Algebra & Number Theory

Volume 11

2017

No. 2

A generalization of Kato's local ε-conjecture for (φ, Γ)-modules over the Robba ring

Kentaro Nakamura

msp

ALGEBRA AND NUMBER THEORY 11:2 (2017)
dx.doi.org/10.2140/ant.2017.11.319

A generalization of Kato's local ε-conjecture for (φ, Γ)-modules over the Robba ring

Kentaro Nakamura

We generalize Kato's (commutative) p-adic local ε-conjecture for families of (φ, Γ)-modules over the Robba rings. In particular, we prove the essential parts of the generalized local ε-conjecture for families of trianguline (φ, Γ)-modules. The key ingredients are the author's previous work on the Bloch–Kato exponential map for (φ, Γ)-modules and the recent results of Kedlaya, Pottharst and Xiao on the finiteness of cohomology of (φ, Γ)-modules.

1. Introduction 319
2. Cohomology and Bloch–Kato exponential of (φ, Γ)-modules 325
3. Local ε-conjecture for (φ, Γ)-modules over the Robba ring 350
4. Rank-one case 368
Appendix: Explicit calculations of $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L)$ and $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(1))$ 397
Acknowledgements 402
References 402

1. Introduction

1A. ***Introduction.*** Since the works of Kisin [2003], Colmez [2008], and Bellaïche and Chenevier [2009], among others, the theory of (φ, Γ)-modules over the (relative) Robba ring has become one of the main focuses in the theory of p-adic Galois representations. In particular, the trianguline representation, which is a class of p-adic Galois representations defined using (φ, Γ)-modules over the Robba ring, is important since the rigid analytic families of p-adic Galois representations associated to Coleman–Mazur eigencurves (or more general eigenvarieties) turn out to be trianguline.

The recent works of Pottharst [2013] and Kedlaya, Pottharst and Xiao [Kedlaya et al. 2014] established the fundamental theorems (comparison with Galois cohomology, finiteness, base change property, Tate duality, Euler–Poincaré formula) in

MSC2010: primary 11F80; secondary 11F85, 11S25.
Keywords: p-adic Hodge theory, (φ, Γ)-module, B-pair.

the theory of the cohomology of (φ, Γ)-modules over the relative Robba ring over $\mathbb{Q}_p$-affinoid algebras. As is suggested and actually given in [Kedlaya et al. 2014; Pottharst 2012], their results are expected to have many applications in number theory (e.g., eigenvarieties, nonordinary case of Iwasawa theory; see Remarks 1.6 and 1.7 below).

On the other hand, in [Nakamura 2014a], we generalized the theory of Bloch–Kato exponential maps and Perrin-Riou's exponential maps in the framework of (φ, Γ)-modules over the Robba ring. Since these maps are very important tools in Iwasawa theory, we expect that the results of [Nakamura 2014a] also have many applications.

As an application of both theories, the purpose of this article is to generalize Kato's p-adic local ε-conjecture [1993b] in the framework of (φ, Γ)-modules over the relative Robba ring over $\mathbb{Q}_p$-affinoid algebras, and prove the essential parts of its generalized version of the conjecture for rigid analytic families of trianguline (φ, Γ)-modules.

In this introduction, we briefly explain these conjectures; see Section 3 for the precise definitions. Let $G_{\mathbb{Q}_p}$ be the absolute Galois group of $\mathbb{Q}_p$. The main objects of Kato's local ε-conjecture are the pairs (Λ, T), where Λ is a semilocal ring such that $\Lambda/\mathfrak{m}_\Lambda$ is a finite ring of order a power of p (where $\mathfrak{m}_\Lambda$ is the Jacobson radical of Λ) and T is a Λ-representation of $G_{\mathbb{Q}_p}$, i.e., a finite projective Λ-module with a continuous Λ-linear $G_{\mathbb{Q}_p}$-action. Let $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T)$ be the complex of continuous cochains of $G_{\mathbb{Q}_p}$ with values in T. By the classical theory of Galois cohomology of $G_{\mathbb{Q}_p}$, this complex is a perfect complex of Λ-modules which satisfies the base change property, Tate duality, and other properties. This fact enables us to define the determinant

$$\mathrm{Det}_\Lambda(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T)),$$

which is a (graded) invertible Λ-module. Modifying this module by multiplying a kind of $\det_\Lambda(T)$, one can canonically define a graded invertible Λ-module

$$\Delta_\Lambda(T),$$

called the fundamental line of the pair (Λ, T), which is compatible with base change and Tate duality.

Our main objects are the pairs (A, M), where A is a $\mathbb{Q}_p$-affinoid and M is a (φ, Γ)-module over the relative Robba ring $\mathcal{R}_A$ over A. By the results of [Kedlaya et al. 2014], we can similarly attach a graded invertible A-module

$$\Delta_A(M),$$

called the fundamental line for (A, M), which is also compatible with base change and Tate duality. For a pair (Λ, T) as in the previous paragraph and a continuous homomorphism $f : \Lambda \to A$, there exists a canonical comparison isomorphism

$$\Delta_\Lambda(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_A(\boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A))$$

by the result of [Pottharst 2013]. The following conjecture is Kato's conjecture if $(B, N) = (\Lambda, T)$, and our new conjecture if $(B, N) = (A, M)$.

Conjecture 1.1. (See Conjecture 3.8 for the precise formulation.) *We can uniquely define a B-linear isomorphism*

$$\varepsilon_{B,\zeta}(N) : \mathbf{1}_B \xrightarrow{\sim} \Delta_B(N),$$

for each pair (B, N) of type (Λ, T) or (A, M) and for each $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$, which is compatible with any base changes $B \to B'$, exact sequences $0 \to N_1 \to N_2 \to N_3 \to 0$, and Tate duality, and satisfies the following:

(v) *For any $f : \Lambda \to A$ as above, we have*

$$\varepsilon_{\Lambda,\zeta}(T) \otimes \mathrm{id}_A = \varepsilon_{A,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A))$$

under the canonical isomorphism $\Delta_\Lambda(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_A(\boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A))$.

(vi) *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let N be a de Rham representation of $G_{\mathbb{Q}_p}$ or de Rham (φ, Γ)-module over $\mathcal{R}_L$. Then we have*

$$\varepsilon_{L,\zeta}(N) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(N),$$

where the isomorphism

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(N) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(N)$$

is called the de Rham ε-isomorphism which is defined using the Bloch–Kato exponential and the dual exponential of N and the local factors (L-factor, ε-constant) associated to $\boldsymbol{D}_{\mathrm{pst}}(N)$ and $\boldsymbol{D}_{\mathrm{pst}}(N^*)$.

Remark 1.2. To define condition (vi) for de Rham (φ, Γ)-modules, we need to generalize the Bloch–Kato exponential for (φ, Γ)-modules, which was one of the main themes of [Nakamura 2014a].

Roughly speaking, this conjecture says that the local factor which appears in the functional equation of the L-functions of a motif p-adically interpolate to all the families of p-adic Galois representations and also rigid-analytically interpolate to all the families of (φ, Γ)-modules in a compatible way. In fact, Kato [1993a] formulated a conjecture, called the generalized Iwasawa main conjecture, which asserts the existence of a compatible family of the zeta-isomorphisms

$$z_\Lambda(\mathbb{Z}[1/S], T) : \mathbf{1}_\Lambda \xrightarrow{\sim} \Delta^{\mathrm{global}}_\Lambda(T)$$

for any Λ-representation T of $G_{\mathbb{Q},S}$ (S is a finite set of primes) which interpolate the special values of L-functions of a motif. Kato [1993b] also formulated another conjecture, called the global ε-conjecture, which asserts the functional equation between $z_\Lambda(\mathbb{Z}[1/S], T)$ and $z_\Lambda(\mathbb{Z}[1/S], T^*)$, whose local factor at p is $\varepsilon_{\Lambda,\zeta}(T|_{G_{\mathbb{Q}_p}})$. Kato [1993b] (see also [Venjakob 2013]) proved the local (and even the global) ε-conjecture for the rank-one case. As a generalization of his theorem, the main theorem of this article is the following.

Theorem 1.3. (See Theorem 3.11 for the precise statement.) *Conjecture 1.1 is true for the rank-one case.*

From this theorem, we can immediately obtain some results for the trianguline case. We say that a (φ, Γ)-module M over $\mathcal{R}_A$ is trianguline if M has a filtration $\mathcal{F}: 0 := M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n := M$ whose graded quotients M_i/M_{i-1} are rank-one (φ, Γ)-modules over $\mathcal{R}_A$ for all $1 \leq i \leq n$. We call the filtration $\mathcal{F}$ a triangulation of M. For such a pair $(M, \mathcal{F})$, we obtain the following theorem, a special case (in particular, the rank-two case) of which will be used in Theorem 3.10 of our next article, [Nakamura 2015].

Corollary 1.4. (See Corollary 3.12 for the precise statement.) *Let M be a trianguline (φ, Γ)-module over $\mathcal{R}_A$ of rank n with a triangulation $\mathcal{F}$ as above. The isomorphism*

$$\varepsilon_{\mathcal{F},A,\zeta}(M) : \mathbf{1}_A \xrightarrow{\boxtimes_{i=1}^{n}\varepsilon_{A,\zeta}(M_i/M_{i-1})} \boxtimes_{i=1}^{n}\Delta_A(M_i/M_{i-1}) \xrightarrow{\sim} \Delta_A(M),$$

defined as the product of the isomorphisms

$$\varepsilon_{A,\zeta}(M_i/M_{i-1}) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M_i/M_{i-1}),$$

which are defined in Theorem 1.3, satisfies (many parts of) Conjecture 1.1; in particular, it satisfies the following:

(vi)′ *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham and trianguline (φ, Γ)-module over $\mathcal{R}_L$. Then, for any triangulation $\mathcal{F}$ of M, we have*

$$\varepsilon_{\mathcal{F},L,\zeta}(M) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

Remark 1.5. Before this article, the local ε-conjecture was proved only for cyclotomic deformations (or more general twists) of crystalline representations [Benois and Berger 2008; Loeffler et al. 2015]. Since the (φ, Γ)-modules associated to any twists of crystalline representations are trianguline, our Corollary 1.4 essentially contains all the known results concerning the local ε-conjecture. See Corollary 3.13 for the comparison of our theorem with the previous known results. Moreover, since any twists of semistable representations are also trianguline, our results also contain the semistable case, which seems to be unknown before this article.

Remark 1.6. Our method and previous known methods for the construction of local ε-isomorphisms cannot be applied to the nontrianguline case. That case is much more difficult but is much more interesting since the Weil–Deligne representation $\boldsymbol{D}_{\mathrm{pst}}(M)$ associated to a nontrianguline and de Rham (φ, Γ)-module M corresponds to a nonprincipal series representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ via the local Langlands correspondence, whose ε-constants are in general difficult to explicitly describe. In our next article, [Nakamura 2015], we construct ε-isomorphisms for all rank-two torsion p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by using Colmez's theory [2010] of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. More precisely, we will show that (a modified version of) the pairing defined in Corollaire VI.6.2 of [Colmez 2010] essentially gives us ε-isomorphisms for the rank-two case. In the trianguline case, by using Dospinescu's result [2014] on the explicit description of locally analytic vectors of Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, we will show that the ε-isomorphisms constructed in [Nakamura 2015] coincide with those constructed in this article. More interestingly, for the de Rham and nontrianguline case, we will show, by using Emerton's theorem [2006] on the compatibility of classical and p-adic Langlands correspondence, that the ε-isomorphisms defined in [Nakamura 2015] satisfy the suitable interpolation property (i.e., condition (vi) of Conjecture 1.1) for the critical range of Hodge–Tate weights. Moreover, as an application, we will prove a functional equation of Kato's Euler systems associated to Hecke eigen elliptic cusp newforms.

Remark 1.7. Other than the application to Theorem 3.10 of [Nakamura 2015], our Corollary 1.4 should be applicable to some Iwasawa theoretic studies of Galois representations over eigenvarieties. For example, the rank-two case of the local ε-isomorphism constructed in Corollary 1.4 should be the p-th local factor of the conjectural functional equation satisfied by the conjectural zeta element over the Coleman–Mazur eigencurve, whose existence is conjectured in (for example) [Hansen 2016, Conjecture 1.3.3]. Since our article is long enough, we don't study this problem in this article, but we hope to study it in future works.

1B. ***Structure of the paper.*** In Section 2, we recall the results of [Kedlaya et al. 2014; Pottharst 2013; Nakamura 2014a]. After recalling the definition of (φ, Γ)-modules over the relative Robba ring, we recall the main results of [Kedlaya et al. 2014; Pottharst 2013] on the cohomology of (φ, Γ)-modules, i.e., comparison with Galois cohomology, finiteness, base change property, Euler–Poincaré formula, Tate duality, and the classification of rank-one objects, all of which are essential for the formulation of our conjecture. We next recall the result of [Nakamura 2014a] on the theory of the Bloch–Kato exponential map of (φ, Γ)-modules. Since the result of [Nakamura 2014a] is not sufficient for our purpose, we slightly generalize the result. In particular, we show the existence of Bloch–Kato fundamental exact sequences involving $\boldsymbol{D}_{\mathrm{cris}}(M)$ (Lemma 2.20), establishing Bloch–Kato duality for the finite cohomology of (φ, Γ)-modules (Proposition 2.24). The explicit formulae of our Bloch–Kato exponential maps (Proposition 2.23) are frequently used in later sections.

In Section 3, using the preliminaries recalled in Section 2, we formulate our ε-conjecture and state our main theorem of this paper. Since the conjecture is formulated by using the notion of determinant, we first recall this notion in Section 3A. In Section 3B, using the determinant of cohomology of (φ, Γ)-modules, we define a graded invertible A-module $\Delta_A(M)$, called the fundamental line, for any (φ, Γ)-module M over $\mathcal{R}_A$. In Section 3C, for any de Rham (φ, Γ)-module M, we define a trivialization (called a de Rham ε-isomorphism) of the fundamental line using the Bloch–Kato fundamental exact sequence, Deligne–Langlands–Fontaine–Perrin-Riou's ε-constants and the "gamma-factor" associated to $\boldsymbol{D}_{\mathrm{pst}}(M)$. In Section 3D, we formulate our conjecture and compare our conjecture with Kato's conjecture, and state our main theorem of this article, which solves the conjecture for all rank-one (φ, Γ)-modules.

Section 4 is the main part of this paper, where we prove the conjecture for the rank-one case. In Section 4A, using the theory of analytic Iwasawa cohomology [Kedlaya et al. 2014; Pottharst 2012], and using the standard technique of p-adic Fourier transform, we construct our ε-isomorphism for all rank-one (φ, Γ)-modules. In Section 4B, we show that our ε-isomorphism defined in Section 4A specializes to the de Rham ε-isomorphism defined in Section 3B at each de Rham point. In Section 4B1, we first verify this condition (which we call the de Rham condition) for the "generic" rank-one de Rham (φ, Γ)-modules by establishing a kind of explicit reciprocity law (Proposition 4.11, 4.16). In the process of proving this, we prove a proposition (Proposition 4.13) on the compatibility of our ε-isomorphism with a natural differential operator. Using the result in the generic case and the density argument, we prove the compatibility of our ε-isomorphism with Tate duality and compare our ε-isomorphism with Kato's ε-isomorphism. In Section 4B2, we verify the de Rham condition via explicit calculations for the exceptional case which includes the case of $\mathcal{R}$, $\mathcal{R}(1)$ (the (φ, Γ)-modules corresponding to $\mathbb{Q}_p$, $\mathbb{Q}_p(1)$, respectively).

In the Appendix, we explicitly calculate the cohomologies $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}(1))$ and $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R})$, which will be used in Section 4B2. Finally, we remark that, in our proof, we don't use any previous known results (e.g., [Kato 1993b; Benois and Berger 2008; Loeffler et al. 2015]) on the local ε-conjecture. Our proof essentially follows from the results in Section 2 of this article and those of [Nakamura 2014a] on the explicit definition of the exponential and the dual exponential maps for (φ, Γ)-modules. We believe that our proof is the most simple and the most natural one.

1C. ***Notation.*** Throughout this paper, we fix a prime number p. The letter A will always denote a $\mathbb{Q}_p$-affinoid algebra; we use $\mathrm{Max}(A)$ to denote the associated rigid analytic space. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and consider any finite extension K of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$. Let $|-| : \bar{\mathbb{Q}}_p^\times \to \mathbb{Q}_{>0}$ be the absolute value such that $|p| = p^{-1}$. For $n \geq 0$, let us denote by μ_{p^n} the set of p^n-th power roots of unity in $\bar{\mathbb{Q}}_p$, and put $\mu_{p^\infty} := \bigcup_{n\geq 1} \mu_{p^n}$. For a finite extension K of $\mathbb{Q}_p$, put $K_n := K(\mu_{p^n})$ for $\infty \geq n \geq 0$. Let us denote by $\chi : \Gamma_{\mathbb{Q}_p} := \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Z}_p^\times$ the cyclotomic character given by $\gamma(\zeta) = \zeta^{\chi(\gamma)}$ for $\gamma \in \Gamma$ and $\zeta \in \mu_{p^\infty}$. Set $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$, $H_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$, and $\Gamma_K := \mathrm{Gal}(K_\infty/K)$.

We let k be the residue field of K, with $F := W(k)[1/p]$. Put $\mathbb{Z}_p(1) := \varprojlim_{n\geq 0} \mu_{p^n}$. For $k \in \mathbb{Z}$, define $\mathbb{Z}_p(k) := \mathbb{Z}_p(1)^{\otimes k}$ equipped with a natural action of Γ_K. For a $\mathbb{Z}_p[G_K]$-module N, let us define $N(k) := N \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)$. When we fix a generator $\zeta = \{\zeta_{p^n}\}_{n\geq 0} \in \mathbb{Z}_p(1)$, we put $\boldsymbol{e}_1 := \zeta$ and $\boldsymbol{e}_k := \boldsymbol{e}_1^{\otimes k} \in \mathbb{Z}$. For a continuous G_K-module N, let us denote by $C^\bullet_{\mathrm{cont}}(G_K, N)$ the complex of continuous cochains of G_K with values in N. Define $\mathrm{H}^i(K, N) := \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_K, N))$. For a group G, denote by G_{tor} the subgroup of G consisting of all torsion elements in G. If G is a finite group, let $|G|$ be the order of G.

For a commutative ring R, let us denote by $\boldsymbol{P}_{\mathrm{fg}}(R)$ the category of finitely generated projective R-modules. For $N \in \boldsymbol{P}_{\mathrm{fg}}(R)$, denote by $\mathrm{rk}_R\, N$ the rank of N and let $N^\vee := \mathrm{Hom}_R(N, R)$. Let $[-, -] : N_1 \times N_2 \to R$ be a perfect pairing. Then we always identify N_2 with $N_1^\vee$ by the isomorphism $N_2 \xrightarrow{\sim} N_1^\vee : x \mapsto (y \mapsto [y, x])$. Let us denote by $\boldsymbol{D}^-(R)$ the derived category of bounded-below complexes of R-modules. For $a_1 \leq a_2 \in \mathbb{Z}$, let us denote by $\boldsymbol{D}^{[a_1,a_2]}_{\mathrm{perf}}(R)$ (resp. $\boldsymbol{D}^b_{\mathrm{perf}}(R)$) the full subcategory of $\boldsymbol{D}^-(R)$ consisting of the complexes of R-modules which are quasi-isomorphic to a complex $P^\bullet$ of $\boldsymbol{P}_{\mathrm{fg}}(R)$ concentrated in degrees in $[a_1, a_2]$ (resp. bounded degree). There exists a duality functor

$$\boldsymbol{R}\,\mathrm{Hom}_R(-, R) : \boldsymbol{D}^{[a_1,a_2]}_{\mathrm{perf}}(R) \to \boldsymbol{D}^{[-a_2,-a_1]}_{\mathrm{perf}}(R)$$

characterized by $\boldsymbol{R}\,\mathrm{Hom}_R(P^\bullet, R) := \mathrm{Hom}_R(P^{-\bullet}, R)$ for any bounded complex $P^\bullet$ of $\boldsymbol{P}_{\mathrm{fg}}(R)$. Define the notion $\chi_R(-)$ of Euler characteristic for any objects of $\boldsymbol{D}^b_{\mathrm{perf}}(R)$, which is characterized by

$$\chi_R(P^\bullet) := \sum_{i\in\mathbb{Z}} (-1)^i\, \mathrm{rk}_R\, P^i \in \mathrm{Map}(\mathrm{Spec}(R), \mathbb{Z})$$

for any bounded complex $P^\bullet$ of $\boldsymbol{P}_{\mathrm{fg}}(R)$.

2. Cohomology and Bloch–Kato exponential of (φ, Γ)-modules

2A. ***Cohomology of (φ, Γ)-modules.*** In this subsection, we recall the definition of (families of) (φ, Γ)-modules and the definition of their cohomologies following [Kedlaya et al. 2014], and then recall the results of their article on the finiteness of the cohomology.

Put $\omega := p^{-1/(p-1)} \in \mathbb{R}_{>0}$. For $r \in \mathbb{Q}_{>0}$, define the r-Gauss norm $|-|_r$ on $\mathbb{Q}_p[T^{\pm}]$ by the formula $\left|\sum_i a_i T^i\right|_r := \max_i\{|a_i|\omega^{ir}\}$. For $0 < s \le r \in \mathbb{Q}_{>0}$, we write $A^1[s,r]$ for the rigid analytic annulus defined over $\mathbb{Q}_p$ in the variable T with radii $|T| \in [\omega^r, \omega^s]$; its ring of analytic functions, denoted by $\mathcal{R}^{[s,r]}$, is the completion of $\mathbb{Q}_p[T^{\pm}]$ with respect to the norm $|\cdot|_{[s,r]} := \max\{|\cdot|_r, |\cdot|_s\}$. We also allow r (but not s) to be ∞, in which case $A^1[s,r]$ is interpreted as the rigid analytic disc in the variable T with radii $|T| \le \omega^s$; its ring of analytic functions $\mathcal{R}^{[s,r]} = \mathcal{R}^{[s,\infty]}$ is the completion of $\mathbb{Q}_p[T]$ with respect to $|\cdot|_s$. Let A be a $\mathbb{Q}_p$-affinoid algebra. Denote by $\mathcal{R}_A^{[s,r]}$ the ring of rigid analytic functions on the relative annulus (or disc if $r = \infty$) $\mathrm{Max}(A) \times A^1[s,r]$; its ring of analytic functions is $\mathcal{R}_A^{[s,r]} := \mathcal{R}^{[s,r]} \widehat{\otimes}_{\mathbb{Q}_p} A$. Put

$$\mathcal{R}_A^r := \bigcap_{0<s\le r} \mathcal{R}_A^{[s,r]} \quad \text{and} \quad \mathcal{R}_A := \bigcup_{0<r} \mathcal{R}_A^r.$$

Let k' be the residue field of K_∞, with $F' := W(k')[1/p]$. Put $\tilde{e}_K := [K_\infty : F'_\infty]$.

For $0 < s \le r$, we let $\mathcal{R}^{[s,r]}(\pi_K)$ be the formal substitution of T by π_K in the ring $\mathcal{R}_{F'}^{[s/\tilde{e}_K, r/\tilde{e}_K]}$; we set $\mathcal{R}_A^{[s,r]}(\pi_K) := \mathcal{R}^{[s,r]}(\pi_K) \widehat{\otimes}_{\mathbb{Q}_p} A$. We define $\mathcal{R}_A^r(\pi_K)$, $\mathcal{R}_A(\pi_K)$ similarly; the latter is referred to as the relative Robba ring over A for K.

By the theory of fields of norms, there exists a constant $C(K) > 0$ such that, for any $0 < r \le C(K)$, we can equip $\mathcal{R}_A^r(\pi_K)$ with a finite étale $\mathcal{R}_A^r(\pi_{\mathbb{Q}_p})$ algebra free of rank $[K_\infty : \mathbb{Q}_{p,\infty}]$ with the Galois group $H_{\mathbb{Q}_p}/H_K$. More generally, for any finite extensions $L \supseteq K \supseteq \mathbb{Q}_p$, we can naturally equip $\mathcal{R}_A^r(\pi_L)$ with a structure of finite étale $\mathcal{R}_A^r(\pi_K)$-algebra free of rank $[L_\infty : K_\infty]$ with the Galois group H_K/H_L for any $0 < r \le \min\{C(K), C(L)\}$.

There are commuting A-linear actions of Γ_K on $\mathcal{R}_A^{[s,r]}(\pi_K)$ and of an operator

$$\varphi : \mathcal{R}_A^{[s,r]}(\pi_K) \to \mathcal{R}_A^{[s/p,r/p]}(\pi_K)$$

for $0 < s \le r \le C(K)$. The actions on the coefficients F' are the natural ones, i.e., Γ_K through its quotient $\mathrm{Gal}(F'/F)$ and φ by the canonical lift of the p-th Frobenius on k'. For $0 < s \le r \le C(K)$, φ makes $\mathcal{R}_A^{[s/p,r/p]}(\pi_K)$ into a free $\mathcal{R}_A^{[s,r]}(\pi_K)$-module of rank p, and we obtain a Γ_K-equivariant left inverse

$$\psi : \mathcal{R}_A^{[s/p,r/p]}(\pi_K) \to \mathcal{R}_A^{[s,r]}(\pi_K)$$

by the formula

$$\tfrac{1}{p}\varphi^{-1} \circ \mathrm{Tr}_{\mathcal{R}_A^{[s/p,r/p]}(\pi_K)/\varphi(\mathcal{R}_A^{[s,r]}(\pi_K))}.$$

The map ψ naturally extends to the maps $\mathcal{R}_A^{r/p}(\pi_K) \to \mathcal{R}_A^r(\pi_K)$ for $0 < r \le C(K)$ and $\mathcal{R}_A(\pi_K) \to \mathcal{R}_A(\pi_K)$.

Remark 2.1. In fact, these rings are constructed using Fontaine's rings of p-adic periods. We don't have any canonical choice of the parameter π_K for general K, but the ring $\mathcal{R}_A(\pi_K)$ and the actions of φ, Γ_K don't depend on the choice of π_K. More precisely, $\mathcal{R}(\pi_K)$ is defined as a subring of the ring $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ of p-adic periods defined in [Berger 2002], and this subring does not depend on the choice of π_K, and the actions of φ, Γ_K are induced by the natural actions of φ, G_K on $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$.

However, for unramified K, once we fix a $\mathbb{Z}_p$-basis $\zeta := \{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1) := \varprojlim_{n\geq 0} \mu_{p^n}$, we have a natural choice of π_K as follows. Let $\bar{\mathbb{Z}}_p$ be the integral closure of $\mathbb{Z}_p$ in $\bar{\mathbb{Q}}_p$, let $\widetilde{\mathbb{E}}^+ := \varprojlim_{n\geq 0} \bar{\mathbb{Z}}_p/p\bar{\mathbb{Z}}_p$ be the projective limit with respect to the p-th power map, and let $[-] : \widetilde{\mathbb{E}}^+ \to W(\widetilde{\mathbb{E}}^+)$ be the Teichmüller lift to the ring $W(\widetilde{\mathbb{E}}^+)$ of Witt vectors. Under the fixed ζ, we can choose

$$\pi_K = \pi_{\mathbb{Q}_p} = \pi_\zeta := [(\bar{\zeta}_{p^n})_{n\geq 0}] - 1 \in W(\widetilde{\mathbb{E}}^+) \subseteq \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}},$$

and then φ and $\Gamma_{\mathbb{Q}_p}$ act by $\varphi(\pi_\zeta) = (1+\pi_\zeta)^p - 1$ and $\gamma(\pi_\zeta) = (1+\pi_\zeta)^{\chi(\gamma)} - 1$ for $\gamma \in \Gamma_{\mathbb{Q}_p}$.

Notation 2.2. From Section 3, we will concentrate on the case $K = \mathbb{Q}_p$ and fix $\zeta := \{\zeta_{p^n}\}_{n\geq 0}$ as above. Then we use the notation $\Gamma := \Gamma_{\mathbb{Q}_p}$, $\pi := \pi_\zeta$ and omit $(\pi_{\mathbb{Q}_p})$ from the notation of Robba rings by writing, for example, $\mathcal{R}_A^{[s,r]}$ instead of $\mathcal{R}_A^{[s,r]}(\pi_{\mathbb{Q}_p})$. In this case, $\mathcal{R}_A^{[s/p,r/p]} = \bigoplus_{0\leq i\leq p-1}(1+\pi)^i\varphi(\mathcal{R}_A^{[s,r]})$, so if $f = \sum_{i=0}^{p-1}(1+\pi)^i\varphi(f_i)$ then $\psi(f) = f_0$. We define the special element $t = \log(1+\pi) \in \mathcal{R}_A^{\infty}$. We have $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$ for $\gamma \in \Gamma$.

We first recall the definitions of φ-modules over $\mathcal{R}_A(\pi_K)$ following [Kedlaya et al. 2014, Definition 2.2.5].

Definition 2.3. Choose $0 < r_0 \leq C(K)$. A φ-module over $\mathcal{R}_A^{r_0}(\pi_K)$ is a finite projective $\mathcal{R}_A^{r_0}(\pi_K)$-module M^{r_0} equipped with a $\mathcal{R}_A^{r_0/p}(\pi_K)$-linear isomorphism $\varphi^*M^{r_0} \xrightarrow{\sim} M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^{r_0/p}(\pi_K)$. A φ-module M over $\mathcal{R}_A(\pi_K)$ is a base change to $\mathcal{R}_A(\pi_K)$ of a φ-module over some $\mathcal{R}_A^{r_0}(\pi_K)$.

For a φ-module M^{r_0} over $\mathcal{R}_A^{r_0}(\pi_K)$ and for $0 < s \leq r \leq r_0$, we set

$$M^{[s,r]} = M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^{[s,r]}(\pi_K) \quad \text{and} \quad M^s = M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^{s}(\pi_K).$$

For $0 < s \leq r_0$, the given isomorphism $\varphi^*(M^{r_0}) \xrightarrow{\sim} M^{r_0/p}$ induces a φ-semilinear map

$$\varphi : M^s \to \varphi^*M^s \xrightarrow{\sim} \varphi^*M^{r_0} \otimes_{\mathcal{R}_A^{r_0/p}(\pi_K)} \mathcal{R}_A^{s/p}(\pi_K) \xrightarrow{\sim} M^{r_0/p} \otimes_{\mathcal{R}_A^{r_0/p}(\pi_K)} \mathcal{R}_A^{s/p}(\pi_K) = M^{s/p},$$

where the first map, $M^s \hookrightarrow \varphi^*M^s$, is given by

$$x \mapsto x \otimes 1 \in M^s \otimes_{\mathcal{R}_A^s(\pi_K),\varphi} \mathcal{R}_A^{s/p}(\pi_K) =: \varphi^*M^s,$$

the second isomorphism is just the associativity of tensor products, and the third isomorphism is the base change of the given isomorphism $\varphi^*M^{r_0} \xrightarrow{\sim} M^{r_0/p}$. This map φ also induces an A-linear homomorphism

$$\psi : M^{s/p} = \varphi(M^s) \otimes_{\varphi(\mathcal{R}_A^s(\pi_K))} \mathcal{R}_A^{s/p}(\pi_K) \to M^s$$

given by $\psi(\varphi(m)\otimes f)=m\otimes\psi(f)$ for $m\in M^s$ and $f\in\mathcal{R}_A^{s/p}(\pi_K)$. For a φ-module M over $\mathcal{R}_A(\pi_K)$, the maps $\varphi: M^s\to M^{s/p}$ and $\psi: M^{s/p}\to M^s$ naturally extend to $\varphi: M\to M$ and $\psi: M\to M$.

We recall the definition of (φ,Γ)-modules over $\mathcal{R}_A(\pi_K)$ following [Kedlaya et al. 2014, Definition 2.2.12].

Definition 2.4. Choose $0<r_0\leq C(K)$. A (φ,Γ)-module over $\mathcal{R}_A^{r_0}(\pi_K)$ is a φ-module over $\mathcal{R}_A^{r_0}(\pi_K)$ equipped with a commuting semilinear continuous action of Γ_K. A (φ,Γ)-module over $\mathcal{R}_A(\pi_K)$ is a base change of a (φ,Γ)-module over $\mathcal{R}_A^{r_0}(\pi_K)$ for some $0<r_0\leq C(K)$.

We can generalize these notions for general rigid analytic space as in [Kedlaya et al. 2014, Definition 6.1.1]

Definition 2.5. Let X be a rigid analytic space over $\mathbb{Q}_p$. A (φ,Γ)-module over $\mathcal{R}_X(\pi_K)$ is a compatible family of (φ,Γ)-modules over $\mathcal{R}_A(\pi_K)$ for each affinoid $\mathrm{Max}(A)$ of X.

For (φ,Γ)-modules M, N over $\mathcal{R}_X(\pi_K)$, we define $M\otimes N:=M\otimes_{\mathcal{R}_X(\pi_K)}N$ to be the tensor product equipped with the diagonal action of (φ,Γ_K). We also define $M^\vee:=\mathrm{Hom}_{\mathcal{R}_X(\pi_K)}(M,\mathcal{R}_X(\pi_K))$ to be the dual (φ,Γ)-module.

For a (φ,Γ)-module M over $\mathcal{R}_A(\pi_K)$, we define

$$r_M:=\mathrm{rk}_{\mathcal{R}_A(\pi_K)}M\in\mathrm{Map}(\mathrm{Spec}(\mathcal{R}_A(\pi_K)),\mathbb{Z}_{\geq 0})$$

to be the rank of M, where $\mathrm{Map}(-,-)$ is the set of continuous maps and $\mathbb{Z}_{\geq 0}$ is equipped with the discrete topology. We will see later (in Remark 2.16) that r_M is in fact in $\mathrm{Map}(\mathrm{Spec}(A),\mathbb{Z}_{\geq 0})$, i.e., we have $r_M=\mathrm{pr}\circ f_M$ for unique $f_M\in\mathrm{Map}(\mathrm{Spec}(A),\mathbb{Z}_{\geq 0})$, where $\mathrm{pr}:\mathrm{Spec}(\mathcal{R}_A(\pi_K))\to\mathrm{Spec}(A)$ is the natural projection. We also let $r_M:=f_M$.

The importance of (φ,Γ)-modules follows from the next theorem.

Theorem 2.6 [Kedlaya and Liu 2010, Theorem 3.11]. *Let V be a vector bundle over X equipped with a continuous $\mathcal{O}_X$-linear action of G_K. Then there is functorially associated to V a (φ,Γ)-module $\boldsymbol{D}_{\mathrm{rig}}(V)$ over $\mathcal{R}_X(\pi_K)$. The rule $V\mapsto\boldsymbol{D}_{\mathrm{rig}}(V)$ is fully faithful and exact, and it commutes with base change in X.*

For example, we have a canonical isomorphism $\boldsymbol{D}_{\mathrm{rig}}(A(k))=\mathcal{R}_A(\pi_K)(k)$ for $k\in\mathbb{Z}$.

From Section 3, we will concentrate on the case where $K=\mathbb{Q}_p$ and M is a rank-one (φ,Γ)-module over $\mathcal{R}_X$. Here, we recall the result of [Kedlaya et al. 2014] concerning the classification of rank-one (φ,Γ)-modules. Actually, they obtained a similar result for general K, but we don't recall it since we don't use it.

Definition 2.7. For a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$, we define $\mathcal{R}_X(\delta)$ to be the rank-one (φ, Γ)-module $\mathcal{R}_X \cdot \boldsymbol{e}_\delta$ over $\mathcal{R}_X$ with $\varphi(\boldsymbol{e}_\delta) = \delta(p)\boldsymbol{e}_\delta$ and $\gamma(\boldsymbol{e}_\delta) = \delta(\chi(\gamma))\boldsymbol{e}_\delta$ for $\gamma \in \Gamma$.

Theorem 2.8 [Kedlaya et al. 2014, Theorem 6.1.10]. *Let M be a rank-one (φ, Γ)-module over $\mathcal{R}_X$. Then there exist a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and an invertible sheaf $\mathcal{L}$ on X, the pair of which is unique up to isomorphism, such that $M \xrightarrow{\sim} \mathcal{R}_X(\delta) \otimes_{\mathcal{O}_X} \mathcal{L}$.*

Notation 2.9. (i) For $\delta, \delta' : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$, we fix isomorphisms

$$\mathcal{R}_X(\delta) \otimes \mathcal{R}_X(\delta') \xrightarrow{\sim} \mathcal{R}_X(\delta\delta') \quad \text{by } \boldsymbol{e}_\delta \otimes \boldsymbol{e}_{\delta'} \mapsto \boldsymbol{e}_{\delta\delta'},$$
$$\mathcal{R}_X(\delta)^\vee \xrightarrow{\sim} \mathcal{R}_X(\delta^{-1}) \quad \text{by } \boldsymbol{e}_\delta^\vee \mapsto \boldsymbol{e}_{\delta^{-1}}.$$

(ii) For $k \in \mathbb{Z}$, we define a continuous homomorphism $x^k : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times : y \mapsto y^k$. Define $|x| : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times : p \mapsto p^{-1}$, $a \mapsto 1$ for $a \in \mathbb{Z}_p^\times$. Then the homomorphism $x|x|$ corresponds to the Tate twist, i.e., we have an isomorphism $\mathcal{R}_X(1) \xrightarrow{\sim} \mathcal{R}_X(x|x|)$. When we fix a generator $\zeta \in \mathbb{Z}_p(1)$, we identify $\mathcal{R}_X(1) = \mathcal{R}_X(x|x|)$ by $\boldsymbol{e}_1 \mapsto \boldsymbol{e}_{x|x|}$.

We next recall some cohomology theories concerning (φ, Γ)-modules. Denote by Δ the largest p-power torsion subgroup of Γ_K. Fix $\gamma \in \Gamma_K$, whose image in Γ_K/Δ is a topological generator. For a Δ-module M, put $M^\Delta = \{m \in M \mid \sigma(m) = m \text{ for all } \sigma \in \Delta\}$.

Definition 2.10. For a (φ, Γ)-module M over $\mathcal{R}_A(\pi_K)$, we define the complexes $C^\bullet_{\varphi,\gamma}(M)$ and $C^\bullet_{\psi,\gamma}(M)$ of A-modules concentrated in degree $[0, 2]$, and define a morphism Ψ_M between them as follows:

$$\begin{array}{ccccccc} C^\bullet_{\varphi,\gamma}(M) = \big[M^\Delta & \xrightarrow{(\gamma-1,\varphi-1)} & M^\Delta \oplus M^\Delta & \xrightarrow{(\varphi-1)\oplus(1-\gamma)} & M^\Delta\big] \\ \Psi_M \big\downarrow \qquad \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id}\oplus -\psi & & \big\downarrow -\psi \\ C^\bullet_{\psi,\gamma}(M) = \big[M^\Delta & \xrightarrow{(\gamma-1,\psi-1)} & M^\Delta \oplus M^\Delta & \xrightarrow{(\psi-1)\oplus(1-\gamma)} & M^\Delta\big] \end{array} \tag{1}$$

The map Ψ_M is a quasi-isomorphism by Proposition 2.3.4 of [Kedlaya et al. 2014].

