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Variation of anticyclotomic Iwasawa invariants in Hida families

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Building on the construction of big Heegner points in the quaternionic setting by Longo and Vigni, and their relation to special values of Rankin–Selberg L-functions established by Castella and Longo, we obtain anticyclotomic analogues of the results of Emerton, Pollack and Weston on the variation of Iwasawa invariants in Hida families. In particular, combined with the known cases of the anticyclotomic Iwasawa main conjecture in weight 2, our results yield a proof of the main conjecture for p-ordinary newforms of higher weights and trivial nebentypus.

Introduction

In the remarkable paper [Emerton et al. 2006], Emerton, Pollack and Weston obtained striking results on the behavior of the cyclotomic Iwasawa invariants attached to p-ordinary modular forms as they vary in Hida families. In particular, combined with Greenberg's conjecture on the vanishing of the μ-invariant, their main result *reduces* the proof of the main conjecture to the weight two case. In this paper, we develop analogous results for newforms base-changed to imaginary quadratic fields in the definite anticyclotomic setting. In particular, combined with

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Vatsal's result [2003] on the vanishing of the anticyclotomic μ-invariant, and the known cases of the anticyclotomic main conjecture in weight 2 (thanks to the works of Bertolini and Darmon [2005], Pollack and Weston [2011], and Skinner and Urban [2014]), our results yield a proof of Iwasawa's main conjecture for p-ordinary modular forms of higher weights $k \geqslant 2$ and trivial nebentypus in the anticyclotomic setting.

Let us begin by recalling the setup of [Emerton et al. 2006], but adapted to the context at hand. Let

$$\bar{\rho} : G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$$

be a continuous Galois representation defined over a finite field $\mathbb{F}$ of characteristic $p > 3$, and assume that $\bar{\rho}$ is odd and irreducible. After the proof of Serre's conjecture [Khare and Wintenberger 2009], we know that $\bar{\rho}$ is modular, meaning that $\bar{\rho}$ is isomorphic to the mod p Galois representation $\bar{\rho}_{f_0}$ associated to an elliptic newform f_0. Throughout this paper, it will be assumed that $\bar{\rho} \simeq \bar{\rho}_{f_0}$ for some newform f_0 of weight 2 and trivial nebentypus.

Let $N(\bar{\rho})$ be the tame conductor of $\bar{\rho}$, and let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant prime $-D_K < 0$ to $pN(\bar{\rho})$. The field K then determines a decomposition

$$N(\bar{\rho}) = N(\bar{\rho})^+ \cdot N(\bar{\rho})^-$$

with $N(\bar{\rho})^+$ (resp. $N(\bar{\rho})^-$) only divisible by primes which are split (resp. inert) in K. We similarly define the decomposition $M = M^+ \cdot M^-$ for any positive integer M prime to D_K.

As in [Pollack and Weston 2011], we consider the following conditions on a pair $(\bar{\rho}, N^-)$, where N^- is a fixed square-free product of an odd number of primes inert in K:

Assumption (CR). (1) $\bar{\rho}$ is irreducible;

(2) $N(\bar{\rho})^- \mid N^-$;

(3) $\bar{\rho}$ is ramified at every prime $\ell \mid N^-$ such that $\ell \equiv \pm 1 \pmod{p}$.

Let $\mathcal{H}(\bar{\rho})$ be the set of all p-ordinary and p-stabilized newforms with mod p Galois representation isomorphic to $\bar{\rho}$, and let $\Gamma := \mathrm{Gal}(K_\infty/K)$ denote the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K. Associated with each $f \in \mathcal{H}(\bar{\rho})$ of tame level N_f with $N_f^- = N^-$, defined over say a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, there is a p-adic L-function

$$L_p(f/K) \in \mathcal{O}[\![\Gamma]\!]$$

constructed by Bertolini and Darmon [1996] in weight two, and by Chida and Hsieh [2016] for higher weights. The p-adic L-function $L_p(f/K)$ is characterized, as

χ runs over the p-adic characters of Γ corresponding to certain algebraic Hecke characters of K, by an interpolation property of the form

$$\chi(L_p(f/K)) = C_p(f,\chi)\cdot E_p(f,\chi)\cdot \frac{L(f/K,\chi,k/2)}{\Omega_{f,N^-}},$$

where $C_p(f,\chi)$ is an explicit nonzero constant, $E_p(f,\chi)$ is a p-adic multiplier, and Ω_{f,N^-} is a complex period making the above ratio algebraic. (Of course, implicit in all the above is a fixed choice of complex and p-adic embeddings $\mathbb{C} \overset{\iota_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$.)

The anticyclotomic Iwasawa main conjecture gives an arithmetic interpretation of $L_p(f/K)$. More precisely, let

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}_F(V_f) \simeq \mathrm{GL}_2(F)$$

be a self-dual twist of the p-adic Galois representation associated to f, fix an $\mathcal{O}$-stable lattice $T_f \subseteq V_f$, and set $A_f := V_f/T_f$. Let $D_p \subseteq G_{\mathbb{Q}}$ be the decomposition group corresponding to our fixed embedding ι_p, and let $\varepsilon_{\mathrm{cyc}}$ be the p-adic cyclotomic character. Since f is p-ordinary, there is a unique one-dimensional D_p-invariant subspace $F_p^+V_f \subseteq V_f$ where the inertia group at p acts via $\varepsilon_{\mathrm{cyc}}^{k/2}\psi$, with ψ a finite order character. Let $F_p^+A_f$ be the image of $F_p^+V_f$ in A_f and set $F_p^-A_f := A_f/F_p^+A_f$. Following the terminology in [Pollack and Weston 2011], the *minimal Selmer group* of f is defined by

$$\mathrm{Sel}(K_\infty, f) := \ker\Biggl\{H^1(K_\infty, A_f) \to \prod_{w\nmid p} H^1(K_{\infty,w}, A_f) \times \prod_{w\mid p} H^1(K_{\infty,w}, F_p^-A_f)\Biggr\},$$

where w runs over the places of K_∞. By standard arguments (see [Greenberg 1989], for example), one knows that the Pontryagin dual of $\mathrm{Sel}(K_\infty, f)$ is finitely generated over the anticyclotomic Iwasawa algebra $\Lambda := \mathcal{O}[[\Gamma]]$. The *anticyclotomic main conjecture* is then the following:

Conjecture 1. *The Pontryagin dual* $\mathrm{Sel}(K_\infty, f)^\vee$ *is* Λ*-torsion, and*

$$Ch_\Lambda(\mathrm{Sel}(K_\infty, f)^\vee) = (L_p(f/K)).$$

For newforms f of weight 2 corresponding to elliptic curves $E/\mathbb{Q}$ with ordinary reduction at p, and under rather stringent assumptions on $\bar\rho_f$ which were later relaxed by Pollack and Weston [2011], one of the divisibilities predicted by Conjecture 1 was obtained by Bertolini and Darmon [2005] using Heegner points and Kolyvagin's method of Euler systems. More recently, after the work of Chida and Hsieh [2015] the divisibility

$$Ch_\Lambda(\mathrm{Sel}(K_\infty, f)^\vee) \supseteq (L_p(f/K))$$

is known for newforms f of weight $k \leqslant p-2$ and trivial nebentypus, provided the pair $(\bar\rho_f, N_f^-)$ satisfies a mild strengthening of Hypotheses (CR). This restriction to small weights comes from the use of Ihara's lemma [Diamond and Taylor 1994], and it seems difficult to directly extend their arguments in [Chida and Hsieh 2015] to higher weights. Instead, as we shall explain in the following paragraphs, in this paper we will complete the proof of Conjecture 1 to all weights $k \equiv 2 \pmod{p-1}$ by a different approach, using Howard's big Heegner points in Hida families [Howard 2007], as extended by Longo and Vigni [2011] to quaternionic Shimura curves.

Associated with every $f \in \mathcal{H}(\bar{\rho})$ there are *anticyclotomic Iwasawa invariants* $\mu^{\mathrm{an}}(K_\infty, f)$, $\lambda^{\mathrm{an}}(K_\infty, f)$, $\mu^{\mathrm{alg}}(K_\infty, f)$, and $\lambda^{\mathrm{alg}}(K_\infty, f)$. The analytic (resp. algebraic) λ-invariants are the number of zeros of $L_p(f/K)$ (resp. of a generator of the characteristic ideal of $\mathrm{Sel}(K_\infty, f)^\vee$), while the μ-invariants are defined as the exponent of the highest power of ϖ (with $\varpi \in \mathcal{O}$ any uniformizer) dividing the same objects. Our main results on the variation of these invariants are summarized in the following. (Recall that we assume $\bar{\rho} \simeq \bar{\rho}_{f_0}$ for some newform f_0 of weight 2 and trivial nebentypus.)

Theorem 2. *Assume in addition that*:

- $\bar{\rho}$ *is irreducible*;
- $\bar{\rho}$ *is p-ordinary*, *"nonanomalous" and p-distinguished*:

$$\bar{\rho}|_{D_p} \simeq \begin{pmatrix} \bar{\varepsilon} & * \\ 0 & \bar{\delta} \end{pmatrix},$$

  with $\bar{\varepsilon}, \bar{\delta} : D_p \to \mathbb{F}^\times$ *characters such that* $\bar{\delta}$ *is unramified*, $\bar{\delta}(\mathrm{Frob}_p) \neq \pm 1$ *and* $\bar{\delta} \neq \bar{\varepsilon}$;
- $N(\bar{\rho})^-$ *is the square-free product of an odd number of primes.*

Let $\mathcal{H}^-(\bar{\rho}) := \mathcal{H}^{N(\bar{\rho})^-}(\bar{\rho})$ *consist of all newforms* $f \in \mathcal{H}(\bar{\rho})$ *with* $N_f^- = N(\bar{\rho})^-$, *and fix* $* \in \{\mathrm{alg}, \mathrm{an}\}$. *Then the following hold*:

(1) *For all* $f \in \mathcal{H}^-(\bar{\rho})$, *we have*

$$\mu^*(K_\infty, f) = 0.$$

(2) *Let* $f_1, f_2 \in \mathcal{H}^-(\bar{\rho})$ *lie on the branches* $\mathbb{T}(\mathfrak{a}_1)$, $\mathbb{T}(\mathfrak{a}_2)$ (*defined in* §*1D*), *respectively. Then*

$$\lambda^*(K_\infty, f_1) - \lambda^*(K_\infty, f_2) = \sum_{\ell | N_{f_1}^+ N_{f_2}^+} e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1),$$

where the sum is over the split primes in K *which divide the tame level of* f_1 *or* f_2, *and* $e_\ell(\mathfrak{a}_j)$ *is an explicit nonnegative invariant of the branch* $\mathbb{T}(\mathfrak{a}_j)$ *and the prime* ℓ.

Provided that p splits in K, and under the same hypotheses on $\bar{\rho}$ as in Theorem 2, the work of Skinner and Urban [2014] establishes one of the divisibilities in their "three-variable" Iwasawa main conjecture. Combining their work with our Theorem 2, and making use of the aforementioned results of Bertolini and Darmon [2005] and Pollack and Weston [2011] in weight 2, we obtain many new cases of Conjecture 1 (cf., Corollary 5.5):

Corollary 3. *Suppose that $\bar{\rho}$ is as in Theorem 2 and that p splits in K. Then the anticyclotomic Iwasawa main conjecture holds for every $f \in \mathcal{H}^{-}(\bar{\rho})$ of weight $k \equiv 2 \pmod{p-1}$ and trivial nebentypus.*

Let us briefly explain the new ingredients in the proof of Theorem 2. As it will be clear to the reader, the results contained in Theorem 2 are anticyclotomic analogues of the results of Emerton, Pollack and Weston [Emerton et al. 2006] in the cyclotomic setting. In fact, on the algebraic side the arguments of *loc.cit.* carry over almost verbatim, and our main innovations in this paper are in the development of anticyclotomic analogues of their results on the *analytic* side. Indeed, the analytic results of [Emerton et al. 2006] are based on the study of certain two-variable p-adic L-functions à la Mazur and Kitagawa, whose construction relies on the theory of modular symbols on classical modular curves. In contrast, we need to work on a family of Shimura curves associated with definite quaternion algebras, for which cusps are not available. In the cyclotomic case, modular symbols are useful in two ways: They yield a concrete realization of the degree-one compactly supported cohomology of open modular curves, and provide a powerful tool for studying the arithmetic properties of critical values of the L-functions attached to modular forms. Our basic observation is that in the present anticyclotomic setting, Heegner points on definite Shimura curves provide a similarly convenient way of describing the *central* critical values of the Rankin L-series $L(f/K, \chi, s)$.

Also fundamental for the method of [Emerton et al. 2006] is the possibility to "deform" modular symbols in Hida families. In our anticyclotomic context, the construction of big Heegner points in Hida families was obtained in the work [Longo and Vigni 2011] of one of us in collaboration with Vigni, while the relation between these points and Rankin–Selberg L-values was established in the work [Castella and Longo 2016] by two of us. With these key results at hand, and working over appropriate quotients of the Hecke algebras considered in [Emerton et al. 2006] via the Jacquet–Langlands correspondence, we are then able to develop analogues of the arguments of loc. cit. in our setting, making use of the ramification hypotheses on $\bar{\rho}$ to ensure a multiplicity one property of certain Hecke modules, similarly as in the works of Pollack and Weston [2011] and one of us [Kim 2017].

