leads to integration of compactly supported (n,n)-forms on X^{an}. The space of currents $D^{\cdot,\cdot}(X^{\mathrm{an}})$ is defined as the topological dual of the space of compactly supported forms.

A major result of Chambert-Loir and Ducros is the Poincaré–Lelong formula for the meromorphic section of a line bundle endowed with a continuous metric $\|\cdot\|$. Note that in this situation, $c_1(L, \|\cdot\|)$ is only a current, while a smooth metric allows one to define the first Chern form in $A^{1,1}(X^{\mathrm{an}})$. For a smooth metric, $c_1(L, \|\cdot\|)^n$

is a form of top degree and hence defines a signed measure called the Monge–Ampère measure of $(L, \|\cdot\|)$. In arithmetic, metrics are often induced by proper algebraic models over the valuation ring. Such metrics are called algebraic. They are continuous on X^{an}, but not smooth. This makes it difficult to define the Monge–Ampère measure as a wedge product of currents. In the complex situation, one needs Bedford–Taylor theory to define such a wedge product. In the nonarchimedean situation, Chambert-Loir and Ducros use an approximation process by smooth metrics to define this top-dimensional wedge product of first Chern currents.

The main theorem in [Chambert-Loir and Ducros 2012] shows that the Monge–Ampère measure of a line bundle endowed with a formal metric is equal to the Chambert-Loir measure. The latter was introduced in [Chambert-Loir 2006] before a definition of first Chern current was available. It is defined as a discrete measure on the Berkovich space using degrees of the irreducible components of the special fibre. Chambert-Loir measures play a prominent role in nonarchimedean equidistribution results. For example, they occur in the nonarchimedean version of Yuan's equidistribution theorem, which has applications to the geometric Bogomolov conjecture.

In the thesis of Christensen [2013] a different approach to a first Chern form was given. Christensen studied the example E^2 for a Tate elliptic curve E and he defined the first Chern form as a tropical divisor on the skeleton of E^2. Then he showed that the 2-fold tropical self-intersection of this divisor gives the Chambert-Loir measure.

In this paper, we combine both approaches. We enrich the theory of differential forms given in [Chambert-Loir and Ducros 2012] by enlarging the space of smooth forms to the space of δ-forms. They behave as forms and they have the advantage that we can define a first Chern δ-form for a line bundle endowed with a formal metric. This leads to a direct definition of the Monge–Ampère measure as a wedge product of δ-forms and to an approach to nonarchimedean Arakelov theory.

This will be explained in more detail now. Throughout this paper K denotes an algebraically closed field endowed with a nontrivial nonarchimedean complete absolute value. Note that this is no restriction of generality as for many problems including the ones discussed in this paper such a setup can always be achieved by base change. This is similar to the archimedean case where analysis is usually performed over the complex numbers. For sake of simplicity, we assume in the introduction that tropical cycles have constant weights as usual in tropical geometry (see Section 1 for details and for a generalization to smooth weights). A δ-preform on $\mathbb{R}^r$ is a supercurrent α on $\mathbb{R}^r$ of the form

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i} \tag{0.0.1}$$

for finitely many superforms α_i and tropical cycles C_i on $\mathbb{R}^r$. Using the wedge product of superforms and the stable intersection product of tropical cycles, we get a wedge product of δ-preforms. Since the supercurrents of integration δ_{C_i} are d'-closed and d''-closed, we can extend the differential operators d' and d'' to δ-preforms leading to a differential bigraded $\mathbb{R}$-algebra. We refer to Section 2 for precise definitions and generalizations allowing smooth tropical weights.

In Section 3, we extend all these notions from $\mathbb{R}^r$ to a fixed tropical cycle C of $\mathbb{R}^r$. The balancing condition is equivalent to closedness of the supercurrent δ_C, which means that C is boundaryless in the sense that no boundary integral shows up in the theorem of Stokes over C. Therefore we may view a tropical cycle as a combinatorial analogue of a complex analytic space. Using integration over C, we will see that a piecewise smooth form η on the support of C induces a supercurrent $[\eta]$ on C. We apply this to a piecewise smooth function ϕ on C. In tropical geometry, ϕ plays the role of a Cartier divisor on C and has an associated tropical Weil divisor $\phi \cdot C$. The latter is also called the corner locus of ϕ as it is a tropical cycle of codimension 1 with support equal to the singular locus of ϕ. We show in Corollary 3.19 the following *tropical Poincaré–Lelong formula*:

Theorem 0.1. *Let ϕ be a piecewise smooth function on C and let $\delta_{\phi\cdot C}$ be the supercurrent of integration over the corner locus $\phi \cdot C$. Then we have*

$$d'd''[\phi] - [d'd''\phi] = \delta_{\phi\cdot C}$$

as supercurrents on C.

This is a statement about integration of superforms on tropical currents and its proof relies on Stokes theorem. In fact, we prove a more general statement in Theorem 3.16 involving integration of δ-preforms on C.

Let X be an n-dimensional algebraic variety over K. We now define δ-forms on X^{an} similarly as differential forms, but replacing superforms by the more general δ-preforms. This means that a δ-form is given locally with respect to the Berkovich analytic topology on very affine open subsets by pull-backs of δ-preforms with respect to the tropicalization maps. The δ-preforms have to agree on overlaps which involves a quite complicated restriction process which is explained in Section 4. Moreover, we will show that δ-forms are bigraded, have a wedge product and differential operators d', d'' extending the corresponding structures for differential forms on X^{an}. There is also a pull-back with respect to morphisms and so we see that δ-forms behave as differential forms on complex manifolds.

In Section 5, we study integration of compactly supported δ-forms of bidegree (n, n) on X^{an}. To define the integral of such a δ-form α, we choose a dense open subset U of X with a closed embedding $U \hookrightarrow \mathbb{G}_m^r$ such that α is given on U^{an} by the pull-back of a δ-preform α_U on $\mathbb{R}^r$ with respect to the tropicalization map $\mathrm{trop}_U : U^{\mathrm{an}} \to \mathbb{R}^r$. Using the corresponding tropical variety $\mathrm{Trop}(U)$, we set

$$\int_{X^{\mathrm{an}}} \alpha := \int_{|\mathrm{Trop}(U)|} \alpha_U.$$

In Section 6, we introduce δ-currents as continuous linear functionals on the space of compactly supported δ-forms. By integration, every δ-form α induces a δ-current $[\alpha]$. Similarly, we get a current of integration δ_Z for every cycle Z on X. As a major result, we show in Corollary 6.15 that $[\alpha]$ is a signed Radon measure on X^{an} for every α of bidegree (n,n). We deduce in Proposition 6.16 that every continuous real function g on X^{an} induces a δ-current $[g]$ on X^{an} which is defined at a compactly supported δ-form α of bidegree (n,n) by integrating g with respect to the corresponding Radon measure.

Now let f be a rational function on X which is not identically zero. By integration again, we will get a δ-current $[-\log|f|]$ on X^{an}.

Theorem 0.2. *Let* $\mathrm{cyc}(f)$ *be the Weil divisor associated to* f*. Then we have the Poincaré–Lelong equation*

$$\delta_{\mathrm{cyc}(f)} = d'd''[\log|f|]$$

of δ*-currents on* X^{an}.

This is demonstrated as Theorem 7.2. The Poincaré–Lelong equation of Chambert-Loir and Ducros is the special case of our formula where one evaluates the δ-currents at differential forms. The generalization to δ-forms is not obvious and needs a more tropical adaptation of their beautiful arguments. In Section 7, we introduce the first Chern δ-current $[c_1(L, \|\cdot\|)]$ of a continuously metrized line bundle $(L, \|\cdot\|)$ on X. As usual, we mean here continuity with respect to the Berkovich topology on X^{an}. In Corollary 7.8, we deduce from Theorem 0.2 that a nonzero meromorphic section s of L satisfies the Poincaré–Lelong equation

$$d'd''[-\log\|s\|] = [c_1(L, \|\cdot\|)] - \delta_{\mathrm{cyc}(s)} \tag{0.2.1}$$

for δ-currents on X^{an}.

In Section 8, we define piecewise smooth and piecewise linear metrics on L. We show in Proposition 8.11 that a metric is piecewise linear if and only if it is induced by a formal model of the line bundle. In Section 9, we introduce piecewise smooth forms on X^{an}. For a piecewise smooth metric $\|\cdot\|$ on L, the first Chern δ-current $[c_1(L, \|\cdot\|)]$ has a canonical decomposition into a sum of a piecewise smooth form and a residual current. If $\|\cdot\|$ is smooth, then $c_1(L, \|\cdot\|)$ is a differential form on X^{an}. We say that a piecewise smooth metric $\|\cdot\|$ is a δ-metric if the first Chern δ-current $[c_1(L, \|\cdot\|)]$ is induced by a δ-form $c_1(L, \|\cdot\|)$ (see Definition 9.9 for a more precise definition). In this situation, we call $c_1(L, \|\cdot\|)$ the first Chern δ-form of $(L, \|\cdot\|)$. We will see in Remark 9.16 that every piecewise linear metric is a δ-metric. Canonical metrics on line bundles exist on line bundles on abelian varieties, on line bundles which are algebraically equivalent to zero and on line bundles on toric varieties. It follows from our considerations in Section 8 that all these canonical metrics are δ-metrics (see Example 9.17).

In Section 10, we consider a proper algebraic variety X over K of dimension n with a line bundle L endowed with an algebraic metric $\|\cdot\|$. This means that the metric is induced by an algebraic model of L. Based on the formal GAGA principle, we show in Proposition 8.13 that an algebraic metric is the same as a formal metric and hence this is also the same as a piecewise linear metric. As a consequence, we note that $\|\cdot\|$ is a δ-metric and hence $c_1(L, \|\cdot\|)$ is a well-defined δ-form. We deduce that $c_1(L, \|\cdot\|)^n$ is a δ-form of bidegree (n, n) on X^{an}, which we may view as a signed Radon measure on X^{an} by the above. We call it the Monge–Ampère measure associated to $(L, \|\cdot\|)$. Our Theorem 10.5 can be expressed as follows:

Theorem 0.3. *Under the assumptions above, the Monge–Ampère measure associated to $(L, \|\cdot\|)$ is equal to the Chambert-Loir measure associated to $(L, \|\cdot\|)$.*

As mentioned before, this theorem was first proved by Chambert-Loir and Ducros in a slightly different setting (for discrete valuations, but their method works also for algebraically closed fields). However, they have a different construction of the Monge–Ampère measure. Since algebraic metrics are usually not smooth, they have only a first Chern current $c_1(L, \|\cdot\|)$ available. In general, the wedge product of currents is not well defined. In the present situation, they can use a rather complicated approximation process by smooth metrics to make sense of the wedge product $c_1(L_1, \|\cdot\|_1)^n$ as a current leading to their Monge–Ampère measure. Our Monge–Ampère measure is defined directly as a wedge product of δ-forms based on tropical intersection theory instead of the approximation process. This means that our proof is more influenced by tropical methods.

In Section 11, we define a Green current for a cycle Z on the algebraic variety X over K as a δ-current g_Z such that

$$d'd''g_Z = [\omega_Z] - \delta_Z$$

for a δ-form ω_Z on X^{an}. By the Poincaré–Lelong equation (0.2.1), a nonzero meromorphic section s of L induces a Green current $g_Y := -\log\|s\|$ for the Weil divisor Y of s. Here, we assume that $\|\cdot\|$ is a δ-metric on the line bundle L of X. In case of proper intersection, we define $g_Y * g_Z := g_Y \wedge \delta_Z + \omega_Y \wedge g_Z$ as in the archimedean theory of Gillet–Soulé. It is an easy consequence of the Poincaré–Lelong equation that $g_Y * g_Z$ is a Green current for the cycle $Y \cdot Z$. We show the usual properties for such $*$-products. Most difficult is the proof of the commutativity of the $*$-product of two Green currents for properly intersecting divisors. It relies on the study of piecewise smooth forms and the tropical Poincaré–Lelong formula in Theorem 0.1.

In Section 12, we define the local height of a proper n-dimensional variety X over K with respect to properly intersecting Cartier divisors $D_0, \dots, D_n$ endowed with δ-metrics on $O(D_0), \dots, O(D_n)$ as follows: Let g_{Y_j} be the Green current for the Weil divisor Y_j associated to D_j as above; then the local height is given by

$$\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X) := g_{Y_0} * \cdots * g_{Y_n}(1).$$

We show that these local heights are multilinear and symmetric in the metrized Cartier divisors $\hat{D}_0, \dots, \hat{D}_n$, functorial with respect to morphisms and satisfy an induction formula useful to decrease the dimension of X. For algebraic metrics, local heights of proper varieties are also defined using intersection theory on a suitable proper model (see [Gubler 1998, §9]).

Theorem 0.4. *Suppose that the metrics on $O(D_0), \dots, O(D_n)$ are all algebraic. Then the local height $\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X)$ based on the $*$-product of Green currents is equal to the local height of X given by intersection theory of divisors on K°-models.*

The proof uses the observation that the induction formula holds for both definitions of local heights and then Theorem 0.3 gives the claim (see Remark 12.7 for more details and the proof).

In the introduction, we have presented the whole theory of δ-forms based on δ-preforms as in (0.0.1) using tropical cycles with constant weights. However, the theory can be extended to δ-forms locally given by δ-preforms allowing tropical cycles with smooth weights. This will be done throughout the whole paper which leads to slightly more complications, but it increases the class of δ-metrics at the end which makes it worthwhile. Observe that tropical cycles which arise as tropicalizations from varieties always have integer weights. Therefore tropical cycles are always considered with constant weights when they serve, as in Section 3, as underlying spaces for supercurrents and δ-preforms.

Notation and terminology. Throughout this paper K denotes an algebraically closed field endowed with a complete nontrivial nonarchimedean absolute value $|\cdot|$, valuation ring K°, and corresponding valuation $v = -\log|\cdot|$. Let $\Gamma := v(K^\times)$ be the value group.

In $A \subset B$, A is strictly smaller than B. The complement of A in B is denoted by $B \setminus A$. The zero is included in $\mathbb{N}$ and in $\mathbb{R}_+$.

The group of multiplicative units in a ring A with 1 is denoted by $A^\times$. An (algebraic) variety over a field is an irreducible separated reduced scheme of finite type. The terminology from convex geometry is explained in the Appendix.

1. Tropical intersection theory with smooth weights

In tropical geometry, a tropical cycle is given by a polyhedral complex whose maximal faces are weighted by integers satisfying a balancing condition along the faces of codimension 1. In this section, we generalize the notion of a tropical cycle allowing smooth real functions on the maximal faces as weights. This is similar to the tropical fans with polynomial weights introduced by Esterov [2012] and François [2013]. We generalize basic facts from stable tropical intersection theory and introduce the corner locus of a piecewise smooth function.

Throughout this section N and N' denote free $\mathbb{Z}$-modules of finite rank r and r'. We write $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$. Every integral $\mathbb{R}$-affine polyhedron σ of dimension n in the $\mathbb{R}$-vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ determines an affine subspace $\mathbb{A}_\sigma$ with underlying vector space $\mathbb{L}_\sigma$ and a lattice $N_\sigma = \mathbb{L}_\sigma \cap N$ in $\mathbb{L}_\sigma$ (see A.2 in the Appendix). A smooth function $f : \sigma \to \mathbb{R}$ on a polyhedron σ in $N_{\mathbb{R}}$ is the restriction of a smooth function on some open neighbourhood of σ in $\mathbb{A}_\sigma$. For further notation borrowed from convex geometry, we refer to the Appendix.

Definition 1.1. (i) A polyhedral complex $\mathscr{C}$ of pure dimension n is called *weighted (with smooth weights)* if each polyhedron $\sigma \in \mathscr{C}_n$ is endowed with a smooth weight function $m_\sigma : \sigma \to \mathbb{R}$. If all m_σ are constant functions, then we call them *constant weights*. The *support* $|(\mathscr{C}, m)|$ of a weighted polyhedral complex $(\mathscr{C}, m)$ of pure dimension n is the closed set

$$|(\mathscr{C}, m)| = \bigcup_{\sigma \in \mathscr{C}_n} \operatorname{supp}(m_\sigma).$$

The support $|\mathscr{C}|$ of a polyhedral complex $\mathscr{C}$ is the support of $(\mathscr{C}, m)$, where $m = 1$ is the trivial weight function. We have $|(\mathscr{C}, m)| \subseteq |\mathscr{C}|$.

(ii) Let $C = (\mathscr{C}, m)$ be an integral $\mathbb{R}$-affine polyhedral complex of pure dimension n with smooth weights in $N_{\mathbb{R}}$. For each codimension-one face τ of a polyhedron $\sigma \in \mathscr{C}_n$ we choose a representative $\omega_{\sigma,\tau} \in N_\sigma$ of the generator of the one-dimensional lattice N_σ / N_τ pointing in the direction of σ. Then we say that the weighted polyhedral complex C satisfies the *balancing condition* if we have

$$\sum_{\substack{\sigma \in \mathscr{C}_n \\ \sigma \succ \tau}} m_\sigma(\omega)\omega_{\sigma,\tau} \in \mathbb{L}_\tau \tag{1.1.1}$$

for all $\tau \in \mathscr{C}_{n-1}$ and all $\omega \in \tau$, where $\sigma \succ \tau$ means that τ is a face of σ. This is a straightforward generalization of the balancing condition for polyhedral complexes with integer weights [Gubler 2013, 13.9].

(iii) A *tropical cycle* $C = (\mathscr{C}, m)$ *of dimension* n in $N_{\mathbb{R}}$ is a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension n which satisfies the balancing condition (1.1.1). In the following, we identify two tropical cycles $(\mathscr{C}, m)$ and $(\mathscr{C}', m')$ of dimension n if $|(\mathscr{C}, m)| = |(\mathscr{C}', m')|$ and if $m_\sigma = m_{\sigma'}$ on the intersection of the relative interiors of σ and σ' for all $\sigma \in \mathscr{C}_n$ and $\sigma' \in \mathscr{C}'_n$. A tropical cycle

$(\mathscr{C}, m)$ whose underlying polyhedral complex is a rational polyhedral fan in the vector space $N_{\mathbb{R}}$ is called a *tropical fan*.

(iv) Let $C = (\mathscr{C}, m)$ be a tropical cycle of dimension n. Given an integral $\mathbb{R}$-affine subdivision $\mathscr{C}'$ of $\mathscr{C}$, there is a unique family of weight functions m' such that $(\mathscr{C}', m')$ is a tropical cycle and $m_\sigma|_{\sigma'} = m'_{\sigma'}$ holds for all $\sigma' \in \mathscr{C}'_n$ and $\sigma \in \mathscr{C}_n$ such that $\sigma' \subseteq \sigma$. If a tropical cycle C in $N_{\mathbb{R}}$ is defined by a weighted integral $\mathbb{R}$-affine polyhedral complex $(\mathscr{C}, m)$, we call $\mathscr{C}$ a *polyhedral complex of definition for the tropical cycle C*.

(v) The set of tropical cycles with smooth weights of pure dimension n in $N_{\mathbb{R}}$ defines an abelian group $\mathrm{TZ}_n(N_{\mathbb{R}})$ where the group law is given by the addition of multiplicity functions on a common refinement of the integral $\mathbb{R}$-affine polyhedral complexes. We denote by $\mathrm{TZ}^k(N_{\mathbb{R}}) = \mathrm{TZ}_{r-k}(N_{\mathbb{R}})$ the *group of tropical cycles of codimension k*.

Remark 1.2 (reduction from smooth to constant weight functions). In tropical geometry, one usually considers tropical cycles with integer weights. However it causes no problems to work instead with tropical cycles with constant but not necessarily integer weights.

Many properties of these tropical cycles with integer or constant weights extend even to tropical cycles with smooth weights by the following local argument in $\omega \in |\mathscr{C}|$. We replace $\mathscr{C}$ by the rational polyhedral fan of local cones in ω (see A.6) and we endow the local cone of $\sigma \in \mathscr{C}_n$ by the constant weight $m_\sigma(\omega)$. By definition, these constant weights on the rational cones satisfy the balancing condition. We illustrate the use of this reduction process in Remark 1.4(ii).

1.3. In tropical geometry, there is a *stable tropical intersection product* of tropical cycles with integer weights. The astonishing fact is that this product is well-defined as a tropical cycle in contrast to algebraic intersection theory or homology, where an equivalence relation is needed. Constructions of a stable tropical intersection product of tropical cycles with integer weights have been given by Mikhalkin [2006] and Allermann and Rau [2010]. In both cases the construction is reduced to the case of tropical fans. For tropical fans with integer weights, Mikhalkin uses the fan displacement rule from [Fulton and Sturmfels 1997], whereas Allermann and Rau use reduction to the diagonal and intersections with tropical Cartier divisors. It is shown in [Katz 2012, §5; Rau 2009, Theorem 1.5.17] that both definitions agree. This is based on a result of Fulton and Sturmfels [1997, Theorem 3.1] which shows that the space of tropical fans, with integer weights and with a given complete rational polyhedral fan Σ as a polyhedral complex of definition, is canonically isomorphic to the Chow cohomology ring of the complete toric variety Y_Σ associated to Σ. Then the product in Chow cohomology leads to the stable intersection product of tropical fans with integer weights and the usual properties in Chow cohomology lead to corresponding properties in stable tropical intersection theory. By passing to a smooth rational polyhedral fan subdividing Σ, which means that Y_Σ is smooth, we may use the usual Chow groups instead of the Chow cohomology groups from [Fulton 1984, Chapter 17].

Remark 1.4 (stable tropical intersection theory). As an application of the reduction principle described in Remark 1.2, we get a stable tropical intersection theory for tropical cycles with smooth weights. The reduction process leads to tropical fans with constant weight functions. These weights are not necessarily integers, but it is still possible to apply 1.3 by using Chow cohomology with real coefficients. We list here the main properties:

(i) There exists a natural bilinear pairing

$$\mathrm{TZ}^k(N_\mathbb{R}) \times \mathrm{TZ}^l(N_\mathbb{R}) \to \mathrm{TZ}^{k+l}(N_\mathbb{R}), \quad (C_1, C_2) \mapsto C_1 \cdot C_2$$

which is called the *stable intersection product for tropical cycles*. It is associative and commutative and respects supports in the sense that we have $|C_1 \cdot C_2| \subseteq |C_1| \cap |C_2|$.

(ii) The concrete construction of the stable intersection product for tropical cycles C_1 and C_2 of codimension l_1 and l_2 in $N_\mathbb{R}$ is based on the fan displacement rule (see [Fulton and Sturmfels 1997, §4]). We choose a common polyhedral complex of definition $\mathscr{C}$ for C_1 and C_2 and write $C_i = (\mathscr{C}, m_i)$ $(i = 1, 2)$ for suitable families of weight functions $m_i = (m_{i,\sigma})_{\sigma \in \mathscr{C}^{l_i}}$. Let $\mathscr{D}$ denote the polyhedral subcomplex of $\mathscr{C}$ which is generated by $\mathscr{C}^{l_1+l_2}$. We choose a generic vector $v \in N_\mathbb{R}$ for $\mathscr{D}$, a small $\varepsilon > 0$, and equip $\mathscr{D}$ with the family of weight functions $m = (m_\tau)_{\tau \in \mathscr{C}^{l_1+l_2}}$, where $m_\tau : \tau \to \mathbb{R}$ is given by

$$m_\tau(\omega) = \sum_{\substack{(\sigma_1,\sigma_2) \in \mathscr{C}^{l_1} \times \mathscr{C}^{l_2} \\ \tau = \sigma_1 \cap \sigma_2 \\ \sigma_1 \cap (\sigma_2 + \varepsilon v) \neq \varnothing}} [N : N_{\sigma_1} + N_{\sigma_2}]\, m_{1,\sigma_1}(\omega)\, m_{2,\sigma_2}(\omega). \tag{1.4.1}$$

We will show that $D = (\mathscr{D}, m)$ is a tropical cycle whose construction is independent of the choice of the generic vector v and a sufficiently small $\varepsilon > 0$. We use D as the definition of the stable intersection product $C_1 \cdot C_2$.

The proof illustrates the reduction to constant weights given in Remark 1.2. For $\omega \in |\mathscr{C}|$, let $\mathscr{C}_\omega$ be the rational polyhedral fan of local cones in ω of the polyhedra in $\mathscr{C}$. First, we note that $\sigma \mapsto \rho := \mathrm{LC}_\sigma(\omega)$ is a bijective map from the set of polyhedra in $\mathscr{C}$ containing ω onto $\mathscr{C}_\omega$. For $i = 1, 2$ and $\sigma \in \mathscr{C}_n$ with $\omega \in \sigma$, we endow the local cone $\rho = \mathrm{LC}_\sigma(\omega)$ with the constant weight $m_{i,\rho}(\omega) := m_{i,\sigma}(\omega)$. Since the weight functions $m_{i,\sigma}$ pointwise satisfy the balancing condition, we get a tropical fan $(\mathscr{C}_\omega, m_i(\omega))$ with real weights.

We claim that $m_\tau(\omega)$ from (1.4.1) is the same as the weight of the stable intersection product $(\mathscr{C}_\omega, m_1(\omega)) \cdot (\mathscr{C}_\omega, m_2(\omega))$ in $\tau \in \mathscr{C}^{l_1+l_2}$ obtained from Chow cohomology as in 1.3. To see this, note that for a generic vector $v \in N_R$, we choose $\varepsilon > 0$ so small that the condition $\sigma_1 \cap (\sigma_2 + \varepsilon v) \neq \varnothing$ is equivalent to $\rho_1 \cap (\rho_2 + v) \neq \varnothing$ for the corresponding cones ρ_1, ρ_2. Then (1.4.1) agrees with the formula in [Fulton and Sturmfels 1997, Theorem on p. 336] for the product in Chow cohomology of proper toric varieties. By definition, the same formula is used for the stable intersection product of tropical fans with constant weights, proving our local claim. It is well known in tropical geometry, and follows from the comparison with Chow cohomology in [Fulton and Sturmfels 1997], that the definition of the stable tropical intersection product of tropical fans with real weights is independent of the choice of generic vector v, and hence the definition of $D = (\mathscr{D}, m)$ is independent of the choice of generic vector $v \in N_{\mathbb{R}}$ and sufficiently small $\varepsilon > 0$.

It is easily seen that the definition of D is compatible with subdivisions and hence $C_1 \cdot C_2$ is a well-defined tropical cycle. The properties in (i) follow from the corresponding properties of the stable tropical intersection product of tropical fans with real weights.

(iii) Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. Let $C' = (\mathscr{C}', m')$ be a weighted integral $\mathbb{R}$-affine polyhedral complex in $N'_{\mathbb{R}}$ of pure dimension n. After a suitable refinement we can assume that

$$F_*\mathscr{C}' := \{F(\tau') \mid \exists \sigma' \in \mathscr{C}'_n \text{ such that } \tau' \preccurlyeq \sigma' \text{ and } F|_{\sigma'} \text{ is injective}\} \tag{1.4.2}$$

is a polyhedral complex in $N_{\mathbb{R}}$. We equip $F_*\mathscr{C}'$ with the family of weight functions

$$m_\nu : \nu \to \mathbb{R}, \quad m_\nu(\omega) = \sum_{\substack{\sigma' \in \mathscr{C}'_n \\ F(\sigma') = \nu}} [N_\nu : \mathbb{L}_F(N'_{\sigma'})] m'_{\sigma'}((F|_{\sigma'})^{-1}(\omega)) \tag{1.4.3}$$

for ν in $(F_*\mathscr{C}')_n$, where $\mathbb{L}_F$ denotes the linear morphism defined by the affine morphism F. The weighted integral $\mathbb{R}$-affine polyhedral complex

$$F_*C' = (F_*\mathscr{C}', m)$$

in $N_{\mathbb{R}}$ of pure dimension n is called the *direct image of C' under F*.

(iv) Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. There is a natural push-forward morphism

$$F_* : \mathrm{TZ}_n(N'_{\mathbb{R}}) \to \mathrm{TZ}_n(N_{\mathbb{R}}), \quad C' \mapsto F_*C',$$

which satisfies $|F_*C'| \subseteq F\big(|C'|\big)$. Given a tropical cycle C' in $\mathrm{TZ}_n(N'_{\mathbb{R}})$, we write $C' = (\mathscr{C}', m')$ for a polyhedral complex of definition $\mathscr{C}'$ such that $F_*\mathscr{C}'$ from (1.4.2) is a polyhedral complex in $N_{\mathbb{R}}$. One defines the direct image $F_*C' = (F_*\mathscr{C}', m)$ as in (iii) and verifies that F_*C' is again a tropical cycle. The formation of F_* is functorial in F. For further details see [Allermann and Rau 2010, §7] or [Gubler 2013, 13.16].

(v) Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. There is a natural pull-back

$$F^* : \mathrm{TZ}^l(N_{\mathbb{R}}) \to \mathrm{TZ}^l(N'_{\mathbb{R}}), \quad C \mapsto F^*(C)$$

which satisfies $|F^*C| \subseteq F^{-1}(|C|)$. The formation of F^* is functorial in F. For a tropical cycle C in $\mathrm{TZ}^l(N_{\mathbb{R}})$, there is a complete polyhedral complex of definition $\mathscr{C}$ and we write $C = (\mathscr{C}, m)$. After passing to a subdivision, there is a complete, integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}'$ of $N'_{\mathbb{R}}$ such that for every $\gamma' \in \mathscr{C}'$, there is a $\sigma \in \mathscr{C}$ with $F(\gamma') \subseteq \sigma$. We choose a generic vector $v \in N_{\mathbb{R}}$ and a sufficiently small $\varepsilon > 0$. For $\gamma' \in (\mathscr{C}')^l$, $\sigma \in \mathscr{C}^l$ with $F(\gamma') \subseteq \sigma$ and $\sigma' \in (\mathscr{C}')^0$ with $\gamma' \subseteq \sigma'$, we define

$$m^{\gamma'}_{\sigma',\sigma} := \begin{cases} [N : \mathbb{L}_F(N') + N_\sigma] & \text{if } F(\sigma') \text{ meets } \sigma + \varepsilon v, \\ 0 & \text{otherwise.} \end{cases}$$

These coefficients may depend on the choice of the generic vector v, but the following smooth weight function $m_{\gamma'}$ on $\gamma' \in (\mathscr{C}')^l$ does not:

$$m_{\gamma'}(\omega') := \sum_{\sigma',\sigma} m^{\gamma'}_{\sigma',\sigma} m_\sigma(F(\omega')), \tag{1.4.4}$$

where (σ', σ) ranges over all pairs in $(\mathscr{C}')^0 \times \mathscr{C}^l$ with $\gamma' \subseteq \sigma'$, $F(\gamma') \subseteq \sigma$ and where $\omega' \in \gamma'$. By [Fulton and Sturmfels 1997, 4.5–4.7], $(\mathscr{C}')^{\geq l}$ equipped with the smooth weight functions $m_{\gamma'}$ is a tropical cycle in $\mathrm{TZ}^l(N'_{\mathbb{R}})$, which we define as $F^*(C)$.

Let p_1 (resp. p_2) be the projection of $N'_{\mathbb{R}} \times N_{\mathbb{R}}$ to $N'_{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$) and let Γ_F be the graph of F in $N'_{\mathbb{R}} \times N_{\mathbb{R}}$. Using the stable tropical intersection product from (ii) and [Fulton and Sturmfels 1997, 4.5–4.7], we deduce

$$F^*(C) = (p_1)_*(p_2^*(C) \cdot \Gamma_F). \tag{1.4.5}$$

Proposition 1.5. *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map.*

(i) *For tropical cycles C and D on $N_{\mathbb{R}}$ we have*

$$F^*(C \cdot D) = F^*(C) \cdot F^*(D).$$

(ii) *For tropical cycles C on $N_{\mathbb{R}}$ and C' on $N'_{\mathbb{R}}$ we have*

$$F_*\big(F^*(C) \cdot C'\big) = C \cdot F_*(C').$$

Proof. We reduce as in Remark 1.2 to the case where our tropical cycles are tropical fans with constant weight functions. Since both sides of the claims are linear in the weights of the tropical fans, we may assume that the weights are integers. In this situation, the claims were proven by L. Allermann [2012, Theorem 3.3]. ∎

Definition 1.6. Let Ω be an open subset of an integral $\mathbb{R}$-affine polyhedral set P in $N_{\mathbb{R}}$. We call $\phi : \Omega \to \mathbb{R}$ *piecewise smooth* if there is an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ with support P and smooth functions $\phi_\sigma : \sigma \cap \Omega \to \mathbb{R}$ for every $\sigma \in \mathscr{C}$ such that $\phi|_\sigma = \phi_\sigma$ on $\sigma \cap \Omega$. In this situation, we call $\mathscr{C}$ a *polyhedral complex of definition for the piecewise smooth function* ϕ. We call ϕ *piecewise linear* if each ϕ_σ extends to an integral $\mathbb{R}$-affine function on $\mathbb{A}_\sigma$.

Remark 1.7. The balancing condition (1.1.1) for smooth weights shows easily that a tropical cycle of codimension 0 in $N_{\mathbb{R}}$ is the same as a piecewise smooth function defined on the whole space $N_{\mathbb{R}}$.

Proposition 1.8. *Let ϕ be a piecewise smooth function on the open subset Ω of the integral $\mathbb{R}$-affine polyhedral set P in $N_{\mathbb{R}}$ and let $\widetilde{\Omega}$ be any open subset of $N_{\mathbb{R}}$ with $\widetilde{\Omega} \cap P = \Omega$. Then there is a piecewise smooth function on $\widetilde{\Omega}$ which restricts to ϕ on Ω.*

Proof. We first show the claim in the special case when $\Omega = P$ is the support of a rational polyhedral fan $\mathscr{C}$ of definition for ϕ and $\widetilde{\Omega} = N_{\mathbb{R}}$. After passing to a subdivision of $\mathscr{C}$, we can easily find a complete rational polyhedral fan $\mathscr{C}'$ in $N_{\mathbb{R}}$ which contains $\mathscr{C}$. After suitable subdivisions of $\mathscr{C}'$ (and $\mathscr{C}$) we may furthermore assume that all cones in $\mathscr{C}'$ are simplicial. Now we will extend ϕ inductively by ascending dimension from the cones in $\mathscr{C}$ to the cones in $\mathscr{C}'$.

Let σ be a cone in $\mathscr{C}'$ of dimension m. We are looking for an extension $\tilde{\phi}$ of ϕ to σ. By our inductive procedure, we can assume that ϕ is defined already on all faces of codimension one of σ. After a linear change of coordinates, we may assume that σ is the standard cone $\mathbb{R}_+^m$ in $\mathbb{R}^m$. Let us assume that ϕ is given on the face $\{x_i = 0\}$ of σ by the smooth function $\phi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. For any $1 \leq i_1 < \cdots < i_k \leq m$, the restriction of ϕ to the face $\{x_{i_1} = \cdots = x_{i_k} = 0\}$ of σ is given by a smooth function $\phi_{i_1 \cdots i_k}$ depending only on the coordinates x_j with $j \notin \{i_1, \ldots, i_k\}$ which agrees with the restrictions of the functions $\phi_{i_1}, \ldots, \phi_{i_k}$ to this face of codimension k. We consider all these functions $\phi_{i_1 \cdots i_k}$ as functions on σ depending only on the coordinates x_j. Then an elementary combinatorial argument shows that

$$\tilde{\phi} := \sum_i \phi_i - \sum_{i<j} \phi_{ij} + \cdots + (-1)^{k+1} \sum_{i_1 < \cdots < i_k} \phi_{i_1 \cdots i_k} \pm \cdots + (-1)^{n+1} \phi_{1 \cdots n}$$

is a smooth extension of ϕ to σ.

Finally, we prove the claim in general. There is a finite open covering $(\Omega_i)_{i \in I}$ of Ω such that $\Omega_i = \Omega_i \cap (\mathrm{LC}_{\omega_i}(P) + \omega_i)$ for the local cone $\mathrm{LC}_{\omega_i}(P)$ of P at a suitable ω_i. Let us choose an open covering $(\widetilde{\Omega}_i)_{i \in I}$ of $\widetilde{\Omega}$ such that $\widetilde{\Omega}_i \cap P = \Omega_i$. There is a partition of unity $(\rho_j)_{j \in J}$ on $\widetilde{\Omega}$ such that every ρ_j has compact support in $\widetilde{\Omega}_{i(j)}$ for a suitable $i(j) \in I$. We choose $\nu_j \in C^\infty(N_{\mathbb{R}})$ with compact support in $\widetilde{\Omega}_{i(j)}$ such that $\nu_j \equiv 1$ on $\mathrm{supp}(\rho_j)$. Then the special case above shows that the piecewise smooth function $\nu_j\phi$ on P has a piecewise smooth extension $\tilde{\phi}_j$ to $N_{\mathbb{R}}$. Even if J is infinite, we note that only finitely many rational fans of definition occur and the above construction gives piecewise smooth extensions $\tilde{\phi}_j$ with finitely many integral $\mathbb{R}$-affine polyhedral complexes of definition. By passing to a common refinement, we may assume that they are all equal to a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$. We conclude that $\tilde{\phi} = \sum_{j\in J} \tilde{\rho}_j\tilde{\phi}_j$ is a piecewise smooth extension of ϕ to $\widetilde{\Omega}$ with $\mathscr{D}$ as a polyhedral complex of definition. □

Remark 1.9. Let Σ be a rational polyhedral fan of $N_{\mathbb{R}}$ and let $\phi : |\Sigma| \to \mathbb{R}$ be a piecewise linear function with polyhedral complex of definition Σ. Then ϕ is the restriction of a piecewise linear function on $N_{\mathbb{R}}$ with a complete rational polyhedral fan of definition. The argument is a little different: By toric resolution of singularities, one can subdivide Σ until we get a subcomplex of a smooth rational polyhedral fan Σ' of $N_{\mathbb{R}}$ (see A.7 for the connection to toric varieties). We may assume that $\phi(0) = 0$. Let λ be a primitive lattice vector contained in an edge of Σ' with $\lambda \notin |\Sigma|$ and let $\phi(\lambda) \in \mathbb{Z}$. Then there is a unique piecewise linear function ϕ' on $N_{\mathbb{R}}$ with $\phi' = \phi$ on $|\Sigma|$ and $\phi'(\lambda) = \phi(\lambda)$ for all primitive lattice vectors λ as above.

Similarly to [Esterov 2012; François 2013], we introduce the corner locus of a piecewise smooth function.

Definition 1.10 (corner locus). Let $C = (\mathscr{C}, m)$ be a tropical cycle with smooth weights of dimension n. We consider a piecewise smooth function $\phi : |\mathscr{C}| \to \mathbb{R}$ with polyhedral complex of definition $\mathscr{C}$. Given $\tau \in \mathscr{C}_{n-1}$ we choose for each $\sigma \in \mathscr{C}_n$ with $\tau \prec \sigma$ an $\omega_{\sigma,\tau} \in N_\sigma$ as in Definition 1.1(ii). For ω in τ, we define

$$\omega_\tau := \sum_{\substack{\sigma\in\mathscr{C}_n \\ \tau\prec\sigma}} m_\sigma(\omega)\omega_{\sigma,\tau} \in \mathbb{L}_\tau.$$

Note that ω_τ depends on the choice of ω. Viewing $\omega_{\sigma,\tau}$ and ω_τ as tangential vectors at ω, we denote the corresponding derivatives by

$$\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} := \langle d\phi_\sigma, \omega_{\sigma,\tau}\rangle, \quad \text{and} \quad \frac{\partial\phi_\tau}{\partial\omega_\tau} := \langle d\phi_\tau, \omega_\tau\rangle,$$

respectively. It is straightforward to check that the definition of the weight function

$$m_\tau : \tau \to \mathbb{R}, \quad m_\tau(\omega) := \Bigg(\sum_{\substack{\sigma\in\mathscr{C}_n \\ \tau\prec\sigma}} m_\sigma(\omega)\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}}(\omega)\Bigg) - \frac{\partial\phi_\tau}{\partial\omega_\tau}(\omega) \tag{1.10.1}$$

does not depend on the choice of the $\omega_{\sigma,\tau}$. The *corner locus* $\phi\cdot C$ *of* ϕ is by definition the weighted polyhedral subcomplex $\mathscr{C}'$ of $\mathscr{C}$ generated by $\mathscr{C}_{n-1}$ endowed with the smooth weight functions m_τ defined in (1.10.1).

Remark 1.11. Let $\phi : |\mathscr{C}| \to \mathbb{R}$ be a piecewise linear function on a tropical cycle $C = (\mathscr{C}, m)$ with integral weights. Then the corner locus $\phi \cdot \mathscr{C}$ is a tropical cycle with integral weights which is the tropical Weil divisor of ϕ on C in the sense of Allermann and Rau [2010, 6.5].

Esterov [2012, Theorem 2.7] showed that the corner locus of a piecewise polynomial function on a tropical cycle with polynomial weights is again a tropical cycle of the same kind. We have here a similar result for tropical cycles with smooth weights:

Proposition 1.12. *The corner locus $\phi \cdot C$ of a piecewise smooth function $\phi : |\mathscr{C}| \to \mathbb{R}$ on a tropical cycle $C = (\mathscr{C}, m)$ of dimension n is a tropical cycle with smooth weights of dimension $n-1$. The corner locus is defined independently of the choice of the polyhedral complex $\mathscr{C}$ and $\phi \cdot C$ depends only on the function $\phi|_{|C|}$.*

Proof. This follows from Remark 1.11 as explained in Remark 1.2. □

Proposition 1.13. *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. Let ϕ be a piecewise smooth function on an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ on $N_{\mathbb{R}}$. Suppose that $C' = (\mathscr{C}', m')$ is a tropical cycle on $N'_{\mathbb{R}}$ with smooth weights m' such that $F(|\mathscr{C}'|) \subseteq |\mathscr{C}|$. Then we have the projection formula $F_*(F^*(\phi) \cdot C') = \phi \cdot F_*(C')$, where $F^*(\phi)$ is the piecewise smooth function on $|\mathscr{C}'|$ obtained by $\phi \circ F$.*

Proof. This follows locally as in [Allermann and Rau 2010, Proposition 4.8] using Remarks 1.2 and 1.11, and a linearization procedure (see proof of Proposition 1.14). □

Proposition 1.14. *Let C and C' be tropical cycles on $N_{\mathbb{R}}$ with smooth weights. Let $\mathscr{C}$ be a polyhedral complex of definition for C and $\phi : |\mathscr{C}| \to \mathbb{R}$ and $\psi : |\mathscr{C}| \to \mathbb{R}$ piecewise smooth functions. Then we have the associativity law*

$$\phi \cdot (C \cdot C') = (\phi \cdot C) \cdot C' \tag{1.14.1}$$

and the commutativity law

$$\phi \cdot (\psi \cdot C) = \psi \cdot (\phi \cdot C) \tag{1.14.2}$$

as identities of tropical cycles on $N_{\mathbb{R}}$.

Proof. Using Remark 1.11 it is shown in [Allermann and Rau 2010, Lemma 9.7, Proposition 6.7] that (1.14.1) and (1.14.2) hold for tropical cycles C, C' with integral weights and piecewise linear functions $\phi, \psi : |\mathscr{C}| \to \mathbb{R}$ with integral slopes. As both sides of (1.14.1) and (1.14.2) are linear in weights and slopes, both formulas extend by linearity to tropical cycles with constant weight functions and piecewise linear functions with arbitrary real slopes.

To reduce to the above situation, we use the procedure described in Remark 1.2. We may assume that C and C' are tropical cycles of pure dimension n and n'

respectively. Let $\mathscr{C}$ be an integral $\mathbb{R}$-affine polyhedral complex such that $\mathscr{C}_{\leq n}$ and $\mathscr{C}_{\leq n'}$ are polyhedral complexes of definition for C and C'. We write $C = (\mathscr{C}_{\leq n}, m)$, $C' = (\mathscr{C}'_{\leq n'}, m')$, and $C \cdot C' = (\mathscr{C}_{\leq l}, m'')$ with $l := n + n' - r$. Given $\omega \in |\mathscr{C}|$ we denote by $\mathscr{C}_\omega$ the rational polyhedral fan of local cones of $\mathscr{C}$ in ω. There is a bijective correspondence between the polyhedra $\sigma \in \mathscr{C}$ with $\omega \in \sigma$ and the cones σ_ω in $\mathscr{C}_\omega$. Each $\sigma \in \mathscr{C}$ with $\omega \in \sigma$ determines a canonical isomorphism of affine spaces $I_\omega : \mathbb{L}_{\sigma_\omega} \xrightarrow{\sim} \mathbb{A}_\sigma$ with $I_\omega(0) = \omega$. We obtain tropical fans with constant weight functions $C_\omega = (\mathscr{C}_{\omega,\leq n}, m(\omega))$, $C'_\omega = (\mathscr{C}_{\omega,\leq n'}, m'(\omega))$, and $(C \cdot C')_\omega = (\mathscr{C}_{\omega,\leq n}, m''(\omega))$. We have $C_\omega \cdot C'_\omega = (C \cdot C')_\omega$ by our construction of the stable tropical intersection product with smooth weights. There is a unique piecewise linear function $\phi_\omega : |\mathscr{C}_\omega| \to \mathbb{R}$ such that for all $\sigma_\omega \in \mathscr{C}_\omega$ the $\mathbb{R}$-linear function ϕ_{σ_ω} on $\mathbb{L}_{\sigma_\omega}$ determined by $\phi_\omega|_{\sigma_\omega} = \phi_{\sigma_\omega}|_{\sigma_\omega}$ satisfies

$$(d\phi_{\sigma_\omega})(0) = (I_\omega^* d\phi)(0)$$

in $\mathbb{L}^*_{\sigma_\omega}$. We write

$$\phi \cdot (C \cdot C') = (\mathscr{C}_{\leq l-1}, m_1), \quad \phi_\omega \cdot (C_\omega \cdot C'_\omega) = (\mathscr{C}_{\omega,\leq l-1}, m_{\omega,1}),$$
$$(\phi \cdot C) \cdot C' = (\mathscr{C}_{\leq l-1}, m_2), \quad (\phi_\omega \cdot C_\omega) \cdot C'_\omega = (\mathscr{C}_{\omega,\leq l-1}, m_{\omega,2}).$$

The local nature of our definitions yields

$$m_{i,\sigma}(\omega) = m_{\omega,i,\sigma_\omega}(0)$$

for $i = 1, 2$ and all $\sigma \in \mathscr{C}_{\leq n+n'-r-1}$ with $\omega \in \sigma$. Formula (1.14.1) for constant weight functions and piecewise linear functions with arbitrary real slopes gives $m_{\omega,1,\sigma_\omega}(0) = m_{\omega,2,\sigma_\omega}(0)$. Hence $m_1 = m_2$ and (1.14.1) is proven in general. The reduction of (1.14.2) to the case of constant weight functions and piecewise linear functions proceeds in exactly the same way. □

Corollary 1.15. *Let $F : N'_\mathbb{R} \to N_\mathbb{R}$ be an integral $\mathbb{R}$-affine map. We consider a tropical cycle $C = (\mathscr{C}, m)$ with smooth weights on $N_\mathbb{R}$ and a piecewise smooth function $\phi : |\mathscr{C}| \to \mathbb{R}$. We write $F^*C = (\mathscr{C}', m')$, where $F(|\mathscr{C}'|) \subseteq |\mathscr{C}|$. Then ϕ induces a piecewise smooth function $F^*(\phi) : |\mathscr{C}'| \to \mathbb{R}$ and we have*

$$F^*(\phi) \cdot F^*(C) = F^*(\phi \cdot C),$$

i.e., *the formation of the corner locus is compatible with pull-back.*

Proof. Using (1.4.5) giving pull-back as a stable intersection with the graph, the claim follows by applying Proposition 1.13 and (1.14.1) in Proposition 1.14. □

2. The algebra of delta-preforms

In this section we define polyhedral supercurrents on an open subset $\widetilde{\Omega}$ in $N_\mathbb{R}$ for some free $\mathbb{Z}$-module N of finite rank. The polyhedral supercurrents are special supercurrents in the sense of Lagerberg. We show that an analogue of Stokes' theorem holds for polyhedral supercurrents with respect to the polyhedral derivatives d'_{P} and d''_{P}. Then we introduce the algebra of δ-preforms on $\widetilde{\Omega}$ which is going to play a central role in this paper. These δ-preforms are special polyhedral supercurrents defined by tropical cycles and superforms. We show that δ-preforms admit products and pull-back morphisms, satisfy a projection formula and that the polyhedral derivative of a δ-preform coincides with its derivative in the sense of supercurrents.

Throughout this section N and N' denote free $\mathbb{Z}$-modules of finite rank r and r'. We write $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$. We refer to the Appendix for the notation from convex geometry.

2.1. Given an open subset $\widetilde{\Omega}$ in $N_{\mathbb{R}}$, we denote by $A^{p,q}(\widetilde{\Omega})$ the space of superforms of type (p,q) on $\widetilde{\Omega}$, by $A_c^{p,q}(\widetilde{\Omega})$ the space of superforms with compact support of type (p,q) on $\widetilde{\Omega}$, and by $D_{k,l}(\widetilde{\Omega}) = D^{r-k,r-l}(\widetilde{\Omega})$ the space of supercurrents of type (k,l) on $\widetilde{\Omega}$ in the sense of Lagerberg [2012] (see also [Chambert-Loir and Ducros 2012; Gubler 2016]). We have seen in the introduction that $A := \bigoplus_{p,q} A^{p,q}$ defines a sheaf of differential bigraded $\mathbb{R}$-algebras with respect to the differentials d' and d''. The bigraded sheaf $D := \bigoplus_{p,q} D^{p,q}$ contains A as a bigraded subsheaf and has canonical differentials d' and d'' extending those of A.

The sheaf $A^{p,q}$ comes with a natural operator $J^{p,q} : A^{p,q} \to A^{q,p}$ which extends to $J^{p,q} : D^{p,q} \to D^{q,p}$. The first one induces an involution $J := \bigoplus_{p,q} J^{p,q}$ on A which is determined by the fact that it is an endomorphism of sheaves of $A^{0,0}$-algebras and that $d' \circ J = J \circ d''$. The extension of J to supercurrents is determined by

$$\langle J(T), \alpha \rangle = (-1)^r \langle T, J(\alpha) \rangle$$

for $\alpha \in A^{r-p,r-q}(\widetilde{\Omega})$ and $T \in D^{p,q}(\widetilde{\Omega})$. Sections of $A^{p,p}$ (resp. $D^{p,p}$) which are invariant under the action of $(-1)^p J^{p,p}$ are called *symmetric superforms* (resp. *symmetric supercurrents*). Sections of $A^{p,p}$ (resp. $D^{p,p}$) which are invariant under the action of $(-1)^{p+1} J^{p,p}$ are called *antisymmetric superforms* (resp. *antisymmetric supercurrents*).

2.2. Let $\widetilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$. An integral $\mathbb{R}$-affine polyhedron Δ of dimension n in $N_{\mathbb{R}}$ determines a canonical calibration

$$\mu_\Delta \in \left|\bigwedge^n \mathbb{L}_\Delta\right| = \mathrm{Or}(\mathbb{A}_\Delta) \times^{\pm 1} \bigwedge^n \mathbb{L}_\Delta$$

as in [Chambert-Loir and Ducros 2012, (1.3.5)]. Given a superform $\alpha \in A_c^{n,n}(\widetilde{\Omega})$ the integral

$$\int_\Delta \alpha = \int_{N_{\mathbb{R}}} \langle \alpha, \mu_\Delta \rangle$$

was defined in [Chambert-Loir and Ducros 2012, §1.5] (see also [Gubler 2016, §3]). The polyhedron Δ determines a continuous functional

$$A_c^{n,n}(\widetilde{\Omega}) \to \mathbb{R}, \quad \alpha \mapsto \int_\Delta \alpha \tag{2.2.1}$$

and a symmetric supercurrent $\delta_\Delta \in D_{n,n}(\widetilde{\Omega})$.

For $\Omega := \widetilde{\Omega} \cap \Delta$, we define $A_\Delta^{p,q}(\Omega)$ as the space of superforms on the open subset $\widetilde{\Omega} \cap \operatorname{relint}(\Delta)$ of the affine space $\mathbb{A}_\Delta$ given by restriction of elements in $A^{p,q}(\widetilde{\Omega})$. A partition of unity argument shows that this definition is independent of the choice of $\widetilde{\Omega}$.

For a superform $\alpha \in A_\Delta^{p,q}(\widetilde{\Omega} \cap \Delta)$, the supercurrent

$$\alpha \wedge \delta_\Delta \in D_{n-p,n-q}(\widetilde{\Omega})$$

is defined by $\langle \alpha \wedge \delta_\Delta, \beta\rangle = \langle \delta_\Delta, \alpha \wedge \beta\rangle$ for all $\beta \in A_c^{n-p,n-q}(\widetilde{\Omega})$.

Definition 2.3 (polyhedral supercurrents). Let $\widetilde{\Omega}$ be an open subset of $N_\mathbb{R}$. A supercurrent $\alpha \in D(\widetilde{\Omega})$ is called *polyhedral* if there exists an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ and a family $(\alpha_\Delta)_{\Delta\in\mathscr{C}}$ of superforms $\alpha_\Delta \in A_\Delta(\widetilde{\Omega} \cap \Delta)$ such that

$$\alpha = \sum_{\Delta\in\mathscr{C}} \alpha_\Delta \wedge \delta_\Delta \tag{2.3.1}$$

holds in $D(\widetilde{\Omega})$. In this case we say that $\mathscr{C}$ is *a polyhedral complex of definition for* α. The *polyhedral derivatives* $d'_\mathrm{P}(\alpha)$ and $d''_\mathrm{P}(\alpha)$ of a polyhedral supercurrent (2.3.1) are the polyhedral supercurrents defined by the formulas

$$d'_\mathrm{P}(\alpha) = \sum_{\Delta\in\mathscr{C}} d'(\alpha_\Delta) \wedge \delta_\Delta, \quad d''_\mathrm{P}(\alpha) = \sum_{\Delta\in\mathscr{C}} d''(\alpha_\Delta) \wedge \delta_\Delta.$$

Remark 2.4. (i) We observe that the family of forms $(\alpha_\Delta)_{\Delta\in\mathscr{C}}$ in (2.3.1) is uniquely determined by α and $\mathscr{C}$. Furthermore the support $\operatorname{supp}(\alpha)$ of a polyhedral supercurrent α is the union of the supports of the forms α_Δ for all $\Delta \in \mathscr{C}$.

(ii) It is straightforward to check that the definitions of the polyhedral derivatives $d'_\mathrm{P}(\alpha)$ and $d''_\mathrm{P}(\alpha)$ do not depend on the choice of the polyhedral complex of definition $\mathscr{C}$.

(iii) We do not claim that the polyhedral derivatives of a polyhedral supercurrent α coincide with derivative of a α in the sense of supercurrents. In fact the derivatives of a polyhedral supercurrent in the sense of supercurrents are in general not even polyhedral.

Definition 2.5. Let $\widetilde{\Omega}$ denote an open subset of $N_\mathbb{R}$. Let $P \subseteq \widetilde{\Omega}$ be an integral $\mathbb{R}$-affine polyhedral set in $N_\mathbb{R}$. We choose an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ whose support is P.

(i) Let $\alpha \in D_{0,0}(\widetilde{\Omega})$ be a polyhedral supercurrent such that $\mathrm{supp}(\alpha) \cap P$ is compact. After suitable refinements, we may assume that α admits a polyhedral complex of definition $\mathscr{D}$ such that $\mathscr{D}$ is a subcomplex of $\mathscr{C}$. In this situation we write

$$\alpha = \sum_{\Delta \in \mathscr{D}} \alpha_\Delta \wedge \delta_\Delta \tag{2.5.1}$$

as in (2.3.1) and define the integral of α over P as

$$\int_P \alpha = \sum_{\Delta \in \mathscr{D}} \int_\Delta \alpha_\Delta. \tag{2.5.2}$$

(ii) Let $\beta \in D_{1,0}(\widetilde{\Omega})$ be a polyhedral supercurrent with $\mathrm{supp}(\beta) \cap P$ compact. Proceeding as in (i), we get $\beta = \sum_{\Delta \in \mathscr{D}} \beta_\Delta \wedge \delta_\Delta$ for a suitable subcomplex $\mathscr{D}$ of $\mathscr{C}$ and we define the integral of β over the boundary of P as

$$\int_{\partial P} \beta = \sum_{\Delta \in \mathscr{D}} \int_{\partial \Delta} \beta_\Delta, \tag{2.5.3}$$

where the boundary integrals on the right are defined as in [Chambert-Loir and Ducros 2012, §1.5; Gubler 2016, 2.6]. We define the boundary integral (2.5.3) for a polyhedral supercurrent $\beta \in D_{0,1}(\widetilde{\Omega})$ with $\mathrm{supp}(\beta) \cap P$ compact by the same formula.

Remark 2.6. (i) The definitions in (2.5.2) and (2.5.3) do not depend on the choice of the polyhedral complex $\mathscr{D}$.

(ii) On the Borel algebra $\mathbb{B}(P)$, we get signed measures

$$\mu_{P,\alpha} : \mathbb{B}(P) \to \mathbb{R}, \quad \mu_{P,\alpha}(M) = \sum_{\Delta \in \mathscr{D}} \int_{M \cap \Delta} \alpha_\Delta$$

and

$$\mu_{\partial P,\beta} : \mathbb{B}(P) \to \mathbb{R}, \quad \mu_{\partial P,\beta}(M) = \sum_{\Delta \in \mathscr{D}} \int_{M \cap \partial \Delta} \beta_\Delta.$$

(iii) We recall from A.5 that $\mathrm{relint}(P)$ denotes the set of regular points of a polyhedral set P. Then $\mathrm{supp}(\beta) \cap P \subseteq \mathrm{relint}(P)$ implies

$$\int_{\partial P} \beta = 0 \tag{2.6.1}$$

as an immediate consequence of the definitions.

Proposition 2.7 (Stokes' formula for polyhedral supercurrents). *Let $\widetilde{\Omega}$ denote an open subset and P an integral $\mathbb{R}$-affine polyhedral subset in $N_{\mathbb{R}}$ with $P \subseteq \widetilde{\Omega}$. Then we have*

$$\int_P d'_{\mathrm{P}} \alpha = \int_{\partial P} \alpha, \quad \int_P d''_{\mathrm{P}} \beta = \int_{\partial P} \beta$$

for all polyhedral supercurrents $\alpha \in D_{1,0}(\widetilde{\Omega})$ *and* $\beta \in D_{0,1}(\widetilde{\Omega})$ *with* $\operatorname{supp}(\alpha) \cap P$ *and* $\operatorname{supp}(\beta) \cap P$ *compact.*

Proof. We choose a polyhedral complex of definition $\mathscr{C}$ for α such that a subcomplex $\mathscr{D}$ has support P. By linearity it is sufficient to treat the case $\alpha = \alpha_\Delta \wedge \delta_\Delta$ for a superform $\alpha_\Delta \in A_c^{n-1,n}(\widetilde{\Omega} \cap \Delta)$ and $\Delta \in \mathscr{D}_n$. We get

$$\int_P d'_{\mathrm{P}}(\alpha) = \int_P d'(\alpha_\Delta) \wedge \delta_\Delta = \int_\Delta d'(\alpha_\Delta) = \int_{\partial \Delta} \alpha_\Delta = \int_{\partial P} \alpha,$$

using Stokes' formula for superforms on polyhedra (see [Chambert-Loir and Ducros 2012, (1.5.7)] or [Gubler 2016, 2.9]). The formula for β follows in the same way. □

Remark 2.8. Let $\widetilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$. An integral $\mathbb{R}$-affine polyhedral complex $C = (\mathscr{C}, m)$ with smooth weights of pure dimension n and a superform $\alpha \in A^{p,q}(\widetilde{\Omega})$ determine a polyhedral supercurrent

$$\alpha \wedge \delta_C = \sum_{\Delta \in \mathscr{C}_n} (m_\Delta \cdot \alpha|_\Delta) \wedge \delta_\Delta \in D_{n-p,n-q}(\widetilde{\Omega}).$$

In particular we get the polyhedral supercurrents $[\alpha] = \alpha \wedge \delta_{N_{\mathbb{R}}} \in D_{r-p,r-q}(\widetilde{\Omega})$ and $\delta_C = 1 \wedge \delta_C \in D_{n,n}(\widetilde{\Omega})$.

Definition 2.9 (δ-preforms). (i) Let $\widetilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$. A supercurrent $\alpha \in D_{r-p,r-q}(\widetilde{\Omega})$ is called a *δ-preform of type* (p, q) if there exist a finite set I, a family $(C_i)_{i \in I}$ of tropical cycles with smooth weights $C_i = (\mathscr{C}_i, m_i)$ of codimension n_i in $N_{\mathbb{R}}$, and a family $(\alpha_i)_{i \in I}$ of superforms $\alpha_i \in A^{p_i,q_i}(\widetilde{\Omega})$ such that $p_i + n_i = p$ and $q_i + n_i = q$ for all $i \in I$ and

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i} \tag{2.9.1}$$

holds in $D_{r-p,r-q}(\widetilde{\Omega})$. The *support* of a δ-preform is the support of its underlying supercurrent.

(ii) The δ-preforms define a subspace $P^{p,q}(\widetilde{\Omega})$ in $D_{r-p,r-q}(\widetilde{\Omega})$. We put

$$P^n(\widetilde{\Omega}) = \bigoplus_{p+q=n} P^{p,q}(\widetilde{\Omega})$$

and $P(\widetilde{\Omega}) = \bigoplus_{n \in \mathbb{N}} P^n(\widetilde{\Omega})$. We denote by $P_c(\widetilde{\Omega})$ the subspace of $P(\widetilde{\Omega})$ given by the δ-preforms with compact support. A δ-preform $\alpha \in P^{p,p}(\widetilde{\Omega})$ of type (p, p) is called *symmetric* (resp. *antisymmetric*), if the underlying supercurrent of α is symmetric (resp. antisymmetric).

(iii) We say that a δ-preform α has *codimension* l, if it admits a presentation (2.9.1) where all the tropical cycles $\mathscr{C}_i$ are of pure codimension l. The δ-preforms of type $(p+l, q+l)$ of codimension l define a subspace of $D^{p+l,q+l}(\widetilde{\Omega})$ which we denote by $P^{p,q,l}(\widetilde{\Omega})$. As an immediate consequence of our definitions, we have the direct sum

$$P^n(\widetilde{\Omega}) = \bigoplus_{p+q+2l=n} P^{p,q,l}(\widetilde{\Omega}).$$

Example 2.10. It follows from Remark 1.7 that a δ-preform of codimension 0 on $\widetilde{\Omega}$ is the same as a superform on $\widetilde{\Omega}$ with piecewise smooth coefficients.

Remark 2.11. Let

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i} \in P^{p,q,l}(\widetilde{\Omega})$$

be a δ-preform as in (2.9.1). Let $\mathscr{C}$ be a common polyhedral complex of definition for the tropical cycles $(C_i)_{i\in I}$. Then the supercurrent α is polyhedral and $\mathscr{C}$ is a polyhedral complex of definition for α. In fact we have $C_i = (\mathscr{C}, m_i)$ for suitable families of weight functions $m_{i,\Delta}$ on polyhedra Δ in $\mathscr{C}_{r-l}$ and define

$$\alpha_\Delta := \sum_{i\in I} m_{i,\Delta} \cdot (\alpha_i|_\Delta) \in A_\Delta^{p,q}(\widetilde{\Omega} \cap |\Delta|).$$

Then we get

$$\delta_{C_i} = \sum_{\Delta \in \mathscr{C}_{r-l}} m_{i,\Delta} \wedge \delta_\Delta$$

and

$$\alpha = \sum_{\Delta \in \mathscr{C}_{r-l}} \alpha_\Delta \wedge \delta_\Delta.$$

In order to compare δ-preforms in $P^{p,q,l}(\widetilde{\Omega})$, presented as in (2.9.1),

$$\alpha = \sum_{i\in I} \alpha_i \wedge \delta_{C_i}, \quad \beta = \sum_{j \in J} \beta_j \wedge \delta_{D_j},$$

we choose a common polyhedral complex of definition $\mathscr{C}$ for the finite families $(C_i)_{i\in I}$ and $(D_j)_{j\in J}$ of tropical cycles and obtain

$$\alpha = \beta \iff \alpha_\Delta = \beta_\Delta \quad \text{for all } \Delta \in \mathscr{C}_{r-l}. \tag{2.11.1}$$

Proposition 2.12. *Let $\widetilde{\Omega}$ denote an open subset of $N_\mathbb{R}$. Presenting δ-preforms as in* (2.9.1), *we can perform the following constructions*:

(i) *We have a canonical $C^\infty(\widetilde{\Omega})$-linear map*

$$A^{p,q}(\widetilde{\Omega}) \to P^{p,q,0}(\widetilde{\Omega}), \quad \alpha \mapsto \alpha \wedge \delta_{N_\mathbb{R}}$$

and a $C^\infty(N_\mathbb{R})$-linear isomorphism

$$\mathrm{TZ}^l(N_\mathbb{R}) \xrightarrow{\sim} P^{0,0,l}(N_\mathbb{R}), \quad C \mapsto 1 \wedge \delta_C.$$

(ii) *There are well-defined $C^\infty(\widetilde{\Omega})$-bilinear products*

$$\wedge : P^{p,q,l}(\widetilde{\Omega}) \otimes_{\mathbb{R}} P^{p',q',l'}(\widetilde{\Omega}) \to P^{p+p',q+q',l+l'}(\widetilde{\Omega}),$$
$$\left(\sum_{i\in I} \alpha_i \wedge \delta_{C_i}\right) \wedge \left(\sum_{j\in J} \beta_j \wedge \delta_{D_j}\right) = \sum_{(i,j)\in I\times J} (\alpha_i \wedge \beta_j) \wedge \delta_{C_i \cdot D_j}.$$

(iii) *An integral $\mathbb{R}$-affine map $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ induces a natural pull-back*

$$F^* : P^{p,q,k}(\widetilde{\Omega}) \to P^{p,q,k}(\widetilde{\Omega}'), \quad \sum_{i\in I} \alpha_i \wedge \delta_{C_i} \mapsto \sum_{i\in I} (F^*\alpha_i) \wedge \delta_{F^*C_i}$$

for any open subset $\widetilde{\Omega}'$ of $F^{-1}(\widetilde{\Omega})$.

(iv) *The pull-back morphism F^* in (iii) satisfies*

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$$

for all $\alpha, \beta \in P(\widetilde{\Omega})$.

Proof. The proof of (i) is straightforward. For (ii), we have to show that the definition

$$\alpha \wedge \beta := \sum_{(i,j)\in I\times J} (\alpha_i \wedge \beta_j) \wedge \delta_{C_i \cdot D_j} \tag{2.12.1}$$

is independent of the presentations

$$\alpha = \sum_{i\in I} \alpha_i \wedge \delta_{C_i} \in P^{p,q,l}(\widetilde{\Omega}), \quad \beta = \sum_{j\in J} \beta_j \wedge \delta_{D_j} \in P^{p',q',l'}(\widetilde{\Omega})$$

given as in (2.9.1). We choose a common polyhedral complex of definition $\mathscr{C}$ for all tropical cycles C_i and D_j. Using Remark 2.11, we represent the δ-preforms as polyhedral supercurrents

$$\alpha = \sum_{\sigma\in\mathscr{C}^l} \alpha_\sigma \wedge \delta_\sigma, \quad \beta = \sum_{\sigma'\in\mathscr{C}^{l'}} \beta_{\sigma'} \wedge \delta_{\sigma'}. \tag{2.12.2}$$

We choose a generic vector v and $\varepsilon > 0$ as in 1.4(ii). From (2.12.1) and (1.4.1), we deduce

$$\alpha \wedge \beta = \sum_{\tau\in\mathscr{C}^{l+l'}} \sum_{\sigma,\sigma'} [N : N_\sigma + N_{\sigma'}] \cdot \alpha_\sigma \wedge \beta_{\sigma'} \wedge \delta_\tau, \tag{2.12.3}$$

where σ, σ' ranges over all pairs in $\mathscr{C}^l \times \mathscr{C}^{l'}$ with $\sigma \cap \sigma' = \tau$ and $\sigma \cap (\sigma' + \varepsilon v) \neq \varnothing$. Then (ii) follows from (2.12.3) and from the uniqueness of the representations in (2.12.2). Bilinearity is obvious.

Similarly we show (iii). Given a δ-preform α as above, we have to prove that

$$F^*(\alpha) := \sum_{i\in I} (F^*\alpha_i) \wedge \delta_{F^*C_i} \tag{2.12.4}$$

is independent of the representation of α in (2.12.2). There is a complete, integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}'$ of $N'_{\mathbb{R}}$ and a complete, common polyhedral complex of definition $\mathscr{C}$ for all tropical cycles C_i satisfying the following compatibility property: for every $\sigma' \in \mathscr{C}'$, there is a $\sigma \in \mathscr{C}$ with $F(\sigma') \subseteq \sigma$. Using the coefficients $m^{\gamma'}_{\sigma',\sigma}$ from Remark 1.4(v), we deduce from (2.12.4) and (1.4.4) that

$$F^*(\alpha) = \sum_{\gamma' \in (\mathscr{C}')^l} \sum_{\sigma',\sigma} m^{\gamma'}_{\sigma',\sigma} \cdot F^*\alpha_\sigma \wedge \delta_{\gamma'}, \tag{2.12.5}$$

where σ', σ ranges over all pairs in $(\mathscr{C}')^0 \times \mathscr{C}^l$ with $\gamma' \subseteq \sigma'$, $F(\gamma') \subseteq \sigma$. Then (iii) follows from (2.12.5) and uniqueness of the representation (2.12.2).

Note that (iv) is a direct consequence of our definitions. $\square$

Remark 2.13. Let P be an integral $\mathbb{R}$-affine polyhedral subset in $N_{\mathbb{R}}$ of dimension n. Let $C = (\mathscr{C}, m)$ be a tropical cycle with $|C| = P$ or $|\mathscr{C}| = P$ and $\alpha \in P^{\cdot}_c(N_{\mathbb{R}})$. Observe that $\int_P \alpha$ is in general different from $\int_{N_{\mathbb{R}}} \alpha \wedge \delta_C$ as the latter integral takes the multiplicities of C into account.