For $i \in \mathbb{Z}_{\geq 0}$, define $\mathrm{H}^i_{\varphi,\gamma}(M)$ for the i-th cohomology of $C^\bullet_{\varphi,\gamma}(M)$, called the (φ, Γ)-cohomology of M. We similarly define $\mathrm{H}^i_{\psi,\gamma}(M)$ to be the i-th cohomology of $C^\bullet_{\psi,\gamma}(M)$, called the (ψ, Γ)-cohomology of M. In this article, we freely identify $C^\bullet_{\varphi,\gamma}(M)$ (resp. $\mathrm{H}^i_{\varphi,\gamma}(M)$) with $C^\bullet_{\psi,\gamma}(M)$ (resp. $\mathrm{H}^i_{\psi,\gamma}(M)$) via the quasi-isomorphism Ψ_M.

More generally, for $h = \varphi, \psi$ and any module N with commuting actions of h and Γ, we similarly define the complexes $C^\bullet_{h,\gamma}(N)$ and denote the resulting cohomology by $\mathrm{H}^i_{h,\gamma}(N)$. We denote by $[x, y] \in \mathrm{H}^1_{h,\gamma}(N)$ (resp. $[z] \in \mathrm{H}^2_{h,\gamma}(N)$) the element represented by a 1-cocycle $(x, y) \in N^\Delta \oplus N^\Delta$ (resp. by $z \in N^\Delta$). The functor

$N \mapsto C^\bullet_{h,\gamma}(N)$ from the category of topological A-modules which are Hausdorff with commuting continuous actions of h, Γ_K to the category of complexes of A-modules is independent of the choice of γ up to canonical isomorphism; i.e., for another choice $\gamma' \in \Gamma_K$, we have a canonical isomorphism

$$\begin{array}{ccccccc} C^\bullet_{h,\gamma}(N) = \big[N^\Delta & \xrightarrow{(\gamma-1,h-1)} & N^\Delta \oplus N^\Delta & \xrightarrow{(h-1)\oplus(1-\gamma)} & N^\Delta\big] \\ \iota_{\gamma,\gamma'}\downarrow \qquad \downarrow \mathrm{id} & & \downarrow \frac{\gamma'-1}{\gamma-1}\oplus \mathrm{id} & & \downarrow \frac{\gamma'-1}{\gamma-1} \\ C^\bullet_{h,\gamma'}(N) = \big[N^\Delta & \xrightarrow{(\gamma'-1,h-1)} & N^\Delta \oplus N^\Delta & \xrightarrow{(h-1)\oplus(1-\gamma')} & N^\Delta\big] \end{array} \tag{2}$$

For a commutative ring R, let us denote by $\boldsymbol{D}^-(R)$ the derived category of bounded-below complexes of R-modules. We use the same notation, $C^\bullet_{h,\gamma}(N) \in \boldsymbol{D}^-(A)$, for the object represented by this complex.

Let V be a finite projective A-module with a continuous A-linear action of G_K. Let us denote by $C^\bullet_{\mathrm{cont}}(G_K, V)$ the complex of continuous G_K-cochains with values in V, and let $\mathrm{H}^i(K, V)$ be the cohomology. By Theorem 2.8 of [Pottharst 2013], we have a functorial isomorphism

$$C^\bullet_{\mathrm{cont}}(G_K, V) \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(V))$$

in $\boldsymbol{D}^-(A)$ and a functorial A-linear isomorphism

$$\mathrm{H}^i(K, V) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(V)).$$

Definition 2.11. For (φ, Γ)-modules M, N over $\mathcal{R}_A(\pi_K)$, we have a natural A-bilinear cup product morphism

$$C^\bullet_{\varphi,\gamma}(M) \times C^\bullet_{\varphi,\gamma}(N) \to C^\bullet_{\varphi,\gamma}(M \otimes N);$$

see Definition 2.3.11 of [Kedlaya et al. 2014]. This induces an A-bilinear graded commutative cup product pairing

$$\cup : \mathrm{H}^i_{\varphi,\gamma}(M) \times \mathrm{H}^j_{\varphi,\gamma}(N) \to \mathrm{H}^{i+j}_{\varphi,\gamma}(M \otimes N).$$

For example, this is defined by the formulae

$$x \cup [y] := [x \otimes y] \quad \text{for } i = 0,\ j = 2,$$
$$[x_1, y_1] \cup [x_2, y_2] := [x_1 \otimes \gamma(y_2) - y_1 \otimes \varphi(x_2)] \quad \text{for } i = j = 1.$$

Remark 2.12. The definition of the cup product for $\mathrm{H}^1_{\varphi,\gamma}(-) \times \mathrm{H}^1_{\varphi,\gamma}(-) \to \mathrm{H}^2_{\varphi,\gamma}(-)$, given in our previous paper, [Nakamura 2014a], is (-1) times the above definition. The above one seems to be the standard one in the literature. All the results of [Nakamura 2014a] hold without any changes when we use the above definition, except Lemmas 2.13 and 2.14, where we need to multiply by (-1) for the commutative diagrams there to be commutative.

Definition 2.13. Let us denote by $M^* := M^\vee(1)$ the Tate dual of M. Using the cup product, the evaluation map $\mathrm{ev} : M^* \otimes M \to \mathcal{R}_A(\pi_K)(1) : f \otimes x \mapsto f(x)$, the comparison isomorphism $\mathrm{H}^2(K, A(1)) \xrightarrow{\sim} \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_A(\pi_K)(1))$ and Tate's trace map $\mathrm{H}^2(K, A(1)) \xrightarrow{\sim} A$, one gets the Tate duality pairings

$$\begin{aligned} C^\bullet_{\varphi,\gamma}(M^*) \times C^\bullet_{\varphi,\gamma}(M) \to C^\bullet_{\varphi,\gamma}(M^* \otimes M) \to C^\bullet_{\varphi,\gamma}(\mathcal{R}_A(\pi_K)(1))) \\ \to \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_A(\pi_K)(1))[-2] \xrightarrow{\sim} \mathrm{H}^2(K, A(1))[-2] \xrightarrow{\sim} A[-2] \end{aligned}$$

and

$$\langle -, - \rangle : \mathrm{H}^i_{\varphi,\gamma}(M^*) \times \mathrm{H}^{2-i}_{\varphi,\gamma}(M) \to A.$$

Remark 2.14. In the Appendix, we explicitly describe the isomorphism

$$\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_A(1)) \xrightarrow{\sim} \mathrm{H}^2(G_{\mathbb{Q}_p}, A(1)) \xrightarrow{\sim} A$$

using the residue map; see Proposition 5.2.

One of the main results of [Kedlaya et al. 2014] which is crucial to formulating our conjecture is the following.

Theorem 2.15 [Kedlaya et al. 2014, Theorems 4.4.3, 4.4.4]. *Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$.*

(1) $C^\bullet_{\varphi,\gamma}(M) \in \boldsymbol{D}^{[0,2]}_{\mathrm{perf}}(A)$. *In particular, the cohomology groups $\mathrm{H}^i_{\varphi,\gamma}(M)$ are finite A-modules.*

(2) *Let $A \to A'$ be a continuous morphism of $\mathbb{Q}_p$-affinoid algebras. Then the canonical morphism $C^\bullet_{\varphi,\gamma}(M) \otimes^{\boldsymbol{L}}_A A' \to C^\bullet_{\varphi,\gamma}(M \widehat{\otimes}_A A')$ is a quasi-isomorphism. In particular, if A' is flat over A, we have $\mathrm{H}^i_{\varphi,\gamma}(M) \otimes_A A' \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(M \widehat{\otimes}_A A')$.*

(3) (Euler–Poincaré characteristic formula) *We have $\chi_A(C^\bullet_{\varphi,\gamma}(M)) = -[K : \mathbb{Q}_p] \cdot r_M$.*

(4) (Tate duality) *The Tate duality pairing defined in Definition 2.13 induces a quasi-isomorphism*

$$C^\bullet_{\varphi,\gamma}(M) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}_A(C^\bullet_{\varphi,\gamma}(M^*), A)[-2].$$

Remark 2.16. By the equality of (3), the rank $r_M \in \mathrm{Map}(\mathrm{Spec}(\mathcal{R}_A(\pi_K)), \mathbb{Z}_{\geq 0})$ is contained in $\mathrm{Map}(\mathrm{Spec}(A), \mathbb{Z}_{\geq 0})$.

Let X be a rigid analytic space over $\mathbb{Q}_p$ and let M be a (φ, Γ)-module over $\mathcal{R}_X(\pi_K)$. By (1) and (2) of the above theorem, the correspondence $U \mapsto \mathrm{H}^i_{\varphi,\gamma}(M|_U)$ for each affinoid open U in X defines a coherent $\mathcal{O}_X$-module for each $i \in [0, 2]$, which we also denote by $\mathrm{H}^i_{\varphi,\gamma}(M)$.

2B. ***Bloch–Kato exponential for (φ,Γ)-modules.*** For any $\mathbb{Q}_p$-representation V of G_K, Bloch and Kato [1990] defined the diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to \mathrm{H}^0(K,V) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(V)^{\varphi=1} & \xrightarrow{x\mapsto \bar{x}} & t_V(K) & \xrightarrow{\exp_V} & \mathrm{H}^1_e(K,V) \to 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto x} & & \downarrow{\scriptstyle x\mapsto (0,x)} & & \downarrow{\scriptstyle x\mapsto x} \\ 0 \to \mathrm{H}^0(K,V) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(V) & \xrightarrow{f} & \boldsymbol{D}^K_{\mathrm{cris}}(V)\oplus t_V(K) & \xrightarrow{g} & \mathrm{H}^1_f(K,V) \to 0 \end{array} \tag{3}$$

with

$$f(x,y) = ((1-\varphi)x, \bar{x}) \quad \text{and} \quad g = \exp_{f,V} \oplus \exp_V,$$

which is associated to the tensor product of V (over $\mathbb{Q}_p$) with the Bloch–Kato fundamental exact sequences

$$\begin{array}{ccccccc} 0 \to \mathbb{Q}_p & \xrightarrow{x\mapsto (x,x)} & \boldsymbol{B}^{\varphi=1}_{\mathrm{cris}} \oplus \boldsymbol{B}^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto x-y} & \boldsymbol{B}_{\mathrm{dR}} & \to 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (x,y)\mapsto (x,y)} & & \downarrow{\scriptstyle x\mapsto (0,x)} & \\ 0 \to \mathbb{Q}_p & \xrightarrow{x\mapsto (x,x)} & \boldsymbol{B}_{\mathrm{cris}} \oplus \boldsymbol{B}^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto ((1-\varphi)x, x-y)} & \boldsymbol{B}_{\mathrm{cris}} \oplus \boldsymbol{B}_{\mathrm{dR}} & \to 0 \end{array}$$

in which $\boldsymbol{B}_{\mathrm{cris}}$ and $\boldsymbol{B}_{\mathrm{dR}}$ are Fontaine's rings of p-adic periods. We set $\boldsymbol{D}^K_{\mathrm{cris}}(V) := (\boldsymbol{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, $t_V(K) := (\boldsymbol{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} / (\boldsymbol{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$,

$$\mathrm{H}^1_e(K,V) := \mathrm{Im}(\exp_V : t_V(K) \to \mathrm{H}^1(K,V))$$

and

$$\mathrm{H}^1_f(K,V) := \mathrm{Im}(\exp_{f,V} \oplus \exp_V : \boldsymbol{D}^K_{\mathrm{cris}}(V) \oplus t_V(K) \to \mathrm{H}^1(K,V)).$$

The boundary map

$$\exp_V : t_V(K) \to \mathrm{H}^1_e(K,V)$$

is called the Bloch–Kato exponential, and its definition is generalized to (φ,Γ)-modules over the Robba ring in [Nakamura 2014a]. To formulate the local ε-conjecture, we also need another boundary map,

$$\exp_{f,V} : \boldsymbol{D}^K_{\mathrm{cris}}(V) \to \mathrm{H}^1_f(K,V),$$

which is not studied in [Nakamura 2014a].

The aim of this subsection is to define the map $\exp_{f,M}$ for all the (φ,Γ)-modules M over the Robba ring purely in terms of (φ,Γ)-modules (Propositions 2.21 and 2.23), to prove Bloch–Kato duality for them (Proposition 2.24), to compare our maps $\exp_M$ and $\exp_{f,M}$ with the Bloch–Kato maps for the étale case (Proposition 2.26), all of which we need in order to generalize the local ε-conjecture for (φ,Γ)-modules. The explicit formulae for the maps $\exp_M$ and $\exp_{f,M}$ (Proposition 2.23) is especially important in the proof of our main theorem

(Theorem 1.3). We apologize to the readers that the arguments are slightly longer than §2 of [Nakamura 2014a], but we think that these arguments are needed. This is because, to define the map $\exp_{f,M}$, we need some additional arguments (Lemmas 2.17, 2.18 and 2.20), and, to obtain the precise explicit formulae for the maps $\exp_M$ and $\exp_{f,M}$, it seems to be safer not to omit any steps of the proofs.

Define $n(K)\geq 1$ to be the minimal integer n such that $1/p^{n-1}\leq \tilde{e}_K C(K)$, and put

$$\mathcal{R}_A^{(n)}(\pi_K)=\mathcal{R}_A^{1/(p^{n-1}\tilde{e}_K)}(\pi_K)$$

for $n\geq n(K)$. For $n\geq n(K)$, one has a Γ_K-equivariant A-algebra homomorphism

$$\iota_n:\mathcal{R}_A^{(n)}(\pi_K)\to (K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]$$

such that

$$\iota_n(\pi)=\zeta_{p^n}\cdot\exp\Bigl(\frac{t}{p^n}\Bigr)-1 \qquad\text{and}\qquad \iota_n(a)=\varphi^{-n}(a)\quad (a\in F').$$

For $n\geq n(K)$, we have the commutative diagrams

$$\begin{array}{ccc}\mathcal{R}_A^{(n)}(\pi_K) & \xrightarrow{\iota_n} & (K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]\\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\mathrm{can}}\\ \mathcal{R}_A^{(n+1)}(\pi_K) & \xrightarrow{\iota_{n+1}} & (K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!]\end{array}$$

and

$$\begin{array}{ccc}\mathcal{R}_A^{(n+1)}(\pi_K) & \xrightarrow{\iota_{n+1}} & (K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!]\\ \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}}\\ \mathcal{R}_A^{(n)}(\pi_K) & \xrightarrow{\iota_n} & (K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]\end{array}$$

in which can is the canonical injection and $\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}$ is defined by

$$\sum_{k\geq 0}a_kt^k\mapsto\sum_{k\geq 0}\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}(a_k)t^k.$$

Let M be a (φ,Γ)-module over $\mathcal{R}_A(\pi_K)$ obtained as a base change of a (φ,Γ)-module M^{r_0} over $\mathcal{R}_A^{r_0}(\pi_K)$ for some $0<r_0\leq c(K)$. Define $n(M)\in\mathbb{Z}_{\geq n(K)}$ to be the minimal integer such that $1/p^{n-1}\leq\tilde{e}_K r_0$. Put $M^{(n)}=M^{1/(p^{n-1}\tilde{e}_K)}$ for $n\geq n(M)$. Then φ and ψ induce $\varphi:M^{(n)}\to M^{(n+1)}$ and $\psi:M^{(n+1)}\to M^{(n)}$, respectively. Define

$$\boldsymbol{D}^+_{\mathrm{dif},n}(M)=M^{(n)}\otimes_{\mathcal{R}_A^{(n)}(\pi_K),\iota_n}(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]\quad\bigl(\text{resp. }\boldsymbol{D}_{\mathrm{dif},n}(M)=\boldsymbol{D}^+_{\mathrm{dif},n}(M)[1/t]\bigr),$$

which is a finite projective $(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]$-module (resp. $(K_n\otimes_{\mathbb{Q}_p}A)((t))$-module) with a semilinear action of Γ_K. We also let $\iota_n:M^{(n)}\to\boldsymbol{D}^+_{\mathrm{dif},n}(M)$ be the map defined by $x\mapsto x\otimes 1$.

Using the base change of the Frobenius structure $\varphi^*M^{(n)} \xrightarrow{\sim} M^{(n+1)}$ by the map ι_{n+1}, we obtain a Γ_K-equivariant $(K_{n+1} \otimes_{\mathbb{Q}_p} A)[\![t]\!]$-linear isomorphism

$$\begin{aligned} &\boldsymbol{D}^+_{\mathrm{dif},n}(M) \otimes_{(K_n \otimes_{\mathbb{Q}_p} A)[\![t]\!]} (K_{n+1} \otimes_{\mathbb{Q}_p} A)[\![t]\!] \\ &\qquad \xrightarrow{\sim} \varphi^*(M^{(n)}) \otimes_{\mathcal{R}_A^{(n+1)}(\pi_K),\iota_{n+1}} (K_{n+1} \otimes_{\mathbb{Q}_p} A)[\![t]\!] \\ &\qquad \xrightarrow{\sim} M^{(n+1)} \otimes_{\mathcal{R}_A^{(n+1)}(\pi_K),\iota_{n+1}} (K_{n+1} \otimes_{\mathbb{Q}_p} A)[\![t]\!] = \boldsymbol{D}^+_{\mathrm{dif},n+1}(M). \end{aligned}$$

Using this isomorphism, we obtain Γ_K-equivariant $(K_n \otimes_{\mathbb{Q}_p} A)[\![t]\!]$-linear morphisms

$$\mathrm{can} : \boldsymbol{D}^+_{\mathrm{dif},n}(M) \xrightarrow{x \mapsto x \otimes 1} \boldsymbol{D}^+_{\mathrm{dif},n}(M) \otimes_{(K_n \otimes_{\mathbb{Q}_p} A)[\![t]\!]} (K_{n+1} \otimes_{\mathbb{Q}_p} A)[\![t]\!] \xrightarrow{\sim} \boldsymbol{D}^+_{\mathrm{dif},n+1}(M)$$

and

$$\begin{aligned} \frac{1}{p} \cdot \mathrm{Tr}_{K_{n+1}/K_n} : \boldsymbol{D}^+_{\mathrm{dif},n+1}(M) &\xrightarrow{\sim} \boldsymbol{D}^+_{\mathrm{dif},n}(M) \otimes_{(K_n \otimes_{\mathbb{Q}_p} A)[\![t]\!]} (K_{n+1} \otimes_{\mathbb{Q}_p} A)[\![t]\!] \\ &\xrightarrow{x \otimes f \mapsto \frac{1}{p} \cdot \mathrm{Tr}_{K_{n+1}/K_n}(f) x} \boldsymbol{D}^+_{\mathrm{dif},n}(M). \end{aligned}$$

These naturally induce $\mathrm{can} : \boldsymbol{D}_{\mathrm{dif},n}(M) \to \boldsymbol{D}_{\mathrm{dif},n+1}(M)$ and $\frac{1}{p} \cdot \mathrm{Tr}_{K_{n+1}/K_n} : \boldsymbol{D}_{\mathrm{dif},n+1}(M) \to \boldsymbol{D}_{\mathrm{dif},n}(M)$, and we have the commutative diagrams

$$\begin{array}{ccc} M^{(n)} & \xrightarrow{\iota_n} & \boldsymbol{D}^+_{\mathrm{dif},n}(M) \\ \big\downarrow \varphi & & \big\downarrow \mathrm{can} \\ M^{(n+1)} & \xrightarrow{\iota_{n+1}} & \boldsymbol{D}^+_{\mathrm{dif},n+1}(M) \end{array}$$

and

$$\begin{array}{ccc} M^{(n+1)} & \xrightarrow{\iota_{n+1}} & \boldsymbol{D}^+_{\mathrm{dif},n+1}(M) \\ \big\downarrow \psi & & \big\downarrow \frac{1}{p} \cdot \mathrm{Tr}_{K_{n+1}/K_n} \\ M^{(n)} & \xrightarrow{\iota_n} & \boldsymbol{D}^+_{\mathrm{dif},n}(M) \end{array}$$

Put $\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M) := \varinjlim_{n \geq n(M)} \boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)$, where the transition map is $\mathrm{can} : \boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M) \to \boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M)$. Then we have

$$\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M) = \boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M) \otimes_{(K_n \otimes_{\mathbb{Q}_p} A)[\![t]\!]} (K_\infty \otimes_{\mathbb{Q}_p} A)[\![t]\!]$$

for any $n \geq n(M)$, where we define $(K_\infty \otimes_{\mathbb{Q}_p} A)[\![t]\!] = \bigcup_{m \geq 1} (K_m \otimes_{\mathbb{Q}_p} A)[\![t]\!]$.

For an $A[\Gamma_K]$-module N, we define a complex of A-modules concentrated in degree [0, 1] by

$$C^\bullet_\gamma(N) = \big[N^\Delta \xrightarrow{\gamma - 1} N^\Delta\big]$$

and denote by $\mathrm{H}^i_\gamma(N)$ the cohomology of $C^\bullet_\gamma(N)$. If N is a topological Hausdorff A-module with a continuous action of Γ_K, the complex $C^\bullet_\gamma(N)$ is also independent of the choice of γ up to canonical isomorphism.

Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$. For $n \geq n(M)$ and $M_0 = M, M[1/t]$, we define a complex $\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)})$ concentrated in degree [0, 2] by

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) := \big[M_0^{(n),\Delta} \xrightarrow{(\gamma-1)\oplus(\varphi-1)} M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} \xrightarrow{(\varphi-1)\oplus(1-\gamma)} M_0^{(n+1),\Delta}\big].$$

Of course, we have $\varinjlim_n \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) = C^{\bullet}_{\varphi,\gamma}(M_0)$, where the transition map is the natural one induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$. We define another complex

$$C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) := \varinjlim_{n,\varphi} \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}),$$

where the transition map is the natural one induced by $\varphi : M_0^{(n)} \to M_0^{(n+1)}$. We similarly define

$$C^{(\varphi),\bullet}_{\gamma}(M_0) := \varinjlim_{n,\varphi} C^{\bullet}_{\gamma}(M_0^{(n)})$$

and denote by $\mathrm{H}^{(\varphi),i}_{\varphi,\gamma}(M_0)$ (resp. $\mathrm{H}^{(\varphi),i}_{\gamma}(M_0)$) the cohomology of $C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ (resp. $C^{(\varphi),\bullet}_{\gamma}(M_0)$). For $n \geq n(M)$, we equip $C^{\bullet}_{\gamma}(M_0^{(n)})$ with a structure of a complex of F-vector spaces by $ax := \varphi^n(a)x$ for $a \in F$, $x \in C^{\bullet}_{\gamma}(M_0^{(n)})$. Then $C^{(\varphi),\bullet}_{\gamma}(M_0)$ (resp. $\mathrm{H}^{(\varphi),i}_{\gamma}(M_0)$) is also naturally equipped with a structure of a complex of F-vector spaces (resp. an F-vector space).

By the compatibility of $\varphi : M^{(n)} \hookrightarrow M^{(n+1)}$ and $\mathrm{can} : \boldsymbol{D}^{+}_{\mathrm{dif},n}(M) \hookrightarrow \boldsymbol{D}^{+}_{\mathrm{dif},n+1}(M)$ with respect to the map $\iota_n : M^{(n)} \to \boldsymbol{D}^{+}_{\mathrm{dif},n}(M)$, the map ι_n induces canonical maps

$$\iota : C^{(\varphi),\bullet}_{\gamma}(M) \to C^{\bullet}_{\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif}}(M)) \quad \text{and} \quad \iota : C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \to C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M)),$$

which are $(F \otimes_{\mathbb{Q}_p} A)$-linear.

Lemma 2.17. *For $n \geq n(M)$, the natural maps*

$$C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M)), \quad C^{\bullet}_{\gamma}(M_0^{(n)}) \to C^{\bullet}_{\gamma}(M_0^{(n+1)})$$

and

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n+1)})$$

for $M_0 = M, M[1/t]$, which are induced by φ, are quasi-isomorphism. Similarly, the maps

$$C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)), \quad C^{\bullet}_{\gamma}(M_0^{(n)}) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$$

and

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) \to C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$$

for $M_0 = M, M[1/t]$ are quasi-isomorphism.

Proof. The latter statement is trivial if we can prove the first statement. Let's prove the first statement. We first note that $\gamma - 1 : (M_0^{(n)})^{\psi=0} \to (M_0^{(n)})^{\psi=0}$ is an isomorphism for $n \geq n(M)+1$ by Theorem 3.1.1 of [Kedlaya et al. 2014] (precisely, this fact for $M_0 = M[1/t]$ follows from the proof of this theorem). Taking the base change of this isomorphism by the map $\iota_n : \mathcal{R}_A^{(n)}(\pi_K) \to (K_n \otimes_{\mathbb{Q}_p} A)[[t]]$, we also have that

$$\gamma - 1 : (\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))^{\frac{1}{p}\cdot\mathrm{Tr}_{K_n/K_{n-1}}=0} \to (\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))^{\frac{1}{p}\cdot\mathrm{Tr}_{K_n/K_{n-1}}=0}$$

is an isomorphism for $n \geq n(M)+1$. Using these facts, we prove the lemma as follows. Here, we only prove that the map $C^\bullet_\gamma(M_0^{(n)}) \to C^\bullet_\gamma(M_0^{(n+1)})$ induced by $\varphi : M_0^{(n)} \to M_0^{(n+1)}$ is an quasi-isomorphism for $n \geq n(M)$ since the other cases can be proved in the same way. Since we have a Γ_K-equivariant decomposition $M_0^{(n+1)} = \varphi(M_0^{(n)}) \oplus (M_0^{(n+1)})^{\psi=0}$, we obtain a decomposition

$$C^\bullet_\gamma(M_0^{(n+1)}) = \varphi(C^\bullet_\gamma(M_0^{(n)})) \oplus C^\bullet_\gamma((M_0^{(n+1)})^{\psi=0}).$$

Since the complex $C^\bullet_\gamma((M_0^{(n+1)})^{\psi=0})$ is acyclic by the above remark and $\varphi : M_0^{(n)} \to M_0^{(n+1)}$ is an injection, the map $\varphi : C^\bullet_\gamma(M_0^{(n)}) \to C^\bullet_\gamma(M_0^{(n+1)})$ is a quasi-isomorphism. □

For another canonical map, $C^\bullet_\gamma(M_0^{(n)}) \to C^\bullet_\gamma(M_0)$, which is induced by the canonical inclusion $M^{(n)} \hookrightarrow M$, we can show the following lemma.

Lemma 2.18. *For $n \geq n(M)$ and $M_0 = M, M[1/t]$, the inclusion*

$$\mathrm{H}^0_\gamma(M_0^{(n)}) \hookrightarrow \mathrm{H}^0_\gamma(M_0)$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is an isomorphism.

Proof. It suffices to show that $\mathrm{H}^0_\gamma(M_0^{(n)}) \hookrightarrow \mathrm{H}^0_\gamma(M_0^{(n+1)})$ is an isomorphism for each $n \geq n(M)$. We first prove this claim when A is a finite $\mathbb{Q}_p$-algebra. In this case, we may assume $A = \mathbb{Q}_p$. Since we have an inclusion $\iota_n : \mathrm{H}^0_\gamma(M_0^{(n)}) \hookrightarrow \mathrm{H}^0_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M))$ and the latter is a finite-dimensional $\mathbb{Q}_p$-vector space, $\mathrm{H}^0_\gamma(M_0^{(n)})$ is also finite-dimensional. Since $\varphi : C^\bullet_\gamma(M_0^{(n)}) \to C^\bullet_\gamma(M_0^{(n+1)})$ is a quasi-isomorphism for $n \geq n(M)$ by the above lemma, we get an isomorphism $\varphi : \mathrm{H}^0_\gamma(M_0^{(n)}) \xrightarrow{\sim} \mathrm{H}^0_\gamma(M_0^{(n+1)})$. In particular, the dimension of $\mathrm{H}^0_\gamma(M_0^{(n)})$ is independent of $n \geq n(M)$. Hence, the canonical inclusion $\mathrm{H}^0_\gamma(M_0^{(n)}) \hookrightarrow \mathrm{H}^0_\gamma(M_0^{(n+1)})$ is an isomorphism.

We next prove the claim for general A. By Lemma 6.4 of [Kedlaya and Liu 2010], there exists a strict inclusion $A \hookrightarrow \prod_{i=1}^k A_i$ of topological rings, in which each A_i is a finite algebra over a complete discretely valued field. If we similarly define the rings $\mathcal{R}_{A_i}^{(n)}(\pi_K)$, $\mathcal{R}_{A_i}(\pi_K)$, we can generalize the notions concerning (φ, Γ)-modules for $\mathcal{R}_{A_i}(\pi_K)$. In particular, the above claim holds for $M_{0,i} := M_0 \widehat{\otimes}_A A_i$ for each i.

Consider the following canonical diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \longrightarrow & M_0^{(n)} & \longrightarrow & M_0^{(n+1)} & \longrightarrow & M_0^{(n+1)}/M_0^{(n)} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n)} & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n+1)} & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n+1)}/M_{0,i}^{(n)} & \longrightarrow 0 \end{array}$$

If we can show that the right vertical arrow is an injection, then the claim for A follows from the claim for each A_i by a simple diagram chase. To show that the right vertical arrow is an injection, we may assume that $M = \mathcal{R}_A(\pi_K)$ since $M^{(n)}$ is finite projective over $\mathcal{R}_A^{(n)}(\pi_K)$ for each n. Then the natural map

$$\mathcal{R}_A^{(n+1)}(\pi_K)[1/t]/\mathcal{R}_A^{(n)}(\pi_K)[1/t] \to \prod_{i=1}^{k} \mathcal{R}_{A_i}^{(n+1)}(\pi_K)[1/t]/\mathcal{R}_{A_i}^{(n)}(\pi_K)[1/t]$$

is an injection since the inclusion $A \hookrightarrow \prod_{i=1}^k A_i$ is strict, which proves the claim for general A, hence proves the lemma. □

Remark 2.19. We don't know whether the natural map $\mathrm{H}^1_\gamma(M_0^{(n)}) \to \mathrm{H}^1_\gamma(M_0)$ induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is an isomorphism or not.

For the (φ, Γ)-cohomology, we can prove the following lemma.

Lemma 2.20. (1) *For $n \geq n(M)$ and for $M_0 = M$, $M[1/t]$, the map*

$$\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to C^\bullet_{\varphi,\gamma}(M_0)$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is a quasi-isomorphism.

(2) *In $\boldsymbol{D}^-(A)$, the isomorphism*

$$C^\bullet_{\varphi,\gamma}(M_0) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0),$$

which is obtained as the composition of the inverse of the isomorphism in (1), $\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(M_0)$, *with the isomorphism* $\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ *in Lemma 2.17, is independent of the choice of $n \geq n(M)$.*

Proof. For $n \geq n(M)$, we define a map $f_\bullet : \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})[+1]$ by

$$f_1 : M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} \to M_0^{(n+1),\Delta} : (x, y) \mapsto y,$$
$$f_2 : M_0^{(n+1),\Delta} \to M_0^{(n+1),\Delta} \oplus M_0^{(n+2),\Delta} : x \mapsto (x, 0).$$

This gives a homotopy between

$$\varphi : \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})$$

and

$$\mathrm{can} : \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$. Hence, can : $\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})$ is also an isomorphism by Lemma 2.17, and, by taking the limit, the map $\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to C^\bullet_{\varphi,\gamma}(M_0)$ is also an isomorphism, which proves (1).

In a similar way, we can show that the map can : $C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \to C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ induced by the canonical inclusions can : $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$ for any $n \geq n(M)$ is homotopic to the identity map. Hence, we obtain the following commutative diagram in $\boldsymbol{D}^-(A)$ for any $n \geq n(M)$:

$$\begin{array}{ccc} \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) & \longrightarrow & C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \\ \big\downarrow \text{can} & & \big\downarrow \text{id} \\ \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)}) & \longrightarrow & C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \end{array}$$

From this we obtain the second statement in the lemma. □

We define a morphism

$$f : C^\bullet_{\varphi,\gamma}(M_0) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$$

in $\boldsymbol{D}^-(A)$ as the composition of the isomorphism $C^\bullet_{\varphi,\gamma}(M_0) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ in Lemma 2.20(2) with the map $C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$, which is induced by

$$\begin{array}{ccccc} \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) = \big[M_0^{(n),\Delta} & \xrightarrow{(\gamma-1)\oplus(\varphi-1)} & M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} & \xrightarrow{(\varphi-1)\oplus(1-\gamma)} & M_0^{(n+1),\Delta}\big] \\ \big\downarrow \qquad\qquad \big\downarrow \text{id} & & \big\downarrow (x,y)\mapsto x & & \\ C^\bullet_{\gamma}(M_0^{(n)}) = \big[M_0^{(n),\Delta} & \xrightarrow{\gamma-1} & M_0^{(n),\Delta}\big] & & \end{array}$$

We define

$$g : C^\bullet_{\varphi,\gamma}(M) \xrightarrow{f} C^{(\varphi),\bullet}_{\gamma}(M) \xrightarrow{\iota} C^\bullet_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M))$$

and let

$$\text{can} : C^{(\varphi),\bullet}_{\gamma}(M_0) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$$

be the map induced by the canonical inclusion can : $M_0^{(n)} \to M_0^{(n+1)}$ for each $n \geq n(M)$. Under this notation, we prove the following proposition, which is a modified version of Theorem 2.8 of [Nakamura 2014a].

Proposition 2.21. *We have a functorial map between the two distinguished triangles*

$$\begin{array}{ccccccc} C^\bullet_{\varphi,\gamma}(M) & \xrightarrow{d_1} & \begin{matrix} C^\bullet_{\varphi,\gamma}(M[1/t]) \\ \oplus C^\bullet_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \end{matrix} & \xrightarrow{d_2} & C^\bullet_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M)) & \xrightarrow{[+1]} & \\ \big\downarrow \text{id} & & \big\downarrow f\oplus\text{id} & & \big\downarrow x\mapsto(0,x) & & \\ C^\bullet_{\varphi,\gamma}(M) & \xrightarrow{d_3} & \begin{matrix} C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \\ \oplus C^\bullet_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \end{matrix} & \xrightarrow{d_4} & \begin{matrix} C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \\ \oplus C^\bullet_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M)) \end{matrix} & \xrightarrow{[+1]} & \end{array} \qquad (4)$$

with

$$d_1(x) = (x, g(x)), \qquad d_2(x, y) = g(x) - y,$$
$$d_3(x) = (f(x), g(x)), \quad d_4(x, y) = ((\mathrm{can} - 1)x, g(x) - y).$$

Remark 2.22. In §2 of [Nakamura 2014a], we (essentially) proved that the top horizontal line in the proposition is a distinguished triangle. For the application to the local ε-conjecture, we also need the bottom triangle, which involves $\boldsymbol{D}^K_{\mathrm{cris}}(M) := \mathrm{H}^0_\gamma(M[1/t])$.

Proof of Proposition 2.21. We first show that the top horizontal line is a distinguished triangle. Actually, this is the content of Theorem 2.8 of [Nakamura 2014a], but we briefly recall the proof since we also use it to prove that the bottom line is a distinguished triangle. In this proof, we assume $\Delta = \{1\}$ for simplicity; the general case follows by just taking the Δ-fixed parts.

