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Test vectors and central L-values for GL(2)

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We determine local test vectors for Waldspurger functionals for GL_2, in the case where both the representation of GL_2 and the character of the degree two extension are ramified, with certain restrictions. We use this to obtain an explicit version of Waldspurger's formula relating twisted central L-values of automorphic representations on GL_2 with certain toric period integrals. As a consequence, we generalize an average value formula of Feigon and Whitehouse, and obtain some nonvanishing results.

1. Introduction

1A. ***Global results.*** Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let L/F be a quadratic extension and Ω an idèle class character of $L^\times$ such that $\Omega|_{\mathbb{A}_F^\times} = \omega_\pi$, the central character of π. We are interested in the central value of the L-function

$$L(s, \pi_L \otimes \Omega) = L(s, \pi \times \theta_\Omega),$$

where π_L denotes the base change of π to $\mathrm{GL}_2(\mathbb{A}_L)$ and θ_Ω denotes the theta series on $\mathrm{GL}_2(\mathbb{A}_F)$ associated to Ω. Note this contains the following interesting special case: when Ω is trivial, then $L(s, \pi_L \otimes \Omega) = L(s, \pi)L(s, \pi \otimes \eta)$, where $\eta = \eta_{L/F}$ denotes the quadratic character of $\mathbb{A}_F^\times$ associated to L via class field theory. Assume

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that ω_π is trivial or η. Then $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = \pm 1$, even though $\pi_L \otimes \Omega$ need not be self-dual (cf. [Jacquet and Chen 2001]). In the case where $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = -1$, the central value $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = 0$. Henceforth, assume $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = +1$.

Let D be a quaternion algebra over F containing L such that π has a Jacquet–Langlands transfer to an automorphic representation π' of $D^\times(\mathbb{A}_F)$. We allow for the possibility that $D = M_2(F)$ and $\pi' = \pi$, so there is always at least one such π'. Embed $L^\times$ as a torus T inside $D^\times$. The period integrals we are interested in are

$$P_D(\phi) = \int_{Z(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} \phi(t)\Omega^{-1}(t)\, dt, \tag{1-1}$$

where $\phi \in \pi'$ and Z denotes the center of $D^\times$ (with dt as in Section 7). If $F = \mathbb{Q}$ and L is imaginary quadratic, then this period simplifies to a finite sum over certain "CM points".

When ω_π is trivial, a beautiful theorem of Waldspurger [1985] states that

$$\frac{|P_D(\phi)|^2}{(\phi, \phi)} = \zeta(2) \prod_v \alpha_v(L, \Omega, \phi) \frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L(1, \pi, \mathrm{Ad})} \tag{1-2}$$

for any $\phi \in \pi'$. Here $(\cdot\,,\cdot)$ is a certain inner product on π' and the factors $\alpha_v(L, \Omega, \phi)$ are certain local integrals which equal 1 at almost all places. For all but one D, $P_D \equiv 0$ for local reasons. Namely, the linear functional P_D factors into a product of local linear functionals $P_{D,v}$. There is a unique $D \supset L$ for which all $P_{D,v} \neq 0$, and this D is determined by local epsilon factors in work of Tunnell [1983] and Saito [1993]. Fixing this D, one now gets the nonvanishing criterion: $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) \neq 0$ if and only if $P_D \neq 0$.

It is useful to have a more explicit version of this formula for certain applications like equidistribution, nonvanishing, subconvexity, p-adic L-functions, etc.; see, e.g., [Popa 2006; Martin and Whitehouse 2009; Feigon and Whitehouse 2009; Hsieh 2014]. In particular, it is not even obvious from (1-2) that $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) \geq 0$, as predicted by the grand Riemann hypothesis. This positivity result was subsequently shown by Jacquet and Chen [2001] using a trace formula identity.

Explicit versions of (1-2) have been considered by many authors under various assumptions; see, e.g., [Gross 1987; Zhang 2001; Xue 2006; Popa 2006; Martin and Whitehouse 2009; Murase 2010; Hida 2010; Hsieh 2014]. These explicit formulas rely on picking out a suitable *test vector* ϕ in (1-2). All of these works rely on the theta correspondence (as did [Waldspurger 1985]), except for [Martin and Whitehouse 2009], which uses the trace formula identity from [Jacquet and Chen 2001]. The only assumption in [Martin and Whitehouse 2009] is that π and Ω have disjoint ramification, i.e., for any finite place v of F, π and Ω are not both ramified at v. In this case one has a natural choice for the test vector ϕ from the work of Gross and Prasad [1991] on *local* test vectors. In [Martin and Whitehouse

2009], it was noted that this restriction of disjoint ramification is not essential to the method and could be removed if one had a reasonable way to define the test vector ϕ in a more general setting.

The main local results of this paper (see Theorems 1.6 and 1.7 below) are the existence and characterization of suitable local test vectors in the case of joint ramification under certain conditions. This allows us to extend the formula of [Martin and Whitehouse 2009] to these cases. To be precise, for a finite place v of F, let $c(\pi_v)$ be the (exponent of) the conductor of π_v and $c(\Omega_v)$ be the (exponent of) the "F-conductor" of Ω (see (2-19)). Then we make the following assumption:

$$\textit{If } v<\infty \textit{ is inert in } L \textit{ and } c(\pi_v),\ c(\Omega_v)>0, \textit{ then we have } c(\Omega_v)\geq c(\pi_v). \tag{1-3}$$

In particular, if the level $N=\prod_{v<\infty}\varpi_v^{c(\pi_v)}$ of π is squarefree, there is no condition on Ω. We note that a consequence of our determination of test vectors is that assumption (1-3) implies that D and Ω do not have joint ramification at any finite place.

Theorems 1.6 and 1.7 below give suitable local test vectors ϕ_v under assumption (1-3), which yields the desired global test vector ϕ. Here suitable essentially means that the local test vectors can be described purely in terms of ramification data, and do not require more refined information about local representations. This is crucial for global applications. Note that it is not even a priori clear if suitable test vectors should exist in general.

Let us now describe the L-value formula more precisely. Denote the absolute value of the discriminants of F and L by Δ and Δ_L. Let $e(L_v/F_v)$ be the ramification degree of L_v/F_v. Let S_{inert} be the set of places of F inert in L. Let $S(\pi)$ be the set of finite places of F where π is ramified, $S(\Omega)$ the set of finite places of F where Ω is ramified, $S_1(\pi)$ the set of places in $S(\pi)$ where $c(\pi_v)=1$ and $S_2(\pi)$ the set of places in $S(\pi)$ where $c(\pi_v)\geq 2$. Finally, let $S_0(\pi)=S_2(\pi)\cup\{v\in S_1(\pi): L_v/F_v$ is ramified and Ω_v is unramified$\}$, and denote by $c(\Omega)$ the absolute norm of the conductor of Ω.

Theorem 1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character and Ω a character of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$. Assume $\epsilon\left(\frac{1}{2},\pi_L\otimes\Omega\right)=1$ and that π and Ω satisfy (1-3). Then, with the test vector $\phi\in\pi'$ defined in Section 7A and archimedean factors $C_v(L,\pi,\Omega)$ defined in Section 7B, we have*

$$\begin{aligned}\frac{|P_D(\phi)|^2}{(\phi,\phi)} &= \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\,L_{S(\Omega)}(1,\eta)L_{S(\pi)\cup S(\Omega)}(1,\eta)L_{S(\pi)\cap S(\Omega)}(1,1_F)L^{S(\pi)}(2,1_F)\\ &\qquad\times\prod_{v\in S(\pi)\cap S(\Omega)^c} e(L_v/F_v)\prod_{v\mid\infty}C_v(L,\pi,\Omega)\cdot\frac{L^{S_0(\pi)}\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{L^{S_0(\pi)}(1,\pi,\mathrm{Ad})}.\end{aligned}$$

Here $(\cdot\,,\cdot)$ *is the standard inner product on* π' *with respect to the measure on* $D^\times(\mathbb{A}_F)$ *which is the product of local Tamagawa measures.*

After our paper was originally completed, the paper [Cai et al. 2014] appeared, which gives a similar formula using a less explicit choice of test vector.

Note ϕ is specified up to a scalar, and the left-hand side is invariant under scaling. As in [Martin and Whitehouse 2009, Theorem 4.2], one can rewrite this formula using the Petersson norm of a normalized newform in π instead of $L(1, \pi, \mathrm{Ad})$. See (8-19) for when π corresponds to a holomorphic Hilbert modular form. If $F = \mathbb{Q}$ and π corresponds to a holomorphic new form of squarefree level N with $N \mid c(\Omega)$, then the above formula simplifies considerably:

Corollary 1.2. *Let* f *be a normalized holomorphic modular eigenform of weight* k *and squarefree level* N. *Let* S *be the set of primes* $p \mid N$ *which split in* L. *Let* Ω *be any ideal class character of* L *such that* $N \mid c(\Omega)$ *and* $\epsilon\left(\frac{1}{2}, f \times \Omega\right) = 1$. *Then*

$$\frac{|P_D(\phi)|^2}{(\phi,\phi)} = \frac{C_\infty(L, f, \Omega)}{2^{k+1}\sqrt{c(\Omega)\Delta_L}} L_{S(\Omega)}(1,\eta)^2 \prod_{p\mid N}(1+p^{-1})^{\epsilon_p} \times \frac{L^S\left(\frac{1}{2}, f \times \Omega\right)}{\langle f, f\rangle},$$

where ϵ_p *is* $+1$ *if* p *splits in* L *and* -1 *otherwise, and* $\langle\cdot\,,\cdot\rangle$ *is the Petersson inner product.*

In the setting of the corollary, $C_\infty(L, f, \Omega)$ is also easier to describe. If L is real quadratic, then $C_\infty(L, f, \Omega) = 2^k$. If L is imaginary quadratic, it is described by beta functions, and if we also assume Ω_∞ is trivial, then

$$C_\infty(L, f, \Omega) = \frac{\left(\frac{1}{2}k-1\right)!^2}{\pi(k-1)!}.$$

We prove Theorem 1.1 by computing local spectral distributions appearing in the trace formula identity of [Jacquet and Chen 2001], just as in [Martin and Whitehouse 2009]. For simplicity, we only do this when $\omega_\pi = 1$, though the case of $\omega_\pi = \eta$ should be similar. (One needs either $\omega_\pi = 1$ or $\omega_\pi = \eta$ to use the identity from [Jacquet and Chen 2001].) Note this formula is considerably more general than the one in [Martin and Whitehouse 2009] (for trivial central character) and one expects that it should generalize the applications of the previously mentioned formulas. For instance, we obtain the following generalization of an average value result of Feigon and Whitehouse [2009, Theorem 1.1] by computing the geometric side of a certain trace formula.

Theorem 1.3. *Let* F *be a totally real number field with* $d = [F : \mathbb{Q}]$. *Let* $\mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})$ *be the set of cuspidal automorphic representations of* $\mathrm{GL}_2(\mathbb{A}_F)$ *associated to the holomorphic Hilbert modular eigen newforms of weight* $2\boldsymbol{k}$ *and level* $\mathfrak{N}$, *with* $\boldsymbol{k} = (k_1, \dots, k_d) \neq (1, \dots, 1)$ *and* $\mathfrak{N}$ *squarefree. Let* L *be a totally imaginary* *quadratic extension of F, which is inert and unramified above each place $\mathfrak{p} \mid \mathfrak{N}$. Fix a unitary character Ω of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$, and let $\mathfrak{C}$ be the norm of its conductor in F. Suppose $\mathfrak{N} = \mathfrak{N}_0\mathfrak{N}_1$ and $\mathfrak{C} = \mathfrak{C}_0\mathfrak{N}_1$ with $\mathfrak{N}_0$, $\mathfrak{N}_1$ and $\mathfrak{C}_0$ all coprime. Assume $\mathfrak{N}_1$ is odd, and that the number of primes dividing $\mathfrak{N}_0$ has the same parity as d. Further assume that for each infinite place v of F, we have $k_v > |m_v|$, where $\Omega_v(z) = (z/\bar{z})^{m_v}$.*

Then, if

$$|\mathfrak{N}_0| > d_{L/F}(|\mathfrak{C}_0|/|\mathfrak{N}_1|)^{h_F},$$

where h_F is the class number of F, we have

$$\prod_{v\mid\infty}\binom{2k_v-2}{k_v-m_v-1}\sum_{\mathfrak{N}'}\sum_{\pi\in\mathcal{F}(\mathfrak{N}',2\boldsymbol{k})}\frac{L\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{L^{S(\mathfrak{N})}(1,\pi,\mathrm{Ad})} = 2^{2-d}\Delta^{3/2}|\mathfrak{N}|\,L_{S(\mathfrak{N}_0)}(2,1_F)L_{S(\mathfrak{N}_1)}(1,1_F)L^{S(\mathfrak{C}_0)}(1,\eta),$$

where $\mathfrak{N}'$ runs over ideals dividing $\mathfrak{N}$ which are divisible by $\mathfrak{N}_0$, and $S(\mathfrak{J})$ denotes the set of all primes dividing $\mathfrak{J}$.

The parity condition guarantees the sign $\epsilon\left(\frac{1}{2},\pi_L\otimes\Omega\right)$ of the relevant functional equation is $+1$ for $\pi\in\mathcal{F}(\mathfrak{N},\boldsymbol{k})$. Without a condition to the effect that $\mathfrak{N}$ (or $\mathfrak{N}_0$) is large, one does not expect a nice explicit formula, but rather just an asymptotic in $\mathfrak{N}$, which miraculously stabilizes for $\mathfrak{N}$ large (cf. [Michel and Ramakrishnan 2012; Feigon and Whitehouse 2009]). Hence the condition above on the size of $\mathfrak{N}_0$ means we are in the *stable range*. The other assumptions in the theorem allow for simplifications of the trace formula we will use, but are not necessary to express such averages as the geometric side of an appropriate trace formula.

Theorem 1.3 specializes to [Feigon and Whitehouse 2009, Theorem 1.1] in the case that $\mathfrak{N}$ and $\mathfrak{C}$ are coprime, i.e., $\mathfrak{N}=\mathfrak{N}_0$. This case $\mathfrak{N}=\mathfrak{N}_0$ is particularly nice as one can transfer the problem to a trace formula computation on a quaternion algebra that only picks up forms of exact level $\mathfrak{N}$. Additionally, one can rewrite the formula in terms of the complete adjoint L-value at 1, as in [Feigon and Whitehouse 2009]. However, this is impossible to manage in general, and the primary difficulty in going from Theorem 1.1 to Theorem 1.3 is to determine the contribution to the spectral side of the relevant trace formula coming from the oldforms. (In general, it is not easy to isolate the newforms in such formulas — see, e.g., [Knightly and Li 2010] or [Nelson 2013] — and the issue for us is that the contribution from the oldforms is now weighted by local adjoint L-factors.)

Still, one can use the above formula together with formulas for smaller levels to get both explicit bounds and asymptotics for average values over just the forms of exact level $\mathfrak{N}$. We do this in the case $\mathfrak{N}_1$ is prime. This immediately implies $L\left(\frac{1}{2},\pi_L\otimes\Omega\right)\neq 0$ for some $\pi_L\in\mathcal{F}(\mathfrak{N},2\boldsymbol{k})$.

Theorem 1.4. *With assumptions as in Theorem 1.3, and further assuming that* $\mathfrak{N}_1 = \mathfrak{p}$ *is an odd prime and* $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$, *we have*

$$|\mathfrak{p}| - \frac{1}{1-2|\mathfrak{p}|^{-1}+|\mathfrak{p}|^{-2}} \le \Sigma(\mathfrak{N}) \le |\mathfrak{p}| - \frac{1}{1+2|\mathfrak{p}|^{-1}+|\mathfrak{p}|^{-2}},$$

where $\Sigma(\mathfrak{N})$ *is equal to*

$$\frac{2^{d-2}}{\Delta^{3/2}|\mathfrak{N}_0| L(1, 1_{F_\mathfrak{p}}) L_{S(\mathfrak{N}_0)}(2, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta)} \times \prod_{v|\infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\pi \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})} \frac{L\big(\frac{1}{2}, \pi_L \otimes \Omega\big)}{L(1, \pi, \mathrm{Ad})}.$$

In particular, $\Sigma(\mathfrak{N}) \sim |\mathfrak{p}| - 1 + O(|\mathfrak{p}|^{-1})$ *as* $|\mathfrak{N}_0\mathfrak{p}| \to \infty$ *such that* $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$. *Furthermore, with* $\mathfrak{p}$ *fixed, we have*

$$\lim_{|\mathfrak{N}_0| \to \infty} \Sigma(\mathfrak{N}) = |\mathfrak{p}| - 1.$$

In both of these asymptotics, $\mathfrak{N}_0$ *travels along squarefree ideals coprime to* $\mathfrak{C}$ *which are products of unramified primes and satisfy our previous parity assumption.*

Note the above theorem implies the nonvanishing of $\Sigma(\mathfrak{N})$, and therefore at least one of these central values, provided $|\mathfrak{p}| > \frac{1}{2}(3+\sqrt{5})$ and $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$, or $\mathfrak{p}$ is arbitrary and $|\mathfrak{N}_0|$ is sufficiently large.

We remark that the bounds come from having to estimate the p-th Hecke eigenvalues $\{a_p, a_p^{-1}\}$ of the oldforms of level $\mathfrak{N}_0$. The latter asymptotic comes from an asymptotic for a weighted analogue of Theorem 1.3 in the case of disjoint ramification (see [Feigon and Whitehouse 2009, Theorem 1.2]) to pick off the contribution from the oldforms. One should be able to prove a version of Theorem 1.3 involving weighting by Hecke eigenvalues (namely, extend [Feigon and Whitehouse 2009, Theorem 6.1] to the case of joint ramification) whereby one could inductively obtain asymptotics for the average values $\Sigma(\mathfrak{N})$ in the case where $\gcd(\mathfrak{N}, \mathfrak{C})$ has an arbitrary number of prime factors. (We remark Sugiyama and Tsuzuki [2016] have recently obtained asymptotics for weighted averages using a different relative trace formula approach when Ω is trivial, but $\mathfrak{N}$ need not be squarefree.)

Note that in previous studies of such averages, $\mathfrak{N}$ is typically required to be prime (e.g., [Ramakrishnan and Rogawski 2005]) or have an odd number of prime factors (e.g., [Feigon and Whitehouse 2009]) to force the sign of the functional equation to be $+1$ if, say, d is odd. However, allowing for joint ramification we can treat levels $\mathfrak{N}$ with an arbitrary number of prime divisors, though we do not always get an exact formula in this situation.

Lastly, we include another application of Theorem 1.3 when $\mathfrak{N} = \mathfrak{N}_0$ (i.e., [Feigon and Whitehouse 2009, Theorem 1.1]). Here, having an exact formula for the average value over newforms allows us to deduce the nonvanishing mod p of the algebraic part $L^{\mathrm{alg}}\big(\frac{1}{2}, \pi_L \otimes \Omega\big)$ (see (8-18)) of the central value for p suitably large.

Theorem 1.5. *With notation and assumptions as in Theorem 1.3, suppose that* $|\mathfrak{N}| > d_{L/F}|\mathfrak{C}|^{h_F}$, *that* $\mathfrak{N}$ *is coprime to* $\mathfrak{C}$, *and that* m_v *is even for each* $v \mid \infty$. *Let* p *be an odd rational prime satisfying* $p > q+1$ *for all primes* $q \in S(\Omega)$, *and* $\mathcal{P}$ *a prime of* $\overline{\mathbb{Q}}$ *above* p. *Then there exists* $\pi \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})$ *such that*

$$L^{\mathrm{alg}}\big(\tfrac{1}{2}, \pi_L \otimes \Omega\big) \not\equiv 0 \mod \mathcal{P}.$$

This generalizes a theorem of Michel and Ramakrishnan [2012] on the case $F = \mathbb{Q}$ and $\mathfrak{N} = N$ is prime. The parity condition on m_v ensures that Ω is algebraic and that the above central value is critical.

As in [Feigon and Whitehouse 2009], one should be able to use Theorem 1.1 to get estimates on more general averages of L-values, and apply this to subconvexity and equidistribution problems, but we do not address this here. Theorem 1.1 has also been used in very recent works of Hamieh [2014] on valuations of Rankin–Selberg L-values in anticyclotomic towers and Van Order [2014] on constructing p-adic L-functions.

We remark that similar L-value formulas have been recently proven in certain cases of joint ramification with L totally imaginary, namely in Hida [2010] for $F = \mathbb{Q}$ and in Hsieh [2014] for Hilbert modular forms of squarefree level (these works have some additional conditions, but they do not assume trivial central character). In general, when the joint ramification does not satisfy (1-3), this problem appears considerably more complicated.

1B. ***Local results.*** Now, we pass to the local situation and discuss the local test vectors in some detail.

Let F be a p-adic field and L a quadratic separable extension of F (either a field or $F \oplus F$). We may then embed $L^\times$ as a torus $T(F)$ of $\mathrm{GL}_2(F)$. All such embeddings are conjugate in $\mathrm{GL}_2(F)$, so the choice of embedding will be merely one of convenience. Consider an (infinite-dimensional) irreducible admissible representation π of $\mathrm{GL}_2(F)$. We do not assume that the central character ω_π is trivial. A basic question to ask is the following: which characters of $T(F)$ appear as quotients in $\pi|_{T(F)}$? Let Ω be a character of $T(F)$. If Ω is an irreducible constituent of $\pi|_{T(F)}$, i.e., if

$$\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0,$$

then we must have $\Omega|_{Z(F)} = \omega_\pi$, where Z denotes the center of GL_2. Hence we will assume $\Omega|_{Z(F)} = \omega_\pi$.

Let D be the unique quaternion division algebra over F, and let π' be the Jacquet–Langlands transfer to $D^\times(F)$ when it exists. If π' exists and $T(F)$ embeds into $D^\times(F)$, put $A(\pi) = \{\pi, \pi'\}$. Otherwise, put $A(\pi) = \{\pi\}$. From [Waldspurger 1985], one knows that

$$\sum_{\tau \in A(\pi)} \dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\tau, \Omega) = 1.$$

In other words, Ω is a constituent of $\pi|_{T(F)}$ if and only if it does not occur in that of $\pi'|_{T(F)}$ (when this makes sense), and it occurs with multiplicity at most one. Further, Tunnell [1983] and Saito [1993] gave a local ϵ-factor criterion:

$$\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = \tfrac{1}{2}\big(1 + \varepsilon\big(\tfrac{1}{2}, \pi_L \otimes \Omega\big)\omega_\pi(-1)\big).$$

Applications to a global L-value formula (discussed in Section 1A) require finer information than this. Namely, suppose $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$ and let $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$ be nonzero. Then one would like to have a *test vector* for ℓ, i.e., an element $\phi \in \pi$ such that $\ell(\phi) \neq 0$. For the applications, we will need ϕ to satisfy two further conditions:

(i) $\phi \in V_\pi^K$ for a compact subgroup K of $\mathrm{GL}_2(F)$ with $\dim(V_\pi^K) = 1$.

(ii) The compact subgroup K above depends only on the ramification data attached to π and Ω.

Let us note that, if $\ell \neq 0$, then some translate of the new vector of π is always a test vector for ℓ. Hence, we can always find a test vector satisfying the first condition above. Under some restriction on the conductors of π and Ω, we will obtain a test vector satisfying the second condition as well.

Specifically, let $\mathfrak{o}$ be the ring of integers of F, $\mathfrak{p}$ its maximal ideal and ϖ a uniformizer. Let $c(\pi)$ be the exponent of the conductor of π as defined in Section 2A, and let

$$K_1(\mathfrak{p}^{c(\pi)}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}) : c \in \mathfrak{p}^{c(\pi)},\ d \in 1 + \mathfrak{p}^{c(\pi)} \right\}.$$

Let $c(\Omega)$ be the conductor of Ω as defined in (2-19). Gross and Prasad [1991] determine a test vector when $c(\pi) = 0$ (π is unramified) or $c(\Omega) = 0$ (Ω is unramified). In particular, when $c(\pi) = 0$ so $A(\pi) = \{\pi\}$, the vector they obtain can be described as a translate of the new vector.

We will now describe test vectors when π and Ω are both ramified. We will distinguish the split and field case.

1B1. *The split case.* In the following, $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

Theorem 1.6. *Suppose $L = F \oplus F$ and let $T(F) \cong L^\times$ be the diagonal torus in $\mathrm{GL}_2(F)$. Let π be any infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$ with central character ω_π and conductor $\mathfrak{p}^{c(\pi)}$, $c(\pi) \geq 0$. Let*

$\Omega(\mathrm{diag}(x, y)) = \Omega_1(x)\Omega_2(y)$ *be a character of* $T(F)$ *such that* $\Omega_1\Omega_2 = \omega_\pi$. *Without loss of generality, assume that* $c(\Omega_1) \geq c(\Omega_2)$. *Write* $\Omega_1 = |\cdot|^{1/2-s_0}\mu$ *for some* $s_0 \in \mathbb{C}$ *and some unitary character* μ *of* $F^\times$ *such that* $\mu(\varpi) = 1$. *Then* $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$, *and for nonzero* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$, *the subgroup* $hK_1(\mathfrak{p}^{c(\pi)})h^{-1}$ *fixes a* 1*-dimensional space of* π *consisting of test vectors for* ℓ, *where*

$$h = \begin{cases} \begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ 0 & 1 \end{bmatrix} & \text{if } c(\mu) = 0 \\ & \quad \text{or } L(s, \pi \otimes \mu^{-1}) \text{ does not have a pole at } s = s_0; \\ w\begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ 0 & 1 \end{bmatrix} & \text{if } c(\mu) > 0 \text{ and } L(s, \pi \otimes \mu^{-1}) \text{ has a pole at } s = s_0, \\ & \quad \text{but } L(1-s, \tilde{\pi} \otimes \mu) \text{ does not have a pole at } s = s_0. \end{cases}$$

In particular, if both Ω *and* π *are unitary, then we are always in the first case above.*

The proof of the above theorem uses the theory of zeta integrals for GL_2 representations given by their Whittaker models. The zeta integral $Z(s_0, *, \mu^{-1})$ (defined in (3-1)) divided by the L-value $L(s_0, \pi \otimes \mu^{-1})$ gives a concrete realization of a nonzero $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. One checks that the newform in the Whittaker model translated by the matrix h in the statement of the above theorem is a test vector for ℓ.

Note that we do not give a compact subgroup that fixes a 1-dimensional space of π consisting of test vectors for ℓ when both $L(s, \pi \otimes \mu^{-1})$ and $L(1-s, \tilde{\pi} \otimes \mu)$ have a pole at $s = s_0$.

1B2. *The field case.*

Theorem 1.7. *Suppose* L *is a field. Let* π *be any infinite-dimensional, irreducible, admissible representation of* $\mathrm{GL}_2(F)$ *with central character* ω_π *and conductor* $\mathfrak{p}^{c(\pi)}$. *Let* Ω *be a character on* $L^\times$ *such that* $\Omega|_{F^\times} = \omega_\pi$. *Assume that* $c(\Omega) \geq c(\pi) > 0$. *Embed* $L^\times$ *as a torus* $T(F)$ *in* $\mathrm{GL}_2(F)$ *as in Section 2C. Then* $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$, *and for a nonzero* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$, *the subgroup*

$$\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{c(\Omega)} \\ \mathfrak{p}^{c(\pi)-c(\Omega)} & 1+\mathfrak{p}^{c(\pi)} \end{bmatrix} \cap \mathrm{GL}_2(F) = hK_1(\mathfrak{p}^{c(\pi)})h^{-1}, \quad h = \begin{bmatrix} \varpi^{c(\Omega)-c(\pi)} & 0 \\ 0 & 1 \end{bmatrix} w,$$

fixes a 1*-dimensional space of* π *consisting of test vectors for* ℓ.

If π has trivial central character, then we can replace the compact subgroup in the statement of the above theorem by

$$\begin{bmatrix} \varpi^{c(\Omega)} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{c(\pi)} & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} \varpi^{-c(\Omega)} & \\ & 1 \end{bmatrix},$$

since the Atkin–Lehner element normalizes the group $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{c(\pi)} & \mathfrak{o}^\times \end{smallmatrix}\right]$.

The proof of the theorem breaks up into several cases depending on the type of the representation π. Although the proofs are quite different in all cases, it turns out that one of the key ingredients of the proof is that a function roughly of the form $x \mapsto \Omega(1+x\beta)$ (see Section 2B for details on notation) is an additive character of $\mathfrak{o}$ of a specific conductor. The condition $c(\Omega) \geq c(\pi)$ is required to make this key ingredient work. Also, in certain cases we obtain test vectors for more general situations than the one mentioned above.

Principal series. If π is a principal series representation, then we realize it in its induced model and explicitly define a linear functional

$$\ell(f) = \int_{Z(F)\backslash T(F)} f(t)\Omega^{-1}(t)\,dt.$$

It is easy to see that $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. We are able to show, for any $c(\pi), c(\Omega) \geq 0$, that $\ell \neq 0$. See (4-4) and (4-5) for details. It is not clear if the explicit test vector for ℓ obtained in (4-4) belongs to a 1-dimensional subspace of π of vectors right-invariant under a compact subgroup. It is also not clear how to obtain a component that is right invariant under a conjugate of $K_1(\mathfrak{p}^{c(\pi)})$. To obtain a test vector with the right invariance mentioned in the statement of the theorem, we evaluate ℓ at a translate of the newform of π by h and show that that is nonzero. For this, we need $c(\Omega) \geq c(\pi) > 0$. If we replace the h in the statement of the theorem by $h = \left[\begin{smallmatrix} \varpi^s & \\ & 1 \end{smallmatrix}\right]$, $s = c(\pi) - c(\Omega) - v(\boldsymbol{a})$, where $\boldsymbol{a}$ depends on a particular embedding of $T(F)$ in $\mathrm{GL}_2(F)$, then we can extend the result to the case $c(\Omega) \geq 2c(\chi_1)$ (see Proposition 4.2). Here, $\pi = \chi_1 \times \chi_2$ and $c(\chi_1) \leq c(\chi_2)$.

Twists of the Steinberg representation. If π is a twist of the Steinberg representation by a ramified character χ, then realizing it as a subrepresentation of the reducible induced representation $\chi|\cdot|^{1/2} \times \chi|\cdot|^{-1/2}$, we see that we get the same linear functional and the same nonvanishing of the translate of newform as in the irreducible principal series case.

If π is a twist of the Steinberg representation by an unramified character χ, then we use the fact that such representations are characterized by the existence of a unique (up to constant) vector that is right invariant under the Iwahori subgroup I and is an eigenvector of the Atkin–Lehner operator with eigenvalue $-\chi(\varpi)$. If we assume that $c(\Omega) \geq c(\pi)$, then [Waldspurger 1985] implies the existence of a nonzero $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. As in Section 2D, we can then realize π as a subrepresentation of the space of smooth functions $B : \mathrm{GL}_2(F) \to \mathbb{C}$ satisfying $B(tg) = \Omega(t)B(g)$. In this latter space, we look for a vector B with three properties: one that is right invariant under I, is zero when averaged over $\mathrm{GL}_2(\mathfrak{o})/I$, and is an eigenvector for the Atkin–Lehner operator with eigenvalue $-\chi(\varpi)$. Using a double coset decomposition for $T(F)\backslash \mathrm{GL}_2(F)/I$, we obtain in Lemma 4.4 the explicit values of such a B for all $g \in \mathrm{GL}_2(F)$. This gives us $B(h) \neq 0$, for h defined in the statement of the theorem. The advantage of the above method is twofold. It gives us the explicit values of the newform in the Waldspurger model and it also gives another proof of the uniqueness of the Waldspurger model. One can also obtain an independent proof of existence using the methods of [Pitale 2011], but we do not do that here.

Supercuspidal representations. In the case that π is an irreducible supercuspidal representation we may appeal to Mackey theory. We begin with the explicit construction of supercuspidal representations of $\mathrm{GL}_2(F)$ by induction from an open subgroup that is compact modulo the center. Suppose that J is such a subgroup and $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)} \rho$. We first describe the situation when π is minimal, i.e., when the conductor of π cannot be lowered upon twisting by a character.

We say that ρ and Ω intertwine on $T(F)gJ$ if $\mathrm{Hom}_{J \cap g^{-1}T(F)g}(\rho, \Omega^g) \neq 0$. Understanding $\mathrm{Hom}_{T(F)}(\pi, \Omega)$ then reduces to understanding the double cosets $T(F)\backslash \mathrm{GL}_2(F)/J$ on which ρ and Ω intertwine. We do this in two steps. The first step is to consider a larger subgroup $K_{\mathfrak{A}} \supseteq J$ where $K_{\mathfrak{A}}$ is one of two subgroups depending on J. There is a unique double coset $T(F)h_0K_{\mathfrak{A}}$ that depends only on $c(\pi)$ and $c(\Omega)$ containing a $T(F)\backslash \mathrm{GL}_2(F)/J$ double coset on which ρ and Ω can possibly intertwine. This double coset decomposes as the disjoint union of finitely many $T(F)\backslash \mathrm{GL}_2(F)/J$ double cosets

$$T(F)h_0K_{\mathfrak{A}} = \bigsqcup_i T(F)h_iJ.$$

When $c(\Omega) > \left[\frac{1}{2}c(\pi)\right]$, we describe this decomposition explicitly, show that one may choose the representatives h_i to be diagonal matrices, and show for each i that

$$(J \cap h_i^{-1}T(F)h_i)\ker\rho/Z(F)\ker\rho \cong (J \cap \overline{N})/(\ker\rho \cap \overline{N}),$$

where $\overline{N}$ is the subgroup of lower triangular unipotent matrices. It suffices to examine $\rho|_{J\cap\overline{N}}$, which decomposes as a direct sum of characters. We show that there is a unique i_0 such that ρ and Ω intertwine on $T(F)h_{i_0}J$. We conclude that there exists a nonzero linear functional $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. We describe the translate of the newvector in the induced model explicitly, and show that this translate is a test vector.

Finally, we deal with the case of an irreducible supercuspidal representation τ that is not minimal. In this case $\tau \cong \pi \otimes \chi$, where π is a minimal supercuspidal representation and χ is a character of $F^\times$. We construct a vector $\varphi_\chi \in \pi$ so that $\varphi_\chi \otimes \chi$ is a translate of the newvector in τ. Using the results of the minimal case, we show that φ_χ is a test vector for $\Omega \otimes \chi^{-1}$.

Similarly to the irreducible principal series case, if we replace h in the statement of the theorem by $h = \left[\begin{smallmatrix} \varpi^s & \\ & 1 \end{smallmatrix}\right]$, $s = c(\pi) - c(\Omega) - v(\boldsymbol{a})$, then in the *minimal* supercuspidal case, we can extend the result to the case $c(\Omega) \geq \left\lfloor \frac{3}{4}c(\pi)\right\rfloor + 1$.

1B3. *Relation to test vectors of Gross–Prasad.* We recall some results of Gross and Prasad [1991]. For simplicity assume that $\omega_\pi = 1$, that L/F is unramified and that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{T(F)}(\pi, \Omega) = 1.$$

For an order R of $M_2(F)$, let $d(R)$ be the exponent of its reduced discriminant and $c(R)$ be the smallest $c \geq 0$ such that $\mathfrak{o} + \varpi^c \mathfrak{o}_L \subset R$. It is clear that $R^\times$ can only fix a test vector if $c(R) \geq c(\Omega)$. Moreover, if we want $R^\times$ to fix a line in π, it is reasonable to try R with $d(R) = c(\pi)$. Thus one might consider orders with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$. If either $c(\Omega) = 0$ or $c(\pi) = 0$, then there is a unique-up-to-$L^\times$-conjugacy order R with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$, and [Gross and Prasad 1991] shows that $R^\times$ fixes a line consisting of test vectors. If $c(\pi) = 0$ then R is a maximal order, but in general R is not an Eichler order.

When $c(\Omega) > 0$ and $c(\pi) > 0$, the invariants $c(R)$ and $d(R)$ no longer specify R uniquely up to conjugacy by $L^\times$. However, with the above assumptions, Theorem 1.7 can be interpreted as follows: when $c(\Omega) \geq c(\pi)$, there is an Eichler order R with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$ such that $R^\times$ fixes a line in π which consists of test vectors. Moreover, this R can be described uniquely up to $L^\times$-conjugacy as the intersection of two maximal ideals R_1 and R_2, with $c(R_1) = c(\Omega)$ and $c(R_2) = c(\Omega) - c(\pi)$, which are the maximal possible distance apart in the Bruhat–Tits tree, i.e., $d(R_1, R_2) = c(\pi)$. This provides an intrinsic description of our test vectors, i.e., one without reference to a specific embedding of $L^\times$ in $\mathrm{GL}_2(F)$. It would be interesting to know whether other Eichler orders R satisfying $c(R) = c(\Omega)$ and $d(R) = c(\pi)$ also pick out test vectors.

Note that if $c(\pi) > 2c(\Omega)$, there is no Eichler order with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$, which suggests that the case when π is highly ramified, in comparison with Ω, is more complicated than the reverse situation.

1C. *Outline.* Our paper consists of two parts, one local and one global.

In the first (local) part of the paper we prove our results on local test vectors, which we treat in three separate cases. Section 2 contains our local notation and embedding of $L^\times$ into $\mathrm{GL}_2(F)$. Then in Section 3 we treat the case where L/F is split, using zeta integrals. This proves Theorem 1.6. Now assume L/F is inert. In Section 4, we treat the case of principal series and Steinberg representations. In Section 5, we treat the case of supercuspidal representations. These two sections complete Theorem 1.7. Finally, in Section 6 we compute certain local spectral distributions associated to our local test vectors.

The global part of the paper consists of two sections. In Section 7, we use the local spectral calculations of Section 6 to prove our L-value formula (Theorem 1.1). In Section 8, we deduce our results on average values and nonvanishing (Theorems 1.3, 1.4 and 1.5).

2. Local setup

Let F be a nonarchimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$ and ϖ a generator of $\mathfrak{p}$. Denote by q the size of the residue field and by v the normalized valuation map on F.

For a character χ of $F^\times$, let $c(\chi)$ be the exponent of its conductor, i.e., $c(\chi) \geq 0$ is minimal such that χ is trivial on $(1+\mathfrak{p}^{c(\chi)}) \cap \mathfrak{o}^\times$.

2A. ***Subgroups and representations of* $\mathbf{GL_2}$.** We use the following compact subgroups of $\mathrm{GL}_2(F)$. Put $K_1(\mathfrak{o}) = K_2(\mathfrak{o}) = \mathrm{GL}_2(\mathfrak{o})$. For $n > 0$, put

$$K_1(\mathfrak{p}^n) = \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{bmatrix}, \tag{2-1}$$

$$K_2(\mathfrak{p}^n) = \begin{bmatrix} 1+\mathfrak{p}^n & \mathfrak{o} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{bmatrix}. \tag{2-2}$$

For $s \in \mathbb{Z}$, $n \geq 0$, let

$$K_1^{(s)}(\mathfrak{p}^n) = \begin{bmatrix} \varpi^s & \\ & 1 \end{bmatrix} K_1(\mathfrak{p}^n) \begin{bmatrix} \varpi^{-s} & \\ & 1 \end{bmatrix}. \tag{2-3}$$

We also have the Iwahori subgroup

$$I = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{bmatrix} \cap \mathrm{GL}_2(\mathfrak{o}). \tag{2-4}$$

Let (π, V) be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$. For $n \geq 0$, denote by V^n the subspace of $K_1(\mathfrak{p}^n)$-fixed vectors. By [Jacquet et al. 1981], one knows $V^n \neq 0$ for some n. Further, if $c(\pi)$ is the minimal n such that $V^n \neq 0$, then $\dim(V^{c(\pi)}) = 1$. Call the ideal $\mathfrak{p}^{c(\pi)}$ the *conductor* of π. If $c(\pi) = 0$, then π is unramified.

Such a π is a principal series, twist of Steinberg (special), or supercuspidal representation. Let χ_1, χ_2 be two characters of $F^\times$. The representation $\pi = \chi_1 \times \chi_2$ is the standard induced representation of $\mathrm{GL}_2(F)$ consisting of locally constant functions $f : \mathrm{GL}_2(F) \to \mathbb{C}$ such that

$$f\left(\begin{bmatrix} a & b \\ & d \end{bmatrix} g\right) = \chi_1(a)\chi_2(d)\left|ad^{-1}\right|^{1/2} f(g),$$
$$\text{for all } g \in \mathrm{GL}_2(F),\ a, d \in F^\times,\ b \in F. \tag{2-5}$$

This is irreducible if and only if $\chi_1\chi_2 \neq |\cdot|^{\pm 1}$, in which case we say $\chi_1 \times \chi_2$ is a principal series representation. For a character χ of $F^\times$, the twist of the Steinberg representation by χ, which we denote by $\chi\,\mathrm{St}_{\mathrm{GL}_2}$, is the unique irreducible subrepresentation of the induced representation $\chi|\cdot|^{1/2} \times \chi|\cdot|^{-1/2}$. The supercuspidal representations are described in Section 5.

2B. ***The degree-two extension.*** As in [Furusawa 1993], we fix three elements $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in F$ such that $\boldsymbol{d} = \boldsymbol{b}^2 - 4\boldsymbol{ac} \neq 0$. We let $L = F(\sqrt{\boldsymbol{d}})$ if $\boldsymbol{d} \notin F^{\times 2}$, and $L = F \oplus F$ otherwise. In the latter case we consider F diagonally embedded in L. Let $z \mapsto \bar{z}$ be the obvious involution on L whose fixed point set is F. We define the Legendre symbol as

$$\left(\frac{L}{\mathfrak{p}}\right) = \begin{cases} -1 & \text{if } L/F \text{ is an unramified field extension,} \\ \ \ 0 & \text{if } L/F \text{ is a ramified field extension,} \\ \ \ 1 & \text{if } L = F \oplus F. \end{cases} \tag{2-6}$$

We make the following assumptions:

- $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{o}$ and $\boldsymbol{c} \in \mathfrak{o}^\times$.
- If $\boldsymbol{d} \notin F^{\times 2}$, then $\boldsymbol{d}$ is a generator of the discriminant of L/F.
- If $\boldsymbol{d} \in F^{\times 2}$, then $\boldsymbol{d} \in \mathfrak{o}^\times$.

We define elements β and ξ_0 of L by

$$\beta = \begin{cases} \dfrac{\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}} & \text{if } L \text{ is a field,} \\ \left(\dfrac{\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}}, \dfrac{\boldsymbol{b}-\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}}\right) & \text{if } L = F \oplus F, \end{cases} \tag{2-7}$$

$$\xi_0 = \begin{cases} \dfrac{-\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2} & \text{if } L \text{ is a field,} \\ \left(\dfrac{-\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2}, \dfrac{-\boldsymbol{b}-\sqrt{\boldsymbol{d}}}{2}\right) & \text{if } L = F \oplus F. \end{cases} \tag{2-8}$$

If L is a field, let $\mathfrak{o}_L$ be its ring of integers, ϖ_L a uniformizer, and v_L the normalized valuation. If $L = F \oplus F$, put $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$ and $\varpi_L = (\varpi, 1)$. By [Pitale and Schmidt 2009, Lemma 3.1.1], in either case,

$$\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\beta = \mathfrak{o} + \mathfrak{o}\xi_0. \tag{2-9}$$

Lemma 2.1. *Suppose L is a field. The possible valuations of β and $\boldsymbol{a}$ are*

$$v_L(\beta) = v(\boldsymbol{a}) = 0 \qquad \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \tag{2-10}$$

$$v_L(\beta) = v(\boldsymbol{a}) \in \{0, 1\} \quad \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = 0. \tag{2-11}$$

Proof. Consider the identity

$$\frac{\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}} \cdot \frac{\boldsymbol{b}-\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}} = \frac{\boldsymbol{a}}{\boldsymbol{c}}. \tag{2-12}$$

If $\left(\frac{L}{\mathfrak{p}}\right) = -1$, we get the result by observing that d is a nonsquare unit. If $\left(\frac{L}{\mathfrak{p}}\right) = 0$, we get the result since $1, \beta$ is an integral basis. □

Fix the ideal in $\mathfrak{o}_L$ given by

$$\mathfrak{P}_L := \mathfrak{p}\mathfrak{o}_L = \begin{cases} \mathfrak{p}_L & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \mathfrak{p}_L^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \mathfrak{p} \oplus \mathfrak{p} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{2-13}$$

Here $\mathfrak{p}_L$ is the maximal ideal of $\mathfrak{o}_L$ when L is a field. We have $\mathfrak{P}_L^n \cap \mathfrak{o} = \mathfrak{p}^n$ for all $n \geq 0$.

Under our stated assumptions, it makes sense to consider the quadratic equation $\boldsymbol{c}u^2 + \boldsymbol{b}u + \boldsymbol{a} = 0$ over the residue class field $\mathfrak{o}/\mathfrak{p}$. The number of solutions of this equation is $\left(\frac{L}{\mathfrak{p}}\right) + 1$. In the ramified case we will fix an element $u_0 \in \mathfrak{o}$ such that

$$\boldsymbol{c}u_0^2 + \boldsymbol{b}u_0 + \boldsymbol{a} \in \mathfrak{p}; \tag{2-14}$$

see [Pitale and Schmidt 2009, Lemma 3.1.1]. Further, note that in the ramified case we have

$$\boldsymbol{b} + 2\boldsymbol{c}u_0 \in \mathfrak{p}. \tag{2-15}$$

This follows from the fact that u_0 is a double root of $\boldsymbol{c}u^2 + \boldsymbol{b}u + \boldsymbol{a}$ over $\mathfrak{o}/\mathfrak{p}$.

2C. *The torus.* We now specify an embedding of $L^\times$ as a torus in GL_2 for convenience of calculations. With $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as above, let

$$S = \begin{bmatrix} \boldsymbol{a} & \frac{1}{2}\boldsymbol{b} \\ \frac{1}{2}\boldsymbol{b} & \boldsymbol{c} \end{bmatrix}, \qquad \xi = \begin{bmatrix} \frac{1}{2}\boldsymbol{b} & \boldsymbol{c} \\ -\boldsymbol{a} & -\frac{1}{2}\boldsymbol{b} \end{bmatrix}.$$

Then $F(\xi) = F \cdot I_2 + F \cdot \xi$ is a 2-dimensional F-algebra isomorphic to L. If L is a field, then an isomorphism is given by $x + y\xi \mapsto x + y\sqrt{\boldsymbol{d}}/2$. If $L = F \oplus F$, then an isomorphism is given by $x + y\xi \mapsto \big(x + y\sqrt{\boldsymbol{d}}/2, x - y\sqrt{\boldsymbol{d}}/2\big)$. The determinant map on $F(\xi)$ corresponds to the norm map on L. Let

$$T(F) = \{g \in \mathrm{GL}_2(F) : {}^t gSg = \det(g)S\}. \tag{2-16}$$

One can check that $T(F) = F(\xi)^\times$. Note that $T(F) \cong L^\times$ via the isomorphism $F(\xi) \cong L$. Under the same isomorphism the group $T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is isomorphic to $\mathfrak{o}_L^\times$. Note that $T(F)$ consists of all matrices

$$t(x, y) = \begin{bmatrix} x + \frac{1}{2}y\boldsymbol{b} & \boldsymbol{c}y \\ -\boldsymbol{a}y & x - \frac{1}{2}y\boldsymbol{b} \end{bmatrix},$$
$$\text{for all } x, y \in F,\ \det(g) = x^2 - \tfrac{1}{4}y^2(\boldsymbol{b}^2 - 4\boldsymbol{a}\boldsymbol{c}) \neq 0. \tag{2-17}$$

We give a useful structural lemma here.

Lemma 2.2. *Let L/F be a field extension. For any $m, n \geq 0$, we have*

$$T(F)\begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} K_1(\mathfrak{p}^n) = T(F)\begin{bmatrix} \varpi^{m-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix} w K_1(\mathfrak{p}^n). \tag{2-18}$$

Proof. Set $y = \varpi^{-m}$ and $x = \frac{1}{2} y \boldsymbol{b}$. Then

$$\begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} = \frac{-1}{\boldsymbol{a}\varpi^{-v(\boldsymbol{a})}} t(x, y) \begin{bmatrix} \varpi^{m-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix} wk,$$

with

$$k = \begin{bmatrix} \frac{\boldsymbol{a}\varpi^{-v(\boldsymbol{a})}}{\boldsymbol{c}} & \frac{\boldsymbol{b}\varpi^{m-v(\boldsymbol{a})}}{\boldsymbol{c}} \\ & 1 \end{bmatrix} \in K_1(\mathfrak{p}^n) \quad \text{for all } n \geq 0,$$

since $v(\boldsymbol{b}) \geq 1$ whenever $v(\boldsymbol{a}) = 1$. □

2D. *The Waldspurger model.* Let Ω be any character of $L^\times$, which we may view as a character of the torus $T(F)$. Define

$$c(\Omega) := \min\{m \geq 0 : \Omega|_{(1+\mathfrak{P}_L^m)\cap\mathfrak{o}_L^\times} \equiv 1\}. \tag{2-19}$$

Note that this is the (exponent of the) conductor of Ω only in the case L/F is an unramified field extension. Let $\mathcal{B}(\Omega)$ be the space of all locally constant functions $B : \mathrm{GL}_2(F) \to \mathbb{C}$ satisfying

$$B(tg) = \Omega(t)B(g) \quad \text{for all } t \in T(F),\ g \in \mathrm{GL}_2(F). \tag{2-20}$$

Let (π, V) be any infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$. We say that π has an Ω-*Waldspurger model* if π is isomorphic to a subrepresentation of $\mathcal{B}(\Omega)$. We call a linear functional ℓ on π an Ω-*Waldspurger functional* if it satisfies

$$\ell(\pi(t)v) = \Omega(t)\ell(v) \quad \text{for all } t \in T(F),\ v \in V. \tag{2-21}$$

If π has an Ω-Waldspurger model then we obtain an Ω-Waldspurger functional ℓ by $\ell(B) = B(1)$. On the other hand, if π has an Ω-Waldspurger functional ℓ, we obtain an Ω-Waldspurger model for π by the map $v \mapsto B_v$, where $B_v(g) = \ell(\pi(g)v)$. Observe that a necessary condition for an Ω-Waldspurger model or functional to exist is that $\Omega|_{F^\times} = \omega_\pi$, the central character of π.

If π has an Ω-Waldspurger functional ℓ, we say that $v \in V$ is a *test vector* for ℓ if $\ell(v) \neq 0$. From the discussion above, this is equivalent to $B_v(1) \neq 0$. Suppose B_0 is the newform in an Ω-Waldspurger model of π. Lemma 2.2 states that, in the field case, for $m \geq 0$, the vector

$$\pi\Big(\begin{bmatrix} \varpi^{m-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix} w\Big) B_0$$

is a test vector if and only if $\pi\left(\begin{bmatrix}\varpi^{-m} & \\ & 1\end{bmatrix}\right)B_0$ is also a test vector. This will be used in the proof of Theorem 1.7 below.

Criteria for existence of Waldspurger functionals, which must be unique up to scalars, are given in Section 1B.

3. Zeta integrals and test vectors for split Waldspurger models

In this section we show that any irreducible admissible representation π of $\mathrm{GL}_2(F)$ has a split Ω-Waldspurger model for every character Ω of $L^\times = F^\times \oplus F^\times$. Under certain restrictions on the poles of the L-function of π, we also determine test vectors for the Waldspurger functional that are right invariant under certain conjugates of the compact group $K_1(\mathfrak{p}^{c(\pi)})$. The conjugating elements depend only on $c(\pi)$ and $c(\Omega)$.

Let π be any irreducible admissible representation of $\mathrm{GL}_2(F)$ with central character ω_π (not assumed to be trivial). Let π be given by its Whittaker model $\mathcal{W}(\pi,\psi)$, where ψ is a nontrivial character of F with conductor $\mathfrak{o}$. For any $W \in \mathcal{W}(\pi,\psi)$ and a unitary character μ of $F^\times$, define the zeta integral

$$Z(s, W, \mu^{-1}) := \int_{F^\times} W\left(\begin{bmatrix} x & \\ & 1\end{bmatrix}\right)|x|^{s-1/2}\mu^{-1}(x)\, d^\times x, \tag{3-1}$$

where $d^\times x$ is the Haar measure on $F^\times$ giving $\mathfrak{o}^\times$ volume $1-q^{-1}$. Since μ is unitary, there is an $r \in \mathbb{R}$ not depending on μ such that $Z(s, W, \mu^{-1})$ converges absolutely for $\Re(s) > r$. By the theory of L-functions, we have

$$\frac{Z(s, W, \mu^{-1})}{L(s, \mu^{-1}\otimes\pi)} \in \mathbb{C}[q^{-s}, q^s] \tag{3-2}$$

and the functional equation

$$\frac{Z(1-s, \pi(w)W, \mu\omega_\pi^{-1})}{L(1-s, \mu\otimes\tilde{\pi})} = \varepsilon(s, \mu^{-1}\otimes\pi, \psi)\frac{Z(s, W, \mu^{-1})}{L(s, \mu^{-1}\otimes\pi)} \tag{3-3}$$

for any $W \in \mathcal{W}(\pi,\psi)$. Here $w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$. Please refer to Theorem 6.12 of [Gelbart 1975] for details.

Let W_0 be the unique $K_1(\mathfrak{p}^{c(\pi)})$-right invariant vector in $\mathcal{W}(\pi,\psi)$ such that $W_0(1) = 1$. The formula for $W_0\left(\begin{bmatrix} x & \\ & 1\end{bmatrix}\right)$ in various cases is given in Table 1 (see, e.g., [Schmidt 2002]).

Proposition 3.1. *Let π be any irreducible, admissible representation of* $\mathrm{GL}_2(F)$ *with central character ω_π and conductor $\mathfrak{p}^{c(\pi)}$. Let W_0 be the newform in the Whittaker model $\mathcal{W}(\pi,\psi)$ of π such that $W_0(1) = 1$. Let μ be a unitary character of $F^\times$.*

(i) *If $c(\mu)=0$ then, for any π, we have*

$$Z\Bigl(s,\pi\Bigl(\begin{bmatrix}1 & \varpi^{-c(\mu)}\\ & 1\end{bmatrix}\Bigr)W_0,\mu^{-1}\Bigr)=\Bigl(1-\frac{1}{q}\Bigr)L(s,\mu^{-1}\otimes\pi).$$

(ii) *If $c(\mu)>0$ then*

$$Z\Bigl(s,\pi\Bigl(\begin{bmatrix}1 & \varpi^{-c(\mu)}\\ & 1\end{bmatrix}\Bigr)W_0,\mu^{-1}\Bigr)=q^{-c(\mu)/2}\mu(\varpi^{-c(\mu)})\varepsilon\Bigl(\frac{1}{2},\mu,\psi\Bigr).$$

Proof. If $c(\mu)=0$, then the values of the newform W_0 from Table 1 and the normalization of the measure give us (i). We have, for any $k\in\mathbb{Z}$ and any π,

$$\begin{aligned}
&Z\Bigl(s,\pi\Bigl(\begin{bmatrix}1 & \varpi^{k}\\ & 1\end{bmatrix}\Bigr)W_0,\mu^{-1}\Bigr)\\
&\qquad=\int_{F^\times}\psi(a\varpi^k)W_0\Bigl(\begin{bmatrix}a & \\ & 1\end{bmatrix}\Bigr)|a|^{s-1/2}\mu^{-1}(a)\,d^\times a\\
&\qquad=\sum_{j\in\mathbb{Z}}\int_{\mathfrak{o}^\times}\psi(a\varpi^{j+k})W_0\Bigl(\begin{bmatrix}a\varpi^j & \\ & 1\end{bmatrix}\Bigr)|a\varpi^j|^{s-1/2}\mu^{-1}(a\varpi^j)\,d^\times a\\
&\qquad=\sum_{j\in\mathbb{Z}}q^{-j(s-1/2)}\mu^{-1}(\varpi^j)W_0\Bigl(\begin{bmatrix}\varpi^j & \\ & 1\end{bmatrix}\Bigr)\int_{\mathfrak{o}^\times}\psi(a\varpi^{j+k})\mu^{-1}(a)\,d^\times a.
\end{aligned}$$

If $c(\mu)>0$, then, by the definition of the epsilon factor for μ (see [Schmidt 2002, equation (5)]), we have

$$\int_{\mathfrak{o}^\times}\psi(a\varpi^{j+k})\mu^{-1}(a)\,d^\times a
=\begin{cases}q^{-c(\mu)/2}\mu(\varpi^{j+k})\varepsilon\bigl(\frac{1}{2},\mu,\psi\bigr) & \text{if } j+k=-c(\mu),\\ 0 & \text{if } j+k\neq -c(\mu).\end{cases}\tag{3-4}$$

Now the proposition follows since $W_0(1)=1$. □

π	$W_0\Bigl(\begin{bmatrix}x & \\ & 1\end{bmatrix}\Bigr)$
$\chi_1\times\chi_2$, with χ_1,χ_2 unramified, $\chi_1\chi_2^{-1}\neq|\cdot|^{\pm 1}$	$|x|^{1/2}\Bigl(\sum_{k+l=v(x)}\chi_1(\varpi^k)\chi_2(\varpi^l)\Bigr)1_{\mathfrak{o}}(x)$
$\chi_1\times\chi_2$, with χ_1 unramified, χ_2 ramified	$|x|^{1/2}\chi_1(x)1_{\mathfrak{o}}(x)$
$\chi\,\mathrm{St}_{\mathrm{GL}_2}$, with χ unramified	$|x|\chi(x)1_{\mathfrak{o}}(x)$
$L(s,\pi)=1$	$1_{\mathfrak{o}^\times}(x)$

Table 1. Whittaker newform values.

Proof of Theorem 1.6. For any $W \in \mathcal{W}(\pi, \psi)$, define

$$\ell(W) := \frac{Z(s_0, W, \mu^{-1})}{L(s_0, \mu^{-1} \otimes \pi)}. \tag{3-5}$$

The well-definedness of ℓ for all s_0 and μ follows from (3-2). By [Gelbart 1975, Theorem 6.12], ℓ is nonzero. The definition of the zeta integral and $\Omega_1\Omega_2 = \omega_\pi$ gives us the transformation property

$$\ell\left(\pi\left(\begin{bmatrix} x & \\ & y \end{bmatrix}\right)W\right) = \Omega_1(x)\Omega_2(y)\ell(W), \quad x, y \in F^\times.$$

Hence, we get $\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0$. The 1-dimensionality follows from [Waldspurger 1985]. Note that, if $c(\mu) = 0$ or if $L(s, \mu^{-1} \otimes \pi)$ does not have a pole at $s = s_0$, then we have

$$\ell\left(\pi\left(\begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ & 1 \end{bmatrix}\right)W_0\right) \neq 0$$

by Proposition 3.1. If $c(\mu) > 0$ and $L(s, \mu^{-1} \otimes \pi)$ has a pole at $s = s_0$, then

$$\ell\left(\pi\left(\begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ & 1 \end{bmatrix}\right)W_0\right) = 0$$

by Proposition 3.1. In this case, if we assume that $L(1 - s, \mu \otimes \tilde{\pi})$ does not have a pole at $s = s_0$, then we can use the local functional equation (3-3), which gives us the test vectors for ℓ. The uniqueness statement follows from the uniqueness of W_0. If Ω and π are unitary, then $s_0 = \frac{1}{2}$ and one can check that $L(s, \mu^{-1} \otimes \pi)$ does not have a pole at $s = \frac{1}{2}$. □

4. Nonsupercuspidal representations

Here we assume that L is a field and prove Theorem 1.7 when π is not supercuspidal.

Let us define Haar measures dg on $\mathrm{GL}_2(F)$ such that $\mathrm{GL}_2(\mathfrak{o})$ has volume 1; $d^\times x$ on $F^\times = Z(F)$, the center of $\mathrm{GL}_2(F)$, such that $\mathfrak{o}^\times$ has volume 1 (note this is different from Section 3); and dt on $T(F) = L^\times$ such that the volume of $\mathfrak{o}_L^\times$ is 1.

4A. ***Irreducible principal series representation.*** Let π be a *ramified* irreducible principal series representation of $\mathrm{GL}_2(F)$ given by

$$\begin{gathered} \pi = \chi_1 \times \chi_2, \qquad \chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}, \qquad c(\chi_2) \geq c(\chi_1), \\ c(\pi) = c(\chi_1) + c(\chi_2) > 0, \qquad \omega_\pi = \chi_1\chi_2. \end{gathered} \tag{4-1}$$

Recall that π consists of locally constant functions f on $\mathrm{GL}_2(F)$ satisfying (2-5). The unique, up to scalars, right $K_1(\mathfrak{p}^{c(\pi)})$-invariant vector f_0 in π is given by the formula

$$f_0(g) = \begin{cases} |a/d|^{1/2}\chi_1(a)\chi_2(d) & \text{if } g \in \begin{bmatrix} a & * \\ & d \end{bmatrix}\gamma_{c(\chi_2)}K_1(\mathfrak{p}^{c(\pi)}), \\ 0 & \text{if } g \notin B(F)\gamma_{c(\chi_2)}K_1(\mathfrak{p}^{c(\pi)}), \end{cases} \tag{4-2}$$

where $\gamma_{c(\chi_2)} = \left[\begin{smallmatrix} 1 & \\ \varpi^{c(\chi_2)} & 1 \end{smallmatrix}\right]$ and $B(F)$ is the Borel subgroup of $\mathrm{GL}_2(F)$ consisting of upper triangular matrices. See [Schmidt 2002] for details.

Let Ω be a character of $L^\times$ such that $\Omega|_{F^\times} = \omega_\pi$. Let $\mathcal{B}(\Omega)$ be the space of all locally constant functions $B : \mathrm{GL}_2(F) \to \mathbb{C}$ satisfying (2-20) defined in Section 2D. Define an intertwining operator $\mathcal{A} : \pi \to \mathcal{B}(\Omega)$ by the formula

$$(\mathcal{A}(f))(g) := \int_{Z(F)\backslash T(F)} f(tg)\Omega^{-1}(t)\,dt, \qquad f \in \pi,\ g \in \mathrm{GL}_2(F). \tag{4-3}$$

Since $Z(F)\backslash T(F)$ is compact and $\Omega|_{F^\times} = \omega_\pi$, this integral is well defined and convergent. Note $Z(F)\backslash T(F)$ is isomorphic to $Z(\mathfrak{o})\backslash T(\mathfrak{o})$ if $\left(\frac{L}{\mathfrak{p}}\right) = -1$, and to $Z(\mathfrak{o})\backslash T(\mathfrak{o}) \sqcup \varpi_L(Z(\mathfrak{o})\backslash T(\mathfrak{o}))$ if $\left(\frac{L}{\mathfrak{p}}\right) = 0$.

Next we show that $\mathcal{A}$ is nonzero for all Ω and, assuming $c(\Omega) \geq 2c(\chi_1)$, obtain $g \in \mathrm{GL}_2(F)$ such that $(\mathcal{A}(f_0))(g) \neq 0$. First observe that we can write $\mathrm{GL}_2(F) = M_2(F)T(F)$, where $M_2(F) = \left\{\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right] : a, b \in F\right\} \cap \mathrm{GL}_2(F)$ is the mirabolic subgroup of $\mathrm{GL}_2(F)$ and $M_2(F) \cap T(F) = \{1\}$. Hence, the function $\hat{f}$ defined by

$$\hat{f}\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} t\right) = |a|^{1/2}\chi_1(a)\Omega(t), \quad \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in M_2(F),\ t \in T(F), \tag{4-4}$$

is a well-defined element of π and, for $t \in T(F)$, satisfies $\pi(t)\hat{f} = \Omega(t)\hat{f}$, which implies

$$\mathcal{A}(\hat{f}) \neq 0. \tag{4-5}$$

For the computation of $\mathcal{A}$ applied to the newvector f_0, we need to know when the argument tg of f_0 is in the support of f_0 for certain elements $g \in \mathrm{GL}_2(F)$. We obtain that information in the following lemma.

Lemma 4.1. *Let $t = t(x, y) \in T(F)$. For $s \in \mathbb{Z}$, we have the following decomposition of $t\left[\begin{smallmatrix} \varpi^s & \\ & 1 \end{smallmatrix}\right]$ as bk with $b \in B(F)$ and $k \in \mathrm{GL}_2(\mathfrak{o})$.*

(i) *If $x - \frac{1}{2}\boldsymbol{b}y \in \varpi^{-l}\mathfrak{o}^\times$, $l \geq 0$, $\boldsymbol{a}\varpi^{s+l}y \in \mathfrak{o}$, then*

$$t\begin{bmatrix} \varpi^s & \\ & 1 \end{bmatrix} = \begin{bmatrix} \det(t)\varpi^s/\left(x - \frac{1}{2}\boldsymbol{b}y\right) & \varpi^{-l}\boldsymbol{c}y/\left(x - \frac{1}{2}\boldsymbol{b}y\right) \\ 0 & \varpi^{-l} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}\varpi^{s+l}y & \varpi^l\left(x - \frac{1}{2}\boldsymbol{b}y\right) \end{bmatrix}.$$

(ii) *If* $x-\frac{1}{2}\boldsymbol{b}y\in\varpi^{-l}\mathfrak{o}^\times$, $l\ge 0$, $\boldsymbol{a}\varpi^{s+l}y\notin\mathfrak{o}$, *then*

$$t\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}=\begin{bmatrix}\det(t)/(\boldsymbol{a}y) & -\varpi^s\big(x+\frac{1}{2}\boldsymbol{b}y\big)\\ 0 & \boldsymbol{a}\varpi^s y\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & \big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\boldsymbol{a}\varpi^s y)\end{bmatrix}.$$

(iii) *If* $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$, $\big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\varpi^s\boldsymbol{a}y)\in\mathfrak{o}$, *then*

$$t\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}=\begin{bmatrix}\det(t)/(\boldsymbol{a}y) & -\varpi^s\big(x+\frac{1}{2}\boldsymbol{b}y\big)\\ 0 & \varpi^s\boldsymbol{a}y\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & \big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\varpi^s\boldsymbol{a}y)\end{bmatrix}.$$

(iv) *If* $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$, $\big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\varpi^s\boldsymbol{a}y)\notin\mathfrak{o}$, *then*

$$t\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}=\begin{bmatrix}\det(t)\varpi^s/\big(x-\frac{1}{2}\boldsymbol{b}y\big) & \boldsymbol{c}y\\ 0 & x-\frac{1}{2}\boldsymbol{b}y\end{bmatrix}\begin{bmatrix}1 & 0\\ -\boldsymbol{a}\varpi^s y/\big(x-\frac{1}{2}\boldsymbol{b}y\big) & 1\end{bmatrix}.$$

Proof. The lemma is obtained by direct computation. □

Proposition 4.2. *Let* $c(\Omega)\ge 2c(\chi_1)$ *and set* $s=c(\pi)-c(\Omega)-v(\boldsymbol{a})$. *Then*

$$(\mathcal{A}(f_0))\left(\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}\right)\neq 0.$$

Proof. Since $\Omega|_{F^\times}=\omega_\pi$ and $c(\Omega)\ge 2c(\chi_1)$, we have $c(\Omega)>0$. Let us first compute the part of the integral $(\mathcal{A}(f_0))\big(\big[\begin{smallmatrix}\varpi^s & \\ & 1\end{smallmatrix}\big]\big)$ over $Z(\mathfrak{o})\backslash T(\mathfrak{o})$. The argument of f_0 is given by $t\big[\begin{smallmatrix}\varpi^s & \\ & 1\end{smallmatrix}\big]$, where

$$t=\begin{bmatrix}x+\frac{1}{2}\boldsymbol{b}y & \boldsymbol{c}y\\ -\boldsymbol{a}y & x-\frac{1}{2}\boldsymbol{b}y\end{bmatrix}\in T(\mathfrak{o}),$$

i.e., y, $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{o}$ and $x^2-\frac{1}{4}y^2\boldsymbol{d}\in\mathfrak{o}^\times$. We write $t\big[\begin{smallmatrix}\varpi^s & \\ & 1\end{smallmatrix}\big]$ as bk with $b\in B(F)$ and $k\in\mathrm{GL}_2(\mathfrak{o})$ according to Lemma 4.1. Since $t\in T(\mathfrak{o})$, we must have $l=0$ in parts (i) and (ii) of Lemma 4.1, and $\boldsymbol{a}$, $y\in\mathfrak{o}^\times$ in parts (iii) and (iv) of Lemma 4.1. The support of f_0 is $B(F)\gamma_{c(\chi_2)}K_1(\mathfrak{p}^{c(\pi)})$ and an element $k\in\mathrm{GL}_2(\mathfrak{o})$ lies in the support if and only if the (2, 1) entry of k has (strictly positive) valuation $c(\chi_2)$ if $c(\chi_1)>0$ and $\ge c(\chi_2)$ if $c(\chi_1)=0$. Hence, the k obtained in parts (ii) and (iii) of Lemma 4.1 is never in the support of f_0. Since $s<c(\chi_2)-v(\boldsymbol{a})$, the k obtained in part (iv) of Lemma 4.1 is not in the support of f_0 as well. Hence, the only possibility is part (i) of Lemma 4.1. First suppose that $c(\chi_1)>0$. By successive change of variable

$x \to x + \frac{1}{2}\boldsymbol{b}y$ and $y \to xy$, we have

$$\int_{Z(\mathfrak{o})\backslash T(\mathfrak{o})} f_0\left(t\begin{bmatrix} \varpi^s & \\ & 1\end{bmatrix}\right)\Omega^{-1}(t)\,dt$$
$$= \int_{y\in\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}\mathfrak{o}^\times} f_0\left(\begin{bmatrix} \varpi^s(1+\boldsymbol{b}y+\boldsymbol{ac}y^2) & \boldsymbol{c}y \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^s & 1\end{bmatrix}\right) \times \Omega^{-1}(1+\boldsymbol{c}y\beta)\,dy$$
$$= q^{-s/2}\chi_1(\varpi^s)\int_{y\in\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}\mathfrak{o}^\times} \chi_1(1+\boldsymbol{b}y+\boldsymbol{ac}y^2)\,f_0\left(\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^s & 1\end{bmatrix}\right) \times \Omega^{-1}(1+\boldsymbol{c}y\beta)\,dy$$
$$= (1-q^{-1})q^{s/2-c(\chi_2)+v(\boldsymbol{a})}\chi_1(\varpi^s) \times \int_{\mathfrak{o}^\times}\chi_1(1+\boldsymbol{b}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y+\boldsymbol{ac}\varpi^{2(c(\chi_2)-s-v(\boldsymbol{a}))}y^2) \times f_0\left(\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^{c(\chi_2)-v(\boldsymbol{a})} & 1\end{bmatrix}\right)\Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y\beta)\,d^\times y. \quad (4\text{-}6)$$

We get the factor $(1-q^{-1})$ above by the normalization of measures. Now, we have

$$\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^{c(\chi_2)-v(\boldsymbol{a})} & 1\end{bmatrix} = \begin{bmatrix} -\varpi^{v(\boldsymbol{a})}/(\boldsymbol{a}y) & 0 \\ 0 & 1\end{bmatrix}\gamma_{c(\chi_2)}\begin{bmatrix} -\boldsymbol{a}y\varpi^{-v(\boldsymbol{a})} & \\ & 1\end{bmatrix}.$$

Hence the integral (4-6) is equal to

$$(1-q^{-1})q^{s/2-c(\chi_2)+v(\boldsymbol{a})}\chi_1(\varpi^s) \times \int_{\mathfrak{o}^\times}\chi_1(1+\boldsymbol{b}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y+\boldsymbol{ac}\varpi^{2(c(\chi_2)-s-v(\boldsymbol{a}))}y^2) \times \chi_1(-\varpi^{v(\boldsymbol{a})}/(\boldsymbol{a}y))\Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y\beta)\,d^\times y \quad (4\text{-}7)$$

Using $c(\chi_2)-s-v(\boldsymbol{a}) = c(\Omega)-c(\chi_1) \geq c(\chi_1)$, we get

$$(1-q^{-1})q^{(c(\chi_1)-c(\chi_2)-c(\Omega)+v(\boldsymbol{a}))/2}\chi_1(-\varpi^{c(\pi)-c(\Omega)}/\boldsymbol{a}) \times \int_{\mathfrak{o}^\times}\chi_1^{-1}(y)\Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\Omega)-c(\chi_1)}y\beta)\,d^\times y.$$

Since $c(\Omega) \geq 2c(\chi_1)$, the map $y \mapsto \Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\Omega)-c(\chi_1)}y\beta)$ is an additive character of $\mathfrak{o}$ of conductor $c(\chi_1)$. This character extends to a character $\widehat{\psi}$ of F with conductor $c(\chi_1)$.

Hence, using (3-4), we get

$$\int_{Z(\mathfrak{o})\backslash T(\mathfrak{o})} f_0\left(t\begin{bmatrix} \varpi^{c(\pi)-c(\Omega)-v(\boldsymbol{a})} & \\ & 1\end{bmatrix}\right)\Omega^{-1}(t)\,dt$$
$$= (1-q^{-1})q^{(-c(\chi_2)-c(\Omega)+v(\boldsymbol{a}))/2}\chi_1(-\varpi^{c(\chi_2)-c(\Omega)}/\boldsymbol{a})\varepsilon\left(\tfrac{1}{2},\chi_1,\widehat{\psi}\right). \quad (4\text{-}8)$$

If $c(\chi_1)=0$, the integral is much simpler. We get

$$\int_{Z(\mathfrak{o})\backslash T(\mathfrak{o})} f_0\left(t\begin{bmatrix} \varpi^{s} & \\ & 1\end{bmatrix}\right)\Omega^{-1}(t)\,dt$$
$$= \int_{y\in\mathfrak{p}^{c(\Omega)}} f_0\left(\begin{bmatrix} \varpi^{s}(1+\boldsymbol{b}y+\boldsymbol{a}\boldsymbol{c}y^2) & \boldsymbol{c}y \\ 0 & 1\end{bmatrix}\right)\Omega^{-1}(1+\boldsymbol{c}y\beta)\,dy$$
$$= \chi_1(\varpi^s)q^{-s/2-c(\Omega)}. \quad (4\text{-}9)$$

If L/F is a ramified field extension, then it is also necessary to integrate over $\varpi_L(Z(\mathfrak{o})\backslash T(\mathfrak{o}))$. Let

$$t = \begin{bmatrix} x+\frac{1}{2}\boldsymbol{b}y & \boldsymbol{c}y \\ -\boldsymbol{a}y & x-\frac{1}{2}\boldsymbol{b}y\end{bmatrix} \in \varpi_L T(\mathfrak{o}).$$

Hence, we have

$$x^2-\tfrac{1}{4}y^2\boldsymbol{d} = \left(x+\tfrac{1}{2}\boldsymbol{b}y\right)\left(x-\tfrac{1}{2}\boldsymbol{b}y\right)+\boldsymbol{a}\boldsymbol{c}y^2 \in \varpi\,\mathfrak{o}^\times.$$

We claim that, for $r<c(\chi_2)-v(\boldsymbol{a})$, the element $t\left[\begin{smallmatrix}\varpi^r & \\ & 1\end{smallmatrix}\right]$ is never in the support of f_0. We look at the four possibilities from Lemma 4.1. We know that the values of x, y satisfying the conditions of parts (ii) and (iii) never give elements in the support of f_0.

- Suppose $x-\frac{1}{2}\boldsymbol{b}y\in\varpi^{-l}\mathfrak{o}^\times$ with $l\geq 0$ and $\boldsymbol{a}y\varpi^{r+l}\in\mathfrak{o}$. To prove the claim, it is enough to show that $v(y)\leq -l$. Suppose $y\in\mathfrak{p}^{-l+1}$. Then we have $\varpi^l y\in\mathfrak{p}$. By assumption, we have $\varpi^l\left(x-\frac{1}{2}\boldsymbol{b}y\right)\in\mathfrak{o}^\times$. Hence $\varpi^l\left(x+\frac{1}{2}\boldsymbol{b}y\right)\in\mathfrak{o}^\times$. But then we get $\varpi^{2l}\left(x^2-\frac{1}{4}y^2\boldsymbol{d}\right)\in\mathfrak{o}^\times$, which is a contradiction.
- Suppose $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$, $\left(x-\frac{1}{2}\boldsymbol{b}y\right)/(\varpi^r\boldsymbol{a}y)\notin\mathfrak{o}$. To prove the claim, it is enough to show that $v(y)\leq 0$. Suppose $y\in\mathfrak{p}$. Then $x+\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$. But then we get $\left(x^2-\frac{1}{4}y^2\boldsymbol{d}\right)\in\mathfrak{p}^2$, a contradiction.

Hence, for $r<c(\chi_2)-v(\boldsymbol{a})$, we have

$$\int_{\varpi_L(Z(\mathfrak{o})\backslash T(\mathfrak{o}))} f_0\left(t\begin{bmatrix} \varpi^{r} & \\ & 1\end{bmatrix}\right)\Omega^{-1}(t)\,dt = 0. \quad (4\text{-}10)$$

This completes the proof of the proposition by observing that $s<c(\chi_2)-v(\boldsymbol{a})$. □

Observe that in the above proof, we have used $c(\Omega) \geq 2c(\chi_1)$ at two crucial steps to simplify the integral. In the case $c(\Omega) < 2c(\chi_1)$, it is not clear if the statement of the proposition still remains valid.

Proof of Theorem 1.7 for principal series representations. By the definition (4-3) of $\mathcal{A}$ and (4-5), the linear functional on π given by $\ell(f) = (\mathcal{A}(f))(1)$ is a nonzero functional satisfying $\ell(\pi(t)f) = \Omega(t)\ell(f)$ for all $t \in T(F)$ and $f \in \pi$. Hence, $\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0$. The 1-dimensionality follows from [Waldspurger 1985]. Since $c(\Omega) \geq c(\pi)$, we can apply Lemma 2.2 together with Proposition 4.2 to obtain the existence of the required test vector. The uniqueness follows from the uniqueness of f_0. □

4B. ***Steinberg representation.*** Let $\pi = \chi|\cdot|^{1/2} \times \chi|\cdot|^{-1/2}$. Let V_0 be the unique invariant (infinite-dimensional) subspace of π, so $\pi|_{V_0}$ is the twisted Steinberg representation $\chi\,\mathrm{St}_{\mathrm{GL}_2}$. If we set $\chi_1 = \chi|\cdot|^{1/2}$ and $\chi_2 = \chi|\cdot|^{-1/2}$, then V_0 is characterized as the kernel of the intertwining operator $M : \chi_1 \times \chi_2 \to \chi_2 \times \chi_1$, given by

$$(M(f))(g) = \int_F f\left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) dx.$$

Case 1: χ ramified. If χ is a ramified character, then f_0, defined as in (4-2), is in V_0 and is, in fact, the unique (up to a constant) newform in $\chi\,\mathrm{St}_{\mathrm{GL}_2}$ (see [Schmidt 2002]). Hence the proof of Proposition 4.2 is valid in this case without any modification.

Case 2: χ unramified. If χ is unramified, then the vector f_0, defined as in (4-2), is a spherical vector in $\chi_1 \times \chi_2$, hence clearly not the newform of $\chi\,\mathrm{St}_{\mathrm{GL}_2}$, which has conductor $\mathfrak{p}$. Any vector in $\chi\,\mathrm{St}_{\mathrm{GL}_2}$ which is right $K_1(\mathfrak{p})$-invariant is also right I-invariant, where I is the Iwahori subgroup defined in (2-4). It is known (see [Schmidt 2002]) that the newform in the induced model is given by

$$f_0(g) = \begin{cases} |a/d|\chi(ad)q & \text{if } g \in \begin{bmatrix} a & * \\ & d \end{bmatrix} I, \\ -|a/d|\chi(ad) & \text{if } g \in \begin{bmatrix} a & * \\ & d \end{bmatrix} wI. \end{cases} \tag{4-11}$$

We can try to compute $(\mathcal{A}(f_0))(g)$ (defined in (4-3)) in this case for various values of g. But instead, we use a double coset decomposition and properties of the Steinberg representation to obtain the test vector. This has the added advantage of obtaining a new proof of the uniqueness (up to a constant) of the Waldspurger functional, and also gives us the explicit formula for $B(g)$, where B is the newform in the corresponding Waldspurger model and g is any element of $\mathrm{GL}_2(F)$. By

[Sugano 1985, Lemma 2-4], there is the disjoint double coset decomposition

$$\mathrm{GL}_2(F) = \bigsqcup_{r=0}^{\infty} T(F) \begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} \mathrm{GL}_2(\mathfrak{o}). \tag{4-12}$$

Note that, by the Iwasawa decomposition of $\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p})$, we have

$$\mathrm{GL}_2(\mathfrak{o}) = wI \sqcup \bigsqcup_{u \in \mathfrak{o}/\mathfrak{p}} \begin{bmatrix} 1 & \\ u & 1 \end{bmatrix} I, \qquad w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}. \tag{4-13}$$

For $u \in \mathfrak{o}$ and $r \geq 0$ set $\beta_{u,r} := \boldsymbol{a}\varpi^{2r} + \boldsymbol{b}\varpi^r u + \boldsymbol{c}u^2$. Arguing as in Lemma 3.1 of [Pitale 2011], we have

$$T(F) \begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} wI = T(F) \begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ u & 1 \end{bmatrix} I \iff \beta_{u,r} \in \mathfrak{o}^\times. \tag{4-14}$$

Lemma 3.2 of [Pitale 2011] tells us exactly when $\beta_{u,r} \in \mathfrak{o}^\times$. Putting everything together, we get the following proposition.

Proposition 4.3. *For $r > 0$, we have*

$$T(F) \begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} \mathrm{GL}_2(\mathfrak{o}) = T(F) \begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} I \sqcup T(F) \begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} wI.$$

For $r = 0$, $\left(\frac{L}{\mathfrak{p}}\right) = -1$, we have

$$T(F)\mathrm{GL}_2(\mathfrak{o}) = T(F)I = T(F)wI.$$

For $r = 0$, $\left(\frac{L}{\mathfrak{p}}\right) = 0$, we have

$$T(F)\mathrm{GL}_2(\mathfrak{o}) = T(F)wI \sqcup T(F) \begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} I,$$

where u_0 is chosen as in (2-14).

The twisted Steinberg representation is characterized as the representation π with a newform v_0 which is invariant under I and satisfies the two conditions

$$\sum_{\gamma \in \mathrm{GL}_2(\mathfrak{o})/I} \pi(\gamma) v_0 = 0, \qquad \pi\left(\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}\right) v_0 = -\chi(\varpi) v_0.$$

These conditions follow from the action of the Atkin–Lehner element and the fact that π does not have a vector invariant under $\mathrm{GL}_2(\mathfrak{o})$. See Proposition 3.1.2 of [Schmidt 2002]. Let Ω be a character of $L^\times$ with $\Omega|_{F^\times} = \omega_\pi$.

Let $B : \mathrm{GL}_2(F) \to \mathbb{C}$ be a function that satisfies $B(tgk) = \Omega(t)B(g)$ for all $t \in T(F)$, $g \in \mathrm{GL}_2(F)$, $k \in I$ and

$$\sum_{u \in \mathfrak{o}/\mathfrak{p}} B\left(g\begin{bmatrix} 1 & \\ u & 1 \end{bmatrix}\right) = -B(gw), \tag{4-15}$$

$$B\left(g\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}\right) = -\chi(\varpi)B(g) \tag{4-16}$$

for all $g \in \mathrm{GL}_2(F)$. If π has a Ω-Waldspurger model, then B will precisely be the unique (up to scalars) newform of π in the Ω-Waldspurger model; otherwise B will be 0.

Lemma 4.4. (i) *If* $c(\Omega) \geq 2$, *then*

$$B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right) = 0 \quad \text{if } r \leq c(\Omega) - 2. \tag{4-17}$$

(ii) *For* $r > 0$, *we have*

$$B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}\right) = \begin{cases} -qB\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right) & \text{if } r \geq c(\Omega), \\ 0 & \text{if } r < c(\Omega). \end{cases} \tag{4-18}$$

(iii) *For* $r \geq \max\{c(\Omega) - 1, 0\}$, *we have*

$$B\left(\begin{bmatrix} \varpi^{r+1} & \\ & 1 \end{bmatrix} w\right) = \frac{\chi(\varpi)}{q} B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right). \tag{4-19}$$

(iv) *If* $\left(\frac{L}{\mathfrak{p}}\right) = 0$, *then*

$$B\left(\begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix}\right) = \begin{cases} -qB(w) & \text{if } c(\Omega) = 0, \\ 0 & \text{if } c(\Omega) > 0. \end{cases} \tag{4-20}$$

(v) *If* $c(\Omega) = 0$ *and* $\Omega = \chi \circ N_{L/F}$, *then*

$$B(w) = 0. \tag{4-21}$$

Proof. We illustrate the proof of (i) and (ii) in detail here. Let $u, v \in \mathfrak{p}^{c(\Omega)-1}$ be such that $\Omega(1 + u + v\beta) \neq 1$. Take $y = v/\boldsymbol{c}$, $x = 1 + u + \frac{1}{2}\boldsymbol{b}y$ and, for $r \leq c(\Omega) - 2$, let

$$k = \begin{bmatrix} 1+u & \boldsymbol{a}/\boldsymbol{c}v\varpi^r \\ -\varpi^{-r}v & 1+u+\boldsymbol{b}/\boldsymbol{c}v \end{bmatrix} \in I.$$

Then

$$B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right) = B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} wk\right) = B\left(t(x, y)\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right)$$
$$= \Omega(1 + u + v\beta) B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right).$$

This gives us (4-17) and completes the proof of (i).

Next, we give the proof of (ii). Substitute $g = \left[\begin{smallmatrix} \varpi^r & \\ & 1 \end{smallmatrix}\right]$ in (4-15) to get

$$\sum_{u \in \mathfrak{o}/\mathfrak{p}} B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ u & 1 \end{bmatrix}\right) = -B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right).$$

For $u \neq 0$, setting $x = \boldsymbol{b}/2\varpi^r + \boldsymbol{c}u$, $y = \varpi^r$, we get

$$\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ u & 1 \end{bmatrix}\begin{bmatrix} -\boldsymbol{c} & \boldsymbol{b}\varpi^r + \boldsymbol{c}u \\ & -\beta_{u,r} \end{bmatrix} = t(x, y)\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w.$$

Since $r > 0$ by assumption, $\beta_{u,r} \in \mathfrak{o}^\times$. Hence, for $u \neq 0$ we have

$$B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ u & 1 \end{bmatrix}\right) = \Omega(u + \varpi^r \beta) B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right).$$

This gives us

$$B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}\right) = -\left(\sum_{u \in (\mathfrak{o}/\mathfrak{p})^\times} \Omega(u + \varpi^r \beta) + \Omega(1)\right) B\left(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix} w\right).$$

Using (4-17) and the definition of $c(\Omega)$ we get the result for $r \geq c(\Omega)$ or $r \leq c(\Omega) - 2$. For $r = c(\Omega) - 1$, using Lemma 3.4 of [Pitale 2011], we see that the expression in the parentheses on the right-hand side above is 0. This completes the proof of (ii).

Using (4-15), (4-16) and similar calculations as above, we get the remaining results. □

Proof of Theorem 1.7 for twists of Steinberg representations. Let D be the quaternion division algebra over F and $N_{D/F}$ be the reduced norm. Since $\pi = \chi\,\mathrm{St}$ corresponds to the 1-dimensional representation $\pi' = \chi \circ N_{D/F}$ of $D^\times(F)$, one knows by [Waldspurger 1985] that π has an Ω-Waldspurger model if and only if $\Omega \neq \chi \circ N_{L/F}$. Since $c(\Omega) \geq c(\pi)$, this must be the case, i.e., $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$. The case of ramified χ follows exactly as in the principal series case. For χ unramified, the result follows from Lemma 4.4. □

5. Supercuspidal representations

Throughout this section we continue to assume that L/F is a field.

5A. ***Chain orders and strata.*** This section contains a summary of the facts about chain orders and fundamental strata that we will use to construct test vectors for the supercuspidal representations π of $\mathrm{GL}_2(F)$, all of which can be found in [Bushnell and Henniart 2006, Chapter 4].