Proposition 2.14 (projection formula). *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map and C' a tropical cycle of dimension n on $N'_{\mathbb{R}}$. Let P be an integral $\mathbb{R}$-affine polyhedral subset and $\widetilde{\Omega}$ an open subset of $N_{\mathbb{R}}$ with $P \subseteq \widetilde{\Omega}$. Let $\alpha \in P(\widetilde{\Omega})$ be a δ-preform such that* $\operatorname{supp}(F^*(\alpha) \wedge \delta_{C'}) \cap F^{-1}(P)$ *is compact. Then* $\operatorname{supp}(\alpha \wedge \delta_{F_*(C')}) \cap P$ *is compact. If $\alpha \in P^{n,n}(\widetilde{\Omega})$, then*

$$\int_P \alpha \wedge \delta_{F_*(C')} = \int_{F^{-1}(P)} F^*(\alpha) \wedge \delta_{C'}. \tag{2.14.1}$$

If $\alpha \in P^{n-1,n}(\widetilde{\Omega})$, then

$$\int_{\partial P} \alpha \wedge \delta_{F_*(C')} = \int_{\partial F^{-1}(P)} F^*(\alpha) \wedge \delta_{C'}. \tag{2.14.2}$$

Proof. We consider first the case where $\alpha \in P^{n,n}(\widetilde{\Omega})$. We may assume without loss of generality that $\alpha \in P^{p,p,l}(\widetilde{\Omega})$, where $n = p + l$. We write

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i}$$

for suitable $\alpha_i \in A^{p,p}(\widetilde{\Omega})$ and $C_i \in \mathrm{TZ}^l(N_{\mathbb{R}})$ as in (2.9.1). We get

$$\alpha \wedge \delta_{F_*(C')} = \sum_{i \in I} \alpha_i \wedge \delta_{F_*(F^*C_i \cdot C')}, \qquad F^*(\alpha) \wedge \delta_{C'} = \sum_{i \in I} F^*(\alpha_i) \wedge \delta_{F^*C_i \cdot C'}$$

by the projection formula, Proposition 1.5(ii). We choose common polyhedral complexes of definition $\mathscr{C}'$ in $N'_{\mathbb{R}}$ for C' and F^*C_i for all $i \in I$ and $\mathscr{C}$ in $N_{\mathbb{R}}$ for F_*C' and C_i for all $i \in I$. We may assume that $F_*(\mathscr{C}')$ is a subcomplex of $\mathscr{C}$. After further refinements we can find polyhedral subcomplexes $\mathscr{D}'$ of $\mathscr{C}'$ with support $F^{-1}(P)$ and $\mathscr{D}$ of $\mathscr{C}$ with support P. Then $F_*\mathscr{D}'$ is a subcomplex of $\mathscr{D}$ and we write

$$\sum_{i\in I} \alpha_i \wedge \delta_{F_*(F^*C_i\cdot C')} = \sum_{\sigma\in\mathscr{C}_p} \alpha_\sigma \wedge \delta_\sigma, \tag{2.14.3}$$

$$\sum_{i\in I} F^*(\alpha_i) \wedge \delta_{F^*C_i\cdot C'} = \sum_{\sigma'\in\mathscr{C}'_p} \alpha_{\sigma'} \wedge \delta_{\sigma'}. \tag{2.14.4}$$

Consider $\sigma\in\mathscr{C}_p$. Given $\sigma'\in\mathscr{C}'_p$ with $F(\sigma')=\sigma$ there is a unique form $\tilde{\alpha}_{\sigma'}\in A_\sigma(\sigma)$ such that $F^*(\tilde{\alpha}_{\sigma'})=\alpha_{\sigma'}$ in $A_{\sigma'}(\sigma')$. From (1.4.3), (2.14.3) and (2.14.4) we get

$$\alpha_\sigma = \sum_{\substack{\sigma'\in\mathscr{C}'_p\\ F(\sigma')=\sigma}} [N_\sigma : \mathbb{L}_F(N'_{\sigma'})]\cdot\tilde{\alpha}_{\sigma'} \tag{2.14.5}$$

and $\alpha_{\sigma'}=0$ for all $\sigma'\in\mathscr{C}'_p$ with $\dim F(\sigma')<p$. If $\sigma\in\mathscr{D}_p$, which means $\sigma\subseteq P$, then only the $\sigma'\in\mathscr{D}'_p$ contribute to the sum in (2.14.5). Since $\sigma'\in\mathscr{D}'_p$ is equivalent to $\sigma'\subseteq F^{-1}(P)$, we deduce from (2.14.5) and compactness of $\operatorname{supp}(F^*(\alpha)\wedge\delta_{C'})\cap F^{-1}(P)$ that $\operatorname{supp}(\alpha\wedge\delta_{F_*(C')})\cap P$ is compact. The above formulas show that

$$\int_P \alpha\wedge\delta_{F_*(C')} = \sum_{\sigma\in\mathscr{D}_p}\int_\sigma \alpha_\sigma = \sum_{\sigma\in\mathscr{D}_p}\sum_{\substack{\sigma'\in\mathscr{C}'_p\\ F(\sigma')=\sigma}} [N_\sigma : \mathbb{L}_F(N'_{\sigma'})]\int_\sigma \tilde{\alpha}_{\sigma'}$$

and hence the transformation formula of integration theory (see [Chambert-Loir and Ducros 2012, (1.5.8); Gubler 2016, Proposition 3.10]) gives

$$\int_P \alpha\wedge\delta_{F_*(C')} = \sum_{\sigma\in\mathscr{D}_p}\sum_{\substack{\sigma'\in\mathscr{C}'_p\\ F(\sigma')=\sigma}}\int_{\sigma'}\alpha_{\sigma'} = \sum_{\sigma'\in\mathscr{D}'_p}\int_{\sigma'}\alpha_{\sigma'} = \int_{F^{-1}(P)} F^*(\alpha)\wedge\delta_{C'}.$$

This proves (2.14.1). Formula (2.14.2) is proved in exactly the same way using the transformation formula for boundary integrals in [Chambert-Loir and Ducros 2012, (1.5.8)]. □

2.15. Given a tropical cycle $C=(\mathscr{C},m)$ with constant weight functions, it follows from Stokes' theorem that the supercurrent δ_C is closed under d' and d'' [Gubler 2016, Proposition 3.8]. The following proposition shows that this is no longer true for tropical cycles with smooth weights.

Proposition 2.16. *Let $C=(\mathscr{C},m)$ be a tropical cycle with smooth weights of pure dimension n in $N_\mathbb{R}$. Then we have*

$$d'\delta_C = d'm\wedge\delta_{\mathscr{C}},\quad d''\delta_C = d''m\wedge\delta_{\mathscr{C}}$$

in $D.(N_{\mathbb{R}})$, *where the polyhedral supercurrent* $d'm \wedge \delta_{\mathscr{C}}$ *is defined by*

$$\langle d'm \wedge \delta_{\mathscr{C}}, \alpha\rangle = \sum_{\sigma \in \mathscr{C}_n} \int_\sigma d'm_\sigma \wedge \alpha$$

and the supercurrent $d''m \wedge \delta_{\mathscr{C}}$ *is defined analogously.*

Proof. This is a direct consequence of Stokes' formula for superforms on polyhedra [Chambert-Loir and Ducros 2012, lemme (1.5.7)], [Gubler 2016, 2.9] and the balancing condition (1.1.1). □

Remark 2.17. It follows from Proposition 2.16 that the subspace $P^{\cdot}(N_{\mathbb{R}})$ of $D^{\cdot}(N_{\mathbb{R}})$ of δ-preforms is not closed under the differentials d' and d'' in the sense of supercurrents. We will address this problem again in 4.6.

Proposition 2.18. *Let* $\widetilde{\Omega}$ *denote an open subset of* $N_{\mathbb{R}}$. *Then we have*

$$d'(\beta) = d'_{\mathrm{P}}(\beta), \quad d''(\beta) = d''_{\mathrm{P}}(\beta)$$

for all δ*-preforms* $\beta \in P(\widetilde{\Omega})$.

Proof. It is sufficient to treat the case $\beta = \alpha \wedge \delta_C$ for a superform $\alpha \in A^{p,q}(\widetilde{\Omega})$ and a tropical cycle $C = (\mathscr{C}, m)$ of pure dimension n on $N_{\mathbb{R}}$. We have

$$\beta = \sum_{\sigma \in \mathscr{C}_n} (m_\sigma \cdot \alpha|_\sigma) \wedge \delta_\sigma .$$

From Proposition 2.16 we get

$$d'\beta = d'\alpha \wedge \delta_C + (-1)^{p+q} \alpha \wedge d'm \wedge \delta_{\mathscr{C}} = \sum_{\sigma \in \mathscr{C}_n} \big(m_\sigma \cdot d'\alpha|_\sigma + d'm_\sigma \wedge \alpha|_\sigma\big) \wedge \delta_\sigma .$$

Then Leibniz's rule shows

$$d'\beta = \sum_{\sigma \in \mathscr{C}_n} d'(m_\sigma \cdot \alpha|_\sigma) \wedge \delta_\sigma = d'_{\mathrm{P}}(\beta)$$

which proves the first equality. The second claim is proved similarly. □

3. Supercurrents and delta-preforms on tropical cycles

In this section, we introduce supercurrents and δ-preforms on a given tropical cycle $C = (\mathscr{C}, m)$ of pure dimension n with *constant weight functions*. Similarly to complex manifolds, such tropical cycles have no boundary as $d'\delta_C = d''\delta_C = 0$. In the applications, C will be the tropical variety of a closed subvariety of a multiplicative torus. We build upon the results in Section 2. We will obtain the formulas of Stokes and Green. The main result is the tropical Poincaré–Lelong equation which will be used in Section 9 for the first Chern δ-current of a metrized line bundle.

3.1. The space $A^{p,q}_{\mathscr{C}}(\Omega)$ of (p,q)-superforms on an open subset Ω in $|\mathscr{C}|$ is defined as follows. We choose an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ such that $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. Elements in $A^{p,q}_{\mathscr{C}}(\Omega)$ are represented by elements in $A^{p,q}(\widetilde{\Omega})$ where two such elements are identified if they induce the same element in $A^{p,q}_{\Delta}(\Omega \cap \Delta)$ (see 2.2) for all maximal polyhedra Δ in $\mathscr{C}$. A partition of unity argument shows that this definition is independent of the choice of $\tilde{\Omega}$. Observe furthermore that $A^{p,q}_{\mathscr{C}}(\Omega)$ depends only on the support $|\mathscr{C}|$ of $\mathscr{C}$. We will often omit the polyhedral complex $\mathscr{C}$ from our notation and write simply $A^{p,q}(\Omega)$ instead of $A^{p,q}_{\mathscr{C}}(\Omega)$ when $\mathscr{C}$ or at least $|\mathscr{C}|$ is clear from the context. The spaces $A^{p,q}(\Omega)$ define a sheaf on $|\mathscr{C}|$. Hence the support of a superform in $A^{p,q}(\Omega)$ is defined as a closed subset of Ω. We denote by $A^{p,q}_{c}(\Omega)$ the space of superforms on Ω with compact support.

Definition 3.2. We define the *space of supercurrents* $D^{\mathscr{C}}_{p,q}(\Omega)$ of type (p,q) on an open subset Ω in $|\mathscr{C}|$ as follows. An element in $D^{\mathscr{C}}_{p,q}(\Omega)$ is given by a linear form $T \in \operatorname{Hom}_{\mathbb{R}}(A^{p,q}_{c}(\Omega), \mathbb{R})$ such that we can find an open set $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ and a supercurrent $T' \in D_{p,q}(\widetilde{\Omega})$ such that $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$ and $T(\eta|_{\Omega}) = T'(\eta)$ for all $\eta \in A^{p,q}_{c}(\widetilde{\Omega})$. As in 3.1 we often omit $\mathscr{C}$ from the notation and write $D_{p,q}(\Omega)$ instead of $D^{\mathscr{C}}_{p,q}(\Omega)$. We also use the grading $D^{p,q}(\Omega) := D_{n-p,n-q}(\Omega)$.

Remark 3.3. In the situation of Definition 3.2 we fix an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. It follows from a partition of unity argument that in the definition of $D_{p,q}(\Omega)$ we may use this $\widetilde{\Omega}$. We may identify $D_{p,q}(\Omega)$ with a subspace of $D_{p,q}(\tilde{\Omega})$ using the canonical map $T \mapsto T'$. Indeed, this map is well defined and injective since $T(\eta|_{\Omega}) = T'(\eta)$ holds for all $\eta \in A^{p,q}_{c}(\widetilde{\Omega})$. Furthermore the differentials d' and d'' on $D(\widetilde{\Omega})$ induce well-defined differentials d' and d'' on $D(\Omega)$.

A polyhedral supercurrent α' on $\widetilde{\Omega}$ is in $D(\Omega)$ if and only if $\operatorname{supp}(\alpha')$ is contained in Ω. The corresponding element α in $D(\Omega)$ is called a *polyhedral supercurrent on* Ω. Using Definition 2.3, the polyhedral derivatives $d'_{\mathrm{P}}\alpha$ and $d''_{\mathrm{P}}\alpha$ are again polyhedral supercurrents on Ω. Definition 2.5 yields integrals $\int_P \alpha = \int_P \alpha'$ and boundary integrals $\int_{\partial P} \alpha = \int_{\partial P} \alpha'$ of polyhedral supercurrents α in $D_0(\Omega)$ and $D_1(\Omega)$, respectively, over an integral $\mathbb{R}$-affine polyhedral subset P of Ω, provided that $\operatorname{supp}(\alpha) \cap P$ is compact.

Definition 3.4. Let Ω be an open subset of $|\mathscr{C}|$ and consider an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. For any δ-preform $\tilde{\alpha} \in P(\widetilde{\Omega})$ on $\widetilde{\Omega}$, the supercurrent $\tilde{\alpha} \wedge \delta_C$ on $\widetilde{\Omega}$ lies in the subspace $D(\Omega)$ of $D(\widetilde{\Omega})$. We will denote the corresponding element in $D(\Omega)$ by $\tilde{\alpha}|_{\Omega}$. A supercurrent $\alpha \in D(\Omega)$ is called a *δ-preform on* Ω if there is an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$ and a $\tilde{\alpha} \in P(\widetilde{\Omega})$ with $\alpha = \tilde{\alpha} \wedge \delta_C$. The space of δ-preforms on Ω is denoted by $P(\Omega)$ and the subspace of compactly supported δ-preforms is denoted by $P_c(\Omega)$. Note that these spaces depend also on the weights m of the tropical cycle $C = (\mathscr{C}, m)$ and not only on the open subset Ω of $|\mathscr{C}|$.

Remark 3.5. (i) A partition of unity argument again shows that $P(\Omega)$ is the image of the natural morphism

$$P(\widetilde{\Omega}) \to D(\Omega), \quad \tilde{\alpha} \mapsto \tilde{\alpha}|_\Omega := \tilde{\alpha} \wedge \delta_C$$

for any open subset $\widetilde{\Omega}$ of $N_\mathbb{R}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. We give $P(\Omega)$ the unique structure as a bigraded algebra such that the surjective map $P(\widetilde{\Omega}) \to P(\Omega)$ is a homomorphism of bigraded algebras. Similarly, we define the grading by codimension on $P(\Omega)$. For δ-preforms $\alpha = \tilde{\alpha} \wedge \delta_C$ and $\alpha' = \tilde{\alpha}' \wedge \delta_C$ on Ω, their product is given by the formula

$$\alpha \wedge \alpha' = \tilde{\alpha} \wedge \tilde{\alpha}' \wedge \delta_C.$$

(ii) By Remarks 2.11 and 3.3, $\alpha = \tilde{\alpha} \wedge \delta_C \in P(\Omega)$ is a polyhedral supercurrent on Ω. After possibly passing to a subdivision of $\mathscr{C}$, we have

$$\alpha = \sum_{\Delta \in \mathscr{C}} \alpha_\Delta \wedge \delta_\Delta \in D(\Omega),$$

with $\alpha_\Delta \in A_\Delta(\Omega \cap \Delta)$. It follows from Proposition 2.18 that

$$d'_{\mathrm{P}}\alpha = d'\alpha \quad \text{and} \quad d''_{\mathrm{P}}\alpha = d''\alpha, \tag{3.5.1}$$

where we use the polyhedral derivative introduced in Definition 2.3 on the left-hand sides, and the derivative of currents in $D(\Omega)$ on the right-hand sides.

(iii) Now we assume that $\alpha \in P^{n,n}(\Omega)$ and that P is an integral $\mathbb{R}$-affine polyhedral subset of Ω such that $\mathrm{supp}(\alpha) \cap P$ is compact. By passing again to a subdivision, we may assume that $\mathscr{C}$ has a subcomplex $\mathscr{D}$ with $|\mathscr{D}| = P$. Using the definition of the integral of polyhedral supercurrents on Ω in Remark 3.3 and a decomposition of α as above, (2.5.2) gives

$$\int_P \alpha = \sum_{\Delta \in \mathscr{D}} \int_\Delta \alpha_\Delta.$$

A similar formula holds for the boundary integral $\int_{\partial P} \alpha$ for $\alpha \in P^{n-1,n}(\Omega)$ or $\alpha \in P^{n,n-1}(\Omega)$.

Proposition 3.6 (Stokes' formula for δ-preforms). *Let P be an integral $\mathbb{R}$-affine polyhedral subset of the open subset Ω of $|\mathscr{C}|$. Then we have*

$$\int_P d'\alpha = \int_{\partial P} \alpha, \quad \int_P d''\beta = \int_{\partial P} \beta$$

for all δ-preforms $\alpha \in P^{n-1,n}(\Omega)$ and $\beta \in P^{n,n-1}(\Omega)$ with $\mathrm{supp}(\alpha) \cap P$ *and* $\mathrm{supp}(\beta) \cap P$ *compact.*

Proof. This follows from Proposition 2.7 and (3.5.1). □

The following result will be important in the construction of δ-forms on algebraic varieties.

Lemma 3.7. *Let Ω be an open subset of $|\mathscr{C}|$. Given d'-closed (resp. d''-closed) δ-preforms γ and γ' on Ω, their product $\gamma \wedge \gamma'$ is again d'-closed (resp. d''-closed).*

Proof. Consider an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$ and d'-closed δ-preforms γ and γ' on Ω. We may assume that γ (resp. γ') is of codimension l (resp. l') and that γ has degree $k+2l$ (resp. $k'+2l$). We choose δ-preforms $\tilde{\gamma} = \sum_i \alpha_i \wedge \delta_{C_i}$ and $\tilde{\gamma}' := \sum_j \alpha'_j \wedge \delta_{C'_j}$ for superforms $\alpha_i \in A^k(\widetilde{\Omega})$, $\alpha'_j \in A^{k'}(\widetilde{\Omega})$ and tropical cycles $C_i = (\mathscr{C}_i, m_i)$, $C'_j = (\mathscr{C}'_j, m'_j)$ of codimension l, l' with smooth weight functions such that $\gamma = \tilde{\gamma} \wedge \delta_C$ and $\gamma' = \tilde{\gamma}' \wedge \delta_C$. We have to show that the supercurrent

$$\gamma \wedge \gamma' = \tilde{\gamma} \wedge \tilde{\gamma}' \wedge \delta_C \in D(\Omega)$$

is d'-closed. After suitable refinements we may assume that the polyhedral complexes $\mathscr{C}_i, \mathscr{C}'_j$ and $\mathscr{C}$ are all subcomplexes of a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ in $N_{\mathbb{R}}$. We choose generic vectors $v, w \in N_{\mathbb{R}}$ in order to compute stable tropical intersection products as in Remark 1.4 for tropical cycles with polyhedral complex of definition $\mathscr{D}$. We have $C'_j \cdot C = (\mathscr{D}_{\leq n-l'}, m''_j)$. For $\rho \in \mathscr{D}_{n-l'}$ and $\omega \in \rho$, we have

$$m''_{j\rho}(\omega) = \sum_{\rho = \sigma' \cap \Delta} c_{\sigma'\Delta}\, m'_{j\sigma'}(\omega)\, m_\Delta$$

for small $\varepsilon > 0$, where (σ', Δ) ranges over $\mathscr{D}^{l'} \times \mathscr{D}_n$ and $c_{\sigma'\Delta} = [N : N_{\sigma'} + N_\Delta]$ if $\sigma' \cap (\Delta + \varepsilon v) \neq \varnothing$ and $c_{\sigma'\Delta} = 0$ otherwise. In the same way we write

$$C_i \cdot C'_j \cdot C = (\mathscr{D}_{\leq n-l-l'}, m'''_{ij}).$$

For $\tau \in \mathscr{D}_{n-l-l'}$ and $\omega \in \tau$, we have

$$m'''_{ij\tau}(\omega) = \sum_{\tau = \sigma \cap \rho} c_{\sigma\rho}\, m_{i\sigma}(\omega)\, m''_{j\rho}(\omega)$$

for small $\varepsilon > 0$, where (σ, ρ) ranges over $\mathscr{D}^l \times \mathscr{D}_{n-l'}$ and $c_{\sigma\rho} = [N : N_\sigma + N_\rho]$ if $\sigma \cap (\rho + \varepsilon w) \neq \varnothing$ and $c_{\sigma\rho} = 0$ otherwise. Combining the last two formulas, we get

$$m'''_{ij\tau}(\omega) = \sum_{\tau = \sigma \cap \sigma' \cap \Delta} c_{\sigma\sigma'\Delta}\, m_{i\sigma}(\omega)\, m'_{j\sigma'}(\omega)\, m_\Delta \tag{3.7.1}$$

where $(\sigma, \sigma', \Delta)$ ranges over $\mathscr{D}^l \times \mathscr{D}^{l'} \times \mathscr{D}_n$ and

$$c_{\sigma\sigma'\Delta} = c_{\sigma, \sigma' \cap \Delta} \cdot c_{\sigma'\Delta}. \tag{3.7.2}$$

We observe that by associativity and commutativity $C_i \cdot (C'_j \cdot C) = C'_j \cdot (C_i \cdot C)$. This implies

$$c_{\sigma\sigma'\Delta} = c_{\sigma', \sigma \cap \Delta} \cdot c_{\sigma\Delta}. \tag{3.7.3}$$

Now we use $d'_{\mathrm{P}}(\tilde{\gamma} \wedge \delta_C) = d'\gamma = 0$ in $D(\Omega)$. For every $\rho \in \mathscr{D}_{n-l}$, we get

$$\sum_i \sum_{\rho = \sigma \cap \Delta} c_{\sigma\Delta}\, d'(m_{i\sigma} m_\Delta \alpha_i) = 0 \tag{3.7.4}$$

on $\Omega \cap \rho$. Similarly, we use $d'_{\mathrm{P}}(\tilde{\gamma}' \wedge \delta_C) = d'\gamma' = 0$. For every $\rho' \in \mathscr{D}_{n-l'}$, this gives

$$\sum_j \sum_{\rho' = \sigma' \cap \Delta} c_{\sigma'\Delta}\, d'(m'_{j\sigma'} m_\Delta \alpha'_j) = 0 \tag{3.7.5}$$

on $\Omega \cap \rho'$. We have to show that

$$d'(\gamma \wedge \gamma') = d'(\tilde{\gamma} \wedge \tilde{\gamma}' \wedge \delta_C) = \sum_{ij} d'(\alpha_i \wedge \alpha'_j \wedge \delta_{C_i \cdot C'_j \cdot C}) \tag{3.7.6}$$

vanishes in $D(\Omega)$. Since d' agrees with d'_{P} on δ-preforms, we deduce

$$d'(\alpha_i \wedge \alpha'_j \wedge \delta_{C_i \cdot C'_j \cdot C}) = \sum_\tau d'(m'''_{ij\tau} \alpha_i \wedge \alpha'_j) \wedge \delta_\tau. \tag{3.7.7}$$

By (3.7.1) and Leibniz's rule, we can split this into the sum of

$$\sum_\tau \sum_{\tau = \sigma \cap \sigma' \cap \Delta} c_{\sigma\sigma'\Delta} m_\Delta\, d'(m_{i\sigma} \alpha_i) \wedge m'_{j\sigma'} \alpha'_j \wedge \delta_\tau \tag{3.7.8}$$

and

$$(-1)^k \sum_\tau \sum_{\tau = \sigma \cap \sigma' \cap \Delta} c_{\sigma\sigma'\Delta} m_\Delta m_{i\sigma} \alpha_i \wedge d'(m'_{j\sigma'} \alpha'_j) \wedge \delta_\tau. \tag{3.7.9}$$

Note that here and in the following, we use our standing assumption that the weight m_Δ of C is constant. From (3.7.3) and (3.7.4) we get

$$\begin{aligned}\sum_{ij} \sum_\tau \sum_{\tau = \sigma \cap \sigma' \cap \Delta} & c_{\sigma\sigma'\Delta} m_\Delta\, d'(m_{i\sigma} \alpha_i) \wedge m'_{j\sigma'} \alpha'_j \\ &= \sum_j \sum_\tau \sum_{\tau = \sigma' \cap \rho} c_{\sigma'\rho} \Big(\sum_{\rho = \sigma \cap \Delta} \sum_i c_{\sigma\Delta} m_\Delta\, d'(m_{i\sigma} \alpha_i) \Big) \wedge m'_{j\sigma'} \alpha'_j = 0.\end{aligned}$$

In the same way we get

$$\sum_{ij} \sum_\tau \sum_{\tau = \sigma \cap \sigma' \cap \Delta} c_{\sigma\sigma'\Delta} m_\Delta m_{i\sigma} \alpha_i \wedge d'(m'_{j\sigma'} \alpha'_j) = 0$$

from (3.7.2) and (3.7.5). These two equations and (3.7.6)–(3.7.9) prove the vanishing of $d'(\gamma \wedge \gamma')$. In the same way, one derives $d''(\gamma \wedge \gamma') = 0$ from the vanishing of $d''(\gamma)$ and $d''(\gamma')$. $\square$

Corollary 3.8. *Let Ω be an open subset of $|\mathscr{C}|$. We consider $\beta = \eta \wedge \gamma \in P^k(\Omega)$ and $\beta' = \eta' \wedge \gamma' \in P^{k'}(\Omega)$ for superforms $\eta, \eta' \in A(\Omega)$ and δ-preforms $\gamma, \gamma' \in P(\Omega)$. If $d'\gamma = d'\gamma' = 0$, then $d'\beta$ is again a δ-preform with*

$$d'\beta = d'\eta \wedge \gamma \quad \textit{and} \quad d'(\beta \wedge \beta') = d'\beta \wedge \beta' + (-1)^k \beta \wedge d'\beta'.$$

If $d''\gamma = d''\gamma' = 0$, then $d''\beta$ is again a δ-preform with

$$d''\beta = d''\eta \wedge \gamma \quad \textit{and} \quad d''(\beta \wedge \beta') = d''\beta \wedge \beta' + (-1)^k \beta \wedge d''\beta'.$$

Proof. Given a superform $\eta \in A^p(\Omega)$ and a supercurrent $T \in D(\Omega)$, we have

$$d'(\eta \wedge T) = d'\eta \wedge T + (-1)^p \eta \wedge d'T. \tag{3.8.1}$$

This implies the first formula and hence $d'\beta$ is a preform. Combined with Lemma 3.7, we deduce the second formula as well. Similarly, we prove the corresponding claims for d''. □

Proposition 3.9 (Green's formula for δ-preforms). *Let Ω be an open subset of $|\mathscr{C}|$ and let P be an integral $\mathbb{R}$-affine polyhedral subset of Ω. We consider symmetric δ-preforms $\beta_i \in P^{p_i,p_i}(\Omega)$ for $i = 1, 2$ with $p_1 + p_2 = n - 1$ such that $\beta_i = \eta_i \wedge \gamma_i$ for superforms $\eta_i \in A(\Omega)$ and δ-preforms $\gamma_i \in P(\Omega)$ with $d'\gamma = d'\gamma' = d''\gamma = d''\gamma' = 0$. Then we have*

$$\int_P \big(\beta_1 \wedge d'd''\beta_2 - \beta_2 \wedge d'd''\beta_1\big) = \int_{\partial P} \big(\beta_1 \wedge d''\beta_2 - \beta_2 \wedge d''\beta_1\big),$$

if we assume furthermore that $\operatorname{supp}(\beta_1) \cap \operatorname{supp}(\beta_2) \cap P$ *is compact.*

Proof. As in [Chambert-Loir and Ducros 2012, lemme (1.3.8)], the formula is obtained as a direct consequence of Proposition 3.6 and the Leibniz formula in Corollary 3.8. □

Definition 3.10. Let P be an integral $\mathbb{R}$-affine polyhedral subset in $N_{\mathbb{R}}$. A *piecewise smooth superform* α on an open subset Ω of P is given by an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ with support P and smooth superforms $\alpha_\Delta \in A_\Delta(\Omega \cap \Delta)$ for every $\Delta \in \mathscr{D}$ such that α_Δ restricts to α_ρ for every closed face ρ of Δ. In this case we call $\mathscr{D}$ a *polyhedral complex of definition* for α. The *support* of a piecewise smooth superform α as above is the union of the supports of the forms α_Δ for all Δ in $\mathscr{D}$. We identify two superforms α, α' on Ω if they have the same support and if $\alpha_\Delta = \alpha'_{\Delta'}$ on $\Delta \cap \Delta' \cap \Omega$ for all polyhedra Δ, Δ' of the underlying polyhedral complexes $\mathscr{D}$, $\mathscr{D}'$.

Remark 3.11 (properties of piecewise smooth superforms). Let Ω denote an open subset of an integral $\mathbb{R}$-affine polyhedral subset P in $N_{\mathbb{R}}$.

(i) The space of piecewise smooth superforms on Ω is denoted by $\mathrm{PS}(\Omega)$. It comes with a natural bigrading and has a natural wedge product. We conclude that $\mathrm{PS}^{\cdot,\cdot}(\Omega)$ is a bigraded $\mathbb{R}$-algebra which contains $A^{\cdot,\cdot}(\Omega)$ as a subalgebra. We denote by $\mathrm{PS}_c^{\cdot,\cdot}(\Omega)$ the subspace of $\mathrm{PS}^{\cdot,\cdot}(\Omega)$ given by piecewise smooth superforms with compact support.

(ii) Let N' be also a free abelian group of finite rank and let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. Suppose that Ω' is an open subset of an integral $\mathbb{R}$-affine polyhedral subset Q in $N'_{\mathbb{R}}$ with $F(Q) \subseteq P$ and $F(\Omega') \subseteq \Omega$. For a piecewise smooth superform α on Ω, there is an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}'$ with $|\mathscr{D}'| = Q$ and a polyhedral complex of definition $\mathscr{D}$ for α such that for every $\Delta' \in \mathscr{D}'$, there is a $\Delta \in \mathscr{D}$ with $F(\Delta') \subseteq \Delta$. Then we define a piecewise smooth superform $F^*(\alpha) = \alpha'$ on Ω' with $\mathscr{D}'$ as a polyhedral complex of definition by setting $\alpha'_{\Delta'} := F^*(\alpha_\Delta) \in A_{\Delta'}(\Omega' \cap \Delta')$ for every $\Delta' \in \mathscr{D}'$. In this way, we get a well-defined graded $\mathbb{R}$-algebra homomorphism

$$F^* : \mathrm{PS}^{\cdot,\cdot}(\Omega) \to \mathrm{PS}^{\cdot,\cdot}(\Omega').$$

In particular, we can restrict α to an open subset of an integral $\mathbb{R}$-affine polyhedral subset of P.

(iii) Let $\alpha \in \mathrm{PS}^{p,q}(\Omega)$ be given by an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ and smooth superforms $\alpha_\Delta \in A^{p,q}(\Omega \cap \Delta)$ for every $\Delta \in \mathscr{D}$. Then the superforms $d'\alpha_\Delta \in A_\Delta^{p+1,q}(\Omega \cap \Delta)$, with Δ ranging over $\mathscr{D}$, define an element in $\mathrm{PS}^{p+1,q}(\Omega)$ which we denote by $d'_{\mathrm{P}}\alpha$. Similarly, we define $d''_{\mathrm{P}}\alpha \in \mathrm{PS}^{p,q+1}(\Omega)$. One verifies immediately that $\mathrm{PS}^{\cdot,\cdot}(W)$ is a differential graded $\mathbb{R}$-algebra. with respect to the differentials

$$d'_{\mathrm{P}} : \mathrm{PS}^{p,q}(\Omega) \to \mathrm{PS}^{p+1,q}(\Omega), \quad d''_{\mathrm{P}} : \mathrm{PS}^{p,q}(\Omega) \to \mathrm{PS}^{p,q+1}(\Omega). \tag{3.11.1}$$

(iv) The elements of $\mathrm{PS}^{0,0}(\Omega)$ are the piecewise smooth functions on the open subset Ω of P from Definition 1.6.

3.12. Now we apply the above to an open subset Ω of the polyhedral set $P := |\mathscr{C}|$ for the given tropical cycle $C = (\mathscr{C}, m)$ with constant weight functions. A piecewise smooth superform α as above defines a polyhedral supercurrent

$$[\alpha] := \sum_{\Delta \in \mathscr{C}_n} \alpha_\Delta \wedge \delta_\Delta$$

and the derivatives in (3.11.1) coincide — as suggested by the notation — with the polyhedral derivatives introduced in Definition 2.3. Note that these differentials of piecewise smooth superforms are not compatible with the corresponding differentials of the associated supercurrents. We define the *d'-residue of* α by

$$\mathrm{Res}_{d'}(\alpha) := d'[\alpha] - [d'_{\mathrm{P}}\alpha].$$

Similarly, we define residues with respect to the differential operators d'' and $d'd''$.

3.13. Given $\alpha \in \mathrm{PS}(\Omega)$ and a polyhedral supercurrent β on the open subset Ω of $|\mathscr{C}|$, there is natural bilinear product $\alpha \wedge \beta$ which is defined as a polyhedral supercurrent on Ω as follows. After passing to a subdivision of $\mathscr{C}$, we may assume that $\mathscr{C}$ is a polyhedral complex of definition for α and β. Then $\mathscr{C}$ is a polyhedral complex of definition for $\alpha \wedge \beta$ and for every $\Delta \in \mathscr{C}$ we set

$$(\alpha \wedge \beta)_\Delta := \alpha_\Delta \wedge \beta_\Delta \in A_\Delta(\Omega \wedge \Delta),$$

where α, β are given on Ω by $\alpha_\Delta, \beta_\Delta \in A_\Delta(\Omega \wedge \Delta)$. For $\alpha \in \mathrm{PS}^k(\Omega)$, the Leibniz-type formula

$$d'_{\mathrm{P}}(\alpha \wedge \beta) = d'_{\mathrm{P}}\alpha \wedge \beta + (-1)^k \alpha \wedge d'_{\mathrm{P}}\beta \tag{3.13.1}$$

is a direct consequence of our definitions. An analogous formula holds for d''_{P}.

There is no obvious product on the space of polyhedral currents which extends the given products on the subspaces $P(\Omega)$ and $\mathrm{PS}(\Omega)$. The next remark shows that such a product exists for a canonical subspace $\mathrm{PSP}(\Omega)$ of the space of polyhedral currents.

Remark 3.14. The linear subspace $\mathrm{PSP}(\Omega)$ of $D(\Omega)$, generated by currents of the form $\alpha \wedge \beta$ with $\alpha \in \mathrm{PS}(\Omega)$ and with $\beta \in P(\Omega)$, will play a role later. Note that $\mathrm{PSP}(\Omega)$ has a unique structure as a bigraded differential $\mathbb{R}$-algebra with respect d'_{P} and d''_{P} extending the corresponding structures on $\mathrm{PS}(\Omega)$ and $P(\Omega)$. To see that the wedge product is well defined, we can use the same arguments as for $P(\Omega)$. The crucial point is that for a piecewise smooth form α as in 3.12 and $\tau \preccurlyeq \Delta \in \mathscr{C}$, the restriction of α_Δ to τ is α_τ. This allows us to use the arguments in Proposition 2.12 which show that $\wedge$ is well defined on $\mathrm{PSP}(\Omega)$.

If $F : N'_{\mathbb{R}} \to N_R$ is an integral $\mathbb{R}$-affine map and if $\widetilde{\Omega}'$ is an open subset of the preimage of the open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$, then we have a unique pull-back F^* : $\mathrm{PSP}(\widetilde{\Omega}) \to \mathrm{PSP}(\widetilde{\Omega}')$ which extends the pull-back maps on piecewise smooth forms and on δ-preforms and which is compatible with the bigrading and the wedge product. Again, the argument is the same as in the proof of Proposition 2.12. Moreover, it is clear that the projection formulas in Proposition 2.14 hold more generally for $\alpha \in \mathrm{PSP}(\widetilde{\Omega})$.

3.15. Recall that $C = (\mathscr{C}, m)$ is a tropical cycle on $N_{\mathbb{R}}$ of pure dimension n and with constant weight functions. Let ϕ be a piecewise smooth function on $|\mathscr{C}|$. We have seen in Proposition 1.12 that the corner locus $\phi \cdot C$ is again a tropical cycle. It induces a polyhedral supercurrent $\delta_{\phi \cdot C} \in D_{n-1,n-1}(|\mathscr{C}|)$ on $|\mathscr{C}|$. By Proposition 1.8, there is a piecewise smooth function $\tilde{\phi}$ on $N_{\mathbb{R}}$ extending ϕ. We have

$$\delta_{\phi \cdot C} = \delta_{\tilde{\phi} \cdot N_{\mathbb{R}}} \wedge \delta_C$$

and hence $\delta_{\phi \cdot C}$ is a δ-preform in $P^{1,1}(|\mathscr{C}|)$. By Remark 3.5, we obtain a δ-preform $\delta_{\phi \cdot C} \wedge \beta \in P^{p,q,l+1}(|\mathscr{C}|)$ for any $\beta \in P^{p,q,l}(|\mathscr{C}|)$.

The following *tropical Poincaré–Lelong formula* and its Corollary 3.19 compute the $d'd''$-residue of ϕ.

Theorem 3.16. *We consider a δ-preform $\omega \in P^{p,q,l}(|\mathscr{C}|)$ such that $d'\omega = 0 = d''\omega$. Let $\eta \in A^{n-p-l-1,n-q-l-1}(|\mathscr{C}|)$ be a superform such that $\beta = \eta \wedge \omega$ has compact support. Then we have*

$$\int_{|\mathscr{C}|} \phi\, d'd''\beta - \int_{|\mathscr{C}|} d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta = \int_{|\mathscr{C}|} \delta_{\phi\cdot C} \wedge \beta, \tag{3.16.1}$$

where we use the integral of polyhedral supercurrents on $|\mathscr{C}|$ defined in Remark 3.3.

Proof. We may assume after suitable refinements that $\mathscr{C}$ is also a polyhedral complex of definition for ϕ and ω. From (3.13.1) and (3.5.1), we get

$$d''_{\mathrm{P}}(\phi\, d'\beta) + d'_{\mathrm{P}}(d'_{\mathrm{P}}\phi \wedge \beta) = \phi\, d''d'\beta + d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta = d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta - \phi\, d'd''\beta.$$

Let P denote the polyhedral set $|\mathscr{C}|$. Stokes' formula for polyhedral supercurrents, Proposition 2.7, yields

$$\int_P \left(d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta - \phi\, d'd''\beta\right) = \int_{\partial P} \phi \wedge d'\beta + \int_{\partial P} d''_{\mathrm{P}}\phi \wedge \beta. \tag{3.16.2}$$

We write

$$\omega = \sum_{i\in I} \omega_i \wedge \delta_{C_i}$$

for tropical cycles $C_i = (\mathscr{C}_{\leq n-l}, m_i)$ with suitable smooth weight functions m_i and superforms ω_i. Then we have

$$\begin{aligned}\int_{\partial P} \phi \wedge d'\beta &= \sum_{i\in I} \int_{\partial P} \phi \wedge d'\eta \wedge \omega_i \wedge \delta_{C_i}\\ &= \sum_{i\in I}\sum_{\sigma\in\mathscr{C}_{n-l}} \int_{\partial\sigma} m_{i\sigma}\phi_\sigma\, d'\eta \wedge \omega_i.\end{aligned}$$

For each $\sigma \in \mathscr{C}_{n-l}$ and each face $\tau \in \mathscr{C}_{n-l-1}$ we choose an element $\omega_{\sigma,\tau}$ as in (1.1.1). We observe that the elements $\omega_{\tau,\sigma}$ used in [Gubler 2016, 2.8] to define the boundary integrals $\int_{\partial\sigma}$ satisfy $\omega_{\tau,\sigma} = -\omega_{\sigma,\tau}$. The definition of the boundary integral uses the contraction $\langle\,\cdot\,,\omega_{\tau,\sigma}\rangle_{\{n-l\}}$ of the involved superform of type $(n-l, n-l)$ given by inserting $\omega_{\tau,\sigma}$ for the $(n-l)$-th variable and leads to

$$\int_{\partial P} \phi \wedge d'\beta = -\sum_{i\in I}\sum_{\tau\in\mathscr{C}_{n-l-1}}\sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \langle m_{i\sigma}\phi_\sigma\, d'\eta\wedge\omega_i, \omega_{\sigma,\tau}\rangle_{\{n-l\}}.$$

Given $i \in I$ and $\tau \in \mathscr{C}_{n-l-1}$, the balancing condition (1.1.1) for C_i gives us the vector field

$$\omega_{i\tau} := \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} m_{i\sigma}\omega_{\sigma,\tau} : \tau \to \mathbb{L}_\tau.$$

We observe that $\phi_\sigma|_\tau = \phi_\tau$ for all $\tau \prec \sigma$ yielding

$$\int_{\partial P} \phi \wedge d'\beta = -\sum_{i\in I} \sum_{\tau\in\mathscr{C}_{n-1}} \int_\tau \langle \phi_\tau \, d'\eta \wedge \omega_i, \omega_{i\tau}\rangle_{\{n-l\}} = 0$$

as a superform contracted with a vector field with values in $\mathbb{L}_\tau$ restricts to zero on τ. Using this in (3.16.2), we obtain

$$\int_{|\mathscr{C}|} \phi \, d'd''\beta - \int_{|\mathscr{C}|} d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta = -\int_{\partial|\mathscr{C}|} d''_{\mathrm{P}}\phi \wedge \beta. \tag{3.16.3}$$

Our claim is then a consequence of the following lemma. □

Lemma 3.17. *Let ϕ be a piecewise smooth function on $|\mathscr{C}|$. For any δ-preform $\beta \in P_c^{n-1,n-1}(|\mathscr{C}|)$ with compact support, we have*

$$\int_{\partial|\mathscr{C}|} d''_{\mathrm{P}}\phi \wedge \beta = -\int_{|\mathscr{C}|} \delta_{\phi\cdot C} \wedge \beta, \quad \int_{\partial|\mathscr{C}|} d'_{\mathrm{P}}\phi \wedge \beta = \int_{|\mathscr{C}|} \delta_{\phi\cdot C} \wedge \beta. \tag{3.17.1}$$

Proof. We prove only the first formula. The second formula follows by applying the first one to $J^*(\beta)$ and using the symmetry of the supercurrent of integration. We use the notation introduced in the proof of Theorem 3.16. We may assume that $\beta \in P^{n-l-1,n-l-1,l}(|\mathscr{C}|)$ and that

$$\beta = \sum_{i\in I} \eta_i \wedge \delta_{C_i}$$

for tropical cycles $C_i = (\mathscr{C}_{\leq n-l}, m_i)$ with suitable smooth weight functions m_i and superforms $\eta_i \in A^{n-l-1,n-l-1}(|\mathscr{C}|)$. Since β is a δ-preform on C, we may assume that there is a tropical cycle $\widetilde{C}_i$ of codimension l in $N_{\mathbb{R}}$ such that $C_i = \widetilde{C}_i.C$ for every $i \in I$. Recall that $\partial/\partial\omega_{\sigma,\tau}$ denotes the partial derivative along the tangential vector $\omega_{\sigma,\tau}$. An exercise in linear algebra gives

$$\langle d''\phi_\sigma \wedge \eta_i, \omega_{\sigma,\tau}\rangle_{\{2n-2l-1\}} = \frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} \wedge \eta_i + d''\phi_\sigma \wedge \langle \eta_i, \omega_{\sigma,\tau}\rangle_{\{2n-2l-2\}}$$

for all $i \in I$, $\sigma \in \mathscr{C}_{n-l}$ and all faces τ of σ of codimension one. Furthermore one sees easily that

$$\int_\tau d''\phi_\tau \wedge \langle \eta_i, \omega_{i\tau}\rangle_{\{2n-2l-2\}} = -\int_\tau \frac{\partial\phi_\tau}{\partial\omega_{i\tau}} \wedge \eta_i.$$

Let $\phi \cdot C_i = (\mathscr{C}_{\leq n-l-1}, m_i)$ denote the corner locus of ϕ on C_i introduced in Definition 1.10. Using the last two formulas and the definition of the weight functions $m_{i\tau}$ of the corner locus in (1.10.1), we get

$$
\begin{aligned}
\sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \langle m_{i\sigma}\, d''\phi_\sigma\wedge\eta_i,\omega_{\sigma,\tau}\rangle_{\{2n-2l-1\}}
&= \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \left(m_{i\sigma}\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}}\wedge\eta_i + d''\phi_\tau\wedge\left\langle \eta_i, \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} m_{i\sigma}\omega_{\sigma,\tau}\right\rangle_{\{2n-2l-2\}}\right)\\
&= \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \left(m_{i\sigma}\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}}\wedge\eta_i - \frac{\partial\phi_\tau}{\partial\omega_{i\tau}}\wedge\eta_i\right)\\
&= \int_\tau \Big(\sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} m_{i\sigma}\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} - \frac{\partial\phi_\tau}{\partial\omega_{i\tau}}\Big)\wedge\eta_i\\
&= \int_\tau m_{i\tau}\eta_i
\end{aligned}
$$

for all $i\in I$ and $\tau\in\mathscr{C}_{n-l-1}$. For the polyhedral set $P:=|\mathscr{C}|$, we have

$$
\begin{aligned}
\int_{\partial P} d''_{\mathrm{P}}\phi\wedge\beta &= \sum_{i\in I}\sum_{\sigma\in\mathscr{C}_{n-l}}\int_{\partial\sigma} m_{i\sigma}\, d''\phi_\sigma\wedge\eta_i\\
&= -\sum_{i\in I}\sum_{\tau\in\mathscr{C}_{n-l-1}}\sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}}\int_\tau \langle m_{i\sigma}\, d''\phi_\sigma\wedge\eta_i,\omega_{\sigma,\tau}\rangle_{\{2n-2l-1\}}\\
&= -\sum_{i\in I}\sum_{\tau\in\mathscr{C}_{n-l-1}}\int_\tau m_{i\tau}\eta_i\\
&= -\sum_{i\in I}\int_P \eta_i\wedge\delta_{\phi\cdot C_i}.
\end{aligned}
$$

We get $\delta_{\phi\cdot C_i}=\delta_{\phi\cdot\widetilde{C}_i\cdot C}=\delta_{\widetilde{C}_i}\wedge\delta_{\phi\cdot C}$ from Proposition 1.14. Hence

$$
\sum_{i\in I}\int_P \eta_i\wedge\delta_{\phi\cdot C_i} = \int_P\Big(\sum_{i\in I}\eta_i\wedge\delta_{\widetilde{C}_i}\Big)\wedge\delta_{\phi\cdot C} = \int_P \delta_{\phi\cdot C}\wedge\beta
$$

yields our claim. □

Remark 3.18. In the situation of Lemma 3.17 we consider a δ-preform $\beta\in P^{n-1,n-1}(|\mathscr{C}|)$ on C. However we do no longer assume that β has compact support. Instead we make the weaker assumption that the polyhedral supercurrents $d''_{\mathrm{P}}\phi\wedge\beta\in D_{1,0}(|\mathscr{C}|)$ (resp. $d'_{\mathrm{P}}\phi\wedge\beta\in D_{0,1}(|\mathscr{C}|)$) and $\delta_{\phi\cdot C}\wedge\beta\in D_{0,0}(|\mathscr{C}|)$ have compact support. Then the first (resp. second) formula in (3.17.1) still hold for β. In order to prove this, one chooses a function $f\in A^0_c(|\mathscr{C}|)$ which is equal to 1 on the above compact supports and applies Lemma 3.17 to $f\cdot\beta$.

Corollary 3.19. *Let $C = (\mathscr{C}, m)$ be a tropical cycle with constant weight functions of pure dimension n on $N_{\mathbb{R}}$ and $\phi : |\mathscr{C}| \to \mathbb{R}$ a piecewise smooth function with corner locus $\phi \cdot C$. Then we have*

$$d'd''[\phi] - [d'_{\mathrm{P}}d''_{\mathrm{P}}\phi] = \delta_{\phi\cdot C} \tag{3.19.1}$$

in $D^{\mathscr{C}}_{n-1,n-1}(|\mathscr{C}|)$.

Proof. Both sides of (3.19.1) have support in $|\mathscr{C}|$. Hence it suffices to show that

$$\big(d'd''[\phi] - [d'_{\mathrm{P}}d''_{\mathrm{P}}\phi]\big)(\alpha) = \delta_{\phi\cdot C}(\alpha)$$

holds for all $\alpha \in A_c^{n-1,n-1}(|\mathscr{C}|)$, and this is a special case of Theorem 3.16. □

Corollary 3.20. *Let $\phi : |\mathscr{C}| \to \mathbb{R}$ a piecewise linear function on C. Then we have*

$$d'd''[\phi] = \delta_{\phi\cdot C} \tag{3.20.1}$$

in $D^{\mathscr{C}}_{n-1,n-1}(N_{\mathbb{R}})$.

Proof. This follows from Corollary 3.19. □

4. Delta-forms on algebraic varieties

Let X be an algebraic variety over K of dimension n and X^{an} the associated Berkovich space.

We introduce the algebra $B(W)$ of δ-forms on an open subset W of X^{an}. We use tropicalizations as in [Chambert-Loir and Ducros 2012] and [Gubler 2016] to pull-back algebras of δ-preforms to X^{an}. After a suitable sheafification process we obtain the sheaves of algebras B and P of δ-forms and generalized δ-forms. We show that B is a sheaf of bigraded differential $\mathbb{R}$-algebras with respect to natural differentials d' and d''.

4.1. Consider a *tropical chart* (V, φ_U) on X as in [Gubler 2016, 4.15]. It consists of a very affine Zariski open U in X. Recall that U is called *very affine* if U has a closed immersion into a multiplicative torus. Then there is a canonical torus T_U with cocharacter group

$$N_U = \mathrm{Hom}_{\mathbb{Z}}(\mathscr{O}(U)^\times/K^\times, \mathbb{Z}),$$

and a canonical closed embedding $\varphi_U : U \to T_U$, unique up to translation (see [Gubler 2016, 4.12, 4.13] for details). We get a tropicalization map

$$\mathrm{trop}_U : U^{\mathrm{an}} \xrightarrow{\varphi_U^{\mathrm{an}}} T_U^{\mathrm{an}} \xrightarrow{\mathrm{trop}} N_{U,\mathbb{R}}$$

associated with φ_U. The second ingredient of a tropical chart is an open subset $V \subseteq U^{\mathrm{an}}$ for which there is an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$.

The set $\mathrm{trop}_U(U^{\mathrm{an}})$ is the support of a canonical tropical cycle $\mathrm{Trop}(U) = (\mathrm{Trop}(U), m_U)$ with integral weights. It is the *tropical variety* associated to the closed subvariety U of T_U equipped with its canonical tropical weights (see [Gubler 2013, §3, §13]). Note that $V = \mathrm{trop}_U^{-1}(\Omega)$ for the open subset $\Omega := \widetilde{\Omega} \cap \mathrm{Trop}(U)$ of $\mathrm{Trop}(U)$.

Definition 4.2. Let $f : X' \to X$ be a morphism of algebraic varieties over K. We say that charts (V, φ_U) and $(V', \varphi_{U'})$ of X and X' respectively are *compatible with respect to* f, if we have $f(U') \subseteq U$ and $f^{\mathrm{an}}(V') \subseteq V$.

4.3. Let $f : X' \to X$ be a morphism of algebraic varieties over K. Given compatible charts (V, φ_U) and $(V', \varphi_{U'})$ of X and X', we obtain a commutative diagram

$$\begin{tikzcd}
V' \arrow[r, hook] \arrow[d, "f^{\mathrm{an}}|_{V'}"] & (U')^{\mathrm{an}} \arrow[r, "\varphi_{U'}"] \arrow[d, "f^{\mathrm{an}}|_{U'_{\mathrm{an}}}"] & T_{U'} \arrow[r, "\mathrm{trop}"] \arrow[d, "\psi"] & N_{U',\mathbb{R}} \arrow[d, "F"] \\
V \arrow[r, hook] & U^{\mathrm{an}} \arrow[r, "\varphi_U"] & T_U \arrow[r, "\mathrm{trop}"] & N_{U,\mathbb{R}}
\end{tikzcd}$$

where $\psi : T_{U'} \to T_U$ is the canonical affine homomorphism of tori induced by $\mathscr{O}^\times(U) \to \mathscr{O}^\times(U')$ and $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ is the induced canonical integral Γ-affine map. These maps are unique up to translation, but this ambiguity will never play a role. If Ω' is the open subset of $\mathrm{Trop}(U')$ with $\mathrm{trop}_{U'}^{-1}(\Omega') = V'$, then $\Omega' \subseteq F^{-1}(\Omega) \cap \mathrm{Trop}(U')$.

We define $\deg(f) = [K(X') : K(X)]$ if f is dominant and the extension of function fields is finite. Otherwise we set $\deg(f) = 0$. Let Y be the schematic image of f and $f' : X' \to Y$ the induced morphism. Then a formula of Sturmfels and Tevelev [2008] which was generalized by Baker, Payne and Rabinoff [2016, Section 7] to the present setting gives

$$F_* \,\mathrm{Trop}(U') = \deg(f') \cdot \mathrm{Trop}(\overline{f(U')}) \tag{4.3.1}$$

as an equality of tropical cycles (see [Gubler 2013, Theorem 13.17]).

Definition 4.4. Let us consider a tropical chart (V, φ_U) of X. As above, we consider the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. We choose an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U)$ and a δ-preform $\tilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$. For any morphism $f : X' \to X$ of varieties over K and a tropical chart $(V', \varphi_{U'})$ of X' compatible with (V, φ_U), we define $\Omega' := \mathrm{trop}_{U'}(V')$. We choose an open subset $\widetilde{\Omega}'$ of $F^{-1}(\widetilde{\Omega})$ with $\widetilde{\Omega}' \cap \mathrm{Trop}(U') = \Omega'$. By Proposition 2.12, we have $F^*(\tilde{\alpha}) \in P^{p,q}(\widetilde{\Omega}')$. We denote by $N^{p,q}(V, \varphi_U)$ the subspace given by elements $\tilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$ such that we have $F^*(\tilde{\alpha})|_{\Omega'} = 0 \in P^{p,q}(\Omega')$ for all compatible pairs of charts as above (see Definition 3.4 for the definition of the restriction). We define

$$P^{p,q}(V, \varphi_U) := P^{p,q}(\widetilde{\Omega}) / N^{p,q}(V, \varphi_U).$$

A partition of unity argument shows that this definition is independent of the choice of $\widetilde{\Omega}$. We call an element in $P^{p,p}(V,\varphi_U)$ *symmetric* (resp. *antisymmetric*) if it can be represented by a symmetric (resp. antisymmetric) δ-preform in $P^{p,p}(\widetilde{\Omega})$. We define

$$P^{p,q,l}(V,\varphi_U) := P^{p,q,l}(\widetilde{\Omega})\big/\big(P^{p,q,l}(\widetilde{\Omega}) \cap N^{p,q}(V,\varphi_U)\big)$$

using the δ-preforms on $\widetilde{\Omega}$ of codimension l from Definition 2.9.

Remark 4.5. (i) The $\wedge$-product descends to the space

$$P(V,\varphi_U) := \bigoplus_{p,q\geq 0} P^{p,q}(V,\varphi_U)$$

and we get a bigraded anticommutative $\mathbb{R}$-algebra which contains $A(\Omega)$ as a bigraded subalgebra.

(ii) If $(V',\varphi_{U'})$ and (V,φ_U) are compatible charts with respect to $f: X' \to X$ as in Definition 4.2, then we get a canonical bigraded homomorphism

$$f^*: P(V,\varphi_U) \to P(V',\varphi_{U'})$$

of bigraded $\mathbb{R}$-algebras which is defined for $\alpha \in P^{p,q}(V,\varphi_U)$ as follows: By definition, α is represented by some $\tilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$. Let $\Omega' := \operatorname{trop}_{U'}(V')$ and choose an open subset $\widetilde{\Omega}'$ of $F^{-1}(\widetilde{\Omega})$ with $\Omega' = \widetilde{\Omega}' \cap \operatorname{Trop}(U')$. Then we define $f^*(\alpha) \in P^{p,q}(V',\varphi_{U'})$ as the class of $F^*(\tilde{\alpha}) \in P^{p,q}(\widetilde{\Omega}')$. If $X = X'$ and $f = \mathrm{id}$, then $(V',\varphi_{U'})$ is a tropical subchart of (V,φ_U) and we write $\alpha|_{V'}$ for the pull-back of $\alpha \in P^{p,q}(V,\varphi_U)$.

Note that the definition of $f^*(\alpha)$ does not depend on the choice of the representative $\tilde{\alpha}$.

However, the elements of $P^{p,q}(V,\varphi_U)$ do not only depend on the *restriction*

$$\alpha|_\Omega := \tilde{\alpha}|_\Omega = \tilde{\alpha} \wedge \delta_{\operatorname{Trop}(U)} \in P^{p,q}(\Omega) \subseteq D^{p,q}(\Omega) \tag{4.5.1}$$

to Ω as Example 4.22 below shows that it might happen that two different elements $\alpha, \beta \in P^{p,q}(V,\varphi_U)$ satisfy $\alpha|_\Omega = \beta|_\Omega \in P^{p,q}(\Omega)$. The purpose of our definition of $P(V,\varphi_U)$ is to have a pull-back as above at hand. Here we use the fact that we always have a pull-back from tropical cycles on $N_{U,\mathbb{R}}$ to tropical cycles on $N_{U',\mathbb{R}}$, but there is a pull-back available from tropical cycles on $\operatorname{Trop}(U)$ to tropical cycles on $\operatorname{Trop}(U')$ only if these tropical varieties are smooth (see [François and Rau 2013]). To have a pull-back available, we consider all morphisms $f: X' \to X$ of varieties over K in the definition of $N^{p,q}(V,\varphi_U)$ and not only open immersions.

4.6. As mentioned already in Remark 2.17, we have the problem that the differential operators d' and d'' are not defined on the algebra $P(V,\varphi_U)$. For α in $P^{p,q}(V,\varphi_U)$ and every compatible tropical chart $(V',\varphi_{U'})$ with respect to $f: X' \to X$, we use the above notation. We get a δ-preform $f^*(\alpha)|_{\Omega'} = F^*(\tilde{\alpha})|_{\Omega'} \in P^{p,q}(\Omega')$. Recall that

$f^*(\alpha)|_{\Omega'}$ is a supercurrent on Ω'. We differentiate it in the sense of supercurrents to get $d'[f^*(\alpha)|_{\Omega'}] \in D(\Omega')$, but it need not be a δ-preform on Ω'. In the following construction, we pass to a convenient subalgebra of $P(V, \varphi_U)$ which is invariant under d' and d''.

As an initial step, we consider the elements ω of $P^{p,q}(V, \varphi_U)$ and $P^{p,q,l}(V, \varphi_U)$, respectively, satisfying the *closedness condition*

$$d'[f^*(\omega)|_{\Omega'}] = d''[f^*(\omega)|_{\Omega'}] = 0 \in D(\Omega') \tag{4.6.1}$$

for every tropical chart $(V', \varphi_{U'})$ which is compatible with (V, φ_U) with respect to $f : X' \to X$. These elements form subspaces $Z^{p,q}(V, \varphi_U)$ of $P^{p,q}(V, \varphi_U)$ and $Z^{p,q,l}(V, \varphi_U)$ of $P^{p,q,l}(V, \varphi_U)$, respectively, and we define

$$Z(V, \varphi_U) := \bigoplus_{p,q \geq 0} Z^{p,q}(V, \varphi_U) = \bigoplus_{p,q,l \geq 0} Z^{p,q,l}(V, \varphi_U)$$

as usual.

Proposition 4.7. *Using the notation above, $Z(V, \varphi_U)$ is a bigraded $\mathbb{R}$-subalgebra of $P(V, \varphi_U)$.*

Proof. The only nontrivial point is that $Z(V, \varphi_U)$ is closed under the $\wedge$-product. This is a direct consequence of Lemma 3.7 applied to δ-preforms on the tropical cycle $\mathrm{Trop}(U')$ for any tropical chart $(V', \varphi_{U'})$ compatible with (V, φ_U). □

Example 4.8. Every tropical cycle $C = (\mathscr{C}, m)$ on $N_{U,\mathbb{R}}$ with *constant weight functions* induces an element in $Z(V, \varphi_U)$. Indeed, if $(V', \varphi_{U'})$ is a tropical chart on X' compatible with (V, φ_U) as above, then $F^*(\delta_C)|_{\Omega'}$ is given by the restriction of $\delta_{F^*(C)\cdot \mathrm{Trop}(U')}$ to Ω'. Since $F^*(C) \cdot \mathrm{Trop}(U')$ is a tropical cycle with constant weight functions, the associated current is d'- and d''-closed [Gubler 2016, Proposition 3.8].

Definition 4.9. Let $\mathrm{AZ}(V, \varphi_U)$ be the subalgebra of $P(V, \varphi_U)$ generated by $A(\Omega)$ and $Z(V, \varphi_U)$. An element $\beta \in \mathrm{AZ}(V, \varphi_U)$ has the form

$$\beta = \sum_{i \in I} \alpha_i \wedge \omega_i \tag{4.9.1}$$

for a finite set I with all $\alpha_i \in A(\Omega)$ and $\omega_i \in Z(V, \varphi_U)$. We define

$$d'\beta := \sum_{i \in I} d'(\alpha_i) \wedge \omega_i, \quad d''\beta := \sum_{i \in I} d''(\alpha_i) \wedge \omega_i.$$

It follows from the closedness condition (4.6.1) that $d'\beta$ and $d''\beta$ are well-defined elements in $\mathrm{AZ}(V, \varphi_U)$. By definition, we have

$$Z(V, \varphi_U) = \{\alpha \in \mathrm{AZ}(V, \varphi_U) \mid d'(\alpha) = d''(\alpha) = 0\}.$$

An element in $\mathrm{AZ}(V, \varphi_U)$ is called *symmetric* (resp. *antisymmetric*) if it is symmetric (resp. antisymmetric) in $P(V, \varphi_U)$.

The following result shows that $\mathrm{AZ}(V, \varphi_U)$ is a good analogue of the algebra of complex differential forms.

Proposition 4.10. *The space* $\mathrm{AZ}(V, \varphi_U)$ *is a bigraded differential* $\mathbb{R}$*-algebra with respect to the differentials* d' *and* d''.

Proof. This follows easily from Leibniz's rule 3.8(ii) and Proposition 4.7. □

Proposition 4.11. *Let* $f : X' \to X$ *be a morphism of varieties over* K. *Let* (V, φ_U) *and* $(V', \varphi_{U'})$ *be tropical charts of* X *and* X' *respectively which are compatible with respect to* f. *Then the pull-back homomorphism* $f^* : P(V, \varphi_U) \to P(V', \varphi_{U'})$ *maps* $Z(V, \varphi_U)$ *to* $Z(V', \varphi_{U'})$ *and* $\mathrm{AZ}(V, \varphi_U)$ *to* $\mathrm{AZ}(V', \varphi_{U'})$.

Proof. This follows directly from the definitions. We leave the details to the reader. □

Proposition 4.12. *Let* (V, φ_U) *be a tropical chart of* X *and* $\Omega := \mathrm{trop}_U(V)$. *Let* $(\Omega_i)_{i\in I}$ *be a finite open covering of* Ω. *For* $i \in I$, *let* $V_i := \mathrm{trop}_U^{-1}(\Omega_i)$ *and let* $\alpha_i \in P(V_i, \varphi_U)$. *For all* $i, j \in I$, *we assume that* $\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j}$. *Then there is a unique* $\alpha \in P(V, \varphi_U)$ *with* $\alpha|_{V_i} = \alpha_i$ *for every* $i \in I$. *If* $\alpha_i \in \mathrm{AZ}(V_i, \varphi_U)$ *for every* $i \in I$ *then* $\alpha \in \mathrm{AZ}(V, \varphi_U)$.

Proof. It is a straightforward consequence of our definitions that α is unique. In order to construct α we choose for each $i \in I$ an open subset $\widetilde{\Omega}_i$ in $N_{U,\mathbb{R}}$ such that $\widetilde{\Omega}_i \cap \mathrm{Trop}(U) = \Omega_i$ and a δ-preform $\tilde{\alpha}_i \in P(\widetilde{\Omega}_i)$ which represents α_i. Let $(\phi_i)_{i\in I}$ be a smooth partition of unity on $\widetilde{\Omega} = \bigcup_{i\in I} \widetilde{\Omega}_i$ with respect to the covering $(\widetilde{\Omega}_i)_{i\in I}$. Observe that we may choose the same index set I as we do not require that the ϕ_i have compact support. Then by our assumptions $\tilde{\alpha} := \sum_{i\in I} \phi_i \tilde{\alpha}_i \in P(\widetilde{\Omega})$ induces the desired element α in $P(V, \varphi_U)$. If

$$\alpha_i = \sum_{j\in I_i} \beta_{ij} \wedge \omega_{ij} \in \mathrm{AZ}(V_i, \varphi_U)$$

as in (4.9.1), we choose representatives $\tilde{\beta}_{ij} \in A(\widetilde{\Omega}_i)$ of $\beta_{ij} \in A(\Omega_i)$ and $\tilde{\omega}_{ij}$ in $P(\widetilde{\Omega}_i)$ of $\omega_{ij} \in Z(V, \varphi_U)$. Then we may choose $\tilde{\alpha}_i$ as $\sum_{j\in I_i} \phi_i \beta_{ij} \wedge \tilde{\omega}_{ij}$ and

$$\tilde{\alpha} = \sum_{i\in I} \phi_i \tilde{\alpha}_i = \sum_{i\in I} \sum_{j\in I_i} \phi_i \beta_{ij} \wedge \tilde{\omega}_{ij}$$

shows $\alpha \in \mathrm{AZ}(V, \varphi_U)$, using the finiteness of I. □

Recall that the tropical charts (V, φ_U) of X form a basis for X^{an} [Gubler 2016, Proposition 4.16]. Hence we can use the algebras $P(V, \varphi_U)$ and $\mathrm{AZ}(V, \varphi_U)$ to define sheaves on X^{an} as follows:

Definition 4.13. For a fixed open subset W in X^{an}, the set of all tropical charts (V, φ_U) on X with $W \subseteq V$ is ordered with respect to compatibility and forms a directed set. Then we get presheaves

$$W \mapsto \varinjlim P(V, \varphi_U), \quad W \mapsto \varinjlim \mathrm{AZ}(V, \varphi_U) \tag{4.13.1}$$

of real vector spaces on X^{an}, where the limit is taken over this directed set with respect to the pull-back maps considered in Proposition 4.11. The associated sheaves P and B on X^{an} are by definition the *sheaf of generalized δ-forms* and the *subsheaf of δ-forms*. On an open subset W of X^{an} the space of δ-forms

$$B(W) = \bigoplus_{p,q \geq 0} B^{p,q}(W) = \bigoplus_{p,q,l \geq 0} B^{p,q,l}(W)$$

and the space of generalized δ-forms

$$P(W) = \bigoplus_{p,q \geq 0} P^{p,q}(W) = \bigoplus_{p,q,l \geq 0} P^{p,q,l}(W)$$

carry natural gradings by the (p, q)-type of the underlying currents and the codimension of the underlying tropical cycles (as defined in Definitions 2.9 and 4.4). The wedge product on the spaces $\mathrm{AZ}(V, \varphi_U)$ (resp. $P(V, \varphi_U)$) induces a product on $B(W)$ (resp. $P(W)$). Moreover, the differential operators d', d'' on $\mathrm{AZ}(V, \varphi_U)$ induce differential operators d', d'' on $B(W)$. The symmetric and antisymmetric elements in $P(V, \varphi_U)$ define subsheaves of (generalized) symmetric and antisymmetric δ-forms in $B^{p,q}$ and $P^{p,q}$ for all $p, q \geq 0$.

4.14. We conclude that a *δ-form* β of bidegree (p, q) on an open subset W of X^{an} is given by a covering $(V_i)_{i \in I}$ of W by tropical charts (V_i, φ_{U_i}) of X^{an} and elements $\beta_i \in \mathrm{AZ}^{p,q}(V_i, \varphi_{U_i})$ such that

$$\beta_i|_{V_i \cap V_j} = \beta_j|_{V_i \cap V_j}$$

holds for all $i, j \in I$. If β' is another δ-form of bidegree (p, q) on W given by $\beta'_j \in \mathrm{AZ}^{p,q}(V'_j, \varphi_{U'_j})$ with respect to the tropical charts $(V'_j, \varphi_{U'_j})_{j \in J}$ covering W, then β and β' define the same δ-forms if and only if

$$\beta_i|_{V_i \cap V'_j} = \beta'_j|_{V_i \cap V'_j}$$

holds for all $i \in I$ and $j \in J$. A similar description holds for generalized δ-forms.

Proposition 4.15. (i) *The sheaves P and B are sheaves of bigraded anticommutative $\mathbb{R}$-algebras.*

(ii) *We have natural monomorphisms of sheaves of bigraded $\mathbb{R}$-algebras $A \to B$ and $B \to P$.*

(iii) *The differentials $d', d'' : B \to B$ turn (B, d', d'') into a sheaf of bigraded differential $\mathbb{R}$-algebras.*

Proof. Only the injectivity of the natural morphism $A \to B$ does not follow directly from what we have shown before. The injectivity of $A \to B$ can be checked on the presheaves (4.13.1). For each tropical chart (V, φ_U) of X the natural map from $A(V)$ to $\mathrm{AZ}(V, \varphi_U)$ is injective as the associated map $A(\Omega) \to \mathrm{AZ}(V, \varphi_U) \to D(\Omega)$ for $\Omega = \mathrm{trop}_U(V)$ is injective. This directly yields our claim. $\square$

4.16. Let $f : X' \to X$ be a morphism of varieties over K. For an open subset W of X^{an} and an open subset W' of $f^{-1}(W)$, we have a canonical pull-back morphism $f^* : P(W) \to P(W')$ which respects products and the bigrading. Furthermore it induces a homomorphism $f^* : B(W) \to B(W')$ of bigraded $\mathbb{R}$-algebras which commutes with the differentials d' and d'' on B. They are induced by the pull-back $f^* : P(V, \varphi_U) \to P(V', \varphi_{U'})$ for compatible charts $(V', \varphi_{U'})$ on W' and (V, φ_U) on W given in Proposition 4.11.