For $n \geq n(M)$, we have the exact sequence of A-modules

$$0 \to M^{(n)} \xrightarrow{c_1} M^{(n)}[1/t] \oplus \prod_{m \geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \xrightarrow{c_2} \bigcup_{k \geq 0} \prod_{m \geq n} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \to 0 \tag{5}$$

with

$$c_1(x) = (x, (\iota_m(x))_{m \geq n}) \quad \text{and} \quad c_2(x, (y_m)_{m \geq n}) = (\iota_m(x) - y_m)_{m \geq n}$$

by Lemma 2.9 of [Nakamura 2014a] (precisely, we proved it when A is a finite $\mathbb{Q}_p$-algebra, but we can prove it for general A in the same way). For $n \geq n(M)$ and $k \geq 0$, we define a complex $\widetilde{C}^\bullet_{\varphi,\gamma}\big(\frac{1}{t^k}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\big)$ concentrated in degree in [0, 2] by

$$\Big[\prod_{m \geq n} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \xrightarrow{b_0} \prod_{m \geq n} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \oplus \prod_{m \geq n+1} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \xrightarrow{b_1} \prod_{m \geq n+1} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\Big] \tag{6}$$

with

$$b_0((x_m)_{m \geq n}) = \big(((\gamma - 1)x_m)_{m \geq n}, (x_{m-1} - x_m)_{m \geq n+1}\big)$$

and

$$b_1((x_m)_{m \geq n}, (y_m)_{m \geq n+1}) = ((x_{m-1} - x_m) - (\gamma - 1)y_m)_{m \geq n+1}.$$

Put $\widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M)) = \bigcup_{k \geq 0} \widetilde{C}^\bullet_{\varphi,\gamma}\big(\frac{1}{t^k}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\big)$. By the exact sequence (5), we obtain the following exact sequence of complexes of A-modules:

$$0 \to \widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M)) \to 0. \tag{7}$$

Moreover, the map $C^\bullet_\gamma(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M))$, which is defined by

$$\begin{array}{ccccc} \boldsymbol{D}^+_{\mathrm{dif},n}(M) & \xrightarrow{\gamma-1} & \boldsymbol{D}^+_{\mathrm{dif},n}(M) & & \\ \big\downarrow{\scriptstyle x\mapsto (x)_{m\geq n}} & & \big\downarrow{\scriptstyle x\mapsto ((x)_{m\geq n},0)} & & \\ \prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) & \to & \begin{array}{c}\prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \\ \oplus \prod_{m\geq n+1} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\end{array} & \to & \prod_{m\geq n+1} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \end{array} \tag{8}$$

and the similar map $C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))$ are easily seen to be quasi-isomorphisms since we have the exact sequence

$$0 \to \boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M) \xrightarrow{x\mapsto (x)_{m\geq n}} \prod_{m\geq n} \boldsymbol{D}^{(+)}_{\mathrm{dif},m}(M) \xrightarrow{(x_m)_{m\geq n}\mapsto (x_{m-1}-x_m)_{m\geq n+1}} \prod_{m\geq n+1} \boldsymbol{D}^{(+)}_{\mathrm{dif},m}(M) \to 0. \tag{9}$$

Put $\widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) := \varinjlim_{n,a^\bullet} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))$, where the transition map

$$a^\bullet : \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M))$$

is defined by

$$\begin{aligned} a^0((x_m)_{m\geq n}) &= (x_m)_{m\geq n+1}, \\ a^1((x_m)_{m\geq n}, (y_m)_{m\geq n+1}) &= ((x_m)_{m\geq n+1}, (y_m)_{m\geq n+2}), \\ a^2((x_m)_{m\geq n+1}) &= (x_m)_{m\geq n+2}. \end{aligned}$$

We also define $\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) := \varinjlim_{n,(a')^\bullet} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))$, where the transition map $(a')^\bullet$ is defined by

$$\begin{aligned} (a')^0((x_m)_{m\geq n}) &= (x_{m-1})_{m\geq n+1}, \\ (a')^1((x_m)_{m\geq n}, (y_m)_{m\geq n+1}) &= ((x_{m-1})_{m\geq n+1}, (y_{m-1})_{m\geq n+2}), \\ (a')^2((x_m)_{m\geq n+1}) &= (x_{m-1})_{m\geq n+2}. \end{aligned}$$

Then it is easy to see that the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \xrightarrow{\sim} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))$ defined in (8) is compatible with the transition maps $a^\bullet$, $(a')^\bullet$ and $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \hookrightarrow C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M))$, hence induces quasi-isomorphisms

$$C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \xrightarrow{\sim} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)), \quad C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \xrightarrow{\sim} \widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)). \tag{10}$$

For $\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$, we also have a left inverse

$$\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \to C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \tag{11}$$

of the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \to \widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$, which is obtained as the limit of the map

$$\begin{array}{ccccc} \prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) & \longrightarrow & \begin{array}{c}\prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\\ \oplus\prod_{m\geq n+1} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\end{array} & \longrightarrow & \prod_{m\geq n+1} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \\ \Big\downarrow{\scriptstyle (x_m)_{m\geq n}\mapsto x_n} & & \Big\downarrow{\scriptstyle ((x_m)_{m\geq n},(y_m)_{m\geq n+1})\mapsto x_n} & & \\ \boldsymbol{D}^+_{\mathrm{dif},n}(M) & \xrightarrow{\gamma-1} & \boldsymbol{D}^+_{\mathrm{dif},n}(M) & & \end{array}$$

Taking the limits of the map $\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) : x \mapsto (\iota_m(x))_{m\geq n_0}$ ($n_0 = n, n+1$), we obtain the maps

$$C^\bullet_{\varphi,\gamma}(M) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \quad \text{and} \quad C^{(\varphi),\bullet}_{\varphi,\gamma}(M) \to \widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)). \tag{12}$$

Taking the limit of the exact sequence (7) with respect to the transition map induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$ and $a_\bullet$, and taking the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \xrightarrow{\sim} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$ in (10), we obtain the following exact triangle, which is the top horizontal line in the proposition:

$$C^\bullet_{\varphi,\gamma}(M) \xrightarrow{d_1} C^\bullet_{\varphi,\gamma}(M[1/t]) \oplus C^\bullet_\gamma(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \xrightarrow{d_2} C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M)) \xrightarrow{[+1]} .$$

On the other hand, since we have

$$C^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) = \mathrm{Cone}\big(1-\varphi : C^\bullet_\gamma(M^{(n)}[1/t]) \to C^\bullet_\gamma(M^{(n+1)}[1/t])\big)[-1]$$

for $n \geq n(M)$ (where we define $\mathrm{Cone}(f : M^\bullet \to N^\bullet)[-1]^n = M^n \oplus N^{n-1}$ and $d : M^n \oplus N^{n-1} \to M^{n+1} \oplus N^n : (x, y) \mapsto (d_M(x), -f(x) - d_N(y))$), taking the limit of the exact sequence (7) with respect to the transition map induced by $a'_\bullet$ and $\varphi : M_0^{(n)} \hookrightarrow M_0^{(n+1)}$, and taking the left inverse $\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \to C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$ in (11), and identifying $C^\bullet_{\varphi,\gamma}(M) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M)$ by Lemma 2.20(2), we obtain the following exact triangle, which is the bottom horizontal line in the proposition:

$$\begin{aligned} C^\bullet_{\varphi,\gamma}(M) &\xrightarrow{d_3} C^{(\varphi),\bullet}_\gamma(M[1/t]) \oplus C^\bullet_\gamma(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \\ &\xrightarrow{d_4} C^{(\varphi),\bullet}_\gamma(M[1/t]) \oplus C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M)) \xrightarrow{[+1]} . \end{aligned}$$

Here $d_3(x) = (f(x), g(x))$ and $d_4(x, y) = ((\mathrm{can}-1)(x), g(x) - y)$, which proves the proposition. □

We next recall some notions concerning p-adic Hodge theory for (φ, Γ)-modules over the Robba ring. For a (φ, Γ)-module M over $\mathcal{R}_A(\pi_K)$, let us define

$$\boldsymbol{D}^K_{\mathrm{dR}}(M) := \mathrm{H}^0_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M)) \quad \text{and} \quad \boldsymbol{D}^K_{\mathrm{dR}}(M)^i := \boldsymbol{D}^K_{\mathrm{dR}}(M) \cap t^i \boldsymbol{D}^+_{\mathrm{dif}}(M)$$

for $i \in \mathbb{Z}$, and

$$\boldsymbol{D}^K_{\mathrm{cris}}(M) := \mathrm{H}^0_\gamma(M[1/t]).$$

By Lemma 2.17, $\varphi : C^{\bullet}_{\gamma}(M[1/t]) \to C^{\bullet}_{\gamma}(M[1/t])$ induces a φ-semilinear automorphism

$$\varphi : \boldsymbol{D}^{K}_{\mathrm{cris}}(M) \xrightarrow{\sim} \boldsymbol{D}^{K}_{\mathrm{cris}}(M).$$

More precisely, by Lemma 2.18, we have $\boldsymbol{D}_{\mathrm{cris}}(M) = \mathrm{H}^0_{\gamma}(M^{(n)}[1/t])$, and φ induces an automorphism $\varphi : \mathrm{H}^0_{\gamma}(M^{(n)}[1/t]) \xrightarrow{\varphi} \mathrm{H}^0_{\gamma}(M^{(n+1)}[1/t]) = \mathrm{H}^0_{\gamma}(M^{(n)}[1/t])$ for $n \geq n(M)$. Using these facts, we define an isomorphism

$$j_1 : \boldsymbol{D}^{K}_{\mathrm{cris}}(M) = \mathrm{H}^0_{\gamma}(M^{(n)}[1/t]) \xrightarrow{\varphi^n} \mathrm{H}^0_{\gamma}(M^{(n)}[1/t]) \xrightarrow{\sim} \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]),$$

which does not depend on the choice of n. Then the map $\iota : C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \to C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M))$ induces an $(F \otimes_{\mathbb{Q}_p} A)$-linear injection

$$\iota : \boldsymbol{D}^{K}_{\mathrm{cris}}(M) \xrightarrow{j_1} \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]) \xrightarrow{\iota} \boldsymbol{D}^{K}_{\mathrm{dR}}(M).$$

We define another isomorphism

$$j_2 : \boldsymbol{D}^{K}_{\mathrm{cris}}(M) \xrightarrow{j_1} \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]) \xrightarrow{\mathrm{can}} \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]),$$

where $\mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]) \xrightarrow{\mathrm{can}} \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t])$ is the map induced by

$$\mathrm{can} : C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \to C^{(\varphi),\bullet}_{\gamma}(M[1/t]),$$

which is an isomorphism by Lemma 2.20. Then we obtain the commutative diagram

$$\begin{array}{ccc} \boldsymbol{D}^{K}_{\mathrm{cris}}(M) & \xrightarrow{1-\varphi} & \boldsymbol{D}^{K}_{\mathrm{cris}}(M) \\ \big\downarrow{\scriptstyle j_1} & & \big\downarrow{\scriptstyle j_2} \\ \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]) & \xrightarrow{\mathrm{can}-\mathrm{id}} & \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]) \end{array}$$

Let us denote by

$$\exp_M : \boldsymbol{D}^{K}_{\mathrm{dR}}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M), \quad \exp_{f,M} : \boldsymbol{D}^{K}_{\mathrm{cris}}(M) \xrightarrow{j_2} \mathrm{H}^{(\varphi),0}_{\gamma}(M[1/t]) \to \mathrm{H}^1_{\varphi,\gamma}(M)$$

the boundary maps obtained by taking the cohomology of the exact triangles in Proposition 2.21. We define

$$\mathrm{H}^1_{\varphi,\gamma}(M)_e = \mathrm{Im}\big(\boldsymbol{D}^{K}_{\mathrm{dR}}(M) \xrightarrow{\exp_M} \mathrm{H}^1_{\varphi,\gamma}(M)\big)$$

and

$$\mathrm{H}^1_{\varphi,\gamma}(M)_f = \mathrm{Im}\big(\boldsymbol{D}^{K}_{\mathrm{cris}}(M) \oplus \boldsymbol{D}^{K}_{\mathrm{dR}}(M) \xrightarrow{\exp_{f,M} \oplus \exp_M} \mathrm{H}^1_{\varphi,\gamma}(M)\big).$$

We call the latter group the finite cohomology. Put $t_M(K) := \boldsymbol{D}^K_{\mathrm{dR}}(M)/\boldsymbol{D}^K_{\mathrm{dR}}(M)^0$. By Proposition 2.21, we obtain the diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{H}^0_{\varphi,\gamma}(M) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(M)^{\varphi=1} & \xrightarrow{x\mapsto\overline{\iota(x)}} & t_M(K) & \xrightarrow{\exp_M} & \mathrm{H}^1_{\varphi,\gamma}(M)_e \longrightarrow 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto x} & & \downarrow{\scriptstyle x\mapsto(0,x)} & & \downarrow{\scriptstyle x\mapsto x} \\ 0 \longrightarrow \mathrm{H}^0_{\varphi,\gamma}(M) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(M) & \xrightarrow{d_5} & \boldsymbol{D}^K_{\mathrm{cris}}(M)\oplus t_M(K) & \xrightarrow{d_6} & \mathrm{H}^1_{\varphi,\gamma}(M)_f \longrightarrow 0 \end{array} \tag{13}$$

with

$$d_5(x, y) = ((1-\varphi)x, \overline{\iota(x)}) \quad \text{and} \quad d_6 = \exp_{f,M} \oplus \exp_M,$$

where we also define $\exp_M : t_M(K) \to \mathrm{H}^1_{\varphi,\gamma}(M)$, which is naturally induced by $\exp_M : \boldsymbol{D}^K_{\mathrm{dR}}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M)$.

By the proof of Proposition 2.21, we obtain the following explicit formulae for $\exp_M$ and $\exp_{f,M}$, which are very important in the proof of our main theorem (Theorem 1.3).

Proposition 2.23. (1) *For $x \in \boldsymbol{D}^K_{\mathrm{dR}}(M)$, take $\tilde{x} \in M^{(n)}[1/t]^{\Delta}$ $(n \geq n(M))$ such that*

$$\iota_m(\tilde{x}) - x \in \boldsymbol{D}^+_{\mathrm{dif},m}(M)$$

for any $m \geq n$ (such an $\tilde{x}$ exists by the exact sequence (5) *in the proof of Proposition 2.21). Then we have*

$$\exp_M(x) = [(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}] \in \mathrm{H}^1_{\varphi,\gamma}(M).$$

(2) *For $x \in \boldsymbol{D}^K_{\mathrm{cris}}(M)$, take $\tilde{x} \in M^{(n)}[1/t]^{\Delta}$ $(n \geq n(M))$ such that*

$$\iota_n(\tilde{x}) \in \boldsymbol{D}^+_{\mathrm{dif},n}(M)$$

and

$$\iota_{n+k}(\tilde{x}) - \sum_{l=1}^{k} \iota_{n+l}(\varphi^n(x)) \in \boldsymbol{D}^+_{\mathrm{dif},n+k}(M)$$

for any $k \geq 1$ (we remark that we have $\varphi^n(x) \in M^{(n)}[1/t]$ by Lemma 2.18 and that such an $\tilde{x}$ exists by the exact sequence (5)*). Then we have*

$$\exp_{f,M}(x) = [(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x} + \varphi^n(x)] \in \mathrm{H}^1_{\varphi,\gamma}(M).$$

Proof. These formulae directly follow from simple but a little bit long diagram chases in the proof of Proposition 2.21. For the convenience of the reader, we give a proof of these formulae.

We first prove formula (1). By the proof of Proposition 2.21, the above exact triangle in this proposition is obtained by taking the limit of the composition of the quasi-isomorphism

$$\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \xrightarrow{\sim} \mathrm{Cone}\big(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] := C^\bullet_1$$

(obtained by the exact sequence (7)) with the inverse of the quasi-isomorphism

$$C^\bullet_2 := \mathrm{Cone}\big(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^\bullet_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to C^\bullet_{\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] \xrightarrow{\sim} C^\bullet_1$$

induced by the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) : x \mapsto (x)_{m\geq n}$ of (10).

By definition of $\exp_M(-)$, for $x \in \mathrm{H}^0_\gamma(\boldsymbol{D}_{\mathrm{dif},n}(M))$, these quasi-isomorphisms send $\exp_M(x)$ $\big($which we see as an element of $\mathrm{H}^1(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}))\big)$ to the element $[0, 0, x] \in \mathrm{H}^1(C^\bullet_2)$ represented by

$$(0, 0, x) \in \widetilde{C}^1_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^1_\gamma(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \oplus C^0_\gamma(\boldsymbol{D}_{\mathrm{dif},n}(M)).$$

Take $\tilde{x} \in M^{(n)}[1/t]^\Delta$ satisfying the condition in (1). Then it suffices to show that $[(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}] \in \mathrm{H}^1(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}))$ and $[0, 0, x] \in \mathrm{H}^1(C^\bullet_2)$ are the same element in $\mathrm{H}^1(C^\bullet_1)$. By definition, $[(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}]$ is sent to

$$\big[((\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}), \big((\iota_m((\gamma-1)\tilde{x}))_{m\geq n}, (\iota_m((\varphi-1)\tilde{x}))_{m\geq n+1}\big), 0\big]$$

and $[0, 0, x]$ is sent to

$$[0, 0, (-x)_{m\geq n}]$$

in $\mathrm{H}^1(C^\bullet_1)$. Both are represented by elements of

$$\widetilde{C}^1_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \widetilde{C}^1_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \oplus \widetilde{C}^0_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))$$

(we note the sign; for $f : C^\bullet \to D^\bullet$, we define $D^{\bullet-1} \to \mathrm{Cone}(C^\bullet \to D^\bullet)[-1]$ by $x \mapsto (-x, 0)$ and $\mathrm{Cone}(C^\bullet \to D^\bullet)[-1] \to C^\bullet$ by $(x, y) \mapsto y$). Then it is easy to check that the difference of these two elements is the coboundary of the element

$$(\tilde{x}, (\iota_m(\tilde{x}) - x)_{m\geq n}) \in C^0_1 = M^{(n)}[1/t]^\Delta \oplus \prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M)^\Delta,$$

which proves (1).

We next prove (2). The bottom exact triangle in Proposition 2.21 is obtained by taking the limit of the composition of the quasi-isomorphism $\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \xrightarrow{\sim} C^\bullet_1$ defined above with the quasi-isomorphism

$$C^\bullet_1 \xrightarrow{\sim} \mathrm{Cone}\big(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^\bullet_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to C^\bullet_{\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] := C^\bullet_3$$

induced by the map $\prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \to \boldsymbol{D}^+_{\mathrm{dif},n}(M) : (x_m)_{m\geq n} \to x_n$, with the inverse of the quasi-isomorphism

$$C_3^\bullet \xrightarrow{\sim} \mathrm{Cone}\big(C_\gamma^\bullet(M^{(n)}[1/t]) \oplus C_\gamma^\bullet(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \\ \to C_\gamma^\bullet(M^{(n+1)}[1/t]) \oplus C_\gamma^\bullet(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] := C_4^\bullet,$$

which is naturally obtained by the identity

$$\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t])) = \mathrm{Cone}\big(C_\gamma^\bullet(M^{(n)}[1/t]) \xrightarrow{1-\varphi} C_\gamma^\bullet(M^{(n+1)}[1/t])\big)[-1].$$

For $x' \in \mathrm{H}^0_\gamma(M^{(n+1)}[1/t])$, the image of x' by the first boundary map of the cone $C_4^\bullet$ is equal to $[0, 0, x', 0] \in \mathrm{H}^1(C_4^\bullet)$, which is represented by the element

$$(0, 0, x', 0) \in C^1_\gamma(M^{(n)}[1/t]) \oplus C^1_\gamma(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \oplus C^0_\gamma(M^{(n+1)}[1/t]) \oplus C^0_\gamma(\boldsymbol{D}_{\mathrm{dif},n}(M)).$$

Take $\tilde{x}' \in M^{(n)}[1/t]^\Delta$ such that

$$\iota_n(\tilde{x}') \in \boldsymbol{D}^+_{\mathrm{dif},n}(M) \quad \text{and} \quad \iota_{n+k}(\tilde{x}') - \sum_{l=1}^{k} \iota_{n+l}(x') \in \boldsymbol{D}^+_{\mathrm{dif},n+k}(M) \quad \text{for any } k \geq 1.$$

Then, by definition of the map $j_2 : \boldsymbol{D}^K_{\mathrm{cris}}(M) \xrightarrow{\sim} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t])$ and $\exp_{f,M}$, it suffices to show that the element $[(\gamma-1)\tilde{x}', (\varphi-1)\tilde{x}'+x'] \in \mathrm{H}^1(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}))$ is sent to $[0, 0, x', 0] \in \mathrm{H}^1(C_4^\bullet)$ by the above quasi-isomorphisms. By definition, the element $[(\gamma-1)\tilde{x}', (\varphi-1)\tilde{x}'+x']$ is sent to

$$[(\gamma-1)\tilde{x}', \iota_n((\gamma-1)\tilde{x}'), (\varphi-1)\tilde{x}'+x', 0] \in \mathrm{H}^1(C_4^\bullet)$$

by the above quasi-isomorphism. Then it is easy to check that the difference of this element with $[0, 0, x', 0]$ is the coboundary of the element

$$(\tilde{x}', \iota_n(\tilde{x}')) \in C_4^0 = M^{(n)}[1/t]^\Delta \oplus \boldsymbol{D}^+_{\mathrm{dif},n}(M)^\Delta,$$

which proves formula (2). □

We next generalize the Bloch–Kato duality concerning the finite cohomology for (φ, Γ)-modules. Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a (φ, Γ)-module over $\mathcal{R}_L(\pi_K)$. We say that M is de Rham if the equality $\dim_K \boldsymbol{D}^K_{\mathrm{dR}}(M) = [L:\mathbb{Q}_p]\cdot r_M$ holds. When M is de Rham, we have a natural L-bilinear perfect pairing

$$[-,-]_{\mathrm{dR}} : \boldsymbol{D}^K_{\mathrm{dR}}(M^*) \times \boldsymbol{D}^K_{\mathrm{dR}}(M) \xrightarrow{(f,x)\mapsto f(x)} \boldsymbol{D}^K_{\mathrm{dR}}(\mathcal{R}_L(1)) \\ = L \otimes_{\mathbb{Q}_p} K\frac{1}{t}\boldsymbol{e}_1 \xrightarrow{\frac{a}{t}\boldsymbol{e}_1 \mapsto \frac{1}{[K:\mathbb{Q}_p]}(\mathrm{id}\otimes \mathrm{tr}_{K/\mathbb{Q}_p})(a)} L, \quad (14)$$

which induces natural isomorphisms

$$\boldsymbol{D}^K_{\mathrm{dR}}(M) \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{dR}}(M^*)^\vee \quad \text{and} \quad \boldsymbol{D}^K_{\mathrm{dR}}(M)^0 \xrightarrow{\sim} t_{M^*}(K)^\vee.$$

We remark that, as in the étale case, we have

$$\mathrm{H}^1_{\varphi,\gamma}(M)_e = \mathrm{Ker}(\mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M[1/t]))$$

and

$$\mathrm{H}^1_{\varphi,\gamma}(M)_f = \mathrm{Ker}(\mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^{(\varphi),1}_{\gamma}(M[1/t]))$$

under the assumption that M is de Rham.

Proposition 2.24. *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L(\pi_K)$. Then $\mathrm{H}^1_{\varphi,\gamma}(M)_f$ is the orthogonal complement of $\mathrm{H}^1_{\varphi,\gamma}(M^*)_f$ with respect to the Tate duality pairing*

$$\langle -, - \rangle : \mathrm{H}^1_{\varphi,\gamma}(M^*) \times \mathrm{H}^1_{\varphi,\gamma}(M) \to L.$$

Proof. We remark that we have $\dim_L \mathrm{H}^1_{\varphi,\gamma}(M)_f = \dim_L(t_M(K)) + \dim_L \mathrm{H}^0_{\varphi,\gamma}(M)$ by the bottom exact sequence of (13). Using this formula for M, M^*, it is easy to check that we have $\dim_L \mathrm{H}^1_{\varphi,\gamma}(M)_f + \dim_L \mathrm{H}^1_{\varphi,\gamma}(M^*)_f = \dim_L \mathrm{H}^1_{\varphi,\gamma}(M)$ under the assumption that M is de Rham. Hence, it suffices to show that we have $\langle x, y \rangle = 0$ for any $x \in \mathrm{H}^1_{\varphi,\gamma}(M^*)_f$ and $y \in \mathrm{H}^1_{\varphi,\gamma}(M)_f$ by comparing the dimensions. By definition of $\mathrm{H}^1_{\varphi,\gamma}(-)_f$, this claim follows from Lemma 2.25 below. □

Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$ (we don't need to assume that M is de Rham). Using the isomorphism $j_2 : \boldsymbol{D}^K_{\mathrm{cris}}(M^*) \xrightarrow{\sim} \mathrm{H}^{(\varphi),0}_{\gamma}(M^*[1/t])$, define an A-bilinear pairing

$$\begin{aligned} h(-, -) : (\boldsymbol{D}^K_{\mathrm{cris}}(M^*) \oplus \boldsymbol{D}^K_{\mathrm{dR}}(M^*)) \times (\mathrm{H}^{(\varphi),1}_{\gamma}(M[1/t]) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M))) \\ \to \mathrm{H}^{(\varphi),1}_{\gamma}(M^* \otimes M[1/t]) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M^* \otimes M)) \end{aligned}$$

by

$$h((x, y), ([z], [w])) := ([j_2(x) \otimes z], [y \otimes w]).$$

Lemma 2.25. *For $(x, y) \in \boldsymbol{D}^K_{\mathrm{cris}}(M^*) \oplus \boldsymbol{D}^K_{\mathrm{dR}}(M^*)$ and $z \in \mathrm{H}^1_{\varphi,\gamma}(M)$, we have*

$$f_2(h((x, y), g(z))) = (\exp_{f,M^*}(x) + \exp_{M^*}(y)) \cup z \in \mathrm{H}^2_{\varphi,\gamma}(M^* \otimes M),$$

where

$$g : \mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^{(\varphi),1}_{\gamma}(M) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}^+_{\mathrm{dR}}(M))$$

is induced by d_3 and

$$f_2 : \mathrm{H}^{(\varphi),1}_{\gamma}(M^* \otimes M[1/t]) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M^* \otimes M)) \to \mathrm{H}^2_{\varphi,\gamma}(M^* \otimes M)$$

is the second boundary map of the bottom exact triangle of Proposition 2.21.

Proof. The equality $\exp_{M^*}(y)\cup z = f_2(h((0, y), g(z)))$, $y \in \boldsymbol{D}^K_{\mathrm{dR}}(M^*)$, $z \in \mathrm{H}^1_{\varphi,\gamma}(M)$, is proved in Lemma 2.13 of [Nakamura 2014a]. Hence, it suffices to show the equality

$$\exp_{f,M^*}(x)\cup z = f_2(h((x, 0), g(z)))$$

for $x \in \boldsymbol{D}^K_{\mathrm{cris}}(M^*)$, whose proof is also just a diagram chase similar to the proof of Proposition 2.23, hence we omit the proof. □

Finally, we compare our exponential map with the Bloch–Kato exponential map for p-adic representations V. Here, we assume that $A = \mathbb{Q}_p$ for simplicity. We can do the same things for any $L = A$ a finite $\mathbb{Q}_p$-algebra.

We want to compare the diagram (3) for V with the diagram (13) for $M = \boldsymbol{D}_{\mathrm{rig}}(V)$. More generally, as in §2.4 of [Nakamura 2014a], we compare a similar diagram defined below for a B-pair $W = (W_e, W^+_{\mathrm{dR}})$ with the diagram (13) for the associated (φ, Γ)-module $\boldsymbol{D}_{\mathrm{rig}}(W)$. For the definitions of B-pairs and the definition of the functor $W \mapsto \boldsymbol{D}_{\mathrm{rig}}(W)$, which gives an equivalence between the category of B-pairs and that of (φ, Γ)-modules over $\mathcal{R}(\pi_K)$, see [Nakamura 2014a, §2.5; Berger 2008a].

Let $W = (W_e, W^+_{\mathrm{dR}})$ be a B-pair for K. Put $W_{\mathrm{cris}} := \boldsymbol{B}_{\mathrm{cris}} \otimes_{\boldsymbol{B}_e} W_e$, which is naturally equipped with an action of φ. Since we have an exact sequence

$$0 \to \boldsymbol{B}^{\varphi=1}_{\mathrm{cris}} \to \boldsymbol{B}_{\mathrm{cris}} \xrightarrow{1-\varphi} \boldsymbol{B}_{\mathrm{cris}} \to 0,$$

we have a natural quasi-isomorphism (the vertical arrows) between the following two complexes of G_K-modules concentrated in degree [0, 1]:

$$\begin{array}{ccc} \big[W_e \oplus W^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto x-y} & W_{\mathrm{dR}}\big] \\ \Big\downarrow{\scriptstyle (x,y)\mapsto(x,y)} & & \Big\downarrow{\scriptstyle x\mapsto(0,x)} \\ \big[W_{\mathrm{cris}} \oplus W^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto((1-\varphi)x,x-y)} & W_{\mathrm{cris}} \oplus W_{\mathrm{dR}}\big] \end{array}$$

Put

$$C^\bullet_{\mathrm{cont}}(G_K, W) := \mathrm{Cone}\big(C^\bullet_{\mathrm{cont}}(G_K, W_e) \oplus C^\bullet_{\mathrm{cont}}(G_K, W^+_{\mathrm{dR}}) \to C^\bullet_{\mathrm{cont}}(G_K, W_{\mathrm{dR}})\big)[-1]$$

and

$$C^\bullet_{\mathrm{cont}}(G_K, W)' := \mathrm{Cone}\big(C^\bullet_{\mathrm{cont}}(G_K, W_{\mathrm{cris}}) \oplus C^\bullet_{\mathrm{cont}}(G_K, W^+_{\mathrm{dR}}) \to C^\bullet_{\mathrm{cont}}(G_K, W_{\mathrm{cris}}) \oplus C^\bullet_{\mathrm{cont}}(G_K, W_{\mathrm{dR}})\big)[-1].$$

We identify

$$\mathrm{H}^i(K, W) := \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_K, W)) = \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_K, W)')$$

by the above quasi-isomorphism. Put $\boldsymbol{D}^K_{\mathrm{cris}}(W) := \mathrm{H}^0(K, W_{\mathrm{cris}})$, $\boldsymbol{D}^K_{\mathrm{dR}}(W) := \mathrm{H}^0(K, W_{\mathrm{dR}})$, and $\boldsymbol{D}^K_{\mathrm{dR}}(W)^i := \boldsymbol{D}^K_{\mathrm{dR}}(W) \cap t^i W^+_{\mathrm{dR}}$ for $i \in \mathbb{Z}$. Taking the cohomology of the mapping cones, we obtain the similar diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{H}^0(K,W) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(W)^{\varphi=1} & \xrightarrow{x\mapsto \bar{x}} & t_W(K) & \xrightarrow{\exp_W} & \mathrm{H}^1_e(K,W) \longrightarrow 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x\mapsto x} & & \downarrow{\scriptstyle x\mapsto (0,x)} & & \downarrow{\scriptstyle x\mapsto x} \\ 0 \longrightarrow \mathrm{H}^0(K,W) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(W) & \xrightarrow{f} & \begin{array}{c}\boldsymbol{D}^K_{\mathrm{cris}}(W)\\ \oplus\, t_W(K)\end{array} & \xrightarrow{g} & \mathrm{H}^1_f(K,W) \longrightarrow 0 \end{array} \tag{15}$$

with

$$f(x,y) = ((1-\varphi)x, \bar{x}) \quad \text{and} \quad g = \exp_{f,W} \oplus \exp_W .$$

By definition, it is clear that the diagram (15) for the associated B-pair $W(V) := (\boldsymbol{B}_e \otimes_{\mathbb{Q}_p} V, \boldsymbol{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is canonically isomorphic to the diagram (3) for V defined by Bloch–Kato.

Our comparison result is the following.

Proposition 2.26. (1) *We have the following functorial isomorphisms*:

(i) $\mathrm{H}^i(K,W) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(W))$,

(ii) $\boldsymbol{D}^K_{\mathrm{dR}}(W)^j \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{dR}}(\boldsymbol{D}_{\mathrm{rig}}(W))^j$ *for* $j \in \mathbb{Z}$,

(iii) $\boldsymbol{D}^K_{\mathrm{cris}}(W) \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W))$.

(2) *The isomorphisms in* (1) *induce an isomorphism from the diagram* (15) *for* W *to the diagram* (13) *for* $\boldsymbol{D}_{\mathrm{rig}}(W)$.

Proof. We have already proved (i), (ii) of (1) and the comparison of the top exact sequence in (15) for W with that in (13) for $\boldsymbol{D}_{\mathrm{rig}}(W)$; see Theorem 2.21 of [Nakamura 2014a] or the references in the proof of this theorem.

Moreover, the isomorphism (iii) may be well known to the experts, but we give a proof of it since we couldn't find suitable references. In this proof, we freely use the notation in §2.5 of [Nakamura 2014a] or in [Berger 2008a]; please see these references. We first note that the inclusion $(\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e)^{G_K} \hookrightarrow \boldsymbol{D}^K_{\mathrm{cris}}(W)$ induced by the natural inclusion $\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}} := \bigcap_{n\geq 0} \varphi^n(\boldsymbol{B}^+_{\mathrm{cris}}) \hookrightarrow \boldsymbol{B}^+_{\mathrm{cris}}$ is an isomorphism since $\boldsymbol{D}^K_{\mathrm{cris}}(W)$ is a finite-dimensional $\mathbb{Q}_p$-vector space on which φ acts as an automorphism. Moreover, in the same way as the proof of Proposition 3.4 of [Berger 2002], we can show that the natural inclusion $(\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e)^{G_K} \hookrightarrow (\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e)^{G_K}$ is also an isomorphism. Since we have

$$\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e = \widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathcal{R}(\pi_K)[1/t]} \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$$

by definition of $\boldsymbol{D}_{\mathrm{rig}}(W)$, it suffices to show that the natural inclusion

$$\boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W)) \hookrightarrow \left(\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathcal{R}(\pi_K)[1/t]} \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]\right)^{G_K} =: D_0$$

is an isomorphism. Moreover, it suffices to show that $D_0 \subseteq \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$. This claim is proved as follows. Define $\mathcal{R}(\pi_K) \otimes_F D_0 \subseteq \widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}} \otimes_F D_0$, which are

(φ, Γ)-modules over $\mathcal{R}(\pi_K)$ (resp. (φ, G_K)-modules over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$). Then, by Théorème 1.2 of [Berger 2009], the natural map

$$\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}} \otimes_F D_0 \to \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathcal{R}(\pi_K)[1/t]} \boldsymbol{D}_{\mathrm{rig}}(W)[1/t] : a \otimes x \mapsto a \cdot x$$

(which is actually an inclusion) of (φ, G_K)-modules factors through $\mathcal{R}(\pi_K) \otimes_F D_0 \to \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$. In particular we have $D_0 \subseteq \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$, which proves the claim.

We next prove that the bottom exact sequence in (15) for W is isomorphic to that in (13) for $\boldsymbol{D}_{\mathrm{rig}}(W)$ by the isomorphisms in (1) of this proposition. Since the other commutativities are clear, or were already proved in Theorem 2.21 of [Nakamura 2014a], it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W)) & \xrightarrow{\exp_{f, \boldsymbol{D}_{\mathrm{rig}}(W)}} & \mathrm{H}^1_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(W)) \\ \downarrow \wr & & \downarrow \wr \\ \boldsymbol{D}^K_{\mathrm{cris}}(W) & \xrightarrow{\exp_{f,W}} & \mathrm{H}^1(K, W) \end{array} \tag{16}$$

In the same way as the proof of Theorem 2.21 of [Nakamura 2014a], we assume that $\Delta = \{1\}$, and using the canonical identifications

$$\mathrm{H}^1(K, W) \xrightarrow{\sim} \mathrm{Ext}^1(B, W), \quad \mathrm{H}^1_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(W)) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}(\pi_K), \boldsymbol{D}_{\mathrm{rig}}(W))$$

(where we denote by $B = (\boldsymbol{B}_e, \boldsymbol{B}^+_{\mathrm{dR}})$ the trivial B-pair). It suffices to show that, for $a \in \boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W))$, the extension corresponding to $\exp_{f, \boldsymbol{D}_{\mathrm{rig}}(W)}(a)$ is sent to the extension corresponding to $\exp_{f,W}(a)$ by the inverse functor $W(-)$ of $\boldsymbol{D}_{\mathrm{rig}}(-)$. We prove this claim as follows. Take $n \geq 1$ sufficiently large such that $a \in (\boldsymbol{D}^{(n)}_{\mathrm{rig}}(W)[1/t])^{\Gamma_K}$. Take $\tilde{a} \in \boldsymbol{D}^{(n)}_{\mathrm{rig}}[1/t]$ satisfying the condition in (2) of Proposition 2.23. Then, by (2) of Proposition 2.23, the extension D_a corresponding to $\exp_{f, \boldsymbol{D}_{\mathrm{rig}}(W)}(a)$ is written by

$$\left[0 \to \boldsymbol{D}_{\mathrm{rig}}(W) \xrightarrow{x \mapsto (x,0)} \boldsymbol{D}_{\mathrm{rig}}(W) \oplus \mathcal{R}(\pi_K)\boldsymbol{e} \xrightarrow{(x, y\boldsymbol{e}) \mapsto y} \mathcal{R}(\pi_K) \to 0\right]$$

such that

$$\varphi((x, y\boldsymbol{e})) = (\varphi(x) + \varphi(y)((\varphi - 1)\tilde{a} + \varphi^n(a)), \varphi(y)\boldsymbol{e})$$

and

$$\gamma((x, y\boldsymbol{e})) = (\gamma(x) + \gamma(y)(\gamma - 1)\tilde{a}, \gamma(y)\boldsymbol{e}).$$

(Here, we remark that there is a mistake in [Nakamura 2014a]; in the proof of Theorem 2.21 of [Nakamura 2014a], D_a should be defined by

$$\varphi((x, y\boldsymbol{e})) = (\varphi(x) + \varphi(y)(\varphi - 1)\tilde{a}, \varphi(y)\boldsymbol{e})$$

and

$$\gamma((x, y\boldsymbol{e})) = (\gamma(x) + \gamma(y)(\gamma - 1)\tilde{a}, \gamma(y)\boldsymbol{e}).)$$

On the other hand, by definition of $\exp_{f,W}$, the extension

$$W_a := \big(W_{e,a},\ W^+_{\mathrm{dR},a} := W^+_{\mathrm{dR}} \oplus \boldsymbol{B}^+_{\mathrm{dR}}\boldsymbol{e}_{\mathrm{dR}}\big)$$

corresponding to $\exp_{f,W}(a)$ is defined by

$$g(x, y\boldsymbol{e}_{\mathrm{dR}}) = (g(x), g(y)\boldsymbol{e}_{\mathrm{dR}})$$

for $x \in W^+_{\mathrm{dR}},\ y \in \boldsymbol{B}^+_{\mathrm{dR}},\ g \in G_K$, and $W_{e,a}$ is defined as the kernel of the surjection

$$W_{\mathrm{cris},a} := W_{\mathrm{cris}} \oplus \boldsymbol{B}_{\mathrm{cris}}\boldsymbol{e}_{\mathrm{cris}} \to W_{\mathrm{cris},a} : (x, y\boldsymbol{e}_{\mathrm{cris}}) \mapsto ((\varphi-1)x+\varphi(y)a, (\varphi-1)y\boldsymbol{e}_{\mathrm{cris}})$$

on which G_K acts by $g(\boldsymbol{e}_{\mathrm{cris}}) = \boldsymbol{e}_{\mathrm{cris}}$ (actually, this is equal to the kernel of the surjection

$$W_{\mathrm{rig},a} := W_{\mathrm{rig}} \oplus \widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t]\boldsymbol{e}_{\mathrm{cris}} \to W_{\mathrm{rig},a} : (x, y\boldsymbol{e}_{\mathrm{cris}}) \mapsto ((\varphi-1)x+\varphi(y)a, (\varphi-1)y\boldsymbol{e}_{\mathrm{cris}}),$$

where we define $W_{\mathrm{rig}} := \widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e$), and the isomorphism $\boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}_e} W_{e,a} \xrightarrow{\sim} \boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}$ is defined by

$$\boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}_e} W_{e,a} = \boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}_{\mathrm{cris}}} W_{\mathrm{cris},a} \xrightarrow{(x, y\boldsymbol{e}_{\mathrm{cris}}) \mapsto (x, y\boldsymbol{e}_{\mathrm{dR}})} \boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}.$$

Then, by definition of the functor $\boldsymbol{D}_{\mathrm{rig}}(-)$ in §2.2 of [Berger 2008a] (where the notation $D(-)$ is used), $\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}} \otimes_{\mathcal{R}^{(n)}(\pi_K)} \boldsymbol{D}^{(n)}_{\mathrm{rig}}(W_a)$ is equal to

$$\big\{x \in \widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_{e,a} \mid \iota_m(x) \in W^+_{\mathrm{dR},a} \text{ for any } m \geq n\big\}. \tag{17}$$

Since we have

$$\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_{e,a} = \widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}[1/t] \otimes_{\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t]} W_{\mathrm{rig},a},$$

and $\varphi^{-m}(\boldsymbol{e}_{\mathrm{cris}}) = \boldsymbol{e}_{\mathrm{cris}} - \sum_{k=1}^m \varphi^{-k}(a)$ for $m \geq 1$ and we have $\iota_{n+k} \circ \varphi^n = \varphi^{-k}$, it is easy to see that the group (17) is equal to

$$\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}} \otimes_{\mathcal{R}^{(n)}(\pi_K)} \boldsymbol{D}^{(n)}_{\mathrm{rig}}(W) \oplus \widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}(\tilde{a} + \varphi^n(\boldsymbol{e}_{\mathrm{cris}})),$$

which is easily seen to be isomorphic to $\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}} \otimes_{\mathcal{R}^{(n)}(\pi_K)} D^{(n)}_a$ as a (φ, G_K)-module. Therefore, we obtain the isomorphism

$$\boldsymbol{D}_{\mathrm{rig}}(W_a) \xrightarrow{\sim} D_a$$

as an extension by Théorème 1.2 of [Berger 2009], which proves the proposition. □

3. Local ε-conjecture for (φ, Γ)-modules over the Robba ring

From now on, we assume that $K = \mathbb{Q}_p$, and we freely omit the notation $\mathbb{Q}_p$, i.e., we use the notation Γ, $\mathcal{R}_A$, $\boldsymbol{D}_{\mathrm{dR}}(M)$, $\boldsymbol{D}_{\mathrm{cris}}(M)$, $t_M, \ldots$ instead of $\Gamma_{\mathbb{Q}_p}$, $\mathcal{R}_A(\pi_{\mathbb{Q}_p})$, $\boldsymbol{D}^{\mathbb{Q}_p}_{\mathrm{dR}}(M)$, $\boldsymbol{D}^{\mathbb{Q}_p}_{\mathrm{cris}}(M)$, $t_M(\mathbb{Q}_p), \ldots$. Moreover, since Kato's and our conjectures are formulated after fixing a $\mathbb{Z}_p$-basis $\zeta = \{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1)$, we also fix a parameter $\pi := \pi_\zeta$ of $\mathcal{R}_A$ and let $t = \log(1+\pi)$ as in Notation 2.2.