Let $\mathfrak{A}$ be a chain order in $\mathrm{M}_2(F)$. Up to $\mathrm{GL}_2(F)$-conjugacy, $\mathfrak{A}$ must be either $\mathfrak{M} = \mathrm{M}_2(\mathfrak{o})$ or $\mathfrak{J} = \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{smallmatrix}\right]$, so we always take $\mathfrak{A}$ to be $\mathfrak{M}$ or $\mathfrak{J}$.

Write $e_{\mathfrak{A}} = 1$ if $\mathfrak{A} = \mathfrak{M}$ and $e_{\mathfrak{A}} = 2$ if $\mathfrak{A} = \mathfrak{J}$. For more intrinsic definitions, see [Bushnell and Henniart 2006]. Let $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$, the Jacobson radical of $\mathfrak{A}$. There is an element $\Pi \in \mathrm{GL}_2(F)$ such that $\mathfrak{P} = \Pi\mathfrak{A}$, and one has

$$\operatorname{rad} \mathfrak{M} = \varpi \mathfrak{M}, \qquad \operatorname{rad} \mathfrak{J} = \begin{bmatrix} & 1 \\ \varpi & \end{bmatrix} \mathfrak{J}.$$

Let $\mathfrak{P}^n = \Pi^n \mathfrak{A}$ for $n \in \mathbb{Z}$. Let $U_{\mathfrak{A}}^0 = U_{\mathfrak{A}} := \mathfrak{A}^\times$, $U_{\mathfrak{A}}^n := 1 + \mathfrak{P}^n$ for $n \geq 1$, and $K_{\mathfrak{A}} = \{g \in \mathrm{GL}_2(F) : g\mathfrak{A}g^{-1} = \mathfrak{A}\}$. Then

$$K_{\mathfrak{A}} = \begin{cases} Z(F)\mathrm{GL}_2(\mathfrak{o}) & \text{if } \mathfrak{A} = \mathfrak{M}, \\ \left\langle \begin{bmatrix} & 1 \\ \varpi & \end{bmatrix} \right\rangle \ltimes \mathfrak{J}^\times & \text{if } \mathfrak{A} = \mathfrak{J}. \end{cases}$$

We fix a character $\psi_1 : F \to \mathbb{C}^\times$ so that the conductor of ψ_1 is $\mathfrak{p}$. For $\alpha \in \mathrm{M}_2(F)$, define a function of $U_{\mathfrak{A}}$ by $\psi_\alpha(x) = \psi_1(\operatorname{Tr} \alpha(x-1))$. Then for $1 \leq m \leq n \leq 2m$, there is an isomorphism

$$\begin{aligned} \mathfrak{P}^{-n}/\mathfrak{P}^{-m} &\to \left(U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1}\right)^\wedge, \\ \alpha + \mathfrak{P}^{-m} &\mapsto \psi_\alpha. \end{aligned}$$

The normalized level of π is defined to be

$$\ell(\pi) = \min\{n/e_{\mathfrak{A}} : \pi|_{U_{\mathfrak{A}}^{n+1}} \text{ contains the trivial character}\}.$$

A stratum in $\mathrm{M}_2(F)$ is a triple $(\mathfrak{A}, n, \alpha)$ where $\mathfrak{A}$ is a chain order in $\mathrm{M}_2(F)$ with radical $\mathfrak{P}$, n is an integer and $\alpha \in \mathfrak{P}^{-n}$. For $n \geq 1$ one associates to a stratum the character ψ_α of $U_{\mathfrak{A}}^n$ which is trivial on $U_{\mathfrak{A}}^{n+1}$.

We say that a smooth representation π contains the stratum $(\mathfrak{A}, n, \alpha)$ if $\pi|_{U_{\mathfrak{A}}^n}$ contains ψ_α. A fundamental stratum is one such that $\alpha + \mathfrak{P}^{1-n}$ contains no nilpotent elements. If an irreducible smooth representation π of $\mathrm{GL}_2(F)$ contains a stratum $(\mathfrak{A}, n, \alpha)$, then $(\mathfrak{A}, n, \alpha)$ is fundamental if and only if $\ell(\pi) = n/e_{\mathfrak{A}}$ [Bushnell and Henniart 2006, 12.9 Theorem].

Suppose that $(\mathfrak{A}, n, \alpha)$ is a fundamental stratum with $e_{\mathfrak{A}} = 1$. Write $\alpha = \varpi^{-n}\alpha_0$ for $\alpha_0 \in \mathfrak{A}$. Let $f_\alpha(t) \in \mathfrak{o}[t]$ be the characteristic polynomial of α_0, and let $\tilde{f}_\alpha \in \boldsymbol{k}[t]$ be its reduction modulo $\mathfrak{p}$. Here $\boldsymbol{k}$ is the residue class field. If $\tilde{f}_\alpha$ has two solutions in $\boldsymbol{k}$, then $(\mathfrak{A}, n, \alpha)$ is said to be a split fundamental stratum. If $\tilde{f}_\alpha$ is irreducible, then the stratum $(\mathfrak{A}, n, \alpha)$ is said to be unramified simple. On the other hand, if $(\mathfrak{A}, n, \alpha)$ is a fundamental stratum with $e_{\mathfrak{A}} = 2$, and n odd, then $(\mathfrak{A}, n, \alpha)$ is said to be ramified simple. A simple stratum is either a simple unramified stratum or a simple ramified stratum. Suppose that $(\mathfrak{A}, n, \alpha)$ is a simple stratum with α_0 as above and let $E = F[\alpha_0]$. Bushnell and Henniart define what it means for α to be minimal [Bushnell and Henniart 2006, 13.4 Definition], and when this is the case $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$ [Bushnell and Henniart 2006, 13.4 Lemma]. If $(\mathfrak{A}, n, \alpha)$ is a simple stratum with $\mathfrak{A} = \mathfrak{M}$, then $\alpha_0 \in \mathfrak{M}$ but $\alpha_0 \notin \mathfrak{P}$.

Define π to be *minimal* if, for all characters χ of $F^\times$, $\ell(\pi \otimes \chi) \geq \ell(\pi)$. Every irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ is either minimal, or isomorphic to the twist of a minimal irreducible supercuspidal representation. Every minimal irreducible smooth representation of $\mathrm{GL}_2(F)$ contains exactly one of the following: a ramified simple stratum, an unramified simple stratum, or a split fundamental stratum [Bushnell and Henniart 2006, 13.3 Corollary]. If π contains a split fundamental stratum, then π is not supercuspidal.

5B. *Construction of minimal supercuspidals.* In this section we review the construction of minimal irreducible supercuspidal representations. See [Bushnell and Henniart 2006, Section 19] for more details. In each case we describe a distinguished vector v_0 in the inducing representation. This vector v_0 will be used to construct a test vector for π.

We remark that if a representation π contains a simple stratum $(\mathfrak{A}, n, \alpha)$, then it contains all $\mathrm{GL}_2(F)$-conjugates of $(\mathfrak{A}, n, \alpha)$. Therefore, we may always take $\mathfrak{A}$ to be either $\mathfrak{M}$ or $\mathfrak{J}$. Since $K_{\mathfrak{A}}$ normalizes $U_{\mathfrak{A}}$, we may also consider α up to $K_{\mathfrak{A}}$-conjugacy.

For the rest of Section 5 we assume that all supercuspidal representations are irreducible.

5B1. $\mathfrak{A} = \mathfrak{M}$, $\ell(\pi) = 2r + 1$. Suppose that π is a minimal supercuspidal representation containing the simple stratum given by $(\mathfrak{M}, 2r + 1, \alpha)$. Then $E = F[\alpha]$ is an unramified quadratic extension of F, and $\pi \cong c\text{-}\mathrm{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)} \lambda$, where $J_\alpha = E^\times U_{\mathfrak{M}}^{r+1}$ and λ is a character.

We have that $\lambda|_{U_{\mathfrak{M}}^{r+1}} = \psi_\alpha$ with $\alpha \in \mathfrak{P}_{\mathfrak{M}}^{-2r-1}$ and α is minimal. One may take $\alpha_0 = \varpi^{2r+1}\alpha$ to be in rational canonical form, i.e.,

$$\alpha_0 = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix}, \tag{5-1}$$

for $a_i \in \mathfrak{o}$, $i = 0, 1$. Then

$$1 + \begin{bmatrix} \mathfrak{p}^{r+1} & \mathfrak{p}^{2r+2} \\ \mathfrak{p}^{2r+2} & \mathfrak{p}^{2r+2} \end{bmatrix} \subseteq \ker \psi_\alpha.$$

5B2. $\mathfrak{A}=\mathfrak{J}$, $\ell(\pi)=\frac{1}{2}(2r+1)$. Suppose that π is a minimal supercuspidal representation containing the simple stratum $(\mathfrak{J}, 2r+1, \alpha)$, and let $E=F[\alpha]$. In this case E/F is a ramified extension, and $\ell(\pi)e(E/F)=2r+1$. Then $\pi=c\text{-Ind}_{J_\alpha}^{\mathrm{GL}_2(F)}\lambda$, where $J_\alpha=E^\times U_{\mathfrak{J}}^{r+1}$ and λ is a character. Observe

$$U_{\mathfrak{J}}^{r+1}=1+\mathfrak{P}^{r+1}=\begin{cases}1+\begin{bmatrix}\mathfrak{p}^{r/2+1} & \mathfrak{p}^{r/2}\\ \mathfrak{p}^{r/2+1} & \mathfrak{p}^{r/2+1}\end{bmatrix} & \text{if } r \text{ is even,}\\ 1+\begin{bmatrix}\mathfrak{p}^{(r+1)/2} & \mathfrak{p}^{(r+1)/2}\\ \mathfrak{p}^{(r+3)/2} & \mathfrak{p}^{(r+1)/2}\end{bmatrix} & \text{if } r \text{ is odd.}\end{cases}$$

Note that

$$U_{\mathfrak{J}}^{2r+2}=1+\begin{bmatrix}\mathfrak{p}^{r+1} & \mathfrak{p}^{r+1}\\ \mathfrak{p}^{r+2} & \mathfrak{p}^{r+1}\end{bmatrix}\subseteq\ker\lambda.$$

Let $\alpha_0=\varpi^{r+1}\alpha$ be of the form (5-1), where now $a_0\in\varpi\mathfrak{o}^\times$ and $a_1\in\mathfrak{p}$, and $k:=\left[\frac{1}{2}r\right]+1$. Then

$$1+\begin{bmatrix}\mathfrak{p}^{k} & \mathfrak{p}^{r+1}\\ \mathfrak{p}^{r+2} & \mathfrak{p}^{r+1}\end{bmatrix}\subseteq\ker\lambda.$$

5B3. $\mathfrak{A}=\mathfrak{M}$, $\ell(\pi)=2r>0$. Now suppose π contains an unramified simple stratum $(\mathfrak{M}, 2r, \alpha)$ for some $\alpha\in\mathfrak{P}^{-2r}$ so that $\ell(\pi)=2r>0$ and $e(E/F)=1$, where as before $E=F[\alpha]$. Continue to assume that $\alpha_0=\varpi^{2r}\alpha$ has the form (5-1). In this case, π is not induced from a character, and E is an unramified quadratic extension of F. We describe a representation ρ of $J_\alpha=E^\times U_{\mathfrak{M}}^r$ so that π is compactly induced from ρ, and we follow Kutzko [1978, §1] since his construction is more convenient for our applications.

Write $U_E^1=U_{\mathfrak{M}}^1\cap E^\times$. Since $\alpha_0\in\mathfrak{M}\setminus\mathfrak{P}$ and $\mathfrak{o}_E=\mathfrak{o}[\alpha_0]$ (see Section 5A), a simple argument shows $U_E^1\subset 1+\mathfrak{p}_E$. The opposite inclusion is obvious; therefore, $U_E^1=1+\mathfrak{p}_E$. We similarly note that $E^\times\cap U_{\mathfrak{M}}=:U_E\cong\mathfrak{o}_E^\times$. There is a character χ of $E^\times$ such that $\chi(1+x)=\psi_1\circ\mathrm{Tr}_{E/F}(\alpha x)$ for all $x\in\mathfrak{p}_E^{r+1}$. Define a character

$$\lambda: H_\alpha^1:=U_E^1U_{\mathfrak{M}}^{r+1}\to\mathbb{C}^\times$$

by $\lambda(ux)=\chi(u)\psi_\alpha(x)$ for $u\in U_E^1$ and $x\in U_{\mathfrak{M}}^{r+1}$.

Let $A=\left\{\left[\begin{smallmatrix}x & \\ & 1\end{smallmatrix}\right]:x\in F^\times\right\}$, and set $A^n=A\cap U_{\mathfrak{M}}^n$ for $n\geq 0$. The character λ can be extended to a character $\tilde{\lambda}$ of $A^rH_\alpha^1$ by $\tilde{\lambda}(yx)=\lambda(x)$ for $y\in A^rU_{\mathfrak{M}}^{2r+1}$ and $x\in H_\alpha^1$ [Kutzko 1978, Definition 1.8].

Let $J_\alpha^1=U_E^1U_{\mathfrak{M}}^r$, and define $\eta=\mathrm{Ind}_{A^rH_\alpha^1}^{J_\alpha^1}\tilde{\lambda}$. Then η is an irreducible representation of J_α^1 of dimension q. There is an irreducible representation ρ of J_α such that $\pi\cong\mathrm{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)}\rho$, and $\rho|_{J_\alpha^1}\cong\eta$ [Kutzko 1978, Lemma 1.10 and Proposition 1.15]. Note that $U_{\mathfrak{M}}^{2r+1}\subset\ker\rho$. We must compute $\rho|_{A^r}\cong\eta|_{A^r}$. We have $[J_\alpha^1:A^rH_\alpha^1]=q$, and an irredundant set of coset representatives is given by $\left\{\left[\begin{smallmatrix}1 & \\ a & 1\end{smallmatrix}\right]:a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}\right\}$.

It is a simple computation to show that $\eta|_{A^r} = \operatorname{Ind}_{A^r H_\alpha^1}^{J_\alpha^1} \tilde{\lambda}|_{A^r}$ is isomorphic to the regular representation of A^r/A^{r+1}. In particular it contains the trivial character with multiplicity one, so there is a vector $v_0 \in \rho$ that is unique up to scalars which is fixed by $1 + \left[\begin{smallmatrix} \mathfrak{p}^r & \mathfrak{p}^{2r+1} \\ \mathfrak{p}^{2r+1} & \mathfrak{p}^{2r+1} \end{smallmatrix}\right]$.

Sometimes it will be convenient to consider the corresponding vector $f_0 \in \eta$ given by

$$f_0(k) = \begin{cases} \tilde{\lambda}(k) & \text{if } k \in A^r H_\alpha^1, \\ 0 & \text{otherwise.} \end{cases} \tag{5-2}$$

5B4. *Depth zero supercuspidals.* Now, consider a depth zero supercuspidal representation, i.e., $\ell(\pi) = 0$. Then π is induced from a representation ρ of $K_{\mathfrak{M}}$ that is inflated from a cuspidal representation $\tilde{\rho}$ of $\mathrm{GL}_2(\mathfrak{o}/\mathfrak{p})$, i.e., ρ is trivial on $U_{\mathfrak{M}}^1 = 1 + \mathfrak{p}\mathrm{M}_2(\mathfrak{o})$ and it factors through $\tilde{\rho}$. The cuspidal representations $\tilde{\rho}$ are parameterized by Galois conjugacy classes of regular characters $\theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^1$. Such a character θ can also be regarded as a character of $\mathfrak{o}_E^\times$ that is trivial on $1 + \mathfrak{p}_E$, where E/F is the unique unramified quadratic extension. Embed $\mathfrak{o}_E^\times$ in $\mathrm{GL}_2(\mathfrak{o})$, and identify $\mathbb{F}_{q^2}^\times$ with the image of $\mathfrak{o}_E^\times$ under the reduction map modulo $\mathfrak{p}$. The following proposition gives the character table for $\tilde{\rho}$, which is a well-known result (see, e.g., [Bushnell and Henniart 2006, 6.4.1]).

Proposition 5.1. *The character table of* $\tilde{\rho}$ *is given by*

$$\begin{aligned} \operatorname{Tr} \tilde{\rho}(z) &= (q-1)\theta(z), & & z \in Z; \\ \operatorname{Tr} \tilde{\rho}(zu) &= -\theta(z), & & z \in Z,\ u \in N,\ u \neq 1; \\ \operatorname{Tr} \tilde{\rho}(y) &= -(\theta(y) + \theta^q(y)), & & y \in \mathbb{F}_{q^2}^\times \setminus Z. \end{aligned}$$

If g *is not conjugate to an element of* $\mathbb{F}_{q^2}^\times \cup ZN$, *then* $\operatorname{Tr} \tilde{\rho}(g) = 0$.

From the character table one sees that the restriction of ρ to A^0 is isomorphic to the regular representation of A^0/A^1. In particular there is a vector $v_0 \in \rho$ such that $\rho(a)v_0 = v_0$ for $a \in A^0$.

5C. ***Remarks on minimal supercuspidals.*** We consider a minimal supercuspidal representation $\pi = c\text{-}\operatorname{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)} \rho$, where $E = F[\alpha]$ for $\alpha \in \mathfrak{P}^{-n}$ such that π contains the simple stratum $(\mathfrak{A}, n, \alpha)$. When $e(E/F)\ell(\pi)$ is odd, then $\rho = \lambda$ is a character which restricts to ψ_α. When $e(E/F)\ell(\pi)$ is even, then ρ is not a character, and if additionally $\ell(\pi) > 0$, then we sometimes identify $\rho|_{J_\alpha^1}$ with η as described above.

From the discussion in the previous section, we may always take $J = J_\alpha$ of the form $E^\times\big(1 + \mathfrak{P}^{[(e_{\mathfrak{A}}\ell(\pi)+1)/2]}\big)^\times$, where we take $E = F$ if $\ell(\pi) = 0$. In all cases, we have $E^\times \subset K_{\mathfrak{A}}$, so $J \subset K_{\mathfrak{A}}$.

Definition 5.2. Suppose $\pi = c\text{-}\operatorname{Ind}_J^{\mathrm{GL}_2(F)} \rho$ is an irreducible minimal supercuspidal representation.

(i) If $\rho=\lambda$ is a character, define v_0 to be the unique vector up to scalar multiples in ρ. That is, $\rho(k)v_0=\lambda(k)$ for $k\in J$. Then according to the constructions in Sections 5B1 and 5B2, $\rho(a)v_0=\lambda(a)=1$ for $a\in A\cap J$.

(ii) Suppose $\dim_{\mathbb{C}}\rho>1$. Define $v_0\in\rho$ to be the vector described in Sections 5B3 and 5B4 such that $\rho(a)v_0=v_0$ for $a\in A\cap J$.

Let $N=\left\{\left[\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right]\right\}\subset \mathrm{GL}_2(F)$, $\bar{N}=\left\{\left[\begin{smallmatrix}1&\\x&1\end{smallmatrix}\right]\right\}\subset \mathrm{GL}_2(F)$ and $\bar{N}^r=\bar{N}\cap U^r_{\mathfrak{M}}$ for $r\geq 0$.

Lemma 5.3. *Suppose that π is a minimal supercuspidal representation and that $\ell(\pi)=2r$. Write $\pi=c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$, where ρ is not a character.*

(i) *If $\ell(\pi)=0$, then $\rho|_{\bar{N}^0}=\bigoplus_{i=1}^{q-1}\psi_i$, where ψ_i runs over all the nontrivial characters of $\bar{N}^0/\bar{N}^1$.*

(ii) *If $\ell(\pi)>0$ and $J=J_\alpha$, then $\rho|_{\bar{N}\cap J_\alpha}=\bigoplus_j\psi_j$, where the sum runs over all characters ψ_j of $\bar{N}\cap J_\alpha$ such that $\psi_j|_{\bar{N}\cap H_\alpha}=\psi_\alpha|_{\bar{N}\cap H_\alpha}$, and $H_\alpha:=E^\times U^{r+1}_{\mathfrak{M}}$.*

Proof. The first part may be deduced from Proposition 5.1. Now, suppose that $\ell(\pi)>0$. Since $J_\alpha=E^\times U^r_{\mathfrak{M}}$, we have $\bar{N}\cap J_\alpha=\bar{N}^r$, and similarly $\bar{N}\cap H_\alpha=\bar{N}^{r+1}$. Also, recall that $\rho|_{J^1_\alpha}\cong\eta=\mathrm{Ind}_{A^rH^1_\alpha}^{J^1_\alpha}\tilde{\lambda}$, where $\tilde{\lambda}$ is obtained from λ by extending it trivially to A^r.

A set of irredundant coset representatives for $A^rH^1_\alpha\backslash J^1_\alpha$ is given by

$$\left\{\begin{bmatrix}1&\\a&1\end{bmatrix}:a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}\right\}.$$

Let ψ' be one of the characters ψ_j of $\bar{N}^r$ as in (ii). For $a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}$, define

$$f_a(y)=\begin{cases}\psi'\left(\begin{bmatrix}1&\\a&1\end{bmatrix}\right)\tilde{\lambda}(x) & \text{if } y=x\begin{bmatrix}1&\\a&1\end{bmatrix},\ x\in A^rH^1_\alpha,\\ 0 & \text{otherwise.}\end{cases}\tag{5-3}$$

This is well defined since ψ' and $\tilde{\lambda}$ agree on $\bar{N}^{r+1}$. Note the f_a span η, and when $a=0$, (5-3) agrees with (5-2). Define $f_{\psi'}=\sum_{a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}}f_a$. Then we have

$$\eta\left(\begin{bmatrix}1&\\x&1\end{bmatrix}\right)f_{\psi'}=\psi'(x)f_{\psi'}.$$

From the explicit basis we computed for η, one sees that each of these characters appear with multiplicity one. This proves the lemma. □

Later, it will be useful to have a case-by-case description of the kernel of ρ. We summarize what we know about the kernel from the previous section in Table 2. The quantities in the latter two columns will be denoted i and i' in Propositions 5.5 and 5.9, and are included here for the later convenience of the reader.

$\ell(\pi)$	J	$\subset \ker\rho$	$\left[\frac{\ell(\pi)+3/2}{2}\right]$	$\left[\ell(\pi)+\frac{3}{2}\right]-\left[\frac{\ell(\pi)}{2}\right]-1$
$2r+1$	$E^\times(1+\mathfrak{P}^{r+1})$	$1+\begin{bmatrix}\mathfrak{p}^{r+1} & \mathfrak{p}^{2r+2}\\ \mathfrak{p}^{2r+2} & \mathfrak{p}^{2r+2}\end{bmatrix}$	$r+1$	$r+1$
$2r>0$	$E^\times(1+\mathfrak{P}^{r})$	$1+\begin{bmatrix}\mathfrak{p}^{r+1} & \mathfrak{p}^{2r+1}\\ \mathfrak{p}^{2r+1} & \mathfrak{p}^{2r+1}\end{bmatrix}$	r	r
0	$Z(F)\mathrm{GL}_2(\mathfrak{o})$	$1+\mathfrak{P}$	0	0
$\frac{2r+1}{2}$	$E^\times(1+\mathfrak{P}^{r+1})$	$1+\begin{bmatrix}\mathfrak{p}^{[r/2]+1} & \mathfrak{p}^{r+1}\\ \mathfrak{p}^{r+2} & \mathfrak{p}^{r+1}\end{bmatrix}$	$\left[\frac{r}{2}\right]+1$	$r-\left[\frac{r}{2}\right]+1$

Table 2. Data for minimal supercuspidal representations.

5D. ***Mackey theory.*** In this section we describe the strategy to obtain the desired test vector for π. Consider a minimal supercuspidal representation π of $\mathrm{GL}_2(F)$. There is an open subgroup J of $\mathrm{GL}_2(F)$ that contains the center $Z(F)$, is compact modulo $Z(F)$ and has an irreducible representation ρ of J with $\pi \cong c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$. As before, let $\Omega : T(F) \to \mathbb{C}^\times$ be a character such that $\Omega|_{Z(F)} = \omega_\pi$.

Consider the space

$$\mathrm{Hom}_{T(F)}(\pi, \Omega) \cong \mathrm{Hom}_{\mathrm{GL}_2(F)}\big(c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho,\ \mathcal{B}(\Omega)\big). \tag{5-4}$$

See Section 2D for the definition of $\mathcal{B}(\Omega)$ and details of the above isomorphism. Following the proof of Proposition 1.6 of [Bushnell and Henniart 1998] and [Kutzko 1977], define $\mathscr{H}(\mathrm{GL}_2(F),\ \rho,\ \Omega)$ to be the space of functions

$$f : \mathrm{GL}_2(F) \to \mathrm{Hom}_{\mathbb{C}}(\rho, \mathbb{C})$$

satisfying

$$f(tgk) = \Omega(t) f(g) \circ \rho(k), \quad t \in T(F),\ g \in \mathrm{GL}_2(F),\ k \in J.$$

Then for $\varphi \in c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$ and $f \in \mathscr{H}(\mathrm{GL}_2(F),\ \rho,\ \Omega)$, the convolution $f * \varphi$ defined as

$$f*\varphi(g) = \int_{\mathrm{GL}_2(F)/Z(F)} f(x)\varphi(x^{-1}g)\,d\bar{x}, \quad g \in \mathrm{GL}_2(F),$$

gives a function in the space $\mathcal{B}(\Omega)$. Furthermore, $\mathrm{GL}_2(F)$ acts on $\mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$ through the convolution by $(g\cdot f)*\varphi = f*(g\cdot\varphi)$, and there is a $\mathrm{GL}_2(F)$ homomorphism

$$\mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}\big(c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho,\ \mathcal{B}(\Omega)\big),$$
$$f \mapsto (\varphi \mapsto f*\varphi).$$

This is in fact an isomorphism. By [Waldspurger 1985],

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathrm{GL}_2(F)}\bigl(c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho,\ \mathcal{B}(\Omega)\bigr) \le 1.$$

Hence, there is at most one double coset $T(F)h_0J$ which has nontrivial intersection with the support of any $f \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$, and each f is uniquely determined by its value at h_0 (see 1.8 of [Bushnell and Henniart 1998]). Suppose that $f \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$ has support in a double coset $T(F)h_0J$, and that $f(h_0) = \ell_0 \in \operatorname{Hom}(\rho, \mathbb{C})$. For $k \in J \cap h_0^{-1}T(F)h_0$, define $\Omega^{h_0}(k) = \Omega(h_0kh_0^{-1})$. Then ℓ_0 has the property that for $k \in J \cap h_0^{-1}T(F)h_0$,

$$\ell_0(\rho(k)v) = \Omega^{h_0}(k)\,\ell_0(v).$$

Therefore,

$$\ell_0 \in \operatorname{Hom}_{J\cap h_0^{-1}T(F)h_0}(\rho,\ \Omega^{h_0}). \tag{5-5}$$

When the Hom space in (5-5) is not 0, we say that π and Ω *intertwine* on h_0. If this is the case, then the double coset $T(F)h_0J$ supports a nonzero function in $\mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$, and the Hom space in (5-4) is not zero.

5E. ***Test vectors for minimal supercuspidal representations.*** By [Henniart 2002, Section A.3], if π is a minimal representation with level $\ell(\pi)$, then $c(\pi) = 2\ell(\pi)+2$. Let $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$ as described above. Let $v_0 \in \rho$ be the vector described in Definition 5.2. Assume that $c(\Omega) \ge c(\pi)$. Set

$$m_0 = \bigl[\ell(\pi) + \tfrac{3}{2}\bigr] - c(\Omega) - v(\boldsymbol{a}). \tag{5-6}$$

In the next proposition, we determine a double coset representative h_0 of $T(F)\backslash\mathrm{GL}_2(F)/J$ such that $\operatorname{Hom}_{J\cap h_0^{-1}T(F)h_0}(\rho,\ \Omega^{h_0}) \ne 0$. We remark that this result depends on our choice of inducing subgroup J, and in particular the quadratic extension $E = F[\alpha] = F[\alpha_0]$, where α_0 is always assumed to be of the form (5-1). For $m \in \mathbb{Z}$ and $z \in \mathfrak{o}^\times$, we define $g(m, z) := \left[\begin{smallmatrix} z\varpi^m & \\ & 1\end{smallmatrix}\right]$.

Lemma 5.4. *For $z \in \mathfrak{o}^\times$, $T(F)\cap g(m_0, z)Jg(m_0, z)^{-1} = F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\bigr)$.*

Proof. We give the details when $\mathfrak{A} = \mathfrak{M}$. First, suppose $\ell(\pi) = 0$. In this case $J = Z(F)\mathrm{GL}_2(\mathfrak{o})$ and $m_0 = 1 - c(\Omega) - v(\boldsymbol{a})$. Furthermore, for $z \in \mathfrak{o}^\times$, $g(m_0, z)Jg(m_0, z)^{-1} = g(m_0, 1)Jg(m_0, 1)^{-1}$. Let $t' \in T(F)\cap g(m_0, 1)Jg(m_0, 1)^{-1}$. Since $J = Z(F)\mathrm{GL}_2(\mathfrak{o})$, there is a unique integer k such that

$$\varpi^k t' \in T(F) \cap g(m_0, 1)\mathrm{GL}_2(\mathfrak{o})g(m_0, 1)^{-1}.$$

Let $t = \varpi^k t' = \left[\begin{smallmatrix} x+\boldsymbol{b}y & \boldsymbol{c}y \\ \boldsymbol{a}y & x\end{smallmatrix}\right]$. Then

$$g(m_0, 1)^{-1}tg(m_0, 1) = \begin{bmatrix} x+\boldsymbol{b}y & \boldsymbol{c}y\varpi^{-m_0} \\ \boldsymbol{a}y\varpi^{m_0} & x\end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}).$$

Therefore, $y \in \mathfrak{p}^{-m_0-v(\boldsymbol{a})} = \mathfrak{p}^{c(\Omega)-1} \subset \mathfrak{p}$. Since $g(m_0,1)^{-1}tg(m_0,1) \in \mathrm{GL}_2(\mathfrak{o})$, we have $x \in \mathfrak{o}^\times$. This proves that $T(F) \cap g(m_0,z)Jg(m_0,z)^{-1} \subseteq F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-1}\bigr)$. The other inclusion is straightforward. This completes the proof for $\ell(\pi)=0$.

Now, assume $\ell(\pi)>0$. Note that $t \in T(F)\cap g(m_0,z)Jg(m_0,z)^{-1}$ if and only if $wt \in T(F)\cap g(m_0,z)Jg(m_0,z)^{-1}$ for all $w \in Z(F)$.

Suppose that $t' \in T(F)\cap g(m_0,z)Jg(m_0,z)^{-1}$, where $t' = \left[\begin{smallmatrix} x'+\boldsymbol{b}y' & \boldsymbol{c}y' \\ -\boldsymbol{a}y' & x' \end{smallmatrix}\right]$. Let $k = \max\{-v(x'), -v(y')-m_0-v(\boldsymbol{a})\}$, $x = x'\varpi^k$, $y = y'\varpi^k$ and $t = \varpi^k t'$. Then

$$g(m_0,z)^{-1}tg(m_0,z) = \begin{bmatrix} x+\boldsymbol{b}y & z^{-1}\boldsymbol{c}y\varpi^{-m_0} \\ -z\boldsymbol{a}y\varpi^{m_0} & x \end{bmatrix}.$$

Let $i = \bigl[\frac{1}{2}(\ell(\pi)+1)\bigr]$, so $J = E^\times U_{\mathfrak{M}}^i$. There is a $u \in U_{\mathfrak{M}}^i$ such that

$$g(m_0,z)^{-1}tg(m_0,z)u \in E^\times.$$

Since $g(m_0,z)^{-1}tg(m_0,z) \in \mathrm{M}_2(\mathfrak{o})$ and $u \in U_{\mathfrak{M}}^i$, this implies that

$$a_0 z^{-1}\boldsymbol{c}y\varpi^{-m_0} \equiv -z\boldsymbol{a}y\varpi^{m_0} \mod \mathfrak{p}^i$$

(see (5-1) for a_0). As $y\varpi^{-m_0} \in \mathfrak{p}^{-2m_0-v(\boldsymbol{a})}$, we have $y \in \mathfrak{p}^{i-m_0-v(\boldsymbol{a})} = \mathfrak{p}^{c(\Omega)-[\ell(\pi)/2]-1}$. But this means that $v(\boldsymbol{a}y\varpi^{m_0})>0$, and hence $v(x)=0$ by our choice of k. Therefore, we have $t \in \mathfrak{o}^\times(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1})$. The discussion above shows that $t' \in F^\times(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1})$. The inclusion

$$T(F)\cap g(m_0,z)Jg(m_0,z)^{-1} \supseteq F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\bigr)$$

is straightforward. □

Proposition 5.5. *Let $i = \bigl[\frac{1}{2}\bigl(\ell(\pi)+\frac{3}{2}\bigr)\bigr]$. There is a unique $z_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that we have*

$$\mathrm{Hom}_{J\cap g(m_0,z_0)^{-1}T(F)g(m_0,z_0)}(\rho, \Omega^{g(m_0,z_0)}) \neq 0 \quad \textit{for } g(m_0,z_0) := \begin{bmatrix} z_0\varpi^{m_0} & \\ & 1 \end{bmatrix}.$$

Proof. We give the details when $\ell(\pi)>0$ and $\mathfrak{A}=\mathfrak{M}$. In this case, we have $m_0 = \ell(\pi)+1-c(\Omega)-v(\boldsymbol{a})$ and $i = \bigl[\frac{1}{2}(\ell(\pi)+1)\bigr]$. By Lemma 5.4,

$$T(F)\cap g(m_0,z)Jg(m_0,z)^{-1} = F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\bigr).$$

Since $\Omega|_{F^\times} = \omega_\pi$, intertwining only depends on $\Omega|_{1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}}$. Recall the definition of ξ from Section 2. The function given by $y \mapsto \Omega\bigl(1+y\bigl(\xi-\frac{1}{2}\boldsymbol{b}\bigr)\bigr)$ is an additive character of $\mathfrak{p}^{c(\Omega)-[\ell(\pi)/2]-1}/\mathfrak{p}^{c(\Omega)}$.

On the other hand, we have an isomorphism

$$\mathfrak{p}^{c(\Omega)-[\ell(\pi)/2]-1}/\mathfrak{p}^{c(\Omega)} \to \bar{N}^i U_{\mathfrak{M}}^{\ell(\pi)+1}/U_{\mathfrak{M}}^{\ell(\pi)+1},$$
$$y \mapsto g(m_0,z)^{-1}\bigl(1+y\bigl(\xi-\tfrac{1}{2}\boldsymbol{b}\bigr)\bigr)g(m_0,z).$$

Therefore, $\Omega^{g(m_0,z)}$ determines a character of $\overline{N}^i U_{\mathfrak{M}}^{\ell(\pi)+1}/U_{\mathfrak{M}}^{\ell(\pi)+1} \cong \overline{N}^i/\overline{N}^{\ell(\pi)+1}$. As z runs over $\mathfrak{o}^\times/(1+\mathfrak{p}^{\ell(\pi)+1-i})$, the character determined by $\Omega^{g(m_0,z)}$ runs over all characters of $\overline{N}^i/\overline{N}^{\ell(\pi)+1}$ that are nontrivial on $\overline{N}^{\ell(\pi)}$. In an abuse of notation we also refer to the character of $\overline{N}^i/\overline{N}^{\ell(\pi)+1}$ by $\Omega^{g(m_0,z)}$.

If $\ell(\pi)$ is odd, then ρ is a character of J. Then $\rho|_{\overline{N}^i} = \psi'$, i.e., ρ restricts to a single character, and $\ell(\pi)+1-i = i$, giving the proposition in this case.

Otherwise $\ell(\pi) > 0$ is even and, according to Lemma 5.3, $\rho|_{\overline{N}^i}$ is the direct sum of characters all of which restrict to the same character ψ' on $\overline{N}^{i+1}$. In this case there is a unique $z_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that

$$\rho|_{\overline{N}^i} \cong \bigoplus_{\substack{z\in\mathfrak{o}^\times/(1+\mathfrak{p}^{i+1}) \\ z\equiv z_0 \pmod{\mathfrak{p}^i}}} \Omega^{g(m_0,z)},$$

proving the proposition. The other cases follow similarly. □

Remark 5.6. Let us comment on the choice of the m_0 in (5-6). Put $g_m = g(m,1)$. One can exhibit the following double coset decomposition:

$$\mathrm{GL}_2(F) = \begin{cases} \bigsqcup_{m\geq v(\boldsymbol{a})} T(F)g_{-m}K_{\mathfrak{M}} & \text{if } \mathfrak{A}=\mathfrak{M}, \\ \bigsqcup_{m\geq 0} T(F)g_{-m}K_{\mathfrak{J}} & \text{if } \mathfrak{A}=\mathfrak{J},\ \left(\frac{L}{\mathfrak{p}}\right)=-1, \\ \bigsqcup_{m\geq v(\boldsymbol{a})} T(F)g_{-m}K_{\mathfrak{J}} \sqcup T(F)\begin{bmatrix} 1 & \\ u_0 & 1\end{bmatrix}K_{\mathfrak{J}} & \text{if } \mathfrak{A}=\mathfrak{J},\ \left(\frac{L}{\mathfrak{p}}\right)=0. \end{cases}$$

Then, still assuming $c(\Omega) \geq c(\pi)$, one can prove that if $f \in \mathscr{H}(\mathrm{GL}_2(F),\rho,\Omega)$ is nonzero and is supported on the double coset $T(F)g_{-m}K_{\mathfrak{A}}$, one must have $-m = m_0$. Thus it makes sense to look for intertwining on an element of $T(F)g_{-m}K_{\mathfrak{A}}$. The decomposition above involves negative powers of the uniformizer in the double coset representatives, whereas (4-12) uses positive exponents in the representatives. The difference in the indices occurs because for $m \geq v(\boldsymbol{a})$, g_{-m} and $g_{m-v(\boldsymbol{a})}$ represent the same double coset.

Next, we define a vector in π which will be a test vector for a Ω-Waldspurger functional and have the desired right-invariance. Recall v_0 from Definition 5.2. Define $\varphi_0 \in \pi$ by

$$\varphi_0(g) = \begin{cases} \rho(k_1)v_0 & \text{if } g = k_1 g_{m_0}^{-1} k_2,\ k_1 \in J,\ k_2 \in K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2}), \\ 0 & \text{otherwise.} \end{cases} \tag{5-7}$$

See (2-3) for the definition of $K_1^{(s)}(\mathfrak{p}^n)$. The vector φ_0 is well defined because of the inclusion

$$J \cap g_{m_0}^{-1} K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2}) g_{m_0} \subseteq \mathrm{Stab}(v_0).$$

Since φ_0 is a translate of the newform in π, we see that φ_0 is the unique (up to scalar) $K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{c(\pi)})$ fixed vector in π.

For $z \in \mathfrak{o}^\times$, define

$$\varphi_z(g) = \begin{cases} \rho(k)v_0 & \text{if } g = kg(m_0, z)^{-1},\ k \in J, \\ 0 & \text{otherwise.} \end{cases} \tag{5-8}$$

Proposition 5.7. *Suppose π is a minimal supercuspidal representation. Let i and z_0 be as in Proposition 5.5. Then*

(i) $\varphi_0 = \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)} \varphi_z$, *and*

(ii) $\ell(\varphi_0) = \ell(\varphi_{z_0})$ *for* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$.

Proof. The space $\big(g_{m_0} J g_{m_0}^{-1} \cap K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2})\big)\backslash K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2})$ has an irredundant set of coset representatives given by $\{g(0, z) : z \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)\}$. This shows (i). It is a straightforward computation to show that the double cosets $T(F)g(m_0, z)J$ for $z \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ are disjoint. Hence, z_0 is the unique element in $\mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that the double coset $T(F)g(m_0, z_0)J$ is in the support of a nonzero $f \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$. By the discussion in Section 5D, this gives (ii). □

Proposition 5.8. *Let π be a minimal supercuspidal representation. There is a nonzero $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$ satisfying $\ell(\varphi_0) \neq 0$.*

Proof. Let $z_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ be as in Proposition 5.5 and ℓ_0 be a nonzero element of the space $\mathrm{Hom}_{J\cap g(m_0,z_0)^{-1}T(F)g(m_0,z_0)}(\rho, \Omega^{g(m_0,z_0)})$. Define

$$\xi = 1_{T(F)g(m_0,z_0)J} \otimes \ell_0 \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega).$$

As in Section 5D, define $\ell(\varphi) = \xi * \varphi(1) \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. After appropriate normalization of measures, $\ell(\varphi_0) = \ell(\varphi_{z_0}) = \ell_0(v_0)$.

When ρ is a character, it follows immediately that $\ell(\varphi_0) \neq 0$. However, when ρ is not a character, it must be shown that $v_0 \notin \ker \ell_0$.

Suppose that $\ell(\pi) = 2r > 0$ and write $J = J_\alpha$. Recall that under the identification

$$\rho|_{J_\alpha^1} \cong \eta = \mathrm{Ind}_{A^r H_\alpha^1}^{J_\alpha^1} \tilde{\lambda},$$

the vector v_0 is identified with f_0 defined by (5-2). Consider any character ψ' which is a summand of $\rho|_{\bar{N}^r}$, and the vectors $f_a \in \eta$ defined by (5-3) with respect to ψ'. We may take ℓ_0 to be given by

$$\ell_0\bigg(\sum_{a \in \mathfrak{p}^r/\mathfrak{p}^{r+1}} c_a f_a\bigg) := \sum_{a \in \mathfrak{p}^r/\mathfrak{p}^{r+1}} c_a.$$

Indeed, with this definition, note that for $f \in \eta$ and $x \in \mathfrak{p}^r$,

$$\ell_0\Big(\eta\Big(\Big[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\Big]\Big) f\Big) = \psi'\Big(\Big[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\Big]\Big)\ell_0(f)$$
$$= \Omega^{g(m_0,z)}\Big(\Big[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\Big]\Big)\ell_0(f),$$

viewing $\Omega^{g(m_0,z)}$ as a character of $\bar{N}^r$ for some choice of $z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{i+1})$ corresponding to ψ' as in the proof of Proposition 5.5. Hence $\ell_0(v_0) = \ell_0(f_0) = 1 \neq 0$.

Finally, suppose that $\ell(\pi) = 0$, so $c(\pi) = 2$ and $s = 1 - c(\Omega) - v(\boldsymbol{a}) < 0$. Let $h = g_s$. The linear functional ℓ_0 is the projection onto one of the irreducible summands of $\rho|_{\bar{N}^0}$. Let $a \in \mathfrak{o}^\times$, and denote by ψ_a the character of $\bar{N}^0$ given by $\psi_a\big(\big[\begin{smallmatrix} 1 & \\ u & 1 \end{smallmatrix}\big]\big) = \psi_1(au)$. Denote by v_a the vector in ρ such that $\rho\big(\big[\begin{smallmatrix} 1 & \\ u & 1 \end{smallmatrix}\big]\big)v_a = \psi_a(u)v_a$.

Now, write $v_0 = \sum c_a v_a$, where $c_a \in \mathbb{C}$ and a runs over $\mathfrak{o}^\times/(1+\mathfrak{p})$. For $b \in \mathfrak{o}^\times$, we have $\rho(g(0,b))v_0 = v_0$. However, $\rho(g(0,b))v_a = v_{ba}$. Therefore, $c_a = c_{ba}$ for all $b \in \mathfrak{o}^\times$. Therefore, v_0 has a nonzero component in each summand of $\rho|_{\bar{N}^0}$. □

Proof of Theorem 1.7 for minimal supercuspidal representations. Since $c(\Omega) \geq c(\pi)$, we can apply Lemma 2.2 together with Proposition 5.8 and the definition (5-7) to see that $\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0$ and that φ_0 is a test vector with the required properties. □

5F. *Nonminimal representations.* In this section we consider a nonminimal supercuspidal representation τ and let Ω be a character of $T(F)$ such that $\Omega|_{Z(F)} = \omega_\tau$ and $c(\Omega) \geq c(\tau)$. There exists a minimal supercuspidal representation π and a (ramified) character χ of $F^\times$ such that $\tau \cong \pi \otimes \chi$. Identify τ with $\pi \otimes \chi$. Since π is minimal and τ is not, Proposition 3.4 of [Tunnell 1978] tells us

$$c(\tau) = 2c(\chi) > c(\pi).$$

Then $c(\Omega \otimes \chi^{-1}) \geq c(\Omega) > c(\pi)$.

Observe that the considerations of the previous section guarantee the existence of a vector in π that is a test vector for an $(\Omega \otimes \chi^{-1})$-Waldspurger functional and is the unique vector (up to scalars) in π that is right invariant under the corresponding conjugate of $K_1(\mathfrak{p}^{c(\pi)})$. To get the desired test vector for τ, we actually need a vector in π with respect to $\Omega \otimes \chi^{-1}$, but which transforms according to $\chi^{-1} \circ \det$ under right translation by a conjugate of $K_1(\mathfrak{p}^{c(\tau)})$. In the next proposition, we obtain a vector φ_χ with the correct right-transformation property, and then show that it is a test vector for the appropriate linear functional.

Proposition 5.9. *Suppose that* $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$. *Let* $s = 2c(\chi) - c(\Omega) - v(\boldsymbol{a})$, $b = \big[\ell(\pi) + \frac{3}{2}\big] - c(\chi)$, $m_0 = \big[\ell(\pi) + \frac{3}{2}\big] - c(\Omega) - v(\boldsymbol{a})$, $i = \big[\frac{1}{2}\big(\ell(\pi) + \frac{3}{2}\big)\big]$ *and* $i' = \big[\ell(\pi) + \frac{3}{2}\big] - \big[\frac{1}{2}\ell(\pi)\big] - 1$.

(i) *There is a unique* $u \in \mathfrak{o}^\times/((1+\mathfrak{p}^i)\cap\mathfrak{o}^\times)$, *and* $v_\chi \in \rho$ *depending on* u, *which is unique up to scaling*, *such that for all* $x \in \mathfrak{p}^{i'}$,

$$\rho\left(\begin{bmatrix} 1+x\varpi^{-b} & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi = \chi^{-1}(1+x\varpi^{-b})v_\chi. \tag{5-9}$$

(ii) *Suppose* u *and* v_χ *satisfy* (5-9). *Let*

$$\varphi_\chi(g) = \begin{cases} (\chi^{-1}\circ\det)(k_1)\rho(k_2)v_\chi & \text{if } g = k_2 g_\chi k_1,\ k_2\in J,\ k_1 \in K_1^{(s)}(\mathfrak{p}^{2c(\chi)}), \\ 0 & \text{otherwise}, \end{cases}$$

where $g_\chi = \left[\begin{smallmatrix} \varpi^{-m_0} & 0 \\ u^{-1}\varpi^{c(\chi)-s} & 1 \end{smallmatrix}\right]$. *Then* φ_χ *is well defined*, *and is the unique vector* (*up to scalars*) *in* π *such that*, *for* $k \in K_1^{(s)}(\mathfrak{p}^{2c(\chi)})$, $\pi(k)\varphi_\chi = (\chi^{-1}\circ\det)(k)\varphi_\chi$.

Note that $i = i'$ when $\ell(\pi)\in\mathbb{Z}$ or $\ell(\pi) = \frac{1}{2}(2r+1)$ with r even; otherwise they are off by 1 (see Table 2).

Proof. Observe that for $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \in K_1^{(s)}(\mathfrak{p}^{2c(\chi)})$, we have

$$g_\chi \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} g_\chi^{-1} \in \begin{bmatrix} a_{11} & 0 \\ (a_{11}-1)u^{-1}\varpi^b & 1 \end{bmatrix} + \mathfrak{P}^{\ell(\pi)e_{\mathfrak{A}}+1}. \tag{5-10}$$

We remark that $b \le 0$. If $g_\chi \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] g_\chi^{-1} \in J$, then $a_{11} \equiv 1 \mod \mathfrak{p}^{i'-b}$. To show that φ_χ is well defined, we must check that ρ^{g_χ} and χ agree on $g_\chi^{-1} J g_\chi \cap K_1^{(s)}(\mathfrak{p}^{2c(\chi)})$. This is precisely the condition (5-9). Once this is established, uniqueness then follows since $\varphi_\chi \otimes \chi$ is a translate of the newform for τ. Therefore, part (ii) of the proposition will follow from part (i).

First, suppose $\rho = \lambda$ is a character. As in Section 5B we have

$$\rho\left(\begin{bmatrix} 1+x\varpi^{-b} & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi = \lambda\left(\begin{bmatrix} 1 & 0 \\ xu^{-1} & 1 \end{bmatrix}\right).$$

As a function of x, both sides of (5-9) are nontrivial characters of $\mathfrak{p}^{i'}/\mathfrak{p}^{[\ell(\pi)+3/2]}$ (see Table 2). Therefore, there is a unique $u \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that (5-9) holds. This proves part (i) when ρ is a character.

If ρ is not a character, then $b < 0$. By Section 5B,

$$\rho\left(\begin{bmatrix} 1+x\varpi^{-b} & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi = \rho\left(\begin{bmatrix} 1 & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi.$$

Suppose that π is a depth zero supercuspidal representation. Let $u = 1$. By Lemma 5.3 there is a unique up to scalar $v_\chi \in \rho$ such that, for $x \in \mathfrak{o}$,

$$\rho\left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\right)v_\chi = \chi^{-1}(1+x\varpi^{c(\chi)-1})v_\chi.$$

Finally, suppose that $\ell(\pi) = 2r > 0$, and $\pi = c\text{-Ind}_J^{\mathrm{GL}_2(F)}\rho$, where ρ is not a character of J. By Lemma 5.3, $\rho|_{\overline{N}\cap H_\alpha}$ is a multiple of $\psi_\alpha|_{\overline{N}\cap H_\alpha}$. There exists a unique $u \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that (5-9) holds for $x \in \mathfrak{p}^{i'}$. By Lemma 5.3, with this choice of u there is a unique up to scalar multiple $v_\chi \in \rho$ such that (5-9) holds. This completes the proof of (i) of the proposition. □

The next lemma gives a double coset decomposition of the support of φ_χ.

Lemma 5.10. *Let* $g_\chi = \begin{bmatrix} \varpi^{-m_0} & 0 \\ u^{-1}\varpi^{c(\chi)-s} & 1 \end{bmatrix}$ *as in Proposition 5.9. Then*

$$Jg_\chi K_1^{(s)}(\mathfrak{p}^{2c(\chi)}) = \bigsqcup_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1})} Jg_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix}.$$

Proof. Recall

$$K_1^{(s)}(\mathfrak{p}^{2c(\chi)}) = \begin{bmatrix} 1 & \mathfrak{p}^s \\ \mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})} & 1+\mathfrak{p}^{2c(\chi)} \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & \\ & 1 \end{bmatrix}.$$

We have, for $z \in \mathfrak{o}^\times$,

$$g_\chi \begin{bmatrix} 1 & \mathfrak{p}^s \\ \mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})} & 1+\mathfrak{p}^{2c(\chi)} \end{bmatrix} g_\chi^{-1} = \begin{bmatrix} 1+\mathfrak{p}^{c(\chi)} & \mathfrak{p}^{2c(\chi)-[\ell(\pi)+3/2]} \\ \mathfrak{p}^{[\ell(\pi)+3/2]} & 1+\mathfrak{p}^{c(\chi)} \end{bmatrix}, \tag{5-11}$$

$$g_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix} g_\chi^{-1} = \begin{bmatrix} z & \\ u^{-1}\varpi^{c(\chi)-s+m_0}(z-1) & 1 \end{bmatrix}. \tag{5-12}$$

Using the description of J in Table 2, we see that the right-hand side of (5-11) lies in J. Also, the right-hand side of (5-12) lies in J if and only if $z \in 1+\mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1}$. This completes the proof of the lemma. □

The next two lemmas give a useful decomposition of g_χ. Fix g_χ and u to be as in Proposition 5.9, and set $y_0 = -u^{-1}\boldsymbol{a}^{-1}\varpi^{c(\Omega)+v(\boldsymbol{a})-c(\chi)}$. For $y \in F$, define $t_y = t\big(1+\frac{1}{2}\boldsymbol{b}y, y\big) = \left[\begin{smallmatrix} 1+\boldsymbol{b}y & \boldsymbol{c}y \\ -\boldsymbol{a}y & 1 \end{smallmatrix}\right]$.

Lemma 5.11. *We have* $g_\chi = k_0 g_{m_0}^{-1} t_{y_0}$, *where* $k_0 \in U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$.

Proof. Write $g_\chi = g_{m_0}^{-1} g$, where $g = \left[\begin{smallmatrix} 1 & \\ -\boldsymbol{a}y_0 & 1 \end{smallmatrix}\right]$. Let $k_0^{-1} = g_{m_0}^{-1} t_{y_0} g^{-1} g_{m_0}$. We see that

$$k_0^{-1} = \begin{bmatrix} 1+\boldsymbol{b}y_0+\boldsymbol{a}\boldsymbol{c}y_0^2 & \boldsymbol{c}y_0\varpi^{-m_0} \\ 0 & 1 \end{bmatrix}.$$

So $k_0 \in U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ and $g_\chi = k_0 g_{m_0}^{-1} t_{y_0}$. □

Lemma 5.12. *For each* $z \in \mathfrak{o}^\times$, *there exists* $k_z \in U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ *such that*

$$g_{m_0}^{-1} t_{y_0} \begin{bmatrix} z & \\ & 1 \end{bmatrix} = k_z g_{m_0}^{-1} \begin{bmatrix} z & \\ & 1 \end{bmatrix} t_{zy_0}. \tag{5-13}$$

Proof. Write $g_\chi = k_0 g_{m_0}^{-1} t_{y_0}$ as in Lemma 5.11. Then

$$\begin{aligned} g_{m_0}^{-1} t_{y_0}\begin{bmatrix} z & \\ & 1 \end{bmatrix} &= k_0^{-1} g_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix} = k_0^{-1} g_{m_0}^{-1}\begin{bmatrix} 1 & \\ -\boldsymbol{a} y_0 & 1 \end{bmatrix}\begin{bmatrix} z & \\ & 1 \end{bmatrix} \\ &= k_0^{-1}\begin{bmatrix} z & \\ & 1 \end{bmatrix} g_{m_0}^{-1}\begin{bmatrix} 1 & \\ -\boldsymbol{a} z y_0 & 1 \end{bmatrix} = k_0^{-1}\begin{bmatrix} z & \\ & 1 \end{bmatrix} k_0' g_{m_0}^{-1} t_{zy_0} \\ &= k_z \begin{bmatrix} z & \\ & 1 \end{bmatrix} g_{m_0}^{-1} t_{zy_0}. \end{aligned}$$

For the second to last equality we have used a decomposition similar to Lemma 5.11, and k_0' is the corresponding element of $U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$. For the last equality we use the fact that the subgroup $U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ is normalized by A^0. □

Let us remark here that $U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ lies in the kernel of ρ (see Table 2 for details). For any $g \in \mathrm{GL}_2(F)$ and $v \in \rho$, define

$$\varphi_{g,v}(h) = \begin{cases} \rho(k)v & \text{if } h = kg,\ k \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for any $z \in \mathfrak{o}^\times$, the support of $\pi\left(\begin{bmatrix} z^{-1} & \\ & 1 \end{bmatrix}\right)\varphi_{g_\chi, v_\chi}$ is exactly $J g_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix}$. Then

$$\begin{aligned} \varphi_\chi &= \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1})} \chi^{-1}(z)\, \pi\left(\begin{bmatrix} z^{-1} & \\ & 1 \end{bmatrix}\right)\varphi_{g_\chi, v_\chi} \\ &= q^{-[\ell(\pi)/2]-1} \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)})} \chi(z)\, \pi\left(\begin{bmatrix} z & \\ & 1 \end{bmatrix}\right)\varphi_{g_\chi, v_\chi} \\ &= q^{-[\ell(\pi)/2]-1} \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)})} \chi(z)\, \pi\left(t_{z^{-1}y_0}^{-1}\right)\varphi_{g(m_0,z)^{-1}, v_\chi}. \end{aligned} \tag{5-14}$$

The first equality follows from the double coset decomposition in Lemma 5.10. To get the second equality, note that for $z \in 1 + \mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1}$, we can apply (5-9) to the right-hand side of (5-12). Finally, the third equality follows from Lemmas 5.11 and 5.12.

Write $c_0 = \left[\frac{1}{2}(c(\chi)+1)\right]$. For $x \in \mathfrak{p}^{c_0}$, $x \mapsto \chi(1+x)$ is an additive character of $\mathfrak{p}^{c_0}/\mathfrak{p}^{c(\chi)}$.

Proposition 5.13. *Suppose* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega \otimes \chi^{-1})$. *Then there is a unique* $w_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-c_0})$ *satisfying the following conditions*:

(i) *If*

$$\sum_{z \in (1+\mathfrak{p}^{c_0})/(1+\mathfrak{p}^{c(\chi)})} \chi(zw)\, \ell\left(\pi\left(t_{(zw)^{-1}y_0}^{-1}\right)\varphi_{g(m_0,zw)^{-1}, v_\chi}\right) \neq 0,$$

then $w \equiv w_0 \mod \mathfrak{p}^{c(\chi)-c_0}$.

(ii) $\ell(\varphi_{g(m_0,zw_0)^{-1}, v_\chi}) \neq 0$.

Proof. Recall $y_0 = -u^{-1}\boldsymbol{a}^{-1}\varpi^{c(\Omega)+v(\boldsymbol{a})-c(\chi)}$. First, note $\chi\circ\det$ is trivial on elements t_{zy_0} for $z \in \mathfrak{o}^\times$. We can define an additive character of $\mathfrak{o}$ by $\psi_\Omega(x) := \Omega(t_{xy_0})$. By (5-14), we have

$$\ell(\varphi_\chi) = q^{-[\ell(\pi)/2]-1} \times \sum_{w\in\mathfrak{o}^\times/(1+\mathfrak{p}^{c_0})} \sum_{z\in(1+\mathfrak{p}^{c_0})/(1+\mathfrak{p}^{c(\chi)})} \chi^{-1}(wz)\psi_\Omega^{-1}(wz)\ell(\varphi_{g(-m_0,zw),v_\chi}). \tag{5-15}$$

If $z \in 1+\mathfrak{p}^{c_0}$, then $\rho(g(0,z^{-1}))v_\chi = v_\chi$, and $\varphi_{g(-m_0,zw),v_\chi} = \varphi_{g(-m_0,w),v_\chi}$. Consider the inner sum of (5-15):

$$\sum_{z\in(1+\mathfrak{p}^{c_0})/(1+\mathfrak{p}^{c(\chi)})} \chi^{-1}(wz)\psi_\Omega^{-1}(wz) = \chi^{-1}(w)\psi_\Omega^{-1}(w) \sum_{x\in\mathfrak{p}^{c_0}/\mathfrak{p}^{c(\chi)}} \chi^{-1}(1+x)\psi_\Omega^{-1}(wx). \tag{5-16}$$

This sum does not equal zero if and only if $\chi^{-1}(1+x) = \psi_\Omega(xw)$ for all $x \in \mathfrak{p}^{c_0}$, and this occurs for exactly one element $w = w_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-c_0})$. This proves the first part.

Consider an element $t \in T(F)\cap g_{m_0} J g_{m_0}^{-1} = F^\times\big(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\big)$ (see Lemma 5.4). Since $2c(\chi) > c(\pi) = 2\ell(\pi)+2$, we see

$$\big(c(\Omega) - \big[\tfrac{1}{2}\ell(\pi)\big] - 1\big) - (c(\Omega) - c(\chi)) = c(\chi) - \big[\tfrac{1}{2}\ell(\pi)\big] - 1 > c_0.$$

Therefore, there exist $x \in \mathfrak{p}^{c_0}$ and $z \in F^\times$ such that $t = zt_{xy_0}$, and, by the remarks after (5-16),

$$\chi^{-1}(1+w_0^{-1}x) = \psi_\Omega(x) = \Omega(t_{xy_0}) = \Omega(z)^{-1}\Omega(t).$$

By (5-9) we have

$$\rho\left(\begin{bmatrix} 1 & \\ u^{-1}x & 1 \end{bmatrix}\right)v_\chi = \chi^{-1}(1+x\varpi^{-b})v_\chi.$$

Therefore, for all $t \in T(F)\cap g_{m_0} J g_{m_0}^{-1} = T(F)\cap g_{m_0,w_0} J g_{m_0,w_0}^{-1}$,

$$\rho(g_{m_0,w_0}^{-1}\, t\, g_{m_0,w_0})v_\chi = \Omega(t)v_\chi = \Omega(t)(\chi^{-1}\circ\det)(t)v_\chi.$$

This implies that there is $\ell_0 \in \mathrm{Hom}_{g(m_0,w_0)^{-1}T(F)g(m_0,w_0)\cap J}(\rho, (\Omega\otimes\chi^{-1})^{g(m_0,w_0)})$ such that $\ell_0(v_\chi) \neq 0$, and from the discussion in Section 5D, after a normalization, $\ell(\varphi_{g(m_0,w_0)^{-1},v_\chi}) = \ell_0(v_\chi)$. □

Proof of Theorem 1.7 for nonminimal supercuspidal representations. We compute, by Proposition 5.13,

$$\begin{aligned}\ell(\varphi_\chi) &= q^{-[\ell(\pi)/2]-1}\ell(\varphi_{g(m_0,w_0)^{-1},v_\chi}) \sum \chi^{-1}(ww_0^{-1})\psi_\Omega^{-1}(ww_0^{-1}) \\ &= q^{-[\ell(\pi)/2]-1-[c(\chi)/2]}\psi_\Omega^{-1}(w_0^{-1})G(\chi,\psi_\Omega^{-1})\ell(\varphi_{g(m_0,w_0)^{-1},v_\chi}) \neq 0,\end{aligned}$$

where the sum is over $w \in (1+\mathfrak{p}^{c(\chi)-c_0})/(1+\mathfrak{p}^{c_0})$, w_0 is the unique element of $\mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-c_0})$ such that $\chi^{-1}(1+z) = \psi_\Omega(zw_0^{-1})$ for all $z \in \mathfrak{p}^{c_0}$, and $G(\chi, \psi_\Omega^{-1})$ is the Gauss sum for the pair χ, ψ_Ω^{-1}. For the last equality see [Bushnell and Henniart 2006, 23.6 Proposition]. This shows that $\mathrm{Hom}_{T(F)}(\tau, \Omega) \neq 0$. The 1-dimensionality follows from [Waldspurger 1985]. Since $c(\Omega) \geq c(\tau)$, we can apply Lemma 2.2 to obtain the test vector with the required properties. The uniqueness of the test vector follows from the uniqueness of the newform in π. □

6. Local spectral distributions

Now we return to the setting where F is a p-adic field and L is a quadratic separable extension as in Section 2B. Let π be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$, and Ω a character of $L^\times$ such that $\Omega|_{F^\times} = \omega_\pi$. In this section, we calculate certain local spectral distributions $\tilde{J}_\pi(f)$ defined by Jacquet and Chen [2001] for certain test functions $f \in C_c^\infty(\mathrm{GL}_2(F))$. These are used in Section 7 to generalize the global L-value formula previously obtained in [Martin and Whitehouse 2009]. For simplicity, we prove this global L-value formula when the central character of our automorphic representation is trivial, so we may as well assume $\omega_\pi = 1$ in this section also. We also assume that π and Ω are unitary, since the global objects in the following sections are unitary as well.

The calculation of $\tilde{J}_\pi(f)$ is contained in [Martin and Whitehouse 2009] in the cases where F is archimedean, L/F is split, or π and Ω have disjoint ramification. Hence, we assume L/F is a quadratic extension of nonarchimedean fields and $c(\pi), c(\Omega) > 0$. In particular, either $L(s, \pi) = 1$ or $\pi = \chi\,\mathrm{St}_{\mathrm{GL}_2}$, where, for the rest of this section, χ denotes an unramified quadratic character. Further, we assume $c(\Omega) \geq c(\pi)$ to use our determination of test vectors.

Write $L^\times = F(\xi)^\times$, where $\xi = \frac{1}{2}\sqrt{\boldsymbol{d}}$. Let $T = T(F)$ be the torus in $\mathrm{GL}_2(F)$ isomorphic to $L^\times$ defined in (2-16). Here it is convenient to take a slightly different parameterization for T than the one given by $t(x, y)$ in (2-17). Namely, we map

$$x + y\xi_0 \mapsto \begin{bmatrix} x & \boldsymbol{c}y \\ -\boldsymbol{a}y & x - \boldsymbol{b}y \end{bmatrix}, \tag{6-1}$$

where

$$\xi_0 = \xi - \tfrac{1}{2}\boldsymbol{b} = \tfrac{1}{2}\big(\sqrt{\boldsymbol{d}} - \boldsymbol{b}\big).$$

By [Tunnell 1983; Saito 1993] or Theorem 1.7, the assumption $c(\Omega) \geq c(\pi)$ implies $\dim_{\mathbb{C}} \mathrm{Hom}_T(\pi, \Omega) = 1$. Fix a nonzero linear functional $\ell \in \mathrm{Hom}_T(\pi, \Omega)$.

Consider the Kirillov model for π and the inner product on π given by

$$(\phi_1, \phi_2) = \int_{F^\times} \phi_1(a)\overline{\phi_2(a)}\, d^\times a,$$

where $d^\times a$ is the Haar measure giving $\mathrm{vol}(\mathfrak{o}^\times) = 1$. This inner product is $\mathrm{GL}_2(F)$-invariant. Let e be the unique (up to scalars) vector in π such that $\pi(t)e = \Omega(t)e$ for $t \in T$, which we normalize so that $(e, e) = 1$. Let dg denote the local Tamagawa measure on $\mathrm{GL}_2(F)$. Then the local distribution we are interested in is defined in [Jacquet and Chen 2001] by

$$\tilde{J}_\pi(f) = (\pi(f)e, e) = \int_{\mathrm{GL}_2(F)} f(g)(\pi(g)e, e)\, dg, \quad f \in C_c^\infty(\mathrm{GL}_2(F)). \tag{6-2}$$

Put $s = c(\Omega) - c(\pi)$, $h = \begin{bmatrix} \varpi^s & \\ & 1 \end{bmatrix} w$, and

$$K' = hK_1(\mathfrak{p}^{c(\pi)})h^{-1} = \begin{bmatrix} 1+\mathfrak{p}^{c(\pi)} & \mathfrak{p}^{c(\Omega)} \\ \mathfrak{p}^{c(\pi)-c(\Omega)} & \mathfrak{o}^\times \end{bmatrix}. \tag{6-3}$$

Then Theorem 1.7 says there is a unique (up to scalars) test vector $\phi \in \pi$ which is right invariant by K' such that $\ell(\phi) \neq 0$. Let ϕ_0 be the newvector in π normalized so that $\phi_0(1) = 1$. Then we may take $\phi = \pi(h)\phi_0$.

Observe that Ω is trivial on $T \cap ZK'$, where $Z = Z(T)$, since ϕ is fixed by ZK'. Consider the vector $e' \in \pi$ given by

$$e' = \sum_{t \in T/(T\cap ZK')} \Omega^{-1}(t)\pi(t)\phi. \tag{6-4}$$

Note the index set for the sum is finite, so e' is well defined. Then for any $t \in T$, we have $\pi(t)e' = \Omega(t)e'$, and $\ell(e') \neq 0$. In other words, we may assume

$$e = \frac{e'}{(e', e')^{1/2}}.$$

We take for our test function $f = 1_{K'}/\mathrm{vol}(K')$, so our calculations do not in fact depend on the normalization of dg in (6-2). Then

$$\tilde{J}_\pi(f) = \mathrm{vol}(K')^{-1} \int_{K'} (\pi(k)e, e)\, dk = \mathrm{vol}(K')^{-1} \int_{K'} \frac{(\pi(k)e', e')}{(e', e')}\, dk.$$

Note, using the $\mathrm{GL}_2(F)$-invariance of the inner product, we get

$$\begin{aligned}(e', e') &= \sum_{t \in T/(T\cap ZK')} \Omega^{-1}(t)(\phi, \pi(t^{-1})e') \\ &= |T/(T \cap ZK')|\,(\phi, e').\end{aligned}$$

Since $\pi(f)$ is simply orthogonal projection onto $\langle \phi \rangle = \pi^{K'}$,

$$\mathrm{vol}(K')^{-1} \int_{K'} (\pi(k)e', e')\, dk = (\pi(f)e', e') = \frac{(e', \phi)(\phi, e')}{(\phi, \phi)}.$$

Hence

$$\tilde{J}_\pi(f) = \frac{1}{|T/(T \cap ZK')|} \frac{(e', \phi)}{(\phi, \phi)}. \tag{6-5}$$

Note that

$$(\phi, \phi) = (\phi_0, \phi_0) = \begin{cases} L(2, 1_F) & \text{if } \pi = \chi \operatorname{St}_{\mathrm{GL}_2}, \\ 1 & \text{if } L(s, \pi) = 1, \end{cases} \tag{6-6}$$

so it remains to compute $|T/(T \cap ZK')|$ and (e', ϕ). (Recall χ denotes an unramified character.) Only the latter computation is involved. This requires knowing some facts about values of the Whittaker newform and determining a set of representatives for $T/(T \cap ZK')$. We first tackle these two tasks, and then compute (e', ϕ), and hence $\tilde{J}_\pi(f)$, under our above assumptions.

Whittaker values. Assume π has trivial central character and let ψ be a nontrivial additive character of F of conductor $\mathfrak{o}$. Let $\mathcal{W}(\pi, \psi)$ be the Whittaker model for π with respect to ψ. Let W_0 be the newform normalized so that $W_0(1) = 1$, and therefore $\phi_0(a) = W_0\big(\big[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\big]\big)$.

We are interested in certain values of the Whittaker newform when the local L-factor of π has degree 1 or 0. For this, we recall (see Table 1 in Section 3) that $\phi_0(a) = \chi(a)|a| 1_{\mathfrak{o}}(a)$ when $\pi = \chi \operatorname{St}_{\mathrm{GL}_2}$ and $\phi_0(a) = 1_{\mathfrak{o}^\times}(a)$ when $L(s, \pi) = 1$. From this, one obtains the following result on Whittaker newform values.

Lemma 6.1. (i) *If $u, v \in \mathfrak{o}^\times$, then*

$$W_0\left(g \begin{bmatrix} u & \\ & v \end{bmatrix} w\right) = W_0(gw).$$

(ii) *If $\pi = \chi \operatorname{St}_{\mathrm{GL}_2}$ with χ unramified, then for $j \in \mathbb{Z}$,*

$$W_0\left(\begin{bmatrix} \varpi^j & \\ & 1 \end{bmatrix} w\right) = \begin{cases} -\chi(\varpi)^j q^{-j-1} & \text{if } j \geq -1, \\ 0 & \text{else}. \end{cases}$$

If $L(s, \pi) = 1$, then for any $j \in \mathbb{Z}$,

$$W_0\left(\begin{bmatrix} \varpi^j & \\ & 1 \end{bmatrix} w\right) = \begin{cases} \epsilon\left(\frac{1}{2}, \pi\right) & \text{if } j = -c(\pi), \\ 0 & \text{else}. \end{cases}$$

(iii) *If $\pi = \chi \operatorname{St}_{\mathrm{GL}_2}$ with χ unramified, for $j \geq 0 \geq k$, we have*

$$\int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} \varpi^j u & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi^k & 1 \end{bmatrix}\right) d^\times u = -q^{-1}(\chi(\varpi) q^{-1})^{j-2k}.$$

If $L(s, \pi) = 1$, then for all $j, k \in \mathbb{Z}$,

$$\int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} \varpi^j u & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ \varpi^k & 1 \end{bmatrix}\right) d^\times u = \begin{cases} 1 & \text{if } j = 0 \text{ and } k \geq c(\pi), \\ (1-q)^{-1} & \text{if } j = 0 \text{ and } k = c(\pi) - 1, \\ 0 & \text{else}. \end{cases}$$

Proof. Part (i) follows simply from the facts that W_0 is right invariant by $K_1(\mathfrak{p}^{c(\pi)})$ and $\omega_\pi = 1$. The proof of parts (ii) and (iii) follows from the functional equation (3-3) with $\mu = 1$ by comparing coefficients of q^s. □

The toric quotient. We identify $t = x + y\xi_0 \in L^\times$ with its image in T via (6-1). Since we have assumed that $c(\Omega) \geq c(\pi)$, we have $t = x + y\xi_0 \in K'$ if and only if $x \in 1 + \mathfrak{p}^{c(\pi)}$ and $y \in \mathfrak{p}^{c(\Omega)}$.

Lemma 6.2. *We have*

$$|T/(T \cap ZK')| = \begin{cases} q^{c(\Omega)}(1+q^{-1}) & \text{if } L/F \text{ is unramified,} \\ 2q^{c(\Omega)} & \text{if } L/F \text{ is ramified.} \end{cases} \tag{6-7}$$

Furthermore, if L/F is unramified or $v(\boldsymbol{a}) = 1$, then a complete set of representatives of $T/(T \cap ZK')$ is given by

$$\{1 + y\xi_0 : y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)}\} \cup \{x + \xi_0 : x \in \mathfrak{p}/\mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})}\}, \tag{6-8}$$

while if L/F is ramified and $v(\boldsymbol{a}) = 0$, then a complete set of representatives of $T/(T \cap ZK')$ is given by

$$\{1 + y\xi_0 : y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)},\ y \not\equiv u_0' \mod \mathfrak{p}\} \\ \cup \{1 + (u_0' + y)\xi_0 : y \in \mathfrak{p}/\mathfrak{p}^{c(\Omega)+1}\} \cup \{x + \xi_0 : x \in \mathfrak{p}/\mathfrak{p}^{c(\Omega)}\}, \tag{6-9}$$

where $u_0' = -u_0/\boldsymbol{a} \in \mathfrak{o}^\times$ with u_0 as in (2-14).

Proof. We obtain the set of representatives of $T/(T \cap ZK')$, from which (6-7) follows. Given an arbitrary $t = x + y\xi_0 \in T$, we may multiply t by an element of Z to assume that $x, y \in \mathfrak{o}$ and either $x = 1$ or $y = 1$. Further, if x and y are both units, we may assume $x = 1$. So we may consider a set of representatives of the form $x + \xi_0$ and $1 + y\xi_0$, where $x \in \mathfrak{p}$ and $y \in \mathfrak{o}$. Observe $\xi_0^2 = -\boldsymbol{ac} - \boldsymbol{b}\xi_0$. For $t, t' \in T$, write $t \sim t'$ if $t = t_0 t'$ for some $t_0 \in T \cap ZK'$.

First we observe that $x + \xi_0 \sim 1 + y\xi_0$, where $x \in \mathfrak{p}$ and $y \in \mathfrak{o}$, is not possible. If it were, there would exist $u \in 1 + \mathfrak{p}^{c(\pi)}$, $r \in \mathfrak{p}^{c(\Omega)}$ and $z \in F^\times$ such that

$$zx + z\xi_0 = (u + r\xi_0)(1 + y\xi_0) = u - \boldsymbol{ac}ry + (uy + r - \boldsymbol{b}ry)\xi_0.$$

Since $u - \boldsymbol{ac}ry \in \mathfrak{o}^\times$, we see $v(z) < 0$, but $z = uy + r - \boldsymbol{b}ry \in \mathfrak{o}$, a contradiction.

Now consider $x_1 + \xi_0 \sim x_2 + \xi_0$ for $x_1, x_2 \in \mathfrak{p}$. Then, for some $u \in 1 + \mathfrak{p}^{c(\pi)}$, $r \in \mathfrak{p}^{c(\Omega)}$ and $z \in F^\times$, we have

$$zx_1 + z\xi_0 = (u + r\xi_0)(x_2 + \xi_0) = ux_2 - \boldsymbol{ac}r + (u + rx_2 - \boldsymbol{b}r)\xi_0.$$

Hence $z = u + rx_2 - \boldsymbol{b}r \in 1 + \mathfrak{p}^{c(\pi)}$ and

$$zx_1 = ux_1 + rx_1x_2 - \boldsymbol{b}rx_1 = ux_2 - \boldsymbol{ac}r,$$

which implies

$$u(x_2 - x_1) = r(\boldsymbol{ac} - \boldsymbol{b}x_1 + x_1 x_2).$$

In particular, we must have $x_1 \equiv x_2 \mod \mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})}$.

In fact, if $v(\boldsymbol{a}) = 0$, we have $x_1 + \xi_0 \sim x_2 + \xi_0$ if and only if $x_1 \equiv x_2 \mod \mathfrak{p}^{c(\Omega)}$. Similarly, if $v(\boldsymbol{a}) = 1$, then $v(\boldsymbol{b}) > 0$ and $x_1 + \xi_0 \sim x_2 + \xi_0$ if and only if $x_1 \equiv x_2 \mod \mathfrak{p}^{c(\Omega)+1}$. The rest of the cases are computed similarly. □

Let us remark that the coset representatives in the previous lemma depend only on $c(\Omega)$ since we are in the case $c(\Omega) \geq c(\pi)$.

Projection onto the test vector. Put $e(L/F) = 1$ if L/F is unramified, $e(L/F) = 2$ if L/F is ramified. Denote by η the quadratic character of $F^\times$ associated to L/F.

Proposition 6.3. *If* $c(\pi) \geq 2$, *then*

$$\tilde{J}_\pi(f) = q^{-c(\Omega)} \frac{L(1, 1_F)L(1, \eta)}{e(L/F)}. \tag{6-10}$$

If $c(\pi) = 1$, *then*

$$\tilde{J}_\pi(f) = q^{-c(\Omega)} \frac{L(1, 1_F)L(1, \eta)}{e(L/F)L(2, 1_F)}. \tag{6-11}$$

Proof. By (6-5), (6-6) and (6-7), this proposition is equivalent to the statement that

$$(e', \phi) = L(1, 1_F).$$

To show this, first observe

$$(e', \phi) = \sum_{t \in T/(T \cap ZK')} \Omega^{-1}(t)(\pi(t)\phi, \phi) = \sum_{t \in T/(T \cap ZK')} \Omega^{-1}(t)(\pi(h^{-1}th)\phi_0, \phi_0).$$

Recall that $h = \left[\begin{smallmatrix} \varpi^s & \\ & 1 \end{smallmatrix}\right] w$ with $s = c(\Omega) - c(\pi)$. We give the details of the case $c(\pi) \geq 2$ here. The other case is computed similarly. Hence, assume that $c(\pi) \geq 2$, so $L(s, \pi) = 1$. Then, for $g \in \mathrm{GL}(2)$,

$$(\pi(g)\phi_0, \phi_0) = \int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix} g\right) d^\times u.$$

First suppose $t = x + \xi_0$, where $x \in \mathfrak{p}$ and $v(x) \leq c(\Omega) + v(\boldsymbol{a})$. Note

$$\begin{aligned} h^{-1}th &= \begin{bmatrix} x - \boldsymbol{b} & \varpi^s \boldsymbol{a} \\ -\varpi^{-s}\boldsymbol{c} & x \end{bmatrix} \\ &= \varpi^{-s}\boldsymbol{c} \begin{bmatrix} 1 & (\boldsymbol{b} - x)\varpi^s/\boldsymbol{c} \\ & 1 \end{bmatrix} \begin{bmatrix} \det(t)\varpi^{2s}/\boldsymbol{c}^2 & \\ & 1 \end{bmatrix} w \begin{bmatrix} 1 & -\varpi^s x/\boldsymbol{c} \\ & 1 \end{bmatrix}. \end{aligned}$$

Since the rightmost matrix lies in $K_1(\mathfrak{p}^{c(\pi)})$, we have

$$(\pi(h^{-1}th)\phi_0, \phi_0) = W_0\left(\begin{bmatrix} \det(t)\varpi^{2s}/\boldsymbol{c}^2 & \\ & 1 \end{bmatrix} w\right) = 0,$$

where the last equality follows from Lemma 6.1(ii). Now suppose $t = 1 + y\xi_0$, where $y \in \mathfrak{o}$. If $y = 0$, then

$$(\pi(h^{-1}th)\phi_0, \phi_0) = (\phi_0, \phi_0) = 1.$$

Otherwise, assume $v(y) < c(\Omega)$ and write

$$h^{-1}th = \begin{bmatrix} 1 & \varpi^s \boldsymbol{a} y \\ & 1 \end{bmatrix} \begin{bmatrix} \det(t) & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ -\varpi^{-s}\boldsymbol{c}y & 1 \end{bmatrix}.$$

Then, by Lemma 6.1(iii),

$$\begin{aligned}(\pi(h^{-1}th)\phi_0, \phi_0) &= \int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} \det(t)u & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\varpi^{-s}\boldsymbol{c}y & 1 \end{bmatrix}\right) d^\times u \\ &= \begin{cases} (1-q)^{-1} & \text{if } v(y) = c(\Omega) - 1, \\ 0 & \text{else.} \end{cases}\end{aligned}$$

Observe that $\int_{1+\varpi^k \mathfrak{o}_L} \Omega^{-1}(u)\, d^\times u = 0$ for $0 < k < c(\Omega)$, together with $\Omega^{-1}|_{\mathfrak{o}^\times} = 1$, implies

$$\sum_{y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)} : v(y) \geq k} \Omega^{-1}(1 + y\xi_0) = 0.$$

Hence, for $0 < k \leq c(\Omega)$, we have

$$\sum_{y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)} : v(y) = k} \Omega^{-1}(1 + y\xi_0) = \begin{cases} 0 & \text{if } 0 < k < c(\Omega) - 1, \\ -1 & \text{if } k = c(\Omega) - 1 \text{ and } c(\Omega) > 1, \\ 1 & \text{if } k = c(\Omega). \end{cases}$$

Summing up gives the desired calculation

$$(e', \phi) = 1 + (1-q)^{-1} \sum_{y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)} : v(y) = c(\Omega) - 1} \Omega(1 + y\xi_0)^{-1} = \frac{1}{1 - q^{-1}},$$

since here $c(\Omega) \geq 2$. □

7. A central-value formula

In this section we work globally. Specifically, let L/F be a quadratic extension of number fields, $\mathbb{A}$ the adèles of F and $\mathbb{A}_L$ the adèles of L. Let Δ and Δ_L be the absolute values of the discriminants of F and L, and let $\eta = \eta_{L/F}$ be the quadratic idèle class character associated to L/F via class field theory.

Set $G = \mathrm{GL}(2)/F$. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character, and Ω a unitary character of $\mathbb{A}_L^\times/L^\times\mathbb{A}^\times$. Assume the sign of the functional equation $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = 1$, where π_L is the base change of π to L. Then by [Waldspurger 1985; Tunnell 1983; Saito 1993], one knows that there is a unique quaternion algebra (possibly the split matrix algebra) D/F in which L embeds, such that π has a Jacquet–Langlands transfer to a representation π' of $D^\times(\mathbb{A})$ and the local Hom spaces $\mathrm{Hom}_{L_v^\times}(\pi'_v, \Omega_v) \neq 0$ for all places v, and in fact have dimension 1. Fix this D and π', and write G' for $D^\times$, regarded as an algebraic group over F. Let T be a torus in G' whose F-points are isomorphic to $L^\times$, and view Ω as a character of $T(\mathbb{A})/Z(\mathbb{A})$, where Z is the center of G'.

Let ψ be the standard additive character on $\mathbb{A}/F$, i.e., the composition of the trace map with the standard additive character on $\mathbb{A}_\mathbb{Q}$. Let S be a finite set of places of F containing all archimedean places, such that, for all $v \notin S$, ψ, π and Ω are unramified and L is not ramified at or above v.

Put on $G'(\mathbb{A})$ the product of the local Tamagawa measures times $L^S(2, 1_F)$, i.e., take the local Tamagawa measure dg_v for $v \in S$ and dg_v normalized so that $G(\mathfrak{o}_v) \cong G'(\mathfrak{o}_v)$ has volume 1 if $v \notin S$ (see, e.g., [Jacquet and Chen 2001, Section 2] for the definition of local Tamagawa measures). Note we will renormalize our measure on $G'(\mathbb{A})$ later in Section 7C.

Jacquet and Chen [2001] prove a formula for a distribution appearing on the spectral side of the relative trace formula,

$$J_{\pi'}(f) = \sum_\phi \int_{T(\mathbb{A})/Z(\mathbb{A})T(F)} \pi'(f)\phi(t)\Omega(t)^{-1}\,dt \int_{T(\mathbb{A})/Z(\mathbb{A})T(F)} \overline{\phi(t)\Omega(t)^{-1}}\,dt, \quad (7\text{-}1)$$

where ϕ runs over an orthonormal basis for the space of π'. Here $T(\mathbb{A})$ and $Z(\mathbb{A})$ are given the product of local Tamagawa measures, $T(F)$ has the counting measure, and dt is the quotient measure.

Let S_{inert} be the set of finite places v in S such that L_v/F_v is inert (ramified or unramified). For $v \in S_{\text{inert}}$, as in (6-2), define

$$\tilde{J}_{\pi'_v}(f_v) = \int_{G'(F_v)} f_v(g)(\pi'_v(g)e'_v, e'_v)\,dg_v,$$

where e'_v is a norm 1 vector such that $\pi'_v(t)e'_v = \Omega_v(t)e'_v$ for all $t \in T(F_v)$. For $v \in S - S_{\text{inert}}$, set

$$\tilde{J}_{\pi'_v}(f_v) = \sum_W \left(\int_{F_v^\times} \pi'_v(f_v)W\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)\Omega\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)^{-1} d^\times a \right.$$
$$\left. \times \overline{\int_{F_v^\times} W\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)\Omega\left(\begin{bmatrix} a & \\ & 1\end{bmatrix}\right)^{-1} d^\times a}\right),$$

where $d^\times a$ is the local Tamagawa measure and W runs over an orthonormal basis for the local Whittaker model $\mathcal{W}(\pi_v, \psi_v)$.

With the above normalizations, the formula of Jacquet and Chen is as follows.

Theorem 7.1 [Jacquet and Chen 2001]. *Let S be a set of places containing all infinite places and all places at which L, π or Ω is ramified. Let*

$$f = \prod f_v \in C_c^\infty(G'(\mathbb{A}_F))$$

with f_v the unit element of the Hecke algebra for $v \notin S$. Then

$$J_{\pi'}(f) = \frac{1}{2} \prod_S \tilde{J}_{\pi'_v}(f_v) \prod_{v \in S_{\text{inert}}} 2\epsilon(1, \eta_v, \psi_v) L(0, \eta_v) \times \frac{L_S(1, \eta) L^S\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L^S(1, \pi, \mathrm{Ad})}.$$

Note that if $\pi'(f)$ is an orthogonal projection onto a 1-dimensional subspace $\langle \phi \rangle$, then

$$J_{\pi'}(f) = \frac{\left| \int_{T(\mathbb{A})/Z(\mathbb{A})T(F)} \phi(t) \Omega(t)^{-1} \, dt \right|^2}{(\phi, \phi)}. \tag{7-2}$$

This expression is written to be invariant under replacing ϕ by a scalar multiple.

7A. ***Choice of test vector.*** To obtain an explicit L-value formula, we choose $f = \prod f_v$ so that it picks out a global test vector $\phi = \bigotimes \phi_v$ as follows.

First suppose v is a finite place of F. We denote by $\mathfrak{o}_v$, $\mathfrak{o}_{L_v}$, $\mathfrak{p}_v$ and ϖ_v what was denoted in previous sections by these symbols without the subscript v for the local field F_v. Since we have assumed that the central character is trivial, we may work with the congruence subgroups

$$K_{0,v}(\mathfrak{p}_v^n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G(\mathfrak{o}_v) : c \in \mathfrak{p}_v^n \right\}.$$

We assume that at any finite $v \in S_{\text{inert}}$ such that $c(\Omega_v) > 0$, we have $c(\Omega_v) \geq c(\pi_v)$. Recall that, if L_v/F_v is split or $0 \leq c(\pi_v) \leq c(\Omega_v)$, then we can identify $G'(F_v)$ with $G(F_v)$.

For $v \notin S$, let f_v be the characteristic function of $G(\mathfrak{o}_v)$. Then $\pi_v \cong \pi'_v$, and $\pi'_v(f_v)$ is orthogonal projection onto the local newvector ϕ_v.