We conclude this introduction with an overview of the contents of the paper. In the next section, we briefly recall the Hida theory that we need, following the exposition in [Emerton et al. 2006, §1] for the most part. In Section 2, we describe a key extension of the construction of big Heegner points of [Longo and Vigni 2011] to "imprimitive" branches of the Hida family. In Section 3, we construct two-variable p-adic L-functions attached to a Hida family and to each of its irreducible components (or branches), and prove Theorem 3.10 relating the two. This theorem is the key technical result of this paper, and the analytic part of Theorem 2 follows easily from this. In Section 4, we deduce the algebraic part of Theorem 2 using the residual Selmer groups studied in [Pollack and Weston 2011, §3.2]. Finally, in Section 5 we give the applications of our results to the anticyclotomic Iwasawa main conjecture.

1. Hida theory

Throughout this section, we fix a positive integer N admitting a factorization

$$N = N^+N^-$$

with $(N^+, N^-) = 1$ and N^- equal to the square-free product of an *odd* number of primes. We also fix a prime $p \nmid 6N$.

1A. ***Hecke algebras.*** For each integer $k \geqslant 2$, denote by $\mathfrak{h}_{N,r,k}$ the $\mathbb{Z}_p$-algebra generated by the Hecke operators T_ℓ for $\ell \nmid Np$, the operators U_ℓ for $\ell \mid Np$, and the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/p^r\mathbb{Z})^\times$, acting on the space $S_k(\Gamma_{0,1}(N, p^r), \overline{\mathbb{Q}}_p)$ of cusp forms of weight k on $\Gamma_{0,1}(N, p^r) := \Gamma_0(N) \cap \Gamma_1(p^r)$. For $k = 2$, we abbreviate $\mathfrak{h}_{N,r} := \mathfrak{h}_{N,r,2}$.

Let $e^{\mathrm{ord}} := \lim_{n\to\infty} U_p^{n!}$ be Hida's ordinary projector, and define

$$\mathfrak{h}^{\mathrm{ord}}_{N,r,k} := e^{\mathrm{ord}}\mathfrak{h}_{N,r,k}, \qquad \mathfrak{h}^{\mathrm{ord}}_{N,r} := e^{\mathrm{ord}}\mathfrak{h}_{N,r}, \qquad \mathfrak{h}^{\mathrm{ord}}_{N} := \varprojlim_r \mathfrak{h}^{\mathrm{ord}}_{N,r},$$

where the limit is over the projections induced by the natural restriction maps.

Denote by $\mathbb{T}^{N^-}_{N,r,k}$ the quotient of $\mathfrak{h}^{\mathrm{ord}}_{N,r,k}$ acting faithfully on the subspace of $e^{\mathrm{ord}}S_k(\Gamma_{0,1}(N, p^r), \overline{\mathbb{Q}}_p)$ consisting of forms which are new at all primes dividing N^-. Set $\mathbb{T}^{N^-}_{N,r} := \mathbb{T}^{N^-}_{N,r,2}$ and define

$$\mathbb{T}^{N^-}_{N} := \varprojlim_r \mathbb{T}^{N^-}_{N,r}.$$

Each of these Hecke algebras is equipped with natural $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-algebra structures via the diamond operators, and by a well-known result of Hida, $\mathfrak{h}^{\mathrm{ord}}_N$ is finite and flat over $\mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$.

1B. ***Galois representations on Hecke algebras.*** For each positive integer $M \mid N$ we may consider the new quotient $\mathbb{T}^{\mathrm{new}}_M$ of $\mathfrak{h}^{\mathrm{ord}}_M$, and the Galois representation

$$\rho_M : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}^{\mathrm{new}}_M \otimes \mathcal{L})$$

described in [Emerton et al. 2006, Theorem 2.2.1], where $\mathcal{L}$ denotes the fraction field of $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$.

Let $\mathbb{T}'_N$ be the $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$-subalgebra of $\mathbb{T}_N^{N^-}$ generated by the image under the natural projection $\mathfrak{h}_N^{\mathrm{ord}} \to \mathbb{T}_N^{N^-}$ of the Hecke operators of level prime to N. As in [Emerton et al. 2006, Proposition 2.3.2], one can show that the canonical map

$$\mathbb{T}'_N \to \prod_M \mathbb{T}_M^{\mathrm{new}},$$

where the product is over all integers $M \geqslant 1$ with $N^- \mid M \mid N$, becomes an isomorphism after tensoring with $\mathcal{L}$. Taking the product of the Galois representations ρ_M we thus obtain

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}'_N \otimes \mathcal{L}).$$

For any maximal ideal $\mathfrak{m}$ of $\mathbb{T}'_N$, let $(\mathbb{T}'_N)_{\mathfrak{m}}$ denote the localization of $\mathbb{T}'_N$ at $\mathfrak{m}$ and let

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GL}_2((\mathbb{T}'_N)_{\mathfrak{m}} \otimes \mathcal{L})$$

be the resulting Galois representation. If the residual representation $\bar{\rho}_{\mathfrak{m}}$ is irreducible, then $\rho_{\mathfrak{m}}$ admits an integral model (still denoted in the same manner)

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GL}_2((\mathbb{T}'_N)_{\mathfrak{m}})$$

which is unique up to isomorphism.

1C. ***Residual representations.*** Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an odd irreducible Galois representation defined over a finite field $\mathbb{F}$ of characteristic $p > 3$. As in the introduction, we assume that $\bar{\rho} \simeq \bar{\rho}_{f_0}$ for some newform f_0 of weight 2, level N, and trivial nebentypus. Consider the following three conditions we may impose on the pair $(\bar{\rho}, N^-)$:

Assumption (SU). (1) $\bar{\rho}$ is *p-ordinary*: the restriction of $\bar{\rho}$ to a decomposition group $D_p \subseteq G_{\mathbb{Q}}$ at p has a one-dimensional unramified quotient over $\mathbb{F}$;

(2) $\bar{\rho}$ is *p-distinguished*: $\bar{\rho}|_{D_p} \sim \left(\begin{smallmatrix} \bar{\varepsilon} & * \\ 0 & \bar{\delta} \end{smallmatrix}\right)$ with $\bar{\varepsilon} \neq \bar{\delta}$;

(3) $\bar{\rho}$ is ramified at every prime $\ell \mid N^-$.

Fix once and for all a representation $\bar{\rho}$ satisfying Assumption (SU), together with a p-stabilization of $\bar{\rho}$ in the sense of [Emerton et al. 2006, Definition 2.2.10]. Let $\overline{V}$ be the two-dimensional $\mathbb{F}$-vector space which affords $\bar{\rho}$, and for any finite set of primes Σ that does not contain p or any factor of N^-, define

$$N(\Sigma) := N(\bar{\rho}) \prod_{\ell \in \Sigma} \ell^{m_\ell}, \tag{1}$$

where $N(\bar{\rho})$ is the tame conductor of $\bar{\rho}$, and $m_\ell := \dim_{\mathbb{F}} \overline{V}_{I_\ell}$.

Remark 1.1. By Assumption (SU) we have the divisibility $N^- \mid N(\bar\rho)$; we will further assume that $(N^-, N(\bar\rho)/N^-) = 1$.

Combining [Emerton et al. 2006, Theorem 2.4.1] and [Emerton et al. 2006, Proposition 2.4.2] with the fact that $\bar\rho$ is ramified at the primes dividing N^-, one can see that there exist unique maximal ideals $\mathfrak{n}$ and $\mathfrak{m}$ of $\mathbb{T}^{N^-}_{N(\Sigma)}$ and $\mathbb{T}'_{N(\Sigma)}$, respectively, such that

- $\mathfrak{n} \cap \mathbb{T}'_{N(\Sigma)} = \mathfrak{m}$;
- $(\mathbb{T}'_{N(\Sigma)})_{\mathfrak{m}} \simeq (\mathbb{T}^{N^-}_{N(\Sigma)})_{\mathfrak{n}}$ by the natural map on localizations;
- $\bar\rho_{\mathfrak{m}} \simeq \bar\rho$.

Define the ordinary Hecke algebra $\mathbb{T}_\Sigma$ attached to $\bar\rho$ and Σ by

$$\mathbb{T}_\Sigma := (\mathbb{T}'_{N(\Sigma)})_{\mathfrak{m}}.$$

Thus $\mathbb{T}_\Sigma$ is a local factor of $\mathbb{T}'_{N(\Sigma)}$, and we let

$$\rho_\Sigma : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}_\Sigma)$$

denote the Galois representation deduced from $\rho_{\mathfrak{m}}$.

Adopting the terminology of [Emerton et al. 2006, §2.4], we shall refer to $\mathrm{Spec}(\mathbb{T}_\Sigma)$ as "the Hida family" $\mathcal{H}^-(\bar\rho)$ attached to $\bar\rho$ (and our chosen p-stabilization) that is minimally ramified outside Σ.

Remark 1.2. Note that by Assumption (SU), all the p-stabilized newforms in $\mathcal{H}^-(\bar\rho)$ have tame level divisible by N^-.

1D. ***Branches of the Hida family.*** If $\mathfrak{a}$ is a minimal prime of $\mathbb{T}_\Sigma$ (for a finite set of primes Σ as above), we put $\mathbb{T}(\mathfrak{a}) := \mathbb{T}_\Sigma/\mathfrak{a}$ and let

$$\rho(\mathfrak{a}) : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}(\mathfrak{a}))$$

be the Galois representation induced by ρ_Σ. As in [Emerton et al. 2006, Proposition 2.5.2], one can show that there is a unique divisor $N(\mathfrak{a})$ of $N(\Sigma)$ and a unique minimal prime $\mathfrak{a}' \subseteq \mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}$ above $\mathfrak{a}$ such that the diagram

$$\begin{tikzcd}
\mathbb{T}_\Sigma \arrow[r] \arrow[d] & \mathbb{T}'_{N(\Sigma)} \arrow[r] & \prod_{N^- \mid M \mid N(\Sigma)} \mathbb{T}^{\mathrm{new}}_M \arrow[d] \\
\mathbb{T}_\Sigma/\mathfrak{a} \arrow[r, "="] & \mathbb{T}(\mathfrak{a}) \arrow[r] & \mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}'
\end{tikzcd}$$

commutes. We call $N(\mathfrak{a})$ the *tame conductor* of $\mathfrak{a}$ and set

$$\mathbb{T}(\mathfrak{a})^\circ := \mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}'.$$

In particular, note that $N^- \mid N(\mathfrak{a})$ by construction, and that the natural map $\mathbb{T}(\mathfrak{a}) \to \mathbb{T}(\mathfrak{a})^\circ$ is an embedding of local domains.

1E. *Arithmetic specializations.* For any finite $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$-algebra $\mathbb{T}$, we say that a height one prime $\wp$ of $\mathbb{T}$ is an *arithmetic prime* of $\mathbb{T}$ if $\wp$ is the kernel of a $\mathbb{Z}_p$-algebra homomorphism $\mathbb{T}\to\overline{\mathbb{Q}}_p$ such that the composite map

$$1+p\mathbb{Z}_p\to\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]^\times\to\mathbb{T}^\times\to\overline{\mathbb{Q}}_p^\times$$

is given by $\gamma\mapsto\gamma^{k-2}$ on some open subgroup of $1+p\mathbb{Z}_p$, for some integer $k\geqslant 2$. We then say that $\wp$ has *weight* k.

Let $\mathfrak{a}\subseteq\mathbb{T}_\Sigma$ be a minimal prime as above. For each $n\geqslant 1$, let $\boldsymbol{a}_n\in\mathbb{T}(\mathfrak{a})^\circ$ be the image of T_n under the natural projection $\mathfrak{h}^{\mathrm{ord}}_{N(\Sigma)}\to\mathbb{T}(\mathfrak{a})^\circ$, and form the q-expansion

$$\boldsymbol{f}(\mathfrak{a})=\sum_{n\geqslant 1}\boldsymbol{a}_n q^n\in\mathbb{T}(\mathfrak{a})^\circ[\![q]\!].$$

By [Hida 1986, Theorem 1.2], if $\wp$ is an arithmetic prime of $\mathbb{T}(\mathfrak{a})$ of weight k, then there is a unique height one prime $\wp'$ of $\mathbb{T}(\mathfrak{a})^\circ$ such that

$$\boldsymbol{f}_\wp(\mathfrak{a}):=\sum_{n\geqslant 1}(\boldsymbol{a}_n \bmod \wp')q^n\in\mathcal{O}^\circ_\wp[\![q]\!],$$

where $\mathcal{O}^\circ_\wp:=\mathbb{T}(\mathfrak{a})^\circ/\wp'$, is the q-expansion of a p-ordinary eigenform $f_\wp$ of weight k and tame level $N(\mathfrak{a})$ (see [Emerton et al. 2006, Proposition 2.5.6]).