In this section, we formulate a conjecture which is a natural generalization of Kato's (p-adic) local ε-conjecture, where the main objects were p-adic or torsion representations of $G_{\mathbb{Q}_p}$, for (φ, Γ)-modules over the relative Robba ring $\mathcal{R}_A$. Since the article [Kato 1993b], in which the conjecture was stated, remains unpublished, and since the compatibility of our conjecture with his conjecture is an important part of our conjecture, here we also recall his original conjecture.

3A. ***Determinant functor.*** Kato's and our conjectures are formulated using the theory of the determinant functor. In this subsection, we briefly recall this theory following [Knudsen and Mumford 1976] and §2.1 of [Kato 1993a].

Let R be a commutative ring. We define a category $\mathcal{P}_R$ whose objects are pairs (L, r), where L is an invertible R-module and $r : \mathrm{Spec}(R) \to \mathbb{Z}$ is a locally constant function, and whose morphisms are defined by $\mathrm{Mor}_{\mathcal{P}_R}((L, r), (M, s)) := \mathrm{Isom}_R(L, M)$ if $r = s$, and empty otherwise. We call the objects of this category graded invertible R-modules. The category $\mathcal{P}_R$ is equipped with the structure of a (tensor) product defined by $(L, r) \boxtimes (M, s) := (L \otimes_R M, r + s)$ with the natural associativity constraint and the commutativity constraint

$$(L, r) \boxtimes (M, s) \xrightarrow{\sim} (M, s) \boxtimes (L, r) : l \otimes m \mapsto (-1)^{rs} m \otimes l.$$

From now on, we always identify $(L, r)\boxtimes(M, s) = (M, s)\boxtimes(L, r)$ by this constraint isomorphism. The unit object for the product is $\mathbf{1}_R := (R, 0)$. For each (L, r), set $L^\vee := \mathrm{Hom}_R(L, R)$. Then $(L, r)^{-1} := (L^\vee, -r)$ becomes an inverse of (L, r) by the isomorphism $i_{(L,r)} : (L, r) \boxtimes (L^\vee, -r) \xrightarrow{\sim} \mathbf{1}_R$ induced by the evaluation map $L \otimes_R \mathrm{Hom}_R(L, R) \xrightarrow{\sim} R : x \otimes f \mapsto f(x)$. We remark that we have $i_{(L,r)^{-1}} = (-1)^r i_{(L,r)}$. For a ring homomorphism $f : R \to R'$, one has a base change functor $(-) \otimes_R R' : \mathcal{P}_R \to \mathcal{P}_{R'}$ defined by $(L, r) \mapsto (L, r) \otimes_R R' := (L \otimes_R R', r \circ f^*)$, where $f^* : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$.

For a category $\mathcal{C}$, denote by $(\mathcal{C}, \mathrm{is})$ the category whose objects are the same as $\mathcal{C}$ and whose morphisms are all isomorphisms in $\mathcal{C}$. Define a functor

$$\mathrm{Det}_R : (\boldsymbol{P}_{\mathrm{fg}}(R), \mathrm{is}) \to \mathcal{P}_R : P \mapsto (\det_R P, \mathrm{rk}_R P),$$

where $\mathrm{rk}_R : \boldsymbol{P}_{\mathrm{fg}}(R) \to \mathbb{Z}_{\geq 0}$ is the R-rank of P and $\det_R P := \bigwedge_R^{\mathrm{rk}_R P} P$. Note that $\mathrm{Det}_R(0) = \mathbf{1}_R$ is the unit object. For a short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$ in $\boldsymbol{P}_{\mathrm{fg}}(R)$, we always identify $\mathrm{Det}_R(P_1)\boxtimes\mathrm{Det}_R(P_3)$ with $\mathrm{Det}_R(P_2)$ by the functorial isomorphism (put $r_i := \mathrm{rk}_R P_i$)

$$\mathrm{Det}_R(P_1) \boxtimes \mathrm{Det}_R(P_3) \xrightarrow{\sim} \mathrm{Det}_R(P_2) \tag{18}$$

induced by

$$(x_1 \wedge \cdots \wedge x_{r_1}) \otimes (\overline{x_{r_1+1}} \wedge \cdots \wedge \overline{x_{r_2}}) \mapsto x_1 \wedge \cdots \wedge x_{r_1} \wedge x_{r_1+1} \wedge \cdots \wedge x_{r_2},$$

where $x_1, \dots, x_{r_1}$ (resp. $\overline{x_{r_1+1}}, \dots, \overline{x_{r_2}}$) are local sections of P_1 (resp. P_3) and $x_i \in P_2$ ($r_1 + 1 \le i \le r_2$) is a lift of $\overline{x_i} \in P_3$.

For a bounded complex $P^\bullet$ in $\boldsymbol{P}_{\mathrm{fg}}(R)$, define $\mathrm{Det}_R(P^\bullet) \in \mathcal{P}_R$ by

$$\mathrm{Det}_R(P^\bullet) := \boxtimes_{i \in \mathbb{Z}} \mathrm{Det}_R(P^i)^{(-1)^i}.$$

For a short exact sequence $0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$ of bounded complexes in $\boldsymbol{P}_{\mathrm{fg}}(R)$, we define a canonical isomorphism

$$\mathrm{Det}_R(P_1^\bullet) \boxtimes \mathrm{Det}_R(P_3^\bullet) \xrightarrow{\sim} \mathrm{Det}_R(P_2^\bullet) \tag{19}$$

by applying the isomorphism (18) to each exact sequence $0 \to P_1^i \to P_2^i \to P_3^i \to 0$. Moreover, if $P^\bullet$ is an acyclic bounded complex in $\boldsymbol{P}_{\mathrm{fg}}(R)$, we can define a canonical isomorphism

$$h_{P^\bullet} : \mathrm{Det}_R(P^\bullet) \xrightarrow{\sim} \mathbf{1}_R, \tag{20}$$

which is characterized by the following properties: when $P^\bullet := \left[P^i \xrightarrow{f} P^{i+1}\right]$ is concentrated in degree $[i, i+1]$, we define it as the composite

$$\mathrm{Det}_R(P^\bullet) = \mathrm{Det}_R(P^i) \boxtimes \mathrm{Det}_R(P^{i+1})^{-1} \xrightarrow{\mathrm{Det}(f)\boxtimes \mathrm{id}} \mathrm{Det}_R(P^{i+1}) \boxtimes \mathrm{Det}_R(P^{i+1})^{-1} \xrightarrow{\delta_{\mathrm{Det}_R(P^{i+1})}} \mathbf{1}_R$$

when i is even (when i is odd, we similarly define it using $f^{-1} : P^{i+1} \xrightarrow{\sim} P^i$), and for a short exact sequence $0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$ of acyclic bounded complexes of $\boldsymbol{P}_{\mathrm{fg}}(R)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Det}_R(P_1^\bullet) \boxtimes \mathrm{Det}_R(P_3^\bullet) & \xrightarrow{\sim} & \mathrm{Det}_R(P_2^\bullet) \\ \big\downarrow{\scriptstyle h_{P_1^\bullet} \boxtimes h_{P_3^\bullet}} & & \big\downarrow{\scriptstyle h_{P_2^\bullet}} \\ \mathbf{1}_R \boxtimes \mathbf{1}_R & \xrightarrow{=} & \mathbf{1}_R \end{array}$$

The theory of determinants of [Knudsen and Mumford 1976] enables us to uniquely (up to canonical isomorphism) extend $\mathrm{Det}_R(-)$ to a functor

$$\mathrm{Det}_R : (\boldsymbol{D}^b_{\mathrm{perf}}(R), \mathrm{is}) \to \mathcal{P}_R$$

such that the isomorphism (19) extends to the following situation: for any exact sequence $0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$ of complexes of R-modules such that each $P_i^\bullet$ is quasi-isomorphic to a bounded complex in $\boldsymbol{P}_{\mathrm{fg}}(R)$, there exists a canonical isomorphism

$$\mathrm{Det}_R(P_1^\bullet) \boxtimes \mathrm{Det}_R(P_3^\bullet) \xrightarrow{\sim} \mathrm{Det}_R(P_2^\bullet). \tag{21}$$

By this property, if $P^\bullet \in \boldsymbol{D}^b_{\mathrm{perf}}(R)$ satisfies $\mathrm{H}^i(P^\bullet)[0] \in \boldsymbol{D}^b_{\mathrm{perf}}(R)$ for any i, there exists a canonical isomorphism

$$\mathrm{Det}_R(P^\bullet) \xrightarrow{\sim} \boxtimes_{i\in\mathbb{Z}} \mathrm{Det}_R(\mathrm{H}^i(P^\bullet)[0])^{(-1)^i}.$$

For $(L, r) \in \mathcal{P}_R$, define $(L, r)^\vee := (L^\vee, r) \in \mathcal{P}_R$, which induces an antiequivalence $(-)^\vee : \mathcal{P}_R \xrightarrow{\sim} \mathcal{P}_R$. For $P \in \boldsymbol{P}_{\mathrm{fg}}(R)$, we have a canonical isomorphism $\mathrm{Det}_R(P^\vee) \xrightarrow{\sim} \mathrm{Det}_R(P)^\vee$ defined by the isomorphism

$$\det_R(P^\vee) \xrightarrow{\sim} (\det_R P)^\vee :$$
$$f_1 \wedge \cdots \wedge f_r \mapsto \Bigl[x_1 \wedge \cdots \wedge x_r \mapsto \sum_{\sigma\in\mathfrak{S}_r} \mathrm{sgn}(\sigma) f_1(x_{\sigma(1)}) \cdots f_r(x_{\sigma(r)})\Bigr].$$

This naturally extends to $(\boldsymbol{D}^b_{\mathrm{perf}}(R), \mathrm{is})$, i.e., for any $P^\bullet \in \boldsymbol{D}^b_{\mathrm{perf}}(R)$, there exists a canonical isomorphism

$$\mathrm{Det}_R(\boldsymbol{R}\,\mathrm{Hom}_R(P^\bullet, R)) \xrightarrow{\sim} \mathrm{Det}_R(P^\bullet)^\vee. \tag{22}$$

3B. *Fundamental lines.* Both Kato's conjecture and ours concern the existence of a compatible family of canonical trivializations of some graded invertible modules defined by using the determinants of the Galois cohomologies of Galois representations or (φ, Γ)-modules. We call these graded invertible modules the fundamental lines, which we explain in this subsection.

Kato's conjecture concerns pairs (Λ, T) such that:

(i) Λ is a noetherian semilocal ring which is complete with respect to the $\mathfrak{m}_\Lambda$-adic topology (where $\mathfrak{m}_\Lambda$ is the Jacobson radial of Λ) such that $\Lambda/\mathfrak{m}_\Lambda$ is a finite ring with order a power of p.

(ii) T is a Λ-representation of $G_{\mathbb{Q}_p}$, i.e., a finite projective Λ-module equipped with a continuous Λ-linear action of $G_{\mathbb{Q}_p}$.

Our conjecture concerns pairs (A, M) such that:

(i) A is a $\mathbb{Q}_p$-affinoid algebra.

(ii) M is a (φ, Γ)-module over $\mathcal{R}_A$.

For each pair $(B, N) = (\Lambda, T)$ or (A, M) as above, we'll define graded invertible Λ-modules $\Delta_{B,i}(N) \in \mathcal{P}_B$ for $i = 1, 2$ as below, and the fundamental line will be defined as $\Delta_B(N) := \Delta_{B,1}(N) \boxtimes \Delta_{B,2}(N) \in \mathcal{P}_B$.

We first define $\Delta_{\Lambda,i}(T)$ for (Λ, T). Denote by $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T)$ the complex of continuous cochains of $G_{\mathbb{Q}_p}$ with values in T. It is known that $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T) \in \boldsymbol{D}^-(\Lambda)$ is contained in $\boldsymbol{D}^b_{\mathrm{perf}}(\Lambda)$ and that it satisfies properties similar to (1), (2), (3), (4) in Theorem 2.15. In particular, we can define a graded invertible Λ-module

$$\Delta_{\Lambda,1}(T) := \mathrm{Det}_\Lambda(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T))$$

(whose degree is $-r_T := -\mathrm{rk}_\Lambda T$ by the Euler–Poincaré formula) which satisfies the following properties:

(i) For each continuous homomorphism $f : \Lambda \to \Lambda'$, there exists a canonical Λ'-linear isomorphism

$$\Delta_{\Lambda,1}(T) \otimes_\Lambda \Lambda' \xrightarrow{\sim} \Delta_{\Lambda',1}(T \otimes_\Lambda \Lambda').$$

(ii) For each exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of Λ-representations of $G_{\mathbb{Q}_p}$, there exists a canonical Λ-linear isomorphism

$$\Delta_{\Lambda,1}(T_1) \boxtimes \Delta_{\Lambda,1}(T_3) \xrightarrow{\sim} \Delta_{\Lambda,1}(T_2).$$

(iii) The Tate duality $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}_\Lambda(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T^*), \Lambda)[-2]$ and the isomorphism (22) induce a canonical Λ-linear isomorphism

$$\Delta_{\Lambda,1}(T) \xrightarrow{\sim} \Delta_{\Lambda,1}(T^*)^\vee.$$

We next define $\Delta_{\Lambda,2}(T)$ as follows. For $a \in \Lambda^\times$, we define

$$\Lambda_a := \left\{x \in W(\overline{\mathbb{F}}_p)\,\widehat{\otimes}_{\mathbb{Z}_p} \Lambda \mid (\varphi \otimes \mathrm{id}_\Lambda)(x) = (1 \otimes a)x\right\},$$

which is an invertible Λ-module. In the same way as in Theorem 2.8, for any rank-one Λ-representation T_0, there exists a unique (up to isomorphism) pair $(\delta_{T_0}, \mathcal{L}_{T_0})$, where $\delta_{T_0} : \mathbb{Q}_p^\times \to \Lambda^\times$ is a continuous homomorphism and $\mathcal{L}_{T_0}$ is an invertible Λ-module such that $T_0 \xrightarrow{\sim} \Lambda(\tilde{\delta}_{T_0}) \otimes_\Lambda \mathcal{L}_{T_0}$, where we denote by $\tilde{\delta}_{T_0} : G^{\mathrm{ab}}_{\mathbb{Q}_p} \to \Lambda^\times$ the continuous character which satisfies $\tilde{\delta}_{T_0} \circ \mathrm{rec}_{\mathbb{Q}_p} = \delta_{T_0}$. Under these definitions, we define $a(T) := \delta_{\det_\Lambda T}(p) \in \Lambda^\times$, an invertible Λ-module

$$\mathcal{L}_\Lambda(T) := \Lambda_{a(T)} \otimes_\Lambda \det_\Lambda T$$

and a graded invertible Λ-module

$$\Delta_{\Lambda,2}(T) := (\mathcal{L}_\Lambda(T), r_T).$$

Since we have a canonical isomorphism $\Lambda_{a_1} \otimes_\Lambda \Lambda_{a_2} \xrightarrow{\sim} \Lambda_{a_1 a_2} : x \otimes y \mapsto xy$ for any $a_1, a_2 \in \Lambda$, $\Delta_{\Lambda,2}(T)$ also satisfies the following similar properties:

(i) For $f : \Lambda \to \Lambda'$, there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T) \otimes_\Lambda \Lambda' \xrightarrow{\sim} \Delta_{\Lambda,2}(T \otimes_\Lambda \Lambda').$$

(ii) For $0 \to T_1 \to T_2 \to T_3 \to 0$, there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T_1) \boxtimes \Delta_{\Lambda,2}(T_3) \xrightarrow{\sim} \Delta_{\Lambda,2}(T_2).$$

(iii) Let r_T be the rank of T. Then there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T) \xrightarrow{\sim} \Delta_{\Lambda,2}(T^*)^\vee \boxtimes (\Lambda(r_T), 0)$$

which is induced by the product of the isomorphisms

$$\Lambda_{\delta_{\det_\Lambda T}(p)} \xrightarrow{\sim} (\Lambda_{\delta_{\det_\Lambda T^*}(p)})^\vee : x \mapsto [y \mapsto y \otimes x]$$

(note that we have

$$\Lambda_{\delta_{\det_\Lambda T^*}(p)} \otimes \Lambda_{\delta_{\det_\Lambda T}(p)} \xrightarrow{\sim} \Lambda : y \otimes x \mapsto yx$$

since we have $\delta_{\det_\Lambda T}(p) = \delta_{\det_\Lambda T^*}(p)^{-1}$) and the isomorphism $\det_\Lambda T \xrightarrow{\sim} \det_\Lambda (T^*)^\vee \otimes_\Lambda \Lambda(r_T)$ induced by the canonical isomorphism $T \xrightarrow{\sim} (T^*)^\vee(1) : x \mapsto [y \mapsto y(x) \otimes \boldsymbol{e}_{-1}] \otimes \boldsymbol{e}_1$.

Finally, we define

$$\Delta_\Lambda(T) := \Delta_{\Lambda,1}(T) \boxtimes \Delta_{\Lambda,2}(T) \in \mathcal{P}_B.$$

Then $\Delta_\Lambda(T)$ also satisfies properties similar to (i), (ii) for $\Delta_{\Lambda,i}(T)$ and

(iii) there exists a canonical isomorphism

$$\Delta_\Lambda(T) \xrightarrow{\sim} \Delta_\Lambda(T^*)^\vee \boxtimes (\Lambda(r_T), 0).$$

Next, we define the fundamental line $\Delta_A(M)$ for (φ, Γ)-modules M over $\mathcal{R}_A$. Let A be a $\mathbb{Q}_p$-affinoid algebra, and let M be a (φ, Γ)-module over $\mathcal{R}_A$. By our Theorem 2.15 (Kedlaya–Pottharst–Xiao), we can define a graded invertible A-module

$$\Delta_{A,1}(M) := \mathrm{Det}_A\, C^\bullet_{\varphi,\gamma}(M) \in \mathcal{P}_A$$

which satisfies properties similar to (i), (ii), (iii) for $\Delta_{\Lambda,1}(T)$. We next define $\Lambda_{A,2}(M)$ as follows. By our Theorem 2.8 (Kedlaya–Pottharst–Xiao), there exists a unique (up to isomorphism) pair $(\delta_{\det_{\mathcal{R}_A} M}, \mathcal{L}_{\det_{\mathcal{R}_A} M})$, where $\delta_{\det_{\mathcal{R}_A} M} : \mathbb{Q}_p^\times \to A^\times$ is a continuous homomorphism and $\mathcal{L}_{\det_{\mathcal{R}_A} M}$ is an invertible A-module such that $\det_{\mathcal{R}_A} M \xrightarrow{\sim} \mathcal{R}_A(\delta_{\det_{\mathcal{R}_A} M}) \otimes_{\mathcal{R}_A} \mathcal{L}_{\det_{\mathcal{R}_A} M}$. Then we define an A-module

$$\mathcal{L}_A(M) := \left\{x \in \det_{\mathcal{R}_A} M \mid \varphi(x) = \delta_{\det_{\mathcal{R}_A} M}(p)x,\ \gamma(x) = \delta_{\det_{\mathcal{R}_A} M}(\chi(\gamma))x\ (\gamma \in \Gamma)\right\},$$

which is an invertible A-module since it is isomorphic to $\mathcal{L}_{\det_{\mathcal{R}_A} M}$, and we define a graded invertible A-module

$$\Delta_{A,2}(M) := (\mathcal{L}_A(M), r_M) \in \mathcal{P}_A.$$

By definition, it is easy to check that $\Delta_{A,2}(M)$ satisfies properties similar to (i), (ii), (iii) for $\Delta_{\Lambda,2}(T)$. Finally, we define a graded invertible A-module $\Delta_A(M)$, which we call the fundamental line, by

$$\Delta_A(M) := \Delta_{A,1}(M) \boxtimes \Delta_{A,2}(M) \in \mathcal{P}_A,$$

which also satisfies properties similar to (i), (ii), (iii) for $\Delta_\Lambda(T)$.

More generally, let X be a rigid analytic space over $\mathbb{Q}_p$, and let M be a (φ, Γ)-module over $\mathcal{R}_X$. By the base change property (i) of $\Delta_A(M)$, we can also functorially define a graded invertible $\mathcal{O}_X$-module

$$\Delta_X(M) \in \mathcal{P}_{\mathcal{O}_X}$$

on X (we can naturally generalize the notion of graded invertible modules in this setting) such that there exists a canonical isomorphism

$$\Gamma(\mathrm{Max}(A), \Delta_X(M)) \xrightarrow{\sim} \Delta_A(M|_{\mathrm{Max}(A)})$$

for any affinoid open subset $\mathrm{Max}(A) \subseteq X$.

We next compare Kato's fundamental line $\Delta_\Lambda(T)$ with our fundamental line $\Delta_A(M)$. Let $f : \Lambda \to A$ be a continuous ring homomorphism, where Λ is equipped with $\mathfrak{m}_\Lambda$-adic topology and A is equipped with p-adic topology. Let T be a Λ-representation of $G_{\mathbb{Q}_p}$. Let us denote by $M := \boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A)$ the (φ, Γ)-module over $\mathcal{R}_A$ associated to the A-representation $T \otimes_\Lambda A$ of $G_{\mathbb{Q}_p}$. By Theorem 2.8 of [Pottharst 2013], there exists a canonical quasi-isomorphism $\boldsymbol{C}^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T) \otimes^{\boldsymbol{L}}_\Lambda A \xrightarrow{\sim} \boldsymbol{C}^\bullet_{\varphi,\gamma}(M)$, and this induces an A-linear isomorphism

$$\Delta_{\Lambda,1}(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_{A,1}(M).$$

We also have the following isomorphism.

Lemma 3.1. *In the above situation, there exists a canonical A-linear isomorphism*

$$\Delta_{\Lambda,2}(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_{A,2}(M).$$

Proof. By definition, it suffices to show the lemma when T is of rank one. Hence, we may assume that $T = \Lambda(\tilde{\delta}) \otimes_\Lambda \mathcal{L}$ for a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Lambda^\times$ and an invertible Λ-module $\mathcal{L}$ (where $\tilde{\delta}$ is the character of $G^{\mathrm{ab}}_{\mathbb{Q}_p}$ such that $\tilde{\delta} \circ \mathrm{rec}_{\mathbb{Q}_p} = \delta$). Moreover, since we have a canonical isomorphism

$$\boldsymbol{D}_{\mathrm{rig}}((\Lambda(\tilde{\delta}) \otimes_\Lambda \mathcal{L}) \otimes_\Lambda A) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{rig}}(\Lambda(\tilde{\delta}) \otimes_\Lambda A) \otimes_A (\mathcal{L} \otimes_\Lambda A)$$

by the exactness of $\boldsymbol{D}_{\mathrm{rig}}(-)$, it suffices to show the lemma when $\mathcal{L} = \Lambda$.

Since the image of $H_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p,\infty})$ in $G^{\mathrm{ab}}_{\mathbb{Q}_p}$ is the closed subgroup which is topologically generated by $\mathrm{rec}_{\mathbb{Q}_p}(p)$, we have

$$\boldsymbol{D}_{\mathrm{rig}}(\Lambda(\tilde{\delta}) \otimes_\Lambda A) = (W(\overline{\mathbb{F}}_p) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(\tilde{\delta}))^{\mathrm{rec}_{\mathbb{Q}_p}(p)=1} \otimes_\Lambda \mathcal{R}_A,$$

by definition of $\boldsymbol{D}_{\mathrm{rig}}(-)$, and the right-hand side is isomorphic to $\mathcal{R}_A(f \circ \delta)$. Hence, we obtain

$$\begin{aligned}\mathcal{L}_A(M) &= \big((W(\overline{\mathbb{F}}_p) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(\tilde{\delta}))^{\mathrm{rec}_{\mathbb{Q}_p}(p)=1} \otimes_\Lambda \mathcal{R}_A\big)^{\varphi=f(\delta(p)),\Gamma=f\circ\delta\circ\chi} \\ &= \big(W(\overline{\mathbb{F}}_p) \widehat{\otimes}_{\mathbb{Z}_p} \Lambda(\tilde{\delta})\big)^{\mathrm{rec}_{\mathbb{Q}_p}(p)=1} \otimes_\Lambda A = \mathcal{L}_\Lambda(T) \otimes_\Lambda A,\end{aligned}$$

which proves the lemma. □

Taking the products of these two canonical isomorphisms, we obtain the following corollary.

Corollary 3.2. *In the above situation, there exists a canonical isomorphism*

$$\Delta_\Lambda(T)\otimes_\Lambda A \xrightarrow{\sim} \Delta_A(M).$$

Example 3.3. The typical example of the above base change property is the following. For Λ as above, let us denote by X the associated rigid analytic space. More precisely, X is the union of affinoids $\mathrm{Max}(A_n)$ for $n\geq 1$, where A_n is the $\mathbb{Q}_p$-affinoid algebra defined by $A_n := \Lambda[\mathfrak{m}_\Lambda^n/p]^\wedge[1/p]$ (for a ring R, denote by $R^\wedge$ the p-adic completion). Let T be a Λ-representation of $G_{\mathbb{Q}_p}$, and let $M_n := \boldsymbol{D}_{\mathrm{rig}}(T\otimes_\Lambda A_n)$. Since M_n is compatible with the base change with respect to the canonical map $A_n\to A_{n+1}$ for any n, $\{M_n\}_{n\geq 1}$ defines a (φ,Γ)-module $\mathcal{M}$ over $\mathcal{R}_X$. Then the canonical isomorphism $\Delta_\Lambda(T)\otimes_\Lambda A_n \xrightarrow{\sim} \Delta_{A_n}(M_n)$ defined in the above corollary glues to an isomorphism

$$\Delta_\Lambda(T)\otimes_\Lambda \mathcal{O}_X \xrightarrow{\sim} \Delta_X(\mathcal{M}).$$

Moreover, using the terminology of coadmissible modules [Schneider and Teitelbaum 2003], we can define this comparison isomorphism without using sheaves. Let us define $A_\infty := \Gamma(X,\mathcal{O}_X)$ and $\Delta_{A_\infty}(M_\infty) := \varprojlim_n \Delta_{A_n}(M_n)$. Taking the limit of the isomorphism $\Delta_\Lambda(T)\otimes_\Lambda A_n \xrightarrow{\sim} \Delta_{A_n}(M_n)$ we obtain an A_∞-linear isomorphism

$$\Delta_\Lambda(T)\otimes_\Lambda A_\infty \xrightarrow{\sim} \Delta_{A_\infty}(M_\infty).$$

Then the theory of coadmissible modules [Schneider and Teitelbaum 2003, Corollary 3.3] says that to consider the isomorphism $\Delta_\Lambda(T)\otimes_\Lambda \mathcal{O}_X \xrightarrow{\sim} \Delta_X(\mathcal{M})$ is the same as to consider the isomorphism $\Delta_\Lambda(T)\otimes_\Lambda A_\infty \xrightarrow{\sim} \Delta_{A_\infty}(M_\infty)$. In fact, we will frequently use the latter object $\Delta_{A_\infty}(M_\infty)$ in Section 4.

3C. ***de Rham ε-isomorphism.*** In this subsection, we assume that $L=A$ is a finite extension of $\mathbb{Q}_p$. We define a trivialization

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(M):\mathbf{1}_L \xrightarrow{\sim} \Delta_L(M),$$

which we call the de Rham ε-isomorphism, for each de Rham (φ,Γ)-module M over $\mathcal{R}_L$ and for each $\mathbb{Z}_p$-basis $\zeta=\{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1)$.

Let M be a de Rham (φ,Γ)-module over $\mathcal{R}_L$. We first recall the definition of Deligne and Langlands' [Deligne 1973] and Fontaine and Perrin-Riou's [1994] ε-constant associated to M.

We first briefly recall the theory of ε-constants of Deligne and Langlands [Deligne 1973]. Let $W_{\mathbb{Q}_p} \subseteq G_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$. Let E be a field of characteristic zero, and let $V = (V, \rho)$ be an E-representation of $W_{\mathbb{Q}_p}$, i.e., V is a finite-dimensional E-vector space equipped with a smooth E-linear action ρ of $W_{\mathbb{Q}_p}$. Let us denote by $V^\vee$ the dual $(\mathrm{Hom}_E(V, E), \rho^\vee)$ of V. Denote by $E(|x|)$ the rank-one E-representation of $W_{\mathbb{Q}_p}$ corresponding to the continuous homomorphism $|x| : \mathbb{Q}_p^\times \to E^\times : p \mapsto 1/p,\ a \mapsto 1 (a \in \mathbb{Z}_p^\times)$ via the local class field theory. Put $V^\vee(|x|) := V^\vee \otimes_E E(|x|)$. Assume that E is a field which contains $\mathbb{Q}(\zeta_{p^\infty})$. The definition of the ε-constants depends on the choice of an additive character of $\mathbb{Q}_p$ and a Haar measure on $\mathbb{Q}_p$. In this article, we fix the Haar measure dx on $\mathbb{Q}_p$ for which $\mathbb{Z}_p$ has measure 1. For each $\mathbb{Z}_p$-basis $\zeta = \{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1)$, we define an additive character $\psi_\zeta : \mathbb{Q}_p \to E^\times$ such that $\psi_\zeta(1/p^n) := \zeta_{p^n}$ for $n \geq 1$. In this article, we don't recall the precise definition of ε-constants, but we recall here some of their basic properties under the fixed additive character ψ_ζ and the fixed Haar measure dx. We can attach a constant $\varepsilon(V, \psi_\zeta, dx) \in E^\times$ for each V as above which satisfies the following properties (we let $\varepsilon(V, \zeta) := \varepsilon(V, \psi_\zeta, dx)$ for simplicity):

(1) For each exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of finite-dimensional E-vector spaces with continuous actions of $W_{\mathbb{Q}_p}$, we have
$$\varepsilon(V_2, \zeta) = \varepsilon(V_1, \zeta)\varepsilon(V_3, \zeta).$$

(2) For each $a \in \mathbb{Z}_p^\times$, we define $\zeta^a := \{\zeta_{p^n}^a\}_{n\geq 1}$. Then we have
$$\varepsilon(V, \zeta^a) = \det_E V(\mathrm{rec}_{\mathbb{Q}_p}(a))\varepsilon(V, \zeta).$$

(3) $\varepsilon(V, \zeta)\varepsilon(V^\vee(|x|), \zeta^{-1}) = 1$.

(4) $\varepsilon(V, \zeta) = 1$ if V is unramified.

(5) If $\dim_E V$ equals 1 and corresponds to a locally constant homomorphism $\delta : \mathbb{Q}_p^\times \to E^\times$ via the local class field theory, then
$$\varepsilon(V, \zeta) = \delta(p)^{n(\delta)}\Bigg(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times} \delta(i)^{-1}\zeta_{p^{n(\delta)}}^i\Bigg),$$
where $n(\delta) \geq 0$ is the conductor of δ, i.e., the minimal integer $n \geq 0$ such that $\delta|_{(1+p^n\mathbb{Z}_p)\cap\mathbb{Z}_p^\times} = 1$ (then $\delta|_{\mathbb{Z}_p^\times}$ factors through $(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times$).

For a Weil–Deligne representation $W = (V, \rho, N)$ of $W_{\mathbb{Q}_p}$ defined over E, we set
$$\varepsilon(W, \zeta) := \varepsilon((V, \rho), \zeta) \cdot \det_E\big(-\mathrm{Fr}_p \mid V^{L_p}/(V^{N=0})^{I_p}\big),$$
which also satisfies
$$\varepsilon(W, \zeta) \cdot \varepsilon(W^\vee(|x|), \zeta^{-1}) = 1.$$

Next, we define the ε-constant for each de Rham (φ, Γ)-module over $\mathcal{R}_L$ following Fontaine and Perrin-Riou [1994]. Let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$. Then M is potentially semistable by the result of Berger (for example, see Théorème III.2.4 of [Berger 2008b]) based on Crew's conjecture, which was proved by André, Mebkhout, and Kedlaya. Hence, we can define a filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module $\boldsymbol{D}_{\mathrm{pst}}(M) := \bigcup_{K \subseteq \bar{\mathbb{Q}}_p} \boldsymbol{D}_{\mathrm{st}}^K(M|_K)$ which is a free $(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L)$-module whose rank is r_M, where K runs through all the finite extensions of $\mathbb{Q}_p$ and we define $\boldsymbol{D}_{\mathrm{st}}^K(M|_K) := (\mathcal{R}_L(\pi_K)[\log(\pi), 1/t] \otimes_{\mathcal{R}_L} M)^{\Gamma_K = 1}$. Set $\boldsymbol{D}_{\mathrm{st}}(M) := \boldsymbol{D}_{\mathrm{st}}^{\mathbb{Q}_p}(M)$. Following Fontaine, one can define a Weil–Deligne representation $W(M) := (\boldsymbol{D}_{\mathrm{pst}}(M), \rho, N)$ of $W_{\mathbb{Q}_p}$ defined over $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L$ such that N is the natural one and $\rho(g)(x) := \varphi^{v(g)}(g \cdot x)$ for $g \in W_{\mathbb{Q}_p}$ and $x \in W(M)$, where we denote by $g \cdot x$ the natural action of $G_{\mathbb{Q}_p}$ on $W(M)$ and

$$v : W_{\mathbb{Q}_p} \twoheadrightarrow W_{\mathbb{Q}_p}^{\mathrm{ab}} \xrightarrow{\mathrm{rec}_{\mathbb{Q}_p}^{-1}} \mathbb{Q}_p^{\times} \xrightarrow{v_p} \mathbb{Z}.$$

Taking the base change of $W(M)$ by the natural inclusion $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L \hookrightarrow \mathbb{Q}_p^{\mathrm{ab}} \otimes_{\mathbb{Q}_p} L$, and decomposing $\mathbb{Q}_p^{\mathrm{ab}} \otimes_{\mathbb{Q}_p} L \xrightarrow{\sim} \prod_\tau L_\tau$ into a finite product of fields L_τ, we obtain a Weil–Deligne representation $W(M)_\tau$ of $W_{\mathbb{Q}_p}$ defined over L_τ for each τ. Hence, we can define the ε-constant $\varepsilon(W(M)_\tau, \tau(\zeta)) \in L_\tau^{\times}$, where $\tau(\zeta)$ is the image of ζ in L_τ by the projection $\mathbb{Q}_p^{\mathrm{ab}} \otimes_{\mathbb{Q}_p} L \to L_\tau$. Then the product

$$\varepsilon_L(W(M), \zeta) := (\varepsilon(W(M)_\tau, \tau(\zeta)))_\tau \in \prod_\tau L_\tau^{\times}$$

is contained in $L_\infty^{\times} := (\mathbb{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbb{Q}_p} L)^{\times} \subseteq (\mathbb{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L)^{\times}$ since it is easy to check that $\varepsilon_L(W(M), \zeta)$ is fixed by $1 \otimes \varphi \otimes 1$.

Using this definition, for each de Rham (φ, Γ)-module M over $\mathcal{R}_L$, we construct a trivialization $\varepsilon_{L,\zeta}^{\mathrm{dR}}(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(M)$ as follows. We will first define two isomorphisms

$$\theta_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$$

and

$$\theta_{\mathrm{dR},L}(M, \zeta) : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\sim} \Delta_{L,2}(M)$$

(we remark that $\theta_{\mathrm{dR},L}(M, \zeta)$ depends on the choice of ζ), and then define $\varepsilon_{L,\xi}^{\mathrm{dR}}(M)$ as the composite

$$\varepsilon_{L,\xi}^{\mathrm{dR}}(M) : \mathbf{1}_L \xrightarrow{\Gamma_L(M) \cdot \theta_L(M)} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\mathrm{id} \boxtimes \theta_{\mathrm{dR},L}(M,\zeta)} \Delta_{L,1}(M) \boxtimes \Delta_{L,2}(M) = \Delta_L(M),$$

where $\Gamma_L(M) \in \mathbb{Q}^{\times}$ is defined by

$$\Gamma_L(M) := \prod_{r \in \mathbb{Z}} \Gamma^*(r)^{-\dim_L \mathrm{gr}^{-r} \boldsymbol{D}_{\mathrm{dR}}(M)},$$

where we set

$$\Gamma^*(r) := \begin{cases} (r-1)! & (r \geq 1), \\ (-1)^r/(-r)! & (r \leq 0). \end{cases}$$

We first define $\theta_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$. By the result of Section 2B, we have the exact sequence of L-vector spaces

$$0 \to \mathrm{H}^0_{\varphi,\gamma}(M_0) \to \boldsymbol{D}_{\mathrm{cris}}(M_0)_1 \xrightarrow{x \mapsto ((1-\varphi)x, \bar{x})} \boldsymbol{D}_{\mathrm{cris}}(M_0)_2 \oplus t_{M_0} \xrightarrow{\exp_{f,M_0} \oplus \exp_{M_0}} \mathrm{H}^1_{\varphi,\gamma}(M_0)_f \to 0 \quad (23)$$

for $M_0 = M, M^*$, where we let $\boldsymbol{D}_{\mathrm{cris}}(M_0)_i = \boldsymbol{D}_{\mathrm{cris}}(M_0)$ for $i = 1, 2$.