Let $v \in S - S_{\text{inert}}$. Take $g_v \in G(F_v)$ such that $g_v^{-1} T(F_v) g_v$ is the diagonal subgroup of $G(F_v)$. Let f_v be the characteristic function of the subgroup of $G(F_v)$ given by

$$g_v^{-1} \begin{bmatrix} 1 & -\varpi_v^{-c(\Omega_v)} \\ & 1 \end{bmatrix} K_{0,v}(\mathfrak{p}_v^{c(\pi_v)}) \begin{bmatrix} 1 & \varpi_v^{-c(\Omega_v)} \\ & 1 \end{bmatrix} g_v$$

divided by its volume. Then ϕ_v is the unique (up to scalar multiples) vector in π_v fixed by this subgroup.

Consider $v \in S_{\text{inert}}$.

Suppose $c(\pi_v)=0$ or $c(\Omega_v)=0$. Let $R(\pi_v')$ be an order in $D(F_v)$ of reduced discriminant $\mathfrak{p}_v^{c(\pi_v)}$ such that $R(\pi_v')\cap L_v=\mathfrak{o}_v+\varpi_v^{c(\Omega_v)}\mathfrak{o}_{L_v}$ (see [Gross 1988, Proposition 3.4]). Note $R(\pi_v')$ is unique up to $T(F_v)$-conjugacy. In this case, we take f_v to be the characteristic function of $R(\pi_v')^\times$ divided by its volume. Then $\pi_v'(f_v)$ acts as orthogonal projection onto the local Gross–Prasad test vector ϕ_v [Gross and Prasad 1991], except in the case that $c(\pi_v)\geq 2$ and L_v/F_v is ramified. (Note [Gross and Prasad 1991] also assumes F_v has odd residual characteristic if π_v is supercuspidal because of this restriction in [Tunnell 1983], but this hypothesis is no longer needed due to [Saito 1993].) When $c(\Omega_v)=0$, $c(\pi_v)\geq 2$ and L_v/F_v is ramified, $\pi_v'(f_v)$ acts as orthogonal projection onto a 2-dimensional space containing a vector ϕ_v which satisfies $\pi_v'(t_v)\phi_v=\Omega_v(t_v)\phi_v$ for all $t_v\in T(F_v)$ [Gross and Prasad 1991, Remark 2.7]; hence on this space any linear form in $\mathrm{Hom}(\pi_v,\Omega_v)$ is simply a multiple of the map $\phi_v'\mapsto(\phi_v',\phi_v)$.

If $0<c(\pi_v)\leq c(\Omega_v)$, take g_v so that $g_v^{-1}T(F_v)g_v$ is of the form (2-16), and let K_v be such that $g_vK_vg_v^{-1}$ is the subgroup in (6-3). Let f_v be the characteristic function of K_v divided by its volume, so $\pi_v(f_v)$ acts as orthogonal projection onto the line generated by ϕ_v, the unique (up to scalar multiples) vector in π_v fixed by K_v.

Lastly, suppose v is an infinite place of F. Let K_v be a maximal compact subgroup of $G'(F_v)$ whose restriction to $T(F_v)$ remains maximal compact. Let ϕ_v be a vector of minimal weight such that $\pi_v'(t_v)\phi_v=\Omega_v(t_v)\phi_v$ for $t_v\in K_v\cap T(F_v)$. Choose f_v so that $\pi_v'(f_v)$ is orthogonal projection onto $\langle\phi_v\rangle$.

Take $f=\prod f_v$ and $\phi=\otimes\phi_v$, so $\pi(f)$ acts as orthogonal projection onto a finite-dimensional space V containing ϕ. Local considerations show the toric period integral vanishes on the orthogonal complement of $\langle\phi\rangle$ in V, and hence one has (7-2).

7B. ***Archimedean factors.*** Here we recall from [Martin and Whitehouse 2009] certain archimedean constants $C_v(L,\pi,\Omega)$. Let v be an infinite place of F. By assumption, Ω_v is a unitary character of L_v.

First suppose $F_v=\mathbb{R}$ and $L_v=\mathbb{R}\oplus\mathbb{R}$. Write

$$\Omega_v(x_1,x_2)=\left|\frac{x_1}{x_2}\right|^{it}\mathrm{sgn}^{m_v}\left(\frac{x_1}{x_2}\right),$$

where $t\in\mathbb{R}$ and m_v is 0 or 1. If $\pi_v=\mu_v\times\mu_v^{-1}$ is a principal series with Laplacian eigenvalue λ_v, let $\epsilon_v\in\{0,1\}$ such that $\mu_v\Omega_v=|\cdot|^r\mathrm{sgn}^{\epsilon_v}$ for some r. Then we put

$$C_v(L,\pi,\Omega)=\left(\frac{8\pi^2}{\lambda_v}\right)^{\epsilon_v}.$$

If π is a discrete series of weight k_v, put

$$C_v(L,\pi,\Omega)=2^{k_v}.$$

Now suppose $F_v = \mathbb{R}$ and $L_v = \mathbb{C}$. Write $\Omega_v(z) = (z/\bar{z})^{\pm m_v}$, where $m_v \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. If $\pi_v = \mu_v \times \mu_v^{-1}$ is a principal series where μ_v is of the form $|\cdot|^{r_v}\mathrm{sgn}^{\epsilon_v}$, then

$$C_v(L, \pi, \Omega) = (2\pi)^{2m_v} \prod_{j=0}^{m_v-1} (\lambda_v + j(j+1))^{-1},$$

where $\lambda_v = \frac{1}{4} - r_v^2$. If π_v is a discrete series of weight k_v, then

$$C_v(L, \pi, \Omega) = \frac{1}{\pi B(k_v/2 + m_v, k_v/2 - m_v)}$$

if $m_v < \frac{1}{2}(k_v - 1)$ and

$$C_v(L, \pi, \Omega) = \frac{(2\pi)^{2m_v - k_v} k_v!}{m_v! B(k_v/2 + m_v, 1 - k_v/2 + m_v)}$$

if $m_v \geq \frac{1}{2}(k_v - 1)$. Here $B(x, y)$ denotes the beta function.

Lastly suppose $F_v = \mathbb{C}$, so $L_v = \mathbb{C} \oplus \mathbb{C}$. Write Ω_v in the form

$$\Omega_v(z_1, z_2) = (z_1\bar{z}_1)^{it} \left(\frac{z_1}{\bar{z}_1}\right)^{m_v} (z_2\bar{z}_2)^{-it} \left(\frac{z_2}{\bar{z}_2}\right)^{-m_v},$$

where $t \in \mathbb{R}$ and $m_v \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then π_v is a principal series. Let k_v be its weight, λ_v the Laplacian eigenvalue and $\ell_v = \max(k_v, m_v)$. Then

$$C_v(L, \pi, \Omega) = \left(\frac{1}{2} + \ell_v\right) \binom{2\ell_v}{|k_v - m_v|} \prod_{j=k_v+1}^{\ell_v} \frac{4\pi^2}{4\lambda_v + j^2 - 1}.$$

7C. ***Proof of Theorem 1.1.*** We consider a measure on $G'(\mathbb{A})$ which is the product of local Tamagawa measures. Write $\Delta = \Delta_{\mathrm{inert}}\Delta_{\mathrm{split}}$, where Δ_{inert} is the part of Δ coprime to every place over which L/F splits. Then note that

$$\prod_{v \in S_{\mathrm{inert}}} 2\epsilon(1, \eta_v, \psi_v) L(0, \eta_v) = \frac{1}{\sqrt{c(\eta)c(\psi)}} \prod_{v \in S_{\mathrm{inert}}} e(L_v/F_v) = \sqrt{\frac{\Delta_{\mathrm{inert}}}{\Delta_L}} \prod_{v \in S_{\mathrm{inert}}} e(L_v/F_v).$$

Let $v \in S$ be finite. The calculations of $\tilde{J}_{\pi'_v}(f_v)$ below for when L_v/F_v is split, v is infinite, or at most one of π_v and Ω_v is ramified are taken from [Martin and Whitehouse 2009].

Suppose $L_v = F_v \oplus F_v$. Then

$$\tilde{J}_{\pi'_v}(f_v) = \begin{cases} q_v^{-c(\Omega_v)} \dfrac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)}{(W_{\pi_v}, W_{\pi_v})} & \text{if } \Omega_v \text{ is unramified,} \\ q_v^{-c(\Omega_v)} \dfrac{L(1, 1_{F_v})^2}{(W_{\pi_v}, W_{\pi_v})} & \text{if } \Omega_v \text{ is ramified,} \end{cases}$$

where W_{π_v} is the normalized Whittaker newvector. Furthermore,

$$\mathrm{vol}(\mathfrak{o}_v^\times)(W_{\pi_v}, W_{\pi_v}) = \begin{cases} L(1, \pi_v, \mathrm{Ad})L(1, 1_{F_v})/L(2, 1_{F_v}) & \text{if } \pi_v \text{ is unramified,} \\ L(1, \pi_v, \mathrm{Ad}) = L(2, 1_{F_v}) & \text{if } c(\pi_v) = 1, \\ 1 & \text{if } c(\pi_v) > 1. \end{cases}$$

Since we are using local Tamagawa measures, the product over all such v of $\mathrm{vol}(\mathfrak{o}_v^\times)$ is $\sqrt{\Delta_{\mathrm{split}}}$.

Suppose now L_v/F_v is inert. If π'_v is unramified, then $\tilde{J}_{\pi'_v}(f_v)$ is

$$\frac{q_v^{-c(\Omega_v)}}{e(L_v/F_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} L(1, \eta_v)^{\delta_v},$$

where $\delta_v = -1$ if Ω_v is unramified and $\delta_v = 1$ if Ω_v is ramified. If π_v is ramified and Ω_v is unramified, then $\tilde{J}_{\pi'_v}(f_v) = 1$. When both π_v and Ω_v are ramified, $\tilde{J}_{\pi'_v}(f_v)$ is calculated in Proposition 6.3.

Summing up, if π_v is unramified, then, up to factors of the form $\mathrm{vol}(\mathfrak{o}_v^\times)$ and $e(L_v/F_v)$, $\tilde{J}_{\pi'_v}(f_v)$ is

$$q^{-c(\Omega_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} L(1, \eta_v)^{\delta_v}.$$

If $c(\pi_v) = 1$, then, up to factors of the form $\mathrm{vol}(\mathfrak{o}_v^\times)$ and $e(L_v/F_v)$, $\tilde{J}_{\pi'_v}(f_v)$ is

$$\frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)}{L(1, \pi_v, \mathrm{Ad})}$$

if Ω_v is unramified and L_v/F_v is split or unramified; 1 if Ω_v is unramified and L_v/F_v is ramified; and

$$q^{-c(\Omega_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)}{L(1, \pi_v, \mathrm{Ad})} L(1, 1_{F_v})L(1, \eta_v)$$

if Ω_v is ramified.

If $c(\pi_v) \geq 2$, then, up to factors of the form $\mathrm{vol}(\mathfrak{o}_v^\times)$ and $e(L_v/F_v)$, $\tilde{J}_{\pi'_v}(f_v)$ is 1 if Ω_v is unramified and $q^{-c(\Omega_v)}L(1, 1_{F_v})L(1, \eta_v)$ if Ω_v is ramified.

Now suppose $v \mid \infty$. Then from [Martin and Whitehouse 2009] one has

$$\tilde{J}_{\pi'_v}(f_v) = \frac{C_v(L, \pi, \Omega)}{e(L_v/F_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})L(1, \eta_v)}.$$

Combining the above calculations completes the proof of Theorem 1.1.

Remark 7.2. When $S(\pi)\cap S(\Omega)=\varnothing$, Theorem 1.1 is exactly the main theorem of [Martin and Whitehouse 2009], though their choice of measure on $G'(\mathbb{A})$ is slightly different. Our set $S_0(\pi)$ is denoted by $S'(\pi)$ in that paper.

As in [Martin and Whitehouse 2009], one can rewrite this formula using the Petersson norm (ϕ_π, ϕ_π) of the new vector $\phi_\pi \in \pi$ instead of $L(1, \pi, \mathrm{Ad})$. The formula in [Martin and Whitehouse 2009] is also valid when $\omega_\pi = \eta$, and one could treat that case here similarly. The restriction that $\omega_\pi \in \{1, \eta\}$ is not inherent in the method, but is due to this assumption in [Jacquet and Chen 2001].

Remark 7.3. For many applications, one would like a formula for the *complete* ratio of L-values $L\big(\frac{1}{2}, \pi_L \otimes \Omega\big)/L(1, \pi, \mathrm{Ad})$. Theorem 1.1 of course gives this when $S_0=\varnothing$ (e.g., if the conductor $c(\pi)$ of π is squarefree and π and L/F have disjoint ramification). In general, one can of course multiply both sides by the appropriate local factors, but then the rest of the formula will depend on more than just the ramification of π and Ω together with their infinity types. Specifically, for $v \in S_1(\pi)$ and $\pi_v = \chi_v \,\mathrm{St}_v$, the local factor $L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)$ depends on the sign of χ_v when L_v/F_v is ramified. Similarly, for $v \in S_2(\pi)$, the local factor $L(1, \pi_v, \mathrm{Ad})$ depends on more than just the ramification of π_v.

8. An average-value formula

In this section, we prove Theorems 1.3, 1.4 and 1.5. Fix notation as in the first paragraph of Theorem 1.3.

8A. ***The trace formula.*** Let D/F be the quaternion algebra which is ramified precisely at the infinite primes and the primes dividing $\mathfrak{N}_0$. Set $G' = D^\times$ and $G=\mathrm{GL}(2)/F$. Let Z denote the center of either of these. Let ϵ be an element of the normalizer of $T(F)$ inside $G'(F)$ which does not lie in $T(F)$, so $\epsilon^2 \in Z(F)$ and $D(F)=L\oplus\epsilon L$. Then we may write an element of $G'(F)$ in the form

$$\begin{bmatrix} \alpha & \beta\epsilon \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad \alpha, \beta \in L.$$

With this representation,

$$T = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix} \right\}.$$

As in Section 7, let ψ be the standard additive character of $\mathbb{A}/F$, and take the product of the local Tamagawa measures on $T(\mathbb{A})$, $G'(\mathbb{A})$, $G(\mathbb{A})$ and $Z(\mathbb{A})$. For a cuspidal automorphic representation π' of $G'(\mathbb{A})$, let $\mathrm{JL}(\pi')$ denote its Jacquet–Langlands transfer to $G(\mathbb{A})$. Denote by $\mathcal{F}'(\mathfrak{N}, 2\boldsymbol{k})$ the set of cuspidal automorphic representations π' of $G'(\mathbb{A})$ such that $\mathrm{JL}(\pi') \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})$. We call $\mathfrak{N}$ the conductor of π' and write $c(\pi') = \mathfrak{N}$. Subject to assumption (1-3), we note that our choice of D guarantees $\mathrm{Hom}_T(\pi', \Omega) \neq 0$ for all $\pi' \in \mathcal{F}'(\mathfrak{N}, 2\boldsymbol{k})$.

We now recall Jacquet's relative trace formula for G' from [Jacquet 1987]. This is an identity of the form

$$I(f) = J(f), \tag{8-1}$$

where $I(f)$ is a certain geometric distribution, and $J(f)$ is a certain spectral distribution. Specifically, let $f = \prod f_v \in C_c^\infty(G'(\mathbb{A}))$. The geometric (relative) orbital integrals of f are defined by

$$I(0, f) = \int_{T(\mathbb{A})} f(t)\Omega(t)\, dt,$$

$$I(\infty, f) = \int_{T(\mathbb{A})} f\left(t\begin{bmatrix} 0 & \epsilon \\ 1 & 0 \end{bmatrix}\right)\Omega(t)\, dt$$

and

$$I(b, f) = \int_{T(\mathbb{A})/Z(\mathbb{A})} \int_{T(\mathbb{A})} f\left(s\begin{bmatrix} 1 & \epsilon\beta \\ \bar{\beta} & 1 \end{bmatrix}t\right)\Omega(st)\, ds\, dt,$$

where $b = \epsilon N(\beta)$ for $\beta \in L^\times$. Note this latter integral only depends on b and not the choice of a specific β. Then the left-hand (geometric) side of (8-1) is

$$I(f) = \mathrm{vol}(T(\mathbb{A})/Z(\mathbb{A})T(F))(I(0, f) + \delta(\Omega^2)I(\infty, f)) + \sum_{b \in \epsilon N(L^\times)} I(b, f), \tag{8-2}$$

where $\delta(\chi) = 1$ if χ is trivial and $\delta(\chi) = 0$ otherwise.

We now describe $J(f)$, but for simplicity only in the situation that is relevant for us. Namely, for each $v \mid \infty$, fix an embedding $\iota_v : G'(F_v) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ and let π'_{2k_v} be the irreducible $(2k_v - 1)$-dimensional representation of $G'(F_v)$ given by $\pi'_{2k_v} = (\mathrm{Sym}^{2k_v-2} \otimes \det^{1-k_v}) \circ \iota_v$. Hence $\mathrm{JL}(\pi'_{2k_v})$ is the holomorphic discrete series of weight $2k_v$ on $G(F_v)$. The assumption that $|m_v| < k_v$ implies that there is a 1-dimensional subspace of π'_{2k_v} consisting of vectors w_v such that $\pi'_{2k_v}(t)w_v = \Omega_v(t)w_v$ for all $t \in T(F_v)$. Fix such a vector $w_v \in \pi'_{2k_v}$ which satisfies $(w_v, w_v) = 1$. For all $v \mid \infty$, we may take $f_v \in C_c^\infty(G'(\mathbb{R}))$ as in Section 7A, so that

$$\int_{Z(F_v)} f_v(zg)\, dz = \frac{2k_v - 1}{\mathrm{vol}(G'(F_v)/Z(F_v))}\overline{(\pi'_{2k_v}(g)w_v, w_v)}$$

(cf. [Feigon and Whitehouse 2009, Lemma 3.4]).

For a cuspidal automorphic representation π' of $G'(\mathbb{A})/Z(\mathbb{A})$, we consider the spectral distribution

$$J_{\pi'}(f) = \sum_\phi P_D(\pi'(f)\phi)\overline{P_D(\phi)},$$

where ϕ runs over an orthonormal basis for π' and P_D is defined as in (1-1). In general, the spectral side $J(f)$ of (8-1) is a sum over all π' of $J_{\pi'}(f)$ plus a noncuspidal contribution. However, things simplify greatly for our choice of f.

We already specified f_v for $v \mid \infty$. Now let $v < \infty$ and put $m_v = c(\Omega_v)$. For such a v, as in Section 7A, we take f_v to be the characteristic function of $R_v^\times$ divided by its volume, for an order R_v of $G'(F_v)$ chosen as follows. If $v \nmid \mathfrak{N}$, then $G'(F_v) \cong G(F_v)$ and we take R_v to be a maximal order optimally containing $\mathfrak{o}_v + \varpi_v^{m_v}\mathfrak{o}_{L_v}$. If $v \mid \mathfrak{N}_0$, then $G'(F_v)$ is not split and we take R_v to be a maximal order containing $\mathfrak{o}_{L_v}$. If $v \mid \mathfrak{N}_1$, then $G'(F_v) \cong G(F_v)$ and, at least when v is odd, we can take

$$R_v = \left\{ \begin{bmatrix} \alpha & \beta\epsilon_v \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : \mathrm{Tr}(\alpha), \mathrm{Tr}(\beta) \in \mathfrak{o}_v,\ \alpha, \beta \in \mathfrak{p}_v^{1-m_v}\mathfrak{o}_{L_v}, \right. \\ \left. \text{and } \alpha - \beta \in \mathfrak{o}_v + \mathfrak{p}_v^{m_v}\mathfrak{o}_{L_v} \right\}. \tag{8-3}$$

Note that for each $v \nmid \mathfrak{N}_0$, this agrees with our choice of test functions in Section 7A. The difference of the present choice of f_v for $v \mid \mathfrak{N}_0$ is simply out of convenience so we can directly apply local calculations from [Feigon and Whitehouse 2009]. What is important is that one still has $\pi_v(f_v)$ being orthogonal projection onto our local test vector for $v \mid \mathfrak{N}_0$ (cf. [Feigon and Whitehouse 2009, Lemma 3.3]).

Consequently, for this f, assuming $k_v > 1$ for some $v \mid \infty$, the spectral side of (8-1) is given by

$$J(f) = \sum_{\mathfrak{N}'} \sum_{\pi' \in \mathcal{F}'(\mathfrak{N}', 2\boldsymbol{k})} J_{\pi'}(f), \tag{8-4}$$

where $\mathfrak{N}'$ runs over ideals which divide $\mathfrak{N}$ and are divisible by $\mathfrak{N}_0$. This is because, for our choice of f', $\pi'(f')$ is zero unless π' is of weight $2\boldsymbol{k}$ and has conductor dividing $\mathfrak{N}$. Furthermore, by our choice of D, $J_{\pi'}(f)$ vanishes for local reasons if the conductor of π' is not divisible by $\mathfrak{N}_0$ (cf. [Feigon and Whitehouse 2009, Lemmas 3.6 and 3.7]). (The avoidance of the case $k_v = 1$ for all $v \mid \infty$ is purely for simplicity, for in this case there is also contribution from the residual spectrum, which one would treat as in [Feigon and Whitehouse 2009].)

8B. ***Spectral calculations.*** Here we compute the spectral expansion (8-4). For $\pi' \in \mathcal{F}'(\mathfrak{N}, 2\boldsymbol{k})$, we see that $J_{\pi'}(f) = |P_D(\phi)|^2/(\phi, \phi)$. Hence Theorem 1.1 implies

$$J_{\pi'}(f) = \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\, L^{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N})}(1, \eta) L_{S(\mathfrak{C}_0)}(1, \eta)^2 \\ \times \prod_{v \mid \infty} \frac{2k_v - 1}{\pi} \binom{2k_v - 2}{k_v - m_v - 1} \frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L(1, \pi, \mathrm{Ad})}. \tag{8-5}$$

We now need to extend this equality to general $\pi' \in \mathcal{F}'(\mathfrak{N}', 2\boldsymbol{k})$, where $\mathfrak{N}'$ divides $\mathfrak{N}$ and is divisible by $\mathfrak{N}_0$. For $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$, let R'_v be the maximal order of

$G'(F_v) \cong G(F_v)$ which contains R_v given by (8-3). Let $f' = \prod f'_v$, where $f'_v = f_v$ if $v \nmid (\mathfrak{N}')^{-1}\mathfrak{N}$, and f'_v is the characteristic function of $(R'_v)^\times$ divided by its volume if $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$. Now, f' agrees with our choice of test function for π' in Section 7A, and Theorem 1.1 gives

$$J_{\pi'}(f') = \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\, L^{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N})}(1, \eta) L_{S(\mathfrak{C}_0)}(1, \eta)^2 \times \prod_{v \mid (\mathfrak{N}')^{-1}\mathfrak{N}} L(1, \eta_v) \prod_{v\mid\infty} \frac{2k_v - 1}{\pi} \binom{2k_v - 2}{k_v - m_v - 1} \frac{L(\frac{1}{2}, \pi_L \otimes \Omega)}{L(1, \pi, \mathrm{Ad})}. \tag{8-6}$$

From Theorem 7.1, we see that

$$J_{\pi'}(f) = J_{\pi'}(f') \prod_{v\mid(\mathfrak{N}')^{-1}\mathfrak{N}} \frac{\tilde{J}_{\pi'_v}(f_v)}{\tilde{J}_{\pi'_v}(f'_v)}.$$

From [Martin and Whitehouse 2009, Section 2.2.4], we know

$$\tilde{J}_{\pi'_v}(f'_v) = q_v^{-m_v} L(2, 1_{F_v}) L(1, \eta_v) \frac{1}{L(1, \pi_v, \mathrm{Ad})},$$

so it remains to compute $\tilde{J}_{\pi'_v}(f_v)$. Here $v \mid (\mathfrak{N}')^{-1}\mathfrak{N} \supset \mathfrak{N}_1$, so $\pi'_v = \pi_v$ is unramified and $m_v = 1$. We may write $\pi_v = \chi \times \chi^{-1}$, where $\chi = \chi_v$ is an unramified (unitary) character of $F_v^\times$.

Note $\pi_v(f_v)$ is orthogonal projection onto $\pi_v^{R_v^\times}$. Embedding L_v in $M_2(F_v)$ as in (2-16), we may write

$$R_v^\times = K_v := \begin{bmatrix} \mathfrak{o}_v^\times & \mathfrak{p}_v^{m_v} \\ \mathfrak{p}_v^{1-m_v} & \mathfrak{o}_v^\times \end{bmatrix} = \begin{bmatrix} \mathfrak{o}_v^\times & \mathfrak{p}_v \\ \mathfrak{o}_v & \mathfrak{o}_v^\times \end{bmatrix}.$$

Note

$$K_v = h_v \mathrm{GL}_2(\mathfrak{o}_v) h_v^{-1} \cap h_v \begin{bmatrix} \varpi_v^{-1} & \\ & 1 \end{bmatrix} \mathrm{GL}_2(\mathfrak{o}_v) \begin{bmatrix} \varpi_v & \\ & 1 \end{bmatrix} h_v^{-1},$$

where $h_v = \left[\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right]$. So if we put ϕ_0 to be a newvector in π_v and $\phi'_0 = \pi_v(h_v^{-1})\phi_0$, then

$$\pi_v^{K_v} = \langle \pi_v(h_v)\phi_0,\ \pi_v(h_v)\phi'_0 \rangle.$$

Normalize ϕ_0 so that $(\phi_0, \phi_0) = 1$.

Lemma 8.1. *We have*

$$(\phi_0, \phi'_0) = (\phi'_0, \phi_0) = \frac{q_v^{-1/2}}{1 + q_v^{-1}} (\chi(\varpi_v) + \chi(\varpi_v)^{-1}). \tag{8-7}$$

Proof. In the induced model for π_v, we have

$$(\phi'_0, \phi_0) = (\pi(h_v)\phi_0, \phi_0) = \int_{\mathrm{GL}_2(\mathfrak{o}_v)} \phi_0(k h_v)\, dk.$$

We may then use the fact that the subgroup K_v of $\mathrm{GL}_2(\mathfrak{o}_v)$ is normalized by h_v to get the lemma. □

Lemma 8.2. *For $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$, so that π_v is unramified and $m_v = 1$, we have* $\tilde{J}_{\pi_v}(f_v) = q_v^{-1}$.

Proof. Write $\phi_1 = \pi_v(h_v)\phi_0$ and

$$\phi_2 = \frac{\phi_0 - (\phi_0, \phi_1)\phi_1}{(1 - (\phi_0, \phi_0')^2)^{1/2}} = \left(\frac{L(1, \pi_v, \mathrm{Ad})(1 + q_v^{-1})}{L(2, 1_{F_v})}\right)^{1/2} (\phi_0 - (\phi_0, \phi_1)\phi_1),$$

so that $\{\phi_1, \phi_2\}$ forms an orthonormal basis for $\pi_v^{K_v}$. As in Section 6, put

$$e' = \sum_{t \in T_v/(T_v \cap Z_v K_v)} \Omega_v^{-1}(t)\pi_v(t)\phi_1,$$

so

$$\tilde{J}_{\pi_v}(f_v) = \mathrm{vol}(K_v)^{-1} \int_{K_v} \frac{(\pi_v(k)e', e')}{(e', e')}\, dk = \frac{1}{|T_v/(T_v \cap Z_v K_v)|(\phi_1, e')} (\pi_v(f_v)e', e').$$

Since $\pi_v(f_v)e' = (e', \phi_1)\phi_1 + (e', \phi_2)\phi_2$, we have

$$\tilde{J}_{\pi_v}(f_v) = \frac{1}{|T_v/(T_v \cap Z_v K_v)|(\phi_1, e')} \big((e', \phi_1)(\phi_1, e') + (e', \phi_2)(\phi_2, e')\big).$$

From [Martin and Whitehouse 2009, Section 2.2.4], where (ϕ_1, e') is denoted $\langle v_0, e_T''\rangle / \langle v_0, v_0\rangle$, we know $(\phi_1, e') = L(2, 1_{F_v})/L(1, \pi_v, \mathrm{Ad})$. Thus

$$\tilde{J}_{\pi_v}(f_v) = \frac{1}{|T_v/(T_v \cap Z_v K_v)|} \left(\frac{L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} + \frac{L(1, \pi_v, \mathrm{Ad})}{L(2, 1_{F_v})} |(\phi_2, e')|^2\right). \quad (8\text{-}8)$$

Note

$$(\phi_2, e') = \left(\frac{L(1, \pi_v, \mathrm{Ad})(1 + q_v^{-1})}{L(2, 1_{F_v})}\right)^{1/2} \left((\phi_0, e') - (\phi_0', \phi_0)\frac{L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})}\right). \quad (8\text{-}9)$$

Hence it suffices to compute

$$\begin{aligned}(\phi_0, e') &= \sum_{t \in T_v/(T_v \cap Z_v K_v)} \Omega_v(t)(\phi_0, \pi_v(t)\phi_1) \\ &= \sum_{t \in T_v/(T_v \cap Z_v K_v)} \Omega_v^{-1}(t)(\pi_v(h_v^{-1} t h_v)\phi_0', \phi_0).\end{aligned}$$

Using the set of representatives for $T_v/(T_v \cap Z_v K_v)$ given in Lemma 6.2, we see

$$(\phi_0, e') = \sum_{x \in \mathfrak{p}_v/\mathfrak{p}_v} \Omega_v^{-1}(x + \xi_{0,v}) \left(\pi_v\left(\begin{bmatrix} \varpi_v^{-1} x & \boldsymbol{c}\varpi_v^{-1} \\ -\boldsymbol{a} & x - \boldsymbol{b} \end{bmatrix}\right)\phi_0, \phi_0\right)$$
$$+ \sum_{y \in \mathfrak{o}_v/\mathfrak{p}_v} \Omega_v^{-1}(1 + y\xi_{0,v}) \left(\pi_v\left(\begin{bmatrix} \varpi_v^{-1} & \boldsymbol{c}\varpi_v^{-1} y \\ -\boldsymbol{a} y & 1 - \boldsymbol{b} y \end{bmatrix}\right)\phi_0, \phi_0\right). \quad (8\text{-}10)$$

Let ϕ and ϕ' be the realizations of the unit newvectors ϕ_0 and ϕ_0', respectively, in the Kirillov model for π_v with respect to an unramified ψ_v. Recall $\phi(z) = 0$ unless $z \in \mathfrak{o}_v$. Recall also the action of the standard Borel on the Kirillov model is given by

$$\left(\pi_v\begin{bmatrix} a & x \\ & d \end{bmatrix}\phi\right)(z) = \psi_v(xz/d)\phi(az/d).$$

Since v is odd and unramified, we may assume $\boldsymbol{b} = 0$ and $\boldsymbol{a}$ is a unit. For the $x = 0$ term in (8-10), note

$$\pi_v\begin{bmatrix} & \boldsymbol{c}\varpi_v^{-1} \\ -\boldsymbol{a} & \end{bmatrix}\phi(z) = \pi_v\begin{bmatrix} \boldsymbol{c}\varpi_v^{-1} & \\ & \boldsymbol{a} \end{bmatrix}\phi(z) = \phi(\varpi_v^{-1}z) = \phi'(z).$$

For $y \in \mathfrak{o}_v$, we have

$$\begin{aligned}\pi_v\left(\begin{bmatrix} \varpi_v^{-1} & \boldsymbol{c}\varpi_v^{-1}y \\ -\boldsymbol{a}y & 1 \end{bmatrix}\right)\phi(z) &= \pi_v\left(\begin{bmatrix} \varpi^{-1}(1+\boldsymbol{ac}y^2) & \boldsymbol{c}\varpi_v^{-1}y \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\boldsymbol{a}y & 1 \end{bmatrix}\right)\phi(z) \\ &= \psi_v(\boldsymbol{c}\varpi_v^{-1}yz)\phi(\varpi_v^{-1}z) = \phi(\varpi_v^{-1}z) = \phi'(z).\end{aligned}$$

In the last line, we used the facts that $1+\boldsymbol{ac}y^2 \in \mathfrak{o}_v^\times$, ψ is unramified and ϕ vanishes outside of $\mathfrak{o}_v$. Hence (8-10) becomes

$$(\phi_0, e') = \left(\Omega_v^{-1}(\xi_{0,v}) + \sum_{y\in\mathfrak{o}_v/\mathfrak{p}_v} \Omega_v^{-1}(1 + y\xi_{0,v})\right)(\phi_0', \phi_0) = 0, \tag{8-11}$$

as this character sum is zero. Combining (8-7), (8-8) and (8-9) gives

$$\begin{aligned}\tilde{J}_{\pi_v}(f_v) &= \frac{1}{|T_v/(T_v \cap Z_v K_v)|}\left(\frac{L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} + \frac{q_v^{-1}}{1+q_v^{-1}}\big(\chi(\varpi_v) + \chi(\varpi_v)^{-1}\big)^2\right) \\ &= \frac{1+q_v^{-1}}{|T_v/(T_v \cap Z_v K_v)|}.\end{aligned}$$

This, with Lemma 6.2, gives the result. □

Hence,

$$\frac{\tilde{J}_{\pi_v'}(f_v)}{\tilde{J}_{\pi_v'}(f_v')} = \frac{L(1, \pi_v, \mathrm{Ad})}{L(2, 1_{F_v})L(1, \eta_v)} \tag{8-12}$$

for $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$, which yields

$$\begin{aligned}J(f) = {} & \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}L^{S(\mathfrak{N}_0)}(2, 1_F)L_{S(\mathfrak{N})}(1, \eta)L_{S(\mathfrak{C}_0)}(1, \eta)^2 \prod_{v\mid\infty}\frac{2k_v - 1}{\pi}\binom{2k_v-2}{k_v - m_v - 1} \\ & \times \sum_{\mathfrak{N}'}\sum_{\pi\in\mathcal{F}(\mathfrak{N}',2\boldsymbol{k})}\left(\prod_{v\mid(\mathfrak{N}')^{-1}\mathfrak{N}}\frac{L(1, \pi_v, \mathrm{Ad})}{L(2, 1_{F_v})}\right)\frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L(1, \pi, \mathrm{Ad})}.\end{aligned}$$

Here $\mathfrak{N}'$ runs over all divisors of $\mathfrak{N}$ which are divisible by $\mathfrak{N}_0$. Writing

$$\prod_{v|(\mathfrak{N}')^{-1}\mathfrak{N}} \frac{L(1,\pi_v,\mathrm{Ad})}{L(2,1_{F_v})} \cdot \frac{1}{L(1,\pi,\mathrm{Ad})} = \prod_{v|\mathfrak{N}'} \frac{L(2,1_{F_v})}{L(1,\pi_v,\mathrm{Ad})} \cdot \frac{1}{L_{S(\mathfrak{N})}(2,1_F) L^{S(\mathfrak{N})}(1,\pi,\mathrm{Ad})}$$

and observing $L(1,\pi_v,\mathrm{Ad}) = L(2,1_{F_v})$ for $v \mid \mathfrak{N}'$ and $\pi \in \mathcal{F}(\mathfrak{N}', \boldsymbol{k})$ gives

$$J(f) = \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}} \frac{L^{S(\mathfrak{N}_0)}(2,1_F)}{L_{S(\mathfrak{N})}(1,1_F)} L_{S(\mathfrak{C}_0)}(1,\eta)^2 \times \prod_{v|\infty} \frac{2k_v-1}{\pi} \binom{2k_v-2}{k_v-m_v-1} \sum_{\mathfrak{N}'} \sum_{\pi\in\mathcal{F}(\mathfrak{N}',2\boldsymbol{k})} \frac{L\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{L^{S(\mathfrak{N})}(1,\pi,\mathrm{Ad})}. \quad (8\text{-}13)$$

8C. *Geometric calculations.* We now obtain our average value formula from the trace formula (8-1) and spectral calculation (8-5) by computing the geometric side $I(f)$. Most of the calculations we need are done in [Feigon and Whitehouse 2009], with the proviso that our choice of test functions f_v (for $v \nmid \mathfrak{N}_1$) are essentially constant multiples of those therein (the test functions in [Feigon and Whitehouse 2009] also come "preintegrated over the center").

Lemma 8.3. *Let $b \in \epsilon N(L^\times)$. We have the following vanishing of local orbital integrals.*

(i) *If $v \mid \mathfrak{N}_0$, then $I(\infty, f_v) = 0$.*

(ii) *If $v \mid \mathfrak{N}_0$ and $b \notin \mathfrak{p}_v$, then $I(b, f_v) = 0$.*

(iii) *If $v \nmid \mathfrak{N}$ is finite and $v(1-b) > v(\mathfrak{d}_{L/F}c(\Omega))$, then $I(b, f_v) = 0$.*

(iv) *If $v \mid \mathfrak{N}_1$ and $v(1-b) > v(c(\Omega)) - 2$, then $I(b, f_v) = 0$.*

Proof. The first three results are directly from [Feigon and Whitehouse 2009, Lemmas 4.2, 4.10 and 4.11]. So suppose $v \mid \mathfrak{N}_1$ is odd and write $b = \epsilon N(\beta)$ for some $\beta \in L^\times$. For $I(b, f_v)$ to be nonzero we need that, for some $\alpha \in L_v^\times$ and $u \in L_v^1$,

$$\begin{bmatrix} \alpha & \\ & \bar{\alpha} \end{bmatrix} \begin{bmatrix} 1 & \epsilon\beta u \\ \overline{\beta u} & 1 \end{bmatrix} \in R_v^\times,$$

i.e.,

$$N(\alpha)(1-b) \in \mathfrak{o}_v^\times, \quad \mathrm{Tr}(\alpha) \in \mathfrak{o}_v^\times, \quad \alpha \in \mathfrak{p}_v^{1-m_v}\mathfrak{o}_{L_v}, \quad \alpha(1-\beta u) \in \mathfrak{o}_v + \mathfrak{p}_v^{m_v}\mathfrak{o}_{L_v}.$$

Note this implies $v(1-b) = -v(N(\alpha)) \le 2m_v - 2$. Hence if $v(1-b) \ge 2m_v - 1$, then $I(b, f_v) = 0$. □

Proposition 8.4. *If* $|\mathfrak{N}_0| > d_{L/F}(|\mathfrak{C}|/|\mathfrak{N}_1|)^{h_F}$, *then* $I(f) = 2L(1,\eta)I(0,f)$ *and*

$$I(0,f) = \frac{\Delta^2|\mathfrak{N}|}{\sqrt{c(\Omega)\Delta_L}} \frac{L_{S(\mathfrak{C}_0)}(1,\eta)}{L_{S(\mathfrak{N}_0)}(1,1_F)} L(2,1_F) \prod_{v|\infty} \frac{2k_v-1}{2\pi}.$$

Proof. This argument is adapted from the proof of [Feigon and Whitehouse 2009, Lemma 4.21]. By the first part of the previous lemma, we know the global orbital integral $I(\infty, f) = 0$. Arguing as in Feigon and Whitehouse's proof, we see that, if $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{N}_1^{-2}\mathfrak{C}|^{h_F}$, then $I(b,f) = 0$ for all b.

Next we compute $I(0,f)$. For $v \nmid \mathfrak{N}_1$, we recall the following calculations from [Feigon and Whitehouse 2009, Section 4.1]; see [Jacquet and Chen 2001, Section 2; Feigon and Whitehouse 2009, Section 2.1 and proof of Proposition 4.20] for necessary facts about local Tamagawa measures. Due to the difference in our definition of test functions from those in [Feigon and Whitehouse 2009], our local orbital integrals $I(0,f_v)$ (for $v \nmid \mathfrak{N}_1$) will be $\mathrm{vol}(Z_v \cap R_v^\times)/\mathrm{vol}(R_v^\times)$ times theirs for finite v, and $(2k_v-1)/\mathrm{vol}(G'(F_v)/Z(F_v))$ times theirs for infinite v.

For $v \mid \mathfrak{N}_0$,

$$\begin{aligned} I(0,f_v) &= \mathrm{vol}(\mathfrak{o}_{L_v}^\times/\mathfrak{o}_v^\times)\mathrm{vol}(Z_v \cap R_v^\times)/\mathrm{vol}(R_v^\times) \\ &= (q_v-1)L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(\mathfrak{o}_v^\times)^4, \end{aligned}$$

since $\mathrm{vol}(R_v^\times) = L(2,1_{F_v})^{-1}(q_v-1)^{-1}\mathrm{vol}(\mathfrak{o}_v^\times)^4$ and $R_v^\times \cap Z_v = \mathfrak{o}_v^\times$.

For a finite $v \nmid \mathfrak{N}$, we have $\mathrm{vol}(R_v^\times) = L(2,1_{F_v})^{-1}\mathrm{vol}(\mathfrak{o}_v^\times)^4$ and

$$I(0,f_v) = \begin{cases} \mathrm{vol}(\mathfrak{o}_{L_v}^\times/\mathfrak{o}_v^\times)\mathrm{vol}(Z_v \cap R_v^\times)/\mathrm{vol}(R_v^\times) & \\ \quad = L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(\mathfrak{o}_v^\times)^4 & \text{for } m_v = 0, \\ q^{-m_v}L(1,\eta_v)\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(R_v^\times) & \\ \quad = q^{-m_v}L(1,\eta_v)L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(\mathfrak{o}_v^\times)^4 & \text{for } m_v > 0. \end{cases}$$

For $v \mid \infty$,

$$I(0,f_v) = \mathrm{vol}(F_v^\times \backslash L_v^\times)\frac{2k_v-1}{\mathrm{vol}(G'(F_v)/Z(F_v))} = \frac{2k_v-1}{2\pi^2}.$$

Now, for $v \mid \mathfrak{N}_1$, our description of R_v readily implies

$$I(0,f_v) = \mathrm{vol}(\mathfrak{o}_v^\times(1+\mathfrak{p}_v^{m_v}\mathfrak{o}_{L_v}))/\mathrm{vol}(R_v^\times) = q^{-m_v}L(1,\eta_v)\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(R_v)^\times.$$

A simple calculation gives $\mathrm{vol}(R_v^\times) = q_v^{-1}\mathrm{vol}(\mathfrak{o}^\times)^4/L(1,1_{F_v})$. Hence when $v \mid \mathfrak{N}_1$, we have

$$I(0,f_v) = q_v^{1-m_v}L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_L^\times)/\mathrm{vol}(\mathfrak{o}^\times)^4.$$

Putting together the nonarchimedean calculations gives

$$\prod_{v<\infty} I(0,f_v) = \frac{|\mathfrak{N}|}{\sqrt{c(\Omega)}} L_{\mathrm{fin}}(2,1_F) \prod_{v|\mathfrak{N}_0} L(1,1_{F_v})^{-1} \prod_{v|\mathfrak{C}_0} L(1,\eta_v) \prod_{v<\infty} \frac{\mathrm{vol}(\mathfrak{o}_{L_v}^\times)}{\mathrm{vol}(\mathfrak{o}^\times)^4}.$$

Noting that $\prod_{v<\infty} \mathrm{vol}(\mathfrak{o}^\times) = \Delta^{-1/2}$ (and similarly over L), we have

$$\prod_{v<\infty} I(0, f_v) = \frac{\Delta^2|\mathfrak{N}|}{\sqrt{c(\Omega)\Delta_L}} L_{\mathrm{fin}}(2, 1_F) \prod_{v|\mathfrak{N}_0} L(1, 1_{F_v})^{-1} \prod_{v|\mathfrak{C}_0} L(1, \eta_v).$$

Recalling that $L(2, 1_F) = L_{\mathrm{fin}}(2, 1_F)/\pi^d$, we see

$$I(0, f) = \frac{\Delta^2|\mathfrak{N}|}{\sqrt{c(\Omega)\Delta_L}} L(2, 1_F) \prod_{v|\mathfrak{N}_0} L(1, 1_{F_v})^{-1} \prod_{v|\mathfrak{C}_0} L(1, \eta_v) \prod_{v|\infty} \frac{2k_v - 1}{2\pi}. \qquad \square$$

8D. ***Proofs.***

Proof of Theorem 1.3. The result immediately follows from our above calculations of both sides of the equality $J(f) = I(f) = 2L(1, \eta)I(0, f)$. □

Proof of Theorem 1.4. Suppose $\mathfrak{N}_1$ contains exactly one prime $\mathfrak{p}$. Put

$$\Sigma_{\mathfrak{N}}(\mathfrak{N}') = \prod_{v|\infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\pi \in \mathcal{F}(\mathfrak{N}', 2\boldsymbol{k})} \frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L^{S(\mathfrak{N})}(1, \pi, \mathrm{Ad})}.$$

Then Theorem 1.3 reads

$$\Sigma_{\mathfrak{N}}(\mathfrak{N}_0) + \Sigma_{\mathfrak{N}}(\mathfrak{N}) = 2^{2-d}\Delta^{3/2}|\mathfrak{N}|L(1, 1_{F_\mathfrak{p}})L_{S(\mathfrak{N}_0)}(2, 1_F)L^{S(\mathfrak{C}_0)}(1, \eta). \quad (8\text{-}14)$$

Applying our average value formula when $\mathfrak{N} = \mathfrak{N}_0$, we also see

$$\Sigma_{\mathfrak{N}_0}(\mathfrak{N}_0) = 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0|L_{S(\mathfrak{N}_0)}(2, 1_F)L^{S(\mathfrak{C}_0)}(1, \eta) \quad (8\text{-}15)$$

if $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$. (This is precisely [Feigon and Whitehouse 2009, Theorem 1.1].) For $\pi_\mathfrak{p}$ unramified, we have

$$L(1, 1_{F_\mathfrak{p}}) \frac{1}{1 + 2q_\mathfrak{p}^{-1} + q_\mathfrak{p}^{-2}} \le L(1, \pi_\mathfrak{p}, \mathrm{Ad}) \le L(1, 1_{F_\mathfrak{p}}) \frac{1}{1 - 2q_\mathfrak{p}^{-1} + q_\mathfrak{p}^{-2}},$$

which implies

$$\begin{aligned} L(1, 1_{F_\mathfrak{p}}) \frac{1}{(1 + q_\mathfrak{p}^{-1})^2} \Sigma_{\mathfrak{N}_0}(\mathfrak{N}_0) &\le \Sigma_{\mathfrak{N}}(\mathfrak{N}_0) \\ &\le L(1, 1_{F_\mathfrak{p}}) \frac{1}{(1 - q_\mathfrak{p}^{-1})^2} \Sigma_{\mathfrak{N}_0}(\mathfrak{N}_0). \quad (8\text{-}16) \end{aligned}$$

Combining the (in)equalities above gives

$$\begin{aligned} &\Sigma_{\mathfrak{N}}(\mathfrak{N}) \\ &\le 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0|L(1, 1_{F_\mathfrak{p}})L_{S(\mathfrak{N}_0)}(2, 1_F)L^{S(\mathfrak{C}_0)}(1, \eta)\left(|\mathfrak{p}| - \frac{1}{1 + 2|\mathfrak{p}|^{-1} + |\mathfrak{p}|^{-2}}\right), \end{aligned}$$

and a similar lower bound, which are precisely the bounds asserted in Theorem 1.4.

To get an asymptotic, we use a special case of [Feigon and Whitehouse 2009, Theorem 1.2], which is an asymptotic for

$$\sum_{\pi\in\mathcal{F}(\mathfrak{N}_0,2\boldsymbol{k})}\frac{L^{\mathfrak{p}}\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{L^{\mathfrak{p}}(1,\pi,\mathrm{Ad})}=\prod_{v\mid\infty}\binom{2k_v-2}{k_v-m_v-1}^{-1}\frac{\Sigma_{\mathfrak{N}}(\mathfrak{N}_0)}{L_{S(\mathfrak{N}_0)}(2,1_F)}$$

as $|\mathfrak{N}_0|\to\infty$. (Note $L^{\mathfrak{p}}\left(\frac{1}{2},\pi_L\otimes\Omega\right)=L\left(\frac{1}{2},\pi_L\otimes\Omega\right)$ since Ω is ramified at $\mathfrak{p}$.) Specifically, [Feigon and Whitehouse 2009, Theorem 1.2] tells us

$$\Sigma_{\mathfrak{N}}(\mathfrak{N}_0)\sim 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0|L(1,1_{F_{\mathfrak{p}}})L_{S(\mathfrak{N}_0)}(2,1_F)L^{S(\mathfrak{C}_0)}(1,\eta)\tag{8-17}$$

as $|\mathfrak{N}_0|\to\infty$ along a sequence of squarefree ideals $\mathfrak{N}_0$ coprime to $\mathfrak{C}$ satisfying our parity and ramification assumptions. Consequently, we have

$$\Sigma_{\mathfrak{N}}(\mathfrak{N})\sim 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0|L(1,1_{F_{\mathfrak{p}}})L_{S(\mathfrak{N}_0)}(1,\eta)L^{S(\mathfrak{C}_0)}(1,\eta)(|\mathfrak{p}|-1).$$

This gives the asymptotic asserted in the theorem. □

8E. ***Nonvanishing mod p.*** Let $\pi\in\mathcal{F}(\mathfrak{N},2\boldsymbol{k})$, and let $\boldsymbol{f}$ be the corresponding normalized Hilbert modular newform of weight $2\boldsymbol{k}$ and level $\mathfrak{N}$ over F. As before, Ω is a unitary character of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$ such that, for all $v\mid\infty$, $\Omega_v(z)=(z/\bar{z})^{\pm m_v}$ with $0\le m_v<k_v$. Put $\boldsymbol{m}=(m_1,\dots,m_d)$. Then Ω gives rise to a Hilbert modular form $\boldsymbol{g}$ over F of weight $\boldsymbol{m}+1=(m_1+1,\dots,m_d+1)$; see [Shimura 1978, Section 5]. Assume $m_1\equiv m_2\equiv\cdots\equiv m_d \mod 2$. This implies Ω is algebraic, so that the field of rationality $\mathbb{Q}(\boldsymbol{g})\subset\overline{\mathbb{Q}}$ [Shimura 1978, Proposition 2.8].

Put $k_0=\max_{v\mid\infty}k_v$ and $m_0=\max_{v\mid\infty}m_v$. Then Shimura [1978, Theorem 4.1] proved

$$\frac{D(s_0,\boldsymbol{f},\boldsymbol{g})}{\sqrt{\Delta}\,\pi^{2|\boldsymbol{k}|}(\boldsymbol{f},\boldsymbol{f})}\in\overline{\mathbb{Q}}(\boldsymbol{g})=\overline{\mathbb{Q}}$$

for any $s_0\in\mathbb{Z}$ such that $\frac{1}{2}(2k_0+m_0-1)<s_0<\frac{1}{2}(2k_0+m_0+2k_v-m_v)$ for all $v\mid\infty$. Here $D(s,\boldsymbol{f},\boldsymbol{g})$ is the Dirichlet series defined in [Shimura 1978], $(\boldsymbol{f},\boldsymbol{f})$ is the Petersson norm defined as in [Hida 1991], and $|\boldsymbol{k}|=\sum_{v\mid\infty}k_v$. Assume that $m_0\equiv 0 \mod 2$. Then, for $s_0=\frac{1}{2}(2k_0+m_0)$, this means

$$L^{\mathrm{alg}}\left(\tfrac{1}{2},\pi_L\otimes\Omega\right):=\frac{1}{L(1,\eta)}\frac{L_{\mathrm{fin}}\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{\sqrt{\Delta}\,\pi^{2|\boldsymbol{k}|}(\boldsymbol{f},\boldsymbol{f})}\in\overline{\mathbb{Q}}.\tag{8-18}$$

(Note that we normalize the algebraic part of the L-value in a different way than other authors.) Recall the archimedean L-factors are given by

$$L_v\left(\tfrac{1}{2},\pi_L\otimes\Omega\right)=(2\pi)^{-2k_v}4\Gamma(k_v+m_v)\Gamma(k_v-m_v),\quad v\mid\infty.$$

From [Hida and Tilouine 1993, Theorem 7.1; Hida 1991, (7.2c)] (cf. [Getz and Goresky 2012, Theorem 5.16]), we have

$$L(1,\pi,\mathrm{Ad}) = \frac{2^{2|\boldsymbol{k}|-1}}{\Delta^2 h_F |\mathfrak{N}|}(\boldsymbol{f},\boldsymbol{f}). \tag{8-19}$$

Thus,

$$\frac{L\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{L(1,\pi,\mathrm{Ad})} = 2^{2d+1-4|\boldsymbol{k}|}\Delta^{5/2}h_F|\mathfrak{N}|L(1,\eta)L^{\mathrm{alg}}\left(\tfrac{1}{2},\pi_L\otimes\Omega\right)\prod_{v|\infty}\Gamma(k_v+m_v)\Gamma(k_v-m_v).$$

Hence we can rewrite the average value formula from Theorem 1.3 as

$$2^{3d-4|\boldsymbol{k}|-1}\Delta h_F\prod_v(2k_v-2)!\sum_{\pi\in\mathcal{F}(\mathfrak{N},2\boldsymbol{k})}L^{\mathrm{alg}}\left(\tfrac{1}{2},\pi_L\otimes\Omega\right) = \frac{1}{L_{S(\Omega)}(1,\eta)}. \tag{8-20}$$

This immediately implies Theorem 1.5.

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A generalization of Kato's local ε-conjecture for (φ, Γ)-modules over the Robba ring

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We generalize Kato's (commutative) p-adic local ε-conjecture for families of (φ, Γ)-modules over the Robba rings. In particular, we prove the essential parts of the generalized local ε-conjecture for families of trianguline (φ, Γ)-modules. The key ingredients are the author's previous work on the Bloch–Kato exponential map for (φ, Γ)-modules and the recent results of Kedlaya, Pottharst and Xiao on the finiteness of cohomology of (φ, Γ)-modules.

1. Introduction

1A. *Introduction.* Since the works of Kisin [2003], Colmez [2008], and Bellaïche and Chenevier [2009], among others, the theory of (φ, Γ)-modules over the (relative) Robba ring has become one of the main focuses in the theory of p-adic Galois representations. In particular, the trianguline representation, which is a class of p-adic Galois representations defined using (φ, Γ)-modules over the Robba ring, is important since the rigid analytic families of p-adic Galois representations associated to Coleman–Mazur eigencurves (or more general eigenvarieties) turn out to be trianguline.

The recent works of Pottharst [2013] and Kedlaya, Pottharst and Xiao [Kedlaya et al. 2014] established the fundamental theorems (comparison with Galois cohomology, finiteness, base change property, Tate duality, Euler–Poincaré formula) in

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the theory of the cohomology of (φ, Γ)-modules over the relative Robba ring over $\mathbb{Q}_p$-affinoid algebras. As is suggested and actually given in [Kedlaya et al. 2014; Pottharst 2012], their results are expected to have many applications in number theory (e.g., eigenvarieties, nonordinary case of Iwasawa theory; see Remarks 1.6 and 1.7 below).

On the other hand, in [Nakamura 2014a], we generalized the theory of Bloch–Kato exponential maps and Perrin-Riou's exponential maps in the framework of (φ, Γ)-modules over the Robba ring. Since these maps are very important tools in Iwasawa theory, we expect that the results of [Nakamura 2014a] also have many applications.

As an application of both theories, the purpose of this article is to generalize Kato's p-adic local ε-conjecture [1993b] in the framework of (φ, Γ)-modules over the relative Robba ring over $\mathbb{Q}_p$-affinoid algebras, and prove the essential parts of its generalized version of the conjecture for rigid analytic families of trianguline (φ, Γ)-modules.

In this introduction, we briefly explain these conjectures; see Section 3 for the precise definitions. Let $G_{\mathbb{Q}_p}$ be the absolute Galois group of $\mathbb{Q}_p$. The main objects of Kato's local ε-conjecture are the pairs (Λ, T), where Λ is a semilocal ring such that $\Lambda/\mathfrak{m}_\Lambda$ is a finite ring of order a power of p (where $\mathfrak{m}_\Lambda$ is the Jacobson radical of Λ) and T is a Λ-representation of $G_{\mathbb{Q}_p}$, i.e., a finite projective Λ-module with a continuous Λ-linear $G_{\mathbb{Q}_p}$-action. Let $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T)$ be the complex of continuous cochains of $G_{\mathbb{Q}_p}$ with values in T. By the classical theory of Galois cohomology of $G_{\mathbb{Q}_p}$, this complex is a perfect complex of Λ-modules which satisfies the base change property, Tate duality, and other properties. This fact enables us to define the determinant

$$\mathrm{Det}_\Lambda(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T)),$$

which is a (graded) invertible Λ-module. Modifying this module by multiplying a kind of $\det_\Lambda(T)$, one can canonically define a graded invertible Λ-module

$$\Delta_\Lambda(T),$$

called the fundamental line of the pair (Λ, T), which is compatible with base change and Tate duality.

Our main objects are the pairs (A, M), where A is a $\mathbb{Q}_p$-affinoid and M is a (φ, Γ)-module over the relative Robba ring $\mathcal{R}_A$ over A. By the results of [Kedlaya et al. 2014], we can similarly attach a graded invertible A-module

$$\Delta_A(M),$$

called the fundamental line for (A, M), which is also compatible with base change and Tate duality. For a pair (Λ, T) as in the previous paragraph and a continuous homomorphism $f : \Lambda \to A$, there exists a canonical comparison isomorphism

$$\Delta_\Lambda(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_A(\boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A))$$

by the result of [Pottharst 2013]. The following conjecture is Kato's conjecture if $(B, N) = (\Lambda, T)$, and our new conjecture if $(B, N) = (A, M)$.

Conjecture 1.1. (See Conjecture 3.8 for the precise formulation.) *We can uniquely define a B-linear isomorphism*

$$\varepsilon_{B,\zeta}(N) : \mathbf{1}_B \xrightarrow{\sim} \Delta_B(N),$$

for each pair (B, N) of type (Λ, T) or (A, M) and for each $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$, which is compatible with any base changes $B \to B'$, exact sequences $0 \to N_1 \to N_2 \to N_3 \to 0$, and Tate duality, and satisfies the following:

(v) *For any $f : \Lambda \to A$ as above, we have*

$$\varepsilon_{\Lambda,\zeta}(T) \otimes \mathrm{id}_A = \varepsilon_{A,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A))$$

under the canonical isomorphism $\Delta_\Lambda(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_A(\boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A))$.

(vi) *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let N be a de Rham representation of $G_{\mathbb{Q}_p}$ or de Rham (φ, Γ)-module over $\mathcal{R}_L$. Then we have*

$$\varepsilon_{L,\zeta}(N) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(N),$$

where the isomorphism

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(N) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(N)$$

is called the de Rham ε-isomorphism which is defined using the Bloch–Kato exponential and the dual exponential of N and the local factors (L-factor, ε-constant) associated to $\boldsymbol{D}_{\mathrm{pst}}(N)$ and $\boldsymbol{D}_{\mathrm{pst}}(N^*)$.

Remark 1.2. To define condition (vi) for de Rham (φ, Γ)-modules, we need to generalize the Bloch–Kato exponential for (φ, Γ)-modules, which was one of the main themes of [Nakamura 2014a].

Roughly speaking, this conjecture says that the local factor which appears in the functional equation of the L-functions of a motif p-adically interpolate to all the families of p-adic Galois representations and also rigid-analytically interpolate to all the families of (φ, Γ)-modules in a compatible way. In fact, Kato [1993a] formulated a conjecture, called the generalized Iwasawa main conjecture, which asserts the existence of a compatible family of the zeta-isomorphisms

$$z_\Lambda(\mathbb{Z}[1/S], T) : \mathbf{1}_\Lambda \xrightarrow{\sim} \Delta^{\mathrm{global}}_\Lambda(T)$$

for any Λ-representation T of $G_{\mathbb{Q},S}$ (S is a finite set of primes) which interpolate the special values of L-functions of a motif. Kato [1993b] also formulated another conjecture, called the global ε-conjecture, which asserts the functional equation between $z_\Lambda(\mathbb{Z}[1/S], T)$ and $z_\Lambda(\mathbb{Z}[1/S], T^*)$, whose local factor at p is $\varepsilon_{\Lambda,\zeta}(T|_{G_{\mathbb{Q}_p}})$. Kato [1993b] (see also [Venjakob 2013]) proved the local (and even the global) ε-conjecture for the rank-one case. As a generalization of his theorem, the main theorem of this article is the following.

Theorem 1.3. (See Theorem 3.11 for the precise statement.) *Conjecture 1.1 is true for the rank-one case.*

From this theorem, we can immediately obtain some results for the trianguline case. We say that a (φ, Γ)-module M over $\mathcal{R}_A$ is trianguline if M has a filtration $\mathcal{F}: 0 := M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n := M$ whose graded quotients M_i/M_{i-1} are rank-one (φ, Γ)-modules over $\mathcal{R}_A$ for all $1 \leq i \leq n$. We call the filtration $\mathcal{F}$ a triangulation of M. For such a pair $(M, \mathcal{F})$, we obtain the following theorem, a special case (in particular, the rank-two case) of which will be used in Theorem 3.10 of our next article, [Nakamura 2015].

Corollary 1.4. (See Corollary 3.12 for the precise statement.) *Let M be a trianguline (φ, Γ)-module over $\mathcal{R}_A$ of rank n with a triangulation $\mathcal{F}$ as above. The isomorphism*

$$\varepsilon_{\mathcal{F},A,\zeta}(M) : \mathbf{1}_A \xrightarrow{\boxtimes_{i=1}^n \varepsilon_{A,\zeta}(M_i/M_{i-1})} \boxtimes_{i=1}^n \Delta_A(M_i/M_{i-1}) \xrightarrow{\sim} \Delta_A(M),$$

defined as the product of the isomorphisms

$$\varepsilon_{A,\zeta}(M_i/M_{i-1}) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M_i/M_{i-1}),$$

which are defined in Theorem 1.3, satisfies (many parts of) Conjecture 1.1; in particular, it satisfies the following:

(vi)$'$ *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham and trianguline (φ, Γ)-module over $\mathcal{R}_L$. Then, for any triangulation $\mathcal{F}$ of M, we have*

$$\varepsilon_{\mathcal{F},L,\zeta}(M) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

Remark 1.5. Before this article, the local ε-conjecture was proved only for cyclotomic deformations (or more general twists) of crystalline representations [Benois and Berger 2008; Loeffler et al. 2015]. Since the (φ, Γ)-modules associated to any twists of crystalline representations are trianguline, our Corollary 1.4 essentially contains all the known results concerning the local ε-conjecture. See Corollary 3.13 for the comparison of our theorem with the previous known results. Moreover, since any twists of semistable representations are also trianguline, our results also contain the semistable case, which seems to be unknown before this article.

Remark 1.6. Our method and previous known methods for the construction of local ε-isomorphisms cannot be applied to the nontrianguline case. That case is much more difficult but is much more interesting since the Weil–Deligne representation $\boldsymbol{D}_{\mathrm{pst}}(M)$ associated to a nontrianguline and de Rham (φ, Γ)-module M corresponds to a nonprincipal series representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ via the local Langlands correspondence, whose ε-constants are in general difficult to explicitly describe. In our next article, [Nakamura 2015], we construct ε-isomorphisms for all rank-two torsion p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by using Colmez's theory [2010] of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. More precisely, we will show that (a modified version of) the pairing defined in Corollaire VI.6.2 of [Colmez 2010] essentially gives us ε-isomorphisms for the rank-two case. In the trianguline case, by using Dospinescu's result [2014] on the explicit description of locally analytic vectors of Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, we will show that the ε-isomorphisms constructed in [Nakamura 2015] coincide with those constructed in this article. More interestingly, for the de Rham and nontrianguline case, we will show, by using Emerton's theorem [2006] on the compatibility of classical and p-adic Langlands correspondence, that the ε-isomorphisms defined in [Nakamura 2015] satisfy the suitable interpolation property (i.e., condition (vi) of Conjecture 1.1) for the critical range of Hodge–Tate weights. Moreover, as an application, we will prove a functional equation of Kato's Euler systems associated to Hecke eigen elliptic cusp newforms.

Remark 1.7. Other than the application to Theorem 3.10 of [Nakamura 2015], our Corollary 1.4 should be applicable to some Iwasawa theoretic studies of Galois representations over eigenvarieties. For example, the rank-two case of the local ε-isomorphism constructed in Corollary 1.4 should be the p-th local factor of the conjectural functional equation satisfied by the conjectural zeta element over the Coleman–Mazur eigencurve, whose existence is conjectured in (for example) [Hansen 2016, Conjecture 1.3.3]. Since our article is long enough, we don't study this problem in this article, but we hope to study it in future works.

1B. ***Structure of the paper.*** In Section 2, we recall the results of [Kedlaya et al. 2014; Pottharst 2013; Nakamura 2014a]. After recalling the definition of (φ, Γ)-modules over the relative Robba ring, we recall the main results of [Kedlaya et al. 2014; Pottharst 2013] on the cohomology of (φ, Γ)-modules, i.e., comparison with Galois cohomology, finiteness, base change property, Euler–Poincaré formula, Tate duality, and the classification of rank-one objects, all of which are essential for the formulation of our conjecture. We next recall the result of [Nakamura 2014a] on the theory of the Bloch–Kato exponential map of (φ, Γ)-modules. Since the result of [Nakamura 2014a] is not sufficient for our purpose, we slightly generalize the result. In particular, we show the existence of Bloch–Kato fundamental exact sequences involving $\boldsymbol{D}_{\mathrm{cris}}(M)$ (Lemma 2.20), establishing Bloch–Kato duality for the finite cohomology of (φ, Γ)-modules (Proposition 2.24). The explicit formulae of our Bloch–Kato exponential maps (Proposition 2.23) are frequently used in later sections.

In Section 3, using the preliminaries recalled in Section 2, we formulate our ε-conjecture and state our main theorem of this paper. Since the conjecture is formulated by using the notion of determinant, we first recall this notion in Section 3A. In Section 3B, using the determinant of cohomology of (φ, Γ)-modules, we define a graded invertible A-module $\Delta_A(M)$, called the fundamental line, for any (φ, Γ)-module M over $\mathcal{R}_A$. In Section 3C, for any de Rham (φ, Γ)-module M, we define a trivialization (called a de Rham ε-isomorphism) of the fundamental line using the Bloch–Kato fundamental exact sequence, Deligne–Langlands–Fontaine–Perrin-Riou's ε-constants and the "gamma-factor" associated to $\boldsymbol{D}_{\mathrm{pst}}(M)$. In Section 3D, we formulate our conjecture and compare our conjecture with Kato's conjecture, and state our main theorem of this article, which solves the conjecture for all rank-one (φ, Γ)-modules.

Section 4 is the main part of this paper, where we prove the conjecture for the rank-one case. In Section 4A, using the theory of analytic Iwasawa cohomology [Kedlaya et al. 2014; Pottharst 2012], and using the standard technique of p-adic Fourier transform, we construct our ε-isomorphism for all rank-one (φ, Γ)-modules. In Section 4B, we show that our ε-isomorphism defined in Section 4A specializes to the de Rham ε-isomorphism defined in Section 3B at each de Rham point. In Section 4B1, we first verify this condition (which we call the de Rham condition) for the "generic" rank-one de Rham (φ, Γ)-modules by establishing a kind of explicit reciprocity law (Proposition 4.11, 4.16). In the process of proving this, we prove a proposition (Proposition 4.13) on the compatibility of our ε-isomorphism with a natural differential operator. Using the result in the generic case and the density argument, we prove the compatibility of our ε-isomorphism with Tate duality and compare our ε-isomorphism with Kato's ε-isomorphism. In Section 4B2, we verify the de Rham condition via explicit calculations for the exceptional case which includes the case of $\mathcal{R}$, $\mathcal{R}(1)$ (the (φ, Γ)-modules corresponding to $\mathbb{Q}_p$, $\mathbb{Q}_p(1)$, respectively).

In the Appendix, we explicitly calculate the cohomologies $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}(1))$ and $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R})$, which will be used in Section 4B2. Finally, we remark that, in our proof, we don't use any previous known results (e.g., [Kato 1993b; Benois and Berger 2008; Loeffler et al. 2015]) on the local ε-conjecture. Our proof essentially follows from the results in Section 2 of this article and those of [Nakamura 2014a] on the explicit definition of the exponential and the dual exponential maps for (φ, Γ)-modules. We believe that our proof is the most simple and the most natural one.

1C. ***Notation.*** Throughout this paper, we fix a prime number p. The letter A will always denote a $\mathbb{Q}_p$-affinoid algebra; we use $\mathrm{Max}(A)$ to denote the associated rigid analytic space. We fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and consider any finite extension K of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$. Let $|-|:\bar{\mathbb{Q}}_p^\times \to \mathbb{Q}_{>0}$ be the absolute value such that $|p| = p^{-1}$. For $n \geq 0$, let us denote by μ_{p^n} the set of p^n-th power roots of unity in $\bar{\mathbb{Q}}_p$, and put $\mu_{p^\infty} := \bigcup_{n\geq 1} \mu_{p^n}$. For a finite extension K of $\mathbb{Q}_p$, put $K_n := K(\mu_{p^n})$ for $\infty \geq n \geq 0$. Let us denote by $\chi : \Gamma_{\mathbb{Q}_p} := \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Z}_p^\times$ the cyclotomic character given by $\gamma(\zeta) = \zeta^{\chi(\gamma)}$ for $\gamma \in \Gamma$ and $\zeta \in \mu_{p^\infty}$. Set $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$, $H_K := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K_\infty)$, and $\Gamma_K := \mathrm{Gal}(K_\infty/K)$.

We let k be the residue field of K, with $F := W(k)[1/p]$. Put $\mathbb{Z}_p(1) := \varprojlim_{n\geq 0} \mu_{p^n}$. For $k \in \mathbb{Z}$, define $\mathbb{Z}_p(k) := \mathbb{Z}_p(1)^{\otimes k}$ equipped with a natural action of Γ_K. For a $\mathbb{Z}_p[G_K]$-module N, let us define $N(k) := N \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)$. When we fix a generator $\zeta = \{\zeta_{p^n}\}_{n\geq 0} \in \mathbb{Z}_p(1)$, we put $\boldsymbol{e}_1 := \zeta$ and $\boldsymbol{e}_k := \boldsymbol{e}_1^{\otimes k} \in \mathbb{Z}$. For a continuous G_K-module N, let us denote by $C^\bullet_{\mathrm{cont}}(G_K, N)$ the complex of continuous cochains of G_K with values in N. Define $\mathrm{H}^i(K, N) := \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_K, N))$. For a group G, denote by G_{tor} the subgroup of G consisting of all torsion elements in G. If G is a finite group, let $|G|$ be the order of G.

For a commutative ring R, let us denote by $\boldsymbol{P}_{\mathrm{fg}}(R)$ the category of finitely generated projective R-modules. For $N \in \boldsymbol{P}_{\mathrm{fg}}(R)$, denote by $\mathrm{rk}_R\, N$ the rank of N and let $N^\vee := \mathrm{Hom}_R(N, R)$. Let $[-,-] : N_1 \times N_2 \to R$ be a perfect pairing. Then we always identify N_2 with $N_1^\vee$ by the isomorphism $N_2 \xrightarrow{\sim} N_1^\vee : x \mapsto (y \mapsto [y, x])$. Let us denote by $\boldsymbol{D}^-(R)$ the derived category of bounded-below complexes of R-modules. For $a_1 \leq a_2 \in \mathbb{Z}$, let us denote by $\boldsymbol{D}^{[a_1,a_2]}_{\mathrm{perf}}(R)$ (resp. $\boldsymbol{D}^b_{\mathrm{perf}}(R)$) the full subcategory of $\boldsymbol{D}^-(R)$ consisting of the complexes of R-modules which are quasi-isomorphic to a complex $P^\bullet$ of $\boldsymbol{P}_{\mathrm{fg}}(R)$ concentrated in degrees in $[a_1, a_2]$ (resp. bounded degree). There exists a duality functor

$$\boldsymbol{R}\,\mathrm{Hom}_R(-, R) : \boldsymbol{D}^{[a_1,a_2]}_{\mathrm{perf}}(R) \to \boldsymbol{D}^{[-a_2,-a_1]}_{\mathrm{perf}}(R)$$

characterized by $\boldsymbol{R}\,\mathrm{Hom}_R(P^\bullet, R) := \mathrm{Hom}_R(P^{-\bullet}, R)$ for any bounded complex $P^\bullet$ of $\boldsymbol{P}_{\mathrm{fg}}(R)$. Define the notion $\chi_R(-)$ of Euler characteristic for any objects of $\boldsymbol{D}^b_{\mathrm{perf}}(R)$, which is characterized by

$$\chi_R(P^\bullet) := \sum_{i\in\mathbb{Z}} (-1)^i\, \mathrm{rk}_R\, P^i \in \mathrm{Map}(\mathrm{Spec}(R), \mathbb{Z})$$

for any bounded complex $P^\bullet$ of $\boldsymbol{P}_{\mathrm{fg}}(R)$.

2. Cohomology and Bloch–Kato exponential of (φ, Γ)-modules

2A. ***Cohomology of (φ, Γ)-modules.*** In this subsection, we recall the definition of (families of) (φ, Γ)-modules and the definition of their cohomologies following [Kedlaya et al. 2014], and then recall the results of their article on the finiteness of the cohomology.

Put $\omega := p^{-1/(p-1)} \in \mathbb{R}_{>0}$. For $r \in \mathbb{Q}_{>0}$, define the r-Gauss norm $|-|_r$ on $\mathbb{Q}_p[T^{\pm}]$ by the formula $\left|\sum_i a_i T^i\right|_r := \max_i\{|a_i|\omega^{ir}\}$. For $0 < s \le r \in \mathbb{Q}_{>0}$, we write $A^1[s,r]$ for the rigid analytic annulus defined over $\mathbb{Q}_p$ in the variable T with radii $|T| \in [\omega^r, \omega^s]$; its ring of analytic functions, denoted by $\mathcal{R}^{[s,r]}$, is the completion of $\mathbb{Q}_p[T^{\pm}]$ with respect to the norm $|\cdot|_{[s,r]} := \max\{|\cdot|_r, |\cdot|_s\}$. We also allow r (but not s) to be ∞, in which case $A^1[s,r]$ is interpreted as the rigid analytic disc in the variable T with radii $|T| \le \omega^s$; its ring of analytic functions $\mathcal{R}^{[s,r]} = \mathcal{R}^{[s,\infty]}$ is the completion of $\mathbb{Q}_p[T]$ with respect to $|\cdot|_s$. Let A be a $\mathbb{Q}_p$-affinoid algebra. Denote by $\mathcal{R}_A^{[s,r]}$ the ring of rigid analytic functions on the relative annulus (or disc if $r = \infty$) $\mathrm{Max}(A) \times A^1[s,r]$; its ring of analytic functions is $\mathcal{R}_A^{[s,r]} := \mathcal{R}^{[s,r]} \widehat{\otimes}_{\mathbb{Q}_p} A$. Put

$$\mathcal{R}_A^r := \bigcap_{0<s\le r} \mathcal{R}_A^{[s,r]} \quad \text{and} \quad \mathcal{R}_A := \bigcup_{0<r} \mathcal{R}_A^r.$$

Let k' be the residue field of K_∞, with $F' := W(k')[1/p]$. Put $\tilde{e}_K := [K_\infty : F'_\infty]$.

For $0 < s \le r$, we let $\mathcal{R}^{[s,r]}(\pi_K)$ be the formal substitution of T by π_K in the ring $\mathcal{R}_{F'}^{[s/\tilde{e}_K, r/\tilde{e}_K]}$; we set $\mathcal{R}_A^{[s,r]}(\pi_K) := \mathcal{R}^{[s,r]}(\pi_K) \widehat{\otimes}_{\mathbb{Q}_p} A$. We define $\mathcal{R}_A^r(\pi_K)$, $\mathcal{R}_A(\pi_K)$ similarly; the latter is referred to as the relative Robba ring over A for K.

By the theory of fields of norms, there exists a constant $C(K) > 0$ such that, for any $0 < r \le C(K)$, we can equip $\mathcal{R}_A^r(\pi_K)$ with a finite étale $\mathcal{R}_A^r(\pi_{\mathbb{Q}_p})$ algebra free of rank $[K_\infty : \mathbb{Q}_{p,\infty}]$ with the Galois group $H_{\mathbb{Q}_p}/H_K$. More generally, for any finite extensions $L \supseteq K \supseteq \mathbb{Q}_p$, we can naturally equip $\mathcal{R}_A^r(\pi_L)$ with a structure of finite étale $\mathcal{R}_A^r(\pi_K)$-algebra free of rank $[L_\infty : K_\infty]$ with the Galois group H_K/H_L for any $0 < r \le \min\{C(K), C(L)\}$.

There are commuting A-linear actions of Γ_K on $\mathcal{R}_A^{[s,r]}(\pi_K)$ and of an operator

$$\varphi : \mathcal{R}_A^{[s,r]}(\pi_K) \to \mathcal{R}_A^{[s/p,r/p]}(\pi_K)$$

for $0 < s \le r \le C(K)$. The actions on the coefficients F' are the natural ones, i.e., Γ_K through its quotient $\mathrm{Gal}(F'/F)$ and φ by the canonical lift of the p-th Frobenius on k'. For $0 < s \le r \le C(K)$, φ makes $\mathcal{R}_A^{[s/p,r/p]}(\pi_K)$ into a free $\mathcal{R}_A^{[s,r]}(\pi_K)$-module of rank p, and we obtain a Γ_K-equivariant left inverse

$$\psi : \mathcal{R}_A^{[s/p,r/p]}(\pi_K) \to \mathcal{R}_A^{[s,r]}(\pi_K)$$

by the formula

$$\tfrac{1}{p}\varphi^{-1} \circ \mathrm{Tr}_{\mathcal{R}_A^{[s/p,r/p]}(\pi_K)/\varphi(\mathcal{R}_A^{[s,r]}(\pi_K))}.$$

The map ψ naturally extends to the maps $\mathcal{R}_A^{r/p}(\pi_K) \to \mathcal{R}_A^r(\pi_K)$ for $0 < r \le C(K)$ and $\mathcal{R}_A(\pi_K) \to \mathcal{R}_A(\pi_K)$.

Remark 2.1. In fact, these rings are constructed using Fontaine's rings of p-adic periods. We don't have any canonical choice of the parameter π_K for general K, but the ring $\mathcal{R}_A(\pi_K)$ and the actions of φ, Γ_K don't depend on the choice of π_K. More precisely, $\mathcal{R}(\pi_K)$ is defined as a subring of the ring $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ of p-adic periods defined in [Berger 2002], and this subring does not depend on the choice of π_K, and the actions of φ, Γ_K are induced by the natural actions of φ, G_K on $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$.

However, for unramified K, once we fix a $\mathbb{Z}_p$-basis $\zeta := \{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1) := \varprojlim_{n\geq 0}\mu_{p^n}$, we have a natural choice of π_K as follows. Let $\bar{\mathbb{Z}}_p$ be the integral closure of $\mathbb{Z}_p$ in $\bar{\mathbb{Q}}_p$, let $\widetilde{\mathbb{E}}^+ := \varprojlim_{n\geq 0}\bar{\mathbb{Z}}_p/p\bar{\mathbb{Z}}_p$ be the projective limit with respect to the p-th power map, and let $[-] : \widetilde{\mathbb{E}}^+ \to W(\widetilde{\mathbb{E}}^+)$ be the Teichmüller lift to the ring $W(\widetilde{\mathbb{E}}^+)$ of Witt vectors. Under the fixed ζ, we can choose

$$\pi_K = \pi_{\mathbb{Q}_p} = \pi_\zeta := [(\bar{\zeta}_{p^n})_{n\geq 0}] - 1 \in W(\widetilde{\mathbb{E}}^+) \subseteq \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}},$$

and then φ and $\Gamma_{\mathbb{Q}_p}$ act by $\varphi(\pi_\zeta) = (1+\pi_\zeta)^p - 1$ and $\gamma(\pi_\zeta) = (1+\pi_\zeta)^{\chi(\gamma)} - 1$ for $\gamma \in \Gamma_{\mathbb{Q}_p}$.

Notation 2.2. From Section 3, we will concentrate on the case $K = \mathbb{Q}_p$ and fix $\zeta := \{\zeta_{p^n}\}_{n\geq 0}$ as above. Then we use the notation $\Gamma := \Gamma_{\mathbb{Q}_p}$, $\pi := \pi_\zeta$ and omit $(\pi_{\mathbb{Q}_p})$ from the notation of Robba rings by writing, for example, $\mathcal{R}_A^{[s,r]}$ instead of $\mathcal{R}_A^{[s,r]}(\pi_{\mathbb{Q}_p})$. In this case, $\mathcal{R}_A^{[s/p,r/p]} = \bigoplus_{0\leq i\leq p-1}(1+\pi)^i\varphi(\mathcal{R}_A^{[s,r]})$, so if $f = \sum_{i=0}^{p-1}(1+\pi)^i\varphi(f_i)$ then $\psi(f) = f_0$. We define the special element $t = \log(1+\pi) \in \mathcal{R}_A^{\infty}$. We have $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$ for $\gamma \in \Gamma$.

We first recall the definitions of φ-modules over $\mathcal{R}_A(\pi_K)$ following [Kedlaya et al. 2014, Definition 2.2.5].

Definition 2.3. Choose $0 < r_0 \leq C(K)$. A φ-module over $\mathcal{R}_A^{r_0}(\pi_K)$ is a finite projective $\mathcal{R}_A^{r_0}(\pi_K)$-module M^{r_0} equipped with a $\mathcal{R}_A^{r_0/p}(\pi_K)$-linear isomorphism $\varphi^*M^{r_0} \xrightarrow{\sim} M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^{r_0/p}(\pi_K)$. A φ-module M over $\mathcal{R}_A(\pi_K)$ is a base change to $\mathcal{R}_A(\pi_K)$ of a φ-module over some $\mathcal{R}_A^{r_0}(\pi_K)$.

For a φ-module M^{r_0} over $\mathcal{R}_A^{r_0}(\pi_K)$ and for $0 < s \leq r \leq r_0$, we set

$$M^{[s,r]} = M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^{[s,r]}(\pi_K) \quad \text{and} \quad M^s = M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^{s}(\pi_K).$$

For $0 < s \leq r_0$, the given isomorphism $\varphi^*(M^{r_0}) \xrightarrow{\sim} M^{r_0/p}$ induces a φ-semilinear map

$$\varphi : M^s \to \varphi^*M^s \xrightarrow{\sim} \varphi^*M^{r_0} \otimes_{\mathcal{R}_A^{r_0/p}(\pi_K)} \mathcal{R}_A^{s/p}(\pi_K) \xrightarrow{\sim} M^{r_0/p} \otimes_{\mathcal{R}_A^{r_0/p}(\pi_K)} \mathcal{R}_A^{s/p}(\pi_K) = M^{s/p},$$

where the first map, $M^s \hookrightarrow \varphi^*M^s$, is given by

$$x \mapsto x \otimes 1 \in M^s \otimes_{\mathcal{R}_A^s(\pi_K),\varphi} \mathcal{R}_A^{s/p}(\pi_K) =: \varphi^*M^s,$$

the second isomorphism is just the associativity of tensor products, and the third isomorphism is the base change of the given isomorphism $\varphi^*M^{r_0} \xrightarrow{\sim} M^{r_0/p}$. This map φ also induces an A-linear homomorphism

$$\psi : M^{s/p} = \varphi(M^s) \otimes_{\varphi(\mathcal{R}_A^s(\pi_K))} \mathcal{R}_A^{s/p}(\pi_K) \to M^s$$

given by $\psi(\varphi(m)\otimes f)=m\otimes\psi(f)$ for $m\in M^s$ and $f\in\mathcal{R}_A^{s/p}(\pi_K)$. For a φ-module M over $\mathcal{R}_A(\pi_K)$, the maps $\varphi: M^s\to M^{s/p}$ and $\psi: M^{s/p}\to M^s$ naturally extend to $\varphi: M\to M$ and $\psi: M\to M$.

We recall the definition of (φ,Γ)-modules over $\mathcal{R}_A(\pi_K)$ following [Kedlaya et al. 2014, Definition 2.2.12].

Definition 2.4. Choose $0<r_0\le C(K)$. A (φ,Γ)-module over $\mathcal{R}_A^{r_0}(\pi_K)$ is a φ-module over $\mathcal{R}_A^{r_0}(\pi_K)$ equipped with a commuting semilinear continuous action of Γ_K. A (φ,Γ)-module over $\mathcal{R}_A(\pi_K)$ is a base change of a (φ,Γ)-module over $\mathcal{R}_A^{r_0}(\pi_K)$ for some $0<r_0\le C(K)$.

We can generalize these notions for general rigid analytic space as in [Kedlaya et al. 2014, Definition 6.1.1]

Definition 2.5. Let X be a rigid analytic space over $\mathbb{Q}_p$. A (φ,Γ)-module over $\mathcal{R}_X(\pi_K)$ is a compatible family of (φ,Γ)-modules over $\mathcal{R}_A(\pi_K)$ for each affinoid $\mathrm{Max}(A)$ of X.

For (φ,Γ)-modules M, N over $\mathcal{R}_X(\pi_K)$, we define $M\otimes N:=M\otimes_{\mathcal{R}_X(\pi_K)}N$ to be the tensor product equipped with the diagonal action of (φ,Γ_K). We also define $M^\vee:=\mathrm{Hom}_{\mathcal{R}_X(\pi_K)}(M,\mathcal{R}_X(\pi_K))$ to be the dual (φ,Γ)-module.

For a (φ,Γ)-module M over $\mathcal{R}_A(\pi_K)$, we define

$$r_M:=\mathrm{rk}_{\mathcal{R}_A(\pi_K)}M\in\mathrm{Map}(\mathrm{Spec}(\mathcal{R}_A(\pi_K)),\mathbb{Z}_{\ge 0})$$

to be the rank of M, where $\mathrm{Map}(-,-)$ is the set of continuous maps and $\mathbb{Z}_{\ge0}$ is equipped with the discrete topology. We will see later (in Remark 2.16) that r_M is in fact in $\mathrm{Map}(\mathrm{Spec}(A),\mathbb{Z}_{\ge0})$, i.e., we have $r_M=\mathrm{pr}\circ f_M$ for unique $f_M\in\mathrm{Map}(\mathrm{Spec}(A),\mathbb{Z}_{\ge0})$, where $\mathrm{pr}:\mathrm{Spec}(\mathcal{R}_A(\pi_K))\to\mathrm{Spec}(A)$ is the natural projection. We also let $r_M:=f_M$.

The importance of (φ,Γ)-modules follows from the next theorem.

Theorem 2.6 [Kedlaya and Liu 2010, Theorem 3.11]. *Let V be a vector bundle over X equipped with a continuous $\mathcal{O}_X$-linear action of G_K. Then there is functorially associated to V a (φ,Γ)-module $\boldsymbol{D}_{\mathrm{rig}}(V)$ over $\mathcal{R}_X(\pi_K)$. The rule $V\mapsto\boldsymbol{D}_{\mathrm{rig}}(V)$ is fully faithful and exact, and it commutes with base change in X.*

For example, we have a canonical isomorphism $\boldsymbol{D}_{\mathrm{rig}}(A(k))=\mathcal{R}_A(\pi_K)(k)$ for $k\in\mathbb{Z}$.

From Section 3, we will concentrate on the case where $K=\mathbb{Q}_p$ and M is a rank-one (φ,Γ)-module over $\mathcal{R}_X$. Here, we recall the result of [Kedlaya et al. 2014] concerning the classification of rank-one (φ,Γ)-modules. Actually, they obtained a similar result for general K, but we don't recall it since we don't use it.

Definition 2.7. For a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$, we define $\mathcal{R}_X(\delta)$ to be the rank-one (φ, Γ)-module $\mathcal{R}_X \cdot \boldsymbol{e}_\delta$ over $\mathcal{R}_X$ with $\varphi(\boldsymbol{e}_\delta) = \delta(p)\boldsymbol{e}_\delta$ and $\gamma(\boldsymbol{e}_\delta) = \delta(\chi(\gamma))\boldsymbol{e}_\delta$ for $\gamma \in \Gamma$.