2. Big Heegner points

As in Section 1, we fix an integer $N\geqslant 1$ admitting a factorization $N=N^+N^-$ with $(N^+,N^-)=1$ and N^- equal to the square-free product of an *odd* number of primes, and fix a prime $p\nmid 6N$. Also, we let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D_K<0$ prime to Np and such that every prime factor of N^+ (resp. N^-) splits (resp. is inert) in K.

In this section we describe a mild extension of the construction in [Longo and Vigni 2011] (following [Howard 2007]) of big Heegner points attached to K. Indeed, using the results from the preceding section, we can extend the constructions of *loc.cit.* to branches of the Hida family which are *not necessarily* primitive (in the sense of [Hida 1986, §1]). The availability of such an extension is fundamental for the purposes of this paper.

2A. *Definite Shimura curves.* Let B be the definite quaternion algebra over $\mathbb{Q}$ of discriminant N^-. We fix once and for all an embedding of $\mathbb{Q}$-algebras $K\hookrightarrow B$, and use it to identity K with a subalgebra of B. Denote by $z\mapsto\bar z$ the nontrivial automorphism of K, and choose a basis $\{1,j\}$ of B over K such that

- $j^2=\beta\in\mathbb{Q}^\times$ with $\beta<0$;
- $jt=\bar t j$ for all $t\in K$;
- $\beta\in(\mathbb{Z}_q^\times)^2$ for $q\mid pN^+$, and $\beta\in\mathbb{Z}_q^\times$ for $q\mid D_K$.

Fix a square-root $\delta_K = \sqrt{-D_K}$, and define $\boldsymbol{\theta} \in K$ by

$$\boldsymbol{\theta} := \tfrac{1}{2}D' + \delta_K, \quad \text{where } D' := \begin{cases} D_K & \text{if } 2 \nmid D_K; \\ \tfrac{1}{2}D_K & \text{if } 2 \mid D_K. \end{cases}$$

Note that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\boldsymbol{\theta}$, and for every prime $q \mid pN^+$, define $i_q : B_q := B \otimes_{\mathbb{Q}} \mathbb{Q}_q \simeq \mathrm{M}_2(\mathbb{Q}_q)$ by

$$i_q(\boldsymbol{\theta}) = \begin{pmatrix} \mathrm{Tr}(\boldsymbol{\theta}) & -\mathrm{Nm}(\boldsymbol{\theta}) \\ 1 & 0 \end{pmatrix}, \qquad i_q(j) = \sqrt{\beta} \begin{pmatrix} -1 & \mathrm{Tr}(\boldsymbol{\theta}) \\ 0 & 1 \end{pmatrix},$$

where Tr and Nm are the reduced trace and reduced norm maps on B, respectively. On the other hand, for each prime $q \nmid Np$ we fix any isomorphism $i_q : B_q \simeq \mathrm{M}_2(\mathbb{Q}_q)$ with the property that $i_q(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_q) \subset \mathrm{M}_2(\mathbb{Z}_q)$.

For each $r \geqslant 0$, let $R_{N^+,r}$ be the Eichler order of B of level N^+p^r with respect to the above isomorphisms $\{i_q : B_q \simeq \mathrm{M}_2(\mathbb{Q}_q)\}_{q \nmid N^-}$, and let $U_{N^+,r}$ be the compact open subgroup of $\widehat{R}^{\times}_{N^+,r}$ defined by

$$U_{N^+,r} := \left\{ (x_q)_q \in \widehat{R}^{\times}_{N^+,r} \;\middle|\; i_p(x_p) \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{p^r} \right\}.$$

Consider the double coset spaces

$$\widetilde{X}_{N^+,r} = B^{\times} \backslash (\mathrm{Hom}_{\mathbb{Q}}(K, B) \times \widehat{B}^{\times}) / U_{N^+,r}, \tag{2}$$

where $b \in B^{\times}$ acts on $(\Psi, g) \in \mathrm{Hom}_{\mathbb{Q}}(K, B) \times \widehat{B}^{\times}$ by

$$b \cdot (\Psi, g) = (b\Psi b^{-1}, bg)$$

and $U_{N^+,r}$ acts on $\widehat{B}^{\times}$ by right multiplication. As is well known (see, e.g., [Longo and Vigni 2011, §2.1]), $\widetilde{X}_{N^+,r}$ may be naturally identified with the set of K-rational points of certain genus zero curves defined over $\mathbb{Q}$. Nonetheless, there is a nontrivial Galois action on $\widetilde{X}_{N^+,r}$ defined as follows: If $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $P \in \widetilde{X}_{N^+,r}$ is the class of a pair (Ψ, g), then

$$P^{\sigma} := [(\Psi, \widehat{\Psi}(a)g)],$$

where $a \in K^{\times} \backslash \widehat{K}^{\times}$ is chosen so that $\mathrm{rec}_K(a) = \sigma$. It will be convenient to extend this action to an action of $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ in the obvious manner.

Finally, we note that $\widetilde{X}_{N^+,r}$ is also equipped with standard actions of U_p, Hecke operators T_ℓ for $\ell \nmid Np$, and diamond operators $\langle d \rangle$ for $d \in (\mathbb{Z}/p^r\mathbb{Z})^{\times}$ (see [Longo and Vigni 2011, §2.4], for example).

2B. ***Compatible systems of Heegner points.*** For each integer $c \geqslant 1$, let $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ be the order of K of conductor c.

Definition 2.1. We say that a point $P \in \widetilde{X}_{N^+,r}$ is a *Heegner point of conductor c* if P is the class of a pair (Ψ, g) with

$$\Psi(\mathcal{O}_c) = \Psi(K) \cap (B \cap g\widehat{R}_{N^+,r}g^{-1})$$

and

$$\Psi_p((\mathcal{O}_c \otimes \mathbb{Z}_p)^\times \cap (1 + p^r\mathcal{O}_K \otimes \mathbb{Z}_p)^\times) = \Psi_p((\mathcal{O}_c \otimes \mathbb{Z}_p)^\times) \cap g_p U_{N^+,r,p} g_p^{-1},$$

where $U_{N^+,r,p}$ denotes the p-component of $U_{N^+,r}$.

Fix a decomposition $N^+\mathcal{O}_K = \mathfrak{N}^+\overline{\mathfrak{N}^+}$, and for each prime $q \neq p$ define

- $\varsigma_q = 1$, if $q \nmid N^+$;
- $\varsigma_q = \delta_K^{-1}\begin{pmatrix}\boldsymbol{\theta} & \bar{\boldsymbol{\theta}} \\ 1 & 1\end{pmatrix} \in \mathrm{GL}_2(K_\mathfrak{q}) = \mathrm{GL}_2(\mathbb{Q}_q)$, if $q = \mathfrak{q}\bar{\mathfrak{q}}$ splits with $\mathfrak{q} \mid \mathfrak{N}^+$,

and for each $s \geqslant 0$, let

- $\varsigma_p^{(s)} = \begin{pmatrix}\boldsymbol{\theta} & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}p^s & 0 \\ 0 & 1\end{pmatrix} \in \mathrm{GL}_2(K_\mathfrak{p}) = \mathrm{GL}_2(\mathbb{Q}_p)$, if $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K;
- $\varsigma_p^{(s)} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}p^s & 0 \\ 0 & 1\end{pmatrix}$, if p is inert in K.

Set $\varsigma^{(s)} := \varsigma_p^{(s)}\prod_{q\neq p}\varsigma_q$, viewed as an element in $\widehat{B}^\times$ via the isomorphisms $\{i_q : B_q \simeq M_2(\mathbb{Q}_q)\}_{q\nmid N^-}$ introduced in Section 2A. Let $\iota_K : K \hookrightarrow B$ be the inclusion. Then one easily checks (see [Castella and Longo 2016, Theorem 1.2]) that for all $n, r \geqslant 0$ the points

$$\widetilde{P}_{p^n,r} := [(\iota_K, \varsigma^{(n+r)})] \in \widetilde{X}_{N^+,r}$$

are Heegner points of conductor p^{n+r} with the following properties:

- *Field of definition*: $\widetilde{P}_{p^n,r} \in H^0(L_{p^n,r}, \widetilde{X}_{N^+,r})$, where $L_{p^n,r} := H_{p^{n+r}}(\boldsymbol{\mu}_{p^r})$ and H_c is the ring class field of K of conductor c.
- *Galois equivariance*: for all $\sigma \in \mathrm{Gal}(L_{p^n,r}/H_{p^{n+r}})$, we have

  $$\widetilde{P}^\sigma_{p^n,r} = \langle\vartheta(\sigma)\rangle \cdot \widetilde{P}_{p^n,r},$$

  where $\vartheta : \mathrm{Gal}(L_{p^n,r}/H_{p^{n+r}}) \to \mathbb{Z}_p^\times/\{\pm 1\}$ is such that $\vartheta^2 = \varepsilon_{\mathrm{cyc}}$.
- *Horizontal compatibility*: if $r > 1$, then

  $$\sum_{\sigma\in\mathrm{Gal}(L_{p^n,r}/L_{p^{n-1},r})} \widetilde{\alpha}_r(\widetilde{P}^\sigma_{p^n,r}) = U_p \cdot \widetilde{P}_{p^n,r-1},$$

  where $\widetilde{\alpha}_r : \widetilde{X}_{N^+,r} \to \widetilde{X}_{N^+,r-1}$ is the map induced by the inclusion $U_{N^+,r} \subseteq U_{N^+,r-1}$.

- *Vertical Compatibility*: if $n > 0$, then

$$\sum_{\sigma\in\mathrm{Gal}(L_{p^n,r}/L_{p^{n-1},r})} \widetilde{P}^{\sigma}_{p^n,r} = U_p \cdot \widetilde{P}_{p^{n-1},r}.$$

Remark 2.2. We will only consider the points $\widetilde{P}_{p^n,r}$ for a fixed a value of N^- (which amounts to fixing the quaternion algebra $B/\mathbb{Q}$), but it will be fundamental to consider *different* values of N^+, and the relations between the corresponding $\widetilde{P}_{p^n,r}$ (which clearly depend on N^+) under various degeneracy maps.

2C. ***Critical character.*** Factor the p-adic cyclotomic character as

$$\varepsilon_{\mathrm{cyc}} = \varepsilon_{\mathrm{tame}} \cdot \varepsilon_{\mathrm{wild}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times \simeq \boldsymbol{\mu}_{p-1} \times (1 + p\mathbb{Z}_p)$$

and define the *critical character* $\Theta : G_{\mathbb{Q}} \to \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]^\times$ by

$$\Theta(\sigma) = [\varepsilon^{1/2}_{\mathrm{wild}}(\sigma)], \tag{3}$$

where $\varepsilon^{1/2}_{\mathrm{wild}}$ is the unique square root of $\varepsilon_{\mathrm{wild}}$ taking values in $1 + p\mathbb{Z}_p$, and the map $[\,\cdot\,] : 1 + p\mathbb{Z}_p \to \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]^\times$ is given by the inclusion as group-like elements.

2D. ***Big Heegner points.*** Recall the Shimura curves $\widetilde{X}_{N^+,p^r}$ from Section 2A, and set

$$\mathfrak{D}_{N^+,r} := e^{\mathrm{ord}}(\mathrm{Div}(\widetilde{X}_{N^+,r}) \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

By the Jacquet–Langlands correspondence, $\mathfrak{D}_{N^+,r}$ is naturally endowed with an action of the Hecke algebra $\mathbb{T}^{N^-}_{N,r}$. Let $(\mathbb{T}^{N^-}_{N,r})^\dagger$ be the free $\mathbb{T}^{N^-}_{N,r}$-module of rank one equipped with the Galois action via the inverse of the critical character Θ, and set

$$\mathfrak{D}^\dagger_{N^+,r} := \mathfrak{D}_{N^+,r} \otimes_{\mathbb{T}^{N^-}_{N,r}} (\mathbb{T}^{N^-}_{N,r})^\dagger.$$

Let $\widetilde{P}_{p^n,r} \in \widetilde{X}_{N^+,r}$ be the system of Heegner points of Section 2B, and denote by $\mathcal{P}_{p^n,r}$ the image of $e^{\mathrm{ord}}\widetilde{P}_{p^n,r}$ in $\mathfrak{D}_{N^+,r}$. By the Galois equivariance of $\widetilde{P}_{p^n,r}$ (see [Longo and Vigni 2011, §7.1]), we have

$$\mathcal{P}^{\sigma}_{p^n,r} = \Theta(\sigma) \cdot \mathcal{P}_{p^n,r}$$

for all $\sigma \in \mathrm{Gal}(L_{p^n,r}/H_{p^{n+r}})$, and hence $\mathcal{P}_{p^n,r}$ defines an element

$$\mathcal{P}_{p^n,r} \otimes \zeta_r \in H^0(H_{p^{n+r}}, \mathfrak{D}^\dagger_{N^+,r}). \tag{4}$$

In the next section we shall see how this system of points, for varying n and r, can be used to construct various anticyclotomic p-adic L-functions.