Lemma 4.17. *Let (V, φ_U) be a tropical chart on X. Let $(V_i)_{i\in I}$ be an open covering of V by tropical charts (V_i, φ_{U_i}) on X which are compatible with (V, φ_U). There are canonical integral Γ-affine morphisms $F_i : N_{U_i,\mathbb{R}} \to N_{U,\mathbb{R}}$ such that* $\mathrm{trop}_U = F_i \circ \mathrm{trop}_{U_i}$. *We choose open subsets $\widetilde{\Omega}$ in $N_{U,\mathbb{R}}$ and $\widetilde{\Omega}_i$ in $F_i^{-1}(\widetilde{\Omega})$ such that $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$ and $V_i = \mathrm{trop}_U^{-1}(\widetilde{\Omega}_i)$ for all $i \in I$. Let $\tilde{\alpha}_U \in P(\widetilde{\Omega})$ be a δ-preform. Then $\tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)}$ vanishes in $D(\widetilde{\Omega})$ if $F_i^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U_i)}$ vanishes in $D(\widetilde{\Omega}_i)$ for every $i \in I$.*

Proof. We write $\tilde{\alpha}_U = \sum_{j\in J} \alpha_j \wedge \delta_{C_j}$ for suitable superforms $\alpha_j \in A(\widetilde{\Omega})$ and tropical cycles C_j. We have $F_{i*}\,\mathrm{Trop}(U_i) = \mathrm{Trop}(U)$ by (4.3.1). The projection formula (Proposition 1.5) gives

$$F_{i*}(F_i^* C_j \cdot \mathrm{Trop}(U_i)) = C_j \cdot \mathrm{Trop}(U).$$

By the same arguments as in the proof of Proposition 2.14, the vanishing of

$$F_i^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U_i)} = \sum_{j\in J} F_i^*(\alpha_j) \wedge \delta_{F_i^* C_j \cdot \mathrm{Trop}(U_i)}$$

in $D(\widetilde{\Omega}_i)$ for all $i \in I$ yields that

$$\tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)} = \sum_{j\in J} \alpha_j \wedge \delta_{F_{i*}(F_i^* C_j \cdot \mathrm{Trop}(U_i))}$$

vanishes in $D(\widetilde{\Omega})$. $\square$

Proposition 4.18. *Given a tropical chart (V, φ_U) on X, we have by construction natural algebra homomorphisms*

$$\mathrm{trop}_U^* : P^{p,q}(V, \varphi_U) \to P^{p,q}(V), \quad \mathrm{trop}_U^* : \mathrm{AZ}^{p,q}(V, \varphi_U) \to B^{p,q}(V)$$

for all $p, q \geq 0$. These maps are injective.

Proof. We extend the argument in [Chambert-Loir and Ducros 2012, lemme (3.2.2)]. It suffices to show that the first map is injective. Let $\mathrm{trop}_U^*(\alpha_U)$ vanish for some $\alpha_U \in P(V, \varphi_U)$. We obtain an open covering $(V_i)_{i\in I}$ of V by tropical charts (V_i, φ_{U_i}) compatible with (V, φ_U) such that $\alpha_U|_{V_i} = 0$ in $P(V_i, \varphi_{U_i})$ for all $i \in I$. Let α_U be induced by $\tilde{\alpha}_U \in P(\widetilde{\Omega})$ for some open subset $\widetilde{\Omega} \in N_{U,\mathbb{R}}$ with $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$. We have to show that $\tilde{\alpha}_U \in N(V, \varphi_U)$. Let $f : X' \to X$ be a morphism of varieties and $(V', \varphi_{U'})$ a tropical chart on X' which is compatible with (V, φ_U). We obtain a canonical integral Γ-affine morphism $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ such that $\mathrm{trop}_U = F \circ \mathrm{trop}_{U'}$. We choose an open subset $\widetilde{\Omega}'$ in $F^{-1}(\widetilde{\Omega})$ such that $V' = \mathrm{trop}_U^{-1}(\widetilde{\Omega}')$. We have to show that $F^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U')}$ vanishes in $D(\widetilde{\Omega}')$.

For every $i \in I$ we choose an open covering $(V'_{ij})_{j\in J_i}$ of $(f^{\mathrm{an}})^{-1}(V_i) \cap V'$ by tropical charts $(V'_{ij}, \varphi_{U'_{ij}})$ on X' which are compatible with $(V', \varphi_{U'})$ and (V_i, φ_{U_i}). For all $i \in I$ and $j \in J_i$ we obtain a commutative diagram

$$\begin{tikzcd}
(U'_{ij})^{\mathrm{an}} \arrow[r, hook] \arrow[d, "f|_{U'_{ij}}"'] \arrow[drr, "\mathrm{trop}_{U'_{ij}}"] & (U')^{\mathrm{an}} \arrow[d, "f|_{U'}"'] \arrow[drr, "\mathrm{trop}_{U'}"] & & \\
U_i^{\mathrm{an}} \arrow[r, hook] \arrow[drr, "\mathrm{trop}_{U_i}"'] & U^{\mathrm{an}} \arrow[drr, "\mathrm{trop}_U"'] & N_{U'_{ij},\mathbb{R}} \arrow[r, "F'_{ij}"'] \arrow[d, "F_{ij}"] & N_{U',\mathbb{R}} \arrow[d, "F"] \\
 & & N_{U_i,\mathbb{R}} \arrow[r, "F_i"'] & N_{U,\mathbb{R}}
\end{tikzcd}$$

of canonical maps. We choose an open subset $\widetilde{\Omega}'_{ij}$ in $(F'_{ij})^{-1}(\widetilde{\Omega}') \cap (F_{ij})^{-1}(\widetilde{\Omega}_i)$ such that $V'_{ij} = \mathrm{trop}_{U'_{ij}}^{-1}(\widetilde{\Omega}'_{ij})$. We have $(F'_{ij})^* F^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U'_{ij})} = 0$ in $D(\widetilde{\Omega}'_{ij})$ by the commutativity of the above diagram and the fact that $\alpha_U|_{V_i} = 0$ in $P(V_i, \varphi_{U_i})$. Now Lemma 4.17 applied to $F^*(\tilde{\alpha}_U)$ on $(V', \varphi_{U'})$ and the covering $(V'_{ij})_{ij}$ of V' yields the vanishing of $F^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U')}$ in $D(\widetilde{\Omega}')$. □

4.19. Let W be an open subset of X^{an}. By construction, the algebra $A^{\cdot,\cdot}(W)$ of differential forms on W is a bigraded subalgebra of the algebra $B^{\cdot,\cdot}(W)$ of δ-forms. In general, $A^{p,q}(W)$ is a proper subspace of $B^{p,q}(W)$. The situation in degree zero is quite different as we may identify δ-forms of degree 0 with functions. We will show that

$$A^{0,0}(W) = B^{0,0}(W). \tag{4.19.1}$$

Clearly, this is a local statement and so we may consider a tropical chart (V, φ_U) on W. It is enough to show

$$A^{0,0}(\Omega) = \mathrm{AZ}^{0,0}(V, \varphi_U) \tag{4.19.2}$$

for the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. Let $\widetilde{\Omega}$ be any open subset of $N_{U,\mathbb{R}}$. Since pull-back of functions is always well defined, we may identify the elements of $P^{0,0}(V,\varphi_U)$ with some continuous functions on Ω and a partition of unity argument together with Example 2.10 shows

$$P^{0,0}(V,\varphi_U) = \{\phi|_\Omega \mid \phi \in P^{0,0}(\widetilde{\Omega})\} = \{\phi|_\Omega \mid \phi \in \mathrm{PS}^{0,0}(\widetilde{\Omega})\}. \tag{4.19.3}$$

To prove (4.19.2), it is enough to show that the elements of $Z(V,\varphi_U)$ are precisely the locally constant functions on Ω. By (4.19.3), we have to show that $\phi|_\Omega$ is locally constant for any $\phi \in \mathrm{PS}^{0,0}(\widetilde{\Omega})$ with $\phi|_\Omega \in Z(V,\varphi_U)$. This means that ϕ is a continuous function on $\widetilde{\Omega}$ with an integral $\mathbb{R}$-affine complete polyhedral complex $\mathscr{C}$ on $N_\mathbb{R}$ such that $\phi|_{\Omega\cap\Delta}$ is smooth for every $\Delta \in \mathscr{C}$. By refinement, we may assume that a subcomplex $\mathscr{D}$ of $\mathscr{C}$ has support equal to $\mathrm{Trop}(U)$. Then the closedness condition (4.6.1) yields that $[\phi|_\Omega]$ is d'- and d''-closed. We conclude that $\phi|_{\Omega\cap\Delta}$ is constant on every $\Delta \in \mathscr{D}$. By continuity, we deduce that $\phi|_\Omega$ is locally constant proving the claim.

4.20. Let (V,φ_U) be a tropical chart on X and $\Omega = \mathrm{trop}_U(V)$.

(i) If Ω_0 is an open subset of Ω, then $V_0 := \mathrm{trop}_U^{-1}(\Omega_0)$ is an open subset of V and (V_0,φ_U) is a tropical chart of X. We say that $\alpha_U \in P(V,\varphi_U)$ vanishes on the open subset Ω_0 if we have $\alpha_U|_{V_0} = 0$ in $P(V_0,\varphi_U)$ (see Remark 4.5). We define $\mathrm{supp}(\alpha_U)$, the *support of* $\alpha_U \in P(V,\varphi_U)$, as

$$\bigl\{\omega \in \Omega \mid \alpha_U \text{ does not vanish on any open neighbourhood } \Omega_0 \text{ of } \omega \text{ in } \Omega\bigr\},$$

which is a closed subset of Ω.

(ii) A (generalized) δ-form α on an open subset W of X^{an} has a well defined support as a section of the sheaf $B^{p,q}$ (resp. $P^{p,q}$). We denote by $B_c^{p,q}$ (resp. by $P_c^{p,q}$) the subsheaves of forms with compact support.

(iii) Observe that compact support always implies proper support in the sense of [Chambert-Loir and Ducros 2012, (4.2.1)] as our assumptions imply that we have $\partial W = \varnothing$ for each open subset W of X^{an} (using that X^{an} is closed, meaning that it has no boundary, see [Berkovich 1990, Theorem 3.4.1]).

Proposition 4.21. *Let (V,φ_U) be a tropical chart on X. Suppose that a generalized δ-form $\alpha \in P(V)$ is given by $\alpha_U \in P(V,\varphi_U)$. Then α_U is uniquely determined and we have* $\mathrm{trop}_U(\mathrm{supp}(\alpha)) = \mathrm{supp}(\alpha_U)$. *Furthermore α has compact support if and only if α_U has compact support.*

Proof. This uniqueness follows from Proposition 4.18. The second statement follows from Proposition 4.18 by the same arguments as in [Chambert-Loir and Ducros 2012, corollaire (3.2.3)]. The last statement is a direct consequence of the continuity and properness of the tropicalization map trop_U (see [Baker et al. 2016, Remark in 2.3]). □

Example 4.22. We construct a tropical chart (V, φ_U) and a nonzero δ-form $\alpha \in \mathrm{AZ}(V, \varphi_U) \setminus \{0\}$ with $\alpha|_\Omega = 0$ for $\Omega := \mathrm{trop}_U(V)$. This example announced in Remark 4.5 justifies the functorial definition of (generalized) delta-forms in Definition 4.4.

We work over the ground field $K = \mathbb{C}_p$ for some prime number $p \neq 2, 3$ and consider the affine curve X in $\mathbb{A}^2_K$ defined by the affine equation

$$f(x, y) = xy + px^3 + py^3.$$

We consider the very affine open subset $U = X \setminus (\{x = 1\} \cup \{y = 1\})$. The only singularity of the rational cubic X is the origin $0 = (0, 0)$, which is an ordinary double point. The normalization of X may be seen as an open subset of $\mathbb{P}^1_K$ and can be obtained as the blowup of X in $(0, 0)$, as in [Hartshorne 1977, Example I.4.9.1]. This description leads to a surjective morphism

$$\varphi\colon \mathbb{P}^1_K \setminus \{\xi_1, \xi_2, \xi_3\} \to X, \quad u \mapsto \left(x = \frac{-u}{p(1+u^3)}, y = \frac{-u^2}{p(1+u^3)}\right)$$

for a suitable affine coordinate u on $\mathbb{P}^1_K$, where ξ_i are the roots of $u^3 + 1 = 0$. It is clear that all ξ_i have absolute value 1 and we may choose $\xi_1 = -1$. Note that $\varphi^{-1}(\{x = 1\}) = \{\rho_1, \rho_2, \rho_3\}$ for the roots ρ_i of $pu^3 + u + p = 0$ in K and $\varphi^{-1}(X \setminus \{y = 1\}) = \{\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1}\}$. Moreover, we have $\varphi^{-1}(0) = \{0, \infty\}$.

The method of the Newton polygon [Neukirch 1999, Proposition II.6.3] shows that $pu^3 + u + p = 0$ has one root ρ_1 of absolute value $|p|$, and two roots ρ_2, ρ_3 of absolute value $|p|^{-\frac{1}{2}}$. We put

$$(\lambda_0, \lambda_1, \ldots, \lambda_8) = (\xi_1, \xi_2, \xi_3, \rho_1, \rho_2, \rho_3, \rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1})$$

and get

$$W := \varphi^{-1}(U) = \mathbb{P}^1_K \setminus \{(\lambda_i : 1) \mid i = 0, \ldots, 8\}.$$

The abelian group $\mathcal{O}(W)^\times / K^\times$ is free of rank eight with generators $b_i = \dfrac{u-\lambda_i}{u+1}$, $i = 1, \ldots, 8$. We deduce from [Liu 2002, Proposition 7.5.15] that

$$\mathcal{O}(U)^\times = \{f \in \mathcal{O}(W)^\times \mid f(0) = f(\infty)\}.$$

We conclude that $M_U := \mathcal{O}(U)^\times / K^\times$ is a free abelian group of rank seven.

In the following, we would like to describe the canonical tropicalization $\mathrm{Trop}(U)$ in the euclidean space $\mathbb{R}^7$ given by choosing a basis in M_U. This is rather complicated and so we compute the tropicalization $\mathrm{trop}_{x-1,y-1}(U)$ in $\mathbb{R}^2$ using the tropicalization map

$$\mathrm{trop}_{x-1,y-1}\colon U^{\mathrm{an}} \to \mathbb{R}^2, \quad q \mapsto \bigl(-\log|(x-1)(q)|, -\log|(y-1)(q)|\bigr).$$

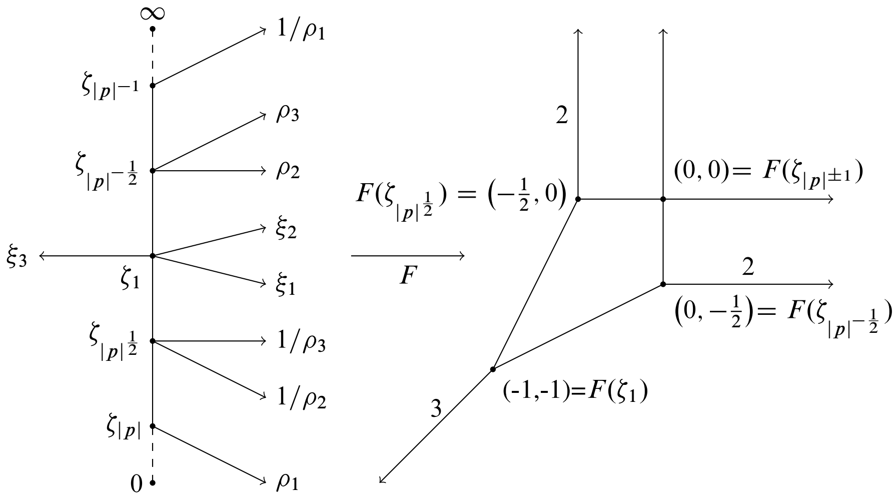

Figure 1. Minimal skeleton $S(W)$ and $\mathrm{Trop}_{x-1,y-1}(U)$.

This will be not enough for our purpose, but we will use the minimal skeleton $S(W)$ of W for the computation and as $S(W)$ also covers $\mathrm{Trop}(U)$, we get a very good picture of the latter. This method to compute tropicalizations is due to [Baker et al. 2013; 2016] and we will refer to these papers for details of the following construction. Skeleta are discussed in [Baker et al. 2013] and we refer to [Baker et al. 2013, Corollary 4.23] for existence and uniqueness of the minimal skeleton $S(W)$ of the smooth curve W. We recall that the skeleton $S(W)$ has a canonical retraction $\tau\colon (\mathbb{P}^1_K)^{\mathrm{an}} \to S(W)$ and hence $S(W)$ is a compact subset of $(\mathbb{P}^1_K)^{\mathrm{an}}$. Similarly as in the examples in [Baker et al. 2016, Section 2], we describe the minimal skeleton $S(W)$ and the tropicalization $\mathrm{Trop}_{x-1,y-1}(U) := \mathrm{trop}_{x-1,y-1}(U^{\mathrm{an}})$ in Figure 1[1]. Using [Gubler et al. 2016, Section 5], there is a map $F : S(W) \to \mathrm{Trop}_{x-1,y-1}(U)$ with $F \circ \tau = \mathrm{trop}_{x-1,y-1} \circ \varphi^{\mathrm{an}}$ such that F maps each segment (resp. leaf) of $S(W)$ by an integral $\mathbb{Q}$-affine map onto a segment (resp. leaf) of $\mathrm{Trop}_{x-1,y-1}(U)$. One computes easily that these affine maps are all integral $\mathbb{Q}$-affine isomorphisms. The polyhedral set $\mathrm{Trop}_{x-1,y-1}(U)$ carries a natural structure of a tropical cycle [Gubler 2013, Theorem 13.11]. All weights are one if not indicated otherwise in Figure 1. For $r > 0$, let $\zeta_r \in (\mathbb{P}^1_K)^{\mathrm{an}}$ be the supremum norm on the closed ball $\{|u| \leq r\}$, where u denotes our distinguished affine coordinate on $\mathbb{P}^1_K$.

Let

$$\widetilde{\Omega} := \left\{(x, y) \in \mathbb{R}^2 \mid x > -\tfrac{1}{2},\ y > -\tfrac{1}{2}\right\}$$

and $\Omega := \widetilde{\Omega} \cap \mathrm{Trop}_{x-1,y-1}(U)$. Let $H\colon N_{U,\mathbb{R}} \to \mathbb{R}^2$ be the canonical affine map with N_U the dual of M_U. Moreover, we have a canonical surjective map G from

[1]Thanks to Christian Vilsmeier for drawing the figure.

the minimal skeleton $S(W)$ onto the canonical tropicalization $\mathrm{Trop}(U)$ which is affine on every segment and every leaf of the minimal skeleton and such that $\mathrm{trop}_U \circ \varphi^{\mathrm{an}} = G \circ \tau$ for the canonical retraction τ onto the skeleton $S(W)$ (see [Gubler et al. 2016, Section 5]). Using our description of $\mathcal{O}(U)^\times$, we deduce that

$$G(\zeta_{|p|^{-1}}) = G \circ \tau(\infty) = \mathrm{trop}_U \circ \varphi^{\mathrm{an}}(\infty) \quad \text{and} \quad G(\zeta_{|p|}) = G \circ \tau(0) = \mathrm{trop}_U \circ \varphi^{\mathrm{an}}(0)$$

are equal as the right-hand sides are given in terms of units on U.

Using the fact that $F = H \circ G$, we conclude that the fibre of the surjective map $H\colon \mathrm{Trop}(U) \to \mathrm{trop}_{x-1,y-1}(U)$ over $(0,0)$ is one single point and that H maps $\Omega' := H^{-1}(\widetilde{\Omega}) \cap \mathrm{Trop}(U)$ homeomorphically and isometrically with respect to lattice length onto Ω. We express this fact by saying that Ω' is unimodular to Ω. This is all we need in the following.

Now we consider the tropical chart (V, φ_U) around the ordinary double point $0 = (0,0)$ of U, where $V := \mathrm{trop}_U^{-1}(\Omega)$. We consider the unique function $\tilde{\phi}$ on $\mathbb{R}^2$ which is linear on each quadrant with $\tilde{\phi}(1,0) = 1$, $\tilde{\phi}(0,1) = -1$ and which is zero in the third quadrant. Let ϕ be the restriction of $\tilde{\phi}$ to Ω. Let $\phi' := \phi \circ H$ as a real function on Ω'. It follows from the tropical Poincaré–Lelong formula in Theorem 0.1 that $d'd''[\phi]$ is the supercurrent on Ω given by $\delta_{\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)}$, where $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ is the corner locus of ϕ. Similarly, $d'd''[\phi'] = \delta_{\phi' \cdot \mathrm{Trop}(U)}$. It is clear that $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ is zero on $\Omega \setminus \{(0,0)\}$ as ϕ is linear there. By definition, the multiplicity of $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ in $(0,0)$ is the sum of the four outgoing slopes, which is zero as well. We conclude that the corner locus $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ is zero. Since we have shown that Ω' is unimodular to Ω, we conclude that the corner locus $\phi' \cdot \mathrm{Trop}(U)$ is zero on Ω' as well.

We note that the corner locus $\tilde{\phi}' \cdot N_{U,\mathbb{R}}$ of the function $\tilde{\phi}' := \tilde{\phi} \circ H$ on $N_{U,\mathbb{R}}$ induces a δ-preform $\delta_{\tilde{\phi}' \cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$ which represents a δ-form α on the tropical chart (V, φ_U). We have $\alpha \in \mathrm{AZ}^{1,1}(V, \varphi_U) \subset P^{1,1}(V, \varphi_U)$. It follows from Proposition 1.14 that

$$\alpha|_\Omega = \delta_{\tilde{\phi}' \cdot N_{U,\mathbb{R}}} \wedge \delta_{\mathrm{Trop}(U)} = \delta_{\phi' \cdot \mathrm{Trop}(U)} = 0. \tag{4.22.1}$$

Now let us consider the open ball $B := \{|u| < |p|^{\frac{1}{2}}\}$ in $\mathbb{P}^1_K$. It is clear that $V'' := B \setminus \{\rho_1\}$ is mapped by F to $\Omega \cap \{x = 0\}$. The coordinate $w := u - \rho_1$ on $U'' := \mathbb{P}^1_K \setminus \{\rho_1, \infty\}$ induces an isomorphism $\varphi_{U''}\colon U'' \to \mathbb{G}_m$. Note that $(V'', \varphi_{U''})$ is a tropical chart. Indeed, we have $V'' = \mathrm{trop}_{U''}^{-1}(\Omega'')$ and $\mathrm{trop}_{U''}(V'') = \Omega''$ for $\Omega'' := \left(\frac{1}{2}, \infty\right)$. The tropical charts $(V'', \varphi_{U''})$ and (V, φ_U) are compatible with respect to the morphism φ and hence there is a canonical affine map $E : \mathbb{R} \to N_{U,\mathbb{R}}$ with $\mathrm{trop}_U \circ \varphi^{\mathrm{an}} = E \circ \mathrm{trop}_w$. We have

$$\varphi^*(\alpha)|_{\Omega''} = E^*(\delta_{\tilde{\phi}' \cdot N_{U,\mathbb{R}}})|_{\Omega''} = \delta_{E^*(\tilde{\phi}') \cdot N_{U'',\mathbb{R}}}|_{\Omega''}.$$

It is clear that $\phi'' := E^*(\tilde{\phi}')$ is a piecewise linear function on Ω'' which is identically zero on $\left(\frac{1}{2}, 1\right]$ and which has slope 1 on $[1, \infty)$. It follows that $\varphi^*(\alpha)|_{\Omega''} = \delta_1$. We conclude that $\alpha \in \mathrm{AZ}^{1,1}(V, \varphi_U)$ is an example with $\alpha|_\Omega = 0$, but $\alpha \neq 0$ as $\varphi^*(\alpha)|_{\Omega''} \neq 0$.

5. Integration of delta-forms

We keep the notation and the hypotheses from the previous section. Our goal is to introduce integration of generalized δ-forms of top degree with compact support. We proceed as in [Gubler 2016, 5.13]. A crucial ingredient in our definition of the integral is Lemma 5.5 which shows that the support of a generalized δ-form of high degree is always concentrated in points of high local dimensions. This allows us to compute the integral with a single chart of integration. We obtain a well defined integral for generalized δ-forms which satisfies a projection formula and the theorem of Stokes.

5.1. Let (V, φ_U) be a tropical chart of X. As before we write $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$ for some open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ and $\Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U)$. Recall $n := \dim(X)$.

(i) An element α_U in $P(V, \varphi_U)$ is represented by a δ-preform $\tilde{\alpha}_U$ in $P(\widetilde{\Omega})$ and determines a δ-preform

$$\alpha_U|_\Omega = \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)} \in P(\Omega) \subseteq D(\Omega)$$

on Ω as in (4.5.1) which does neither depend on the choice of $\tilde{\alpha}_U$ nor on the choice of $\widetilde{\Omega}$. Often, it is convenient to use the notation $\alpha_U|_{\mathrm{Trop}(U)}$ for $\alpha_U|_\Omega$.

(ii) Given α_U in $P^{n,n}(V, \varphi_U)$ and an integral $\mathbb{R}$-affine polyhedral subset P of Ω such that $P \cap \mathrm{supp}(\alpha_U|_\Omega)$ is compact, we define

$$\int_P \alpha_U := \int_P \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)},$$

where the right-hand side is defined as in Remark 3.5. As usual, we extend the integral by 0 to the α_U of other bidegrees.

(iii) If α_U in $P^{n,n}(V, \varphi_U)$ and if the support of $\alpha_U|_\Omega$ is compact, then we can consider $\alpha_U|_\Omega$ as a δ-preform on $\mathrm{Trop}(U)$ with compact support and we write

$$\int_\Omega \alpha_U := \int_{|\mathrm{Trop}(U)|} \alpha_U|_\Omega,$$

again using Remark 3.5.

(iv) Given α_U in $P^{n-1,n}(V, \varphi_U)$ or $P^{n,n-1}(V, \varphi_U)$ and an integral $\mathbb{R}$-affine polyhedral subset P of Ω such that $P \cap \mathrm{supp}(\alpha_U|_\Omega)$ is compact, we define

$$\int_{\partial P} \alpha_U := \int_{\partial P} \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)},$$

where the right-hand side is defined in Remark 3.5. We extend the boundary integral by 0 to the α_U of other bidegrees.

5.2. In the next result, we look at functoriality of the above integrals with respect to a morphism $f : X' \to X$ of algebraic varieties over K. Let (V, φ_U) be a tropical chart of X. Let U' be a very affine open subset of X' with $f(U') \subseteq U$. Recall that there is a canonical integral Γ-affine morphism $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ such that $\mathrm{trop}_U = F \circ \mathrm{trop}_{U'}$. Letting $V' := (f^{\mathrm{an}})^{-1}(V) \cap (U')^{\mathrm{an}}$, we deduce easily that $(V', \varphi_{U'})$ is a tropical chart of X' which is compatible with the tropical chart (V, φ_U). Let P be an integral $\mathbb{R}$-affine polyhedral subset of $\Omega := \mathrm{trop}_U(V)$ and let $Q := F^{-1}(P) \cap \mathrm{Trop}(U')$. We consider $\alpha_U \in P(V, \varphi_U)$ and its pull-back $f^*(\alpha_U) \in P(V', \varphi_{U'})$ (see Remark 4.5). In the following, we will use the degree of a morphism as introduced in 4.3.

Proposition 5.3. *Under the hypothesis of 5.2 and with $n := \dim(X)$, we assume additionally that $Q \cap \mathrm{supp}(f^*(\alpha_U)|_{\mathrm{Trop}(U')})$ is compact. Then the following properties hold*:

(i) *The set $P \cap \mathrm{supp}(\alpha_U|_{\mathrm{Trop}(U)})$ is compact.*

(ii) *If α_U is of bidegree (n, n), then*

$$\deg(f) \cdot \int_P \alpha_U = \int_Q f^*(\alpha_U). \tag{5.3.1}$$

(iii) *If α_U is of bidegree $(n-1, n)$ or $(n, n-1)$, then*

$$\deg(f) \cdot \int_{\partial P} \alpha_U = \int_{\partial Q} f^*(\alpha_U). \tag{5.3.2}$$

Proof. We choose an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U)$. We write $V' = \mathrm{trop}_U^{-1}(\widetilde{\Omega}')$ for some open subset $\widetilde{\Omega}'$ of $N'_{\mathbb{R}}$. Replacing $\widetilde{\Omega}'$ by $\widetilde{\Omega}' \cap F^{-1}(\widetilde{\Omega})$, we may assume that $\widetilde{\Omega}'$ is contained in $F^{-1}(\widetilde{\Omega})$. We write $\Omega' = \widetilde{\Omega}' \cap \mathrm{Trop}(U')$. If $\alpha_U \in P(V, \varphi_U)$ is represented by some element $\tilde{\alpha}_U \in P(\widetilde{\Omega})$, then $f^*(\alpha_U)$ is represented by the element $F^*(\tilde{\alpha}_U)$ in $P(\widetilde{\Omega}')$. We obtain from (4.3.1) and (2.14.1) that $P \cap \mathrm{supp}(\alpha_U|_{\mathrm{Trop}(U)})$ is compact. This proves (i).

If $\alpha_U \in P^{n,n}(V, \varphi_U)$, then we obtain

$$\deg(f) \int_P \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)} = \int_{F^{-1}(P)} F^* \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U')} \tag{5.3.3}$$

if we combine (4.3.1) with the projection formula (2.14.1). By definition, (5.3.1) is a direct consequence of (5.3.3). Equation (5.3.2) is derived in the same way from (4.3.1) and (2.14.2) □

Let W denote an open subset of X^{an}. Note that a generalized δ-form on W is locally given by elements of $P(V, \varphi_U)$ for tropical charts (V, φ_U). The following corollary will be crucial for the definition of the integral of generalized δ-forms.

Corollary 5.4. *We consider very affine open subsets $U' \subseteq U$ in X. Let $\alpha = \operatorname{trop}_U^*(\alpha_U)$ for some $\alpha_U \in P(U^{\mathrm{an}}, \varphi_U)$. Then there is a unique $\alpha_{U'} \in P((U')^{\mathrm{an}}, \varphi_{U'})$ with $\alpha|_{(U')^{\mathrm{an}}} = \operatorname{trop}_{U'}^*(\alpha_{U'})$. If α is of bidegree (n,n) and has compact support in $(U')^{\mathrm{an}}$, then we have*

$$\int_{|\operatorname{Trop}(U)|} \alpha_U = \int_{|\operatorname{Trop}(U')|} \alpha_{U'}.$$

Proof. Let $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ be the canonical affine map with $\operatorname{trop}_U = F \circ \operatorname{trop}_{U'}$ on $(U')^{\mathrm{an}}$. Then $\alpha|_{(U')^{\mathrm{an}}}$ is given by $\alpha_{U'} := F^*(\alpha_U) \in P((U')^{\mathrm{an}}, \varphi_{U'})$. This proves existence, and uniqueness follows from Proposition 4.18. To prove the last claim, we use (5.3.1) for $f = \mathrm{id}$, $P = |\operatorname{Trop}(U)|$ and $Q = |\operatorname{Trop}(U')|$. □

In the following result, we need the local invariant $d(x)$ for $x \in X^{\mathrm{an}}$ [Gubler 2016, 4.2]. This invariant was introduced in [Berkovich 1990, Chapter 9] and was extensively studied in [Ducros 2012]. We note that $d(x) \le m$ if x belongs to a Zariski closed subset of dimension m [Berkovich 1990, Proposition 9.1.3].

Lemma 5.5. *Let W be an open subset of X^{an} and let $\alpha \in P^{p,q}(W)$. If $x \in W$ satisfies $d(x) < \max(p,q)$, then $x \notin \operatorname{supp}(\alpha)$.*

Proof. The proof relies on a result of Ducros [2012, théorème 3.4] which says roughly that in a sufficiently small analytic neighbourhood of x, the dimension of the tropical variety is bounded by $d(x)$. The details are as follows. We choose a tropical chart (V, φ_U) around x such that α is induced by a δ-preform $\sum_{i \in I} \alpha_i \wedge \delta_{C_i}$ on $N_{U,\mathbb{R}}$. By linearity, we may assume that α is induced by $\alpha_1 \wedge \delta_{C_1}$ for a superform α_1 in $A^{p',q'}(N_{U,\mathbb{R}})$ and a tropical cycle C_1 of codimension $c := p - p' = q - q' \ge 0$ in $N_{U,\mathbb{R}}$. By definition of a tropical chart, there is an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ such that $V = \operatorname{trop}_U^{-1}(\widetilde{\Omega})$. By the mentioned result of Ducros (see also [Gubler 2016, Proposition 4.14]), there is a compact neighbourhood V_x of x in V such that $\operatorname{trop}_U(V_x)$ is a polyhedral subset of $N_{U,\mathbb{R}}$ with

$$\dim(\operatorname{trop}_U(V_x)) \le d(x) < \max(p,q). \tag{5.5.1}$$

We will show that $\alpha|_{V_x} = 0$. Let $f : X' \to X$ be a morphism of algebraic varieties over K and $(V', \varphi_{U'})$ a tropical chart of X' with $f^{\mathrm{an}}(V') \subseteq V_x$. By definition, we have $V' = \operatorname{trop}_{U'}^{-1}(\Omega')$ for the open subset $\Omega' := \operatorname{trop}_{U'}(V')$ of $\operatorname{Trop}(U')$. In this situation, we get a commutative diagram

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[d, "f^{\mathrm{an}}|_{V'}"] & (U')^{\mathrm{an}} \arrow[r, "\varphi_{U'}"] \arrow[d, "f^{\mathrm{an}}|_{(U')^{\mathrm{an}}}"] & T_{U'} \arrow[r] \arrow[d] & N_{U',\mathbb{R}} \arrow[d, "F"] \\ V \arrow[r, hook] & U^{\mathrm{an}} \arrow[r, "\varphi_U"] & T_U \arrow[r] & N_{U,\mathbb{R}} \end{tikzcd} \tag{5.5.2}$$

as before. To prove the claim, it is enough to show that

$$f^*(\alpha)|_{\Omega'} := F^*(\alpha_1) \wedge \delta_{F^*(C_1)}|_{\Omega'} = 0,$$

or equivalently

$$F^*(\alpha_1) \wedge \delta_{C'} = 0 \in D(\Omega') \tag{5.5.3}$$

for the tropical cycle $C' := F^*(C_1) \cdot \mathrm{Trop}(U')$ of codimension c in $\mathrm{Trop}(U')$. We note that $\Omega' = \mathrm{trop}_{U'}(V') \subseteq F^{-1}(\mathrm{trop}_U(V_x))$. Let Δ' be a maximal polyhedron from C' with $\Delta' \cap \Omega' \neq \varnothing$. Then $F(\Delta' \cap \Omega') \subseteq \mathrm{trop}_U(V_x)$ and hence

$$F^*(\alpha_1)|_{\Delta' \cap \Omega'} = (F|_{\Delta' \cap \Omega'})^*(\alpha_1|_{F(\Delta') \cap \mathrm{trop}_U(V_x)}). \tag{5.5.4}$$

We will show below that

$$\mathrm{codim}\big(F(\Delta' \cap \Omega'), \mathrm{trop}_U(V_x)\big) \geq c. \tag{5.5.5}$$

Then (5.5.3) follows from (5.5.4) by using (5.5.5) and (5.5.1). This proves $x \notin \mathrm{supp}(\alpha)$.

It remains to prove (5.5.5). By definition of the stable tropical intersection product in Remark 1.4(ii), there are maximal polyhedra Δ'_0 and Δ'_1 of $\mathrm{Trop}(U')$ and $F^*(C_1)$, respectively, such that $\Delta' = \Delta'_0 \cap \Delta'_1$. Moreover, $N_{\Delta'_0,\mathbb{R}}$ and $N_{\Delta'_1,\mathbb{R}}$ intersect transversely in $N_{U',\mathbb{R}}$ which means that

$$N_{\Delta'_0,\mathbb{R}} + N_{\Delta'_1,\mathbb{R}} = N_{U',\mathbb{R}}. \tag{5.5.6}$$

Similarly, the definition of pull-back of tropical cycles in Remark 1.4(v) shows that there is a maximal polyhedron Δ_1 of C_1 with $F(\Delta'_1) \subseteq \Delta_1$ and such that

$$N_{\Delta_1,\mathbb{R}} + \mathbb{L}_F(N_{U',\mathbb{R}}) = N_{U,\mathbb{R}}. \tag{5.5.7}$$

It follows from (5.5.6) and (5.5.7) that $\mathbb{L}_F(N_{\Delta'_0,\mathbb{R}})$ intersects $N_{\Delta_1,\mathbb{R}}$ transversely in $N_{U,\mathbb{R}}$. Since the codimension is decreasing under a surjective linear map, we easily get

$$\mathbb{L}_F(N_{\Delta',\mathbb{R}}) = \mathbb{L}_F(N_{\Delta'_0,\mathbb{R}} \cap N_{\Delta'_1,\mathbb{R}}) = \mathbb{L}_F(N_{\Delta'_0,\mathbb{R}}) \cap N_{\Delta_1,\mathbb{R}}$$

and hence

$$\mathrm{codim}\big(F(\Delta' \cap \Omega'), F(\Delta'_0 \cap \Omega')\big) = \mathrm{codim}\big(\mathbb{L}_F(N_{\Delta',\mathbb{R}}), \mathbb{L}_F(N_{\Delta'_0,\mathbb{R}})\big) = c$$

by transversality. Using $F(\Omega') \subseteq \mathrm{trop}_U(V_x)$, this proves (5.5.5). □

Corollary 5.6. *Let W be an open subset of X^{an} and let U be an open subset of X. If $\alpha \in P^{p,q}(W)$ with $\dim(X \setminus U) < \max(p, q)$, then $\mathrm{supp}(\alpha) \subseteq W \cap U^{\mathrm{an}}$.*

Proof. If $x \in W \setminus U^{\mathrm{an}}$, then the assumptions yield $d(x) \leq \dim(X \setminus U) < \max(p, q)$ and hence the claim follows from Lemma 5.5. □

Proposition 5.7. *Let $\alpha \in P^{p,q}(X^{\mathrm{an}})$ with compact support in the open subset W of X^{an}.*

(a) *There is a nonempty tropical chart (V, φ_U) with* $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V \subseteq U^{\mathrm{an}} \cap W$ *and $\alpha_U \in P^{p,q}(U^{\mathrm{an}}, \varphi_U)$ such that $\alpha = \mathrm{trop}_U^*(\alpha_U)$ on U^{an}.*

(b) *Given U, the element α_U in* (a) *is unique.*

(c) *If α is a δ-form, then we may choose $\alpha_U \in \mathrm{AZ}^{p,q}(U^{\mathrm{an}}, \varphi_U)$.*

(d) *If* $\max(p, q) = \dim(X)$, *then any nonempty very affine open subset U of X with $\alpha|_{U^{\mathrm{an}}} = \mathrm{trop}_U^*(\alpha_U)$ for some $\alpha_U \in P^{p,q}(U^{\mathrm{an}}, \varphi_U)$ satisfies automatically* $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$. *Moreover, α_U has always compact support in* $\mathrm{Trop}(U)$.

Explicitly, if $\mathrm{supp}(\alpha)$ *is covered by nonempty tropical charts $(V_i, \varphi_{U_i})_{i=1,\dots,s}$ in W and if α is given on V_i by $\alpha_i \in P^{p,q}(V_i, \varphi_{U_i})$, then any nonempty very affine open subset U of $U_1 \cap \dots \cap U_s$ and $V = (V_1 \cup \dots \cup V_s) \cap U^{\mathrm{an}}$ fit in (a).*

Proof. Since the support of α is a compact subset of W, it is covered by tropical charts $(V_i, \varphi_{U_i})_{i=1,\dots,s}$ describing α as above. Compactness again shows that for any $i = 1, \dots, s$, there is a relatively compact open subset Ω_i' of Ω_i with corresponding open subset $V_i' := \mathrm{trop}_{U_i}^{-1}(\Omega_i')$ of V_i such that $\mathrm{supp}(\alpha) \subseteq V_1' \cup \dots \cup V_s'$. Let us consider a nonempty very affine open subset U of $U_1 \cap \dots \cap U_s$ of X and the open subsets

$$V' := U^{\mathrm{an}} \cap \bigcup_{i=1}^{s} V_i' \subseteq V := U^{\mathrm{an}} \cap \bigcup_{i=1}^{s} V_i$$

of $W \cap U^{\mathrm{an}}$. We have to show that V and U satisfy (a). Let $F_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical integral Γ-affine map induced by the inclusion $U \subseteq U_i$ (see 4.3). Then the open subsets

$$\Omega' := \mathrm{Trop}(U) \cap \bigcup_{i=1}^{s} F_i^{-1}(\Omega_i') \subseteq \Omega := \mathrm{Trop}(U) \cap \bigcup_{i=1}^{s} F_i^{-1}(\Omega_i)$$

of $\mathrm{Trop}(U)$ satisfy $V = \mathrm{trop}_U^{-1}(\Omega)$ and $V' = \mathrm{trop}_U^{-1}(\Omega')$ which means that (V', φ_U) and (V, φ_U) are compatible tropical charts of X contained in W. Note that the tropical chart $(V_i \cap U^{\mathrm{an}}, \varphi_U)$ is compatible with (V_i, φ_{U_i}) and hence α is given on $(V_i \cap U^{\mathrm{an}}, \varphi_U)$ by $\alpha_i' = \alpha_i|_{V_i \cap U^{\mathrm{an}}} \in P(V_i \cap U^{\mathrm{an}}, \varphi_U)$. Using that Ω_i' is relatively compact in Ω_i, we deduce that the closure S of Ω' in $\mathrm{Trop}(U)$ is contained in Ω. We set $V'' := \mathrm{trop}_U^{-1}(\mathrm{Trop}(U) \setminus S)$ leading to the tropical chart (V'', φ_U). Since α has compact support in W, we may view α as an element of $P^{p,q}(X^{\mathrm{an}})$. By construction, we have $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V'$. Using that V' and V'' are disjoint, we deduce that α is given on the tropical chart (V'', φ_U) by $0 \in P^{p,q}(V'', \varphi_U)$. We note that the tropical charts $(V_i \cap U^{\mathrm{an}}, \varphi_U)_{i=1,\dots,s}$ and (V', φ_U) cover U^{an} and hence we may apply the glueing from Proposition 4.12 to get the desired $\alpha_U \in P^{p,q}(U^{\mathrm{an}}, \varphi_U)$ from (a).

Uniqueness in (b) follows from Proposition 4.18. If $\alpha \in B_c^{p,q}(X^{\mathrm{an}})$, then we may choose $\alpha_i \in \mathrm{AZ}^{p,q}(V_i, \varphi_{U_i})$ and hence we get (c).

If $\max(p,q) = \dim(X)$, then (d) follows from Corollary 5.6 and Proposition 4.21. $\square$

Definition 5.8. Let W be an open subset of X^{an} and let $\alpha \in P_c^{n,n}(W)$, where $n := \dim(X)$. We may view α as a generalized δ-form on X^{an} with compact support contained in W. A nonempty very affine open subset U of X is called a *very affine chart of integration for* α if $\alpha|_{U^{\mathrm{an}}} = \mathrm{trop}_U^*(\alpha_U)$ for some $\alpha_U \in P^{n,n}(U^{\mathrm{an}}, \varphi_U)$. By Proposition 5.7, a chart of integration exists, and α_U is unique and has compact support in $\mathrm{Trop}(U)$. We define *the integral of* α *over* W by

$$\int_W \alpha := \int_{|\mathrm{Trop}(U)|} \alpha_U,$$

where the right-hand side is defined in 5.1. As usual, we extend the integral by 0 to generalized δ-forms of other bidegrees.

Proposition 5.9. *Let W be an open subset of X^{an} and $\alpha \in P_c^{n,n}(W)$ as above.*

(i) *If* $\mathrm{supp}(\alpha)$ *is covered by finitely many nonempty tropical charts (V_i, φ_{U_i}) such that α is given on any V_i by $\alpha_i \in P^{n,n}(V_i, \varphi_{U_i})$, then $U := \bigcap_i U_i$ is a very affine chart of integration for α.*

(ii) *The definition of the integral $\int_W \alpha$ given in Definition 5.8 does not depend on the choice of the very affine chart of integration for α.*

(iii) *The integral defines a linear map $\int_W \colon P_c^{n,n}(W) \to \mathbb{R}$.*

(iv) *If $f : X' \to X$ is a proper morphism of degree* $\deg(f)$ *then the projection formula*

$$\deg(f) \int_W \alpha = \int_{(f^{\mathrm{an}})^{-1}(W)} f^* \alpha \tag{5.9.1}$$

holds for all $\alpha \in P_c^{n,n}(W)$.

Proof. The explicit description of U in Proposition 5.7 proves (i). We show (ii). Let U be a very affine chart of integration for α. Then every nonempty very affine open subset U' of U is a very affine chart of integration and it is enough to show that U' leads to the same integral. By uniqueness in Proposition 5.7, the pull-back of α_U with respect to the canonical affine map $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ is equal to $\alpha_{U'}$ and the claim follows from Corollary 5.4.

Claim (iii) is a direct consequence of our definitions. To prove (iv), we may assume that $\dim(X') = \dim(X) = n$. We choose a very affine chart of integration U for α and a nonempty very affine open subset U' of X' with $f(U') \subseteq U$. Note that $f^*(\alpha)$ is given on $(U')^{\mathrm{an}}$ by $f^*(\alpha_U) \in P^{n,n}(U', \varphi_{U'})$ constructed in Remark 4.5.

Since f^{an} is proper as well, the support of $f^*(\alpha)$ is compact. We conclude that U' is a very affine chart of integration for $f^*(\alpha)$ and

$$\int_{(f^{\mathrm{an}})^{-1}(W)} f^*\alpha = \int_{|\mathrm{Trop}(U')|} f^*(\alpha_U).$$

The projection formula in (iv) is now a direct consequence of (5.3.1). □

In our setting, we have the following version of the *theorem of Stokes*.

Theorem 5.10. *For $\alpha \in B_c^{2n-1}(X)$ we have*

$$\int_{X^{\mathrm{an}}} d'\alpha = \int_{X^{\mathrm{an}}} d''\alpha = 0.$$

Proof. By Proposition 5.7, there is a nonempty very affine open subset U of X such that $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$ and $\alpha_U \in \mathrm{AZ}_c^{2n-1}(U^{\mathrm{an}}, \varphi_U)$ such that $\alpha|_{U^{\mathrm{an}}} = \mathrm{trop}_U^*(\alpha_U)$. Then U is a chart of integration for $d'\alpha$ and $d''\alpha$ using $d'\alpha_U$ and $d''\alpha_U$ on the tropical side for integration. The claim follows from Stokes' formula for δ-preforms on $\mathrm{Trop}(U)$ (see Proposition 3.6) by observing that boundary integrals $\int_{\partial|\mathrm{Trop}(U)|}$ vanish as $\mathrm{Trop}(U)$ satisfies the balancing condition. □

6. Delta-currents

In this section, we define δ-currents on an open subset W of X^{an} for an n-dimensional algebraic variety X over K. We proceed analogously to the case of manifolds in differential geometry, endowing some specific subspaces of the space $B_c(W)$ of δ-forms with compact support in W with the structure of a locally convex topological vector space. Then we define a δ-current as a linear functional on $B_c(W)$ with continuous restrictions to all these subspaces.

6.1. Let (V, φ_U) be a tropical chart of X with $V \subseteq W$ and let $\Omega := \mathrm{trop}_U(V)$ be as usual. We recall from Definition 4.9 that an element $\beta \in \mathrm{AZ}(V, \varphi_U)$ has the form

$$\beta = \sum_{j \in J} \alpha_j \wedge \omega_j \in P(V, \varphi_U) \tag{6.1.1}$$

for a finite set J, $\alpha_j \in A(\Omega)$ and $\omega_j \in Z(V, \varphi_U)$.

Now we *fix* the family $\omega_J := (\omega_j)_{j \in J}$ and define $\mathrm{AZ}(V, \varphi_U, \omega_J)$ as the subspace of $\mathrm{AZ}(V, \varphi_U)$ given by all elements β with a decomposition (6.1.1) for suitable $\alpha_j \in A(\Omega)$. For every $s \in \mathbb{N}$ and every compact subset C of Ω, we have the usual seminorms $p_{C,s}$ on $A(\Omega)$ measuring uniform convergence on C of the derivatives of the coefficients of the superforms up to order s (see for example [Dieudonné 1972, (17.3.1)]). We get seminorms p_{C,s,ω_J} on $\mathrm{AZ}(V, \varphi_U, \omega_J)$ by defining

$$p_{C,s,\omega_J}(\beta) := \inf\Big\{ \max_{j \in J} p_{C,s}(\alpha_j) \;\Big|\; \beta = \sum_{j \in J} \alpha_j \wedge \omega_j,\ \alpha_j \in A(\Omega) \Big\}.$$

Letting $s \in \mathbb{N}$ and the compact subset C of W vary, we get a structure of a locally convex topological vector space on $\mathrm{AZ}(V, \varphi_U, \omega_J)$.

6.2. A δ-form β on W is given by a covering $(V_i, \varphi_{U_i})_{i \in I}$ of W by tropical charts and by $\beta_i \in \mathrm{AZ}(V_i, \varphi_{U_i})$ such that $\beta|_{V_i} = \mathrm{trop}_{U_i}^*(\beta_i)$ for every $i \in I$. Using 6.1, we have a finite tuple ω_{J_i} of elements in $\mathrm{AZ}(V_i, \varphi_{U_i})$ such that $\beta_i \in \mathrm{AZ}(V_i, \varphi_{U_i}, \omega_{J_i})$ for every $i \in I$. Now we fix the covering by tropical charts and all ω_{J_i} and we define $B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ to be the subspace of $B(W)$ given by the elements β such that $\beta|_{V_i} = \mathrm{trop}_{U_i}^*(\beta_i)$ for some $\beta_i \in \mathrm{AZ}(V_i, \varphi_{U_i}, \omega_{J_i})$ and for every $i \in I$. We endow $B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ with the coarsest structure of a locally convex topological vector space such that the canonical linear maps

$$B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I) \to \mathrm{AZ}(V_i, \varphi_{U_i}, \omega_{J_i})$$

are continuous for every $i \in I$. An element $\beta \in B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ given as above is mapped to β_i, which is well defined by Proposition 4.18.

For a compact subset C of W, let $B_C(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ be the subspace of $B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ given by the δ-forms with compact support in C. We endow it with the induced structure of a locally convex topological vector space.

Definition 6.3. A δ-*current* on W is a real linear functional T on $B_c(W)$ such that the restriction of T to $B_C(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ is continuous for every compact subset C of W, for every covering $(V_i, \varphi_{U_i})_{i \in I}$ of W by tropical charts and for every finite tuple ω_{J_i} of elements in $Z(V_i, \varphi_{U_i})$. We denote the space of δ-currents on W by $E(W)$. A δ-current is called *symmetric* (resp. *antisymmetric*) if it vanishes on the subspace of antisymmetric (resp. symmetric) δ-forms in $B_c(W)$.

6.4. Let W be an open subset of X^{an}. Using that $B_c(W) = \bigoplus_{p,q} B_c^{p,q}(W)$ is bigraded, we get $E(W) = \bigoplus_{r,s} E_{r,s}(W)$ as a bigraded $\mathbb{R}$-vector space, where a δ-current in $E_{r,s}(W)$ acts trivially on every $B_c^{p,q}(W)$ with $(p,q) \neq (r,s)$. We set $E^{p,q}(W) := E_{n-p,n-q}(W)$. The definition of δ-currents in 6.3 is local and hence $E^{\cdot,\cdot}$ is a sheaf of bigraded real vector spaces on X^{an}. This follows from standard arguments using partition of unity if W is paracompact, and follows in general from the fact that every compact subset C of W has a paracompact open neighbourhood in W by [Chambert-Loir and Ducros 2012, lemme (2.1.6)]. The argument is similar to that in [Chambert-Loir and Ducros 2012, lemme (4.2.5)] and we leave the details to the reader.

There is a product

$$B^{p,q}(W) \times E^{p',q'}(W) \to E^{p+p',q+q'}(W), \quad (\alpha, T) \mapsto \alpha \wedge T \tag{6.4.1}$$

such that

$$\langle \alpha \wedge T, \beta \rangle = (-1)^{(p+q)(p'+q')} T(\alpha \wedge \beta)$$

for each $\beta \in B_c^{n-p-p',n-q-q'}(W)$.

Proposition 6.5. *Let U be a Zariski open subset of X and let W be an open subset of X^{an}. If $\operatorname{codim}(X \setminus U, X) > \min(p,q)$, then $E^{p,q}(W \cap U^{\mathrm{an}}) = E^{p,q}(W)$.*

Proof. Corollary 5.6 shows that every δ-form on W of bidegree $(n-p, n-q)$ has support in $W \cap U^{\mathrm{an}}$. We conclude that every δ-current T in $E^{p,q}(W \cap U^{\mathrm{an}})$ is a linear functional on $B_c^{n-p,n-q}(W)$. It remains to prove that the restriction of T to $B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ is continuous for every compact subset C of W, for every covering $(V_i, \varphi_{U_i})_{i \in I}$ of W by tropical charts and for every finite tuple ω_{J_i} of elements in $Z(V_i, \varphi_{U_i})$.

We consider the set S of $x \in W$ for which there is $i \in I$ and a compact neighbourhood V_x of x in $W \cap V_i$ with

$$\dim(\operatorname{trop}_{U_i}(V_x)) < \max(n-p, n-q). \tag{6.5.1}$$

Note that $\operatorname{trop}_{U_i}(V_x)$ is a polyhedral subset of $N_{U,\mathbb{R}}$ by [Ducros 2012, théorème 3.2]. Obviously, S is an open subset of W. It follows from the proof of Lemma 5.5 that S is disjoint from the support of any δ-form in $B^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. We conclude that

$$B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I) = B_D^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I) \tag{6.5.2}$$

for the compact subset $D := C \setminus S$ of C. By the proof of Lemma 5.5 again, every $x \in X^{\mathrm{an}} \setminus U^{\mathrm{an}}$ satisfies

$$d(x) \leq \dim(X \setminus U) < \max(n-p, n-q)$$

and has a compact neighbourhood V_x contained in some V_i and satisfying (6.5.1). This proves $D \subseteq W \cap U^{\mathrm{an}}$. Using (6.5.2) and $T \in E^{p,q}(W \cap U^{\mathrm{an}})$, we get the continuity of the restriction of T to $B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. □

Proposition 6.6. *A generalized δ-form $\eta \in P^{p,q}(W)$ determines a δ-current $[\eta] \in E^{p,q}(W)$ such that*

$$\langle [\eta], \beta \rangle = \int_W \eta \wedge \beta$$

for each $\beta \in B_c^{n-p,n-q}(W)$.

Proof. We have to show that the restriction of $[\eta]$ to every subspace

$$B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$$

as in Definition 6.3 is continuous. By passing to a refinement of the covering by tropical charts, we may assume that η is given on V_i by $\eta_i \in P^{p,q}(V_i, \varphi_{U_i})$ for every $i \in I$. Since C is compact, there is a finite subset I_0 of I such that $\bigcup_{i \in I_0} V_i$

covers C. By Proposition 5.9(i), we may use $U := \bigcap_{i\in I_0} U_i$ as a very affine chart of integration for any $\gamma \in B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$.

Similarly to the proof of Proposition 6.5, we consider the set S of $x \in W$ for which there is an $i \in I$ and a compact neighbourhood V_x of x in $W \cap V_i$ with

$$\dim(\mathrm{trop}_{U_i}(V_x)) < n. \tag{6.6.1}$$

It follows again from the proof of Lemma 5.5 that the open subset S of W is disjoint from the support of $\eta \wedge \beta \in P^{n,n}(W)$ for any $\beta \in B^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ and that the compact set $D := C \setminus S$ is contained in $W \cap U^{\mathrm{an}}$.

By definition, $\beta \in B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ is given on V_i by $\beta_i = \sum_{j\in J_i} \alpha_{ij} \wedge \omega_{ij}$ with $\alpha_{ij} \in A(\Omega_i)$ and $\omega_{ij} \in Z(V_i, \varphi_{U_i})$, where $\Omega_i := \mathrm{trop}_{U_i}(V_i)$. For $i \in I_0$, let $F_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical affine map with $\mathrm{trop}_{U_i} = F_i \circ \mathrm{trop}_U$ on U^{an} and let $\Omega_i' := F_i^{-1}(\Omega_i) \cap \mathrm{Trop}(U) = \mathrm{trop}_U(V_i \cap U^{\mathrm{an}})$. The definition of $\int_W \eta \wedge \beta$ uses that $\eta \wedge \beta$ is given on U^{an} by a unique $\gamma_U \in P^{n,n}(U^{\mathrm{an}}, \varphi_U)$ (see Definition 5.8). Moreover, Proposition 5.7 shows that γ_U has compact support in $\bigcup_{i\in I_0} \Omega_i'$ and that γ_U is characterized by the restrictions

$$\gamma_U|_{V_i \cap U^{\mathrm{an}}} = \sum_{j\in J_i} \eta_i \wedge \alpha_{ij} \wedge \omega_{ij}|_{V_i\cap U^{\mathrm{an}}} \in P^{n,n}(V_i, \varphi_{U_i})$$

for every $i \in I_0$. Recall that D is a compact subset of $W \cap U^{\mathrm{an}}$ with $\mathrm{supp}(\gamma) \subseteq D$. By Proposition 4.21, $\mathrm{trop}_U(D)$ is a compact set of $\mathrm{Trop}(U)$ containing the support of γ_U. Then there is an integral $\mathbb{R}$-affine polyhedral subset P of $\mathrm{Trop}(U)$ with $\mathrm{trop}_U(D) \subseteq P$ and hence we have

$$\langle [\eta], \beta\rangle = \int_{X^{\mathrm{an}}} \eta \wedge \beta = \int_{|\mathrm{Trop}(U)|} \gamma_U = \int_P \gamma_U. \tag{6.6.2}$$

We use now that P is independent of the choice of $\beta \in B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. If all the α_{ij} are small with respect to the supremum-norm (of the coefficients), then a partition of unity argument on $\mathrm{Trop}(U)$ shows that (6.6.2) is small, proving the desired continuity. □

Remark 6.7. The maps $P^{p,q}(W) \to E^{p,q}(W)$ induce a map of sheaves $P^{p,q} \to E^{p,q}$, $\alpha \mapsto [\alpha]$ which fits into a commutative diagram

$$\begin{tikzcd} A^{p,q} \arrow[r, hook] \arrow[d, hook, "{[\,\cdot\,]_D}"] & B^{p,q} \arrow[r, hook] \arrow[d, "{[\,\cdot\,]}"] & P^{p,q} \arrow[dl, "{[\,\cdot\,]}"] \\ D^{p,q} & E^{p,q} \arrow[l] & \end{tikzcd} \tag{6.7.1}$$

There is an induced map $P^{p,q}(W) \to D^{p,q}(W)$. For $\beta \in P^{p,q}(W)$, we denote the associated current in $D^{p,q}(W)$ by $[\beta]_D$.

There is no a priori reason that the canonical map from δ-forms to currents or δ-currents is injective. However, we have the following functorial criterion:

Proposition 6.8. *Let W be an open subset of X^{an} and let $\alpha, \beta \in P^{p,q}(W)$. Then $\alpha = \beta$ if and only if $[f^*(\alpha)]_D = [f^*(\beta)]_D \in D^{p,q}(W')$ for all morphisms $f : X' \to X$ from algebraic varieties X' over K and for all open subsets W' of $(X')^{\mathrm{an}}$ with $f(W') \subseteq W$.*

Proof. If $\alpha = \beta$, then all pull-backs and also their associated currents are the same. Conversely, we assume that the associated currents of all pull-backs are the same for α and β. There is an open covering $(V_i)_{i \in I}$ of X^{an} by tropical charts (V_i, φ_{U_i}) such that α, β are given on V_i by $\alpha_i, \beta_i \in P^{p,q}(V_i, \varphi_{U_i})$. Let $f : X' \to X$ be a morphism of varieties over K and let $(V', \varphi_{U'})$ be a tropical chart of X' which is compatible with (V_i, φ_{U_i}). Let Ω' denote the open subset $\mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. It follows from our definitions that $\alpha_i = \beta_i$ in $P(V_i, \varphi_{U_i})$ if we show $f^*(\alpha_i)|_{\Omega'} = f^*(\beta_i)|_{\Omega'} \in D^{p,q}(\Omega')$ for all morphisms f and all charts $(V', \varphi_{U'})$ compatible with (V_i, φ_{U_i}). By assumption, we have $[f^*(\alpha)]_D = [f^*(\beta)]_D$ in $D^{p,q}(V')$. We conclude that $f^*(\alpha_i)|_{\Omega'} = f^*(\beta_i)|_{\Omega'} \in P^{p,q}(\Omega') \subseteq D^{p,q}(\Omega')$ and get $\alpha_i = \beta_i \in P(V_i, \varphi_{U_i})$ proving the claim. □

6.9. As usual, we define the linear differential operators $d' : E^{p,q}(W) \to E^{p+1,q}(W)$ and $d'' : E^{p,q} \to E^{p,q+1}(W)$ by

$$\langle d'T, \beta \rangle := (-1)^{p+q+1} \langle T, d'\beta \rangle, \quad \langle d''T, \beta \rangle := (-1)^{p+q+1} \langle T, d''\beta \rangle.$$

Note that d' and d'' induce continuous linear maps on the locally convex topological vector spaces introduced in 6.2 and hence it is easy to check that d' and d'' are well-defined on δ-currents. Moreover, the natural maps from Remark 6.7 fit into commutative diagrams

$$\begin{array}{ccc} B^{p,q} & \xrightarrow{[\cdot]} & E^{p,q} \\ \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\ B^{p+1,q} & \xrightarrow{[\cdot]} & E^{p+1,q} \end{array} \qquad \begin{array}{ccc} B^{p,q} & \xrightarrow{[\cdot]} & E^{p,q} \\ \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle d''} \\ B^{p,q+1} & \xrightarrow{[\cdot]} & E^{p,q+1} \end{array} \tag{6.9.1}$$

of sheaves. As usual, we define $d := d' + d''$ also on E.

6.10. If $f : X' \to X$ is a proper morphism of algebraic varieties, then we get a *push-forward* $f_* : E_{r,s}(f^{-1}(W)) \to E_{r,s}(W)$ as follows: For $T' \in E_{r,s}(f^{-1}(W))$, the push-forward is the δ-current on W given by

$$\langle f_*(T), \beta \rangle := \langle T, f^*(\beta) \rangle$$

for $\beta \in B_c^{r,s}(W)$. It is easy to see that pull-back of δ-forms induces continuous linear maps between appropriate locally convex topological vector spaces defined in 6.2 and hence the proper push-forward of δ-currents is well defined.

Example 6.11. In Definition 5.8, we introduced $\int_{X^{\mathrm{an}}} \beta$ for $\beta \in P_c^{n,n}(X^{\mathrm{an}})$. Setting $\langle \delta_X, \beta \rangle := \int_{X^{\mathrm{an}}} \beta$, we get the δ-current $\delta_X = [1] \in E^{0,0}(X^{\mathrm{an}})$. We call it the *$\delta$-current of integration* along X. Using linearity in the components and 6.10, we get a δ-current of integration δ_Z for every algebraic cycle Z on X.

Proposition 6.12. *Let $f : X' \to X$ be a proper morphism of algebraic varieties and let Z' be a p-dimensional algebraic cycle on X'. Then we have the equality $f_* \delta_{Z'} = \delta_{f_* Z'}$ in $E_{p,p}(X^{\mathrm{an}})$.*

Proof. This is a direct consequence of the projection formula (5.9.1). □

Proposition 6.13. *Let W be an open subset of X^{an}. We equip the space $C_c(W)$ of continuous functions $f : W \to \mathbb{R}$ with compact support with the supremum norm $|\cdot|_W$ and its subspace $A_c^0(W)$ of smooth functions with compact support with the induced norm. Then for each $\alpha \in P_c^{n,n}(W)$ the map*

$$A_c^0(W) \to \mathbb{R}, \quad f \mapsto \int_W f \cdot \alpha$$

is continuous and extends in a unique way to a continuous map $C_c(W) \to \mathbb{R}$.

Proof. We may assume that α is of codimension l. We observe that the Stone–Weierstraß theorem [Chambert-Loir and Ducros 2012, proposition (3.3.5)] implies that $A_c^0(W)$ is a dense subspace of $C_c(W)$. Consider $f \in A_c^0(W)$ and $\alpha \in P_c^{n,n}(W)$. Our claims are obvious once we have obtained a bound C_α such that the inequality

$$\left| \int_W f \cdot \alpha \right| \leq C_\alpha \cdot |f|_W \tag{6.13.1}$$

holds. We are going to prove this inequality in four steps.

First step: The definition of the bound C_α. We fix a very affine chart of integration U for α which means that there is $\alpha_U \in P_c^{n,n}(U^{\mathrm{an}}, \varphi_U)$ with $\mathrm{trop}_U^*(\alpha_U) = \alpha$ and we set $N := N_U$. Then α_U is represented by a δ-preform $\tilde{\alpha}_U \in P_c^{n,n}(N_{\mathbb{R}})$ of the form

$$\tilde{\alpha}_U = \sum_\sigma \alpha_\sigma \wedge \delta_\sigma \tag{6.13.2}$$

as a polyhedral supercurrent, where σ ranges over $\mathscr{C}^l$ for a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ of N and where $\alpha_\sigma \in A_c^{n-l,n-l}(\sigma)$. The definition of the bound C_α will depend on the choice of U and of the lift $\tilde{\alpha}_U$, but not on the choice of $\mathscr{C}$. The restriction $\alpha_{\sigma\tau}$ of α_σ to an $(n-l)$-dimensional face τ of σ is an element of $A_c^{n-l,n-l}(\tau)$. As this is a superform of top-degree, we have a well-defined compactly supported superform $|\alpha_{\sigma\tau}|$ of degree $(n-l, n-l)$ with continuous coefficient on τ. This single coefficient is independent of the choice of an integral base of $\mathbb{L}_\tau$ and it is given by the absolute value of the coefficient of α.

After passing to a refinement, we may assume that $\mathrm{Trop}(U)$ is given by the tropical cycle $(\mathscr{C}_{\leq n}, m)$. Then we define

$$C_\alpha := \sum_{(\Delta,\sigma)} [N : N_\Delta + N_\sigma] m_\Delta \int_\tau |\alpha_{\sigma\tau}|, \tag{6.13.3}$$

where (Δ, σ) ranges over all elements of $\mathscr{C}_n \times \mathscr{C}^l$ such that $\mathbb{L}_\Delta + \mathbb{L}_\sigma = N_\mathbb{R}$ and such that $\tau := \Delta \cap \sigma$ is $(n-l)$-dimensional. Here, the integral of a superform of top-degree with continuous coefficient is defined as in [Chambert-Loir and Ducros 2012, (1.2.2), (1.4.1)].

Second step: A first estimate for the integral. By definition of a smooth function, there is a covering of W by tropical charts $(V'_j, \varphi_{U'_j})_{j\in J}$ such that $f|_{V'_j} = \mathrm{trop}^*_{U'_j}(\phi'_j)$ for smooth functions ϕ'_j on the open subsets $\Omega'_j = \mathrm{trop}_{U'_j}(V'_j)$ of $\mathrm{Trop}(U'_j)$. Any given compact subset C of W containing the support of α will be covered by $(V'_j)_{j\in J_0}$ for a finite subset J_0 of J. By Proposition 5.9, $U' := U \cap \bigcap_{j\in J_0} U'_j$ is a very affine chart of integration for α and for $f\alpha$. Let $N' := N_{U'}$ and let $F : N'_\mathbb{R} \to N_\mathbb{R}$ be the canonical integral $\mathbb{R}$-affine map. Since the restriction map $\mathscr{O}(U)^\times \to \mathscr{O}(U')^\times$ is injective, it follows that F is surjective. After refining $\mathscr{C}$, there is a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}'$ on $N'_\mathbb{R}$ such that $\mathrm{Trop}(U') = (\mathscr{C}'_{\leq n}, m')$ and such that $\Delta := F(\Delta') \in \mathscr{C}$ for every $\Delta' \in \mathscr{C}'$.