Theorem 2.8 [Kedlaya et al. 2014, Theorem 6.1.10]. *Let M be a rank-one (φ, Γ)-module over $\mathcal{R}_X$. Then there exist a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and an invertible sheaf $\mathcal{L}$ on X, the pair of which is unique up to isomorphism, such that $M \xrightarrow{\sim} \mathcal{R}_X(\delta) \otimes_{\mathcal{O}_X} \mathcal{L}$.*

Notation 2.9. (i) For $\delta, \delta' : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$, we fix isomorphisms

$$\mathcal{R}_X(\delta) \otimes \mathcal{R}_X(\delta') \xrightarrow{\sim} \mathcal{R}_X(\delta\delta') \quad \text{by } \boldsymbol{e}_\delta \otimes \boldsymbol{e}_{\delta'} \mapsto \boldsymbol{e}_{\delta\delta'},$$
$$\mathcal{R}_X(\delta)^\vee \xrightarrow{\sim} \mathcal{R}_X(\delta^{-1}) \quad \text{by } \boldsymbol{e}_\delta^\vee \mapsto \boldsymbol{e}_{\delta^{-1}}.$$

(ii) For $k \in \mathbb{Z}$, we define a continuous homomorphism $x^k : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times : y \mapsto y^k$. Define $|x| : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times : p \mapsto p^{-1}$, $a \mapsto 1$ for $a \in \mathbb{Z}_p^\times$. Then the homomorphism $x|x|$ corresponds to the Tate twist, i.e., we have an isomorphism $\mathcal{R}_X(1) \xrightarrow{\sim} \mathcal{R}_X(x|x|)$. When we fix a generator $\zeta \in \mathbb{Z}_p(1)$, we identify $\mathcal{R}_X(1) = \mathcal{R}_X(x|x|)$ by $\boldsymbol{e}_1 \mapsto \boldsymbol{e}_{x|x|}$.

We next recall some cohomology theories concerning (φ, Γ)-modules. Denote by Δ the largest p-power torsion subgroup of Γ_K. Fix $\gamma \in \Gamma_K$, whose image in Γ_K/Δ is a topological generator. For a Δ-module M, put $M^\Delta = \{m \in M \mid \sigma(m) = m \text{ for all } \sigma \in \Delta\}$.

Definition 2.10. For a (φ, Γ)-module M over $\mathcal{R}_A(\pi_K)$, we define the complexes $C^\bullet_{\varphi,\gamma}(M)$ and $C^\bullet_{\psi,\gamma}(M)$ of A-modules concentrated in degree $[0, 2]$, and define a morphism Ψ_M between them as follows:

$$\begin{array}{ccccccc} C^\bullet_{\varphi,\gamma}(M) = \big[M^\Delta & \xrightarrow{(\gamma-1,\varphi-1)} & M^\Delta \oplus M^\Delta & \xrightarrow{(\varphi-1)\oplus(1-\gamma)} & M^\Delta\big] \\ \Psi_M\big\downarrow \quad\quad \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id}\oplus -\psi & & \big\downarrow -\psi \\ C^\bullet_{\psi,\gamma}(M) = \big[M^\Delta & \xrightarrow{(\gamma-1,\psi-1)} & M^\Delta \oplus M^\Delta & \xrightarrow{(\psi-1)\oplus(1-\gamma)} & M^\Delta\big] \end{array} \tag{1}$$

The map Ψ_M is a quasi-isomorphism by Proposition 2.3.4 of [Kedlaya et al. 2014].

For $i \in \mathbb{Z}_{\geq 0}$, define $\mathrm{H}^i_{\varphi,\gamma}(M)$ for the i-th cohomology of $C^\bullet_{\varphi,\gamma}(M)$, called the (φ, Γ)-cohomology of M. We similarly define $\mathrm{H}^i_{\psi,\gamma}(M)$ to be the i-th cohomology of $C^\bullet_{\psi,\gamma}(M)$, called the (ψ, Γ)-cohomology of M. In this article, we freely identify $C^\bullet_{\varphi,\gamma}(M)$ (resp. $\mathrm{H}^i_{\varphi,\gamma}(M)$) with $C^\bullet_{\psi,\gamma}(M)$ (resp. $\mathrm{H}^i_{\psi,\gamma}(M)$) via the quasi-isomorphism Ψ_M.

More generally, for $h = \varphi, \psi$ and any module N with commuting actions of h and Γ, we similarly define the complexes $C^\bullet_{h,\gamma}(N)$ and denote the resulting cohomology by $\mathrm{H}^i_{h,\gamma}(N)$. We denote by $[x, y] \in \mathrm{H}^1_{h,\gamma}(N)$ (resp. $[z] \in \mathrm{H}^2_{h,\gamma}(N)$) the element represented by a 1-cocycle $(x, y) \in N^\Delta \oplus N^\Delta$ (resp. by $z \in N^\Delta$). The functor

$N \mapsto C^\bullet_{h,\gamma}(N)$ from the category of topological A-modules which are Hausdorff with commuting continuous actions of h, Γ_K to the category of complexes of A-modules is independent of the choice of γ up to canonical isomorphism; i.e., for another choice $\gamma' \in \Gamma_K$, we have a canonical isomorphism

$$\begin{array}{ccccccc} C^\bullet_{h,\gamma}(N) = \big[N^\Delta & \xrightarrow{(\gamma-1,h-1)} & N^\Delta \oplus N^\Delta & \xrightarrow{(h-1)\oplus(1-\gamma)} & N^\Delta\big] \\ {\scriptstyle \iota_{\gamma,\gamma'}}\big\downarrow \quad \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \frac{\gamma'-1}{\gamma-1}\oplus \mathrm{id}} & & \big\downarrow{\scriptstyle \frac{\gamma'-1}{\gamma-1}} \\ C^\bullet_{h,\gamma'}(N) = \big[N^\Delta & \xrightarrow{(\gamma'-1,h-1)} & N^\Delta \oplus N^\Delta & \xrightarrow{(h-1)\oplus(1-\gamma')} & N^\Delta\big] \end{array} \qquad (2)$$

For a commutative ring R, let us denote by $\boldsymbol{D}^-(R)$ the derived category of bounded-below complexes of R-modules. We use the same notation, $C^\bullet_{h,\gamma}(N) \in \boldsymbol{D}^-(A)$, for the object represented by this complex.

Let V be a finite projective A-module with a continuous A-linear action of G_K. Let us denote by $C^\bullet_{\mathrm{cont}}(G_K, V)$ the complex of continuous G_K-cochains with values in V, and let $\mathrm{H}^i(K, V)$ be the cohomology. By Theorem 2.8 of [Pottharst 2013], we have a functorial isomorphism

$$C^\bullet_{\mathrm{cont}}(G_K, V) \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(V))$$

in $\boldsymbol{D}^-(A)$ and a functorial A-linear isomorphism

$$\mathrm{H}^i(K, V) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(V)).$$

Definition 2.11. For (φ, Γ)-modules M, N over $\mathcal{R}_A(\pi_K)$, we have a natural A-bilinear cup product morphism

$$C^\bullet_{\varphi,\gamma}(M) \times C^\bullet_{\varphi,\gamma}(N) \to C^\bullet_{\varphi,\gamma}(M \otimes N);$$

see Definition 2.3.11 of [Kedlaya et al. 2014]. This induces an A-bilinear graded commutative cup product pairing

$$\cup : \mathrm{H}^i_{\varphi,\gamma}(M) \times \mathrm{H}^j_{\varphi,\gamma}(N) \to \mathrm{H}^{i+j}_{\varphi,\gamma}(M \otimes N).$$

For example, this is defined by the formulae

$$x \cup [y] := [x \otimes y] \quad \text{for } i = 0,\ j = 2,$$
$$[x_1, y_1] \cup [x_2, y_2] := [x_1 \otimes \gamma(y_2) - y_1 \otimes \varphi(x_2)] \quad \text{for } i = j = 1.$$

Remark 2.12. The definition of the cup product for $\mathrm{H}^1_{\varphi,\gamma}(-) \times \mathrm{H}^1_{\varphi,\gamma}(-) \to \mathrm{H}^2_{\varphi,\gamma}(-)$, given in our previous paper, [Nakamura 2014a], is (-1) times the above definition. The above one seems to be the standard one in the literature. All the results of [Nakamura 2014a] hold without any changes when we use the above definition, except Lemmas 2.13 and 2.14, where we need to multiply by (-1) for the commutative diagrams there to be commutative.

Definition 2.13. Let us denote by $M^* := M^\vee(1)$ the Tate dual of M. Using the cup product, the evaluation map $\mathrm{ev} : M^* \otimes M \to \mathcal{R}_A(\pi_K)(1) : f \otimes x \mapsto f(x)$, the comparison isomorphism $\mathrm{H}^2(K, A(1)) \xrightarrow{\sim} \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_A(\pi_K)(1))$ and Tate's trace map $\mathrm{H}^2(K, A(1)) \xrightarrow{\sim} A$, one gets the Tate duality pairings

$$\begin{aligned} C^\bullet_{\varphi,\gamma}(M^*) \times C^\bullet_{\varphi,\gamma}(M) \to C^\bullet_{\varphi,\gamma}(M^* \otimes M) \to C^\bullet_{\varphi,\gamma}(\mathcal{R}_A(\pi_K)(1))) \\ \to \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_A(\pi_K)(1))[-2] \xrightarrow{\sim} \mathrm{H}^2(K, A(1))[-2] \xrightarrow{\sim} A[-2] \end{aligned}$$

and

$$\langle -, - \rangle : \mathrm{H}^i_{\varphi,\gamma}(M^*) \times \mathrm{H}^{2-i}_{\varphi,\gamma}(M) \to A.$$

Remark 2.14. In the Appendix, we explicitly describe the isomorphism

$$\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_A(1)) \xrightarrow{\sim} \mathrm{H}^2(G_{\mathbb{Q}_p}, A(1)) \xrightarrow{\sim} A$$

using the residue map; see Proposition 5.2.

One of the main results of [Kedlaya et al. 2014] which is crucial to formulating our conjecture is the following.

Theorem 2.15 [Kedlaya et al. 2014, Theorems 4.4.3, 4.4.4]. *Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$.*

(1) $C^\bullet_{\varphi,\gamma}(M) \in \boldsymbol{D}^{[0,2]}_{\mathrm{perf}}(A)$. *In particular, the cohomology groups $\mathrm{H}^i_{\varphi,\gamma}(M)$ are finite A-modules.*

(2) *Let $A \to A'$ be a continuous morphism of $\mathbb{Q}_p$-affinoid algebras. Then the canonical morphism $C^\bullet_{\varphi,\gamma}(M) \otimes^{\boldsymbol{L}}_A A' \to C^\bullet_{\varphi,\gamma}(M \widehat{\otimes}_A A')$ is a quasi-isomorphism. In particular, if A' is flat over A, we have $\mathrm{H}^i_{\varphi,\gamma}(M) \otimes_A A' \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(M \widehat{\otimes}_A A')$.*

(3) (Euler–Poincaré characteristic formula) *We have $\chi_A(C^\bullet_{\varphi,\gamma}(M)) = -[K : \mathbb{Q}_p] \cdot r_M$.*

(4) (Tate duality) *The Tate duality pairing defined in Definition 2.13 induces a quasi-isomorphism*

$$C^\bullet_{\varphi,\gamma}(M) \xrightarrow{\sim} \boldsymbol{R}\,\mathrm{Hom}_A(C^\bullet_{\varphi,\gamma}(M^*), A)[-2].$$

Remark 2.16. By the equality of (3), the rank $r_M \in \mathrm{Map}(\mathrm{Spec}(\mathcal{R}_A(\pi_K)), \mathbb{Z}_{\geq 0})$ is contained in $\mathrm{Map}(\mathrm{Spec}(A), \mathbb{Z}_{\geq 0})$.

Let X be a rigid analytic space over $\mathbb{Q}_p$ and let M be a (φ, Γ)-module over $\mathcal{R}_X(\pi_K)$. By (1) and (2) of the above theorem, the correspondence $U \mapsto \mathrm{H}^i_{\varphi,\gamma}(M|_U)$ for each affinoid open U in X defines a coherent $\mathcal{O}_X$-module for each $i \in [0, 2]$, which we also denote by $\mathrm{H}^i_{\varphi,\gamma}(M)$.

2B. ***Bloch–Kato exponential for (φ, Γ)-modules.*** For any $\mathbb{Q}_p$-representation V of G_K, Bloch and Kato [1990] defined the diagram with exact rows

$$\begin{array}{ccccccccccc}
0 \to & \mathrm{H}^0(K,V) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(V)^{\varphi=1} & \xrightarrow{x\mapsto \bar{x}} & t_V(K) & \xrightarrow{\exp_V} & \mathrm{H}^1_e(K,V) & \to 0 \\
& \downarrow \mathrm{id} & & \downarrow x\mapsto x & & \downarrow x\mapsto (0,x) & & \downarrow x\mapsto x & \\
0 \to & \mathrm{H}^0(K,V) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(V) & \xrightarrow{f} & \boldsymbol{D}^K_{\mathrm{cris}}(V) \oplus t_V(K) & \xrightarrow{g} & \mathrm{H}^1_f(K,V) & \to 0
\end{array} \tag{3}$$

with

$$f(x,y) = ((1-\varphi)x, \bar{x}) \quad \text{and} \quad g = \exp_{f,V} \oplus \exp_V,$$

which is associated to the tensor product of V (over $\mathbb{Q}_p$) with the Bloch–Kato fundamental exact sequences

$$\begin{array}{ccccccc}
0 \to \mathbb{Q}_p & \xrightarrow{x\mapsto (x,x)} & \boldsymbol{B}^{\varphi=1}_{\mathrm{cris}} \oplus \boldsymbol{B}^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto x-y} & \boldsymbol{B}_{\mathrm{dR}} & \to 0 \\
\downarrow \mathrm{id} & & \downarrow (x,y)\mapsto (x,y) & & \downarrow x\mapsto (0,x) & \\
0 \to \mathbb{Q}_p & \xrightarrow{x\mapsto (x,x)} & \boldsymbol{B}_{\mathrm{cris}} \oplus \boldsymbol{B}^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto ((1-\varphi)x, x-y)} & \boldsymbol{B}_{\mathrm{cris}} \oplus \boldsymbol{B}_{\mathrm{dR}} & \to 0
\end{array}$$

in which $\boldsymbol{B}_{\mathrm{cris}}$ and $\boldsymbol{B}_{\mathrm{dR}}$ are Fontaine's rings of p-adic periods. We set $\boldsymbol{D}^K_{\mathrm{cris}}(V) := (\boldsymbol{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, $t_V(K) := (\boldsymbol{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} / (\boldsymbol{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$,

$$\mathrm{H}^1_e(K,V) := \mathrm{Im}(\exp_V : t_V(K) \to \mathrm{H}^1(K,V))$$

and

$$\mathrm{H}^1_f(K,V) := \mathrm{Im}(\exp_{f,V} \oplus \exp_V : \boldsymbol{D}^K_{\mathrm{cris}}(V) \oplus t_V(K) \to \mathrm{H}^1(K,V)).$$

The boundary map

$$\exp_V : t_V(K) \to \mathrm{H}^1_e(K,V)$$

is called the Bloch–Kato exponential, and its definition is generalized to (φ, Γ)-modules over the Robba ring in [Nakamura 2014a]. To formulate the local ε-conjecture, we also need another boundary map,

$$\exp_{f,V} : \boldsymbol{D}^K_{\mathrm{cris}}(V) \to \mathrm{H}^1_f(K,V),$$

which is not studied in [Nakamura 2014a].

The aim of this subsection is to define the map $\exp_{f,M}$ for all the (φ, Γ)-modules M over the Robba ring purely in terms of (φ, Γ)-modules (Propositions 2.21 and 2.23), to prove Bloch–Kato duality for them (Proposition 2.24), to compare our maps $\exp_M$ and $\exp_{f,M}$ with the Bloch–Kato maps for the étale case (Proposition 2.26), all of which we need in order to generalize the local ε-conjecture for (φ, Γ)-modules. The explicit formulae for the maps $\exp_M$ and $\exp_{f,M}$ (Proposition 2.23) is especially important in the proof of our main theorem

(Theorem 1.3). We apologize to the readers that the arguments are slightly longer than §2 of [Nakamura 2014a], but we think that these arguments are needed. This is because, to define the map $\exp_{f,M}$, we need some additional arguments (Lemmas 2.17, 2.18 and 2.20), and, to obtain the precise explicit formulae for the maps $\exp_M$ and $\exp_{f,M}$, it seems to be safer not to omit any steps of the proofs.

Define $n(K) \geq 1$ to be the minimal integer n such that $1/p^{n-1} \leq \tilde{e}_K C(K)$, and put

$$\mathcal{R}_A^{(n)}(\pi_K) = \mathcal{R}_A^{1/(p^{n-1}\tilde{e}_K)}(\pi_K)$$

for $n \geq n(K)$. For $n \geq n(K)$, one has a Γ_K-equivariant A-algebra homomorphism

$$\iota_n : \mathcal{R}_A^{(n)}(\pi_K) \to (K_n \otimes_{\mathbb{Q}_p} A)[[t]]$$

such that

$$\iota_n(\pi) = \zeta_{p^n} \cdot \exp\Big(\frac{t}{p^n}\Big) - 1 \qquad \text{and} \qquad \iota_n(a) = \varphi^{-n}(a) \quad (a \in F').$$

For $n \geq n(K)$, we have the commutative diagrams

$$\begin{array}{ccc} \mathcal{R}_A^{(n)}(\pi_K) & \xrightarrow{\iota_n} & (K_n \otimes_{\mathbb{Q}_p} A)[[t]] \\ \downarrow \varphi & & \downarrow \mathrm{can} \\ \mathcal{R}_A^{(n+1)}(\pi_K) & \xrightarrow{\iota_{n+1}} & (K_{n+1} \otimes_{\mathbb{Q}_p} A)[[t]] \end{array}$$

and

$$\begin{array}{ccc} \mathcal{R}_A^{(n+1)}(\pi_K) & \xrightarrow{\iota_{n+1}} & (K_{n+1} \otimes_{\mathbb{Q}_p} A)[[t]] \\ \downarrow \psi & & \downarrow \frac{1}{p}\cdot \mathrm{Tr}_{K_{n+1}/K_n} \\ \mathcal{R}_A^{(n)}(\pi_K) & \xrightarrow{\iota_n} & (K_n \otimes_{\mathbb{Q}_p} A)[[t]] \end{array}$$

in which can is the canonical injection and $\frac{1}{p} \cdot \mathrm{Tr}_{K_{n+1}/K_n}$ is defined by

$$\sum_{k\geq 0} a_k t^k \mapsto \sum_{k \geq 0} \frac{1}{p} \cdot \mathrm{Tr}_{K_{n+1}/K_n}(a_k) t^k.$$

Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$ obtained as a base change of a (φ, Γ)-module M^{r_0} over $\mathcal{R}_A^{r_0}(\pi_K)$ for some $0 < r_0 \leq c(K)$. Define $n(M) \in \mathbb{Z}_{\geq n(K)}$ to be the minimal integer such that $1/p^{n-1} \leq \tilde{e}_K r_0$. Put $M^{(n)} = M^{1/(p^{n-1}\tilde{e}_K)}$ for $n \geq n(M)$. Then φ and ψ induce $\varphi : M^{(n)} \to M^{(n+1)}$ and $\psi : M^{(n+1)} \to M^{(n)}$, respectively. Define

$$\boldsymbol{D}^+_{\mathrm{dif},n}(M) = M^{(n)} \otimes_{\mathcal{R}_A^{(n)}(\pi_K),\iota_n} (K_n \otimes_{\mathbb{Q}_p} A)[[t]] \quad \big(\text{resp. } \boldsymbol{D}_{\mathrm{dif},n}(M) = \boldsymbol{D}^+_{\mathrm{dif},n}(M)[1/t]\big),$$

which is a finite projective $(K_n \otimes_{\mathbb{Q}_p} A)[[t]]$-module (resp. $(K_n \otimes_{\mathbb{Q}_p} A)((t))$-module) with a semilinear action of Γ_K. We also let $\iota_n : M^{(n)} \to \boldsymbol{D}^+_{\mathrm{dif},n}(M)$ be the map defined by $x \mapsto x \otimes 1$.

Using the base change of the Frobenius structure $\varphi^*M^{(n)} \xrightarrow{\sim} M^{(n+1)}$ by the map ι_{n+1}, we obtain a Γ_K-equivariant $(K_{n+1}\otimes_{\mathbb{Q}_p} A)[\![t]\!]$-linear isomorphism

$$\begin{aligned}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\otimes_{(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]}(K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!] &\xrightarrow{\sim} \varphi^*(M^{(n)})\otimes_{\mathcal{R}_A^{(n+1)}(\pi_K),\iota_{n+1}}(K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!]\\ &\xrightarrow{\sim} M^{(n+1)}\otimes_{\mathcal{R}_A^{(n+1)}(\pi_K),\iota_{n+1}}(K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!]=\boldsymbol{D}^+_{\mathrm{dif},n+1}(M).\end{aligned}$$

Using this isomorphism, we obtain Γ_K-equivariant $(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]$-linear morphisms

$$\mathrm{can}:\boldsymbol{D}^+_{\mathrm{dif},n}(M)\xrightarrow{x\mapsto x\otimes 1}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\otimes_{(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]}(K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!]\xrightarrow{\sim}\boldsymbol{D}^+_{\mathrm{dif},n+1}(M)$$

and

$$\begin{aligned}\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}:\boldsymbol{D}^+_{\mathrm{dif},n+1}(M)&\xrightarrow{\sim}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\otimes_{(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]}(K_{n+1}\otimes_{\mathbb{Q}_p}A)[\![t]\!]\\ &\xrightarrow{x\otimes f\mapsto\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}(f)x}\boldsymbol{D}^+_{\mathrm{dif},n}(M).\end{aligned}$$

These naturally induce can : $\boldsymbol{D}_{\mathrm{dif},n}(M)\to\boldsymbol{D}_{\mathrm{dif},n+1}(M)$ and $\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}$: $\boldsymbol{D}_{\mathrm{dif},n+1}(M)\to\boldsymbol{D}_{\mathrm{dif},n}(M)$, and we have the commutative diagrams

$$\begin{array}{ccc} M^{(n)} & \xrightarrow{\iota_n} & \boldsymbol{D}^+_{\mathrm{dif},n}(M)\\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mathrm{can}}\\ M^{(n+1)} & \xrightarrow{\iota_{n+1}} & \boldsymbol{D}^+_{\mathrm{dif},n+1}(M)\end{array}$$

and

$$\begin{array}{ccc} M^{(n+1)} & \xrightarrow{\iota_{n+1}} & \boldsymbol{D}^+_{\mathrm{dif},n+1}(M)\\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\frac{1}{p}\cdot\mathrm{Tr}_{K_{n+1}/K_n}}\\ M^{(n)} & \xrightarrow{\iota_n} & \boldsymbol{D}^+_{\mathrm{dif},n}(M)\end{array}$$

Put $\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M):=\varinjlim_{n\geq n(M)}\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)$, where the transition map is can : $\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)\to\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M)$. Then we have

$$\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)=\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)\otimes_{(K_n\otimes_{\mathbb{Q}_p}A)[\![t]\!]}(K_\infty\otimes_{\mathbb{Q}_p}A)[\![t]\!]$$

for any $n\geq n(M)$, where we define $(K_\infty\otimes_{\mathbb{Q}_p}A)[\![t]\!]=\bigcup_{m\geq 1}(K_m\otimes_{\mathbb{Q}_p}A)[\![t]\!]$.

For an $A[\Gamma_K]$-module N, we define a complex of A-modules concentrated in degree [0, 1] by

$$C^\bullet_\gamma(N)=\left[N^\Delta\xrightarrow{\gamma-1}N^\Delta\right]$$

and denote by $\mathrm{H}^i_\gamma(N)$ the cohomology of $C^\bullet_\gamma(N)$. If N is a topological Hausdorff A-module with a continuous action of Γ_K, the complex $C^\bullet_\gamma(N)$ is also independent of the choice of γ up to canonical isomorphism.

Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$. For $n \geq n(M)$ and $M_0 = M, M[1/t]$, we define a complex $\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)})$ concentrated in degree [0, 2] by

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) := \big[M_0^{(n),\Delta} \xrightarrow{(\gamma-1)\oplus(\varphi-1)} M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} \xrightarrow{(\varphi-1)\oplus(1-\gamma)} M_0^{(n+1),\Delta}\big].$$

Of course, we have $\varinjlim_n \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) = C^{\bullet}_{\varphi,\gamma}(M_0)$, where the transition map is the natural one induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$. We define another complex

$$C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) := \varinjlim_{n,\varphi} \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}),$$

where the transition map is the natural one induced by $\varphi : M_0^{(n)} \to M_0^{(n+1)}$. We similarly define

$$C^{(\varphi),\bullet}_{\gamma}(M_0) := \varinjlim_{n,\varphi} C^{\bullet}_{\gamma}(M_0^{(n)})$$

and denote by $\mathrm{H}^{(\varphi),i}_{\varphi,\gamma}(M_0)$ (resp. $\mathrm{H}^{(\varphi),i}_{\gamma}(M_0)$) the cohomology of $C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ (resp. $C^{(\varphi),\bullet}_{\gamma}(M_0)$). For $n \geq n(M)$, we equip $C^{\bullet}_{\gamma}(M_0^{(n)})$ with a structure of a complex of F-vector spaces by $ax := \varphi^n(a)x$ for $a \in F$, $x \in C^{\bullet}_{\gamma}(M_0^{(n)})$. Then $C^{(\varphi),\bullet}_{\gamma}(M_0)$ (resp. $\mathrm{H}^{(\varphi),i}_{\gamma}(M_0)$) is also naturally equipped with a structure of a complex of F-vector spaces (resp. an F-vector space).

By the compatibility of $\varphi : M^{(n)} \hookrightarrow M^{(n+1)}$ and $\mathrm{can} : \boldsymbol{D}^{+}_{\mathrm{dif},n}(M) \hookrightarrow \boldsymbol{D}^{+}_{\mathrm{dif},n+1}(M)$ with respect to the map $\iota_n : M^{(n)} \to \boldsymbol{D}^{+}_{\mathrm{dif},n}(M)$, the map ι_n induces canonical maps

$$\iota : C^{(\varphi),\bullet}_{\gamma}(M) \to C^{\bullet}_{\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif}}(M)) \quad \text{and} \quad \iota : C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \to C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M)),$$

which are $(F \otimes_{\mathbb{Q}_p} A)$-linear.

Lemma 2.17. *For $n \geq n(M)$, the natural maps*

$$C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M)), \quad C^{\bullet}_{\gamma}(M_0^{(n)}) \to C^{\bullet}_{\gamma}(M_0^{(n+1)})$$

and

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n+1)})$$

for $M_0 = M, M[1/t]$, which are induced by φ, are quasi-isomorphism. Similarly, the maps

$$C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)), \quad C^{\bullet}_{\gamma}(M_0^{(n)}) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$$

and

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) \to C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$$

for $M_0 = M, M[1/t]$ are quasi-isomorphism.

Proof. The latter statement is trivial if we can prove the first statement. Let's prove the first statement. We first note that $\gamma - 1 : (M_0^{(n)})^{\psi=0} \to (M_0^{(n)})^{\psi=0}$ is an isomorphism for $n \geq n(M)+1$ by Theorem 3.1.1 of [Kedlaya et al. 2014] (precisely, this fact for $M_0 = M[1/t]$ follows from the proof of this theorem). Taking the base change of this isomorphism by the map $\iota_n : \mathcal{R}_A^{(n)}(\pi_K) \to (K_n \otimes_{\mathbb{Q}_p} A)[\![t]\!]$, we also have that

$$\gamma - 1 : (\boldsymbol{D}_{\mathrm{dif},n}^{(+)}(M))^{\frac{1}{p}\cdot\mathrm{Tr}_{K_n/K_{n-1}}=0} \to (\boldsymbol{D}_{\mathrm{dif},n}^{(+)}(M))^{\frac{1}{p}\cdot\mathrm{Tr}_{K_n/K_{n-1}}=0}$$

is an isomorphism for $n \geq n(M)+1$. Using these facts, we prove the lemma as follows. Here, we only prove that the map $C_\gamma^\bullet(M_0^{(n)}) \to C_\gamma^\bullet(M_0^{(n+1)})$ induced by $\varphi : M_0^{(n)} \to M_0^{(n+1)}$ is an quasi-isomorphism for $n \geq n(M)$ since the other cases can be proved in the same way. Since we have a Γ_K-equivariant decomposition $M_0^{(n+1)} = \varphi(M_0^{(n)}) \oplus (M_0^{(n+1)})^{\psi=0}$, we obtain a decomposition

$$C_\gamma^\bullet(M_0^{(n+1)}) = \varphi(C_\gamma^\bullet(M_0^{(n)})) \oplus C_\gamma^\bullet((M_0^{(n+1)})^{\psi=0}).$$

Since the complex $C_\gamma^\bullet((M_0^{(n+1)})^{\psi=0})$ is acyclic by the above remark and $\varphi : M_0^{(n)} \to M_0^{(n+1)}$ is an injection, the map $\varphi : C_\gamma^\bullet(M_0^{(n)}) \to C_\gamma^\bullet(M_0^{(n+1)})$ is a quasi-isomorphism. □

For another canonical map, $C_\gamma^\bullet(M_0^{(n)}) \to C_\gamma^\bullet(M_0)$, which is induced by the canonical inclusion $M^{(n)} \hookrightarrow M$, we can show the following lemma.

Lemma 2.18. *For $n \geq n(M)$ and $M_0 = M, M[1/t]$, the inclusion*

$$\mathrm{H}_\gamma^0(M_0^{(n)}) \hookrightarrow \mathrm{H}_\gamma^0(M_0)$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is an isomorphism.

Proof. It suffices to show that $\mathrm{H}_\gamma^0(M_0^{(n)}) \hookrightarrow \mathrm{H}_\gamma^0(M_0^{(n+1)})$ is an isomorphism for each $n \geq n(M)$. We first prove this claim when A is a finite $\mathbb{Q}_p$-algebra. In this case, we may assume $A = \mathbb{Q}_p$. Since we have an inclusion $\iota_n : \mathrm{H}_\gamma^0(M_0^{(n)}) \hookrightarrow \mathrm{H}_\gamma^0(\boldsymbol{D}_{\mathrm{dif}}(M))$ and the latter is a finite-dimensional $\mathbb{Q}_p$-vector space, $\mathrm{H}_\gamma^0(M_0^{(n)})$ is also finite-dimensional. Since $\varphi : C_\gamma^\bullet(M_0^{(n)}) \to C_\gamma^\bullet(M_0^{(n+1)})$ is a quasi-isomorphism for $n \geq n(M)$ by the above lemma, we get an isomorphism $\varphi : \mathrm{H}_\gamma^0(M_0^{(n)}) \xrightarrow{\sim} \mathrm{H}_\gamma^0(M_0^{(n+1)})$. In particular, the dimension of $\mathrm{H}_\gamma^0(M_0^{(n)})$ is independent of $n \geq n(M)$. Hence, the canonical inclusion $\mathrm{H}_\gamma^0(M_0^{(n)}) \hookrightarrow \mathrm{H}_\gamma^0(M_0^{(n+1)})$ is an isomorphism.

We next prove the claim for general A. By Lemma 6.4 of [Kedlaya and Liu 2010], there exists a strict inclusion $A \hookrightarrow \prod_{i=1}^k A_i$ of topological rings, in which each A_i is a finite algebra over a complete discretely valued field. If we similarly define the rings $\mathcal{R}_{A_i}^{(n)}(\pi_K)$, $\mathcal{R}_{A_i}(\pi_K)$, we can generalize the notions concerning (φ, Γ)-modules for $\mathcal{R}_{A_i}(\pi_K)$. In particular, the above claim holds for $M_{0,i} := M_0 \widehat{\otimes}_A A_i$ for each i.

Consider the following canonical diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \longrightarrow & M_0^{(n)} & \longrightarrow & M_0^{(n+1)} & \longrightarrow & M_0^{(n+1)}/M_0^{(n)} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n)} & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n+1)} & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n+1)}/M_{0,i}^{(n)} & \longrightarrow 0 \end{array}$$

If we can show that the right vertical arrow is an injection, then the claim for A follows from the claim for each A_i by a simple diagram chase. To show that the right vertical arrow is an injection, we may assume that $M = \mathcal{R}_A(\pi_K)$ since $M^{(n)}$ is finite projective over $\mathcal{R}_A^{(n)}(\pi_K)$ for each n. Then the natural map

$$\mathcal{R}_A^{(n+1)}(\pi_K)[1/t]/\mathcal{R}_A^{(n)}(\pi_K)[1/t] \to \prod_{i=1}^{k} \mathcal{R}_{A_i}^{(n+1)}(\pi_K)[1/t]/\mathcal{R}_{A_i}^{(n)}(\pi_K)[1/t]$$

is an injection since the inclusion $A \hookrightarrow \prod_{i=1}^k A_i$ is strict, which proves the claim for general A, hence proves the lemma. □

Remark 2.19. We don't know whether the natural map $\mathrm{H}^1_\gamma(M_0^{(n)}) \to \mathrm{H}^1_\gamma(M_0)$ induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is an isomorphism or not.

For the (φ, Γ)-cohomology, we can prove the following lemma.

Lemma 2.20. (1) *For $n \geq n(M)$ and for $M_0 = M, M[1/t]$, the map*

$$\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to C^\bullet_{\varphi,\gamma}(M_0)$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is a quasi-isomorphism.

(2) *In $\boldsymbol{D}^-(A)$, the isomorphism*

$$C^\bullet_{\varphi,\gamma}(M_0) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0),$$

which is obtained as the composition of the inverse of the isomorphism in (1), *$\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(M_0)$, with the isomorphism $\widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ in Lemma 2.17, is independent of the choice of $n \geq n(M)$.*

Proof. For $n \geq n(M)$, we define a map $f_\bullet : \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})[+1]$ by

$$f_1 : M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} \to M_0^{(n+1),\Delta} : (x, y) \mapsto y,$$
$$f_2 : M_0^{(n+1),\Delta} \to M_0^{(n+1),\Delta} \oplus M_0^{(n+2),\Delta} : x \mapsto (x, 0).$$

This gives a homotopy between

$$\varphi : \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})$$

and

$$\mathrm{can} : \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M_0^{(n+1)})$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$. Hence, can : $\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) \to \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n+1)})$ is also an isomorphism by Lemma 2.17, and, by taking the limit, the map $\widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) \to C^{\bullet}_{\varphi,\gamma}(M_0)$ is also an isomorphism, which proves (1).

In a similar way, we can show that the map can : $C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \to C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ induced by the canonical inclusions can : $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$ for any $n \geq n(M)$ is homotopic to the identity map. Hence, we obtain the following commutative diagram in $\boldsymbol{D}^-(A)$ for any $n \geq n(M)$:

$$\begin{array}{ccc} \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) & \longrightarrow & C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \\ \big\downarrow \text{can} & & \big\downarrow \text{id} \\ \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n+1)}) & \longrightarrow & C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \end{array}$$

From this we obtain the second statement in the lemma. □

We define a morphism

$$f : C^{\bullet}_{\varphi,\gamma}(M_0) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$$

in $\boldsymbol{D}^-(A)$ as the composition of the isomorphism $C^{\bullet}_{\varphi,\gamma}(M_0) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0)$ in Lemma 2.20(2) with the map $C^{(\varphi),\bullet}_{\varphi,\gamma}(M_0) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$, which is induced by

$$\begin{array}{ccccccc} \widetilde{C}^{\bullet}_{\varphi,\gamma}(M_0^{(n)}) = \big[& M_0^{(n),\Delta} & \xrightarrow{(\gamma-1)\oplus(\varphi-1)} & M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} & \xrightarrow{(\varphi-1)\oplus(1-\gamma)} & M_0^{(n+1),\Delta} \big] \\ \big\downarrow & \big\downarrow \text{id} & & \big\downarrow (x,y)\mapsto x & & \\ C^{\bullet}_{\gamma}(M_0^{(n)}) = \big[& M_0^{(n),\Delta} & \xrightarrow{\gamma-1} & M_0^{(n),\Delta} \big] & & \end{array}$$

We define

$$g : C^{\bullet}_{\varphi,\gamma}(M) \xrightarrow{f} C^{(\varphi),\bullet}_{\gamma}(M) \xrightarrow{\iota} C^{\bullet}_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M))$$

and let

$$\text{can} : C^{(\varphi),\bullet}_{\gamma}(M_0) \to C^{(\varphi),\bullet}_{\gamma}(M_0)$$

be the map induced by the canonical inclusion can : $M_0^{(n)} \to M_0^{(n+1)}$ for each $n \geq n(M)$. Under this notation, we prove the following proposition, which is a modified version of Theorem 2.8 of [Nakamura 2014a].

Proposition 2.21. *We have a functorial map between the two distinguished triangles*

$$\begin{array}{ccccccc} C^{\bullet}_{\varphi,\gamma}(M) & \xrightarrow{d_1} & \begin{array}{c} C^{\bullet}_{\varphi,\gamma}(M[1/t]) \\ \oplus C^{\bullet}_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \end{array} & \xrightarrow{d_2} & C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M)) & \xrightarrow{[+1]} \\ \big\downarrow \text{id} & & \big\downarrow f\oplus\text{id} & & \big\downarrow x\mapsto(0,x) & \\ C^{\bullet}_{\varphi,\gamma}(M) & \xrightarrow{d_3} & \begin{array}{c} C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \\ \oplus C^{\bullet}_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \end{array} & \xrightarrow{d_4} & \begin{array}{c} C^{(\varphi),\bullet}_{\gamma}(M[1/t]) \\ \oplus C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M)) \end{array} & \xrightarrow{[+1]} \end{array} \tag{4}$$

with

$$d_1(x) = (x, g(x)), \qquad d_2(x, y) = g(x) - y,$$
$$d_3(x) = (f(x), g(x)), \quad d_4(x, y) = ((\mathrm{can} - 1)x, g(x) - y).$$

Remark 2.22. In §2 of [Nakamura 2014a], we (essentially) proved that the top horizontal line in the proposition is a distinguished triangle. For the application to the local ε-conjecture, we also need the bottom triangle, which involves $\boldsymbol{D}^K_{\mathrm{cris}}(M) := \mathrm{H}^0_\gamma(M[1/t])$.

Proof of Proposition 2.21. We first show that the top horizontal line is a distinguished triangle. Actually, this is the content of Theorem 2.8 of [Nakamura 2014a], but we briefly recall the proof since we also use it to prove that the bottom line is a distinguished triangle. In this proof, we assume $\Delta = \{1\}$ for simplicity; the general case follows by just taking the Δ-fixed parts.

For $n \geq n(M)$, we have the exact sequence of A-modules

$$0 \to M^{(n)} \xrightarrow{c_1} M^{(n)}[1/t] \oplus \prod_{m \geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \xrightarrow{c_2} \bigcup_{k \geq 0} \prod_{m \geq n} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \to 0 \tag{5}$$

with

$$c_1(x) = (x, (\iota_m(x))_{m \geq n}) \quad \text{and} \quad c_2(x, (y_m)_{m \geq n}) = (\iota_m(x) - y_m)_{m \geq n}$$

by Lemma 2.9 of [Nakamura 2014a] (precisely, we proved it when A is a finite $\mathbb{Q}_p$-algebra, but we can prove it for general A in the same way). For $n \geq n(M)$ and $k \geq 0$, we define a complex $\widetilde{C}^\bullet_{\varphi,\gamma}\bigl(\frac{1}{t^k}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\bigr)$ concentrated in degree in [0, 2] by

$$\Bigl[\prod_{m \geq n} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \xrightarrow{b_0} \prod_{m \geq n} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \oplus \prod_{m \geq n+1} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \xrightarrow{b_1} \prod_{m \geq n+1} \frac{1}{t^k} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\Bigr] \tag{6}$$

with

$$b_0((x_m)_{m \geq n}) = \bigl(((\gamma - 1)x_m)_{m \geq n}, (x_{m-1} - x_m)_{m \geq n+1}\bigr)$$

and

$$b_1((x_m)_{m \geq n}, (y_m)_{m \geq n+1}) = ((x_{m-1} - x_m) - (\gamma - 1)y_m)_{m \geq n+1}.$$

Put $\widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M)) = \bigcup_{k \geq 0} \widetilde{C}^\bullet_{\varphi,\gamma}\bigl(\frac{1}{t^k}\boldsymbol{D}^+_{\mathrm{dif},n}(M)\bigr)$. By the exact sequence (5), we obtain the following exact sequence of complexes of A-modules:

$$0 \to \widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M)) \to 0. \tag{7}$$

Moreover, the map $C^\bullet_\gamma(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M))$, which is defined by

$$\begin{array}{ccccc} \boldsymbol{D}^+_{\mathrm{dif},n}(M) & \xrightarrow{\gamma-1} & \boldsymbol{D}^+_{\mathrm{dif},n}(M) & & \\ \Big\downarrow{\scriptstyle x\mapsto(x)_{m\geq n}} & & \Big\downarrow{\scriptstyle x\mapsto((x)_{m\geq n},0)} & & \\ \prod_{m\geq n}\boldsymbol{D}^+_{\mathrm{dif},m}(M) & \longrightarrow & \begin{array}{c}\prod_{m\geq n}\boldsymbol{D}^+_{\mathrm{dif},m}(M)\\ \oplus\prod_{m\geq n+1}\boldsymbol{D}^+_{\mathrm{dif},m}(M)\end{array} & \longrightarrow & \prod_{m\geq n+1}\boldsymbol{D}^+_{\mathrm{dif},m}(M) \end{array} \tag{8}$$

and the similar map $C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))$ are easily seen to be quasi-isomorphisms since we have the exact sequence

$$0 \to \boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M) \xrightarrow{x\mapsto(x)_{m\geq n}} \prod_{m\geq n}\boldsymbol{D}^{(+)}_{\mathrm{dif},m}(M) \xrightarrow{(x_m)_{m\geq n}\mapsto(x_{m-1}-x_m)_{m\geq n+1}} \prod_{m\geq n+1}\boldsymbol{D}^{(+)}_{\mathrm{dif},m}(M) \to 0. \tag{9}$$

Put $\widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) := \varinjlim_{n,a^\bullet}\widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))$, where the transition map

$$a^\bullet : \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M))$$

is defined by

$$a^0((x_m)_{m\geq n}) = (x_m)_{m\geq n+1},$$
$$a^1((x_m)_{m\geq n}, (y_m)_{m\geq n+1}) = ((x_m)_{m\geq n+1}, (y_m)_{m\geq n+2}),$$
$$a^2((x_m)_{m\geq n+1}) = (x_m)_{m\geq n+2}.$$

We also define $\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) := \varinjlim_{n,(a')^\bullet}\widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))$, where the transition map $(a')^\bullet$ is defined by

$$(a')^0((x_m)_{m\geq n}) = (x_{m-1})_{m\geq n+1},$$
$$(a')^1((x_m)_{m\geq n}, (y_m)_{m\geq n+1}) = ((x_{m-1})_{m\geq n+1}, (y_{m-1})_{m\geq n+2}),$$
$$(a')^2((x_m)_{m\geq n+1}) = (x_{m-1})_{m\geq n+2}.$$

Then it is easy to see that the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \xrightarrow{\sim} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M))$ defined in (8) is compatible with the transition maps $a^\bullet$, $(a')^\bullet$ and $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \hookrightarrow C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif},n+1}(M))$, hence induces quasi-isomorphisms

$$C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \xrightarrow{\sim} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)), \quad C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \xrightarrow{\sim} \widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)). \tag{10}$$

For $\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$, we also have a left inverse

$$\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \to C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \tag{11}$$

of the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \to \widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$, which is obtained as the limit of the map

$$\begin{array}{ccccc} \prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) & \longrightarrow & \begin{array}{c}\prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\\ \oplus\prod_{m\geq n+1} \boldsymbol{D}^+_{\mathrm{dif},m}(M)\end{array} & \longrightarrow & \prod_{m\geq n+1} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \\ \downarrow{\scriptstyle (x_m)_{m\geq n}\mapsto x_n} & & \downarrow{\scriptstyle ((x_m)_{m\geq n},(y_m)_{m\geq n+1})\mapsto x_n} & & \\ \boldsymbol{D}^+_{\mathrm{dif},n}(M) & \xrightarrow{\gamma-1} & \boldsymbol{D}^+_{\mathrm{dif},n}(M) & & \end{array}$$

Taking the limits of the map $\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) : x \mapsto (\iota_m(x))_{m\geq n_0}$ $(n_0 = n, n+1)$, we obtain the maps

$$C^\bullet_{\varphi,\gamma}(M) \to \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \quad \text{and} \quad C^{(\varphi),\bullet}_{\varphi,\gamma}(M) \to \widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M)). \tag{12}$$

Taking the limit of the exact sequence (7) with respect to the transition map induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$ and $a_\bullet$, and taking the quasi-isomorphism $C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \xrightarrow{\sim} \widetilde{C}^\bullet_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$ in (10), we obtain the following exact triangle, which is the top horizontal line in the proposition:

$$C^\bullet_{\varphi,\gamma}(M) \xrightarrow{d_1} C^\bullet_{\varphi,\gamma}(M[1/t]) \oplus C^\bullet_\gamma(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \xrightarrow{d_2} C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M)) \xrightarrow{[+1]} .$$

On the other hand, since we have

$$C^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) = \mathrm{Cone}\big(1-\varphi : C^\bullet_\gamma(M^{(n)}[1/t]) \to C^\bullet_\gamma(M^{(n+1)}[1/t])\big)[-1]$$

for $n \geq n(M)$ (where we define $\mathrm{Cone}(f : M^\bullet \to N^\bullet)[-1]^n = M^n \oplus N^{n-1}$ and $d : M^n \oplus N^{n-1} \to M^{n+1} \oplus N^n : (x, y) \mapsto (d_M(x), -f(x) - d_N(y))$), taking the limit of the exact sequence (7) with respect to the transition map induced by $a'_\bullet$ and $\varphi : M_0^{(n)} \hookrightarrow M_0^{(n+1)}$, and taking the left inverse $\widetilde{C}^{(\varphi),\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M)) \to C^\bullet_\gamma(\boldsymbol{D}^{(+)}_{\mathrm{dif}}(M))$ in (11), and identifying $C^\bullet_{\varphi,\gamma}(M) \xrightarrow{\sim} C^{(\varphi),\bullet}_{\varphi,\gamma}(M)$ by Lemma 2.20(2), we obtain the following exact triangle, which is the bottom horizontal line in the proposition:

$$C^\bullet_{\varphi,\gamma}(M) \xrightarrow{d_3} C^{(\varphi),\bullet}_\gamma(M[1/t]) \oplus C^\bullet_\gamma(\boldsymbol{D}^+_{\mathrm{dif}}(M)) \xrightarrow{d_4} C^{(\varphi),\bullet}_\gamma(M[1/t]) \oplus C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M)) \xrightarrow{[+1]} .$$

Here $d_3(x) = (f(x), g(x))$ and $d_4(x, y) = ((\mathrm{can}-1)(x), g(x) - y)$, which proves the proposition. □

We next recall some notions concerning p-adic Hodge theory for (φ, Γ)-modules over the Robba ring. For a (φ, Γ)-module M over $\mathcal{R}_A(\pi_K)$, let us define

$$\boldsymbol{D}^K_{\mathrm{dR}}(M) := \mathrm{H}^0_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M)) \quad \text{and} \quad \boldsymbol{D}^K_{\mathrm{dR}}(M)^i := \boldsymbol{D}^K_{\mathrm{dR}}(M) \cap t^i \boldsymbol{D}^+_{\mathrm{dif}}(M)$$

for $i \in \mathbb{Z}$, and

$$\boldsymbol{D}^K_{\mathrm{cris}}(M) := \mathrm{H}^0_\gamma(M[1/t]).$$

By Lemma 2.17, $\varphi : C^\bullet_\gamma(M[1/t]) \to C^\bullet_\gamma(M[1/t])$ induces a φ-semilinear automorphism

$$\varphi : \boldsymbol{D}^K_{\mathrm{cris}}(M) \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{cris}}(M).$$

More precisely, by Lemma 2.18, we have $\boldsymbol{D}_{\mathrm{cris}}(M) = \mathrm{H}^0_\gamma(M^{(n)}[1/t])$, and φ induces an automorphism $\varphi : \mathrm{H}^0_\gamma(M^{(n)}[1/t]) \xrightarrow{\varphi} \mathrm{H}^0_\gamma(M^{(n+1)}[1/t]) = \mathrm{H}^0_\gamma(M^{(n)}[1/t])$ for $n \geq n(M)$. Using these facts, we define an isomorphism

$$j_1 : \boldsymbol{D}^K_{\mathrm{cris}}(M) = \mathrm{H}^0_\gamma(M^{(n)}[1/t]) \xrightarrow{\varphi^n} \mathrm{H}^0_\gamma(M^{(n)}[1/t]) \xrightarrow{\sim} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]),$$

which does not depend on the choice of n. Then the map $\iota : C^{(\varphi),\bullet}_\gamma(M[1/t]) \to C^\bullet_\gamma(\boldsymbol{D}_{\mathrm{dif}}(M))$ induces an $(F \otimes_{\mathbb{Q}_p} A)$-linear injection

$$\iota : \boldsymbol{D}^K_{\mathrm{cris}}(M) \xrightarrow{j_1} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]) \xrightarrow{\iota} \boldsymbol{D}^K_{\mathrm{dR}}(M).$$

We define another isomorphism

$$j_2 : \boldsymbol{D}^K_{\mathrm{cris}}(M) \xrightarrow{j_1} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]) \xrightarrow{\mathrm{can}} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]),$$

where $\mathrm{H}^{(\varphi),0}_\gamma(M[1/t]) \xrightarrow{\mathrm{can}} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t])$ is the map induced by

$$\mathrm{can} : C^{(\varphi),\bullet}_\gamma(M[1/t]) \to C^{(\varphi),\bullet}_\gamma(M[1/t]),$$

which is an isomorphism by Lemma 2.20. Then we obtain the commutative diagram

$$\begin{array}{ccc} \boldsymbol{D}^K_{\mathrm{cris}}(M) & \xrightarrow{1-\varphi} & \boldsymbol{D}^K_{\mathrm{cris}}(M) \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]) & \xrightarrow{\mathrm{can}-\mathrm{id}} & \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]) \end{array}$$

Let us denote by

$$\exp_M : \boldsymbol{D}^K_{\mathrm{dR}}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M), \quad \exp_{f,M} : \boldsymbol{D}^K_{\mathrm{cris}}(M) \xrightarrow{j_2} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t]) \to \mathrm{H}^1_{\varphi,\gamma}(M)$$

the boundary maps obtained by taking the cohomology of the exact triangles in Proposition 2.21. We define

$$\mathrm{H}^1_{\varphi,\gamma}(M)_e = \mathrm{Im}\big(\boldsymbol{D}^K_{\mathrm{dR}}(M) \xrightarrow{\exp_M} \mathrm{H}^1_{\varphi,\gamma}(M)\big)$$

and

$$\mathrm{H}^1_{\varphi,\gamma}(M)_f = \mathrm{Im}\big(\boldsymbol{D}^K_{\mathrm{cris}}(M) \oplus \boldsymbol{D}^K_{\mathrm{dR}}(M) \xrightarrow{\exp_{f,M} \oplus \exp_M} \mathrm{H}^1_{\varphi,\gamma}(M)\big).$$

We call the latter group the finite cohomology. Put $t_M(K) := \boldsymbol{D}^K_{\mathrm{dR}}(M)/\boldsymbol{D}^K_{\mathrm{dR}}(M)^0$. By Proposition 2.21, we obtain the diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow \mathrm{H}^0_{\varphi,\gamma}(M) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(M)^{\varphi=1} & \xrightarrow{x\mapsto\overline{\iota(x)}} & t_M(K) & \xrightarrow{\exp_M} & \mathrm{H}^1_{\varphi,\gamma}(M)_e \longrightarrow 0 \\
\Big\downarrow \mathrm{id} & & \Big\downarrow x\mapsto x & & \Big\downarrow x\mapsto(0,x) & & \Big\downarrow x\mapsto x \\
0 \longrightarrow \mathrm{H}^0_{\varphi,\gamma}(M) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(M) & \xrightarrow{d_5} & \boldsymbol{D}^K_{\mathrm{cris}}(M) \oplus t_M(K) & \xrightarrow{d_6} & \mathrm{H}^1_{\varphi,\gamma}(M)_f \longrightarrow 0
\end{array} \tag{13}$$

with

$$d_5(x, y) = ((1-\varphi)x, \overline{\iota(x)}) \quad \text{and} \quad d_6 = \exp_{f,M} \oplus \exp_M,$$

where we also define $\exp_M : t_M(K) \to \mathrm{H}^1_{\varphi,\gamma}(M)$, which is naturally induced by $\exp_M : \boldsymbol{D}^K_{\mathrm{dR}}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M)$.

By the proof of Proposition 2.21, we obtain the following explicit formulae for $\exp_M$ and $\exp_{f,M}$, which are very important in the proof of our main theorem (Theorem 1.3).

Proposition 2.23. (1) *For $x \in \boldsymbol{D}^K_{\mathrm{dR}}(M)$, take $\tilde{x} \in M^{(n)}[1/t]^{\Delta}$ $(n \geq n(M))$ such that*

$$\iota_m(\tilde{x}) - x \in \boldsymbol{D}^+_{\mathrm{dif},m}(M)$$

for any $m \geq n$ (such an $\tilde{x}$ exists by the exact sequence (5) in the proof of Proposition 2.21). Then we have

$$\exp_M(x) = [(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}] \in \mathrm{H}^1_{\varphi,\gamma}(M).$$

(2) *For $x \in \boldsymbol{D}^K_{\mathrm{cris}}(M)$, take $\tilde{x} \in M^{(n)}[1/t]^{\Delta}$ $(n \geq n(M))$ such that*

$$\iota_n(\tilde{x}) \in \boldsymbol{D}^+_{\mathrm{dif},n}(M)$$

and

$$\iota_{n+k}(\tilde{x}) - \sum_{l=1}^{k} \iota_{n+l}(\varphi^n(x)) \in \boldsymbol{D}^+_{\mathrm{dif},n+k}(M)$$

for any $k \geq 1$ (we remark that we have $\varphi^n(x) \in M^{(n)}[1/t]$ by Lemma 2.18 and that such an $\tilde{x}$ exists by the exact sequence (5)). Then we have

$$\exp_{f,M}(x) = [(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x} + \varphi^n(x)] \in \mathrm{H}^1_{\varphi,\gamma}(M).$$

Proof. These formulae directly follow from simple but a little bit long diagram chases in the proof of Proposition 2.21. For the convenience of the reader, we give a proof of these formulae.

We first prove formula (1). By the proof of Proposition 2.21, the above exact triangle in this proposition is obtained by taking the limit of the composition of the quasi-isomorphism

$$\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}) \xrightarrow{\sim} \mathrm{Cone}\big(\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \widetilde{C}^{\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif},n}(M)) \to \widetilde{C}^{\bullet}_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] := C^{\bullet}_1$$

(obtained by the exact sequence (7)) with the inverse of the quasi-isomorphism

$$C^{\bullet}_2 := \mathrm{Cone}\big(\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^{\bullet}_{\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif},n}(M)) \to C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] \xrightarrow{\sim} C^{\bullet}_1$$

induced by the quasi-isomorphism $C^{\bullet}_{\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) \to \widetilde{C}^{\bullet}_{\varphi,\gamma}(\boldsymbol{D}^{(+)}_{\mathrm{dif},n}(M)) : x \mapsto (x)_{m\geq n}$ of (10).

By definition of $\exp_M(-)$, for $x \in \mathrm{H}^0_{\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))$, these quasi-isomorphisms send $\exp_M(x)$ $\big($which we see as an element of $\mathrm{H}^1(\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}))\big)$ to the element $[0, 0, x] \in \mathrm{H}^1(C^{\bullet}_2)$ represented by

$$(0, 0, x) \in \widetilde{C}^{1}_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^{1}_{\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif},n}(M)) \oplus C^{0}_{\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M)).$$

Take $\tilde{x} \in M^{(n)}[1/t]^{\Delta}$ satisfying the condition in (1). Then it suffices to show that $[(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}] \in \mathrm{H}^1(\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}))$ and $[0, 0, x] \in \mathrm{H}^1(C^{\bullet}_2)$ are the same element in $\mathrm{H}^1(C^{\bullet}_1)$. By definition, $[(\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}]$ is sent to

$$\big[((\gamma-1)\tilde{x}, (\varphi-1)\tilde{x}), \big((\iota_m((\gamma-1)\tilde{x}))_{m\geq n}, (\iota_m((\varphi-1)\tilde{x}))_{m\geq n+1}\big), 0\big]$$

and $[0, 0, x]$ is sent to

$$[0, 0, (-x)_{m\geq n}]$$

in $\mathrm{H}^1(C^{\bullet}_1)$. Both are represented by elements of

$$\widetilde{C}^{1}_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \widetilde{C}^{1}_{\varphi,\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif},n}(M)) \oplus \widetilde{C}^{0}_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))$$

(we note the sign; for $f : C^{\bullet} \to D^{\bullet}$, we define $D^{\bullet-1} \to \mathrm{Cone}(C^{\bullet} \to D^{\bullet})[-1]$ by $x \mapsto (-x, 0)$ and $\mathrm{Cone}(C^{\bullet} \to D^{\bullet})[-1] \to C^{\bullet}$ by $(x, y) \mapsto y$). Then it is easy to check that the difference of these two elements is the coboundary of the element

$$(\tilde{x}, (\iota_m(\tilde{x}) - x)_{m\geq n}) \in C^0_1 = M^{(n)}[1/t]^{\Delta} \oplus \prod_{m\geq n} \boldsymbol{D}^{+}_{\mathrm{dif},m}(M)^{\Delta},$$

which proves (1).

We next prove (2). The bottom exact triangle in Proposition 2.21 is obtained by taking the limit of the composition of the quasi-isomorphism $\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}) \xrightarrow{\sim} C^{\bullet}_1$ defined above with the quasi-isomorphism

$$C^{\bullet}_1 \xrightarrow{\sim} \mathrm{Cone}\big(\widetilde{C}^{\bullet}_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^{\bullet}_{\gamma}(\boldsymbol{D}^{+}_{\mathrm{dif},n}(M)) \to C^{\bullet}_{\gamma}(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] := C^{\bullet}_3$$

induced by the map $\prod_{m\geq n} \boldsymbol{D}^+_{\mathrm{dif},m}(M) \to \boldsymbol{D}^+_{\mathrm{dif},n}(M) : (x_m)_{m\geq n} \to x_n$, with the inverse of the quasi-isomorphism

$$C_3^\bullet \xrightarrow{\sim} \mathrm{Cone}\big(C_\gamma^\bullet(M^{(n)}[1/t]) \oplus C_\gamma^\bullet(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \\ \to C_\gamma^\bullet(M^{(n+1)}[1/t]) \oplus C_\gamma^\bullet(\boldsymbol{D}_{\mathrm{dif},n}(M))\big)[-1] := C_4^\bullet,$$

which is naturally obtained by the identity

$$\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t])) = \mathrm{Cone}\big(C_\gamma^\bullet(M^{(n)}[1/t]) \xrightarrow{1-\varphi} C_\gamma^\bullet(M^{(n+1)}[1/t])\big)[-1].$$

For $x' \in \mathrm{H}^0_\gamma(M^{(n+1)}[1/t])$, the image of x' by the first boundary map of the cone $C_4^\bullet$ is equal to $[0, 0, x', 0] \in \mathrm{H}^1(C_4^\bullet)$, which is represented by the element

$$(0, 0, x', 0) \in C^1_\gamma(M^{(n)}[1/t]) \oplus C^1_\gamma(\boldsymbol{D}^+_{\mathrm{dif},n}(M)) \oplus C^0_\gamma(M^{(n+1)}[1/t]) \oplus C^0_\gamma(\boldsymbol{D}_{\mathrm{dif},n}(M)).$$

Take $\tilde{x}' \in M^{(n)}[1/t]^\Delta$ such that

$$\iota_n(\tilde{x}') \in \boldsymbol{D}^+_{\mathrm{dif},n}(M) \qquad \text{and} \qquad \iota_{n+k}(\tilde{x}') - \sum_{l=1}^{k} \iota_{n+l}(x') \in \boldsymbol{D}^+_{\mathrm{dif},n+k}(M) \quad \text{for any } k \geq 1.$$

Then, by definition of the map $j_2 : \boldsymbol{D}^K_{\mathrm{cris}}(M) \xrightarrow{\sim} \mathrm{H}^{(\varphi),0}_\gamma(M[1/t])$ and $\exp_{f,M}$, it suffices to show that the element $[(\gamma-1)\tilde{x}', (\varphi-1)\tilde{x}'+x'] \in \mathrm{H}^1(\widetilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}))$ is sent to $[0, 0, x', 0] \in \mathrm{H}^1(C_4^\bullet)$ by the above quasi-isomorphisms. By definition, the element $[(\gamma-1)\tilde{x}', (\varphi-1)\tilde{x}'+x']$ is sent to

$$[(\gamma-1)\tilde{x}', \iota_n((\gamma-1)\tilde{x}'), (\varphi-1)\tilde{x}' + x', 0] \in \mathrm{H}^1(C_4^\bullet)$$

by the above quasi-isomorphism. Then it is easy to check that the difference of this element with $[0, 0, x', 0]$ is the coboundary of the element

$$(\tilde{x}', \iota_n(\tilde{x}')) \in C_4^0 = M^{(n)}[1/t]^\Delta \oplus \boldsymbol{D}^+_{\mathrm{dif},n}(M)^\Delta,$$

which proves formula (2). □

We next generalize the Bloch–Kato duality concerning the finite cohomology for (φ, Γ)-modules. Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a (φ, Γ)-module over $\mathcal{R}_L(\pi_K)$. We say that M is de Rham if the equality $\dim_K \boldsymbol{D}^K_{\mathrm{dR}}(M) = [L:\mathbb{Q}_p]\cdot r_M$ holds. When M is de Rham, we have a natural L-bilinear perfect pairing

$$[-,-]_{\mathrm{dR}} : \boldsymbol{D}^K_{\mathrm{dR}}(M^*) \times \boldsymbol{D}^K_{\mathrm{dR}}(M) \xrightarrow{(f,x)\mapsto f(x)} \boldsymbol{D}^K_{\mathrm{dR}}(\mathcal{R}_L(1)) \\ = L \otimes_{\mathbb{Q}_p} K\frac{1}{t}\boldsymbol{e}_1 \xrightarrow{\frac{a}{t}\boldsymbol{e}_1 \mapsto \frac{1}{[K:\mathbb{Q}_p]}(\mathrm{id}\otimes \mathrm{tr}_{K/\mathbb{Q}_p})(a)} L, \quad (14)$$

which induces natural isomorphisms

$$\boldsymbol{D}^K_{\mathrm{dR}}(M) \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{dR}}(M^*)^\vee \quad \text{and} \quad \boldsymbol{D}^K_{\mathrm{dR}}(M)^0 \xrightarrow{\sim} t_{M^*}(K)^\vee.$$

We remark that, as in the étale case, we have

$$\mathrm{H}^1_{\varphi,\gamma}(M)_e = \mathrm{Ker}(\mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M[1/t]))$$

and

$$\mathrm{H}^1_{\varphi,\gamma}(M)_f = \mathrm{Ker}(\mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^{(\varphi),1}_{\gamma}(M[1/t]))$$

under the assumption that M is de Rham.

Proposition 2.24. *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L(\pi_K)$. Then $\mathrm{H}^1_{\varphi,\gamma}(M)_f$ is the orthogonal complement of $\mathrm{H}^1_{\varphi,\gamma}(M^*)_f$ with respect to the Tate duality pairing*

$$\langle -, - \rangle : \mathrm{H}^1_{\varphi,\gamma}(M^*) \times \mathrm{H}^1_{\varphi,\gamma}(M) \to L.$$

Proof. We remark that we have $\dim_L \mathrm{H}^1_{\varphi,\gamma}(M)_f = \dim_L(t_M(K)) + \dim_L \mathrm{H}^0_{\varphi,\gamma}(M)$ by the bottom exact sequence of (13). Using this formula for M, M^*, it is easy to check that we have $\dim_L \mathrm{H}^1_{\varphi,\gamma}(M)_f + \dim_L \mathrm{H}^1_{\varphi,\gamma}(M^*)_f = \dim_L \mathrm{H}^1_{\varphi,\gamma}(M)$ under the assumption that M is de Rham. Hence, it suffices to show that we have $\langle x, y \rangle = 0$ for any $x \in \mathrm{H}^1_{\varphi,\gamma}(M^*)_f$ and $y \in \mathrm{H}^1_{\varphi,\gamma}(M)_f$ by comparing the dimensions. By definition of $\mathrm{H}^1_{\varphi,\gamma}(-)_f$, this claim follows from Lemma 2.25 below. □

Let M be a (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$ (we don't need to assume that M is de Rham). Using the isomorphism $j_2 : \boldsymbol{D}^K_{\mathrm{cris}}(M^*) \xrightarrow{\sim} \mathrm{H}^{(\varphi),0}_{\gamma}(M^*[1/t])$, define an A-bilinear pairing

$$h(-, -) : (\boldsymbol{D}^K_{\mathrm{cris}}(M^*) \oplus \boldsymbol{D}^K_{\mathrm{dR}}(M^*)) \times (\mathrm{H}^{(\varphi),1}_{\gamma}(M[1/t]) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(M))) \\ \to \mathrm{H}^{(\varphi),1}_{\gamma}(M^* \otimes M[1/t]) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M^* \otimes M))$$

by

$$h((x, y), ([z], [w])) := ([j_2(x) \otimes z], [y \otimes w]).$$

Lemma 2.25. *For $(x, y) \in \boldsymbol{D}^K_{\mathrm{cris}}(M^*) \oplus \boldsymbol{D}^K_{\mathrm{dR}}(M^*)$ and $z \in \mathrm{H}^1_{\varphi,\gamma}(M)$, we have*

$$f_2(h((x, y), g(z))) = (\exp_{f,M^*}(x) + \exp_{M^*}(y)) \cup z \in \mathrm{H}^2_{\varphi,\gamma}(M^* \otimes M),$$

where

$$g : \mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^{(\varphi),1}_{\gamma}(M) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}^+_{\mathrm{dR}}(M))$$

is induced by d_3 and

$$f_2 : \mathrm{H}^{(\varphi),1}_{\gamma}(M^* \otimes M[1/t]) \oplus \mathrm{H}^1_{\gamma}(\boldsymbol{D}_{\mathrm{dif}}(M^* \otimes M)) \to \mathrm{H}^2_{\varphi,\gamma}(M^* \otimes M)$$

is the second boundary map of the bottom exact triangle of Proposition 2.21.

Proof. The equality $\exp_{M^*}(y)\cup z = f_2(h((0,y),g(z)))$, $y\in \boldsymbol{D}^K_{\mathrm{dR}}(M^*)$, $z\in \mathrm{H}^1_{\varphi,\gamma}(M)$, is proved in Lemma 2.13 of [Nakamura 2014a]. Hence, it suffices to show the equality

$$\exp_{f,M^*}(x)\cup z = f_2(h((x,0),g(z)))$$

for $x\in \boldsymbol{D}^K_{\mathrm{cris}}(M^*)$, whose proof is also just a diagram chase similar to the proof of Proposition 2.23, hence we omit the proof. □

Finally, we compare our exponential map with the Bloch–Kato exponential map for p-adic representations V. Here, we assume that $A=\mathbb{Q}_p$ for simplicity. We can do the same things for any $L=A$ a finite $\mathbb{Q}_p$-algebra.

We want to compare the diagram (3) for V with the diagram (13) for $M=\boldsymbol{D}_{\mathrm{rig}}(V)$. More generally, as in §2.4 of [Nakamura 2014a], we compare a similar diagram defined below for a B-pair $W=(W_e,W^+_{\mathrm{dR}})$ with the diagram (13) for the associated (φ,Γ)-module $\boldsymbol{D}_{\mathrm{rig}}(W)$. For the definitions of B-pairs and the definition of the functor $W\mapsto \boldsymbol{D}_{\mathrm{rig}}(W)$, which gives an equivalence between the category of B-pairs and that of (φ,Γ)-modules over $\mathcal{R}(\pi_K)$, see [Nakamura 2014a, §2.5; Berger 2008a].

Let $W=(W_e,W^+_{\mathrm{dR}})$ be a B-pair for K. Put $W_{\mathrm{cris}} := \boldsymbol{B}_{\mathrm{cris}}\otimes_{\boldsymbol{B}_e} W_e$, which is naturally equipped with an action of φ. Since we have an exact sequence

$$0\to \boldsymbol{B}^{\varphi=1}_{\mathrm{cris}}\to \boldsymbol{B}_{\mathrm{cris}}\xrightarrow{1-\varphi}\boldsymbol{B}_{\mathrm{cris}}\to 0,$$

we have a natural quasi-isomorphism (the vertical arrows) between the following two complexes of G_K-modules concentrated in degree [0, 1]:

$$\begin{array}{ccc} [W_e\oplus W^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto x-y} & W_{\mathrm{dR}}] \\ \downarrow{\scriptstyle (x,y)\mapsto(x,y)} & & \downarrow{\scriptstyle x\mapsto(0,x)} \\ [W_{\mathrm{cris}}\oplus W^+_{\mathrm{dR}} & \xrightarrow{(x,y)\mapsto((1-\varphi)x,x-y)} & W_{\mathrm{cris}}\oplus W_{\mathrm{dR}}] \end{array}$$

Put

$$C^\bullet_{\mathrm{cont}}(G_K,W) := \mathrm{Cone}\big(C^\bullet_{\mathrm{cont}}(G_K,W_e)\oplus C^\bullet_{\mathrm{cont}}(G_K,W^+_{\mathrm{dR}})\to C^\bullet_{\mathrm{cont}}(G_K,W_{\mathrm{dR}})\big)[-1]$$

and

$$C^\bullet_{\mathrm{cont}}(G_K,W)' := \mathrm{Cone}\big(C^\bullet_{\mathrm{cont}}(G_K,W_{\mathrm{cris}})\oplus C^\bullet_{\mathrm{cont}}(G_K,W^+_{\mathrm{dR}}) \to C^\bullet_{\mathrm{cont}}(G_K,W_{\mathrm{cris}})\oplus C^\bullet_{\mathrm{cont}}(G_K,W_{\mathrm{dR}})\big)[-1].$$

We identify

$$\mathrm{H}^i(K,W) := \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_K,W)) = \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_K,W)')$$

by the above quasi-isomorphism. Put $\boldsymbol{D}^K_{\mathrm{cris}}(W) := \mathrm{H}^0(K,W_{\mathrm{cris}})$, $\boldsymbol{D}^K_{\mathrm{dR}}(W) := \mathrm{H}^0(K,W_{\mathrm{dR}})$, and $\boldsymbol{D}^K_{\mathrm{dR}}(W)^i := \boldsymbol{D}^K_{\mathrm{dR}}(W)\cap t^iW^+_{\mathrm{dR}}$ for $i\in\mathbb{Z}$. Taking the cohomology of the mapping cones, we obtain the similar diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{H}^0(K,W) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(W)^{\varphi=1} & \xrightarrow{x\mapsto \bar{x}} & t_W(K) & \xrightarrow{\exp_W} & \mathrm{H}^1_e(K,W) \longrightarrow 0 \\ \big\downarrow \mathrm{id} & & \big\downarrow x\mapsto x & & \big\downarrow x\mapsto (0,x) & & \big\downarrow x\mapsto x \\ 0 \longrightarrow \mathrm{H}^0(K,W) & \xrightarrow{x\mapsto x} & \boldsymbol{D}^K_{\mathrm{cris}}(W) & \xrightarrow{f} & \begin{array}{c}\boldsymbol{D}^K_{\mathrm{cris}}(W)\\ \oplus t_W(K)\end{array} & \xrightarrow{g} & \mathrm{H}^1_f(K,W) \longrightarrow 0 \end{array} \tag{15}$$

with

$$f(x,y) = ((1-\varphi)x, \bar{x}) \quad \text{and} \quad g = \exp_{f,W} \oplus \exp_W .$$

By definition, it is clear that the diagram (15) for the associated B-pair $W(V) := (\boldsymbol{B}_e \otimes_{\mathbb{Q}_p} V, \boldsymbol{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is canonically isomorphic to the diagram (3) for V defined by Bloch–Kato.

Our comparison result is the following.

Proposition 2.26. (1) *We have the following functorial isomorphisms*:

(i) $\mathrm{H}^i(K,W) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(W))$,

(ii) $\boldsymbol{D}^K_{\mathrm{dR}}(W)^j \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{dR}}(\boldsymbol{D}_{\mathrm{rig}}(W))^j$ *for* $j \in \mathbb{Z}$,

(iii) $\boldsymbol{D}^K_{\mathrm{cris}}(W) \xrightarrow{\sim} \boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W))$.

(2) *The isomorphisms in* (1) *induce an isomorphism from the diagram* (15) *for* W *to the diagram* (13) *for* $\boldsymbol{D}_{\mathrm{rig}}(W)$.

Proof. We have already proved (i), (ii) of (1) and the comparison of the top exact sequence in (15) for W with that in (13) for $\boldsymbol{D}_{\mathrm{rig}}(W)$; see Theorem 2.21 of [Nakamura 2014a] or the references in the proof of this theorem.

Moreover, the isomorphism (iii) may be well known to the experts, but we give a proof of it since we couldn't find suitable references. In this proof, we freely use the notation in §2.5 of [Nakamura 2014a] or in [Berger 2008a]; please see these references. We first note that the inclusion $(\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e)^{G_K} \hookrightarrow \boldsymbol{D}^K_{\mathrm{cris}}(W)$ induced by the natural inclusion $\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}} := \bigcap_{n\geq 0} \varphi^n(\boldsymbol{B}^+_{\mathrm{cris}}) \hookrightarrow \boldsymbol{B}^+_{\mathrm{cris}}$ is an isomorphism since $\boldsymbol{D}^K_{\mathrm{cris}}(W)$ is a finite-dimensional $\mathbb{Q}_p$-vector space on which φ acts as an automorphism. Moreover, in the same way as the proof of Proposition 3.4 of [Berger 2002], we can show that the natural inclusion $(\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e)^{G_K} \hookrightarrow (\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e)^{G_K}$ is also an isomorphism. Since we have

$$\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e = \widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathcal{R}(\pi_K)[1/t]} \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$$

by definition of $\boldsymbol{D}_{\mathrm{rig}}(W)$, it suffices to show that the natural inclusion

$$\boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W)) \hookrightarrow \left(\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathcal{R}(\pi_K)[1/t]} \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]\right)^{G_K} =: D_0$$

is an isomorphism. Moreover, it suffices to show that $D_0 \subseteq \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$. This claim is proved as follows. Define $\mathcal{R}(\pi_K) \otimes_F D_0 \subseteq \widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}} \otimes_F D_0$, which are

(φ,Γ)-modules over $\mathcal{R}(\pi_K)$ (resp. (φ, G_K)-modules over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$). Then, by Théorème 1.2 of [Berger 2009], the natural map

$$\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}} \otimes_F D_0 \to \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathcal{R}(\pi_K)[1/t]} \boldsymbol{D}_{\mathrm{rig}}(W)[1/t] : a \otimes x \mapsto a \cdot x$$

(which is actually an inclusion) of (φ, G_K)-modules factors through $\mathcal{R}(\pi_K)\otimes_F D_0 \to \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$. In particular we have $D_0 \subseteq \boldsymbol{D}_{\mathrm{rig}}(W)[1/t]$, which proves the claim.

We next prove that the bottom exact sequence in (15) for W is isomorphic to that in (13) for $\boldsymbol{D}_{\mathrm{rig}}(W)$ by the isomorphisms in (1) of this proposition. Since the other commutativities are clear, or were already proved in Theorem 2.21 of [Nakamura 2014a], it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W)) & \xrightarrow{\exp_{f,\boldsymbol{D}_{\mathrm{rig}}(W)}} & \mathrm{H}^1_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(W)) \\ \downarrow\wr & & \downarrow\wr \\ \boldsymbol{D}^K_{\mathrm{cris}}(W) & \xrightarrow{\exp_{f,W}} & \mathrm{H}^1(K,W) \end{array} \tag{16}$$

In the same way as the proof of Theorem 2.21 of [Nakamura 2014a], we assume that $\Delta = \{1\}$, and using the canonical identifications

$$\mathrm{H}^1(K,W) \xrightarrow{\sim} \mathrm{Ext}^1(B,W), \quad \mathrm{H}^1_{\varphi,\gamma}(\boldsymbol{D}_{\mathrm{rig}}(W)) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{R}(\pi_K), \boldsymbol{D}_{\mathrm{rig}}(W))$$

(where we denote by $B = (\boldsymbol{B}_e, \boldsymbol{B}^+_{\mathrm{dR}})$ the trivial B-pair). It suffices to show that, for $a \in \boldsymbol{D}^K_{\mathrm{cris}}(\boldsymbol{D}_{\mathrm{rig}}(W))$, the extension corresponding to $\exp_{f,\boldsymbol{D}_{\mathrm{rig}}(W)}(a)$ is sent to the extension corresponding to $\exp_{f,W}(a)$ by the inverse functor $W(-)$ of $\boldsymbol{D}_{\mathrm{rig}}(-)$. We prove this claim as follows. Take $n \geq 1$ sufficiently large such that $a \in (\boldsymbol{D}^{(n)}_{\mathrm{rig}}(W)[1/t])^{\Gamma_K}$. Take $\tilde{a} \in \boldsymbol{D}^{(n)}_{\mathrm{rig}}[1/t]$ satisfying the condition in (2) of Proposition 2.23. Then, by (2) of Proposition 2.23, the extension D_a corresponding to $\exp_{f,\boldsymbol{D}_{\mathrm{rig}}(W)}(a)$ is written by

$$\left[0 \to \boldsymbol{D}_{\mathrm{rig}}(W) \xrightarrow{x \mapsto (x,0)} \boldsymbol{D}_{\mathrm{rig}}(W) \oplus \mathcal{R}(\pi_K)\boldsymbol{e} \xrightarrow{(x,y\boldsymbol{e})\mapsto y} \mathcal{R}(\pi_K) \to 0\right]$$

such that

$$\varphi((x, y\boldsymbol{e})) = (\varphi(x) + \varphi(y)((\varphi-1)\tilde{a} + \varphi^n(a)), \varphi(y)\boldsymbol{e})$$

and

$$\gamma((x, y\boldsymbol{e})) = (\gamma(x) + \gamma(y)(\gamma-1)\tilde{a}, \gamma(y)\boldsymbol{e}).$$

(Here, we remark that there is a mistake in [Nakamura 2014a]; in the proof of Theorem 2.21 of [Nakamura 2014a], D_a should be defined by

$$\varphi((x, y\boldsymbol{e})) = (\varphi(x) + \varphi(y)(\varphi-1)\tilde{a}, \varphi(y)\boldsymbol{e})$$

and

$$\gamma((x, y\boldsymbol{e})) = (\gamma(x) + \gamma(y)(\gamma-1)\tilde{a}, \gamma(y)\boldsymbol{e}).)$$

On the other hand, by definition of $\exp_{f,W}$, the extension

$$W_a := \big(W_{e,a},\, W^+_{\mathrm{dR},a} := W^+_{\mathrm{dR}} \oplus \boldsymbol{B}^+_{\mathrm{dR}}\boldsymbol{e}_{\mathrm{dR}}\big)$$

corresponding to $\exp_{f,W}(a)$ is defined by

$$g(x,\, y\boldsymbol{e}_{\mathrm{dR}}) = (g(x),\, g(y)\boldsymbol{e}_{\mathrm{dR}})$$

for $x \in W^+_{\mathrm{dR}},\ y \in \boldsymbol{B}^+_{\mathrm{dR}},\ g \in G_K$, and $W_{e,a}$ is defined as the kernel of the surjection

$$W_{\mathrm{cris},a} := W_{\mathrm{cris}} \oplus \boldsymbol{B}_{\mathrm{cris}}\boldsymbol{e}_{\mathrm{cris}} \to W_{\mathrm{cris},a} : (x,\, y\boldsymbol{e}_{\mathrm{cris}}) \mapsto ((\varphi-1)x+\varphi(y)a,\, (\varphi-1)y\boldsymbol{e}_{\mathrm{cris}})$$

on which G_K acts by $g(\boldsymbol{e}_{\mathrm{cris}}) = \boldsymbol{e}_{\mathrm{cris}}$ (actually, this is equal to the kernel of the surjection

$$W_{\mathrm{rig},a} := W_{\mathrm{rig}} \oplus \widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t]\boldsymbol{e}_{\mathrm{cris}} \to W_{\mathrm{rig},a} : (x,\, y\boldsymbol{e}_{\mathrm{cris}}) \mapsto ((\varphi-1)x+\varphi(y)a,\, (\varphi-1)y\boldsymbol{e}_{\mathrm{cris}}),$$

where we define $W_{\mathrm{rig}} := \widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_e$), and the isomorphism $\boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}_e} W_{e,a} \xrightarrow{\sim} \boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}$ is defined by

$$\boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}_e} W_{e,a} = \boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}_{\mathrm{cris}}} W_{\mathrm{cris},a} \xrightarrow{(x,\, y\boldsymbol{e}_{\mathrm{cris}}) \mapsto (x,\, y\boldsymbol{e}_{\mathrm{dR}})} \boldsymbol{B}_{\mathrm{dR}} \otimes_{\boldsymbol{B}^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}.$$

Then, by definition of the functor $\boldsymbol{D}_{\mathrm{rig}}(-)$ in §2.2 of [Berger 2008a] (where the notation $D(-)$ is used), $\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}} \otimes_{\mathcal{R}^{(n)}(\pi_K)} \boldsymbol{D}^{(n)}_{\mathrm{rig}}(W_a)$ is equal to

$$\big\{x \in \widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_{e,a} \mid \iota_m(x) \in W^+_{\mathrm{dR},a} \text{ for any } m \geq n\big\}. \tag{17}$$

Since we have

$$\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}[1/t] \otimes_{\boldsymbol{B}_e} W_{e,a} = \widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}[1/t] \otimes_{\widetilde{\boldsymbol{B}}^+_{\mathrm{rig}}[1/t]} W_{\mathrm{rig},a},$$

and $\varphi^{-m}(\boldsymbol{e}_{\mathrm{cris}}) = \boldsymbol{e}_{\mathrm{cris}} - \sum_{k=1}^{m} \varphi^{-k}(a)$ for $m \geq 1$ and we have $\iota_{n+k} \circ \varphi^n = \varphi^{-k}$, it is easy to see that the group (17) is equal to

$$\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}} \otimes_{\mathcal{R}^{(n)}(\pi_K)} \boldsymbol{D}^{(n)}_{\mathrm{rig}}(W) \oplus \widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}}(\tilde{a} + \varphi^n(\boldsymbol{e}_{\mathrm{cris}})),$$

which is easily seen to be isomorphic to $\widetilde{\boldsymbol{B}}^{\dagger,r_n}_{\mathrm{rig}} \otimes_{\mathcal{R}^{(n)}(\pi_K)} D^{(n)}_a$ as a (φ, G_K)-module. Therefore, we obtain the isomorphism

$$\boldsymbol{D}_{\mathrm{rig}}(W_a) \xrightarrow{\sim} D_a$$

as an extension by Théorème 1.2 of [Berger 2009], which proves the proposition. □

3. Local ε-conjecture for (φ, Γ)-modules over the Robba ring

From now on, we assume that $K = \mathbb{Q}_p$, and we freely omit the notation $\mathbb{Q}_p$, i.e., we use the notation Γ, $\mathcal{R}_A$, $\boldsymbol{D}_{\mathrm{dR}}(M)$, $\boldsymbol{D}_{\mathrm{cris}}(M)$, $t_M, \ldots$ instead of $\Gamma_{\mathbb{Q}_p}$, $\mathcal{R}_A(\pi_{\mathbb{Q}_p})$, $\boldsymbol{D}^{\mathbb{Q}_p}_{\mathrm{dR}}(M)$, $\boldsymbol{D}^{\mathbb{Q}_p}_{\mathrm{cris}}(M)$, $t_M(\mathbb{Q}_p), \ldots$. Moreover, since Kato's and our conjectures are formulated after fixing a $\mathbb{Z}_p$-basis $\zeta = \{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1)$, we also fix a parameter $\pi := \pi_\zeta$ of $\mathcal{R}_A$ and let $t = \log(1+\pi)$ as in Notation 2.2.

In this section, we formulate a conjecture which is a natural generalization of Kato's (p-adic) local ε-conjecture, where the main objects were p-adic or torsion representations of $G_{\mathbb{Q}_p}$, for (φ, Γ)-modules over the relative Robba ring $\mathcal{R}_A$. Since the article [Kato 1993b], in which the conjecture was stated, remains unpublished, and since the compatibility of our conjecture with his conjecture is an important part of our conjecture, here we also recall his original conjecture.

3A. ***Determinant functor.*** Kato's and our conjectures are formulated using the theory of the determinant functor. In this subsection, we briefly recall this theory following [Knudsen and Mumford 1976] and §2.1 of [Kato 1993a].

Let R be a commutative ring. We define a category $\mathcal{P}_R$ whose objects are pairs (L, r), where L is an invertible R-module and $r : \mathrm{Spec}(R) \to \mathbb{Z}$ is a locally constant function, and whose morphisms are defined by $\mathrm{Mor}_{\mathcal{P}_R}((L, r), (M, s)) := \mathrm{Isom}_R(L, M)$ if $r = s$, and empty otherwise. We call the objects of this category graded invertible R-modules. The category $\mathcal{P}_R$ is equipped with the structure of a (tensor) product defined by $(L, r) \boxtimes (M, s) := (L \otimes_R M, r + s)$ with the natural associativity constraint and the commutativity constraint

$$(L, r) \boxtimes (M, s) \xrightarrow{\sim} (M, s) \boxtimes (L, r) : l \otimes m \mapsto (-1)^{rs} m \otimes l.$$

From now on, we always identify $(L, r)\boxtimes(M, s) = (M, s)\boxtimes(L, r)$ by this constraint isomorphism. The unit object for the product is $\mathbf{1}_R := (R, 0)$. For each (L, r), set $L^\vee := \mathrm{Hom}_R(L, R)$. Then $(L, r)^{-1} := (L^\vee, -r)$ becomes an inverse of (L, r) by the isomorphism $i_{(L,r)} : (L, r) \boxtimes (L^\vee, -r) \xrightarrow{\sim} \mathbf{1}_R$ induced by the evaluation map $L \otimes_R \mathrm{Hom}_R(L, R) \xrightarrow{\sim} R : x \otimes f \mapsto f(x)$. We remark that we have $i_{(L,r)^{-1}} = (-1)^r i_{(L,r)}$. For a ring homomorphism $f : R \to R'$, one has a base change functor $(-) \otimes_R R' : \mathcal{P}_R \to \mathcal{P}_{R'}$ defined by $(L, r) \mapsto (L, r) \otimes_R R' := (L \otimes_R R', r \circ f^*)$, where $f^* : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$.

For a category $\mathcal{C}$, denote by $(\mathcal{C}, \mathrm{is})$ the category whose objects are the same as $\mathcal{C}$ and whose morphisms are all isomorphisms in $\mathcal{C}$. Define a functor

$$\mathrm{Det}_R : (\boldsymbol{P}_{\mathrm{fg}}(R), \mathrm{is}) \to \mathcal{P}_R : P \mapsto (\det_R P, \mathrm{rk}_R P),$$

where $\mathrm{rk}_R : \boldsymbol{P}_{\mathrm{fg}}(R) \to \mathbb{Z}_{\geq 0}$ is the R-rank of P and $\det_R P := \bigwedge_R^{\mathrm{rk}_R P} P$. Note that $\mathrm{Det}_R(0) = \mathbf{1}_R$ is the unit object. For a short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$ in $\boldsymbol{P}_{\mathrm{fg}}(R)$, we always identify $\mathrm{Det}_R(P_1)\boxtimes\mathrm{Det}_R(P_3)$ with $\mathrm{Det}_R(P_2)$ by the functorial isomorphism (put $r_i := \mathrm{rk}_R P_i$)

$$\mathrm{Det}_R(P_1) \boxtimes \mathrm{Det}_R(P_3) \xrightarrow{\sim} \mathrm{Det}_R(P_2) \tag{18}$$

induced by

$$(x_1 \wedge \cdots \wedge x_{r_1}) \otimes (\overline{x_{r_1+1}} \wedge \cdots \wedge \overline{x_{r_2}}) \mapsto x_1 \wedge \cdots \wedge x_{r_1} \wedge x_{r_1+1} \wedge \cdots \wedge x_{r_2},$$

where $x_1, \dots, x_{r_1}$ (resp. $\overline{x_{r_1+1}}, \dots, \overline{x_{r_2}}$) are local sections of P_1 (resp. P_3) and $x_i \in P_2$ $(r_1 + 1 \leq i \leq r_2)$ is a lift of $\overline{x_i} \in P_3$.