3. Anticyclotomic p-adic L-functions

3A. ***Multiplicity one.*** Keep the notation introduced in Section 2. For each integer $k \geqslant 2$, denote by $L_k(R)$ the set of polynomials of degree less than or equal to $k-2$ with coefficients in a ring R, and define

$$\mathfrak{J}_{N^+,r,k} := e^{\mathrm{ord}} H_0(\widetilde{X}_{N^+,r}, \mathcal{L}_k(\mathbb{Z}_p)),$$

where $\mathcal{L}_k(\mathbb{Z}_p)$ is the local system on $\widetilde{X}_{N^+,r}$ associated with $L_k(\mathbb{Z}_p)$. The module $\mathfrak{J}_{N^+,r,k}$ is endowed with an action of the Hecke algebra $\mathbb{T}^{N^-}_{N,r,k}$ and with perfect "intersection pairing":

$$\langle\,\cdot\,,\,\cdot\,\rangle_k : \mathfrak{J}_{N^+,r,k} \times \mathfrak{J}_{N^+,r,k} \to \mathbb{Q}_p \tag{5}$$

(see [Chida and Hsieh 2016, Equation (3.9)]) with respect to which the Hecke operators are self-adjoint.

Theorem 3.1. *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{N^-}_{N,r,k}$ whose residual representation is irreducible and satisfies Assumption (SU). Then $(\mathfrak{J}_{N^+,r,k})_{\mathfrak{m}}$ is free of rank one over $(\mathbb{T}^{N^-}_{N,r,k})_{\mathfrak{m}}$. In particular, there is a $(\mathbb{T}^{N^-}_{N,r,k})_{\mathfrak{m}}$-module isomorphism*

$$(\mathfrak{J}_{N^+,r,k})_{\mathfrak{m}} \overset{\alpha_{N,r,k}}{\simeq} (\mathbb{T}^{N^-}_{N,r,k})_{\mathfrak{m}}.$$

Proof. If $k=2$ and $r=1$, this follows by combining Theorem 6.2 and Proposition 6.5 of [Pollack and Weston 2011]. The general case will be deduced from this case in Section 3C using Hida theory. □

Let $f \in S_k(\Gamma_{0,1}(N, p^r))$ be an N^--new eigenform, and suppose that $\mathfrak{m}$ is the maximal ideal of $\mathbb{T}^{N^-}_{N,r,k}$ containing the kernel of the associated $\mathbb{Z}_p$-algebra homomorphism

$$\pi_f : (\mathbb{T}^{N^-}_{N,r,k})_{\mathfrak{m}} \to \mathcal{O},$$

where $\mathcal{O}$ is the finite extension of $\mathbb{Z}_p$ generated by the Fourier coefficients of f. Composing π_f with an isomorphism $\alpha_{N,r,k}$ as in Theorem 3.1, we obtain an $\mathcal{O}$-valued functional

$$\psi_f : (\mathfrak{J}_{N^+,r,k})_{\mathfrak{m}} \to \mathcal{O}.$$

By the duality (5), the map ψ_f corresponds to a generator g_f of the π_f-isotypical component of $\mathfrak{J}_{N^+,r,k}$, and following [Pollack and Weston 2011, §2.1] and [Chida and Hsieh 2016, §4.1] we define the *Gross period* Ω_{f,N^-} attached to f by

$$\Omega_{f,N^-} := \frac{(f,f)_{\Gamma_0(N)}}{\langle g_f, g_f\rangle_k}. \tag{6}$$

Remark 3.2. By Vatsal's work [2003] (see also [Pollack and Weston 2011, Theorem 2.3] and [Chida and Hsieh 2016, §5.4]), the anticyclotomic p-adic L-functions

$L_p(f/K)$ in Theorem 3.14 below (normalized by the complex period Ω_{f,N^-}) have vanishing μ-invariant. The preceding uniform description of ψ_f for all f with a common maximal ideal $\mathfrak{m}$ will allow us to show that this property is preserved in Hida families.

3B. ***One-variable p-adic L-functions.*** Denote by Γ the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension K_∞/K. For each n, let $K_n \subset K_\infty$ be defined by $\mathrm{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and let Γ_n be the subgroup of Γ such that $\Gamma/\Gamma_n \simeq \mathrm{Gal}(K_n/K)$.

Let $\mathcal{P}_{p^{n+1},r} \otimes \zeta_r \in H^0(H_{p^{n+1+r}}, \mathfrak{D}^\dagger_{N^+,r})$ be the Heegner point of conductor p^{n+1}, and define

$$\mathcal{Q}_{n,r} := \mathrm{Cor}_{H_{p^{n+1+r}}/K_n}(\mathcal{P}_{p^{n+1},r} \otimes \zeta_r) \in H^0(K_n, \mathfrak{D}^\dagger_{N^+,r}); \tag{7}$$

with a slight abuse of notation, we also denote by $\mathcal{Q}_{n,r}$ its image under the natural map

$$H^0(K_n, \mathfrak{D}^\dagger_{N^+,r}) \xrightarrow{\subseteq} \mathfrak{D}_{N^+,r} \longrightarrow \mathfrak{J}_{N^+,r}$$

composed with localization at $\mathfrak{m}$, where $\mathfrak{J}_{N^+,r} := \mathfrak{J}_{N^+,r,2}$.

Definition 3.3. For any open subset $\sigma\Gamma_n$ of Γ, define

$$\mu_r(\sigma\Gamma_n) := U_p^{-n} \cdot \mathcal{Q}^\sigma_{n,r} \in (\mathfrak{J}_{N^+,r})_{\mathfrak{m}}.$$

Proposition 3.4. *The rule μ_r is a measure on Γ.*

Proof. This follows immediately from the "horizontal compatibility" of Heegner points. □

3C. ***Gross periods in Hida families.*** Keep the notation of Section 3A, and let

$$(\mathfrak{J}_{N^+})_{\mathfrak{m}} := \varprojlim_r (\mathfrak{J}_{N^+,r})_{\mathfrak{m}},$$

which is naturally equipped with an action of the big Hecke algebra $\mathbb{T}^{N^-}_N = \varprojlim_r \mathbb{T}^{N^-}_{N,r}$.

Theorem 3.5. *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{N^-}_N$ whose residual representation is irreducible and satisfies Assumption (SU). Then $(\mathfrak{J}_{N^+})_{\mathfrak{m}}$ is free of rank one over $(\mathbb{T}^{N^-}_N)_{\mathfrak{m}}$. In particular, there is a $(\mathbb{T}^{N^-}_N)_{\mathfrak{m}}$-module isomorphism*

$$(\mathfrak{J}_{N^+})_{\mathfrak{m}} \overset{\alpha_N}{\simeq} (\mathbb{T}^{N^-}_N)_{\mathfrak{m}}.$$

Proof. As in [Emerton et al. 2006, Proposition 3.3.1]. Note that the version of Hida's control theorem in our context is provided by [Hida 1988, Theorem 9.4]. □

We can now conclude the proof of Theorem 3.1 just as in [Emerton et al. 2006, §3.3]. For the convenience of the reader, we include the argument here.

Proof of Theorem 3.1. Let $\wp_{N,r,k}$ be the product of all the arithmetic primes of $\mathbb{T}_N^{N^-}$ of weight k which become trivial upon restriction to $1 + p^r\mathbb{Z}_p$. By [Hida 1988, Theorem 9.4], we then have

$$(\mathfrak{J}_{N^+})_{\mathfrak{m}} \otimes \mathbb{T}_N^{N^-}/\wp_{N,r,k} \simeq (\mathfrak{J}_{N^+,r,k})_{\mathfrak{m}_{r,k}}, \tag{8}$$

where $\mathfrak{m}_{r,k}$ is the maximal ideal of $\mathbb{T}_{N,r,k}^{N^-}$ induced by $\mathfrak{m}$. Since $(\mathfrak{J}_{N^+})_{\mathfrak{m}}$ is free of rank one over $\mathbb{T}_N^{N^-}$ by Theorem 3.5, it follows that $(\mathfrak{J}_{N^+,r,k})_{\mathfrak{m}_{r,k}}$ is free of rank one over $\mathbb{T}_N^{N^-}/\wp_{N,r,k} \simeq \mathbb{T}_{N,r,k}^{N^-}$, as was to be shown. $\square$

Remark 3.6. In the above proofs we made crucial use of [Hida 1988, Theorem 9.4], which is stated in the context of totally definite quaternion algebras which are unramified at all finite places, since this is the only relevant case for the study of Hilbert modular forms over totally real number fields of even degree. However, the proofs immediately extend to the (simpler) situation of definite quaternion algebras over $\mathbb{Q}$.

3D. *Two-variable p-adic L-functions.* By the "vertical compatibility" satisfied by Heegner points, the points

$$U_p^{-r} \cdot \mathcal{Q}_{n,r} \in (\mathfrak{J}_{N^+,r})_{\mathfrak{m}}$$

are compatible for varying r, thus defining an element

$$\mathcal{Q}_n := \varprojlim_r U_p^{-r} \cdot \mathcal{Q}_{n,r} \in (\mathfrak{J}_{N^+})_{\mathfrak{m}}.$$

Definition 3.7. For any open subset $\sigma\Gamma_n$ of Γ, define

$$\mu(\sigma\Gamma_n) := U_p^{-n} \cdot \mathcal{Q}_n^{\sigma} \in (\mathfrak{J}_{N^+})_{\mathfrak{m}}.$$

Proposition 3.8. *The rule μ is a measure on Γ.*

Proof. This follows immediately from the "horizontal compatibility" of Heegner points. $\square$

Upon the choice of an isomorphism α_N as in Theorem 3.5, we may regard μ as an element

$$\mathcal{L}(\mathfrak{m}, N) \in (\mathbb{T}_N^{N^-})_{\mathfrak{m}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!].$$

Denoting by $\mathcal{L}(\mathfrak{m}, N)^*$ the image of $\mathcal{L}(\mathfrak{m}, N)$ under the involution induced by $\gamma \mapsto \gamma^{-1}$ on group-like elements, we set

$$L(\mathfrak{m}, N) := \mathcal{L}(\mathfrak{m}, N) \cdot \mathcal{L}(\mathfrak{m}, N)^*,$$

to which we will refer as the *two-variable p-adic L-function attached to* $(\mathbb{T}_N^{N^-})_{\mathfrak{m}}$.

3E. ***Two-variable p-adic L-functions on branches of the Hida family.*** Let $\mathbb{T}_\Sigma$ be the universal p-ordinary Hecke algebra

$$\mathbb{T}_\Sigma := (\mathbb{T}'_{N(\Sigma)})_{\mathfrak{m}} \simeq (\mathbb{T}^{N^-}_{N(\Sigma)})_{\mathfrak{n}} \tag{9}$$

associated with a mod p representation $\bar{\rho}$ and a finite set of primes Σ as in Section 1C.

Remark 3.9. Recall that $N^- \mid N(\bar{\rho})$ by Assumption (SU). Throughout the following, it will be further assumed that every prime factor of $N(\Sigma)/N^-$ splits in K. In particular, every prime $\ell \in \Sigma$ splits in K, and any $f \in \mathcal{H}^-(\bar{\rho}) = \mathrm{Spec}(\mathbb{T}_\Sigma)$ has tame level N_f with

$$N_f^- = N(\bar{\rho})^- = N^-.$$

The construction of the preceding section produces a two-variable p-adic L-function

$$L(\mathfrak{n}, N(\Sigma)) \in (\mathbb{T}^{N^-}_{N(\Sigma)})_{\mathfrak{n}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!],$$

which combined with the isomorphism (9) yields an element

$$L_\Sigma(\bar{\rho}) \in \mathbb{T}_\Sigma \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!].$$

If $\mathfrak{a}$ is a minimal prime of $\mathbb{T}_\Sigma$, we thus obtain an element

$$L_\Sigma(\bar{\rho}, \mathfrak{a}) \in \mathbb{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!]$$

by reducing $L_\Sigma(\bar{\rho})$ mod $\mathfrak{a}$ (see Section 1D). On the other hand, if we let $\mathfrak{m}$ denote the inverse image of the maximal ideal of $\mathbb{T}(\mathfrak{a})^\circ$ under the composite surjection

$$\mathbb{T}^{N^-}_{N(\mathfrak{a})} \to \mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})} \to \mathbb{T}^{\mathrm{new}}_{N(\mathfrak{a})}/\mathfrak{a}' = \mathbb{T}(\mathfrak{a})^\circ, \tag{10}$$

the construction of the preceding section yields an L-function

$$L(\mathfrak{m}, N(\mathfrak{a})) \in (\mathbb{T}^{N^-}_{N(\mathfrak{a})})_{\mathfrak{m}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!]$$

giving rise, via (10), to a second element

$$L(\bar{\rho}, \mathfrak{a}) \in \mathbb{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[\![\Gamma]\!].$$

It is natural to compare $L_\Sigma(\bar{\rho}, \mathfrak{a})$ and $L(\bar{\rho}, \mathfrak{a})$, a task that is carried out in the next section, and provides the key for understanding the variation of *analytic* Iwasawa invariants.