For $V' := \bigcup_{j\in J_0} V_j \cap (U')^{\mathrm{an}}$, note that by Corollary 5.6, $(V', \varphi_{U'})$ is a tropical chart of W containing $C \cap (U')^{\mathrm{an}}$ and the support of α. The pull-backs of the functions ϕ'_j with respect to the canonical affine maps $F_j : N'_\mathbb{R} \to N_{U'_j,\mathbb{R}}$ glue to a well-defined smooth function $f_{U'}$ on $\Omega' := \mathrm{trop}_{U'}(V')$. By definition, we have

$$\int_W f\alpha = \int_{|\mathrm{Trop}(U')|} f_{U'} F^*(\tilde{\alpha}_U)|_{\mathrm{Trop}(U')}. \tag{6.13.4}$$

Using that F is surjective, we deduce from (6.13.2) and (2.12.5) that

$$F^*(\tilde{\alpha}_U) = \sum_{\sigma'} [N : \mathbb{L}_F(N') + N_\sigma] \cdot F^*\alpha_\sigma \wedge \delta_{\sigma'},$$

where σ' ranges over all elements of $(\mathscr{C}')^l$ such that $\sigma := F(\sigma')$ is of codimension l in N. We choose a generic vector $v' \in N'_\mathbb{R}$. It follows from (2.12.3) that

$$F^*(\tilde{\alpha}_U)|_{\mathrm{Trop}(U')} = \sum_{\tau'} \sum_{(\Delta',\sigma')} [N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} F^*\alpha_\sigma \wedge \delta_{\tau'},$$

where τ' ranges over $\mathscr{C}'_{n-l}$ and (Δ', σ') ranges over all pairs in $\mathscr{C}'_n \times (\mathscr{C}')^l$ such that $\tau' = \Delta' \cap \sigma'$ and such that $\Delta' \cap (\sigma' + \varepsilon v') \neq \emptyset$ for all sufficiently small $\varepsilon > 0$. Additionally, we assume that $\sigma := F(\sigma')$ is of codimension l in $N_\mathbb{R}$ as above. For degree reasons, we may restrict the sum to those τ' with $\tau := F(\tau')$ of dimension $n-l$. Note that this is equivalent to restricting our attention to those Δ' with $\Delta := F(\Delta')$

of dimension n. Since α has support in V', the restriction of $F^*\alpha_\sigma$ to σ' has support in $\Omega' \cap \sigma'$. By (6.13.4), we have

$$\int_W f\alpha = \sum_{\tau'} \sum_{(\Delta',\sigma')} [N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} \int_{\tau'} f_{U'} F^*\alpha_\sigma .$$

We deduce the following bound:

$$\left| \int_W f\alpha \right| \\ \leq |f|_W \sum_{\tau'} \sum_{(\Delta',\sigma')} [N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} \int_{\tau'} |F^*\alpha_{\sigma\tau}|. \quad (6.13.5)$$

The transformation formula shows

$$\int_{\tau'} |F^*\alpha_{\sigma\tau}| = [N_\tau : \mathbb{L}_F(N'_{\tau'})] \int_\tau |\alpha_{\sigma\tau}|$$

and hence the sum in (6.13.5) is equal to

$$\sum_{\tau'} \sum_{(\Delta',\sigma')} [N_\tau : \mathbb{L}_F(N'_{\tau'})][N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} \int_\tau |\alpha_{\sigma\tau}|. \quad (6.13.6)$$

Third step: The following basic lattice index identity holds:

$$[N_\tau : \mathbb{L}_F(N'_{\tau'})][N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] \\ = [N : N_\Delta + N_\sigma][N_\Delta : \mathbb{L}_F(N_{\Delta'})]. \quad (6.13.7)$$

In the basic lattice index identity (6.13.7), (Δ', σ') is a pair in $\mathscr{C}'_n \times (\mathscr{C}')^l$ such that $\Delta' \cap (\sigma' + \varepsilon v') \neq \varnothing$ for $\varepsilon > 0$ sufficiently small and such that $\sigma := F(\sigma')$ is of codimension l in N. We have also used $\Delta := F(\Delta')$, $\tau' := \Delta' \cap \sigma'$ and $\tau := F(\tau')$. Since F is a surjective integral $\mathbb{R}$-affine map, all lattice indices in the claim of the third step are finite. Setting $P' := N'_{\Delta'}$ and $Q := N_\sigma$, the basic lattice identity (6.13.7) follows from the projection formula for lattices in Lemma 6.14 below.

Fourth step: The desired inequality (6.13.1) *holds.* To prove (6.13.1), we note that $v := F(v')$ is a generic vector for $\mathscr{C}$. We have $\tau = \Delta \cap \sigma$ and $\Delta \cap (\sigma + \varepsilon v) \neq \varnothing$. The basic lattice index identity (6.13.7) yields that the sum in (6.13.6) is equal to

$$\sum_{\tau'} \sum_{(\Delta',\sigma')} [N : N_\Delta + N_\sigma][N_\Delta : \mathbb{L}_F(N'_{\Delta'})] m_{\Delta'} \int_\tau |\alpha_{\sigma\tau}|. \quad (6.13.8)$$

The Sturmfels–Tevelev multiplicity formula (4.3.1) gives

$$\sum_{\Delta'} [N_\Delta : \mathbb{L}_F(N'_{\Delta'})] m_{\Delta'} = m_\Delta ,$$

where Δ' ranges over all elements of $\mathscr{C}'_n$ mapping onto a given $\Delta \in \mathscr{C}_n$. Using this, one can show that (6.13.8) is equal to

$$\sum_{\tau} \sum_{(\Delta,\sigma)} [N : N_\Delta + N_\sigma] m_\Delta \int_\tau |\alpha_{\sigma\tau}|, \tag{6.13.9}$$

where the sum is over all pairs $(\Delta, \sigma) \in \mathscr{C}_n \times \mathscr{C}^l$ such that $\Delta \cap (\sigma + \varepsilon v) \neq \varnothing$ and $\tau = \Delta \cap \sigma$. Now (6.13.1) follows from (6.13.3)–(6.13.9). □

The basic lattice index identity (6.13.7) is a special case of the following *projection formula for lattices.* Note that it is stronger than the projection formula for tropical cycles in Proposition 1.5. The latter would not give the required bound in the fourth step above.

Lemma 6.14. *Let $F : N' \to N$ be a homomorphism of free abelian groups of finite rank and let $P' \subseteq N'$, $Q \subseteq N$ be subgroups. We assume that* $\mathrm{rk}(F(N')) = \mathrm{rk}(N) = \mathrm{rk}(F(P') + Q)$. *Then we have the equality*

$$\begin{aligned}[] [F(P')_{\mathbb{R}} \cap Q : F(P' \cap F^{-1}(Q))][N' : P' + F^{-1}(Q)][N : F(N') + Q] \\ = [N : F(P')_{\mathbb{R}} \cap N + Q][F(P')_{\mathbb{R}} \cap N : F(P')], \end{aligned} \tag{6.14.1}$$

where all involved lattice indices are finite.

Proof. The assumptions show easily that all lattice indices are finite. Using

$$F(P' \cap F^{-1}(Q)) = F(P') \cap Q$$

and the isomorphism theorem $A/(A \cap B) \cong (A + B)/B$ for abelian groups, we get

$$(F(P')_{\mathbb{R}} \cap Q)/F(P' \cap F^{-1}(Q)) \cong (F(P')_{\mathbb{R}} \cap Q + F(P'))/F(P').$$

Similarly, $F(P')_{\mathbb{R}} \cap Q + F(P') = F(P')_{\mathbb{R}} \cap (F(P') + Q)$ yields

$$(F(P')_{\mathbb{R}} \cap N)/(F(P')_{\mathbb{R}} \cap Q + F(P')) \cong (F(P')_{\mathbb{R}} \cap N + Q)/(F(P') + Q).$$

Multiplying (6.14.1) by $[F(P')_{\mathbb{R}} \cap N + Q : F(P') + Q]$, the above two isomorphisms show that the claim is equivalent to

$$[N' : P' + F^{-1}(Q)][N : F(N') + Q] = [N : F(P') + Q]. \tag{6.14.2}$$

Using $F(P') + Q \cap F(N') = (F(P') + Q) \cap F(N')$, we have

$$N'/(P' + F^{-1}(Q)) \cong F(N')/(F(P') + Q \cap F(N')) \cong (F(N') + Q)/(F(P') + Q)$$

and hence (6.14.2) holds. This proves the claim. □

We recall that on a locally compact Hausdorff space Y, the Riesz representation theorem gives a bijective correspondence between positive (resp. signed) Radon measures on Y and positive (resp. bounded) linear functionals on the space of continuous real functions with compact support on Y, endowed with the supremum norm.

Corollary 6.15. *Let W be an open subset of X^{an}. For each $\alpha \in P_c^{n,n}(W)$ there is a unique signed Radon measure μ_α on W such that*

$$\int_W f \cdot \alpha = \int_W f \, d\mu_\alpha \tag{6.15.1}$$

for all smooth functions f on W with compact support.

Proof. This is a consequence of Proposition 6.13 and Riesz's representation theorem. □

Proposition 6.16. *Let W be an open subset of X^{an} and let f be a continuous function on W. Then the map*

$$[f] : B_c^{n,n}(W) \to \mathbb{R}, \quad \alpha \mapsto \int_W f \, d\mu_\alpha$$

is a δ-current in $E^{0,0}(W)$.

Proof. The integral is well defined by Corollary 6.15 using that $\mathrm{supp}(\alpha)$ is compact. Obviously, $[f]$ is a linear map. We have to show that the restriction of $[f]$ to any subspace $B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ as in 6.2 is continuous. For $i \in I$, let $\Omega_i := \mathrm{trop}_{U_i}(V_i)$. For every $x \in C$, there is an $i(x) \in I$ with $x \in V_{i(x)}$. We choose a polytopal neighbourhood $P_{i(x)}$ of $\mathrm{trop}_{U_{i(x)}}(x)$ in $N_{U_{i(x)},\mathbb{R}}$ such that $P_{i(x)} \cap \mathrm{Trop}(U_{i(x)}) \subseteq \Omega_{i(x)}$ and we denote the interior of $P_{i(x)}$ by $Q_{i(x)}$. There is a finite set Y of X such that the open sets $\mathrm{trop}_{U_{i(x)}}^{-1}(Q_{i(x)})$, $x \in Y$, cover the compact set C. By Proposition 5.9, $U := \bigcap_{x \in Y} U_{i(x)}$ works as a very affine chart of integration for every $\alpha \in B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. Then we have $\alpha_U \in \mathrm{AZ}_c^{n,n}(U, \varphi_U)$ with $\mathrm{trop}_U^*(\alpha_U) = \alpha$. By the Sturmfels–Tevelev multiplicity formula (4.3.1) and by degree reasons, one can show that α_U has support in the compact subset

$$C_U = \bigcup_{x \in Y} \bigcup_{\Delta_{i(x)}} \Delta_{i(x)} \cap F_{i(x)}^{-1}(P_{i(x)})$$

of $\mathrm{Trop}(U)$, where $\Delta_{i(x)}$ ranges over all n-dimensional faces of $\mathrm{Trop}(U)$ such that $\Delta_{i(x)} \cap F_{i(x)}^{-1}(P_{i(x)})$ is mapped onto an n-dimensional face of $\mathrm{Trop}(U_{i(x)})$ by the canonical affine map $F_{i(x)} : N_{U,\mathbb{R}} \to N_{U_{i(x)},\mathbb{R}}$. Using the supremum seminorm $|f|_C$ on C, we get

$$\left| \int_W f \, d\mu_\alpha \right| \leq C_\alpha \cdot |f|_C. \tag{6.16.1}$$

To see this, we note first that $\mathrm{supp}(\mu_\alpha) \subseteq C$. There is a smooth function g on W with $0 \leq g \leq 1$, with $g \equiv 1$ on C and with compact support in a sufficiently small neighbourhood of C [Chambert-Loir and Ducros 2012, corollaire (3.3.4)].

Then (6.16.1) follows from applying (6.13.1) to compactly supported smooth approximations of fg using the Stone–Weierstraß theorem in [Chambert-Loir and Ducros 2012, corollaire (3.3.4)].

Now we deduce the claim from the definition of the bound C_α in (6.13.3): We set $i := i(x)$ for $x \in Y$ and we may assume that $\alpha|_{V_i}$ is given by

$$\sum_{j \in J_i} \alpha_{ij} \wedge \omega_{ij}$$

for $\alpha_{ij} \in A(\Omega_i)$ with coefficients of small supremum seminorm $p_{P_i \cap \mathrm{Trop}(U_i),0}(\alpha_{ij})$. Noting that the ω_{ij} are fixed, this yields that every $\alpha_{\sigma\tau}$ in (6.13.3) has small coefficient. Using that only the compact subset $C_U \cap \tau$ matters for integration, we deduce that C_α is small and hence (6.16.1) shows that $[f]$ is continuous. □

7. The Poincaré–Lelong formula and first Chern delta-currents

The Poincaré–Lelong formula in complex analysis is of fundamental importance for Arakelov theory. Chambert-Loir and Ducros [2012, §4.6] have shown that the Poincaré–Lelong formula holds as an identity between currents on Berkovich spaces while Theorem 7.2 below enhances the Poincaré–Lelong formula as an equality of δ-currents. We use the Poincaré–Lelong formula to define the first Chern δ-current of a continuously metrized line bundle.

7.1. Let X be a variety over K of dimension n and let $f \in K(X) \setminus \{0\}$. In Example 6.11, we introduced the δ-current of integration δ_X leading to the definition of the δ-current δ_Z for any cycle Z on X. Using that for the Weil divisor $\mathrm{cyc}(f)$ of f, we get a δ-current $\delta_{\mathrm{cyc}(f)}$ on X^{an}.

On the other hand, the complement U of the support of the principal Cartier divisor $\mathrm{div}(f)$ is an open dense subset of X. By Proposition 6.5, we get the δ-current $[\log|f|] \in E^{0,0}(U^{\mathrm{an}}) = E^{0,0}(X^{\mathrm{an}})$.

Theorem 7.2. *For a nonzero rational function f on X, the Poincaré–Lelong equation*

$$\delta_{\mathrm{cyc}(f)} = d'd''[\log|f|]$$

holds in $E^{1,1}(X^{\mathrm{an}})$.

Proof. The proof is similar to that in [Chambert-Loir and Ducros 2012, §4.6], but it is more on the tropical side as we do not have integrals of δ-forms over analytic subdomains at hand. We will first do some reduction steps and then we will introduce some notation which allows us to use results from [Chambert-Loir and Ducros 2012]. The claim is local on X^{an} and so we may assume that $X = \mathrm{Spec}(A)$ and $f \in A$. The latter induces a morphism $f : X \to \mathbb{A}^1$. We may assume that the morphism is not constant as otherwise all terms are 0. Since A is a domain, the property S_1 of Serre is satisfied.

Let us recall some results from [Chambert-Loir and Ducros 2012] before we start the actual proof. Let W be an affinoid subdomain of X^{an} and let $g : W \to T^{\mathrm{an}}$ be an analytic map for $T = \mathbb{G}_m^r$. Following [Chambert-Loir and Ducros 2012], we call such a map to a torus an *analytic moment map*. We obtain a continuous map

$$g_{\mathrm{trop}} := \mathrm{trop} \circ g : W \to \mathbb{R}^r.$$

We get an analytic map $h := (f, g) : W \to (\mathbb{A}^1)^{\mathrm{an}} \times T^{\mathrm{an}}$. We denote the fibre of W over $t \in (\mathbb{A}^1)^{\mathrm{an}}$ by W_t with respect to the restriction of f to W. If I is an interval in $(0, \infty)$, then $W_I := |f|^{-1}(I) \cap W$. We observe that W_t and W_I carry natural structures of analytic spaces of dimension $n-1$ and n respectively. It follows from general results of Ducros [2012, théorème 3.2] that the sets $g_{\mathrm{trop}}(W_t)$ and $h_{\mathrm{trop}}(W_I)$ are integral $\mathbb{R}$-affine polyhedral sets of dimension less or equal to $n-1$ and n respectively. These polyhedral sets can be equipped with natural integral weights. A construction of these so called tropical weights can be found in [Gubler 2016, §7] or in [Chambert-Loir and Ducros 2012, §3.5] in the language of calibrations. We observe that the tropical weights take the multiplicities of irreducible components into account. The k-skeleton of a polyhedral set P of dimension at most k is by definition the union of all k-dimensional polyhedra contained in P. By [Chambert-Loir and Ducros 2012, proposition (4.6.6)], there exist a real number $r > 0$ and an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ in $\mathbb{R}^r$ of pure dimension $n-1$ with integer weights m such that all polyhedra in $\mathscr{C}$ are polytopes with the following properties:

(a) For every t in the closed ball in $(\mathbb{A}^1)^{\mathrm{an}}$ with centre 0 and radius r, the $(n-1)$-skeleton of $g_{\mathrm{trop}}(W_t)$ endowed with the canonical tropical weights is equal to $(\mathscr{C}, m)$.

(b) For every closed interval $I \subset (0, r]$, the n-skeleton of $h_{\mathrm{trop}}(W_I)$ endowed with the canonical analytic tropical weights is equal to $(-\log(I), 1) \times (\mathscr{C}, m)$ as a product of weighted polyhedral complexes.

In fact, Chambert-Loir and Ducros formulated this crucial result in terms of canonical calibrations instead of analytic tropical weights. We refer to [Gubler 2016, §7] for the definition and translation of these equivalent notions. The analytic space W_0 coincides with the closed analytic subspace of W determined by the effective Cartier divisor $\mathrm{div}(f|_W)$. Using (a) for $t = 0$, we see that $(\mathscr{C}, m)$ is equal to the $(n-1)$-skeleton of $g_{\mathrm{trop}}(\mathrm{div}(f|_W))$ as a weighted polyhedral complex.

Having recalled these preliminary results, we proceed with the proof. Since the δ-currents $\delta_{\mathrm{cyc}(f)}$ and $d'd''[\log|f|]$ are symmetric, it is enough to check the Poincaré–Lelong equation by evaluating at a symmetric $\alpha \in B_c^{n-1,n-1}(X^{\mathrm{an}})$. The δ-form α is given by tropical charts $(V_i, \varphi_{U_i})_{i \in I}$ covering X^{an} and symmetric

$\alpha_i \in \mathrm{AZ}^{n-1,n-1}(V_i, \varphi_{U_i})$. Since α has compact support, there are finitely many i such that the corresponding V_i's cover $\mathrm{supp}(\alpha)$. In the following, we restrict our attention to these finitely many i's and we number them by $i = 1, \ldots, m$.

Let us consider the very affine open subset $U := U_1 \cap \cdots \cap U_m \setminus \mathrm{supp}(\mathrm{div}(f))$ of X. Let $G_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ (resp. $F : N_{U,\mathbb{R}} \to \mathbb{R}$) be the canonical affine map compatible with trop_U and trop_{U_i} (resp. $-\log|f|$). Let x_0 be the coordinate on $\mathbb{R}$ and let $H_i := (F, G_i) : N_{U,\mathbb{R}} \to \mathbb{R} \times N_{U_i,\mathbb{R}}$.

For every $x \in \mathrm{supp}(\alpha)$, there is an $i \in \{1, \ldots, m\}$ such that $x \in V_i$. We choose an integral Γ-affine polytope Δ_i of maximal dimension in $N_{U_i,\mathbb{R}}$ containing $\mathrm{trop}_{U_i}(x)$ in its interior. We may assume that $\Delta_i \cap \mathrm{Trop}(U_i) \subseteq \mathrm{trop}_{U_i}(V_i)$. Then $W_i := \mathrm{trop}_{U_i}^{-1}(\Delta_i)$ is an affinoid subdomain of X^{an} with $x \in \mathrm{Int}(W_i)$. Renumbering the covering and using again compactness of $\mathrm{supp}(\alpha)$, we may assume that i does not depend on x, which means that the interiors of the affinoid subdomains $W_1, \ldots, W_m$ cover $\mathrm{supp}(\alpha)$. Note that $W := \bigcup_{i=1}^m W_i$ is a compact analytic subdomain of X^{an}.

For every nonempty subset E of $\{1, \ldots, m\}$, the set $W_E := \bigcap_{i \in E} W_i$ is affinoid (using that X^{an} is separated). Note that $U_E := \bigcap_{i \in E} U_i$ is very affine and we set $V_E := \bigcap_{i \in E} V_i$. We choose $r > 0$ sufficiently small such that (a) and (b) above hold for every W_E and moment map $g_E := \varphi_{U_E}$. Note that the union of the integral Γ-affine polyhedral sets

$$\mathrm{trop}_U(W_i \cap U^{\mathrm{an}}) = \mathrm{Trop}(U) \cap G_i^{-1}(\Delta_i) \quad (i = 1, \ldots, m) \tag{7.2.1}$$

is equal to $\mathrm{trop}_U(W \cap U^{\mathrm{an}})$. For every subset E of $\{1, \ldots, m\}$, we have a integral Γ-affine polyhedral set

$$\mathrm{trop}_U(W_E \cap U^{\mathrm{an}}) = \mathrm{Trop}(U) \cap \bigcap_{i \in E} G_i^{-1}(\Delta_i) = \bigcap_{i \in E} \mathrm{trop}_U(W_i \cap U^{\mathrm{an}}). \tag{7.2.2}$$

For $V := \bigcup_i V_i \cap U^{\mathrm{an}}$, it follows from Corollary 5.6 that (V, φ_U) is a tropical chart containing the support of $d''\alpha$. The δ-form α is represented on V by $\alpha_U \in \mathrm{AZ}^{n-1,n-1}(V, \varphi_U)$, i.e., $\alpha = \mathrm{trop}_U^*(\alpha_U)$ on V. In fact, we have seen in Proposition 5.7 that α_U extends by 0 to an element of $\mathrm{AZ}^{n-1,n-1}(U^{\mathrm{an}}, \varphi_U)$, but the support of this extension is not necessarily compact. We conclude that U is a very affine chart of integration for $\log|f|\, d'd''\alpha$ and that

$$\langle d'd''[\log|f|], \alpha\rangle = -\int_{\mathrm{trop}_U(V)} F^*(x_0)\, d'd''\alpha_U. \tag{7.2.3}$$

The minus sign comes from the tropical coordinates $\mathrm{trop}_U^*(F^*(x_0)) = -\log|f|$ as remarked above. Corollary 5.6 shows that the support of $d''\alpha$ does not meet $f^{-1}(0)$. Since the support of $d''\alpha$ is compact, there is a positive $s < r$ such that $|f(x)| > s$ for every $x \in \mathrm{supp}(d''\alpha)$. We consider the analytic subdomain of W,

$W(s) := \{x \in W \mid |f(x)| \geq s\}$, and the affinoid subdomains of W_i and W_E,

$$W_i(s) := \{x \in W_i \mid |f(x)| \geq s\} \quad \text{and} \quad W_E(s) := \{x \in W_E \mid |f(x)| \geq s\}.$$

It follows from (7.2.1) and (7.2.2) that their tropicalizations are integral $\mathbb{R}$-affine polyhedral sets such that the union of all

$$\mathrm{trop}_U(W_i(s) \cap U^{\mathrm{an}}) = \mathrm{Trop}(U) \cap G_i^{-1}(\Delta_i) \cap F^{-1}((-\infty, -\log s]) \tag{7.2.4}$$

for $i = 1, \ldots, m$ is equal to $\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})$ and such that

$$\mathrm{trop}_U(W_E(s) \cap U^{\mathrm{an}}) = \mathrm{Trop}(U) \cap \bigcap_{i \in E} G_i^{-1}(\Delta_i) \cap F^{-1}((-\infty, -\log s]). \tag{7.2.5}$$

In the following, we use integrals and boundary integrals of δ-preforms over integral $\mathbb{R}$-affine polyhedral sets as introduced in Definition 2.5, Remark 3.5 and 5.1. By the choice of s, we have $\mathrm{supp}(d''\alpha) \subseteq W(s) \cap U^{\mathrm{an}}$. We conclude that $\mathrm{supp}(d''\alpha_U) \subseteq \mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})$ and hence

$$\int_{\mathrm{trop}_U(V)} F^*(x_0)\, d'd''\alpha_U = \int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} F^*(x_0)\, d'd''\alpha_U. \tag{7.2.6}$$

By Green's formula (see Proposition 3.9) and using $d'd''F^*(x_0) = 0$, the integrals in (7.2.6) are equal to

$$\int_{\partial(\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}}))} \big(F^*(x_0)\, d''\alpha_U - d''(F^*(x_0)) \wedge \alpha_U\big). \tag{7.2.7}$$

By construction and (7.2.1), we have

$$\mathrm{supp}(\alpha_U) \subseteq \mathrm{relint}(\mathrm{trop}_U(W \cap U^{\mathrm{an}})).$$

By the choice of s, it follows that $\mathrm{supp}(d''\alpha_U) \subseteq \mathrm{relint}(\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}}))$. Applying Remark 2.6(iii) to the integral $\mathbb{R}$-affine polyhedral set $\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})$, it follows that

$$\int_{\partial(\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}}))} F^*(x_0)\, d''\alpha_U = 0. \tag{7.2.8}$$

Combining (7.2.3) and (7.2.6)–(7.2.8) with (7.3.1) below, we get

$$\langle d'd''[\log|f|], \alpha \rangle = \langle \delta_{\mathrm{cyc}(f)}, \alpha \rangle, \tag{7.2.9}$$

proving the claim. □

Lemma 7.3. *In the situation of the proof of Theorem 7.2 above, we have*

$$\int_{\partial(\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}}))} d''(F^*(x_0)) \wedge \alpha_U = \langle \delta_{\mathrm{cyc}(f)}, \alpha \rangle. \tag{7.3.1}$$

Proof. For integers $\ell \geq 1$, there are $\varphi_\ell \in C^\infty(\mathbb{R})$ with $0 \leq \varphi_\ell \leq 1$, $\varphi_\ell(t) = 1$ for $t \leq -\log(s) - 1/\ell$ and $\varphi_\ell(t) = 0$ for $t \geq -\log(s) - 1/(2\ell)$. By construction, $\operatorname{supp}(\varphi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U$ is contained in the relative interior of $\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$ and hence

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} \varphi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U = 0 \tag{7.3.2}$$

as above. Setting $\psi_\ell := 1 - \varphi_\ell$, it follows from (7.3.2) that the left-hand side in (7.3.1) is equal to

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} \psi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U. \tag{7.3.3}$$

Now we use the additivity of measures from Remark 2.6(ii). The decomposition (7.2.4) of the polyhedral set $\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$ and Equation (7.2.5) show that (7.3.3) is equal to

$$\sum_{j=1}^{m} (-1)^{j+1} \sum_{|E|=j} \int_{\partial(\operatorname{trop}_U(W_E(s)\cap U^{\mathrm{an}}))} \psi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U. \tag{7.3.4}$$

We fix $i \in E$. Let $G_E : N_{U,\mathbb{R}} \to N_{U_E,\mathbb{R}}$ and $G_{E,i} : N_{U_E,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical affine maps which are compatible with the given moment maps. Let us consider the closed embedding

$$h_E := (f, g_E) = (f, \varphi_{U_E}) : U_E \setminus \operatorname{div}(f) \to \mathbb{G}_m \times T_{U_E}$$

inducing the tropical variety $h_{E,\mathrm{trop}}(U_E \setminus \operatorname{div}(f))$, which we view as a tropical cycle on $\mathbb{R} \times N_{U_E,\mathbb{R}}$. The affine maps $H_E := (F, G_E) : N_{U,\mathbb{R}} \to \mathbb{R} \times N_{U_E,\mathbb{R}}$ (resp. $H_{E,i} := \mathrm{id}_{\mathbb{R}} \times G_{E,i} : \mathbb{R} \times N_{U_E,\mathbb{R}} \to \mathbb{R} \times N_{U_i,\mathbb{R}}$) are compatible with the moment maps φ_U and h_E (resp. h_E and h_i). The Sturmfels–Tevelev multiplicity formula shows that

$$h_{E,\mathrm{trop}}(U_E \setminus \operatorname{div}(f)) = (H_E)_*(\operatorname{Trop}(U)) \tag{7.3.5}$$

(see [Gubler 2016, Proposition 4.11] for the required generalization of (4.3.1)). For $\alpha_E := \alpha_i|_{V_E} \in \mathrm{AZ}^{n-1,n-1}(V_E, \varphi_{U_E})$, we have $\alpha|_{V_E} = \operatorname{trop}_{U_E}^*(\alpha_E)$ and the definition of α_E does not depend on the choice of $i \in E$. In the following, the weighted integral $\mathbb{R}$-affine polyhedral complex $\Sigma_E(s) := h_{E,\mathrm{trop}}(W_E(s))$ in $\mathbb{R} \times N_{U_E,\mathbb{R}}$ plays a crucial role. Note that we have

$$\Sigma_E(s) = h_{E,\mathrm{trop}}(U_E \setminus \operatorname{div}(f)) \cap \bigcap_{i \in E} H_{E,i}^{-1}\big((-\infty, -\log s] \times \Delta_i\big). \tag{7.3.6}$$

Let $P_E : \mathbb{R} \times N_{U_E,\mathbb{R}} \to N_{U_E,\mathbb{R}}$ denote the canonical projection. By definition, the element α_E of $\mathrm{AZ}^{n-1,n-1}(V_E, \varphi_{U_E})$ is represented by a δ-preform $\tilde{\alpha}_E$ on an open subset $\widetilde{\Omega}_E$ of $N_{U_E,\mathbb{R}}$ with $\widetilde{\Omega}_E \cap \operatorname{Trop}(U_E) = \operatorname{trop}_{U_E}(V_E)$. Recall from (4.5.1), that

$\alpha_U|_\Omega = \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)}$ denotes the δ-preform on $\Omega := \mathrm{Trop}_U(V)$ induced by α_U. Using $\alpha_U = G_E^*(\alpha_E)$, we get

$$\alpha_U|_\Omega = G_E^*(\alpha_E)|_\Omega = G_E^*(\tilde{\alpha}_E) \wedge \delta_{\mathrm{Trop}(U)} = H_E^* P_E^*(\tilde{\alpha}_E) \wedge \delta_{\mathrm{Trop}(U)}.$$

We consider the coordinate x_0 on $\mathbb{R}$ also as a function on $\mathbb{R} \times N_{U_E,\mathbb{R}}$. Using $\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}}) = H_E^{-1}(\Sigma_E(s)) \cap \mathrm{Trop}(U)$ and (7.3.5), the projection formula (2.14.2) shows that

$$\begin{aligned}\int_{\partial(\mathrm{trop}_U(W_E(s)\cap U^{\mathrm{an}}))} & \psi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U \\ &= \int_{\partial(\mathrm{trop}_U(W_E(s)\cap U^{\mathrm{an}}))} H_E^*(\psi_\ell(x_0) d'' x_0) \wedge H_E^* P_E^*(\tilde{\alpha}_E) \wedge \delta_{\mathrm{Trop}(U)} \\ &= \int_{\partial(\Sigma_E(s))} \psi_\ell(x_0)\, d'' x_0 \wedge P_E^*(\tilde{\alpha}_E) \wedge \delta_{h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))}. \qquad (7.3.7)\end{aligned}$$

By construction of the functions φ_ℓ, we have

$$\begin{aligned}\lim_{\ell\to\infty} \int_{\partial(\Sigma_E(s))} & \psi_\ell(x_0)\, d'' x_0 \wedge P_E^*(\tilde{\alpha}_E) \wedge \delta_{h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))} \\ &= \int_{\Sigma_E(s)\cap\{x_0 = -\log|s|\}} P_E^*(\tilde{\alpha}_E) \wedge \delta_{h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))}. \qquad (7.3.8)\end{aligned}$$

By (7.3.6) and [Gubler 2016, §7], the analytic tropical weights on the n-skeleton of the tropicalization $\Sigma_E(s)$ of the affinoid domain $W_E(s)$ are the same as the tropical weights induced by $h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))$. Using that $s < r$ and $I := [s, r]$, it follows from (a) and (b) that the n-skeletons of $\Sigma_E(I) := \{\omega \in \Sigma_E(s) \mid x_0(\omega) \in -\log(I)\}$ and $-\log(I) \times \mathrm{trop}_{U_E}(\mathrm{div}(f) \cap W_E)$ are equal even as a product of weighted polyhedral complexes if we endow $-\log(I)$ with weight 1. Note that these tropicalizations can differ from the n-skeletons only inside the relative boundary. As we have some flexibility in the choice of the polyhedra Δ_i and in the choice of s, we may assume that $\Sigma_E(I) = -\log(I) \times \mathrm{trop}_{U_E}(\mathrm{div}(f) \cap W_E)$ and that this is of pure dimension n. We conclude that (7.3.8) is equal to

$$\int_{\mathrm{trop}_{U_E}(\mathrm{div}(f)\cap W_E \cap U_E^{\mathrm{an}})} \alpha_E. \qquad (7.3.9)$$

Using (7.3.3)–(7.3.9), it follows that the left-hand side of (7.3.1) is equal to

$$\sum_{j=1}^{m} (-1)^{j+1} \sum_{|E|=j} \int_{\mathrm{trop}_{U_E}(\mathrm{div}(f)\cap W_E \cap U_E^{\mathrm{an}})} \alpha_E.$$

Let Y be an irreducible component of $\mathrm{div}(f)$ and let

$$E_Y := \{i \in \{1, \dots, m\} \mid U_i \cap Y \neq \varnothing\}.$$

Then we use the very affine open subset U_{E_Y} to compute the following integrals over Y by performing the above steps backwards:

$$\sum_{j=1}^{m}(-1)^{j+1}\sum_{|E|=j}\int_{\mathrm{trop}_{U_E}(Y\cap W_E\cap U_E^{\mathrm{an}})}\alpha_E=\int_{\mathrm{trop}_{U_{E_Y}}(Y\cap W\cap U_{E_Y}^{\mathrm{an}})}\alpha_{U_{E_Y}}=\int_Y\alpha, \qquad (7.3.10)$$

where we have used in the last step that W covers $\mathrm{supp}(\alpha)$. Using linearity in the irreducible components Y (see [Gubler 2013, Remark 13.12]), we get Equation (7.3.1). □

Remark 7.4. Let f denote a regular function on the affine variety X. The proof of Lemma 7.3 given above shows that Equation (7.3.1) holds more generally for any generalized δ-forms α on X^{an} with compact support. If we permute the roles of d' and d'', we obtain by the same argument that

$$-\int_{\partial(\mathrm{trop}_U(W(s)\cap U^{\mathrm{an}}))}d'F^*(x_0)\wedge\alpha_U=\langle\delta_{\mathrm{cyc}(f)},\alpha\rangle \qquad (7.4.1)$$

holds for all generalized δ-forms $\alpha\in P_c^{n-1,n-1}(X^{\mathrm{an}})$. An elegant way to deduce (7.4.1) is to apply (7.3.1) for $J^*(\alpha)$ and to use the symmetry of the δ-current of integration.

7.5. Let φ denote an invertible analytic function on some open subset W of X^{an}. Given $x\in W$ there exists by [Gubler 2016, Proposition 7.2] an open subset U of X, an algebraic moment map $f:U\to\mathbb{G}_m$ and an open neighbourhood V of x in $U^{\mathrm{an}}\cap W$ such that $-\log|\varphi|$ and $-\log|f|$ agree on V. It follows that the function $-\log|\varphi|$ belongs to $A^0(W)$ and we get

$$d'd''[-\log|\varphi|]=-[d'd''\log|\varphi|]=0 \qquad (7.5.1)$$

from (6.9.1) and the trivial case of the Poincaré–Lelong formula where f is invertible.

7.6. Let L be a line bundle on X and let W be an open subset of X^{an}. We fix an open covering $(U_i)_{i\in I}$ of X, a family $(s_i)_{i\in I}$ of nowhere vanishing sections $s_i\in\Gamma(U_i,L)$, and the 1-cocycle (h_{ij}) with values in $\mathscr{O}_X^\times$ determined by $s_j=h_{ij}s_i$. Recall that a *continuous metric* $\|\cdot\|$ *on* L over W is given by a family $(\rho_i)_{i\in I}$ of continuous functions $\rho_i:U_i^{\mathrm{an}}\cap W\to\mathbb{R}$ such that $\rho_j=|h_{ij}|\rho_i$ on $(U_i\cap U_j)^{\mathrm{an}}\cap W$ for all $i,j\in I$. An analytic section $s\in\Gamma(V,L^{\mathrm{an}})$ on some open subset V of W determines as follows a continuous function $\|s\|:V\to\mathbb{R}$. We write $s=f_is_i$ for some analytic function f_i on $V\cap U_i^{\mathrm{an}}$ and define $\|s\|=|f_i|\cdot\rho_i$ on $V\cap U_i^{\mathrm{an}}$. Observe that we have $\rho_i=\|s_i\|$ on $U_i^{\mathrm{an}}\cap W$.

7.7. Let L be a line bundle on X endowed with a continuous metric $\|\cdot\|$ over the open subset W of X^{an}. Then we define the *first Chern current associated to the metrized line bundle* $(L|_W,\|\cdot\|)$ as the δ-current $[c_1(L|_W,\|\cdot\|)]\in E^{1,1}(W)$ given locally on $W \cap U^{\mathrm{an}}$ by $d'd''[-\log\|s|_{U^{\mathrm{an}}\cap W}\|]$ for any trivialization U of L with nowhere vanishing section $s \in \Gamma(U, L)$. Here, we have used that a continuous function defines a δ-current as explained in Proposition 6.16. Since $d'd''[-\log|\varphi|] = 0$ for an invertible analytic function φ, the δ-current $[c_1(L|_W, \|\cdot\|)]$ is well-defined on W and we may even use analytic trivializations in the definition. Obviously, the formation of the first Chern current is compatible with tensor products of metrized line bundles as usual.

If the metric is smooth then $[c_1(L, \|\cdot\|)]$ is the current associated to the first Chern form $c_1(L|_W, \|\cdot\|)$ defined in [Chambert-Loir and Ducros 2012]. In general, the notion $c_1(L|_W, \|\cdot\|)$ has no meaning as a form and we use brackets to emphasize that $[c_1(L|_W, \|\cdot\|)]$ is a δ-current. In Section 9, we will introduce metrics for which $c_1(L|_W, \|\cdot\|)$ has a meaning as a δ-form.

Corollary 7.8. *Let L be a line bundle on X endowed with a continuous metric $\|\cdot\|$ over the open subset W of X^{an}. For every nontrivial meromorphic section s of L with associated Weil divisor Y, the equality*

$$\big[c_1(L|_W, \|\cdot\|)\big] = d'd''\big[-\log\big\|s|_W\big\|\big] + \delta_Y|_W$$

holds in $E^{1,1}(W)$.

Proof. This can be checked locally on a trivialization U of L with a nowhere vanishing $s_U \in \Gamma(U, L)$. Then there is a rational function f on X with $s = f s_U$ and hence

$$d'd''\big[-\log\big\|s|_{W\cap U^{\mathrm{an}}}\big\|\big] = d'd''\big[-\log\big\|s_U|_{W\cap U^{\mathrm{an}}}\big\|\big] + d'd''\big[-\log\big|f|_{W\cap U^{\mathrm{an}}}\big|\big].$$

Using the definition of $c_1(L|_{W\cap U^{\mathrm{an}}}, \|\cdot\|)$ for the first summand and Theorem 7.2 for the second summand, we get the claim. □

8. Piecewise smooth and formal metrics on line bundles

In this section, X is an algebraic variety over K. In the following, we consider an open subset W of X^{an}.

We first introduce piecewise smooth functions and piecewise linear functions on W. This leads to corresponding notions for metrics on line bundles. We prove that a piecewise linear metric is the same as a formal metric. We show that canonical metrics in various situations are piecewise smooth.

In Definition 1.6, we have defined piecewise smooth functions on an open subset of an integral $\mathbb{R}$-affine polyhedral set. Using tropicalizations and viewing tropical varieties as polyhedral sets, we will define piecewise smooth functions on W as follows:

Definition 8.1. A function $f : W \to \mathbb{R}$ is called *piecewise smooth* if for every $x \in W$ there is a tropical chart (V, φ_U) such that V is an open neighbourhood of

x in W and such that there is a piecewise smooth function ϕ on $\mathrm{trop}_U(V)$ with $f = \phi \circ \mathrm{trop}_U$ on V.

In a similar way, we will define a piecewise linear function on W. We recall from Definition 1.6 that we have defined piecewise linear functions on integral $\mathbb{R}$-affine polyhedral complexes. As we are working with a variety over a valued field, we will take the value group Γ into account and in the definition of piecewise linear functions we will additionally require that the underlying polyhedral complex and the restriction of the functions are both integral Γ-affine. Note however that in Definition 8.1, the underlying polyhedral complex for ϕ is only assumed to be integral $\mathbb{R}$-affine.

Definition 8.2. A function $f : W \to \mathbb{R}$ is called *piecewise linear* if for every $x \in W$ there is a tropical chart (V, φ_U) such that V is an open neighbourhood of x in W and a real function ϕ on $\mathrm{trop}_U(V)$ with $f = \phi \circ \mathrm{trop}_U$ on V. We require that there is an integral Γ-affine polyhedral complex Σ in $N_{U,\mathbb{R}}$ with $\mathrm{trop}_U(V) \subseteq |\Sigma|$ such that ϕ is the restriction of a function on $|\Sigma|$ with integral Γ-affine restrictions to all faces of Σ.

8.3. The space of piecewise smooth functions on W is an $\mathbb{R}$-subalgebra of the $\mathbb{R}$-algebra of continuous functions on W. It contains all smooth functions on W. The space of piecewise linear functions on W is closed under forming max and min. Moreover, it is a subgroup of the space of piecewise smooth functions on W with respect to addition. If $\varphi : X' \to X$ is a morphism and W' is an open subset of $(\varphi^{\mathrm{an}})^{-1}(W)$, then for every piecewise smooth (resp. piecewise linear) function f on W, the restriction of $f \circ \varphi$ to W' is a piecewise smooth (resp. piecewise linear) function on W'.

In the following result, we need the G-topology on W. It is a Grothendieck topology build up from analytic subdomains of W and it is closely related to the Grothendieck topology of the underlying rigid analytic space ([Berkovich 1993, §1.3, §1.6]).

Proposition 8.4. *Let $f : W \to \mathbb{R}$ be a continuous function. Then f is piecewise smooth* (*resp. piecewise linear*) *if and only if there is a* G-*covering $(W_i)_{i \in I}$ by analytic* (*resp. strict analytic*) *subdomains W_i of W and analytic moment maps $\varphi_i : W_i \to (T_i)^{\mathrm{an}}$ to tori $T_i := \mathrm{Spec}(K[M_i])$ such that $f = \phi_i \circ \varphi_{i,\mathrm{trop}}$ on W_i for a smooth* (*resp. integral Γ-affine*) *function $\phi_i : N_{i,\mathbb{R}} \to \mathbb{R}$, where $N_i := \mathrm{Hom}(M_i, \mathbb{Z})$ as usual.*

Proof. First, we assume that f is piecewise smooth (resp. piecewise linear). For any $x \in W$, there is a tropical chart (V, φ_U) in W containing x such that $f = \phi \circ \mathrm{trop}_U$ on V for a piecewise smooth (resp. integral Γ-affine function) ϕ on the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. There are finitely many integral $\mathbb{R}$-affine (resp. Γ-affine) polytopes Δ_i in $N_{U,\mathbb{R}}$ containing $\mathrm{trop}_U(x)$ such that $\bigcup_i \Delta_i$ is a neighbourhood of $\mathrm{trop}_U(x)$ in Ω and such that $\phi|_{\Delta_i} = \phi_i|_{\Delta_i}$ for a smooth (resp. integral Γ-affine) function $\phi_i : N_{U,\mathbb{R}} \to \mathbb{R}$. Note that the affinoid (resp. strictly affinoid) subdomains $W_i(x) := \mathrm{trop}_U^{-1}(\Delta_i)$ of W contain x and cover a neighbourhood of x. Letting x vary over W, we get a G-covering of W with the desired properties.

To prove the converse, we assume that f is given on a G-covering $(W_i)_{i\in I}$ of W by smooth (resp. integral Γ-affine) functions $\phi_i : N_{i,\mathbb{R}} \to \mathbb{R}$ with respect to analytic moment maps $\varphi_i : W_i \to (T_i)^{\mathrm{an}}$. Piecewise smoothness (resp. piecewise linearity) is a local condition and so we have to check that f is piecewise smooth in a neighbourhood of $x \in X^{\mathrm{an}}$. There is a finite $I_0 \subseteq I$ such that the sets $(W_i)_{i\in I_0}$ cover a sufficiently small strict affinoid neighbourhood W' of x in W. By shrinking W, we may assume that $x \in W_i$ for every $i \in I_0$. In the following, we restrict our attention to elements $i \in I_0$. The definition of an analytic (resp. of a strict analytic) domain shows that we may assume that all $W_i' := W_i \cap W'$ are affinoid (resp. strict affinoid) subdomains of W. Any analytic function on a neighbourhood of x in W_i' can be approximated uniformly on a sufficiently small neighbourhood of x by rational functions on X. By shrinking W again, this shows that we may assume that $\varphi_i|_{W_i'}$ is induced by the restriction of an algebraic moment map $\varphi_i' : U_i \to T_i$ for a dense open subset U_i of X with $W_i' \subseteq (U_i)^{\mathrm{an}}$ (see [Gubler 2016, Proposition 7.2] for a similar argument). Similarly, we may assume that there are affinoid coordinates $(x_{ij})_{j\in J_i}$ on W_i' which extend to rational functions on X. Clearly, we may assume that $|x_{ij}(x)| = 1$ for $i \in I_0$ and $j \in J_i$. There is a tropical chart (V, φ_U) with $x \in V \subseteq W'$, $U \subseteq \bigcap_{i\in I_0} U_i$ and such that all the functions x_{ij} are in $\mathscr{O}(U)^\times$. We may assume that $\mathrm{trop}_U(x) = 0$ and hence there is an open neighbourhood $\widetilde{\Omega}$ of 0 in $N_{U,\mathbb{R}}$ with $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$. By [Gubler 2016, 4.12, Proposition 4.16], $\varphi_i'|_U$ is the composition of an affine homomorphism $\psi_i : T_U \to T_i$ with φ_U. By shrinking V and using the Bieri–Groves theorem [Gubler 2013, Theorem 3.3], we may assume that there are finitely many rational cones $(\Delta_j)_{j\in J}$ in $N_{U,\mathbb{R}}$ such that

$$\mathrm{trop}_U(V) = \widetilde{\Omega} \cap \bigcup_{j\in J} \Delta_j. \tag{8.4.1}$$

For every $i \in I_0$ and every $j \in J_i$, there is a linear form $u_{ij} \in M_U$ with $-\log|x_{ij}| = u_{ij} \circ \mathrm{trop}_U$ on U^{an}. The definition of affinoid coordinates yields

$$W_i' \cap U^{\mathrm{an}} = \mathrm{trop}_U^{-1}(\sigma_i) \tag{8.4.2}$$

for

$$\sigma_i := \{\omega \in N_{U,\mathbb{R}} \mid u_{ij}(\omega) \ge r_{ij} \ \forall j \in J_i\}$$

and suitable $r_{ij} \in \mathbb{R}$. Note that σ_i is an integral $\mathbb{R}$-affine polyhedron. In the piecewise linear case, we may choose always $r_{ij} = 0$ and hence σ_i is a rational cone. Using that the sets $W_i' \cap U^{\mathrm{an}}$ cover V and equations (8.4.1), (8.4.2), we get the decomposition

$(\sigma_i \cap \Delta_j \cap \widetilde{\Omega})_{i \in I_0, j \in J}$ of $\mathrm{trop}_U(V)$. On $\sigma_i \cap \Delta_j \cap \widetilde{\Omega}$, we choose the smooth (resp. integral Γ-affine) function $\phi'_{ij} := \phi_i \circ \psi_i$. Using (8.4.2), we see that these functions paste to a continuous piecewise smooth (resp. continuous piecewise linear) function ϕ' on $\mathrm{trop}_U(V)$ with $\phi' \circ \mathrm{trop}_U = f$ on V. This proves easily that f is piecewise smooth (resp. piecewise linear) on W. □

Definition 8.5. Let L be a line bundle on X and let W be an open subset of X^{an}. A metric $\|\cdot\|$ on $L|_W$ is called *piecewise smooth* (resp. *piecewise linear*) if for every $x \in W$, there is a tropical chart (V, φ_U) with $x \in V \subseteq W$ and a nowhere vanishing section $s \in \Gamma(U, L)$ such that $-\log\|s|_V\|$ is piecewise smooth (resp. piecewise linear) on V.

8.6. Since $-\log|f|$ is smooth for an invertible regular function f and even the pull-back of a linear function with respect to a suitable tropicalization, the last definition does neither depend on the choice of the trivialization s nor on the choice of the tropical chart (V, φ_U). Moreover, we may also use analytic trivializations in the definition. By Proposition 8.4, the definition of a piecewise linear metric agrees with the definition of PL-metrics in [Chambert-Loir and Ducros 2012, §6.2].

Note that every piecewise linear metric is piecewise smooth. It follows from 8.3 that every piecewise smooth metric is continuous, that the tensor product of piecewise linear (resp. piecewise smooth) metrics is again a piecewise linear (resp. piecewise smooth) metric and that the dual metric of a piecewise linear (resp. piecewise smooth) metric is piecewise linear (resp. piecewise smooth). Moreover, the pull-back of a piecewise linear (resp. piecewise smooth) metric on $L|_W$ with respect to a morphism $\varphi : X' \to X$ is a piecewise linear (resp. piecewise smooth) metric on $\varphi^*(L)|_{W'}$ for any open subset W' of $\varphi^{-1}(W)$.

8.7. Recall that K° is the valuation ring of the given nonarchimedean absolute value $|\ |$ on K. Raynaud introduced an *admissible formal scheme over K°* as a formal scheme $\mathscr{X}$ over the valuation ring K° which is locally isomorphic to $\mathrm{Spf}(A)$ for a flat K°-algebra A of topologically finite type over K° (see [Bosch and Lütkebohmert 1993, §1] for details). For simplicity, we require additionally that $\mathscr{X}$ has a locally finite atlas of admissible affine formal schemes over K°. Then $\mathscr{X}$ has a generic fibre $\mathscr{X}_\eta$ (resp. a special fibre $\mathscr{X}_s$) which is a paracompact strictly analytic Berkovich space over K (resp. an algebraic scheme over the residue field $\tilde{K}$) locally isomorphic to $\mathscr{M}(\mathscr{A})$ (resp. $\mathrm{Spec}(A \otimes_{K^\circ} \tilde{K})$) for the strict affinoid algebra $\mathscr{A} := A \otimes_{K^\circ} K$ (see [Berkovich 1993, §1.6] for the equivalence to rigid analytic spaces over K with an affinoid covering of finite type).

A *formal K°-model of* X is an admissible formal scheme $\mathscr{X}$ over K° with an isomorphism $\mathscr{X}_\eta \cong X^{\mathrm{an}}$. For a line bundle L on X, we define a *formal K°-model of* L as a line bundle $\mathscr{L}$ on a formal K°-model $\mathscr{X}$ of X with an isomorphism $\mathscr{L}_\eta \cong L^{\mathrm{an}}$ over $\mathscr{X}_\eta \cong X^{\mathrm{an}}$. For simplicity, we usually identify $\mathscr{L}_\eta$ with L^{an}.

8.8. Let L be a line bundle on X. A *formal metric* on L is a metric $\|\cdot\|_{\mathscr{L}}$ associated to a formal K°-model $\mathscr{L}$ of L in the following way: If $\mathscr{L}$ admits a formal trivialization over $\mathscr{U}$ and if $s \in \Gamma(\mathscr{U}, \mathscr{L})$ corresponds under this trivialization to the function $\gamma \in \mathscr{O}_{\mathscr{X}}(\mathscr{U})$, then $\|s(x)\| = |\gamma(x)|$ for all $x \in \mathscr{U}_\eta$. This definition is independent of the choice of the trivialization and shows immediately that formal metrics are continuous. The tensor product and the pull-back of formal metrics are again formal metrics.

Proposition 8.9. *Every line bundle L on X has a formal K°-model and hence a formal metric.*

Proof. This follows as in [Gubler 1998, Proposition 7.6] based on the theorem of Raynaud that every paracompact analytic space has a formal K°-model (see [Bosch 2014, Theorem 8.4.3]). The argument for paracompact strictly K-analytic spaces was first given in [Chambert-Loir and Ducros 2012, proposition (6.2.13)]. □

Proposition 8.10. *Let $\|\cdot\|$ be a formal metric on the line bundle L on X. Then there is an admissible formal K°-model $\mathscr{X}$ of X with reduced special fibre $\mathscr{X}_s$ and a K°-model $\mathscr{L}$ of L on $\mathscr{X}$ such that $\|\cdot\| = \|\cdot\|_{\mathscr{L}}$. Moreover, the invertible sheaf associated to $\mathscr{L}$ is always canonically isomorphic to the sheaf on $\mathscr{X}$ given by $\{s \in \Gamma(L, \mathscr{U}_\eta) \mid \|s(s)\| \le 1\ \forall x \in \mathscr{U}_\eta\}$ on a formal open subset $\mathscr{U}$ of $\mathscr{X}$.*

Proof. This follows as in [Gubler 1998, Lemma 7.4 and Proposition 7.5]. □

Proposition 8.11. *Let $\|\cdot\|$ be a metric on the line bundle L on X. Then the following properties are equivalent*:

(a) $\|\cdot\|$ *is a formal metric*;

(b) $\|\cdot\|$ *is a piecewise linear metric*;

(c) *there is a* G*-covering $(W_i)_{i \in I}$ of X^{an} by strict analytic subdomains W_i of X^{an} and trivializations $s_i \in \Gamma(W_i, L^{\mathrm{an}})$ with $\|s_i(x)\| = 1$ for all $x \in W_i$, $i \in I$.*

Proof. If we use again Raynaud's theorem to generalize to paracompact X^{an}, the equivalence of (a) and (c) is proved as in [Gubler 1998, Lemma 7.4 and Proposition 7.5]. The implication (a) ⇒ (c) can also be found in [Chambert-Loir and Ducros 2012, exemple (6.2.10)]. It remains to see the equivalence of (b) and (c). Suppose that (b) holds. Then there is a locally finite covering of X by trivializations U_i of L such that $-\log\|s_i\|$ is piecewise linear on $(U_i)^{\mathrm{an}}$ for every $i \in I$. By Proposition 8.4, there is a G-covering W_{ij} of $(U_i)^{\mathrm{an}}$ and analytic moment maps $\varphi_{ij} : W_{ij} \to (T_{ij})^{\mathrm{an}}$ such that $-\log\|s_i\| = \phi_{ij} \circ \varphi_{ij,\mathrm{trop}}$ on W_{ij} for an integral Γ-affine function ϕ_{ij} on $N_{ij,\mathbb{R}}$. The definition of integral Γ-affine functions shows that there is an invertible analytic function γ_{ij} on W_{ij} such that $\|s_i\| = |\gamma_{ij}|$ on W_{ij}. Using the trivialization $\gamma_{ij}^{-1} s_i$ on W_{ij}, we get (b) ⇒ (c). The converse is an immediate application of Proposition 8.4. □

8.12. If X is proper over K, then an *algebraic K°-model* of X is an integral scheme $\mathfrak{X}$ which is of finite type, flat and proper over K° and with a fixed isomorphism between the generic fibre $\mathscr{X}_\eta$ and X. We use the isomorphism to identify $\mathfrak{X}_\eta$ and X. An *algebraic K°-model* of L is a line bundle $\mathfrak{L}$ on an algebraic K°-model $\mathfrak{X}$ of X together with a fixed isomorphism between $\mathfrak{L}_\eta$ and L. We define an *algebraic metric* on L as in 8.8 by using an algebraic K°-model $\mathfrak{L}$ of L.

Proposition 8.13. *On a line bundle on a proper variety over K, a metric is algebraic if and only if it is formal.*

Proof. Passing to the formal completion along the special fibre, it is clear that every algebraic metric is a formal metric. Using [Gubler 2003, Proposition 10.5], the converse is true if X is projective. The same argument shows that the converse is also true for proper X if the formal GAGA theorem in [EGA III$_1$ 1961, Theorem 5.1.4] holds over K° and if X has an algebraic K°-model. In [EGA III$_1$ 1961, Theorem 5.1.4], the base has to be noetherian and hence it applies only for discrete valuation rings. The required generalization is now given in [Fujiwara and Kato 2014, Theorem I.10.1.2]. The existence of an algebraic K°-model follows from Nagata's compactification theorem. This was proved by Nagata in the noetherian case and proved by Conrad in general (based on notes of Deligne, see [Temkin 2011] for another proof and references). □

Corollary 8.14. *Let L be a line bundle on a proper variety over K. Then L has an algebraic metric.*

Proof. This follows from Proposition 8.9 and Proposition 8.13. □

We will show now that many important metrics are piecewise smooth.

Example 8.15. Let L be a line bundle on the abelian variety A over K. Choosing a rigidification of L at $0 \in A$ and assuming L symmetric (resp. odd), the theorem of the cube allows one to identify $[m]^*(L)$ with $L^{\otimes m^2}$ (resp. $L^{\otimes m}$). There is a unique continuous metric $\|\cdot\|_{\mathrm{can}}$ on L^{an} with $[m]^*\|\cdot\|_{\mathrm{can}} = \|\cdot\|_{\mathrm{can}}^{\otimes m^2}$ (resp. $[m]^*\|\cdot\|_{\mathrm{can}} = \|\cdot\|_{\mathrm{can}}^{\otimes m}$) for all $m \in \mathbb{Z}$. In general, $L^{\otimes 2}$ is the tensor product of a symmetric and an odd line bundle, unique up to 2-torsion in $\mathrm{Pic}(A)$, and hence we get a *canonical metric* $\|\cdot\|_{\mathrm{can}}$ on L which is unique up to multiples from $|K^\times|$ if we vary rigidifications. We claim that $\|\cdot\|_{\mathrm{can}}$ is locally on X^{an} the tensor product of a smooth metric and a piecewise linear metric. In particular, we deduce that $\|\cdot\|_{\mathrm{can}}$ is a piecewise smooth metric.

To prove the claim, we use the Raynaud extension of A to describe the canonical metric on L (see [Gubler 2010, §4] for details). The Raynaud extension is an exact sequence

$$1 \to T^{\mathrm{an}} \to E \xrightarrow{q} B^{\mathrm{an}} \to 0 \tag{8.15.1}$$

of commutative analytic groups, where $T = \operatorname{Spec}(K[M])$ is a multiplicative torus of rank r and B is an abelian variety of good reduction. Moreover, there is a lattice P in E with $E/P = A^{\mathrm{an}}$. More precisely P is a discrete subgroup of $E(K)$ which is mapped by a canonical map, $\mathrm{val} : E \to N_{\mathbb{R}}$, isomorphically onto a complete lattice Λ of $N_{\mathbb{R}}$, where N is the dual of M. The map val is locally over B a tropicalization. Note that the Raynaud extension is algebraizable, but the quotient homomorphism $p : E \to A^{\mathrm{an}}$ is only defined in the analytic category.

Let $\mathscr{B}$ be the abelian scheme over K° with generic fibre B. By [loc. cit.] there exists a line bundle $\mathscr{H}$ on $\mathscr{B}$ such that $q^*((\mathscr{H}_\eta)^{\mathrm{an}}) = p^*(L^{\mathrm{an}})$. Here, and in the following, we use rigidified line bundles to identify isomorphic line bundles. Then $q^*\|\cdot\|_{\mathscr{H}}$ is a formal metric on $p^*(L^{\mathrm{an}})$. On $p^*(L^{\mathrm{an}})$, we have a canonical P-action α over the canonical action of P on E by translation. By [loc. cit.] there is a 1-cocycle Z in $Z^1(P, (\mathbb{R}^\times)^E)$ such that

$$(q^*\|\alpha_\gamma(w)\|_{\mathscr{H}})_{\gamma\cdot x} = Z_\gamma(x)^{-1} \cdot (q^*\|w\|_{\mathscr{H}})_x \tag{8.15.2}$$

for all $\gamma \in P,\ x \in E$ and $w \in (p^*L^{\mathrm{an}})_x$. The cocycle Z depends only on the map val, which means that there is a unique function $z_\lambda : N_{\mathbb{R}} \to \mathbb{R}$ with

$$z_\lambda(\mathrm{val}(x)) = -\log(Z_\gamma(x)) \quad (\gamma \in P,\ x \in E,\ \lambda = \mathrm{val}(\gamma)).$$

Moreover, there is a canonical symmetric bilinear form $b : \Lambda \times \Lambda \to \mathbb{Z}$ associated to L such that

$$z_\lambda(\omega) = z_\lambda(0) + b(\omega, \lambda) \quad (\omega \in N_{\mathbb{R}},\ \lambda \in \Lambda).$$

The cocycle condition

$$Z_{\rho\gamma}(x) = Z_\rho(\gamma x) Z_\gamma(x) \quad (\rho, \gamma \in P,\ x \in E)$$

shows that

$$z_{\lambda+\mu}(0) = z_\lambda(0) + z_\mu(0) + b(\lambda, \mu) \quad (\lambda, \mu \in \Lambda),$$

which means that $\lambda \mapsto z_\lambda(0)$ is a quadratic function on Λ. There is a unique extension to a quadratic function $q_0 : N_{\mathbb{R}} \to \mathbb{R}$. We define a metric $\|\cdot\|$ on $p^*(L^{\mathrm{an}})$ by $\|\cdot\| := e^{-q_0\circ\mathrm{val}} q^*\|\cdot\|_{\mathscr{H}}$. Using (8.15.2) and that q_0 is a quadratic function with associated bilinear form b, it follows easily that $\|\cdot\|$ descends to the canonical metric on L. We conclude from the descent with respect to the local isomorphism p that the canonical metric on L is locally on A^{an} the tensor product of a smooth metric with a piecewise linear metric. This proves the claim.

Example 8.16. Let L be a line bundle on a proper smooth algebraic variety over K which is algebraically equivalent to zero. Let A denote the Albanese variety of X (see [Grothendieck 1966, théorème 2.1, corollaire 3.2]). We fix some $x \in X(K)$ and obtain a universal morphism $\psi : X \to A$ from X to the abelian variety A with

$\psi(x) = 0$. Furthermore L is in a canonical way the pull-back of an odd line bundle on A along ψ. It follows that L carries a canonical metric $\|\cdot\|_{\mathrm{can}}$, unique up to multiples from $|K^\times|$. By [Gubler 2010, Example 3.7], there is an integer $N \geq 1$ such that $\|\cdot\|_{\mathrm{can}}^{\otimes N}$ is an algebraic metric and hence piecewise linear. We conclude that $\|\cdot\|_{\mathrm{can}}$ is a piecewise smooth metric.

Example 8.17. Let L be a line bundle on a complete toric variety X over K. Similarly as in the case of abelian varieties and using rigidifications, we have $[m]^*(L) = L^{\otimes m}$ and there is a unique metric $\|\cdot\|_{\mathrm{can}}$ on L with $[m]^*\|\cdot\|_{\mathrm{can}} = \|\cdot\|_{\mathrm{can}}^{\otimes m}$ for all integers $m \in \mathbb{Z}$ (see [Maillot 2000, Section 3]). There is a canonical algebraic K°-model $\mathscr{X}$ of K° and a canonical algebraic K°-model $\mathscr{L}$ by using the same rational polyhedral fan and the same piecewise linear function. Since $\|\cdot\|_{\mathrm{can}} = \|\cdot\|_{\mathscr{L}}$, the canonical metric on L is algebraic and hence a piecewise linear metric.

8.18. Finally, we consider the case where our variety X is defined over a ground field F which is equipped with the trivial valuation. If L is a line bundle on X, then we choose an algebraically closed extension field K endowed with a nontrivial complete absolute value extending the trivial absolute value of F. Then $F \subseteq K^\circ$ and the line bundle $L \otimes_F K^\circ$ on $X \otimes_F K^\circ$ is a canonical algebraic K°-model of the line bundle L_K on X_K. We conclude that L has a canonical metric $\|\cdot\|_{\mathrm{can}}$.

The metric $\|\cdot\|_{\mathrm{can}}$ has the following intrinsic description. Let $U = \mathrm{Spec}(A)$ be an affine open subset of X which is a trivialization of L given by the nowhere vanishing section $s \in \Gamma(U, L)$. We consider the formal affinoid subdomain $U^\circ := \{x \in U^{\mathrm{an}} \mid |f(x)| \leq 1\ \forall f \in A\}$ of X^{an}. Note that U° is the set of points in U^{an} with reduction contained in U (see [Gubler 2013, §4] for more details). It follows that $\|s(x)\|_{\mathrm{can}} = 1$ for all $x \in U^\circ$. Since X is proper, such trivializations U° cover X^{an} leading to a description of $\|\cdot\|_{\mathrm{can}}$ which is independent of K.

For simplicity, we have considered only varieties in this paper. We may also consider continuous metrics on L^{an} for a line bundle over a separated scheme X of finite type over the ground field F. For such schemes X, the intrinsic description above shows in particular that we still have a canonical metric $\|\cdot\|_{\mathrm{can}}$ on L in the case of a trivially valued F.

9. Piecewise smooth forms and delta-metrics

We consider again an algebraic variety X over K of dimension n. In this section, we first study piecewise smooth forms on an open subset W of X^{an}. This leads to a decomposition of the first Chern current of a piecewise smoothly metrized line bundle $(L|_W, \|\cdot\|)$ into the sum of a piecewise smooth form and a residual current. We show that the residual current is induced by a generalized δ-form. If the first Chern current of $(L|_W, \|\cdot\|)$ is induced by a δ-form on W, then $\|\cdot\|$ is called a δ-metric and the δ-form is called the first Chern δ-form. We show that many important metrics are δ-metrics. In the following sections, we will use δ-metrics for our approach to nonarchimedean Arakelov theory.

9.1. In Definition 3.10, we defined the space $\mathrm{PS}(\Omega)$ of piecewise smooth superforms on an open subset Ω of a polyhedral subset. If (V, φ_U) is a tropical chart, then we apply this definition for the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. If $\alpha \in \mathrm{PS}(\Omega)$ and $(V', \varphi_{U'})$ is a tropical chart with $V' \subseteq V$ and $U' \subseteq U$, then we define $\alpha|_{V'}$ as the piecewise smooth form on $\Omega' := \mathrm{trop}_{U'}(V')$ given by pull-back of α with respect to the canonical affine map $N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$.

Definition 9.2. A *piecewise smooth form* on an open subset W of X^{an} may be defined in a similar way as a differential form in $A(W)$: A piecewise smooth form α is given by an open covering $(V_i, \varphi_{U_i})_{i\in I}$ of W by tropical charts and piecewise smooth superforms α_i on $\Omega_i := \mathrm{trop}_{U_i}(V_i)$ such that $\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i\cap V_j}$ for all $i, j \in I$. A superform α' given by the covering $(V_j', \varphi_{U_j'})_{j\in J}$ and piecewise smooth superforms α_j' on $\Omega_j' := \mathrm{trop}_{U_j'}(V_j')$ will be identified with α if and only if $\alpha_i|_{V_i\cap V_j'} = \alpha_j'|_{V_i\cap V_j'}$ for every $i \in I$ and every $j \in J$.

9.3. We denote the space of piecewise smooth forms on W by $\mathrm{PS}(W)$. It comes with a bigrading and is canonically equipped with a $\wedge$-product. We conclude easily that $\mathrm{PS}^{\cdot,\cdot}(W)$ is a bigraded $A^{\cdot,\cdot}(W)$-algebra on X^{an}. It is clear that $\mathrm{PS}^{0,0}(W)$ is the space of piecewise smooth functions on W. It coincides with the space $P^{0,0}(W)$ of generalized δ-preforms of degree zero. The equality

$$\mathrm{PS}^{0,0}(W) = P^{0,0}(W) \tag{9.3.1}$$

is in fact a direct consequence of (4.19.3).

If $\varphi : X' \to X$ is a morphism of algebraic varieties over K, then the pull-back of piecewise smooth superforms from Definition 3.10 carries over to define a *pull-back* $f^* : \mathrm{PS}^{p,q}(W) \to \mathrm{PS}^{p,q}(W')$ for any open subset W' of $(X')^{\mathrm{an}}$ with $f(W') \subseteq W$. In the special case of $X = X'$, $f = \mathrm{id}$ and W' an open subset of W, we denote the pull-back by $\alpha|_{W'}$ and call it the *restriction of* α *to* W'.

9.4. In (3.11.1), we introduced differentials of piecewise smooth forms on open subsets of polyhedral sets. If $\alpha \in \mathrm{PS}^{p,q}(W)$ is given as in Definition 9.2, then the polyhedral differential $d_{\mathrm{P}}'\alpha \in \mathrm{PS}^{p+1,q}(W)$ is locally defined by $d_{\mathrm{P}}'\alpha_i \in \mathrm{PS}^{p+1,q}(\Omega_i)$. Similarly, we define $d_{\mathrm{P}}''\alpha \in \mathrm{PS}^{p,q+1}(W)$. Then $\mathrm{PS}^{\cdot,\cdot}(W)$ is a differential graded $\mathbb{R}$-algebra with respect to the polyhedral differentials d_{P}' and d_{P}''.

9.5. The bigraded differential $\mathbb{R}$-algebras $\mathrm{PS}(W)$ of piecewise smooth forms and $P(W)$ of generalized δ-forms are not directly comparable except that they both contain $A(W)$ as a bigraded differential $\mathbb{R}$-subalgebra. We construct a bigraded differential $\mathbb{R}$-algebra $\mathrm{PSP}(W)$ containing both spaces as follows.

Recall from Remark 3.14 that we have obtained a bigraded differential $\mathbb{R}$-algebra $\mathrm{PSP}(\widetilde{\Omega})$ with respect to $d'_{\mathrm{P}}, d''_{\mathrm{P}}$ for any open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$. We repeat now the construction of generalized δ-forms in Section 4 building upon the spaces $\mathrm{PSP}(\widetilde{\Omega})$ instead of $P(\widetilde{\Omega})$. This leads first to spaces $\mathrm{PSP}(V, \varphi_U)$ for tropical charts (V, φ_U) of X and then to the desired space $\mathrm{PSP}(W)$. Note that $\mathrm{PSP}(W)$ is a differential bigraded $\mathbb{R}$-algebra with respect to the polyhedral differential operators d'_{P} and d''_{P} which extends the corresponding structure on the subalgebra $P(W)$. To see that $\mathrm{PS}(W)$ is a graded subalgebra of $\mathrm{PSP}(W)$, we use the obvious generalization of Proposition 1.8 from piecewise smooth functions to piecewise smooth forms. Obviously, $\mathrm{PSP}(W)$ is generated by the subalgebras $\mathrm{PS}(W)$ and $P(W)$. Moreover, the polyhedral differentials d'_{P} and d''_{P} agree with the corresponding differential operators on $\mathrm{PS}(W)$.

All properties of generalized δ-forms from Section 4 and Section 5 extend immediately to the sheaves PSP. Hence we have an integral $\int_W \alpha$ for any $\alpha \in \mathrm{PSP}_c^{n,n}(W)$. As a special case, we obtain such an integral for a piecewise smooth form with compact support on W. As in 6.4, this leads to a δ-current $[\alpha] \in E^{p,q}(W)$ for any $\alpha \in \mathrm{PSP}^{p,q}(W)$. In particular, this applies to a piecewise smooth α.

Remark 9.6. Note that the polyhedral differential $d'_{\mathrm{P}}\alpha$ of a piecewise smooth form α, or more generally of any $\alpha \in \mathrm{PSP}(W)$, is not compatible with the corresponding differential of the associated δ-current. We define the *d'-residue* by

$$\mathrm{Res}_{d'}(\alpha) := d'[\alpha] - [d'_{\mathrm{P}}\alpha].$$

Similarly, we define residues with respect to d'' and $d'd''$.

9.7. Now we consider a line bundle L on X endowed with a piecewise smooth metric $\|\cdot\|$ over the open subset W of X^{an}. We are going to obtain a canonical decomposition of the Chern current $[c_1(L|_W, \|\cdot\|)] \in E^{1,1}(X^{\mathrm{an}})$ (see 7.7) into a *piecewise smooth part* $c_1(L|_W, \|\cdot\|)_{\mathrm{ps}} \in \mathrm{PS}^{1,1}(W)$ and a *residual part* $[c_1(L|_W, \|\cdot\|)]_{\mathrm{res}} \in E^{1,1}(W)$.