For a bounded complex $P^\bullet$ in $\boldsymbol{P}_{\mathrm{fg}}(R)$, define $\mathrm{Det}_R(P^\bullet) \in \mathcal{P}_R$ by

$$\mathrm{Det}_R(P^\bullet) := \boxtimes_{i \in \mathbb{Z}} \mathrm{Det}_R(P^i)^{(-1)^i}.$$

For a short exact sequence $0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$ of bounded complexes in $\boldsymbol{P}_{\mathrm{fg}}(R)$, we define a canonical isomorphism

$$\mathrm{Det}_R(P_1^\bullet) \boxtimes \mathrm{Det}_R(P_3^\bullet) \xrightarrow{\sim} \mathrm{Det}_R(P_2^\bullet) \tag{19}$$

by applying the isomorphism (18) to each exact sequence $0 \to P_1^i \to P_2^i \to P_3^i \to 0$. Moreover, if $P^\bullet$ is an acyclic bounded complex in $\boldsymbol{P}_{\mathrm{fg}}(R)$, we can define a canonical isomorphism

$$h_{P^\bullet} : \mathrm{Det}_R(P^\bullet) \xrightarrow{\sim} \mathbf{1}_R, \tag{20}$$

which is characterized by the following properties: when $P^\bullet := \left[P^i \xrightarrow{f} P^{i+1}\right]$ is concentrated in degree $[i, i+1]$, we define it as the composite

$$\mathrm{Det}_R(P^\bullet) = \mathrm{Det}_R(P^i) \boxtimes \mathrm{Det}_R(P^{i+1})^{-1} \xrightarrow{\mathrm{Det}(f) \boxtimes \mathrm{id}} \mathrm{Det}_R(P^{i+1}) \boxtimes \mathrm{Det}_R(P^{i+1})^{-1} \xrightarrow{\delta_{\mathrm{Det}_R(P^{i+1})}} \mathbf{1}_R$$

when i is even (when i is odd, we similarly define it using $f^{-1} : P^{i+1} \xrightarrow{\sim} P^i$), and for a short exact sequence $0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$ of acyclic bounded complexes of $\boldsymbol{P}_{\mathrm{fg}}(R)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Det}_R(P_1^\bullet) \boxtimes \mathrm{Det}_R(P_3^\bullet) & \xrightarrow{\sim} & \mathrm{Det}_R(P_2^\bullet) \\ \downarrow{\scriptstyle h_{P_1^\bullet} \boxtimes h_{P_3^\bullet}} & & \downarrow{\scriptstyle h_{P_2^\bullet}} \\ \mathbf{1}_R \boxtimes \mathbf{1}_R & \xrightarrow{=} & \mathbf{1}_R \end{array}$$

The theory of determinants of [Knudsen and Mumford 1976] enables us to uniquely (up to canonical isomorphism) extend $\mathrm{Det}_R(-)$ to a functor

$$\mathrm{Det}_R : (\boldsymbol{D}^b_{\mathrm{perf}}(R), \mathrm{is}) \to \mathcal{P}_R$$

such that the isomorphism (19) extends to the following situation: for any exact sequence $0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$ of complexes of R-modules such that each $P_i^\bullet$ is quasi-isomorphic to a bounded complex in $\boldsymbol{P}_{\mathrm{fg}}(R)$, there exists a canonical isomorphism

$$\mathrm{Det}_R(P_1^\bullet) \boxtimes \mathrm{Det}_R(P_3^\bullet) \xrightarrow{\sim} \mathrm{Det}_R(P_2^\bullet). \tag{21}$$

By this property, if $P^\bullet \in \boldsymbol{D}^b_{\mathrm{perf}}(R)$ satisfies $\mathrm{H}^i(P^\bullet)[0] \in \boldsymbol{D}^b_{\mathrm{perf}}(R)$ for any i, there exists a canonical isomorphism

$$\mathrm{Det}_R(P^\bullet) \xrightarrow{\sim} \boxtimes_{i\in\mathbb{Z}} \mathrm{Det}_R(\mathrm{H}^i(P^\bullet)[0])^{(-1)^i}.$$

For $(L, r) \in \mathcal{P}_R$, define $(L, r)^\vee := (L^\vee, r) \in \mathcal{P}_R$, which induces an antiequivalence $(-)^\vee : \mathcal{P}_R \xrightarrow{\sim} \mathcal{P}_R$. For $P \in \boldsymbol{P}_{\mathrm{fg}}(R)$, we have a canonical isomorphism $\mathrm{Det}_R(P^\vee) \xrightarrow{\sim} \mathrm{Det}_R(P)^\vee$ defined by the isomorphism

$$\det_R(P^\vee) \xrightarrow{\sim} (\det_R P)^\vee :$$
$$f_1 \wedge \cdots \wedge f_r \mapsto \left[x_1 \wedge \cdots \wedge x_r \mapsto \sum_{\sigma \in \mathfrak{S}_r} \mathrm{sgn}(\sigma) f_1(x_{\sigma(1)}) \cdots f_r(x_{\sigma(r)}) \right].$$

This naturally extends to $(\boldsymbol{D}^b_{\mathrm{perf}}(R), \mathrm{is})$, i.e., for any $P^\bullet \in \boldsymbol{D}^b_{\mathrm{perf}}(R)$, there exists a canonical isomorphism

$$\mathrm{Det}_R(\boldsymbol{R}\,\mathrm{Hom}_R(P^\bullet, R)) \xrightarrow{\sim} \mathrm{Det}_R(P^\bullet)^\vee. \tag{22}$$

3B. ***Fundamental lines.*** Both Kato's conjecture and ours concern the existence of a compatible family of canonical trivializations of some graded invertible modules defined by using the determinants of the Galois cohomologies of Galois representations or (φ, Γ)-modules. We call these graded invertible modules the fundamental lines, which we explain in this subsection.

Kato's conjecture concerns pairs (Λ, T) such that:

(i) Λ is a noetherian semilocal ring which is complete with respect to the $\mathfrak{m}_\Lambda$-adic topology (where $\mathfrak{m}_\Lambda$ is the Jacobson radial of Λ) such that $\Lambda/\mathfrak{m}_\Lambda$ is a finite ring with order a power of p.

(ii) T is a Λ-representation of $G_{\mathbb{Q}_p}$, i.e., a finite projective Λ-module equipped with a continuous Λ-linear action of $G_{\mathbb{Q}_p}$.

Our conjecture concerns pairs (A, M) such that:

(i) A is a $\mathbb{Q}_p$-affinoid algebra.

(ii) M is a (φ, Γ)-module over $\mathcal{R}_A$.

For each pair $(B, N) = (\Lambda, T)$ or (A, M) as above, we'll define graded invertible Λ-modules $\Delta_{B,i}(N) \in \mathcal{P}_B$ for $i = 1, 2$ as below, and the fundamental line will be defined as $\Delta_B(N) := \Delta_{B,1}(N) \boxtimes \Delta_{B,2}(N) \in \mathcal{P}_B$.

We first define $\Delta_{\Lambda,i}(T)$ for (Λ, T). Denote by $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T)$ the complex of continuous cochains of $G_{\mathbb{Q}_p}$ with values in T. It is known that $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T) \in \boldsymbol{D}^-(\Lambda)$ is contained in $\boldsymbol{D}^b_{\mathrm{perf}}(\Lambda)$ and that it satisfies properties similar to (1), (2), (3), (4) in Theorem 2.15. In particular, we can define a graded invertible Λ-module

$$\Delta_{\Lambda,1}(T) := \mathrm{Det}_\Lambda(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T))$$

(whose degree is $-r_T := -\mathrm{rk}_\Lambda T$ by the Euler–Poincaré formula) which satisfies the following properties:

(i) For each continuous homomorphism $f : \Lambda \to \Lambda'$, there exists a canonical Λ'-linear isomorphism

$$\Delta_{\Lambda,1}(T) \otimes_\Lambda \Lambda' \xrightarrow{\sim} \Delta_{\Lambda',1}(T \otimes_\Lambda \Lambda').$$

(ii) For each exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of Λ-representations of $G_{\mathbb{Q}_p}$, there exists a canonical Λ-linear isomorphism

$$\Delta_{\Lambda,1}(T_1) \boxtimes \Delta_{\Lambda,1}(T_3) \xrightarrow{\sim} \Delta_{\Lambda,1}(T_2).$$

(iii) The Tate duality $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T) \xrightarrow{\sim} \boldsymbol{R}\,\mathrm{Hom}_\Lambda(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T^*), \Lambda)[-2]$ and the isomorphism (22) induce a canonical Λ-linear isomorphism

$$\Delta_{\Lambda,1}(T) \xrightarrow{\sim} \Delta_{\Lambda,1}(T^*)^\vee.$$

We next define $\Delta_{\Lambda,2}(T)$ as follows. For $a \in \Lambda^\times$, we define

$$\Lambda_a := \left\{x \in W(\overline{\mathbb{F}}_p) \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \Lambda \mid (\varphi \otimes \mathrm{id}_\Lambda)(x) = (1 \otimes a)x\right\},$$

which is an invertible Λ-module. In the same way as in Theorem 2.8, for any rank-one Λ-representation T_0, there exists a unique (up to isomorphism) pair $(\delta_{T_0}, \mathcal{L}_{T_0})$, where $\delta_{T_0} : \mathbb{Q}_p^\times \to \Lambda^\times$ is a continuous homomorphism and $\mathcal{L}_{T_0}$ is an invertible Λ-module such that $T_0 \xrightarrow{\sim} \Lambda(\tilde{\delta}_{T_0}) \otimes_\Lambda \mathcal{L}_{T_0}$, where we denote by $\tilde{\delta}_{T_0} : G^{\mathrm{ab}}_{\mathbb{Q}_p} \to \Lambda^\times$ the continuous character which satisfies $\tilde{\delta}_{T_0} \circ \mathrm{rec}_{\mathbb{Q}_p} = \delta_{T_0}$. Under these definitions, we define $a(T) := \delta_{\det_\Lambda T}(p) \in \Lambda^\times$, an invertible Λ-module

$$\mathcal{L}_\Lambda(T) := \Lambda_{a(T)} \otimes_\Lambda \det_\Lambda T$$

and a graded invertible Λ-module

$$\Delta_{\Lambda,2}(T) := (\mathcal{L}_\Lambda(T), r_T).$$

Since we have a canonical isomorphism $\Lambda_{a_1} \otimes_\Lambda \Lambda_{a_2} \xrightarrow{\sim} \Lambda_{a_1 a_2} : x \otimes y \mapsto xy$ for any $a_1, a_2 \in \Lambda$, $\Delta_{\Lambda,2}(T)$ also satisfies the following similar properties:

(i) For $f : \Lambda \to \Lambda'$, there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T) \otimes_\Lambda \Lambda' \xrightarrow{\sim} \Delta_{\Lambda,2}(T \otimes_\Lambda \Lambda').$$

(ii) For $0 \to T_1 \to T_2 \to T_3 \to 0$, there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T_1) \boxtimes \Delta_{\Lambda,2}(T_3) \xrightarrow{\sim} \Delta_{\Lambda,2}(T_2).$$

(iii) Let r_T be the rank of T. Then there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T) \xrightarrow{\sim} \Delta_{\Lambda,2}(T^*)^\vee \boxtimes (\Lambda(r_T), 0)$$

which is induced by the product of the isomorphisms

$$\Lambda_{\delta_{\det_\Lambda T}(p)} \xrightarrow{\sim} (\Lambda_{\delta_{\det_\Lambda T^*}(p)})^\vee : x \mapsto [y \mapsto y \otimes x]$$

(note that we have

$$\Lambda_{\delta_{\det_\Lambda T^*}(p)} \otimes \Lambda_{\delta_{\det_\Lambda T}(p)} \xrightarrow{\sim} \Lambda : y \otimes x \mapsto yx$$

since we have $\delta_{\det_\Lambda T}(p) = \delta_{\det_\Lambda T^*}(p)^{-1}$) and the isomorphism $\det_\Lambda T \xrightarrow{\sim} \det_\Lambda (T^*)^\vee \otimes_\Lambda \Lambda(r_T)$ induced by the canonical isomorphism $T \xrightarrow{\sim} (T^*)^\vee(1) : x \mapsto [y \mapsto y(x) \otimes \boldsymbol{e}_{-1}] \otimes \boldsymbol{e}_1$.

Finally, we define

$$\Delta_\Lambda(T) := \Delta_{\Lambda,1}(T) \boxtimes \Delta_{\Lambda,2}(T) \in \mathcal{P}_B.$$

Then $\Delta_\Lambda(T)$ also satisfies properties similar to (i), (ii) for $\Delta_{\Lambda,i}(T)$ and

(iii) there exists a canonical isomorphism

$$\Delta_\Lambda(T) \xrightarrow{\sim} \Delta_\Lambda(T^*)^\vee \boxtimes (\Lambda(r_T), 0).$$

Next, we define the fundamental line $\Delta_A(M)$ for (φ, Γ)-modules M over $\mathcal{R}_A$. Let A be a $\mathbb{Q}_p$-affinoid algebra, and let M be a (φ, Γ)-module over $\mathcal{R}_A$. By our Theorem 2.15 (Kedlaya–Pottharst–Xiao), we can define a graded invertible A-module

$$\Delta_{A,1}(M) := \mathrm{Det}_A\, C^\bullet_{\varphi,\gamma}(M) \in \mathcal{P}_A$$

which satisfies properties similar to (i), (ii), (iii) for $\Delta_{\Lambda,1}(T)$. We next define $\Lambda_{A,2}(M)$ as follows. By our Theorem 2.8 (Kedlaya–Pottharst–Xiao), there exists a unique (up to isomorphism) pair $(\delta_{\det_{\mathcal{R}_A} M}, \mathcal{L}_{\det_{\mathcal{R}_A} M})$, where $\delta_{\det_{\mathcal{R}_A} M} : \mathbb{Q}_p^\times \to A^\times$ is a continuous homomorphism and $\mathcal{L}_{\det_{\mathcal{R}_A} M}$ is an invertible A-module such that $\det_{\mathcal{R}_A} M \xrightarrow{\sim} \mathcal{R}_A(\delta_{\det_{\mathcal{R}_A} M}) \otimes_{\mathcal{R}_A} \mathcal{L}_{\det_{\mathcal{R}_A} M}$. Then we define an A-module

$$\mathcal{L}_A(M) := \left\{x \in \det_{\mathcal{R}_A} M \mid \varphi(x) = \delta_{\det_{\mathcal{R}_A} M}(p)x,\ \gamma(x) = \delta_{\det_{\mathcal{R}_A} M}(\chi(\gamma))x\ (\gamma \in \Gamma)\right\},$$

which is an invertible A-module since it is isomorphic to $\mathcal{L}_{\det_{\mathcal{R}_A} M}$, and we define a graded invertible A-module

$$\Delta_{A,2}(M) := (\mathcal{L}_A(M), r_M) \in \mathcal{P}_A.$$

By definition, it is easy to check that $\Delta_{A,2}(M)$ satisfies properties similar to (i), (ii), (iii) for $\Delta_{\Lambda,2}(T)$. Finally, we define a graded invertible A-module $\Delta_A(M)$, which we call the fundamental line, by

$$\Delta_A(M) := \Delta_{A,1}(M) \boxtimes \Delta_{A,2}(M) \in \mathcal{P}_A,$$

which also satisfies properties similar to (i), (ii), (iii) for $\Delta_\Lambda(T)$.

More generally, let X be a rigid analytic space over $\mathbb{Q}_p$, and let M be a (φ,Γ)-module over $\mathcal{R}_X$. By the base change property (i) of $\Delta_A(M)$, we can also functorially define a graded invertible $\mathcal{O}_X$-module

$$\Delta_X(M)\in\mathcal{P}_{\mathcal{O}_X}$$

on X (we can naturally generalize the notion of graded invertible modules in this setting) such that there exists a canonical isomorphism

$$\Gamma(\mathrm{Max}(A),\Delta_X(M))\xrightarrow{\sim}\Delta_A(M|_{\mathrm{Max}(A)})$$

for any affinoid open subset $\mathrm{Max}(A)\subseteq X$.

We next compare Kato's fundamental line $\Delta_\Lambda(T)$ with our fundamental line $\Delta_A(M)$. Let $f:\Lambda\to A$ be a continuous ring homomorphism, where Λ is equipped with $\mathfrak{m}_\Lambda$-adic topology and A is equipped with p-adic topology. Let T be a Λ-representation of $G_{\mathbb{Q}_p}$. Let us denote by $M:=\boldsymbol{D}_{\mathrm{rig}}(T\otimes_\Lambda A)$ the (φ,Γ)-module over $\mathcal{R}_A$ associated to the A-representation $T\otimes_\Lambda A$ of $G_{\mathbb{Q}_p}$. By Theorem 2.8 of [Pottharst 2013], there exists a canonical quasi-isomorphism $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p},T)\otimes^{\boldsymbol{L}}_\Lambda A\xrightarrow{\sim}C^\bullet_{\varphi,\gamma}(M)$, and this induces an A-linear isomorphism

$$\Delta_{\Lambda,1}(T)\otimes_\Lambda A\xrightarrow{\sim}\Delta_{A,1}(M).$$

We also have the following isomorphism.

Lemma 3.1. *In the above situation, there exists a canonical A-linear isomorphism*

$$\Delta_{\Lambda,2}(T)\otimes_\Lambda A\xrightarrow{\sim}\Delta_{A,2}(M).$$

Proof. By definition, it suffices to show the lemma when T is of rank one. Hence, we may assume that $T=\Lambda(\tilde{\delta})\otimes_\Lambda\mathcal{L}$ for a continuous homomorphism $\delta:\mathbb{Q}_p^\times\to\Lambda^\times$ and an invertible Λ-module $\mathcal{L}$ (where $\tilde{\delta}$ is the character of $G^{\mathrm{ab}}_{\mathbb{Q}_p}$ such that $\tilde{\delta}\circ\mathrm{rec}_{\mathbb{Q}_p}=\delta$). Moreover, since we have a canonical isomorphism

$$\boldsymbol{D}_{\mathrm{rig}}((\Lambda(\tilde{\delta})\otimes_\Lambda\mathcal{L})\otimes_\Lambda A)\xrightarrow{\sim}\boldsymbol{D}_{\mathrm{rig}}(\Lambda(\tilde{\delta})\otimes_\Lambda A)\otimes_A(\mathcal{L}\otimes_\Lambda A)$$

by the exactness of $\boldsymbol{D}_{\mathrm{rig}}(-)$, it suffices to show the lemma when $\mathcal{L}=\Lambda$.

Since the image of $H_{\mathbb{Q}}:=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p,\infty})$ in $G^{\mathrm{ab}}_{\mathbb{Q}_p}$ is the closed subgroup which is topologically generated by $\mathrm{rec}_{\mathbb{Q}_p}(p)$, we have

$$\boldsymbol{D}_{\mathrm{rig}}(\Lambda(\tilde{\delta})\otimes_\Lambda A)=(W(\overline{\mathbb{F}}_p)\,\widehat{\otimes}_{\mathbb{Z}_p}\,\Lambda(\tilde{\delta}))^{\mathrm{rec}_{\mathbb{Q}_p}(p)=1}\otimes_\Lambda\mathcal{R}_A,$$

by definition of $\boldsymbol{D}_{\mathrm{rig}}(-)$, and the right-hand side is isomorphic to $\mathcal{R}_A(f\circ\delta)$. Hence, we obtain

$$\begin{aligned}\mathcal{L}_A(M)&=\bigl((W(\overline{\mathbb{F}}_p)\,\widehat{\otimes}_{\mathbb{Z}_p}\,\Lambda(\tilde{\delta}))^{\mathrm{rec}_{\mathbb{Q}_p}(p)=1}\otimes_\Lambda\mathcal{R}_A\bigr)^{\varphi=f(\delta(p)),\Gamma=f\circ\delta\circ\chi}\\&=\bigl(W(\overline{\mathbb{F}}_p)\,\widehat{\otimes}_{\mathbb{Z}_p}\,\Lambda(\tilde{\delta})\bigr)^{\mathrm{rec}_{\mathbb{Q}_p}(p)=1}\otimes_\Lambda A=\mathcal{L}_\Lambda(T)\otimes_\Lambda A,\end{aligned}$$

which proves the lemma. □

Taking the products of these two canonical isomorphisms, we obtain the following corollary.

Corollary 3.2. *In the above situation, there exists a canonical isomorphism*

$$\Delta_\Lambda(T)\otimes_\Lambda A \xrightarrow{\sim} \Delta_A(M).$$

Example 3.3. The typical example of the above base change property is the following. For Λ as above, let us denote by X the associated rigid analytic space. More precisely, X is the union of affinoids $\mathrm{Max}(A_n)$ for $n\geq 1$, where A_n is the $\mathbb{Q}_p$-affinoid algebra defined by $A_n := \Lambda[\mathfrak{m}_\Lambda^n/p]^\wedge[1/p]$ (for a ring R, denote by $R^\wedge$ the p-adic completion). Let T be a Λ-representation of $G_{\mathbb{Q}_p}$, and let $M_n := \boldsymbol{D}_{\mathrm{rig}}(T\otimes_\Lambda A_n)$. Since M_n is compatible with the base change with respect to the canonical map $A_n\to A_{n+1}$ for any n, $\{M_n\}_{n\geq 1}$ defines a (φ,Γ)-module $\mathcal{M}$ over $\mathcal{R}_X$. Then the canonical isomorphism $\Delta_\Lambda(T)\otimes_\Lambda A_n \xrightarrow{\sim} \Delta_{A_n}(M_n)$ defined in the above corollary glues to an isomorphism

$$\Delta_\Lambda(T)\otimes_\Lambda \mathcal{O}_X \xrightarrow{\sim} \Delta_X(\mathcal{M}).$$

Moreover, using the terminology of coadmissible modules [Schneider and Teitelbaum 2003], we can define this comparison isomorphism without using sheaves. Let us define $A_\infty := \Gamma(X,\mathcal{O}_X)$ and $\Delta_{A_\infty}(M_\infty) := \varprojlim_n \Delta_{A_n}(M_n)$. Taking the limit of the isomorphism $\Delta_\Lambda(T)\otimes_\Lambda A_n \xrightarrow{\sim} \Delta_{A_n}(M_n)$ we obtain an A_∞-linear isomorphism

$$\Delta_\Lambda(T)\otimes_\Lambda A_\infty \xrightarrow{\sim} \Delta_{A_\infty}(M_\infty).$$

Then the theory of coadmissible modules [Schneider and Teitelbaum 2003, Corollary 3.3] says that to consider the isomorphism $\Delta_\Lambda(T)\otimes_\Lambda \mathcal{O}_X \xrightarrow{\sim} \Delta_X(\mathcal{M})$ is the same as to consider the isomorphism $\Delta_\Lambda(T)\otimes_\Lambda A_\infty \xrightarrow{\sim} \Delta_{A_\infty}(M_\infty)$. In fact, we will frequently use the latter object $\Delta_{A_\infty}(M_\infty)$ in Section 4.

3C. ***de Rham ε-isomorphism.*** In this subsection, we assume that $L=A$ is a finite extension of $\mathbb{Q}_p$. We define a trivialization

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(M):\mathbf{1}_L \xrightarrow{\sim} \Delta_L(M),$$

which we call the de Rham ε-isomorphism, for each de Rham (φ,Γ)-module M over $\mathcal{R}_L$ and for each $\mathbb{Z}_p$-basis $\zeta=\{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1)$.

Let M be a de Rham (φ,Γ)-module over $\mathcal{R}_L$. We first recall the definition of Deligne and Langlands' [Deligne 1973] and Fontaine and Perrin-Riou's [1994] ε-constant associated to M.

We first briefly recall the theory of ε-constants of Deligne and Langlands [Deligne 1973]. Let $W_{\mathbb{Q}_p} \subseteq G_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$. Let E be a field of characteristic zero, and let $V = (V, \rho)$ be an E-representation of $W_{\mathbb{Q}_p}$, i.e., V is a finite-dimensional E-vector space equipped with a smooth E-linear action ρ of $W_{\mathbb{Q}_p}$. Let us denote by $V^\vee$ the dual $(\mathrm{Hom}_E(V, E), \rho^\vee)$ of V. Denote by $E(|x|)$ the rank-one E-representation of $W_{\mathbb{Q}_p}$ corresponding to the continuous homomorphism $|x| : \mathbb{Q}_p^\times \to E^\times : p \mapsto 1/p,\ a \mapsto 1 (a \in \mathbb{Z}_p^\times)$ via the local class field theory. Put $V^\vee(|x|) := V^\vee \otimes_E E(|x|)$. Assume that E is a field which contains $\mathbb{Q}(\zeta_{p^\infty})$. The definition of the ε-constants depends on the choice of an additive character of $\mathbb{Q}_p$ and a Haar measure on $\mathbb{Q}_p$. In this article, we fix the Haar measure dx on $\mathbb{Q}_p$ for which $\mathbb{Z}_p$ has measure 1. For each $\mathbb{Z}_p$-basis $\zeta = \{\zeta_{p^n}\}_{n\geq 0}$ of $\mathbb{Z}_p(1)$, we define an additive character $\psi_\zeta : \mathbb{Q}_p \to E^\times$ such that $\psi_\zeta(1/p^n) := \zeta_{p^n}$ for $n \geq 1$. In this article, we don't recall the precise definition of ε-constants, but we recall here some of their basic properties under the fixed additive character ψ_ζ and the fixed Haar measure dx. We can attach a constant $\varepsilon(V, \psi_\zeta, dx) \in E^\times$ for each V as above which satisfies the following properties (we let $\varepsilon(V, \zeta) := \varepsilon(V, \psi_\zeta, dx)$ for simplicity):

(1) For each exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of finite-dimensional E-vector spaces with continuous actions of $W_{\mathbb{Q}_p}$, we have

$$\varepsilon(V_2, \zeta) = \varepsilon(V_1, \zeta)\varepsilon(V_3, \zeta).$$

(2) For each $a \in \mathbb{Z}_p^\times$, we define $\zeta^a := \{\zeta_{p^n}^a\}_{n\geq 1}$. Then we have

$$\varepsilon(V, \zeta^a) = \det_E V(\mathrm{rec}_{\mathbb{Q}_p}(a))\varepsilon(V, \zeta).$$

(3) $\varepsilon(V, \zeta)\varepsilon(V^\vee(|x|), \zeta^{-1}) = 1.$

(4) $\varepsilon(V, \zeta) = 1$ if V is unramified.

(5) If $\dim_E V$ equals 1 and corresponds to a locally constant homomorphism $\delta : \mathbb{Q}_p^\times \to E^\times$ via the local class field theory, then

$$\varepsilon(V, \zeta) = \delta(p)^{n(\delta)}\biggl(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times} \delta(i)^{-1}\zeta_{p^{n(\delta)}}^i\biggr),$$

where $n(\delta) \geq 0$ is the conductor of δ, i.e., the minimal integer $n \geq 0$ such that $\delta|_{(1+p^n\mathbb{Z}_p)\cap\mathbb{Z}_p^\times} = 1$ (then $\delta|_{\mathbb{Z}_p^\times}$ factors through $(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times$).

For a Weil–Deligne representation $W = (V, \rho, N)$ of $W_{\mathbb{Q}_p}$ defined over E, we set

$$\varepsilon(W, \zeta) := \varepsilon((V, \rho), \zeta) \cdot \det_E\bigl(-\mathrm{Fr}_p \mid V^{L_p}/(V^{N=0})^{I_p}\bigr),$$

which also satisfies

$$\varepsilon(W, \zeta) \cdot \varepsilon(W^\vee(|x|), \zeta^{-1}) = 1.$$

Next, we define the ε-constant for each de Rham (φ, Γ)-module over $\mathcal{R}_L$ following Fontaine and Perrin-Riou [1994]. Let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$. Then M is potentially semistable by the result of Berger (for example, see Théorème III.2.4 of [Berger 2008b]) based on Crew's conjecture, which was proved by André, Mebkhout, and Kedlaya. Hence, we can define a filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module $\boldsymbol{D}_{\mathrm{pst}}(M) := \bigcup_{K\subseteq\bar{\mathbb{Q}}_p} \boldsymbol{D}^K_{\mathrm{st}}(M|_K)$ which is a free $(\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L)$-module whose rank is r_M, where K runs through all the finite extensions of $\mathbb{Q}_p$ and we define $\boldsymbol{D}^K_{\mathrm{st}}(M|_K) := (\mathcal{R}_L(\pi_K)[\log(\pi), 1/t] \otimes_{\mathcal{R}_L} M)^{\Gamma_K=1}$. Set $\boldsymbol{D}_{\mathrm{st}}(M) := \boldsymbol{D}^{\mathbb{Q}_p}_{\mathrm{st}}(M)$. Following Fontaine, one can define a Weil–Deligne representation $W(M) := (\boldsymbol{D}_{\mathrm{pst}}(M), \rho, N)$ of $W_{\mathbb{Q}_p}$ defined over $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L$ such that N is the natural one and $\rho(g)(x) := \varphi^{v(g)}(g \cdot x)$ for $g \in W_{\mathbb{Q}_p}$ and $x \in W(M)$, where we denote by $g \cdot x$ the natural action of $G_{\mathbb{Q}_p}$ on $W(M)$ and

$$v : W_{\mathbb{Q}_p} \twoheadrightarrow W^{\mathrm{ab}}_{\mathbb{Q}_p} \xrightarrow{\mathrm{rec}^{-1}_{\mathbb{Q}_p}} \mathbb{Q}_p^\times \xrightarrow{v_p} \mathbb{Z}.$$

Taking the base change of $W(M)$ by the natural inclusion $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L \hookrightarrow \mathbb{Q}_p^{\mathrm{ab}} \otimes_{\mathbb{Q}_p} L$, and decomposing $\mathbb{Q}_p^{\mathrm{ab}} \otimes_{\mathbb{Q}_p} L \xrightarrow{\sim} \prod_\tau L_\tau$ into a finite product of fields L_τ, we obtain a Weil–Deligne representation $W(M)_\tau$ of $W_{\mathbb{Q}_p}$ defined over L_τ for each τ. Hence, we can define the ε-constant $\varepsilon(W(M)_\tau, \tau(\zeta)) \in L_\tau^\times$, where $\tau(\zeta)$ is the image of ζ in L_τ by the projection $\mathbb{Q}_p^{\mathrm{ab}} \otimes_{\mathbb{Q}_p} L \to L_\tau$. Then the product

$$\varepsilon_L(W(M), \zeta) := (\varepsilon(W(M)_\tau, \tau(\zeta)))_\tau \in \prod_\tau L_\tau^\times$$

is contained in $L_\infty^\times := (\mathbb{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbb{Q}_p} L)^\times \subseteq (\mathbb{Q}_p(\zeta_{p^\infty}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} L)^\times$ since it is easy to check that $\varepsilon_L(W(M), \zeta)$ is fixed by $1 \otimes \varphi \otimes 1$.

Using this definition, for each de Rham (φ, Γ)-module M over $\mathcal{R}_L$, we construct a trivialization $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(M)$ as follows. We will first define two isomorphisms

$$\theta_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$$

and

$$\theta_{\mathrm{dR},L}(M, \zeta) : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\sim} \Delta_{L,2}(M)$$

(we remark that $\theta_{\mathrm{dR},L}(M, \zeta)$ depends on the choice of ζ), and then define $\varepsilon^{\mathrm{dR}}_{L,\xi}(M)$ as the composite

$$\varepsilon^{\mathrm{dR}}_{L,\xi}(M) : \mathbf{1}_L \xrightarrow{\Gamma_L(M)\cdot\theta_L(M)} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$$
$$\xrightarrow{\mathrm{id}\boxtimes\theta_{\mathrm{dR},L}(M,\zeta)} \Delta_{L,1}(M) \boxtimes \Delta_{L,2}(M) = \Delta_L(M),$$

where $\Gamma_L(M) \in \mathbb{Q}^\times$ is defined by

$$\Gamma_L(M) := \prod_{r\in\mathbb{Z}} \Gamma^*(r)^{-\dim_L \mathrm{gr}^{-r} \boldsymbol{D}_{\mathrm{dR}}(M)},$$

where we set

$$\Gamma^*(r) := \begin{cases} (r-1)! & (r \geq 1), \\ (-1)^r/(-r)! & (r \leq 0). \end{cases}$$

We first define $\theta_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$. By the result of Section 2B, we have the exact sequence of L-vector spaces

$$0 \to \mathrm{H}^0_{\varphi,\gamma}(M_0) \to \boldsymbol{D}_{\mathrm{cris}}(M_0)_1 \xrightarrow{x \mapsto ((1-\varphi)x, \bar{x})} \boldsymbol{D}_{\mathrm{cris}}(M_0)_2 \oplus t_{M_0} \xrightarrow{\exp_{f,M_0} \oplus \exp_{M_0}} \mathrm{H}^1_{\varphi,\gamma}(M_0)_f \to 0 \quad (23)$$

for $M_0 = M, M^*$, where we let $\boldsymbol{D}_{\mathrm{cris}}(M_0)_i = \boldsymbol{D}_{\mathrm{cris}}(M_0)$ for $i = 1, 2$.

Using Tate duality, the de Rham duality

$$\boldsymbol{D}_{\mathrm{dR}}(M)^0 \xrightarrow{\sim} t^\vee_{M^*} : x \mapsto [\bar{y} \mapsto [y, x]_{\mathrm{dR}}]$$

(here $y \in \boldsymbol{D}_{\mathrm{dR}}(M^*)$ is a lift of $\bar{y}$) and Proposition 2.24, we define a map

$$\exp^*_{M^*} : \mathrm{H}^1_{\varphi,\gamma}(M)_{/f} := \mathrm{H}^1_{\varphi,\gamma}(M)/\mathrm{H}^1_{\varphi,\gamma}(M)_f \xrightarrow{x \mapsto [y \mapsto \langle y, x \rangle]} \mathrm{H}^1_{\varphi,\gamma}(M^*)^\vee_f \xrightarrow{\exp^\vee_{M^*}} t^\vee_{M^*} \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(M)^0$$

which is called the dual exponential map and was studied in §2.4 of [Nakamura 2014a]. Using this map, as the dual of the exact sequence (23) for $M_0 = M^*$, we obtain an exact sequence

$$0 \to \mathrm{H}^1_{\varphi,\gamma}(M)_{/f} \xrightarrow{\exp^\vee_{f,M^*} \oplus \exp^*_{M^*}} \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2 \oplus \boldsymbol{D}_{\mathrm{dR}}(M)^0 \xrightarrow{(*)} \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1 \to \mathrm{H}^2_{\varphi,\gamma}(M) \to 0, \quad (24)$$

where the map $\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2 \to \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1$ in $(*)$ is the dual of $(1-\varphi)$. Therefore, as the composite of the exact sequences (23) for $M_0 = M$ and (24), we obtain the exact sequence

$$0 \to \mathrm{H}^0_{\varphi,\gamma}(M) \to \boldsymbol{D}_{\mathrm{cris}}(M)_1 \xrightarrow{x \mapsto ((1-\varphi)x, \bar{x})} \boldsymbol{D}_{\mathrm{cris}}(M)_2 \oplus t_M \to \mathrm{H}^1_{\varphi,\gamma}(M) \to \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2 \oplus \boldsymbol{D}_{\mathrm{dR}}(M)^0 \to \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1 \to \mathrm{H}^2_{\varphi,\gamma}(M) \to 0. \quad (25)$$

Applying the trivialization (20) to this exact sequence and using the canonical isomorphisms

$$i_{\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M)_1)} : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M)_2) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M)_1)^{-1} \xrightarrow{\sim} \mathbf{1}_L,$$
$$i_{\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1)} : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_2) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee_1)^{-1} \xrightarrow{\sim} \mathbf{1}_L,$$

and

$$\mathrm{Det}_L(\boldsymbol{D}^0_{\mathrm{dR}}(M)) \boxtimes \mathrm{Det}_L(t_M) \xrightarrow{\sim} \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)),$$

we obtain a canonical isomorphism

$$\theta_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)).$$

Next, we define an isomorphism $\theta_{\mathrm{dR},L}(M,\zeta) : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\sim} \Delta_{L,2}(M)$. To define this, we show the following lemma.

Lemma 3.4. *Let $\{h_1, h_2, \dots, h_{r_M}\}$ be the set of Hodge–Tate weights of M (with multiplicity). Put $h_M := \sum_{i=1}^{r_M} h_i$. For any $n \geq n(M)$ such that $\varepsilon_L(W(M),\zeta) \in L_n := \mathbb{Q}_p(\zeta_{p^n}) \otimes_{\mathbb{Q}_p} L$, the map*

$$\mathcal{L}_L(M) \to \boldsymbol{D}_{\mathrm{dif},n}(\det_{\mathcal{R}_L} M) = L_n((t)) \otimes_{\iota_n, \mathcal{R}_L^{(n)}} (\det_{\mathcal{R}_L} M)^{(n)} : \\ x \mapsto \frac{1}{\varepsilon_L(W(M),\zeta)} \cdot \frac{1}{t^{h_M}} \otimes \varphi^n(x)$$

induces an isomorphism

$$f_{M,\zeta} : \mathcal{L}_L(M) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} M),$$

and doesn't depend on the choice of n.

Proof. The independence of n follows from the definition of the transition map $\boldsymbol{D}_{\mathrm{dif},n}(-) \hookrightarrow \boldsymbol{D}_{\mathrm{dif},n+1}(-)$.

We show that $f_{M,\zeta}$ is an isomorphism. Comparing the dimensions, it suffices to show that the image of the map in the lemma is contained in $\boldsymbol{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} M)$, i.e., is fixed by the action of Γ. Since we have $\varepsilon_L(W(M),\zeta)/\varepsilon_L(W(\det_{\mathcal{R}_L} M),\zeta) \in L^\times (\subseteq L_\infty^\times)$, it suffices to show the claim when M is of rank one. We assume that M is of rank one. By the classification of rank-one de Rham (φ,Γ)-modules, there exists a locally constant homomorphism $\tilde{\delta} : \mathbb{Q}_p^\times \to L^\times$ such that $M \xrightarrow{\sim} \mathcal{R}_L(\tilde{\delta} \cdot x^{h_M})$. The corresponding representation $W(M)$ of $W_{\mathbb{Q}_p}$ is given by the homomorphism $\tilde{\delta} \cdot |x|^{h_M} : \mathbb{Q}_p^\times \to L^\times$ via the local class field theory. By the property (2) of ε-constants, we have $\gamma(\varepsilon_L(\boldsymbol{D}_{\mathrm{pst}}(M),\zeta)) = \tilde{\delta}(\chi(\gamma))\varepsilon_L(W(M),\zeta)$ for $\gamma \in \Gamma$, which proves the claim since we have $\gamma(\varphi^n(x)) = \tilde{\delta}(\chi(\gamma))\chi(\gamma)^{h_M}\varphi^n(x)$ for $x \in \mathcal{L}_L(M)$, $\gamma \in \Gamma$, by definition. □

Since we have a canonical isomorphism $\boldsymbol{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} M) \xrightarrow{\sim} \det_L \boldsymbol{D}_{\mathrm{dR}}(M)$, the isomorphism $f_{M,\zeta}$ induces an isomorphism $f_{M,\zeta} : \Delta_{L,2}(M) \xrightarrow{\sim} \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$. We define the isomorphism $\theta_{\mathrm{dR},L}(M,\zeta)$ as the inverse

$$\theta_{\mathrm{dR},L}(M,\zeta) := f_{M,\zeta}^{-1} : \mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M)) \xrightarrow{\sim} \Delta_{L,2}(M).$$

Remark 3.5. The isomorphism $f_{M,\zeta}$, and hence the isomorphism $\theta_{\mathrm{dR},L}(M,\zeta)$, depends on the choice of ζ. If we choose another $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1)$ which can be written as $\zeta^a := \{\zeta_{p^n}^a\}_{n\geq 0}$ for unique $a \in \mathbb{Z}_p^\times$, then f_{M,ζ^a} is defined using $\varepsilon_L(W(M),\zeta^a)$ and the parameter π_{ζ^a} (see Remark 2.1) and $t_a := \log(1+\pi_{\zeta^a})$. Since we have $\varepsilon_L(W(M),\zeta^a)=\det W(M)(\mathrm{rec}_{\mathbb{Q}_p}(a))\varepsilon_L(W(M),\zeta)$ and $\pi_{\zeta^a}=(1+\pi)^a-1$ and $t_a=at$, we have $f_{M,\zeta^a}=f_{M,\zeta}/\delta_{\det_{\mathcal{R}_L}M}(a)$, and hence we also have

$$\theta_{\mathrm{dR},L}(M,\zeta^a)=\delta_{\det_{\mathcal{R}_L}M}(a)\cdot\theta_{\mathrm{dR},L}(M,\zeta),$$

and obtain

$$\varepsilon^{\mathrm{dR}}_{L,\zeta^a}(M)=\delta_{\det_{\mathcal{R}_L}M}(a)\cdot\varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

Remark 3.6. Kato [1993b] and Fukaya and Kato [2006] defined their de Rham ε-isomorphism $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)':\mathbf{1}_L\xrightarrow{\sim}\Delta_L(V)$ (using a different notation) for each de Rham L-representation V of $G_{\mathbb{Q}_p}$ using the original Bloch–Kato exponential map. Using Proposition 2.26, we can compare our $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V))$ with their $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'$ under the canonical isomorphism $\Delta_L(V)\xrightarrow{\sim}\Delta_L(\boldsymbol{D}_{\mathrm{rig}}(V))$ defined in Corollary 3.2. We remark that ours and theirs are different since they used (in our notation) the ε-constant $\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(V),\rho),\zeta)$ associated to the representation $(\boldsymbol{D}_{\mathrm{pst}}(V),\rho)$ of $W_{\mathbb{Q}_p}$ instead of $W(V)$. Since one has

$$\varepsilon_L(W(V),\zeta)=\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(V),\rho),\zeta)\cdot\det_L(-\varphi\mid\boldsymbol{D}_{\mathrm{st}}(V)/\boldsymbol{D}_{\mathrm{cris}}(V)),$$

the correct relation between ours and theirs is

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V))=\det_L(-\varphi\mid\boldsymbol{D}_{\mathrm{st}}(V)/\boldsymbol{D}_{\mathrm{cris}}(V))\cdot\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'. \tag{26}$$

Moreover, we insist that ours is the correct one, since we show in Lemma 3.7 below that our $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ is compatible with exact sequences (but $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'$ may not satisfy this compatibility).

Finally in this subsection, we prove a lemma on the compatibility of the de Rham ε-isomorphism with exact sequences and the Tate duality.

Lemma 3.7. (1) *For any exact sequence* $0\to M_1\to M_2\to M_3\to 0$, *we have*

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(M_2)=\varepsilon^{\mathrm{dR}}_{L,\zeta}(M_1)\boxtimes\varepsilon^{\mathrm{dR}}_{L,\zeta}(M_3)$$

under the canonical isomorphism $\Delta_L(M_2)\xrightarrow{\sim}\Delta_L(M_1)\boxtimes\Delta_L(M_3)$.

(2) *One has the following commutative diagram of isomorphisms*

$$\begin{array}{ccc}\Delta_L(M) & \xrightarrow{\mathrm{can}} & \Delta_L(M^*)^\vee\boxtimes(L(r_M),0)\\ {\scriptstyle\varepsilon^{\mathrm{dR}}_{L,\zeta^{-1}}(M)}\uparrow & & \downarrow{\scriptstyle\varepsilon^{\mathrm{dR}}_{L,\zeta}(M^*)^\vee\boxtimes[\boldsymbol{e}_{r_M}\mapsto 1]}\\ \mathbf{1}_L & \xrightarrow{\mathrm{can}} & \mathbf{1}_L\boxtimes\mathbf{1}_L\end{array}$$

Proof. We first prove (1). The proof is identical to that of Proposition 3.3.8 of [Fukaya and Kato 2006], but we give a proof for convenience of the readers. We first remark that we have

$$\Gamma_L(M)\cdot\Gamma_L(M^*)=(-1)^{h_M+\dim_L t_M} \tag{27}$$

since we have

$$\Gamma^*(r)\cdot\Gamma^*(1-r)=\begin{cases}(-1)^{r-1} & (r\geq 1),\\ (-1)^{r} & (r\leq 0).\end{cases}$$

We next remark that one has the commutative diagram

$$\begin{array}{ccc}\mathbf{1}_L & \xrightarrow{\theta_L(M)} & \Delta_{L,1}(M)\boxtimes\mathrm{Det}_L(M)\\ {\scriptstyle(-1)^{\dim_L t_M}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{can}}\\ \mathbf{1}_L & \xleftarrow{\theta_L(M^*)^\vee} & \Delta_{L,1}(M^*)^\vee\boxtimes\mathrm{Det}_L(M^*)^\vee\end{array}\tag{28}$$

in which the right vertical arrow is induced by the Tate duality, since one has the commutative diagram

$$\begin{array}{ccccc}t_M & \xrightarrow{-\exp_M} & \mathrm{H}^1_{\varphi,\gamma}(M) & \xrightarrow{\exp^*_{M^*}} & \boldsymbol{D}_{\mathrm{dR}}(M)^0\\ {\scriptstyle\bar{x}\mapsto[y,\mapsto[y,x]_{\mathrm{dR}}]}\big\downarrow & & {\scriptstyle x\mapsto[y\mapsto\langle y,x\rangle]}\big\downarrow & & \big\downarrow{\scriptstyle x\mapsto[\bar{y}\mapsto[y,x]_{\mathrm{dR}}]}\\ \boldsymbol{D}_{\mathrm{dR}}(M^*)^\vee & \xrightarrow{(\exp^*_M)^\vee} & \mathrm{H}^1_{\varphi,\gamma}(M^*)^\vee & \xrightarrow{(\exp_{M^*})^\vee} & (t_{M^*})^\vee\end{array}$$

Finally, we remark that one has the commutative diagram

$$\begin{array}{ccc}\mathrm{Det}_L(M) & \xrightarrow{\theta_{\mathrm{dR},L}(M,\zeta^{-1})} & \Delta_{L,2}(M)\\ {\scriptstyle(-1)^{h_M}\cdot\mathrm{can}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{can}}\\ \mathrm{Det}_L(M^*)^\vee & & \\ =\mathrm{Det}_L(M^*)^\vee\boxtimes\mathbf{1}_L & \xleftarrow{\theta_{\mathrm{dR},L}(M^*,\zeta)^\vee\boxtimes[\boldsymbol{e}_{r_M}\mapsto 1]} & \Delta_{L,2}(M^*)^\vee\boxtimes(L(r_M),0)\end{array}\tag{29}$$

in which the vertical maps can are also defined by the duality, since we have

$$\varepsilon_L(W(V),\zeta^{-1})\cdot\varepsilon_L(W(V^*),\zeta)=1.$$

Then (1) follows from the commutative diagrams (27), (28) and (29).

We next prove (1). We first define an isomorphism

$$\theta_L(M)':\mathbf{1}_L\xrightarrow{\sim}\Delta_{L,1}(M)\boxtimes\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))$$

in the same way as $\theta_L(M)$ using the exact sequence

$$0\to\mathrm{H}^0_{\varphi,\gamma}(M)\to\boldsymbol{D}_{\mathrm{cris}}(M)_1\xrightarrow{x\mapsto((1-\varphi^{-1})x,\bar{x})}\boldsymbol{D}_{\mathrm{cris}}(M)_2\oplus t_M\xrightarrow{\exp_{f,M}\oplus\exp_M}\mathrm{H}^1_{\varphi,\gamma}(M)_f\to 0\tag{30}$$

(we use φ^{-1} instead of φ) and (24), and define

$$\theta_{\mathrm{dR},L}(M,\zeta)':\mathrm{Det}_L(\boldsymbol{D}_{\mathrm{dR}}(M))\xrightarrow{\sim}\Delta_{L,2}(M)$$

in the same way as $\theta_{\mathrm{dR},L}(M,\zeta)'$ using the constant

$$\varepsilon_L(W(M),\zeta)\cdot\det_L(-\varphi\mid \boldsymbol{D}_{\mathrm{cris}}(M))=\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M),\rho),\zeta)\cdot\det_L(-\varphi\mid \boldsymbol{D}_{\mathrm{st}}(M))$$

instead of $\varepsilon_L(W(V),\zeta)$. Since we have $\theta_L(M)'=\theta_L(M)\cdot\det_L(-\varphi^{-1}\mid \boldsymbol{D}_{\mathrm{cris}}(M))$, $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ can be defined using the triple $(\Gamma_L(M),\theta_L(M)',\theta_{\mathrm{dR},L}(M,\zeta)')$ instead of $(\Gamma_L(M),\theta_L(M),\theta_{\mathrm{dR},L}(M,\zeta))$.

Let $0\to M_1\to M_2\to M_3\to 0$ be an exact sequence as in (1). Since one has $\Gamma(M_2)=\Gamma(M_1)\cdot\Gamma(M_3)$, it suffices to show that both $\theta_L(-)'$ and $\theta_{\mathrm{dR},L}(-)'$ are compatible with the exact sequence.

Since we have

$$\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M_2),\rho),\zeta)=\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M_1),\rho),\zeta)\cdot\varepsilon_L((\boldsymbol{D}_{\mathrm{pst}}(M_3),\rho),\zeta)$$

and

$$\det_L(-\varphi\mid \boldsymbol{D}_{\mathrm{st}}(M_2))=\det_L(-\varphi\mid \boldsymbol{D}_{\mathrm{st}}(M_1))\cdot\det_L(-\varphi\mid \boldsymbol{D}_{\mathrm{st}}(M_3))$$

(since $\boldsymbol{D}_{\mathrm{pst}}(-)$ and $\boldsymbol{D}_{\mathrm{st}}(-)$ are exact for de Rham (φ,Γ)-modules), the isomorphism $\theta_{\mathrm{dR},L}(-)'$ is compatible with the exact sequence.

We remark that the functor $\boldsymbol{D}_{\mathrm{cris}}(-)$ is not exact (in general) for de Rham (φ,Γ)-modules, but we have the exact sequence

$$\begin{aligned}0\to \boldsymbol{D}_{\mathrm{cris}}(M_1)\to \boldsymbol{D}_{\mathrm{cris}}(M_2)\to \boldsymbol{D}_{\mathrm{cris}}(M_3)&\\ \xrightarrow{(*)}\boldsymbol{D}_{\mathrm{cris}}(M_1^*)^\vee\to \boldsymbol{D}_{\mathrm{cris}}(M_2^*)^\vee\to \boldsymbol{D}_{\mathrm{cris}}(M_3^*)^\vee\to 0&\end{aligned}$$

such that the boundary map $(*)$ satisfies the commutative diagram

$$\begin{array}{ccc}\boldsymbol{D}_{\mathrm{cris}}(M_3) & \xrightarrow{(*)} & \boldsymbol{D}_{\mathrm{cris}}(M_1^*)^\vee\\ \downarrow{\scriptstyle\varphi^{-1}} & & \downarrow{\scriptstyle\varphi^\vee}\\ \boldsymbol{D}_{\mathrm{cris}}(M_3) & \xrightarrow{(*)} & \boldsymbol{D}_{\mathrm{cris}}(M_1^*)^\vee\end{array}$$

from which the compatibility of $\theta_L(-)'$ with the exact sequence follows, which finishes the proof of the lemma. □

3D. ***Formulation of the local ε-conjecture.*** In this subsection, using the definitions in the previous subsections, we formulate the following conjecture, which we call the local ε-conjecture. This conjecture is a combination of Kato's original ε-conjecture for (Λ,T) with our conjecture for (A,M). To state both situations at the same time, we use the notation (B,N) for (Λ,T) or (A,M), and $f:B\to B'$ for $f:\Lambda\to\Lambda'$ or $f:A\to A'$.

Conjecture 3.8. *We can uniquely define a B-linear isomorphism*

$$\varepsilon_{B,\zeta}(N):\mathbf{1}_B\xrightarrow{\sim}\Delta_B(N)$$

for each pair (B, N) as above and for each $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$ satisfying the following conditions:

(i) *Let $f : B \to B'$ be a continuous homomorphism. Then we have*

$$\varepsilon_{B,\zeta}(N) \otimes \mathrm{id}_{B'} = \varepsilon_{B',\zeta}(N \otimes_B B')$$

under the canonical isomorphism $\Delta_B(N) \otimes_B B' \xrightarrow{\sim} \Delta_{B'}(N \otimes_B B')$.

(ii) *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence. Then we have*

$$\varepsilon_{B,\zeta}(N_1) \boxtimes \varepsilon_{B,\zeta}(N_3) = \varepsilon_{B,\zeta}(N_2)$$

under the canonical isomorphism $\Delta_B(N_1) \boxtimes \Delta_B(N_3) \xrightarrow{\sim} \Delta_B(N_2)$.

(iii) *For any $a \in \mathbb{Z}_p^\times$, we have*

$$\varepsilon_{B,\zeta^a}(N) = \delta_{\det_B(N)}(a) \cdot \varepsilon_{B,\zeta}(N).$$

(iv) *One has the following commutative diagram of isomorphisms*:

$$\begin{array}{ccc} \Delta_B(N) & \xrightarrow{\text{can}} & \Delta_B(N^*)^\vee \boxtimes (L(r_N), 0) \\ {\scriptstyle \varepsilon_{B,\zeta^{-1}}(N)} \uparrow & & \downarrow {\scriptstyle \varepsilon_{B,\zeta}(N^*)^\vee \boxtimes [\boldsymbol{e}_{r_N} \mapsto 1]} \\ \mathbf{1}_B & \xrightarrow{\text{can}} & \mathbf{1}_B \boxtimes \mathbf{1}_B \end{array}$$

(v) *Let $f : \Lambda \to A$ be a continuous homomorphism, and let $M := \boldsymbol{D}_{\mathrm{rig}}(T \otimes_\Lambda A)$ be the associated (φ, Γ)-module obtained by the base change of T with respect to f. Then we have*

$$\varepsilon_{\Lambda,\zeta}(T) \otimes \mathrm{id}_A = \varepsilon_{A,\zeta}(M)$$

under the canonical isomorphism $\Delta_\Lambda(T) \otimes_\Lambda A \xrightarrow{\sim} \Delta_A(M)$ of Corollary 3.2.

(vi) *Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$. Then we have*

$$\varepsilon_{L,\zeta}(M) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

Remark 3.9. Kato's original conjecture [1993b] is the restriction of Conjecture 3.8 to the pairs (Λ, T). As explained in Remark 3.6, we insist that condition (v) should be stated using $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V))$ $\big($or $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V) := \varepsilon^{\mathrm{dR}}_{L,\zeta}(V)' \cdot \det_L(-\varphi \mid \boldsymbol{D}_{\mathrm{st}}(V)/\boldsymbol{D}_{\mathrm{cris}}(V))\big)$ instead of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'$.

Remark 3.10. In Kato's conjecture, the uniqueness of the ε-isomorphism was not explicitly predicted. Recently, it has been shown that the de Rham points (even crystalline points) are Zariski dense in "universal" families of p-adic representations, or (φ, Γ)-modules in many cases ([Colmez 2008; Kisin 2010] for the two-dimensional case, [Chenevier 2013; Nakamura 2014b] for general case), hence we add the uniqueness assertion in our conjecture.

Kato [1993b] proved his conjecture for the rank-one case (note that one has $\boldsymbol{D}_{\mathrm{st}}(V)=\boldsymbol{D}_{\mathrm{cris}}(V)$ for the rank-one case, hence one also has $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)'=\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)$). As a generalization of his theorem, our main theorem of this article is the following, whose proof is given in the next section.

Theorem 3.11. *Conjecture 3.8 is true for the rank-one case. More precisely, we can uniquely define a B-linear isomorphism $\varepsilon_{B,\zeta}(N):\mathbf{1}_B\xrightarrow{\sim}\Delta_B(N)$ for each pair (B,N) such that N is of rank one and for each $\mathbb{Z}_p$-basis ζ of $\mathbb{Z}_p(1)$ satisfying the conditions* (i), (iii), (iv), (v), (vi).

Before passing to the proof of this theorem in the next section, we prove two easy corollaries concerning the trianguline case. We say that a (φ,Γ)-module M over $\mathcal{R}_A$ is trianguline if M has a filtration $\mathcal{F}:0:=M_0\subseteq M_1\subseteq\cdots\subseteq M_n:=M$ whose graded quotients M_i/M_{i-1} are rank-one (φ,Γ)-modules over $\mathcal{R}_A$ for all $1\leq i\leq n$. We call the filtration $\mathcal{F}$ a triangulation of M.

Corollary 3.12. *Let M be a trianguline (φ,Γ)-module over $\mathcal{R}_A$ of rank n with a triangulation $\mathcal{F}$ as above. The isomorphism*

$$\varepsilon_{\mathcal{F},A,\zeta}(M):\mathbf{1}_A\xrightarrow{\boxtimes_{i=1}^n\varepsilon_{A,\zeta}(M_i/M_{i-1})}\boxtimes_{i=1}^n\Delta_A(M_i/M_{i-1})\xrightarrow{\sim}\Delta_A(M)$$

defined as the product of the isomorphisms $\varepsilon_{A,\zeta}(M_i/M_{i-1}):\mathbf{1}_A\xrightarrow{\sim}\Delta_A(M_i/M_{i-1})$, which are defined in Theorem 3.11, satisfies the following properties:

(i)′ *For any $f:A\to A'$, we have*

$$\varepsilon_{\mathcal{F},A,\zeta}(M)\otimes\mathrm{id}_{A'}=\varepsilon_{\mathcal{F}',A',\zeta}(M\otimes_A A'),$$

where $\mathcal{F}'$ is the base change of the triangulation $\mathcal{F}$ by f.

(iii)′ *For any $a\in\mathbb{Z}_p^\times$, we have*

$$\varepsilon_{\mathcal{F},A,\zeta^a}(M)=\delta_{\det_A(M)}(a)\cdot\varepsilon_{\mathcal{F},A,\zeta}(M).$$

(iv)′ *One has the commutative diagram of isomorphisms*

$$\begin{array}{ccc}\Delta_A(M) & \xrightarrow{\mathrm{can}} & \Delta_A(M^*)^\vee\boxtimes(A(r_M),0)\\ {\scriptstyle \varepsilon_{\mathcal{F},A,\zeta}(M)}\uparrow & & \downarrow{\scriptstyle \varepsilon_{\mathcal{F}^*,A,\zeta}(M^*)^\vee\boxtimes[\boldsymbol{e}_{r_M}\mapsto(-1)^{r_M}]}\\ \mathbf{1}_A & \xrightarrow{\mathrm{can}} & \mathbf{1}_A\boxtimes\mathbf{1}_A\end{array}$$

in which $\mathcal{F}^*$ is the Tate dual of the triangulation $\mathcal{F}$.

(vi)′ *Let $L=A$ be a finite extension of $\mathbb{Q}_p$, and let M be a de Rham and trianguline (φ,Γ)-module over $\mathcal{R}_L$. Then, for any triangulation $\mathcal{F}$ of M, we have*

$$\varepsilon_{\mathcal{F},L,\zeta}(M)=\varepsilon^{\mathrm{dR}}_{L,\zeta}(M).$$

In particular, in this case, $\varepsilon_{\mathcal{F},L,\zeta}(M)$ does not depend on $\mathcal{F}$.

Proof. This corollary immediately follows from Theorem 3.11 since $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ is multiplicative with respect to exact sequences by (1) of Lemma 3.7. □

Finally, we compare Corollary 3.12 with the previous known results on Kato's ε-conjecture for the cyclotomic deformations of crystalline ones. Let F be a finite unramified extension of $\mathbb{Q}_p$. Let V be a crystalline L-representation of G_F, and let $T \subseteq V$ be a G_F-stable $\mathcal{O}_L$-lattice of V. In [Benois and Berger 2008] and [Loeffler et al. 2015], they defined ε-isomorphisms for some twists of T. Here, for simplicity, we only recall the result of [Benois and Berger 2008] under the additional assumption that $F = \mathbb{Q}_p$, since other cases can be proven in the same way. Let $\mathcal{O}_L[\![\Gamma]\!]$ be the Iwasawa algebra with coefficients in $\mathcal{O}_L$. We define an $\mathcal{O}_L[\![\Gamma]\!]$-representation $\mathbf{Dfm}(T) := T \otimes_{\mathcal{O}_L} \mathcal{O}_L[\![\Gamma]\!]$ on which $G_{\mathbb{Q}_p}$ acts by $g(x \otimes y) := g(x) \otimes [\bar{g}]^{-1} y$ for any $g \in G_{\mathbb{Q}_p},\ x \in T,\ y \in \mathcal{O}_L[\![\Gamma]\!]$. In [Benois and Berger 2008], by studying the associated Wach modules very carefully, they essentially showed that Perrin-Riou's big exponential map induces an ε-isomorphism, which we denote by

$$\varepsilon^{\mathrm{BB}}_{\mathcal{O}_L[\![\Gamma]\!],\zeta}(\mathbf{Dfm}(T)) : \mathbf{1}_{\mathcal{O}_L[\![\Gamma]\!]} \xrightarrow{\sim} \Delta_{\mathcal{O}_L[\![\Gamma]\!]}(\mathbf{Dfm}(T)),$$

satisfying the conditions in Conjecture 3.8. Let $\boldsymbol{D}_{\mathrm{rig}}(V)$ be the (φ, Γ)-module over $\mathcal{R}_L$ associated to V. Then, applying Example 3.3 to $(\Lambda, T) = (\mathcal{O}_L[\![\Gamma]\!], T)$, we obtain a canonical isomorphism

$$\Delta_{\mathcal{O}_L[\![\Gamma]\!]}(\mathbf{Dfm}(T)) \otimes_{\mathcal{O}_L[\![\Gamma]\!]} \mathcal{R}^{\infty}_L(\Gamma) \xrightarrow{\sim} \Delta_{\mathcal{R}^{\infty}(\Gamma)}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))) \tag{31}$$

(see the next section for the definitions of $\mathcal{R}^{\infty}_L(\Gamma)$ and $\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))$). Since $\boldsymbol{D}_{\mathrm{rig}}(V)$ is crystalline, after extending scalars, we may assume that it is trianguline with a triangulation $\mathcal{F}$. Then $\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))$ is also trianguline with a triangulation $\mathcal{F}' := \mathbf{Dfm}(\mathcal{F})$. Hence, by Corollary 3.12, we obtain an isomorphism

$$\varepsilon_{\mathcal{F}',\mathcal{R}^{\infty}_L(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))) : \mathbf{1}_{\mathcal{R}^{\infty}_L(\Gamma)} \xrightarrow{\sim} \Delta_{\mathcal{R}^{\infty}(\Gamma)}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))).$$

Under this situation, we easily obtain the following corollary.

Corollary 3.13. *Under the isomorphism* (31), *we have*

$$\varepsilon^{\mathrm{BB}}_{\mathcal{O}_L[\![\Gamma]\!],\zeta}(\mathbf{Dfm}(T)) \otimes \mathrm{id}_{\mathcal{R}^{\infty}_L(\Gamma)} = \varepsilon_{\mathcal{F}',\mathcal{R}^{\infty}_L(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V))).$$

In particular, the isomorphism $\varepsilon_{\mathcal{F}',\mathcal{R}^{\infty}_L(\Gamma),\zeta}(\mathbf{Dfm}(\boldsymbol{D}_{\mathrm{rig}}(V)))$ *does not depend on* $\mathcal{F}$.

Proof. By [Benois and Berger 2008] and Theorem 3.11, the base changes of both sides in Corollary 3.13 by the continuous L-algebra morphism $f_\delta : \mathcal{R}^{\infty}_L(\Gamma) \to L : [\gamma] \to \delta(\gamma)^{-1}$ are equal to $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\boldsymbol{D}_{\mathrm{rig}}(V(\delta)))$ for any potentially crystalline character $\delta : \Gamma \to L^{\times}$. Since the points corresponding to such characters are Zariski dense in the rigid analytic space associated to $\mathrm{Spf}(\mathcal{O}_L[\![\Gamma]\!])$, we obtain the equality in the corollary. □

4. Rank-one case

Kato [1993b] proved his ε-conjecture using the theory of Coleman homomorphism, which interpolates the exponential maps and the dual exponential maps of rank-one de Rham p-adic representations of $G_{\mathbb{Q}_p}$. In particular, the so-called explicit reciprocity law, which is the explicit formula of its interpolation property, was very important in his proof.

In this final section, we first construct the ε-isomorphism

$$\varepsilon_{A,\zeta}(M) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M)$$

for any rank-one (φ, Γ)-module M by interpreting the theory of Coleman homomorphism in terms of p-adic Fourier transforms (e.g., Amice transforms, Colmez transforms), which seems to be standard for experts of the theory of (φ, Γ)-modules. Then we prove that this isomorphism satisfies the de Rham condition (vi) by establishing the "explicit reciprocity law" of our Coleman homomorphism using our theory of Bloch–Kato exponential maps developed in Section 2B.

4A. ***Construction of the ε-isomorphism.*** We first recall the theory of analytic Iwasawa cohomology of (φ, Γ)-modules over the Robba ring after [Pottharst 2012; Kedlaya et al. 2014]. Let $\Lambda(\Gamma) := \mathbb{Z}_p[\![\Gamma]\!]$ be the Iwasawa algebra of Γ with coefficients in $\mathbb{Z}_p$, and let $\mathfrak{m}$ be the Jacobson radical of $\Lambda(\Gamma)$. For each $n \geq 1$, define a $\mathbb{Q}_p$-affinoid algebra $\mathcal{R}^{[1/p^n,\infty]}(\Gamma) := (\Lambda(\Gamma)[\mathfrak{m}^n/p])^\wedge[1/p]$, where, for any ring R, we denote by $R^\wedge$ the p-adic completion of R. Let $X_n := \mathrm{Max}(\mathcal{R}^{[1/p^n,\infty]}(\Gamma))$ be the associated affinoid. Define $X := \bigcup_{n\geq 1} X_n$, which is a disjoint union of open unit discs. For $n \geq 1$, consider the rank-one (φ, Γ)-module

$$\mathbf{Dfm}_n := \mathcal{R}^{[1/p^n,\infty]}(\Gamma) \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{R}\boldsymbol{e} = \mathcal{R}_{\mathcal{R}^{[1/p^n,\infty]}(\Gamma)}\boldsymbol{e}$$

with

$$\varphi(1 \mathbin{\widehat{\otimes}} \boldsymbol{e}) = 1 \mathbin{\widehat{\otimes}} \boldsymbol{e} \qquad \text{and} \qquad \gamma(1 \mathbin{\widehat{\otimes}} \boldsymbol{e}) = [\gamma]^{-1} \mathbin{\widehat{\otimes}} \boldsymbol{e} \quad \text{for } \gamma \in \Gamma.$$

Put $\mathbf{Dfm} := \varprojlim_n \mathbf{Dfm}_n$; this is a (φ, Γ)-module over the relative Robba ring over X. For M a (φ, Γ)-module over $\mathcal{R}_A$, we define the cyclotomic deformation of M by

$$\mathbf{Dfm}(M) := \varprojlim_n \mathbf{Dfm}_n(M)$$

with

$$\mathbf{Dfm}_n(M) := M \mathbin{\widehat{\otimes}}_{\mathcal{R}} \mathbf{Dfm}_n \xrightarrow{\sim} M \mathbin{\widehat{\otimes}}_A \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\boldsymbol{e},$$

which is a (φ, Γ)-module over the relative Robba ring over $\mathrm{Max}(A) \times X$. This (φ, Γ)-module is the universal cyclotomic deformation of M in the sense that, for each continuous homomorphism $\delta_0 : \Gamma \to A^\times$, we have a natural isomorphism

$$\mathbf{Dfm}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} M(\delta_0) : (x \mathbin{\widehat{\otimes}} \eta\boldsymbol{e}) \otimes a \mapsto f_{\delta_0}(\eta) a x \boldsymbol{e}_{\delta_0}$$

for $x \in M,\ \eta\boldsymbol{e} \in \mathcal{R}_A^\infty(\Gamma)\boldsymbol{e}$ and $a \in A$, where

$$f_{\delta_0} : \mathcal{R}_A^\infty(\Gamma) \to A$$

is the continuous A-algebra homomorphism defined by

$$f_{\delta_0}([\gamma]) := \delta_0(\gamma)^{-1}$$

for $\gamma \in \Gamma$ (and recall that $M(\delta_0) := M \otimes_A A\boldsymbol{e}_{\delta_0} = M\boldsymbol{e}_{\delta_0}$ is defined by $\varphi(x\boldsymbol{e}_{\delta_0}) = \varphi(x)\boldsymbol{e}_{\delta_0}$ and $\gamma(x\boldsymbol{e}_{\delta_0}) := \delta_0(\gamma)\gamma(x)\boldsymbol{e}_{\delta_0}$ for $x \in M$ and $\gamma \in \Gamma$).

By Theorem 4.4.8 of [Kedlaya et al. 2014], we have a natural quasi-isomorphism of $\mathcal{R}_A^\infty(\Gamma)$-modules

$$g_\gamma : C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_\psi(M) := \left[M^\Delta \xrightarrow{\psi-1} M^\Delta\right],$$

where the latter complex is concentrated in degree $[1, 2]$. This quasi-isomorphism is obtained as a composite of (a system of) quasi-isomorphisms

$$C^\bullet_{\psi,\gamma}(\mathbf{Dfm}_n(M)) \xrightarrow{\sim} C^\bullet_\psi(M) \mathbin{\widehat{\otimes}}_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma),$$

which are naturally induced by the following diagrams of $\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)$-modules for $n \geq 1$ with exact rows:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{\gamma-1} & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{f_\gamma} & M \mathbin{\widehat{\otimes}}_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma) & \longrightarrow 0 \\ & {\scriptstyle \psi-1}\downarrow & & {\scriptstyle \psi-1}\downarrow & & {\scriptstyle \psi-1}\downarrow & \\ 0 \longrightarrow & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{\gamma-1} & \mathbf{Dfm}_n(M)^\Delta & \xrightarrow{f_\gamma} & M \mathbin{\widehat{\otimes}}_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma) & \longrightarrow 0 \end{array} \tag{32}$$

Here

$$f_\gamma\left(\sum_i x_i \mathbin{\widehat{\otimes}} \eta_i \boldsymbol{e}\right) := \frac{1}{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))} \sum_i x_i \mathbin{\widehat{\otimes}} \eta_i$$

for $x_i \in M,\ \eta_i\boldsymbol{e} \in \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\boldsymbol{e}$, with the inverse of the natural quasi-isomorphism

$$C^\bullet_\psi(M) \xrightarrow{\sim} \varprojlim_n C^\bullet_\psi\left(M \otimes_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\right) \xrightarrow{\sim} \varprojlim_n C^\bullet_\psi\left(M \mathbin{\widehat{\otimes}}_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\right)$$

(see Theorem 4.4.8 of [Kedlaya et al. 2014] and Theorem 2.8(3) of [Pottharst 2012] for the proof). This quasi-isomorphism is canonical in the sense that, for another $\gamma' \in \Gamma$ whose image in Γ/Δ is a topological generator, we have the commutative diagram

$$\begin{array}{ccc} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) & \xrightarrow{g_\gamma} & C^\bullet_\psi(M) \\ {\scriptstyle \iota_{\gamma,\gamma'}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ C^\bullet_{\psi,\gamma'}(\mathbf{Dfm}(M)) & \xrightarrow{g_{\gamma'}} & C^\bullet_\psi(M) \end{array} \tag{33}$$

For $\delta_0 : \Gamma \to A^\times$, using the natural isomorphism $\mathbf{Dfm}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} M(\delta_0)$ and the quasi-isomorphism g_γ, we obtain the quasi-isomorphism

$$g_{\gamma,\delta_0} : C^\bullet_{\psi,\gamma}(M(\delta_0)) \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A) \\ \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} C^\bullet_{\psi}(M) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A, \quad (34)$$

where the second isomorphism follows from the fact that any (φ, Γ)-module M_0 over $\mathcal{R}_{A_0}$ is flat over A_0 for any A_0 (see Corollary 2.1.7 of [Kedlaya et al. 2014]). This quasi-isomorphism can be written in a more explicit way as follows. To recall this, we see A as an $\mathcal{R}_A^\infty(\Gamma)$-module by the map f_{δ_0}. Then we can take the projective resolution of A

$$0 \to \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{2,\gamma}} A \to 0,$$

where

$$p_{\delta_0} := \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \delta_0^{-1}(\sigma)[\sigma] \in \mathcal{R}_A^\infty(\Gamma)$$

(this is an idempotent) and

$$d_{1,\gamma}(\eta) := (\delta_0(\gamma)[\gamma] - 1)\eta \quad \text{and} \quad d_{2,\gamma}(\eta) := \frac{1}{|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma))} f_{\delta_0}(\eta).$$

This resolution induces a canonical isomorphism

$$C^\bullet_\psi(M) \otimes_{\mathcal{R}_A^\infty(\Gamma)} \big[\mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0}\big] \xrightarrow{\sim} C^\bullet_\psi(M) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A.$$

Moreover, using the isomorphism

$$M \otimes_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{\sim} M(\delta)^\Delta : m \otimes \lambda p_{\delta_0} \mapsto \lambda p_{\delta_0}(m\boldsymbol{e}_{\delta_0}),$$

we obtain a natural isomorphism

$$C^\bullet_\psi(M) \otimes_{\mathcal{R}_A^\infty(\Gamma)} \big[\mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0}\big] \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(M(\delta_0)).$$

Composing both, we obtain a natural quasi-isomorphism

$$C^\bullet_\psi(M) \otimes^{\boldsymbol{L}}_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(M(\delta_0)),$$

which is easily seen to be equal to g_{γ,δ_0}.

Using the theory of analytic Iwasawa cohomology recalled as above, we can describe the fundamental line $\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M))$ as follows. The quasi-isomorphism $g_\gamma : C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_\psi(M)$ and the quasi-isomorphism $C^\bullet_{\varphi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M))$ induce a natural isomorphism in $\mathcal{P}_{\mathcal{R}_A^\infty(\Gamma)}$

$$\Delta_{\mathcal{R}_A^\infty(\Gamma),1}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M))) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)).$$

Moreover, since we have

$$\begin{aligned}\Delta_{\mathcal{R}_A^\infty(\Gamma),2}(\mathbf{Dfm}(M)) &= \varprojlim_n \Delta_{\mathcal{R}_A^{[1/p^n,\infty]}(\Gamma),2}(\mathbf{Dfm}_n(M))\\ &\xrightarrow{\sim} \varprojlim_n\big(\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^{[1/p^n,\infty]}(\Gamma)\boldsymbol{e}^{\otimes r_M}\big)\\ &= \Delta_{A,2}(M)\boldsymbol{e}^{\otimes r_M}\otimes_A \mathcal{R}_A^\infty(\Gamma) \xrightarrow{\sim} \Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma),\end{aligned}$$

where the last isomorphism is just the division by $\boldsymbol{e}^{\otimes r_M}$, we obtain a canonical isomorphism

$$\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)) \boxtimes (\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma)). \tag{35}$$

Under this canonical isomorphism, we will first define an isomorphism

$$\theta_\zeta(M) : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M))^{-1} \xrightarrow{\sim} (\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma)),$$

and then define $\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M))$ as the composite

$$\begin{aligned}\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M)) : \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)} &\xrightarrow{\mathrm{can}} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M))^{-1}\\ &\xrightarrow{\mathrm{id}\boxtimes\theta_\zeta(M)} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(M)) \boxtimes (\Delta_{A,2}(M)\otimes_A \mathcal{R}_A^\infty(\Gamma))\\ &\xrightarrow{\sim} \Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(M))\end{aligned}$$

for the following special rank-one (φ, Γ)-modules M.

For $\lambda \in A^\times$, define the "unramified" continuous homomorphism $\delta_\lambda : \mathbb{Q}_p^\times \to A^\times$ by $\delta_\lambda(p) := \lambda$ and $\delta_\lambda|_{\mathbb{Z}_p^\times} := 1$. We define an isomorphism $\theta_\zeta(M)$ for $M = \mathcal{R}_A(\delta_\lambda)$ by the following steps, which are based on the reinterpretation of the theory of the Coleman homomorphism in terms of the p-adic Fourier transform.

Let $\mathrm{LA}(\mathbb{Z}_p, A)$ be the set of A-valued locally analytic functions on $\mathbb{Z}_p$, and define the action of (φ, ψ, Γ) on it by

$$\varphi(f)|_{\mathbb{Z}_p^\times} := 0, \qquad \varphi(f)(y) := f\left(\frac{y}{p}\right) \quad (y \in p\mathbb{Z}_p),$$
$$\psi(f)(y) := f(py), \qquad \gamma(f)(y) := \frac{1}{\chi(\gamma)} f\left(\frac{y}{\chi(\gamma)}\right) \quad (\gamma \in \Gamma).$$

One has a (φ, ψ, Γ)-equivariant A-linear surjection, which we call the Colmez transform,

$$\mathrm{Col} : \mathcal{R}_A \to \mathrm{LA}(\mathbb{Z}_p, A) \tag{36}$$

defined by

$$\mathrm{Col}(f(\pi))(y) := \mathrm{Res}_0\left((1+\pi)^y f(\pi)\frac{d\pi}{(1+\pi)}\right),$$

where $\mathrm{Res}_0 : \mathcal{R}_A \to A$ is defined by $\mathrm{Res}_0\big(\sum_{n\in\mathbb{Z}} a_n\pi^n\big) := a_{-1}$ (note that Col depends on the choice of the parameter π, i.e., the choice of ζ). By this map, we obtain the short exact sequence

$$0 \to \mathcal{R}_A^\infty \to \mathcal{R}_A \xrightarrow{\mathrm{Col}} \mathrm{LA}(\mathbb{Z}_p, A) \to 0. \tag{37}$$

Twisting the action of (φ, ψ, Γ) by δ_λ, we obtain the (φ, ψ, Γ)-equivariant exact sequence

$$0 \to \mathcal{R}_A^\infty(\delta_\lambda) \to \mathcal{R}_A(\delta_\lambda) \xrightarrow{\mathrm{Col}\otimes \boldsymbol{e}_{\delta_\lambda}} \mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda) \to 0,$$

from which we obtain the exact sequence of complexes of $\mathcal{R}_A^\infty(\Gamma)$-modules

$$0 \to C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)) \to C_\psi^\bullet(\mathcal{R}_A(\delta_\lambda)) \to C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \to 0. \tag{38}$$

For each $k \geq 0$, we define the algebraic function

$$y^k : \mathbb{Z}_p \to A : a \mapsto a^k.$$

Then $Ay^k\boldsymbol{e}_{\delta_\lambda} \subseteq \mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)$ is a ψ-stable sub-$\mathcal{R}_A^\infty(\Gamma)$-module. By Lemme 2.9 of [Chenevier 2013], the natural inclusion

$$C_\psi^\bullet\Bigg(\bigoplus_{0=k}^{N} Ay^k\boldsymbol{e}_{\delta_\lambda}\Bigg) \hookrightarrow C_\psi^\bullet(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \tag{39}$$

is a quasi-isomorphism for sufficiently large N.

Set $P_i^k := Ay^k\boldsymbol{e}_{\delta_\lambda}$ for $i = 1, 2$. Since we have $Ay^k\boldsymbol{e}_{\delta_\lambda}[0] \in \boldsymbol{D}_{\mathrm{perf}}^{[-1,0]}(\mathcal{R}_A^\infty(\Gamma))$ for any $k \geq 0$, the natural exact sequence

$$0 \to P_1^k[-1] \to C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda}) \to P_2^k[-2] \to 0$$

induces a canonical isomorphism

$$g_k : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda})) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(P_2^k) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(P_1^k)^{-1} \xrightarrow{i_{\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(P_1^k)}} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}.$$

We remark that, if the complex $C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda})$ is acyclic, then the composite of this isomorphism with the inverse of the canonical trivialization isomorphism

$$h_{\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda}))} : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_\psi^\bullet(Ay^k\boldsymbol{e}_{\delta_\lambda})) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}$$

is the identity map. Hence, if we define the isomorphism

$$g^N := \boxtimes_{0=k}^N g_k : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}\Bigg(C_\psi^\bullet\Bigg(\bigoplus_{k=0}^{N} Ay^k\boldsymbol{e}_{\delta_\lambda}\Bigg)\Bigg) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}, \tag{40}$$

then, by (39) and (40) (for sufficiently large N), we obtain an isomorphism

$$\iota_0 : \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda))) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)}, \tag{41}$$

which is independent of the choice of (sufficiently large) N.

Since $C^\bullet_\psi(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda))$, $C^\bullet_\psi(\mathcal{R}_A(\delta_\lambda))$ are both perfect complexes, we also have

$$C^\bullet_\psi(\mathcal{R}_A^\infty(\Gamma)) \in \boldsymbol{D}^b_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$$

by the exact sequence (38), and then we obtain an isomorphism

$$\begin{aligned}\iota_1 : {}&\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A(\delta_\lambda)))\\ &\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda))) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathrm{LA}(\mathbb{Z}_p, A)(\delta_\lambda)))\\ &\xrightarrow{\mathrm{id}\boxtimes\iota_0} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda))) \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(\mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1}[0])^{-1},\end{aligned} \tag{42}$$

where the last isomorphism is the one naturally induced by the exact sequence

$$0 \to \mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1} \to \mathcal{R}_A^\infty(\delta_\lambda) \xrightarrow{\psi-1} \mathcal{R}_A^\infty(\delta_\lambda) \to 0$$

(where the surjectivity is proved in Lemme 2.9(v) of [Chenevier 2013]).

We next consider the complex $C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda))$. For a $\mathcal{R}_A^\infty(\Gamma)$-module M with linear actions of φ and ψ, define a complex

$$C^\bullet_{\tilde{\psi}}(M) := \left[M \xrightarrow{\psi} M\right] \in \boldsymbol{D}^{[1,2]}(\mathcal{R}_A^\infty(\Gamma)),$$

and define a map of complexes $\alpha_M : C^\bullet_\psi(M) \to C^\bullet_{\tilde{\psi}}(M)$ by

$$\begin{array}{ccc} C^\bullet_\psi(M) : \big[M & \xrightarrow{\psi-1} & M\big] \\ \Big\downarrow \alpha_M \quad \Big\downarrow 1-\varphi & & \Big\downarrow \mathrm{id}_M \\ C^\bullet_{\tilde{\psi}}(M) : \big[M & \xrightarrow{\psi} & M\big] \end{array} \tag{43}$$

For $N \geq 0$, set $D_N := \bigoplus_{0\leq k\leq N} At^k \boldsymbol{e}_{\delta_\lambda}$. Since $At^k\boldsymbol{e}_{\delta_\lambda}[0] \in \boldsymbol{D}^{[-1,0]}_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$, we can define a canonical isomorphism

$$\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(D_N)) \xrightarrow{\sim} \mathbf{1}_{\mathcal{R}_A^\infty(\Gamma)} \tag{44}$$

in the same way as the isomorphism (40). Then the natural exact sequence $0 \to C^\bullet_\psi(D_N) \to C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)) \to C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)/D_N) \to 0$ induces a canonical isomorphism

$$\begin{aligned}&\mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)))\\ &\quad\xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(D_N)) \boxtimes \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)/D_N))\\ &\qquad\qquad \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)/D_N)),\end{aligned} \tag{45}$$

where the last isomorphism is induced by the isomorphism (44).

Since the map $1-\varphi:\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N\to\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N$ is an isomorphism for sufficiently large N by Lemme 2.9(ii) of [Chenevier 2013], the map $\alpha_{(\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N)}$ is also an isomorphism for sufficiently large N. Hence, for sufficiently large N, we obtain a canonical isomorphism

$$\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\psi}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N))\xrightarrow{\sim}\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N)).\tag{46}$$

Since the complex $C_{\tilde{\psi}}^{\bullet}(D_N)$ is acyclic (since $\psi:At^k\boldsymbol{e}_{\delta_\lambda}\to At^k\boldsymbol{e}_{\delta_\lambda}$ is an isomorphism for any $k\geq 0$), the natural exact sequence $0\to C_{\tilde{\psi}}^{\bullet}(D_N)\to C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda))\to C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N)\to 0$ induces a canonical isomorphism

$$\begin{aligned}&\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N))\\&\quad\xrightarrow{\sim}\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(D_N))\boxtimes\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda)/D_N))\\&\qquad\xrightarrow{\sim}\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda))),\end{aligned}\tag{47}$$

where the first isomorphism is induced by the inverse of the isomorphism

$$h_{C_{\tilde{\psi}}^{\bullet}(D_N)}:\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(D_N))\xrightarrow{\sim}\mathbf{1}_{\mathcal{R}_A^{\infty}(\Gamma)}.$$

Moreover, the exact sequence $0\to\mathcal{R}_A^{\infty}(\delta_\lambda)^{\psi=0}\to\mathcal{R}_A^{\infty}(\delta_\lambda)\xrightarrow{\psi}\mathcal{R}_A^{\infty}(\delta_\lambda)\to 0$ and the isomorphism

$$\mathcal{R}_A^{\infty}(\Gamma)\boldsymbol{e}_{\delta_\lambda}\xrightarrow{\sim}\mathcal{R}_A^{\infty}(\delta_\lambda)^{\psi=0}:\lambda\boldsymbol{e}_{\delta_\lambda}\mapsto(\lambda\cdot(1+\pi)^{-1})\boldsymbol{e}_{\delta_\lambda}\tag{48}$$

(note that this isomorphism depends on the choice of ζ) naturally induces the isomorphism

$$\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\tilde{\psi}}^{\bullet}(\mathcal{R}_A^{\infty}(\delta_\lambda)))^{-1}\xrightarrow{\sim}\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(\mathcal{R}_A^{\infty}(\delta_\lambda)^{\psi=0})\xrightarrow{\sim}(\mathcal{R}_A^{\infty}(\Gamma)\boldsymbol{e}_{\delta_\lambda},1).\tag{49}$$

Finally, as the composites of the inverses of the isomorphisms (42), (45), (46), (47), and the isomorphism (49), we define the desired isomorphism

$$\begin{aligned}&\theta_\zeta(\mathcal{R}_A(\delta_\lambda)):\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\psi}^{\bullet}(\mathcal{R}_A(\delta_\lambda)))^{-1}\\&\qquad\xrightarrow{\sim}(\mathcal{R}_A^{\infty}(\Gamma)\boldsymbol{e}_{\delta_\lambda},1)=\Delta_{A,2}(\mathcal{R}_A(\delta_\lambda))\otimes_A\mathcal{R}_A^{\infty}(\Gamma).\end{aligned}$$

Definition 4.1. Using the isomorphism (35), for $M=\mathcal{R}_A(\delta_\lambda)$, we define the ε-isomorphism by

$$\begin{aligned}\varepsilon_{\mathcal{R}_A^{\infty}(\Gamma),\zeta}(\mathbf{Dfm}(M)):\mathbf{1}_{\mathcal{R}_A^{\infty}(\Gamma)}&\xrightarrow{\mathrm{can}}\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\psi}^{\bullet}(M))\boxtimes\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\psi}^{\bullet}(M))^{-1}\\&\xrightarrow{\mathrm{id}\boxtimes\theta_\zeta(M)}\mathrm{Det}_{\mathcal{R}_A^{\infty}(\Gamma)}(C_{\psi}^{\bullet}(M))\boxtimes(\Delta_{A,2}(M)\otimes_A\mathcal{R}_A^{\infty}(\Gamma))\\&\xrightarrow{\sim}\Delta_{\mathcal{R}_A^{\infty}(\Gamma)}(\mathbf{Dfm}(M)).\end{aligned}$$

Before defining the ε-isomorphism for the general rank-one case, we check that the isomorphism $\varepsilon_{\mathcal{R}_A^{\infty}(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda)))$ defined above satisfies the properties (i) and (iii) in Conjecture 3.8

For the property (i), it is clear that, for each continuous homomorphism $f: A \to A'$ (and set $\lambda' = f(\lambda)$), we have

$$\varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes \mathrm{id}_{A'} = \varepsilon_{\mathcal{R}_{A'}^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_{A'}(\delta_{\lambda'})))$$

under the canonical isomorphism

$$\begin{aligned}\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes_A A' &\xrightarrow{\sim} \Delta_{\mathcal{R}_{A'}^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda) \widehat{\otimes}_A A')) \\ &\xrightarrow{\sim} \Delta_{\mathcal{R}_{A'}^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_{A'}(\delta_{\lambda'}))),\end{aligned}$$

where the last isomorphism is induced by the isomorphism

$$\mathcal{R}_A(\delta_\lambda) \widehat{\otimes}_A A' \xrightarrow{\sim} \mathcal{R}_{A'}(\delta_{\lambda'}) : g(\pi)\boldsymbol{e}_{\delta_\lambda} \widehat{\otimes} a \mapsto ag^f(\pi)\boldsymbol{e}_{\delta_{\lambda'}};$$

here we define

$$g^f(\pi) := \sum_{n\in\mathbb{Z}} f(a_n)\pi^n \in \mathcal{R}_{A'} \quad \text{for } g(\pi) = \sum_{n\in\mathbb{Z}} a_n\pi^n \in \mathcal{R}_A.$$

The property (iii) easily follows from (48) since one has $(1+\pi_{\zeta^a}) = (1+\pi_\zeta)^a = [\sigma_a]\cdot(1+\pi_\zeta)$ for $a \in \mathbb{Z}_p^\times$.