3F. *Comparison.* Write $\Sigma = \{\ell_1, \dots, \ell_n\}$ and for each $\ell = \ell_i \in \Sigma$, let e_ℓ be the valuation of $N(\Sigma)/N(\mathfrak{a})$ at ℓ, and define the reciprocal Euler factor $E_\ell(\mathfrak{a}, X) \in \mathbb{T}(\mathfrak{a})^\circ[X]$ by

$$E_\ell(\mathfrak{a}, X) := \begin{cases} 1 & \text{if } e_\ell = 0; \\ 1 - (T_\ell \bmod \mathfrak{a}')\Theta^{-1}(\ell)X & \text{if } e_\ell = 1; \\ 1 - (T_\ell \bmod \mathfrak{a}')\Theta^{-1}(\ell)X + \ell X^2 & \text{if } e_\ell = 2. \end{cases}$$

Also, writing $\ell = \mathfrak{l}\bar{\mathfrak{l}}$, define $E_\ell(\mathfrak{a}) \in \mathbb{T}(\mathfrak{a})^\circ \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]]$ by

$$E_\ell(\mathfrak{a}) := E_\ell(\mathfrak{a}, \ell^{-1}\gamma_{\mathfrak{l}}) \cdot E_\ell(\mathfrak{a}, \ell^{-1}\gamma_{\bar{\mathfrak{l}}}), \tag{11}$$

where $\gamma_{\mathfrak{l}}, \gamma_{\bar{\mathfrak{l}}}$ are arithmetic Frobenius maps at $\mathfrak{l}, \bar{\mathfrak{l}}$ in Γ, respectively, and put $E_\Sigma(\mathfrak{a}) := \prod_{\ell \in \Sigma} E_\ell(\mathfrak{a})$.

Recall that $N^- \mid N(\mathfrak{a}) \mid N(\Sigma)$ and set

$$N(\mathfrak{a})^+ := N(\mathfrak{a})/N^-, \qquad N(\Sigma)^+ := N(\Sigma)/N^-,$$

both of which consist entirely of prime factors which split in K. The purpose of this section is to prove the following result.

Theorem 3.10. *There is an isomorphism of $\mathbb{T}(\mathfrak{a})^\circ$-modules*

$$\mathbb{T}(\mathfrak{a})^\circ \otimes_{(\mathbb{T}^{N^-}_{N(\Sigma)})_{\mathfrak{n}}} (\mathfrak{J}_{N(\Sigma)^+})_{\mathfrak{n}} \simeq \mathbb{T}(\mathfrak{a})^\circ \otimes_{(\mathbb{T}^{N^-}_{N(\mathfrak{a})})_{\mathfrak{m}}} (\mathfrak{J}_{N(\mathfrak{a})^+})_{\mathfrak{m}}$$

such that the map induced on the corresponding spaces of measures valued in these modules sends $L_\Sigma(\bar{\rho}, \mathfrak{a})$ to $E_\Sigma(\mathfrak{a}) \cdot L(\bar{\rho}, \mathfrak{a})$.

Proof. The proof follows closely the constructions and arguments in [Emerton et al. 2006, §3.8], suitably adapted to the quaternionic setting at hand. Let $r \geqslant 1$. If M is any positive integer with $(M, pN^-) = 1$, and $d' \mid d$ are divisors of M, we have degeneracy maps

$$B_{d,d'} : \widetilde{X}_{M,r} \to \widetilde{X}_{M/d,r}$$

induced by $(\Psi, g) \mapsto (\Psi, \pi_{d'} g)$, where $\pi_{d'} \in \widehat{B}^\times$ has local component $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell^{\mathrm{val}_\ell(d')} \end{smallmatrix}\right)$ at every prime $\ell \mid d'$ and 1 outside d'. We thus obtain a map on homology

$$(B_{d,d'})_* : e^{\mathrm{ord}} H_0(\widetilde{X}_{M,r}, \mathbb{Z}_p) \to e^{\mathrm{ord}} H_0(\widetilde{X}_{M/d,r}, \mathbb{Z}_p)$$

and we may define

$$\epsilon_r : e^{\mathrm{ord}} H_0(\widetilde{X}_{N(\Sigma)^+,r}, \mathbb{Z}_p) \to e^{\mathrm{ord}} H_0(\widetilde{X}_{N(\mathfrak{a})^+,r}, \mathbb{Z}_p) \tag{12}$$

by $\epsilon_r := \epsilon(\ell_n) \circ \cdots \circ \epsilon(\ell_1)$, where for every $\ell = \ell_i \in \Sigma$ we put

$$\epsilon(\ell) := \begin{cases} 1 & \text{if } e_\ell = 0; \\ (B_{\ell,1})_* - (B_{\ell,\ell})_* \ell^{-1} T_\ell & \text{if } e_\ell = 1; \\ (B_{\ell^2,1})_* - (B_{\ell^2,\ell})_* \ell^{-1} T_\ell + (B_{\ell^2,\ell^2})_* \ell^{-1} \langle \ell \rangle_{N(\mathfrak{a})p} & \text{if } e_\ell = 2. \end{cases}$$

As before, let M be a positive integer with $(M, pN^-) = 1$ all of whose prime factors split in K, and let $\ell \nmid Mp$ be a prime which also splits in K. We shall adopt the following simplifying notation for the system of points $\widetilde{P}_{p^n,r} \in \widetilde{X}_{N^+,r}$ constructed in Section 2B:

$$P := \widetilde{P}^{(M)}_{p^n,r} \in \widetilde{X}_{M,r}, \quad P^{(\ell)} := \widetilde{P}^{(M\ell)}_{p^n,r} \in \widetilde{X}_{M\ell,r}, \quad P^{(\ell^2)} := \widetilde{P}^{(M\ell^2)}_{p^n,r} \in \widetilde{X}_{M\ell^2,r}.$$

It is easy to check that we have the following relations in $\widetilde{X}_{M,r}$:

$$(B_{\ell,1})_*(P^{(\ell)}) = P, \qquad (B_{\ell,\ell})_*(P^{(\ell)}) = P^{\sigma_{\mathfrak{l}}}, \quad (B_{\ell^2,1})_*(P^{(\ell^2)}) = P,$$
$$(B_{\ell^2,\ell})_*(P^{(\ell^2)}) = P^{\sigma_{\mathfrak{l}}}, \quad (B_{\ell^2,\ell^2})_*(P^{(\ell^2)}) = P^{\sigma_{\mathfrak{l}}^2},$$

where $\sigma_{\mathfrak{l}} \in \mathrm{Gal}(L_{p^n,r}/K)$ is a Frobenius element at a prime $\mathfrak{l} \mid \ell$. Letting $\mathcal{P}$ denote the image of $e^{\mathrm{ord}}P$ in $\mathfrak{D}_{M,r}$, and defining $\mathcal{P}^{(\ell)} \in \mathfrak{D}_{M\ell,r}$ and $\mathcal{P}^{(\ell^2)} \in \mathfrak{D}_{M\ell^2,r}$ similarly, it follows that

$$\begin{aligned}
(B_{\ell,1})_*(\mathcal{P}^{(\ell)} \otimes \zeta_r) &= \mathcal{P} \otimes \zeta_r,\\
(B_{\ell,\ell})_*(\mathcal{P}^{(\ell)} \otimes \zeta_r) &= \mathcal{P}^{\sigma_{\mathfrak{l}}} \otimes \zeta_r \ = \Theta^{-1}(\sigma_{\mathfrak{l}}) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_{\mathfrak{l}}},\\
(B_{\ell^2,1})_*(\mathcal{P}^{(\ell^2)} \otimes \zeta_r) &= \mathcal{P} \otimes \zeta_r,\\
(B_{\ell^2,\ell})_*(\mathcal{P}^{(\ell^2)} \otimes \zeta_r) &= \mathcal{P}^{\sigma_{\mathfrak{l}}} \otimes \zeta_r \ = \Theta^{-1}(\sigma_{\mathfrak{l}}) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_{\mathfrak{l}}},\\
(B_{\ell^2,\ell^2})_*(\mathcal{P}^{(\ell^2)} \otimes \zeta_r) &= \mathcal{P}^{\sigma_{\mathfrak{l}}^2} \otimes \zeta_r = \Theta^{-2}(\sigma_{\mathfrak{l}}) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_{\mathfrak{l}}}
\end{aligned}$$

as elements in $\mathfrak{D}^{\dagger}_{M,r}$. Finally, setting $\mathcal{Q} := \mathrm{Cor}_{H_{p^{n+1+r}}/K_n}(\mathcal{P}) \in H^0(K_n, \mathfrak{D}^{\dagger}_{M,r})$, and defining $\mathcal{Q}^{(\ell)} \in H^0(K_n, \mathfrak{D}^{\dagger}_{M\ell,r})$ and $\mathcal{Q}^{(\ell^2)} \in H^0(K_n, \mathfrak{D}^{\dagger}_{M\ell^2,r})$ similarly, we see that

$$(B_{\ell,1})_*(\mathcal{Q}^{(\ell)}) = \mathcal{Q}, \qquad (B_{\ell,\ell})_*(\mathcal{Q}^{(\ell)}) = \Theta^{-1}(\sigma_{\mathfrak{l}}) \cdot \mathcal{Q}^{\sigma_{\mathfrak{l}}},$$
$$(B_{\ell^2,1})_*(\mathcal{Q}^{(\ell^2)}) = \mathcal{Q},$$
$$(B_{\ell^2,\ell})_*(\mathcal{Q}^{(\ell^2)}) = \Theta^{-1}(\sigma_{\mathfrak{l}}) \cdot \mathcal{Q}^{\sigma_{\mathfrak{l}}}, \quad (B_{\ell^2,\ell^2})_*(\mathcal{Q}^{(\ell^2)}) = \Theta^{-2}(\sigma_{\mathfrak{l}}) \cdot \mathcal{Q}^{\sigma_{\mathfrak{l}}^2}$$

in $H^0(K_n, \mathfrak{D}^{\dagger}_{M,r})$. Each of these equalities is checked by an explicit calculation. For example, for the second one:

$$\begin{aligned}
(B_{\ell,\ell})_*(\mathcal{Q}^{(\ell)}) &= (B_{\ell,\ell})_*(\mathrm{Cor}_{H_{p^{n+1+r}}/K_n}(\mathcal{P}^{(\ell)} \otimes \zeta_r))\\
&= (B_{\ell,\ell})_*\Biggl(\Biggl(\sum_{\sigma \in \mathrm{Gal}(H_{p^{n+1+r}}/K_n)} \Theta(\tilde{\sigma}^{-1}) \cdot (\mathcal{P}^{(\ell)})^{\tilde{\sigma}}\Biggr) \otimes \zeta_r\Biggr)\\
&= \sum_{\sigma \in \mathrm{Gal}(H_{p^{n+1+r}}/K_n)} \Theta(\tilde{\sigma}^{-1}) \cdot (B_{\ell,\ell})_*((\mathcal{P}^{(\ell)})^{\tilde{\sigma}} \otimes \zeta_r)\\
&= \sum_{\sigma \in \mathrm{Gal}(H_{p^{n+1+r}}/K_n)} \Theta(\tilde{\sigma}^{-1})\Theta^{-1}(\sigma_{\mathfrak{l}}) \cdot (\mathcal{P}^{\tilde{\sigma}} \otimes \zeta_r)^{\sigma_{\mathfrak{l}}}\\
&= \Theta^{-1}(\sigma_{\mathfrak{l}}) \cdot \mathcal{Q}^{\sigma_{\mathfrak{l}}}.
\end{aligned}$$

Now let $\mathcal{Q}_{n,r} \in \mathfrak{J}_{N(\Sigma)^+,r}$ be as in (7) with $N = N(\Sigma)$. Using the above formulae, we easily see that of any finite order character χ of Γ of conductor p^n, the effect of ϵ_r on the element $\sum_{\sigma\in\Gamma/\Gamma_n} \chi(\sigma)\mathcal{Q}_{n,r}^{\sigma}$ is given by multiplication by

$$\prod_{e_{\ell_i}=1} (1-(\chi\Theta)^{-1}(\sigma_{\mathfrak{l}_i})\ell_i^{-1}T_{\ell_i}) \prod_{e_{\ell_i}=2} (1-(\chi\Theta)^{-1}(\sigma_{\mathfrak{l}_i})\ell_i^{-1}T_{\ell_i} + (\chi\Theta)^{-2}(\sigma_{\mathfrak{l}_i})\ell_i^{-1}\langle\ell_i\rangle_{N(\mathfrak{a})p}).$$

Similarly, we see ϵ_r has the effect of multiplying the element $\sum\limits_{\sigma\in\Gamma/\Gamma_n} \chi^{-1}(\sigma)\mathcal{Q}_{n,r}^{\sigma}$ by

$$\prod_{e_{\ell_i}=1} (1-(\chi^{-1}\Theta)^{-1}(\sigma_{\mathfrak{l}_i})\ell_i^{-1}T_{\ell_i}) \prod_{e_{\ell_i}=2} (1-(\chi^{-1}\Theta)^{-1}(\sigma_{\mathfrak{l}_i})\ell_i^{-1}T_{\ell_i} + (\chi^{-1}\Theta)^{-2}(\sigma_{\mathfrak{l}_i})\ell_i^{-1}\langle\ell_i\rangle_{N(\mathfrak{a})p}).$$

Hence, using the relations

$$\chi(\sigma_{\bar{\mathfrak{l}}_i}) = \chi^{-1}(\sigma_{\mathfrak{l}_i}), \qquad \Theta(\sigma_{\mathfrak{l}_i}) = \Theta(\sigma_{\bar{\mathfrak{l}}_i}) = \theta(\ell_i), \qquad \theta^2(\ell_i) = \langle\ell_i\rangle_{N(\mathfrak{a})p},$$

it follows that the effect of ϵ_r on the product of the above two elements is given by multiplication by

$$\prod_{\substack{\mathfrak{l}_i|\ell_i\\ e_{\ell_i}=1}} (1-\chi(\sigma_{\mathfrak{l}_i})\theta^{-1}(\ell_i)\ell_i^{-1}T_{\ell_i}) \prod_{\substack{\mathfrak{l}_i|\ell_i\\ e_{\ell_i}=2}} (1-\chi(\sigma_{\mathfrak{l}_i})\theta^{-1}(\ell_i)\ell_i^{-1}T_{\ell_i} + \chi^2(\sigma_{\mathfrak{l}_i})\ell_i^{-1}).$$

Taking the limit over r, we thus obtain a $\mathbb{T}(\mathfrak{a})^{\circ}$-linear map

$$\mathbb{T}(\mathfrak{a})^{\circ} \otimes_{(\mathbb{T}^{N^-}_{N(\Sigma)})_{\mathfrak{n}}} (\mathfrak{J}_{N(\Sigma)^+})_{\mathfrak{n}} \to \mathbb{T}(\mathfrak{a})^{\circ} \otimes_{(\mathbb{T}^{N^-}_{N(\mathfrak{a})})_{\mathfrak{m}}} (\mathfrak{J}_{N(\mathfrak{a})^+})_{\mathfrak{m}} \tag{13}$$

having an effect on the corresponding measures as stated in Theorem 3.10. Hence to conclude the proof it remains to show that (13) is an isomorphism.