Let (U, s) be a trivialization of L, i.e., U is an open subset of X and s is a nowhere vanishing section in $\Gamma(U, L)$. Then $-\log\|s\|$ is a piecewise smooth function on $U^{\mathrm{an}} \cap W$ and hence $-d'_{\mathrm{P}}d''_{\mathrm{P}} \log\|s|_{U^{\mathrm{an}}\cap W}\| \in \mathrm{PS}^{1,1}(U^{\mathrm{an}} \cap W)$. Note that this piecewise smooth form is independent of the choice of s by the same argument as in 7.7 and hence we obtain a globally defined element of $\mathrm{PS}^{1,1}(W)$ which we denote by $c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}$. Recall from 9.5 that we denote the associated δ-current on W by $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}]$. The same argument shows that the residues $\mathrm{Res}_{d'd''}(-\log\|s|_{U^{\mathrm{an}}\cap W}\|)$ paste together to give a global δ-current $[c_1(L|_W, \|\cdot\|)]_{\mathrm{res}} \in E^{1,1}(W)$ and we have

$$[c_1(L|_W, \|\cdot\|)] = [c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}] + [c_1(L|_W, \|\cdot\|)]_{\mathrm{res}}. \tag{9.7.1}$$

Proposition 9.8. *Let $\|\cdot\|$ be a piecewise smooth metric on $L|_W$. Then there is a unique $\beta \in P^{1,1}(W)$ with*

$$[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)]_{\mathrm{res}} \in E^{1,1}(W')$$

for every morphism $\varphi : X' \to X$ from any algebraic variety X' over K and for every open subset W' of $\varphi^{-1}(W)$. The generalized δ-form β has codimension 1 (see Definition 4.13) *and will be denoted by $c_1(L|_W, \|\cdot\|)_{\mathrm{res}}$.*

Proof. Note that uniqueness follows from Proposition 6.8. By definition of a piecewise smooth metric, there is an open covering $(V_i)_{i\in I}$ of W by tropical charts (V_i, φ_{U_i}), nowhere vanishing sections $s_i \in \Gamma(V_i, L^{\mathrm{an}})$ and piecewise smooth functions ϕ_i on $\Omega_i := \mathrm{trop}_{U_i}(V_i)$ with $-\log\|s_i\| = \phi_i \circ \mathrm{trop}_{U_i}$ on V_i. Passing to a refinement of the open covering, we may assume that ϕ_i is defined on $\mathrm{Trop}(U_i)$. By Proposition 1.8, there is a piecewise smooth function $\tilde{\phi}_i$ on $N_{U_i,\mathbb{R}}$ restricting to ϕ_i. By Proposition 1.12, the corner locus $C_i := \tilde{\phi}_i \cdot N_{U_i,\mathbb{R}}$ of $\tilde{\phi}_i$ is a tropical cycle of codimension 1.

The δ-preform δ_{C_i} represents an element $\beta_i \in P(V_i, \varphi_{U_i})$ of codimension 1 (see Definition 4.4). We have seen in Remark 4.5 that there is a pull-back $f^*(\beta_i) \in P^{1,1}(V', \varphi_{U'})$ for every morphism $f : X' \to X$ of algebraic varieties over K and every tropical chart $(V', \varphi_{U'})$ of X' compatible with (V_i, φ_{U_i}). For the open subset $\Omega' := \mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$, we have $f^*(\beta_i)|_{\Omega'} \in P^{1,1}(\Omega') \subseteq D^{1,1}(\Omega')$ (see (4.5.1)). Let $F : N_{U',\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical affine map with $\mathrm{trop}_{U_i} = F \circ \mathrm{trop}_{U'}$ on $(U')^{\mathrm{an}}$. By Proposition 1.14 and Corollary 1.15, $F^*(C_i) \cdot \mathrm{Trop}(U')$ is the corner locus of $\phi' := \phi_i \circ F|_{\mathrm{Trop}(U')}$ and hence we get

$$f^*(\beta_i)|_{\Omega'} = F^*(\delta_{C_i}) \wedge \delta_{\mathrm{Trop}(U')} = \delta_{\phi'\cdot\mathrm{Trop}(U')} \in P^{1,1}(\Omega').$$

Together with the tropical Poincaré–Lelong formula (Corollary 3.19), we get

$$f^*(\beta_i)|_{\Omega'} + [d'_{\mathrm{P}} d''_{\mathrm{P}} \phi'] = d'd''[\phi'] \in D^{1,1}(\Omega'). \tag{9.8.1}$$

It follows from (9.8.1) that $f^*(\beta_i)|_{\Omega'}$ is independent of all choices. This yields that $\beta_i|_{V_i\cap V_j} = \beta_j|_{V_i\cap V_j}$ for all $i, j \in I$. We get a well-defined generalized δ-form $\beta \in P^{1,1}(W)$ of codimension 1 given by $\beta_i \in P^{1,1}(V_i, \varphi_{U_i})$ on V_i for every $i \in I$.

It remains to check that $[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)]_{\mathrm{res}}$ for every morphism $\varphi : X' \to X$ and every open subset W' of $\varphi^{-1}(W)$. This has to be tested on $\alpha \in B_c^{n-1,n-1}(W')$. The claim is local and a partition of unity argument in a paracompact open neighbourhood of $\mathrm{supp}(\alpha)$ shows that we may assume $\mathrm{supp}(\alpha) \subseteq \varphi^{-1}(V_i)$ for some $i \in I$.

There are finitely many tropical charts $(V'_j, \varphi_{U'_j})_{j\in J}$ within W' which cover $\mathrm{supp}(\alpha)$ such that α is given on every V'_j by $\alpha_j \in \mathrm{AZ}^{n-1,n-1}(V'_j, \varphi_{U'_j})$. We choose a nonempty very affine open subset U' of X' contained in every U'_j and in $\varphi^{-1}(U_i)$.

By Proposition 5.9, U' is a very affine chart of integration for both $\varphi^*(\beta) \wedge \alpha$ and $d'd''\alpha$. By construction, $V' := \bigcup_{j \in J} V'_j \cap \varphi^{-1}(V_i) \cap (U')^{\mathrm{an}}$ and $\varphi_{U'}$ form a tropical chart in W'. By Proposition 5.7, α is given on $\Omega' := \mathrm{trop}_{U'}(V')$ by $\alpha_{U'} \in \mathrm{AZ}^{n-1,n-1}(V', \varphi_{U'})$. In the following, we will use only the δ-preform $\alpha' \in P^{n-1,n-1}(\Omega')$ induced by $\alpha_{U'}$. For the tropical cycle $C' := \mathrm{Trop}(U')$ and the canonical affine map $F : N_{U',\mathbb{R}} \to N_{U_i,\mathbb{R}}$, it follows as above that $\varphi^*(\beta)$ is given on V' by the element in $P^{1,1}(V', \varphi_{U'})$ represented by $\delta_{(\phi_i \circ F)\cdot N_{U',\mathbb{R}}} \in P^{1,1}(N_{U',\mathbb{R}})$. For $\phi' := \phi_i \circ F|_{\mathrm{Trop}(U')}$, we have seen that

$$\varphi^*(\beta_i)|_{\Omega'} = (\delta_{(\phi_i \circ F)\cdot N_{U',\mathbb{R}}})|_{\Omega'} = \delta_{\phi' \cdot C'} \in P^{1,1}(\Omega').$$

Note that $\mathrm{supp}(\alpha) \subseteq \bigcup_{j \in J} V'_j \cap \varphi^{-1}(V_i)$. We deduce from the generalizations of Corollary 5.6 and Proposition 4.21 to PSP-forms (see 9.5) that the currents $d'_{\mathrm{P}}\phi' \wedge \alpha'$, $d''_{\mathrm{P}}\phi' \wedge \alpha'$, $d'd''\alpha'$, $\alpha' \wedge \delta_{\phi' \cdot C'}$ have compact support in Ω'. We write $C' = (\mathscr{C}', m')$ for an integral Γ-affine polyhedral complex $\mathscr{C}'$ and a family of integral weights m'. To prove $[c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)]_{\mathrm{res}} = [\varphi^*(\beta)]$, we have to show that

$$\int_{|\mathscr{C}'|} \phi' \wedge d'd''\alpha' = \int_{|\mathscr{C}'|} \delta_{\phi' \cdot C'} \wedge \alpha' + \int_{|\mathscr{C}'|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi' \wedge \alpha' \tag{9.8.2}$$

holds. If α' has compact support in Ω', then this follows from the tropical Poincaré–Lelong formula (9.8.1). In general, we still can deduce from the proof of the tropical Poincaré–Lelong formula in Theorem 3.16 the formula (3.16.3) which here reads as

$$\int_{|\mathscr{C}'|} \phi' \wedge d'd''\alpha' = -\int_{\partial|\mathscr{C}'|} d''_{\mathrm{P}}\phi' \wedge \alpha' + \int_{|\mathscr{C}'|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi' \wedge \alpha'$$

as we have used only that $d'\alpha'$ and $d''_{\mathrm{P}}\phi' \wedge \alpha'$ have compact support. Now (9.8.2) follows from Lemma 3.17 and Remark 3.18 using additionally that $\alpha' \wedge \delta_{\phi' \cdot C'}$ has compact support. □

Definition 9.9. A metric $\|\cdot\|$ on $L|_W$ is called a *δ-metric* if for every $x \in W$, there are a tropical chart (V, φ_U) such that $x \in V \subseteq W$ and a piecewise smooth function ϕ on $\mathrm{Trop}(U)$ satisfying the following properties:

(i) There is a nowhere vanishing section s of L over U such that $\phi \circ \mathrm{trop}_U = -\log\|s\|$ on V.

(ii) There is a superform γ on $N_{U,\mathbb{R}}$ of bidegree $(1, 1)$ with piecewise smooth coefficients such that $d'_{\mathrm{P}} d''_{\mathrm{P}}\phi$ and $\gamma|_{\mathrm{Trop}(U)}$ agree on the open subset $\mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$.

Remark 9.10. Condition (i) just means that the metric is piecewise smooth. Note that a superform on $N_{U,\mathbb{R}}$ with piecewise smooth coefficients is the same as a δ-preform on $N_{U,\mathbb{R}}$ of codimension 0 (see Example 2.10). Using 9.7, we deduce easily that (ii) is equivalent to the condition that $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}]$ is the δ-current associated to a generalized δ-form on W (of codimension 0).

Proposition 9.11. *Let $\|\cdot\|$ be a piecewise smooth metric on $L|_W$. Then $\|\cdot\|$ is a δ-metric if and only if there is a $\beta \in B^{1,1}(W)$ with*

$$[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)] \in E^{1,1}(W')$$

for every morphism $\varphi : X' \to X$ from any algebraic variety X' over K and for every open subset W' of $\varphi^{-1}(W)$.

Proof. Suppose that $\|\cdot\|$ is a δ-metric. By Remark 9.10, there is $\gamma \in P^{1,1}(W)$ of codimension 0 such that $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}] = [\gamma]$. Since γ is of codimension 0, we may handle γ as a piecewise smooth form and hence we get

$$[\varphi^*(\gamma)] = [\varphi^*(c_1(L|_W, \|\cdot\|)_{\mathrm{ps}})] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)_{\mathrm{ps}}] \in E^{1,1}(W').$$

Proposition 9.8 yields that $\beta := c_1(L, \|\cdot\|)_{\mathrm{res}} + \gamma \in P^{1,1}(W)$ and that

$$[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)_{\mathrm{res}}] + [\varphi^*(\gamma)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)] \in E^{1,1}(W')$$

as claimed. It remains to show that $\beta \in B^{1,1}(W)$. Let (V, φ_U) be a tropical chart in W and let ϕ be a piecewise smooth function on $\mathrm{Trop}(U)$ as in Definition 9.9 such that $\beta|_V$ is given by $\beta_V \in P^{1,1}(V, \varphi_U)$. For every tropical chart $(U', \varphi_{U'})$ of an algebraic variety X' over K compatible with (V, φ_U) with respect to the morphism $f : X' \to X$ and for $\Omega' := \mathrm{trop}_{U'}(V')$, the last display yields

$$[f^*(\beta_V)|_{\Omega'}] = d'd''[\phi \circ F] \in D^{1,1}(\Omega'), \tag{9.11.1}$$

where $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ is the canonical affine map. Since this supercurrent is d'-closed and d''-closed on Ω', we conclude that β is given on V by an element of $Z(V, \varphi_U)$. This shows $\beta \in B^{1,1}(W)$.

To prove the converse, we use that $[c_1(L|_W, \|\cdot\|)] = [\beta]$ for some $\beta \in P^{1,1}(W)$. By Proposition 9.8, the δ-current associated to $\beta - c_1(L|_W, \|\cdot\|)_{\mathrm{res}} \in P^{1,1}(W)$ is $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}]$. By Remark 9.10, $\|\cdot\|$ is a δ-metric. □

Definition 9.12. Let $\|\cdot\|$ be a δ-metric on $L|_W$. By Proposition 6.8, the δ-form β in Proposition 9.11 is unique. We call it the *first Chern δ-form* of $(L|_W, \|\cdot\|)$ and we denote it by $c_1(L|_W, \|\cdot\|)$.

9.13. We summarize the above constructions and definitions. A metric $\|\cdot\|$ on $L|_W$ is a δ-metric if and only if every $x \in W$ is contained in a tropical chart (V, φ_U) in W with a piecewise smooth function ϕ on $N_{U,\mathbb{R}}$ and a nowhere vanishing section s of L over U such that

$$-\log\|s\| = \phi \circ \mathrm{trop}_U$$

on V and such that

$$d'_{\mathrm{P}} d''_{\mathrm{P}}(\phi|_{\mathrm{trop}_U(V)}) = \gamma|_{\mathrm{trop}_U(V)}$$

for a superform γ on $N_{\mathbb{R}}$ of bidegree $(1,1)$ with piecewise smooth coefficients. Then the restriction of $c_1(L|_W, \|\cdot\|)_{\mathrm{res}}$ to V is represented by the δ-preform $\delta_{\phi\cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$, $c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}|_V$ is given by γ and $c_1(L|_W, \|\cdot\|)|_V$ is represented by the δ-preform $\gamma + \delta_{\phi\cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$. A piecewise linear metric is a δ-metric as we can choose ϕ integral Γ-affine (use Remark 1.9) and $\gamma = 0$.

9.14. By construction, the δ-current associated to $c_1(L|_W, \|\cdot\|)$ is equal to the first Chern current $[c_1(L|_W, \|\cdot\|)]$ defined in 7.7 which explains the notation used there. It is an immediate consequence of (9.11.1) that the first Chern δ-form $c_1(L|_W, \|\cdot\|)$ is d'-closed and d''-closed.

To be a δ-metric is a local property and respects isometry. The tensor product of δ-metrics is again a δ-metric and the dual metric of a δ-metric is also a δ-metric. If a positive tensor power of a metric $\|\cdot\|$ on $L|_W$ is a δ-metric, then $\|\cdot\|$ is a δ-metric. It is easy to see that the first Chern δ-form $c_1(L|_W, \|\cdot\|)$ is additive in terms of isometry classes $(L|_W, \|\cdot\|)$ for δ-metrics $\|\cdot\|$.

Proposition 9.15. *Let $\varphi : X' \to X$ be a morphism of algebraic varieties and let L be a line bundle on X endowed with a δ-metric $\|\cdot\|$ over the open subset W of X^{an}. Then $\varphi^*\|\cdot\|$ is a δ-metric on $\varphi^*(L)|_{W'}$ and we have*

$$c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|) = \varphi^* c_1(L|_W, \|\cdot\|) \in B^{1,1}(W') \tag{9.15.1}$$

for any open subset W' of $\varphi^{-1}(W)$.

Proof. This follows from 8.6 and Proposition 9.11. □

Remark 9.16. Smooth metrics and piecewise linear metrics are δ-metrics, which is clear from the definitions. It follows from Proposition 8.11 that every formal metric is a δ-metric. In particular, every algebraic metric on a line bundle of a proper variety is a δ-metric.

Example 9.17. All the canonical metrics in Examples 8.15, 8.16 and 8.17 are δ-metrics. Indeed, a positive tensor power of such a metric is locally the tensor product of a formal metric with a smooth metric and hence the claim follows from Remark 9.16.

10. Monge–Ampère measures

We have seen in the previous section that formal metrics are δ-metrics giving rise to a first Chern δ-form. The formalism of δ-forms allows us to define the Monge–Ampère measure as a wedge product of first Chern δ-forms. We recall that Chambert-Loir has introduced discrete measures for formally metrized line bundles on a proper variety which are important for nonarchimedean equidistribution. The main result of this section shows that the Monge–Ampère measure is equal to the Chambert-Loir measure.

In this section X is a proper algebraic variety over K of dimension n.

10.1. Let $\overline{L_1}, \dots, \overline{L_n}$ be line bundles on X endowed with δ-metrics. Then the wedge product $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_n})$ of the first Chern δ-forms is a δ-form of bidegree (n, n). By Corollary 6.15, the δ-current associated to a δ-form on X^{an} of type (n, n) extends to a bounded linear functional on the space of continuous functions and defines a signed Radon measure on X^{an}. The *Monge-Ampère measure* is the signed measure associated to $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_n})$; it is denoted by

$$\mathrm{MA}\big(c_1(\overline{L_1}), \dots, c_1(\overline{L_n})\big).$$

Proposition 10.2. *If $\varphi : X' \to X$ is a morphism of n-dimensional proper varieties over K, then the following projection formula holds*:

$$\varphi_* \,\mathrm{MA}\big(c_1(\varphi^*\overline{L_1}), \dots, c_1(\varphi^*\overline{L_n})\big) = \deg(\varphi)\,\mathrm{MA}\big(c_1(\overline{L_1}), \dots, c_1(\overline{L_n})\big).$$

Proof. The Stone–Weierstraß theorem [Chambert-Loir and Ducros 2012, proposition (3.3.5)] implies that $A^0(X^{\mathrm{an}})$ is a dense subspace of $C(X^{\mathrm{an}})$. For functions in $A^0(X^{\mathrm{an}})$ the desired equality follows from Proposition 9.15 and from the projection formula for δ-forms (5.9.1). This yields our claim. □

Proposition 10.3. *If X is a proper variety of dimension n, then the total mass of* $\mathrm{MA}\big(c_1(L_1, \|\cdot\|_1), \dots, c_1(L_n, \|\cdot\|_n)\big)$ *is equal to* $\deg_{L_1,\dots,L_n}(X)$.

Proof. This follows as in [Chambert-Loir and Ducros 2012, proposition (6.4.3)]. They handled there only the case of smooth metrics, but our formalism of δ-forms allows us to obtain this result more generally for δ-metrics. □

We recall the crucial properties of Chambert-Loir's measures. They were introduced in a slightly different setting by Chambert-Loir [2006].

Proposition 10.4. *There is a unique way to associate to any n-dimensional proper variety X over K and to any family of formally metrized line bundles $\overline{L_1}, \dots, \overline{L_n}$ on X a signed Radon measure $\mu = \mu_{\overline{L_1},\dots,\overline{L_n}}$ on X^{an} such that the following properties hold*:

(a) *The measure μ is multilinear and symmetric in $\overline{L_1}, \dots, \overline{L_n}$.*

(b) *If $\varphi : Y \to X$ is a morphism of n-dimensional proper varieties over K, then the following projection formula holds*:

$$\varphi_*(\mu_{\varphi^*\overline{L_1},\dots,\varphi^*\overline{L_n}}) = \deg(\varphi)\mu_{\overline{L_1},\dots,\overline{L_n}}.$$

(c) *If $\mathscr{X}$ is a formal K°-model of X with reduced special fibre $\mathscr{X}_s$ and if the metric of $\overline{L_j}$ is induced by a formal K°-model $\mathscr{L}_j$ of L_j on $\mathscr{X}$ for every $j = 1, \dots, n$, then*

$$\mu = \sum_Y \deg_{\mathscr{L}_1,\dots,\mathscr{L}_n}(Y)\delta_{\xi_Y},$$

where Y ranges over the irreducible components of $\mathscr{X}_s$ and δ_{ξ_Y} is the Dirac measure in the unique point ξ_Y of X^{an} which reduces to the generic point of Y (see [Berkovich 1990, Proposition 2.4.4]).

(d) *The total mass is given by $\mu(X^{\mathrm{an}}) = \deg_{L_1,\dots,L_n}(X)$.*

Proof. For existence, we refer to [Gubler 2007, §3]. Uniqueness follows from (c) alone as the existence of a simultaneous formal K°-model with reduced special fibre is a consequence of [Gubler 1998, Proposition 7.5]. □

Theorem 10.5. *For formally metrized line bundles $\overline{L_1}, \dots, \overline{L_n}$ on the proper variety X of dimension n, the Monge–Ampère measure* $\mathrm{MA}(c_1(\overline{L_1}), \dots, c_1(\overline{L_n}))$ *agrees with the Chambert-Loir measure $\mu_{\overline{L_1},\dots,\overline{L_n}}$.*

This theorem was first proven by Chambert-Loir and Ducros [2012, §6.9] for their Monge–Ampère measures defined by a tricky approximation process with smooth metrics. Their argument uses Zariski–Riemann spaces, while we use here a more tropical approach related to our δ-forms.

10.6. In Lemma 10.8, we will consider a closed subvariety $\mathscr{U}$ of a torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$ over K°. We will use the following notation: N is the dual of the free abelian group M of finite rank. Let U be the generic fibre of $\mathscr{U}$ and let $\mathscr{U}_s$ be the special fibre.

The tropicalization $\mathrm{trop} : (\mathbb{T}_K)^{\mathrm{an}} \to N_{\mathbb{R}}$ (resp. $\mathrm{trop} : \mathbb{T}_s^{\mathrm{an}} \to N_{\mathbb{R}}$) with respect to the valuation v on K (resp. the trivial valuation on $\tilde{K}$) leads to the tropical variety $\mathrm{Trop}(U)$ (resp. $\mathrm{Trop}(\mathscr{U}_s)$).

The local cone $\mathrm{LC}_0(\mathrm{Trop}(U))$ at 0 is defined as the cone in $N_{\mathbb{R}}$ which agrees with $\mathrm{Trop}(U)$ in a neighbourhood of 0. We endow it with the weights induced by the canonical tropical weights on $\mathrm{Trop}(U)$.

10.7. For the proof of Theorem 10.5, we need a preparatory result. Let L be a line bundle on the proper variety X over K. We consider an algebraic K°-model $(\mathscr{X}, \mathscr{L})$ of (X, L). Then we get an algebraic metric $\|\cdot\|_{\mathscr{L}}$ on L. We have seen in 8.18 that the restriction $\mathscr{L}_s$ of $\mathscr{L}$ to the special fibre $\mathscr{X}_s$ has a canonical metric $\|\cdot\|_{\mathrm{can}}$. Note that the first metric is continuous on the Berkovich space X^{an} with respect to the given valuation v while $\|\cdot\|_{\mathrm{can}}$ is continuous on the Berkovich space $\mathscr{X}_s^{\mathrm{an}}$ with respect to the trivial valuation on the residue field $\tilde{K}$. Since $\mathscr{X}$ is assumed to be proper, we have a reduction map $\pi : X^{\mathrm{an}} \to \mathscr{X}_s$. For $x \in X^{\mathrm{an}}$, $\pi(x)$ is a scheme theoretic point of $\mathscr{X}_s$. Using the trivial valuation on the residue field of $\pi(x)$, we will view $\pi(x)$ as a point of $\mathscr{X}_s^{\mathrm{an}}$.

In the next lemma, we will show that $\|\cdot\|_{\mathrm{can}}$ is piecewise linear in an neighbourhood of $\pi(x)$ in $\mathscr{X}_s^{\mathrm{an}}$. This means that using a trivialization and a tropicalization, the canonical metric is induced by a piecewise linear function on the tropical variety. It will be crucial in the proof of Theorem 10.5 that we can use tropically the same piecewise linear function to describe the formal metric $\|\cdot\|_{\mathscr{L}}$ in a neighbourhood of x in X^{an}. We now make this precise:

Lemma 10.8. *Under the setup given in 10.7, we fix an element $x \in X^{\mathrm{an}}$ and an open neighbourhood $\mathscr{V}$ of $\pi(x)$ in $\mathscr{X}$. Then there is an open neighbourhood $\mathscr{U}$ of $\pi(x)$ in $\mathscr{V}$ and a closed embedding $\mathscr{U} \hookrightarrow \mathbb{T}$ into a torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$ with the following properties* (using the notation from 10.6):

(a) *We have $0 = \mathrm{trop}(x)$ and the weighted local cone in 0 satisfies*

$$\mathrm{LC}_0(\mathrm{Trop}(U)) = \mathrm{Trop}(\mathscr{U}_s).$$

(b) *There is an open neighbourhood $\widetilde{\Omega}$ of 0 in $N_{\mathbb{R}}$ with*

$$\mathrm{LC}_0(\mathrm{Trop}(U)) \cap \widetilde{\Omega} = \mathrm{Trop}(U) \cap \widetilde{\Omega}.$$

(c) *There exist a complete rational polyhedral fan Σ on $N_{\mathbb{R}}$ and a continuous function $\phi : N_{\mathbb{R}} \to \mathbb{R}$ which is piecewise linear with respect to Σ (i.e., for every $\sigma \in \Sigma$, there is $u_\sigma \in M$ with $\phi = u_\sigma$ on σ).*

(d) *$\mathscr{U}$ is a trivialization of $\mathscr{L}$ with respect to a nowhere vanishing section $s \in \Gamma(\mathscr{U}, \mathscr{L})$.*

(e) *We have $-\log\|s\|_{\mathscr{L}} = \phi \circ \mathrm{trop}$ on a neighbourhood of x in X^{an}.*

(f) *We have $-\log\|s\|_{\mathrm{can}} = \phi \circ \mathrm{trop}$ on a neighbourhood of $\pi(x)$ in $\mathscr{X}_s^{\mathrm{an}}$.*

If $\pi(x)$ is the generic point of an irreducible component of $\mathscr{X}_s$, then there is a $\mathscr{U}$ as above with (a)–(d) *and the following stronger properties*:

(e′) *We have $-\log\|s\|_{\mathscr{L}} = \phi \circ \mathrm{trop}$ on $\mathrm{trop}^{-1}(\widetilde{\Omega}) \subseteq U^{\mathrm{an}}$.*

(f′) *The identity $-\log\|s\|_{\mathrm{can}} = \phi \circ \mathrm{trop}$ holds on $\mathscr{U}_s^{\mathrm{an}}$.*

Proof. Let $(\mathscr{U}_i)_{i\in I}$ be a finite affine open covering of $\mathscr{X}$ such that $\mathscr{L}$ is trivial over any $\mathscr{U}_i$. The generic (resp. special) fibre of $\mathscr{U}_i$ is denoted by U_i (resp. $\mathscr{U}_{i,s}$). For every $i \in I$, we choose a nowhere vanishing section $s_i \in \Gamma(\mathscr{U}_i, \mathscr{L})$. Let $I(x) := \{i \in I \mid \pi(x) \in \mathscr{U}_{i,s}\}$. For $i \in I(x)$, let $(x_{ij})_{j\in J_i}$ be a finite set of generators of the K°-algebra $\mathscr{O}(\mathscr{U}_i)$. Replacing x_{ij} by $1+x_{ij}$ if necessary, we may assume that these generators are invertible in $\pi(x)$. For $i \in I$, we have an affinoid subdomain

$$U_i^\circ := \{z \in U_i^{\mathrm{an}} \mid |a(z)| \leq 1 \ \forall a \in \mathscr{O}(\mathscr{U}_i)\} = \{z \in U_i^{\mathrm{an}} \mid \pi(z) \in \mathscr{U}_{i,s}\}$$

of X^{an}. Using the trivial valuation on $\tilde{K}$, we similarly get an affinoid subdomain $\mathscr{U}_{i,s}^\circ := \{z \in \mathscr{U}_{i,s}^{\mathrm{an}} \mid |a(z)| \leq 1 \ \forall a \in \mathscr{O}(\mathscr{U}_{i,s})\}$ of $\mathscr{X}_s^{\mathrm{an}}$. We consider $\pi(x)$ as a point of $\mathscr{X}_s^{\mathrm{an}}$ by using the trivial absolute value on the residue field of $\pi(x)$ and hence we have $I(x) = \{i \in I \mid \pi(x) \in \mathscr{U}_{i,s}^\circ\}$.

It is easy to see that $\pi(x)$ has a very affine open neighbourhood $\mathscr{U}$ in $\mathscr{X}$ such that $\mathscr{U}$ is contained in $\mathscr{U}_i$ for every $i \in I(x)$. Very affine means that there is a closed embedding $\varphi : \mathscr{U} \hookrightarrow \mathbb{T}$ into a torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$. By shrinking $\mathscr{U}$ and by adding new invertible functions to φ, we obtain the following properties:

(i) For every $i, k \in I(x)$, the invertible meromorphic function s_i/s_k on $\mathscr{U}$ is the restriction of a character $\chi^{u_{ik}}$ associated to some $u_{ik} \in M$.

(ii) For every $i \in I(x)$ and every $j \in J_i$, the generator x_{ij} is invertible on $\mathscr{U}$ and equal to the restriction of a character $\chi^{u'_{ij}}$ associated to some $u'_{ij} \in M$.

Note that we have $0 = \mathrm{trop}(\pi(x)) \in \mathrm{Trop}(\mathscr{U}_s)$ since we use the trivial valuation on the residue field of $\pi(x)$. It follows from $\pi(x) \in \mathscr{U}_s$ that $\mathrm{trop}(x) = 0$. By definition, $\mathscr{U}_s$ is the initial degeneration of U at 0 and hence (a) follows from [Gubler 2013, Propositions 10.15, 13.7]. By definition of the local cone, we find an open neighbourhood $\widetilde{\Omega}$ of 0 in $N_{\mathbb{R}}$ with (b).

By construction, $\mathscr{L}$ is trivial over $\mathscr{U}$ and we choose $s := s_k$ for a fixed $k \in I(x)$ in (d). For $i \in I(x)$, we define the rational cone $\sigma_i := \{\omega \in N_{\mathbb{R}} \mid \langle \omega, u'_{ij}\rangle \geq 0 \,\forall j \in J_i\}$ in $N_{\mathbb{R}}$. Then (ii) yields

$$\mathscr{U}^\circ_{i,s} \cap \mathscr{U}_s^{\mathrm{an}} = \mathrm{trop}^{-1}(\sigma_i) \cap \mathscr{U}_s^{\mathrm{an}}. \tag{10.8.1}$$

By the Bieri–Groves theorem, $\mathrm{Trop}(\mathscr{U}_s)$ is the support of a rational polyhedral fan in $N_{\mathbb{R}}$ (see [Gubler 2013, Remark 3.4]). We conclude that there is a complete rational polyhedral fan Σ on $N_{\mathbb{R}}$ and a rational polyhedral subfan Σ_x with

$$|\Sigma_x| = \bigcup_{i \in I(x)} \sigma_i \cap \mathrm{Trop}(\mathscr{U}_s)$$

such that every cone $\sigma \in \Sigma_x$ is contained in σ_i for some $i \in I(x)$. Note that $\|s_i\|_{\mathrm{can}} = 1$ on $\mathscr{U}^\circ_{i,s}$ and hence (i) shows that

$$-\log\|s\|_{\mathrm{can}} = -\log|s_k/s_i| = u_{ki} \circ \mathrm{trop} \tag{10.8.2}$$

on $\mathscr{U}^\circ_{i,s} \cap \mathscr{U}_s^{\mathrm{an}}$. By (10.8.1), there is a continuous function $\phi : |\Sigma_x| \to \mathbb{R}$ with $\phi = u_{ki}$ on every σ. Using Remark 1.9 and passing to a refinement of Σ, we easily extend ϕ to a continuous function on $N_{\mathbb{R}}$ satisfying (c). Since $\mathscr{X}_s$ is proper over $\tilde{K}$, the sets $\mathscr{U}^\circ_{i,s}$, $i \in I$, form an open covering of $\mathscr{X}_s^{\mathrm{an}}$. It follows from (10.8.1) and (10.8.2) that (f) holds in the neighbourhood

$$W := \mathscr{U}_s^{\mathrm{an}} \setminus \bigcup_{i \in I \setminus I(x)} \mathscr{U}^\circ_{i,s}$$

of $\pi(x)$ in $\mathscr{X}_s^{\mathrm{an}}$.

Again (ii) shows

$$U_i^\circ \cap U^{\mathrm{an}} = \mathrm{trop}^{-1}(\sigma_i) \cap U^{\mathrm{an}} \tag{10.8.3}$$

for every $i \in I(x)$. Note that $\|s_i\|_{\mathscr{L}} = 1$ on U_i° and hence (i) shows that

$$-\log\|s\|_{\mathscr{L}} = -\log|s_k/s_i| = u_{ki} \circ \mathrm{trop} \tag{10.8.4}$$

on $U_i^\circ \cap U^{\mathrm{an}}$. Since X is proper, the sets U_i°, $i \in I$, form a compact covering of X^{an}. It follows from (a), (b), (10.8.3) and (10.8.4) that (e) holds in the neighbourhood $\mathrm{trop}^{-1}(\widetilde{\Omega}) \setminus \bigcup_{i \in I \setminus I(x)} U_i^\circ$ of x in X^{an}. This proves (e).

We assume now that $\pi(x)$ is the generic point of an irreducible component Y of $\mathscr{X}_s$. Then we may assume that $\mathscr{U}_s \subseteq Y$. Let $i \in I$ with $\mathscr{U}_{i,s} \cap Y \neq \varnothing$. Since we use the trivial valuation on the residue field of $\pi(x)$, we deduce easily that $\pi(x) \in \mathscr{U}_{i,s}^\circ$. By construction, we get $W = \mathscr{U}_s^{\mathrm{an}}$, proving (f$'$). It remains to show (e$'$). Let $i \in I \setminus I(x)$. By construction, we have $\mathscr{U}_{i,s} \cap Y = \varnothing$. For $y \in U_i^\circ \cap U^{\mathrm{an}}$, we have $\pi(y) \in \mathscr{U}_{i,s}$ and hence $\pi(y) \notin Y$. In particular, we have $y \notin U^\circ$. Using $\mathrm{trop}^{-1}(0) \cap U^{\mathrm{an}} = U^\circ$, we see that $\mathrm{trop}(U_i^\circ \cap U^{\mathrm{an}})$ is a closed subset of $\mathrm{Trop}(U)$ not containing 0. By shrinking $\widetilde{\Omega}$, we may assume that $\widetilde{\Omega}$ is a neighbourhood of 0 which is disjoint from $\mathrm{trop}(U_i^\circ \cap U^{\mathrm{an}})$ for every $i \in I \setminus I(x)$. Then the above proof of (e) shows that (e$'$) holds. $\square$

Proof of Theorem 10.5. Let $\mu^{\mathrm{MA}} := \mathrm{MA}(c_1(\overline{L_1}), \dots, c_1(\overline{L_n}))$. For simplicity, we assume that $L = L_1 = \cdots = L_n$ and that all metrics are induced by the same K°-model $\mathscr{L}$ on $\mathscr{X}$. The general case follows either by the same arguments or by multilinearity. It is more convenient for us to work algebraically and so we use Proposition 8.13 to assume that $\mathscr{X}$ and $\mathscr{L}$ are algebraic K°-models. There is a generically finite surjective morphism $\mathscr{X}' \to \mathscr{X}$ from a proper flat variety $\mathscr{X}'$ over K° with reduced special fibre. This is a consequence of de Jong's pluristable alteration theorem which works over any Henselian valuation ring (see [Berkovich 1999, Lemma 9.2]). Since both sides of the claim satisfy the projection formula, we may prove the claim for $\mathscr{X}'$. This shows that we may assume that $\mathscr{X}$ is an algebraic K°-model of X with reduced special fibre.

We will analyse μ^{MA} in a neighbourhood of $x \in X^{\mathrm{an}}$. Let $\pi(x) \in \mathscr{X}_s$ be the reduction of x. We choose a very affine open neighbourhood $\mathscr{U}$ of $\pi(x)$ in $\mathscr{X}$ as in Lemma 10.8. We will use the closed embedding $\mathscr{U} \hookrightarrow \mathbb{T}$ into the torus $\mathbb{T}$ and the notation from there.

It follows from a theorem of Ducros [2012, théorème 3.4] that x has a compact analytic neighbourhood V such that the germ of $\mathrm{trop}(V)$ in $\mathrm{trop}(x)$ (considering polytopal neighbourhoods) agrees with the germ of $\mathrm{trop}(W)$ in $\mathrm{trop}(x)$ for every compact analytic neighbourhood $W \subseteq V$ of x. Using that $\mathrm{trop}(x) = 0$, we deduce from [Ducros 2012, théorème 3.4 item 1)] that the dimension of the germ is equal to the transcendence degree of $\tilde{K}(\pi(x))$ over $\tilde{K}$.

We first assume that $\pi(x)$ is not the generic point of an irreducible component of $\mathscr{X}_s$. Then the transcendence degree of $\tilde{K}(\pi(x))$ over $\tilde{K}$ is less than $n = \dim(X)$. Using the theorem of Ducros, there is a compact analytic neighbourhood V of x in X^{an} such that $\mathrm{trop}(V)$ has dimension $< n$ and such that Lemma 10.8(e) holds on V. We choose a tropical chart $(V', \varphi_{U'})$ in x which is contained in V and with U' contained in the generic fibre U of $\mathscr{U}$. We describe $c_1(\bar{L})|_{V'}$ using the function ϕ constructed in Lemma 10.8 and the canonical affine map $F : N_{U',\mathbb{R}} \to N_{\mathbb{R}}$. Using that ϕ is piecewise linear, it follows from 9.13 that $c_1(\bar{L})|_{V'} = \mathrm{trop}_{U'}^*(\beta)$ for $\beta \in \mathrm{AZ}^{1,1}(V', \varphi_{U'})$ represented by the δ-preform $\delta_{F^*(\phi)\cdot N_{U',\mathbb{R}}}$ on $N_{U',\mathbb{R}}$. By our construction of products and Corollary 1.15, μ^{MA} is given on V' by $\beta^{\wedge n} \in \mathrm{AZ}^{n,n}(V', \varphi_{U'})$ represented by the δ-preform

$$(\delta_{F^*(\phi)\cdot N_{U',\mathbb{R}}})^{\wedge n} = \delta_{F^*(C)} \tag{10.8.5}$$

on $N_{U',\mathbb{R}}$, where C is the n-codimensional tropical cycle of $N_{\mathbb{R}}$ obtained by the n-fold self-intersection of the tropical divisor $\phi \cdot N_{\mathbb{R}}$. Since $V' \subseteq V$, we have $F(\mathrm{trop}_{U'}(V')) \subseteq \mathrm{trop}(V)$ and hence $\dim(F(\mathrm{trop}_{U'}(V'))) < n$. It follows from the definition of the pull-back and the local nature of stable tropical intersection that $\delta_{F^*(C)\cdot \mathrm{Trop}(U')}$ does not meet the open subset $\mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. A similar argument applies to any tropical chart compatible with $(V', \varphi_{U'})$ and hence (10.8.5) yields $\beta^{\wedge n} = 0$. We conclude that the support of μ^{MA} does not meet V'.

Now we assume that $\pi(x)$ is the generic point of an irreducible component Y of $\mathscr{X}_s$. Then x is the unique point of X^{an} with reduction $\pi(x)$ (see [Berkovich 1990, Proposition 2.4.4]) and we write $x = \xi_Y$. We may assume that the very affine open neighbourhood $\mathscr{U}$ of $\pi(x)$ in $\mathscr{X}$ from Lemma 10.8 has special fibre $\mathscr{U}_s$ disjoint from all other irreducible components Y' of $\mathscr{X}_s$. We conclude that $\pi(\xi_{Y'}) \notin \mathscr{U}_s$ and hence $\mathrm{trop}(\xi_{Y'}) \neq 0 = \mathrm{trop}(x)$. We may choose the neighbourhood $\widetilde{\Omega}$ of 0 in $N_{\mathbb{R}}$ disjoint from all points $\mathrm{trop}(\xi_{Y'})$. We will use in the following that Lemma 10.8(e′) holds on the open subset $V := \mathrm{trop}^{-1}(\widetilde{\Omega})$ of X^{an}. Since no $\xi_{Y'}$ is contained in V, the nongeneric case above shows that the restriction of μ^{MA} to V is supported in ξ_Y.

Now we choose a very affine open subset U' contained in the generic fibre U of $\mathscr{U}$ with $x \in (U')^{\mathrm{an}}$. Let $F : N_{U',\mathbb{R}} \to N_{\mathbb{R}}$ be the canonical affine map. Then

$$V' := \mathrm{trop}_{U'}^{-1}(F^{-1}(\widetilde{\Omega})) = (U')^{\mathrm{an}} \cap V.$$

is an open neighbourhood of x in X^{an} and $(V', \varphi_{U'})$ is a tropical chart. Similarly to the above, μ^{MA} is given on V' by $\beta^{\wedge n} \in \mathrm{AZ}^{n,n}(V', \varphi_{U'})$ represented by the δ-preform in (10.8.5). Since $\mu^{\mathrm{MA}}|_{V'}$ is supported in the single point $x = \xi_Y$, we conclude that the 0-dimensional tropical cycle $F^*(C) \cdot \mathrm{Trop}(U')$ has only one point ω' contained in the open subset $\mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. In fact, we have $\omega' = \mathrm{trop}_{U'}(x)$ with multiplicity $\mu^{\mathrm{MA}}(V')$. The tropical projection formula in Proposition 1.5 and the Sturmfels–Tevelev multiplicity formula [Gubler 2013, Theorem 13.17] give the identity

$$F_*(F^*(C) \cdot \mathrm{Trop}(U')) = C \cdot \mathrm{Trop}(U)$$

of tropical cycles on $N_{\mathbb{R}}$. Using that $\mathrm{trop}_{U'}(V') = F^{-1}(\mathrm{trop}(V)) \cap \mathrm{Trop}(U')$, we deduce that $\mu^{\mathrm{MA}}(V')$ is equal to the multiplicity of $0 = \mathrm{trop}(x) = F(\omega')$ in $C \cdot \mathrm{Trop}(U)$.

By Lemma 10.8, we conclude that $\mu^{\mathrm{MA}}(V')$ is equal to the tropical intersection number of C with $\mathrm{LC}_0(\mathrm{Trop}(U)) = \mathrm{Trop}(\mathscr{U}_s)$.

We recall that C is the n-fold self-intersection of the tropical divisor $\phi \cdot N_{\mathbb{R}}$ and we note that these objects are weighted tropical fans. Now we use Lemma 10.8(f′). This shows that $\mathscr{U}_s$ is a very affine chart of integration for $c_1(\mathscr{L}_s|_Y, \|\cdot\|_{\mathrm{can}})^n$, where this δ-form is represented by the pull-back of δ_C with respect to the canonical affine map $N_{\mathscr{U}_s,\mathbb{R}} \to N_{\mathbb{R}}$. Note that we may perform a base change to omit the trivial valuation which was excluded for simplicity in our paper. The tropical projection formula and the Sturmfels–Tevelev multiplicity formula show

$$\int_{Y^{\mathrm{an}}} c_1(\mathscr{L}_s|_Y, \|\cdot\|_{\mathrm{can}})^n = \deg(C \cdot \mathrm{Trop}(\mathscr{U}_s))$$

as above. By Proposition 10.3, the left-hand side is equal to $\deg_{\mathscr{L}}(Y)$. We have seen above that the right-hand side equals $\mu^{\mathrm{MA}}(V')$. This proves that $\mu^{\mathrm{MA}}|_V$ is a point measure concentrated in $x = \xi_Y$ with total mass $\deg_{\mathscr{L}}(Y)$. By Proposition 10.4(c), the Chambert-Loir measure $\mu_{\overline{L}_1,\ldots,\overline{L}_n}$ is equal to μ^{MA}. □

11. Green currents

In this section X is an algebraic variety over K of dimension n. We introduce Green currents for cycles on X. We define the product $g_Y * g_Z$ for a divisor Y and a cycle Z on X which intersect properly. This operation has the expected properties.

Definition 11.1. Let Z be a cycle of X of codimension p and let g be any δ-current in $E^{p-1,p-1}(X^{\mathrm{an}})$. Then we define

$$\omega(Z, g) := d'd''g + \delta_Z \in E^{p,p}(X^{\mathrm{an}}).$$

If there is a δ-form $\omega_{Z,g} \in B^{p,p}(X^{\mathrm{an}})$ with $\omega(Z, g) = [\omega_{Z,g}]$, then we call g a *Green current* for the cycle Z. We will use often the notation g_Z for such a current and then we set $\omega(g_Z) := \omega(Z, g_Z)$ and $\omega_Z := \omega_{Z,g_Z}$ for simplicity.

11.2. Let $(L, \|\cdot\|)$ be a line bundle on X endowed with a δ-metric and let Z be a cycle of codimension p in X with any current $g_Z \in E^{p-1,p-1}(X^{\mathrm{an}})$. We assume that s is a meromorphic section of L with Cartier divisor D intersecting Z properly. By the Poincaré–Lelong equation in Corollary 7.8 and by the definition of the first Chern δ-form in Definition 9.12, $g_Y := [-\log\|s\|]$ is a Green current for the Weil divisor Y associated to D with $\omega_Y = c_1(L, \|\cdot\|)$.

If Z is a prime cycle of codimension p, then we define $g_Y \wedge \delta_Z \in E^{p,p}(X^{\mathrm{an}})$ as the push-forward of $[-\log\|s\||_Z]$ with respect to the inclusion $i_Z : Z \to X$. In general, we proceed by linearity in the prime components of Z to define $g_Y \wedge \delta_Z \in E^{p,p}(X^{\mathrm{an}})$. This leads to the definition of the $*$-*product*

$$g_Y * g_Z := g_Y \wedge \delta_Z + \omega_Y \wedge g_Z \in E^{p,p}(X^{\mathrm{an}}).$$

Lemma 11.3. *Under the hypothesis above and if Z is prime, then we have the identity*

$$d'd''[-\log\|s\||_Z] = [\omega_Y|_Z] - \delta_{D\cdot Z} \tag{11.3.1}$$

of δ-currents on Z^{an}.

Proof. This follows immediately from the Poincaré–Lelong equation for $s|_Z$ (see Corollary 7.8). We use here $c_1(L|_Z, \|\cdot\|) = c_1(L, \|\cdot\|)|_Z$, which follows from Proposition 9.15. □

Proposition 11.4. *Under the hypothesis in 11.2, we have*

$$\omega(D\cdot Z, g_Y * g_Z) = \omega_Y \wedge \omega(g_Z).$$

If g_Z is a Green current for Z, then $g_Y * g_Z$ is a Green current for $D\cdot Z$.

Proof. Using Lemma 11.3 and linearity in the prime components of Z, we get

$$d'd''[-\log\|s\| \wedge \delta_Z] = \omega_Y \wedge \delta_Z - \delta_{D\cdot Z} \tag{11.4.1}$$

and hence 9.14 and Proposition 4.15(iii) give

$$\omega(D\cdot Z, g_Y * g_Z) = d'd''[-\log\|s\| \wedge \delta_Z] + d'd''(\omega_Y \wedge g_Z) + \delta_{D\cdot Z} = \omega_Y \wedge \omega(g_Z),$$

proving the claim. □

Proposition 11.5. *For $i = 1, 2$, let L_i be a line bundle on X with a δ-metric $\|\cdot\|_i$ and nonzero meromorphic section s_i. We assume that the associated Cartier divisors D_1 and D_2 intersect properly. Let $\eta_{Y_i} := -\log\|s_i\|_i$ and let $g_{Y_i} = [\eta_{Y_i}]$ be the induced Green current for the Weil divisor Y_i of D_i. Then we have the identity*

$$g_{Y_1} * g_{Y_2} - g_{Y_2} * g_{Y_1} = d'[d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2}] + d''[\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2}]$$

of δ-currents on X^{an}.

Note that the piecewise smooth forms $d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2}$ and $\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2}$ of degree 1 are defined on the analytification of a Zariski open and dense subset of X. By 9.5 and Proposition 6.5, they define δ-currents on X^{an}.

Proof. The claim can be checked locally on X. Hence we may assume that X is affine and $L_1 = L_2 = O_X$. For $i = 1, 2$, we may view s_i as a rational function f_i and we have

$$\eta_{Y_i} = -\log|f_i| - \log\|1\|_i.$$

The usual partition of unity argument shows that it is enough to test the claim by evaluating at $\alpha \in B_c^{n-1,n-1}(W)$ for a small open neighbourhood W of a given point x in X^{an}. There are finitely many tropical charts $\{(V_j, \varphi_{U_j})\}_{j=1,\ldots,m}$ in W covering $\mathrm{supp}(\alpha)$ such that $\alpha = \mathrm{trop}_{U_j}^*(\alpha_j)$ on V_j for some element $\alpha_j \in \mathrm{AZ}^{n-1,n-1}(V_j, \varphi_{U_j})$. We will use a Zariski dense very affine open subset U of $U_1 \cap \cdots \cap U_m$ which will serve as a very affine chart of integration for various forms. Now we consider the restriction of the canonical affine map $F_j : N_{U,\mathbb{R}} \to N_{U_j,\mathbb{R}}$ to $\mathrm{Trop}(U)$. Let Ω in $\mathrm{Trop}(U)$ denote the union of the preimages of the open subsets $\Omega_j := \mathrm{trop}_{U_j}(V_j)$ in $\mathrm{Trop}(U_j)$ and put $V = \mathrm{trop}_U^{-1}(\Omega)$. By Proposition 4.12 there exists a unique element $\alpha_U \in \mathrm{AZ}^{n-1,n-1}(V, \varphi_U)$ such that $\alpha|_V = \mathrm{trop}_U^*(\alpha_U)$ and such that α_U coincides for all j on the preimage of Ω_j with the pull-back of α_j. Note that Ω is an open subset of $\mathrm{Trop}(U)$ and (V, φ_U) is a tropical chart for $V := \mathrm{trop}_U^{-1}(\Omega)$. Then α_U has not necessarily compact support, but we can extend α_U by zero to an element in $\mathrm{AZ}^{n-1,n-1}(U^{\mathrm{an}}, \varphi_U)$ using that $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}}$ is a closed subset of V. By abuse of notation, this extension will also be denoted by α_U. Then we have $\alpha = \mathrm{trop}_U^*(\alpha_U)$ on U^{an}. By shrinking W and using an appropriate U, we may assume that

$$-\log\|1\|_i = \phi_i \circ \mathrm{trop}_U$$

on V for a piecewise smooth function ϕ_i on $\mathrm{Trop}(U)$ and $i = 1, 2$. Since we deal with δ-metrics, we may assume that there is a piecewise smooth extension $\tilde{\phi}_i$ of ϕ_i to $N_{U,\mathbb{R}}$ and a superform γ_i on $N_{U,\mathbb{R}}$ of bidegree (1, 1) such that the first Chern δ-form ω_{Y_i} is represented on V by the δ-preform $\gamma_i + \delta_{\tilde{\phi}_i \cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$ and such that γ_i restricts to $d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_i$ on Ω (see 9.13). We have

$$\eta_{Y_i} = -\log|f_i| - \log\|1\|_i = -\log|f_i| + \phi_i \circ \mathrm{trop}_U$$

on V. Using bilinearity of $*$ and of $\wedge$, we may either assume that η_{Y_i} is equal to $-\log\|1\|_i$ or equal to $-\log|f_i|$. Hence we have to consider the following four cases:

<u>Case 1</u>: $s_1 = s_2 = 1$. In this case, the divisors Y_1, Y_2 are zero and $\eta_{Y_i} = -\log\|1\|_i$ for $i = 1, 2$ are piecewise smooth functions on X^{an}. Then we have

$$\langle g_{Y_1} * g_{Y_2}, \alpha \rangle = \langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle = \langle g_{Y_2}, \omega_{Y_1} \wedge \alpha \rangle. \tag{11.5.1}$$

Recall that g_{Y_2} is the current associated to η_{Y_2}. By 9.3, we have $\mathrm{PS}^{0,0}(W) = P^{0,0}(W)$ and hence $\eta_{Y_2}\alpha \in P_c^{n-1,n-1}(W)$. We may view it as a generalized δ-form on X^{an} given on U^{an} by $\phi_2 \alpha_U \in P^{n-1,n-1}(U^{\mathrm{an}}, \varphi_U)$. Since the first Chern δ-form ω_{Y_1} is represented on V by $\delta_{\tilde{\phi}_1 \cdot N_{U,\mathbb{R}}} + \gamma_1 \in P^{1,1}(N_{U,\mathbb{R}})$, we get

$$\langle g_{Y_1} * g_{Y_2}, \alpha \rangle = \int_{|\mathrm{Trop}(U)|} (\delta_{\phi_1 \cdot \mathrm{Trop}(U)} + d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_1) \wedge \phi_2 \alpha_U. \tag{11.5.2}$$

Here, we have used that U is a very affine chart of integration for $\omega_{Y_1} \wedge \eta_{Y_2}\alpha \in P_c^{n,n}(X^{\mathrm{an}})$. Recall from 9.13 that the generalized δ-forms $c_1(L_1, \|\cdot\|_1)_{\mathrm{res}}$ and $c_1(L_1, \|\cdot\|_1)_{\mathrm{ps}}$ are represented on V by $\delta_{\tilde{\phi}_1 \cdot N_{U,\mathbb{R}}}$ and γ_1 in $P^{1,1}(N_{U,\mathbb{R}})$. We conclude that U is a very affine chart of integration for $c_1(L_1, \|\cdot\|_1)_{\mathrm{res}} \wedge \eta_{Y_2}\alpha$ and $c_1(L_1, \|\cdot\|_1)_{\mathrm{ps}} \wedge \eta_{Y_2}\alpha$ in $P_c^{n,n}(X^{\mathrm{an}})$ and hence (11.5.2) yields

$$\langle g_{Y_1} * g_{Y_2}, \alpha \rangle = \int_{|\mathrm{Trop}(U)|} \delta_{\phi_1 \cdot \mathrm{Trop}(U)} \wedge \phi_2 \alpha_U + \int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_1 \wedge \phi_2 \alpha_U. \tag{11.5.3}$$

Since α has compact support in W and $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V$, it follows from Proposition 4.21 and Corollary 5.6 that the integrands in (11.5.3) have compact support in Ω. The generalization of Corollary 5.6 to PSP-forms given in 9.5 shows that $-d''_{\mathrm{P}} \log\|1\|_1 \wedge \alpha$ has compact support contained in U^{an}. Again, we conclude that $d''_{\mathrm{P}}\phi_1 \wedge \alpha_U$ has compact support contained in Ω. Now Leibniz's rule and the theorem of Stokes (Proposition 2.7) for d'_{P} show

$$\begin{aligned}\int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U &= \int_{\partial|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}(\phi_2\alpha_U) \\ &= \int_{\partial|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}\phi_2 \wedge \alpha_U \\ &\quad + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2 d'\alpha_U. \end{aligned} \tag{11.5.4}$$

Recall that $\phi_2\alpha_U$ is a δ-preform on $\mathrm{Trop}(U)$ and hence Remark 3.18 gives

$$\int_{|\mathrm{Trop}(U)|} \delta_{\phi_1\cdot\mathrm{Trop}(U)} \wedge \phi_2\alpha_U + \int_{\partial|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U = 0. \tag{11.5.5}$$

Using (11.5.4) and (11.5.5) in (11.5.3), we get

$$\langle g_{Y_1} * g_{Y_2}, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}\phi_2 \wedge \alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2 d'\alpha_U. \tag{11.5.6}$$

A similar computation where we replace (11.5.4) by an application of Stokes' theorem with respect to d''_{P} shows

$$\langle g_{Y_2} * g_{Y_1}, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}\phi_2 \wedge \alpha_U - \int_{|\mathrm{Trop}(U)|} \phi_1 d'_{\mathrm{P}}\phi_2 \wedge d''\alpha_U. \tag{11.5.7}$$

Using that U is a very affine chart of integration for $d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2} \wedge d'\alpha \in \mathrm{PSP}_c^{n,n}(X^{\mathrm{an}})$ and for $\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2} \wedge d''\alpha_U \in \mathrm{PSP}_c^{n,n}(X^{\mathrm{an}})$, this proves the claim in the first case.

<u>Case 2</u>: $s_1 = 1$ and $\|1\|_2 = 1$. In this case Y_1 is zero and $\eta_{Y_2} = -\log|f_2|$. The following computation is similar to the one in the proof of the Poincaré–Lelong formula (see Theorem 7.2) and we will use the same terminology as there. We have $g_{Y_2} * g_{Y_1} = 0$ as $\delta_{Y_1} = 0$ and $\omega_{Y_2} = c_1(O_X, \|\cdot\|_2) = 0$. It remains to show that

$$\omega_{Y_1} \wedge g_{Y_2} = -g_{Y_1} \wedge \delta_{Y_2} + d'[d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2}] + d''[\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2}]. \tag{11.5.8}$$

It is enough to check the claim locally and by linearity, we may assume that f_2 is a regular function on X. By the first case, we may assume that f_2 is nonconstant. We choose a very affine open subset U, the open subset Ω of $\mathrm{Trop}(U)$ and ϕ_1 as above. We may assume that $\mathrm{supp}(\mathrm{div}(f_2)) \cap U = \varnothing$ and hence $-\log|f_2|$ is induced by an integral Γ-affine function φ_2 on $\mathrm{Trop}(U)$. We use the very affine open U to compute the term $\langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle$. Similarly to (11.5.1) and (11.5.2), we deduce

$$\langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle = \int_{|\mathrm{Trop}(U)|} \delta_{\phi_1 \cdot \mathrm{Trop}(U)} \wedge \varphi_2 \alpha_U + \int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_1 \wedge \varphi_2 \alpha_U. \tag{11.5.9}$$

The same computation as in (11.5.4)–(11.5.6) yields

$$\langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle = \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge \varphi_2 d' \alpha_U. \tag{11.5.10}$$

As in the first case, we have

$$\int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge \varphi_2 d' \alpha_U = \langle d'[d''_{\mathrm{P}} \eta_{Y_1} \wedge \eta_{Y_2}], \alpha \rangle. \tag{11.5.11}$$

Similarly to the proof of the Poincaré–Lelong formula, we may assume that the support of α is covered by the interiors of the affinoid subdomains $W_j := \mathrm{trop}_{U_j}^{-1}(\Delta_j)$ of the tropical chart V_j for $j = 1, \ldots, m$. We set $W := \bigcup_{j=1}^m W_j$. We choose $s > 0$ sufficiently small with $\varphi_2 \leq -\log|s|$ on the compact set $\mathrm{supp}(d''_{\mathrm{P}} \phi_1 \wedge \alpha_U)$. Since W covers $\mathrm{supp}(\alpha)$, the analytic subdomain $W(s) := \{x \in W \mid |f_2(x)| \geq s\}$ of W contains $\mathrm{supp}(d''_{\mathrm{P}} \phi_1 \wedge \alpha)$ and hence we have

$$\int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U = \int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} d''_{\mathrm{P}} \phi_1 \wedge d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U. \tag{11.5.12}$$

By the theorem of Stokes (Proposition 2.7), this is equal to

$$\int_{\partial \mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U + \int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge d'' \alpha_U. \tag{11.5.13}$$

By Corollary 5.6, the support of $d''\alpha$ is contained in U^{an}. We may assume that the compact set $\mathrm{supp}(d''\alpha)$ is contained in $W(s)$. Using that U is a very affine chart of integration for $\eta_{Y_1} \wedge d'_{\mathrm{P}} \eta_{Y_2} \wedge d''\alpha$, we get

$$\int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge d'' \alpha_U = \langle d''[\eta_{Y_1} \wedge d'_{\mathrm{P}} \eta_{Y_2}], \alpha \rangle. \tag{11.5.14}$$

Now we apply Remark 7.4 with f_2 instead of f and the generalized δ-form $\eta_{Y_1} \wedge \alpha$ instead of α and observe that φ_2 corresponds to $F^*(x_0)$ in Remark 7.4. Then Equation (7.4.1) yields

$$\int_{\partial \mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U = -\langle g_{Y_1} \wedge \delta_{Y_2}, \alpha \rangle \tag{11.5.15}$$

as W covers $\mathrm{supp}(\alpha)$. Using (11.5.11)–(11.5.15) in (11.5.10), we get (11.5.8) proving the claim in the second case.

<u>Case 3</u>: $\|1\|_1 = 1$ and $s_2 = 1$. The formula proved in the second case yields

$$g_{Y_2} * g_{Y_1} - g_{Y_1} * g_{Y_2} = d'[d''_{\mathrm{P}} \eta_{Y_2} \wedge \eta_{Y_1}] + d''[\eta_{Y_2} \wedge d'_{\mathrm{P}} \eta_{Y_1}]$$

The (1, 1)-current on the left-hand side is clearly symmetric. Hence the right-hand side is symmetric as well and equals

$$-d''[d'_{\mathrm{P}}\eta_{Y_2}\wedge\eta_{Y_1}]-d'[\eta_{Y_2}\wedge d''_{\mathrm{P}}\eta_{Y_1}]$$

This proves our claim in the third case.

Case 4: $\|1\|_1=1$ and $\|1\|_2=1$. In this case $\eta_{Y_1}=-\log|f_1|$ and $\eta_{Y_2}=-\log|f_2|$. We have to show that

$$g_{Y_1}\wedge\delta_{Y_2}-g_{Y_2}\wedge\delta_{Y_1}=d'[d''_{\mathrm{P}}\eta_{Y_1}\wedge\eta_{Y_2}]+d''[\eta_{Y_1}\wedge d'_{\mathrm{P}}\eta_{Y_2}].$$

Again, we may assume that f_1 and f_2 are regular functions on X. By the previous cases, we may assume that these functions are nonconstant. We use the same notation as above. Here, we choose the very affine open subset U disjoint from $\mathrm{supp}(\mathrm{div}(f_1))\cup\mathrm{supp}(\mathrm{div}(f_2))$. Then φ_1,φ_2 are integral Γ-affine functions on $\mathrm{Trop}(U)$ inducing $-\log|f_1|$, $-\log|f_2|$ on U^{an}. Going the computation in the second case backwards, we see that

$$\langle g_{Y_1}\wedge\delta_{Y_2},\alpha\rangle=-\int_{|\mathrm{Trop}(U)|}d''_{\mathrm{P}}\varphi_1\wedge d'\varphi_2\wedge\alpha_U+\langle d''[\eta_{Y_1}\wedge d'_{\mathrm{P}}\eta_{Y_2}],\alpha\rangle. \quad (11.5.16)$$

Note here that $d''_{\mathrm{P}}\varphi_1\wedge d'_{\mathrm{P}}\varphi_2\wedge\alpha_U$ has compact support in Ω. Indeed, it follows from Corollary 5.6, that $d''_{\mathrm{P}}\log|f_1|\wedge d'_{\mathrm{P}}\log|f_2|\wedge\alpha$ is a well-defined δ-form on X^{an} with compact support in U^{an} (using that the divisors intersect properly) and hence we get compactness in Ω from Proposition 4.21. Interchanging the role of Y_1, Y_2 and also of d', d'' and d'_{P}, d''_{P} in (11.5.16), we get the fourth claim. This proves the proposition. □

In the following, we denote the support of a cycle Z (resp. of a Cartier divisor D) on X by $|Z|$ (resp. $|D|$).

Corollary 11.6. *Let Z be a cycle of X of codimension p and let g_Z be any δ-current in $E^{p-1,p-1}(X)$. For $i=1,2$, let L_i be a line bundle on X with a δ-metric $\|\cdot\|_i$ and nonzero meromorphic section s_i. Let D_i denote the Cartier divisor on X defined by s_i. We assume that $|D_1|\cap|Z|$ and $|D_2|\cap|Z|$ both have codimension ≥ 1 in $|Z|$, and that $|D_1|\cap|D_2|\cap|Z|$ has codimension ≥ 2 in $|Z|$. Let $\eta_{Y_i}:=-\log\|s_i\|_i$ and let $g_{Y_i}=[\eta_{Y_i}]$ be the induced Green current for the Weil divisor Y_i of D_i. Then we have*

$$g_{Y_1}*(g_{Y_2}*g_Z)-g_{Y_2}*(g_{Y_1}*g_Z)\in d'(E^{p,p+1}(X^{\mathrm{an}}))+d''(E^{p+1,p}(X^{\mathrm{an}})).$$

Proof. This follows immediately from Proposition 11.5 applied to the analytifications of the prime components of Z. □

12. Local heights of varieties

In this section, we study the local height of a proper variety X of dimension n over K with respect to metrized line bundles endowed with δ-metrics. If the metrics are formal, then we show that these analytically defined local heights agree with the ones based on divisorial intersection theory on formal models in [Gubler 1998]. In particular, they coincide with the local heights used in Arakelov theory over number fields.

12.1. For $i = 0, \dots, n$, let L_i be a line bundle on X endowed with a δ-metric $\|\cdot\|_i$ and a nonzero meromorphic section s_i. For the associated Cartier divisor $D_i := \operatorname{div}(s_i)$, we consider the metrized Cartier divisor $\hat{D}_i := (D_i, \|\cdot\|_i)$, i.e., a Cartier divisor D_i and a metric $\|\cdot\|_i$ on the associated line bundle $O(D_i)$. Recall from 11.2 that we obtain the Green current $g_{Y_i} := [-\log\|s_i\|_i]$ for the Weil divisor Y_i associated to D_i.

We assume that the Cartier divisors $D_0, \dots, D_n$ intersect properly. Then we define *the local height of X with respect to $\hat{D}_0, \dots, \hat{D}_n$* by

$$\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X) := g_{Y_0} * \cdots * g_{Y_n}(1).$$

12.2. If Z is a cycle on X of dimension t and $\hat{D}_0, \dots, \hat{D}_t$ are δ-metrized Cartier divisors on X with $|D_0|, \dots, |D_t|, |Z|$ intersecting properly, then 12.1 induces a local height $\lambda_{\hat{D}_0,\dots,\hat{D}_t}(Z)$ by linearity in the prime components of Z.

Remark 12.3. The problem with this definition is that it is not functorial as the pull-back of a Cartier divisor is not always well defined as a Cartier divisor. This problem is resolved by using pseudodivisors instead of Cartier divisors (see [Fulton 1984, Chapter 2]). We follow [Gubler 2003] and define a *δ-metrized pseudodivisor* as a triple $(\bar{L}, Z, s)$, where $\bar{L} = (L, \|\cdot\|)$ is a line bundle on X equipped with a δ-metric, Z is a closed subset of X, and s is a nowhere vanishing section of L over $X \setminus Z$. Using the same arguments as in [Gubler 2003], we get a local height $\lambda_{\hat{D}_0,\dots,\hat{D}_t}(Z)$ for δ-metrized pseudodivisors which is well defined under the weaker condition $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \varnothing$.

It is straightforward to show that the local height is linear in Z and multilinear in $\hat{D}_0, \dots, \hat{D}_t$. It follows from Corollary 11.6 along the arguments in [Gubler 2003] that the local height is symmetric in $\hat{D}_0, \dots, \hat{D}_t$.

The next result shows that the *induction formula* holds for local heights.

Proposition 12.4. *Let $\hat{D}_0, \dots, \hat{D}_n$ be δ-metrized pseudodivisors on X with*

$$|D_0| \cap \cdots \cap |D_n| = \varnothing.$$

Then the local height $\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X)$ is equal to

$$\lambda_{\hat{D}_0,\dots,\hat{D}_{n-1}}(Y_n) - \int_{X^{\mathrm{an}}} \log\|s_n\|_n \cdot c_1(\overline{O(D_0)}) \wedge \dots \wedge c_1(\overline{O(D_{n-1})}),$$

where we assume that D_n is a Cartier divisor with associated Weil divisor Y_n and canonical meromorphic section s_n of $O(D_n)$.

Proof. The argument is the same as for [Gubler 2003, Proposition 3.5]. □

Proposition 12.5. *Let $\varphi : X' \to X$ be a morphism of proper varieties over K and let $\hat{D}_0, \dots, \hat{D}_n$ be δ-metrized pseudodivisors on X with $|D_0| \cap \dots \cap |D_n| = \varnothing$. Then the functoriality*

$$\deg(\varphi)\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X) = \lambda_{\varphi^*(\hat{D}_0),\dots,\varphi^*(\hat{D}_n)}(X')$$

holds.

Proof. The proof relies on the induction formula in Proposition 12.4 and the projection formula for integrals (5.9.1). We refer to [Gubler 2003] for the analogous arguments in the archimedean case. □

Proposition 12.6 (metric change formula). *Suppose that the local height $\lambda(X)$ with respect to the δ-metrized pseudodivisors $\hat{D}_0, \dots, \hat{D}_n$ is well defined. Let $\lambda'(X)$ be the local height of X obtained by replacing the metric $\|\cdot\|_0$ on $O(D_0)$ by another δ-metric $\|\cdot\|_0'$. Then $\rho := \log(\|\cdot\|_0'/\|\cdot\|_0)$ is a piecewise smooth function on X^{an} and we have*

$$\lambda(X) - \lambda'(X) = \int_{X^{\mathrm{an}}} \rho \cdot c_1(\overline{O(D_1)}) \wedge \dots \wedge c_1(\overline{O(D_n)}).$$

Proof. This follows from linearity and symmetry of the local height in $\hat{D}_0$ and $\hat{D}_n$ and from the induction formula in Proposition 12.4. □

Remark 12.7. Now suppose that $\hat{D}_0, \dots, \hat{D}_n$ are formally metrized pseudodivisors on X with $|D_0| \cap \dots \cap |D_n| = \varnothing$. Then the intersection theory of divisors on admissible formal K°-models given in [Gubler 1998] induces also a local height of X (see [Gubler 2003]). It also satisfies an induction formula involving Chambert-Loir's measures (see [Gubler 2003, Remark 9.5]). Since the Chambert-Loir measure agrees with the Monge–Ampère measure (see Theorem 10.5), we deduce from the induction formula in Proposition 12.4 that the local height based on intersection theory of divisors agrees with $\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X)$ from Remark 12.3. In particular, this proves Theorem 0.4 stated in the introduction.