Next, we consider a rank-one (φ,Γ)-module over $\mathcal{R}_A$ of the form $\mathcal{R}_A(\delta)$ for a general continuous homomorphism $\delta : \mathbb{Q}_p^\times \to A^\times$. Set

$$\lambda := \delta(p) \quad \text{and} \quad \delta_0 := \delta|_{\mathbb{Z}_p^\times},$$

which we freely see as a homomorphism $\delta_0 : \Gamma \to A^\times$ by identifying $\chi : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$. We define the continuous A-algebra homomorphism

$$f_{\delta_0} : \mathcal{R}_A^\infty(\Gamma) \to A,$$

which is uniquely characterized by $f_{\delta_0}([\gamma]) = \delta_0(\gamma)^{-1}$ for any $\gamma \in \Gamma$. Then we have a canonical isomorphism

$$\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda)) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} \mathcal{R}_A(\delta)$$

defined by

$$(f(\pi)\boldsymbol{e}_{\delta_\lambda} \widehat{\otimes} \eta\boldsymbol{e}) \otimes a := af_{\delta_0}(\eta)f(\pi)\boldsymbol{e}_\delta$$

for $f(\pi) \in \mathcal{R}_A$, $\eta \in \mathcal{R}_A^\infty(\Gamma)$, $a \in A$, which also induces a canonical isomorphism

$$\Delta_{\mathcal{R}_A^\infty(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0}} A \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta)).$$

Definition 4.2. We define the isomorphism

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta))$$

by

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) := \varepsilon_{\mathcal{R}_A^\infty(\Gamma),\zeta}(\mathbf{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes \mathrm{id}_A$$

under the above isomorphism.

Next, we consider a rank-one (φ, Γ)-module of the form $\mathcal{R}_A(\delta) \otimes_A \mathcal{L}$ for an invertible A-module $\mathcal{L}$.

Lemma 4.3. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$ (of any rank), and let $\mathcal{L}$ be an invertible A-module. Then there exist a canonical A-linear isomorphism*

$$\Delta_A(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_A(M).$$

Proof. The natural isomorphism $C^{\bullet}_{\varphi,\gamma}(M \otimes_A \mathcal{L}) \xrightarrow{\sim} C^{\bullet}_{\varphi,\gamma}(M) \otimes_A \mathcal{L}$ induces an isomorphism

$$\Delta_{A,1}(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_{A,1}(M) \boxtimes (\mathcal{L}^{\otimes -r_M}, 0).$$

Since we also have a natural isomorphism $\mathcal{L}_A(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \mathcal{L}_A(M) \otimes_A \mathcal{L}^{\otimes r_M}$, we obtain a natural isomorphism

$$\Delta_{A,2}(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_{A,2}(M) \boxtimes (\mathcal{L}^{\otimes r_M}, 0).$$

Then the isomorphism in the lemma is obtained by taking the products of these isomorphisms with the canonical isomorphism $i_{(\mathcal{L}^{\otimes r_M},0)} : (\mathcal{L}^{\otimes r_M}, 0) \boxtimes (\mathcal{L}^{\otimes -r_M}, 0) \xrightarrow{\sim} \mathbf{1}_A$. □

Definition 4.4. We define the isomorphism

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta) \otimes_A \mathcal{L})$$

by

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) := \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$$

under the above isomorphism $\Delta_A(M \otimes_A \mathcal{L}) \xrightarrow{\sim} \Delta_A(M)$.

Finally, let M be a general rank-one (φ, Γ)-module over $\mathcal{R}_A$. By Theorem 2.8, there exists a unique pair $(\delta, \mathcal{L})$ such that $g : M \xrightarrow{\sim} \mathcal{R}(\delta) \otimes_A \mathcal{L}$. This isomorphism induces an isomorphism $g_* : \Delta_A(M) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta) \otimes_A \mathcal{L})$.

Definition 4.5. Under the above situation, we define

$$\varepsilon_{A,\zeta}(M) := \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) \circ g_* : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M).$$

Lemma 4.6. *The isomorphism $\varepsilon_{A,\zeta}(M)$ is well defined, i.e., does not depend on g.*

Proof. Since we have $\mathrm{Aut}(\mathcal{R}_A(\delta) \otimes_A \mathcal{L}) = A^{\times}$ (where $\mathrm{Aut}(M)$ is the group of automorphisms of M as (φ, Γ)-modules over $\mathcal{R}_A$), it suffices to show the following lemma. □

Lemma 4.7. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$. For $a \in A^{\times}$, let us define $g_a : M \xrightarrow{\sim} M : x \mapsto ax$. Then we have*

$$(g_a)_* = \mathrm{id}_{\Delta_A(M)}.$$

Proof. This lemma immediately follows from the fact that g_a induces $\Delta_{1,A}(M) \xrightarrow{\sim} \Delta_{A,1}(M) : x \mapsto a^{-r_M}x$ (by the Euler–Poincaré formula) and $\Delta_{A,2}(M) \xrightarrow{\sim} \Delta_{A,2}(M) : x \mapsto a^{r_M}x$ by definition. □

Remark 4.8. By definition, it is clear that $\varepsilon_{A,\zeta}(M)$, constructed above, satisfies the conditions (i) and (iii) in Conjecture 3.8. It also seems to be easy to directly prove the conditions (iv), (v) of Conjecture 3.8. However, in the next subsection, we prove the conditions (iv) and (v) using density arguments in the process of verifying the condition (vi).

Remark 4.9. Define $\mathcal{O}_{\mathcal{E}} := \left\{\sum_{n\in\mathbb{Z}} a_n\pi^n \mid a_n \in \mathbb{Z}_p,\ a_{-n} \to 0\ (n \to +\infty)\right\}$, $\mathcal{O}_{\mathcal{E}^+} := \mathbb{Z}_p[\![\pi]\!]$, and $\mathcal{O}_{\mathcal{E}^+,\Lambda} := \mathcal{O}_{\mathcal{E}^+} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda$. Define $\mathcal{C}^0(\mathbb{Z}_p, \Lambda)$ to be the Λ-modules of Λ-valued continuous functions on $\mathbb{Z}_p$. Using the exact sequence

$$0 \to \mathcal{O}_{\mathcal{E}^+,\Lambda} \to \mathcal{O}_{\mathcal{E},\Lambda} \xrightarrow{\mathrm{Col}} \mathcal{C}^0(\mathbb{Z}_p, \Lambda) \to 0,$$

which is the continuous function analogue of the exact sequence (37), and using the equivalence between the category of Λ-representations of $G_{\mathbb{Q}_p}$ with that of étale (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E},\Lambda}$ [Dee 2001], it seems possible to define an ε-isomorphism $\varepsilon_{\Lambda,\zeta}(\Lambda(\tilde{\delta}))$ for any $\tilde{\delta} : G^{\mathrm{ab}}_{\mathbb{Q}_p} \to \Lambda^\times$ in the same way as the definition of $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$. Using this ε-isomorphism, it is clear that our ε-isomorphism $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$ satisfies the condition (v) in Conjecture 3.8. Moreover, it is easy to compare the isomorphism $\varepsilon_{\Lambda,\zeta}(\Lambda(\tilde{\delta}))$ with the one Kato defined [1993b].

4B. ***Verification of the conditions* (iv), (v), (vi).** In this final subsection, we prove that our ε-isomorphism $\varepsilon_{A,\zeta}(M)$, constructed in the previous subsection, satisfies the conditions (iv), (v), (vi) of Conjecture 3.8. Of course, the essential part is to prove the condition (vi); the other conditions follow from it using density arguments.

Therefore, in this subsection, we mainly concentrate on the case where $A = L$ is a finite extension of $\mathbb{Q}_p$. Before verifying the condition (vi), we describe the isomorphism $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(\mathcal{R}_L(\delta))$ for any continuous homomorphism $\delta = \delta_\lambda\delta_0 : \mathbb{Q}_p^\times \to L^\times$ in a more explicit way.

For an $\mathcal{R}_L^\infty(\Gamma)$-module N, define a Γ-module $N(\delta_0) := N\boldsymbol{e}_{\delta_0}$ by $\gamma(x\boldsymbol{e}_{\delta_0}) = \delta_0(\gamma)([\gamma]\cdot x)\boldsymbol{e}_{\delta_0}$ for any $\gamma \in \Gamma$. Then we have a natural quasi-isomorphism

$$N[-1] \otimes^{\boldsymbol{L}}_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} N \otimes_{\mathcal{R}_L^\infty(\Gamma)} \left[\mathcal{R}_L^\infty(\Gamma)p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_L^\infty(\Gamma)p_{\delta_0}\right] \xrightarrow{\sim} C^\bullet_\gamma(N(\delta_0)).$$

Hence, if $N[0] \in \boldsymbol{D}^b_{\mathrm{perf}}(\mathcal{R}_L^\infty(\Gamma))$, then we obtain a natural isomorphism

$$\mathrm{Det}_L(N[-1]) \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} \mathrm{Det}_L(C^\bullet_\gamma(N(\delta_0))) \xrightarrow{\sim} \boxtimes_{i=0,1} \mathrm{Det}_L(\mathrm{H}^i_\gamma(N(\delta_0)))^{(-1)^i}.$$

Moreover, if N is also equipped with a commuting linear action of ψ such that $\boldsymbol{C}^\bullet_\psi(M) \in \boldsymbol{D}^b_{\mathrm{perf}}(\mathcal{R}^\infty_L(\Gamma))$, then we obtain a natural isomorphism

$$\mathrm{Det}_L(C^\bullet_\psi(N)) \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} \mathrm{Det}_L(C^\bullet_{\psi,\gamma}(N(\delta_0))) \xrightarrow{\sim} \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(N(\delta_0)))^{(-1)^i}.$$

In particular, the isomorphism $\bar{\theta}_\zeta(\mathcal{R}_L(\delta)) := \theta_\zeta(\mathcal{R}_L(\delta_\lambda)) \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} \mathrm{id}_L$ can be seen as the isomorphism

$$\bar{\theta}_\zeta(\mathcal{R}_L(\delta)) : \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(\delta)))^{(-1)^{i+1}} \xrightarrow{\sim} (\mathcal{R}^\infty_L(\Gamma)\boldsymbol{e}_{\delta_\lambda}, 1) \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} (L\boldsymbol{e}_\delta, 1), \quad (50)$$

where the last isomorphism is induced by the isomorphism

$$\mathcal{R}^\infty_L(\Gamma)\boldsymbol{e}_{\delta_\lambda} \otimes_{\mathcal{R}^\infty_L(\Gamma), f_{\delta_0}} L \xrightarrow{\sim} L\boldsymbol{e}_\delta : (\eta\boldsymbol{e}_{\delta_\lambda}) \otimes a \mapsto a f_{\delta_0}(\eta)\boldsymbol{e}_\delta.$$

Therefore, to verify the condition (vi) when $\mathcal{R}_L(\delta)$ is de Rham, we need to relate the map $\bar{\theta}_\zeta(\mathcal{R}_L(\delta))$ with the Bloch–Kato exponential map or the dual exponential map.

To do so, we divide into the following three cases:

(1) $\delta \neq x^{-k}, x^{k+1}|x|$ for any $k \in \mathbb{Z}_{\geq 0}$ (which we call the generic case).

(2) $\delta = x^{-k}$ for $k \geq 0$.

(3) $\delta = x^{k+1}|x|$ for $k \geq 0$.

We will first verify the condition (vi) in the generic case by establishing a kind of explicit reciprocity law (see Propositions 4.11 and 4.16). Then we will verify the conditions (iv) and (v) using the generic case by density argument. Finally, we will prove the condition (vi) in the case (2) via direct calculations, and reduce the case (3) to the case (2) using the duality condition (iv).

In the remaining parts, we freely use the results of Colmez and Chenevier concerning the calculations of cohomologies

$$\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(\delta)), \quad \mathrm{H}^i_{\psi,\gamma}(\mathcal{R}^\infty_L(\delta)) \quad \text{and} \quad \mathrm{H}^i_{\psi,\gamma}(\mathrm{LA}(\mathbb{Z}_p, L)(\delta));$$

see Proposition 2.1 and Théorème 2.9 of [Colmez 2008] and Lemme 2.9 and Corollaire 2.11 of [Chenevier 2013].

4B1. *Verification of the condition* (vi) *in the generic case.* In this subsection, we assume that δ is generic. Then we have

$$\mathrm{H}^i_{\psi,\gamma}(Lt^k\boldsymbol{e}_\delta) = \mathrm{H}^i_{\psi,\gamma}(Ly^k\boldsymbol{e}_\delta) = \mathrm{H}^i_{\psi,\gamma}(\mathrm{LA}(\mathbb{Z}_p, L)(\delta)) = 0$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, 1, 2\}$, and

$$\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(\delta)) = \mathrm{H}^i_{\psi,\gamma}(\mathcal{R}^\infty_L(\delta)) = 0$$

for $i = 0, 2$, and

$$\dim_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) = \dim_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L^{\infty}(\delta)) = 1.$$

Then $\iota_{1,\delta} := \iota_1 \otimes_{\mathcal{R}_L^{\infty}(\Gamma), f_{\delta_0}} \mathrm{id}_L$ (see (42)) is the isomorphism

$$(\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)), 1)^{-1} \xrightarrow{\sim} (\mathrm{H}^1_{\gamma}(\mathcal{R}_L^{\infty}(\delta)^{\psi=1}), 1)^{-1} \tag{51}$$

in $\mathcal{P}_L$ induced by the isomorphism

$$\mathrm{H}^1_{\gamma}(\mathcal{R}_L^{\infty}(\delta)^{\psi=1}) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) : [x] \mapsto [x, 0].$$

Then the base change by f_{δ_0} of the isomorphism

$$\mathrm{Det}_{\mathcal{R}_L^{\infty}(\Gamma)}(C^{\bullet}_{\psi}(\mathcal{R}_L^{\infty}(\delta_\lambda)))^{-1} \xrightarrow{\sim} \mathrm{Det}_{\mathcal{R}_L^{\infty}(\Gamma)}(\mathcal{R}_L^{\infty}(\delta_\lambda)^{\psi=0}[0]) \xrightarrow{\sim} (\mathcal{R}_L^{\infty}(\Gamma)\boldsymbol{e}_{\delta_\lambda}, 1),$$

which is induced by (45), (46), (47) and (49), becomes the isomorphism

$$(\mathrm{H}^1_{\gamma}(\mathcal{R}_L^{\infty}(\delta)^{\psi=1}), 1) \xrightarrow{[x]\mapsto[(1-\varphi)x]} (\mathrm{H}^1_{\gamma}(\mathcal{R}_L^{\infty}(\delta)^{\psi=0}), 1) \xrightarrow{\sim} (L\boldsymbol{e}_{\delta}, 1), \tag{52}$$

where the last isomorphism is explicitly defined as follows. For an explicit definition of this isomorphism, it is useful to use the Amice transform. Let $D(\mathbb{Z}_p, L) := \mathrm{Hom}_L^{\mathrm{cont}}(\mathrm{LA}(\mathbb{Z}_p, L), L)$ be the algebra of L-valued distributions on $\mathbb{Z}_p$, where the multiplication is defined by the convolution. By the theorem of Amice, we have an isomorphism of topological L-algebras

$$D(\mathbb{Z}_p, L) \xrightarrow{\sim} \mathcal{R}_L^{\infty} : \mu \mapsto f_\mu(\pi) := \sum_{n\geq 0} \mu\left(\binom{y}{n}\right)\pi^n$$

(which depends on the choice of π, i.e., the choice of ζ), where

$$\binom{y}{n} := \frac{y(y-1)\cdots(y-n+1)}{n!}.$$

Then the action of (φ, Γ, ψ) on $\mathcal{R}_L^{\infty}$ induces the action on $D(\mathbb{Z}_p, L)$ by

$$\int_{\mathbb{Z}_p} f(y)\varphi(\mu)(y) := \int_{\mathbb{Z}_p} f(py)\mu(y), \quad \int_{\mathbb{Z}_p} f(y)\psi(\mu)(y) := \int_{p\mathbb{Z}_p} f\left(\frac{y}{p}\right)\mu(y)$$

and

$$\int_{\mathbb{Z}_p} f(y)\sigma_a(\mu)(y) := \int_{\mathbb{Z}_p} f(ay)\mu(y),$$

where, for $a \in \mathbb{Z}_p^{\times}$, we define $\sigma_a \in \Gamma$ such that $\chi(\sigma_a) = a$.

Using this notion, it is easy to see that the second isomorphism in (52) is defined by

$$\mathrm{H}^1_{\gamma}(\mathcal{R}_L^{\infty}(\delta)^{\psi=0}) \xrightarrow{\sim} L\boldsymbol{e}_{\delta} : [f_\mu \boldsymbol{e}_\delta] \mapsto \frac{\delta(-1)}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))} \cdot \int_{\mathbb{Z}_p^{\times}} \delta^{-1}(y)\mu(y)\boldsymbol{e}_{\delta},$$

where we note that we have an isomorphism

$$D(\mathbb{Z}_p^\times, L)\boldsymbol{e}_\delta \xrightarrow{\sim} \mathcal{R}_L^\infty(\delta)^{\psi=0} : \mu\boldsymbol{e}_\delta \mapsto f_\mu\boldsymbol{e}_\delta,$$

since one has

$$f_{\delta_0}(\lambda) = \int_{\mathbb{Z}_p^\times} \delta_0^{-1}(y)\mu_\gamma(y)$$

for any $\lambda \in \mathcal{R}_L^\infty(\Gamma)$ and any continuous homomorphism $\delta_0 : \mathbb{Z}_p^\times \to L^\times$, where we define $\mu_\gamma \in D(\mathbb{Z}_p^\times, L)$ by $f_{\mu_\gamma}(\pi) = \lambda \cdot (1+\pi)$.

For a Γ-module N, we define $\mathrm{H}^1(\Gamma, N) := N/N_0$, where N_0 is the submodule generated by the set $\{(\gamma - 1)n \mid \gamma \in \Gamma,\ n \in N\}$. Then we have the canonical isomorphism

$$\mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} \mathrm{H}^1_\gamma(\mathcal{R}_L^\infty(\delta)^{\psi=1}) : [f\boldsymbol{e}_\delta] \mapsto [|\Gamma_{\mathrm{tor}}| \log_0(\chi(\gamma)) p_\Delta(f\boldsymbol{e}_\delta)]$$

(where "canonical" means that this is independent of γ, i.e., is compatible with the isomorphisms $\iota_{\gamma,\gamma'}$ for any $\gamma' \in \Gamma$). Composing this with the isomorphism (52), we obtain an isomorphism

$$(\mathrm{H}^1(\Gamma, \mathcal{R}_L(\delta)^{\psi=1}), 1) \xrightarrow{\sim} (L\boldsymbol{e}_\delta, 1) \tag{53}$$

in $\mathcal{P}_L$. Concerning the explicit description of this isomorphism, we obtain the following lemma.

Lemma 4.10. *The isomorphism (53) is induced by the isomorphism*

$$\iota_\delta : \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} L\boldsymbol{e}_\delta : [f_\mu\boldsymbol{e}_\delta] \mapsto \delta(-1) \cdot \int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y).$$

Proof. For $f_\mu\boldsymbol{e}_\delta \in \mathcal{R}_L^\infty(\delta)^{\psi=1}$, we have $(1-\varphi)(f_\mu\boldsymbol{e}_\delta) = ((1-\varphi\psi)f_\mu)\cdot\boldsymbol{e}_\delta$. Then the lemma follows from the formula

$$\int_{\mathbb{Z}_p} f(x)(1-\varphi\psi)\mu(x) = \int_{\mathbb{Z}_p^\times} f(x)\mu(x) \quad \text{for } \mu \in D(\mathbb{Z}_p, L). \qquad \square$$

Next, we furthermore assume that $\mathcal{R}_L(\delta)$ is de Rham. By the classification, it is equivalent to $\delta = \tilde{\delta}x^k$ for $k \in \mathbb{Z}$ and a locally constant homomorphism $\tilde{\delta} : \mathbb{Q}_p^\times \to L^\times$. In the generic case, we have the following isomorphisms of one-dimensional L-vector spaces:

(1) $\exp^*_{\mathcal{R}_L(\delta)^*} : \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))$ if $k \le 0$.

(2) $\exp_{\mathcal{R}_L(\delta)} : \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))$ if $k \ge 1$.

Let us define $n(\delta) \in \mathbb{Z}_{\ge 0}$ as the minimal integer such that $\tilde{\delta}|_{(1+p^n\mathbb{Z}_p)\cap\mathbb{Z}_p^\times}$ is trivial. Then:

(1) $n(\delta) = 0$ if and only if $\mathcal{R}_L(\delta)$ is crystalline.

(2) $\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)=1$ if $n(\delta)=0$.

(3) $\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)=\tilde{\delta}(p)^{n(\delta)}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\tilde{\delta}(i)^{-1}\zeta^i_{p^{n(\delta)}}$ if $n(\delta)\geq 1$.

(4) $\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)\cdot\varepsilon_L(W(\mathcal{R}_L(\delta)^*),\zeta)=\tilde{\delta}(-1)$.

By definition of $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta))$ and $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$, and by Lemma 4.10, to verify the condition (vi), it suffices to show the following two propositions (Proposition 4.11 for $k\leq 0$ and Proposition 4.16 for $k\geq 1$), which can be seen as a kind of explicit reciprocity law.

Proposition 4.11. *If $k\leq 0$, then the map*

$$\mathrm{H}^1(\Gamma,\mathcal{R}_L^\infty(\delta)^{\psi=1})\xrightarrow{\sim}\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))\xrightarrow{\exp^*_{\mathcal{R}_L(\delta)^*}}\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))=\left(\frac{1}{t^k}L_\infty\boldsymbol{e}_\delta\right)^\Gamma$$

(*where the first isomorphism is defined by* $[f\boldsymbol{e}_\delta]\mapsto[|\Gamma_{\mathrm{for}}|\log_0(\chi(\gamma))p_\Delta(f\boldsymbol{e}_\delta),0]$) *sends each element* $[f_\mu\boldsymbol{e}_\delta]\in\mathrm{H}^1(\Gamma,\mathcal{R}_L^\infty(\delta)^{\psi=1})$ *to*

(1) $\dfrac{(-1)^k}{(-k)!}\cdot\dfrac{\delta(-1)}{\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)}\cdot\dfrac{1}{t^k}\cdot\displaystyle\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *if* $n(\delta)\neq 0$,

(2) $\dfrac{(-1)^k}{(-k)!}\cdot\dfrac{\det_L(1-\varphi\mid\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)^*))}{\det_L(1-\varphi\mid\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)))}\cdot\dfrac{\delta(-1)}{t^k}\cdot\displaystyle\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *if* $n(\delta)=0$.

Proof. Here, we prove the proposition only when $k=0$, i.e., $\delta=\tilde{\delta}$ is locally constant. We will prove it for general $k\leq 0$ after some preparations on the differential operator ∂ (the proof for general k will be given after Remark 4.15).

Hence, we assume that $k=0$. For such δ, we define a map

$$g_{\mathcal{R}_L(\delta)}:\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))\to\mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}_L(\delta))):x\mapsto[\log(\chi(\gamma))x],$$

which is easily seen to be an isomorphism. By Proposition 2.16 of [Nakamura 2014a], one has the commutative diagram

$$\begin{array}{ccc}\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \xrightarrow{\exp^*_{\mathcal{R}_L(\delta)^*}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))\\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle g_{\mathcal{R}_L(\delta)}}\\ \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \xrightarrow{\mathrm{can}} & \mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}(\delta)))\end{array}\tag{54}$$

Set $n_0:=\max\{n(\delta),1\}$ if $p\neq 2$, and set $n_0:=\max\{n(\delta),2\}$ if $p=2$. Then the image of $[f_\mu\boldsymbol{e}_\delta]\in\mathrm{H}^1(\Gamma,\mathcal{R}_L^\infty(\delta)^{\psi=1})\xrightarrow{\sim}\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))$ by the canonical map $\mathrm{can}:\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta))\to\mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}(\delta)))$ is equal to

$$[|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_\Delta(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta))]\in\mathrm{H}^1_\gamma(\boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}(\delta))).$$

Hence, it suffices to calculate $g^{-1}_{\mathcal{R}_L(\delta)}([|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_\Delta(\iota_{n_0}(f_\mu \boldsymbol{e}_\delta))])$. By definition of $g_{\mathcal{R}_L(\delta)}$, it is easy to check that we have

$$g^{-1}_{\mathcal{R}_L(\delta)}([|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_\Delta(\iota_{n_0}(f_\mu \boldsymbol{e}_\delta))]) = \frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{1}{[\mathbb{Q}_p(\zeta_{p^{n_0}}):\mathbb{Q}_p]}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\sigma_i(\iota_{n_0}(f_\mu \boldsymbol{e}_\delta)|_{t=0}) =: (*).$$

Concerning the right-hand side, when $n(\delta)\geq 1$ if $p\neq 2$, or $n(\delta)\geq 2$ if $p=2$, one has the following equalities, from which the equality (1) follows in this case:

$$\begin{aligned}
(*) &= \frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{1}{[\mathbb{Q}_p(\zeta_{p^{n(\delta)}}):\mathbb{Q}_p]}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\sigma_i(\iota_{n(\delta)}(f_\mu \boldsymbol{e}_\delta)|_{t=0})\\
&= \frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{p}{(p-1)}\frac{1}{p^{n(\delta)}}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\sigma_i\left(\frac{1}{\delta(p)^{n(\delta)}}\int_{\mathbb{Z}_p}\zeta^y_{p^{n(\delta)}}\mu(y)\boldsymbol{e}_\delta\right)\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\int_{\mathbb{Z}_p}\zeta^{iy}_{p^{n(\delta)}}\mu(y)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\left(\sum_{j\in\mathbb{Z}/p^{n(\delta)}\mathbb{Z}}\zeta^{ij}_{p^{n(\delta)}}\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\right)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{j\in\mathbb{Z}/p^{n(\delta)}\mathbb{Z}}\left(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta^{ij}_{p^{n(\delta)}}\right)\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\sum_{j\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\left(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta^{ij}_{p^{n(\delta)}}\right)\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta\\
&= \frac{1}{(p\delta(p))^{n(\delta)}}\left(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta^{i}_{p^{n(\delta)}}\right)\sum_{j\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(j)^{-1}\int_{j+p^{n(\delta)}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta\\
&= \varepsilon_L(W(\mathcal{R}_L(\delta)^*),\zeta)\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta\\
&= \frac{\delta(-1)}{\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta)}\int_{\mathbb{Z}_p^\times}\delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta.
\end{aligned}$$

Here the second equality follows from

$$\iota_{n(\delta)}(f_\mu)|_{t=0} = f_\mu(\zeta_{p^{n(\delta)}}-1) = \int_{\mathbb{Z}_p}\zeta^y_{p^{n(\delta)}}\mu(y),$$

the third equality follows from

$$\frac{|\Gamma_{\mathrm{for}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{p}{p-1} = 1$$

(for any p), the sixth equality follows from the fact that

$$\left(\sum_{i\in(\mathbb{Z}/p^{n(\delta)}\mathbb{Z})^\times}\delta(i)\zeta_{p^{n(\delta)}}^{ij}\right)=0$$

if $p\mid j$, and the seventh and eighth follow from the property (4) of ε-constants listed before this proposition.

When $n(\delta)=0$, one has $n_0=1$ if $p\neq 2$ and $n_0=2$ if $p=2$. Then one has the following equalities:

$$\begin{aligned}(*)&=\frac{1}{p^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\sigma_i(\iota_{n_0}(f_\mu\boldsymbol{e}_\delta)|_{t=0})\\ &=\frac{1}{p^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\sigma_i\left(\frac{1}{\delta(p)^{n_0}}\int_{\mathbb{Z}_p}\zeta_{p^{n_0}}^{y}\mu(y)\boldsymbol{e}_\delta\right)\\ &=\frac{1}{(p\delta(p))^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\int_{\mathbb{Z}_p}\zeta_{p^{n_0}}^{iy}\mu(y)\boldsymbol{e}_\delta\\ &=\frac{1}{(p\delta(p))^{n_0}}\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\left(\sum_{j\in\mathbb{Z}/p^{n_0}\mathbb{Z}}\zeta_{p^{n_0}}^{ij}\int_{j+p^{n_0}\mathbb{Z}_p}\mu(y)\right)\boldsymbol{e}_\delta\\ &=\frac{1}{(p\delta(p))^{n_0}}\sum_{j\in\mathbb{Z}/p^{n_0}\mathbb{Z}}\left(\sum_{i\in(\mathbb{Z}/p^{n_0}\mathbb{Z})^\times}\zeta_{p^{n_0}}^{ij}\right)\int_{j+p^{n_0}\mathbb{Z}_p}\mu(y)\boldsymbol{e}_\delta.\end{aligned}$$

Here the first equality follows from

$$\frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{1}{[\mathbb{Q}_p(\zeta_{p^{n_0}}):\mathbb{Q}_p]}=\frac{1}{p^{n_0}}$$

for any p.

When $p\neq 2$, the last term is equal to

$$\frac{1}{p\delta(p)}\left((p-1)\int_{p\mathbb{Z}_p}\mu(y)-\int_{\mathbb{Z}_p^\times}\mu(y)\right)\boldsymbol{e}_\delta$$

since $\sum_{i\in(\mathbb{Z}/p\mathbb{Z})^\times}\zeta_p^{ij}=p-1$ if $p\mid j$ and $\sum_{i\in(\mathbb{Z}/p\mathbb{Z})^\times}\zeta_p^{ij}=-1$ if $p\nmid j$.

Since $f_\mu\boldsymbol{e}_\delta\in\mathcal{R}^\infty(\delta)^{\psi=1}$, we have $\psi(f_\mu)=\delta(p)f_\mu$, hence we have

$$\int_{p\mathbb{Z}_p}\mu(y)=\int_{\mathbb{Z}_p}\psi(\mu)(y)=\delta(p)\int_{\mathbb{Z}_p}\mu(y)=\delta(p)\left(\int_{\mathbb{Z}_p^\times}\mu(y)+\int_{p\mathbb{Z}_p}\mu(y)\right),$$

and we have

$$\int_{p\mathbb{Z}_p}\mu(y)=\frac{\delta(p)}{1-\delta(p)}\int_{\mathbb{Z}_p^\times}\mu(y)$$

since we have $\delta(p)\neq 1$ by the generic assumption on δ.

Therefore, we have

$$\frac{1}{p\delta(p)}\left((p-1)\int_{p\mathbb{Z}_p}\mu(y)-\int_{\mathbb{Z}_p^\times}\mu(y)\right)\boldsymbol{e}_\delta=\frac{1}{p\delta(p)}\left((p-1)\frac{\delta(p)}{1-\delta(p)}-1\right)\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta$$
$$=\frac{1}{p\delta(p)}\frac{p\delta(p)-1}{1-\delta(p)}\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta$$
$$=\frac{1-\frac{1}{p\delta(p)}}{1-\delta(p))}\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta,$$

from which we obtain the equality (2) for $p\neq 2$.

When $p=2$, then the last term is equal to

$$\frac{1}{(p\delta(p))^2}\left(2\int_{4\mathbb{Z}_2}\mu(y)-2\int_{2+4\mathbb{Z}_2}\mu(y)\right)\boldsymbol{e}_\delta=\frac{1}{p\delta(p)^2}\left(\int_{4\mathbb{Z}_2}\mu(y)-\int_{2\mathbb{Z}_2}\mu(y)\right)\boldsymbol{e}_\delta$$

since $\sum_{i\in(\mathbb{Z}/4\mathbb{Z})^\times}\zeta_4^{ij}$ is equal to 2 if $j\equiv 0 \pmod 4$, is equal to 0 if $j\equiv 1, 3 \pmod 4$, and is equal to -2 if $j\equiv 2 \pmod 4$. Since we have $\psi(f_\mu)=\delta(p)f_\mu$, we have

$$\int_{4\mathbb{Z}_p}\mu(y)=\int_{2\mathbb{Z}_2}\psi(\mu)(y)=\delta(p)\int_{2\mathbb{Z}_2}\mu(y)=\delta(p)\frac{\delta(p)}{1-\delta(p)}\int_{\mathbb{Z}_2^\times}\mu(y),$$

where the last equality follows from the same argument for $p\neq 2$.

Therefore, we have

$$\frac{1}{p\delta(p)^2}\left(\int_{4\mathbb{Z}_2}\mu(y)-\int_{2\mathbb{Z}_2}\mu(y)\right)\boldsymbol{e}_\delta=\frac{1}{p\delta(p)^2}\left(\delta(p)\frac{\delta(p)}{1-\delta(p)}-\frac{\delta(p)}{1-\delta(p)}\right)\int_{\mathbb{Z}_p^\times}\mu(y)\boldsymbol{e}_\delta$$
$$=\frac{1}{p\delta(p)^2}\frac{2\delta(p)^2-\delta(p)}{1-\delta(p)}\int_{\mathbb{Z}_2^\times}\mu(y)\boldsymbol{e}_\delta$$
$$=\frac{1-\frac{1}{p\delta(p)}}{1-\delta(p)}\int_{\mathbb{Z}_2^\times}\mu(y)\boldsymbol{e}_\delta,$$

from which we obtain the equality (2) for $p=2$. □

To prove the above proposition for general $k\leq 0$, we need to recall and prove some facts on the differential operator ∂ defined in §2.4 of [Colmez 2008], which will be used to reduce the verification of the condition (vi) for general k to that for $k=0, 1$ (even for the nongeneric case).

Let A be a $\mathbb{Q}_p$-affinoid algebra. We define an A-linear differential operator $\partial:\mathcal{R}_A\to\mathcal{R}_A: f(\pi)\mapsto(1+\pi)\frac{df(\pi)}{d\pi}$. Let $\delta:\mathbb{Q}_p^\times\to A^\times$ be a continuous homomorphism. Then ∂ naturally induces an A-linear and (φ,Γ)-equivariant morphism

$$\partial:\mathcal{R}_A(\delta)\to\mathcal{R}_A(\delta x): f(\pi)\boldsymbol{e}_\delta\mapsto\partial(f(\pi))\boldsymbol{e}_{\delta x},$$

which sits in the exact sequence

$$0 \to A(\delta) \xrightarrow{a\boldsymbol{e}_\delta \mapsto a\boldsymbol{e}_\delta} \mathcal{R}_A(\delta) \xrightarrow{\partial} \mathcal{R}_A(\delta x) \xrightarrow{f\boldsymbol{e}_{\delta x} \mapsto \mathrm{Res}_0\left(f\frac{d\pi}{1+\pi}\right)\boldsymbol{e}_{\delta|x|^{-1}}} A(\delta|x|^{-1}) \to 0. \quad (55)$$

By this exact sequence, when $A = L$ is a finite extension of $\mathbb{Q}_p$, we immediately obtain the following lemma.

Lemma 4.12. $\partial : C^\bullet_{\varphi,\gamma}(\mathcal{R}_L(\delta)) \to C^\bullet_{\varphi,\gamma}(\mathcal{R}_L(\delta x))$ *is a quasi-isomorphism except when* $\delta = \mathbf{1}, |x|$.

For the general case, the exact sequence (55) induces the canonical isomorphism

$$\mathrm{Det}_A(C^\bullet_{\varphi,\gamma}(A(\delta))) \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta))^{-1} \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta x)) \boxtimes \mathrm{Det}_A(C^\bullet_{\varphi,\gamma}(A(\delta|x|^{-1})))^{-1} \xrightarrow{\sim} \mathbf{1}_A. \quad (56)$$

For $\delta' = \delta, \delta|x|^{-1}$, since $A(\delta')$ is a free A-module, the complex

$$C^\bullet_{\varphi,\gamma}(A(\delta')) : \left[A(\delta')^\Delta_1 \xrightarrow{(\gamma-1)\oplus(\varphi-1)} A(\delta')^\Delta_2 \oplus A(\delta')^\Delta_3 \xrightarrow{(\varphi-1)\oplus(1-\gamma)} A(\delta')^\Delta_4\right]$$

(where $A(\delta')_i = A(\delta')$ for $i = 1, \dots, 4$) induces the canonical isomorphism

$$\begin{aligned}\mathrm{Det}_A(C^\bullet_{\varphi,\gamma}(A(\delta'))) &= \left(\mathrm{Det}_A(A(\delta')^\Delta_1) \boxtimes \mathrm{Det}_A(A(\delta')^\Delta_3)^{-1}\right) \boxtimes \left(\mathrm{Det}_A(A(\delta')^\Delta_4) \boxtimes \mathrm{Det}_A(A(\delta')^\Delta_2)^{-1}\right) \\ &\xrightarrow{i_{\mathrm{Det}_A(A(\delta')^\Delta_1)} \boxtimes i_{\mathrm{Det}_A(A(\delta')^\Delta_4)}} \mathbf{1}_A.\end{aligned}$$

Applying this isomorphism, the isomorphism (56) becomes the isomorphism $\Delta_{A,1}(\mathcal{R}_A(\delta))^{-1} \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta x)) \xrightarrow{\sim} \mathbf{1}_A$, and then, multiplying by $\Delta_{A,1}(\mathcal{R}_A(\delta))$ on both sides, we obtain the following isomorphism, which we also denote by ∂:

$$\begin{aligned}\partial : \Delta_{A,1}(\mathcal{R}_A(\delta)) &\xrightarrow{\sim} \Delta_{A,1}(\mathcal{R}_A(\delta)) \boxtimes \left(\Delta_{A,1}(\mathcal{R}_A(\delta))^{-1} \boxtimes \Delta_{A,1}(\mathcal{R}_A(\delta x))\right) \\ &\xrightarrow{i_{\Delta_{A,1}(\mathcal{R}_A(\delta))^{-1}}} \Delta_{A,1}(\mathcal{R}_A(\delta x)).\end{aligned}$$

Taking the product of this isomorphism with the isomorphism

$$\Delta_{A,2}(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_{A,2}(\mathcal{R}_A(\delta x)) : a\boldsymbol{e}_\delta \mapsto -a\boldsymbol{e}_{\delta x},$$

we obtain the isomorphism

$$\partial : \Delta_A(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta x)).$$

By definition, it is clear that this isomorphism is compatible with any base change $A \to A'$.

Concerning this isomorphism, we prove the following proposition.

Proposition 4.13. $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta x)) = \partial \circ \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$.

Proof. The proof of this proposition is a typical density argument, which will be used several times later.

Define the unramified homomorphism $\delta_Y : \mathbb{Q}_p^\times \to \Gamma(\mathbb{G}_m^{\mathrm{an}}, \mathcal{O}_{\mathbb{G}_m^{\mathrm{an}}})^\times$ by $\delta_Y(p) := Y$ (where Y is the parameter of $\mathbb{G}_m^{\mathrm{an}}$). Then $\mathcal{R}_A(\delta)$ is obtained as a base change of the "universal" rank-one (φ, Γ)-module $\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))$ over $\mathcal{R}_{X\times\mathbb{G}_m^{\mathrm{an}}}$ (X is the rigid analytic space associated to $\mathbb{Z}_p[[\Gamma]]$). Since the isomorphism $\partial : \Delta_A(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta x))$ is compatible with any base change, it suffices to show the proposition for $\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))$. Since $X \times \mathbb{G}_m^{\mathrm{an}}$ is reduced, it suffices to show it for the Zariski dense subset S_0 of $X \times \mathbb{G}_m^{\mathrm{an}}$ defined by

$$S_0 := \{(\delta_0, \lambda) \in X(L)\times\mathbb{G}_m^{\mathrm{an}}(L) \mid L \text{ is a finite extension of } \mathbb{Q}_p,\ \delta := \delta_\lambda\delta_0 \text{ is generic}\}.$$

For any (δ_0, λ) in $S_0(L)$, $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta))$ corresponds to the isomorphism

$$\iota_\delta : \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} L\boldsymbol{e}_\delta : [f_\mu \boldsymbol{e}_\delta] \mapsto \delta(-1)\cdot\int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$$

by Lemma 4.10 and by the arguments before this lemma. Then the equality $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta x)) = \partial \circ \varepsilon_{L,\zeta}(\mathcal{R}_L(\delta))$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) & \xrightarrow{\iota_\delta} & L\boldsymbol{e}_\delta \\ \partial\big\downarrow & & \big\downarrow \boldsymbol{e}_\delta\mapsto -\boldsymbol{e}_{\delta x} \\ \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta x)^{\psi=1}) & \xrightarrow{\iota_{\delta x}} & L\boldsymbol{e}_{\delta x} \end{array}$$

Finally, this commutativity follows from the formula

$$\int_{\mathbb{Z}_p} f(y)\partial(\mu)(y) = \int_{\mathbb{Z}_p} yf(y)\mu(y)$$

for any $f(y) \in \mathrm{LA}(\mathbb{Z}_p, L)$, which finally proves the proposition. □

We next prove the compatibility of ∂ with the de Rham ε-isomorphism $\varepsilon_{L,\zeta}^{\mathrm{dR}}(\mathcal{R}_L(\delta))$ for de Rham rank-one (φ, Γ)-modules $\mathcal{R}_L(\delta)$ under a condition on the Hodge–Tate weight of $\mathcal{R}_L(\delta)$ as below.

Lemma 4.14. *Let $\mathcal{R}_L(\delta)$ be a de Rham (φ, Γ)-module (here we don't assume that δ is generic). If the Hodge–Tate weight of $\mathcal{R}_L(\delta)$ is not zero, i.e., we have $\delta = \tilde{\delta}x^k$ such that $k \neq 0$, then we have the equality*

$$\varepsilon_{L,\zeta}^{\mathrm{dR}}(\mathcal{R}_L(\delta x)) = \partial \circ \varepsilon_{L,\zeta}^{\mathrm{dR}}(\mathcal{R}_L(\delta)).$$

Proof. Since one has $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) = \left(L_\infty \frac{1}{t^k}\boldsymbol{e}_\delta\right)^\Gamma$ and $\partial(g(t)) = \frac{dg(t)}{dt}$ for $g(t) \in L_\infty((t))$, the differential operator ∂ naturally induces an isomorphism

$$\partial : \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \to \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x)) : \frac{a}{t^k}\boldsymbol{e}_\delta \mapsto (-k)\frac{a}{t^{k+1}}\boldsymbol{e}_{\delta x}$$

under the condition $k \neq 0$. Hence, by definition of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(M)$ using the isomorphisms $\theta_L(M)$ and $\theta_{\mathrm{dR},L}(M,\zeta)$ and the constant $\Gamma_L(M)$ in Section 3B, it suffices to show the following two equalities:

(1) $\theta_L(\mathcal{R}_L(\delta x)) = \partial \circ \theta_L(\mathcal{R}_L(\delta))$.

(2) $\Gamma_L(\mathcal{R}_L(\delta)) \cdot \partial \circ \theta_{\mathrm{dR},L}(\mathcal{R}_L(\delta),\zeta) = \Gamma_L(\mathcal{R}_L(\delta x)) \cdot \theta_{\mathrm{dR},L}(\mathcal{R}_L(\delta x),\zeta) \circ \partial$.

We first prove the equality (2). Since one has $\Gamma_L(\mathcal{R}_L(\delta)) = \Gamma^*(k)$ and $\Gamma_L(\mathcal{R}_L(\delta x)) = \Gamma^*(k+1)$, it suffices to show that the diagram

$$\begin{array}{ccc} \mathcal{L}_L(\mathcal{R}_L(\delta)) = L\boldsymbol{e}_\delta & \xrightarrow{\Gamma^*(k)\cdot f_{\mathcal{R}_L(\delta),\zeta}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \\ \big\downarrow \boldsymbol{e}_\delta \mapsto -\boldsymbol{e}_{\delta x} & & \big\downarrow \partial \\ \mathcal{L}_L(\mathcal{R}_L(\delta x)) = L\boldsymbol{e}_{\delta x} & \xrightarrow{\Gamma^*(k+1)\cdot f_{\mathcal{R}_L(\delta x),\zeta}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x)) \end{array}$$

is commutative, where the map $f_{\mathcal{R}_L(\delta'),\zeta}$ (for $\delta' = \delta, \delta x$) is defined in Lemma 3.4. This commutativity is obvious by definition of $f_{\mathcal{R}_L(\delta_0),\zeta}$ since one has

$$\varepsilon_L(W(\mathcal{R}_L(\delta)),\zeta) = \varepsilon_L(W(\mathcal{R}_L(\delta x)),\zeta)$$

(this is because one has a natural isomorphism $\boldsymbol{D}_{\mathrm{pst}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{pst}}(\mathcal{R}_L(\delta x)) : \frac{a}{t^k}\boldsymbol{e}_\delta \mapsto \frac{a}{t^{k+1}}\boldsymbol{e}_{\delta x}$) and $k \cdot \Gamma^*(k) = \Gamma^*(k+1)$ for $k \neq 0$. We next show the equality (1). Under the assumption that $k \neq 0$, it is easy to see that ∂ induces the isomorphisms

$$\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x)), \quad \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))^0 \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta x))^0$$

and

$$\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta x)), \quad \mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(\delta x))$$

for any $i = 0, 1, 2$ by Lemma 4.12. Hence, by definition of $\theta'_L(\mathcal{R}_L(\delta))$, it suffices to show that the following two diagrams are commutative for $M = \mathcal{R}_L(\delta)$:

$$\begin{array}{ccccccc} \mathrm{H}^0_{\varphi,\gamma}(M) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M) \oplus t_M & \longrightarrow & \mathrm{H}^1_{\varphi,\gamma}(M) \\ \big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial \\ \mathrm{H}^0_{\varphi,\gamma}(M(x)) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M(x)) & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M(x)) \oplus t_{M(x)} & \longrightarrow & \mathrm{H}^1_{\varphi,\gamma}(M(x)) \end{array} \tag{57}$$

and

$$\begin{array}{ccccccc} \mathrm{H}^1_{\varphi,\gamma}(M) & \longrightarrow & \begin{matrix} \boldsymbol{D}_{\mathrm{cris}}(M^*)^\vee \\ \oplus \boldsymbol{D}_{\mathrm{dR}}(M)^0 \end{matrix} & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M)^\vee & \longrightarrow & \mathrm{H}^2_{\varphi,\gamma}(M) \\ \big\downarrow \partial & & \big\downarrow (-\partial^\vee)\oplus\partial & & \big\downarrow -\partial^\vee & & \big\downarrow \partial \\ \mathrm{H}^1_{\varphi,\gamma}(M(x)) & \longrightarrow & \begin{matrix} \boldsymbol{D}_{\mathrm{cris}}(M(x)^*)^\vee \\ \oplus \boldsymbol{D}_{\mathrm{dR}}(M(x))^0 \end{matrix} & \longrightarrow & \boldsymbol{D}_{\mathrm{cris}}(M(x)^*)^\vee & \longrightarrow & \mathrm{H}^2_{\varphi,\gamma}(M(x)) \end{array} \tag{58}$$

Here $\partial^\vee$ is the dual of

$$\partial : \boldsymbol{D}_{\mathrm{cris}}(M(x)^*) = \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta^{-1}|x|)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta^{-1}x|x|)) = \boldsymbol{D}_{\mathrm{cris}}(M^*).$$

For the commutativity of the diagram (57), the only nontrivial part is the commutativity of the diagram

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{cris}}(M) \oplus t_M & \xrightarrow{\exp_{M,f} \oplus \exp_M} & \mathrm{H}^1_{\varphi,\gamma}(M) \\ \partial \downarrow & & \partial \downarrow \\ \boldsymbol{D}_{\mathrm{cris}}(M(x)) \oplus t_{M(x)} & \xrightarrow{\exp_{M(x),f} \oplus \exp_{M(x)}} & \mathrm{H}^1_{\varphi,\gamma}(M(x)) \end{array}$$

but this commutativity easily follows from Proposition 2.23. Using the commutativity of (57) for $M = \mathcal{R}_L(\delta^{-1}|x|)$, to prove the commutativity of (58), it suffices to show the commutativities of the following diagrams:

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{dR}}(M) & \longrightarrow & \boldsymbol{D}_{\mathrm{dR}}(M^*)^\vee \\ \partial \downarrow & & -\partial^\vee \downarrow \\ \boldsymbol{D}_{\mathrm{dR}}(M(x)) & \longrightarrow & \boldsymbol{D}_{\mathrm{dR}}(M(x)^*)^\vee \end{array} \tag{59}$$

and

$$\begin{array}{ccc} \mathrm{H}^i_{\varphi,\gamma}(M) & \longrightarrow & \mathrm{H}^{2-i}_{\varphi,\gamma}(M^*)^\vee \\ \partial \downarrow & & -\partial^\vee \downarrow \\ \mathrm{H}^i_{\varphi,\gamma}(M(x)) & \longrightarrow & \mathrm{H}^{2-i}_{\varphi,\gamma}(M(x)^*)^\vee \end{array} \tag{60}$$

Here the horizontal arrows are isomorphisms obtained by (Tate) duality. Since the commutativity of (59) is easy to check, here we only prove the commutativity of (60). Moreover, we only prove it for $i = 2$ since other cases are proved in the same way. For $i = 2$, it suffices to show the equality

$$[\partial(f)g\boldsymbol{e}_1] = -[f\partial(g)\boldsymbol{e}_1] \in \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))$$

for any $[f\boldsymbol{e}_\delta] \in \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(\delta))$ and $g\boldsymbol{e}_{\delta^{-1}|x|} \in \mathrm{H}^0_{\varphi,\gamma}(\mathcal{R}_L(\delta^{-1}|x|))$. Since we have $\partial(fg) = \partial(f)g + f\partial(g)$, the equality follows from the fact that we have $[\partial(h)\boldsymbol{e}_1] = 0$ in $\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))$ for any $h \in \mathcal{R}_L$. □

Remark 4.15. Proposition 4.13 and Lemma 4.14 and the following proof of Proposition 4.11 should be generalizable to a more general setting. Let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$ of any rank. In §3 of [Nakamura 2014a], we developed the theory of Perrin-Riou's big exponential map for a de Rham (φ, Γ)-module, which is an $\mathcal{R}^\infty_L(\Gamma)$-linear map $\mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(M)) \to \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(\boldsymbol{N}_{\mathrm{rig}}(M)))$, where $\boldsymbol{N}_{\mathrm{rig}}(M) \subseteq M[1/t]$ is a de Rham (φ, Γ)-module equipped with a natural action of the differential operator ∂_M defined by Berger. This big exponential map is defined using the operator ∂_M. Our generalization of Perrin-Riou's $\delta(V)$-theorem [Nakamura 2014a, Theorem 3.21] states that this map gives an isomorphism

$$\mathrm{Exp}_M : \Delta_{\mathcal{R}_L^\infty(\Gamma)}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \Delta_{\mathcal{R}_L^\infty(\Gamma)}(\mathbf{Dfm}(N_{\mathrm{rig}}(M))).$$

Therefore, as a generalization of Proposition 4.13, it seems to be natural to conjecture that the conjectural ε-isomorphisms should satisfy

$$\varepsilon_{\mathcal{R}_L^\infty(\Gamma),\zeta}(\mathbf{Dfm}(N_{\mathrm{rig}}(M))) = \mathrm{Exp}_M \circ \varepsilon_{\mathcal{R}_L^\infty(\Gamma),\zeta}(\mathbf{Dfm}(M)),$$

which we want to study in future works.

Using these results, we prove Proposition 4.11 for general $k \leq 0$ as follows.

Proof of Proposition 4.11 for general $k \leq 0$. Let $\delta = \tilde{\delta}x^k$ be a generic homomorphism such that $k \leq 0$. By the arguments before Proposition 4.11, it suffices to show the equality $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$. This equality follows from the equality $\varepsilon_{L,\zeta}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\tilde{\delta}))$ proved in Proposition 4.11 for $k = 0$, since we have

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \partial^k \circ \varepsilon_{L,\zeta}(\mathcal{R}_L(\tilde{\delta})) \quad \text{and} \quad \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta)) = \partial^k \circ \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\tilde{\delta}))$$

by Proposition 4.13 and Lemma 4.14. □

We next consider the case where $k \geq 1$. To verify the condition (vi), it suffices to show the following proposition.

Proposition 4.16. *If $k \geq 1$, then the map*

$$\mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\exp^{-1}_{\mathcal{R}_L(\delta)}} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta))$$

sends each element $[f_\mu \boldsymbol{e}_\delta] \in \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1})$ to

(1) $(k-1)! \cdot \dfrac{\delta(-1)}{\varepsilon_L(W(\mathcal{R}_L(\delta)), \zeta)} \cdot \dfrac{1}{t^k} \cdot \displaystyle\int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *when* $n(\delta) \neq 0$,

(2) $(k-1)! \cdot \dfrac{\det_L(1-\varphi \mid \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)^*))}{\det_L(1-\varphi \mid \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)))} \cdot \dfrac{\delta(-1)}{t^k} \cdot \displaystyle\int_{\mathbb{Z}_p^\times} \delta^{-1}(y)\mu(y)\boldsymbol{e}_\delta$ *when* $n(\delta) = 0$.

Proof. In the same way as the proof of Proposition 4.11, it suffices to show the proposition for $k = 1$ (i.e., $\delta = \tilde{\delta}x$) using Proposition 4.13 and Lemma 4.14.

Hence, we assume $k = 1$. Then, in a similar way as the proof of Proposition 4.11 (for $k = 0$), we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\tilde{\delta})) & \longleftarrow & \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\tilde{\delta})^{\psi=1}) & \xrightarrow{\iota_{\tilde{\delta}}} & L\boldsymbol{e}_{\tilde{\delta}} \\
\big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \boldsymbol{e}_{\tilde{\delta}} \mapsto -\boldsymbol{e}_\delta \\
\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \longleftarrow & \mathrm{H}^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) & \xrightarrow{\iota_\delta} & L\boldsymbol{e}_\delta
\end{array} \qquad (61)$$

such that all the arrows are isomorphisms by Lemma 4.12. Hence, reducing to the case of $k=0$, it suffices to show that the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(\tilde{\delta})^{\psi=1}) & \longrightarrow & \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\tilde{\delta})) & \xrightarrow{\exp^*_{\mathcal{R}_L(\tilde{\delta}^{-1}x|x|)}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\tilde{\delta})) = (L_{\infty}\boldsymbol{e}_{\tilde{\delta}})^{\Gamma} \\ \partial\downarrow & & \partial\downarrow & & \downarrow a\boldsymbol{e}_{\tilde{\delta}} \mapsto \frac{a}{t}\boldsymbol{e}_{\delta} \\ \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(\delta)^{\psi=1}) & \longrightarrow & \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(\delta)) & \xleftarrow{\exp_{\mathcal{R}_L(\delta)}} & \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\delta)) = \left(L_{\infty}\frac{1}{t}\boldsymbol{e}_{\delta}\right)^{\Gamma} \end{array} \tag{62}$$

The following proof of this commutativity is very similar to that of Theorem 3.10 of [Nakamura 2014a]. Take $[f\boldsymbol{e}_{\tilde{\delta}}] \in \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(\tilde{\delta})^{\psi=1})$. If we define

$$\alpha\boldsymbol{e}_{\tilde{\delta}} := \exp^*_{\mathcal{R}_L(\tilde{\delta}^{-1}x|x|)}([|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))p_{\Delta}(f\boldsymbol{e}_{\tilde{\delta}}), 0]) \in \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(\tilde{\delta})) \subseteq \boldsymbol{D}_{\mathrm{dif}}(\mathcal{R}_L(\tilde{\delta})),$$

then it suffices to show the equality

$$\exp_{\mathcal{R}_L(\delta)}\left(\frac{\alpha}{t}\boldsymbol{e}_{\delta}\right) = |\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))[p_{\Delta}(\partial(f)\boldsymbol{e}_{\delta}), 0].$$

We prove this equality as follows. First, we have an equality

$$\frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}[\iota_n(p_{\Delta}(f\boldsymbol{e}_{\tilde{\delta}}))] = [\alpha\boldsymbol{e}_{\tilde{\delta}}] \in \mathrm{H}^1_{\psi,\gamma}(\boldsymbol{D}^+_{\mathrm{dif}}(\mathcal{R}_L(\tilde{\delta})))$$

for large enough $n \geq 1$ by the explicit definition of $\exp^*_{\mathcal{R}_L(\tilde{\delta}^{-1}x|x|)}$ [Nakamura 2014a, Proposition 2.16]. This equality means that for some $y_n \in \boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_L(\tilde{\delta}))^{\Delta}$ we have

$$\frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\iota_n(p_{\Delta}(f\boldsymbol{e}_{\tilde{\delta}})) - \alpha\boldsymbol{e}_{\tilde{\delta}} = (\gamma-1)y_n.$$

If we set $\nabla_0 := \log([\gamma])/\log(\chi(\gamma)) \in \mathcal{R}_L^{\infty}(\Gamma)$ and define

$$\frac{\nabla_0}{\gamma-1} := \frac{1}{\log(\chi(\gamma))}\sum_{m\geq 1}^{\infty}\frac{(-1)^{m-1}([\gamma]-1)^{m-1}}{m} \in \mathcal{R}_L^{\infty}(\Gamma),$$

then we obtain the equality

$$\begin{aligned} &\iota_n\left(\frac{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}{\log(\chi(\gamma))}\frac{\nabla_0}{\gamma-1}(p_{\Delta}(f\boldsymbol{e}_{\tilde{\delta}}))\right) \\ &\qquad = \frac{1}{\log(\chi(\gamma))}\alpha\boldsymbol{e}_{\tilde{\delta}} + \nabla_0(y_n) \in \frac{1}{\log(\chi(\gamma))}\alpha\boldsymbol{e}_{\tilde{\delta}} + t\boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_L(\tilde{\delta})). \end{aligned} \tag{63}$$

Since we have $f\boldsymbol{e}_{\tilde{\delta}} \in \mathcal{R}_L(\tilde{\delta})^{\psi=1}$, we have

$$(1-\varphi)(p_{\Delta}(f\boldsymbol{e}_{\tilde{\delta}})) \in \mathcal{R}_L(\tilde{\delta})^{\Delta,\psi=0}.$$

Hence, there exists $\beta \in \mathcal{R}_L(\tilde{\delta})^{\Delta,\psi=0}$ such that

$$(1-\varphi)(p_{\Delta}(f\boldsymbol{e}_{\tilde{\delta}})) = (\gamma-1)\beta$$

by (for example) Theorem 3.1.1 of [Kedlaya et al. 2014]. Then, for any $m \geq n+1$, we obtain

$$\begin{aligned}\iota_m\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)-\iota_{m-1}\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)\\ =\iota_m\left((1-\varphi)\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)\right)=\iota_m\left(\frac{\nabla_0}{\gamma-1}((1-\varphi)(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}})))\right)\\ =\iota_m\left(\frac{\nabla_0}{\gamma-1}((\gamma-1)\beta)\right)=\iota_m(\nabla_0(\beta))\in t\boldsymbol{D}^+_{\mathrm{dif},m}(\mathcal{R}_L(\tilde{\delta}))\end{aligned}$$

since we have $\nabla_0(\mathcal{R}_L(\tilde{\delta}))\subseteq t\mathcal{R}_L(\tilde{\delta})$. In particular, we obtain

$$\iota_m\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)-\iota_n\left(\frac{\nabla_0}{\gamma-1}(p_\Delta(f\boldsymbol{e}_{\tilde{\delta}}))\right)\in t\boldsymbol{D}^+_{\mathrm{dif},m}(\mathcal{R}_L(\tilde{\delta})) \tag{64}$$

for any $m \geq n+1$ by induction.

Since the map $\mathcal{R}_L(\tilde{\delta}) \xrightarrow{\sim} \frac{1}{t}\mathcal{R}_L(\delta) : g\boldsymbol{e}_{\tilde{\delta}} \mapsto \frac{g}{t}\boldsymbol{e}_\delta$ is an isomorphism of (φ, Γ)-modules, the facts (63), (64) and the explicit definition of the exponential map (Proposition 2.23(1)) induce the equality

$$\begin{aligned}&\exp_{\mathcal{R}_L(\delta)}\left(\frac{\alpha}{t}\boldsymbol{e}_\delta\right)\\ &\quad=|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))\left[(\gamma-1)\frac{\nabla_0}{\gamma-1}\left(p_\Delta\left(\frac{f}{t}\boldsymbol{e}_\delta\right)\right),(\psi-1)\frac{\nabla_0}{\gamma-1}\left(p_\Delta\left(\frac{f}{t}\boldsymbol{e}_\delta\right)\right)\right]\\ &\quad=|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))\left[\nabla_0\left(p_\Delta\left(\frac{f}{t}\boldsymbol{e}_\delta\right)\right),0\right]\\ &\quad=|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))[p_\Delta(\partial(f)\boldsymbol{e}_\delta),0],\end{aligned}$$

where the last equality follows from the equality $\nabla_0\left(\frac{f}{t}\boldsymbol{e}_\delta\right)=\partial(f)\boldsymbol{e}_\delta$ since we have $\nabla_0\left(\frac{1}{t}\boldsymbol{e}_\delta\right)=0$ by the assumption $k=1$, from which the commutativity of the diagram (62) follows. □

As a corollary of Propositions 4.11 and 4.16, we verify the conditions (iv), (v) by the density argument as follows.

Corollary 4.17. *Let M be a rank-one (φ, Γ)-module over $\mathcal{R}_A$. Then the isomorphism $\varepsilon_{A,\zeta}(M) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(M)$, which is defined in Section 4A, satisfies the conditions* (iv) *and* (v) *of Conjecture 3.8.*

Proof. We first verify the conditions (iv). By the definition of $\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)\otimes_A\mathcal{L})$, it suffices to do this for (φ, Γ)-modules of the form $M=\mathcal{R}_A(\delta)$ (i.e., $\mathcal{L}=A$) since the general case immediately follows from this case by Lemma 4.6. Then, in the same way as the proof of Proposition 4.13, it suffices to verify these conditions for any $\delta=\delta_\lambda\delta_0 : \mathbb{Q}_p^\times \to L^\times$ such that the point $(\delta_0, \lambda)\in X\times\mathbb{G}_m^{\mathrm{an}}$ is contained in the

Zariski dense subset S_1 of $X \times \mathbb{G}_m^{\mathrm{an}}$ defined by

$$S_1 := \{(\delta_0, \lambda) \in X(L) \times \mathbb{G}_m^{\mathrm{an}}(L) \mid [L : \mathbb{Q}_p] < \infty,\ \delta \text{ is generic},\ \mathcal{R}_L(\delta) \text{ is de Rham}\}.$$

For such δ, the conditions (iv) follow from Lemma 3.7 since we have $\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$ by Propositions 4.11 and 4.16.

We next verify the condition (v). Let (Λ, T) be as in Conjecture 3.8(v). We recall that we defined a canonical isomorphism

$$\Delta_\Lambda(T) \otimes_\Lambda A_\infty \xrightarrow{\sim} \Delta_{A_\infty}(M_\infty)$$

(see Example 3.3 for definition and notation). Since any continuous map $\Lambda \to A$ factors through $\Lambda \to A_\infty \to A$, it suffices to show the equality

$$\varepsilon_{\Lambda,\zeta}(T) \otimes \mathrm{id}_{A_\infty} = \varepsilon_{A_\infty,\zeta}(M_\infty) \quad \Big(:= \varprojlim_n \varepsilon_{A_n,\zeta}(M_n)\Big). \tag{65}$$

Since condition (v) is local for $\mathrm{Spf}(\Lambda)$, it suffices to verify it for Λ-representations of the form $\Lambda(\tilde{\delta})$ for some $\tilde{\delta} : G^{\mathrm{ab}}_{\mathbb{Q}_p} \to \Lambda^\times$. Let us decompose $\delta = \tilde{\delta} \circ \mathrm{rec}_{\mathbb{Q}_p}$ into $\delta = \delta_\lambda \delta_0$. Since $\Lambda/\mathfrak{m}_\Lambda$ is a finite ring, there exists $k \geq 1$ such that $\lambda^k \equiv 1 \pmod{\mathfrak{m}_\Lambda}$. Then we can define a continuous $\mathbb{Z}_p$-algebra homomorphism

$$\Lambda_k := \varprojlim_n \mathbb{Z}_p[Y]/(p, (Y^k - 1))^n \to \Lambda : Y \mapsto \lambda.$$

Hence, the Λ-representation $\Lambda(\tilde{\delta})$ is obtained by a base change of the "universal" $\mathbb{Z}_p[\![\Gamma]\!] \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_k$-representation T_k^{univ}, which corresponds to the homomorphism

$$\delta_k^{\mathrm{univ}} : \mathbb{Q}_p^\times \to (\mathbb{Z}_p[\![\Gamma]\!] \widehat{\otimes}_{\mathbb{Z}_p} \Lambda_k)^\times : p \mapsto 1 \widehat{\otimes} Y,\ a \mapsto [\sigma_a^{-1}] \widehat{\otimes} 1$$

for $a \in \mathbb{Z}_p^\times$. Hence, it suffices to verify the equality (65) for this universal one. In this case, since the associated rigid space is an admissible open of $X \times \mathbb{G}_m^{\mathrm{an}}$ defined by

$$Z_k := \{(\delta_0, \lambda) \in X \times \mathbb{G}_m^{\mathrm{an}} \mid |\lambda^k - 1| < 1\},$$

and the associated (φ, Γ)-module is isomorphic to the restriction of the universal one $\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))$ defined in the proof of Proposition 4.13, it suffices to show the equality

$$\varepsilon_{\mathbb{Z}_p[\![\Gamma]\!]\widehat{\otimes}_{\mathbb{Z}_p}\Lambda_k,\zeta}(T_k^{\mathrm{univ}}) \otimes \mathrm{id}_{\Gamma(Z_k,\mathcal{O}_{Z_k})} = \varepsilon_{\Gamma(Z_k,\mathcal{O}_{Z_k}),\zeta}(\mathbf{Dfm}(\mathcal{R}_{\mathbb{G}_m^{\mathrm{an}}}(\delta_Y))|_{Z_k}).$$

Since both sides satisfy the condition (vi) for any point $(\delta_0, \lambda) \in Z_k \cap S_1$ by Kato's theorem [1993b] and by Propositions 4.11 and 4.16, and since the set $Z_k \cap S_1$ is Zariski dense in Z_k, the equality above follows by the density argument. $\square$

4B2. *Verification of the condition* (vi): *the exceptional case.* Finally, we verify the condition (vi) in the exceptional case, i.e., $\delta = x^{-k}$ or $\delta = x^{k+1}|x|$ for $k \in \mathbb{Z}_{\geq 0}$.

We first reduce all the exceptional cases to the case $\delta = x|x|$.

Lemma 4.18. *We assume that the equality*

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(x|x|)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(x|x|))$$

holds. Then the other equalities

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(\delta))$$

also hold for all $\delta = x^{k+1}|x|, x^{-k}$ *for* $k \geq 0$.

Proof. The equality for $\delta = x^0$ follows from that for $\delta = x|x|$ by the compatibility of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(-)$ and $\varepsilon_{L,\zeta}(-)$ with the Tate duality, which is proved in Lemma 3.7 and Corollary 4.17. Then the equality for $\delta = x^{k+1}|x|$ (resp. $\delta = x^{-k}$) follows from that for $\delta = x|x|$ (resp. $\delta = x^0$) by the compatibility of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(-)$ and $\varepsilon_{L,\zeta}(-)$ with ∂, which is proved in Lemma 4.14 and Proposition 4.13. □

Finally, it remains to show the equality

$$\varepsilon_{L,\zeta}(\mathcal{R}_L(1)) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(1))$$

(we identify $\mathcal{R}_L(x|x|) = \mathcal{R}_L(1) : f\boldsymbol{e}_{x|x|} \mapsto f\boldsymbol{e}_1$). Since $\mathcal{R}_L(1)$ is étale, this equality immediately follows from Kato's result since we have $\varepsilon_{L,\zeta}(\mathcal{R}_L(1)) = \varepsilon_{\mathcal{O}_L,\zeta}(\mathcal{O}_L(1)) \otimes \mathrm{id}_L$ under the canonical isomorphism

$$\Delta_L(\mathcal{R}_L(1)) \xrightarrow{\sim} \Delta_{\mathcal{O}_L}(\mathcal{O}_L(1)) \otimes_{\mathcal{O}_L} L$$

by Corollary 4.17. However, here we give another proof of this equality only using the framework of (φ, Γ)-modules.

In the remaining part of this section, we prove this equality by explicit calculations. First, it is easy to see that the inclusion

$$C^\bullet_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) \hookrightarrow C^\bullet_{\psi,\gamma}(\mathrm{LA}(\mathbb{Z}_p, L)(1))$$

induced by the natural inclusion $L \cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1 \hookrightarrow \mathrm{LA}(\mathbb{Z}_p, L)(1)$ (here, $1_{\mathbb{Z}_p}$ is the constant function on $\mathbb{Z}_p$ with the constant value 1) is quasi-isomorphism. This quasi-isomorphism and the quasi-isomorphism

$$C^\bullet_{\gamma}(\mathcal{R}^\infty_L(1)^{\psi=1}) \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathcal{R}^\infty_L(1)),$$

and the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{R}^\infty_L(1) \to \mathcal{R}_L(1) \to \mathrm{LA}(\mathbb{Z}_p, L)(1) \to 0$$

induce the isomorphisms

$$\alpha_0 : \mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) \xrightarrow{\sim} \mathrm{H}^1(\Gamma, \mathcal{R}^\infty_L(1)^{\psi=1}),$$
$$\alpha_1 : \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1)) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) :$$
$$[f_1\boldsymbol{e}_1, f_2\boldsymbol{e}_2] \mapsto \left(\mathrm{Res}_0\left(f_1\frac{d\pi}{1+\pi}\right)\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1, \mathrm{Res}_0\left(f_2\frac{d\pi}{1+\pi}\right)\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1\right),$$
$$\alpha_2 : \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)) \xrightarrow{\sim} \mathrm{H}^2_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) : [f\boldsymbol{e}_1] \mapsto \mathrm{Res}_0\left(f\frac{d\pi}{1+\pi}\right)\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1.$$

Therefore, the isomorphism

$$\bar{\theta}_\zeta(\mathcal{R}_L(1)) : \boxtimes_{i=1}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(1)))^{(-1)^{i+1}} \xrightarrow{\sim} (L(1), 1),$$

defined in (50), is the composition of the isomorphisms β_0, β_1 and $\iota_{x|x|}$:

$$\begin{aligned}&\boxtimes_{i=1}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(\mathcal{R}_L(1)))^{(-1)^{i+1}}\\ &\xrightarrow{\beta_0} \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1))^{(-1)^{i+1}} \boxtimes (\mathrm{H}^1(\Gamma, \mathcal{R}^\infty_L(1)^{\psi=1}), 1)\\ &\xrightarrow{\beta_1} (\mathrm{H}^1(\Gamma, \mathcal{R}^\infty_L(1)^{\psi=1}), 1) \xrightarrow{\iota_{x|x|}} (L(1), 1).\end{aligned}$$

Here β_0 is induced by α_i $(i = 0, 1, 2)$, and β_1 is induced by the canonical isomorphism

$$\beta_1 : \boxtimes_{i=0}^2 \mathrm{Det}_L(\mathrm{H}^i_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1))^{(-1)^{i-1}} \xrightarrow{\sim} \mathbf{1}_L,$$

which is the base change by $f_{x|x|} : \mathcal{R}^\infty_L(\Gamma) \to L : [\gamma] \mapsto \chi(\gamma)^{-1}$ of the isomorphism (40) for $M = \mathcal{R}_L$.

By definition, the isomorphism β_1 is explicitly described as in the following lemma, which easily follows from the definition (hence, we omit the proof).

Lemma 4.19. *If we define* $\tilde{f}_0 := 1_{\mathbb{Z}_p}\boldsymbol{e}_1$ (*resp.* $\tilde{f}_{1,1} := (1_{\mathbb{Z}_p}\boldsymbol{e}_1, 0)$, $\tilde{f}_{1,2} := (0, 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$, *resp.* $\tilde{f}_2 := 1_{\mathbb{Z}_p}\boldsymbol{e}_1$) *for the basis of* $\mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$ (*resp.* $\mathrm{H}^1_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$, *resp.* $\mathrm{H}^2_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1)$), *then the canonical trivialization*

$$\beta_1 : (\mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1), 1)^{-1} \boxtimes (\det_L \mathrm{H}^1_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1), 2) \boxtimes (\mathrm{H}^2_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1), 1)^{-1} \xrightarrow{\sim} \mathbf{1}_L$$

satisfies the equality

$$\beta_1(\tilde{f}_0^\vee \otimes (\tilde{f}_{1,1}\wedge\tilde{f}_{1,2}) \otimes \tilde{f}_2^\vee) = 1.$$

Lemma 4.20. *The isomorphism*

$$\mathrm{H}^0_{\psi,\gamma}(L\cdot 1_{\mathbb{Z}_p}\boldsymbol{e}_1) \xrightarrow{\alpha_0} \mathrm{H}^1(\Gamma, \mathcal{R}^\infty_L(1)^{\psi=1}) \xrightarrow{\iota_{x|x|}} L\boldsymbol{e}_1$$

sends the element $\tilde{f}_0$ *to* $-\boldsymbol{e}_1 \in L(1)$.

Proof. Since we have $\mathrm{Col}\big(\frac{1+\pi}{\pi}\big) = 1_{\mathbb{Z}_p}$ and $\psi\big(\frac{1+\pi}{\pi}\boldsymbol{e}_1\big) = \frac{1+\pi}{\pi}\boldsymbol{e}_1$, we have

$$\alpha_0(\tilde{f}_0) = \left[\frac{1}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}(\gamma-1)\left(\frac{1+\pi}{\pi}\boldsymbol{e}_1\right)\right]$$

by definition of the boundary map.

Since we have

$$(\gamma-1)\left(\frac{1+\pi}{\pi}\boldsymbol{e}_1\right) = \partial\left(\log\left(\frac{\gamma(\pi)}{\pi}\right)\right)\boldsymbol{e}_1 \quad \text{and} \quad \log\left(\frac{\gamma(\pi)}{\pi}\right)\boldsymbol{e}_{|x|} \in \mathcal{R}_L^{\infty}(|x|)^{\psi=1},$$

and have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(|x|)^{\psi=1}) & \xrightarrow{\iota_{|x|}} & L\boldsymbol{e}_{|x|} \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle \boldsymbol{e}_{|x|}\mapsto -\boldsymbol{e}_1} \\ \mathrm{H}^1(\Gamma, \mathcal{R}_L^{\infty}(1)^{\psi=1}) & \xrightarrow{\iota_{x|x|}} & L\boldsymbol{e}_1 \end{array} \tag{66}$$

we obtain an equality

$$\begin{aligned}\iota_{x|x|}(\alpha_0(\tilde{f}_0)) &= \frac{1}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}\iota_{x|x|}\left(\left[\partial\left(\log\left(\frac{\gamma(\pi)}{\pi}\right)\right)\boldsymbol{e}_1\right]\right) \\ &= -\frac{1}{|\Gamma_{\mathrm{tor}}|\log_0(\chi(\gamma))}\int_{\mathbb{Z}_p^{\times}}\mu_\gamma(y)\boldsymbol{e}_1\end{aligned}$$

by Lemma 4.10, where we define $\mu_\gamma \in \mathcal{D}(\mathbb{Z}_p, L)$ such that $f_{\mu_\gamma}(\pi) = \log(\gamma(\pi)/\pi)$.

We calculate $\int_{\mathbb{Z}_p^{\times}}\mu_\gamma(y)$ as follows. Since we have $\psi(\mu_\gamma) = \frac{1}{p}\mu_\gamma$, we obtain

$$\int_{p\mathbb{Z}_p}\mu_\gamma(y) = \int_{\mathbb{Z}_p}\psi(\mu_\gamma)(y) = \frac{1}{p}\int_{\mathbb{Z}_p}\mu_\gamma(y).$$

Hence, we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p^{\times}}\mu_\gamma(y) &= \int_{\mathbb{Z}_p}\mu_\gamma(y) - \int_{p\mathbb{Z}_p}\mu_\gamma(y) = \int_{\mathbb{Z}_p}\mu_\gamma(y) - \frac{1}{p}\int_{\mathbb{Z}_p}\mu_\gamma(y) \\ &= \frac{p-1}{p}\int_{\mathbb{Z}_p}\mu_\gamma(y) = \frac{p-1}{p}\log\left(\frac{\gamma(\pi)}{\pi}\right)|_{\pi=0} = \frac{p-1}{p}\log(\chi(\gamma)).\end{aligned}$$

Hence,

$$\iota_{x|x|}(\alpha_0(\tilde{f}_0)) = -\frac{\log(\chi(\gamma))}{|\Gamma_{\mathrm{for}}|\log_0(\chi(\gamma))}\frac{p-1}{p}\boldsymbol{e}_1 = -\boldsymbol{e}_1$$

(for any prime p), which proves the lemma. □

In the Appendix, we define a canonical basis $\{f_{1,1}, f_{1,2}\}$ of $\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))$, $f_2 \in \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1))$, $e_0 \in \mathrm{H}^0_{\psi,\gamma}(\mathcal{R}_L)$ and $\{e_{1,1}, e_{1,2}\}$ of $\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L)$; see the Appendix for the definition.

Corollary 4.21. *The isomorphism*

$$\bar{\theta}_\zeta(\mathcal{R}_L(1)) : (\det_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1)), 2) \boxtimes (\mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)), 1)^{-1} \xrightarrow{\sim} (L\boldsymbol{e}_1, 1)$$

sends the element $(f_{1,1} \wedge f_{1,2}) \otimes f_2^\vee$ *to* $-\frac{p-1}{p}\boldsymbol{e}_1$.

Proof. By definition, we have

$$\alpha_1(f_{1,1}) = \frac{p-1}{p}\log(\chi(\gamma))\tilde{f}_{1,1},$$
$$\alpha_1(f_{1,2}) = \frac{p-1}{p}\tilde{f}_{1,2},$$
$$\alpha_2(f_2) = \frac{p-1}{p}\log(\chi(\gamma))\tilde{f}_2.$$

Then the corollary follows from the previous lemmas. □

Finally, since one has $\Gamma_L(\mathcal{R}_L(1)) = 1$ and $\theta_{\mathrm{dR},L}(\mathcal{R}_L(1), \zeta)$ corresponds to the isomorphism

$$\mathcal{L}_L(\mathcal{R}_L(1)) = L\boldsymbol{e}_1 \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(1)) = \frac{1}{t}L\boldsymbol{e}_1 : a\boldsymbol{e}_1 \mapsto \frac{a}{t}\boldsymbol{e}_1,$$

it suffices to show the following lemma.

Lemma 4.22. *The isomorphism*

$$\theta_L(\mathcal{R}_L(1)) : (\det_L \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1)), 2) \boxtimes (\mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)), 1)^{-1} \xrightarrow{\sim} (\boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(1)), 1) = \left(L\frac{1}{t}\boldsymbol{e}_1, 1\right)$$

sends the element $(f_{1,1} \wedge f_{1,2}) \otimes f_2^\vee$ *to* $-\frac{p-1}{pt}\boldsymbol{e}_1$.

Proof. By definition, the above isomorphism is the one which is naturally induced by the exact sequence

$$0 \to \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(1)) \xrightarrow{(1-\varphi)\oplus\mathrm{can}} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L(1)) \oplus \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_L(1)) \xrightarrow{\exp_{f,\mathcal{R}_L(1)} \oplus \exp_{\mathcal{R}_L(1)}} \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))_f \to 0$$

and the isomorphisms

$$\exp^\vee_{f,\mathcal{R}_L} : \mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))/\mathrm{H}^1_{\psi,\gamma}(\mathcal{R}_L(1))_f \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^\vee$$

and

$$\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^\vee \xrightarrow{\sim} \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)),$$

which is the dual of the natural isomorphism $\mathrm{H}^0_{\psi,\gamma}(\mathcal{R}_L) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$.

We have $\exp_{\mathcal{R}_L(1)}\left(\frac{1}{t}\boldsymbol{e}_1\right) = f_{1,2}$ by the proof of Lemma 5.1. Since we have

$$\exp_{f,\mathcal{R}_L}(1) = e_{1,2}$$

for $d_0 := 1 \in L = \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$ by the explicit definition of $\exp_f$ (Proposition 2.23(2)), and since we have $\langle f_{1,1}, e_{1,2}\rangle = 1$ by Lemma 5.4, we obtain

$$\exp^{\vee}_{f,\mathcal{R}_L}(f_{1,1}) = -d_0^{\vee} \in \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^{\vee}$$

(we should be careful with the sign). Since the natural isomorphism $\mathrm{H}^0_{\psi,\gamma}(\mathcal{R}_L) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$ sends e_0 to $d_0 \in L = \boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)$, we obtain

$$\boldsymbol{D}_{\mathrm{cris}}(\mathcal{R}_L)^{\vee} \to \mathrm{H}^2_{\psi,\gamma}(\mathcal{R}_L(1)) : d_0^{\vee} \mapsto f_2$$

by Lemma 5.4. The lemma follows from these calculations and a diagram chase. □

Appendix: Explicit calculations of $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L)$ and $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(1))$

In this appendix, we compare $\mathrm{H}^i(\mathbb{Q}_p, L(k))$ with $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(k))$ explicitly for $k=0, 1$, and define a canonical basis of $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_L(k))$, which is used to compare $\varepsilon_{L,\zeta}(\mathcal{R}_L(1))$ with $\varepsilon^{\mathrm{dR}}_{L,\zeta}(\mathcal{R}_L(1))$ in Corollary 4.21 and Lemma 4.22. All the results in this appendix seem to be known (see for example [Benois 2000]), but here we give another proof of these results in the framework of (φ, Γ)-modules over the Robba ring. Of course, we may assume that $L = \mathbb{Q}_p$ by base change.

We first consider $\mathrm{H}^i_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$. If we identify by

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G^{\mathrm{ab}}_{\mathbb{Q}_p}, \mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^{\times}, \mathbb{Q}_p) : \tau \mapsto \tau \circ \mathrm{rec}_{\mathbb{Q}_p},$$

then this has a basis $\{[\mathrm{ord}_p], [\log]\}$ defined by

$$\mathrm{ord}_p : \mathbb{Q}_p^{\times} \to \mathbb{Q}_p : p \mapsto 1,\ a \mapsto 0 \qquad \text{for } a \in \mathbb{Z}_p^{\times},$$
$$\log : \mathbb{Q}_p^{\times} \to \mathbb{Q}_p : p \mapsto 0,\ a \mapsto \log(a) \quad \text{for } a \in \mathbb{Z}_p^{\times}.$$

We define a basis e_0 of $\mathrm{H}^0_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ and $\{e_{1,1}, e_{1,2}\}$ of $\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ by

$$e_0 = 1 \in \mathcal{R}_{\mathbb{Q}_p}, \quad e_{1,1} := [\log(\chi(\gamma)), 0], \quad e_{1,2} := [0, 1].$$

The basis is independent of the choice of γ, i.e., is compatible with the comparison isomorphism $\iota_{\gamma,\gamma'}$. We can easily check that the canonical isomorphism $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) \xrightarrow{\sim} \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ sends $[\log]$ to $e_{1,1}$ and $[\mathrm{ord}_p]$ to $e_{1,2}$.

We next consider $\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$. Let us denote by

$$\kappa : \mathbb{Q}_p^{\times} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$$

the Kummer map. Composing this with the canonical isomorphism

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)),$$

we obtain a homomorphism

$$\kappa_0 : \mathbb{Q}_p^{\times} \to \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)).$$

We define a homomorphism

$$\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \to \mathbb{Q}_p \oplus \mathbb{Q}_p :$$
$$[f_1\boldsymbol{e}_1, f_2\boldsymbol{e}_1] \mapsto \left(\frac{p}{p-1}\cdot\frac{1}{\log(\chi(\gamma))}\cdot \mathrm{Res}_0\left(f_1\frac{d\pi}{1+\pi}\right), -\frac{p}{p-1}\cdot \mathrm{Res}_0\left(f_2\frac{d\pi}{1+\pi}\right)\right)$$

(we note that $\frac{p-1}{p}\cdot\log(\chi(\gamma)) = |\Gamma_{\mathrm{tor}}|\cdot\log_0(\chi(\gamma))$), which is also independent of the choice of γ, and is an isomorphism. Using this isomorphism, we define a basis $\{f_{1,1}, f_{1,2}\}$ of $\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$ such that $f_{1,1}$ (resp. $f_{1,2}$) corresponds to $(1,0)\in L\oplus L$ (resp. $(0,1)$) by this isomorphism. We want to explicitly describe the map κ_0 using this basis. For this, we first prove the following lemma.