By Theorem 3.5, both the source and the target of this map are free of rank one over $\mathbb{T}(\mathfrak{a})^{\circ}$, and as in [Emerton et al. 2006, p. 559] (using [Hida 1988, Theorem 9.4]), one is reduced to showing the injectivity of the dual map modulo p:

$$\begin{aligned} H^0(\widetilde{X}_{N(\mathfrak{a})^+,1};\mathbb{F}_p)^{\mathrm{ord}}[\mathfrak{m}] &\to (\mathbb{T}^{N^-}_{N(\mathfrak{a})}/\mathfrak{m}) \otimes_{\mathbb{T}^{N^-}_{N(\Sigma)}/\mathfrak{n}} (H^0(\widetilde{X}_{N(\mathfrak{a})^+,1};\mathbb{F}_p)^{\mathrm{ord}}[\mathfrak{m}']) \\ &\to (\mathbb{T}^{N^-}_{N(\mathfrak{a})}/\mathfrak{m}) \otimes_{\mathbb{T}^{N^-}_{N(\Sigma)}/\mathfrak{n}} (H^0(\widetilde{X}_{N(\Sigma)^+,1};\mathbb{F}_p)^{\mathrm{ord}}[\mathfrak{m}']) \\ &\to (\mathbb{T}^{N^-}_{N(\mathfrak{a})}/\mathfrak{m}) \otimes_{\mathbb{T}^{N^-}_{N(\Sigma)}/\mathfrak{n}} (H^0(\widetilde{X}_{N(\Sigma)^+,1};\mathbb{F}_p)^{\mathrm{ord}}[\mathfrak{n}]); \end{aligned} \tag{14}$$

or equivalently (by a version of [Emerton et al. 2006, Lemma 3.8.1]), to showing that the composite of the first two arrows in (14) is injective.

In turn, the latter injectivity follows from Lemma 3.11 below, where the notations are as follows:

- M^+ is any positive integer with $(M^+, pN^-) = 1$;
- $\ell \neq p$ is a prime;
- $n_\ell = 1$ or 2 according to whether or not ℓ divides M^+;
- $N^+ := \ell^{n_\ell} M^+$;

and

$$\epsilon_\ell^* : H^0(\widetilde{X}_{M^+,1}; \mathbb{F}_p)^{\mathrm{ord}}[\mathfrak{m}] \to (\mathbb{T}^{N^-}_{M^+N^-}/\mathfrak{m}) \otimes_{\mathbb{T}'_{N^+N^-}/\mathfrak{m}'} (H^0(\widetilde{X}_{N^+,1}; \mathbb{F}_p)^{\mathrm{ord}}[\mathfrak{m}']) \quad (15)$$

is the map defined by

$$\epsilon_\ell^* := \begin{cases} B^*_{\ell,1} - B^*_{\ell,\ell}\ell^{-1}T_\ell & \text{if } n_\ell = 1; \\ B^*_{\ell^2,1} - B^*_{\ell^2,\ell}\ell^{-1}T_\ell + B^*_{\ell^2,\ell^2}\ell^{-1}\langle \ell \rangle_{N(\mathfrak{a})p} & \text{if } n_\ell = 2. \end{cases}$$

Lemma 3.11. *The map* (15) *is injective.*

Proof. As in the proof of the analogous result [Emerton et al. 2006, Lemma 3.8.2] in the modular curve case, it suffices to show the injectivity of the map

$$(H^0(\widetilde{X}_{M^+,1}; \mathbb{F})^{\mathrm{ord}}[\mathfrak{m}_{\mathbb{F}}])^{n_\ell+1} \xrightarrow{\beta_\ell} H^0(\widetilde{X}_{N^+,1}; \mathbb{F})^{\mathrm{ord}}[\mathfrak{m}'_{\mathbb{F}}]$$

defined by

$$\beta_\ell := \begin{cases} B^*_{\ell,1}\pi_1 + B^*_{\ell,\ell}\pi_2 & \text{if } n_\ell = 1; \\ B^*_{\ell^2,1}\pi_1 + B^*_{\ell^2,\ell}\pi_2 + B^*_{\ell^2,\ell^2}\pi_3 & \text{if } n_\ell = 2. \end{cases}$$

But in our quaternionic setting the proof of this injectivity follows from [Skinner and Wiles 1999, Lemma 3.26] for $n_\ell = 1$ and [*loc.cit.*, Lemma 3.28] for $n_\ell = 2$. □

Applying inductively Lemma 3.11 to the primes in Σ, the proof of Theorem 3.10 follows. □

3G. ***Analytic Iwasawa invariants.*** Upon the choice of an isomorphism

$$\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$$

we may regard the p-adic L-functions $L_\Sigma(\bar{\rho}, \mathfrak{a})$ and $L(\bar{\rho}, \mathfrak{a})$, as well as the Euler factor $E_\Sigma(\bar{\rho}, \mathfrak{a})$, as elements in $\mathbb{T}(\mathfrak{a})^\circ[[T]]$. In this section we apply the main result of the preceding section to study the variation of the Iwasawa invariants attached to the anticyclotomic p-adic L-functions of p-ordinary modular forms.

For any power series $f(T) \in R[[T]]$ with coefficients in a ring R, the *content* of $f(T)$ is defined to be the ideal $I(f(T)) \subseteq R$ generated by the coefficients of $f(T)$. If $\wp$ is a height one prime of $\mathbb{T}_\Sigma$ belonging to the branch $\mathbb{T}(\mathfrak{a})$ (in the sense that $\mathfrak{a}$ is the unique minimal prime of $\mathbb{T}_\Sigma$ contained in $\wp$), we denote by $L(\bar{\rho}, \mathfrak{a})(\wp)$ the element of $\mathcal{O}_\wp[[\Gamma]]$ obtained from $L(\bar{\rho}, \mathfrak{a})$ via reduction modulo $\wp$. In particular, we note that $L(\bar{\rho}, \mathfrak{a})(\wp)$ has unit content if and only if its μ-invariant vanishes.

Theorem 3.12. *The following are equivalent*:

(1) $\mu(L(\bar{\rho}, \mathfrak{a})(\wp)) = 0$ *for some newform* $f_\wp$ *in* $\mathcal{H}^-(\bar{\rho})$;

(2) $\mu(L(\bar{\rho}, \mathfrak{a})(\wp)) = 0$ *for every newform* $f_\wp$ *in* $\mathcal{H}^-(\bar{\rho})$;

(3) $L(\bar{\rho}, \mathfrak{a})$ *has unit content for some irreducible component* $\mathbb{T}(\mathfrak{a})$ *of* $\mathcal{H}^-(\bar{\rho})$;

(4) $L(\bar{\rho}, \mathfrak{a})$ *has unit content for every irreducible component* $\mathbb{T}(\mathfrak{a})$ *of* $\mathcal{H}^-(\bar{\rho})$.

Proof. The argument in [Emerton et al. 2006, Theorem 3.7.5] applies verbatim, replacing the appeal to [*loc.cit.*, Corollary 3.6.3] by our Theorem 3.10 above. □

When any of the conditions in Theorem 3.12 hold, we shall write

$$\mu^{\mathrm{an}}(\bar{\rho}) = 0.$$

For a power series $f(T)$ with unit content and coefficients in a local ring R, the λ-invariant $\lambda(f(T))$ is defined to be the smallest degree in which $f(T)$ has a unit coefficient.

Theorem 3.13. *Assume that* $\mu^{\mathrm{an}}(\bar{\rho}) = 0$.

(1) *Let* $\mathbb{T}(\mathfrak{a})$ *be an irreducible component of* $\mathcal{H}^-(\bar{\rho})$. *As* $\wp$ *varies over the arithmetic primes of* $\mathbb{T}(\mathfrak{a})$, *the* λ-*invariant* $\lambda(L(\bar{\rho}, \mathfrak{a})(\wp))$ *takes on a constant value*, *denoted* $\lambda^{\mathrm{an}}(\bar{\rho}, \mathfrak{a})$.

(2) *For any two irreducible components* $\mathbb{T}(\mathfrak{a}_1)$, $\mathbb{T}(\mathfrak{a}_2)$ *of* $\mathcal{H}^-(\bar{\rho})$, *we have that*

$$\lambda^{\mathrm{an}}(\bar{\rho}, \mathfrak{a}_1) - \lambda^{\mathrm{an}}(\bar{\rho}, \mathfrak{a}_2) = \sum_{\ell \neq p} e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1),$$

where $e_\ell(\mathfrak{a}) = \lambda(E_\ell(\mathfrak{a}))$.

Proof. The first part follows immediately from the definitions. For the second part, the argument in [Emerton et al. 2006, Theorem 3.7.7] applies verbatim, replacing their appeal to [*loc.cit.*, Cor. 3.6.3] by our Theorem 3.10 above. □

By Theorem 3.12 and Theorem 3.13, the Iwasawa invariants of $L(\bar{\rho}, \mathfrak{a})(\wp)$ are well behaved as $\wp$ varies. However, for the applications of these results to the Iwasawa main conjecture it is of course necessary to relate $L(\bar{\rho}, \mathfrak{a})(\wp)$ to p-adic L-functions defined by the interpolation of special values of L-functions. This question was addressed in [Castella and Longo 2016], as we now recall.

Theorem 3.14. *If* $\wp$ *is the arithmetic prime of* $\mathbb{T}(\mathfrak{a})$ *corresponding to a* p-*ordinary* p-*stabilized newform* $f_\wp$ *of weight* $k \geqslant 2$ *and trivial nebentypus*, *then*

$$L(\bar{\rho}, \mathfrak{a})(\wp) = L_p(f_\wp/K),$$

where $L_p(f_\wp/K)$ *is the* p-*adic* L-*function of Chida and Hsieh* [2016]. *In particular*, *if* $\chi : \Gamma \to \mathbb{C}_p^\times$ *is the* p-*adic avatar of an anticyclotomic Hecke character of* K

of infinity type $(m, -m)$ with $-k/2 < m < k/2$, then $L(\bar{\rho}, \mathfrak{a})(\wp)$ interpolates the central critical values

$$\frac{L(f_\wp/K, \chi, k/2)}{\Omega_{f_\wp, N^-}}$$

as χ varies, where $\Omega_{f_\wp,N^-}$ is the complex Gross period (6).

Proof. This is a reformulation of the main result of [Castella and Longo 2016]. (Note that the constant $\lambda_\wp \in F_\wp^\times$ in [Castella and Longo 2016, Theorem. 4.6] is not needed here, since the specialization map of [*loc.cit.*, §3.1] is being replaced by the map $(\mathfrak{J}_{N^+})_{\mathfrak{m}} \to (\mathfrak{J}_{N^+,r,k})_{\mathfrak{m}_{r,k}}$ induced by the isomorphism (8), which preserves integrality.) □

Corollary 3.15. *Let f_1, $f_2 \in \mathcal{H}^-(\bar{\rho})$ be newforms with trivial nebentypus lying in the branches $\mathbb{T}(\mathfrak{a}_1)$, $\mathbb{T}(\mathfrak{a}_2)$, respectively. Then $\mu^{\mathrm{an}}(\bar{\rho}) = 0$ and*

$$\lambda(L_p(f_1/K)) - \lambda(L_p(f_2/K)) = \sum_{\ell \neq p} e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1),$$

where $e_\ell(\mathfrak{a}_j) = \lambda(E_\ell(\mathfrak{a}_j))$.