Appendix: Convex geometry

In this appendix, we gather the notions from convex geometry on a finite dimensional real vector space W coming with an *integral structure*. This means that we consider a free abelian group N of rank r with $W = N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $M := \mathrm{Hom}(N, \mathbb{Z})$ be the dual abelian group and let $V := \mathrm{Hom}(M, \mathbb{R}) = M_{\mathbb{R}}$ be the dual vector space of W. The natural duality between V and W is denoted by $\langle u, \omega \rangle$. Let Γ be a fixed subgroup of $\mathbb{R}$. In the applications, it is usually the value group of a nonarchimedean absolute value.

A.1. Let N' be another free abelian group of finite rank and let $F : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ be an affine map. Then F is called *integral* Γ*-affine* if $F = \mathbb{L}_F + \omega$ with $\omega \in N' \otimes_{\mathbb{Z}} \Gamma \subseteq N'_{\mathbb{R}}$ and with the associated linear map $\mathbb{L}_F$ induced by a homomorphism $N \to N'$.

A.2. A *polyhedron* Δ in W is defined as the intersection of finitely many half spaces $\{\omega \in W \mid \langle u_i, \omega \rangle \geq c_i\}$ with $u_i \in V$ and $c_i \in \mathbb{R}$. If we may choose all $u_i \in M$ and all $c_i \in \Gamma$, then we say that Δ is an *integral* Γ*-affine polyhedron.* A *face* of Δ is either Δ itself or the intersection of Δ with the boundary of a closed half-space containing Δ. We write $\tau \preccurlyeq \Delta$ for a face τ of Δ and we write $\tau \prec \Delta$ if additionally $\tau \neq \Delta$. The *relative interior* of Δ is defined by

$$\mathrm{relint}(\Delta) := \Delta \setminus \bigcup_{\tau \prec \Delta} \tau.$$

Note that every polyhedron is convex. A *polytope* is a bounded polyhedron.

A polyhedron Δ in W generates an affine space $\mathbb{A}_\Delta$ of the same dimension. Recall that an affine space in W is a translate of a linear subspace and $\mathbb{A}_\Delta$ is the intersection of all affine spaces in W which contain Δ. We denote the underlying vector space by $\mathbb{L}_\Delta$. If Δ is integral Γ-affine, then the integral structure of $\mathbb{A}_\Delta$ is given by the complete lattice $N_\Delta := N \cap \mathbb{L}_\Delta$ in $\mathbb{L}_\Delta$.

A.3. A *polyhedral complex* $\mathscr{C}$ in W is a finite set of polyhedra such that

(a) $\Delta \in \mathscr{C} \Rightarrow$ all closed faces of Δ are in $\mathscr{C}$;

(b) $\Delta, \sigma \in \mathscr{C} \Rightarrow \Delta \cap \sigma$ is either empty or a closed face of Δ and σ.

The polyhedral complex $\mathscr{C}$ is called *integral* Γ*-affine* if every $\Delta \in \mathscr{C}$ is integral Γ-affine. The *support* of $\mathscr{C}$ is defined as

$$|\mathscr{C}| := \bigcup_{\Delta \in \mathscr{C}} \Delta.$$

We say that a polyhedral complex $\mathscr{C}$ is *complete* if $|\mathscr{C}| = W$. A *subdivision* of $\mathscr{C}$ is a polyhedral complex $\mathscr{D}$ with $|\mathscr{D}| = |\mathscr{C}|$ and with every $\Delta \in \mathscr{D}$ contained in a polyhedron of $\mathscr{C}$. This has to be distinguished from a *subcomplex* of $\mathscr{C}$ which is a polyhedral complex $\mathscr{D}$ with $\mathscr{D} \subseteq \mathscr{C}$.

A.4. Given a polyhedral complex $\mathscr{C}$ in $N_{\mathbb{R}}$, we denote by $\mathscr{C}_n$ the subset of n-dimensional polyhedra in $\mathscr{C}$ and by $\mathscr{C}^l = \mathscr{C}_{r-l}$ the subset of polyhedra in $\mathscr{C}$ of codimension l in $N_{\mathbb{R}}$. We say that a polyhedral complex $\mathscr{C}$ is of *pure dimension n* (resp. of *pure codimension* l) if all polyhedra in $\mathscr{C}$ which are maximal with respect to $\preccurlyeq$ lie in $\mathscr{C}_n$ (resp. $\mathscr{C}^l$). Given a polyhedral complex $\mathscr{C}$ of pure dimension n and $m \leq n$, we denote by $\mathscr{C}_{\leq m}$ the polyhedral subcomplex of $\mathscr{C}$ of pure dimension m given by all $\sigma \in \mathscr{C}$ with $\dim \sigma \leq m$. We set $\mathscr{C}^{\geq l} = \mathscr{C}_{\leq r-l}$ if $r - l \leq n$. Recall here that r is the rank of N.

Definition A.5. (i) A *polyhedral set* P in $N_{\mathbb{R}}$ (of pure dimension n) is a finite union of polyhedra (of pure dimension n). Equivalently, there exists a polyhedral complex $\mathscr{D}$ (of pure dimension n) whose support is P. The polyhedral set is called *integral* Γ*-affine* if the above polyhedra can be chosen integral Γ-affine.

(ii) Let P be a polyhedral set in $N_{\mathbb{R}}$. A point $x \in P$ is called *regular* if there exists a polyhedron $\Delta \subseteq P$ such that $\operatorname{relint}(\Delta)$ is an open neighbourhood of x in P. We denote by $\operatorname{relint}(P)$ the set of regular points of the polyhedral set P.

A.6. A *cone* σ in W is characterized by $\mathbb{R}_{\geq 0} \cdot \sigma = \sigma$. A cone which is a polyhedron is called a *polyhedral cone*. An integral $\mathbb{R}$-affine polyhedral cone is simply called a *rational polyhedral cone*. A polyhedral cone is called *strictly convex* if it does not contain a line. The *local cone* $\mathrm{LC}_{\omega}(S)$ of $S \subseteq W$ at $\omega \in W$ is defined by

$$\mathrm{LC}_{\omega}(S) := \{\omega' \in W \mid \omega + [0, \varepsilon)\omega' \subseteq S \text{ for some } \varepsilon > 0\}.$$

A.7. A polyhedral complex Σ consisting of strictly convex rational polyhedral cones is called a *rational polyhedral fan*. The theory of toric varieties (see [Kempf et al. 1973; Oda 1988; Fulton 1993; Cox et al. 2011]) gives a bijective correspondence $\Sigma \mapsto Y_{\Sigma}$ between rational polyhedral fans on $N_{\mathbb{R}}$ and normal toric varieties over any field K with open dense torus $\operatorname{Spec}(K[M])$ (up to equivariant isomorphisms restricting to the identity on the torus). Then Σ is complete if and only if Y_{Σ} is a proper variety over K.

A *simplicial cone* in $N_{\mathbb{R}}$ is generated by a part of a basis. A *simplicial fan* is a fan formed by simplicial cones. A *smooth fan* in $N_{\mathbb{R}}$ is a rational polyhedral fan Σ such that every cone $\sigma \in \Sigma$ is generated by a part of a basis of N. In particular, a smooth fan is a simplicial fan. A polyhedral fan Σ is smooth if and only if Y_{Σ} is a smooth variety [Cox et al. 2011, Chapter 1, Theorem 3.12; Fulton 1993, 2.1 Proposition].

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Existence of compatible systems of lisse sheaves on arithmetic schemes

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Deligne conjectured that a single ℓ-adic lisse sheaf on a normal variety over a finite field can be embedded into a compatible system of ℓ'-adic lisse sheaves with various ℓ'. Drinfeld used Lafforgue's result as an input and proved this conjecture when the variety is smooth. We consider an analogous existence problem for a regular flat scheme over $\mathbb{Z}$ and prove some cases using Lafforgue's result and the work of Barnet-Lamb, Gee, Geraghty, and Taylor.

1. Introduction

Deligne [1980] conjectured that all the $\overline{\mathbb{Q}}_\ell$-sheaves on a variety over a finite field are mixed. A standard argument reduces this conjecture to the following one.

Conjecture 1.1 (Deligne). *Let p and ℓ be distinct primes. Let X be a connected normal scheme of finite type over $\mathbb{F}_p$ and $\mathscr{E}$ an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant has finite order. Then the following properties hold*:

(i) *$\mathscr{E}$ is pure of weight* 0.

(ii) *There exists a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that the polynomial* $\det(1-\mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ *has coefficients in E for every $x \in |X|$.*

(iii) *The roots of* $\det(1-\mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ *are λ-adic units for any nonarchimedean place λ of E prime to p.*

(iv) *For a sufficiently large E and for every nonarchimedean place λ of E prime to p, there exists an E_λ-sheaf $\mathscr{E}_\lambda$ compatible with $\mathscr{E}$, that is,* $\det(1-\mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}}) = \det(1-\mathrm{Frob}_x\, t, \mathscr{E}_{\lambda,\bar{x}})$ *for every $x \in |X|$.*

Here $|X|$ denotes the set of closed points of X and $\bar{x}$ is a geometric point above x.

The conjecture for curves is proved by L. Lafforgue [2002]. He also deals with parts (i) and (iii) in general by reducing them to the case of curves (see also [Deligne 2012]). Deligne [2012] proves part (ii), and Drinfeld [2012] proves part (iv) for smooth varieties. They both use Lafforgue's results.

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We can consider similar questions for arbitrary schemes of finite type over $\mathbb{Z}[\ell^{-1}]$. This paper focuses on part (iv), namely the problem of embedding a single lisse $\overline{\mathbb{Q}}_\ell$-sheaf into a compatible system of lisse sheaves. We have the following folklore conjecture in this direction.

Conjecture 1.2. *Let ℓ be a rational prime. Let X be an irreducible regular scheme that is flat and of finite type over $\mathbb{Z}[\ell^{-1}]$. Let E be a finite extension of $\mathbb{Q}$ and λ a prime of E above ℓ. Let $\mathscr{E}$ be an irreducible lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(X)$. Assume the following conditions*:

(i) *The polynomial* $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ *has coefficients in E for every $x \in |X|$.*

(ii) *$\mathscr{E}$ is de Rham at ℓ* (see below for the definition).

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\overline{E}_{\lambda'}$-sheaf on $X[\ell'^{-1}]$ which is compatible with $\mathscr{E}|_{X[\ell'^{-1}]}$.

The conjecture when $\dim X = 1$ is usually rephrased in terms of Galois representations of a number field (see Conjecture 1.3 of [Taylor 2002], for example).

When $\dim X > 1$, a lisse sheaf $\mathscr{E}$ is called *de Rham at ℓ* if for every closed point $y \in X \otimes \mathbb{Q}_\ell$, the representation $i_y^*\rho$ of $\mathrm{Gal}\big(\overline{k(y)}/k(y)\big)$ is de Rham, where i_y is the morphism $\mathrm{Spec}\, k(y) \to X$. Ruochuan Liu and Xinwen Zhu [2017] have shown that this is equivalent to the condition that the lisse E_λ-sheaf $\mathscr{E}|_{X \otimes \mathbb{Q}_\ell}$ on $X \otimes \mathbb{Q}_\ell$ is a de Rham sheaf in the sense of relative p-adic Hodge theory.

Now we discuss our main results. They concern Conjecture 1.2 for schemes over the ring of integers of a totally real or CM field.

Theorem 1.3. *Let ℓ be a rational prime and K a totally real field which is unramified at ℓ. Let X be an irreducible smooth $\mathcal{O}_K[\ell^{-1}]$-scheme such that*

- *the generic fiber is geometrically irreducible*,
- *$X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$, and*
- *X extends to an irreducible smooth $\mathcal{O}_K$-scheme with nonempty fiber over each place of K above ℓ.*

Let E be a finite extension of $\mathbb{Q}$ and λ a prime of E above ℓ. Let $\mathscr{E}$ be a lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(X)$. Suppose that $\mathscr{E}$ satisfies the following assumptions:

(i) *The polynomial* $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ *has coefficients in E for every $x \in |X|$.*

(ii) *For every totally real field L which is unramified at ℓ and every morphism $\alpha\colon$ $\mathrm{Spec}\, L \to X$, the E_λ-representation $\alpha^*\rho$ of* $\mathrm{Gal}(\overline{L}/L)$ *is crystalline at each prime v of L above ℓ, and for each $\tau\colon L \hookrightarrow \overline{E}_\lambda$ it has distinct τ-Hodge–Tate numbers in the range* $[0, \ell - 2]$.

(iii) *ρ can be equipped with a symplectic (resp. orthogonal) structure with multiplier $\mu\colon \pi_1(X) \to E_\lambda^\times$ such that $\mu|_{\pi_1(X_K)}$ admits a factorization*

$$\mu|_{\pi_1(X_K)}\colon \pi_1(X_K) \to \mathrm{Gal}(\bar{K}/K) \xrightarrow{\mu_K} E_\lambda^\times$$

with a totally odd (resp. totally even) character μ_K (see below for the definitions).

(iv) *The residual representation $\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$ is absolutely irreducible. Here ζ_ℓ is a primitive ℓ-th root of unity and $X[\zeta_\ell] = X \otimes_{\mathbb{O}_K} \mathbb{O}_K[\zeta_\ell]$.*

(v) $\ell \geq 2(\operatorname{rank}\mathscr{E} + 1)$.

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\bar{E}_{\lambda'}$-sheaf on $X[\ell'^{-1}]$ which is compatible with $\mathscr{E}|_{X[\ell'^{-1}]}$.

For an E_λ-representation $\rho\colon \pi_1(X) \to \mathrm{GL}(V_\rho)$, a symplectic (resp. orthogonal) structure with multiplier is a pair $(\langle\cdot,\cdot\rangle, \mu)$ consisting of a symplectic (resp. orthogonal) pairing $\langle\cdot,\cdot\rangle\colon V_\rho \times V_\rho \to E_\lambda$ and a continuous homomorphism $\mu\colon \pi_1(X) \to E_\lambda^\times$ satisfying $\langle \rho(g)v, \rho(g)v'\rangle = \mu(g)\langle v, v'\rangle$ for any $g \in \pi_1(X)$ and $v, v' \in V_\rho$.

We show a similar theorem without assuming that K is unramified at ℓ using the potential diagonalizability assumption. See Theorem 4.1 for this statement and Theorem 4.2 for the corresponding statement when K is CM.

The proof of Theorem 1.3 uses Lafforgue's work and the work of Barnet-Lamb, Gee, Geraghty, and Taylor [Barnet-Lamb et al. 2014, Theorem C]. The latter work concerns Galois representations of a totally real field, and it can be regarded as a special case of Conjecture 1.2 when $\dim X = 1$. We remark that their theorem has several assumptions on Galois representations since they use potential automorphy. Hence Theorem 1.3 needs assumptions (ii)–(v) on lisse sheaves.

The main part of this paper is devoted to constructing a compatible system of lisse sheaves on a scheme from those on curves. Our method is modeled after Drinfeld's [2012] result, which we explain now.

For a given lisse sheaf on a scheme, one can obtain a lisse sheaf on each curve on the scheme by restriction. Conversely, Drinfeld [2012, Theorem 2.5] proves that a collection of lisse sheaves on curves on a regular scheme defines a lisse sheaf on the scheme if it satisfies some compatibility and tameness conditions. See also a remark after Theorem 1.4. This method originates from the work of Wiesend [2006] on higher dimensional class field theory [Kerz and Schmidt 2009].

Drinfeld uses this method to reduce part (iv) of Conjecture 1.1 for smooth varieties to the case when $\dim X = 1$, where he can use Lafforgue's result. Similarly, one can use his result to reduce Conjecture 1.2 to the case when $\dim X = 1$.

However, Drinfeld's result cannot be used to reduce Theorem 1.3 to the results of Lafforgue and Barnet-Lamb, Gee, Geraghty, and Taylor since his theorem needs a lisse sheaf on every curve on the scheme as an input. On the other hand, the results of [Lafforgue 2002] and [Barnet-Lamb et al. 2014] only provide compatible systems of lisse sheaves on special types of curves on the scheme: curves over finite fields and *totally real curves*, that is, open subschemes of the spectrum of the ring of integers of a totally real field. Thus the goal of this paper is to deduce Theorem 1.3 using the existence of compatible systems of lisse sheaves on these types of curves.

We now explain our method. Fix a prime ℓ and a finite extension E_λ of $\mathbb{Q}_\ell$. Fix a positive integer r. On a normal scheme X of finite type over $\operatorname{Spec}\mathbb{Z}[\ell^{-1}]$, each lisse E_λ-sheaf $\mathscr{E}$ of rank r defines a polynomial-valued map $f_{\mathscr{E}}\colon |X| \to E_\lambda[t]$ of degree r given by $f_{\mathscr{E},x}(t) = \det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$. Here we say that a polynomial-valued map is of degree r if its values are polynomials of degree r. Moreover, $f_{\mathscr{E}}$ determines $\mathscr{E}$ up to semisimplifications by the Chebotarev density theorem. Conversely, we can ask whether a polynomial-valued map $f\colon |X| \to E_\lambda[t]$ of degree r arises from a lisse sheaf of rank r on X in this way.

Let K be a totally real field. Let X be an irreducible smooth $\mathbb{O}_K$-scheme which has geometrically irreducible generic fiber and satisfies $X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$. In this situation, we show the following theorem.

Theorem 1.4. *A polynomial-valued map f of degree r on $|X|$ arises from a lisse sheaf on X if and only if it satisfies the following conditions*:

(i) *The restriction of f to each totally real curve arises from a lisse sheaf.*

(ii) *The restriction of f to each separated smooth curve over a finite field arises from a lisse sheaf.*

We prove a similar theorem when K is CM (Theorem 3.14).

Drinfeld's theorem involves a similar equivalence, which holds for arbitrary regular schemes of finite type, although his condition (i) is required to hold for arbitrary regular curves and there is an additional tameness assumption in his condition (ii).[1]

If K and X satisfy the assumptions in Theorem 1.3, then we prove a variant of Theorem 1.4, where we require condition (i) to hold only for totally real curves which are unramified over ℓ (Remark 3.13). This variant, combined with the results by Lafforgue and Barnet-Lamb, Gee, Geraghty, and Taylor, implies Theorem 1.3.

One of the main ingredients for the proof of these types of theorems is an approximation theorem: one needs to find a curve passing through given points in given tangent directions and satisfying a technical condition coming from a given étale covering. To prove this Drinfeld uses the Hilbert irreducibility theorem. In our case, we need to further require that such a curve be totally real or CM. For this we use a theorem of Moret-Bailly.

[1] We do not need tameness assumption in condition (ii) in Theorem 1.4. This was pointed out by Drinfeld.

We briefly mention a topic related to Conjecture 1.2. As conjectures on Galois representations suggest, the following stronger statement should hold.

Conjecture 1.5. (With notation as in Conjecture 1.2). *Condition* (ii) *implies condition* (i) *after replacing E by a bigger number field inside E_λ.*

This is an analogue of Conjecture 1.1(ii) and (iii). Even if we assume the conjectures for curves, that is, Galois representations of a number field, no method is known to prove Conjecture 1.5 in full generality. However in [Shimizu 2015] we show the conjecture for smooth schemes assuming conjectures on Galois representations of a number field and the Generalized Riemann Hypothesis. Note that Deligne's [2012] proof of Conjecture 1.1(iii) uses the Riemann Hypothesis for varieties over finite fields, or more precisely, the purity theorem of [Deligne 1980].

We now explain the organization of this paper. In Section 2, we review the theorem of Moret-Bailly and prove an approximation theorem for "schemes with enough totally real curves." We show a similar theorem in the CM case. In Section 3, we prove Theorem 1.4 and its variants using the approximation theorems. Most arguments in Section 3 originate from [Drinfeld 2012]. Finally, we prove the main theorems in Section 4.

Notation. For a number field E and a place λ of E, we denote by $\bar{E}_\lambda$ a fixed algebraic closure of E_λ.

For a scheme X, we denote by $|X|$ the set of closed points of X. We equip finite subsets of $|X|$ with the reduced scheme structure. We denote the residue field of a point x of X by $k(x)$. An étale covering over X means a scheme which is finite and étale over X.

For a number field K and an $\mathcal{O}_K$-scheme X, X_K denotes the generic fiber of X regarded as a K-scheme. In particular, for a K-algebra R, $X_K(R)$ means $\mathrm{Hom}_K(\mathrm{Spec}\, R, X_K)$, not $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Spec}\, R, X_K)$. We also write $X(R)$ instead of $X_K(R)$.

For simplicity, we omit base points of fundamental groups and we often change base points implicitly in the paper.

2. Existence of totally real and CM curves via the theorem of Moret-Bailly

Theorem 2.1 [Moret-Bailly 1989]. *Let K be a number field. We consider a quadruple $(X_K, \Sigma, \{M_v\}_{v\in\Sigma}, \{\Omega_v\}_{v\in\Sigma})$ consisting of*

(i) *a geometrically irreducible, smooth and separated K-scheme X_K,*

(ii) *a finite set Σ of places of K,*

(iii) *a finite Galois extension M_v of K_v for every $v \in \Sigma$, and*

(iv) *a nonempty* $\mathrm{Gal}(M_v/K_v)$*-stable open subset Ω_v of $X_K(M_v)$ with respect to the M_v-topology.*

Then there exist a finite extension L of K and an L-rational point $x \in X_K(L)$ satisfying the following two conditions:

- *For every $v \in \Sigma$, L is M_v-split, that is, $L \otimes_K M_v \cong M_v^{[L:\,K]}$.*
- *The images of x in $X_K(M_v)$ induced from embeddings $L \hookrightarrow M_v$ lie in Ω_v.*

Remark 2.2. Our formulation is slightly different from that of Moret-Bailly, but Theorem 2.1 is a simple consequence of [Moret-Bailly 1989, théorème 1.3]: We can always find an integral model $f: X \to B$ of $X_K \to \operatorname{Spec} K$ over a sufficiently small open subscheme B of $\operatorname{Spec} \mathbb{O}_K$ such that $(X \to B, \Sigma, \{M_v\}_{v\in\Sigma}, \{\Omega_v\}_{v\in\Sigma})$ is an incomplete Skolem datum (see [Moret-Bailly 1989, définition 1.2]). Then Theorem 2.1 follows from théorème 1.3 of [Moret-Bailly 1989] applied to this incomplete Skolem datum.

Since the set Σ can contain infinite places, the above theorem implies the existence of totally real or CM valued points.

Lemma 2.3. (i) *Let K be a totally real field and X_K a geometrically irreducible smooth K-scheme such that $X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$. For any dense open subscheme U_K of X_K, there exists a totally real extension L of K such that $U_K(L) \neq \varnothing$.*

(ii) *Let F be a CM field and Z_F a geometrically irreducible smooth F-scheme. For any dense open subscheme V_F of Z_F, there exists a CM extension L of F such that $V_F(L) \neq \varnothing$.*

Proof. In either setting, we may assume that the scheme is separated over the base field by replacing it by an open dense subscheme.

(i) For every real place $v: K \hookrightarrow \mathbb{R}$, let $U_v = U_K \cap (X_K \otimes_{K,v} \mathbb{R})(\mathbb{R})$. It follows from the assumptions and the implicit function theorem that U_v is a nonempty open subset of $(X_K \otimes_{K,v} \mathbb{R})(\mathbb{R})$ with respect to real topology.

We apply the theorem of Moret-Bailly to the datum $(X_K, \{v\}, \{\mathbb{R}\}_v, \{U_v\}_v)$ to find a finite extension L of K and a point $x \in X_K(L)$ such that $L \otimes_K \mathbb{R} \cong \mathbb{R}^{[L:K]}$ and the images of x induced from real embeddings $L \hookrightarrow \mathbb{R}$ above v lie in U_v. Then L is totally real and $x \in U_K(L)$. Hence $U_K(L) \neq \varnothing$.

(ii) Let F^+ be the maximal totally real subfield of F. Define $Z^+_{F^+}$ (resp. $V^+_{F^+}$) to be the Weil restriction $\operatorname{Res}_{F/F^+} Z_F$ (resp. $\operatorname{Res}_{F/F^+} V_F$). Denote the nontrivial element of $\operatorname{Gal}(F/F^+)$ by c and $Z_F \otimes_{F,c} F$ by c^*Z_F. Then we have $Z^+_{F^+} \otimes_{F^+} F \cong Z_F \times_F c^*Z_F$, and this scheme is geometrically irreducible over F. Thus $Z^+_{F^+}$ is a geometrically irreducible smooth F^+-scheme, and $V^+_{F^+}$ is dense and open in $Z^+_{F^+}$. Moreover, for every real place $F^+ \hookrightarrow \mathbb{R}$, we can extend it to a complex place $F \hookrightarrow \mathbb{C}$ and get $F \otimes_{F^+} \mathbb{R} \cong \mathbb{C}$. Hence we have $Z^+_{F^+}(\mathbb{R}) = Z_F(\mathbb{C}) \neq \varnothing$. Therefore we can apply (i) to the triple $(F^+, Z^+_{F^+}, V^+_{F^+})$ and find a totally real extension L^+ of

F^+ such that $V_F(L^+ \otimes_{F^+} F) = V_{F^+}^+(L^+) \neq \varnothing$. Since $L^+ \otimes_{F^+} F$ is a CM extension of F, this completes the proof. $\square$

This lemma leads to the following definitions.

Definition 2.4. A *totally real curve* is an open subscheme of the spectrum of the ring of integers of a totally real field. A *CM curve* is an open subscheme of the spectrum of the ring of integers of a CM field.

Definition 2.5. Let K be a totally real field and X an irreducible regular $\mathbb{O}_K$-scheme. We say that X is an $\mathbb{O}_K$-scheme *with enough totally real curves* if X is flat and of finite type over $\mathbb{O}_K$ with geometrically irreducible generic fiber and $X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$.

Now we introduce some notation and state our approximation theorems.

Definition 2.6. Let $g \colon X \to Y$ be a morphism of schemes. For $x \in X$, consider the tangent space $T_x X = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$ at x, where $\mathfrak{m}_x$ denotes the maximal ideal of the local ring at x. This contains $T_x(X_{g(x)})$, where $X_{g(x)} = X \otimes_Y k(g(x))$. A one-dimensional subspace l of $T_x X$ is said to be *horizontal* (with respect to g) if l does not lie in the subspace $T_x(X_{g(x)})$.

Definition 2.7. Let X be a connected scheme and Y a generically étale X-scheme. A point $x \in X(L)$ with some field L is said to *be inert in* $Y \to X$ if for each irreducible component Y_α of Y, $(\mathrm{Spec}\, L) \times_{x,X} Y_\alpha$ is nonempty and connected.

Theorem 2.8. *Let K be a totally real field and X an irreducible smooth separated $\mathbb{O}_K$-scheme with enough totally real curves. Consider the following data*:

(i) *a flat $\mathbb{O}_K$-scheme Y which is generically étale over X*;

(ii) *a finite subset $S \subset |X|$ such that $S \to \mathrm{Spec}\, \mathbb{O}_K$ is injective*;

(iii) *a one-dimensional subspace l_s of $T_s X$ for every $s \in S$.*

Then there exist a totally real curve C with fraction field L, a morphism $\varphi \colon C \to X$ and a section $\sigma \colon S \to C$ of φ over S such that $\varphi(\mathrm{Spec}\, L)$ is inert in $Y \to X$ and $\mathrm{Im}(T_{\sigma(s)} C \to T_s X) = l_s$ for every $s \in S$.

Proof. We will use the theorem of Moret-Bailly to find the desired curve. Note that we can replace Y by any dominant étale Y-scheme.

Let v denote the structure morphism $X \to \mathrm{Spec}\, \mathbb{O}_K$. Take an open subscheme $U \subset X$ such that $v(U) \cap v(S) = \varnothing$ and the morphism $Y \times_X U \to U$ is finite and étale. Replacing each connected component of $Y \times_X U$ by its Galois closure, we may assume that each connected component of $Y \times_X U$ is Galois over U. Write $Y \times_X U = \coprod_{1 \leq i \leq k} W_i$ as the disjoint union of connected components and denote by G_i the Galois group of the covering $W_i \to U$. Let $H_{i1}, \ldots, H_{ir_i}$ be all the proper subgroups of G_i.

We will choose a quadruple of the form

$$\bigl(X_K, \{v_{ij}\}_{i,j} \cup \{v(s)\}_{s\in S} \cup \{v_{\infty,i}\}_i, \{K_{v_{ij}}\} \cup \{M_s\} \cup \{\mathbb{R}\}, \{V_{ij}\} \cup \{V_s\} \cup \{V_{\infty,i}\}\bigr).$$

First we choose v_{ij} and V_{ij} ($1 \le i \le k$, $1 \le j \le r_i$) which control the inertness property.

Claim. *There exist a finite set $\{v_{ij}\}_{i,j}$ of finite places of K and a nonempty open subset V_{ij} of $X(K_{v_{ij}})$ with respect to the $K_{v_{ij}}$-topology for each (i, j) such that*:

(i) *For any finite extension L of K and any L-rational point $x \in X(L)$, x is inert in $Y \to X$ if L is $K_{v_{ij}}$-split and if all the images of x under the induced maps $X(L) \to X(K_{v_{ij}})$ lie in V_{ij}.*

(ii) *The v_{ij} are different from any element of $v(S)$ regarded as a finite place of K.*

Proof of Claim. For each $i = 1, \dots, k$ and $j = 1, \dots, r_i$, let $\pi_{H_{ij}}$ denote the induced morphism $W_i/H_{ij} \to X$ and let M_{ij} be the algebraic closure of K in the field of rational functions of W_i/H_{ij}. Then we have a canonical factorization $W_i/H_{ij} \to \operatorname{Spec} \mathbb{O}_{M_{ij}} \to \operatorname{Spec} \mathbb{O}_K$.

If $M_{ij} = K$, then the generic fiber $(W_i/H_{ij})_K$ is geometrically integral over K. It follows from Proposition 3.5.2 of [Serre 2008] that there are infinitely many finite places v_0 of K such that $U(K_{v_0}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_0}))$ is a nonempty open subset of $U(K_{v_0})$. Thus choose such a finite place v_{ij} and put

$$V_{ij} = U(K_{v_{ij}}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_{ij}})).$$

Next consider the case where $M_{ij} \neq K$. The Lang–Weil theorem and the Chebotarev density theorem show that there are infinitely many finite places v_0 of K such that $U(K_{v_0}) \neq \varnothing$ and v_0 does not split completely in M_{ij}, that is, $M_{ij} \otimes_K K_{v_0} \ncong K_{v_0}^{[M_{ij}:K]}$ (see [Serre 2008, Propositions 3.5.1 and 3.6.1], for example). In this case, choose such a finite place v_{ij} and put

$$V_{ij} = U(K_{v_{ij}}).$$

It is easy to see that we can choose v_{ij} satisfying condition (ii). We now show that these v_{ij} and V_{ij} satisfy condition (i). Take L and $x \in X(L)$ as in condition (i). By Lemma 2.9 below, it suffices to prove that $x \notin \pi_{H_{ij}}(W_i/H_{ij}(L))$ for any H_{ij}.

When $M_{ij} = K$, this is obvious because the images of x under the maps $X(L) \to X(K_{v_{ij}})$ lie in $V_{ij} = U(K_{v_{ij}}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_{ij}}))$. When $M_{ij} \neq K$, we know that $M_{ij} \otimes_K K_{v_{ij}}$ is not $K_{v_{ij}}$-split. Since L is assumed to be $K_{v_{ij}}$-split, M_{ij} cannot be embedded into L. On the other hand, we have a canonical factorization $W_i/H_{ij} \to \operatorname{Spec} \mathbb{O}_{M_{ij}}$. Therefore $W_i/H_{ij}(L) = \varnothing$. Thus x is inert in $Y \to X$ in both cases. $\square$

Next we choose a finite Galois extension M_s of $K_{v(s)}$ and a $\operatorname{Gal}(M_s/K_{v(s)})$-stable nonempty open subset V_s of $X(M_s)$ with respect to the M_s-topology to make a totally real curve pass through s in the tangent direction l_s. Here $K_{v(s)}$ denotes the completion of K with respect to the finite place $v(s)$ of K. Let $\hat{\mathbb{O}}_{X,s}$ denote the completed local ring of X at $s \in S$. Since $\hat{\mathbb{O}}_{X,s}$ is regular, we can find a regular one-dimensional closed subscheme $\operatorname{Spec} R_s \subset \operatorname{Spec} \hat{\mathbb{O}}_{X,s}$ which is tangent to l_s and satisfies $R_s \otimes_{\mathbb{O}_K} K \neq \varnothing$ (see [Drinfeld 2012, Lemma A.6]).

It follows from the construction that R_s is a complete discrete valuation ring which is finite and flat over $\mathbb{O}_{K_{v(s)}}$ and has residue field $k(s)$. Let M'_s be the fraction field of R_s. For each $s \in S$ we first choose M_s and a local homomorphism $\mathbb{O}_{X,s} \to \mathbb{O}_{M_s}$. There are two cases.

If l_s is horizontal, then R_s is unramified over $\mathbb{O}_{K_{v(s)}}$ and hence M'_s is Galois over $K_{v(s)}$. Put $M_s := M'_s$ in this case. Then we have a natural local homomorphism $\mathbb{O}_{X,s} \to \hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M_s}$.

If l_s is not horizontal, then R_s is ramified over $\mathbb{O}_{K_{v(s)}}$. Let $K'_{v(s)}$ be the maximal unramified extension of $K_{v(s)}$ in M'_s and M_s the Galois closure of M'_s over $K_{v(s)}$. Then both $K'_{v(s)}$ and M_s have the same residue field $k(s)$.

We construct a local homomorphism $\hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M_s}$ in this setting. Since X is smooth over $\mathbb{O}_K$, the ring $\hat{\mathbb{O}}_{X,s}$ is isomorphic to the ring of formal power series $\mathbb{O}_{K'_{v(s)}}[[t_1, \dots, t_m]]$ for some m and we identify these rings.

Let $u_i \in \mathbb{O}_{M'_s}$ denote the image of t_i under the homomorphism

$$\mathbb{O}_{K'_{v(s)}}[[t_1, \dots, t_m]] = \hat{\mathbb{O}}_{X,s} \to R_s = \mathbb{O}_{M'_s}.$$

Let π (resp. ϖ) be a uniformizer of $\mathbb{O}_{M'_s}$ (resp. $\mathbb{O}_{M_s}$) and consider π-adic expansion $u_i = \sum_{j=0}^{\infty} a_{ij} \pi^j$. Since $\hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M'_s}$ is a local homomorphism, we have $a_{i0} = 0$ for each i.

Consider the differential of $\operatorname{Spec} \mathbb{O}_{M'_s} = \operatorname{Spec} R_s \to \operatorname{Spec} \hat{\mathbb{O}}_{X,s}$ at the closed point. The tangent vector $\partial/\partial\pi$ is sent to $\sum_{i=1}^{m} a_{i1} \partial/\partial t_i$ under this map, and the latter spans the tangent line l_s.

Define a local homomorphism $\hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M_s}$ by sending t_i to $\sum_{j=1}^{\infty} a_{ij} \varpi^j$. Then the image of the differential of the corresponding morphism $\operatorname{Spec} \mathbb{O}_{M_s} \to X$ at the closed point is l_s by the same computation as above.

In either case, we have chosen M_s and a homomorphism $\mathbb{O}_{X,s} \to \mathbb{O}_{M_s}$. Let $\hat{s} \in X(\mathbb{O}_{M_s})$ be the point induced by the homomorphism. Note that $X(\mathbb{O}_{M_s})$ is an open subset of $X(M_s)$ by separatedness. Let $\alpha\colon X(\mathbb{O}_{M_s}) \to X(\mathbb{O}_{M_s}/\mathfrak{m}_{M_s}^2)$ be the reduction map, where $\mathfrak{m}_{M_s}$ denotes the maximal ideal of $\mathbb{O}_{M_s}$. Define $V'_s = \alpha^{-1}(\alpha(\hat{s}))$, which is a nonempty open subset of $X(M_s)$, and put

$$V_s = \bigcup_{\sigma} \sigma(V'_s),$$

where σ runs over all the elements of $\operatorname{Gal}(M_s/K_{v(s)})$. Since $\operatorname{Gal}(M_s/K_{v(s)})$ acts continuously on $X(M_s)$, V_s is a nonempty $\operatorname{Gal}(M_s/K_{v(s)})$-stable open subset of $X(M_s)$.

Finally, let $v_{\infty,1}, \dots, v_{\infty,n}$ be the real places of K and put

$$V_{\infty,i} = X(\mathbb{R})$$

for each $i = 1, \dots, n$. This is nonempty by our assumption.

It follows from the theorem of Moret-Bailly (Theorem 2.1) that there exist a finite extension L of K and an L-rational point $x \in X(L)$ satisfying the following properties:

(i) $L \otimes_K K_{v_{ij}}$ is $K_{v_{ij}}$-split and x goes into V_{ij} under any embedding $L \hookrightarrow K_{v_{ij}}$.

(ii) $L \otimes_K M_s$ is M_s-split and x goes into V_s under any embedding $L \hookrightarrow M_s$.

(iii) L is totally real.

We can spread out the L-rational point $x\colon \operatorname{Spec} L \to X$ to a morphism $\varphi\colon C \to X$, where C is a totally real curve with fraction field L. By property (ii), we can choose C and φ so that all the points of $\operatorname{Spec} \mathcal{O}_L$ above $v(S) \subset \operatorname{Spec} \mathcal{O}_K$ are contained in C. The claim on page 188 shows that x is inert in $Y \to X$. Thus it remains to prove that there exists a section σ of φ over S such that $\operatorname{Im}(T_{\sigma(s)}C \to T_sX) = l_s$ for every $s \in S$.

It follows from property (ii) and the definition of V_s that there exists an embedding $L \hookrightarrow M_s$ such that the image of x under the associated map $X(L) \to X(M_s)$ lies in V'_s. Let $s' \in \operatorname{Spec} \mathcal{O}_L$ be the closed point corresponding to this embedding. Then we have $s' \in C$, $k(s') = k(s)$, and $\operatorname{Im}(T_{s'}C \to T_sX) = l_s$. Hence we can define a desired section of φ over S. □

Lemma 2.9. *Let L be a field, U a locally noetherian connected scheme and $\pi : W \to U$ a Galois covering with Galois group G. For any subgroup $H \subset G$, let π_H denote the induced morphism $W/H \to U$. An L-valued point of X is inert in π if and only if it lies in $U(L) \setminus \bigcup_{H \subsetneq G} \pi_H(W/H(L))$.*

Proof. Let x denote the L-valued point. Choose a point of W above x and fix a geometric point above it. This also defines a geometric point above x and we have a homomorphism $\pi_1(x) \to G$, where $\pi_1(x)$ is the absolute Galois group of L. Let H_0 denote the image of this homomorphism. Then x is inert in π if and only if the homomorphism is surjective, that is, $H_0 = G$. On the other hand, for a subgroup $H \subset G$, the point x lies in $\pi_H(W/H(L))$ if and only if $\operatorname{Spec} L \times_{x,U} W/H \to \operatorname{Spec} L$ has a section, which is equivalent to the condition that some conjugate of H contains H_0. The lemma follows from these two observations. □

For our applications, we need a stronger variant of the theorem.

Corollary 2.10. *Let K be a totally real field and X an irreducible smooth separated $\mathcal{O}_K$-scheme with enough totally real curves. Let U be a nonempty open subscheme of X. Suppose that we are given the following data:*

(i) *a flat $\mathcal{O}_K$-scheme Y which is generically étale over X*;

(ii) *a closed normal subgroup $H \subset \pi_1(U)$ such that $\pi_1(U)/H$ contains an open pro-ℓ subgroup*;

(iii) *a finite subset $S \subset |X|$ such that $S \to \operatorname{Spec} \mathcal{O}_K$ is injective*;

(iv) *a one-dimensional subspace l_s of $T_s X$ for every $s \in S$.*

Then there exist a totally real curve C with fraction field L, a morphism $\varphi\colon C \to X$ with $\varphi^{-1}(U) \neq \varnothing$ and a section $\sigma\colon S \to C$ of φ over S such that

- $\varphi(\operatorname{Spec} L)$ *is inert in* $Y \to X$,
- $\pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ *is surjective, and*
- $\operatorname{Im}(T_{\sigma(s)} C \to T_s X) = l_s$ *for every* $s \in S$.

Proof. As is shown in the proof of Proposition 2.17 of [Drinfeld 2012], we can find an open normal subgroup $G_0 \subset \pi_1(U)/H$ satisfying the following property: Every closed subgroup $G \subset \pi_1(U)/H$ such that the map $G \to (\pi_1(U)/H)/G_0$ is surjective equals $\pi_1(U)/H$.

Let Y' be the Galois covering of U corresponding to G_0. Then we can apply Theorem 2.8 to $(Y \sqcup Y', S, (l_s)_{s \in S})$ and get the desired triple (C, φ, σ). □

We have a similar approximation theorem in the CM case. The proof uses the Weil restriction and is essentially similar to the totally real case, although one has to check that the conditions are preserved under the Weil restriction.

Theorem 2.11. *Let F be a CM field and Z an irreducible smooth separated $\mathcal{O}_F$-scheme with geometrically irreducible generic fiber. Let U be a nonempty open subscheme of Z. Suppose that we are given the following data*:

(i) *a flat $\mathcal{O}_F$-scheme W which is generically étale over Z*;

(ii) *a closed normal subgroup $H \subset \pi_1(U)$ such that $\pi_1(U)/H$ contains an open pro-ℓ subgroup*;

(iii) *a finite subset $S \subset |Z|$ such that $S \to \operatorname{Spec} \mathcal{O}_{F^+}$ is injective*;

(iv) *a one-dimensional subspace l_s of $T_s Z$ for every $s \in S$.*

Then there exist a CM curve C with fraction field L, a morphism $\varphi\colon C \to Z$ with $\varphi^{-1}(U) \neq \varnothing$ and a section $\sigma\colon S \to C$ of φ over S such that

- $\varphi(\operatorname{Spec} L)$ *is inert in* $W \to Z$,
- $\pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ *is surjective, and*
- $\operatorname{Im}(T_{\sigma(s)} C \to T_s Z) = l_s$ *for every* $s \in S$.

Proof. Let F^+ be the maximally totally real subfield of F. Let w (resp. v) denote the structure morphism $Z \to \operatorname{Spec} \mathcal{O}_F$ (resp. $Z \to \operatorname{Spec} \mathcal{O}_{F^+}$). As in the proof of

Corollary 2.10, we may omit the datum (ii) by replacing W by another flat, generically étale Z-scheme and prove the first and third properties of the triple (C, φ, σ).

Define Z^+ to be the Weil restriction $\mathrm{Res}_{\mathbb{O}_F/\mathbb{O}_{F^+}} Z$. Then we have $Z^+ \otimes_{\mathbb{O}_{F^+}} \mathbb{O}_F \cong Z \times_{\mathbb{O}_F} c^*Z$, where c denotes the nontrivial element of $\mathrm{Gal}(F/F^+)$ and c^*Z denotes $Z \otimes_{\mathbb{O}_F, c} \mathbb{O}_F$. It follows from the assumptions that Z^+ is an irreducible smooth $\mathbb{O}_{F^+}$-scheme with enough totally real curves. We will apply the theorem of Moret-Bailly to Z^+ with appropriate data.

We may assume that each connected component of W is a Galois cover over its image in Z by replacing W if necessary. Put $Y = W \times_{\mathbb{O}_F} c^*W$ and regard it as an $\mathbb{O}_{F^+}$-scheme. Then $Y \to Z \times_{\mathbb{O}_F} c^*Z \to Z^+$ is flat and generically étale, and therefore satisfies the same assumptions as $Y \to X$ in Theorem 2.8 and the second paragraph of its proof. Hence, as in the claim on page 188, there exist a finite set $\{v_{ij}\}_{1\le i\le k, 1\le j\le r_i}$ of finite places of F^+ and a nonempty open subset V_{ij} of $Z^+(F^+_{v_{ij}})$ for each (i, j) satisfying the following properties:

(i) For any finite extension L^+ of F^+ and any L^+-rational point $z^+ \in Z^+(L^+)$, z^+ is inert in $Y \to Z^+$ if L^+ is $F^+_{v_{ij}}$-split and if z^+ lands in V_{ij} under any embedding $L^+ \hookrightarrow F^+_{v_{ij}}$.

(ii) The v_{ij} are different from any element of $v(S)$.

Next take any $s \in S$. We will choose a finite Galois extension M_s of $F_{w(s)}$ and a $\mathrm{Gal}(M_s/F^+_{v(s)})$-stable nonempty subset of $Z^+(M_s)$ with respect to the M_s-topology. Here we regard $w(s)$ (resp. $v(s)$) as a finite place of F (resp. F^+). Then M_s is Galois over $F^+_{v(s)}$ since $w(s)$ lies above $v(s)$ and $[F_{w(s)} : F^+_{v(s)}]$ is either 1 or 2.

As in the proof of Theorem 2.8, we can find a finite Galois extension M_s of $F_{w(s)}$ with residue field $k(s)$ and a homomorphism $\mathbb{O}_{Z,s} \to \mathbb{O}_{M_s}$ such that the image of the differential of the corresponding morphism $\mathrm{Spec}\, \mathbb{O}_{M_s} \to Z$ at the closed point is l_s. Denote by $\hat{s} \in Z(\mathbb{O}_{M_s}) \subset Z(M_s)$ the point corresponding to this morphism.

Let $\alpha\colon Z(\mathbb{O}_{M_s}) \to Z(\mathbb{O}_{M_s}/\mathfrak{m}^2_{M_s})$ be the reduction map, where $\mathfrak{m}_{M_s}$ denotes the maximal ideal of $\mathbb{O}_{M_s}$. Define $V'_s = \alpha^{-1}(\alpha(\hat{s}))$, which is a nonempty open subset of $Z(M_s)$, and put $V''_s = \bigcup_\sigma \sigma(V'_s)$, where σ runs over all the elements of $\mathrm{Gal}(M_s/F_{w(s)})$.

Denote by ι a natural embedding $F \hookrightarrow F_{w(s)} \hookrightarrow M_s$. We have $\mathrm{Hom}_{F^+}(F, M_s) = \{\iota, \iota\circ c\}$ and the F^+-homomorphism $(\iota, \iota\circ c)\colon F \to M_s \times M_s$ induces an isomorphism $M_s \otimes_{F^+} F \cong M_s \times M_s$ which sends $a \otimes b$ to $\big(a\iota(b), a\iota(c(b))\big)$. Hence we get identifications

$$Z^+(M_s) = Z(M_s \otimes_{F^+} F) = Z(M_s) \times c^*Z(M_s).$$

Here $Z(M_s)$ denotes $\mathrm{Hom}_{\mathbb{O}_F}(\mathrm{Spec}\, M_s, Z)$ by regarding $\mathrm{Spec}\, M_s$ as an F-scheme via ι.

Define

$$V_s := V_s'' \times c^* V_s'' \subset Z(M_s) \times c^* Z(M_s) = Z^+(M_s).$$

This is a nonempty open subset of $Z^+(M_s)$. Since V_s'' is $\mathrm{Gal}(M_s/F_{w(s)})$-stable and $\mathrm{Gal}(F/F^+) = \{\mathrm{id}, c\}$, V_s is $\mathrm{Gal}(M_s/F^+_{v(s)})$-stable.

Let $v_{\infty,1}, \ldots, v_{\infty,n}$ be the real places of F^+. Then for each $1 \le i \le n$ put

$$V_{\infty,i} = Z^+(\mathbb{R}) = Z(\mathbb{C})$$

via an isomorphism $F \otimes_{F^+} \mathbb{R} \cong \mathbb{C}$. This is a nonempty open set.

We apply Theorem 2.1 to the quadruple

$$\big(Z^+_{F^+}, \{v_{ij}\}_{i,j} \cup \{v(s)\}_{s\in S} \cup \{v_{\infty,i}\}_i, \{F^+_{v_{ij}}\} \cup \{M_s\} \cup \{\mathbb{R}\}, \{V_{ij}\} \cup \{V_s\} \cup \{V_{\infty,i}\}\big)$$

and find a totally real finite extension L^+ of F^+ and an L^+-rational point $z^+ \in Z^+(L^+)$ satisfying the following properties:

(i) z^+ is inert in $Y \to Z^+$.

(ii) L^+ is M_s-split and z^+ goes into V_s via any embedding $L^+ \hookrightarrow M_s$.

Let L be the CM field $L^+ \otimes_{F^+} F$ and $z \in Z(L)$ be the L-rational point corresponding to $z^+ \in Z^+(L^+)$. Then the morphism z is equal to the composite

$$\mathrm{pr}_Z \circ (z^+ \otimes_{F^+} F)\colon\ \mathrm{Spec}\, L \to Z^+ \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F = Z \times_{\mathcal{O}_F} c^* Z \to Z.$$

We can spread out $z\colon\ \mathrm{Spec}\, L \to Z$ to a morphism $\varphi\colon C \to Z$ for some CM curve C with fraction field L. We may assume that C contains all the points of $\mathrm{Spec}\, \mathcal{O}_L$ above $w(S) \subset \mathrm{Spec}\, \mathcal{O}_F$. It follows from property (ii) and the definition of V_s that φ has a section σ over S such that $\mathrm{Im}(T_{\sigma(s)} C \to T_s X) = l_s$ for every $s \in S$.

It remains to prove that $z = \varphi(\mathrm{Spec}\, L)$ is inert in $W \to Z$. Without loss of generality, we may assume that W_F is connected, and thus it suffices to show that $\mathrm{Spec}\, L \times_{z,Z} W = \mathrm{Spec}\, L \times_{z,Z_F} W_F$ is connected. Define the schemes P and Q such that the squares in the following diagram are Cartesian:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathrm{Spec}\, L \arrow[r] \arrow[d, "z^+ \otimes_{F^+} F"] & \mathrm{Spec}\, L^+ \arrow[d, "z^+"] \\
W_F \times_F c^* W_F \arrow[r] & W_F \times_F c^* Z_F \arrow[r] \arrow[d] & Z_F \times_F c^* Z_F \arrow[r] \arrow[d, "\mathrm{pr}_{Z_F}"] & Z^+_{F^+} \\
 & W_F \arrow[r] & Z_F &
\end{tikzcd}$$

Since $W_F \times_F c^* W_F = Y_{F^+}$, we have $P \cong \mathrm{Spec}\, L^+ \times_{z^+, Z^+_{F^+}} Y_{F^+}$. As $Q \cong \mathrm{Spec}\, L \times_{z,Z_F} W_F$, we need to show that Q is connected.

Note that $W_F \times_F c^* Z_F$ is connected; this follows from the fact that $c^* Z_F$ is geometrically connected over F and W_F is connected. Now take any connected component T of $Y_{F^+} = W_F \times_F c^* W_F$. Since $W_F \times_F c^* W_F \to W_F \times_F c^* Z_F$ is an étale covering with connected base, T surjects onto $W_F \times_F c^* Z_F$. At the same time, the subscheme $\operatorname{Spec} L^+ \times_{z^+, Z^+_{F^+}} T \subset P$ is connected because z^+ is inert in $Y \to Z^+$. Since the connected scheme $\operatorname{Spec} L^+ \times_{z^+, Z^+_{F^+}} T$ surjects onto Q, the latter is also connected. $\square$

Remark 2.12. In Theorem 2.8, Corollary 2.10, and Theorem 2.11, we assume that the scheme in question is smooth and separated. If $S = \varnothing$, then we can replace these two assumptions by regularity. In fact, if $S = \varnothing$, we can replace the scheme by an open subscheme, and thus reduce to the separated case. Moreover, the regularity implies that the generic fiber of the scheme is smooth. So we can apply the theorem of Moret-Bailly to our scheme. Note that the smoothness assumption was used only when $S \neq \varnothing$ and l_s is not horizontal for some $s \in S$.

3. Proofs of Theorem 1.4 and its variants

In this section, we prove Theorem 1.4 and its variants following [Drinfeld 2012]. First we set up our notation. Fix a prime ℓ and a finite extension E_λ of $\mathbb{Q}_\ell$. Let $\mathbb{O}$ be the ring of integers of E_λ and $\mathfrak{m}$ its maximal ideal.

Fix a positive integer r. For a normal scheme X of finite type over $\operatorname{Spec} \mathbb{Z}[\ell^{-1}]$, $\mathrm{LS}_r^{E_\lambda}(X)$ denotes the set of equivalence classes of lisse E_λ-sheaves on X of rank r, and $\widetilde{\mathrm{LS}}_r^{E_\lambda}(X)$ denotes the set of maps from the set of closed points of X to the set of polynomials of the form $1 + c_1 t + \cdots + c_r t^r$ with $c_i \in \mathbb{O}$ and $c_r \in \mathbb{O}^\times$. Here we say that two lisse E_λ-sheaves on X are *equivalent* if they have isomorphic semisimplifications. Since the coefficient field E_λ is fixed throughout this section, we simply write $\mathrm{LS}_r(X)$ or $\widetilde{\mathrm{LS}}_r(X)$.

For an element $f \in \widetilde{\mathrm{LS}}_r(X)$, we denote by $f_x(t)$ or $f(x)(t)$ the value of f at $x \in X$; this is a polynomial in t. By the Chebotarev density theorem, we can regard $\mathrm{LS}_r(X)$ as a subset of $\widetilde{\mathrm{LS}}_r(X)$ by attaching to each equivalence class its Frobenius characteristic polynomials. For another scheme Y and a morphism $\alpha \colon Y \to X$, we have a canonical map $\alpha^* \colon \widetilde{\mathrm{LS}}_r(X) \to \widetilde{\mathrm{LS}}_r(Y)$ whose restriction to $\mathrm{LS}_r(X)$ coincides with the pullback map of sheaves $\mathrm{LS}_r(X) \to \mathrm{LS}_r(Y)$. We also denote $\alpha^*(f)$ by $f|_Y$.

Let C be a separated smooth curve over a finite field and $\bar{C}$ the smooth compactification of C. We define $\mathrm{LS}_r^{\mathrm{tame}}(C)$ to be the subset of $\mathrm{LS}_r(C)$ consisting of equivalence classes of lisse E_λ-sheaves on C which are tamely ramified at each point of $\bar{C} \setminus C$. This condition does not depend on the choice of a lisse sheaf in the equivalence class. Let φ be a morphism $C \to X$ and $f \in \widetilde{\mathrm{LS}}_r(X)$. When $\varphi^*(f) \in \mathrm{LS}_r(C)$ (resp. $\varphi^*(f) \in \mathrm{LS}_r^{\mathrm{tame}}(C)$), we simply say that f *arises from* a lisse sheaf (resp. a tame lisse sheaf) over the curve C.

To show Theorem 2.5 of [Drinfeld 2012], which is a prototype of Theorem 1.4, Drinfeld considers a subset $\mathrm{LS}'_r(X)$ of $\widetilde{\mathrm{LS}}_r(X)$ which contains $\mathrm{LS}_r(X)$ and is characterized by a group-theoretic property. He then proves the following three statements for $f \in \widetilde{\mathrm{LS}}_r(X)$, which imply Theorem 2.5 of [Drinfeld 2012].

- If the map f satisfies two conditions[2] similar to those in Theorem 1.4, then $f|_U \in \mathrm{LS}'_r(U)$ for some dense open subscheme $U \subset X$.
- If U is regular, then $\mathrm{LS}'_r(U) = \mathrm{LS}_r(U)$. In particular, the restriction $f|_U \in \mathrm{LS}'_r(U)$ arises from a lisse sheaf.
- If $f|_U$ arises from a lisse sheaf, then so does f under the assumptions that X is regular and that $f|_C$ arises from a lisse sheaf for every regular curve C.

Following Drinfeld, we will introduce the group-theoretic notion of "having a kernel" and prove similar statements: Propositions 3.4, 3.10, and 3.11. Then Theorem 1.4 and its variants will be deduced from them at the end of the section.

Definition 3.1. Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $f \in \widetilde{\mathrm{LS}}_r(X)$. For a nonzero ideal $I \subset \mathbb{O}$, the map f is said to be *trivial modulo* I if it has the value congruent to $(1-t)^r$ modulo I at every closed point of X.

When X is connected, the map f is said to *have a kernel* if there exists a closed normal subgroup $H \subset \pi_1(X)$ satisfying the following conditions:

(i) $\pi_1(X)/H$ contains an open pro-ℓ subgroup.

(ii) For every $n \in \mathbb{N}$, there exists an open subgroup $H_n \subset \pi_1(X)$ containing H such that the pullback of f to X_n is trivial modulo $\mathfrak{m}^n$. Here X_n denotes the covering of X corresponding to H_n.

When X is disconnected, the map f is said to have a kernel if the restriction of f to each connected component of X has a kernel.

Remark 3.2. If f arises from a lisse sheaf on X, it has a kernel. To see this, we may assume that X is connected. Then the kernel of the E_λ-representation of $\pi_1(X)$ corresponding to the lisse sheaf satisfies the conditions.

Remark 3.3. The set $\mathrm{LS}'_r(X)$ defined by Drinfeld [2012, Definition 2.11] consists of the maps f which have a kernel and arise from a lisse sheaf over every regular curve.

Proposition 3.4. *Let K be a totally real field. Let X be an irreducible regular $\mathbb{O}_K[\ell^{-1}]$-scheme with enough totally real curves and $f \in \widetilde{\mathrm{LS}}_r(X)$. Assume that*

(i) *f arises from a lisse sheaf over every totally real curve*, *and*

[2] One needs a tameness assumption in the second condition (it is identical to condition (ii) of Proposition 3.4).

(ii) *there exists a dominant étale morphism $X' \to X$ such that the pullback $f|_{X'}$ arises from a tame lisse sheaf over every separated smooth curve over a finite field.*

Then there exists a dense open subscheme $U \subset X$ such that $f|_U$ has a kernel.

We will first show two lemmas and then prove Proposition 3.4 by induction on the dimension of X. For this we use elementary fibrations, which we recall now.

Definition 3.5. A morphism of schemes $\pi : X \to S$ is called an *elementary fibration* if there exist an S-scheme $\bar{\pi} : \bar{X} \to S$ and a factorization $X \to \bar{X} \xrightarrow{\bar{\pi}} S$ of π such that

(i) the morphism $X \to \bar{X}$ is an open immersion and X is fiberwise dense in $\bar{\pi} : \bar{X} \to S$,

(ii) $\bar{\pi}$ is a smooth and projective morphism whose geometric fibers are nonempty irreducible curves, and

(iii) the reduced closed subscheme $\bar{X} \setminus X$ is finite and étale over S.

The next lemma, which is due to Drinfeld and Wiesend, is a key to our induction argument in the proof of Proposition 3.4.

Lemma 3.6. *Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $f \in \widetilde{\mathrm{LS}}_r(X)$. Suppose that X admits an elementary fibration $X \to S$ with a section $\sigma : S \to X$. Assume that*

(i) *f arises from a tame lisse sheaf over every fiber of $X \to S$, and*

(ii) *there exists a dense open subscheme $V \subset S$ such that $\sigma^*(f)|_V$ has a kernel.*

Then there exists a dense open subscheme $U \subset X$ such that $f|_U$ has a kernel.

Proof. This is shown in the latter part of the proof of Lemma 3.1 of [Drinfeld 2012]. For the reader's convenience, we summarize the proof below.

We may assume that X is connected and normal, and that $V = S$. For every $n \in \mathbb{N}$, consider the functor which attaches to an S-scheme S' the set of isomorphism classes of $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsors on $X \times_S S'$ tamely ramified along $(\bar{X} \setminus X) \times_S S'$ relative to S' with trivialization over the section $S' \hookrightarrow X \times_S S'$. Then this functor is representable by an étale scheme T_n of finite type over S and the morphism $T_{n+1} \to T_n$ is finite for each n. By shrinking S, we may assume that the morphism $T_n \to S$ is finite for each n. We will prove that f has a kernel in this situation.

Since $\sigma^*(f)$ has a kernel by assumption (ii), there exist connected étale coverings S_n of S such that

- the pullback of $\sigma^*(f)$ to S_n is trivial modulo $\mathfrak{m}^n$, and

- for some (or any) geometric point $\bar{s}$ of S, the quotient of the group $\pi_1(S, \bar{s})$ by the intersection of the kernels of its actions on the fibers $(S_n)_{\bar{s}}$, where n runs in $\mathbb{N}$, contains an open pro-ℓ subgroup.

Let $\mathfrak{T}_n$ be the universal tame $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsor over $X \times_S T_n$. Define the X-scheme Y_n to be the Weil restriction $\mathrm{Res}_{X \times_S T_n/X} \mathfrak{T}_n$ and let X_n denote $Y_n \times_S S_n$. We thus have a diagram whose squares are Cartesian

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & X \times_S S_n \arrow[r] \arrow[d] & S_n \arrow[d] \arrow[dl] \\
Y_n \arrow[r] & X \arrow[r] & S
\end{tikzcd}$$

and regard X_n as an étale covering of X. Here the morphism $S_n \to X$ is the composite of $S_n \to S$ and the section $\sigma \colon S \to X$.

It suffices to prove the following two assertions:

(a) The pullback of f to X_n is trivial modulo $\mathfrak{m}^n$.

(b) For some (or any) geometric point $\bar{x}$ of X, the quotient of the group $\pi_1(X, \bar{x})$ by the intersection of the kernels of its actions on the fibers $(X_n)_{\bar{x}}$, where n runs in $\mathbb{N}$, contains an open pro-ℓ subgroup.

In fact, if we take a Galois covering X'_n of X splitting the (possibly disconnected) covering X_n, the corresponding subgroup $H_n := \pi(X'_n, \bar{x}) \subset \pi_1(X, \bar{x})$ and the intersection $H := \bigcap_n H_n$ satisfy the conditions for the map f to have a kernel.

First we prove assertion (a). Take an arbitrary closed point $x \in X_n$. Let $s \in S$ denote the image of x and choose a geometric point $\bar{s}$ above $s \in S$. By assumption (i), the restriction $f|_{X_s}$ arises from a lisse E_λ-sheaf of rank r. Let $\mathscr{F}$ be a locally constant constructible sheaf of free $(\mathbb{O}/\mathfrak{m}^n)$-modules of rank r obtaining from the above lisse sheaf modulo $\mathfrak{m}^n$.

Consider the $X_{\bar{s}}$-scheme $(Y_n)_{\bar{s}}$. The scheme $(X \times_S T_n)_{\bar{s}}$ is the disjoint union of copies of $X_{\bar{s}}$, and $(\mathfrak{T}_n)_{\bar{s}}$ is a disjoint union of the $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsors, each of which lies above a copy of $X_{\bar{s}}$ in $(X \times_S T_n)_{\bar{s}}$. Since the Weil restriction and the base change commute, $(Y_n)_{\bar{s}}$ is the fiber product of the tame $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsors over $X_{\bar{s}}$. Hence $\mathscr{F}|_{(Y_n)_{\bar{s}}}$ is constant, and so is $\mathscr{F}|_{(X_n)_{\bar{s}}}$.

Now let $s' \in S_n$ be the image of x. By the choice of S_n, we have

$$(\sigma^*(f))|_{S_n}(s')(t) \equiv (1-t)^r \mod \mathfrak{m}^n.$$

Since we have shown that $\mathscr{F}|_{(X_n)_{\bar{s}}}$ is constant, it follows that

$$f|_{X_n}(x)(t) \equiv (1-t)^r \mod \mathfrak{m}^n.$$

Finally, we prove assertion (b). Let η be the generic point of S. Choose a geometric point $\bar{\eta}$ above $\eta \in S$ and let $\bar{x}$ denote the geometric point above $\sigma(\eta)$

induced from $\bar{\eta}$. Let H be the intersection of the kernels of actions of $\pi_1(X, \bar{x})$ on the fibers $(X_n)_{\bar{x}}$, where n runs in $\mathbb{N}$. We need to show that $\pi_1(X, \bar{x})/H$ contains an open pro-ℓ subgroup.

Using the fact that the tame fundamental group $\pi_1^{\mathrm{tame}}(X_{\bar{\eta}}, \bar{x})$ is topologically finitely generated, one can prove that the quotient of the group $\pi_1(X, \bar{x})$ by the intersection H_Y of the kernels of its actions on the fibers $(Y_n)_{\bar{x}}$, $n \in \mathbb{N}$ contains an open pro-ℓ subgroup (see the last part of the proof of Lemma 3.1 in [Drinfeld 2012]).

Let H'_S be the intersection of the kernels of actions of $\pi_1(S, \bar{\eta})$ on the fibers $(S_n)_{\bar{\eta}}$, where n runs in $\mathbb{N}$, and let H_S be the inverse image of H'_S with respect to the homomorphism $\pi_1(X, \bar{x}) \to \pi_1(S, \bar{\eta})$. By the choice of S_n, the group $\pi_1(S, \bar{\eta})/H'_S$ contains an open pro-ℓ subgroup. Since we have a surjection

$$\pi_1(X, \bar{x})/(H_Y \cap H_S) \to \pi_1(X, \bar{x})/H$$

and an injection

$$\pi_1(X, \bar{x})/(H_Y \cap H_S) \to \pi_1(X, \bar{x})/H_Y \times \pi_1(S, \bar{\eta})/H'_S,$$

the group $\pi_1(X, \bar{x})/H$ also contains an open pro-ℓ subgroup. □

To use the above lemma, we show that there exists a chain of split fibrations ending with a totally real curve.

Definition 3.7. A sequence of schemes $X_n \to X_{n-1} \to \cdots \to X_1$ is called *a chain of split fibrations* if the morphism $X_{i+1} \to X_i$ is an elementary fibration which admits a section $X_i \to X_{i+1}$ for each $i = 1, \ldots, n-1$.

Lemma 3.8. *Let K be a totally real field and X an n-dimensional irreducible regular $\mathcal{O}_K$-scheme with enough totally real curves. Then there exist an étale X-scheme X_n, a totally real curve X_1 and a chain of split fibrations $X_n \to \cdots \to X_1$.*

Proof. We prove the lemma by induction on $n = \dim X$. When $\dim X = 1$, the lemma holds by assumption. Thus we assume $\dim X \geq 2$.

By induction on $\dim X$, it suffices to prove that after replacing K by a totally real field extension and X by a nonempty étale X-scheme, there exist an irreducible regular $\mathcal{O}_K$-scheme S with enough totally real curves and an elementary fibration $X \to S$ with a section $S \to X$.

It follows from Lemma 2.3(i) that there exists a totally real extension L of K such that $X_K(L) \neq \varnothing$. Replacing K by L and X by a nonempty open subscheme of $X \otimes_{\mathcal{O}_K} \mathcal{O}_L$ that is étale over X, we may further assume that the generic fiber $X_K \to \operatorname{Spec} K$ has a section $x\colon \operatorname{Spec} K \to X_K$. We also denote the image of x in X_K by x.

If $\dim X = 2$, then X_K is a smooth and geometrically connected curve over K. Take the smooth compactification $\overline{X}_K$ of X_K over K. Then the structure morphism

$X_K \to \operatorname{Spec} K$ has the factorization $X_K \subset \overline{X}_K \to \operatorname{Spec} K$ and thus it is an elementary fibration with a section x. After shrinking X, we can spread it out into an elementary fibration $X \to S$ over an open subscheme S of $\operatorname{Spec} \mathbb{O}_K$ such that it admits a section $S \to X$.

Now assume $\dim X \geq 3$. We apply Artin's theorem on elementary fibration [SGA 4_3 1973, Exposé XI, proposition 3.3] to the pair (X_K, x), and by shrinking X if necessary we get an elementary fibration $\pi: X_K \to S_K$ over K with a geometrically irreducible smooth K-scheme S_K. Note that this theorem holds if the base field is perfect and infinite.

Since X_K is smooth over S_K, there exist an open neighborhood V_K of x in X_K and an étale morphism $\alpha: V_K \to \mathbb{A}^1_{S_K}$ such that $\pi|_{V_K}: V_K \to S_K$ admits a factorization

$$V_K \xrightarrow{\alpha} \mathbb{A}^1_{S_K} \to S_K.$$

Take a section $\tau: S_K \to \mathbb{A}^1_{S_K}$ of the projection such that $\alpha(x)$ lies in $\tau(S_K)$.

Consider the connected component S'_K of $S_K \times_{\tau, \mathbb{A}^1_{S_K}} V_K$ that contains the K-rational point $(\pi(x), x)$. This is étale over S_K and satisfies $S'_K(K) \neq \varnothing$. Moreover, S'_K is geometrically integral over K since it is a connected regular K-scheme containing a K-rational point.

We replace S_K by S'_K and X_K by $X_K \times_{S_K} S'_K$. By this replacement, the elementary fibration $\pi: X_K \to S_K$ admits a section and $S_K(K) \neq \varnothing$. After shrinking X, we can spread it out into an elementary fibration $X \to S$ with a section $S \to X$, where S is an irreducible regular scheme which is flat and of finite type over $\mathbb{O}_K$ with geometrically irreducible generic fiber and contains a K-rational point. The existence of a K-rational point implies that S has enough totally real curves. □

Proof of Proposition 3.4. First note that if $\alpha^*(f)|_{U''}$ has a kernel for a nonempty étale X-scheme $\alpha: X'' \to X$ and a dense open subscheme $U'' \subset X''$, then so does $f|_{\alpha(U'')}$.

Let n denote the dimension of X. Replacing X by the image of $X' \to X$, we may assume that $X' \to X$ is surjective.

Take a chain of split fibrations $X_n \to X_{n-1} \to \cdots \to X_1$ with a totally real curve X_1 as in Lemma 3.8. We regard X_1 as an X-scheme via $X_n \to X$ and the sections $X_i \to X_{i+1}$. Put $X'_1 = X' \times_X X_1$. This is a nonempty scheme. For $i = 2, \ldots, n$ we put $X'_i = X_i \times_{X_1} X'_1$ via the morphism $X_i \to X_{i-1} \to \cdots \to X_1$. Then $X'_n \to X'_{n-1} \to \cdots \to X'_1$ is a chain of split fibrations.

Since $f|_{X_1}$ lies in $\mathrm{LS}_r(X_1)$ by assumption (i), we have $f|_{X'_1} = (f_{X_1})|_{X'_1} \in \mathrm{LS}_r(X'_1)$. Then we get the result for $(X'_2, f|_{X'_2})$ by Lemma 3.6. Repeating this argument for the chain of split fibrations $X'_n \to \cdots \to X'_2$ we get the result for $(X'_n, f|_{X'_n})$. Applying the remark at the beginning to the morphism $X'_n \to X$, we get the result for (X, f). □

For the later use, we prove variants of Lemma 3.8. The proof given below is similar to that of Lemma 3.8, but instead of Lemma 2.3 we will use Corollary 2.10 and Theorem 2.11.