Using Tate duality, the de Rham duality

$$\boldsymbol{D}_{\mathrm{dR}}(M)^0 \xrightarrow{\sim} t^\vee_{M^*} : x \mapsto [\bar{y} \mapsto [y, x]_{\mathrm{dR}}]$$

(here $y \in \boldsymbol{D}_{\mathrm{dR}}(M^*)$ is a lift of $\bar{y}$) and Proposition 2.24, we define a map

$$\exp^*_{M^*} : \mathrm{H}^1_{\varphi,\gamma}(M)_{/f} := \mathrm{H}^1_{\varphi,\gamma}(M)/\mathrm{H}^1_{\varphi,\gamma}(M)_f \xrightarrow{x \mapsto [y \mapsto \langle y, x\rangle]} \mathrm{H}^1_{\varphi,\gamma}(M^*)^\vee_f \xrightarrow{\exp^\vee_{M^*}} t^\vee_{M^*} \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(M)^0$$

which is called the dual exponential map and was studied in §2.4 of [Nakamura 2014a]. Using this map, as the dual of the exact sequence (23) for $M_0 = M^*$, we obtain an exact sequence

$$0 \to \mathrm{H}^1_{\varphi,\gamma}(M)_{/f} \xrightarrow{\exp^\vee_{f,M^*} \oplus \exp^*_{M^*}} \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2 \oplus \boldsymbol{D}_{\mathrm{dR}}(M)^0 \xrightarrow{(*)} \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1 \to \mathrm{H}^2_{\varphi,\gamma}(M) \to 0, \quad (24)$$

where the map $\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2 \to \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1$ in $(*)$ is the dual of $(1-\varphi)$. Therefore, as the composite of the exact sequences (23) for $M_0 = M$ and (24), we obtain the exact sequence

$$0 \to \mathrm{H}^0_{\varphi,\gamma}(M) \to \boldsymbol{D}_{\mathrm{cris}}(M)_1 \xrightarrow{x \mapsto ((1-\varphi)x, \bar{x})} \boldsymbol{D}_{\mathrm{cris}}(M)_2 \oplus t_M \to \mathrm{H}^1_{\varphi,\gamma}(M) \to \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2 \oplus \boldsymbol{D}_{\mathrm{dR}}(M)^0 \to \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1 \to \mathrm{H}^2_{\varphi,\gamma}(M) \to 0. \quad (25)$$

Applying the trivialization (20) to this exact sequence and using the canonical isomorphisms

$$i_{\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M)_1)} : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M)_2) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M)_1)^{-1} \xrightarrow{\sim} \mathbf{1}_L,$$
$$i_{\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1)} : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1)^{-1} \xrightarrow{\sim} \mathbf{1}_L,$$

and

$$\mathrm{Det}_L(\boldsymbol{D}^0_{\mathrm{dR}}(M)) \boxtimes \mathrm{Det}_L(t_M) \xrightarrow{\sim} \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)),$$

we obtain a canonical isomorphism

$$\theta_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)).$$

Next, we define an isomorphism $\theta_{\mathrm{dR},L}(M,\zeta) : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\sim} \Delta_{L,2}(M)$. To define this, we show the following lemma.

Lemma 3.4. *Let* $\{h_1, h_2, \ldots, h_{r_M}\}$ *be the set of Hodge–Tate weights of* M (*with multiplicity*). *Put* $h_M := \sum_{i=1}^{r_M} h_i$. *For any* $n \geq n(M)$ *such that* $\varepsilon_L(W(M),\zeta) \in L_n := \mathbb{Q}_p(\zeta_{p^n}) \otimes_{\mathbb{Q}_p} L$, *the map*

$$\mathcal{L}_L(M) \to \boldsymbol{D}_{\mathrm{dif},n}(\det_{\mathcal{R}_L} M) = L_n((t)) \otimes_{\iota_n, \mathcal{R}_L^{(n)}} (\det_{\mathcal{R}_L} M)^{(n)} : \quad x \mapsto \frac{1}{\varepsilon_L(W(M),\zeta)} \cdot \frac{1}{t^{h_M}} \otimes \varphi^n(x)$$

induces an isomorphism

$$f_{M,\zeta} : \mathcal{L}_L(M) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} M),$$

and doesn't depend on the choice of n.

Proof. The independence of n follows from the definition of the transition map $\boldsymbol{D}_{\mathrm{dif},n}(-) \hookrightarrow \boldsymbol{D}_{\mathrm{dif},n+1}(-)$.

We show that $f_{M,\zeta}$ is an isomorphism. Comparing the dimensions, it suffices to show that the image of the map in the lemma is contained in $\boldsymbol{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} M)$, i.e., is fixed by the action of Γ. Since we have $\varepsilon_L(W(M),\zeta)/\varepsilon_L(W(\det_{\mathcal{R}_L} M),\zeta) \in L^\times (\subseteq L_\infty^\times)$, it suffices to show the claim when M is of rank one. We assume that M is of rank one. By the classification of rank-one de Rham (φ,Γ)-modules, there exists a locally constant homomorphism $\tilde{\delta} : \mathbb{Q}_p^\times \to L^\times$ such that $M \xrightarrow{\sim} \mathcal{R}_L(\tilde{\delta} \cdot x^{h_M})$. The corresponding representation $W(M)$ of $W_{\mathbb{Q}_p}$ is given by the homomorphism $\tilde{\delta} \cdot |x|^{h_M} : \mathbb{Q}_p^\times \to L^\times$ via the local class field theory. By the property (2) of ε-constants, we have $\gamma(\varepsilon_L(\boldsymbol{D}_{\mathrm{pst}}(M),\zeta)) = \tilde{\delta}(\chi(\gamma))\varepsilon_L(W(M),\zeta)$ for $\gamma \in \Gamma$, which proves the claim since we have $\gamma(\varphi^n(x)) = \tilde{\delta}(\chi(\gamma))\chi(\gamma)^{h_M}\varphi^n(x)$ for $x \in \mathcal{L}_L(M)$, $\gamma \in \Gamma$, by definition. □

Since we have a canonical isomorphism $\boldsymbol{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} M) \xrightarrow{\sim} \det_L \boldsymbol{D}_{\mathrm{dR}}(M)$, the isomorphism $f_{M,\zeta}$ induces an isomorphism $f_{M,\zeta} : \Delta_{L,2}(M) \xrightarrow{\sim} \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$. We define the isomorphism $\theta_{\mathrm{dR},L}(M,\zeta)$ as the inverse

$$\theta_{\mathrm{dR},L}(M,\zeta) := f_{M,\zeta}^{-1} : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\sim} \Delta_{L,2}(M).$$

Remark 3.5. The isomorphism $f_{M,\zeta}$, and hence the isomorphism $\theta_{\mathrm{dR},L}(M,\zeta)$, depends on the choice of ζ. If we choose another $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1)$ which can be written as $\zeta^a := \{\zeta_{p^n}^a\}_{n\geq 0}$ for unique $a \in \mathbb{Z}_p^\times$, then f_{M,ζ^a} is defined using $\varepsilon_L(W(M),\zeta^a)$ and the parameter π_{ζ^a} (see Remark 2.1) and $t_a := \log(1+\pi_{\zeta^a})$. Since we have $\varepsilon_L(W(M),\zeta^a)=\det W(M)(\mathrm{rec}_{\mathbb{Q}_p}(a))\varepsilon_L(W(M),\zeta)$ and $\pi_{\zeta^a}=(1+\pi)^a-1$ and $t_a=at$, we have $f_{M,\zeta^a}=f_{M,\zeta}/\delta_{\det_{\mathcal{R}_L}M}(a)$, and hence we also have

$$\theta_{\mathrm{dR},L}(M,\zeta^a)=\delta_{\det_{\mathcal{R}_L}M}(a)\cdot\theta_{\mathrm{dR},L}(M,\zeta),$$

and obtain

$$\varepsilon^{\mathrm{dR}}_{L,\zeta^a}(M)=\delta_{\det_{\mathcal{R}_L}M}(a)\cdot\varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

Remark 3.6. Kato [1993b] and Fukaya and Kato [2006] defined their de Rham ε-isomorphism $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)':\mathbf{1}_L\xrightarrow{\sim}\Delta_L(V)$ (using a different notation) for each de Rham L-representation V of $G_{\mathbb{Q}_p}$ using the original Bloch–Kato exponential map. Using Proposition 2.26, we can compare our $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V))$ with their $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'$ under the canonical isomorphism $\Delta_L(V)\xrightarrow{\sim}\Delta_L(\boldsymbol{D}_{\mathrm{rig}}(V))$ defined in Corollary 3.2. We remark that ours and theirs are different since they used (in our notation) the ε-constant $\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(V),\rho),\zeta)$ associated to the representation $(\boldsymbol{D}_{\mathrm{pst}}(V),\rho)$ of $W_{\mathbb{Q}_p}$ instead of $W(V)$. Since one has

$$\varepsilon_L(W(V),\zeta)=\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(V),\rho),\zeta)\cdot\det_L(-\varphi\mid\boldsymbol{D}_{\mathrm{st}}(V)/\boldsymbol{D}_{\mathrm{cris}}(V)),$$

the correct relation between ours and theirs is

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V))=\det_L(-\varphi\mid\boldsymbol{D}_{\mathrm{st}}(V)/\boldsymbol{D}_{\mathrm{cris}}(V))\cdot\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'. \tag{26}$$

Moreover, we insist that ours is the correct one, since we show in Lemma 3.7 below that our $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ is compatible with exact sequences (but $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'$ may not satisfy this compatibility).

Finally in this subsection, we prove a lemma on the compatibility of the de Rham ε-isomorphism with exact sequences and the Tate duality.

Lemma 3.7. (1) *For any exact sequence* $0\to M_1\to M_2\to M_3\to 0$, *we have*

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(M_2)=\varepsilon^{\mathrm{dR}}_{L,\zeta}(M_1)\boxtimes\varepsilon^{\mathrm{dR}}_{L,\zeta}(M_3)$$

under the canonical isomorphism $\Delta_L(M_2)\xrightarrow{\sim}\Delta_L(M_1)\boxtimes\Delta_L(M_3)$.

(2) *One has the following commutative diagram of isomorphisms*

$$\begin{array}{ccc}\Delta_L(M) & \xrightarrow{\mathrm{can}} & \Delta_L(M^*)^\vee\boxtimes(L(r_M),0)\\ {\scriptstyle\varepsilon^{\mathrm{dR}}_{L,\zeta^{-1}}(M)}\big\uparrow & & \big\downarrow{\scriptstyle\varepsilon^{\mathrm{dR}}_{L,\zeta}(M^*)^\vee\boxtimes[\boldsymbol{e}_{r_M}\mapsto 1]}\\ \mathbf{1}_L & \xrightarrow{\mathrm{can}} & \mathbf{1}_L\boxtimes\mathbf{1}_L\end{array}$$

Proof. We first prove (1). The proof is identical to that of Proposition 3.3.8 of [Fukaya and Kato 2006], but we give a proof for convenience of the readers. We first remark that we have

$$\Gamma_L(M)\cdot\Gamma_L(M^*)=(-1)^{h_M+\dim_L t_M} \tag{27}$$

since we have

$$\Gamma^*(r)\cdot\Gamma^*(1-r)=\begin{cases}(-1)^{r-1} & (r\geq 1),\\ (-1)^r & (r\leq 0).\end{cases}$$

We next remark that one has the commutative diagram

$$\begin{array}{ccc}
\mathbf{1}_L & \xrightarrow{\theta_L(M)} & \Delta_{L,1}(M)\boxtimes\mathrm{Det}_L(M)\\
{\scriptstyle (-1)^{\dim_L t_M}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{can}}\\
\mathbf{1}_L & \xleftarrow{\theta_L(M^*)^\vee} & \Delta_{L,1}(M^*)^\vee\boxtimes\mathrm{Det}_L(M^*)^\vee
\end{array}\tag{28}$$

in which the right vertical arrow is induced by the Tate duality, since one has the commutative diagram

$$\begin{array}{ccccc}
t_M & \xrightarrow{-\exp_M} & \mathrm{H}^1_{\varphi,\gamma}(M) & \xrightarrow{\exp^*_{M^*}} & \boldsymbol{D}_{\mathrm{dR}}(M)^0\\
{\scriptstyle \bar{x}\mapsto[y,\mapsto[y,x]_{\mathrm{dR}}]}\big\downarrow & & {\scriptstyle x\mapsto[y\mapsto\langle y,x\rangle]}\big\downarrow & & \big\downarrow{\scriptstyle x\mapsto[\bar{y}\mapsto[y,x]_{\mathrm{dR}}]}\\
\boldsymbol{D}_{\mathrm{dR}}(M^*)^\vee & \xrightarrow{(\exp^*_M)^\vee} & \mathrm{H}^1_{\varphi,\gamma}(M^*)^\vee & \xrightarrow{(\exp_{M^*})^\vee} & (t_{M^*})^\vee
\end{array}$$

Finally, we remark that one has the commutative diagram

$$\begin{array}{ccc}
\mathrm{Det}_L(M) & \xrightarrow{\theta_{\mathrm{dR},L}(M,\zeta^{-1})} & \Delta_{L,2}(M)\\
{\scriptstyle (-1)^{h_M}\cdot\mathrm{can}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{can}}\\
\begin{array}{l}\mathrm{Det}_L(M^*)^\vee\\ =\mathrm{Det}_L(M^*)^\vee\boxtimes\mathbf{1}_L\end{array} & \xleftarrow{\theta_{\mathrm{dR},L}(M^*,\zeta)^\vee\boxtimes[\boldsymbol{e}_{r_M}\mapsto 1]} & \Delta_{L,2}(M^*)^\vee\boxtimes(L(r_M),0)
\end{array}\tag{29}$$

in which the vertical maps can are also defined by the duality, since we have

$$\varepsilon_L(W(V),\zeta^{-1})\cdot\varepsilon_L(W(V^*),\zeta)=1.$$

Then (1) follows from the commutative diagrams (27), (28) and (29).

We next prove (1). We first define an isomorphism

$$\theta_L(M)':\mathbf{1}_L\xrightarrow{\sim}\Delta_{L,1}(M)\boxtimes\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$$

in the same way as $\theta_L(M)$ using the exact sequence

$$0\to\mathrm{H}^0_{\varphi,\gamma}(M)\to\boldsymbol{D}_{\mathrm{cris}}(M)_1\xrightarrow{x\mapsto((1-\varphi^{-1})x,\bar{x})}\boldsymbol{D}_{\mathrm{cris}}(M)_2\oplus t_M\xrightarrow{\exp_{f,M}\oplus\exp_M}\mathrm{H}^1_{\varphi,\gamma}(M)_f\to 0\tag{30}$$

(we use φ^{-1} instead of φ) and (24), and define

$$\theta_{\mathrm{dR},L}(M,\zeta)':\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))\xrightarrow{\sim}\Delta_{L,2}(M)$$

in the same way as $\theta_{\mathrm{dR},L}(M,\zeta)'$ using the constant

$$\varepsilon_L(W(M),\zeta)\cdot \det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{cris}}(M)) = \varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M),\rho),\zeta)\cdot \det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{st}}(M))$$

instead of $\varepsilon_L(W(V),\zeta)$. Since we have $\theta_L(M)' = \theta_L(M)\cdot \det_L(-\varphi^{-1} \mid \boldsymbol{D}_{\mathrm{cris}}(M))$, $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ can be defined using the triple $(\Gamma_L(M), \theta_L(M)', \theta_{\mathrm{dR},L}(M,\zeta)')$ instead of $(\Gamma_L(M), \theta_L(M), \theta_{\mathrm{dR},L}(M,\zeta))$.

Let $0\to M_1\to M_2\to M_3\to 0$ be an exact sequence as in (1). Since one has $\Gamma(M_2)=\Gamma(M_1)\cdot\Gamma(M_3)$, it suffices to show that both $\theta_L(-)'$ and $\theta_{\mathrm{dR},L}(-)'$ are compatible with the exact sequence.

Since we have

$$\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M_2),\rho),\zeta) = \varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M_1),\rho),\zeta)\cdot \varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M_3),\rho),\zeta)$$

and

$$\det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{st}}(M_2)) = \det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{st}}(M_1))\cdot \det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{st}}(M_3))$$

(since $\boldsymbol{D}_{\mathrm{pst}}(-)$ and $\boldsymbol{D}_{\mathrm{st}}(-)$ are exact for de Rham (φ,Γ)-modules), the isomorphism $\theta_{\mathrm{dR},L}(-)'$ is compatible with the exact sequence.

We remark that the functor $\boldsymbol{D}_{\mathrm{cris}}(-)$ is not exact (in general) for de Rham (φ,Γ)-modules, but we have the exact sequence

$$0\to \boldsymbol{D}_{\mathrm{cris}}(M_1)\to \boldsymbol{D}_{\mathrm{cris}}(M_2)\to \boldsymbol{D}_{\mathrm{cris}}(M_3) \xrightarrow{(*)} \boldsymbol{D}_{\mathrm{cris}}(M_1^*)^\vee \to \boldsymbol{D}_{\mathrm{cris}}(M_2^*)^\vee \to \boldsymbol{D}_{\mathrm{cris}}(M_3^*)^\vee \to 0$$

such that the boundary map $(*)$ satisfies the commutative diagram

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{cris}}(M_3) & \xrightarrow{(*)} & \boldsymbol{D}_{\mathrm{cris}}(M_1^*)^\vee \\ \big\downarrow{\scriptstyle \varphi^{-1}} & & \big\downarrow{\scriptstyle \varphi^{\vee}} \\ \boldsymbol{D}_{\mathrm{cris}}(M_3) & \xrightarrow{(*)} & \boldsymbol{D}_{\mathrm{cris}}(M_1^*)^\vee \end{array}$$

from which the compatibility of $\theta_L(-)'$ with the exact sequence follows, which finishes the proof of the lemma. □

3D. ***Formulation of the local ε-conjecture.*** In this subsection, using the definitions in the previous subsections, we formulate the following conjecture, which we call the local ε-conjecture. This conjecture is a combination of Kato's original ε-conjecture for (Λ, T) with our conjecture for (A, M). To state both situations at the same time, we use the notation (B, N) for (Λ, T) or (A, M), and $f: B\to B'$ for $f:\Lambda\to\Lambda'$ or $f: A\to A'$.

Conjecture 3.8. *We can uniquely define a B-linear isomorphism*

$$\varepsilon_{B,\zeta}(N): \mathbf{1}_B \xrightarrow{\sim} \Delta_B(N)$$

for each pair (B, N) as above and for each $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$ satisfying the following conditions:

(i) *Let $f : B \to B'$ be a continuous homomorphism. Then we have*

$$\varepsilon_{B,\zeta}(N) \otimes \mathrm{id}_{B'} = \varepsilon_{B',\zeta}(N \otimes_B B')$$

under the canonical isomorphism $\Delta_B(N) \otimes_B B' \xrightarrow{\sim} \Delta_{B'}(N \otimes_B B')$.

(ii) *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence. Then we have*

$$\varepsilon_{B,\zeta}(N_1) \boxtimes \varepsilon_{B,\zeta}(N_3) = \varepsilon_{B,\zeta}(N_2)$$

under the canonical isomorphism $\Delta_B(N_1) \boxtimes \Delta_B(N_3) \xrightarrow{\sim} \Delta_B(N_2)$.

(iii) *For any $a \in \mathbb{Z}_p^\times$, we have*

$$\varepsilon_{B,\zeta^a}(N) = \delta_{\det_B(N)}(a) \cdot \varepsilon_{B,\zeta}(N).$$

(iv) *One has the following commutative diagram of isomorphisms*:

$$\begin{array}{ccc} \Delta_B(N) & \xrightarrow{\text{can}} & \Delta_B(N^*)^\vee \boxtimes (L(r_N), 0) \\ {\scriptstyle \varepsilon_{B,\zeta^{-1}}(N)}\uparrow & & \downarrow{\scriptstyle \varepsilon_{B,\zeta}(N^*)^\vee \boxtimes [\boldsymbol{e}_{r_N} \mapsto 1]} \\ \mathbf{1}_B & \xrightarrow{\text{can}} & \mathbf{1}_B \boxtimes \mathbf{1}_B \end{array}$$

(v) *Let $f : \Lambda \to A$ be a continuous homomorphism, and let $M := \boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A)$ be the associated (φ, Γ)-module obtained by the base change of T with respect to f. Then we have*

$$\varepsilon_{\Lambda,\zeta}(T) \otimes \mathrm{id}_A = \varepsilon_{A,\zeta}(M)$$

under the canonical isomorphism $\Delta_\Lambda(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_A(M)$ of Corollary 3.2.

(vi) *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$. Then we have*

$$\varepsilon_{L,\zeta}(M) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

Remark 3.9. Kato's original conjecture [1993b] is the restriction of Conjecture 3.8 to the pairs (Λ, T). As explained in Remark 3.6, we insist that condition (v) should be stated using $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V))$ $\big($or $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V) := \varepsilon^{\mathrm{dR}}_{L,\zeta}(V)' \cdot \det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{st}}(V)/\boldsymbol{D}_{\mathrm{cris}}(V))\big)$ instead of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'$.

Remark 3.10. In Kato's conjecture, the uniqueness of the ε-isomorphism was not explicitly predicted. Recently, it has been shown that the de Rham points (even crystalline points) are Zariski dense in "universal" families of p-adic representations, or (φ, Γ)-modules in many cases ([Colmez 2008; Kisin 2010] for the two-dimensional case, [Chenevier 2013; Nakamura 2014b] for general case), hence we add the uniqueness assertion in our conjecture.

Kato [1993b] proved his conjecture for the rank-one case (note that one has $\boldsymbol{D}_{\mathrm{st}}(V) = \boldsymbol{D}_{\mathrm{cris}}(V)$ for the rank-one case, hence one also has $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)' = \varepsilon^{\mathrm{dR}}_{L,\zeta}(V)$). As a generalization of his theorem, our main theorem of this article is the following, whose proof is given in the next section.

Theorem 3.11. *Conjecture 3.8 is true for the rank-one case. More precisely, we can uniquely define a B-linear isomorphism $\varepsilon_{B,\zeta}(N) : \mathbf{1}_B \xrightarrow{\sim} \Delta_B(N)$ for each pair (B, N) such that N is of rank one and for each $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$ satisfying the conditions* (i), (iii), (iv), (v), (vi).

Before passing to the proof of this theorem in the next section, we prove two easy corollaries concerning the trianguline case. We say that a (φ, Γ)-module M over $\mathcal{R}_A$ is trianguline if M has a filtration $\mathcal{F} : 0 := M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n := M$ whose graded quotients M_i/M_{i-1} are rank-one (φ, Γ)-modules over $\mathcal{R}_A$ for all $1 \le i \le n$. We call the filtration $\mathcal{F}$ a triangulation of M.

Corollary 3.12. *Let M be a trianguline (φ, Γ)-module over $\mathcal{R}_A$ of rank n with a triangulation $\mathcal{F}$ as above. The isomorphism*

$$\varepsilon_{\mathcal{F},A,\zeta}(M) : \mathbf{1}_A \xrightarrow{\boxtimes_{i=1}^n \varepsilon_{A,\zeta}(M_i/M_{i-1})} \boxtimes_{i=1}^n \Delta_A(M_i/M_{i-1}) \xrightarrow{\sim} \Delta_A(M)$$

defined as the product of the isomorphisms $\varepsilon_{A,\zeta}(M_i/M_{i-1}) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M_i/M_{i-1})$, which are defined in Theorem 3.11, satisfies the following properties:

(i)′ *For any $f : A \to A'$, we have*

$$\varepsilon_{\mathcal{F},A,\zeta}(M) \otimes \mathrm{id}_{A'} = \varepsilon_{\mathcal{F}',A',\zeta}(M \otimes_A A'),$$

where $\mathcal{F}'$ is the base change of the triangulation $\mathcal{F}$ by f.

(iii)′ *For any $a \in \mathbb{Z}_p^\times$, we have*

$$\varepsilon_{\mathcal{F},A,\zeta^a}(M) = \delta_{\det_A(M)}(a) \cdot \varepsilon_{\mathcal{F},A,\zeta}(M).$$

(iv)′ *One has the commutative diagram of isomorphisms*

$$\begin{array}{ccc} \Delta_A(M) & \xrightarrow{\text{can}} & \Delta_A(M^*)^\vee \boxtimes (A(r_M), 0) \\ {\scriptstyle \varepsilon_{\mathcal{F},A,\zeta}(M)}\uparrow & & \downarrow {\scriptstyle \varepsilon_{\mathcal{F}^*,A,\zeta}(M^*)^\vee \boxtimes [\boldsymbol{e}_{r_M} \mapsto (-1)^{r_M}]} \\ \mathbf{1}_A & \xrightarrow{\text{can}} & \mathbf{1}_A \boxtimes \mathbf{1}_A \end{array}$$

in which $\mathcal{F}^*$ is the Tate dual of the triangulation $\mathcal{F}$.

(vi)′ *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham and trianguline (φ, Γ)-module over $\mathcal{R}_L$. Then, for any triangulation $\mathcal{F}$ of M, we have*

$$\varepsilon_{\mathcal{F},L,\zeta}(M) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

In particular, in this case, $\varepsilon_{\mathcal{F},L,\zeta}(M)$ does not depend on $\mathcal{F}$.

Proof. This corollary immediately follows from Theorem 3.11 since $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ is multiplicative with respect to exact sequences by (1) of Lemma 3.7. □

Finally, we compare Corollary 3.12 with the previous known results on Kato's ε-conjecture for the cyclotomic deformations of crystalline ones. Let F be a finite unramified extension of $\mathbb{Q}_p$. Let V be a crystalline L-representation of G_F, and let $T \subseteq V$ be a G_F-stable $\mathcal{O}_L$-lattice of V. In [Benois and Berger 2008] and [Loeffler et al. 2015], they defined ε-isomorphisms for some twists of T. Here, for simplicity, we only recall the result of [Benois and Berger 2008] under the additional assumption that $F = \mathbb{Q}_p$, since other cases can be proven in the same way. Let $\mathcal{O}_L[[\Gamma]]$ be the Iwasawa algebra with coefficients in $\mathcal{O}_L$. We define an $\mathcal{O}_L[[\Gamma]]$-representation $\mathbf{Dfm}(T) := T \otimes_{\mathcal{O}_L} \mathcal{O}_L[[\Gamma]]$ on which $G_{\mathbb{Q}_p}$ acts by $g(x \otimes y) := g(x) \otimes [\bar{g}]^{-1} y$ for any $g \in G_{\mathbb{Q}_p},\ x \in T,\ y \in \mathcal{O}_L[[\Gamma]]$. In [Benois and Berger 2008], by studying the associated Wach modules very carefully, they essentially showed that Perrin-Riou's big exponential map induces an ε-isomorphism, which we denote by

$$\varepsilon^{\mathrm{BB}}_{\mathcal{O}_L[[\Gamma]],\zeta}(\mathbf{Dfm}(T)) : \mathbf{1}_{\mathcal{O}_L[[\Gamma]]} \xrightarrow{\sim} \Delta_{\mathcal{O}_L[[\Gamma]]}(\mathbf{Dfm}(T)),$$

satisfying the conditions in Conjecture 3.8. Let $\boldsymbol{D}_{\mathrm{rig}}(V)$ be the (φ, Γ)-module over $\mathcal{R}_L$ associated to V. Then, applying Example 3.3 to $(\Lambda, T) = (\mathcal{O}_L[[\Gamma]], T)$, we obtain a canonical isomorphism

$$\Delta_{\mathcal{O}_L[[\Gamma]]}(\mathbf{Dfm}(T)) \otimes_{\mathcal{O}_L[[\Gamma]]} \mathcal{R}^{\infty}_L(\Gamma) \xrightarrow{\sim} \Delta_{\mathcal{R}^{\infty}(\Gamma)}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))) \tag{31}$$

(see the next section for the definitions of $\mathcal{R}^{\infty}_L(\Gamma)$ and $\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))$). Since $\boldsymbol{D}_{\mathrm{rig}}(V)$ is crystalline, after extending scalars, we may assume that it is trianguline with a triangulation $\mathcal{F}$. Then $\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))$ is also trianguline with a triangulation $\mathcal{F}' := \mathbf{Dfm}(\mathcal{F})$. Hence, by Corollary 3.12, we obtain an isomorphism

$$\varepsilon_{\mathcal{F}',\mathcal{R}^{\infty}_L(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))) : \mathbf{1}_{\mathcal{R}^{\infty}_L(\Gamma)} \xrightarrow{\sim} \Delta_{\mathcal{R}^{\infty}(\Gamma)}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))).$$

Under this situation, we easily obtain the following corollary.

Corollary 3.13. *Under the isomorphism* (31), *we have*

$$\varepsilon^{\mathrm{BB}}_{\mathcal{O}_L[[\Gamma]],\zeta}(\mathbf{Dfm}(T)) \otimes \mathrm{id}_{\mathcal{R}^{\infty}_L(\Gamma)} = \varepsilon_{\mathcal{F}',\mathcal{R}^{\infty}_L(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))).$$

In particular, the isomorphism $\varepsilon_{\mathcal{F}',\mathcal{R}^{\infty}_L(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V)))$ *does not depend on* $\mathcal{F}$.

Proof. By [Benois and Berger 2008] and Theorem 3.11, the base changes of both sides in Corollary 3.13 by the continuous L-algebra morphism $f_\delta : \mathcal{R}^{\infty}_L(\Gamma) \to L : [\gamma] \mapsto \delta(\gamma)^{-1}$ are equal to $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V(\delta)))$ for any potentially crystalline character $\delta : \Gamma \to L^{\times}$. Since the points corresponding to such characters are Zariski dense in the rigid analytic space associated to $\mathrm{Spf}(\mathcal{O}_L[[\Gamma]])$, we obtain the equality in the corollary. □

4. Rank-one case

Kato [1993b] proved his ε-conjecture using the theory of Coleman homomorphism, which interpolates the exponential maps and the dual exponential maps of rank-one de Rham p-adic representations of $G_{\mathbb{Q}_p}$. In particular, the so-called explicit reciprocity law, which is the explicit formula of its interpolation property, was very important in his proof.

In this final section, we first construct the ε-isomorphism

$$\varepsilon_{A,\zeta}(M) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M)$$

for any rank-one (φ, Γ)-module M by interpreting the theory of Coleman homomorphism in terms of p-adic Fourier transforms (e.g., Amice transforms, Colmez transforms), which seems to be standard for experts of the theory of (φ, Γ)-modules. Then we prove that this isomorphism satisfies the de Rham condition (vi) by establishing the "explicit reciprocity law" of our Coleman homomorphism using our theory of Bloch–Kato exponential maps developed in Section 2B.

4A. ***Construction of the ε-isomorphism.*** We first recall the theory of analytic Iwasawa cohomology of (φ, Γ)-modules over the Robba ring after [Pottharst 2012; Kedlaya et al. 2014]. Let $\Lambda(\Gamma) := \mathbb{Z}_p[\![\Gamma]\!]$ be the Iwasawa algebra of Γ with coefficients in $\mathbb{Z}_p$, and let $\mathfrak{m}$ be the Jacobson radical of $\Lambda(\Gamma)$. For each $n \geq 1$, define a $\mathbb{Q}_p$-affinoid algebra $\mathcal{R}^{[1/p^n,\infty]}(\Gamma) := (\Lambda(\Gamma)[\mathfrak{m}^n/p])^\wedge[1/p]$, where, for any ring R, we denote by $R^\wedge$ the p-adic completion of R. Let $X_n := \mathrm{Max}(\mathcal{R}^{[1/p^n,\infty]}(\Gamma))$ be the associated affinoid. Define $X := \bigcup_{n\geq 1} X_n$, which is a disjoint union of open unit discs. For $n \geq 1$, consider the rank-one (φ, Γ)-module

$$\mathbf{Dfm}_n := \mathcal{R}^{[1/p^n,\infty]}(\Gamma) \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{R}\boldsymbol{e} = \mathcal{R}_{\mathcal{R}^{[1/p^n,\infty]}(\Gamma)}\boldsymbol{e}$$

with

$$\varphi(1 \mathbin{\widehat{\otimes}} \boldsymbol{e}) = 1 \mathbin{\widehat{\otimes}} \boldsymbol{e} \qquad \text{and} \qquad \gamma(1 \mathbin{\widehat{\otimes}} \boldsymbol{e}) = [\gamma]^{-1} \mathbin{\widehat{\otimes}} \boldsymbol{e} \quad \text{for } \gamma \in \Gamma.$$

Put $\mathbf{Dfm} := \varprojlim_n \mathbf{Dfm}_n$; this is a (φ, Γ)-module over the relative Robba ring over X. For M a (φ, Γ)-module over $\mathcal{R}_A$, we define the cyclotomic deformation of M by

$$\mathbf{Dfm}(M) := \varprojlim_n \mathbf{Dfm}_n(M)$$

with

$$\mathbf{Dfm}_n(M) := M \mathbin{\widehat{\otimes}}_{\mathcal{R}} \mathbf{Dfm}_n \xrightarrow{\sim} M \mathbin{\widehat{\otimes}}_A \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\boldsymbol{e},$$

which is a (φ, Γ)-module over the relative Robba ring over $\mathrm{Max}(A) \times X$. This (φ, Γ)-module is the universal cyclotomic deformation of M in the sense that, for each continuous homomorphism $\delta_0 : \Gamma \to A^\times$, we have a natural isomorphism

$$\mathbf{Dfm}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} M(\delta_0) : (x \mathbin{\widehat{\otimes}} \eta\boldsymbol{e}) \otimes a \mapsto f_{\delta_0}(\eta) a x \boldsymbol{e}_{\delta_0}$$

for $x \in M,\ \eta \boldsymbol{e} \in \mathcal{R}_A^\infty(\Gamma)\boldsymbol{e}$ and $a \in A$, where

$$f_{\delta_0} : \mathcal{R}_A^\infty(\Gamma) \to A$$

is the continuous A-algebra homomorphism defined by

$$f_{\delta_0}([\gamma]) := \delta_0(\gamma)^{-1}$$

for $\gamma \in \Gamma$ (and recall that $M(\delta_0) := M \otimes_A A\boldsymbol{e}_{\delta_0} = M\boldsymbol{e}_{\delta_0}$ is defined by $\varphi(x\boldsymbol{e}_{\delta_0}) = \varphi(x)\boldsymbol{e}_{\delta_0}$ and $\gamma(x\boldsymbol{e}_{\delta_0}) := \delta_0(\gamma)\gamma(x)\boldsymbol{e}_{\delta_0}$ for $x \in M$ and $\gamma \in \Gamma$).

By Theorem 4.4.8 of [Kedlaya et al. 2014], we have a natural quasi-isomorphism of $\mathcal{R}_A^\infty(\Gamma)$-modules

$$g_\gamma : C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_\psi(M) := \big[M^\Delta \xrightarrow{\psi-1} M^\Delta\big],$$

where the latter complex is concentrated in degree $[1,2]$. This quasi-isomorphism is obtained as a composite of (a system of) quasi-isomorphisms

$$C^\bullet_{\psi,\gamma}(\mathbf{Dfm}_n(M)) \xrightarrow{\sim} C^\bullet_\psi(M)\,\widehat{\otimes}_{\mathcal{R}_A^\infty(\Gamma)}\,\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma),$$

which are naturally induced by the following diagrams of $\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)$-modules for $n \geq 1$ with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{\gamma-1} & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{f_\gamma} & M\,\widehat{\otimes}_{\mathcal{R}_A^\infty(\Gamma)}\,\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma) & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle \psi-1} & & \Big\downarrow{\scriptstyle \psi-1} & & \Big\downarrow{\scriptstyle \psi-1} & \\
0 \longrightarrow & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{\gamma-1} & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{f_\gamma} & M\,\widehat{\otimes}_{\mathcal{R}_A^\infty(\Gamma)}\,\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma) & \longrightarrow 0
\end{array} \tag{32}$$

Here

$$f_\gamma\Big(\sum_i x_i \,\widehat{\otimes}\, \eta_i \boldsymbol{e}\Big) := \frac{1}{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))} \sum_i x_i \,\widehat{\otimes}\, \eta_i$$

for $x_i \in M,\ \eta_i\boldsymbol{e} \in \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\boldsymbol{e}$, with the inverse of the natural quasi-isomorphism

$$C^\bullet_\psi(M) \xrightarrow{\sim} \varprojlim_n C^\bullet_\psi\big(M \otimes_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\big) \xrightarrow{\sim} \varprojlim_n C^\bullet_\psi\big(M\,\widehat{\otimes}_{\mathcal{R}_A^\infty(\Gamma)}\,\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\big)$$

(see Theorem 4.4.8 of [Kedlaya et al. 2014] and Theorem 2.8(3) of [Pottharst 2012] for the proof). This quasi-isomorphism is canonical in the sense that, for another $\gamma' \in \Gamma$ whose image in Γ/Δ is a topological generator, we have the commutative diagram

$$\begin{array}{ccc}
C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) & \xrightarrow{g_\gamma} & C^\bullet_\psi(M) \\
\Big\downarrow{\scriptstyle \iota_{\gamma,\gamma'}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
C^\bullet_{\psi,\gamma'}(\mathbf{Dfm}(M)) & \xrightarrow{g_{\gamma'}} & C^\bullet_\psi(M)
\end{array} \tag{33}$$

For $\delta_0 : \Gamma \to A^\times$, using the natural isomorphism $\mathbf{Dfm}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} M(\delta_0)$ and the quasi-isomorphism g_γ, we obtain the quasi-isomorphism

$$\begin{aligned} g_{\gamma,\delta_0} : C^\bullet_{\psi,\gamma}(M(\delta_0)) &\xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A) \\ &\xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} C^\bullet_{\psi}(M) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A, \end{aligned} \tag{34}$$

where the second isomorphism follows from the fact that any (φ, Γ)-module M_0 over $\mathcal{R}_{A_0}$ is flat over A_0 for any A_0 (see Corollary 2.1.7 of [Kedlaya et al. 2014]). This quasi-isomorphism can be written in a more explicit way as follows. To recall this, we see A as an $\mathcal{R}_A^\infty(\Gamma)$-module by the map f_{δ_0}. Then we can take the projective resolution of A

$$0 \to \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{2,\gamma}} A \to 0,$$

where

$$p_{\delta_0} := \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \delta_0^{-1}(\sigma)[\sigma] \in \mathcal{R}_A^\infty(\Gamma)$$

(this is an idempotent) and

$$d_{1,\gamma}(\eta) := (\delta_0(\gamma)[\gamma] - 1)\eta \quad \text{and} \quad d_{2,\gamma}(\eta) := \frac{1}{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))} f_{\delta_0}(\eta).$$

This resolution induces a canonical isomorphism

$$C^\bullet_\psi(M) \otimes_{\mathcal{R}_A^\infty(\Gamma)} \left[\mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0}\right] \xrightarrow{\sim} C^\bullet_\psi(M) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A.$$

Moreover, using the isomorphism

$$M \otimes_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{\sim} M(\delta)^\Delta : m \otimes \lambda p_{\delta_0} \mapsto \lambda p_{\delta_0}(m\boldsymbol{e}_{\delta_0}),$$

we obtain a natural isomorphism

$$C^\bullet_\psi(M) \otimes_{\mathcal{R}_A^\infty(\Gamma)} \left[\mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0}\right] \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(M(\delta_0)).$$

Composing both, we obtain a natural quasi-isomorphism

$$C^\bullet_\psi(M) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(M(\delta_0)),$$

which is easily seen to be equal to g_{γ,δ_0}.