Proof. By [Chida and Hsieh 2016, Theorem. 5.7] (extending Vatsal's result [2003] to higher weights), if $f \in \mathcal{H}^-(\bar{\rho})$ has weight $k \leqslant p+1$ and trivial nebentypus, then $\mu(L_p(f/K)) = 0$. By Theorems 3.12 and 3.14, this implies $\mu^{\mathrm{an}}(\bar{\rho}) = 0$. The result thus follows from Theorem 3.13, using Theorem 3.14 again to replace $\lambda^{\mathrm{an}}(\bar{\rho}, \mathfrak{a}_j)$ by $\lambda(L_p(f_j/K))$. □

4. Anticyclotomic Selmer groups

We keep the notation of the previous sections. In particular, $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ is an odd irreducible Galois representation satisfying Assumption (SU) and isomorphic to $\bar{\rho}_{f_0}$ for some newform f_0 of weight 2, $\mathcal{H}^-(\bar{\rho})$ is the associated Hida family, and Σ is a finite set of primes split in the imaginary quadratic field K.

For each $f \in \mathcal{H}^-(\bar{\rho})$, let V_f denote the self-dual Tate twist of the p-adic Galois representation associated to f, fix an $\mathcal{O}$-stable lattice $T_f \subseteq V_f$, and set $A_f := V_f/T_f$. Since f is p-ordinary, there is a unique one-dimensional $G_{\mathbb{Q}_p}$-invariant subspace $F_p^+ V_f \subseteq V_f$ where the inertia group at p acts via $\varepsilon_{\mathrm{cyc}}^{k/2}\psi$, with ψ a finite order character. Let $F_p^+ A_f$ be the image of $F_p^+ V_f$ in A_f, and as recalled in the Introduction define the *minimal Selmer group* of f by

$$\mathrm{Sel}(K_\infty, f) := \ker\left\{ H^1(K_\infty, A_f) \to \prod_{w \nmid p} H^1(K_{\infty,w}, A_f) \times \prod_{w \mid p} H^1(K_{\infty,w}, F_p^- A_f) \right\},$$

where w runs over the places of K_∞ and we set $F_p^- A_f := A_f / F_p^+ A_f$.

It is well known that $\mathrm{Sel}(K_\infty, f)$ is cofinitely generated over Λ. When it is also Λ-cotorsion, we define the μ-invariant $\mu(\mathrm{Sel}(K_\infty, f))$ (resp. λ-invariant

$\lambda(\mathrm{Sel}(K_\infty, f)))$ to the largest power of ϖ dividing (resp. the number of zeros of) the characteristic power series of the Pontryagin dual of $\mathrm{Sel}(K_\infty, f)$.

A distinguishing feature of the anticyclotomic setting (in comparison with cyclotomic Iwasawa theory) is the presence of primes which split infinitely in the corresponding $\mathbb{Z}_p$-extension. Indeed, being inert in K, all primes $\ell \mid N^-$ are infinitely split in K_∞/K. As a result, the above Selmer group differs in general from the *Greenberg Selmer group* of f, defined as

$$\mathfrak{Sel}(K_\infty, f) := \ker\left\{H^1(K_\infty, A_f) \to \prod_{w\nmid p} H^1(I_{\infty,w}, A_f) \times \prod_{w\mid p} H^1(K_{\infty,w}, F_p^- A_f)\right\},$$

where $I_{\infty,w} \subseteq G_{K_\infty}$ denotes the inertia group at w.

If S is a finite set of primes in K, we let $\mathrm{Sel}^S(K_\infty, f)$ and $\mathfrak{Sel}^S(K_\infty, f)$ be the "S-primitive" Selmer groups defined as above by omitting the local conditions at the primes in S (except those above p, when any such prime is in S). Moreover, if S consists of the primes dividing a rational integer M, we replace the superscript S by M in the above notation.

Immediately from the definitions, we see that there is as exact sequence

$$0 \to \mathrm{Sel}(K_\infty, f) \to \mathfrak{Sel}(K_\infty, f) \to \prod_{\ell\mid N^-} \mathcal{H}_\ell^{\mathrm{un}}, \tag{16}$$

where

$$\mathcal{H}_\ell^{\mathrm{un}} := \ker\left\{\prod_{w\mid \ell} H^1(K_{\infty,w}, A_f) \to \prod_{w\mid \ell} H^1(I_{\infty,w}, A_f)\right\}$$

is the set of unramified cocycles. In [Pollack and Weston 2011, §§3, 5], Pollack and Weston carried out a careful analysis of the difference between $\mathrm{Sel}(K_\infty, f)$ and $\mathfrak{Sel}(K_\infty, f)$. Even though [loc. cit.] is mostly concerned with cases in which f is of weight 2, many of their arguments apply more generally. In fact, the next result follows essentially from their work.

Theorem 4.1. *Assume that $\bar{\rho}$ satisfies Hypotheses (SU). Then the following are equivalent*:

(1) $\mathrm{Sel}(K_\infty, f_0)$ *is Λ-cotorsion with μ-invariant zero for some newform* $f_0 \in \mathcal{H}^-(\bar{\rho})$;

(2) $\mathrm{Sel}(K_\infty, f)$ *is Λ-cotorsion with μ-invariant zero for all newforms* $f \in \mathcal{H}^-(\bar{\rho})$;

(3) $\mathfrak{Sel}(K_\infty, f)$ *is Λ-cotorsion with μ-invariant zero for all newforms* $f \in \mathcal{H}^-(\bar{\rho})$.

Moreover, *in that case* $\mathrm{Sel}(K_\infty, f) \simeq \mathfrak{Sel}(K_\infty, f)$.

Proof. Assume f_0 is a newform in $\mathcal{H}^-(\bar{\rho})$ for which $\mathrm{Sel}(K_\infty, f_0)$ is Λ-cotorsion with μ-invariant zero, and set $N^+ := N(\Sigma)/N^-$. By [Pollack and Weston 2011, Proposition 5.1], we then have the exact sequences

$$0 \to \mathrm{Sel}(K_\infty, f_0) \to \mathrm{Sel}^{N^+}(K_\infty, f_0) \to \prod_{\ell \mid N^+} \mathcal{H}_\ell \to 0; \tag{17}$$

$$0 \to \mathfrak{Sel}(K_\infty, f_0) \to \mathfrak{Sel}^{N^+}(K_\infty, f_0) \to \prod_{\ell \mid N^+} \mathcal{H}_\ell \to 0, \tag{18}$$

where $\mathcal{H}_\ell$ is the product of $H^1(K_{\infty,w}, A_{f_0})$ over the places $w \mid \ell$ in K_∞. Since every prime $\ell \mid N^+$ splits in K (see Remark 3.9), the Λ-cotorsionness and the vanishing of the μ-invariant of $\mathcal{H}_\ell$ can be deduced from [Greenberg and Vatsal 2000, Proposition 2.4]. Since $\mathrm{Sel}(K_\infty, f_0)[\varpi]$ is finite by assumption, it thus follows from (17) that $\mathrm{Sel}^{N^+}(K_\infty, f_0)[\varpi]$ is finite. Combined with (16) and [Pollack and Weston 2011, Corollary 5.2], the same argument using (18) shows that then $\mathfrak{Sel}^{N^+}(K_\infty, f_0)[\varpi]$ is also finite.

On the other hand, following the arguments in the proof [Pollack and Weston 2011, Proposition 3.6] we see that for any $f \in \mathcal{H}(\bar{\rho})$ we have the isomorphisms

$$\mathrm{Sel}^{N^+}(K_\infty, \bar{\rho}) \simeq \mathrm{Sel}^{N^+}(K_\infty, f)[\varpi]; \quad \mathfrak{Sel}^{N^+}(K_\infty, \bar{\rho}) \simeq \mathfrak{Sel}^{N^+}(K_\infty, f)[\varpi].$$

As a result, the argument in the previous paragraph implies that, for any newform $f \in \mathcal{H}^-(\bar{\rho})$, both $\mathrm{Sel}^{N^+}(K_\infty, f)[\varpi]$ and $\mathfrak{Sel}^{N^+}(K_\infty, f)[\varpi]$ are finite, from where (using (17) and (18) with f in place of f_0) the Λ-cotorsionness and the vanishing of both the μ-invariant of $\mathrm{Sel}(K_\infty, f)$ and of $\mathfrak{Sel}(K_\infty, f)$ follows. In view of (16) and [Pollack and Weston 2011, Lemma 3.4], the result follows. $\square$

Let w be a prime of K_∞ above $\ell \neq p$ and denote by $G_w \subseteq G_{K_\infty}$ its decomposition group. Let $\mathbb{T}(\mathfrak{a})$ be the irreducible component of $\mathbb{T}_\Sigma$ passing through f, and define

$$\delta_w(\mathfrak{a}) := \dim_{\mathbb{F}} A_f^{G_w}/\varpi.$$

(Note that this is well defined by [Emerton et al. 2006, Lemma 4.3.1].) Assume $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ splits in K and put

$$\delta_\ell(\mathfrak{a}) := \sum_{w \mid \ell} \delta_w(\mathfrak{a}), \tag{19}$$

where the sum is over the (finitely many) primes w of K_∞ above ℓ.

In view of Theorem 4.1, we write $\mu^{\mathrm{alg}}(\bar{\rho}) = 0$ whenever any of the μ-invariants appearing in that result vanish. In that case, for any newform f in $\mathcal{H}^-(\bar{\rho})$ we may consider the λ-invariants $\lambda(\mathrm{Sel}(K_\infty, f)) = \lambda(\mathfrak{Sel}(K_\infty, f))$.

Theorem 4.2. *Let $\bar{\rho}$ and Σ be as above, and assume that $\mu^{\mathrm{alg}}(\bar{\rho}) = 0$. If f_1 and f_2 are any two newforms in $\mathcal{H}^-(\bar{\rho})$ lying in the branches $\mathbb{T}(\mathfrak{a}_1)$ and $\mathbb{T}(\mathfrak{a}_2)$, respectively, then*

$$\lambda(\mathrm{Sel}(K_\infty, f_1)) - \lambda(\mathrm{Sel}(K_\infty, f_2)) = \sum_{\ell \neq p} \delta_\ell(\mathfrak{a}_1) - \delta_\ell(\mathfrak{a}_2).$$

Proof. Since we have the divisibilities $N^-|N(\mathfrak{a}_i)|N(\Sigma)$ with the quotient $N(\Sigma)/N^-$ only divisible by primes that are split in K, the arguments of [Emerton et al. 2006, §4] apply verbatim (cf., [Pollack and Weston 2011, Theorem 7.1]). $\square$

5. Applications to the main conjecture

5A. ***Variation of anticyclotomic Iwasawa invariants.*** Recall the definition of the analytic invariant $e_\ell(\mathfrak{a}) = \lambda(E_\ell(\mathfrak{a}))$, where $E_\ell(\mathfrak{a})$ is the Euler factor from Section 3F, and of the algebraic invariant $\delta_\ell(\mathfrak{a})$ introduced in (19).

Lemma 5.1. *Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be minimal primes of $\mathbb{T}_\Sigma$. For any prime $\ell \neq p$ split in K,*

$$\delta_\ell(\mathfrak{a}_1) - \delta_\ell(\mathfrak{a}_2) = e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1).$$

Proof. Let $\mathfrak{a}$ be a minimal prime of $\mathbb{T}_\Sigma$, let f be a newform in the branch $\mathbb{T}(\mathfrak{a})$, and let $\wp_f \subseteq \mathfrak{a}$ be the corresponding height one prime. Since $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ splits in K, we have

$$\bigoplus_{w|\ell} H^1(K_{\infty,w}, A_f) = \Big(\bigoplus_{w|\mathfrak{l}} H^1(K_{\infty,w}, A_f)\Big) \oplus \Big(\bigoplus_{w|\bar{\mathfrak{l}}} H^1(K_{\infty,w}, A_f)\Big)$$

and [Greenberg and Vatsal 2000, Proposition 2.4] immediately implies that

$$Ch_\Lambda\Big(\bigoplus_{w|\ell} H^1(K_{\infty,w}, A_f)^\vee\Big) = E_\ell(f, \ell^{-1}\gamma_{\mathfrak{l}}) \cdot E_\ell(f, \ell^{-1}\gamma_{\bar{\mathfrak{l}}}),$$

where $E_\ell(f, \ell^{-1}\gamma_{\mathfrak{l}}) \cdot E_\ell(f, \ell^{-1}\gamma_{\bar{\mathfrak{l}}})$ is the specialization of $E_\ell(\mathfrak{a})$ at $\wp_f$. The result thus follows from [Emerton et al. 2006, Lemma 5.1.5]. $\square$

Theorem 5.2. *Suppose that $\bar{\rho}$ satisfies Assumption (SU). If for some newform $f_0 \in \mathcal{H}^-(\bar{\rho})$ we have the equalities*

$$\mu(\mathrm{Sel}(K_\infty, f_0)) = \mu(L_p(f_0/K)) = 0 \quad \textit{and} \quad \lambda(\mathrm{Sel}(K_\infty, f_0)) = \lambda(L_p(f_0/K)),$$

then the equalities

$$\mu(\mathrm{Sel}(K_\infty, f)) = \mu(L_p(f/K)) = 0 \quad \textit{and} \quad \lambda(\mathrm{Sel}(K_\infty, f)) = \lambda(L_p(f/K))$$

hold for all newforms $f \in \mathcal{H}^-(\bar{\rho})$.