Lemma 3.9. (i) (With notation as in Lemma 3.8). *Suppose that we are given a connected étale covering $Y \to X$. Then there exist an étale X-scheme X_n, a totally real curve X_1, and a chain of split fibrations $X_n \to \cdots \to X_1$ such that $X_1 \times_X Y$ is connected. Here $X_1 \to X$ is the composite of sections $X_{i+1} \to X_i$ and $X_n \to X$.*

(ii) *Let F be a CM field and Z an n-dimensional irreducible regular $\mathcal{O}_F$-scheme with geometrically irreducible generic fiber. Let $Y \to Z$ be a connected étale covering. Then there exist an étale Z-scheme Z_n, a CM curve Z_1, and a chain of split fibrations $Z_n \to \cdots \to Z_1$ such that $Z_1 \times_Z Y$ is connected. Here $Z_1 \to Z$ is the composite of sections $Z_{i+1} \to Z_i$ and $Z_n \to Z$.*

Proof. First we prove (i) by induction on $n = \dim X$. Since the claim is obvious when $\dim X = 1$, we assume $\dim X \geq 2$.

By induction on $\dim X$, it suffices to prove that after replacing K by a totally real field extension, X by a nonempty étale X-scheme, and the covering $Y \to X$ by its pullback, there exist an irreducible regular $\mathcal{O}_K$-scheme S with enough totally real curves and an elementary fibration $X \to S$ with a section $S \to X$ such that $S \times_X Y$ is connected. The construction of such an S will be the same as that of Lemma 3.8.

It follows from Corollary 2.10 and Remark 2.12 that there exist a totally real extension L of K and an L-rational point $x \in X(L)$ such that $\operatorname{Spec} L \times_{x,X} Y$ is connected. Note that $Y \otimes_{\mathcal{O}_K} \mathcal{O}_L$ is connected because

$$Y \otimes_{\mathcal{O}_K} \mathcal{O}_L \to X \otimes_{\mathcal{O}_K} \mathcal{O}_L$$

is an étale covering with connected base and

$$\operatorname{Spec} L \times_{x,(X \otimes_{\mathcal{O}_K} \mathcal{O}_L)} (Y \otimes_{\mathcal{O}_K} \mathcal{O}_L) = \operatorname{Spec} L \times_{x,X} Y$$

is connected. Thus replacing K by L, X by a nonempty open subscheme of $X \otimes_{\mathcal{O}_K} \mathcal{O}_L$ that is étale over X, and Y by its pullback, we may further assume that the generic fiber $X_K \to \operatorname{Spec} K$ has a section $x \colon \operatorname{Spec} K \to X_K$ such that $\operatorname{Spec} K \times_{x,X} Y$ is connected. We also denote the image of x in X_K by x.

If $\dim X = 2$, the morphism $X_K \to \operatorname{Spec} K$ is an elementary fibration with a section x. After shrinking X, we can spread it out into an elementary fibration $X \to S$ over an open subscheme S of $\operatorname{Spec} \mathcal{O}_K$ such that it admits a section $S \to X$. By construction, $S \times_X Y$ is connected.

Now assume $\dim X \geq 3$. We apply Artin's theorem on elementary fibration to the pair (X_K, x), and by shrinking X if necessary we get an elementary fibration $\pi \colon X_K \to S_K$ over K with a geometrically irreducible smooth K-scheme S_K.

By smoothness, there exist an open neighborhood V_K of x in X_K and an étale morphism $\alpha\colon V_K \to \mathbb{A}^1_{S_K}$ such that $\pi|_{V_K}\colon V_K \to S_K$ admits a factorization

$$V_K \xrightarrow{\alpha} \mathbb{A}^1_{S_K} \to S_K.$$

Take a section $\tau\colon S_K \to \mathbb{A}^1_{S_K}$ of the projection such that $\alpha(x)$ lies in $\tau(S_K)$.

Consider the connected component S'_K of $S_K \times_{\tau,\mathbb{A}^1_{S_K}} V_K$ that contains the K-rational point $(\pi(x), x)$. As was shown in Lemma 3.8, S'_K is étale over S_K and geometrically integral over K, and $S'_K(K) \neq \varnothing$.

The section τ defines the morphism $S'_K \to X_K \times_{S_K} S'_K \to X_K$. The composite of this morphism and $(\pi(x), x)\colon \operatorname{Spec} K \to S'_K$ coincides with $x\colon \operatorname{Spec} K \to X_K$. Since $S'_K \times_X Y \to S'_K$ is an étale covering with connected base and

$$\operatorname{Spec} K \times_{(\pi(x),x),S'_K} (S'_K \times_X Y) = \operatorname{Spec} K \times_{x,X} Y$$

is connected, it follows that $S'_K \times_X Y$ is connected.

We replace S_K by S'_K, X_K by $X_K \times_{S_K} S'_K$ and Y_K by $Y_K \times_{S_K} S'_K$. By this replacement, the elementary fibration $\pi\colon X_K \to S_K$ admits a section such that $S_K(K) \neq \varnothing$ and $S_K \times_{X_K} Y_K$ is connected. As is discussed in the last paragraph of the proof of Lemma 3.8, after shrinking X, we can spread out $X_K \to S_K$ and $Y_K \to S_K$ into an elementary fibration $X \to S$ with a section $S \to X$ and a covering $Y \to X$, where S is an irreducible regular $\mathcal{O}_K$-scheme with enough totally real curves. Since $S \times_X Y$ is connected by construction, this S works.

For part (ii), it is easy to verify that the same argument works if we apply Theorem 2.11 instead of Corollary 2.10. $\square$

Next we show that if f has a kernel and arises from a lisse sheaf over every totally real curve then it actually arises from a lisse sheaf.

Proposition 3.10. *Let K be a totally real field and X an irreducible smooth separated $\mathcal{O}_K[\ell^{-1}]$-scheme with enough totally real curves. Suppose that $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfies the following conditions*:

(i) *f arises from a lisse sheaf over every totally real curve.*

(ii) *f has a kernel.*

Then $f \in \mathrm{LS}_r(X)$.

Proof. We follow Section 4 of [Drinfeld 2012]. Since f has a kernel, we take a closed subgroup H of $\pi_1(X)$ as in the definition of having a kernel. In particular, $\pi_1(X)/H$ contains an open pro-ℓ subgroup.

By Corollary 2.10, there exists a totally real curve C with a morphism $\varphi\colon C \to X$ such that $\varphi_*\colon \pi_1(C) \to \pi_1(X)/H$ is surjective. By assumption (i), for any such pair

(C, φ), the pullback $\varphi^*(f)$ arises from a semisimple representation $\rho_C\colon \pi_1(C) \to \mathrm{GL}_r(E_\lambda)$. Define

$$H_C := \mathrm{Ker}\big(\varphi_*\colon \pi_1(C) \to \pi_1(X)/H\big).$$

Then condition (ii) in the definition of having a kernel (Definition 3.1), together with the Chebotarev density theorem, shows that $\mathrm{Ker}\,\rho_C$ contains H_C. See [Drinfeld 2012, Lemma 4.1] for details. Thus we regard ρ_C as a representation

$$\rho_C\colon \pi_1(X) \to \pi_1(X)/H \to \mathrm{GL}_r(E_\lambda)$$

of $\pi_1(X)$. Note that $\rho_C|_{\pi_1(C)}$ gives the original representation of $\pi_1(C)$.

We will show that the lisse sheaf on X corresponding to this representation gives f. For this, we need to show that

$$\det\big(1 - t\rho_C(\mathrm{Frob}_x)\big) = f_x(t)$$

for all closed points $x \in X$. We know that this equality holds for each closed point $x \in \varphi(C)$ such that $\varphi^{-1}(x)$ contains a point whose residue field is equal to $k(x)$.

Take any closed point $x \in X$. We will first construct a curve C' passing through x and some finitely many points on C specified below. We will then construct a lisse sheaf on C' whose Frobenius polynomial at x is $f_x(t)$, and prove that the lisse sheaf on C' extends over X and the corresponding representation of $\pi_1(X)$ coincides with ρ_C.

We use a lemma by Faltings; define T_0 to be the set of closed points of C which have the same image in $\mathrm{Spec}\,\mathbb{O}_K$ as that of x. By the theorem of Hermite, the Chebotarev density theorem, and the Brauer–Nesbitt theorem, there exists a finite set $T \subset |C| \setminus T_0$ satisfying the following properties:

(i) $T \to \mathrm{Spec}\,\mathbb{O}_K$ is injective.

(ii) For any semisimple representations $\rho_1, \rho_2\colon \pi_1(C) \to \mathrm{GL}_r(E_\lambda)$, the equality $\mathrm{tr}\,\rho_1(\mathrm{Frob}_y) = \mathrm{tr}\,\rho_2(\mathrm{Frob}_y)$ for all $y \in T$ implies $\rho_1 \cong \rho_2$.

See [Faltings 1983, Satz 5] or [Deligne 1985, théorème 3.1] for details.

By Corollary 2.10 applied to $S = \varphi(T) \cup \{x\}$, there exists a totally real curve C' with a morphism $\varphi'\colon C' \to X$ such that the map $\varphi'_*\colon \pi_1(C') \to \pi_1(X)/H$ is surjective and for each $y \in \varphi(T) \cup \{x\}$ there exists a point in $\varphi'^{-1}(y)$ whose residue field is equal to $k(y)$. As discussed before, this pair (C', φ') also defines a semisimple representation

$$\rho_{C'}\colon \pi_1(X) \to \pi_1(X)/H \to \mathrm{GL}_r(E_\lambda)$$

such that $\det(1 - t\rho_{C'}(\mathrm{Frob}_y)) = f_y(t)$ for each $y \in \varphi(T) \cup \{x\}$. Note that the surjectivity of φ_* implies that $\rho_{C'}|_{\pi_1(C)}$ is semisimple.

It follows from property (ii) of T that $\rho_C|_{\pi_1(C)}$ and $\rho_{C'}|_{\pi_1(C)}$ are isomorphic as representations of $\pi_1(C)$. Since the map $\varphi_*\colon \pi_1(C) \to \pi_1(X)/H$ is surjective, we have $\rho_C \cong \rho_{C'}$ as representations of $\pi_1(X)/H$ and thus they are also isomorphic as representations of $\pi_1(X)$. In particular, we have

$$\det\bigl(1 - t\rho_C(\mathrm{Frob}_x)\bigr) = \det\bigl(1 - t\rho_{C'}(\mathrm{Frob}_x)\bigr) = f_x(t).$$

Hence f comes from the lisse sheaf on X corresponding to ρ_C. $\square$

We now prove the last proposition of our three key ingredients for Theorem 1.4. This concerns extendability of a lisse sheaf on a dense open subset to the whole scheme. In the proof we use the Zariski–Nagata purity theorem; thus the regularity assumption for X is crucial. We further need to assume that X is smooth as we use Corollary 2.10 to find a totally real curve passing through a given point in a given tangent direction.

Proposition 3.11. *Let K be a totally real field and X an irreducible smooth separated $\mathcal{O}_K[\ell^{-1}]$-scheme with enough totally real curves. Suppose that $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfies the following conditions*:

(i) *f arises from a lisse sheaf over every totally real curve.*

(ii) *There exists a dense open subscheme $U \subset X$ such that $f|_U \in \mathrm{LS}_r(U)$.*

Then $f \in \mathrm{LS}_r(X)$.

Proof. We follow Section 5.2 of [Drinfeld 2012]. Let $\mathscr{E}_U$ be the semisimple lisse E_λ-sheaf on U corresponding to $f|_U$. First we show that $\mathscr{E}_U$ extends to a lisse E_λ-sheaf on X.

Suppose the contrary. Since X is regular, the Zariski–Nagata purity theorem implies that there exists an irreducible divisor D of X contained in $X \setminus U$ such that $\mathscr{E}_U$ is ramified along D. Then by a specialization argument [Drinfeld 2012, Corollary 5.2], we can find a closed point $x \in X \setminus U$ and a one-dimensional subspace $l \subset T_x X$ satisfying the following property:

($*$) Consider a triple (C, c, φ) consisting of a regular curve C, a closed point $c \in C$, and a morphism $\varphi\colon C \to X$ such that $\varphi(c) = x$, $\varphi^{-1}(U) \neq \varnothing$, and $\mathrm{Im}\bigl(T_c C \to T_x X \otimes_{k(x)} k(c)\bigr) = l \otimes_{k(x)} k(c)$. For any such triple, the pullback of $\mathscr{E}_U$ to $\varphi^{-1}(U)$ is ramified at c.

Let H be the kernel of the representation $\rho_U\colon \pi_1(U) \to \mathrm{GL}_r(E_\lambda)$ corresponding to $\mathscr{E}_U$. The group $\pi_1(U)/H \cong \mathrm{Im}\,\rho_U$ contains an open pro-ℓ subgroup because $\mathrm{Im}\,\rho_U$ is a compact open subgroup of $\mathrm{GL}_r(E_\lambda)$. Therefore by Corollary 2.10 we can find a totally real curve C, a closed point $c \in C$, and a morphism $\varphi\colon C \to X$ such that

- $\varphi(c) = x$ and $k(c) \cong k(x)$,

- $\varphi^{-1}(U) \neq \varnothing$ and $\varphi_*: \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, and
- $\mathrm{Im}(T_c C \to T_x X) = l$.

Since $\varphi_*: \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, the pullback of $\mathscr{E}_U$ to $\varphi^{-1}(U)$ is semisimple. Thus this lisse E_λ-sheaf has no ramification at c by assumption (i), which contradicts property $(*)$. Hence $\mathscr{E}_U$ extends to a lisse E_λ-sheaf $\mathscr{E}$ on X.

Let f' be the element of $\mathrm{LS}_r(X)$ corresponding to $\mathscr{E}$. We know $f|_U = f'|_U$. Take any closed point $x \in X$. It suffices to show that $f(x) = f'(x)$. We can find a totally real curve C', a closed point $c' \in C'$, and a morphism $\varphi': C' \to X$ such that $\varphi'(c') = x$, $k(c') = k(x)$, and $\varphi'^{-1}(U) \neq \varnothing$. Then

$$\varphi'^*(f)|_{\varphi'^{-1}(U)} = (f|_U)|_{\varphi'^{-1}(U)} = (f'|_U)|_{\varphi'^{-1}(U)} = \varphi'^*(f')|_{\varphi'^{-1}(U)}.$$

Since $\varphi'^{-1}(U) \neq \varnothing$, the homomorphism $\pi_1(\varphi'^{-1}(U)) \to \pi_1(C')$ is surjective and thus $\varphi'^*(f) = \varphi'^*(f')$. In particular, $f(x) = \varphi'^*(f)(c') = \varphi'^*(f')(c') = f'(x)$. $\square$

Proof of Theorem 1.4. First note that a polynomial-valued map f of degree r in the theorem lies in $\widetilde{\mathrm{LS}}_r(X)$. One direction of the equivalence is obvious, and thus it suffices to prove that if f satisfies conditions (i) and (ii), then f lies in $\mathrm{LS}_r(X)$.

First assume that X is separated. Let k_λ be the residue field of E_λ and N be the cardinality of $\mathrm{GL}_r(k_\lambda)$. Put $X' := X \otimes_{\mathbb{Z}} \mathbb{Z}[N^{-1}]$. Then $X' \to X$ is a dominant étale morphism and satisfies the following property:

> The pullback $f|_{X'}$ arises from a tame lisse sheaf over every separated smooth curve over a finite field.

Thus by Proposition 3.4, there exists an open dense subscheme U of X such that $f|_U$ has a kernel. Therefore $f|_U$ lies in $\mathrm{LS}_r(U)$ by Proposition 3.10 and we have $f \in \mathrm{LS}_r(X)$ by Proposition 3.11.

In the general case, we consider a covering $X = \bigcup_i U_i$ by open separated subschemes. Then we can apply the above discussion to each $f|_{U_i}$ and obtain a lisse E_λ-sheaf $\mathscr{E}_i$ on U_i that represents $f|_{U_i}$. Since U_i is normal, we can replace $\mathscr{E}_i$ by its semisimplification and assume that each $\mathscr{E}_i$ is semisimple.

Put $U = \bigcap_i U_i$. This is nonempty, and the restrictions $\mathscr{E}_i|_U$ are isomorphic to each other. Thus $\{\mathscr{E}_i\}_i$ glues to a lisse E_λ-sheaf on X and this sheaf represents f. $\square$

We end this section with variants of Theorem 1.4. Condition (i) in Theorem 3.12 or Remark 3.13 is weaker than that of Theorem 1.4 since they concern only totally real curves with additional properties. This weaker condition is essential to use the result of [Barnet-Lamb et al. 2014] in the proof of our main theorems in the next section. Theorem 3.14 is a variant in the CM case.

Theorem 3.12. *Let K be a totally real field. Let X be an irreducible smooth $\mathscr{O}_K[\ell^{-1}]$-scheme with enough totally real curves. An element $f \in \widetilde{\mathrm{LS}}_r(X)$ belongs to $\mathrm{LS}_r(X)$ if and only if it satisfies the following conditions*:

(i) *There exists a connected étale covering $Y \to X$ such that f arises from a lisse sheaf over every totally real curve C with the property that $C \times_X Y$ is connected.*

(ii) *The restriction of f to each separated smooth curve over a finite field arises from a lisse sheaf.*

Proof. Recall that Theorem 1.4 is deduced from Propositions 3.4, 3.10, and 3.11 and that these propositions have the same condition (i) that f arises from a lisse sheaf over every totally real curve. Consider the variant statements of Propositions 3.4, 3.10, and 3.11 where we replace condition (i) by

(i′) f arises from a lisse sheaf over every totally real curve C such that $C \times_X Y$ is connected.

It suffices to prove that these variants also hold; then the theorem is deduced from them in the same way as Theorem 1.4.

The variant of Proposition 3.4 is proved in the same way as Proposition 3.4 if one uses Lemma 3.9(i) instead of Lemma 3.8. For the variants of Propositions 3.10 and 3.11, the same proof works; observe that whenever one uses Corollary 2.10 in the proof to find a totally real curve C, one can impose the additional condition that $C \times_X Y$ is connected by adding the covering $Y \to X$ to the input of Corollary 2.10. □

Remark 3.13. We need another variant of Theorem 1.4 to prove Theorem 1.3: With the notation as in Theorem 3.12, suppose further that

- K is unramified at ℓ, and
- X extends to an irreducible smooth $\mathcal{O}_K$-scheme X' with nonempty fiber over each place of K above ℓ.

Then condition (i) in Theorem 3.12 can be replaced by

(i′) There exists a connected étale covering $Y \to X$ such that f arises from a lisse sheaf over every totally real curve C with the properties that
  - $C \times_X Y$ is connected and that
  - the fraction field of C is unramified at ℓ.

This statement is proved in the same way as Theorem 3.12; it suffices to prove variants of Propositions 3.4, 3.10, and 3.11 where condition (i) in these propositions is replaced by the following condition:

The map f arises from a lisse sheaf over every totally real curve C such that $C \times_X Y$ is connected and the fraction field of C is unramified at ℓ.

For the proof of the variant of Proposition 3.4, we also need to consider the variant of Lemma 3.9(i) where we further require that the fraction field of X_1 is unramified at ℓ.

We now explain how to prove the variants of Lemma 3.9(i) and Propositions 3.4, 3.10, and 3.11. By the additional condition on X, for each place v of K above ℓ, there exist a finite unramified extension L of K_v and a morphism $\operatorname{Spec} \mathcal{O}_L \to X'$. We denote the image of the closed point of $\operatorname{Spec} \mathcal{O}_L$ by s_v. Since L is unramified over K_v, we can find a horizontal one-dimensional subspace l_v of $T_{s_v} X'$ with respect to $X' \to \operatorname{Spec} \mathcal{O}_K$.

If we add $\{s_v\}_{v|\ell}$ and l_v to the input when we use Corollary 2.10, the fraction field of the resulting totally real curve is unramified over K at each v, hence unramified at ℓ. Thus we can prove the variant of Lemma 3.9(i) in the same way as Lemma 3.9(i), and the arguments given in Theorem 3.12 work for the current variants of Propositions 3.4, 3.10, and 3.11. Hence the statement of this remark follows.

Theorem 3.14. *Let F be a CM field. Let Z be an irreducible smooth $\mathcal{O}_F[\ell^{-1}]$-scheme with geometrically irreducible generic fiber. An element $f \in \widetilde{\mathrm{LS}}_r(Z)$ belongs to $\mathrm{LS}_r(Z)$ if and only if it satisfies the following conditions*:

(i) *There exists a connected étale covering $Y \to Z$ such that f arises from a lisse sheaf over every CM curve C with the property that $C \times_Z Y$ is connected.*

(ii) *The restriction of f to each separated smooth curve over a finite field arises from a lisse sheaf.*

Proof. We can prove variants of Propositions 3.4, 3.10, and 3.11 for the CM case using Theorem 2.11 and Lemma 3.9(ii). Then the theorem is deduced from them in the same way as Theorems 1.4 and 3.12. □

4. Proofs of the main theorems

In this section, we prove theorems on the existence of the compatible system of a lisse sheaf. Theorem 4.1 concerns the totally real case and Theorem 4.2 concerns the CM case. Theorem 1.3 in the introduction is proved after Theorem 4.1. Following the discussion in [Drinfeld 2012, Section 2.3], we deduce these main theorems from Theorems 3.12, 3.14, and theorems in [Lafforgue 2002; Barnet-Lamb et al. 2014].

As we mentioned in the introduction, some of the assumptions in the main theorems come from the potential diagonalizability condition, which is introduced in [Barnet-Lamb et al. 2014, Section 1.4]. We first review this notion; see [loc. cit.] for details.

Let L be a finite extension of $\mathbb{Q}_\ell$. Let E_λ be a finite extension of $\mathbb{Q}_\ell$. We say that an $\bar{E}_\lambda$-representation ρ of $\mathrm{Gal}(\bar{L}/L)$ is *potentially diagonalizable* if it is potentially crystalline and there is a finite extension L' of L such that $\rho|_{\mathrm{Gal}(\bar{L}/L')}$ lies on the same irreducible component of the universal crystalline lifting ring of the residual representation $\bar{\rho}|_{\mathrm{Gal}(\bar{L}/L')}$ with fixed Hodge–Tate numbers as a sum of characters lifting $\bar{\rho}|_{\mathrm{Gal}(\bar{L}/L')}$.

There are two important examples of this notion (see [Barnet-Lamb et al. 2014, Lemma 1.4.3]): Ordinary representations are potentially diagonalizable. When L is unramified over $\mathbb{Q}_\ell$, a crystalline representation is potentially diagonalizable if for each $\tau\colon L \hookrightarrow \bar{E}_\lambda$ the τ-Hodge–Tate numbers lie in the range $[a_\tau, a_\tau + \ell - 2]$ for some integer a_τ.

We first prove our main theorem for the totally real case.

Theorem 4.1. *Let ℓ be a rational prime. Let K be a totally real field and X an irreducible smooth $\mathcal{O}_K[\ell^{-1}]$-scheme with enough totally real curves. Let E be a finite extension of $\mathbb{Q}$ and λ a prime of E above ℓ. Let $\mathcal{E}$ be a lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(X)$. Suppose that $\mathcal{E}$ satisfies the following assumptions*:

(i) *The polynomial* $\det(1 - \mathrm{Frob}_x\, t, \mathcal{E}_{\bar{x}})$ *has coefficients in E for every $x \in |X|$.*

(ii) *For every totally real field L and every morphism $\alpha\colon$ $\mathrm{Spec}\, L \to X$, the E_λ-representation $\alpha^*\rho$ of* $\mathrm{Gal}(\bar{L}/L)$ *is potentially diagonalizable at each prime v of L above ℓ and for each $\tau\colon L \hookrightarrow \bar{E}_\lambda$ it has distinct τ-Hodge–Tate numbers.*

(iii) *ρ can be equipped with a symplectic (resp. orthogonal) structure with multiplier $\mu\colon \pi_1(X) \to E_\lambda^\times$ such that $\mu|_{\pi_1(X_K)}$ admits a factorization*

$$\mu|_{\pi_1(X_K)}\colon \pi_1(X_K) \to \mathrm{Gal}(\bar{K}/K) \xrightarrow{\mu_K} E_\lambda^\times$$

with a totally odd (resp. totally even) character μ_K.

(iv) *The residual representation $\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$ is absolutely irreducible.*

(v) $\ell \geq 2(\mathrm{rank}\,\mathcal{E} + 1)$.

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\bar{E}_{\lambda'}$-sheaf on $X[\ell'^{-1}]$ which is compatible with $\mathcal{E}|_{X[\ell'^{-1}]}$.

Proof. Replacing X by $X[\ell'^{-1}]$, we may assume that ℓ' is invertible in $\mathcal{O}_X$. Let r be the rank of $\mathcal{E}$. Take an arbitrary extension M of $E_{\lambda'}$ of degree $r!$. By assumption (i), we regard the map $f\colon x \mapsto \det(1 - \mathrm{Frob}_x\, t, \mathcal{E}_{\bar{x}})$ as an element of $\widetilde{\mathrm{LS}}_r^M(X)$ via the embedding $E \hookrightarrow E_{\lambda'} \hookrightarrow M$.

We will apply Theorem 3.12 to $f \in \widetilde{\mathrm{LS}}_r^M(X)$. Here we use the prime ℓ' and the field M (we used ℓ and E_λ in Section 3).

First we show that the map f satisfies condition (i) in Theorem 3.12. Let Y be the connected étale covering $Y \to X[\zeta_\ell]$ that corresponds to $\mathrm{Ker}\,\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$. We regard Y as a connected étale covering over X via $Y \to X[\zeta_\ell] \to X$. We will prove that this $Y \to X$ satisfies condition (i). Take any totally real curve $\varphi\colon C \to X$ such that $C \times_X Y$ is connected.

To show that $\varphi^*(f)$ arises from a lisse M-sheaf on C, it suffices to prove that there exists a lisse $\bar{E}_{\lambda'}$-sheaf on C which is compatible with $\varphi^*\mathscr{E}$; this follows from [Drinfeld 2012, Lemma 2.7]. Namely, let $\rho'_C\colon \pi_1(C) \to \mathrm{GL}_r(\bar{E}_{\lambda'})$ denote the semisimplification of the corresponding $\bar{E}_{\lambda'}$-representation. Since

$$\det\bigl(1 - t\rho'_C(\mathrm{Frob}_x)\bigr) = f_x(t) \in E_{\lambda'}[t]$$

for every closed point $x \in C$, the character of ρ'_C is defined over $E_{\lambda'}$ by the Chebotarev density theorem. It follows from $[M : E_{\lambda'}] = r!$ that the Brauer obstruction of ρ'_C in the Brauer group $\mathrm{Br}(E_{\lambda'})$ vanishes in $\mathrm{Br}(M)$ and ρ'_C can be defined over M. This means $\varphi^*(f) \in \mathrm{LS}_r^M(C)$.

We will construct a lisse $\bar{E}_{\lambda'}$-sheaf on C which is compatible with $\varphi^*\mathscr{E}$. For this we will apply Theorem C from [Barnet-Lamb et al. 2014] to the $\bar{E}_\lambda$-representation $\varphi_L^*\rho$ of $\mathrm{Gal}(\bar{L}/L)$, where L denotes the fraction field of C and $\varphi_L\colon \operatorname{Spec} L \to X$ denotes $\varphi|_{\operatorname{Spec} L}$.

We need to see that the Galois representation $\varphi_L^*\rho$ satisfies the assumptions in Theorem C. By assumptions (ii) and (v) it remains to check that

(a) $\varphi_L^*\rho$ can be equipped with a symplectic (resp. orthogonal) structure with totally odd (resp. totally even) multiplier, and

(b) the residual representation $(\varphi_L^*\bar{\rho})|_{\mathrm{Gal}(\bar{L}/L(\zeta_\ell))}$ is absolutely irreducible.

Assumption (a) follows from assumption (iii). To see (b), recall that $C \times_X Y$ is connected. Hence $C \times_X X[\zeta_\ell]$ is connected with fraction field $L(\zeta_\ell)$, and $C \times_X Y \to C \times_X X[\zeta_\ell]$ is a connected étale covering. It follows from the definition of Y that $\mathrm{Im}(\varphi_L^*\bar{\rho})|_{\mathrm{Gal}(\bar{L}/L(\zeta_\ell))}$ coincides with $\mathrm{Im}\,\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$, and thus $(\varphi_L^*\bar{\rho})|_{\mathrm{Gal}(\bar{L}/L(\zeta_\ell))}$ is absolutely irreducible by assumption (iv).

Hence by [Barnet-Lamb et al. 2014, Theorem C] we obtain an $\bar{E}_{\lambda'}$-representation of $\mathrm{Gal}(\bar{L}/L)$. The proof of the theorem, which uses potential automorphy and Brauer's theorem, shows that this representation is unramified at each closed point of C, and thus it gives rise to a lisse $\bar{E}_{\lambda'}$-sheaf on C which is compatible with $\varphi^*\mathscr{E}$. Hence f satisfies condition (i) in Theorem 3.12.

Next we show that f satisfies condition (ii) in Theorem 3.12. Let C be a separated smooth curve over $\mathbb{F}_p$ for some prime p and denote the structure morphism $C \to \operatorname{Spec} \mathbb{F}_p$ by α. Let $\varphi\colon C \to X$ be a morphism. Note that p is different from ℓ and ℓ'.

We write the semisimplification of $\varphi^*\mathscr{E}$ as $\bigoplus_i \mathscr{E}_i^{\oplus r_i}$, where $\mathscr{E}_i$ are distinct irreducible lisse $\bar{E}_\lambda$-sheaves on C. Then there exist an irreducible lisse $\bar{E}_\lambda$-sheaf $\mathscr{F}_i$ on C and a lisse $\bar{E}_\lambda$-sheaf $\mathscr{G}_i$ of rank 1 on $\operatorname{Spec} \mathbb{F}_p$ such that $\mathscr{F}_i$ has determinant of finite order and $\mathscr{E}_i \cong \mathscr{F}_i \otimes \alpha^*\mathscr{G}_i$ (see [Deligne 1980, Section I.3] or [Deligne 2012, Section 0.4], for example).

By théorèm VII.6 of [Lafforgue 2002], for each closed point $x \in C$, the roots of $\det(1 - \mathrm{Frob}_x\, t, \mathcal{F}_{i,\bar{x}})$ are algebraic numbers that are λ'-adic units. Moreover, there exists an irreducible lisse $\bar{E}_{\lambda'}$-sheaf $\mathcal{F}'_i$ on C which is compatible with $\mathcal{F}_i$.

We will show that there exists a lisse $\bar{E}_{\lambda'}$-sheaf $\mathcal{G}'_i$ on $\operatorname{Spec} \mathbb{F}_p$ which is compatible with $\mathcal{G}_i$. Note that the lisse $\bar{E}_{\lambda}$-sheaf $\mathcal{G}_i$ is determined by the value of the corresponding character of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ at the geometric Frobenius. Denote this value by $\beta_i \in \bar{E}_{\lambda}^{\times}$. It suffices to prove that β_i is an algebraic number that is a λ'-adic unit. Since the roots of $\det(1 - \mathrm{Frob}_x\, t, \mathcal{E}_{\bar{x}})$ and $\det(1 - \mathrm{Frob}_x\, t, \mathcal{F}_{i,\bar{x}})$ are all algebraic numbers, so is β_i.

We prove that β_i is a λ'-adic unit. To see this, take a closed point x of C. Then by Corollary 2.10 we can find a totally real curve C' and a morphism $\varphi' \colon C' \to X$ such that $\varphi(x) \in \varphi'(C')$ and $C' \times_X Y$ is connected. As discussed before, Theorem C of [Barnet-Lamb et al. 2014] implies that there exists a lisse $\bar{E}_{\lambda'}$-sheaf on C' whose Frobenius characteristic polynomial map is $\varphi'^*(f)$. Thus for each closed point $y \in C'$ the roots of $\varphi'^*(f)(y)$ are algebraic numbers that are λ'-adic units. Considering a point $y \in \varphi'^{-1}(\varphi(x))$, we conclude that some power of β_i is a λ'-adic unit and thus so is β_i. Hence there exists a lisse $\bar{E}_{\lambda'}$-sheaf $\mathcal{G}'_i$ on $\operatorname{Spec} \mathbb{F}_p$ which is compatible with $\mathcal{G}_i$.

The Frobenius characteristic polynomial map associated with the semisimple lisse $\bar{E}_{\lambda'}$-sheaf $\bigoplus_i (\mathcal{F}'_i \otimes \alpha^* \mathcal{G}'_i)^{\oplus r_i}$ is $\varphi^*(f)$. As discussed before, this sheaf can be defined over M. Thus f satisfies condition (ii) in Theorem 3.12.

Therefore by Theorem 3.12 there exists a lisse M-sheaf on X which is compatible with $\mathcal{E}$. □

Proof of Theorem 1.3. All the discussions in the proof of Theorem 4.1 also work in this setting by using Remark 3.13 instead of Theorem 3.12. □

We also have a theorem for the CM case.

Theorem 4.2. *Let ℓ be a rational prime, E a finite extension of $\mathbb{Q}$, and λ a prime of E above ℓ. Let F be a CM field with $\zeta_\ell \notin F$ and Z an irreducible smooth $\mathcal{O}_F[\ell^{-1}]$-scheme with geometrically irreducible generic fiber. Let $\mathcal{E}$ be a lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(Z)$. Suppose that $\mathcal{E}$ satisfies the following assumptions*:

(i) *The polynomial* $\det(1 - \mathrm{Frob}_x\, t, \mathcal{E}_{\bar{x}})$ *has coefficients in E for every $x \in |X|$.*

(ii) *For any CM field L with $\zeta_\ell \notin L$ and any morphism $\alpha \colon \operatorname{Spec} L \to Z$, the E_λ-representation $\alpha^*\rho$ of* $\mathrm{Gal}(\bar{L}/L)$ *satisfies the following two conditions*:
  - (a) *$\alpha^*\rho$ is potentially diagonalizable at each prime v of L above ℓ and for each $\tau \colon L \hookrightarrow \bar{E}_\lambda$ it has distinct τ-Hodge–Tate numbers.*
  - (b) *$\alpha^*\rho$ is totally odd and polarizable* (in the sense of [Barnet-Lamb et al. 2014, Section 2.1]).

(iii) *The residual representation $\bar{\rho}|_{\pi_1(Z[\zeta_\ell])}$ is absolutely irreducible.*

(iv) $\ell \geq 2(\operatorname{rank} \mathscr{E} + 1)$.

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\bar{E}_{\lambda'}$-sheaf on $Z[\ell'^{-1}]$ which is compatible with $\mathscr{E}|_{Z[\ell'^{-1}]}$.

Proof. In the same way as Theorem 4.1, the theorem is deduced from Theorem 3.14, Theorem 5.5.1 of [Barnet-Lamb et al. 2014], théorèm VII.6 of [Lafforgue 2002], and the following remark: If $Y \to Z[\zeta_\ell]$ denotes the connected étale covering defined by $\operatorname{Ker} \bar{\rho}_{\pi_1(Z[\zeta_\ell])}$ and C is a CM curve with a morphism to Z such that $C \times_Z Y$ is connected, then $C \times_Z Z[\zeta_\ell] = C \otimes_{\mathbb{O}_F} \mathbb{O}_F(\zeta_\ell)$ is connected. In particular, the fraction field of C does not contain ζ_ℓ. □

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Logarithmic good reduction, monodromy and the rational volume

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Let R be a strictly local ring complete for a discrete valuation, with fraction field K and residue field of characteristic $p > 0$. Let X be a smooth, proper variety over K. Nicaise conjectured that the rational volume of X is equal to the trace of the tame monodromy operator on ℓ-adic cohomology if X is cohomologically tame. He proved this equality if X is a curve. We study his conjecture from the point of view of logarithmic geometry, and prove it for a class of varieties in any dimension: those having logarithmic good reduction.

1. Introduction

Let R be a strictly local ring complete for a discrete valuation, with fraction field K. Assume that the residue field k of R is algebraically closed, of characteristic $p > 0$. Fix a prime number $\ell \neq p$. Let K^s be a separable closure of K, and let K^t be the tame closure of K inside K^s. Let $P = \mathrm{Gal}(K^s/K^t)$ be the wild inertia group.

Let φ be a topological generator of the tame inertia group $I^t = \mathrm{Gal}(K^t/K)$, which is procyclic; we refer to φ as the *tame monodromy operator*.

Let X be a proper, smooth K-variety. One cannot simply "count" the number of rational points on X, but one can use other measures for the number of rational points on X. One such measure, the *rational volume*, can be constructed using a so-called *weak Néron model* of X, of which we recall the definition.

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Definition 1.1. A *weak Néron model* for X is a smooth, separated R-scheme of finite type $\mathcal{X}$, endowed with an isomorphism $\mathcal{X} \times_R K \cong X$, such that the natural map $\mathcal{X}(R) \to \mathcal{X}(K) = X(K)$ is a bijection.

Such a model always exists: one can simply take the smooth locus of a Néron smoothening of any proper R-model of X; see [Bosch et al. 1990, Theorem 3.1.3].

Definition 1.2. The *rational volume* $\mathrm{s}(X)$ of X is the ℓ-adic Euler characteristic of the special fibre of a weak Néron model for X.

That the integer $\mathrm{s}(X)$ does not depend on the choice of a weak Néron model is a deep result; the only known proof for this fact uses the theory of motivic integration. For more details, we refer to the foundational paper of Loeser and Sebag [2003], and subsequent work by Nicaise and Sebag [2007], Nicaise [2011] and Esnault and Nicaise [2011, §3] on the *motivic Serre invariant*. This is the class in $K_0^R(\mathrm{Var}_k)/(\mathbb{L}-1)$ of the special fibre of a weak Néron model; here $K_0^R(\mathrm{Var}_k)$ denotes a "modified" Grothendieck ring of varieties over k (see [Esnault and Nicaise 2011, §2] for the precise definition), and $\mathbb{L}$ is the class of $\mathbb{A}_k^1$ in this ring. The motivic Serre invariant is already independent of the choice of such a model (see, e.g., [Esnault and Nicaise 2011, Theorem 3.6]), and hence so is its realization $\mathrm{s}(X)$.

One has $\mathrm{s}(X) = 0$ if $X(K) = \varnothing$, since then X (viewed as an R-scheme) is a weak Néron model for itself. Nicaise [2011, §6] asked whether the rational volume has a cohomological interpretation similar to the Grothendieck–Lefschetz formula: if the variety X is cohomologically tame, i.e., if the wild inertia subgroup P acts trivially on the étale cohomology groups $H^i(X \times_K K^s, \mathbb{Q}_\ell)$, and if moreover $X(K^t) \neq \varnothing$, do we have the equality

$$\mathrm{s}(X) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\varphi \mid H^i(X \times_K K^t, \mathbb{Q}_\ell))\,? \tag{1}$$

The left-hand side of this formula is defined from the special fibre of an integral model of X, whereas the right-hand side (which is a priori an element of $\mathbb{Q}_\ell$) involves only the generic fibre. In a groundbreaking paper, Nicaise and Sebag [2007] proved a formula of this type for nonarchimedean analytic spaces; their formula is an arithmetic analogue of the one obtained by Denef and Loeser [2002].

Subsequently, Nicaise proved the equality (1) in equal characteristic zero [Nicaise 2011, §6]. However, the situation becomes much more subtle if the residual characteristic is positive: Nicaise [2011, §7] proved the equality (1) if X is a curve, and Halle and Nicaise [2016, Chapter 8] handled the case of a semiabelian variety.

In this paper, we study the conjectural formula (1) in the framework of logarithmic geometry in order to prove it for a large class of varieties in any dimension, those with *log good reduction*. Our main result can be stated as follows.

Theorem 1.3. *Let X be a proper, smooth K-variety. Assume that there exists a flat, proper R-model $\mathcal{X}$ of X such that $\mathcal{X}^\dagger$ is log smooth over $R^\dagger$. Then (1) holds, i.e.,*

$$\mathrm{s}(X) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\varphi \mid H^i(X \times_K K^t, \mathbb{Q}_\ell)).$$

Here the log scheme $\mathcal{X}^\dagger$ stands for the model $\mathcal{X}$ equipped with the natural log structure, i.e., the divisorial log structure induced by the special fibre $\mathcal{X}_s$, and $R^\dagger$ is the "log ring" given by the inclusion $R \setminus \{0\} \hookrightarrow R$. We want to stress that the underlying scheme $\mathcal{X}$ may very well be singular, even if $\mathcal{X}^\dagger$ is log smooth.

By work of Nakayama [1998, Corollary 0.1.1], varieties with log good reduction are automatically cohomologically tame, i.e., they satisfy the major hypothesis in Nicaise's conjecture. Nicaise also assumed the existence of a K^t-point in the statement of his conjecture, but we will not need this assumption: our method works for all varieties with log good reduction, with or without a K^t-point. Nevertheless, it would be interesting to know whether the log good reduction hypothesis implies the existence of a K^t-point in general, and as far as we know, this question is open.

Conversely, it is known in some cases that the cohomological tameness assumption implies the existence of a proper, log smooth model. Indeed, if X is a cohomologically tame curve of genus at least 2, then X has log good reduction, by work of T. Saito [1987; 2004] and Stix [2005, Theorem 1.2]. Hence we recover Nicaise's [2011, §7] results for curves. Moreover, in their recent preprint, A. Bellardini and the author [2015] managed to prove the existence of a proper log smooth model for any cohomologically tame abelian variety. Hence our Theorem 1.3 also implies the result of Halle and Nicaise [2016, Chapter 8] on the validity of the trace formula (1) for abelian varieties.

Our proof is very much inspired by the strategy used by Nicaise and Sebag [2007] and Nicaise [2011]; however, the technicalities become much more complicated, and a new idea will be needed to finish the proof. Let $\mathcal{X}$ be a model of X over R such that the log scheme $\mathcal{X}^\dagger$ is *log regular* in the sense of Kato [1994]. One can associate with $\mathcal{X}^\dagger$ its *fan* $F(\mathcal{X})$, which is a finite monoidal space. There exists a morphism of monoidal spaces $\pi : (\mathcal{X}, \mathcal{M}_X^\sharp) \to F(\mathcal{X})$ which we call the *characteristic morphism*; here $\mathcal{M}_X$ is the sheaf of monoids defining the log structure on $\mathcal{X}^\dagger$, and $\mathcal{M}_X^\sharp = \mathcal{M}_X / \mathcal{O}_X^\times$. With each $x \in F(\mathcal{X})$, one then associates the locally closed subset $\pi^{-1}(x) = U(x)$ of $\mathcal{X}$. These sets give the so-called *logarithmic stratification* $(U(x))_{x \in F(\mathcal{X})}$ of $\mathcal{X}$.

Inspired by the computations on strict normal crossings models in [Nicaise and Sebag 2007; Nicaise 2013], we give explicit formulae for both the trace of the tame monodromy operator and the rational volume in terms of the logarithmic stratification. The trace of the monodromy operator can be computed using the logarithmic description of the sheaves of tame nearby cycles, first given by Nakayama [1998] in the log smooth case and later generalized by Vidal [2004]. We obtain

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi^d \mid H^i(X\times_K K^t,\mathbb{Q}_\ell)) = \sum_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)'\mid d}} s(x)'\chi(U(x)), \tag{2}$$

where $F(X)^{(1)}$ denotes the set of height 1 points in the fan $F(\mathcal{X})$, $s(x)\in\mathbb{N}$ is an integer which can be read off from the log structure and $s(x)'$ denotes the biggest prime-to-p divisor of $s(x)$. The rational volume can be computed using the formalism of log blow-ups developed by Kato [1994, §10] (see also Nizioł's work [2006]). This gives

$$\mathrm{s}(X) = \sum_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)=1}} \chi(U(x)). \tag{3}$$

Using the equalities (2) and (3), one finds

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi \mid H^i(X\times_K K^t,\mathbb{Q}_\ell)) - \mathrm{s}(X) = \sum_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)=p^r,\ r\geq 1}} \chi(U(x)). \tag{4}$$

The result then follows from the fact that each term $\chi(U(x))$ in the right-hand side of the equality (4) vanishes. In fact, we prove the more general result that whenever $x\in F(\mathcal{X})^{(1)}$ is a point for which p divides $s(x)$, then $\chi(U(x))=0$. This is perhaps the principal novelty of this paper. It is only at this point that the logarithmic smoothness assumption needs to be invoked; the previous steps only require the (significantly weaker) assumption that there exists a log regular model. The key observation involves the sheaves of logarithmic 1-forms on the relevant strata.

The paper is organized as follows. For the convenience of the reader, we recall a few important notions from logarithmic geometry in Section 2. In Section 3, we prove some technical results on sheaves of tame nearby cycles in the logarithmic setting; we use these to compute the tame monodromy zeta function and to prove the equality (2). In Section 4, we use resolution of toric singularities à la Kato–Nizioł to prove the formula (3). In Section 5, we obtain the crucial new ingredient needed to finish the proof.

2. Preliminaries on logarithmic geometry

For basic definitions and properties of monoids and logarithmic schemes, we refer to Kato's foundational text [1989] and to the detailed treatments given by Ogus [2016] and Gabber and Ramero [2013, Chapter 9]. In this section, we briefly recall a few notions which play an important role in this paper.

Notation. We denote a log scheme by $(X,\mathcal{M}_X)$, where X is the underlying scheme and $\mathcal{M}_X$ is the sheaf of monoids defining the log structure on X, endowed with a morphism of sheaves $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$. One can define log structures both on Zariski sites and étale sites; to keep things simple, we work with Zariski log structures throughout, since these are sufficient for our purposes.

Given a monoid P, we write $P^\sharp$ for the associated sharp monoid $P/P^\times$, P^{gp} for the group envelope of P and P^{sat} for the saturation of P in P^{gp}.

Divisorial log structures. Let X be a locally Noetherian scheme and let $D \hookrightarrow X$ be a divisor on X. Let $j : U \hookrightarrow X$ be the corresponding open immersion, where $U = X \setminus D$. Then

$$\mathcal{M}_X = \mathcal{O}_X \cap j_\star \mathcal{O}_U^\times \hookrightarrow \mathcal{O}_X$$

defines a log structure on X, the *divisorial log structure* induced by D.

A special case of this construction is the following. Let S be a trait, i.e., $S = \operatorname{Spec} R$ where R is a discrete valuation ring. Let π be a uniformizer. Then $R \setminus \{0\} \hookrightarrow R$ defines a log structure on R, *the standard log structure*, which is exactly the divisorial log structure induced by the divisor given by $\pi = 0$. We denote the corresponding log scheme by $S^\dagger$. If X is a flat R-scheme, then the special fibre X_s is a divisor; we denote by $X^\dagger$ the log scheme obtained by equipping X with the divisorial log structure induced by X_s. This yields a well-defined morphism $X^\dagger \to S^\dagger$ of log schemes. (Sometimes we simply write $R^\dagger$ instead of $S^\dagger$.)

Fibre products. Fibre products in the category $\mathsf{Log}^{\mathsf{fs}}$ of fs log schemes will be denoted by $\times^{\mathrm{fs}}$. If $S = \operatorname{Spec} R$ is a trait and $\pi \in R$ a uniformizer, we define for $d \in \mathbb{Z}_{>0}$ the scheme $S(d) = \operatorname{Spec} R[T]/(T^d - \pi)$. Then $u_d : \mathbb{N} \hookrightarrow \frac{1}{d}\mathbb{N}$ gives a chart for the morphism $S(d)^\dagger \to S^\dagger$. Given any log scheme $X^\dagger$ over $S^\dagger$, define

$$X(d)^\dagger = X^\dagger \times^{\mathsf{fs}}_{S^\dagger} S(d)^\dagger. \tag{5}$$

Recall that fibre products in $\mathsf{Log}^{\mathsf{fs}}$ do not commute with the forgetful functor from $\mathsf{Log}^{\mathsf{fs}}$ to the category of schemes. A simple example illustrating this phenomenon is the following: the underlying scheme of $S(d)^\dagger \times^{\mathsf{fs}}_{S^\dagger} S(d)^\dagger$ is $S(d) \times \operatorname{Spec} \mathbb{Z}[\mathbb{Z}/d\mathbb{Z}]$, which is (geometrically) a disjoint union of d copies of $S(d)$; on the other hand, the fibre product of schemes $S(d) \times_S S(d)$ is not normal.

Log regularity and log smoothness. Kato [1994, §2] introduced the notion of *log regularity*. We recall its definition for the convenience of the reader.

Definition 2.1. Let $(X, \mathcal{M}_X)$ be an fs log scheme. Given any point x of X, let $I(x, \mathcal{M}_X)$ be the ideal of $\mathcal{O}_{X,x}$ generated by $\mathcal{M}^+_{X,x} = \mathcal{M}_{X,x} \setminus \mathcal{O}^\times_{X,x}$. Let $C_{X,x}$ be the closed subscheme of $\operatorname{Spec} \mathcal{O}_{X,x}$ defined by $I(x, \mathcal{M}_X)$. Then $(X, \mathcal{M}_X)$ is *log regular* at x if $C_{X,x}$ is regular (as a scheme) and

$$\dim \mathcal{O}_{X,x} = \dim C_{X,x} + \operatorname{rank}_{\mathbb{Z}} (\mathcal{M}^\sharp_{X,x})^{\mathrm{gp}}.$$

A *log regular scheme* is a log scheme which is everywhere log regular. Whenever we say (or assume) that a log scheme is log regular, this implies that the log structure is fs.

Let us also recall the notion of a log smooth morphism of log schemes. Developing this notion was in some sense the main motivation for the theory of logarithmic geometry; it allows one to treat certain singular objects as if they were smooth.

Definition 2.2. Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of fs log schemes. Then f is *log smooth* (resp. *log étale*) if étale locally on X and Y, there exists a chart for f, given by maps $P_X \to \mathcal{O}_X$, $Q_Y \to \mathcal{O}_Y$ and $u : Q \to P$, such that the kernel and the torsion part of the cokernel (resp. the kernel and cokernel) of $u^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ are finite groups of order invertible on X, and the map $X \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$ is classically smooth (at the level of the underlying schemes).

Recall that a log smooth log scheme over a log regular scheme is again log regular, and that log regular schemes are normal and Cohen–Macaulay.

The logarithmic stratification. The *characteristic sheaf* of a log scheme $(X, \mathcal{M}_X)$ is the quotient $\mathcal{M}_X^\sharp = \mathcal{M}_X / \mathcal{O}_X^\times$. With a log regular scheme, one can associate a finite monoidal space, the *fan* $F(X)$, as follows: as a set,

$$F(X) = \{x \in X \mid I(x, \mathcal{M}_X) = \mathfrak{m}_x\},$$

where $\mathfrak{m}_x$ denotes the maximal ideal in $\mathcal{O}_{X,x}$. Endow the set $F(X)$ with the topology induced by the topology on X and with the inverse image of the sheaf $\mathcal{M}_X^\sharp$. This is a fan, as proved by Kato [1994, Proposition 10.1]. If X is quasicompact, then $F(X)$ is easily seen to be a *finite* monoidal space.

Given a log scheme $(X, \mathcal{M}_X)$, one often considers the *rank stratification of* X, constructed starting from the characteristic sheaf: write

$$X_i = \{x \in X : \operatorname{rank}_{\mathbb{Z}} (\mathcal{M}_{X,x}^\sharp)^{\mathrm{gp}} \geq i\}.$$

The sets $X_i \setminus X_{i-1}$ are locally closed and give the rank stratification of X. In this paper, we use a finer stratification obtained from the fan of a log regular scheme $(X, \mathcal{M}_X)$. Kato [1994, §10.2] defines the *characteristic morphism*: let $\pi : (X, \mathcal{M}_X^\sharp) \to F(X)$ map a point $x \in X$ to the point $\pi(x)$ of $F(X) \subseteq X$ which corresponds to the prime ideal $I(x, \mathcal{M}_X)$ of $\mathcal{O}_{X,x}$. This is an open morphism of monoidal spaces. Given $x \in F(X)$, let $U(x) := \pi^{-1}(x)$. This is a locally closed subset of X. This gives the *logarithmic stratification* $(U(x))_{x \in F(X)}$ of X.

When endowed with the reduced scheme structure, each stratum is irreducible and regular [Gabber and Ramero 2013, Corollary 9.5.52(iii)]. For each $x \in F(X)$, we denote by $\overline{U(x)}$ the Zariski closure of the stratum $U(x)$ inside X, again equipped with the reduced scheme structure. The *height* $h(x)$ of a point x in $F(X)$ is defined as the dimension of the monoid $\mathcal{M}_{F(X),x}$. The set of points of $F(X)$ of height equal to i is denoted by $F(X)^{(i)}$. We have the equality of locally closed, reduced subschemes

$$X_i \setminus X_{i-1} = \bigsqcup_{x \in F(X)^{(i)}} U(x).$$

3. Nearby cycles and the monodromy zeta function

The main goal of this section is to calculate the tame monodromy zeta function of a smooth, proper K-variety X, starting from a proper R-model $\mathcal{X}$ such that $\mathcal{X}^\dagger$ is log regular. To do so, we use the "logarithmic" description of the sheaves of tame nearby cycles, first given by Nakayama [1998] and later generalized by Vidal [2004]; along the way, we prove some technical results on nearby cycles.

Let us first introduce some easy terminology.

Definition 3.1. A morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is *vertical* if for every $x \in X$, the induced homomorphism $\mathcal{M}_{Y,f(x)} \to \mathcal{M}_{X,x}$ is vertical; this means that the image of $\mathcal{M}_{Y,f(x)}$ is not contained in any proper face of $\mathcal{M}_{X,x}$.

A log scheme $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ is vertical over $R^\dagger$ if and only if $X = \mathcal{X} \times_R K$ is precisely the locus of triviality of the log structure. In particular, if $\mathcal{X}$ is a flat R-scheme, then $\mathcal{X}^\dagger$ is vertical over $R^\dagger$. We then simply say that "the log structure is vertical".

Definition 3.2. Let $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ be a vertical log scheme over $R^\dagger$ which is log regular. Let $x \in F(\mathcal{X})^{(1)}$ be a height 1 point in the fan of $\mathcal{X}$. Let $\mathbb{N} \to P$ be a chart for the log structure around x, where P is an fs monoid. Then x corresponds to a height 1 prime ideal $\mathfrak{p}$ in P. Now $\mathbb{N} \to P$ induces a homomorphism

$$\mathbb{N} \to P/(P \setminus \mathfrak{p}) \cong \mathbb{N},$$

which is multiplication by a positive integer $s(x)$. We call this integer the *saturation index* of $x \in F(\mathcal{X})^{(1)}$. The saturation index of $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ over $R^\dagger$ is the least common multiple of the integers $s(x)$ for $x \in F(\mathcal{X})^{(1)}$.

Let us now recall the definition of the sheaves of nearby cycles. Let $S = \operatorname{Spec} R$, let η be its generic point and let s be its closed point. Let η^t be a tame closure of η and let S^t be the normalization of S inside η^t. Let Λ be $\mathbb{Z}/n\mathbb{Z}$, where n is an integer prime to p, or one of $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$. Let $\mathcal{X}$ be a scheme over S and write $X = \mathcal{X}_\eta$. Consider the Cartesian squares

$$\begin{array}{ccccc} \mathcal{X}_s & \xrightarrow{i^t} & \mathcal{X}^t & \xleftarrow{j^t} & X^t \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S^t & \longleftarrow & \eta^t \end{array}$$

Given $\mathcal{F}$ in $D^+(X, \Lambda)$, the derived category corresponding to bounded-below complexes of sheaves of Λ-modules on X, the corresponding complex of *tame nearby cycles* is

$$R\psi^t_{\mathcal{X}}(\mathcal{F}) := (i^t)^\star R(j^t)_\star(\mathcal{F}|_{X^t}) \in D^+(\mathcal{X}_s, \Lambda).$$

Now let X be any smooth, proper variety over K and assume that there exists a proper, flat model $\mathcal{X}$ of X over R such that $\mathcal{X}^\dagger$ is *log regular*. We analyze the behaviour of the complex $R\psi^t_{\mathcal{X}}(\Lambda)$ on the logarithmic strata defined previously. We start with the following local computation of the trace of the tame monodromy operator φ.

Lemma 3.3. *With the above notation and assumptions, fix a point $x \in F(\mathcal{X})$. Take $z \in U(x)$ and let $\bar{z}$ be any geometric point lying over z. Then*

$$\operatorname{Tr}(\varphi^d \mid R\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}}) = \begin{cases} s(x)' & \text{if } x \in F(\mathcal{X})^{(1)} \text{ and } s(x)' \mid d, \\ 0 & \text{else.} \end{cases}$$

Here $s(x)'$ denotes the biggest prime-to-p divisor of $s(x)$.

Proof. Nakayama [1998, Theorem 3.5] gave a “logarithmic” description of the complex of nearby cycles in the case of log good reduction. This result was generalized by Vidal [2004, §1.4]; recall that the complex of nearby cycles is automatically tame in the case of log good reduction [Nakayama 1998, Theorem 3.2]. Their result is the following. Let $\mathcal{C}$ be the locally constant sheaf of finitely generated abelian groups on $\mathcal{X}_s$ given by

$$\mathcal{C} = \operatorname{coker}\bigl(f^\star(\mathcal{M}^\sharp_s)^{\mathrm{gp}} \xrightarrow{\alpha} (\mathcal{M}^\sharp_{\mathcal{X}_s})^{\mathrm{gp}}\bigr) \text{ modulo torsion.}$$

For every integer $q \geq 1$, there is a natural, Galois equivariant isomorphism

$$R^0\psi^t_{\mathcal{X}}(\Lambda) \otimes \textstyle\bigwedge^q(\mathcal{C}|_{\mathcal{X}_{\bar{s}}} \otimes \Lambda(-1)) \longrightarrow R^q\psi^t_{\mathcal{X}}(\Lambda) \tag{6}$$

since the log structure is vertical. The stalk of $R^0\psi^t_{\mathcal{X}}(\Lambda)$ at a classical geometric point $\bar{z}$ lying over $\bar{s}$ is noncanonically isomorphic to $\Lambda[E_{\bar{z}}]$, where $E_{\bar{z}}$ is the cokernel of the induced morphism of logarithmic inertia groups $I^{\log}_{\bar{z}} \to I^{\log}_{\bar{s}}$; this morphism is nothing but $\operatorname{Hom}(\alpha_{\bar{z}}, \widehat{\mathbb{Z}}'(1))$, with α as above.

If $h(x) = 1$, then $\mathcal{C}_{\bar{z}} = 0$ and therefore

$$\operatorname{Tr}(\varphi^d \mid R\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}}) = \operatorname{Tr}(\varphi^d \mid R^0\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}}).$$

Since, moreover,

$$R^0\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}} \cong \Lambda[E_{\bar{z}}],$$

where

$$E_{\bar{z}} = \operatorname{coker}(\operatorname{Hom}(\alpha_{\bar{z}}, \widehat{\mathbb{Z}}'(1))$$

has order $s(x)'$, and the tame inertia group I^t acts through

$$I^t \cong I^{\log}_{\bar{s}} \hookrightarrow \Lambda[I^{\log}_{\bar{s}}] \longrightarrow \Lambda[E_{\bar{z}}],$$

the trace of φ^d on $R^0\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}}$ equals $s(x)'$ if $s(x)'$ divides d, and to 0 otherwise.

If $h(x) \geq 2$, then

$$n := \operatorname{rank}_{\mathbb{Z}} \mathcal{C}_{\bar{z}} = h(x) - 1 \geq 1.$$

Hence, (6) yields

$$\begin{aligned}\operatorname{Tr}(\varphi^d \mid R\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}}) &= \sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi^d \mid R^i\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}})\\ &= \sum_{i\geq 0}(-1)^i \binom{n}{i} \operatorname{Tr}(\varphi^d \mid R^0\psi^t_{\mathcal{X}}(\Lambda)_{\bar{z}})\\ &= 0.\end{aligned}$$

□

We continue with a technical lemma on monoids.

Lemma 3.4. *Let P be an fs monoid. Let $\alpha : \mathbb{N} \to P$ be a vertical homomorphism and let $d \in \mathbb{Z}_{>0}$. Let $Q = P \oplus_{\mathbb{N}} \frac{1}{d}\mathbb{N}$. The submonoid Q^{sat} of Q^{gp} is generated by $Q^\times$, the d-torsion $Q^{\mathrm{gp}}[d]$ and $I \subseteq Q^{\mathrm{sat}}$, the radical of the ideal generated by $P \setminus P^\times$.*

Proof. Let $e = \alpha(1)$. Since the homomorphism α is vertical, e is not contained in any proper face of P. This implies that every nonmaximal face of Q is of the form $F \oplus \{0\}$, where F is a nonmaximal face of P. In particular, we have an isomorphism

$$P^\times \to Q^\times : p \mapsto (p, 0).$$

Take $\left(p, \frac{m}{d}\right) \in Q^{\mathrm{sat}}$, where $p \in P^{\mathrm{gp}}$ and $m \in \mathbb{Z}$. There exists an $N \in \mathbb{Z}_{>0}$ such that $N\left(p, \frac{m}{d}\right) \in Q$. Multiplying by d if necessary, we can assume that $d \mid N$; hence

$$N\left(p, \frac{m}{d}\right) = \left(Np, \frac{Nm}{d}\right) = \left(Np + \frac{Nm}{d}e, 0\right),$$

and therefore

$$Np + \frac{Nm}{d}e = \frac{N}{d}(dp + me) \in P.$$

Since P is saturated, $dp+me \in P$. Conversely, if $dp+me \in P$, clearly $\left(p, \frac{m}{d}\right) \in Q^{\mathrm{sat}}$.

If $dp + me \in P \setminus P^\times$, then $d\left(p, \frac{m}{d}\right) = (dp + me, 0)$ lies in the ideal of Q^{sat} generated by the image of the maximal ideal of P; hence $\left(p, \frac{m}{d}\right)$ lies in I, so we are done.

Let us therefore assume that $dp + me \in P^\times$. Then $\left(p, \frac{m}{d}\right) \in (Q^{\mathrm{sat}})^\times$. Let us consider the image $\bar{x}$ of $x = \left(p, \frac{m}{d}\right)$ in $Q^{\mathrm{sat}}/Q^\times$. It is clear that $\bar{x} \in (Q^{\mathrm{sat}}/Q^\times)^\times$. Hence there exists $y \in Q^{\mathrm{sat}}$ such that $\bar{x} + \bar{y} = \bar{0}$ in $Q^{\mathrm{sat}}/Q^\times$, i.e., $x + y \in Q^\times$. Since $dx \in Q$ and $dy \in Q$, we obtain $dx \in Q^\times$. Using the isomorphism $Q^\times \cong P^\times$

mentioned above, we see that there exists $p' \in P^\times$ such that $d\big(p, \frac{m}{d}\big) = d(p', 0)$. Now $\big(p - p', \frac{m}{d}\big)$ is clearly d-torsion; hence,

$$\Big(p, \frac{m}{d}\Big) = (p', 0) + \Big(p - p', \frac{m}{d}\Big)$$

is the sum of an element of $Q^\times$ and an element of $Q^{\mathrm{gp}}[d]$, so we are done. □

This leads to the following result.

Proposition 3.5. *Assume that $\mathcal{X}^\dagger$ (as above) is log regular. Fix $x \in F(\mathcal{X})$ and let $d \in \mathbb{Z}_{>0}$ be prime to p. The reduced inverse image of $U(x)$ in $\mathcal{X}(d)^\dagger$ is a finite étale cover of $U(x)$. The degree of this cover divides d and $\gcd_{y \in x^{(1)}} s(y)$, where $x^{(1)}$ is the set of generizations of x in $F(\mathcal{X})^{(1)}$.*

Proof. On an affine neighbourhood of $\operatorname{Spec} A$ around x, we have an fs chart for the morphism $\mathcal{X}^\dagger \to R^\dagger$ given by a commutative diagram of the form

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{1 \mapsto \pi} & R \\ \downarrow & & \downarrow \\ P & \longrightarrow & A \end{array}$$

Similarly, a chart for the log structure on $\mathcal{X}(d)^\dagger$ is given by the diagram

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & R(d) \\ \downarrow & & \downarrow \\ Q^{\mathrm{sat}} & \longrightarrow & B \end{array}$$

where

$$Q = P \oplus_{\mathbb{N}} \tfrac{1}{d}\mathbb{N} \quad \text{and} \quad B = A \otimes_{R[P]} R[Q^{\mathrm{sat}}].$$

Now $U(x) \cap \operatorname{Spec} A$ is nothing but

$$\operatorname{Spec} A \times_{\operatorname{Spec} R[P]} \operatorname{Spec}(R[P]/(\mathfrak{p}))_{\mathfrak{p}},$$

where $\mathfrak{p}$ is the prime ideal of P which corresponds to x. Here $(\mathfrak{p})$ denotes the ideal of $R[P]$ generated by $\mathfrak{p}$, and ${}_{\mathfrak{p}}$ denotes localization with respect to the multiplicative subset of $R[P]/(\mathfrak{p})$ generated by $P \setminus \mathfrak{p}$. We may and will assume that $\mathfrak{p}$ is the maximal ideal of P. Hence we can identify $U(x) \cap \operatorname{Spec} A$ with the spectrum of $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^\times]$. The (reduced) inverse image of $U(x)$ in $\mathcal{X}(d)^\dagger$ can be described in a similar way. The fact that this inverse image is étale over $U(x)$ is now an immediate consequence of Lemma 3.4 (recall that we have chosen d prime to p).

We now analyze the degree of this étale cover. We can safely assume that P^{gp} is torsion free. Choose an isomorphism $P^{\mathrm{gp}} \cong \mathbb{Z}^r$, for some $r > 0$. Let v be the image of 1 under the composition $\mathbb{N} \to P \to P^{\mathrm{gp}} \to \mathbb{Z}^r$. Hence we get an identification

$$Q^{\mathrm{gp}} \cong (\mathbb{Z}^r \oplus \mathbb{Z})/\langle (v, -d) \rangle.$$

Write $v = \lambda v_0$, where $\lambda \in \mathbb{Z}$ and v_0 is a *primitive vector*. The order of the torsion in Q^{gp} thus divides $\gcd(d, \lambda)$. The statement about the degree of the étale cover now follows from Lemma 3.4, together with the observation that λ divides the saturation index $s(y)$ for any $y \in x^{(1)}$. Indeed, if $\mathfrak{p}$ is the height 1 prime ideal of P corresponding to y, then $s(y)$ can be identified with the image of 1 under $P \to P/(P \setminus \mathfrak{p}) \cong \mathbb{N}$. Consider the homomorphisms

$$\mathbb{N} \to P \hookrightarrow P^{\mathrm{gp}} \to P^{\mathrm{gp}}/(P \setminus \mathfrak{p})^{\mathrm{gp}} \cong \mathbb{Z}.$$

Since the image of 1 in P^{gp} is divisible by λ, so is the integer $s(y)$, since this is simply the image of 1 in the quotient $P^{\mathrm{gp}}/(P \setminus \mathfrak{p})^{\mathrm{gp}} \cong \mathbb{Z}$. □

We can now prove the following result about sheaves of tame nearby cycles, generalizing the results of [Nicaise 2013, §2.5].

Theorem 3.6. *As above, assume that $\mathcal{X}^\dagger$ is log regular and fix $x \in F(\mathcal{X})$. Write $\Lambda = \mathbb{Q}_\ell$ for some prime $\ell \neq p$. The sheaves $R^n\psi^t_{\mathcal{X}}(\Lambda)$ are lisse on $U(x)$ and tamely ramified along the boundary components of any normal compactification. They become constant on a finite étale cover of degree dividing* $\gcd_{y \in x^{(1)}} s(y)'$.

Proof. Let $\tilde{s}$ be a log geometric point lying above s. Write $\mathcal{X}_{\tilde{s}} = \mathcal{X}_s \times_s^{\mathrm{fs}} \tilde{s}$. We know by [Illusie 2002, Corollary 8.4] that the tame nearby cycles complex is given by the formula

$$R\psi^t_{\mathcal{X}}(\Lambda) = R\tilde{\varepsilon}_\star(\Lambda|_{\mathcal{X}_{\tilde{s}}}),$$

where $\tilde{\varepsilon}$ is the composition

$$\mathcal{X}_{\tilde{s}}^{\mathrm{ket}} \xrightarrow{\varepsilon} \mathcal{X}_{\tilde{s}}^{\mathrm{et}} \xrightarrow{\alpha} \mathcal{X}_s^{\mathrm{et}}.$$

Here ε is the canonical map from the Kummer étale site to the classical étale site; of course α need not be an isomorphism at the level of the underlying schemes. Hence,

$$R^0\psi^t_{\mathcal{X}}(\Lambda) = \alpha_\star(\Lambda|_{X_{\tilde{s}}}).$$

By [Vidal 2004, Proposition 1.3.4.1], α is the composition of a surjective closed immersion β_1 and a finite morphism β_2. More precisely, α factors as

$$\mathcal{X}_{\tilde{s}} \xrightarrow{\beta_1} \mathcal{X}_{\tilde{s}_n} \xrightarrow{\beta_2} \mathcal{X}_s.$$

Here $\mathcal{X}_{\tilde{s}_n}$ denotes the log scheme $\mathcal{X}_s \times_s^{\mathrm{fs}} \tilde{s}_n$, which is nothing but the special fibre of $\mathcal{X}(n)^\dagger$; the integer n is the biggest prime-to-p divisor of the saturation index of $\mathcal{X}$ over R. The degree $d_x = \sum_{\alpha(y)=x}[\kappa(y) : \kappa(x)]$ of β_2 in a point x equals the cardinality of $E_{\bar{x}} = \operatorname{coker}\operatorname{Hom}(\varphi_{\bar{x}}, \widehat{\mathbb{Z}}'(1))$, where φ is the induced map $f^\star(\mathcal{M}_s^\sharp)^{\mathrm{gp}} \to (\mathcal{M}_{\mathcal{X}_s}^\sharp)^{\mathrm{gp}}$.

Let us now fix $x \in F(\mathcal{X})$. We know that

$$R^0\psi_{\mathcal{X}}^t(\Lambda) = \alpha_\star(\Lambda|_{X_{\bar{s}}}) = (\beta_2)_\star(\Lambda|_{X_{\bar{s}_n}}),$$

so the restriction of $R^0\psi_{\mathcal{X}}^t(\Lambda)$ to $U(x)$ becomes constant on the inverse image of $U(x)$ in $\mathcal{X}(n)$, which is étale by Proposition 3.5. The statement about the degree is a consequence of the fact that n is the biggest prime-to-p divisor of $\operatorname{lcm}_{y \in x^{(1)}} s(y)$.