Using the theory of analytic Iwasawa cohomology recalled as above, we can describe the fundamental line $\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M))$ as follows. The quasi-isomorphism $g_\gamma : C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_\psi(M)$ and the quasi-isomorphism $C^\bullet_{\varphi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M))$ induce a natural isomorphism in $\mathcal{P}_{\mathcal{R}_A^\infty(\Gamma)}$

$$\Delta_{\mathcal{R}_A^\infty(\Gamma),1}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M))) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)).$$

Moreover, since we have

$$\begin{aligned}\Delta_{\mathcal{R}_A^\infty(\Gamma),2}(\mathbf{Dfm}(M)) &= \varprojlim_n \Delta_{\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma),2}(\mathbf{Dfm}_n(M))\\ &\xrightarrow{\sim} \varprojlim_n\big(\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\boldsymbol{e}^{\otimes r_M}\big)\\ &= \Delta_{A,2}(M)\boldsymbol{e}^{\otimes r_M}\otimes_A \mathcal{R}_A^\infty(\Gamma)\xrightarrow{\sim}\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma),\end{aligned}$$

where the last isomorphism is just the division by $\boldsymbol{e}^{\otimes r_M}$, we obtain a canonical isomorphism

$$\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(M))\boxtimes(\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma)). \tag{35}$$

Under this canonical isomorphism, we will first define an isomorphism

$$\theta_\zeta(M):\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(M))^{-1}\xrightarrow{\sim}(\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma)),$$

and then define $\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M))$ as the composite

$$\begin{aligned}\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M)):\mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)} &\xrightarrow{\text{can}} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(M))\boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(M))^{-1}\\ &\xrightarrow{\mathrm{id}\boxtimes\theta_\zeta(M)} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(M))\boxtimes(\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma))\\ &\xrightarrow{\sim}\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M))\end{aligned}$$

for the following special rank-one (φ,Γ)-modules M.

For $\lambda\in A^\times$, define the "unramified" continuous homomorphism $\delta_\lambda:\mathbb{Q}_p^\times\to A^\times$ by $\delta_\lambda(p):=\lambda$ and $\delta_\lambda|_{\mathbb{Z}_p^\times}:=1$. We define an isomorphism $\theta_\zeta(M)$ for $M=\mathcal{R}_A(\delta_\lambda)$ by the following steps, which are based on the reinterpretation of the theory of the Coleman homomorphism in terms of the p-adic Fourier transform.

Let $\mathrm{LA}(\mathbb{Z}_p,A)$ be the set of A-valued locally analytic functions on $\mathbb{Z}_p$, and define the action of (φ,ψ,Γ) on it by

$$\varphi(f)|_{\mathbb{Z}_p^\times}:=0,\qquad \varphi(f)(y):=f\left(\frac{y}{p}\right)\quad(y\in p\mathbb{Z}_p),$$
$$\psi(f)(y):=f(py),\qquad \gamma(f)(y):=\frac{1}{\chi(\gamma)}f\left(\frac{y}{\chi(\gamma)}\right)\quad(\gamma\in\Gamma).$$

One has a (φ,ψ,Γ)-equivariant A-linear surjection, which we call the Colmez transform,

$$\mathrm{Col}:\mathcal{R}_A\to\mathrm{LA}(\mathbb{Z}_p,A) \tag{36}$$

defined by

$$\mathrm{Col}(f(\pi))(y):=\mathrm{Res}_0\left((1+\pi)^y f(\pi)\frac{d\pi}{(1+\pi)}\right),$$

where $\mathrm{Res}_0 : \mathcal{R}_A \to A$ is defined by $\mathrm{Res}_0\big(\sum_{n\in\mathbb{Z}} a_n\pi^n\big) := a_{-1}$ (note that Col depends on the choice of the parameter π, i.e., the choice of ζ). By this map, we obtain the short exact sequence

$$0 \to \mathcal{R}_A^\infty \to \mathcal{R}_A \xrightarrow{\mathrm{Col}} \mathrm{LA}(\mathbb{Z}_p, A) \to 0. \tag{37}$$

Twisting the action of (φ, ψ, Γ) by δ_λ, we obtain the (φ, ψ, Γ)-equivariant exact sequence

$$0 \to \mathcal{R}_A^\infty(\delta_\lambda) \to \mathcal{R}_A(\delta_\lambda) \xrightarrow{\mathrm{Col}\otimes \boldsymbol{e}_{\delta_\lambda}} \mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda) \to 0,$$

from which we obtain the exact sequence of complexes of $\mathcal{R}_A^\infty(\Gamma)$-modules

$$0 \to C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)) \to C_\psi^\bullet(\mathcal{R}_A(\delta_\lambda)) \to C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \to 0. \tag{38}$$

For each $k \geq 0$, we define the algebraic function

$$y^k : \mathbb{Z}_p \to A : a \mapsto a^k.$$

Then $Ay^k\boldsymbol{e}_{\delta_\lambda} \subseteq \mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)$ is a ψ-stable sub-$\mathcal{R}_A^\infty(\Gamma)$-module. By Lemme 2.9 of [Chenevier 2013], the natural inclusion

$$C_\psi^\bullet\left(\bigoplus_{0=k}^{N} Ay^k\boldsymbol{e}_{\delta_\lambda}\right) \hookrightarrow C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \tag{39}$$

is a quasi-isomorphism for sufficiently large N.

Set $P_i^k := Ay^k\boldsymbol{e}_{\delta_\lambda}$ for $i = 1, 2$. Since we have $Ay^k\boldsymbol{e}_{\delta_\lambda}[0] \in \boldsymbol{D}_{\mathrm{perf}}^{[-1,0]}(\mathcal{R}_A^\infty(\Gamma))$ for any $k \geq 0$, the natural exact sequence

$$0 \to P_1^k[-1] \to C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda}) \to P_2^k[-2] \to 0$$

induces a canonical isomorphism

$$g_k : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda})) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(P_2^k) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(P_1^k)^{-1} \xrightarrow{i_{\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(P_1^k)}} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}.$$

We remark that, if the complex $C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda})$ is acyclic, then the composite of this isomorphism with the inverse of the canonical trivialization isomorphism

$$h_{\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda}))} : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda})) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}$$

is the identity map. Hence, if we define the isomorphism

$$g^N := \boxtimes_{0=k}^N g_k : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}\left(C_\psi^\bullet\left(\bigoplus_{k=0}^{N} Ay^k\boldsymbol{e}_{\delta_\lambda}\right)\right) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}, \tag{40}$$

then, by (39) and (40) (for sufficiently large N), we obtain an isomorphism

$$\iota_0 : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda))) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}, \tag{41}$$

which is independent of the choice of (sufficiently large) N.

Since $C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda))$, $C_\psi^\bullet(\mathcal{R}_A(\delta_\lambda))$ are both perfect complexes, we also have

$$C_\psi^\bullet(\mathcal{R}_A^\infty(\Gamma)) \in \boldsymbol{D}^b_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$$

by the exact sequence (38), and then we obtain an isomorphism

$$\begin{aligned}\iota_1 : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}&(C_\psi^\bullet(\mathcal{R}_A(\delta_\lambda)))\\ &\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda))) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)))\\ &\xrightarrow{\mathrm{id}\boxtimes\iota_0} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda))) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(\mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1}[0])^{-1},\end{aligned} \tag{42}$$

where the last isomorphism is the one naturally induced by the exact sequence

$$0 \to \mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1} \to \mathcal{R}_A^\infty(\delta_\lambda) \xrightarrow{\psi-1} \mathcal{R}_A^\infty(\delta_\lambda) \to 0$$

(where the surjectivity is proved in Lemme 2.9(v) of [Chenevier 2013]).

We next consider the complex $C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda))$. For a $\mathcal{R}_A^\infty(\Gamma)$-module M with linear actions of φ and ψ, define a complex

$$C_{\tilde{\psi}}^\bullet(M) := \big[M \xrightarrow{\psi} M\big] \in \boldsymbol{D}^{[1,2]}(\mathcal{R}_A^\infty(\Gamma)),$$

and define a map of complexes $\alpha_M : C_\psi^\bullet(M) \to C_{\tilde{\psi}}^\bullet(M)$ by

$$\begin{array}{ccccc} C_\psi^\bullet(M) : \big[& M & \xrightarrow{\psi-1} & M & \big] \\ \downarrow{\scriptstyle \alpha_M} & \downarrow{\scriptstyle 1-\varphi} & & \downarrow{\scriptstyle \mathrm{id}_M} & \\ C_{\tilde{\psi}}^\bullet(M) : \big[& M & \xrightarrow{\psi} & M & \big] \end{array} \tag{43}$$

For $N \geq 0$, set $D_N := \bigoplus_{0\leq k\leq N} At^k \boldsymbol{e}_{\delta_\lambda}$. Since $At^k\boldsymbol{e}_{\delta_\lambda}[0] \in \boldsymbol{D}^{[-1,0]}_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$, we can define a canonical isomorphism

$$\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(D_N)) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)} \tag{44}$$

in the same way as the isomorphism (40). Then the natural exact sequence $0 \to C_\psi^\bullet(D_N) \to C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)) \to C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)/D_N) \to 0$ induces a canonical isomorphism

$$\begin{aligned}\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}&(C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)))\\ &\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(D_N)) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)/D_N))\\ &\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)/D_N)),\end{aligned} \tag{45}$$

where the last isomorphism is induced by the isomorphism (44).

Since the map $1-\varphi : \mathcal{R}_A^\infty(\delta_\lambda)/D_N \to \mathcal{R}_A^\infty(\delta_\lambda)/D_N$ is an isomorphism for sufficiently large N by Lemme 2.9(ii) of [Chenevier 2013], the map $\alpha_{(\mathcal{R}_A^\infty(\delta_\lambda)/D_N)}$ is also an isomorphism for sufficiently large N. Hence, for sufficiently large N, we obtain a canonical isomorphism

$$\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)/D_N)) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda)/D_N)). \tag{46}$$

Since the complex $C^\bullet_{\tilde{\psi}}(D_N)$ is acyclic (since $\psi : At^k\boldsymbol{e}_{\delta_\lambda} \to At^k\boldsymbol{e}_{\delta_\lambda}$ is an isomorphism for any $k \geq 0$), the natural exact sequence $0 \to C^\bullet_{\tilde{\psi}}(D_N) \to C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda)) \to C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda)/D_N) \to 0$ induces a canonical isomorphism

$$\begin{aligned}\mathrm{Det}&_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda)/D_N))\\ &\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(D_N)) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda)/D_N))\\ &\qquad\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda))),\end{aligned} \tag{47}$$

where the first isomorphism is induced by the inverse of the isomorphism

$$h_{C^\bullet_{\tilde{\psi}}(D_N)} : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(D_N)) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}.$$

Moreover, the exact sequence $0 \to \mathcal{R}_A^\infty(\delta_\lambda)^{\psi=0} \to \mathcal{R}_A^\infty(\delta_\lambda) \xrightarrow{\psi} \mathcal{R}_A^\infty(\delta_\lambda) \to 0$ and the isomorphism

$$\mathcal{R}_A^\infty(\Gamma)\boldsymbol{e}_{\delta_\lambda} \xrightarrow{\sim} \mathcal{R}_A^\infty(\delta_\lambda)^{\psi=0} : \lambda\boldsymbol{e}_{\delta_\lambda} \mapsto (\lambda\cdot(1+\pi)^{-1})\boldsymbol{e}_{\delta_\lambda} \tag{48}$$

(note that this isomorphism depends on the choice of ζ) naturally induces the isomorphism

$$\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\tilde{\psi}}(\mathcal{R}_A^\infty(\delta_\lambda)))^{-1} \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(\mathcal{R}_A^\infty(\delta_\lambda)^{\psi=0}) \xrightarrow{\sim} (\mathcal{R}_A^\infty(\Gamma)\boldsymbol{e}_{\delta_\lambda}, 1). \tag{49}$$

Finally, as the composites of the inverses of the isomorphisms (42), (45), (46), (47), and the isomorphism (49), we define the desired isomorphism

$$\begin{aligned}\theta_\zeta(\mathcal{R}_A(\delta_\lambda)) : \mathrm{Det}&_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A(\delta_\lambda)))^{-1}\\ &\xrightarrow{\sim} (\mathcal{R}_A^\infty(\Gamma)\boldsymbol{e}_{\delta_\lambda}, 1) = \Delta_{A,2}(\mathcal{R}_A(\delta_\lambda)) \otimes_A \mathcal{R}_A^\infty(\Gamma).\end{aligned}$$

Definition 4.1. Using the isomorphism (35), for $M = \mathcal{R}_A(\delta_\lambda)$, we define the ε-isomorphism by

$$\begin{aligned}\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M)) : \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)} &\xrightarrow{\text{can}} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M))^{-1}\\ &\xrightarrow{\mathrm{id}\boxtimes\theta_\zeta(M)} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)) \boxtimes (\Delta_{A,2}(M) \otimes_A \mathcal{R}_A^\infty(\Gamma))\\ &\xrightarrow{\sim} \Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M)).\end{aligned}$$

Before defining the ε-isomorphism for the general rank-one case, we check that the isomorphism $\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda)))$ defined above satisfies the properties (i) and (iii) in Conjecture 3.8

For the property (i), it is clear that, for each continuous homomorphism $f : A \to A'$ (and set $\lambda' = f(\lambda)$), we have

$$\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes \mathrm{id}_{A'} = \varepsilon_{\mathcal{R}_{A'}^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_{A'}(\delta_{\lambda'})))$$

under the canonical isomorphism

$$\begin{aligned}\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes_A A' &\xrightarrow{\sim} \Delta_{\mathcal{R}_{A'}^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda)\,\widehat{\otimes}_A\, A'))\\ &\xrightarrow{\sim} \Delta_{\mathcal{R}_{A'}^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_{A'}(\delta_{\lambda'}))),\end{aligned}$$

where the last isomorphism is induced by the isomorphism

$$\mathcal{R}_A(\delta_\lambda)\,\widehat{\otimes}_A\, A' \xrightarrow{\sim} \mathcal{R}_{A'}(\delta_{\lambda'}) : g(\pi)\boldsymbol{e}_{\delta_\lambda}\,\widehat{\otimes}\, a \mapsto a g^f(\pi)\boldsymbol{e}_{\delta_{\lambda'}};$$

here we define

$$g^f(\pi) := \sum_{n\in\mathbb{Z}} f(a_n)\pi^n \in \mathcal{R}_{A'} \quad \text{for } g(\pi) = \sum_{n\in\mathbb{Z}} a_n\pi^n \in \mathcal{R}_A.$$

The property (iii) easily follows from (48) since one has $(1+\pi_{\zeta^a}) = (1+\pi_\zeta)^a = [\sigma_a]\cdot(1+\pi_\zeta)$ for $a \in \mathbb{Z}_p^\times$.

Next, we consider a rank-one (φ, Γ)-module over $\mathcal{R}_A$ of the form $\mathcal{R}_A(\delta)$ for a general continuous homomorphism $\delta : \mathbb{Q}_p^\times \to A^\times$. Set

$$\lambda := \delta(p) \quad \text{and} \quad \delta_0 := \delta|_{\mathbb{Z}_p^\times},$$

which we freely see as a homomorphism $\delta_0 : \Gamma \to A^\times$ by identifying $\chi : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$. We define the continuous A-algebra homomorphism

$$f_{\delta_0} : \mathcal{R}_A^\infty(\Gamma) \to A,$$

which is uniquely characterized by $f_{\delta_0}([\gamma]) = \delta_0(\gamma)^{-1}$ for any $\gamma \in \Gamma$. Then we have a canonical isomorphism

$$\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda)) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} \mathcal{R}_A(\delta)$$

defined by

$$(f(\pi)\boldsymbol{e}_{\delta_\lambda}\,\widehat{\otimes}\,\eta\boldsymbol{e}) \otimes a := a f_{\delta_0}(\eta) f(\pi)\boldsymbol{e}_\delta$$

for $f(\pi) \in \mathcal{R}_A,\ \eta \in \mathcal{R}_A^\infty(\Gamma),\ a \in A$, which also induces a canonical isomorphism

$$\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta)).$$

Definition 4.2. We define the isomorphism

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta))$$

by

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) := \varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes \mathrm{id}_A$$

under the above isomorphism.

Next, we consider a rank-one (φ, Γ)-module of the form $\mathcal{R}_A(\delta) \otimes_A \mathcal{L}$ for an invertible A-module $\mathcal{L}$.

Lemma 4.3. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$ (of any rank), and let $\mathcal{L}$ be an invertible A-module. Then there exist a canonical A-linear isomorphism*

$$\Delta_A(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_A(M).$$

Proof. The natural isomorphism $C^\bullet_{\varphi,\gamma}(M \otimes_A \mathcal{L}) \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(M) \otimes_A \mathcal{L}$ induces an isomorphism

$$\Delta_{A,1}(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_{A,1}(M) \boxtimes (\mathcal{L}^{\otimes -r_M}, 0).$$

Since we also have a natural isomorphism $\mathcal{L}_A(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \mathcal{L}_A(M) \otimes_A \mathcal{L}^{\otimes r_M}$, we obtain a natural isomorphism

$$\Delta_{A,2}(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_{A,2}(M) \boxtimes (\mathcal{L}^{\otimes r_M}, 0).$$

Then the isomorphism in the lemma is obtained by taking the products of these isomorphisms with the canonical isomorphism $i_{(\mathcal{L}^{\otimes r_M},0)} : (\mathcal{L}^{\otimes r_M}, 0) \boxtimes (\mathcal{L}^{\otimes -r_M}, 0) \xrightarrow{\sim} \mathbf{1}_A$. □

Definition 4.4. We define the isomorphism

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta) \otimes_A \mathcal{L})$$

by

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) := \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$$

under the above isomorphism $\Delta_A(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_A(M)$.

Finally, let M be a general rank-one (φ, Γ)-module over $\mathcal{R}_A$. By Theorem 2.8, there exists a unique pair $(\delta, \mathcal{L})$ such that $g : M \xrightarrow{\sim} \mathcal{R}(\delta) \otimes_A \mathcal{L}$. This isomorphism induces an isomorphism $g_* : \Delta_A(M) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta) \otimes_A \mathcal{L})$.

Definition 4.5. Under the above situation, we define

$$\varepsilon_{A,\zeta}(M) := \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) \circ g_* : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M).$$

Lemma 4.6. *The isomorphism $\varepsilon_{A,\zeta}(M)$ is well defined, i.e., does not depend on g.*

Proof. Since we have $\mathrm{Aut}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) = A^\times$ (where $\mathrm{Aut}(M)$ is the group of automorphisms of M as (φ, Γ)-modules over $\mathcal{R}_A$), it suffices to show the following lemma. □

Lemma 4.7. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$. For $a \in A^\times$, let us define $g_a : M \xrightarrow{\sim} M : x \mapsto ax$. Then we have*

$$(g_a)_* = \mathrm{id}_{\Delta_A(M)}.$$

Proof. This lemma immediately follows from the fact that g_a induces $\Delta_{1,A}(M)\xrightarrow{\sim}\Delta_{A,1}(M): x\mapsto a^{-r_M}x$ (by the Euler–Poincaré formula) and $\Delta_{A,2}(M)\xrightarrow{\sim}\Delta_{A,2}(M): x\mapsto a^{r_M}x$ by definition. □

Remark 4.8. By definition, it is clear that $\varepsilon_{A,\zeta}(M)$, constructed above, satisfies the conditions (i) and (iii) in Conjecture 3.8. It also seems to be easy to directly prove the conditions (iv), (v) of Conjecture 3.8. However, in the next subsection, we prove the conditions (iv) and (v) using density arguments in the process of verifying the condition (vi).

Remark 4.9. Define $\mathcal{O}_{\mathcal{E}} := \bigl\{\sum_{n\in\mathbb{Z}} a_n\pi^n \mid a_n\in\mathbb{Z}_p,\ a_{-n}\to 0\ (n\to+\infty)\bigr\}$, $\mathcal{O}_{\mathcal{E}^+} := \mathbb{Z}_p[\![\pi]\!]$, and $\mathcal{O}_{\mathcal{E}^+,\Lambda} := \mathcal{O}_{\mathcal{E}^+}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda$. Define $\mathcal{C}^0(\mathbb{Z}_p,\Lambda)$ to be the Λ-modules of Λ-valued continuous functions on $\mathbb{Z}_p$. Using the exact sequence

$$0\to\mathcal{O}_{\mathcal{E}^+,\Lambda}\to\mathcal{O}_{\mathcal{E},\Lambda}\xrightarrow{\mathrm{Col}}\mathcal{C}^0(\mathbb{Z}_p,\Lambda)\to 0,$$

which is the continuous function analogue of the exact sequence (37), and using the equivalence between the category of Λ-representations of $G_{\mathbb{Q}_p}$ with that of étale (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E},\Lambda}$ [Dee 2001], it seems possible to define an ε-isomorphism $\varepsilon_{\Lambda,\zeta}(\Lambda(\tilde{\delta}))$ for any $\tilde{\delta}: G^{\mathrm{ab}}_{\mathbb{Q}_p}\to\Lambda^\times$ in the same way as the definition of $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$. Using this ε-isomorphism, it is clear that our ε-isomorphism $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$ satisfies the condition (v) in Conjecture 3.8. Moreover, it is easy to compare the isomorphism $\varepsilon_{\Lambda,\zeta}(\Lambda(\tilde{\delta}))$ with the one Kato defined [1993b].

4B. ***Verification of the conditions* (iv), (v), (vi).** In this final subsection, we prove that our ε-isomorphism $\varepsilon_{A,\zeta}(M)$, constructed in the previous subsection, satisfies the conditions (iv), (v), (vi) of Conjecture 3.8. Of course, the essential part is to prove the condition (vi); the other conditions follow from it using density arguments.

Therefore, in this subsection, we mainly concentrate on the case where $A = L$ is a finite extension of $\mathbb{Q}_p$. Before verifying the condition (vi), we describe the isomorphism $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)):\mathbf{1}_L\xrightarrow{\sim}\Delta_L(\mathcal{R}_L(\delta))$ for any continuous homomorphism $\delta=\delta_\lambda\delta_0:\mathbb{Q}_p^\times\to L^\times$ in a more explicit way.

For an $\mathcal{R}_L^\infty(\Gamma)$-module N, define a Γ-module $N(\delta_0) := N\boldsymbol{e}_{\delta_0}$ by $\gamma(x\boldsymbol{e}_{\delta_0}) = \delta_0(\gamma)([\gamma]\cdot x)\boldsymbol{e}_{\delta_0}$ for any $\gamma\in\Gamma$. Then we have a natural quasi-isomorphism

$$N[-1]\otimes^{\boldsymbol{L}}_{\mathcal{R}_L^\infty(\Gamma),f_{\delta_0}} L\xrightarrow{\sim} N\otimes_{\mathcal{R}_L^\infty(\Gamma)}\bigl[\mathcal{R}_L^\infty(\Gamma)p_{\delta_0}\xrightarrow{d_{1,\gamma}}\mathcal{R}_L^\infty(\Gamma)p_{\delta_0}\bigr]\xrightarrow{\sim} C^\bullet_\gamma(N(\delta_0)).$$

Hence, if $N[0]\in\boldsymbol{D}^b_{\mathrm{perf}}(\mathcal{R}_L^\infty(\Gamma))$, then we obtain a natural isomorphism

$$\begin{aligned}\mathrm{Det}_L(N[-1])\otimes_{\mathcal{R}_L^\infty(\Gamma),f_{\delta_0}} L &\xrightarrow{\sim}\mathrm{Det}_L(C^\bullet_\gamma(N(\delta_0)))\\ &\xrightarrow{\sim}\boxtimes_{i=0,1}\mathrm{Det}_L(\mathrm{H}^i_\gamma(N(\delta_0)))^{(-1)^i}.\end{aligned}$$

Moreover, if N is also equipped with a commuting linear action of ψ such that $C^\bullet_\psi(M) \in \boldsymbol{D}^b_{\mathrm{perf}}(\mathcal{R}^\infty_L(\Gamma))$, then we obtain a natural isomorphism

$$\mathrm{Det}_L(C^\bullet_\psi(N)) \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} \mathrm{Det}_L(C^\bullet_{\psi,\gamma}(N(\delta_0))) \xrightarrow{\sim} \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(N(\delta_0)))^{(-1)^i}.$$

In particular, the isomorphism $\bar{\theta}_\zeta(\mathcal{R}_L(\delta)) := \theta_\zeta(\mathcal{R}_L(\delta_\lambda)) \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} \mathrm{id}_L$ can be seen as the isomorphism

$$\bar{\theta}_\zeta(\mathcal{R}_L(\delta)) : \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(\delta)))^{(-1)^{i+1}} \xrightarrow{\sim} (\mathcal{R}^\infty_L(\Gamma)\boldsymbol{e}_{\delta_\lambda}, 1) \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} (L\boldsymbol{e}_\delta, 1), \quad (50)$$

where the last isomorphism is induced by the isomorphism

$$\mathcal{R}^\infty_L(\Gamma)\boldsymbol{e}_{\delta_\lambda} \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} L\boldsymbol{e}_\delta : (\eta \boldsymbol{e}_{\delta_\lambda}) \otimes a \mapsto a f_{\delta_0}(\eta)\boldsymbol{e}_\delta.$$

Therefore, to verify the condition (vi) when $\mathcal{R}_L(\delta)$ is de Rham, we need to relate the map $\bar{\theta}_\zeta(\mathcal{R}_L(\delta))$ with the Bloch–Kato exponential map or the dual exponential map.

To do so, we divide into the following three cases:

(1) $\delta \neq x^{-k}, x^{k+1}|x|$ for any $k \in \mathbb{Z}_{\geq 0}$ (which we call the generic case).

(2) $\delta = x^{-k}$ for $k \geq 0$.

(3) $\delta = x^{k+1}|x|$ for $k \geq 0$.

We will first verify the condition (vi) in the generic case by establishing a kind of explicit reciprocity law (see Propositions 4.11 and 4.16). Then we will verify the conditions (iv) and (v) using the generic case by density argument. Finally, we will prove the condition (vi) in the case (2) via direct calculations, and reduce the case (3) to the case (2) using the duality condition (iv).

In the remaining parts, we freely use the results of Colmez and Chenevier concerning the calculations of cohomologies

$$\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(\delta)), \quad \mathrm{H}^i_{\psi,\gamma}(\mathcal{R}^\infty_L(\delta)) \quad \text{and} \quad \mathrm{H}^i_{\psi,\gamma}(\mathrm{LA}(\mathbb{Z}_p, L)(\delta));$$

see Proposition 2.1 and Théorème 2.9 of [Colmez 2008] and Lemme 2.9 and Corollaire 2.11 of [Chenevier 2013].

4B1. *Verification of the condition* (vi) *in the generic case.* In this subsection, we assume that δ is generic. Then we have

$$\mathrm{H}^i_{\psi,\gamma}(Lt^k\boldsymbol{e}_\delta) = \mathrm{H}^i_{\psi,\gamma}(Ly^k\boldsymbol{e}_\delta) = \mathrm{H}^i_{\psi,\gamma}(\mathrm{LA}(\mathbb{Z}_p, L)(\delta)) = 0$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, 1, 2\}$, and

$$\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(\delta)) = \mathrm{H}^i_{\psi,\gamma}(\mathcal{R}^\infty_L(\delta)) = 0$$

for $i = 0, 2$, and

$$\dim_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) = \dim_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}^\infty_L(\delta)) = 1.$$

Then $\iota_{1,\delta} := \iota_1 \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} \mathrm{id}_L$ (see (42)) is the isomorphism

$$(\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)), 1)^{-1} \xrightarrow{\sim} (\mathrm{H}^1_\gamma(\mathcal{R}^\infty_L(\delta)^{\psi=1}), 1)^{-1} \tag{51}$$

in $\mathcal{P}_L$ induced by the isomorphism

$$\mathrm{H}^1_\gamma(\mathcal{R}^\infty_L(\delta)^{\psi=1}) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) : [x] \mapsto [x, 0].$$

Then the base change by f_{δ_0} of the isomorphism

$$\mathrm{Det}_{\mathcal{R}^\infty_L(\Gamma)}(C^\bullet_\psi(\mathcal{R}^\infty_L(\delta_\lambda)))^{-1} \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}^\infty_L(\Gamma)}(\mathcal{R}^\infty_L(\delta_\lambda)^{\psi=0}[0]) \xrightarrow{\sim} (\mathcal{R}^\infty_L(\Gamma)\boldsymbol{e}_{\delta_\lambda}, 1),$$

which is induced by (45), (46), (47) and (49), becomes the isomorphism

$$(\mathrm{H}^1_\gamma(\mathcal{R}^\infty_L(\delta)^{\psi=1}), 1) \xrightarrow{[x]\mapsto[(1-\varphi)x]} (\mathrm{H}^1_\gamma(\mathcal{R}^\infty_L(\delta)^{\psi=0}), 1) \xrightarrow{\sim} (L\boldsymbol{e}_\delta, 1), \tag{52}$$

where the last isomorphism is explicitly defined as follows. For an explicit definition of this isomorphism, it is useful to use the Amice transform. Let $D(\mathbb{Z}_p, L) := \mathrm{Hom}^{\mathrm{cont}}_L(\mathrm{LA}(\mathbb{Z}_p, L), L)$ be the algebra of L-valued distributions on $\mathbb{Z}_p$, where the multiplication is defined by the convolution. By the theorem of Amice, we have an isomorphism of topological L-algebras

$$D(\mathbb{Z}_p, L) \xrightarrow{\sim} \mathcal{R}^\infty_L : \mu \mapsto f_\mu(\pi) := \sum_{n\geq 0} \mu\left(\binom{y}{n}\right)\pi^n$$

(which depends on the choice of π, i.e., the choice of ζ), where

$$\binom{y}{n} := \frac{y(y-1)\cdots(y-n+1)}{n!}.$$

Then the action of (φ, Γ, ψ) on $\mathcal{R}^\infty_L$ induces the action on $D(\mathbb{Z}_p, L)$ by

$$\int_{\mathbb{Z}_p} f(y)\varphi(\mu)(y) := \int_{\mathbb{Z}_p} f(py)\mu(y), \quad \int_{\mathbb{Z}_p} f(y)\psi(\mu)(y) := \int_{p\mathbb{Z}_p} f\left(\frac{y}{p}\right)\mu(y)$$

and

$$\int_{\mathbb{Z}_p} f(y)\sigma_a(\mu)(y) := \int_{\mathbb{Z}_p} f(ay)\mu(y),$$

where, for $a \in \mathbb{Z}_p^\times$, we define $\sigma_a \in \Gamma$ such that $\chi(\sigma_a) = a$.

Using this notion, it is easy to see that the second isomorphism in (52) is defined by

$$\mathrm{H}^1_\gamma(\mathcal{R}^\infty_L(\delta)^{\psi=0}) \xrightarrow{\sim} L\boldsymbol{e}_\delta : [f_\mu \boldsymbol{e}_\delta] \mapsto \frac{\delta(-1)}{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))} \cdot \int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta,$$

where we note that we have an isomorphism

$$D(\mathbb{Z}_p^\times, L)\boldsymbol{e}_\delta \xrightarrow{\sim} \mathcal{R}_L^\infty(\delta)^{\psi=0} : \mu\boldsymbol{e}_\delta \mapsto f_\mu\boldsymbol{e}_\delta,$$

since one has

$$f_{\delta_0}(\lambda) = \int_{\mathbb{Z}_p^\times} \delta_0^{-1}(y)\mu_\gamma(y)$$

for any $\lambda \in \mathcal{R}_L^\infty(\Gamma)$ and any continuous homomorphism $\delta_0 : \mathbb{Z}_p^\times \to L^\times$, where we define $\mu_\gamma \in D(\mathbb{Z}_p^\times, L)$ by $f_{\mu_\gamma}(\pi) = \lambda \cdot (1+\pi)$.

For a Γ-module N, we define $\mathrm{H}^1(\Gamma, N) := N/N_0$, where N_0 is the submodule generated by the set $\{(\gamma - 1)n \mid \gamma \in \Gamma,\ n \in N\}$. Then we have the canonical isomorphism

$$\mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} \mathrm{H}^1_\gamma(\mathcal{R}_L^\infty(\delta)^{\psi=1}) : [f\boldsymbol{e}_\delta] \mapsto [|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma)) p_\Delta(f\boldsymbol{e}_\delta)]$$

(where "canonical" means that this is independent of γ, i.e., is compatible with the isomorphisms $\iota_{\gamma,\gamma'}$ for any $\gamma' \in \Gamma$). Composing this with the isomorphism (52), we obtain an isomorphism

$$(\mathrm{H}^1(\Gamma, \mathcal{R}_L(\delta)^{\psi=1}), 1) \xrightarrow{\sim} (L\boldsymbol{e}_\delta, 1) \tag{53}$$

in $\mathcal{P}_L$. Concerning the explicit description of this isomorphism, we obtain the following lemma.

Lemma 4.10. *The isomorphism* (53) *is induced by the isomorphism*

$$\iota_\delta : \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} L\boldsymbol{e}_\delta : [f_\mu \boldsymbol{e}_\delta] \mapsto \delta(-1) \cdot \int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y).$$

Proof. For $f_\mu\boldsymbol{e}_\delta \in \mathcal{R}_L^\infty(\delta)^{\psi=1}$, we have $(1-\varphi)(f_\mu\boldsymbol{e}_\delta) = ((1-\varphi\psi)f_\mu)\cdot\boldsymbol{e}_\delta$. Then the lemma follows from the formula

$$\int_{\mathbb{Z}_p} f(x)(1-\varphi\psi)\mu(x) = \int_{\mathbb{Z}_p^\times} f(x)\mu(x) \quad \text{for } \mu \in D(\mathbb{Z}_p, L). \qquad \square$$

Next, we furthermore assume that $\mathcal{R}_L(\delta)$ is de Rham. By the classification, it is equivalent to $\delta = \tilde{\delta}x^k$ for $k \in \mathbb{Z}$ and a locally constant homomorphism $\tilde{\delta} : \mathbb{Q}_p^\times \to L^\times$. In the generic case, we have the following isomorphisms of one-dimensional L-vector spaces:

(1) $\exp^*_{\mathcal{R}_L(\delta)^*} : \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))$ if $k \leq 0$.

(2) $\exp_{\mathcal{R}_L(\delta)} : \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))$ if $k \geq 1$.

Let us define $n(\delta) \in \mathbb{Z}_{\geq 0}$ as the minimal integer such that $\tilde{\delta}|_{(1+p^n\mathbb{Z}_p)\cap\mathbb{Z}_p^\times}$ is trivial. Then:

(1) $n(\delta) = 0$ if and only if $\mathcal{R}_L(\delta)$ is crystalline.

(2) $\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)=1$ if $n(\delta)=0$.

(3) $\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)=\tilde{\delta}(p)^{n(\delta)}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\tilde{\delta}(i)^{-1}\zeta^i_{p^{n(\delta)}}$ if $n(\delta)\geq 1$.

(4) $\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)\cdot\varepsilon_L(W(\mathcal{R}_L(\delta)^*),\zeta)=\tilde{\delta}(-1)$.

By definition of $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta))$ and $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$, and by Lemma 4.10, to verify the condition (vi), it suffices to show the following two propositions (Proposition 4.11 for $k\leq 0$ and Proposition 4.16 for $k\geq 1$), which can be seen as a kind of explicit reciprocity law.

Proposition 4.11. *If $k\leq 0$, then the map*

$$\mathrm{H}^1(\Gamma,\mathcal{R}^\infty_L(\delta)^{\psi=1})\xrightarrow{\sim}\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))\xrightarrow{\exp^*_{\mathcal{R}_L(\delta)^*}}\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))=\left(\frac{1}{t^k}L_\infty\boldsymbol{e}_\delta\right)^\Gamma$$

(*where the first isomorphism is defined by* $[f\boldsymbol{e}_\delta]\mapsto[|\Gamma_{\mathrm{for}}|\log_0(\chi(\gamma))p_\Delta(f\boldsymbol{e}_\delta),0]$) *sends each element* $[f_\mu\boldsymbol{e}_\delta]\in\mathrm{H}^1(\Gamma,\mathcal{R}^\infty_L(\delta)^{\psi=1})$ *to*

(1) $\dfrac{(-1)^k}{(-k)!}\cdot\dfrac{\delta(-1)}{\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)}\cdot\dfrac{1}{t^k}\cdot\displaystyle\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *if* $n(\delta)\neq 0$,

(2) $\dfrac{(-1)^k}{(-k)!}\cdot\dfrac{\det_L(1-\varphi\mid\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)^*))}{\det_L(1-\varphi\mid\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)))}\cdot\dfrac{\delta(-1)}{t^k}\cdot\displaystyle\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *if* $n(\delta)=0$.

Proof. Here, we prove the proposition only when $k=0$, i.e., $\delta=\tilde{\delta}$ is locally constant. We will prove it for general $k\leq 0$ after some preparations on the differential operator ∂ (the proof for general k will be given after Remark 4.15).