Lemma 5.1. *For each $a\in\mathbb{Z}_p^\times$, we have $\kappa_0(a) = \log(a)\cdot f_{1,2}$.*

Proof. By the classical explicit calculation of the exponential map, we have

$$\kappa(a) = \exp_{\mathbb{Q}_p(1)}\left(\frac{\log(a)}{t}\boldsymbol{e}_1\right).$$

Since we have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)) & \xrightarrow{\exp_{\mathbb{Q}_p(1)}} & \mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p(1)) \\ \wr\downarrow & & \wr\downarrow \\ \boldsymbol{D}_{\mathrm{dR}}(\mathcal{R}_{\mathbb{Q}_p}(1)) & \xrightarrow{\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}} & \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \end{array}$$

by Proposition 2.26, it suffices to show that

$$\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}\left(\frac{1}{t}\boldsymbol{e}_1\right) = f_{1,2}.$$

We show this equality as follows. We first take some $f\in(\mathcal{R}^\infty_{\mathbb{Q}_p})^\Delta$ such that $f(\zeta_{p^n}-1) = 1/p^n$ for any $n\geq 0$, which is possible since we have an isomorphism $\mathcal{R}^\infty_{\mathbb{Q}_p}/t \xrightarrow{\sim} \prod_{n\geq 0}\mathbb{Q}_p(\zeta_{p^n}) : \bar{f}\mapsto (f(\zeta_{p^n}-1))_{n\geq 0}$ by Lazard's theorem [1962]. Then the element $\frac{f}{t}\boldsymbol{e}_1 \in \left(\frac{1}{t}\mathcal{R}_{\mathbb{Q}_p}(1)\right)^\Delta$ satisfies

$$\iota_n\left(\frac{f}{t}\boldsymbol{e}_1\right) - \frac{1}{t}\boldsymbol{e}_1 \in \boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_{\mathbb{Q}_p}(1))$$

for any $n\geq 1$, since we have

$$\iota_n\left(\frac{f}{t}\boldsymbol{e}_1\right) \equiv p^n\cdot\frac{f(\zeta_{p^n}-1)}{t}\boldsymbol{e}_1 = \frac{1}{t}\boldsymbol{e}_1 \pmod{\boldsymbol{D}^+_{\mathrm{dif},n}(\mathcal{R}_{\mathbb{Q}_p}(1))}.$$

By the explicit definition of $\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}$ (Proposition 2.23(1)), we have

$$\exp_{\mathcal{R}_{\mathbb{Q}_p}(1)}\Big(\frac{1}{t}\boldsymbol{e}_1\Big) = \Big[(\gamma-1)\Big(\frac{f}{t}\boldsymbol{e}_1\Big), (\varphi-1)\Big(\frac{f}{t}\boldsymbol{e}_1\Big)\Big] \in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)).$$

Hence, it suffices to show that

$$\mathrm{Res}_0\Big(\frac{\gamma(f)-f}{t}\cdot\frac{d\pi}{1+\pi}\Big)=0$$

and

$$\mathrm{Res}_0\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{t}\cdot\frac{d\pi}{1+\pi}\Big)=-\frac{p-1}{p}.$$

Here, we only calculate

$$\mathrm{Res}_0\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{t}\cdot\frac{d\pi}{1+\pi}\Big)$$

(the calculation of

$$\mathrm{Res}_0\Big(\frac{\gamma(f)-f}{t}\cdot\frac{d\pi}{1+\pi}\Big)$$

is similar). By definition of f, we have

$$\frac{\varphi(f)(\zeta_{p^n}-1)}{p}-f(\zeta_{p^n}-1)=\frac{f(\zeta_{p^{n-1}}-1)}{p}-f(\zeta_{p^n}-1)=\frac{1}{p}\cdot\frac{1}{p^{n-1}}-\frac{1}{p^n}=0$$

for each $n\geq 1$. Hence, we have

$$\Big(\frac{\varphi(f)}{p}-f\Big)\in\Big(\prod_{n\geq 1}^{\infty}\frac{Q_n(\pi)}{p}\Big)\mathcal{R}^{\infty}_{\mathbb{Q}_p}$$

by the theorem of Lazard [1962], where we define $Q_n(\pi):=\varphi^{n-1}\big(\varphi(\pi)/\pi\big)$ for each $n\geq 1$. Since we have $t=\pi\prod_{n\geq 1}\big(Q_n(\pi)/p\big)$, we obtain the equality

$$\begin{aligned}\mathrm{Res}_0\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{t}\cdot\frac{d\pi}{1+\pi}\Big)&=\Big(\Big(\frac{\varphi(f)}{p}-f\Big)\cdot\frac{1}{\prod_{n\geq 1}^{\infty}\frac{Q_n(\pi)}{p}}\cdot\frac{1}{1+\pi}\Big)\Big|_{\pi=0}\\&=\Big(\frac{\varphi(f)}{p}-f\Big)\Big|_{\pi=0}=\frac{f(0)}{p}-f(0)=-\frac{p-1}{p},\end{aligned}$$

where the second equality follows from the fact that $\dfrac{Q_n(0)}{p}=1$ for $n\geq 1$, which proves the lemma. □

Before calculating $\kappa_0(p)\in\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$, we explicitly describe Tate's trace map in terms of (φ,Γ)-modules. We note that we normalize Tate's trace map

$$\mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1))\xrightarrow{\sim}\mathbb{Q}_p$$

so that the cup product pairing

$$\langle -, - \rangle : \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \times \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) \xrightarrow{\cup} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$$

satisfies

$$\langle \kappa(a), [\tau] \rangle = \tau(a)$$

for $a \in \mathbb{Q}_p^\times$ and $[\tau] \in \mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Q}_p) = \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p)$ (we remark that this normalization coincides with the one used in §2.4 of [Nakamura 2014a] and with -1 times the one in [Kato 1993a, Chapter II, §1.4]).

Proposition 5.2. *The map* $\iota_\gamma : \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \xrightarrow{\sim} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$, *which is the composition of the canonical isomorphism* $\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \xrightarrow{\sim} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1))$ *with Tate's trace map is explicitly defined by*

$$\iota_\gamma([f\boldsymbol{e}_1]) = \frac{p}{p-1} \cdot \frac{1}{\log(\chi(\gamma))} \mathrm{Res}_0\left(f \frac{d\pi}{1+\pi}\right).$$

Proof. Since the map

$$\iota : \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \xrightarrow{\sim} \mathbb{Q}_p : [f\boldsymbol{e}_1] \mapsto \mathrm{Res}_0\left(f \frac{d\pi}{1+\pi}\right)$$

is a well-defined isomorphism, there exists a unique $\alpha \in \mathbb{Q}_p^\times$ such that $\iota_\gamma = \alpha \cdot \iota$. We calculate α as follows.

We recall that the element $[\log(\chi(\gamma)), 0] \in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p})$ is the image of $[\log] \in \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p)$ by the comparison isomorphism. By the proof of Lemma 5.1, for each $a \in \mathbb{Z}_p^\times$, we have

$$\kappa_0(a) = \log(a)\left[(\gamma - 1)\left(\frac{f}{t}\boldsymbol{e}_1\right), (\varphi - 1)\left(\frac{f}{t}\boldsymbol{e}_1\right)\right] \in \mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)),$$

where $f \in \mathcal{R}^\infty_{\mathbb{Q}_p}$ is an element defined in the proof of Lemma 5.1. Since the cup products are compatible with the comparison isomorphism (see Remark 2.12), we have

$$\iota_\gamma\big(\kappa_0(a) \cup [\log(\chi(\gamma)), 0]\big) = \langle \kappa(a), [\log] \rangle = \log(a). \tag{67}$$

By definition of the cup product, we have

$$\begin{aligned}\kappa_0(a) \cup [\log(\chi(\gamma)), 0] &= \log(a)\left[(\varphi - 1)\left(\frac{f}{t}\boldsymbol{e}_1\right) \otimes \varphi(\log(\chi(\gamma)))\right] \\ &= -\log(a)\log(\chi(\gamma))\left[(\varphi - 1)\left(\frac{f}{t}\boldsymbol{e}_1\right)\right] \in \mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)).\end{aligned}$$

Since $\mathrm{Res}_0\Big((\varphi-1)\Big(\frac{f}{t}\Big)\cdot\frac{d\pi}{1+\pi}\Big)=-\frac{p-1}{p}$ by the proof of Lemma 5.1, we obtain

$$\begin{aligned}\iota_\gamma\big(\kappa_0(a)\cup[\log(\chi(\gamma)),0]\big)&=\alpha\cdot\iota\big(\kappa_0(a)\cup[\log(\chi(\gamma)),0]\big)\\&=-\alpha\cdot\log(\chi(\gamma))\cdot\log(a)\cdot\iota\Big(\Big[(\varphi-1)\Big(\frac{f}{t}\boldsymbol{e}_1\Big)\Big]\Big)\\&=\alpha\cdot\log(\chi(\gamma))\cdot\log(a)\cdot\frac{p-1}{p}.\end{aligned}$$

Comparing this equality with the equality (67), we obtain

$$\alpha=\frac{p}{p-1}\cdot\frac{1}{\log(\chi(\gamma))},$$

which proves the proposition. □

Finally, we prove the following lemma, which completes the calculation of the map $\kappa_0:\mathbb{Q}_p^\times\to\mathbb{Q}_p\oplus\mathbb{Q}_p$.

Lemma 5.3. $\kappa_0(p)=f_{1,1}$.

Proof. Take $f_{1,1}=[f_1\boldsymbol{e}_1,\,f_2\boldsymbol{e}_1]\in\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1))$ to be a representative of $f_{1,1}$. By definition of the cup product, we have

$$\begin{aligned}\iota_\gamma(f_{1,1}\cup e_{1,1})&=\iota_\gamma\big(f_{1,1}\cup[\log(\chi(\gamma)),0]\big)\\&=-\iota_\gamma\big([f_2\boldsymbol{e}_1\otimes\varphi(\log(\chi(\gamma)))]\big)=-\frac{p}{p-1}\mathrm{Res}_0\Big(f_2\frac{d\pi}{1+\pi}\Big)=0,\end{aligned}$$

and

$$\begin{aligned}\iota_\gamma(f_{1,1}\cup e_{1,2})&=\iota_\gamma(f_{1,1}\cup[0,1])\\&=\iota_\gamma([f_1\boldsymbol{e}_1\otimes\gamma(1)])=\frac{p}{p-1}\cdot\frac{1}{\log(\chi(\gamma))}\cdot\mathrm{Res}_0\Big(f_1\frac{d\pi}{1+\pi}\Big)=1\end{aligned}$$

by Proposition 5.2. Since $\kappa(p)\in\mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p(1))$ satisfies the similar formulae

$$\langle\kappa(p),[\mathrm{ord}_p]\rangle=1,\quad\langle\kappa(p),[\log]\rangle=0,$$

we obtain the equality

$$\kappa_0(p)=f_{1,1}.$$

□

Using these lemmas, we obtain the following result. We define the basis f_2 of $\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))$ by $f_2:=\iota_\gamma^{-1}(1)$.

Lemma 5.4. *Tate's duality pairings*

$$\langle-,-\rangle:\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_L(1))\times\mathrm{H}^1_{\varphi,\gamma}(\mathcal{R}_L)\xrightarrow{\cup}\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))\xrightarrow{\iota_\gamma}L$$

and

$$\langle-,-\rangle:\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))\times\mathrm{H}^0_{\varphi,\gamma}(\mathcal{R}_L)\xrightarrow{\cup}\mathrm{H}^2_{\varphi,\gamma}(\mathcal{R}_L(1))\xrightarrow{\iota_\gamma}L$$

satisfy

$$\langle f_{1,1}, e_{1,1}\rangle = 0, \quad \langle f_{1,1}, e_{1,2}\rangle = 1$$
$$\langle f_{1,2}, e_{1,1}\rangle = 1, \quad \langle f_{1,2}, e_{1,2}\rangle = 0,$$
$$\langle f_2, e_0\rangle = 1.$$

Proof. That we have $\langle f_{1,1}, e_{1,1}\rangle = 0$ and $\langle f_{1,1}, e_{1,2}\rangle = 1$ is proved in Lemma 5.3. We prove the formula for $f_{1,2}$. By Lemma 5.1, we have an equality $f_{1,2} = \kappa_0(a)/\log(a)$ for any nontorsion $a \in \mathbb{Z}_p^\times$. Hence, we obtain

$$\langle f_{1,2}, e_{1,1}\rangle = \frac{1}{\log(a)}\langle \kappa(a), [\log]\rangle = 1,$$
$$\langle f_{1,2}, e_{1,2}\rangle = \frac{1}{\log(a)}\langle \kappa(a), [\mathrm{ord}_p]\rangle = 0$$

by the compatibility of the cup products. Finally, that $\langle f_2, e_0\rangle = 1$ is trivial by definition. □

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First covering of the Drinfel'd upper half-plane and Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Lue Pan

For an odd prime p, we construct some admissible Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that conjecturally should correspond to some 2-dimensional tamely ramified, potentially Barsotti–Tate representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ via the p-adic local Langlands correspondence. To achieve this, we generalize Breuil's work in the semistable case and work on the first covering of the Drinfel'd upper half-plane. Our main tool is an explicit semistable model of the first covering.

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1. Introduction

Breuil [2004] constructed some Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, which conjecturally should be nonzero and admissible and correspond to 2-dimensional semistable, noncrystalline representations of $G_{\mathbb{Q}_p}$ under the p-adic local Langlands correspondence. Here $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, where $\overline{\mathbb{Q}_p}$ is some fixed algebraic closure of $\mathbb{Q}_p$. Later on Colmez [2004] found the relationship between these Banach representations and (ϕ, Γ)-modules and proved their admissibility. Breuil and Mézard [2010] also proved the admissibility in some cases by explicitly computing the mod p reductions of these Banach representations. The aim of this paper is to generalize Breuil's work to some 2-dimensional tamely ramified, potentially Barsotti–Tate representations of $G_{\mathbb{Q}_p}$.

First we recall some of Breuil's [2004] construction. Let E be a finite extension of $\mathbb{Q}_p$ and k an integer greater than 2. Up to a twist by some character, all 2-dimensional semistable, noncrystalline E-representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, k-1)$ are classified by the "$\mathcal{L}$-invariant" [Breuil 2004, exemple 1.3.5]. We use $V(k, \mathcal{L})$ to denote this Galois representation. Here $\mathcal{L}$ is an element in E and basically tells you the position of the Hodge filtration on the Weil–Deligne representation associated to $V(k, \mathcal{L})$. Notice that this Weil–Deligne representation does not depend on $\mathcal{L}$. So via the classical local Langlands correspondence, all $V(k, \mathcal{L})$ correspond to the same smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is a twist of St, the usual Steinberg representation.

Breuil's idea is that for each $\mathcal{L}$, there should exist a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm on $\underline{\mathrm{Sym}}^{k-2} E^2 \otimes \mathrm{St}$; here $\underline{\mathrm{Sym}}^{k-2} E^2$ is a twist of the algebraic representation $\mathrm{Sym}^{k-2} E^2$. Different $\mathcal{L}$ should give different noncommensurable unit balls of $\underline{\mathrm{Sym}}^{k-2} E^2 \otimes \mathrm{St}$. If we take the completion, we get a Banach representation $B(k, \mathcal{L})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ for each $\mathcal{L}$. Moreover, we hope this representation is admissible in the sense of [Schneider and Teitelbaum 2002] and the correspondence between $V(k, \mathcal{L})$ and $B(k, \mathcal{L})$ is compatible with the mod p correspondence defined by Breuil [2003].

So how to construct these $B(k, \mathcal{L})$? For simplicity, I assume $E = \mathbb{Q}_p$ and k is even. The strategy of Breuil is to realize the unit ball $O(k, \mathcal{L})^U$ of the dual representation of $B(k, \mathcal{L})$ in $O(k) = \Gamma(\Omega, \mathcal{O}(k))$, where $\mathcal{O}(k)$ is a coherent sheaf on the Drinfel'd upper half-plane Ω over $\mathbb{Q}_p$. Concretely, $\mathcal{O}(2)$ is the sheaf of rigid differential forms and $\mathcal{O}(2n) = \mathcal{O}(2)^{\otimes n}$. Here Ω is considered as a rigid analytic space and $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on everything. We note that the de Rham cohomology of Ω is nothing but $\mathrm{St}^\vee$, the algebraic dual representation of St [Schneider and Stuhler 1991, Theorem 1]. The construction of $O(k, \mathcal{L})^U$, as far as I understand, has the following two important properties:

(1) $O(k, \mathcal{L})^U$ is "globally bounded" and hence compact. In other words, it is contained in $\Gamma(\widehat{\Omega}, \omega^{\otimes k/2})$, where $\widehat{\Omega}$ is a semistable model of Ω and ω is an

integral structure of $\mathcal{O}(2)$. This guarantees that the dual of $O(k,\mathcal{L})^U$ is indeed a Banach representation (after inverting p).

(2) If $f \in O(k)$ comes from a modular form of weight k (see [Breuil 2004, section 5] for the precise meaning), then $f \in O(k,\mathcal{L}_0)^U$ if and only the $\mathcal{L}$-invariant of f is $\mathcal{L}_0$.

Now consider the case where the Galois representation is tamely ramified. We will see later that the situation is very similar. Fix E a finite extension of $\mathbb{Q}_p$ large enough and let O_E be its ring of integers. This time we need to work on the first covering of Drinfel'd upper half-plane. According to Drinfel'd, there is a universal p-divisible group X over $\widehat{\Omega}\hat{\otimes}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ and O_D acts on it, where O_D is the ring of integers inside the quaternion algebra D over $\mathbb{Q}_p$. Fix a uniformizer $\Pi \in O_D$ and define $\mathcal{X}_n$ as the generic fiber of $X[\Pi^n]$. The first covering $\Sigma_1 = \mathcal{X}_1 - \mathcal{X}_0$, also carries the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $O_D^\times$. It was shown by Drinfel'd [1976] that the action of $O_D^\times$ can be extended to $D^\times$. This is a left action and we will keep this convention in this paper unless explicitly inverting it. One remark is that the actions of $\mathbb{Q}_p^\times$ inside $D^\times$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ become the same once we invert the action of $D^\times$.

First we note that the (E-coefficient) de Rham cohomology $H^1_{\mathrm{dR}}(\Sigma_1,E) \overset{\mathrm{def}}{=} H^1_{\mathrm{dR}}(\Sigma_1)\otimes_{\mathbb{Q}_p} E$ of Σ_1 has the following decomposition. Let $\psi : \mathbb{Q}_p^\times \to O_E^\times$ be a unitary character of level 0 in the sense that $1+p\mathbb{Z}_p$ is contained in the kernel of ψ. We will view it as a character of $\mathbb{Q}_p^\times \subset D^\times$. In the following theorem, we invert the action of $D^\times$ so that it acts on the cohomology on the left. We denote the ψ-isotypic component of $H^1_{\mathrm{dR}}(\Sigma_1,E)$ by $H^1_{\mathrm{dR}}(\Sigma_1,E)^\psi$.

Theorem 1.1. *As a representation of* $D^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$,

$$H^1_{\mathrm{dR}}(\Sigma_1,E)^\psi \simeq \bigoplus_{\pi\in\mathcal{A}^0(D^\times)(\psi^\vee)_0} (\pi\otimes \mathrm{JL}(\pi))^\vee \otimes_E D_\pi,$$

where $\cdot^\vee$ *denotes the algebraic dual representation*, $\mathcal{A}^0(D^\times)(\psi^\vee)_0$ *is the space of admissible irreducible representations of* $D^\times$ *of level* 0 *over* E *that are not characters and with central character* $\psi^\vee$ (see [Bushnell and Henniart 2006, Chapter 13]), $\mathrm{JL}(\pi)$ *is the representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *associated to* π *by the Jacquet–Langlands correspondence*, *and* D_π *is a two-dimensional vector space over* E.

Remark 1.2. In fact, we can define more structures on D_π. Roughly speaking, we may find a finite extension F of $\mathbb{Q}_p$ such that

$$F\otimes_{\mathbb{Q}_p} D_\pi \simeq F\otimes_{F_0} D_{\mathrm{crys},\pi},$$

where F_0 is the maximal unramified extension of $\mathbb{Q}_p$ inside F and $D_{\mathrm{crys},\pi}$ is a $(\varphi,N,F/\mathbb{Q}_p,E)$-module (see Section 13 for the notation here). Then up to some unramified character, the Weil–Deligne representation associated to $D_{\mathrm{crys},\pi}$

corresponds to $\mathrm{JL}(\pi)$ under the classical local Langlands correspondence. See Theorem 1.10 below.

Explicitly, any $\pi \in \mathcal{A}^0(D^\times)(\psi^\vee)_0$ is an induced representation

$$\pi \simeq \mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \Xi,$$

where $\Xi : O_D^\times \mathbb{Q}_p^\times \to O_E^\times$ is a character which extends $\psi^\vee$ and is trivial on $1+\Pi O_D$. It is clear that π has an integral structure π_0 over O_E.

As we noted before, we need to construct a $\mathrm{GL}_2(\mathbb{Q}_p)\times D^\times$-equivariant formal model $\widehat{\Sigma_1^{\mathrm{nr}}}$ of Σ_1. This will be done by using Raynaud's theory of $\mathbb{F}$-vector space schemes. As Breuil did in the case of the Drinfel'd upper half-plane, we can define a $\mathrm{GL}_2(\mathbb{Q}_p)\times D^\times$-equivariant integral model ω^1 of $\Omega^1_{\Sigma_1}$ on this formal model, where $\Omega^1_{\Sigma_1}$ is the sheaf of differential forms (see Remark 14.2). Consider the composition of the following maps:

$$H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \to H^0(\Sigma_1, \Omega^1_{\Sigma_1}) \to H^1_{\mathrm{dR}}(\Sigma_1).$$

We will show that this map is injective (Proposition 14.6), so that $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)$ can be viewed as a subspace in the de Rham cohomology. Rewrite Theorem 1.1 as

$$H^1_{\mathrm{dR}}(\Sigma_1, E)(\pi^\vee) = H^1_{\mathrm{dR}}(\Sigma_1, E)^{\psi}(\pi^\vee) \simeq \mathrm{JL}(\pi)^\vee \otimes D_\pi,$$

where $(\cdot)(\pi^\vee) = \mathrm{Hom}_{E[D^\times]}(\pi^\vee, \cdot)$. For any line $\mathcal{L}$ inside D_π (the $\mathcal{L}$-invariant in our case), we may view $\mathrm{JL}(\pi)^\vee \otimes \mathcal{L}$ as a subspace inside $H^1_{\mathrm{dR}}(\Sigma_1, E)(\pi^\vee)$ by the above isomorphism. We can now define the (dual) of our Banach space representations:

Definition 1.3. $M(\pi, \mathcal{L}) \overset{\mathrm{def}}{=} \big(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E\big)(\pi_0^\vee) \cap \big(\mathrm{JL}(\pi)^\vee \otimes \mathcal{L}\big).$

Recall that π_0 is some integral structure of π. Notice that $M(\pi, \mathcal{L})$ is contained in $\big(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E\big)(\pi^\vee)$, a natural subspace of $\big(H^0(\Sigma_1, \Omega^1_{\Sigma_1}) \otimes_{\mathbb{Q}_p} E\big)(\pi^\vee)$. This last space has a natural Fréchet space structure over E. The induced topology on $M(\pi, \mathcal{L})$ makes it into a compact topological space, and thus allows us to introduce:

Definition 1.4. $B(\pi, \mathcal{L}) = \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M(\pi, \mathcal{L}), E).$

This is a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Remark 1.5. The argument of [Breuil 2004, lemme 4.1.1] shows that $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)$ and $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega'^1)$ are commensurable, where ω'^1 is any other $\mathrm{GL}_2(\mathbb{Q}_p)\times D^\times$-equivariant integral model of $\Omega^1_{\Sigma_1}$. Hence $B(\pi, \mathcal{L})$ is independent of the choice of ω^1.

Now we can state the main result of this paper. Assume p is an odd prime.

Theorem 1.6. (1) *$B(\pi, \mathcal{L})$ is nonzero and admissible as a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$. *In fact, its mod p reduction can be computed explicitly.*

(2) *$B(\pi, \mathcal{L})$ is a unitary completion of* $\mathrm{JL}(\pi)$.

The computation will give us an interesting $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant short exact sequence (Corollaries 16.28 and 17.5):

Corollary 1.7. *The sequence*

$$0 \to \widehat{\mathrm{JL}(\pi)} \to H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)^d_E(\pi) \to B(\pi, \mathcal{L}) \to 0,$$

is exact, where $\widehat{\mathrm{JL}(\pi)}$ *is the universal unitary completion of* $\mathrm{JL}(\pi)$ (see [Emerton 2005]), *and*

$$H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)^d_E = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}\big(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1), E\big).$$

Note that the kernel and the middle term are independent of $\mathcal{L}$ while the map between them depends on $\mathcal{L}$.

Remark 1.8. Unfortunately, we have to assume $p \geq 3$ in the proof of Theorem 1.6 (for example in the proof of Lemma 16.4). However Theorem 1.1 is also true for $p = 2$.

Now we explain the strategy of proving Theorem 1.1. By twisting with some unramified unitary characters, it suffices to deal with the case where the central character ψ satisfies $\psi(p) = 1$. This suggests we descend Σ_1 from $\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$ to $\mathbb{Q}_{p^2}$, the unramified quadratic extension of $\mathbb{Q}_p$, by taking the "p-invariant" of Σ_1 (see Section 7). We use Σ_1^p to denote this rigid analytic space. One warning here: even though Σ_1^p has a structure map to $\mathbb{Q}_{p^2}$, I will view it as a rigid space over $\mathbb{Q}_p$. A semistable model of Σ_1^p is very helpful (see Theorem 8.4):

Theorem 1.9. $\Sigma_1^p \times_{\mathbb{Q}_p} F$ *has an explicit* $D^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant semistable model* $\widehat{\Sigma^{(0)}_{1,O_F}}$ *over* O_F, *where* $F \simeq \mathbb{Q}_{p^2}[(-p)^{1/(p^2-1)}]$.

Similar results have been obtained before by Teitelbaum [1990].

Denote the generic fiber of this semistable model by $\Sigma^{(0)}_{1,F} = \Sigma_1^p \times_{\mathbb{Q}_p} F$. With the help of the semistable model, we can compute its de Rham cohomology. Let $\chi(E)$ be the character group of $O_D^\times/(1 + \Pi O_D)$ with values in $E^\times$. Recall that $O_D^\times$ acts on $\Sigma^{(0)}_{1,F}$. We have the following result (see Section 12, especially Corollary 12.10 and Remark 12.11):

Theorem 1.10. *For any* $\chi \in \chi(E)$ *such that* $\chi \neq \chi^p$, *we have a* $\mathrm{Gal}(F/\mathbb{Q}_p) \times O_D^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism*:

$$F \otimes_{F_0} D_{\mathrm{crys},\chi} \otimes_E \big(\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_{\chi^{-1}}\big)^\vee \xrightarrow{\sim} \big(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E\big)^\chi,$$

where $F_0 \simeq \mathbb{Q}_{p^2}$, c-Ind *is the induction with compact support*, $\cdot^\vee$ *means the algebraic dual*, $\rho_{\chi^{-1}}$ *is a cuspidal representation of* $\mathrm{GL}_2(\mathbb{F}_p)$ *over* E *defined via Deligne–Lusztig theory*, *and* $D_{\mathrm{crys},\chi}$ *is a free* $F_0 \otimes E$*-module of rank* 2 *with an action of* $\mathrm{Gal}(F/\mathbb{Q}_p)$. *In addition*, *we can define a Frobenius operator* φ *acting on it. It is explicitly described in Proposition 12.8.*

Take $\pi = \mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \chi$, where χ is viewed as a character of $O_D^\times \mathbb{Q}_p^\times$ that is trivial on p. Then $\mathrm{JL}(\pi) \simeq \text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$. Theorem 1.1 follows from the above theorem by taking the $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariants. There is some inverse involved since we invert the action of $D^\times$ in Theorem 1.1.

It is clear from the theorem that $D_{\mathrm{crys},\chi}$ is a $(\varphi, N, F/\mathbb{Q}_p, E)$-module. A line $\mathcal{L}$ inside D_π, or equivalently, a $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariant "line" inside $F \otimes_{F_0} D_{\mathrm{crys},\chi}$, essentially gives a filtration and makes $D_{\mathrm{crys},\chi}$ into a filtered $(\varphi, N, F/\mathbb{Q}_p, E)$-module. See Section 13 for more details.

After fixing some basis for $D_{\mathrm{crys},\chi}$ (see Proposition 12.8), any line $\mathcal{L}$ can be identified with an element b inside E or ∞. Assume $b \in O_E$ for the moment. We will write

$$M(\chi, [1, b]) = M(\mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \chi, \mathcal{L}),$$

and similarly $B(\chi, [1, b]) = B(\mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \chi, \mathcal{L})$.

Some notation here: Fix a $\mathbb{Z}_p$-linear embedding of $W(\mathbb{F}_{p^2})$, the Witt vector of $\mathbb{F}_{p^2}$ into O_D. Then any $\chi \in \chi(E)$ can be viewed as a character of $\mathbb{F}_{p^2}^\times$ by composing this embedding with the Teichmüller character. Also fix an embedding τ of $W(\mathbb{F}_{p^2})$ into E. Similarly the Teichmüller character gives us a character $\chi_\tau : \mathbb{F}_{p^2}^\times \to E^\times$.

Definition 1.11. We define m as the unique integer in $\{0, \dots, p^2 - 2\}$ such that

$$\chi = \chi_\tau^{-m} : \mathbb{F}_{p^2}^\times \to O_E^\times.$$

We will write $m = i + (p+1)j$, where $i \in \{0, \dots, p\}$, $j \in \{0, \dots, p-2\}$ and $[-mp]$ as the unique integer in $\{0, \dots, p^2 - 2\}$ congruent to $-mp$ modulo $p^2 - 1$.

Let $\sigma_i(j)$ be the following representation of $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\sigma_i(j) = \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} (\mathrm{Sym}^i \mathbb{F}_p^2) \otimes O_E/p \otimes \det{}^j,$$

where $\mathrm{Sym}^i \mathbb{F}_p^2$ is the i-th symmetric power of the natural representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the canonical basis of $\mathbb{F}_p^2$, viewed as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ trivial on $p^{\mathbb{Z}}$.

Using our explicit semistable model, we can compute the mod p reduction of $M(\chi, [1, b])$ (Corollary 16.29, Remark 16.30, Corollary 17.6).

Theorem 1.12. *Let T be the usual Hecke operator* (defined in [Breuil 2007]) *and let $c(\chi, b) = (-1)^{j+1}\tau(w_1^{-i})b \in O_E/p$, where $\tau(w_1)$ satisfies $\tau(w_1)^{p+1} = -1$ and is independent of $\chi, \mathcal{L}$.*

(1) *Assume $p^2 - 1 - m \geq [-mp], i \in \{2, \dots, p-1\}$. As a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$,

$$0 \to \{X \in \sigma_{i-2}(j+1) \mid c(\chi, b)X = T(X)\} \to M(\chi, [1, b])/p \to \{X \in \sigma_{p-1-i}(i+j) \mid X = c(\chi, b)T(X)\} \to 0.$$

(2) *Assume* $p^2-1-m \le [-mp]$, $i \in \{2, \dots, p-1\}$.

$$0 \to \left\{X \in \sigma_{p-1-i}(i+j) \mid X = c(\chi, b)T(X)\right\} \to M(\chi, [1, b])/p \\ \to \left\{X \in \sigma_{i-2}(j+1) \mid c(\chi, b)X = T(X)\right\} \to 0.$$

(3) *Assume* $i = p$. *Then*

$$M(\chi, [1, b])/p \simeq \left\{X \in \sigma_{p-2}(j+1) \mid -c(\chi, b)X + T(X) - c(\chi, b)T^2(X) = 0\right\}.$$

(4) *Assume* $i = 1$. *Then*

$$M(\chi, [1, b])/p \simeq \{X \in \sigma_{p-2}(j+1),\ X + c(\chi, b)T(X) + T^2(X) = 0\}.$$

Thus in any case, $B(\chi, [1, b])$ *is nonzero and admissible.*

Remark 1.13. In a recent paper Gabriel Dospinescu and Arthur-César Le Bras [Dospinescu and Le Bras 2015] independently use a very similar method to construct some locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and verify the compatibilities with the p-adic local Langlands correspondence, and thus generalize Breuil's [2004] work in this direction. Their method works for all the coverings of the Drinfel'd upper half-plane and relies on the previous work of Colmez on the relationship between Banach space representations and (ϕ, Γ)-modules. Combining their results with some known results of p-adic local Langlands correspondence, they can also prove Theorem 1.1 and Theorem 1.6. However, it seems that Corollary 1.7 does not follow directly from their work.

We give a brief outline of this paper. The goal of the next eight sections (Sections 2–9) is to explicitly write down a semistable model of Σ_1. Our strategy is to apply Raynaud's [1974] theory of $\mathbb{F}$-vector spaces schemes to X_1. We will collect some basic facts about the Drinfel'd upper half-plane in Section 2 and review Raynaud's theory in Section 3. To compute the data in Raynaud's theory, we need the existence of some "polarization" of X_1 (Proposition 5.1), which comes from a formal polarization of X (Section 4). Using this, a formal model is obtained in Section 5. By comparing the invariant differential forms of X_1 computed in two different ways, we write down the local equation of this formal model in Section 6. From this, it's not too hard to work out a semistable model in Section 8 and make clear how $\mathrm{GL}_2(\mathbb{Z}_p)$, $O_D^\times$, and $\mathrm{Gal}(F/\mathbb{Q}_p)$ act on it in Section 9.

In Sections 10–12 we compute the de Rham cohomology of $\Sigma_{1,F}^{(0)}$. Using our semistable models, this can be expressed by the crystalline cohomology of the irreducible components of the special fiber, which is well-understood via Deligne–Lusztig theory. The main result is Corollary 12.10, which describes the structure of the de Rham cohomology.

In Section 13 we classify all possible filtrations on $D_{\mathrm{crys},\chi}$ with Hodge–Tate weights (0, 1). We use this result to define $M(\chi, [1, b])$ in Section 14.

Sections 15, 16, and 17 contain the computation of $M(\chi, [1, b])/p$. In Section 15, we compute $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)/p$ (not exactly this space, see the precise statement there). Roughly speaking, the method is by carefully studying the shape of differential forms on each irreducible component of the special fiber. The main result is Proposition 15.13 which says that this space is an extension of two inductions. Sections 16 and 17 treat different cases of computations of $M(\chi, [1, b])/p$ according to the value of i, but their strategies are the same: First we interpret $M(\chi, [1, b])$ as the kernel of a map θ_b from $(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \otimes O_E/p)^\chi$ to a space J_2. Then we compute the mod p reduction $\bar{\theta}_b$ of this map explicitly and show that $\bar{\theta}_b$ is in fact surjective. Hence θ_b has to be surjective as well since both $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)$ and J_2 are p-adically complete. Therefore $M(\chi, [1, b])$ is just the kernel of $\bar{\theta}_b$.

Notation. Throughout this paper, fix an odd prime number p.

Let $\mathbb{Q}_p^{\mathrm{nr}}$ be the maximal unramified extension of $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$ be its p-adic completion. We will write $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$, the ring of Witt vectors of $\mathbb{F}_{p^2}$ and fix an embedding of it into $\mathbb{Q}_p^{\mathrm{nr}}$. Denote the fractional field of $\mathbb{Z}_{p^2}$ by $\mathbb{Q}_{p^2}$. We use F_0 to denote the unique unramified quadratic extension of $\mathbb{Q}_p$. Hence the fixed embedding of $\mathbb{Q}_{p^2}$ into $\mathbb{Q}_p^{\mathrm{nr}}$ gives us an isomorphism between F_0 and $\mathbb{Q}_{p^2}$. Later on, F_0 will appear as some intermediate field extension when we try to compute a semistable model. Let O_{F_0} be the ring of integers inside F_0. Frequently we will identify F_0 with $\mathbb{Q}_{p^2}$ by this fixed isomorphism.

We denote by D the quaternion algebra of $\mathbb{Q}_p$ and fix a uniformizer $\Pi \in D$ such that $\Pi^2 = p$. We will also fix a $\mathbb{Z}_p$-linear embedding of $\mathbb{Z}_{p^2}$ into O_D, hence an isomorphism:

$$O_D/\Pi O_D \simeq \mathbb{F}_{p^2}.$$

Let E be a finite extension of $\mathbb{Q}_p$ such that $\mathrm{Hom}_{\mathbb{Q}_p}(F_0, E) \neq 0$. We use $\tau, \bar{\tau}$ to denote the embeddings of F_0 into E and O_E to denote its ring of integers. For any O_{F_0}-module A, we denote $A \otimes_{O_{F_0},\tau} O_E$ by A_τ and $A \otimes_{O_{F_0},\bar{\tau}} O_E$ by $A_{\bar{\tau}}$.

For $K = E, F_0$, we use $\chi(K)$ to denote the character group of $O_D^\times/(1+\Pi O_D) = (O_D/\Pi)^\times$ with values in $K^\times$.

For any integer n, we will use $[n]$ to denote the unique integer in $\{0, 1, \ldots, p^2-2\}$ congruent to n modulo $p^2 - 1$.

For any ring A and integer n, we use $\mu_n(A)$ to denote $\{a \in A \mid a^n = 1\}$.

For any abelian group M, we denote the p-adic completion of M by $M^\wedge$.

We use $\mathrm{Sym}^i \mathbb{F}_p^2$ to denote the i-th symmetric power of the natural representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the canonical basis of $\mathbb{F}_p^2$ for i nonnegative. Explicitly, we can identify $\mathrm{Sym}^i \mathbb{F}_p^2$ with $\bigoplus_{r=0}^i \mathbb{F}_p x^r y^{i-r}$, where the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^i y^{i-r} = (ax+cy)^r (bx+dy)^{i-r}.$$

Sometimes we will also view it as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ by abuse of notation.

Also, we define an induced representation of $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\sigma_i = \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{Sym}^i\,\mathbb{F}_p^2) \otimes O_E/p,$$

where the induction has no restriction on the support and we view $\mathrm{Sym}^i\,\mathbb{F}_p^2$ as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ trivial on $p^{\mathbb{Z}}$. We define σ_{-1} as 0 and

$$\sigma_i(j) = \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{Sym}^i\,\mathbb{F}_p^2) \otimes O_E/p \otimes \det{}^j,$$

where det is the determinant map.

We recall the definition of Hecke operator T here. See Section 3.2 of [Breuil 2007] for more details. Let $\sigma = \mathrm{Sym}^r\,\mathbb{F}_p^2 \otimes \det^m$, $0 \le r \le p-2$ be an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\mathbb{F}_p$. I would like to view it as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ with p acting trivially. We use V_σ to denote the underlying representation space. Hence,

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\sigma = \{ f : \mathrm{GL}_2(\mathbb{Q}_p) \to V_\sigma \mid f(hg) = \sigma(h)(f(g)),\ h \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \}.$$

Note that we put no restriction on the support. Following [Breuil 2007], denote by

$$[g, v] : \mathrm{GL}_2(\mathbb{Q}_p) \to V_\sigma$$

the following element of $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\sigma$:

$$[g, v](g') = \begin{cases} \sigma(g'g)v & \text{if } g' \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times g^{-1}, \\ 0 & \text{if } g' \notin \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times g^{-1}. \end{cases}$$

We have $g([g', v]) = [gg', v]$ and $[gh, v] = [g, \sigma(h)v]$ if $h \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. It is clear that every element in $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\sigma$ can be written uniquely as an infinite sum of $[g_i, v_i]$ such that no two g_i are within the same coset $\mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. Identify V_σ with $\bigoplus_{k=0}^r \mathbb{F}_p x^k y^{r-k}$. We define $\varphi_r : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{End}_{\mathbb{F}_p}(V_\sigma, V_\sigma)$ as follows:

$$\begin{aligned}
&\varphi_r(g) = 0 \quad \text{if } g \notin \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p), \\
&\varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(x^k y^{r-k}) = 0 \quad \text{if } k \neq 0, \\
&\varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(y^r) = y^r, \\
&\varphi_r(h_1 g h_2) = \sigma(h_1) \circ \varphi_r(g) \circ \sigma(h_2), \qquad h_1, h_2 \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times.
\end{aligned}$$

The Hecke operator T_{φ_r} (or T for simplicity) is defined as:

$$T([g, v]) = \sum_{g' \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \in \mathrm{GL}_2(\mathbb{Q}_p)/(\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times)} [gg', \varphi_r(g'^{-1})(v)].$$

2. Some facts about the Drinfel'd upper half-plane

Let Ω be the p-adic upper half-plane (or Drinfel'd upper half plane) over $\mathbb{Q}_p$. It is a rigid analytic space over $\mathbb{Q}_p$ and its $\mathbb{C}_p$-points are $\mathbb{C}_p - \mathbb{Q}_p$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. There is a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Ω. On the set of $\mathbb{C}_p$-points, it is given by

$$z \mapsto z|_g = \frac{az+c}{bz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p).$$

Ω has a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant formal model $\widehat{\Omega}$ over $\mathbb{Z}_p$, which is described in detail in [Boutot and Carayol 1991]. One warning here: in this paper, the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Ω is a right action rather than a left action used in Drinfel'd's original paper [1976] and in [Boutot and Carayol 1991]. Our action is the inverse of their action. I apologize here if this causes any confusion.

Let me recall some facts we need to use later. There exists an open covering $\{\widehat{\Omega}_e\}_e$ on $\widehat{\Omega}$ indexed by the set of edges of the Bruhat–Tits tree I of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Two different $\widehat{\Omega}_e$ and $\widehat{\Omega}_{e'}$ have nonempty intersection if and only e and e' share a vertex s. When this happens, $\widehat{\Omega}_e \cap \widehat{\Omega}_{e'}$ only depends on the vertex s. We call it $\widehat{\Omega}_s$. For two adjacent vertices s, s', we denote the unique edge connecting them by $[s, s']$. Explicitly, ($\widehat{\ }$ is for p-adic completion)

$$\widehat{\Omega}_{s'} \simeq \mathrm{Spf}\, O_\eta \overset{\mathrm{def}}{=} \mathrm{Spf}\, \mathbb{Z}_p\Big[\eta, \frac{1}{\eta-\eta^p}\Big]^{\widehat{}} \tag{1}$$

$$\widehat{\Omega}_{s} \simeq \mathrm{Spf}\, O_\zeta \overset{\mathrm{def}}{=} \mathrm{Spf}\, \mathbb{Z}_p\Big[\zeta, \frac{1}{\zeta-\zeta^p}\Big]^{\widehat{}} \tag{2}$$

$$\widehat{\Omega}_{[s,s']} \simeq \mathrm{Spf}\, O_{\zeta,\eta} \overset{\mathrm{def}}{=} \mathrm{Spf}\, \frac{\mathbb{Z}_p[\zeta,\eta]}{\zeta\eta-p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\widehat{}}. \tag{3}$$

The inclusion from $\widehat{\Omega}_s$ (resp. $\widehat{\Omega}_{s'}$) to $\widehat{\Omega}_{[s,s']}$ under these isomorphisms is just ζ (resp. η) goes to p/η (resp. p/ζ).

The set of vertices of the tree is in bijection with $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash\mathrm{GL}_2(\mathbb{Q}_p)$. Clearly there is a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on this set and it extends to an action on the set of edges. In fact, this action can be identified with the action on the open covering $\{\widehat{\Omega}_e\}_e$. When s is the vertex that corresponds to the coset $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$, which I call the central vertex s_0', we can choose the isomorphism (1) such that the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on it is given by

$$\eta \mapsto \frac{a\eta+c}{b\eta+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

From the explicit description of $\widehat{\Omega}_{[s,s']}$ and $\widehat{\Omega}_s$ above, it is clear the special fiber of $\widehat{\Omega}$ is a tree of rational curves over $\mathbb{F}_p$ intersecting at all $\mathbb{F}_p$-rational points. The set of irreducible components (singular points) is nothing but the set of vertices (edges) of the tree. The dual graph of the special fiber of $\widehat{\Omega}$ is just the Bruhat–Tits tree.

In [Drinfel'd 1976], it was shown that there exists a universal family of formal groups X of height 4 over $\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$, where $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is the p-adic completion of the ring of integers inside the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}}$ of $\mathbb{Q}_p$. We denote by D the "unique" quaternion algebra over $\mathbb{Q}_p$, and O_D the ring of integers inside D. Then from Drinfel'd's construction, we know that O_D acts on the universal formal group on the left.

Fix a uniformizer Π inside O_D such that $\Pi^2 = p$. Define $X_n = X[\Pi^n]$. They are finite formal group schemes over $X_0 = \widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Let $\mathcal{X}_n$ be the rigid space associated to X_n, or equivalently, $\mathcal{X}_n$ is the generic fiber of X_n. These $\mathcal{X}_n$ are étale coverings of $\mathcal{X}_0 = \Omega\,\hat{\otimes}\,\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$. Then $O_D/(\Pi^n)$ acts on it and we have equivariant inclusions $\mathcal{X}_{n-1} \hookrightarrow \mathcal{X}_n$. Now define

$$\Sigma_n = \mathcal{X}_n - \mathcal{X}_{n-1}.$$

It can be shown that Σ_n is a finite étale Galois covering over $\mathcal{X}_0$ with Galois group $(O_D/(\Pi^n))^\times$.

It is important that all the spaces $(X_n, \mathcal{X}_n, \Sigma_n)$ we construct here have a natural $\mathrm{GL}_2(\mathbb{Q}_p)$ action and all the maps here are $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. On $X_0 = \Omega\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$, $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\widehat{\Omega}$ as we described before and acts on $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ via $\widetilde{\mathrm{Fr}}^{v_p(\det(g))}$, where $\widetilde{\mathrm{Fr}}$ is the (lift of the) arithmetic Frobenius and v_p is the usual p-adic valuation on $\mathbb{Q}_p$. One can show that the action of $\mathbb{Z}_p^\times$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ on Σ_n is inverse to the action of $\mathbb{Z}_p^\times$ in $O_D^\times$.

Now we want to describe the action of Π on the tangent space T of X. It is easy to see from the construction that T is a rank 2 vector bundle on $\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Moreover, T splits canonically into a direct sum of two line bundles T_0, T_1 by considering the action of $\mathbb{Z}_{p^2}$ inside O_D (recall that we fix such an embedding in the previous section). Each eigenspace of this action is a line bundle because X is "special" in the sense of Drinfel'd. Π interchanges T_0, T_1 and under the isomorphisms (1)–(3), we can write it down explicitly. But before doing that, I must introduce the notion of odd and even vertices.

Definition 2.1. A class $[g]$ in $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \backslash \mathrm{GL}_2(\mathbb{Q}_p)$ is called odd (resp. even) if $v_p(\det(g))$ is odd (resp. even).

Notice that this is well defined. And we will call a vertex in the tree (or an irreducible component of the special fiber of $\widehat{\Omega}$) even or odd according to the corresponding class.

Back to the discussion of the tangent space. I should mention that all T_0, T_1, Π descend naturally to $\widehat{\Omega}$, and I still call them T_0, T_1, Π by abuse of notation. Suppose s is an odd vertex and s' is adjacent to s, and so must be even. On $\widehat{\Omega}_{[s,s']}$, both

T_0, and T_1 are trivial. If we choose appropriate bases e_0, e_1 of them, then under the isomorphisms (1)–(3), Π becomes

$$\Pi_0 : T_0 \to T_1, \quad e_0 \mapsto \zeta e_1, \tag{4}$$

$$\Pi_1 : T_1 \to T_0, \quad e_1 \mapsto \eta e_0. \tag{5}$$

Identify Π_0, Π_1 with global sections of $T_0^* \otimes T_1$ and $T_1^* \otimes T_0$, where T_i^* denotes the dual of T_i, $i = 0, 1$ (the cotangent space). Then the explicit description of Π tells us that on an odd component of the special fiber, Π_0 has a simple zero at each intersection point with other irreducible components. Since each irreducible component is a rational curve over $\mathbb{F}_p$ and intersects other components at $\mathbb{F}_p$-rational points, Π_0 corresponds to the divisor that is the sum of all points of $\mathbb{P}^1(\mathbb{F}_p)$. On the other hand, Π_1 is zero on an odd component because $\eta = p/\zeta = 0$ (we are working over the special fiber, so already modulo p). On an even component, a similar argument shows that Π_0 is zero and Π_1 is the sum of all points of $\mathbb{P}^1(\mathbb{F}_p)$ as a divisor.

Restricting everything to the central vertex s'_0, we have an isomorphism $\widehat{\Omega}_{s'_0} \simeq \operatorname{Spf} \widehat{\mathbb{Z}_p}[\eta, 1/(\eta - \eta^p)]^{\wedge}$, and $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on it via

$$\eta \mapsto \frac{a\eta + c}{b\eta + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

The action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on T_0^* is easier to describe than the action on T_0. Using the same basis as in the last paragraph and denoting the dual basis element of e_0 by e_0^*, we have

$$g : T_0^* \to T_0^*, \quad f(\eta)e_0^* \mapsto \frac{1}{b\eta+d} f\Big(\frac{a\eta+c}{b\eta+d}\Big) e_0^* \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p). \tag{6}$$

Most details here can be found in [Boutot and Carayol 1991], especially the first chapter about Deligne's functor (and notice the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ here is the inverse of the action there).

3. Raynaud's theory of $\mathbb{F}$-vector space schemes

We want to write down the equation defining X_1. Recall that there exists an action of $O_D/(\Pi)$ on X_1. But $\mathbb{F} \overset{\text{def}}{=} O_D/(\Pi)$ is a finite field which is isomorphic to $\mathbb{F}_{p^2}$. So X_1 is an "$\mathbb{F}$-vector space scheme" in the sense of Raynaud. Let's recall Raynaud's theory of $\mathbb{F}$-vector space schemes in our situation. The reference is the first section of [Raynaud 1974].

Definition 3.1. Let S be a scheme and $\mathbb{F}$ a finite field. An $\mathbb{F}$-vector space scheme is a group scheme G over S with an embedding of $\mathbb{F}$ into the endomorphism ring of G (over S).

Although the definition here is different from Raynaud's original definition, it's clear that they are equivalent. Now let G be an $\mathbb{F}$-vector space scheme; we also use G to denote the group scheme in the definition by abuse of notation. The action of $\lambda \in \mathbb{F}$ is denoted by $[\lambda]$. Following Raynaud, we assume G is finite, flat and of finite presentation over S.

Let $\mathcal{A}$ be the bialgebra of G and $\mathcal{I}$ be the augmentation ideal. Then $\mathbb{F}^\times$ acts on $\mathcal{A}$ and $\mathcal{I}$. In Raynaud's paper, he defined a ring "D". Since we already use D for the quaternion algebra, I will use D_R for Raynaud's "D". In our case, we can think of D_R as $\mathbb{Z}_{p^2}$, the quadratic extension of $\mathbb{Z}_p$ in $\mathbb{Z}_p^{\mathrm{nr}}$. Although this ring is much bigger than D_R, both of them give the same result here. Under the hypothesis $(*)$ in Raynaud's paper and fixing a map $S \to \operatorname{Spec} D_R$, we have a canonical decomposition of $\mathcal{I}$:

$$\mathcal{I} = \bigoplus_{\chi \in M} \mathcal{I}_\chi,$$

where M is the set of characters of $\mathbb{F}^\times$ with value in $D_R^\times$, and $\mathcal{I}_\chi$ is defined as the "χ-isotypic component". More precisely, for every open set V on S, $H^0(V, \mathcal{I}_\chi)$ is the set of elements $a \in H^0(V, \mathcal{I})$, such that $[\lambda]a = \chi(\lambda)a$ for any $\lambda \in \mathbb{F}^\times$.

Definition 3.2. Let χ_1, χ_2 be the characters of $\mathbb{F}^\times = (O_D/\Pi)^\times$ with values in $D_R^\times = \mathbb{Z}_{p^2}^\times$ such that the composition

$$\mathbb{F}_{p^2}^\times \simeq (O_D/\Pi)^\times \xrightarrow{\chi_i} \mathbb{Z}_{p^2}^\times$$

is the Teichmüller character if $i = 1$ and its Galois twist if $i = 2$. Here, the first isomorphism is the one we fixed in the beginning. They are the fundamental characters defined in Raynaud's paper.

It is clear that $\chi_1^p = \chi_2$ and $\chi_2^p = \chi_1$. Every character χ in M can be written uniquely as

$$\chi = \chi_1^{n_1} \chi_2^{n_2}, \quad 0 \le n_1, n_2 \le p-1, \ (n_1, n_2) \ne (0,0).$$

Now, it is easy see to that given two characters χ, χ' in M, we have two $\mathcal{O}_S$-linear maps

$$\begin{cases} c_{\chi,\chi'} : \mathcal{I}_{\chi\chi'} \to \mathcal{I}_\chi \otimes \mathcal{I}_{\chi'}, \\ d_{\chi,\chi'} : \mathcal{I}_\chi \otimes \mathcal{I}_{\chi'} \to \mathcal{I}_{\chi\chi'} \end{cases}$$

coming from the comultiplication and multiplication structure of $\mathcal{A}$. A slight generalization of this (or equivalently iterate this $p-1$ times) gives us

$$\begin{cases} c_i : \mathcal{I}_{\chi_{i+1}} \to \mathcal{I}_{\chi_i}^{\otimes p}, \\ d_i : \mathcal{I}_{\chi_i}^{\otimes p} \to \mathcal{I}_{\chi_{i+1}} \end{cases}$$

for $i = 1, 2$, and we identify χ_3 as χ_1.

Under the hypothesis (∗∗) in Raynaud's paper, which says that each $\mathcal{I}_\chi$ is an invertible sheaf on S, we have the following classification theorem of $\mathbb{F}$-vector space schemes.

Theorem 3.3 [Raynaud 1974]. *Let S be a D_R-scheme. The map*

$$G \mapsto \left(\mathcal{I}_{\chi_i},\ c_i : \mathcal{I}_{\chi_{i+1}} \to \mathcal{I}_{\chi_i}^{\otimes p},\ d_i : \mathcal{I}_{\chi_i}^{\otimes p} \to \mathcal{I}_{\chi_{i+1}}\right)_{i=1,2}$$

defines a bijection between the isomorphism classes of $\mathbb{F}$-vector space schemes over S satisfying (∗∗) and the isomorphism classes of $(\mathcal{L}_1, \mathcal{L}_2, c_1, c_2, d_1, d_2)$, where:

(1) *$\mathcal{L}_i$ is an invertible sheaf on S for any $i = 1, 2$.*

(2) *The c_i and d_i are $\mathcal{O}_S$-linear maps*

$$\begin{cases} c_i : \mathcal{L}_{\chi_{i+1}} \to \mathcal{L}_{\chi_i}^{\otimes p} \\ d_i : \mathcal{L}_{\chi_i}^{\otimes p} \to \mathcal{L}_{\chi_{i+1}} \end{cases}$$

such that $d_i \circ c_i = w \operatorname{Id}_{\mathcal{L}_{i+1}}$. Here w is a constant in D_R that only depends on $\mathbb{F}$ and can be expressed using Gauss sums. More precisely, if we identify D_R with $\mathbb{Z}_{p^2}$, then $w \in \mathbb{Z}_p \subset \mathbb{Z}_{p^2}$ with p-adic valuation 1. And if we write $w = pu$, then $u \equiv -1 \pmod{p}$.

The inverse map in the theorem is as follows: we define

$$\mathcal{A} = \bigoplus_{0 \le a_i \le p-1} (\mathcal{L}_1^{\otimes a_1} \otimes \mathcal{L}_2^{\otimes a_2})$$

and equip it with the multiplication and comultiplication structure using d_i, c_i. $\mathcal{A}$ is now a bialgebra and thus defines a group scheme over S. The action of $\mathbb{F}^\times$ is defined as the character χ_i on $\mathcal{L}_i$ and more generally as the character $\chi_1^{a_1}\chi_2^{a_2}$ on $\mathcal{L}_1^{\otimes a_1} \otimes \mathcal{L}_2^{\otimes a_2}$. We now define the action of 0 in $\mathbb{F}$ to be trivial on $\mathcal{A}$. The properties of c_i and d_i guarantee that we indeed get a $\mathbb{F}$-vector space scheme. As a corollary, we have a description of the invariant differential forms of G:

Corollary 3.4. $\omega_{G/S} \simeq \mathcal{I}/\mathcal{I}^2 = (\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p})) \oplus (\mathcal{L}_2/d_1(\mathcal{L}_1^{\otimes p}))$.

Remark 3.5. When S is an affine scheme, say $\operatorname{Spec}(A)$, and $\mathcal{L}_i$ is free over S for all i, we have an explicit description of $\mathcal{A}$. Suppose x_i is a basis of $\mathcal{L}_i$. Under such basis, d_i becomes an element v_i inside A, namely $d_i(x_i^{\otimes p}) = v_i x_{i+1}$. Then the bialgebra $\mathcal{A}$ is isomorphic to $A[x_1, x_2]/(x_1^p - v_1 x_2, x_2^p - v_2 x_1)$ as an A-algebra.

Remark 3.6. The Cartier dual of an $\mathbb{F}$-vector space scheme is also an $\mathbb{F}$-vector space scheme by the dual action of $\mathbb{F}$. On the level of bialgebra, the action of $\mathbb{F}$ is given by its transpose. If G corresponds to $(\mathcal{L}_i, c_i, d_i)$, the Cartier dual G^* corresponds to $(\mathcal{L}_i^*, d_i^*, c_i^*)$, where $\mathcal{L}_i^* = \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{O}_S)$ and d_i^* (resp. c_i^*) is the transpose of d_i (resp. c_i).

4. Some results about the formal polarization

We want to apply Raynaud's theory to our situation. Although our base scheme is a formal scheme, the argument of Raynaud can be extended naturally to this situation. As we remarked in the beginning of the previous section, $X_1 = X[\Pi]$ is a $\mathbb{F}$-vector space scheme over $X_0 = \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Q}_p^{\mathrm{nr}}}$, where $\mathbb{F} = O_D/(\Pi)$. Using that its generic fiber $\mathcal{X}_1$ is étale over $\Omega \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ and applying Proposition 1.2.2. in Raynaud's paper, we know that X_1 satisfies hypothesis $(**)$. So the classification theorem tells us there exist 2 invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$, and maps

$$c_1 : \mathcal{L}_2 \mapsto \mathcal{L}_1^{\otimes p}, \quad c_2 : \mathcal{L}_1 \mapsto \mathcal{L}_2^{\otimes p}, \tag{7}$$

$$d_1 : \mathcal{L}_1^{\otimes p} \mapsto \mathcal{L}_2, \quad d_2 : \mathcal{L}_2^{\otimes p} \mapsto \mathcal{L}_1, \tag{8}$$

such that $d_1 \circ c_1 = w \,\mathrm{Id}_{\mathcal{L}_2}$, and $d_2 \circ c_2 = w \,\mathrm{Id}_{\mathcal{L}_1}$.

In order to determine c_i, d_i, we need the existence of "formal $*$-polarization" of the universal formal group X, which is a lemma in the proof of Proposition 4.3. of [Drinfel'd 1976], and proved in detail in [Boutot and Carayol 1991, chapitre III lemma 4.2.]. I would like to recall it here without proof.

Lemma 4.1. *Suppose $t \in D$ such that $t^2 \in pO_D^\times$. There exists a symmetric isomophism $\lambda : X \to X^*$, where X^* is the Cartier dual of X, such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & X^* \\ {\scriptstyle t^{-1}\bar{d}t}\downarrow & & \downarrow{\scriptstyle d^*} \\ X & \xrightarrow{\lambda} & X^* \end{array}$$

commutes for any $d \in O_D$, where $\bar{d}$ is the canonical involution of d in D, and d^* is the dual morphism of the endomorphism d. Here symmetric means $\lambda = \lambda^*$ under the canonical identification between X and X^{**}.

Remark 4.2. This isomorphism is not unique, but is unique up to $\mathbb{Z}_p^\times$-action. From now on, we will fix one such isomorphism λ that is defined in [Drinfel'd 1976] and [Boutot and Carayol 1991]. So we also fix such a t.

How does this isomorphism behave under the action $\mathrm{GL}_2(\mathbb{Q}_p)$? Recall that $X_0 = \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$.

Lemma 4.3. *Suppose $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $\det(g) \in p^{\mathbb{Z}}$; then g "commutes" with λ. More precisely, by abuse of notation, let $g : X_0 \to X_0$ be the automorphism of X_0 induced by g. Then there exists a natural isomorphism $\mu_g : X \to g^*X$ over X_0, where g^*X is the pull-back of X under $g : X_0 \to X_0$ by the equivariance of the $\mathrm{GL}_2(\mathbb{Q}_p)$ action. Denote by μ_g^* the dual morphism of μ_g. We have the following* *commutative diagram*:

$$\begin{array}{ccccc}
X^* & \xleftarrow{\mu_g^*} & (g^*X)^* \simeq g^*X^* & \longrightarrow & X^* \\
\uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle g^*\lambda} & & \uparrow{\scriptstyle\lambda} \\
X & \xrightarrow{\mu_g} & g^*X & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
X_0 & = & X_0 & \xrightarrow{g} & X_0
\end{array}$$

In general, for any $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, *we have the same diagram but replace the upper left square by*

$$\begin{array}{ccc}
X^* & \xleftarrow{\mu_g^*} & g^*X^* \\
\uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle g^*\lambda} \\
X & \xrightarrow[\mu_{g\cdot p^n/\det(g)}]{} & g^*X
\end{array}$$

where $n = v_p(\det(g))$. *Notice that this makes sense since* $\mathbb{Z}_p^\times$ *has trivial action on* X_0, *so* $g^*X = (g \cdot p^n/\det(g))^*X$.

Proof. Since I will use some formulas in [Drinfel'd 1976] and [Boutot and Carayol 1991], I think it's better not to translate their left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ to right action here. Hence I will follow their convention in this proof.

It's clear that we only need to prove the general case. Thanks to Drinfel'd's lemma (the lemma on strictness for p-divisible groups in the appendix of [Drinfel'd 1976]), it suffices to verify this commutative diagram after we reduce modulo p. But by Drinfel'd's construction of the universal p-divisible group, $X \times \mathbb{F}_p$ is quasi-isogenous of degree 0 to a constant p-divisible group $\Phi_{X_0\times\mathbb{F}_p}$ over $X_0 \times \mathbb{F}_p$. Here, recall that Φ is a p-divisible group defined over $\overline{\mathbb{F}}_p$, and $\Phi_{X_0\times\mathbb{F}_p} \overset{\text{def}}{=} \Phi \times_{\overline{\mathbb{F}}_p} X_0$. $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on Φ as quasi-isogenies. A detailed description of Φ can be found in [Boutot and Carayol 1991, chapitre III 4.3] or the proof of Proposition 4.3. of [Drinfel'd 1976]. Besides, the construction of the "formal polarization" λ tells us that λ actually comes from a "formal polarization" λ_0 of Φ that makes the following diagram commutative:

$$\begin{array}{ccc}
X \times \mathbb{F}_p & \xrightarrow{\bar\lambda} & X^* \times \mathbb{F}_p \\
{\scriptstyle\rho}\downarrow & & \uparrow{\scriptstyle\rho^*} \\
\Phi_{X_0\times\mathbb{F}_p} & \xrightarrow{\lambda_{0,X_0\times\mathbb{F}_p}} & \Phi^*_{X_0\times\mathbb{F}_p}
\end{array}$$

where $\bar\lambda \overset{\text{def}}{=} \lambda \pmod{p}$, ρ is the quasi-isogeny and ρ^* is its dual. From the definition of the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we know how ρ changes under this action (basically the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Φ with some twist of Frobenius, see [Drinfel'd 1976, Section 2] or [Boutot and Carayol 1991, chapitre II section 9]). Thus we can translate the diagram of X into a diagram of Φ. It turns out that it suffices to verify that the following diagram is commutative:

$$\begin{array}{ccc} \Phi^* & \xleftarrow{(\mathrm{Frob}_\Phi^{-n}\circ g)^*} & (\mathrm{Fr}^{-n})^*\Phi^* \\ \uparrow{\scriptstyle \lambda_0} & & \uparrow{\scriptstyle (\mathrm{Fr}^{-n})^*\lambda_0} \\ \Phi & \xrightarrow{\mathrm{Frob}_\Phi^{-n}\circ(g\cdot p^n/\det(g))} & (\mathrm{Fr}^{-n})^*\Phi \end{array}$$

Here $\mathrm{Fr}:\mathrm{Spec}(\overline{\mathbb{F}}_p)\to\mathrm{Spec}(\overline{\mathbb{F}}_p)$ is the arithmetic Frobenius and $\mathrm{Frob}_\Phi:(\mathrm{Fr}^{-1})^*\Phi\to\Phi$ is the Frobenius morphism over $\mathrm{Spec}(\overline{\mathbb{F}}_p)$. I would like to decompose the diagram as the following diagram (and invert the arrow on the bottom line):

$$\begin{array}{ccccc} \Phi^* & \xleftarrow{g^*(\det(g))^{-1}} & \Phi^* & \xleftarrow{(\det(g)\,\mathrm{Frob}_\Phi^{-n})^*} & (\mathrm{Fr}^{-n})^*\Phi^* \\ \uparrow{\scriptstyle \lambda_0} & & \uparrow{\scriptstyle \lambda_0} & & \uparrow{\scriptstyle (\mathrm{Fr}^{-n})^*\lambda_0} \\ \Phi & \xleftarrow{g^{-1}} & \Phi & \xleftarrow{(\det(g)/p^n)\,\mathrm{Frob}_\Phi^n} & (\mathrm{Fr}^{-n})^*\Phi \end{array}$$

First we look at the right square:

$$(\det(g)\,\mathrm{Frob}_\Phi^{-n})^* = \left(\frac{\det(g)}{p^n}\right)(p^n\,\mathrm{Frob}_\Phi^{-n})^* = \left(\frac{\det(g)}{p^n}\right)(\mathrm{Ver}_\Phi^n)^* = \left(\frac{\det(g)}{p^n}\right)\mathrm{Frob}_{\Phi^*}^n,$$

where Ver_{Φ^*} is the Verschiebung morphism. Now it is easy to see the diagram commutes from the basic property of the Frobenius morphism.

As for the left square, the commutativity in fact comes from our explicit choice of Φ, λ_0 and the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. See the remarque in [Boutot and Carayol 1991, chapitre III 4.3] which says the Rosati involution associated to λ_0 is nothing but the canonical involution on $\mathrm{M}_2(\mathbb{Q}_p)$. □

Remark 4.4. When $g\in\mathrm{SL}_2(\mathbb{Q}_p)$, the calculation above is essentially given in [Boutot and Carayol 1991, chapitre III 4.5].

5. Structure of $\mathcal{X}_1$ and a formal model of Σ_1

Now let's see how the discussion above helps us study c_i, d_i in (7), (8). The main result is the following:

Proposition 5.1. *There exists an isomorphism λ_1 from $X_1 = X[\Pi]$ to $X[\Pi]^*$, the Cartier dual of $X[\Pi]$, such that*:

(1) *The following diagram commutes for any $d \in O_D^\times$:*

$$\begin{array}{ccc} X[\Pi] & \xrightarrow{\lambda_1} & X[\Pi]^* \\ \bar{d}\downarrow & & \downarrow d^* \\ X[\Pi] & \xrightarrow{\lambda_1} & X[\Pi]^* \end{array}$$

Recall that $\bar{d}$ is the canonical involution of d in D.

(2) $\lambda_1^* = \lambda_1 \circ [-1] = [-1]^* \circ \lambda_1$, *where* $[-1]$ *denotes the action of* $-1 \in O_D$.

Proof. We can take $t = \Pi$ in Lemma 4.1. Then if we restrict to the p torsion points of X, we certainly get an isomophism:

$$\lambda_p : X[p] = X[\Pi^{-1}\bar{p}\Pi] \to X^*[p^*] = X^*[p].$$

Notice that $X^*[p]$ is canonically isomorphic to $(X[p])^*$, the Cartier dual of $X[p]$. The inclusion of $X[\Pi]$ into $X[p]$ induces a canonical isomorphism

$$j : X^*[p]/X^*[\Pi^*] = X^*[p]/((X^*[p])[\Pi^*]) \xrightarrow{\sim} (X[p])^*/((X[p])^*[\Pi^*]) \xrightarrow{\sim} X[\Pi]^*.$$

Since $\Pi^2 = p$, the map $\Pi^* : X^*[p] \to X^*[p]$ gives us an isomorphism

$$h : X^*[p]/X^*[\Pi^*] \xrightarrow{\sim} X^*[\Pi^*].$$

Finally, we restrict λ to the Π torsion points of X and get an isomorphism

$$\lambda_\Pi : X[\Pi] = X[\Pi^{-1}\bar{\Pi}\Pi] \to X^*[\Pi^*].$$

Now, we define $\lambda_1 = j \circ h^{-1} \circ \lambda_\Pi : X[\Pi] \to X[\Pi]^*$.

What is the Rosati involution associated to λ_1? I claim the following diagram commutes:

$$\begin{array}{ccccccc} X[\Pi] & \xrightarrow{\lambda_\Pi} & X^*[\Pi^*] & \xleftarrow{h} & X^*[p]/X^*[\Pi^*] & \xrightarrow{j} & X[\Pi]^* \\ \Pi^{-1}\bar{d}\Pi\downarrow & & d^*\downarrow & & (\Pi d\Pi^{-1})^*\downarrow & & \downarrow(\Pi d\Pi^{-1})^* \\ X[\Pi] & \xrightarrow{\lambda_\Pi} & X^*[\Pi^*] & \xleftarrow{h} & X^*[p]/X^*[\Pi^*] & \xrightarrow{j} & X[\Pi]^* \end{array}$$

The left-most square is commutative because we have a similar diagram for λ and λ_Π is a restriction of λ. The right-most diagram is commutative because j comes from the canonical quotient map $X^*[p] \simeq (X[p])^* \to X[\Pi]^*$ and this certainly commutes with the dual endomorphism of O_D. As for the middle square, notice that h is induced by the map $\Pi^* : X^*[p] \to X^*[p]$ and everything is clear.

Since $\Pi^{-1}\bar{d}\Pi \equiv d \pmod{\Pi O_D}$ and everything in the diagram above is killed by Π or Π^*, we can replace $\Pi^{-1}\bar{d}\Pi$ by d and $(\Pi d\Pi^{-1})^*$ by $\bar{d}^*$, and hence get the desired commutative diagram in part (1).

As for part (2), we use G, H to denote $X[p]$, $X[\Pi]$ respectively. Then $G^* = X^*[p]$. We can decompose $-\Pi : G \to G$ as

$$G \xrightarrow{q} G/H \xrightarrow{h_{-\Pi}} H \xrightarrow{i} G,$$

where i (resp. q) is the canonical inclusion of H to G (resp. canonical quotient map of G to G/H). The induced isomorphism is $h_{-\Pi}$.

Notice that $\Pi^{-1}\bar{\Pi}\Pi = -\Pi$ and G is killed by p. We have the following diagram, which is a restriction of the diagram of Lemma 4.1 to G with $d = \Pi$:

$$\begin{array}{ccc} G & \xrightarrow{-\Pi} & G \\ \downarrow{\lambda_p} & & \downarrow{\lambda_p} \\ G^* & \xrightarrow{\Pi^*} & G^* \end{array}$$

Similarly we can decompose Π^* as we did for $-\Pi$ and have the commutative diagram

$$\begin{array}{ccccccc} G & \xrightarrow{q} & G/H & \xrightarrow{h_{-\Pi}} & H & \xrightarrow{i} & G \\ \downarrow{\lambda_p} & & \downarrow{\lambda_{G/H}} & & \downarrow{\lambda_H} & & \downarrow{\lambda_p} \\ G^* & \xrightarrow{i^*} & H^* & \xrightarrow{h_{\Pi^*}} & (G/H)^* & \xrightarrow{q^*} & G^* \end{array}$$

such that the composition of all three maps in the bottom line is Π^*. The map h_{Π^*} is induced from Π^*. Thus it's easy to see $([-1] \circ h_{-\Pi})^* = h_{\Pi^*}$ and its dual $h_{\Pi^*}^* = [-1] \circ h_{-\Pi}$.

Since λ is symmetric, so is λ_p and we certainly have $\lambda_{G/H}^* = \lambda_H$. Now it's not hard to see that our λ_1 is nothing but $h_{\Pi^*}^{-1} \circ \lambda_H$. So,

$$\begin{aligned} \lambda_1^* &= (h_{\Pi^*}^{-1} \circ \lambda_H)^* = \lambda_H^* \circ (h_{\Pi^*}^{-1})^* = \lambda_{G/H} \circ (h_{\Pi^*}^*)^{-1} \\ &= \lambda_{G/H} \circ ([-1] \circ h_{-\Pi})^{-1} = \lambda_{G/H} \circ h_{-\Pi}^{-1} \circ [-1]^{-1} = \lambda_1 \circ [-1]. \end{aligned}$$

The last identity comes from the middle square of the diagram above. □

Corollary 5.2. *The isomorphism λ_1 induces isomorphisms*

$$\lambda_{\mathcal{L}_1} : \mathcal{L}_2^* \xrightarrow{\sim} \mathcal{L}_1, \quad \lambda_{\mathcal{L}_2} : \mathcal{L}_1^* \xrightarrow{\sim} \mathcal{L}_2.$$

Moreover, $\lambda_{\mathcal{L}_1} = -\lambda_{\mathcal{L}_2}^*$ if p is odd and $\lambda_{\mathcal{L}_1} = \lambda_{\mathcal{L}_2}^*$ if $p = 2$.

Proof. Using Theorem 3.3, we can identify $X_1 = X[\Pi]$ with $(\mathcal{L}_1, \mathcal{L}_2, c_1, c_2, d_1, d_2)$, and the final remark there tells us we can identify $X[\Pi]^*$ with $(\mathcal{L}_1^*, \mathcal{L}_2^*, d_1^*, d_2^*, c_1^*, c_2^*)$.

Now λ_1 gives us an isomorphism from $X[\Pi]$ to $X[\Pi]^*$ but this is not $\mathbb{F} = O_D/(\Pi)$-equivariant. For a character χ of $\mathbb{F}^\times$, considered as a character of $O_D^\times$, we have

$$\chi(\bar{d}) = \chi(d^p) = \chi^p(d)$$

for any $d \in O_D^\times$. This is because when we restrict the canonical involution to a quadratic unramified extension of $\mathbb{Z}_p$ inside O_D, it is nothing but the nontrivial Galois action. So modulo the uniformizer, it becomes the Frobenius automorphism.

Take $\chi = \chi_1$, one of the fundamental characters; then $\chi_1^p = \chi_2$, so $\chi_1(\bar{d}) = \chi_2(d)$. Similarly, we have $\chi_2(\bar{d}) = \chi_1(d)$. From these identities and the commutative diagram in Proposition 5.1, it is easy to see λ_1 really induces isomorphisms

$$\lambda_{\mathcal{L}_1} : \mathcal{L}_2^* \xrightarrow{\sim} \mathcal{L}_1, \quad \lambda_{\mathcal{L}_2} : \mathcal{L}_1^* \xrightarrow{\sim} \mathcal{L}_2.$$

The last identity comes from the consideration that the difference between λ_1 and λ_1^* is the action of -1. And we know $\chi_1(-1) = \chi_2(-1) = -1$ if p is odd and 1 otherwise. □

From now on, I will assume p is odd.

Corollary 5.3. *Under the isomorphism $\lambda_{\mathcal{L}_1}$, we have $-d_1 = c_2^*$. More precisely, we have the following commutative diagram*:

$$\begin{array}{ccc}
\mathcal{L}_1^{\otimes p} & \xrightarrow{-d_1} & \mathcal{L}_2 \\
{\scriptstyle \lambda_{\mathcal{L}_1}^{\otimes p}} \uparrow & & \uparrow {\scriptstyle \lambda_{\mathcal{L}_1}^*} \\
(\mathcal{L}_2^*)^{\otimes p} & \xrightarrow[c_2^*]{} & \mathcal{L}_1^*
\end{array}$$

Proof. It is easy to see λ_1 induces a similar diagram by replacing $-d_1$ with d_1 and $\lambda_{\mathcal{L}_1}^*$ with $\lambda_{\mathcal{L}_2}$. Now the corollary follows from $\lambda_{\mathcal{L}_1} = -\lambda_{\mathcal{L}_2}^*$. □

Corollary 5.4. *Under the isomorphism $\lambda_{\mathcal{L}_1}$, we can identify $d_1 : \mathcal{L}_1^{\otimes p} \to \mathcal{L}_2$ with a global section of $\mathcal{L}_1^{\otimes -p-1}$. Similarly, we can identify d_2 with a global section of $\mathcal{L}_1^{\otimes p+1}$. The canonical pairing*

$$H^0(X_0, \mathcal{L}_1^{\otimes -p-1}) \times H^0(X_0, \mathcal{L}_1^{\otimes p+1}) \to H^0(X_0, \mathcal{O}_{X_0})$$

sends (d_1, d_2) to the constant $-w = -pu$, where w is the constant in Theorem 3.3, and u is w/p.

Proof. Recall that $d_2 \circ c_2 = w \, \mathrm{Id}_{\lambda_{\mathcal{L}_1}}$. Then everything follows from Corollary 5.3. □

Corollary 5.5. *Recall that the bialgebra of X_1 is isomorphic as an $\mathcal{O}_{X_0}$-module to $\bigoplus_{0 \le i, j \le p-1} \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j}$. The isomorphism $\lambda_{\mathcal{L}_1}$ gives a global section $\widetilde{\lambda_{\mathcal{L}_1}}$ of $\mathcal{L}_1 \otimes \mathcal{L}_2$. Then as a global section of X_1, we have*

$$\widetilde{\lambda_{\mathcal{L}_1}}^p = -w \widetilde{\lambda_{\mathcal{L}_1}},$$

where everything is computed inside the bialgebra of X_1.

Proof. We only need to verify this locally. Suppose $\mathcal{L}_1, \mathcal{L}_2$ are free over an open set U and generated by x_1, x_2 such that $x_1 \otimes x_2 = \widetilde{\lambda_{\mathcal{L}_1}}$, or equivalently they are dual to each other under $\lambda_{\mathcal{L}_1}$. Now d_1, d_2 are given by two elements $v_1, v_2 \in H^0(U, \mathcal{O}_{X_0})$. So $x_1^p = v_1 x_2$, and $x_2^p = v_2 x_1$ (see Remark 3.5). But from the last corollary, we have $v_1 v_2 = -w$. Thus the product of these two equations is just what we want. □

Remark 5.6. Perhaps it is better to remark here that $\mathcal{L}_1, \mathcal{L}_2$ are nontrivial on the formal model but we'll see later that they become trivial on the generic fiber (Lemma 10.1).

Now we can describe a formal model of Σ_1. Recall that $\Sigma_1 = \mathcal{X}_1 - \mathcal{X}_0$, where $\mathcal{X}_1, \mathcal{X}_0$ are the rigid analytic spaces associated to X_1, X_0.

Proposition 5.7. *Let* $\mathcal{A} = \bigoplus_{0 \le i,j \le p-1} \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j}$ *be the bialgebra of* X_1. *Then* $\mathcal{A}/(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)$ *(the closed subscheme defined by the ideal sheaf* $(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)$*) is a formal model of* Σ_1. *We will use* $\widehat{\Sigma_1^{\mathrm{nr}}}$ *to denote this formal model.*

Proof. It suffices to check this locally on X_0, so we can assume $\mathcal{L}_1, \mathcal{L}_2$ are free. A point x on Σ_1 gives a morphism $x : \mathcal{A} \to \mathbb{C}_p$. If it does not factor through $\mathcal{A}/(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)$, $x(\widetilde{\lambda_{\mathcal{L}_1}})$ has to be 0 because last corollary tells us $(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)\widetilde{\lambda_{\mathcal{L}_1}} = 0$. But

$$x_1^{p+1} = x_1^p x_1 = v_1 x_2 x_1 = v_1 \widetilde{\lambda_{\mathcal{L}_1}},$$

so $x(x_1) = 0$ and $x(x_2) = 0$ by the same argument. Therefore x factors through $\mathcal{A}$ modulo the ideal sheaf generated by x_1, x_2 which is the augmentation ideal. Therefore x is in $\mathcal{X}_0$. The converse is trivial. □

It's easy to see its underlying algebra of $\widehat{\Sigma_1^{\mathrm{nr}}}$ is just

$$\bigoplus_{\substack{0 \le i,j \le p-1, \\ (i,j) \ne (p-1,p-1)}} \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j}.$$

Remark 5.8. There exist natural actions of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $O_D^\times$ on $\widehat{\Sigma_1^{\mathrm{nr}}}$. The action of $O_D^\times$ is clear. To see the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, notice that $\widetilde{\lambda_{\mathcal{L}_1}}$ is a global section of a trivial line bundle on X_0, but $H^0(X_0, \mathcal{O}_{X_0})$ is canonically isomorphic to $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ (I will prove this later; see Lemma 14.7). So $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\widetilde{\lambda_{\mathcal{L}_1}}$ as a scalar. Recall that $\widetilde{\lambda_{\mathcal{L}_1}}^p + w\widetilde{\lambda_{\mathcal{L}_1}} = 0$. This implies $\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant. The same argument shows that the action of $O_D^\times$ can be extended to $D^\times$.

But how does $\mathrm{GL}_2(\mathbb{Q}_p)$ act on $\widetilde{\lambda_{\mathcal{L}_1}}$? Here is a direct consequence of Lemma 4.3:

Proposition 5.9. *With* $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ *and* $n = v_p(\det(g))$,

$$g(\widetilde{\lambda_{\mathcal{L}_1}}) = \chi_1(\det(g)/p^n)^{-1} \widetilde{\lambda_{\mathcal{L}_1}}.$$

6. Local equation of X_1 and $\widehat{\Sigma_1^{\mathrm{nr}}}$

In order to get a semistable model of $\widehat{\Sigma_1^{\mathrm{nr}}}$, we need to know the local equation defining it. Recall that in Section 2, we described an open covering $\{\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}\}_e$ of X_0 such that

$$\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \operatorname{Spf} \frac{\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\zeta, \eta]}{\zeta\eta - p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\wedge}.$$

We try to write down the equation of $\widehat{\Sigma_1^{\mathrm{nr}}}$ above each $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Our first observation is:

Lemma 6.1. *Any line bundle $\mathcal{L}$ over*

$$\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \operatorname{Spf} \frac{\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\zeta, \eta]}{\zeta\eta - p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\wedge}$$

is trivial.

Proof. Recall (see (3))

$$O_{\zeta,\eta} = \frac{\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\zeta, \eta]}{\zeta\eta - p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\wedge}.$$

The special fiber of $\operatorname{Spf} O_{\zeta,\eta}$ is $\operatorname{Spec} \overline{\mathbb{F}}_p[\zeta, \eta, 1/(1-\zeta^{p-1}), 1/(1-\eta^{p-1})]/(\zeta\eta)$. I claim every line bundle $\bar{\mathcal{L}}$ over it is trivial. Let $\bar{L}$ be $H^0(\operatorname{Spec} O_{\zeta,\eta}/p, \bar{\mathcal{L}})$. Then we have the exact sequence

$$0 \to \bar{L} \to \bar{L}/(\zeta\bar{L}) \oplus \bar{L}/(\eta\bar{L}) \xrightarrow{-} \bar{L}/(\zeta\bar{L} + \eta\bar{L}) \to 0,$$

where the inclusion is the canonical morphism and $-$ is defined by taking their difference. This sequence is exact because $\bar{L}$ is locally free and thus flat over $O_{\zeta,\eta}/p$. Notice that $\bar{L}/(\zeta\bar{L})$ defines a line bundle on

$$\operatorname{Spec} O_{\zeta,\eta}/(p, \zeta) = \operatorname{Spec} \overline{\mathbb{F}}_p\Big[\eta, \frac{1}{1-\eta^{p-1}}\Big],$$

and hence has to be trivial. The same result holds for $\bar{L}/(\eta\bar{L})$. Also $\bar{L}/(\zeta\bar{L} + \eta\bar{L})$ is nothing but $\overline{\mathbb{F}}_p$. Using these, it's not hard to find an element that generates $\bar{L}$. So $\bar{\mathcal{L}}$ is trivial.

Now we can find an element in $H^0(\operatorname{Spf} O_{\zeta,\eta}, \mathcal{L})$ that generates $\mathcal{L}/p$. But $H^0(\operatorname{Spf} O_{\zeta,\eta}, \mathcal{L})$ is p-adically complete, so this element actually generates the whole $H^0(\operatorname{Spf} O_{\zeta,\eta}, \mathcal{L})$. Therefore $\mathcal{L}$ is trivial. (Here we use the fact that a surjective map between two line bundles has to be an isomorphism.) □

Thanks to this lemma, the restriction of $\mathcal{L}_1$ on $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is trivial. We fix an isomorphism between $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ and $\operatorname{Spf} O_{\zeta,\eta}$. Suppose x_1 is a generator of

$H^0(\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}, \mathcal{L}_1)$, and $x_2 \in H^0(\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}, \mathcal{L}_2)$ is the dual basis under the isomorphism $\lambda_{\mathcal{L}_1}$ defined in the previous section. Let v_1, v_2 be the elements given by d_1, d_2 under the basis x_1, x_2. Then we know that locally X_1 is defined by $x_1^p = v_1 x_2$, $x_2^p = v_2 x_1$.

How to determine v_1, v_2? Our strategy is to compare the invariant differential forms of X_1 computed in two different ways. First recall that the tangent space T of the universal formal group over X_0 is a rank 2 vector bundle over X_0 that naturally splits into a direct sum of two line bundles T_0, T_1. So the sheaf of invariant differential forms is its dual, namely $T_0^* \oplus T_1^*$. The action of Π on T_0 sends T_0 (resp. T_1) into T_1 (resp. T_0), which we denoted by Π_0 (resp. Π_1) in Section 2. Thus Π_0^* (resp. Π_1^*) sends T_1^* (resp. T_0^*) to T_0^* (resp. T_1^*) and the sheaf of invariant forms ω_{X_1/X_0} of $X_1 = X[\Pi]$ is

$$T_0^*/\Pi_0^* T_1^* \oplus T_1^*/\Pi_1^* T_0^*.$$

On the other hand, using Corollary 3.4, we know that this is also

$$\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p}) \oplus \mathcal{L}_2/d_1(\mathcal{L}_1^{\otimes p}).$$

It is natural to guess:

Lemma 6.2. $T_0^*/\Pi_0^* T_1^* \simeq \mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p}), \quad T_1^*/\Pi_1^* T_0^* \simeq \mathcal{L}_2/d_1(\mathcal{L}_1^{\otimes p}).$

Proof. If we restrict the action of O_D to $\mathbb{Z}_{p^2}$, it acts by identity on T_0 and by conjugation on T_1. Recall that we fix an embedding of $\mathbb{Z}_{p^2}$ into O_D in the beginning. This is just the definition of X being "special". Now our desired identification follows from a simple comparison of the action of $\mathbb{Z}_{p^2}^\times$ in both ways. □

Recall that all irreducible components of the special fiber of X_0 are isomorphic to $\mathbb{P}^1_{\mathbb{F}_p}$ such that the singular points are exactly $\mathbb{P}^1(\mathbb{F}_p)$. From the explicit description (4), (5) of Π_0, Π_1 and the discussion in Section 2, we know that on an odd component of the special fiber s, $T_0^*/\Pi_0^* T_1^*$ is isomorphic to $\bigoplus_{P \in s_{\mathrm{sing}}} i_{P*}\overline{\mathbb{F}}_p$, where s_{sing} is the set of singular points of the special fiber on s, and $i_P : P \to s$ is the embedding.

Restrict $\mathcal{L}_1, \mathcal{L}_2, d_2 : \mathcal{L}_2^{\otimes p} \to \mathcal{L}_1$ to s. From

$$\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p}) \simeq T_0^*/\Pi_0^* T_1^* \simeq \bigoplus_{P \in s_{\mathrm{sing}}} i_{P*}\overline{\mathbb{F}}_p \qquad (\text{on } s), \tag{9}$$

it's easy to see $\deg(\mathcal{L}_1|_s) - \deg(\mathcal{L}_2^{\otimes p}|_s) = p + 1$. But $\mathcal{L}_2 \simeq \mathcal{L}_1^*$, so

$$\deg(\mathcal{L}_2|_s) = -\deg(\mathcal{L}_1|_s). \tag{10}$$

This implies:

Lemma 6.3. *For any odd component* s, $\deg(\mathcal{L}_1|_s) = 1$. *Similarly*, $\deg(\mathcal{L}_2|_{s'}) = 1$ *for any even component* s'.

Now we would like to choose some good basis of $\mathcal{L}_1$ so that v_1, v_2 have a good form. Using the isomorphism $\widehat{\Omega}_e \simeq \operatorname{Spf} O_{\zeta,\eta}$, we can identify two irreducible components of its special fiber with $\operatorname{Spec} O_{\zeta,\eta}/(\zeta)$ and $\operatorname{Spec} O_{\zeta,\eta}/(\eta)$. Assume the second one is odd and we use s to denote the corresponding component in the special fiber of X_0 and use s' for the other component. Moreover $\operatorname{Spec} O_{\zeta,\eta}/(\eta) = \operatorname{Spec} \overline{\mathbb{F}}_p[\zeta, 1/(1-\zeta^{p-1})]$ hence has an obvious embedding into $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ which can be identified as the embedding into s.

Choose a global section $\widetilde{x_1}$ of $\mathcal{L}_1|_s$ such that it has a simple zero at infinity under the identification above. It is a basis of $H^0(\operatorname{Spec} O_{\zeta,\eta}/(\eta), \mathcal{L}_1)$. Then under this basis,

$$d_2 : \mathcal{L}_2^{\otimes p} \simeq \mathcal{L}_1^{*\otimes p} \to \mathcal{L}_1, \quad \widetilde{x_1}^{*\,\otimes p} \mapsto c(\zeta^p - \zeta)\widetilde{x_1}$$

for some constant $c \in \overline{\mathbb{F}}_p^{\times}$, where $\widetilde{x_1}^*$ is the dual basis of $\widetilde{x_1}^*$.