Hence, $R^n\psi_{\mathcal{X}}(\Lambda)$ becomes constant on the same cover, for any $n \geq 0$, since the sheaf $\mathcal{C}$ in the proof of Lemma 3.3 is constant on $U(x)$. It follows that the sheaves $R^n\psi_{\mathcal{X}}^t(\Lambda)$ are lisse, and tamely ramified along the boundary of any normal compactification. $\square$

We are now ready to calculate the trace of the tame monodromy operator.

Theorem 3.7. *Let $\mathcal{X}$ be a proper, flat R-model such that $\mathcal{X}^\dagger$ is log regular. Then*

$$\operatorname{Tr}(\varphi^d \mid H^\star(X \times_K K^t, \mathbb{Q}_\ell)) = \sum_{\substack{x \in F(\mathcal{X})^{(1)} \\ s(x)' \mid d}} s(x)'\chi(U(x)). \tag{7}$$

for every positive integer d.

Proof. We prove a more general result: if Z is a subscheme of $\mathcal{X}_s$, then

$$\sum_{m \geq 0}(-1)^m \operatorname{Tr}(\varphi^d \mid \mathbb{H}_c^m(Z, R\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)|_Z)) = \sum_{\substack{x \in F(\mathcal{X})^{(1)} \\ s(x)' \mid d}} s(x)'\chi(U(x) \cap Z). \tag{8}$$

In particular, (7) follows from (8) by taking $Z = \mathcal{X}_s$ and applying the spectral sequence for nearby cycles, since $\mathcal{X}$ is assumed to be proper over S. To prove the equality (8), note that both sides are additive with respect to partitioning Z into subvarieties; this allows us to assume that Z is contained in $U(x)$, for some $x \in F(\mathcal{X})$. By the spectral sequence for hypercohomology, the left-hand side of (8) equals

$$\sum_{p,q \geq 0}(-1)^{p+q} \operatorname{Tr}(\varphi^d \mid H_c^p(Z, R^q\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)|_Z)).$$

We can assume that Z is normal and choose a normal compactification Z^{c}. We know (Theorem 3.6) that the sheaves $R^n\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)|_Z$ are lisse on Z, and tamely ramified along each of the irreducible components of $Z^c \setminus Z$. Let $z \in Z$ and let $\bar{z}$ be a geometric point lying over z. Using Nakayama's description of the action of the monodromy operator on the stalks $R^n\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)_{\bar{z}}$ — see the proof of Lemma 3.3 —

we see that this action has finite order. Hence, we can apply [Nicaise and Sebag 2007, Lemma 5.1], which gives

$$\sum_{p\geq 0}(-1)^p \operatorname{Tr}(\varphi^d \mid H_c^p(Z, R^q\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)|_Z)) = \chi(Z)\cdot \operatorname{Tr}(\varphi^d \mid R^q\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)_{\bar{z}}).$$

Using Lemma 3.3, we see that the only contributions to the left-hand side of the equality (8) will come from the height 1 points in the fan $F(\mathcal{X})$: we get

$$\begin{aligned}\sum_{m\geq 0}(-1)^m \operatorname{Tr}(\varphi^d \mid \mathbb{H}_c^m(Z, R\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)|_Z)) &= \chi(Z)\cdot \sum_{q\geq 0}(-1)^q \operatorname{Tr}(\varphi^d \mid R^q\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)_{\bar{z}})\\ &= s(x)'\chi(Z)\end{aligned}$$

if the point $x\in F(\mathcal{X})^{(1)}$ satisfies $s(x)' \mid d$; in all other cases, we get no contribution. This concludes the proof of the theorem. □

As a corollary, we get a formula for the tame monodromy zeta function of X:

$$\begin{aligned}\zeta_X(t) &= \det(t\cdot \mathrm{Id}-\varphi \mid H^\star(X\times_K K^t, \mathbb{Q}_\ell))\\ &= \prod_{m\geq 0}\det(t\cdot \mathrm{Id}-\varphi \mid H^m(X\times_K K^t, \mathbb{Q}_\ell))^{(-1)^{m+1}}.\end{aligned} \tag{9}$$

The result is an A'Campo type formula [A'Campo 1975, §1], generalizing the results of [Nicaise 2013, §2.6].

Corollary 3.8. *We have*

$$\zeta_X(t) = \prod_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)'|d}} (t^{s(x)'}-1)^{-\chi(U(x))}. \tag{10}$$

Proof. As in the proof of Theorem 3.7, we can actually prove a more general result: if Z is any subscheme of $\mathcal{X}_s$, then the equality

$$\det(t\cdot \mathrm{Id}-\varphi \mid \mathbb{H}_c^\star(Z, R\psi_{\mathcal{X}}^t(\mathbb{Q}_\ell)|_Z)) = \prod_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)'|d}} (t^{s(x)'}-1)^{-\chi(U(x)\cap Z)} \tag{11}$$

holds. This formula can easily be deduced from the equality (8) using the fact that, for every field F of characteristic zero, every finite dimensional F-vector space V and every endomorphism g of V, we have the classical identity

$$\det(\mathrm{Id}-t\cdot g \mid V)^{-1} = \exp\biggl(\sum_{d>0}\operatorname{Tr}(g^d \mid V)\frac{t^d}{d}\biggr)$$

in $F[\![t]\!]$. (For similar arguments, we refer to [Deligne 1974, Section 1.5; A'Campo 1975, §1].) □

4. Log blow-ups and the rational volume

As in the previous section, we assume that we are given a smooth, proper K-variety X for which there exists a proper, flat R-model $\mathcal{X}$ such that $\mathcal{X}^\dagger$ is log regular. We now compute the rational volume of X in terms of the logarithmic stratification.

We start with a simple lemma.

Lemma 4.1. *Under the above assumptions, the generic fibre $X = \mathcal{X} \times_R K$ is smooth. Moreover, the following statements are equivalent:*

(A) *the scheme $\mathcal{X}$ is regular and $\mathcal{X}_s$ is a divisor with strict normal crossings;*

(B) *for every point $x \in F(\mathcal{X})$, we have $\mathcal{M}_{F(\mathcal{X}),x} \cong \mathcal{M}^\sharp_{\mathcal{X},x} \cong \mathbb{N}^{h(x)}$.*

As before, $h(x)$ denotes the height of the point x in the fan $F(\mathcal{X})$.

Proof. The fact that the generic fibre is smooth follows from the verticality of the log structure. The underlying scheme $\mathcal{X}$ is regular if and only if for every $x \in F(\mathcal{X})$, the monoid $\mathcal{M}_{F(\mathcal{X}),x}$ is free and finitely generated by [Gabber and Ramero 2013, Corollary 9.5.35].

It remains to check the statement about the special fibre. Assume that (B) holds and fix any $x \in F(\mathcal{X})$. Locally around x, we have a chart given by

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathcal{O}^\times_{\mathcal{X},x} \oplus \mathbb{N}^{h(x)} \\ \downarrow & & \downarrow{\scriptstyle e_i \mapsto x_i} \\ R & \longrightarrow & \mathcal{O}_{\mathcal{X},x} \end{array}$$

where $(e_i)_{1 \le i \le h(x)}$ is the standard basis for $\mathbb{N}^{h(x)}$ and

$$\pi = u \prod_{i=1}^{h(x)} x_i^{n_i} \quad \text{in } \mathcal{O}_{\mathcal{X},x} \tag{12}$$

for some $u \in \mathcal{O}^\times_{\mathcal{X},x}$ and positive integers $(n_i)_{1 \le i \le h(x)}$. Since

$$I(x, \mathcal{M}_\mathcal{X}) = (x_1, \dots, x_{h(x)}),$$

and $C_{X,x} = \mathcal{O}_{\mathcal{X},x}/I(x, \mathcal{M}_\mathcal{X})$ is a regular local ring by [Kato 1994, Definition 2.1], we see that $\mathcal{X}$ has strict normal crossings at x by [Liu 2002, Proposition 4.2.15]. $\square$

In the situation of Lemma 4.1, we have

$$\mathcal{X}_s = \sum_{x \in F(\mathcal{X})^{(1)}} s(x)\, \overline{U(x)}. \tag{13}$$

Let Y be a smooth, projective variety over K. If there exists an *sncd model* $\mathcal{Y}$ of Y as in Lemma 4.1, then the rational volume can be read off from the formula (13) following [Nicaise 2013, Proposition 4.2.1]:

$$\mathrm{s}(Y)=\sum_{\substack{y\in F(\mathcal{Y})^{(1)}\\ s(y)=1}}\chi(U(y)). \tag{14}$$

Indeed, $\mathcal{Y}$ is a regular and proper model of Y; its smooth locus (over R) is a weak Néron model for Y. The smooth locus of $\mathcal{Y}_s$ is precisely the open subscheme

$$\bigsqcup_{\substack{y\in F(\mathcal{Y})^{(1)}\\ s(y)=1}} U(y),$$

whence equality (14) follows.

We can now state and prove the main result of this section.

Theorem 4.2. *Let $\mathcal{X}$ be a proper, flat R-scheme such that $\mathcal{X}^\dagger$ is log regular. Let $X=\mathcal{X}\times_R K$ be the generic fibre. Then*

$$\mathrm{s}(X)=\sum_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)=1}}\chi(U(x)). \tag{15}$$

The idea is simple: we consider a subdivision $\varphi : F'\to F(\mathcal{X})$ of the fan in the sense of [Kato 1994, Definition 9.6] such that $\varphi^\star\mathcal{X}$ satisfies the equivalent conditions of Lemma 4.1, and in particular such that the underlying scheme of $\varphi^\star\mathcal{X}$ is (classically) regular. This is possible by [Kato 1994, §10.4]. We can then compute the rational volume starting from the modified model $\varphi^\star\mathcal{X}$ using the formula (14). The key technical ingredient needed for this computation is the fact that the log blow-up $\mathrm{Bl}_\varphi : \varphi^\star\mathcal{X}\to\mathcal{X}$ is a piecewise trivial fibration in tori; this is the content of the following lemma.

Lemma 4.3. *Let $\mathcal{X}$ be a flat R-scheme such that $\mathcal{X}^\dagger$ is log regular. Consider a subdivision $\varphi : F'\to F(\mathcal{X})$ of its fan. Fix $x'\in F'$ and $x\in\mathcal{X}$ such that $\varphi(x')=\pi(x)$, i.e., $x\in U(\varphi(x'))\subseteq\mathcal{X}$. Then we have*

$$U(x')\cap\mathrm{Bl}_\varphi^{-1}(x)\cong\mathbb{G}_{m,\kappa(x)}^{h(\varphi(x'))-h(x')}$$

where $\mathrm{Bl}_\varphi^{-1}(x)$ is the fibre of the log blow-up $\varphi^\star\mathcal{X}\to\mathcal{X}$ above x.

Proof. Fix $x'\in F'$ and $x\in\mathcal{X}$ such that $x\in U(\varphi(x'))$. The statement is local on $\mathcal{X}$. Hence, by shrinking $\mathcal{X}$ if necessary, we may assume that $F(\mathcal{X})=\operatorname{Spec}P$ and $F'=F(\varphi^\star\mathcal{X})=\operatorname{Spec}P'$, where P and P' are sharp fs monoids.

We can also assume that the natural morphism $P\to\mathcal{M}^\sharp_{\mathcal{X}}$ lifts to a homomorphism of sheaves $P\to\mathcal{M}_X$ such that the composition $P\to\mathcal{M}_X\to\mathcal{O}_X$ gives a chart for the log structure on $\mathcal{X}$. Define Q by

$$Q=P^{\mathrm{gp}}\times_{(P')^{\mathrm{gp}}}P';$$

this is the submonoid of P^{gp} consisting of the elements whose images in $(P')^{\mathrm{gp}}$ lie inside P'. The obvious map $P^{\mathrm{gp}} \to (P')^{\mathrm{gp}}$ is surjective; Q does not need to be sharp, but by [Gabber and Ramero 2013, Lemma 9.6.12] one has $Q^\sharp \cong (P')^\sharp$. The log blow-up $\varphi^\star\mathcal{X}$ is given by $\varphi^\star\mathcal{X} = \mathcal{X} \times_{\operatorname{Spec} R[P]} \operatorname{Spec} R[Q]$; see the construction in [Gabber and Ramero 2013, Proposition 9.6.14]. The log structure on $\varphi^\star\mathcal{X}$ is given by the chart $Q \to \mathcal{O}_{\varphi^\star\mathcal{X}}$.

Let $\psi : P \to Q$ be the natural injection, let $\mathfrak{q}$ be the prime ideal of Q corresponding to x' and let $\mathfrak{p} = \psi^{-1}(\mathfrak{q})$ be the prime ideal of P corresponding to $\varphi(x')$. The stratum $U(\varphi(x'))$ can then be described as

$$\mathcal{X} \times_{\operatorname{Spec} R[P]} \operatorname{Spec}(R[P]/(\mathfrak{p}))_{\mathfrak{p}},$$

and its Zariski closure $\overline{U(\varphi(x'))}$ is simply

$$\mathcal{X} \times_{\operatorname{Spec} R[P]} \operatorname{Spec} R[P]/(\mathfrak{p});$$

here $(\mathfrak{p})$ denotes the ideal of $R[P]$ generated by the elements of $\mathfrak{p}$, and the subscript $_{\mathfrak{p}}$ denotes localization with respect to the multiplicative subset of $R[P]/(\mathfrak{p})$ generated by the image of $P \setminus \mathfrak{p}$. Similarly, the stratum $U(x')$ is now equal to

$$\varphi^\star\mathcal{X} \times_{\operatorname{Spec} R[Q]} \operatorname{Spec}(R[Q]/(\mathfrak{q}))_{\mathfrak{q}}.$$

We have the pullback diagram

$$\begin{tikzcd}
U(x') \arrow[r] \arrow[d] & U(\varphi(x')) \arrow[d] \\
\operatorname{Spec}(R[Q]/(\mathfrak{q}))_{\mathfrak{q}} \arrow[r] & \operatorname{Spec}(R[P]/(\mathfrak{p}))_{\mathfrak{p}}
\end{tikzcd}$$

and hence, taking fibres over $x \in U(\varphi(x'))$, the pullback diagram

$$\begin{tikzcd}
U(x') \cap \mathrm{Bl}_\varphi^{-1}(x) \arrow[r] \arrow[d] & \operatorname{Spec} \kappa(x) \arrow[d] \\
\operatorname{Spec}(\kappa(x)[Q]/(\mathfrak{q}))_{\mathfrak{q}} \arrow[r, "\star"] & \operatorname{Spec}(\kappa(x)[P]/(\mathfrak{p}))_{\mathfrak{p}}
\end{tikzcd}$$

Therefore it now suffices to prove that the fibres of

$$\operatorname{Spec}(\kappa(x)[Q]/(\mathfrak{q}))_{\mathfrak{q}} \to \operatorname{Spec}(\kappa(x)[P]/(\mathfrak{p}))_{\mathfrak{p}}$$

are (split) tori of the required rank. Notice that

$$\kappa(x)[Q]/(\mathfrak{q}) \cong \kappa(x)[Q \setminus \mathfrak{q}]$$

and hence that

$$(\kappa(x)[Q]/(\mathfrak{q}))_{\mathfrak{q}} \cong \kappa(x)[(Q \setminus \mathfrak{q})^{\mathrm{gp}}].$$

Similarly, we obtain that

$$(\kappa(x)[P]/(\mathfrak{p}))_{\mathfrak{p}} \cong \kappa(x)[(P \setminus \mathfrak{p})^{\mathrm{gp}}].$$

We have a commutative diagram

$$\begin{tikzcd}
(\kappa(x)[P]/(\mathfrak{p}))_{\mathfrak{p}} \arrow[r] \arrow[d] & (\kappa(x)[Q]/(\mathfrak{q}))_{\mathfrak{q}} \arrow[d] \\
\kappa(x)[(P \setminus \mathfrak{p})^{\mathrm{gp}}] \arrow[r] & \kappa(x)[(Q \setminus \mathfrak{q})^{\mathrm{gp}}]
\end{tikzcd}$$

in which the vertical maps are the isomorphisms mentioned above, the top horizontal arrow is the one which induces the map $\star$ in the previous diagram and the bottom horizontal arrow is induced by the injective homomorphism $P \setminus \mathfrak{p} \to Q \setminus \mathfrak{q}$ obtained by restricting $\psi : P \to Q$.

Now $P \setminus \mathfrak{p}$ and $Q \setminus \mathfrak{q}$, being submonoids of fs monoids, are themselves fs; the quotient $(Q \setminus \mathfrak{q})^{\mathrm{gp}}/(P \setminus \mathfrak{p})^{\mathrm{gp}}$ is a subgroup of $P^{\mathrm{gp}}/(P \setminus \mathfrak{p})^{\mathrm{gp}}$, since Q (and thus $Q \setminus \mathfrak{q}$) is a submonoid of P. The latter group is a free abelian group of finite type, and hence so is its subgroup $(Q \setminus \mathfrak{q})^{\mathrm{gp}}/(P \setminus \mathfrak{p})^{\mathrm{gp}}$. This proves that the fibres of $\star$ are split tori; it remains to compute their rank. We have the equalities

$$h(\varphi(x')) - h(x') = \dim Q_{\mathfrak{q}} - \dim P_{\mathfrak{p}} = \mathrm{rank}_{\mathbb{Z}}(Q_{\mathfrak{q}}^{\sharp})^{\mathrm{gp}} - \mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}}^{\sharp})^{\mathrm{gp}}$$

(the last one uses the fact that P and Q are fine). It is now easy to see that

$$\mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}}^{\sharp})^{\mathrm{gp}} = \mathrm{rank}_{\mathbb{Z}}(P \setminus \mathfrak{p})^{\mathrm{gp}},$$

and

$$\mathrm{rank}_{\mathbb{Z}}(Q_{\mathfrak{q}}^{\sharp})^{\mathrm{gp}} = \mathrm{rank}_{\mathbb{Z}}(Q \setminus \mathfrak{q})^{\mathrm{gp}},$$

so we are done. □

We are now in the position to prove Theorem 4.2.

Proof of Theorem 4.2. Choose a subdivision $\varphi : F' \to F(\mathcal{X})$ such that

$$\mathcal{M}_{F',x'} \cong \mathbb{N}^{h(x')}$$

for every $x' \in F'$. Let $\mathrm{Bl}_{\varphi} : \varphi^{\star}\mathcal{X} \to \mathcal{X}$ be the log blow-up associated with φ — this is a "resolution of toric singularities à la Kato" [Kato 1994, §10.4]. The log scheme $\varphi^{\star}\mathcal{X}$ satisfies the equivalent conditions of Lemma 4.1; hence, (14) applies to $\varphi^{\star}\mathcal{X}$. Since the Euler characteristic is additive with respect to partitions into locally closed subsets and the Euler characteristic of a torus is 0, the result follows from Lemma 4.3. □

5. Proof of the main theorem

We now finish the proof of Theorem 1.3. From (7) and (15), one immediately obtains the following expression for the so-called "error term" [Nicaise 2011, Definition 6.7], generalizing [Nicaise 2011, Theorem 7.3] in the case of curves:

$$\varepsilon(X) := \sum_{i\geq 0}(-1)^i \operatorname{Tr}(\varphi \mid H^i(X\times_K K^t, \mathbb{Q}_\ell)) - \mathrm{s}(X) = \sum_{\substack{x\in F(\mathcal{X})^{(1)}\\ s(x)=p^r,\ r\geq 1}} \chi(U(x)).$$

This formula is valid for any log regular model $\mathcal{X}$, and $\varepsilon(X)$ does not vanish in general. However, it vanishes if X admits a log smooth model. This is an immediate consequence of the following result, which is interesting in its own right.

Proposition 5.1. *Let $\mathcal{X}$ be a proper R-model of X such that $\mathcal{X}^\dagger$ is log smooth over $R^\dagger$. Let $x \in F(\mathcal{X})^{(1)}$ be such that p divides $s(x)$. Then $\chi(U(x)) = 0$.*

In the case of curves, we know that cohomological tameness implies logarithmic good reduction, by work of Stix [2005, Theorem 1.2]. Moreover, Saito's criterion [1987] for cohomological tameness of curves gives a precise description of the irreducible components of the special fibre of a log smooth model for which the multiplicity is divisible by the residual characteristic p. Each such component is a copy of $\mathbb{P}^1$, which intersects exactly two other components, both of which have multiplicity prime to p. The above result should be seen as a partial generalization of this description.

Proof. Choose $x \in F(\mathcal{X})$ and $y \in \mathcal{X}$ such that $y \in \overline{U(x)}$. Choose a chart à la Kato (Definition 2.2) for the log structure around y, i.e., an étale neighbourhood $V = \operatorname{Spec} A$ of y and a homomorphism of monoids $\mathbb{N} \to P$ such that

$$\begin{tikzcd} P \arrow[r, "\varphi"] & A \\ \mathbb{N} \arrow[u, "u"] \arrow[r, "1\mapsto \pi"] & R \arrow[u] \end{tikzcd}$$

commutes, where P is an fs monoid and $u^{\mathrm{gp}} : \mathbb{N}^{\mathrm{gp}} \to P^{\mathrm{gp}}$ has the property that its kernel and the torsion part of its cokernel C are finite groups, the order of which is invertible on R. We can safely assume that P is toric (i.e., P^{gp} is torsion free).

The point $x \in F(\mathcal{X})$ corresponds to a prime ideal $\mathfrak{p} \subseteq P$ and $\overline{U(x)} \cap V$, seen as a reduced, closed subscheme of V, can be described as the fibre product

$$V \times_{\operatorname{Spec} R[P]} \operatorname{Spec} R[P\setminus \mathfrak{p}] \cong \operatorname{Spec} A/(\mathfrak{p}),$$

where $(\mathfrak{p})$ is the ideal of A generated by $\{\varphi(p) \mid p \in \mathfrak{p}\}$. Now $\overline{U(x)}$ becomes a log scheme for the log structure defined locally by $P\setminus\mathfrak{p} \to A/(\mathfrak{p})$, and this log scheme is log regular by [Kato 1994, Proposition 7.2]. Since k is perfect, it is even log smooth over k (equipped with the trivial log structure). The log structure on $\overline{U(x)}$ is the one induced by the reduced Weil divisor Δ supported on the union of the strata $\overline{U(x')}$, where x' is a (strict) specialization of x in the monoidal space $F(\mathcal{X})$.

From now on, assume that $x \in F(\mathcal{X})^{(1)}$ and that p divides $m = s(x)$. In the local ring $\mathcal{O}_{\mathcal{X},x}$, we have $\pi = uf^m$, where u is a unit and f is a local equation of the irreducible component $\overline{U(x)}$ of $\mathcal{X}_s$. We study the meromorphic one-form $\operatorname{dlog} u$ on $\overline{U(x)}$. If $\pi = vg^m$ is another such factorization in $\mathcal{O}_{\mathcal{X},x}$, we have $\operatorname{dlog} u = \operatorname{dlog} v$ on $\overline{U(x)}$ since the residue field k has characteristic p and p divides m. Hence, $\operatorname{dlog} u$ is a well-defined meromorphic 1-form on the irreducible component $\overline{U(x)}$. Locally around y, it can be described in terms of the log structure as follows:

$$\begin{tikzcd}
 & C & \\
(P \setminus \mathfrak{p})^{\mathrm{gp}} \arrow[r] & P^{\mathrm{gp}} \arrow[u, two heads] \arrow[r, two heads, "r"] & \mathbb{N}^{\mathrm{gp}} \\
 & \mathbb{N}^{\mathrm{gp}} \arrow[u, "u^{\mathrm{gp}}"] &
\end{tikzcd}$$

Let $a = u(1) \in P$, so $\varphi(a) = \pi$ and $r(a) = m \in \mathbb{N}$. Choose $b \in P$ such that $r(b) = 1$, i.e., $\varphi(b)$ is a uniformizer in the discrete valuation ring $\mathcal{O}_{\mathcal{X},x}$. We have $c := mb - a \in (P \setminus \mathfrak{p})^{\mathrm{gp}}$. Taking $u = \varphi(c) \in \mathcal{O}_{\mathcal{X},x}^{\times}$ and $f = \varphi(b)$, we have $\pi = uf^m$.

The sheaf $\Omega^1_{\overline{U(x)}}(\log \Delta)$ is locally free (by log regularity) and

$$\Omega^1_{\overline{U(x)}}(\log \Delta)\big|_V \cong (P \setminus \mathfrak{p})^{\mathrm{gp}} \otimes \mathcal{O}_{\overline{U(x)} \cap V}.$$

Now $\operatorname{dlog} u$ (when restricted to $\overline{U(x)}$) is a section of $\Omega^1_{\overline{U(x)}}(\log \Delta)$, and under the above isomorphism, it corresponds to $c \otimes 1$. This section does not vanish at any point of $\overline{U(x)}$; for that to happen, c would need to be divisible by p in the free abelian group (of finite type) $(P \setminus \mathfrak{p})^{\mathrm{gp}}$. Hence $a = mb - c$ would be divisible by p. This means that the cokernel C of u^{gp} has p-torsion, contradicting our assumptions.

The conclusion is that the 1-form $\operatorname{dlog} u$ is a nowhere vanishing (holomorphic) section of the vector bundle $\Omega^1_{\overline{U(x)}}(\log \Delta)$ on the log regular scheme $\overline{U(x)}$. Thus the top Chern class of this vector bundle vanishes. By [Saito 1991, "Lemma 0"], which we state below for the reader's convenience, this implies the result, since $U(x) = \overline{U(x)} \setminus \Delta$. $\square$

Lemma 5.2. *Let k be an algebraically closed field and let $(X, \mathcal{M}_X)$ be a log regular scheme over k (where k has the trivial log structure). Let U be the locus of triviality* *of the log structure on X. Then*

$$\chi(U) = \deg c_{X,\mathcal{M}_X},$$

where $c_{X,\mathcal{M}_X}$ is the top Chern class of the vector bundle $\Omega^1_X(\log \mathcal{M}_X)$ on X.

To prove this formula, one can use log blow-ups to reduce the statement to the case of a smooth k-variety equipped with a divisorial log structure, induced by a strict normal crossings divisor. In that setting, the formula is rather well-known and can be checked by an explicit computation.

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Finite phylogenetic complexity and combinatorics of tables

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In algebraic statistics, Jukes–Cantor and Kimura models are of great importance. Sturmfels and Sullivant generalized these models by associating to any finite abelian group G a family of toric varieties $X(G, K_{1,n})$. We investigate the generators of their ideals. We show that for any finite abelian group G there exists a constant ϕ, depending only on G, such that the ideals of $X(G, K_{1,n})$ are generated in degree at most ϕ.

1. Introduction

The aim of this article is to prove the finiteness of an intriguing invariant of finite abelian groups, called *phylogenetic complexity*. The invariant was introduced in a seminal paper by Sturmfels and Sullivant [2005], where it appeared in relation to phylogenetic models. In short, to a Markov process encoded by an abelian group G on a tree T one associates a toric variety $X(G, T)$, of particular relevance in algebraic statistics [Eriksson et al. 2005; Pachter and Sturmfels 2005]. The setting above is known as a *group-based model*.

We do not describe the relations to phylogenetics in this paper, referring the interested reader to [Allman and Rhodes 2003; Casanellas 2012; Donten-Bury and Michałek 2012; Michałek 2015]. Instead, in precise, purely mathematical language we present a natural construction of *a family of lattice polytopes* $P_{G,n}$ associated to any finite abelian group G — Definition 2.1. These polytopes should be considered as the simplest combinatorial objects encoding the group action.

Associating to interesting combinatorial objects a polytope and investigating its properties is nowadays a well-developed and powerful tool on the edge of combinatorics and toric geometry [Sturmfels 1996; Ohsugi and Hibi 1998; Herzog and Hibi 2002; Sturmfels and Sullivant 2008]. However, our knowledge of properties of the polytopes $P_{G,n}$ associated to such basic objects as finite abelian groups is still very limited. This may be even more surprising, as for various groups G, these polytopes relate not only to phylogenetics, but also mathematical physics through

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conformal blocks and moduli spaces [Sturmfels and Xu 2010; Manon 2012; 2013; Kubjas and Manon 2014].

Phylogenetic complexity governs the degrees of generators of the ideal of the variety $X(G, T)$. Using the language of toric geometry, one is interested in the generators of integral relations among the vertices of $P_{G,n}$. For the introduction to toric geometry we refer the reader to [Fulton 1993; Cox et al. 2011]. The objects that encode the group action and correspond to vertices of $P_{G,n}$ are called *flows*.

Definition 1.1 [Buczyńska and Wiśniewski 2007; Michałek 2014]. Let G be a finite abelian group and $n \in \mathbb{N}$. A *flow* is a sequence of n elements of G summing up to $0 \in G$, the neutral element of G. The set of flows is equipped with a group structure via the coordinatewise action. The group of flows $\mathfrak{G}$ is (noncanonically) isomorphic to G^{n-1}.

Hence, in our article we study possible relations among n-tuples of elements of G summing up to 0. Let T_0 and T_1 be two matrices or *tables* of the same size, whose rows are flows. These two tables are *compatible* if and only if, for each $1 \leq i \leq n$, the i-th column of T_0 and the i-th column of T_1 are the same multisets — see Example 2.3. Compatible tables correspond to binomials in the ideal $I(X(G, K_{1,n}))$, where $K_{1,n}$ is a star (also called a claw-tree) — the unique tree with one inner vertex and n leaves.

Definition 1.2 [Sturmfels and Sullivant 2005]. Let T be a tree, let $K_{1,n}$ be the star with n leaves, and let $\phi(G, T)$ be the maximal degree of a generator in a minimal generating set of $I(X(G, T))$. Let $\phi(G, n) = \phi(G, K_{1,n})$. We define the *phylogenetic complexity* of G to be $\phi(G) = \sup_{n\in\mathbb{N}} \phi(G, n)$.

The main theorem of the present article is the following:

Theorem 3.12. *For any finite abelian group G, the phylogenetic complexity is finite.*

Let us briefly summarize the state of the art. We maintain the convention that G is a finite abelian group.

Prior to the present work $\phi(G)$ was shown in [Sturmfels and Sullivant 2005] to be 2 for $\mathbb{Z}_2$ and conjectured to be $\leq |G|$ for all G and exactly 4 for the biologically relevant case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ [Conjectures 29 and 30]. It was proved in [Michałek 2017] to be finite for all $\mathbb{Z}_p$, where p is prime, and equal to 3 for $\mathbb{Z}_3$.

If one considers the *projective* scheme $X_p(G, T)$ instead of $X(G, T)$, the analog of our phylogenetic complexity is ≤ 4 for $\mathbb{Z}_2 \times \mathbb{Z}_2$ (in other words, $X_p(\mathbb{Z}_2 \times \mathbb{Z}_2, T)$ can be defined by an ideal generated in degree at most 4, for any T) and is finite for all G; both results are proved in [Michałek 2013]. The case of $\mathbb{Z}_3$ was solved in [Donten-Bury 2016] with the answer 3.

Draisma and Eggermont [2015] considered a generalization of group-based models: the group G of symmetries acts on a finite alphabet that need not coincide with G. They showed that the Zariski closure of the model can be *set-theoretically* defined by polynomial equations whose degree is bounded by a constant depending only on G. Our present results can be regarded as stronger, but for a smaller class. Obtaining finiteness results on an ideal-theoretic level for equivariant models would be a major achievement, far extending the results of [Draisma and Kuttler 2014]. However, this is beyond any of the methods described in this paper, where we focus on group-based models.

Casanellas et al. [2015b; 2015a] produced a collection of explicit equations that describe the phylogenetic variety on a Zariski-open subset of interest, and showed that the corresponding degree is $\leq |G|$. (In [Michałek 2014] that degree had been shown to be 4 for $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

Finiteness also plays an increasingly important role in the context of toric varieties; see [Draisma et al. 2015].

Finally, we would like to mention the reduction that we use from the very beginning, previously obtained by Sturmfels and Sullivant [2005]. Although, in general, one is interested in arbitrary trees, it is enough to consider claw-trees. This is due to the construction of toric fiber products [Sullivant 2007].

The structure of the article is as follows. In Section 2 we describe the basic notation. In particular, we recall how one encodes binomials in $I(X(G, K_{1,n}))$ as special pairs of tables with group elements. Section 3 contains the main result. First, in Section 3A, we present the sketch of the proof, without any technical details and then the complete proof in Section 3B. We hope that some of the ideas of the paper can be made effective. In particular, in future work we plan to prove [Sturmfels and Sullivant 2005, Conjecture 30].

2. Binomials, tables and moves

This section records definitions and notation needed in the rest of the paper.

Let G be a finite abelian group and let $n \in \mathbb{N}$. In Definition 1.1, we introduced the most important algebrocombinatorial objects in our setting: n-tuples of group elements summing to 0, called *flows*. From the point of view of toric geometry and phylogenetics, flows correspond to *monomials* parametrizing our variety $X(G, K_{1,n})$ [Sturmfels and Sullivant 2005; Michałek 2011]. Relations among flows — which are described by *compatible* tables — encode the binomials in $I(X(G, K_{1,n}))$. It is a standard approach in toric geometry to represent the parametrizing monomials by their exponents, as points in a lattice. The polytope, that is the convex hull of such points, captures the geometry of the parametrized variety. For the sake of completeness we present the polytopes corresponding to $X(G, K_{1,n})$.

Definition 2.1 (polytope $P_{G,n}$). Consider the lattice $M \cong \mathbb{Z}^{|G|}$ with a basis corresponding to elements of G. Consider M^n with the basis $e_{(i,g)}$ indexed by pairs

$(i, g) \in [n] \times G$. We define an injective map of sets $\mathfrak{G} \to M^n$, by

$$(g_1, \dots, g_n) \longmapsto \sum_{i=1}^{n} e_{(i,g_i)}.$$

The image of this map defines the vertices of the polytope $P_{G,n}$.

Example 2.2 [Michałek 2017]. For $G = (\mathbb{Z}_2, +)$ and $n = 3$, we have four flows:

$$(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Hence, the polytope $P_{\mathbb{Z}_2,3}$ has the following four corresponding vertices:

$$(1, 0, 1, 0, 1, 0), (1, 0, 0, 1, 0, 1), (0, 1, 1, 0, 0, 1), (0, 1, 0, 1, 1, 0) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2,$$

where $(1, 0) \in \mathbb{Z}^2$ corresponds to $0 \in \mathbb{Z}_2$ and $(0, 1) \in \mathbb{Z}^2$ corresponds to $1 \in \mathbb{Z}_2$.

A more sophisticated example is presented in [Michałek 2011, Example 4.1]. Binomials may be identified with a pair of tables of the same size T_0 and T_1 of elements of G, regarded up to row permutation. Each row of such tables has to be a flow. The identification is as follows. Every binomial is a pair of monomials; the variables in such monomials correspond to flows, given by a collection of n elements in G. Every monomial is viewed as a table, whose rows are the variables appearing in the monomial; the number of rows of the corresponding table is the degree of the monomial. Consequently, a binomial is identified with the pair of tables encoding the two monomials respectively.

A binomial belongs to $I(X(G, K_{1,n}))$ if and only if the two tables are *compatible*, i.e., for each i the i-th column of T_0 and the i-th column of T_1 are equal as multisets.

In order to generate a binomial — represented by a pair of tables T_0 and T_1 — by binomials of degree at most d we are allowed to select a subset of rows in T_0 of cardinality at most d and replace it with a compatible set of rows, repeating this procedure until both tables are equal.

Example 2.3 [Michałek 2017]. For $G = (\mathbb{Z}_2, +)$ and $n = 6$ consider the following two compatible tables:

$$T_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Note that the red subtable of T_0 is compatible with the table

$$T' = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Hence, we may *exchange* them obtaining:

$$\widetilde{T}_0 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that T_0 and $\widetilde{T}_0$ are *compatible*. Now, the brown subtable of $\widetilde{T}_0$ is compatible with the table

$$T'' = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, we exchange them obtaining T_1. Hence we have a sequence of tables $T_0 \rightsquigarrow \widetilde{T}_0 \rightsquigarrow T_1$. More specifically, we started from a degree three binomial given by the pair T_0, T_1 and we generated it using degree two binomials, called *quadratic moves*; see also Example 2.5.

In what follows, *quadratic moves*, i.e., binomials of degree two will play a crucial role. First, let us give the precise definition and an illustrative example.

Definition 2.4 (quadratic moves). Let T be a table — whose rows are flows — of elements of G; let r_i and r_j be two rows of T. For any subsequence $\{r_{i,l_1}, \ldots r_{i,l_t}\}$ of r_i, we define two rows s_i and s_j whose elements are the following:

(i) $s_{i,k} = r_{i,k}$ if $k \neq l_1, \ldots, l_t$, otherwise $s_{i,k} = r_{j,k}$.

(ii) $s_{j,k} = r_{j,k}$ if $k \neq l_1, \ldots, l_t$, otherwise $s_{j,k} = r_{i,k}$.

The transformation of r_i and r_j into s_i and s_j described above is a *quadratic move* if $\sum_{k=1}^{t} r_{i,l_k} = \sum_{k=1}^{t} r_{j,l_k}$; in other words, if the differences sum to $0 \in G$. We note that this condition is equivalent to the fact that s_i and s_j are flows.

To illustrate the definition of quadratic moves, we consider the following example, to be compared with Example 2.3.

Example 2.5. Let $G = (\mathbb{Z}_2, +)$. Let T be the following 2×3 table of elements in $\mathbb{Z}_2$:

$$T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The two rows r_1 and r_2 are flows, since their elements sum up to the $0 \in \mathbb{Z}_2$. We exchange the red subsequence of elements in the first row with the blue subsequence of elements in the second row. The rows s_1 and s_2, corresponding to the chosen (red) subsequence as in Definition 2.4, are the two rows of the following table:

$$\widetilde{T} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

This is a quadratic move, since s_1 and s_2 are still flows. Hence, the table T is transformed into the table $\widetilde{T}$ by the quadratic move above. Note that quadratic moves preserve, up to permutation, each column of a table. In particular, T and $\widetilde{T}$ are two compatible tables, i.e., their columns are the same as multisets.

3. Finite phylogenetic complexity for abelian groups

The aim of this section is to use the combinatorics of tables to prove finiteness of the phylogenetic complexity of a group-based model for any finite abelian group G.

3A. ***Idea of the proof.*** Before going into technical details, let us present here the basic ideas of Theorem 3.12.

The general strategy is to prove that the function $\phi(G, n)$ is eventually constant for large n. Hence, we start with two compatible $d \times n$ tables T_0 and T_1 for large n and we want to transform T_0 to T_1. The main objective is the proof of Lemma 3.11: *One can transform T_0 and T_1, independently, using quadratic moves, in such a way that there exist two columns c_j, c_{j+1} on which both tables exactly agree.* Once this aim is achieved, the induction becomes clear — the precise argument is presented in the last paragraph of the proof of Theorem 3.12. The most involved part is to show Lemma 3.11. First, we pass to subtables. For a table T, we denoted by T' the subtable containing all rows, but only those columns where a given element $g \in G$ is one of the (possibly many) most frequent group elements. This is not a severe restriction — see Remark 3.6. Such a “reference” element g is crucial throughout the proof. Note also that, due to compatibility, the indices of columns of the subtables T_0' and T_1' are the same, as the most frequent elements of any i-th column in T_0 and T_1 coincide. In particular, T_0' and T_1' are compatible (although their rows do not have to be flows any more). In the proof, it is shown that it is easier to move elements that are *frequent* in a table than those that are *rare*; the latter ones are called *dots*. A precise definition, independent on the choice of T_0 or T_1, of frequent and rare elements is given in Definition 3.2.

Equipped with these definitions, it is enough to prove Lemma 3.10: *One can transform T_0 and T_1, independently, using quadratic moves, in such a way that there exist two columns c_j, c_{j+1} such that any row in T_0 or T_1 contains at most one dot in columns c_j and c_{j+1}.* Indeed, once the above statement is proven, as the tables are considered up to row permutations, we can make all dots in both columns in T_0 exactly equal to corresponding dots in T_1. Then Lemma 3.11 follows as the entries that are not dots can also be adjusted — details are in the proof of the lemma.

Hence, the hard part of the proof of Theorem 3.12 lies in the proof of Lemma 3.10. Here the ideas are as follows. First, (as we passed from T to a subtable T' where a given element is one of most frequent in every column) we will be passing to

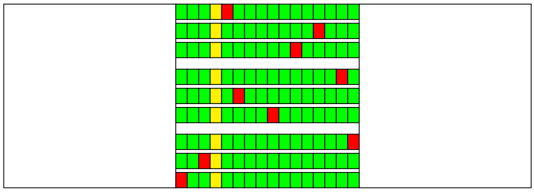


Figure 1. The subdivision algorithm.

thinner and thinner subtables. However, due to technical reasons, we must also allow their horizontal subdivisions, which motivate the following definition.

Definition 3.1 (vertical stripe). Given any table T, we define a *vertical stripe* to be

- a choice of some number of consecutive columns of T,
- a subdivision of rows into parts in the chosen columns.

Less formally, a vertical stripe is a collection of disjoint subtables in the same columns, that cover all rows of T.

Two examples of vertical stripes are presented in Figure 1. One consists of the whole colored part, where the subdivision into three subtables is given by two thick white horizontal stripes. The second stripe is the yellow one with the subdivision into nine subtables.

We would like to find a vertical stripe with (at least) two columns that has at most one dot in each row. Instead, we consider more general subtables that make vertical stripes: each subtable has k columns with at most s dots in each row. Further, we need to control how many distinct elements r of G appear as dots in the subtable. These subtables do not have to contain all rows, but appear in collections that form a vertical stripe, i.e., the collection covers all rows.

Figure 1 pictures the subdivision algorithm devised in Lemma 3.11 for T_0' and T_1'. We start with a vertical stripe — here represented by the colored part of the table. It consists of three subtables, divided by two large horizontal white stripes. In each of the subtables, we fix the same partition of columns into $t = 16$ vertical and three horizontal parts (given numbers are just examples). The new, finer horizontal subdivision is depicted with thin white stripes. In each horizontal part, we discard at most one of the subtables — these are the red squares. The yellow part drawn in the center of the picture is a vertical stripe, consisting of subtables that are not discarded in any of the horizontal parts.

The main point is that, for large k, we may decrease s or r by subdividing each subtable into $t|G|$ small subtables: columns are divided into $t \gg 0$ parts and rows into $|G|$ parts. In particular, we have t vertical parts, each consisting of $|G|$ small subtables stacked one under another. After quadratic moves, we may assume that each small subtable in almost all of the t vertical parts either has smaller number of dots in each row (decreasing s) or smaller number of distinct group elements corresponding to dots (decreasing r). As t is always much greater than the number of horizontal subdivisions (which is always some power of $|G|$) we are able to choose a whole vertical stripe (with much smaller number of columns) such that in each subtable s or r has been decreased. Further, we are able to do it in parallel in T_0' and T_1' — details are in the proof of Lemma 3.10.

We hope this discussion could shed some light on Definition 3.7. We mention here a technical remark; since we work with vertical stripes, once we focus on one subtable, we have to make sure we *do not* change the structure of other subtables. This feature is reflected in (ii) of Definition 3.7, where we restrict to quadratic moves that only modify a small part of the table. We are finally able to list the main steps towards the proof of Lemma 3.10:

(i) Bound the number of dots in each row (Lemma 3.4).

(ii) Prove that we may always subdivide a subtable, as described above, decreasing s or r (Lemma 3.9).

(iii) Show that the subdivision process can be done in parallel in T_0 and T_1 (Lemma 3.10).

3B. *Proof.* We start from the definition of frequent elements in a given table T with respect to a function F. Let $F(G)$ be a function of the cardinality of the group G. We assume $F(G) > |G|^2 + 3|G|$.

Definition 3.2 (F_T and dots). The set of $F(G)$-*frequent elements*, or frequent elements, in a given $d \times n$ table T is defined by

$$F_T = \{h \in G \mid \text{the number of copies of } h \text{ in } T > F(G) \cdot d\}.$$

Note that if an element is frequent, then there exists a row, where it appears at least $F(G)$ times. The elements $g \in G$ that are not in F_T are called *dots* $\bullet$.

The frequent elements have a key role in allowing quadratic moves in the table. Let us start with three basic — yet useful — lemmas.

Lemma 3.3. *Let f, f' be flows. Let I be a subset of indices and suppose $|I| \geq |G|$. There exists a (nonempty) subset $I' \subset I$ such that a quadratic move of f and f' on I' can be performed.*

Proof. Since we have $|G|$ differences, possibly repeated, of the form $f_i - f_i'$ for $i \in I$, we may find a nonempty subset I', such that $\sum_{i \in I'} (f_i - f_i') = 0 \in G$. □

Lemma 3.4. *Let T be a given table of elements of G, then we may assume that each row in T has at most $|G|(F(G)+1)$ dots.*

Proof. Note that there exists a row containing at most $|G|F(G)$ dots. Assuming the contrary, we would have at least $(|G|F(G)+1)d$ dots in T. This would imply that there would be a dot in F_T — a contradiction. Let us consider a row $r_{\max}$ with the largest number of dots. If $r_{\max}$ contains at most $|G|(F(G)+1)$ dots, this finishes the proof. Otherwise, we pick a row $r_{\min}$ with the smallest number of dots; they are at most $|G|F(G)$. Now, there exist $|G|$ dots of $r_{\max}$ in the same columns as $|G|$ elements of $r_{\min}$ which are in F_T. Exchanging a subset of them we decrease the number of rows with the largest number of dots. Repeating the process, we obtain T with all rows containing at most $|G|(F(G)+1)$ dots. □

Lemma 3.5. *Let $z \in \mathbb{N}$. For any $\epsilon > 0$, there exists $n = n(z)$ such that in any $(0,1)$-table T of size $d \times n$, whose columns contain at least $\epsilon \cdot d$ zeros each, there exists a row with at least z zeros.*

Proof. Setting $n > z/\epsilon$ we may conclude by double counting zeros column-wise and row-wise. □

Remark 3.6. Let T be a $d \times n$ table whose entries are elements of G. In each column c_i we select the elements that appear a maximal number of times; these elements are the *most frequent elements* in c_i. Among all the columns, we select those where a *reference* element $g \in G$ appears as one of the most frequent elements.

This is not a severe restriction, as n is very large and we would restrict to a subtable with at least $n/|G|$ columns, for some $g \in G$. Such a reference element g will be important throughout the proof.

We now introduce a crucial property $S(\cdot)$ for our inductive proof.

Definition 3.7 (property $S(\cdot)$). Let $s, r, t, k \in \mathbb{N}$, let T be a $d \times n$ table whose entries are elements of G, and Q a $d' \times k$-subtable of T. Moreover, let us assume that the following hold:

(a) $g \in G$ is one of the most frequent elements in every column of T.

(b) There are at most s dots in every row of Q.

(c) There exists a subset $H \subset G$ of cardinality r, such that each dot of Q is in H.

We say that the property $S(s, r, t, k, T, Q)$ holds for the pair $Q \subset T$ if

(i) $s = 1$ and $k \geq 2$, or

(ii) $s > 1$ and we can transform T into another table $\widetilde{T}$ (transforming Q into $\widetilde{Q}$) such that we may subdivide the first $t \cdot \lfloor k/t \rfloor$ columns of $\widetilde{Q}$ into t consecutive subtables Q_i, each consisting of $\tilde{k} = \lfloor k/t \rfloor$ columns and d' rows that satisfy:

  (1) If $r = 1$ then each Q_i *except one* has the property $S(s-1, r, |G|t, \tilde{k}, T, Q_i)$.

  (2) If $r > 1$ then for every Q_i *except one* we can subdivide the rows into $|G|$ parts Q_{ij}, such that for every j either $S(s-1, r, |G|t, \tilde{k}, T, Q_{ij})$ or $S(s, r-1, |G|t, \tilde{k}, T, Q_{ij})$ holds.

Further, the transformation may only use quadratic moves that do not change dots that are in the columns of Q and in rows outside Q (i.e., it cannot move dots in the same vertical stripe, but outside Q).

Remark 3.8. Condition (a) above is not restrictive, according to Remark 3.6, as we will be applying the definition to subtables of T_0 and T_1 for which g is one of the most frequent elements in each column.

In the next lemma, we show that one can transform and divide Q into smaller subtables decreasing either s or r, provided k is sufficiently large. This is achieved with special quadratic moves.

Lemma 3.9. *For every $s, r, t \in \mathbb{N}$, every k sufficiently large, and every pair $Q \subset T$ satisfying the assumptions in Definition 3.7, the property $S(s, r, t, k, T, Q)$ holds.*

Proof. The proof is by induction on s. For $s = 1$ the claim is true for $k \geq 2$ by Definition 3.7. Assume that the claim is true for s. We show the statement for $s+1$.

If $s+1 > |G|$, let us set $k > t \cdot \tilde{k}$, where $\tilde{k}$ is an integer such that the property $S(s, r, |G|t, \tilde{k}, \cdot, \cdot)$ holds for arbitrary pairs of tables and subtables in the last two arguments which satisfy the assumptions in Definition 3.7. Let us fix an arbitrary pair of tables $Q \subset T$ satisfying the assumptions in Definition 3.7 for $s+1$ and r. In particular, each row of Q has at most $s+1$ dots. We fix a partition of Q into equal-sized subtables Q_j, each consisting of $\lfloor k/t \rfloor$ consecutive columns. If all the Q_j contain only rows with strictly less than $s+1$ dots, we are done. Otherwise, we choose a subtable Q_{i_0} with a maximal number of rows containing $s+1$ dots. Every Q_j has at most as many rows with $s+1$ dots as Q_{i_0}. Hence, for any subtable Q_j different from Q_{i_0} we can pair each row of Q_j with $s+1$ dots with a row of Q_j without any dots (the latter corresponding to a row of Q_{i_0} with $s+1$ dots). The structure of T is as follows.

$$\begin{array}{r}
Q\left\{\begin{matrix} \\ \\ \\ \\ \end{matrix}\right. \\
T\setminus Q\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\
\\
\end{array}
\begin{array}{c}
\begin{bmatrix}
\bullet\quad\bullet\quad\bullet & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\
\bullet\quad\bullet\quad\bullet & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\
\cdots\ \cdots\ \cdots & \bullet\quad\bullet\quad\bullet & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\
\cdots\ \cdots\ \cdots & \bullet\quad\bullet\quad\bullet & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\
\cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\
\cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots & \cdots\ \cdots\ \cdots \\
\underbrace{\bullet\quad\bullet\quad\bullet}_{Q_{i_0}} & \underbrace{\cdots\ \cdots\ \cdots}_{Q_j} & \underbrace{\cdots\ \cdots\ \cdots}_{Q\setminus(Q_{i_0}\cup Q_j)} & \underbrace{\cdots\ \cdots\ \cdots}_{T\setminus Q}
\end{bmatrix}
\end{array}$$

The arrows below describe the pairing between a row with $s+1$ dots with a row without any dots in the subtable Q_j.

$$Q_j = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\bullet & \cdots & \bullet & \bullet & \cdots & \cdots \\
\bullet & \bullet & \cdots & \cdots & \bullet & \cdots
\end{bmatrix}$$

For each such pair we use Lemma 3.3, as $s+1 > |G|$, to make a quadratic move reducing the number of dots that a row of Q_j may have. Hence, by induction, for any $Q_j \neq Q_{i_0}$ the property $S(s, r, |G|t, \lfloor k/t \rfloor, T, Q_j)$ holds, as $k > t \cdot \tilde{k}$. Thus $S(s+1, r, t, k, T, Q)$ holds by Definition 3.7.

If $s+1 \leq |G|$, we proceed by induction on r.

If $r = 1$, let us set $k > t \cdot \tilde{k}$ as before. First, suppose there is only one vertical part Q_{i_0} which contains rows with $s+1$ dots. Since all the other parts Q_j have rows with at most s dots, by induction they satisfy $S(s, r, |G|t, \lfloor k/t \rfloor, T, Q_j)$, hence we may conclude this case. Otherwise, as long as there are two parts Q_{i_0} and Q_{j_0} with rows r_i and r_j respectively with $s+1$ dots, we proceed as follows. Let us fix one dot in r_i and one in r_j. Let g_i and g_j be the elements of the rows r_i and r_j in the same columns as the chosen dots. If $g_i = g_j$ we can make a quadratic move exchanging both chosen dots and the g_i. Suppose $g_i \neq g_j$.

$$\begin{array}{c|cccc|cccc|ccc|cccc|}
 & \bullet & \bullet & \bullet & \bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
Q & \bullet & \bullet & \bullet & {\color{red}\bullet} & g_i & \cdots & \cdots & \cdots & g_j & \cdots & \cdots & g_j & g_j & \cdots & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & \cdots & \cdots & g_j & {\color{red}\bullet} & \bullet & \bullet & \bullet & \cdots & \cdots & g_i & \cdots & \cdots & g_i & g_i \\
\hline
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
T \setminus Q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\multicolumn{1}{c}{} & \multicolumn{4}{c}{Q_{i_0}} & \multicolumn{4}{c}{Q_{j_0}} & \multicolumn{3}{c}{Q \setminus (Q_{i_0} \cup Q_{j_0})} & \multicolumn{4}{c}{T \setminus Q}
\end{array}$$

As g_i is not a dot, there has to exist a row r_t of T with more than $F(G)$ copies of g_i. We make a quadratic move between r_j and r_t not involving the $2s+2$ columns of dots in Q_{i_0} and Q_{j_0} in the rows r_i and r_j. This procedure allows us to put at least $F(G) - 3|G|$ copies of g_i in the row r_j, without moving dots in Q — we need to subtract $|G|$ by Lemma 3.3 and $2|G| \geq 2(s+1)$ to avoid the dots. Now, we can make the same quadratic move for g_j and r_i. The result of these moves is in the table above, where the red bullets ${\color{red}\bullet}$ are the chosen dots.

After performing these quadratic moves, if there is a column c_t containing g_j and g_i in rows r_i and r_j, then we make a quadratic move, exchanging the chosen dots and the elements of c_t. Otherwise, applying Lemma 3.5 for $\epsilon = 1/|G|$ to a subtable of T of columns containing g_i in the row r_j, we may find a row r_t containing at least $|G|$ copies of g, as long as $F(G) - 3|G| > |G|^2$. Then we move some copies of g to the row r_i by Lemma 3.3. Analogously for g_j, we may move some copies of g to the row r_j. Here are, depicted in red, the copies of g and, in blue, the quadratic move putting those copies of g in r_i and r_j respectively.

$$\begin{array}{cc}
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & {\color{red}g} & {\color{red}g} & {\color{red}g} & {\color{red}g} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & {\color{blue}\downarrow} & {\color{blue}\downarrow} & {\color{blue}\downarrow} & {\color{blue}\downarrow} \\
{\color{red}\bullet} & g_i & g_j & g_j & \cdots & g_j & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
g_j & {\color{red}\bullet} & \cdots & \cdots & \cdots & \cdots & g_i & g_i & \cdots & g_i \\
\cdots & \cdots & {\color{blue}\uparrow} & {\color{blue}\uparrow} & {\color{blue}\uparrow} & {\color{blue}\uparrow} & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & {\color{red}g} & {\color{red}g} & {\color{red}g} & {\color{red}g} & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
&
\begin{array}{c}
r_t \\ \\ r_i \\ \\ r_j \\ \\ r_{t'}
\end{array}
\end{array}$$

Applying the blue quadratic move above, we obtain a column c_i that has g in r_i and g_i in r_j. In the same way, we obtain a column c_j that has g in r_j and g_j in r_i. Now, we perform a quadratic move in the subtable below, exchanging the chosen dots:

$$\begin{bmatrix} \bullet & g_i & g_j & g \\ g_j & \bullet & g & g_i \end{bmatrix}.$$

Thus, we reduce the number of dots in both rows. This concludes the case $r = 1$.

Assume $r > 1$. Let us set $k > t \cdot \tilde{k}$, where $\tilde{k}$ is such that both of the properties $S(s, r, |G|t, k, T, \cdot)$ and $S(s+1, r-1, |G|t, k, T, \cdot)$ hold. Suppose that there is only one Q_{i_0} such that there exists a row with $s+1$ dots corresponding to r distinct group elements. Then the rows of every other part Q_j can be partitioned into at most $|G|$ parts $Q_{j,l}$ such that

(i) all the rows in $Q_{j,1}$ have at most s dots,

(ii) all the dots in $Q_{j,l}$ for $l > 1$ correspond to at most $r-1$ distinct group elements.

We conclude by induction in the case when there is only one part Q_{i_0}. We will reduce every other case to this one. Assume that there are two parts Q_{i_0} and Q_{j_0} such that there exist rows r_i and r_j with $s+1$ dots corresponding to r distinct group elements. As both rows r_i and r_j contain dots corresponding to the same r elements of the group G, we can choose one dot in each row corresponding to the same element. Now, repeating the procedure described in the case $r = 1$, we reduce the number of dots in r_i and r_j. This concludes the proof. □

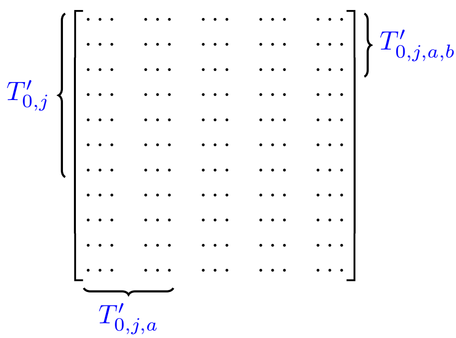
By Lemma 3.9 for any s, r, t we set $K(s, r, t)$ such that for all $k \geq K(s, r, t)$ the property $S(s, r, t, k, \cdot, \cdot)$ holds.

Lemma 3.10. *Let T_0 and T_1 be tables which are compatible and have at least $|G|K\big(|G|(F(G)+1), |G|, 3\big)$ columns. Then, we may transform them using quadratic moves into tables $\widetilde{T}_0$ and $\widetilde{T}_1$ such that the following holds: there exists j such that no row in $\widetilde{T}_0$ nor in $\widetilde{T}_1$ has a dot in both the j-th and the $(j+1)$-st columns.*

Proof. Let us restrict T_0 and T_1 to the subtables T_0' and T_1' containing all rows and those columns that have g as the most frequent element. By Lemma 3.4, we may assume that the upper bound on the number of dots in T_0' and T_1' in each row is $B = |G|(F(G)+1)$. By Remark 3.6 and the assumption on the size of T_0 and T_1, we can assume that T_0' and T_1' have at least $k_0 = K(B, |G|, 3)$ columns. Hence, in particular, the properties $S(B, |G|, 3, k_0, T_0', T_0')$ and $S(B, |G|, 3, k_0, T_1', T_1')$ hold. In the rest of the proof we transform both tables T_0' and T_1' using quadratic moves, at each step passing to a smaller vertical stripe such that each subtable in it satisfies the property $S(s, r, \cdot)$ with smaller and smaller s or r.

Starting with T_0' and T_1' we apply the following algorithm, depicted in Figure 1.

Input of step i: The input of the i-th step of the algorithm is two compatible tables with corresponding distinguished $k_i = \lfloor k_{i-1}/(3|G|^{i-1}) \rfloor$ consecutive columns forming a vertical stripe. The vertical stripe has at most $|G|^i$ parts (subtables). In the table T'_0 the parts are $T'_{0,j}$. For a given part $T'_{0,j}$, let $s_{0,i,j}$ be the maximal number of dots that a row may have. Let $r_{0,i,j}$ be the number of distinct group elements corresponding to dots of $T'_{0,j}$. Properties $S(s_{0,i,j}, r_{0,i,j}, 3|G|^i, k_i, T'_0, T'_{0,j})$ hold. Likewise the parts $T'_{1,j}$ of T'_1 satisfy $S(s_{1,i,j}, r_{1,i,j}, 3|G|^i, k_i, T'_1, T'_{1,j})$. Moreover, $s_{0,i,j} + r_{0,i,j} \leq B + |G| - i$ and $s_{1,i,j} + r_{1,i,j} \leq B + |G| - i$.
Output of step i: The output of the i-th step of the algorithm is the input of the $(i+1)$-st step.
Termination: The algorithm stops when all $s_{0,i,j}, s_{1,i,j} = 1$.
Procedure for step i: In the i-th step we subdivide the k_i columns into $3|G|^i$ parts, subdividing each $T'_{0,j}$ into parts $T'_{0,j,a}$, as in Definition 3.7. Now, the algorithm transforms $T'_{0,j,a}$ using Definition 3.7. Hence, we obtain a subdivision of rows of $T'_{0,j,a}$ into at most $|G|$ parts $T'_{0,j,a,b}$. Here are the parts of T'_0 highlighted in blue (the left and right brackets select horizontal parts and the bottom bracket selects vertical parts):



For each j, for every a *except one* and for every b the subtable $T'_{0,j,a,b}$ satisfies either property (1) or (2) below:

$$S(s_{0,i,j} - 1, r_{0,i,j}, 3|G|^{i+1}, k_{i+1}, T'_0, T'_{0,j,a,b}), \tag{1}$$

$$S(s_{0,i,j}, r_{0,i,j} - 1, 3|G|^{i+1}, k_{i+1}, T'_0, T'_{0,j,a,b}). \tag{2}$$

As each of the $|G|^i$ horizontal parts in T'_0 can exclude one $T'_{0,j,a}$, and each of the $|G|^i$ horizontal parts in T'_1 can exclude one $T'_{1,j,a}$ we may find an index a_0 such that the following conditions hold for every j and b:

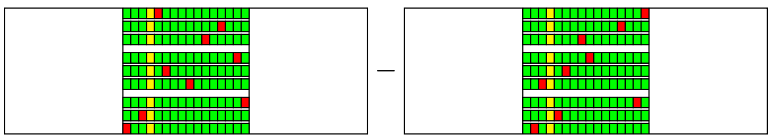
(i) (1) or (2), with a_0 in place of a.

(ii) (3) or (4), given by

$$S(s_{1,i,j} - 1, r_{1,i,j}, 3|G|^{i+1}, k_{i+1}, T'_1, T'_{1,j,a_0,b}), \tag{3}$$

$$S(s_{1,i,j}, r_{1,i,j} - 1, 3|G|^{i+1}, k_{i+1}, T'_1, T'_{1,j,a_0,b}). \tag{4}$$

(Less formally, since the number of vertical stripes is much larger than the number of discarded subtables in each subdivision, we can choose two corresponding vertical stripes in both of the tables. This is pictured below.) The choice of the a_0-th vertical stripe and the subdivisions $T'_{0,j,a_0,b}$, $T'_{1,j,a_0,b}$ are the output of the i-th step of the algorithm and the input of the $(i+1)$-th step. The algorithm terminates when we reach $s = 1$. The procedure terminates in a finite number of steps as at each step either s or r decreases. Moreover, at every step of the algorithm, we have collections of subtables satisfying property $S(\cdot)$. This implies that, at the last step, $k \geq 2$. Thus the algorithm provides the desired pairs of columns.

$\tilde{T}_0 - \tilde{T}_1 =$  $\square$

Lemma 3.11. *Let T_0 and T_1 be two compatible tables with n columns, for n sufficiently large. We can transform T_0 and T_1 using quadratic moves such that the following holds: there exists j such that the j-th and $(j+1)$-st columns in T_0 equal the j-th and $(j+1)$-st columns in T_1 respectively.*

Proof. We restrict to subtables T'_0 and T'_1 where g is the most frequent element, as in Remark 3.6. By Lemma 3.10, we may assume that in every row of T'_0 and T'_1 we have only one dot in the first two columns. Now, we can permute rows in such a way that the dots are equal in the corresponding entries. The elements in the rows which are not dots are not necessarily the same in each row. We show that given any pair of distinct elements $g_i, g_j \in F_{T'_0}$ in the first column and in the rows r_i, r_j respectively, we can exchange them.

Since g_i and g_j are in $F_{T'_0}$, we can find two rows, say r_s and r_t respectively, such that we have at least $F(G)$ copies of g_i and g_j in r_s and r_t respectively — see the table below. By Lemma 3.3, we can move at least $F(G) - |G| - 2$ copies of g_i to the row r_j and at least $F(G) - |G| - 2$ copies of g_j to r_i; here we subtract two because we are avoiding the first two columns. If there is a column c_t containing g_i and g_j in its j-th and i-th rows respectively, then we exchange them by a quadratic move on the column c_t and the first column. Otherwise, we proceed as follows. We restrict to a subtable containing columns where the row r_j has g_i as its entries. By Lemma 3.5, for $\epsilon = 1/|G|$, in this subtable we may find $|G|$ copies of g in some row r_t. Then we move some copies of g to the row r_i applying Lemma 3.3. Analogously for g_j, we may move some copies of g to the row r_j. Below are depicted in red the copies of g and in blue the quadratic moves putting those copies of g in r_i and r_j respectively.

$$T_0' = \begin{bmatrix} \bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bullet & \cdots & g & g & g & g & \cdots & \cdots & \cdots & \cdots \\ \bullet & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots & \cdots & \cdots \\ g_i & \cdots & \cdots & \cdots & \cdots & \cdots & g_j & g_j & g_j & g_j \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_j & \bullet & g_i & g_i & g_i & g_i & \cdots & \cdots & \cdots & \cdots \\ \cdots & \bullet & \cdots & \cdots & \cdots & \cdots & \uparrow & \uparrow & \uparrow & \uparrow \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & g & g & g & g \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Now, we perform a quadratic move exchanging g_i and g_j in a suitable subtable of T_0':

$$\begin{bmatrix} g_i & g_j & g \\ g_j & g & g_i \end{bmatrix}.$$

Such moves allow us to adjust all elements in the first two columns that are *not* dots. This concludes the proof. ☐

Theorem 3.12. *For any finite abelian group G, the phylogenetic complexity $\phi(G)$ of G is finite.*

Proof. Let G be a finite abelian group. Fix $N \gg |G|$. Once N is fixed, the phylogenetic complexity $\phi(G, N)$ is finite by the Hilbert basis theorem. Assume $n > N$. We will show that $\phi(G, n) \leq \phi(G, n-1)$. This implies that they are equal.

Let B be a binomial in $I(X(G, K_{1,n}))$ identified with a compatible pair of $d \times n$ tables T_0 and T_1, as described in Section 2. By Lemma 3.11, we may assume there exist two columns c_j and c_{j+1} in T_0 and their corresponding columns c_j' and c_{j+1}' in T_1, for some $1 \leq j \leq n$, such that, for each row, c_j has the same entries as c_j', and c_{j+1} has the same entries as c_{j+1}'. Note that in Lemma 3.11 we use quadratic moves to transform two given tables T_0 and T_1 into a pair of tables such that they satisfy the condition on columns above.

Now, summing coordinatewise the columns c_j and c_{j+1} in T_0, and c_j' and c_{j+1}' in T_1, we obtain a new pair of tables $\widehat{T_0}$ and $\widehat{T_1}$ with $n-1$ columns. The new pair $\widehat{T_0}, \widehat{T_1}$ is identified with a binomial $\widehat{B} \in I(X(G, K_{1,n-1}))$. By definition, this binomial is generated by binomials of degree at most $\phi(G, n-1)$. Hence, we may transform $\widehat{T_0}$ into $\widehat{T_1}$ by exchanging in every step at most $\phi(G, n-1)$ rows. Each of these steps lifts to an exchange among at most $\phi(G, n-1)$ rows in tables T_0 and T_1. After applying all the steps, the resulting tables $\widetilde{T_0}$ and $\widetilde{T_1}$ still do not have to be equal. However, they only differ possibly on the columns c_j and c_{j+1}. Without loss of generality we may assume $j = 1$. Thus the tables $\widetilde{T_0}$ and $\widetilde{T_1}$ are as follows:

$$\widetilde{T_0} - \widetilde{T_1} = \begin{bmatrix} a_{j_1} & b_{j_1} & \cdots & \cdots & \cdots \\ a_{j_2} & b_{j_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j_d} & b_{j_d} & \cdots & \cdots & \cdots \end{bmatrix} - \begin{bmatrix} a_{k_1} & b_{k_1} & \cdots & \cdots & \cdots \\ a_{k_2} & b_{k_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k_d} & b_{k_d} & \cdots & \cdots & \cdots \end{bmatrix},$$

where columns *different* from the first two are identical. Suppose there exists l such that $a_{j_l} \neq a_{k_l}$ (and $b_{j_l} \neq b_{k_l}$). Then $a_{j_l} + b_{j_l} = a_{k_l} + b_{k_l}$, since the l-th rows of $\widetilde{T}_0$ and $\widetilde{T}_1$ are identical except in the first two columns and, moreover, every row is a flow. On the other hand, there exists s such that $a_{k_l} = a_{j_s}$ and $b_{k_l} = b_{j_s}$. Thus we make a quadratic move between a_{j_l}, b_{j_l} and a_{j_s}, b_{j_s}. This concludes the proof. □

4. Open questions

In this last section, we collect some well-known open questions regarding group-based models for the convenience of the reader. We start from the central conjecture in this context.

Conjecture 4.1 [Sturmfels and Sullivant 2005, Conjecture 29]. For G, any finite abelian group, $\phi(G) \leq |G|$.

Taking into account the inductive approach presented in this article, it seems crucial to first understand the simplest tree $K_{1,3}$.

Conjecture 4.2. For G, any finite abelian group, $\phi(G, 3) \leq |G|$.

Notice that our main theorem — Theorem 3.12 — can be restated as follows: the function $\phi(G, \cdot)$ is eventually constant. The ensuing result would be a desired strengthening of ours.

Conjecture 4.3 [Michałek 2013, Conjecture 9.3]. $\phi(G, n+1) = \max(2, \phi(G, n))$.

We are grateful to Seth Sullivant for noticing that this is equivalent to $\phi(G, \cdot)$ being constant, apart from the case when $G = \mathbb{Z}_2$ and $n = 3$, when the associated variety is the whole projective space. Conjecture 4.3 also implies the following.

Conjecture 4.4 [Sturmfels and Sullivant 2005, Conjecture 30]. The phylogenetic complexity of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is 4.

Yet another direction would be trying to find combinatorial analogs of Δ-modules presented in [Snowden 2013; Sam and Snowden 2016]. We have not pursued this approach, however we present some similarities. First, in the class of equivariant models one can apply such techniques to prove finiteness on the set-theoretic level [Draisma and Eggermont 2015]. Second, one of the properties of equivariant models — a flattening — is mimicked for group-based models (on the algebra level though, but not on the level of varieties). This is the addition of two group elements that turns a flow of length $n+1$ to a flow of length n. The latter was a crucial property that allowed us to obtain the result: generation using the "simple" equations (in our case, quadratic moves) and induced equations for smaller n. It would be very desirable to introduce a general setting for polytopes and toric varieties, which would still allow to obtain finiteness results on the ideal-theoretic level.

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