Hence, we assume that $k=0$. For such δ, we define a map

$$g_{\mathcal{R}_L(\delta)}:\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))\to\mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}_L(\delta))):x\mapsto[\log(\chi(\gamma))x],$$

which is easily seen to be an isomorphism. By Proposition 2.16 of [Nakamura 2014a], one has the commutative diagram

$$\begin{array}{ccc}\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \xrightarrow{\exp^*_{\mathcal{R}_L(\delta)^*}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))\\ \mathrm{id}\downarrow & & \downarrow g_{\mathcal{R}_L(\delta)}\\ \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \xrightarrow{\mathrm{can}} & \mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}(\delta)))\end{array}\qquad(54)$$

Set $n_0:=\max\{n(\delta),1\}$ if $p\neq 2$, and set $n_0:=\max\{n(\delta),2\}$ if $p=2$. Then the image of $[f_\mu\boldsymbol{e}_\delta]\in\mathrm{H}^1(\Gamma,\mathcal{R}^\infty_L(\delta)^{\psi=1})\xrightarrow{\sim}\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))$ by the canonical map $\mathrm{can}:\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))\to\mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}(\delta)))$ is equal to

$$[|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_\Delta(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta))]\in\mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}(\delta))).$$

Hence, it suffices to calculate $g^{-1}_{\mathcal{R}_L(\delta)}([|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_\Delta(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta))])$. By definition of $g_{\mathcal{R}_L(\delta)}$, it is easy to check that we have

$$g^{-1}_{\mathcal{R}_L(\delta)}([|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_\Delta(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta))]) = \frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{1}{[\mathbb{Q}_p(\zeta_{p^{n_0}}):\mathbb{Q}_p]}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\sigma_i(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta)|_{t=0}) =: (*).$$

Concerning the right-hand side, when $n(\delta)\geq 1$ if $p\neq 2$, or $n(\delta)\geq 2$ if $p=2$, one has the following equalities, from which the equality (1) follows in this case:

$$\begin{aligned}
(*) &= \frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{1}{[\mathbb{Q}_p(\zeta_{p^{n(\delta)}}):\mathbb{Q}_p]}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\sigma_i(\iota_{n(\delta)}(f_\mu\boldsymbol{e}_\delta)|_{t=0})\\
&= \frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{p}{(p-1)}\frac{1}{p^{n(\delta)}}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\sigma_i\biggl(\frac{1}{\delta(p)^{n(\delta)}}\int_{\mathbb{Z}_p}\zeta^y_{p^{n(\delta)}}\mu(y)\boldsymbol{e}_\delta\biggr)\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\int_{\mathbb{Z}_p}\zeta^{iy}_{p^{n(\delta)}}\mu(y)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\biggl(\sum_{j\in\mathbb{Z}/p^{n(\delta)}\mathbb{Z}}\zeta^{ij}_{p^{n(\delta)}}\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\biggr)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{j\in\mathbb{Z}/p^{n(\delta)}\mathbb{Z}}\biggl(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta^{ij}_{p^{n(\delta)}}\biggr)\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{j\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\biggl(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta^{ij}_{p^{n(\delta)}}\biggr)\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\biggl(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta^{i}_{p^{n(\delta)}}\biggr)\sum_{j\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(j)^{-1}\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta\\
&= \varepsilon_L(W(\mathcal{R}_L(\delta)^*),\zeta)\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta\\
&= \frac{\delta(-1)}{\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)}\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta.
\end{aligned}$$

Here the second equality follows from

$$\iota_{n(\delta)}(f_\mu)|_{t=0} = f_\mu(\zeta_{p^{n(\delta)}}-1) = \int_{\mathbb{Z}_p}\zeta^y_{p^{n(\delta)}}\mu(y),$$

the third equality follows from

$$\frac{|\Gamma_{\mathrm{for}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{p}{p-1} = 1$$

(for any p), the sixth equality follows from the fact that

$$\left(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta_{p^{n(\delta)}}^{ij}\right)=0$$

if $p\mid j$, and the seventh and eighth follow from the property (4) of ε-constants listed before this proposition.

When $n(\delta)=0$, one has $n_0=1$ if $p\neq 2$ and $n_0=2$ if $p=2$. Then one has the following equalities:

$$\begin{aligned}(*)&=\frac{1}{p^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\sigma_i(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta)|_{t=0})\\&=\frac{1}{p^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\sigma_i\left(\frac{1}{\delta(p)^{n_0}}\int_{\mathbb{Z}_p}\zeta_{p^{n_0}}^{y}\mu(y)\boldsymbol{e}_\delta\right)\\&=\frac{1}{(p\delta(p))^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\int_{\mathbb{Z}_p}\zeta_{p^{n_0}}^{iy}\mu(y)\boldsymbol{e}_\delta\\&=\frac{1}{(p\delta(p))^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\left(\sum_{j\in\mathbb{Z}/p^{n_0}\mathbb{Z}}\zeta_{p^{n_0}}^{ij}\int_{j+p^{n_0}\mathbb{Z}_p}\mu(y)\right)\boldsymbol{e}_\delta\\&=\frac{1}{(p\delta(p))^{n_0}}\sum_{j\in\mathbb{Z}/p^{n_0}\mathbb{Z}}\left(\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\zeta_{p^{n_0}}^{ij}\right)\int_{j+p^{n_0}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta.\end{aligned}$$

Here the first equality follows from

$$\frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{1}{[\mathbb{Q}_p(\zeta_{p^{n_0}}):\mathbb{Q}_p]}=\frac{1}{p^{n_0}}$$

for any p.

When $p\neq 2$, the last term is equal to

$$\frac{1}{p\delta(p)}\left((p-1)\int_{p\mathbb{Z}_p}\mu(y)-\int_{\mathbb{Z}_p^\times}\mu(y)\right)\boldsymbol{e}_\delta$$

since $\sum_{i\in(\mathbb{Z}/p\mathbb{Z})^\times}\zeta_p^{ij}=p-1$ if $p\mid j$ and $\sum_{i\in(\mathbb{Z}/p\mathbb{Z})^\times}\zeta_p^{ij}=-1$ if $p\nmid j$.

Since $f_\mu\boldsymbol{e}_\delta\in\mathcal{R}^\infty(\delta)^{\psi=1}$, we have $\psi(f_\mu)=\delta(p)f_\mu$, hence we have

$$\int_{p\mathbb{Z}_p}\mu(y)=\int_{\mathbb{Z}_p}\psi(\mu)(y)=\delta(p)\int_{\mathbb{Z}_p}\mu(y)=\delta(p)\left(\int_{\mathbb{Z}_p^\times}\mu(y)+\int_{p\mathbb{Z}_p}\mu(y)\right),$$

and we have

$$\int_{p\mathbb{Z}_p}\mu(y)=\frac{\delta(p)}{1-\delta(p)}\int_{\mathbb{Z}_p^\times}\mu(y)$$

since we have $\delta(p)\neq 1$ by the generic assumption on δ.

Therefore, we have

$$\begin{aligned}\frac{1}{p\delta(p)}\left((p-1)\int_{p\mathbb{Z}_p}\mu(y)-\int_{\mathbb{Z}_p^\times}\mu(y)\right)\boldsymbol{e}_\delta &= \frac{1}{p\delta(p)}\left((p-1)\frac{\delta(p)}{1-\delta(p)}-1\right)\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta\\ &= \frac{1}{p\delta(p)}\frac{p\delta(p)-1}{1-\delta(p)}\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta\\ &= \frac{1-\frac{1}{p\delta(p)}}{1-\delta(p))}\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta,\end{aligned}$$

from which we obtain the equality (2) for $p\neq 2$.

When $p=2$, then the last term is equal to

$$\frac{1}{(p\delta(p))^2}\left(2\int_{4\mathbb{Z}_2}\mu(y)-2\int_{2+4\mathbb{Z}_2}\mu(y)\right)\boldsymbol{e}_\delta = \frac{1}{p\delta(p)^2}\left(\int_{4\mathbb{Z}_2}\mu(y)-\int_{2\mathbb{Z}_2}\mu(y)\right)\boldsymbol{e}_\delta$$

since $\sum_{i\in(\mathbb{Z}/4\mathbb{Z})^\times}\zeta_4^{ij}$ is equal to 2 if $j\equiv 0 \pmod 4$, is equal to 0 if $j\equiv 1, 3 \pmod 4$, and is equal to -2 if $j\equiv 2 \pmod 4$. Since we have $\psi(f_\mu)=\delta(p)f_\mu$, we have

$$\int_{4\mathbb{Z}_p}\mu(y)=\int_{2\mathbb{Z}_2}\psi(\mu)(y)=\delta(p)\int_{2\mathbb{Z}_2}\mu(y)=\delta(p)\frac{\delta(p)}{1-\delta(p)}\int_{\mathbb{Z}_2^\times}\mu(y),$$

where the last equality follows from the same argument for $p\neq 2$.

Therefore, we have

$$\begin{aligned}\frac{1}{p\delta(p)^2}\left(\int_{4\mathbb{Z}_2}\mu(y)-\int_{2\mathbb{Z}_2}\mu(y)\right)\boldsymbol{e}_\delta &= \frac{1}{p\delta(p)^2}\left(\delta(p)\frac{\delta(p)}{1-\delta(p)}-\frac{\delta(p)}{1-\delta(p)}\right)\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta\\ &= \frac{1}{p\delta(p)^2}\frac{2\delta(p)^2-\delta(p)}{1-\delta(p)}\int_{\mathbb{Z}_2^\times}\mu(y)\boldsymbol{e}_\delta\\ &= \frac{1-\frac{1}{p\delta(p)}}{1-\delta(p)}\int_{\mathbb{Z}_2^\times}\mu(y)\boldsymbol{e}_\delta,\end{aligned}$$

from which we obtain the equality (2) for $p=2$. □

To prove the above proposition for general $k\leq 0$, we need to recall and prove some facts on the differential operator ∂ defined in §2.4 of [Colmez 2008], which will be used to reduce the verification of the condition (vi) for general k to that for $k=0, 1$ (even for the nongeneric case).

Let A be a $\mathbb{Q}_p$-affinoid algebra. We define an A-linear differential operator $\partial:\mathcal{R}_A\to\mathcal{R}_A: f(\pi)\mapsto(1+\pi)\frac{df(\pi)}{d\pi}$. Let $\delta:\mathbb{Q}_p^\times\to A^\times$ be a continuous homomorphism. Then ∂ naturally induces an A-linear and (φ,Γ)-equivariant morphism

$$\partial:\mathcal{R}_A(\delta)\to\mathcal{R}_A(\delta x): f(\pi)\boldsymbol{e}_\delta\mapsto\partial(f(\pi))\boldsymbol{e}_{\delta x},$$

which sits in the exact sequence

$$0 \to A(\delta) \xrightarrow{a\boldsymbol{e}_\delta \mapsto a\boldsymbol{e}_\delta} \mathcal{R}_A(\delta) \xrightarrow{\partial} \mathcal{R}_A(\delta x) \xrightarrow{f\boldsymbol{e}_{\delta x} \mapsto \mathrm{Res}_0\left(f\frac{d\pi}{1+\pi}\right)\boldsymbol{e}_{\delta|x|^{-1}}} A(\delta|x|^{-1}) \to 0. \quad (55)$$

By this exact sequence, when $A = L$ is a finite extension of $\mathbb{Q}_p$, we immediately obtain the following lemma.

Lemma 4.12. $\partial : C^\bullet_{\varphi,\gamma}(\mathcal{R}_L(\delta)) \to C^\bullet_{\varphi,\gamma}(\mathcal{R}_L(\delta x))$ *is a quasi-isomorphism except when* $\delta = \mathbf{1}, |x|$.

For the general case, the exact sequence (55) induces the canonical isomorphism

$$\mathrm{Det}_A(C^\bullet_{\varphi,\gamma}(A(\delta))) \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta))^{-1} \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta x)) \boxtimes \mathrm{Det}_A(C^\bullet_{\varphi,\gamma}(A(\delta|x|^{-1})))^{-1} \xrightarrow{\sim} \mathbf{1}_A. \quad (56)$$

For $\delta' = \delta, \delta|x|^{-1}$, since $A(\delta')$ is a free A-module, the complex

$$C^\bullet_{\varphi,\gamma}(A(\delta')) : \left[A(\delta')^\Delta_1 \xrightarrow{(\gamma-1)\oplus(\varphi-1)} A(\delta')^\Delta_2 \oplus A(\delta')^\Delta_3 \xrightarrow{(\varphi-1)\oplus(1-\gamma)} A(\delta')^\Delta_4\right]$$

(where $A(\delta')_i = A(\delta')$ for $i = 1, \ldots, 4$) induces the canonical isomorphism

$$\begin{aligned}\mathrm{Det}_A(C^\bullet_{\varphi,\gamma}(A(\delta'))) &= \left(\mathrm{Det}_A(A(\delta')^\Delta_1) \boxtimes \mathrm{Det}_A(A(\delta')^\Delta_3)^{-1}\right) \boxtimes \left(\mathrm{Det}_A(A(\delta')^\Delta_4) \boxtimes \mathrm{Det}_A(A(\delta')^\Delta_2)^{-1}\right) \\ &\xrightarrow{i_{\mathrm{Det}_A(A(\delta')^\Delta_1)} \boxtimes i_{\mathrm{Det}_A(A(\delta')^\Delta_4)}} \mathbf{1}_A.\end{aligned}$$

Applying this isomorphism, the isomorphism (56) becomes the isomorphism $\Delta_{A,1}(\mathcal{R}_A(\delta))^{-1} \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta x)) \xrightarrow{\sim} \mathbf{1}_A$, and then, multiplying by $\Delta_{A,1}(\mathcal{R}_A(\delta))$ on both sides, we obtain the following isomorphism, which we also denote by ∂:

$$\begin{aligned}\partial : \Delta_{A,1}(\mathcal{R}_A(\delta)) &\xrightarrow{\sim} \Delta_{A,1}(\mathcal{R}_A(\delta)) \boxtimes \left(\Delta_{A,1}(\mathcal{R}_A(\delta))^{-1} \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta x))\right) \\ &\xrightarrow{i_{\Delta_{A,1}(\mathcal{R}_A(\delta))^{-1}}} \Delta_{A,1}(\mathcal{R}_A(\delta x)).\end{aligned}$$

Taking the product of this isomorphism with the isomorphism

$$\Delta_{A,2}(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_{A,2}(\mathcal{R}_A(\delta x)) : a\boldsymbol{e}_\delta \mapsto -a\boldsymbol{e}_{\delta x},$$

we obtain the isomorphism

$$\partial : \Delta_A(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta x)).$$

By definition, it is clear that this isomorphism is compatible with any base change $A \to A'$.

Concerning this isomorphism, we prove the following proposition.

Proposition 4.13. $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta x)) = \partial \circ \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$.

Proof. The proof of this proposition is a typical density argument, which will be used several times later.

Define the unramified homomorphism $\delta_Y : \mathbb{Q}_p^\times \to \Gamma(\mathbb{G}_m^{\mathrm{an}}, \mathcal{O}_{\mathbb{G}_m^{\mathrm{an}}})^\times$ by $\delta_Y(p) := Y$ (where Y is the parameter of $\mathbb{G}_m^{\mathrm{an}}$). Then $\mathcal{R}_A(\delta)$ is obtained as a base change of the "universal" rank-one (φ, Γ)-module $\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))$ over $\mathcal{R}_{X\times\mathbb{G}_m^{\mathrm{an}}}$ (X is the rigid analytic space associated to $\mathbb{Z}_p[[\Gamma]]$). Since the isomorphism $\partial : \Delta_A(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta x))$ is compatible with any base change, it suffices to show the proposition for $\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))$. Since $X \times \mathbb{G}_m^{\mathrm{an}}$ is reduced, it suffices to show it for the Zariski dense subset S_0 of $X \times \mathbb{G}_m^{\mathrm{an}}$ defined by

$$S_0 := \{(\delta_0, \lambda) \in X(L)\times\mathbb{G}_m^{\mathrm{an}}(L) \mid L \text{ is a finite extension of } \mathbb{Q}_p,\ \delta := \delta_\lambda\delta_0 \text{ is generic}\}.$$

For any (δ_0, λ) in $S_0(L)$, $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta))$ corresponds to the isomorphism

$$\iota_\delta : \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} L\boldsymbol{e}_\delta : [f_\mu \boldsymbol{e}_\delta] \mapsto \delta(-1)\cdot\int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$$

by Lemma 4.10 and by the arguments before this lemma. Then the equality $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta x)) = \partial \circ \varepsilon_{L,\zeta}(\mathcal{R}_L(\delta))$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) & \xrightarrow{\iota_\delta} & L\boldsymbol{e}_\delta \\ \partial\downarrow & & \downarrow \boldsymbol{e}_\delta \mapsto -\boldsymbol{e}_{\delta x} \\ \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta x)^{\psi=1}) & \xrightarrow{\iota_{\delta x}} & L\boldsymbol{e}_{\delta x} \end{array}$$

Finally, this commutativity follows from the formula

$$\int_{\mathbb{Z}_p} f(y)\partial(\mu)(y) = \int_{\mathbb{Z}_p} yf(y)\mu(y)$$

for any $f(y) \in \mathrm{LA}(\mathbb{Z}_p, L)$, which finally proves the proposition. □

We next prove the compatibility of ∂ with the de Rham ε-isomorphism $\varepsilon_{L,\zeta}^{\mathrm{dR}}(\mathcal{R}_L(\delta))$ for de Rham rank-one (φ, Γ)-modules $\mathcal{R}_L(\delta)$ under a condition on the Hodge–Tate weight of $\mathcal{R}_L(\delta)$ as below.

Lemma 4.14. *Let $\mathcal{R}_L(\delta)$ be a de Rham (φ, Γ)-module (here we don't assume that δ is generic). If the Hodge–Tate weight of $\mathcal{R}_L(\delta)$ is not zero, i.e., we have $\delta = \tilde{\delta}x^k$ such that $k \neq 0$, then we have the equality*

$$\varepsilon_{L,\zeta}^{\mathrm{dR}}(\mathcal{R}_L(\delta x)) = \partial \circ \varepsilon_{L,\zeta}^{\mathrm{dR}}(\mathcal{R}_L(\delta)).$$

Proof. Since one has $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) = \left(L_\infty \frac{1}{t^k}\boldsymbol{e}_\delta\right)^\Gamma$ and $\partial(g(t)) = \frac{dg(t)}{dt}$ for $g(t) \in L_\infty((t))$, the differential operator ∂ naturally induces an isomorphism

$$\partial : \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \to \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x)) : \frac{a}{t^k}\boldsymbol{e}_\delta \mapsto (-k)\frac{a}{t^{k+1}}\boldsymbol{e}_{\delta x}$$

under the condition $k \neq 0$. Hence, by definition of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ using the isomorphisms $\theta_L(M)$ and $\theta_{\mathrm{dR},L}(M,\zeta)$ and the constant $\Gamma_L(M)$ in Section 3B, it suffices to show the following two equalities:

(1) $\theta_L(\mathcal{R}_L(\delta x)) = \partial \circ \theta_L(\mathcal{R}_L(\delta))$.

(2) $\Gamma_L(\mathcal{R}_L(\delta)) \cdot \partial \circ \theta_{\mathrm{dR},L}(\mathcal{R}_L(\delta),\zeta) = \Gamma_L(\mathcal{R}_L(\delta x)) \cdot \theta_{\mathrm{dR},L}(\mathcal{R}_L(\delta x),\zeta) \circ \partial$.

We first prove the equality (2). Since one has $\Gamma_L(\mathcal{R}_L(\delta)) = \Gamma^*(k)$ and $\Gamma_L(\mathcal{R}_L(\delta x)) = \Gamma^*(k+1)$, it suffices to show that the diagram

$$\begin{array}{ccc} \mathcal{L}_L(\mathcal{R}_L(\delta)) = L\boldsymbol{e}_\delta & \xrightarrow{\Gamma^*(k)\cdot f_{\mathcal{R}_L(\delta),\zeta}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \\ \big\downarrow \boldsymbol{e}_\delta \mapsto -\boldsymbol{e}_{\delta x} & & \big\downarrow \partial \\ \mathcal{L}_L(\mathcal{R}_L(\delta x)) = L\boldsymbol{e}_{\delta x} & \xrightarrow{\Gamma^*(k+1)\cdot f_{\mathcal{R}_L(\delta x),\zeta}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x)) \end{array}$$

is commutative, where the map $f_{\mathcal{R}_L(\delta'),\zeta}$ (for $\delta' = \delta, \delta x$) is defined in Lemma 3.4. This commutativity is obvious by definition of $f_{\mathcal{R}_L(\delta_0),\zeta}$ since one has

$$\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta) = \varepsilon_L(W(\mathcal{R}_L(\delta x)),\zeta)$$

(this is because one has a natural isomorphism $\boldsymbol{D}_{\mathrm{pst}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{pst}}(\mathcal{R}_L(\delta x)) : \frac{a}{t^k}\boldsymbol{e}_\delta \mapsto \frac{a}{t^{k+1}}\boldsymbol{e}_{\delta x}$) and $k \cdot \Gamma^*(k) = \Gamma^*(k+1)$ for $k \neq 0$. We next show the equality (1). Under the assumption that $k \neq 0$, it is easy to see that ∂ induces the isomorphisms

$$\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x)), \quad \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))^0 \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x))^0$$

and

$$\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta x)), \quad \mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(\delta x))$$

for any $i = 0, 1, 2$ by Lemma 4.12. Hence, by definition of $\theta'_L(\mathcal{R}_L(\delta))$, it suffices to show that the following two diagrams are commutative for $M = \mathcal{R}_L(\delta)$:

$$\begin{array}{ccccccc} \mathrm{H}^0_{\varphi,\gamma}(M) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M) \oplus t_M & \longrightarrow & \mathrm{H}^1_{\varphi,\gamma}(M) \\ \big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial \\ \mathrm{H}^0_{\varphi,\gamma}(M(x)) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M(x)) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M(x)) \oplus t_{M(x)} & \longrightarrow & \mathrm{H}^1_{\varphi,\gamma}(M(x)) \end{array} \tag{57}$$

and

$$\begin{array}{ccccccc} \mathrm{H}^1_{\varphi,\gamma}(M) & \longrightarrow & \begin{array}{c}\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee \\ \oplus \boldsymbol{D}_{\mathrm{dR}}(M)^0\end{array} & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M)^\vee & \longrightarrow & \mathrm{H}^2_{\varphi,\gamma}(M) \\ \big\downarrow \partial & & \big\downarrow (-\partial^\vee)\oplus\partial & & \big\downarrow -\partial^\vee & & \big\downarrow \partial \\ \mathrm{H}^1_{\varphi,\gamma}(M(x)) & \longrightarrow & \begin{array}{c}\boldsymbol{D}_{\mathrm{cris}}(M(x)^*)^\vee \\ \oplus \boldsymbol{D}_{\mathrm{dR}}(M(x))^0\end{array} & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M(x)^*)^\vee & \longrightarrow & \mathrm{H}^2_{\varphi,\gamma}(M(x)) \end{array} \tag{58}$$

Here $\partial^\vee$ is the dual of

$$\partial : \boldsymbol{D}_{\mathrm{cris}}(M(x)^*) = \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta^{-1}|x|)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta^{-1}x|x|)) = \boldsymbol{D}_{\mathrm{cris}}(M^*).$$

For the commutativity of the diagram (57), the only nontrivial part is the commutativity of the diagram

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{cris}}(M) \oplus t_M & \xrightarrow{\exp_{M,f} \oplus \exp_M} & \mathrm{H}^1_{\varphi,\gamma}(M) \\ \partial \big\downarrow & & \partial \big\downarrow \\ \boldsymbol{D}_{\mathrm{cris}}(M(x)) \oplus t_{M(x)} & \xrightarrow{\exp_{M(x),f} \oplus \exp_{M(x)}} & \mathrm{H}^1_{\varphi,\gamma}(M(x)) \end{array}$$

but this commutativity easily follows from Proposition 2.23. Using the commutativity of (57) for $M = \mathcal{R}_L(\delta^{-1}|x|)$, to prove the commutativity of (58), it suffices to show the commutativities of the following diagrams:

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{dR}}(M) & \longrightarrow & \boldsymbol{D}_{\mathrm{dR}}(M^*)^\vee \\ \partial \big\downarrow & & -\partial^\vee \big\downarrow \\ \boldsymbol{D}_{\mathrm{dR}}(M(x)) & \longrightarrow & \boldsymbol{D}_{\mathrm{dR}}(M(x)^*)^\vee \end{array} \tag{59}$$

and

$$\begin{array}{ccc} \mathrm{H}^i_{\varphi,\gamma}(M) & \longrightarrow & \mathrm{H}^{2-i}_{\varphi,\gamma}(M^*)^\vee \\ \partial \big\downarrow & & -\partial^\vee \big\downarrow \\ \mathrm{H}^i_{\varphi,\gamma}(M(x)) & \longrightarrow & \mathrm{H}^{2-i}_{\varphi,\gamma}(M(x)^*)^\vee \end{array} \tag{60}$$

Here the horizontal arrows are isomorphisms obtained by (Tate) duality. Since the commutativity of (59) is easy to check, here we only prove the commutativity of (60). Moreover, we only prove it for $i = 2$ since other cases are proved in the same way. For $i = 2$, it suffices to show the equality

$$[\partial(f)g\boldsymbol{e}_1] = -[f\partial(g)\boldsymbol{e}_1] \in \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))$$

for any $[f\boldsymbol{e}_\delta] \in \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(\delta))$ and $g\boldsymbol{e}_{\delta^{-1}|x|} \in \mathrm{H}^0_{\varphi,\gamma}(\mathcal{R}_L(\delta^{-1}|x|))$. Since we have $\partial(fg) = \partial(f)g + f\partial(g)$, the equality follows from the fact that we have $[\partial(h)\boldsymbol{e}_1] = 0$ in $\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))$ for any $h \in \mathcal{R}_L$. □

Remark 4.15. Proposition 4.13 and Lemma 4.14 and the following proof of Proposition 4.11 should be generalizable to a more general setting. Let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$ of any rank. In §3 of [Nakamura 2014a], we developed the theory of Perrin-Riou's big exponential map for a de Rham (φ, Γ)-module, which is an $\mathcal{R}^\infty_L(\Gamma)$-linear map $\mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(M)) \to \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(\boldsymbol{N}_{\mathrm{rig}}(M)))$, where $\boldsymbol{N}_{\mathrm{rig}}(M) \subseteq M[1/t]$ is a de Rham (φ, Γ)-module equipped with a natural action of the differential operator ∂_M defined by Berger. This big exponential map is defined using the operator ∂_M. Our generalization of Perrin-Riou's $\delta(V)$-theorem [Nakamura 2014a, Theorem 3.21] states that this map gives an isomorphism

$$\mathrm{Exp}_M : \Delta_{\mathcal{R}_L^\infty(\Gamma)}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \Delta_{\mathcal{R}_L^\infty(\Gamma)}(\mathbf{Dfm}(\boldsymbol{N}_{\mathrm{rig}}(M))).$$

Therefore, as a generalization of Proposition 4.13, it seems to be natural to conjecture that the conjectural ε-isomorphisms should satisfy

$$\varepsilon_{\mathcal{R}_L^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{N}_{\mathrm{rig}}(M))) = \mathrm{Exp}_M \circ \varepsilon_{\mathcal{R}_L^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M)),$$

which we want to study in future works.

Using these results, we prove Proposition 4.11 for general $k \leq 0$ as follows.

Proof of Proposition 4.11 for general $k \leq 0$. Let $\delta = \tilde{\delta}x^k$ be a generic homomorphism such that $k \leq 0$. By the arguments before Proposition 4.11, it suffices to show the equality $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$. This equality follows from the equality $\varepsilon_{L,\zeta}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\tilde{\delta}))$ proved in Proposition 4.11 for $k = 0$, since we have

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \partial^k \circ \varepsilon_{L,\zeta}(\mathcal{R}_L(\tilde{\delta})) \quad \text{and} \quad \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta)) = \partial^k \circ \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\tilde{\delta}))$$

by Proposition 4.13 and Lemma 4.14. □

We next consider the case where $k \geq 1$. To verify the condition (vi), it suffices to show the following proposition.

Proposition 4.16. *If $k \geq 1$, then the map*

$$\mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\exp^{-1}_{\mathcal{R}_L(\delta)}} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))$$

sends each element $[f_\mu \boldsymbol{e}_\delta] \in \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1})$ to

(1) $(k-1)! \cdot \dfrac{\delta(-1)}{\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)} \cdot \dfrac{1}{t^k} \cdot \displaystyle\int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *when* $n(\delta) \neq 0$,

(2) $(k-1)! \cdot \dfrac{\det_L(1-\varphi \mid \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)^*))}{\det_L(1-\varphi \mid \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)))} \cdot \dfrac{\delta(-1)}{t^k} \cdot \displaystyle\int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *when* $n(\delta) = 0$.

Proof. In the same way as the proof of Proposition 4.11, it suffices to show the proposition for $k = 1$ (i.e., $\delta = \tilde{\delta}x$) using Proposition 4.13 and Lemma 4.14.

Hence, we assume $k = 1$. Then, in a similar way as the proof of Proposition 4.11 (for $k = 0$), we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\tilde{\delta})) & \longleftarrow & \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\tilde{\delta})^{\psi=1}) & \xrightarrow{\iota_{\tilde{\delta}}} & L\boldsymbol{e}_{\tilde{\delta}} \\ \downarrow \partial & & \downarrow \partial & & \downarrow \boldsymbol{e}_{\tilde{\delta}} \mapsto -\boldsymbol{e}_\delta \\ \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \longleftarrow & \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) & \xrightarrow{\iota_\delta} & L\boldsymbol{e}_\delta \end{array} \tag{61}$$

such that all the arrows are isomorphisms by Lemma 4.12. Hence, reducing to the case of $k=0$, it suffices to show that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(\tilde{\delta})^{\psi=1}) & \longrightarrow & \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\tilde{\delta})) & \xrightarrow{\exp^*_{\mathcal{R}_L(\tilde{\delta}^{-1}x|x|)}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\tilde{\delta})) = (L_\infty \boldsymbol{e}_{\tilde{\delta}})^{\Gamma} \\ \partial\downarrow & & \partial\downarrow & & \downarrow a\boldsymbol{e}_{\tilde{\delta}} \mapsto \frac{a}{t}\boldsymbol{e}_\delta \\ \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(\delta)^{\psi=1}) & \longrightarrow & \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \xleftarrow{\exp_{\mathcal{R}_L(\delta)}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) = \left(L_\infty \frac{1}{t}\boldsymbol{e}_\delta\right)^{\Gamma} \end{array} \tag{62}$$

The following proof of this commutativity is very similar to that of Theorem 3.10 of [Nakamura 2014a]. Take $[f\boldsymbol{e}_{\tilde{\delta}}] \in \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(\tilde{\delta})^{\psi=1})$. If we define

$$\alpha\boldsymbol{e}_{\tilde{\delta}} := \exp^*_{\mathcal{R}_L(\tilde{\delta}^{-1}x|x|)}([|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma)) p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}), 0]) \in \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\tilde{\delta})) \subseteq \boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}_L(\tilde{\delta})),$$

then it suffices to show the equality

$$\exp_{\mathcal{R}_L(\delta)}\left(\frac{\alpha}{t}\boldsymbol{e}_\delta\right) = |\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))[p_\Delta(\partial(f)\boldsymbol{e}_\delta), 0].$$

We prove this equality as follows. First, we have an equality

$$\frac{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))}{\log(\chi(\gamma))}[\iota_n(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))] = [\alpha\boldsymbol{e}_{\tilde{\delta}}] \in \mathrm{H}^1_{\psi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(\mathcal{R}_L(\tilde{\delta})))$$

for large enough $n \geq 1$ by the explicit definition of $\exp^*_{\mathcal{R}_L(\tilde{\delta}^{-1}x|x|)}$ [Nakamura 2014a, Proposition 2.16]. This equality means that for some $y_n \in \boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_L(\tilde{\delta}))^{\Delta}$ we have

$$\frac{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))}{\log(\chi(\gamma))}\iota_n(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}})) - \alpha\boldsymbol{e}_{\tilde{\delta}} = (\gamma-1)y_n.$$

If we set $\nabla_0 := \log([\gamma])/\log(\chi(\gamma)) \in \mathcal{R}_L^{\infty}(\Gamma)$ and define

$$\frac{\nabla_0}{\gamma-1} := \frac{1}{\log(\chi(\gamma))}\sum_{m\geq 1}^{\infty}\frac{(-1)^{m-1}([\gamma]-1)^{m-1}}{m} \in \mathcal{R}_L^{\infty}(\Gamma),$$

then we obtain the equality

$$\begin{aligned} &\iota_n\left(\frac{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right) \\ &\qquad = \frac{1}{\log(\chi(\gamma))}\alpha\boldsymbol{e}_{\tilde{\delta}} + \nabla_0(y_n) \in \frac{1}{\log(\chi(\gamma))}\alpha\boldsymbol{e}_{\tilde{\delta}} + t\boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_L(\tilde{\delta})). \end{aligned} \tag{63}$$

Since we have $f\boldsymbol{e}_{\tilde{\delta}} \in \mathcal{R}_L(\tilde{\delta})^{\psi=1}$, we have

$$(1-\varphi)(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}})) \in \mathcal{R}_L(\tilde{\delta})^{\Delta,\psi=0}.$$

Hence, there exists $\beta \in \mathcal{R}_L(\tilde{\delta})^{\Delta,\psi=0}$ such that

$$(1-\varphi)(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}})) = (\gamma-1)\beta$$

by (for example) Theorem 3.1.1 of [Kedlaya et al. 2014]. Then, for any $m \geq n+1$, we obtain

$$\begin{aligned}\iota_m\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)-\iota_{m-1}\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)&\\ =\iota_m\left((1-\varphi)\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)\right)&=\iota_m\left(\frac{\nabla_0}{\gamma-1}((1-\varphi)(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}})))\right)\\ =\iota_m\left(\frac{\nabla_0}{\gamma-1}((\gamma-1)\beta)\right)&=\iota_m(\nabla_0(\beta))\in t\boldsymbol{D}^+_{\mathrm{dif},m}(\mathcal{R}_L(\tilde{\delta}))\end{aligned}$$

since we have $\nabla_0(\mathcal{R}_L(\tilde{\delta}))\subseteq t\mathcal{R}_L(\tilde{\delta})$. In particular, we obtain

$$\iota_m\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)-\iota_n\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)\in t\boldsymbol{D}^+_{\mathrm{dif},m}(\mathcal{R}_L(\tilde{\delta})) \tag{64}$$

for any $m \geq n+1$ by induction.

Since the map $\mathcal{R}_L(\tilde{\delta}) \xrightarrow{\sim} \frac{1}{t}\mathcal{R}_L(\delta) : g\boldsymbol{e}_{\tilde{\delta}} \mapsto \frac{g}{t}\boldsymbol{e}_\delta$ is an isomorphism of (φ,Γ)-modules, the facts (63), (64) and the explicit definition of the exponential map (Proposition 2.23(1)) induce the equality

$$\begin{aligned}&\exp_{\mathcal{R}_L(\delta)}\left(\frac{\alpha}{t}\boldsymbol{e}_\delta\right)\\ &\quad=|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))\left[(\gamma-1)\frac{\nabla_0}{\gamma-1}\left(p_\Delta\left(\frac{f}{t}\boldsymbol{e}_\delta\right)\right),(\psi-1)\frac{\nabla_0}{\gamma-1}\left(p_\Delta\left(\frac{f}{t}\boldsymbol{e}_\delta\right)\right)\right]\\ &\quad=|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))\left[\nabla_0\left(p_\Delta\left(\frac{f}{t}\boldsymbol{e}_\delta\right)\right),0\right]\\ &\quad=|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))[p_\Delta(\partial(f)\boldsymbol{e}_\delta),0],\end{aligned}$$

where the last equality follows from the equality $\nabla_0\left(\frac{f}{t}\boldsymbol{e}_\delta\right)=\partial(f)\boldsymbol{e}_\delta$ since we have $\nabla_0\left(\frac{1}{t}\boldsymbol{e}_\delta\right)=0$ by the assumption $k=1$, from which the commutativity of the diagram (62) follows. □

As a corollary of Propositions 4.11 and 4.16, we verify the conditions (iv), (v) by the density argument as follows.

Corollary 4.17. *Let M be a rank-one (φ,Γ)-module over $\mathcal{R}_A$. Then the isomorphism $\varepsilon_{A,\zeta}(M):\mathbf{1}_A \xrightarrow{\sim} \Delta_A(M)$, which is defined in Section 4A, satisfies the conditions* (iv) *and* (v) *of Conjecture 3.8.*

Proof. We first verify the conditions (iv). By the definition of $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)\otimes_A\mathcal{L})$, it suffices to do this for (φ,Γ)-modules of the form $M=\mathcal{R}_A(\delta)$ (i.e., $\mathcal{L}=A$) since the general case immediately follows from this case by Lemma 4.6. Then, in the same way as the proof of Proposition 4.13, it suffices to verify these conditions for any $\delta=\delta_\lambda\delta_0:\mathbb{Q}_p^\times\to L^\times$ such that the point $(\delta_0,\lambda)\in X\times\mathbb{G}_m^{\mathrm{an}}$ is contained in the

Zariski dense subset S_1 of $X \times \mathbb{G}_m^{\mathrm{an}}$ defined by

$$S_1 := \{(\delta_0, \lambda) \in X(L) \times \mathbb{G}_m^{\mathrm{an}}(L) \mid [L : \mathbb{Q}_p] < \infty,\ \delta \text{ is generic},\ \mathcal{R}_L(\delta) \text{ is de Rham}\}.$$

For such δ, the conditions (iv) follow from Lemma 3.7 since we have $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$ by Propositions 4.11 and 4.16.

We next verify the condition (v). Let (Λ, T) be as in Conjecture 3.8(v). We recall that we defined a canonical isomorphism

$$\Delta_\Lambda(T) \otimes_\Lambda A_\infty \xrightarrow{\sim} \Delta_{A_\infty}(M_\infty)$$

(see Example 3.3 for definition and notation). Since any continuous map $\Lambda \to A$ factors through $\Lambda \to A_\infty \to A$, it suffices to show the equality

$$\varepsilon_{\Lambda,\zeta}(T) \otimes \mathrm{id}_{A_\infty} = \varepsilon_{A_\infty,\zeta}(M_\infty) \quad \Big(:= \varprojlim_n \varepsilon_{A_n,\zeta}(M_n) \Big). \tag{65}$$

Since condition (v) is local for $\mathrm{Spf}(\Lambda)$, it suffices to verify it for Λ-representations of the form $\Lambda(\tilde{\delta})$ for some $\tilde{\delta} : G^{\mathrm{ab}}_{\mathbb{Q}_p} \to \Lambda^\times$. Let us decompose $\delta = \tilde{\delta} \circ \mathrm{rec}_{\mathbb{Q}_p}$ into $\delta = \delta_\lambda \delta_0$. Since $\Lambda/\mathfrak{m}_\Lambda$ is a finite ring, there exists $k \geq 1$ such that $\lambda^k \equiv 1 \pmod{\mathfrak{m}_\Lambda}$. Then we can define a continuous $\mathbb{Z}_p$-algebra homomorphism

$$\Lambda_k := \varprojlim_n \mathbb{Z}_p[Y]/(p, (Y^k - 1))^n \to \Lambda : Y \mapsto \lambda.$$

Hence, the Λ-representation $\Lambda(\tilde{\delta})$ is obtained by a base change of the "universal" $\mathbb{Z}_p[\![\Gamma]\!] \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \Lambda_k$-representation T_k^{univ}, which corresponds to the homomorphism

$$\delta_k^{\mathrm{univ}} : \mathbb{Q}_p^\times \to (\mathbb{Z}_p[\![\Gamma]\!] \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \Lambda_k)^\times : p \mapsto 1 \mathbin{\widehat{\otimes}} Y,\ a \mapsto [\sigma_a^{-1}] \mathbin{\widehat{\otimes}} 1$$

for $a \in \mathbb{Z}_p^\times$. Hence, it suffices to verify the equality (65) for this universal one. In this case, since the associated rigid space is an admissible open of $X \times \mathbb{G}_m^{\mathrm{an}}$ defined by

$$Z_k := \{(\delta_0, \lambda) \in X \times \mathbb{G}_m^{\mathrm{an}} \mid |\lambda^k - 1| < 1\},$$

and the associated (φ, Γ)-module is isomorphic to the restriction of the universal one $\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))$ defined in the proof of Proposition 4.13, it suffices to show the equality

$$\varepsilon_{\mathbb{Z}_p[\![\Gamma]\!] \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_k,\zeta}(T_k^{\mathrm{univ}}) \otimes \mathrm{id}_{\Gamma(Z_k, \mathcal{O}_{Z_k})} = \varepsilon_{\Gamma(Z_k,\mathcal{O}_{Z_k}),\zeta}(\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))|_{Z_k}).$$

Since both sides satisfy the condition (vi) for any point $(\delta_0, \lambda) \in Z_k \cap S_1$ by Kato's theorem [1993b] and by Propositions 4.11 and 4.16, and since the set $Z_k \cap S_1$ is Zariski dense in Z_k, the equality above follows by the density argument. □

4B2. *Verification of the condition* (vi): *the exceptional case.* Finally, we verify the condition (vi) in the exceptional case, i.e., $\delta = x^{-k}$ or $\delta = x^{k+1}|x|$ for $k \in \mathbb{Z}_{\geq 0}$.