Proof. Let f be any newform in $\mathcal{H}^-(\bar{\rho})$. Since the algebraic and analytic μ-invariants of f_0 both vanish, the vanishing of $\mu(\mathrm{Sel}(K_\infty, f))$ and $\mu(L_p(f/K))$ follows from Theorems 4.1 and 3.12, respectively. On the other hand, combining Theorems 3.13 and 4.2, and Lemma 5.1, we see that

$$\lambda(\mathrm{Sel}(K_\infty, f)) - \lambda(\mathrm{Sel}(K_\infty, f_0)) = \lambda(L_p(f/K)) - \lambda(L_p(f_0/K)),$$

and hence the equality $\lambda(\mathrm{Sel}(K_\infty, f_0)) = \lambda(L_p(f_0/K))$ implies the same equality for f. $\square$

5B. ***Applications to the main conjecture.*** As an immediate consequence of Weierstrass preparation theorem, Theorem 5.2 together with one the divisibilities predicted by the anticyclotomic main conjecture implies the full anticyclotomic main conjecture.

Theorem 5.3 (Skinner–Urban). *Let $f \in S_k(\Gamma_0(N))$ be a newform of weight $k \equiv 2 \pmod{p-1}$ and trivial nebentypus. Suppose that $\bar{\rho}_f$ satisfies Assumption (SU) and that p splits in K. Then*

$$(L_p(f/K)) \supseteq Ch_\Lambda(\mathrm{Sel}(K_\infty, f)^\vee).$$

Proof. This follows from specializing the divisibility in [Skinner and Urban 2014, Theorem 3.26] to the anticyclotomic line. Indeed, let $\boldsymbol{f} = \sum_{n\geqslant 1} \boldsymbol{a}_n(\boldsymbol{f})q^n \in \mathbb{I}[\![q]\!]$ be the Λ-adic form with coefficients in $\mathbb{I} := \mathbb{T}(\mathfrak{a})^\circ$ associated with the branch of the Hida family containing f, let Σ be a finite set of primes as in Section 3E, let $\Sigma' \supseteq \Sigma$ be a finite set of primes of K containing Σ and all primes dividing $pN(\mathfrak{a})D_K$, and assume that Σ' contains at least one prime $\ell \neq p$ that splits in K. Under these assumptions, in [Skinner and Urban 2014, Theorem 3.26] it is shown that

$$(\mathfrak{L}_p^{\Sigma'}(\boldsymbol{f}/K)) \supseteq Ch_{\Lambda_{\boldsymbol{f}}(L_\infty)}(\mathfrak{Sel}^{\Sigma'}(L_\infty, A_{\boldsymbol{f}})^\vee), \tag{20}$$

where $L_\infty = K_\infty K_{\mathrm{cyc}}$ is the $\mathbb{Z}_p^2$-extension of K, $\Lambda_{\boldsymbol{f}}(L_\infty)$ is the three-variable Iwasawa algebra $\mathbb{I}[\![\mathrm{Gal}(L_\infty/K)]\!]$, and $\mathfrak{L}_p^{\Sigma'}(\boldsymbol{f}/K)$ and $\mathfrak{Sel}^{\Sigma'}(L_\infty, A_{\boldsymbol{f}})$ are the "Σ'-primitive" p-adic L-function and Selmer group defined in [Skinner and Urban 2014, §3.4.5] and [Skinner and Urban 2014, §§3.1.3, 3.1.10], respectively.

Recall the character $\Theta : G_{\mathbb{Q}} \to \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]^\times$ from Section 2C, regarded as a character on $\mathrm{Gal}(L_\infty/K)$, and let

$$\mathrm{Tw}_{\Theta^{-1}} : \Lambda_{\boldsymbol{f}}(L_\infty) \to \Lambda_{\boldsymbol{f}}(L_\infty)$$

be the $\mathbb{I}$-linear isomorphism induced by $\mathrm{Tw}_{\Theta^{-1}}(g) = \Theta^{-1}(g)g$ for $g \in \mathrm{Gal}(L_\infty/K)$. Choose a topological generator $\gamma \in \mathrm{Gal}(K_{\mathrm{cyc}}/K)$, and expand

$$\mathrm{Tw}_{\Theta^{-1}}(\mathfrak{L}_p^{\Sigma'}(\boldsymbol{f}/K)) = \mathfrak{L}_{p,0}^{\Sigma'}(\boldsymbol{f}/K) + \mathfrak{L}_{p,1}^{\Sigma'}(\boldsymbol{f}/K)(\gamma - 1) + \cdots$$

with $\mathfrak{L}_{p,i}^{\Sigma'}(\boldsymbol{f}/K) \in \Lambda_{\boldsymbol{f}}(K_\infty) = \mathbb{I}[\![\Gamma]\!]$. In particular, note that $\mathfrak{L}_{p,0}^{\Sigma'}(\boldsymbol{f}/K)$ is the restriction of the twisted three-variable p-adic L-function $\mathrm{Tw}_{\Theta^{-1}}(\mathfrak{L}_p^{\Sigma'}(\boldsymbol{f}/K))$ to the "self-dual" plane.

Because of our assumptions on f, the Λ-adic form $\boldsymbol{f}$ has trivial tame character, and hence denoting by Frob_ℓ an arithmetic Frobenius at any prime $\ell \nmid N(\mathfrak{a})p$, the Galois representation

$$\rho(\mathfrak{a}) : G_{\mathbb{Q}} \to \mathrm{GL}(T_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{T}(\mathfrak{a})^\circ)$$

considered in Section 1D (which is easily seen to agree with the twisted representation considered in [Skinner and Urban 2014, p. 37]) satisfies

$$\det(X - \mathrm{Frob}_\ell \,|T_{\boldsymbol{f}}) = X^2 - \boldsymbol{a}_\ell(\boldsymbol{f})X + \Theta^2(\ell)\ell.$$

The twist $T_{\boldsymbol{f}}^\dagger := T_{\boldsymbol{f}} \otimes \Theta^{-1}$ is therefore self-dual. Thus combining [Rubin 2000, Lemma 6.1.2] with a straightforward variant of [Skinner and Urban 2014, Proposition 3.9] having $\mathrm{Gal}(K_\infty/K)$ in place of $\mathrm{Gal}(K_{\mathrm{cyc}}/K)$, we see that divisibility (20) implies that

$$(\mathfrak{L}_{p,0}^{\Sigma'}(\boldsymbol{f}/K)) \supseteq Ch_{\Lambda_{\boldsymbol{f}}(K_\infty)}(\mathfrak{Sel}^{\Sigma'}(K_\infty, A_{\boldsymbol{f}}^\dagger)^\vee). \tag{21}$$

(Here, as above, $A_{\boldsymbol{f}}$ denotes the Pontryagin dual $T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{I}, \mathbb{Q}_p/\mathbb{Z}_p)$, and $A_{\boldsymbol{f}}^\dagger$ is the corresponding twist.) We next claim that, setting $\Sigma'' := \Sigma' \setminus \Sigma$, we have

$$(\mathfrak{L}_{p,0}^{\Sigma'}(\boldsymbol{f}/K)) = \biggl(L_\Sigma(\bar{\rho}, \mathfrak{a}) \cdot \prod_{v \in \Sigma'', v \nmid p} E_v(\mathfrak{a})\biggr), \tag{22}$$

where $L_\Sigma(\bar{\rho}, \mathfrak{a})$ is the two-variable p-adic L-function constructed in Section 3D, and if v lies over the rational prime ℓ, $E_v(\mathfrak{a})$ is the Euler factor given by

$$E_v(\mathfrak{a}) = \det(\mathrm{Id} - \mathrm{Frob}_v\, X|(V_{\boldsymbol{f}}^\dagger)_{I_v})_{X=\ell^{-1}\mathrm{Frob}_v},$$

where $V_{\boldsymbol{f}} := T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathrm{Frac}(\mathbb{I})$, and Frob_v is an arithmetic Frobenius at v. (Note that for $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ split in K, $E_{\mathfrak{l}}(\mathfrak{a}) \cdot E_{\bar{\mathfrak{l}}}(\mathfrak{a})$ is simply the Euler factor (11).) Indeed, combined with Theorems 3.10 and 3.14, equality (22) specialized to any arithmetic prime $\wp \subseteq \mathbb{T}(\mathfrak{a})$ of weight 2 is shown in [Skinner and Urban 2014, (12.3)], from where the claim follows easily from the density of these primes. (See also [Pollack and Weston 2011, Theorem 6.8] for the comparison between the different periods involved in the two constructions, which differ by a p-adic unit under our assumptions.)

Finally, (21) and (22) combined with Theorem 3.10 and [Greenberg and Vatsal 2000, Propositions 2.3,8] imply that

$$(L(\bar{\rho}, \mathfrak{a})) \supseteq Ch_{\Lambda_{\boldsymbol{f}}(K_\infty)}(\mathfrak{Sel}(K_\infty, A_{\boldsymbol{f}}^\dagger)^\vee),$$

from where the result follows by specializing at $\wp_f$ using Theorem 3.14 and Theorem 4.1. $\square$

In the opposite direction, we have the following result:

Theorem 5.4 (Bertolini–Darmon). *Let $f = \sum_{n=1}^\infty a_n(f)q^n$ be a p-ordinary newform of weight 2, level N, and trivial nebentypus. Suppose that $\bar{\rho}_f$ satisfies Assumption (CR) and that*

$$a_p(f) \not\equiv \pm 1 \pmod{p}. \tag{PO}$$

Then

$$(L_p(f/K)) \subseteq Ch_\Lambda(\mathrm{Sel}(K_\infty, f)^\vee).$$

Proof. This is the main result of [Bertolini and Darmon 2005], as extended by Pollack and Weston [Pollack and Weston 2011] to newforms of weight 2 not necessarily defined over $\mathbb{Q}$ and under the stated hypotheses (weaker that in [Bertolini and Darmon 2005]) on $\bar{\rho}_f$. See also [Kim et al. 2017] for a detailed discussion on the additional "nonanomalous" hypothesis (PO) on f. □

Before we combine the previous two theorems with our main results in this paper, we note that condition (PO) in Theorem 5.4 can be phrased in terms of the Galois representation ρ_f associated to f. Indeed, let $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be a p-ordinary newform as above, defined over a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Then

$$\rho_f|_{D_p} \simeq \begin{pmatrix} \varepsilon & * \\ 0 & \delta \end{pmatrix}$$

on a decomposition group $D_p \subseteq G_{\mathbb{Q}}$ at p, with $\delta : D_p \to \mathcal{O}^\times$ an unramified character sending Frob_p to the unit root α_p of $X^2 - a_p(f)X + p$. Since clearly $\alpha \equiv a_p(f) \pmod{p}$, we see that condition (PO) amounts to the requirement that

$$\delta(\mathrm{Frob}_p) \not\equiv \pm 1 \pmod{p}. \tag{PO}$$

Now we are finally in a position to prove our main application to the anticyclotomic Iwasawa main conjecture for p-ordinary newforms.

Corollary 5.5. *Suppose that $\bar{\rho}$ satisfies Assumptions* (SU) *and* (PO) *and that p splits in K, and let f be a newform in $\mathcal{H}^-(\bar{\rho})$ of weight $k \equiv 2 \pmod{p-1}$ and trivial nebentypus. Then the anticyclotomic Iwasawa main conjecture holds for f.*

Proof. After Theorems 5.2 and 5.3, to check the anticyclotomic main conjecture for *any* newform f as in the statement, it suffices to check the three equalities

$$\mu(\mathrm{Sel}(K_\infty, f_0)) = \mu(L_p(f_0/K)) = 0, \quad \lambda(\mathrm{Sel}(K_\infty, f_0)) = \lambda(L_p(f_0/K)) \tag{23}$$

hold for *some* $f_0 \in \mathcal{H}^-(\bar{\rho})$ of weight $k \equiv 2 \pmod{p-1}$ and trivial nebentypus.

Let $\mathbb{T}(\mathfrak{a})$ be the irreducible component of $\mathcal{H}^-(\bar{\rho})$ containing f, and let $f_0 \in S_2(\Gamma_0(Np))$ be the p-stabilized newform corresponding to an arithmetic prime $\wp \subseteq \mathbb{T}(\mathfrak{a})$ of weight 2 and trivial nebentypus. By Assumption (PO), the form f_0 is necessarily the p-stabilization of a p-ordinary newform $f_0^\sharp \in S_2(\Gamma_0(N))$ (see, e.g., [Howard 2007, Lemma 2.1.5]). From the combination of Theorems 5.3 and 5.4, the anticyclotomic Iwasawa main conjecture holds for $f_0^\sharp$, and since we clearly have

$$L_p(f_0/K) = L_p(f_0^\sharp/K) \quad \text{and} \quad \mathrm{Sel}(K_\infty, f_0) \simeq \mathrm{Sel}(K_\infty, f_0^\sharp)$$

(note that the latter isomorphism relies on the absolute irreducibility of $\bar{\rho}$), the anticyclotomic Iwasawa main conjecture holds for f_0 as well. In particular, equalities (23) hold for this f_0, and the result follows. □

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