We first reduce all the exceptional cases to the case $\delta = x|x|$.

Lemma 4.18. *We assume that the equality*

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(x|x|)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(x|x|))$$

holds. Then the other equalities

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$$

also hold for all $\delta = x^{k+1}|x|, x^{-k}$ *for* $k \geq 0$.

Proof. The equality for $\delta = x^0$ follows from that for $\delta = x|x|$ by the compatibility of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(-)$ and $\varepsilon_{L,\zeta}(-)$ with the Tate duality, which is proved in Lemma 3.7 and Corollary 4.17. Then the equality for $\delta = x^{k+1}|x|$ (resp. $\delta = x^{-k}$) follows from that for $\delta = x|x|$ (resp. $\delta = x^0$) by the compatibility of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(-)$ and $\varepsilon_{L,\zeta}(-)$ with ∂, which is proved in Lemma 4.14 and Proposition 4.13. □

Finally, it remains to show the equality

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(1)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(1))$$

(we identify $\mathcal{R}_L(x|x|) = \mathcal{R}_L(1) : f\boldsymbol{e}_{x|x|} \mapsto f\boldsymbol{e}_1$). Since $\mathcal{R}_L(1)$ is étale, this equality immediately follows from Kato's result since we have $\varepsilon_{L,\zeta}(\mathcal{R}_L(1)) = \varepsilon_{\mathcal{O}_L,\zeta}(\mathcal{O}_L(1)) \otimes \mathrm{id}_L$ under the canonical isomorphism

$$\Delta_L(\mathcal{R}_L(1)) \xrightarrow{\sim} \Delta_{\mathcal{O}_L}(\mathcal{O}_L(1)) \otimes_{\mathcal{O}_L} L$$

by Corollary 4.17. However, here we give another proof of this equality only using the framework of (φ, Γ)-modules.

In the remaining part of this section, we prove this equality by explicit calculations. First, it is easy to see that the inclusion

$$C^\bullet_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) \hookrightarrow C^\bullet_{\psi,\gamma}(\mathrm{LA}(\mathbb{Z}_p, L)(1))$$

induced by the natural inclusion $L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1 \hookrightarrow \mathrm{LA}(\mathbb{Z}_p, L)(1)$ (here, $1_{\mathbb{Z}_p}$ is the constant function on $\mathbb{Z}_p$ with the constant value 1) is quasi-isomorphism. This quasi-isomorphism and the quasi-isomorphism

$$C^\bullet_{\gamma}(\mathcal{R}^\infty_L(1)^{\psi=1}) \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathcal{R}^\infty_L(1)),$$

and the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{R}^\infty_L(1) \to \mathcal{R}_L(1) \to \mathrm{LA}(\mathbb{Z}_p, L)(1) \to 0$$

induce the isomorphisms

$$\alpha_0 : \mathrm{H}^0_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) \xrightarrow{\sim} \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(1)^{\psi=1}),$$
$$\alpha_1 : \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1)) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) :$$
$$[f_1\boldsymbol{e}_1, f_2\boldsymbol{e}_2] \mapsto \left(\mathrm{Res}_0\left(f_1\frac{d\pi}{1+\pi}\right)\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1, \mathrm{Res}_0\left(f_2\frac{d\pi}{1+\pi}\right)\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1\right),$$
$$\alpha_2 : \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)) \xrightarrow{\sim} \mathrm{H}^2_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) : [f\boldsymbol{e}_1] \mapsto \mathrm{Res}_0\left(f\frac{d\pi}{1+\pi}\right)\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1.$$

Therefore, the isomorphism

$$\bar{\theta}_\zeta(\mathcal{R}_L(1)) : \boxtimes_{i=1}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(1)))^{(-1)^{i+1}} \xrightarrow{\sim} (L(1), 1),$$

defined in (50), is the composition of the isomorphisms β_0, β_1 and $\iota_{x|x|}$:

$$\boxtimes_{i=1}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(1)))^{(-1)^{i+1}}$$
$$\xrightarrow{\beta_0} \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1))^{(-1)^{i+1}} \boxtimes (\mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(1)^{\psi=1}), 1)$$
$$\xrightarrow{\beta_1} (\mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(1)^{\psi=1}), 1) \xrightarrow{\iota_{x|x|}} (L(1), 1).$$

Here β_0 is induced by α_i ($i = 0, 1, 2$), and β_1 is induced by the canonical isomorphism

$$\beta_1 : \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1))^{(-1)^{i-1}} \xrightarrow{\sim} \mathbf{1}_L,$$

which is the base change by $f_{x|x|} : \mathcal{R}_L^{\infty}(\Gamma) \to L : [\gamma] \mapsto \chi(\gamma)^{-1}$ of the isomorphism (40) for $M = \mathcal{R}_L$.

By definition, the isomorphism β_1 is explicitly described as in the following lemma, which easily follows from the definition (hence, we omit the proof).

Lemma 4.19. *If we define $\tilde{f}_0 := 1_{\mathbb{Z}_p}\boldsymbol{e}_1$ (resp. $\tilde{f}_{1,1} := (1_{\mathbb{Z}_p}\boldsymbol{e}_1, 0)$, $\tilde{f}_{1,2} := (0, 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$, resp. $\tilde{f}_2 := 1_{\mathbb{Z}_p}\boldsymbol{e}_1$) for the basis of $\mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$ (resp. $\mathrm{H}^1_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$, resp. $\mathrm{H}^2_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$), then the canonical trivialization*

$$\beta_1 : (\mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1), 1)^{-1} \boxtimes (\det_L \mathrm{H}^1_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1), 2) \boxtimes (\mathrm{H}^2_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1), 1)^{-1} \xrightarrow{\sim} \mathbf{1}_L$$

satisfies the equality

$$\beta_1(\tilde{f}_0^{\vee} \otimes (\tilde{f}_{1,1} \wedge \tilde{f}_{1,2}) \otimes \tilde{f}_2^{\vee}) = 1.$$

Lemma 4.20. *The isomorphism*

$$\mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) \xrightarrow{\alpha_0} \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(1)^{\psi=1}) \xrightarrow{\iota_{x|x|}} L\boldsymbol{e}_1$$

sends the element $\tilde{f}_0$ to $-\boldsymbol{e}_1 \in L(1)$.

Proof. Since we have $\mathrm{Col}\big(\frac{1+\pi}{\pi}\big) = 1_{\mathbb{Z}_p}$ and $\psi\big(\frac{1+\pi}{\pi}\boldsymbol{e}_1\big) = \frac{1+\pi}{\pi}\boldsymbol{e}_1$, we have

$$\alpha_0(\tilde{f}_0) = \left[\frac{1}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}(\gamma-1)\left(\frac{1+\pi}{\pi}\boldsymbol{e}_1\right)\right]$$

by definition of the boundary map.

Since we have

$$(\gamma-1)\left(\frac{1+\pi}{\pi}\boldsymbol{e}_1\right) = \partial\left(\log\left(\frac{\gamma(\pi)}{\pi}\right)\right)\boldsymbol{e}_1 \quad \text{and} \quad \log\left(\frac{\gamma(\pi)}{\pi}\right)\boldsymbol{e}_{|x|} \in \mathcal{R}_L^{\infty}(|x|)^{\psi=1},$$

and have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(|x|)^{\psi=1}) & \xrightarrow{\iota_{|x|}} & L\boldsymbol{e}_{|x|} \\ \big\downarrow \partial & & \big\downarrow \boldsymbol{e}_{|x|}\mapsto -\boldsymbol{e}_1 \\ \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(1)^{\psi=1}) & \xrightarrow{\iota_{x|x|}} & L\boldsymbol{e}_1 \end{array} \tag{66}$$

we obtain an equality

$$\begin{aligned}\iota_{x|x|}(\alpha_0(\tilde{f}_0)) &= \frac{1}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}\iota_{x|x|}\left(\left[\partial\left(\log\left(\frac{\gamma(\pi)}{\pi}\right)\right)\boldsymbol{e}_1\right]\right) \\ &= -\frac{1}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}\int_{\mathbb{Z}_p^\times}\mu_\gamma(y)\boldsymbol{e}_1\end{aligned}$$

by Lemma 4.10, where we define $\mu_\gamma \in \mathcal{D}(\mathbb{Z}_p, L)$ such that $f_{\mu_\gamma}(\pi) = \log(\gamma(\pi)/\pi)$.

We calculate $\int_{\mathbb{Z}_p^\times}\mu_\gamma(y)$ as follows. Since we have $\psi(\mu_\gamma) = \frac{1}{p}\mu_\gamma$, we obtain

$$\int_{p\mathbb{Z}_p}\mu_\gamma(y) = \int_{\mathbb{Z}_p}\psi(\mu_\gamma)(y) = \frac{1}{p}\int_{\mathbb{Z}_p}\mu_\gamma(y).$$

Hence, we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p^\times}\mu_\gamma(y) &= \int_{\mathbb{Z}_p}\mu_\gamma(y) - \int_{p\mathbb{Z}_p}\mu_\gamma(y) = \int_{\mathbb{Z}_p}\mu_\gamma(y) - \frac{1}{p}\int_{\mathbb{Z}_p}\mu_\gamma(y) \\ &= \frac{p-1}{p}\int_{\mathbb{Z}_p}\mu_\gamma(y) = \frac{p-1}{p}\log\left(\frac{\gamma(\pi)}{\pi}\right)\Big|_{\pi=0} = \frac{p-1}{p}\log(\chi(\gamma)).\end{aligned}$$

Hence,

$$\iota_{x|x|}(\alpha_0(\tilde{f}_0)) = -\frac{\log(\chi(\gamma))}{|\Gamma_{\mathrm{for}}|\log_0(\chi(\gamma))}\frac{p-1}{p}\boldsymbol{e}_1 = -\boldsymbol{e}_1$$

(for any prime p), which proves the lemma. □

In the Appendix, we define a canonical basis $\{f_{1,1}, f_{1,2}\}$ of $\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))$, $f_2 \in \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1))$, $e_0 \in \mathrm{H}^0_{\psi,\gamma}(\mathcal{R}_L)$ and $\{e_{1,1}, e_{1,2}\}$ of $\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L)$; see the Appendix for the definition.

Corollary 4.21. *The isomorphism*

$$\bar{\theta}_{\zeta}(\mathcal{R}_L(1)) : (\det_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1)), 2) \boxtimes (\mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)), 1)^{-1} \xrightarrow{\sim} (L\boldsymbol{e}_1, 1)$$

sends the element $(f_{1,1} \wedge f_{1,2}) \otimes f_2^{\vee}$ *to* $-\frac{p-1}{p}\boldsymbol{e}_1$.

Proof. By definition, we have

$$\alpha_1(f_{1,1}) = \frac{p-1}{p}\log(\chi(\gamma))\tilde{f}_{1,1},$$
$$\alpha_1(f_{1,2}) = \frac{p-1}{p}\tilde{f}_{1,2},$$
$$\alpha_2(f_2) = \frac{p-1}{p}\log(\chi(\gamma))\tilde{f}_2.$$

Then the corollary follows from the previous lemmas. □

Finally, since one has $\Gamma_L(\mathcal{R}_L(1)) = 1$ and $\theta_{\mathrm{dR},L}(\mathcal{R}_L(1), \zeta)$ corresponds to the isomorphism

$$\mathcal{L}_L(\mathcal{R}_L(1)) = L\boldsymbol{e}_1 \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(1)) = \frac{1}{t}L\boldsymbol{e}_1 : a\boldsymbol{e}_1 \mapsto \frac{a}{t}\boldsymbol{e}_1,$$

it suffices to show the following lemma.

Lemma 4.22. *The isomorphism*

$$\theta_L(\mathcal{R}_L(1)) : (\det_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1)), 2) \boxtimes (\mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)), 1)^{-1} \xrightarrow{\sim} (\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(1)), 1) = \left(L\frac{1}{t}\boldsymbol{e}_1, 1\right)$$

sends the element $(f_{1,1} \wedge f_{1,2}) \otimes f_2^{\vee}$ *to* $-\frac{p-1}{pt}\boldsymbol{e}_1$.

Proof. By definition, the above isomorphism is the one which is naturally induced by the exact sequence

$$0 \to \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(1)) \xrightarrow{(1-\varphi)\oplus\mathrm{can}} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(1)) \oplus \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(1)) \xrightarrow{\exp_{f,\mathcal{R}_L(1)} \oplus \exp_{\mathcal{R}_L(1)}} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))_f \to 0$$

and the isomorphisms

$$\exp^{\vee}_{f,\mathcal{R}_L} : \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))/\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))_f \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^{\vee}$$

and

$$\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^{\vee} \xrightarrow{\sim} \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)),$$

which is the dual of the natural isomorphism $\mathrm{H}^0_{\psi,\gamma}(\mathcal{R}_L) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$.

We have $\exp_{\mathcal{R}_L(1)}\left(\frac{1}{t}\boldsymbol{e}_1\right) = f_{1,2}$ by the proof of Lemma 5.1. Since we have

$$\exp_{f,\mathcal{R}_L}(1) = e_{1,2}$$

for $d_0 := 1 \in L = \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$ by the explicit definition of $\exp_f$ (Proposition 2.23(2)), and since we have $\langle f_{1,1}, e_{1,2}\rangle = 1$ by Lemma 5.4, we obtain

$$\exp^{\vee}_{f,\mathcal{R}_L}(f_{1,1}) = -d_0^{\vee} \in \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^{\vee}$$

(we should be careful with the sign). Since the natural isomorphism $\mathrm{H}^0_{\psi,\gamma}(\mathcal{R}_L) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$ sends e_0 to $d_0 \in L = \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$, we obtain

$$\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^{\vee} \to \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)) : d_0^{\vee} \mapsto f_2$$

by Lemma 5.4. The lemma follows from these calculations and a diagram chase. □

Appendix: Explicit calculations of $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L)$ and $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(1))$

In this appendix, we compare $\mathrm{H}^i(\mathbb{Q}_p, L(k))$ with $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(k))$ explicitly for $k = 0, 1$, and define a canonical basis of $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(k))$, which is used to compare $\varepsilon_{L,\zeta}(\mathcal{R}_L(1))$ with $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(1))$ in Corollary 4.21 and Lemma 4.22. All the results in this appendix seem to be known (see for example [Benois 2000]), but here we give another proof of these results in the framework of (φ, Γ)-modules over the Robba ring. Of course, we may assume that $L = \mathbb{Q}_p$ by base change.

We first consider $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$. If we identify by

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G^{\mathrm{ab}}_{\mathbb{Q}_p}, \mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^{\times}, \mathbb{Q}_p) : \tau \mapsto \tau \circ \mathrm{rec}_{\mathbb{Q}_p},$$

then this has a basis $\{[\mathrm{ord}_p], [\log]\}$ defined by

$$\mathrm{ord}_p : \mathbb{Q}_p^{\times} \to \mathbb{Q}_p : p \mapsto 1,\ a \mapsto 0 \quad \text{for } a \in \mathbb{Z}_p^{\times},$$
$$\log : \mathbb{Q}_p^{\times} \to \mathbb{Q}_p : p \mapsto 0,\ a \mapsto \log(a) \quad \text{for } a \in \mathbb{Z}_p^{\times}.$$

We define a basis e_0 of $\mathrm{H}^0_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ and $\{e_{1,1}, e_{1,2}\}$ of $\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ by

$$e_0 = 1 \in \mathcal{R}_{\mathbb{Q}_p}, \quad e_{1,1} := [\log(\chi(\gamma)), 0], \quad e_{1,2} := [0, 1].$$

The basis is independent of the choice of γ, i.e., is compatible with the comparison isomorphism $\iota_{\gamma,\gamma'}$. We can easily check that the canonical isomorphism $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) \xrightarrow{\sim} \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ sends $[\log]$ to $e_{1,1}$ and $[\mathrm{ord}_p]$ to $e_{1,2}$.

We next consider $\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$. Let us denote by

$$\kappa : \mathbb{Q}_p^{\times} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$$

the Kummer map. Composing this with the canonical isomorphism

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)),$$

we obtain a homomorphism

$$\kappa_0 : \mathbb{Q}_p^{\times} \to \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)).$$

We define a homomorphism

$$\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \to \mathbb{Q}_p \oplus \mathbb{Q}_p :$$
$$[f_1\boldsymbol{e}_1, f_2\boldsymbol{e}_1] \mapsto \left(\frac{p}{p-1}\cdot\frac{1}{\log(\chi(\gamma))}\cdot \mathrm{Res}_0\Big(f_1\frac{d\pi}{1+\pi}\Big), -\frac{p}{p-1}\cdot \mathrm{Res}_0\Big(f_2\frac{d\pi}{1+\pi}\Big)\right)$$

(we note that $\frac{p-1}{p}\cdot\log(\chi(\gamma)) = |\Gamma_{\mathrm{tor}}|\cdot\log_0(\chi(\gamma))$), which is also independent of the choice of γ, and is an isomorphism. Using this isomorphism, we define a basis $\{f_{1,1}, f_{1,2}\}$ of $\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$ such that $f_{1,1}$ (resp. $f_{1,2}$) corresponds to $(1, 0)\in L\oplus L$ (resp. $(0, 1)$) by this isomorphism. We want to explicitly describe the map κ_0 using this basis. For this, we first prove the following lemma.

Lemma 5.1. *For each $a \in \mathbb{Z}_p^\times$, we have $\kappa_0(a) = \log(a)\cdot f_{1,2}$.*

Proof. By the classical explicit calculation of the exponential map, we have

$$\kappa(a) = \exp_{\mathbb{Q}_p(1)}\Big(\frac{\log(a)}{t}\boldsymbol{e}_1\Big).$$

Since we have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)) & \xrightarrow{\exp_{\mathbb{Q}_p(1)}} & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \\ \sim\downarrow & & \sim\downarrow \\ \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_{\mathbb{Q}_p}(1)) & \xrightarrow{\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}} & \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \end{array}$$

by Proposition 2.26, it suffices to show that

$$\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}\Big(\frac{1}{t}\boldsymbol{e}_1\Big) = f_{1,2}.$$

We show this equality as follows. We first take some $f \in (\mathcal{R}^\infty_{\mathbb{Q}_p})^\Delta$ such that $f(\zeta_{p^n}-1) = 1/p^n$ for any $n\geq 0$, which is possible since we have an isomorphism $\mathcal{R}^\infty_{\mathbb{Q}_p}/t \xrightarrow{\sim} \prod_{n\geq 0}\mathbb{Q}_p(\zeta_{p^n}) : \bar{f} \mapsto (f(\zeta_{p^n}-1))_{n\geq 0}$ by Lazard's theorem [1962]. Then the element $\frac{f}{t}\boldsymbol{e}_1 \in \big(\frac{1}{t}\mathcal{R}_{\mathbb{Q}_p}(1)\big)^\Delta$ satisfies

$$\iota_n\Big(\frac{f}{t}\boldsymbol{e}_1\Big) - \frac{1}{t}\boldsymbol{e}_1 \in \boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_{\mathbb{Q}_p}(1))$$

for any $n\geq 1$, since we have

$$\iota_n\Big(\frac{f}{t}\boldsymbol{e}_1\Big) \equiv p^n\cdot\frac{f(\zeta_{p^n}-1)}{t}\boldsymbol{e}_1 = \frac{1}{t}\boldsymbol{e}_1 \pmod{\boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_{\mathbb{Q}_p}(1))}.$$

By the explicit definition of $\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}$ (Proposition 2.23(1)), we have

$$\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}\Big(\frac{1}{t}\boldsymbol{e}_1\Big)=\Big[(\gamma-1)\Big(\frac{f}{t}\boldsymbol{e}_1\Big),(\varphi-1)\Big(\frac{f}{t}\boldsymbol{e}_1\Big)\Big]\in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)).$$

Hence, it suffices to show that

$$\mathrm{Res}_0\Big(\frac{\gamma(f)-f}{t}\cdot\frac{d\pi}{1+\pi}\Big)=0$$

and

$$\mathrm{Res}_0\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{t}\cdot\frac{d\pi}{1+\pi}\Big)=-\frac{p-1}{p}.$$

Here, we only calculate

$$\mathrm{Res}_0\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{t}\cdot\frac{d\pi}{1+\pi}\Big)$$

(the calculation of

$$\mathrm{Res}_0\Big(\frac{\gamma(f)-f}{t}\cdot\frac{d\pi}{1+\pi}\Big)$$

is similar). By definition of f, we have

$$\frac{\varphi(f)(\zeta_{p^n}-1)}{p}-f(\zeta_{p^n}-1)=\frac{f(\zeta_{p^{n-1}}-1)}{p}-f(\zeta_{p^n}-1)=\frac{1}{p}\cdot\frac{1}{p^{n-1}}-\frac{1}{p^n}=0$$

for each $n\geq 1$. Hence, we have

$$\Big(\frac{\varphi(f)}{p}-f\Big)\in\Big(\prod_{n\geq 1}^{\infty}\frac{Q_n(\pi)}{p}\Big)\mathcal{R}^{\infty}_{\mathbb{Q}_p}$$

by the theorem of Lazard [1962], where we define $Q_n(\pi):=\varphi^{n-1}\big(\varphi(\pi)/\pi\big)$ for each $n\geq 1$. Since we have $t=\pi\prod_{n\geq 1}\big(Q_n(\pi)/p\big)$, we obtain the equality

$$\begin{aligned}\mathrm{Res}_0\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{t}\cdot\frac{d\pi}{1+\pi}\Big)&=\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{\prod_{n\geq 1}^{\infty}\frac{Q_n(\pi)}{p}}\cdot\frac{1}{1+\pi}\Big)\Big|_{\pi=0}\\&=\Big(\frac{\varphi(f)}{p}-f\Big)\Big|_{\pi=0}=\frac{f(0)}{p}-f(0)=-\frac{p-1}{p},\end{aligned}$$

where the second equality follows from the fact that $\dfrac{Q_n(0)}{p}=1$ for $n\geq 1$, which proves the lemma. □

Before calculating $\kappa_0(p)\in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$, we explicitly describe Tate's trace map in terms of (φ,Γ)-modules. We note that we normalize Tate's trace map

$$\mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1))\xrightarrow{\sim}\mathbb{Q}_p$$

so that the cup product pairing

$$\langle -, - \rangle : \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \times \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) \xrightarrow{\cup} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

satisfies

$$\langle \kappa(a), [\tau] \rangle = \tau(a)$$

for $a \in \mathbb{Q}_p^\times$ and $[\tau] \in \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Q}_p) = \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p)$ (we remark that this normalization coincides with the one used in §2.4 of [Nakamura 2014a] and with -1 times the one in [Kato 1993a, Chapter II, §1.4]).

Proposition 5.2. *The map* $\iota_\gamma : \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \xrightarrow{\sim} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$, *which is the composition of the canonical isomorphism* $\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \xrightarrow{\sim} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1))$ *with Tate's trace map is explicitly defined by*

$$\iota_\gamma([f\boldsymbol{e}_1]) = \frac{p}{p-1} \cdot \frac{1}{\log(\chi(\gamma))} \mathrm{Res}_0\left(f \frac{d\pi}{1+\pi}\right).$$

Proof. Since the map

$$\iota : \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \xrightarrow{\sim} \mathbb{Q}_p : [f\boldsymbol{e}_1] \mapsto \mathrm{Res}_0\left(f \frac{d\pi}{1+\pi}\right)$$

is a well-defined isomorphism, there exists a unique $\alpha \in \mathbb{Q}_p^\times$ such that $\iota_\gamma = \alpha \cdot \iota$. We calculate α as follows.

We recall that the element $[\log(\chi(\gamma)), 0] \in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ is the image of $[\log] \in \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p)$ by the comparison isomorphism. By the proof of Lemma 5.1, for each $a \in \mathbb{Z}_p^\times$, we have

$$\kappa_0(a) = \log(a)\left[(\gamma-1)\left(\frac{f}{t}\boldsymbol{e}_1\right), (\varphi-1)\left(\frac{f}{t}\boldsymbol{e}_1\right)\right] \in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)),$$

where $f \in \mathcal{R}^\infty_{\mathbb{Q}_p}$ is an element defined in the proof of Lemma 5.1. Since the cup products are compatible with the comparison isomorphism (see Remark 2.12), we have

$$\iota_\gamma\big(\kappa_0(a) \cup [\log(\chi(\gamma)), 0]\big) = \langle \kappa(a), [\log] \rangle = \log(a). \tag{67}$$

By definition of the cup product, we have

$$\begin{aligned}\kappa_0(a) \cup [\log(\chi(\gamma)), 0] &= \log(a)\left[(\varphi-1)\left(\frac{f}{t}\boldsymbol{e}_1\right) \otimes \varphi(\log(\chi(\gamma)))\right] \\ &= -\log(a)\log(\chi(\gamma))\left[(\varphi-1)\left(\frac{f}{t}\boldsymbol{e}_1\right)\right] \in \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)).\end{aligned}$$

Since $\mathrm{Res}_0\Big((\varphi-1)\Big(\frac{f}{t}\Big)\cdot\frac{d\pi}{1+\pi}\Big)=-\frac{p-1}{p}$ by the proof of Lemma 5.1, we obtain

$$\begin{aligned}\iota_\gamma\big(\kappa_0(a)\cup[\log(\chi(\gamma)),0]\big)&=\alpha\cdot\iota\big(\kappa_0(a)\cup[\log(\chi(\gamma)),0]\big)\\&=-\alpha\cdot\log(\chi(\gamma))\cdot\log(a)\cdot\iota\bigg(\Big[(\varphi-1)\Big(\frac{f}{t}\boldsymbol{e}_1\Big)\Big]\bigg)\\&=\alpha\cdot\log(\chi(\gamma))\cdot\log(a)\cdot\frac{p-1}{p}.\end{aligned}$$

Comparing this equality with the equality (67), we obtain

$$\alpha=\frac{p}{p-1}\cdot\frac{1}{\log(\chi(\gamma))},$$

which proves the proposition. □

Finally, we prove the following lemma, which completes the calculation of the map $\kappa_0:\mathbb{Q}_p^\times\to\mathbb{Q}_p\oplus\mathbb{Q}_p$.

Lemma 5.3. $\kappa_0(p)=f_{1,1}$.

Proof. Take $f_{1,1}=[f_1\boldsymbol{e}_1,\,f_2\boldsymbol{e}_1]\in\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$ to be a representative of $f_{1,1}$. By definition of the cup product, we have

$$\begin{aligned}\iota_\gamma(f_{1,1}\cup e_{1,1})&=\iota_\gamma\big(f_{1,1}\cup[\log(\chi(\gamma)),0]\big)\\&=-\iota_\gamma\big([f_2\boldsymbol{e}_1\otimes\varphi(\log(\chi(\gamma)))]\big)=-\frac{p}{p-1}\,\mathrm{Res}_0\Big(f_2\frac{d\pi}{1+\pi}\Big)=0,\end{aligned}$$

and

$$\begin{aligned}\iota_\gamma(f_{1,1}\cup e_{1,2})&=\iota_\gamma(f_{1,1}\cup[0,1])\\&=\iota_\gamma([f_1\boldsymbol{e}_1\otimes\gamma(1)])=\frac{p}{p-1}\cdot\frac{1}{\log(\chi(\gamma))}\cdot\mathrm{Res}_0\Big(f_1\frac{d\pi}{1+\pi}\Big)=1\end{aligned}$$

by Proposition 5.2. Since $\kappa(p)\in\mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p(1))$ satisfies the similar formulae

$$\langle\kappa(p),[\mathrm{ord}_p]\rangle=1,\quad\langle\kappa(p),[\log]\rangle=0,$$

we obtain the equality

$$\kappa_0(p)=f_{1,1}.$$ □

Using these lemmas, we obtain the following result. We define the basis f_2 of $\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))$ by $f_2:=\iota_\gamma^{-1}(1)$.

Lemma 5.4. *Tate's duality pairings*

$$\langle-,-\rangle:\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_L(1))\times\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_L)\xrightarrow{\cup}\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))\xrightarrow{\iota_\gamma}L$$

and

$$\langle-,-\rangle:\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))\times\mathrm{H}^0_{\varphi,\gamma}(\mathcal{R}_L)\xrightarrow{\cup}\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))\xrightarrow{\iota_\gamma}L$$

satisfy

$$\langle f_{1,1}, e_{1,1}\rangle = 0, \quad \langle f_{1,1}, e_{1,2}\rangle = 1$$
$$\langle f_{1,2}, e_{1,1}\rangle = 1, \quad \langle f_{1,2}, e_{1,2}\rangle = 0,$$
$$\langle f_2, e_0\rangle = 1.$$

Proof. That we have $\langle f_{1,1}, e_{1,1}\rangle = 0$ and $\langle f_{1,1}, e_{1,2}\rangle = 1$ is proved in Lemma 5.3. We prove the formula for $f_{1,2}$. By Lemma 5.1, we have an equality $f_{1,2} = \kappa_0(a)/\log(a)$ for any nontorsion $a \in \mathbb{Z}_p^\times$. Hence, we obtain

$$\langle f_{1,2}, e_{1,1}\rangle = \frac{1}{\log(a)}\langle \kappa(a), [\log]\rangle = 1,$$
$$\langle f_{1,2}, e_{1,2}\rangle = \frac{1}{\log(a)}\langle \kappa(a), [\mathrm{ord}_p]\rangle = 0$$

by the compatibility of the cup products. Finally, that $\langle f_2, e_0\rangle = 1$ is trivial by definition. □

Acknowledgements

The author thanks Seidai Yasuda for introducing him to Kato's global and local ε-conjectures. He also thanks Iku Nakamura for constantly encouraging him. This work is supported in part by the Grant-in-aid (no. S-23224001) for Scientific Research, JSPS.

References

[Bellaïche and Chenevier 2009] J. Bellaïche and G. Chenevier, *Families of Galois representations and Selmer groups*, Astérisque **324**, 2009. MR Zbl

[Benois 2000] D. Benois, "On Iwasawa theory of crystalline representations", *Duke Math. J.* **104**:2 (2000), 211–267. MR Zbl

[Benois and Berger 2008] D. Benois and L. Berger, "Théorie d'Iwasawa des représentations cristallines, II", *Comment. Math. Helv.* **83**:3 (2008), 603–677. MR Zbl

[Berger 2002] L. Berger, "Représentations p-adiques et équations différentielles", *Invent. Math.* **148**:2 (2002), 219–284. MR Zbl

[Berger 2008a] L. Berger, "Construction de (ϕ, Γ)-modules: représentations p-adiques et B-paires", *Algebra Number Theory* **2**:1 (2008), 91–120. MR Zbl

[Berger 2008b] L. Berger, "Équations différentielles p-adiques et (ϕ, N)-modules filtrés", pp. 13–38 in *Représentations p-adiques de groupes p-adiques, I; Représentations galoisiennes et* (ϕ, Γ)*-modules*, Astérisque **319**, 2008. MR Zbl

[Berger 2009] L. Berger, "Presque $\mathbb{C}_p$-représentations et (ϕ, Γ)-modules", *J. Inst. Math. Jussieu* **8**:4 (2009), 653–668. MR Zbl

[Bloch and Kato 1990] S. Bloch and K. Kato, "L-functions and Tamagawa numbers of motives", pp. 333–400 in *The Grothendieck Festschrift, Vol. I*, edited by P. Cartier et al., Progr. Math. **86**, Birkhäuser, Boston, 1990. MR Zbl

[Chenevier 2013] G. Chenevier, "Sur la densité des représentations cristallines de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$", *Math. Ann.* **355**:4 (2013), 1469–1525. MR Zbl

[Colmez 2008] P. Colmez, "Représentations triangulines de dimension 2", pp. 213–258 in *Représentations p-adiques de groupes p-adiques, I; Représentations galoisiennes et (ϕ, Γ)-modules*, Astérisque **319**, 2008. MR Zbl

[Colmez 2010] P. Colmez, *Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (ϕ, Γ)*-modules*, Astérisque **330**, 2010. MR Zbl

[Dee 2001] J. Dee, "Φ-Γ modules for families of Galois representations", *J. Algebra* **235**:2 (2001), 636–664. MR Zbl

[Deligne 1973] P. Deligne, "Les constantes des équations fonctionnelles des fonctions L", pp. 501–597 in *Modular functions of one variable, II* (Antwerp, 1972), edited by P. Deligne and W. Kuyk, Lecture Notes in Math. **349**, Springer, 1973. MR Zbl

[Dospinescu 2014] G. Dospinescu, "Equations différentielles p-adiques et modules de Jacquet analytiques", pp. 359–374 in *Automorphic forms and Galois representations, Vol. 1*, edited by F. Diamond et al., London Math. Soc. Lecture Note Ser. **414**, Cambridge Univ. Press, 2014. MR Zbl

[Emerton 2006] M. Emerton, "A local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$", *Pure Appl. Math. Q.* **2**:2 (2006), 279–393. MR Zbl

[Fontaine and Perrin-Riou 1994] J.-M. Fontaine and B. Perrin-Riou, "Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L", pp. 599–706 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Fukaya and Kato 2006] T. Fukaya and K. Kato, "A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory", pp. 1–85 in *Proceedings of the St. Petersburg Mathematical Society, Vol. XII*, edited by N. N. Uraltseva, Amer. Math. Soc. Transl. Ser. 2 **219**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Hansen 2016] D. Hansen, "Iwasawa theory of overconvergent modular forms, I: critical-slope p-adic L-functions", preprint, 2016, available at http://www.math.columbia.edu/~hansen/bigzetaone.pdf.

[Kato 1993a] K. Kato, "Lectures on the approach to Iwasawa theory for Hasse–Weil L-functions via B_{dR}, I", pp. 50–163 in *Arithmetic algebraic geometry* (Trento, 1991), edited by E. Ballico, Lecture Notes in Math. **1553**, Springer, 1993. MR Zbl

[Kato 1993b] K. Kato, "Lectures on the approach to Iwasawa theory for Hasse–Weil L-functions via B_{dR}, II: local main conjecture", preprint, University of Chicago, 1993.

[Kedlaya and Liu 2010] K. Kedlaya and R. Liu, "On families of ϕ, Γ-modules", *Algebra Number Theory* **4**:7 (2010), 943–967. MR Zbl

[Kedlaya et al. 2014] K. S. Kedlaya, J. Pottharst, and L. Xiao, "Cohomology of arithmetic families of (φ, Γ)-modules", *J. Amer. Math. Soc.* **27**:4 (2014), 1043–1115. MR Zbl

[Kisin 2003] M. Kisin, "Overconvergent modular forms and the Fontaine–Mazur conjecture", *Invent. Math.* **153**:2 (2003), 373–454. MR Zbl

[Kisin 2010] M. Kisin, *Deformations of* $G_{\mathbb{Q}_p}$ *and* $\mathrm{GL}_2(\mathbb{Q}_p)$ *representations*, Astérisque **330**, 2010. MR Zbl

[Knudsen and Mumford 1976] F. F. Knudsen and D. Mumford, "The projectivity of the moduli space of stable curves, I: preliminaries on "det" and "Div"", *Math. Scand.* **39**:1 (1976), 19–55. MR Zbl

[Lazard 1962] M. Lazard, "Les zéros des fonctions analytiques d'une variable sur un corps valué complet", *Inst. Hautes Études Sci. Publ. Math.* **14** (1962), 47–75. MR Zbl

[Loeffler et al. 2015] D. Loeffler, O. Venjakob, and S. L. Zerbes, “Local epsilon isomorphisms”, *Kyoto J. Math.* **55**:1 (2015), 63–127. MR Zbl

[Nakamura 2014a] K. Nakamura, “Iwasawa theory of de Rham (φ, Γ)-modules over the Robba ring”, *J. Inst. Math. Jussieu* **13**:1 (2014), 65–118. MR Zbl

[Nakamura 2014b] K. Nakamura, “Zariski density of crystalline representations for any p-adic field”, *J. Math. Sci. Univ. Tokyo* **21**:1 (2014), 79–127. MR Zbl

[Nakamura 2015] K. Nakamura, “Local ε-isomorphisms for rank two p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a functional equation of Kato’s Euler system”, preprint, 2015. arXiv

[Pottharst 2012] J. Pottharst, “Cyclotomic Iwasawa theory of motives”, preprint, 2012, available at https://vbrt.org/writings/cyc.pdf.

[Pottharst 2013] J. Pottharst, “Analytic families of finite-slope Selmer groups”, *Algebra Number Theory* **7**:7 (2013), 1571–1612. MR Zbl

[Schneider and Teitelbaum 2003] P. Schneider and J. Teitelbaum, “Algebras of p-adic distributions and admissible representations”, *Invent. Math.* **153**:1 (2003), 145–196. MR Zbl

[Venjakob 2013] O. Venjakob, “On Kato’s local ϵ-isomorphism conjecture for rank-one Iwasawa modules”, *Algebra Number Theory* **7**:10 (2013), 2369–2416. MR Zbl

Communicated by John Henry Coates
Received 2014-08-08 Revised 2016-10-11 Accepted 2016-11-13

nkentaro@cc.saga-u.ac.jp *Department of Mathematics, Saga University, 1 Honjo-machi, Saga 840-8502, Japan*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2017 is US $325/year for the electronic version, and $520/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 11 No. 2 2017

Test vectors and central L-values for GL(2) 253
DANIEL FILE, KIMBALL MARTIN and AMEYA PITALE

A generalization of Kato's local ε-conjecture for (φ, Γ)-modules over the Robba ring 319
KENTARO NAKAMURA

First covering of the Drinfel'd upper half-plane and Banach representations of $GL_2(\mathbb{Q}_p)$ 405
LUE PAN

1937-0652(2017)11:2;1-J