Notice that $\widetilde{x_1}$ is only defined up to a constant. If we replace $\widetilde{x_1}$ by $d\widetilde{x_1}$, then the constant c is replaced by $d^{-p-1}c$. We can choose $d = c^{1/(p+1)}$ to eliminate c. More precisely, we can choose a section, which I still call $\widetilde{x_1}$ by abuse of notation, such that under this basis, d_2 is just multiplication by $\zeta^p - \zeta$.

We can do a similar thing for s', which means we can choose a basis $\widetilde{x_2}$ of $\mathcal{L}_2|_{\operatorname{Spec} O_{\zeta,\eta}/(\zeta)}$ such that under this basis, d_1 is multiplication by $c'(\eta^p - \eta)$. Here we choose $\widetilde{x_2}$ so that $\widetilde{x_1}, \widetilde{x_2}^*$ can glue to a global basis $\overline{x_1}$ of $\mathcal{L}_1|_{\operatorname{Spec} O_{\zeta,\eta}/(p)}$ (see the proof of Lemma 6.1). A priori we know nothing about the constant c'.

Now we can lift $\overline{x_1}$ to a global basis x_1 of $\mathcal{L}_1|_{\operatorname{Spec} O_{\zeta,\eta}}$, so it determines a basis x_2 of $\mathcal{L}_2|_{\operatorname{Spec} O_{\zeta,\eta}}$ under the isomorphism $\lambda_{\mathcal{L}_1}$. And d_1, d_2 are given by two numbers v_1, v_2. The explicit description (4), (5) and Lemma 6.2 imply that

$$v_2 = \zeta u_2, \quad v_1 = \eta u_1 \tag{11}$$

for some units $u_1, u_2 \in O_{\zeta,\eta}^{\times}$. Note that $u_1 u_2 = -u$ because $v_1 v_2 = -w = -pu$, by Corollary 5.4, and $\eta\zeta = p$. From our choice of x_1, x_2, we have

$$v_2 \equiv \zeta^p - \zeta (\text{mod } \eta), \quad v_1 \equiv \eta^p - \eta (\text{mod } \zeta), \tag{12}$$

so

$$u_2 \equiv \zeta^{p-1} - 1 \ (\text{mod } \eta), \tag{13}$$

$$u_1 \equiv c'(\eta^{p-1} - 1) \ (\text{mod } \zeta). \tag{14}$$

This is because (ζ, η) is a regular sequence in $O_{\zeta,\eta}$. In fact $O_{\zeta,\eta}$ is normal. When we take the product of the identities above considered in $O_{\zeta,\eta}/(\zeta, \eta) \simeq \overline{\mathbb{F}}_p$, the left-hand side is $u_1 u_2 = -u$, which is 1 modulo p (see Theorem 3.3), while the right-hand side is just c'. Therefore:

Lemma 6.4. $c' = 1.$

Notice that $u_2 \equiv u_1^{-1} (\mathrm{mod}\ p)$, and $(\zeta) \cap (\eta) = (p)$ in $O_{\zeta,\eta}$. It's not hard to see that:

Lemma 6.5. $$u_1 \equiv -\frac{\eta^{p-1}-1}{\zeta^{p-1}-1} \ (\mathrm{mod}\ p), \quad u_2 \equiv -\frac{\zeta^{p-1}-1}{\eta^{p-1}-1} \ (\mathrm{mod}\ p).$$

Now if we replace our x_1 by rx_1 for some unit $r \in O_{\zeta,\eta}^{\times}$, then x_2 is replaced by $r^{-1}x_2$ and u_1 (resp. u_2) is replaced by $r^{p+1}u_1$ (resp. $r^{-p-1}u_2$). Write

$$u_1 = -\frac{\eta^{p-1}-1}{\zeta^{p-1}-1} r_1;$$

then $r_1 \equiv 1 (\mathrm{mod}\ p)$. Thus $r_1^{1/(p+1)}$ exists in $O_{\zeta,\eta}$. Hence we can modify our x_1 to make $u_1 = -(\eta^{p-1}-1)/(\zeta^{p-1}-1)$. In summary:

Proposition 6.6. *We can choose appropriate bases x_1, x_2 of $\mathcal{L}_1, \mathcal{L}_2$ over*

$$\widehat{\Omega}_e \,\hat{\otimes}\, \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \mathrm{Spf}\, O_{\zeta,\eta}$$

such that they are dual to each other under $\lambda_{\mathcal{L}_1}$, and under these bases,

$$d_1 : \mathcal{L}_1^{\otimes p} \to \mathcal{L}_2, \quad x_1^{\otimes p} \mapsto -\frac{\eta^p - \eta}{\zeta^{p-1}-1} x_2, \tag{15}$$

$$d_2 : \mathcal{L}_2^{\otimes p} \to \mathcal{L}_1, \quad x_2^{\otimes p} \mapsto u \frac{\zeta^p - \zeta}{\eta^{p-1}-1} x_1. \tag{16}$$

Corollary 6.7. *The restriction of X_1 to $\widehat{\Omega}_e \,\hat{\otimes}\, \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \mathrm{Spf}\, O_{\zeta,\eta}$ is defined by*

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2] \Big/ \left(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1} x_2,\ x_2^p - u \frac{\zeta^p-\zeta}{\eta^{p-1}-1} x_1 \right). \tag{17}$$

Similarly, the restriction of $\widehat{\Sigma_1^{\mathrm{nr}}}$ to $\widehat{\Omega}_e \,\hat{\otimes}\, \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is defined by

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2] \Big/ \left(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1} x_2,\ x_2^p - u \frac{\zeta^p-\zeta}{\eta^{p-1}-1} x_1,\ (x_1x_2)^{p-1} + pu \right). \tag{18}$$

Proof. The first statement follows from the above discussion. As for $\widehat{\Sigma_1^{\mathrm{nr}}}$, notice that x_1x_2 is just $\widetilde{\lambda_{\mathcal{L}_1}}$ defined in Corollary 5.5. So this is the definition of $\widehat{\Sigma_1^{\mathrm{nr}}}$. □

Fix a $\tilde{u}_1 = (-u)^{1/(p^2-1)}$ in $\mathbb{Z}_p$. If we replace x_1 by $\tilde{u}_1 x_1$, and x_2 by $\tilde{u}_1^p x_2$, then our new x_1, x_2 are dual to each other under $\tilde{u}_1^{-p-1}\lambda_{\mathcal{L}_1}$. Under this basis, $x_1x_2 = \tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}$.

Corollary 6.8. *The restriction of X_1 to $\widehat{\Omega}_e \,\hat{\otimes}\, \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \mathrm{Spf}\, O_{\zeta,\eta}$ is defined by*

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2] \Big/ \left(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1} x_2,\ x_2^p + \frac{\zeta^p-\zeta}{\eta^{p-1}-1} x_1 \right).$$

Similarly, the restriction of $\widehat{\Sigma_1^{\mathrm{nr}}}$ to $\widehat{\Omega}_e \,\hat{\otimes}\, \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is defined by

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2] \Big/ \left(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1} x_2,\ x_2^p + \frac{\zeta^p-\zeta}{\eta^{p-1}-1} x_1,\ (x_1x_2)^{p-1} - p \right).$$

Suppose $e=[s,s']$ and we have (1), (2), and (3). Then $\widehat{\Omega}_{s'}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is obtained by inverting η in $O_{\zeta,\eta}$ and taking the p-adic completion. Therefore, we have:

Corollary 6.9. *The restriction of X_1 to $\widehat{\Omega}_{s'}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}\simeq \operatorname{Spf}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\eta,1/(\eta^p-\eta)]^{\wedge}$ is defined by*

$$\operatorname{Spf}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}\Big[\eta,\frac{1}{\eta^p-\eta}\Big]^{\wedge}[x_1,x_2]\Big/\Big(x_1^p+\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}x_2,\,x_2^p+\frac{(p/\eta)^p-(p/\eta)}{\eta^{p-1}-1}x_1\Big).$$

Similarly, the restriction of $\widehat{\Sigma_1^{\mathrm{nr}}}$ to $\widehat{\Omega}_{s'}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is defined by

$$\operatorname{Spf}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}\Big[\eta,\frac{1}{\eta^p-\eta}\Big]^{\wedge}[x_1,x_2]\Big/\Big(x_1^p+\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}x_2,\,x_2^p+\frac{(p/\eta)^p-(p/\eta)}{\eta^{p-1}-1}x_1,\,(x_1x_2)^{p-1}-p\Big).$$

7. The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Sigma_1^{\mathrm{nr}}}$ and a descent $\widehat{\Sigma}_1$ to $\mathbb{Z}_{p^2}$

Recall that we fix an embedding $\mathbb{Z}_{p^2}\hookrightarrow\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. In this section, I want to describe the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Sigma_1^{\mathrm{nr}}}$. As a corollary, we can descend the formal model from $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ to $\mathbb{Z}_{p^2}$ by taking the "p-invariants", where p is considered as an element in $\mathrm{GL}_2(\mathbb{Q}_p)$. This descent is not quite canonical. On the other hand, as we explained in the introduction, it suffices to prove Theorem 1.1 when the central character is trivial on p, and this is exactly the descent we are considering here.

Denote the canonical morphism $\widehat{\Sigma_1^{\mathrm{nr}}}\to X_0$ by π and $\pi^{-1}(\widehat{\Omega}_e\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}})$ by $\widehat{\Sigma_{1,e}^{\mathrm{nr}}}$, $\pi^{-1}(\widehat{\Omega}_s\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}})$ by $\widehat{\Sigma_{1,s}^{\mathrm{nr}}}$, for edge e and vertex s. Then $\{\widehat{\Sigma_{1,e}^{\mathrm{nr}}}\}_e$ is an open covering of $\widehat{\Sigma_1^{\mathrm{nr}}}$, such that each open set has a nice description as in the previous section. Then the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on this covering can be identified with the action on the Bruhat–Tits tree.

Now let s_0' be the central vertex defined in Section 2. Then, $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\widehat{\Sigma_{1,s_0'}^{\mathrm{nr}}}$. I want to write down explicitly this action under the identification in Corollary 6.9. Since π is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant, we only need to describe the action on x_1,x_2. However it's clear from the equation in Corollary 6.9 that x_2 can be expressed using x_1 because $\eta^p-\eta$ is invertible. So it suffices to describe the action on x_1.

We first observe that $T_0^*/(\Pi_0^*T_1^*)\simeq\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p})$ is a free O_η/p-module of rank one with a basis x_1. Recall $O_\eta=\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\eta,1/(\eta^p-\eta)]^{\wedge}$. In Section 2, we gave an explicit description (6) of the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on T_0^*, which is given by

$$f(\eta)e_0^*\mapsto\frac{1}{b\eta+d}f\Big(\frac{a\eta+c}{b\eta+d}\Big)e_0^*,\quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

for some basis e_o^*. So if we write $x_1 = f(\eta)e_0^*$, for some $f(\eta) \in (O_\eta/p)^\times$, then the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on x_1 in O_η/p is

$$g(x_1) = \frac{1}{b\eta+d} f\Big(\frac{a\eta+c}{b\eta+d}\Big) f(\eta)^{-1} x_1. \tag{19}$$

Notice that on $\widehat{\Sigma^{\mathrm{nr}}_{1,s}}$,

$$x_1^{p+1} \equiv (\eta^p - \eta)x_1x_2 = (\eta^p - \eta)(-u)^{-1/(p-1)}\widetilde{\lambda_{\mathcal{L}_1}} \pmod p.$$

Thanks to Proposition 5.9, we know how $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on the right-hand side:

$$\begin{aligned} &g\big((\eta^p - \eta)(-u)^{-1/(p+1)}\widetilde{\lambda_{\mathcal{L}_1}}\big) \\ &\qquad = \Big(\Big(\frac{a\eta+c}{b\eta+d}\Big)^p - \frac{a\eta+c}{b\eta+d}\Big)(-u)^{-1/(p-1)}(ad-bc)^{-1}\widetilde{\lambda_{\mathcal{L}_1}}. \end{aligned}$$

Here we use the fact $\chi_1(\det(g)) \equiv \det(g) \pmod p$. An easy computation shows this is just $(1/(b\eta+d))^{p+1}(\eta^p - \eta)(-u)^{-1/(p-1)}\widetilde{\lambda_{\mathcal{L}_1}}$.

But from (19),

$$g(x_1)^{p+1} = \Big(\frac{1}{b\eta+d}\Big)^{p+1}\Big(f\Big(\frac{a\eta+c}{b\eta+d}\Big)f(\eta)^{-1}\Big)^{p+1} x_1^{p+1}.$$

Comparing both expressions, we have

$$f\Big(\frac{a\eta+c}{b\eta+d}\Big)^{p+1} = f(\eta)^{p+1} \quad \text{for any } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

Since $f(\eta) \in (O_\eta/p)^\times = \overline{\mathbb{F}}_p[\eta, 1/(\eta^p - \eta)]^\times$, it can only have poles and zeros at $\mathbb{F}_p$-rational points. Now $\mathrm{GL}_2(\mathbb{Z}_p)$ acts transitively on these points, so $f(\eta)$ has to be a constant. In other words:

Proposition 7.1. *The action of* $\mathrm{GL}_2(\mathbb{Z}_p)$ *on the special fiber of* $\widehat{\Sigma^{\mathrm{nr}}_{1,s_0'}}$ *is given by*

$$g(x_1) \equiv \frac{1}{b\eta+d}x_1 \pmod p, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

Corollary 7.2. *This action factors through* $\mathrm{GL}_2(\mathbb{F}_p)$.

What's the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\widehat{\Sigma^{\mathrm{nr}}_{1,s_0'}}$? Using Proposition 7.1 we can write

$$g(x_1)^{p+1} = \Big(\frac{1}{b\eta+d}\Big)^{p+1} x_1^{p+1}(1 + ph(\eta))$$

for some $h(\eta) \in O_\eta$ which only depends on g. Then:

Proposition 7.3. $$g(x_1) = \frac{1}{b\eta+d}x_1(1+ph(\eta))^{1/(p+1)},$$

where $(1+ph(\eta))^{1/(p+1)} = 1 + \frac{1}{p+1}ph(\eta) + \cdots$ *is the binomial expansion.*

Now let e_0 be the edge that connects the central vertex s'_0 and the vertex s_0 that corresponds to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \cdot w$, where $w = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. Then w acts on $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$. What is it?

We fix an isomorphism of $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$ with the explicit formal scheme described above. On $\widehat{\Omega}_{e_0} \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \operatorname{Spf} O_{\zeta,\eta}$, the action of w is given by

$$\eta \mapsto \frac{p}{-\eta} = -\zeta, \quad \zeta \mapsto \frac{p}{-\zeta} = -\eta,$$

and acts as the (lift) of arithmetic Frobenius on $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Notice that w interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$ because it acts semilinearly (over $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$). Using this, it's not hard to see w has the form

$$x_1 \mapsto w_1 x_2, \quad x_2 \mapsto w_2 x_1,$$

where $w_1, w_2 \in O_{\zeta,\eta}^\times$.

An easy computation shows that w_1, w_2 must satisfy the following relation:

$$w_1^p = -w_2. \tag{20}$$

Since $w \in \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid \det(g) \in p^{\mathbb{Z}}\}$, we can apply Proposition 5.9, which tells us $x_1 x_2 = \widetilde{\lambda_{\mathcal{L}_1}}$ is invariant by w. So,

$$w_1 w_2 = 1. \tag{21}$$

Combining these together, we get:

Lemma 7.4. *The action of $w = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ on $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$ is given by*

$$x_1 \mapsto w_1 x_2, \quad x_2 \mapsto w_1^{-1} x_1,$$

where $w_1 \in \mathbb{Z}_{p^2}^\times$ satisfies $w_1^{p+1} = -1$.

Now we are ready to prove the main result of this section:

Proposition 7.5. *$\widehat{\Sigma^{\mathrm{nr}}_1}$ can be descended to a formal scheme $\widehat{\Sigma_1}$ over $\mathbb{Z}_{p^2}$. In fact, $\widehat{\Sigma_1} = \widehat{\Sigma^{\mathrm{nr}}_1}^p$, the formal scheme defined by the $p \in \mathrm{GL}_2(\mathbb{Q}_p)$-invariant sections of $\widehat{\Sigma^{\mathrm{nr}}_1}$.*

Proof. It suffices to prove this locally, so we only need to work on $\widehat{\Sigma^{\mathrm{nr}}_{1,e}}$. Since $\mathrm{GL}_2(\mathbb{Q}_p)$ acts transitively on this covering, and p is in the center of $\mathrm{GL}_2(\mathbb{Q}_p)$, we can just work with $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$. $\widehat{\Omega}_{e_0} \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ certainly descends to $\mathbb{Z}_{p^2}$. The question is whether the descents of $\mathcal{L}_1, \mathcal{L}_2, d_1, d_2$ are effective. We show this by explicit computations.

Choose $c \in \mathbb{Z}_p^{\mathrm{nr}}$ such that $c^{p+1} = v_1 w_1^{-1}$, where v_1 is a choice of $(p-1)$-th root of -1, then c is a root of unity, and $\widetilde{\mathrm{Fr}}(c) = c^p$. Define $e = c x_1$, $e' = c^{-1} x_2$. We have

$$w^2(e) = w(\widetilde{\mathrm{Fr}}(c) w_1 x_2) = w(c^p w_1 x_2) = c^{p^2} w_1^p w_1^{-1} x_1 = c^{p^2-1} w_1^{p-1} e = -e.$$

Similarly, $w^2(e') = -e'$. Notice that $p = -w^2$ and -1 acts on x_1 as $\chi_1(-1)^{-1} = -1$ (the action of $\mathbb{Z}_p^\times$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ is the inverse of the action of it in $O_D^\times$). So e and e' are invariant by p, and $\mathcal{L}_1$, $\mathcal{L}_2$ can be descended to $\mathbb{Z}_{p^2}$.

What about d_1, d_2? Now

$$d_1 : e^{\otimes p} \mapsto -c^{p+1} \frac{\eta^p - \eta}{\zeta^{p-1} - 1} e'.$$

Since $c^{p+1} = (-1)^{1/(p-1)} w_1^{-1} \in \mathbb{Z}_{p^2}$, d_1 is defined over $\mathbb{Z}_{p^2}$. A similar argument works for d_2. □

Remark 7.6. Sometimes e also denotes an edge of a graph. I hope that it is clear from the context whether e refers to an edge or a section of $\mathcal{L}_1$ (locally).

Corollary 7.7. (1) *The action of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *can also be defined over* $\widehat{\Sigma}_1$.

(2) $\widehat{\Sigma}_1$ *has an open covering* $\{\widehat{\Sigma_{1,e}}\}_e$ *indexed by the edges of the Bruhat–Tits tree, such that this identification is* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant.*

(3) $\widehat{\Sigma_{1,e}}$ *is isomorphic to*

$$\operatorname{Spf} O_{e,e'} = \operatorname{Spf} \frac{\mathbb{Z}_{p^2}\Big[\zeta, \eta, \dfrac{1}{1-\zeta^{p-1}}, \dfrac{1}{1-\eta^{p-1}}, e, e'\Big]^{\wedge}}{\Big(e^p + v_1 w_1^{-1} \dfrac{\eta^p - \eta}{\zeta^{p-1}-1} e',\ e'^p + v_1^{-1} w_1 \dfrac{\zeta^p - \zeta}{\eta^{p-1}-1} e,\ (ee')^{p-1} - p,\ \eta\zeta - p\Big)},$$

where $w_1 = (-1)^{1/(p+1)}$ *is a* (p^2-1)*-th root of unity, and* v_1 *is a choice of* $(p-1)$*-th root of* -1.

(4) *The action of* w *on* $\widehat{\Sigma_{1,e_0}}$ *is given by*

$$e \mapsto v_1 e', \quad e' \mapsto v_1^{-1} e.$$

Remark 7.8. The reason that everything can be defined over $\mathbb{Z}_{p^2}$, I believe, is that the universal formal group can be defined over $\mathbb{Z}_{p^2}$. This is because when we formulate the moduli functor it represents, the "unique" 2-dimensional special formal group of height 4 and all endomorphisms can be defined over $\mathbb{F}_{p^2}$.

8. A semistable model of $\widehat{\Sigma}_1$

In this section, our goal is to work out a semistable model of $\widehat{\Sigma}_1$ as a formal scheme over $\mathbb{Z}_p$ (not $\mathbb{Z}_{p^2}$!). Notice that $\widehat{\Sigma}_1$ has a structural map to $\operatorname{Spec} \mathbb{Z}_{p^2}$. Hence if we change our base from $\mathbb{Z}_p$ to O_{F_0}, then

$$\widehat{\Sigma}_1 \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} O_{F_0} \simeq \widehat{\Sigma}_1 \sqcup \widehat{\Sigma}_1'. \tag{22}$$

Here $\widehat{\Sigma}_1'$ is the same scheme $\widehat{\Sigma}_1$ but with twisted map to O_{F_0}. Recall that F_0 is the unique unramified quadratic extension of $\mathbb{Q}_p$, and we fix an isomorphism between it and $\mathbb{Q}_{p^2}$ in the beginning. Hence we may identify $\widehat{\Sigma}_1$ as a formal scheme over O_{F_0}.

Therefore we only need to work over the scheme $\widehat{\Sigma_1}$ as a scheme over Spec $\mathbb{Z}_{p^2}$, and use the equation above to translate everything into the $\mathbb{Z}_p$-scheme $\widehat{\Sigma_1}$. I hope this won't cause too much confusion.

I say a formal scheme X is a semistable curve over Spec R, where R is a complete discrete valuation ring, if:

(1) The generic fiber of X is smooth over the generic fiber of Spec R.

(2) The special fiber of X is reduced.

(3) Each irreducible component of the special fiber of X is a divisor on X.

(4) Each singular point has an étale neighborhood that is étale over
$$\operatorname{Spec} R[x, y]/(xy - \pi_R),$$
where π_R is a uniformizer of R.

Back to our situation; we first work locally on $\widehat{\Sigma_1}$, so we just work with $\widehat{\Sigma_{1,e}}$. Moreover we can assume $e = e_0$ defined in the previous section and use the results there.

First notice that in $O_{e,e'}$ (see the notation in Corollary 7.7), $ee' = \tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}$ (see the equation before Corollary 6.8 and recall in the proof of Proposition 7.5, $x_1x_2 = ee'$), so it is a globally defined section on $\widehat{\Sigma_1}$, and satisfies $(ee')^{p-1} - p$. Now if we do base change from $\mathbb{Z}_{p^2}$ to $\mathbb{Z}_{p^2}[p^{1/(p-1)}]$, the generic fiber of Spf $O_{e,e'}[p^{1/(p-1)}]$ will split into $p-1$ connected components. Each connected component corresponds to a choice of $(p-1)$-th root of p. Adjoining $ee'/p^{1(p-1)} = \tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}/p^{1(p-1)}$ into $O_{e,e'}[p^{1/(p-1)}]$, which I would like to call $O^1_{e,e'}$, the formal scheme also splits into $p-1$ connected components, namely,
$$O^1_{e,e'} = \prod_{\varpi_1^{p-1}=p} O^1_{e,e',\varpi_1}.$$

Explicitly, O^1_{e,e',ϖ_1} is
$$\frac{\mathbb{Z}_{p^2}[p^{1/(p-1)}]\Big[\eta, \zeta, e, e', \dfrac{1}{\eta^{p-1}-1}, \dfrac{1}{\zeta^{p-1}-1}\Big]^{\widehat{}}}{\Big(e^p + v_1 w_1^{-1}\dfrac{\eta^p-\eta}{\zeta^{p-1}-1}e',\ e'^p + v_1^{-1}w_1\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1}e,\ ee' - \varpi_1\Big)}.$$

Now, we have (write ϖ_1 as $p^{1/(p-1)}$)
$$\begin{aligned} e^{p+1} &= e^p \cdot e = -v_1 w_1^{-1}\frac{\eta^p - \eta}{\zeta^{p-1} - 1}e'e = -w_1^{-1}\frac{\eta^p - \eta}{\zeta^{p-1} - 1}v_1 p^{1/(p-1)} \\ &= -w_1^{-1}\frac{\eta^p - \eta}{\zeta^{p-1} - 1}(-p)^{1/(p-1)}. \end{aligned}$$

Recall v_1 is a $(p-1)$-th root of -1. This clearly shows that if we adjoin a (p^2-1)-th root of $-p$, then the normalization of this ring contains $e/((-p)^{1/(p-1)})^{1/(p+1)} = e/(-p)^{1/(p^2-1)}$. Similarly, $e'/(-p)^{1/(p^2-1)}$ is also contained in the normalization.

Definition 8.1. Let ϖ be a fixed choice of $(-p)^{1/(p^2-1)}$. Define $F = F_0[\varpi]$, and O_F as the ring of integers inside F.

We change our base from Spec $\mathbb{Z}_{p^2}$ to Spec O_F via the fixed identification between $\mathbb{Q}_{p^2}$ and F_0, and take the normalization of $O_{e,e',\varpi_1}[\varpi]$ (it's not hard to verify it's integral). Denote the normalization by $\widetilde{O_{e,e',\varpi_1}}[\varpi]$. I claim basically this is just adjoining e/ϖ, e'/ϖ.

Lemma 8.2. $\widetilde{O_{e,e',\varpi_1}}[\varpi] =$

$$\frac{O_F\Big[\eta, \zeta, \dfrac{1}{\eta^{p-1}-1}, \dfrac{1}{\zeta^{p-1}-1}, \dfrac{e}{\varpi}, \dfrac{e'}{\varpi}\Big]^{\wedge}}{\left(\left(\dfrac{e}{\varpi}\right)^{p+1} + v_1 w_1^{-1}\xi\dfrac{\eta^p-\eta}{\zeta^{p-1}-1}, \left(\dfrac{e'}{\varpi}\right)^{p+1} + v_1^{-1}w_1\xi\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1}, \dfrac{e}{\varpi}\dfrac{e'}{\varpi} - \xi\varpi^{p-1}\right)},$$

where $\xi = \dfrac{\varpi_1}{\varpi^{p+1}}$ *is a* $(p-1)$*-th root of* -1.

Proof. It's clear both sides become the same after inverting p and certainly the right-hand side is contained in the left-hand side. Thus it suffices to prove the right-hand side is normal. First, since the generic fiber is smooth, there is no singular point on the generic fiber. Now if we modulo ϖ, the uniformizer, it's easy to see the only singular point is the maximal ideal $(e/\varpi, e'/\varpi, \varpi)$. We only need to show $(e/\varpi, e'/\varpi)$ is a regular sequence. Simple calculations indicate that the right-hand side is p-torsion free, so e/ϖ is not a zero divisor. In fact this already proves that the right-hand side is integral. Modulo e/ϖ, the right-hand side becomes $\mathbb{Z}_{p^2}[\varpi]/(\varpi^{p-1})[\zeta, e'/\varpi]/((e'/\varpi)^{p+1} + a(\zeta^p - \zeta))$ for some unit a. The element e'/ϖ is clearly neither a zero divisor, nor a unit. So we're done. □

Remark 8.3. The special fiber of $\widetilde{O_{e,e',\varpi_1}}[\varpi]$ has two irreducible components, defined by $e/\varpi = 0$ and $e'/\varpi = 0$. Each one maps to an irreducible component of the special fiber of $\widehat{\Omega}_{e_0} \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spec}\, O_F$, and has the form

$$\mathbb{F}_{p^2}\Big[x, y, \frac{1}{y^{p-1}-1}\Big]\Big/(y^p - y - cx^{p+1}),$$

where c is some root of unity. So each irreducible component is smooth and is an open set of an Artin–Schreier curve. In fact, if we do not split these connected components, then the special fiber is isomorphic to

$$\mathbb{F}_{p^2}\Big[x, y, \frac{1}{y^{p-1}-1}\Big]\Big/\big((y^p - y)^{p-1} + w_1^{-2}x^{p^2-1}\big),$$

which is an open set of a twist of Deligne–Lusztig variety of $\mathrm{GL}_2(\mathbb{F}_p)$ (see [Deligne and Lusztig 1976, Section 2]). More precisely, if we invert x and define $X = 1/x$, $Y = y/x$, this curve now has the form $(XY^p - YX^p)^{p-1} = -w_1^{-2}$.

Notice that $\widetilde{O_{e,e',\varpi_1}}[\varpi]$ is not semistable, because locally the singular point is defined by $(e/\varpi)(e'/\varpi) - \varpi^{p-1}\xi$, where ξ is some unit. To get a semistable model, keep blowing up the singular points until our scheme becomes regular. In fact, we need to blow up $[(p-1)/2]$ times. On the level of special fiber, this singular point will be replaced by $p-2$ rational curves in this process. After this, we finally get our desired semistable model of $\widehat{\Sigma_{1,e_0}} \times_{\mathrm{Spec}\,\mathbb{Z}_{p^2}} \mathrm{Spec}\, O_F$.

So far we have been working locally on $\widehat{\Sigma}_1$, but our construction above can be done globally. First, we change the base to $\mathrm{Spec}\, O_F$ and adjoin $u_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}/\varpi^{p-1}$ (equivalently, $\widetilde{\lambda_{\mathcal{L}_1}}/\varpi^{p-1}$). Here, since the difference between ϖ^{p-1} and a $(p-1)$-th root of p is a $(p-1)$-th root of -1, it doesn't matter which one we use. Then our formal scheme will split into $p-1$ connected components, indexed by $(p-1)$-th roots of -1. Now take the normalization of each connected component. Call the total space $\widetilde{\Sigma_{1,O_F}}$. For each component, it is clear from the above explicit local description that the dual graph of its special fiber is the same as $\widehat{\Omega}$'s, which is nothing but the Bruhat–Tits tree. Finally, blow up each singular point to get rid of singularities and we end up with a semistable model of $\widehat{\Sigma}_1 \times_{\mathrm{Spec}\,\mathbb{Z}_{p^2}} \mathrm{Spec}\, O_F$.

Theorem 8.4. $\widehat{\Sigma}_1$ *(over* $\mathrm{Spec}\,\mathbb{Z}_{p^2}$*) has a semistable model* $\widehat{\Sigma_{1,O_F}}$ *over* O_F*, such that*:

(1) $\widehat{\Sigma_{1,O_F}}$ *has* $(p-1)$ *connected components, indexed by* $(p-1)$*-th roots of* -1.

(2) *The dual graph of the special fiber of each connected component is the graph adding* $p-2$ *vertices to each edge of the Bruhat–Tits tree.*

(3) *Vertices that come from the Bruhat–Tits tree correspond to some Artin–Schreier curves* ($y^{p+1} = c(x^p - x)$ *in* $\mathbb{P}^2$*, where* $c \in \mathbb{F}_{p^2}^\times$*). Singular points are points with* $y = 0$*. If we put the* $p-1$ *connected components together, then a dense open set of it is isomorphic to the Deligne–Lusztig variety of* $\mathrm{GL}_2(\mathbb{F}_p)$ *over any algebraically closed field.*

(4) *Other vertices correspond to rational curves. Singular points are zero and infinity.*

Proof. We only need to prove our assertion for the special fiber. In the previous discussion, we already know the dual graph of the special fiber of each connected component of $\widetilde{\Sigma_{1,O_F}}$ is the Bruhat–Tits tree. Since blow-ups replace each singular point by $p-2$ rational curves, everything is clear. □

Let $\hat{\pi}$ and $\tilde{\pi}$ be the canonical maps from $\widehat{\Sigma_{1,O_F}}$ and $\widetilde{\Sigma_{1,O_F}}$ to $\widehat{\Omega} \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spec}\, O_F$. For each edge e of the Bruhat–Tits tree, we can define $\widehat{\Sigma_{1,O_F,e}}$ and $\widetilde{\Sigma_{1,O_F,e}}$ as

$\hat{\pi}^{-1}(\widehat{\Omega}_e \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F)$ and $\tilde{\pi}^{-1}(\widehat{\Omega}_e \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F)$, respectively. Similarly we can define $\widehat{\Sigma_{1,O_F,s}} = \widetilde{\Sigma_{1,O_F,s}}$ for each vertex s. Define $\widehat{\Sigma_{1,O_F,e,\xi}}$, $\widehat{\Sigma_{1,O_F,s,\xi}}$, $\widetilde{\Sigma_{1,O_F,e,\xi}}$, $\widetilde{\Sigma_{1,O_F,s,\xi}}$, where ξ is a $(p-1)$-th root of -1, as the corresponding connected component of $\widehat{\Sigma_{1,O_F,e}}$, $\widehat{\Sigma_{1,O_F,s}}$, $\widetilde{\Sigma_{1,O_F,e}}$, $\widetilde{\Sigma_{1,O_F,s}}$. Note that in the notation of Lemma 8.2, $\widetilde{\Sigma_{1,O_F,e,\xi}} = \operatorname{Spf} \widetilde{O_{e,e',\varpi^{p+1}\xi}}[\varpi]$.

In Lemma 8.2, we have an explicit description of $\widetilde{\Sigma_{1,O_F,e}}$. To simplify notation, I will use $\tilde{e}$, $\tilde{e}'$ for e/ϖ, e'/ϖ. Now let s' be an even vertex (for example, the central vertex s_0'). It's not hard to see that

$$\widehat{\Sigma_{1,O_F,s',\xi}} = \widetilde{\Sigma_{1,O_F,s',\xi}} \simeq \operatorname{Spf} O_F\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p+1} + v_1 w_1^{-1}\xi \frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big), \tag{23}$$

$$\widehat{\Sigma_{1,O_F,s'}} = \widetilde{\Sigma_{1,O_F,s'}} \simeq \operatorname{Spf} O_F\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p^2-1} + w_1^2\Big(\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big)^{p-1}\Big). \tag{24}$$

If s is an odd vertex, then similarly we have

$$\widehat{\Sigma_{1,O_F,s,\xi}} = \widetilde{\Sigma_{1,O_F,s,\xi}} \simeq \operatorname{Spf} O_F\Big[\zeta, \frac{1}{\zeta^p-\zeta}, \tilde{e}'\Big]\Big/\Big(\tilde{e}'^{p+1} + v_1^{-1} w_1\xi \frac{\zeta^p-\zeta}{(p/\zeta)^{p-1}-1}\Big), \tag{25}$$

$$\widehat{\Sigma_{1,O_F,s}} = \widetilde{\Sigma_{1,O_F,s}} \simeq \operatorname{Spf} O_F\Big[\zeta, \frac{1}{\zeta^p-\zeta}, \tilde{e}'\Big]\Big/\Big(\tilde{e}'^{p^2-1} + w_1^{-2}\Big(\frac{\zeta^p-\zeta}{(p/\zeta)^{p-1}-1}\Big)^{p-1}\Big). \tag{26}$$

Remark 8.5. If we view $\widehat{\Sigma_1}$ as a $\mathbb{Z}_p$-scheme, then $\widehat{\Sigma_1} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F$ has a semistable model over $\operatorname{Spec} O_F$, which I call $\widehat{\Sigma^{(0)}_{1,O_F}}$. It is canonically isomorphic to $\widehat{\Sigma_{1,O_F}} \sqcup \widehat{\Sigma'_{1,O_F}}$, where $\widehat{\Sigma'_{1,O_F}}$ is isomorphic with $\widehat{\Sigma_{1,O_F}}$ as a scheme, but the structure morphism to $\operatorname{Spec} O_F$ is twisted: $O_F \to O_F$ is the unique automorphism that fixes ϖ and acts as Frobenius on O_{F_0}. We use g_φ to denote it as an element in $\operatorname{Gal}(F/\mathbb{Q}_p)$.

From now on, I will use the exponent (0) for everything that is base changed from $\mathbb{Z}_p$ to O_F. For example, we can define $\widehat{\Sigma^{(0)}_{1,O_F,s}}$, $\widehat{\Sigma^{(0)}_{1,O_F,s,\xi}}, \dots$. Also we use the exponent $'$ for things with same underlying scheme but with twisted structure morphism to O_F. For example $\widehat{\Sigma'_{1,O_F,s}}$, $\widehat{\Sigma'_{1,O_F,s,\xi}}, \dots$. Under this notation, we have $\widehat{\Sigma^{(0)}_{1,O_F,s}} = \widehat{\Sigma_{1,O_F,s}} \sqcup \widehat{\Sigma'_{1,O_F,s}}, \dots$.

9. The action of $\mathrm{GL}_2(\mathbb{Z}_p)$, $\mathrm{Gal}(F/F_0)$, $O_D^\times$ on $\widetilde{\Sigma_{1,O_F}}$ and $\widehat{\Sigma_{1,O_F}}$

By acting on the first factor, we have an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Sigma_1} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F$ which extends naturally to our semistable $\widehat{\Sigma^{(0)}_{1,O_F}}$. Since $\mathrm{GL}_2(\mathbb{Q}_p)$ will interchange $\widehat{\Sigma_{1,O_F}}$ and $\widehat{\Sigma'_{1,O_F}}$, it does not act on $\widehat{\Sigma_{1,O_F}}$. The reason is that $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathbb{Z}_{p^2}$ by $\widetilde{\mathrm{Fr}}^{v_p(\det(g))}$. However, $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\widehat{\Sigma_{1,O_F}}$.

So how does $\mathrm{GL}_2(\mathbb{Z}_p)$ act on the central component $\widehat{\Sigma_{1,O_F,s_0'}}$ of $\widehat{\Sigma_{1,O_F}}$? We have an explicit description above (23), (24). We will fix this identification from now on.

$$\widehat{\Sigma_{1,O_F,s_0',\xi}} = \mathrm{Spf}\, O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p+1} + v_1 w_1^{-1}\xi\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big), \quad (27)$$

$$\widehat{\Sigma_{1,O_F,s_0'}} = \mathrm{Spf}\, O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p^2-1} + w_1^2(\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1})^{p-1}\Big). \quad (28)$$

Proposition 9.1. (1) *The action of* $\mathrm{GL}_2(\mathbb{Z}_p)$ *on* $\widetilde{\Sigma_{1,O_F,s_0'}} = \widehat{\Sigma_{1,O_F,s_0'}}$ *is given by*

$$g(\tilde{e}) \equiv \frac{1}{b\eta+d}\tilde{e} \pmod p, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p). \quad (29)$$

So it factors through $\mathrm{GL}_2(\mathbb{F}_p)$ *when acting on the special fiber.*

(2) $g \in \mathrm{GL}_2(\mathbb{Z}_p)$ *maps* $\widetilde{\Sigma_{1,O_F,s_0',\xi}}$ *to* $\widetilde{\Sigma_{1,O_F,s_0',\xi\chi_1(\det(g))}}$.

Proof. Since $\tilde{e} = e/\varpi$, we can apply Proposition 7.1 here and everything is clear except for the claim that how it interchanges connected components. Notice that the "ξ" component is defined by $\tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}} - \varpi^{p+1}\xi$. So our claim follows from Proposition 5.9. □

Corollary 9.2. *The identification of the special fiber of* $\widetilde{\Sigma_{1,O_F,s_0'}}$ *with a Deligne–Lusztig variety is* $\mathrm{GL}_2(\mathbb{F}_p)$*-equivariant.*

We will come back to this point later when we review Deligne–Lusztig theory.

For $\widehat{\Sigma_{1,O_F}}$, since we change our base from O_{F_0} to O_F, there is a natural action of $\mathrm{Gal}(F/F_0)$.

Definition 9.3. $\tilde{\omega}_2 : \mathrm{Gal}(F/F_0) \to O_{F_0}^\times$ is the character given by $\tilde{\omega}_2(g) = \dfrac{g(\varpi)}{\varpi}$.

Any other character is a multiple of $\tilde{\omega}_2$.

Remark 9.4. Another equivalent definition of $\tilde{\omega}_2$ is as follows: By local class field theory, it suffices to give a character of $F_0^\times$. This character is trivial on $p^{\mathbb{Z}}$, and on $O_{F_0}^\times$ it is given by first reducing modulo p, then taking the inverse of the Teichmüller character. Our convention on the local Artin map is that uniformizers correspond to arithmetic Frobenius elements.

Remark 9.5. Recall that we defined two characters χ_1, χ_2 of $(O_D/\Pi)^\times$ (see Definition 3.2). Using the above remark, the relation of χ_1 and $\tilde{\omega}_2$ can be described in the following diagram:

$$\begin{array}{ccc} \mathbb{Z}_{p^2}^\times \simeq O_{F_0}^\times & \xrightarrow{\mathrm{Art}_{F_0}} & \mathrm{Gal}(\overline{F_0}/F_0)^{\mathrm{ab}} \\ \downarrow & & \downarrow \tilde{\omega}_2^{-1} \\ O_D^\times & \xrightarrow{\chi_1} & \mathbb{Z}_{p^2}^\times \simeq O_{F_0}^\times \end{array}$$

where the left arrow is our fixed embedding of $\mathbb{Z}_{p^2}$ into O_D, Art_{F_0} is the Artin map in local class field theory, the isomorphism between $\mathbb{Z}_{p^2}^\times$ and $O_{F_0}^\times$ is the one we fixed in the beginning.

Under the isomorphisms (23)–(26), we have:

Proposition 9.6. *The action of $g \in \mathrm{Gal}(F/F_0)$ is given by*

$$g(\tilde{e}) = \tilde{\omega}_2(g)^{-1}\tilde{e}, \quad g(\tilde{e}') = \tilde{\omega}_2(g)^{-1}\tilde{e}'. \tag{30}$$

This is trivial because $\tilde{e} = e/\varpi$, and $\tilde{e}' = e'/\varpi$.

The last group action we want to consider here is the action of $O_D^\times$.

Proposition 9.7. *Under the isomorphisms (23)–(26), for $d \in O_D^\times$,*

$$d(\tilde{e}) = \chi_1(d)\tilde{e}, \quad d(\tilde{e}') = \chi_2(d)\tilde{e}'. \tag{31}$$

Remark 9.8. The action of $O_D^\times$ on $\widehat{\Sigma'_{1,O_F}}$ is a twist of what we considered above:

$$d(\tilde{e}) = \widetilde{\mathrm{Fr}}(\chi_1(d))\tilde{e} = \chi_2(d)\tilde{e} = \chi_1(d)^p\tilde{e}, \qquad \forall d \in O_D^\times. \tag{32}$$

$$d(\tilde{e}') = \widetilde{\mathrm{Fr}}(\chi_2(d))\tilde{e}' = \chi_1(d)\tilde{e}' = \chi_2(d)^p\tilde{e}', \quad \forall d \in O_D^\times. \tag{33}$$

Here I identify $\widehat{\Sigma'_{1,O_F}}$ with $\widehat{\Sigma_{1,O_F}}$ but with twisted structure morphism. And by saying $\chi_2(d)$ I consider it as an element in the "O_F" coming from the structure map, not the $\mathbb{Z}_{p^2}$ coming from the original scheme $\widehat{\Sigma_1}$. However, the action of $\mathrm{Gal}(F/F_0)$ is the same, not twisted. Another way to see this is using a $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ with $v_p(\det(g))$ odd, then g sends $\widehat{\Sigma_{1,O_F,s}}$ to $\widehat{\Sigma'_{1,O_F,sg}}$. Finally, $g_\varphi \in \mathrm{Gal}(F/\mathbb{Q}_p)$ interchanges $\widehat{\Sigma_{1,O_F}}$ and $\widehat{\Sigma'_{1,O_F}}$ by acting as Frobenius endomorphism on O_{F_0} but fixes other things under the isomorphisms (23)–(26).

10. Another admissible open covering of the Drinfel'd upper half-plane and the generic fiber of $\widehat{\Sigma_{1,O_F}}$

In this section, we work on the generic fiber of everything we considered before. The main result of this section is a description of the generic fiber $\Sigma_{1,F}$ of $\widehat{\Sigma_{1,O_F}}$ (and a similar result for the generic fiber $\Sigma_{1,F}^{(0)}$ of $\widehat{\Sigma_{1,O_F}^{(0)}}$).

Recall that Σ_1 is the generic fiber of $\widehat{\Sigma_1^{\mathrm{nr}}}$. The latter is defined by two line bundles, $\mathcal{L}_1$, $\mathcal{L}_2$, and maps

$$d_1 : \mathcal{L}_1^{\otimes p} \to \mathcal{L}_2, \quad d_2 : \mathcal{L}_2^{\otimes p} \to \mathcal{L}_1 \tag{34}$$

(see the beginning of Section 4). Denote by $\mathcal{L}_{1,\eta}$, $\mathcal{L}_{2,\eta}$, $d_{1,\eta}$, $d_{2,\eta}$ the restriction of the corresponding item to Σ_1, the generic fiber.

First we observe:

Lemma 10.1. *Any line bundle over $\mathcal{X}_0$, the generic fiber of the Drinfel'd upper half-plane (and base changed to $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$), is trivial.*

To do this we need another admissible open covering of $\mathcal{X}_0$, which is described in [Drinfel'd 1974] ("topological" analog) and in [Schneider and Stuhler 1991] in detail. Let me recall it now.

Define

$$U_n(\mathbb{C}_p)=\{z\in C_p \mid |z|\le p^n,\ |z-a|\ge p^{-n},\ \forall a\in\mathbb{Q}_p\}, \tag{35}$$

where $|\cdot|$ is the canonical norm on $\mathbb{C}_p$ such that $|p|=p^{-1}$. Notice that we only need finitely many a to define this set, so U_n can be identified as an open set of $\mathbb{P}^1$ by removing some open discs. Therefore U_n is an affinoid space. In fact, we can identify it as an affinoid subdomain of a closed unit ball.

Remark 10.2. Another way to construct U_n is by using the formal model we already have. We can define a distance of two vertices of the Bruhat–Tits tree by counting the number of edges on the unique path between these two vertices. For example, two adjacent vertices have distance 1 and any vertex has distance 0 with itself. Now define Z_n as the set of vertices having distance $\le n$ from the central vertex. Let Ω_{U_n} be the union of Ω_e such that e is an edge between two vertices in Z_n and $\Omega_{U_0}=\Omega_{s_0'}$. Then U_n is the generic fiber of Ω_{U_n}.

It is clear $U_n\subset U_{n+1}$ and $\bigcup U_n=\Omega$, the Drinfel'd upper half-plane. Also it's not hard to verify the open covering $\{U_n\}$ is admissible. Let O_{U_n} be the ring of rigid analytic functions on U_n (over $\mathbb{Q}_p$). The key property we need is:

Lemma 10.3. *The image of the canonical inclusion $\phi_n: O_{U_{n+1}}\to O_{U_n}$ is dense under the canonical topology on O_{U_n}.*

Proof. Choose $a_1,\dots,a_m\in\mathbb{Q}_p$ such that $\{B(a_i,p^{-n})\}_i$ is an open covering of $p^{-n}\mathbb{Z}_p$ in $\mathbb{Q}_p$, where $B(a_i,p^{-n})$ is the open ball centered at a_i of radius p^{-n} in $\mathbb{Q}_p$. Now when we define U_n, we can use $a_1,\dots,a_m$ rather than all $a\in\mathbb{Q}_p$. Thus,

$$O_{U_n} =\left\{F(z)=\sum_{k=0}^{+\infty} b_{0,k}(p^nz)^k+\sum_{i=1}^{m}\sum_{k=0}^{+\infty} b_{i,k}\left(\frac{p^n}{z-a_i}\right)^k \,\middle|\, b_{i,k}\in\mathbb{Q}_p,\ \lim_{k\to+\infty} b_{i,k}=0,\forall i\right\}.$$

We define a norm $|\cdot|_n$ on O_{U_n} by $|F(z)|_n=\sup_{i,k}|b_{i,k}|$. This is nothing but the supremum norm: $|f|_n=\sup_{x\in\operatorname{Spm} O_{U_n}}|f(x)|$. Now the $\mathbb{Q}_p$-algebra generated by $z,1/(z-a_i)$ $(i=1,\dots,m)$ is dense in O_{U_n}. But these functions are defined over Ω and so live in $O_{U_{n+1}}$. □

Remark 10.4. Notice that in fact p^nz, $p^n/(z-a_i)$ $(i=1,\dots,m)$ are affinoid generators of O_{U_n} over $\mathbb{Q}_p$ in the sense there exists a surjective map from the Tate algebra $\mathbb{Q}_p\langle T_0,\dots,T_m\rangle$ to O_{U_n} that sends T_0 to p^nz and other T_i to $p^n/(z-a_i)$. If we restrict p^nz or $p^n/(z-a_i)$ to U_{n-1}, by definition of U_{n-1}, its norm is less than 1 (in fact $\le p^{-1}$). From this description, it's easy to see U_{n-1} is relatively compact in U_n. See [Bosch 2014, §6.3] for a precise definition. A direct corollary of this is that the inclusion map $O_{U_n} \to O_{U_{n-1}}$ is a strictly completely continuous map in the sense of [Bosch 2014, §6.4 Definition 1]. Another consequence is that Ω is a Stein-space as defined in [Kiehl 1967].

Now we return to the proof of Lemma 10.1. We still need one more lemma:

Lemma 10.5. *Any line bundle on U_n is trivial.*

Proof. It suffices to prove O_{U_n} is a principal ideal domain. It's obvious that O_{U_n} is regular and hence normal. So we only need to show every maximal ideal of O_{U_n} is principal. But we know U_n is an affinoid subdomain of a (one dimensional) closed unit ball by removing several open discs centered at $\mathbb{Q}_p$-points, with radius$\in p^{\mathbb{Z}}$. Our claim follows from the fact that $\mathbb{Q}_p\langle T\rangle$, the Tate algebra, is a PID [Bosch 2014, §2.2 Corollary 10]. □

Proof of Lemma 10.1. I learned this argument from [Kiehl 1967, proof of Satz 2.4]. Since $\{U_n\}_n$ is an admissible open covering of Ω and every line bundle on U_n is trivial, a line bundle on Ω is equivalent with a $1-$cocycle: $\{f_{ij}\}_{i<j},\ f_{ij} \in O_{U_i}^{\times}$, such that

$$f_{ij}\phi_{ji}(f_{jk}) = f_{ik}$$

for $i < j < k$, where ϕ_{ji} is the canonical inclusion from O_{U_j} to O_{U_i}. It's easy to see that $f_{12}, f_{23}, \ldots$ determine all f_{ij}. Two cocycles $\{f_{i(i+1)}\}, \{f'_{i(i+1)}\}$ define the same line bundle if and only if there exists $\{g_i\},\ g_i \in O_{U_i}^{\times}$, such that

$$f_{i(i+1)}g_i\phi_i(g_{i+1})^{-1} = f'_{i(i+1)}, \quad \forall i \geq 1.$$

Now let $\{f_{i(i+1)}\}$ be a fixed cocycle. Define $g'_1 = 1 \in O_{U_1}$. Thanks to Lemma 10.3, we can find $g'_{i+1} \in O_{U_i},\ i \geq 1$ by induction, satisfying

$$|1 - g'_i f_{i(i+1)}\phi_i(g'_{i+1})^{-1}|_i < \frac{1}{2^i}.$$

This implies, after modifying our cocycle, we can assume $|1 - f_{i(i+1)}|_i < \frac{1}{2^i}$. Now define $g_i = \prod_{j=i}^{\infty}\phi_{ji}(f_{j(j+1)})^{-1}$. Here ϕ_{ii} is the identity map. Notice that $|f|_j \geq |\phi_{ji}(f)|_i$ for $f \in O_{U_j}$; see Remark 10.4. So the infinite product makes sense by our assumption. But now $f_{i(i+1)}g_i\phi_i(g_{i+1})^{-1} = 1$. Therefore it corresponds to a trivial line bundle. □

Although our proof is working over the base field $\mathbb{Q}_p$, the argument still works if we change the base to other fields.

Corollary 10.6. *$\mathcal{L}_{1,\eta}$ and $\mathcal{L}_{2,\eta}$ are trivial line bundles.*

Now let E_1 be a basis of $\mathcal{L}_{1,\eta}$ and E_1^* the dual basis of E_1 under the isomorphism $\lambda_{\mathcal{L}_1}$. Then d_1, d_2 become two elements U_1, U_2 in $H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ such that $\mathcal{X}_1$ is now defined by

$$\mathcal{O}_{\mathcal{X}_0}[E_1, E_1^*]/(E_1^p - U_1E_1^*, (E_1^*)^p - U_2E_1).$$

We know $E_1E_1^* = \widetilde{\lambda_{\mathcal{L}_1}}$, so $U_1U_2 = -w$ (see Corollary 5.5). Σ_1 is

$$\mathcal{O}_{\mathcal{X}_0}[E_1, E_1^*]/\big(E_1^p - U_1E_1^*, (E_1^*)^p - U_2E_1, (E_1E_1^*)^{p-1} + w\big).$$

Since w is invertible on the generic fiber, so is U_1. We can write $E_1^* = E_1^pU_1^{-1}$.

Proposition 10.7. $\Sigma_1 = \mathcal{O}_{\mathcal{X}_0}[E_1]/(E_1^{p^2-1} + U_1^{p-1}w)$.

In other words, Σ_1 is $\mathcal{X}_0$ adjoined with a (p^2-1)-th root of a rigid analytic function on $\mathcal{X}_0$.

Remark 10.8. If we are careful enough in the beginning and take E_1 to be $p \in \mathrm{GL}_2(\mathbb{Q}_p)$-invariant, we can descend our description to O_{F_0}. This means we have the same description of the generic fiber $\Sigma_{1,F}$ of $\widehat{\Sigma_{1,O_F}}$.

Corollary 10.9. *$\Sigma_{1,F}$ is a Stein-space.*

Proof. As we remarked before (Remark 10.4), U_n is relatively compact in U_{n+1}. It's easy to see the open set of $\Sigma_{1,F}$ above U_n, which we denote by $V_{n,F}$ is an affinoid space and relatively compact in $V_{n+1,F}$. □

11. De Rham cohomology of $\Sigma_{1,F}$ and $\Sigma_{1,F}^{(0)}$

Let $\Omega^1_{\Sigma_{1,F}}$ be the sheaf of holomorphic differential forms on $\Sigma_{1,F}$ and $\Omega^0_{\Sigma_{1,F}} = \mathcal{O}_{\Sigma_{1,F}}$. Then we can consider the de Rham complex:

$$0 \to \Omega^0_{\Sigma_{1,F}} \xrightarrow{d} \Omega^1_{\Sigma_{1,F}}, \tag{36}$$

where d is the usual derivation. Define the de Rham cohomology:

Definition 11.1. $H^i_{\mathrm{dR}}(\Sigma_{1,F}) \overset{\mathrm{def}}{=} i$-th hypercohomology of the de Rham complex.

Remark 11.2. In a pair of papers Große-Klönne [2000; 2004] introduced a theory of de Rham cohomology for rigid analytic spaces. His approach uses the overconvergent de Rham complex rather than the usual De Rham complex. However in our case, they are the same since $\Omega_{\Sigma_{1,F}}$ is a Stein space [Große-Klönne 2000, Theorem 3.2].

Thanks to Kiehl [1967, Satz 2.4.2], we know that all higher cohomology groups of $\Omega^0_{\Sigma_{1,F}}, \Omega^1_{\Sigma_{1,F}}$ vanish:

Proposition 11.3 (de Rham cohomology).

$$H^0_{\mathrm{dR}}(\Sigma_{1,F}) = \ker(H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}}) \xrightarrow{d} H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})) = F, \tag{37}$$

$$H^1_{\mathrm{dR}}(\Sigma_{1,F}) = \operatorname{coker}(H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}}) \xrightarrow{d} H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})), \tag{38}$$

$$H^i_{\mathrm{dR}}(\Sigma_{1,F}) = 0, \ \forall i \geq 2. \tag{39}$$

We can put a certain topology on $H^1_{\mathrm{dR}}(\Sigma_{1,F})$. This is done by writing:

$$H^0(\Sigma_{1,F}, \Omega^i_{\Sigma_{1,F}}) = \varprojlim_n H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}}) \quad \text{for } i = 0, 1.$$

See the proof of Corollary 10.9 for the notation. Since each $H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}})$ is a Banach space and has a canonical topology, we can equip $H^0(\Sigma_{1,F}, \Omega^i_{\Sigma_{1,F}})$ with the projective limit topology. Now $V_{n,F}$ is relatively compact in $V_{n+1,F}$. As we observed in Remark 10.4, the transition map from $H^0(V_{n+1,F}, \Omega^i_{\Sigma_{1,F}})$ to $H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}})$ is completely continuous. Using Corollary 16.6 of [Schneider 2002], we have (notice that a completely continuous map between two Banach spaces is compact; see Proposition 18.11 of [Schneider 2002]):

Proposition 11.4. *$H^0(\Sigma_{1,F}, \Omega^i_{\Sigma_{1,F}})$, $i = 0, 1$ is a reflexive Fréchet space.*

See page 55 of [Schneider 2002] for the definition of reflexive.

Proposition 11.5 [Große-Klönne 2004, Corollary 3.2]. *The image of the derivation map $d : H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}}) \to H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})$ is closed.*

Corollary 11.6. *$H^1_{\mathrm{dR}}(\Sigma_{1,F})$ is a Fréchet space.*

But how to compute de Rham cohomology? We need our semistable $\widehat{\Sigma_{1,O_F}}$ constructed in Section 8. Let $E(\widehat{\Sigma_{1,O_F}})$ (resp. $V(\widehat{\Sigma_{1,O_F}})$) be the set of singular points (resp. irreducible components) of the special fiber of $\widehat{\Sigma_{1,O_F}}$. By definition, we can identify them as the set of edges (resp. vertices) of the dual graph of the special fiber. Now fix an orientation for each edge $e \in E(\widehat{\Sigma_{1,O_F}})$, and we use $v^+(e)$ (resp. $v^-(e)$) to denote the target (resp. source) vertex of the orientation.

Definition 11.7. Let U_e (resp. U_v) be the tubular neighborhood of the singular point indexed by e (resp. irreducible component indexed by v).

It is clear that $\{U_v\}_v$ is an admissible open covering of Σ_{1,O_F}. Hence:

Lemma 11.8. *We have a long exact sequence of de Rham cohomologies*:

$$0 \to H^0_{\mathrm{dR}}(\Sigma_{1,F}) \longrightarrow \prod_{v \in V(\widehat{\Sigma_{1,O_F}})} H^0_{\mathrm{dR}}(U_v) \xrightarrow{a} \prod_{e \in E(\widehat{\Sigma_{1,O_F}})} H^0_{\mathrm{dR}}(U_e) \xrightarrow{\partial} H^1_{\mathrm{dR}}(\Sigma_{1,F})$$
$$\longrightarrow \prod_{v \in V(\widehat{\Sigma_{1,O_F}})} H^1_{\mathrm{dR}}(U_v) \xrightarrow{b} \prod_{e \in E(\widehat{\Sigma_{1,O_F}})} H^1_{\mathrm{dR}}(U_e),$$

where the arrows without labels are canonical restriction maps, and a, b are the canonical restriction maps to $v^+(e)$ minus the restriction map to $v^-(e)$ for an element indexed by e.

Here the de Rham cohomologies of U_e, U_v are defined by the same method as above. We note that they are not affinoid but Stein spaces.

We first look at U_e, the tubular neighborhood of a singular point. It's not hard to see from the explicit description in Lemma 8.2 that U_e is an annulus $\{T \mid |\varpi| < |T| < 1\}$. So its de Rham cohomology is: $H^0_{\mathrm{dR}}(U_e) = F$, generated by the constant function; $H^1_{\mathrm{dR}}(U_e) \simeq F$, generated by dT/T, where T is a coordinate of U_e.

In Lemma 8.2, although we haven't resolved the singularities there, $d\tilde{e}/\tilde{e}$ still makes sense on the generic fiber, and it generates all of $H^1_{\mathrm{dR}}(U_e)$ for any singular point e above the singularity there. In fact, the process of resolving the singularities $xy - \varpi^n$ is just "dividing" the annulus into several small annuli. For example, the tubular neighborhood of $xy - \varpi^n$ can be thought as the annulus $\{T \mid |\varpi|^n < T < 1\}$. For any e above this singular point, U_e can be identified as $\{T \mid |\varpi|^{l+1} < T < |\varpi|^l\}$ for some $l < n$.

Recall that $O_D^\times$ acts as characters on $\tilde{e}$, so acts trivially on $H^0_{\mathrm{dR}}(U_e)$, $H^1_{\mathrm{dR}}(U_e)$.

What about U_v? There are two possibilities. One is that v corresponds to a rational curve. U_v is an annulus and the result is the same as U_e. In particular $O_D^\times$ acts trivially on their de Rham cohomologies.

The other one is more interesting. We will compute it in the next section. Some notation here: recall that every such vertex can be indexed by (s, ξ), where s is a vertex of the Bruhat–Tits tree and ξ satisfies $\xi^{p-1} = -1$.

Definition 11.9. From now on we will use (s, ξ) to denote these vertices.

Definition 11.10. Denote the irreducible component indexed by (s, ξ) by $\overline{U_{s,\xi}}$ and its generic fiber by $U_{s,\xi}$. We also denote the smooth loci of $\overline{U_{s,\xi}}$ by $\overline{U^0_{s,\xi}}$ (viewed as a subscheme in the special fiber of $\widehat{\Sigma_{1,O_F}}$). Notice that this is nothing but the special fiber of $\widehat{\Sigma_{1,O_F,s,\xi}} = \widetilde{\Sigma_{1,O_F,s,\xi}}$. Define

$$\overline{U_s} = \bigcup_{\xi^{p-1}=-1} \overline{U_{s,\xi}},$$

and $\overline{U^0_s}$ similarly.

Recall that in the beginning, we fix a finite extension E of $\mathbb{Q}_p$ that is large enough and define $\chi(E)$ as the set of characters of $(O_D/\Pi)^\times$ with values in $E^\times$.

$O_D^\times$ acts naturally on $H^1_{\mathrm{dR}}(\Sigma_{1,F}) \otimes_{\mathbb{Q}_p} E$ by acting on the first factor. Since the action of $O_D^\times$ on $\Sigma_{1,F}$ factors through $O_D^\times/(1 + \Pi O_D)$, we can decompose

$H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E$ as

$$H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E = \bigoplus_{\chi\in\chi(E)} (H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi}, \tag{40}$$

where $(H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} = \{a \mid d(a) = (1\otimes\chi(d))a,\ \forall d\in O_D^{\times}\}$ is the χ-isotypic component.

Now tensor everything in the long exact sequence of Lemma 11.8 with E, and take the χ-isotypic component for a nontrivial character $\chi\in\chi(E)$. As we explained above, $O_D^{\times}$ acts trivially on the cohomology of any annulus, so only the de Rham cohomology of U_s contributes. In other words:

Lemma 11.11. *For a nontrivial character χ,*

$$(H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} \simeq \prod_s (H^1_{\mathrm{dR}}(U_s)\otimes_{\mathbb{Q}_p} E)^{\chi},$$

$$(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} \simeq \prod_s (H^1_{\mathrm{dR}}(U^{(0)}_s)\otimes_{\mathbb{Q}_p} E)^{\chi}$$

$$= \prod_s \big((H^1_{\mathrm{dR}}(U_s)\oplus H^1_{\mathrm{dR}}(U'_s))\otimes_{\mathbb{Q}_p} E\big)^{\chi},$$

where s takes value in the set of vertices of the Bruhat–Tits tree.

It's clear that $\mathrm{GL}_2(\mathbb{Q}_p)$ preserves $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi}$ because the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ commutes with $O_D^{\times}$. Also $g\in\mathrm{GL}_2(\mathbb{Q}_p)$ induces an isomorphism from $U^{(0)}_s$ to $U^{(0)}_{sg}$, hence an isomorphism from $H^1_{\mathrm{dR}}(U^{(0)}_{sg})$ to $H^1_{\mathrm{dR}}(U^{(0)}_s)$. Note that the set of vertices of the Bruhat–Tits tree is nothing but $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}\backslash \mathrm{GL}_2(\mathbb{Q}_p)$. Thus we have:

Proposition 11.12. *As a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E, we have*

$$(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}} (H^1_{\mathrm{dR}}(U^{(0)}_{s'_0})\otimes_{\mathbb{Q}_p} E)^{\chi}$$

for any nontrivial character $\chi\in\chi(E)$. Recall that s'_0 is the central vertex. Here the induction has no restriction on the support.

12. An F_0-structure of $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi}$ and the computation of $H^1_{\mathrm{dR}}(U_{s'_0})$

Recall that F_0 is the maximal unramified extension of $\mathbb{Q}_p$ inside F and we fixed an isomorphism between it and $\mathbb{Q}_{p^2}$ in the beginning.

Following Coleman and Iovita [1999], we can define an F_0/Frobenius structure on the de Rham cohomology $H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})$. This means we can find an F_0-linear subspace H_{F_0} equipped with a $\widetilde{\mathrm{Fr}}$-linear Frobenius morphism, such that $H_{F_0}\otimes_{F_0} F\simeq H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})$. Let's recall their construction in our situation now.

By Lemma 11.11, we only need to define an F_0/Frobenius structure on each $(H^1_{\mathrm{dR}}(U^{(0)}_s)\otimes_{\mathbb{Q}_p} E)^{\chi}$, in fact, each $(H^1_{\mathrm{dR}}(U_{s,\xi})\otimes_{\mathbb{Q}_p} E)^{\chi}$ (using the notation from

Definition 11.7). Theorem C of [Große-Klönne 2002] tells us we have a natural isomorphism between $H^1_{\mathrm{dR}}(U_{s,\xi})$ and $H^1_{\mathrm{rig}}(\overline{U^0_{s,\xi}}/F)$, the rigid cohomology of $\overline{U^0_{s,\xi}}$ with coefficients in F defined in [Berthelot 1986]. Recall that $\overline{U^0_{s,\xi}}$ is an open set of $\overline{U_{s,\xi}}$ by removing $(p+1)$ $\mathbb{F}_p$-rational points (each corresponds to an edge connecting s). Then we have the following exact sequence:

$$0 \to H^1_{\mathrm{rig}}(\overline{U_{s,\xi}}/F) \to H^1_{\mathrm{rig}}(\overline{U^0_{s,\xi}}/F) \to F^{\oplus p+1} \to F \to 0. \tag{41}$$

Explicitly, we can construct an isomorphism $\psi_{s,\xi} : U_{s,\xi} \to F_{1,\xi}$, where $F_{1,\xi}$ is defined as

$$\bigl\{(x,y) \in \mathbb{A}^2_F \;\big|\; y^{p+1} = v_1^{-1} w_1 \xi (x^p - x),\ |x-k| > p^{-1/(p-1)},\ k = 0, 1, \ldots, p-1, \\ |x| < p^{1/(p-1)}\bigr\}$$

for an odd vertex s (even case is similar). If we restrict this isomorphism to the generic fiber of $\widehat{\Sigma_{1,O_F,s,\xi}}$ and use the description in (25), it is given by

$$x \mapsto \zeta, \quad y \mapsto \tilde{e}'(1-(p/\zeta)^{p-1})^{1/(p+1)},$$

where $(1-(p/\zeta)^{p-1})^{1/(p+1)} = 1 - 1/(p+1)(p/\zeta)^{p-1} + \cdots$ is the binomial expansion. The rigid space $F_{1,\xi}$ is clearly an open set of a projective curve $D_{1,\xi}$ in $\mathbb{P}^2_F$ defined by $y^{p+1} = v_1^{-1} w_1 \xi (x^p - x)$. We note that $D_{1,\xi} - F_{1,\xi}$ is a union of $p+1$ closed discs. Each disc is centered at a point with zero y-coordinate. We denote these points by $C_0, \ldots, C_p$. Then, we have

$$0 \to H^1_{\mathrm{dR}}(D_{1,\xi}) \longrightarrow H^1_{\mathrm{dR}}(F_{1,\xi}) \xrightarrow{\mathrm{Res}} \bigoplus_{i=0}^{p} F \xrightarrow{\mathrm{sum}} F \to 0, \tag{42}$$

where Res is the residue map to each C_i, and sum is taking the sum. A proof of this can be found in Section IV of [Coleman 1989]. Notice that $D_{1,\xi}$ has an obvious formal model over O_F (in fact over O_{F_0}!), and its special fiber is nothing but $\overline{U_{1,\xi}}$. So we have a natural isomorphism between $H^1_{\mathrm{dR}}(D_{1,\xi})$ and $H^1_{\mathrm{rig}}(\overline{U_{1,\xi}})$. Using these isomorphisms, we can identify the two exact sequences (41), (42) with each other.

It is not hard to see $O_D^\times$ acts trivially on the residues. For example, near $x = y = 0$, $t = y/(1-x^{p-1})^{1/(p+1)}$ is a local coordinate. $O_D^\times$ acts as a character on y and acts trivially on x, hence acts trivially on dt/t. Therefore if we tensor the exact sequence (41) with E and take the χ-isotypic component, we obtain:

Lemma 12.1. $(H^1_{\mathrm{dR}}(U_s) \otimes_{\mathbb{Q}_p} E)^\chi \simeq (H^1_{\mathrm{rig}}(\overline{U}_s/F) \otimes_{\mathbb{Q}_p} E)^\chi$.

Since we have a natural isomorphism $H^1_{\mathrm{rig}}(\overline{U}_s/F) \simeq H^1_{\mathrm{crys}}(\overline{U}_s/F_0) \otimes_{F_0} F$, there is an F_0/Frobenius structure on $(H^1_{\mathrm{rig}}(\overline{U}_s/F) \otimes_{\mathbb{Q}_p} E)^\chi$ and thus on $(H^1_{\mathrm{dR}}(U_s) \otimes_{\mathbb{Q}_p} E)^\chi$. Here $H^1_{\mathrm{crys}}(\overline{U}_s/F_0)$ is the first crystalline cohomology of U_s tensored with $\mathbb{Q}_p$. Explicitly, as we mentioned above, $D_{1,\xi}$ can be defined over F_0 and its formal model $\widehat{D_{1,O_{F_0},\xi}}$ over O_{F_0} is a smooth lifting of $\overline{U_{s,\xi}}$. So the de Rham cohomology of $\widehat{D_{1,O_{F_0},\xi}}$ can be identified with the crystalline cohomology of $\overline{U_{s,\xi}}$. Thus we obtain an F_0-linear subspace inside $H^1_{\mathrm{dR}}(D_{1,\xi})$. But to get a Frobenius operator, we need to identify it with the crystalline cohomology.

Remark 12.2. For an even vertex s', we can define similar objects:

$$\psi_{s',\xi} : U_{s',\xi} \to F_{0,\xi},\ D_{0,\xi},\ \widehat{D_{0,O_{F_0},\xi}},\ \ldots .$$

In summary, combining the above results with Proposition 11.12, we have:

Proposition 12.3. *$(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^\chi$ has an F_0/Frobenius structure that comes from the crystalline cohomology of the special fiber of $\widehat{\Sigma^{(0)}_{1,O_F}}$. More precisely, under the identification of $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^\chi$ with*

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} (H^1_{\mathrm{dR}}(U^{(0)}_{s'_0}) \otimes_{\mathbb{Q}_p} E)^\chi,$$

the F_0-subspace is

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} (H^1_{\mathrm{crys}}(\overline{U^{(0)}_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^\chi,$$

and the Frobenius operator is defined in the obvious way.

Remark 12.4. We can also define a monodromy operator, but for any χ such that $\chi \neq \chi^p$ it is zero on $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^\chi$. The reason is that the definition of monodromy operator uses the cohomologies of the tubes of the singular points, which do not contribute to the cohomology we are interested in. See [Coleman and Iovita 1999] for the precise definition of monodromy operator.

As we remarked before, $\overline{U_{s'_0}}$ has a close relation with the Deligne–Lusztig variety of $\mathrm{GL}_2(\mathbb{F}_p)$ (Corollary 9.2), which we call DL. In fact, the open set

$$\overline{U^0_{s'_0}} \simeq \mathrm{Spec}\, \mathbb{F}_{p^2}\Big[\eta, \tilde{e}, \frac{1}{\tilde{e}}\Big] / (\tilde{e}^{p^2-1} + w_1^2(\eta^p - \eta)^{p-1})$$

is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariantly isomorphic with DL over the algebraically closed field (or up to taking a transpose of $\mathrm{GL}_2(\mathbb{F}_p)$). So we can apply Deligne–Lusztig theory (established in [Deligne and Lusztig 1976]). Although Deligne and Lusztig [1976] use l-adic cohomology, their results can be applied directly to crystalline cohomology thanks to Katz and Messing [1974] and Gillet and Messing [1987]. Notice that the action of $O_D^\times$ on $\overline{U^0_{s'_0}}$, which factors through $O_D^\times/(1+\Pi O_D)$, can be identified with the inverse of the action of a nonsplit torus $(\mathrm{T}(w))^F$ in [Deligne and Lusztig 1976]) of $\mathrm{GL}_2(\mathbb{F}_p)$.

Theorem 12.5. *Let $\chi(F_0)$ be the character group of $O_D^\times/(1+\Pi O_D)$ with values in F_0* (it's generated by χ_1; see Definition 3.2). *We can decompose*

$$H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0) = \bigoplus_{\chi' \in \chi(F_0)} H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$$

into the sum of different χ'-isotypic components. Each component has a natural action of $\mathrm{GL}_2(\mathbb{F}_p)$. *Then*:

(1) $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}=0$ *if and only if* $\chi'=(\chi')^p$.

(2) *If* $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}\neq 0$, *it's an irreducible representation of* $\mathrm{GL}_2(\mathbb{F}_p)$.

(3) $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}\simeq H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{(\chi')^p}$ *and these are the only isomorphisms among these nonzero representations.*

Definition 12.6. Define ρ_χ as the representation $(H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)\otimes_{F_0}E)^\chi$ of $\mathrm{GL}_2(\mathbb{F}_p)$, for any $\chi\in\chi(E)$. The theorem above guarantees that different choices of embedding $F_0\to E$ give the same representation.

Remark 12.7. $\mathrm{Gal}(F/F_0)$ also acts on $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$. By the results in Section 9, we have

$$H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}=H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\tilde{\omega}_2^{i(\chi')}},$$

the $\tilde{\omega}_2^{i(\chi')}$-isotypic space for $\mathrm{Gal}(F/F_0)$, where $i(\chi')\in\{0,\dots,p^2-2\}$ is defined as the unique integer such that $\chi_1^{-i(\chi')}=\chi'$. Using results in Remark 9.5, another equivalent definition is that $\tilde{\omega}_2^{i(\chi')}$ is the unique character making the following diagram commutative:

$$\begin{array}{ccc}\mathbb{Z}_{p^2}^\times\simeq O_{F_0}^\times & \xrightarrow{\mathrm{Art}_{F_0}} & \mathrm{Gal}(\overline{F}_0/F_0)^{\mathrm{ab}}\\ \downarrow & & \downarrow{\scriptstyle \tilde{\omega}_2^{i(\chi')}}\\ O_D^\times & \xrightarrow{\chi'} & \mathbb{Z}_{p^2}^\times\simeq O_{F_0}^\times\end{array}$$

Recall that $\tilde{\omega}_2$ is defined in Remark 9.5.

Now I want to translate the theorem above to our situation. Fix an embedding $\tau:F_0\to E$, and use $\bar{\tau}$ to denote the conjugate embedding. Let $\chi'\in\chi(F_0)$ be the unique character that satisfies $\tau\circ\chi'=\chi$. Recall that $g_\varphi\in\mathrm{Gal}(F/\mathbb{Q}_p)$ is the unique element that fixes ϖ but acts as Frobenius on F_0.

Proposition 12.8. $D_{\mathrm{crys},\chi}\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'},(H^1_{\mathrm{crys}}(\overline{U^{(0)}_{s'_0}}/F_0)\otimes_{\mathbb{Q}_p}E)^\chi\big)$ *is a free* $F_0\otimes_{\mathbb{Q}_p}E$*-module of rank* 2. $\mathrm{Gal}(F/\mathbb{Q}_p)$ *and the Frobenius operator* φ *act on it naturally. In fact*, $D_{\mathrm{crys},\chi}$ *is of the form*

$$D_{\mathrm{crys},\chi}=(F_0\otimes_{\mathbb{Q}_p}E)\cdot\boldsymbol{e}_1\oplus(F_0\otimes_{\mathbb{Q}_p}E)\cdot\boldsymbol{e}_2,$$

$$\varphi(\boldsymbol{e}_1)=\boldsymbol{e}_2,\quad \varphi(\boldsymbol{e}_2)=(1\otimes c_x)\boldsymbol{e}_1,$$

$$g\cdot\boldsymbol{e}_1=(\tilde{\omega}_2(g)^m\otimes 1)\boldsymbol{e}_1,\quad g\cdot\boldsymbol{e}_2=(\tilde{\omega}_2(g)^{pm}\otimes 1)\boldsymbol{e}_2,\quad \forall g\in\mathrm{Gal}(F/F_0),$$

$$g_\varphi\cdot\boldsymbol{e}_1=\boldsymbol{e}_1,\quad g_\varphi\cdot\boldsymbol{e}_2=\boldsymbol{e}_2,$$

with $c_x\in E$ *and* $v_p(c_x)=1$, $m=i(\chi')$ *defined in Remark 12.7.*

Proof. We can write (using the fact $\overline{U^{(0)}_{s'_0}} = \overline{U_{s'_0}} \sqcup \overline{U'_{s'_0}}$)

$$\begin{aligned}(H^1_{\text{crys}}(\overline{U^{(0)}_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi} &= (H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi} \oplus (H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi} \\ &= H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E \oplus H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E \\ &\quad \oplus H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E \oplus H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E,\end{aligned}$$

where $\overline{\chi'} = (\chi')^p$, the conjugate character, satisfies $\bar{\tau} \circ \overline{\chi'} = \chi$.

Recall that we can identify $\overline{U_{s'_0}}$ with $\overline{U'_{s'_0}}$ but with different structure map to Spec $\mathbb{F}_{p^2}$. Using Remark 9.8, such an identification induces an isomorphism between $H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$ and $H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}}$. By definition,

$$\text{Hom}_{\text{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E\big) \simeq F_0 \otimes_{F_0,\tau} E$$

and similar results for other factors of $(H^1_{\text{crys}}(\overline{U^{(0)}_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi}$ follow from Deligne–Lusztig theory. It's easy to see $D_{\text{crys},\chi} \simeq F_0 \otimes_{\mathbb{Q}_p} E^{\oplus 2}$ from these descriptions.

By Remarks 12.7 and 9.8, $\text{Gal}(F/F_0)$ acts via $\tilde{\omega}_2^m$ (as an F_0-vector space) on $H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E$, and $H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E$ and acts as $\tilde{\omega}_2^{pm}$ on the other two factors since $i(\overline{\chi'}) = i((\chi')^p) = pi(\chi')$. Remark 9.8 also tells us that g_{φ} induces an isomorphism between $H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E$ and $H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E$. Now, choose a generator $\mathbf{f}_1$ of

$$\text{Hom}_{\text{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E\big),$$

and define

$$\boldsymbol{e}_1 = \mathbf{f}_1 + g_{\varphi} \cdot \mathbf{f}_1, \boldsymbol{e}_2 = \varphi(\boldsymbol{e}_1),$$

where φ is the Frobenius operator coming from the crystalline cohomology. We need to verify our claim in the proposition.

First it's easy to see $\boldsymbol{e}_1$ is indeed a generator of

$$\text{Hom}_{\text{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E \oplus H^1_{\text{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E\big)$$

as a free $F_0 \otimes_{\mathbb{Q}_p} E$-module and satisfies $g \cdot \boldsymbol{e}_1 = (\tilde{\omega}_2(g)^m \otimes 1)\boldsymbol{e}_1,\ g \in \text{Gal}(F/F_0)$. Next we verify the desired property of the Frobenius operator φ. It's induced by the Frobenius endomorphism on $\overline{U_{s'_0}}$, which is nothing but raising anything to its p-th power. So it sends $H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$ to $H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{(\chi')^p} = H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\overline{\chi'}}$. Therefore everything is clear except our claim for $\varphi(\boldsymbol{e}_2)$. This can be shown by explicit computations. See the next lemma. □

Lemma 12.9. $$c_x = -p\tau(w_1^{-2i}).$$

Proof. This can be done using Gauss sums. Since $H^1_{\text{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$ is an irreducible representation of $\text{GL}_2(\mathbb{F}_p)$, φ^2 acts as a scalar $\tilde{c_x}$ on it. It's easy to see $c_x = \tau(\tilde{c_x})$.

To compute $\tilde{c_x}$, we only need to restrict to one component. So let ξ be a root of $\xi^{p-1}=-1$. Then $\overline{U_{s_0',\xi}}$ can be identified as the curve in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ defined by $y^{p+1}=v_1w_1^{-1}\xi(x^p-x)$. There is an action of

$$\mu_{p+1}(\mathbb{F}_{p^2})=\{a\in\mathbb{F}_{p^2}^\times \mid a^{p+1}=1\}$$

on it given by

$$a\cdot x=x,\quad a\cdot y=ay,\quad a\in\mu_{p+1}(\mathbb{F}_{p^2}^\times).$$

Let $\tilde{\chi}:\mu_{p+1}(\mathbb{F}_{p^2})\to F_0^\times$ be the Teichmüller character. It's obvious that

$$H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/F_0)^{\chi'}\simeq H^1_{\mathrm{crys}}(\overline{U_{s_0',\xi}}/F_0)^{\tilde{\chi}^{-i}},$$

the $\tilde{\chi}^{-i}$-isotypical component. Here $i\in\{1,\dots,p\}$ is the unique number satisfying $i\equiv m \bmod p+1$.

On the other hand, $\mathbb{F}_p$ also acts on $\overline{U_{s_0',\xi}}$, which comes from the action of an unipotent subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$:

$$b\cdot x=x+1,\quad b\cdot y=y,\quad b\in\mathbb{F}_p.$$

This action commutes with the action of $\mu_{p+1}(\mathbb{F}_{p^2})$. It's easy to see F_0 contains all p-th roots of unity. Let $\psi_p:\mathbb{F}_p\to F_0^\times$ be a nontrivial additive character. We view $\tilde{\rho}=\tilde{\chi}^{-i}\times\psi_p$ as a one dimensional representation of $\widetilde{G}\overset{\text{def}}{=}\mu_{p+1}(\mathbb{F}_{p^2})\times\mathbb{F}_p$.

Using Lemma 1.1. of [Katz 1981], we know that the eigenvalue of φ^2 on $(H^1_{\mathrm{crys}}(\overline{U_{s_0',\xi}}/F_0)\otimes_{F_0}F)^{\tilde{\rho}}$ is (we will see later that this lemma indeed can be applied to our situation)

$$-S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2},\tilde{\rho},1)\overset{\text{def}}{=}-\frac{1}{\#\widetilde{G}}\sum_{g\in\widetilde{G}}\mathrm{tr}(\tilde{\rho}(g))\,\#\mathrm{Fix}(\mathrm{F}_{p^2}g^{-1}),$$

where F_{p^2} is the Frobenius endomorphism of $\overline{U_{s_0',\xi}}$ relative to $\mathbb{F}_{p^2}$ and $\mathrm{Fix}(\mathrm{F}_{p^2}g^{-1})$ is the subset of $\overline{U_{s_0',\xi}}(\overline{\mathbb{F}_{p^2}})$ fixed by $\mathrm{F}_{p^2}g^{-1}$. Following the strategy of lemma 2.1. of [Katz 1981], we can express $S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2},\tilde{\rho},1)$ as the Gauss sum:

$$S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2},\tilde{\rho},1)=(v_1w_1^{-1}\xi)^{-i(p-1)}\sum_{x\in\mathbb{F}_{p^2}^\times}\psi_{p^2}(x)x^{-i(p-1)},$$

where $\psi_{p^2}\overset{\text{def}}{=}\psi_p(\mathrm{tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x))=\psi_p(x^p+x)$. Notice that for any $x\in\mathbb{F}_{p^2}^\times$,

$$\sum_{a\in\mathbb{F}_p^\times}\psi_{p^2}(ax)=\sum_{a\in\mathbb{F}_p^\times}\psi_p(a(x^p+x))=\begin{cases}-1 & \text{if } x^p+x\neq 0,\\ p-1 & \text{if } x^p+x=0.\end{cases}$$

From this, it's easy to see $S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2},\tilde{\rho},1)=w_1^{i(p-1)}p(-1)^i=w_1^{-2i}p$ (recall $v_1^{p-1}=w_1^{p+1}=\xi^{p-1}=-1$). Hence

$$c_x=-p\tau(w_1^{-2i}).$$

□

Corollary 12.10. *We have a* $\mathrm{Gal}(F/\mathbb{Q}_p)\times O_D^\times\times\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism*:

$$F\otimes_{F_0} D_{\mathrm{crys},\chi}\otimes_E \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi \xrightarrow{\sim} (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^\chi, \tag{43}$$

where $\mathrm{Gal}(F/\mathbb{Q}_p)$ *acts on the first two components*, $O_D^\times$ *acts on the second*, *and* $\mathrm{GL}_2(\mathbb{Q}_p)$ *acts on the third. Moreover*, $D_{\mathrm{crys},\chi}\otimes_E \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ *maps to the* F_0 *subspace we constructed in Proposition 12.3.*

Here we extend ρ_χ to a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by p acting trivially and $\mathrm{GL}_2(\mathbb{Z}_p)$ acting through $\mathrm{GL}_2(\mathbb{F}_p)$.

Remark 12.11. It's easy to see the dual representation of ρ_χ is $\rho_{\chi^{-1}}$, we use $\langle\cdot,\cdot\rangle$ to denote the pairing of them. Then we can construct a pairing:

$$\begin{aligned}\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}}\times \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi &\to E\\ (f_1,f_2)&\mapsto \Sigma_{[g]\in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash \mathrm{GL}_2(\mathbb{Q}_p)}\langle f_1(g),f_2(g)\rangle,\end{aligned}$$

where is the compact induction. More precisely,

$$\begin{aligned}&\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}}\\ &\quad=\Big\{f:\mathrm{GL}_2(\mathbb{Q}_p)\to\rho_{\chi^{-1}}\;\Big|\; f \text{ has compact support mod } \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times,\\ &\qquad\qquad f(kg)=\rho_{\chi^{-1}}(k)f(g),\; k\in\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times,\; g\in\mathrm{GL}_2(\mathbb{Q}_p)\Big\},\end{aligned}$$

and $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ is defined similarly without any restrictions on the support. The sum makes sense because it only has finitely many nonzero terms.

This pairing induces an isomorphism $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi\simeq(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}})^\vee$, the algebraic dual representation. We can rewrite the result in Corollary 12.10 as a $\mathrm{Gal}(F/\mathbb{Q}_p)\times O_D^\times\times\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism (Theorem 1.10):

$$F\otimes_{F_0} D_{\mathrm{crys},\chi}\otimes_E (\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}})^\vee \xrightarrow{\sim} (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^\chi. \tag{44}$$

By Corollary 11.6, there is a natural Fréchet space structure on the right-hand side of the above map. In fact, we can describe this topology directly on the left-hand side. Choosing a family of representatives of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\setminus\mathrm{GL}_2(\mathbb{Q}_p)$, we have a noncanonical isomorphism between $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ and $\prod_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ as E-vector spaces. The topology is nothing but the weakest topology on this product such that each projection to ρ_χ is continuous under the canonical (Banach space) topology on ρ_χ.

13. Some considerations from Galois representations

Let's recall what we have on $D_{\mathrm{crys},\chi}$ (see Proposition 12.8 for more details):

- Frobenius operator φ: an F_0-semilinear, E-linear automorphism;

- monodromy operator N, which is zero here;
- an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, which is F_0-semilinear, E-linear commuting with φ and N.

So if we have a decreasing filtration on $D_F = F \otimes_{F_0} D_{\mathrm{crys},\chi}$, such that $\mathrm{Fil}^i D_F$ is zero if $i \gg 0$ and is equal to D_F if $i \ll 0$ and preserved by the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, $D_{\mathrm{crys},\chi}$ is called a filtered $(\varphi, N, F/\mathbb{Q}_p, E)$-module of rank 2. Moreover, if the underlying (φ, N, F, E)-module is weakly admissible, $D_{\mathrm{crys},\chi}$ is called weakly admissible. See Definitions 2.7 and 2.8 of [Savitt 2005] for the precise definition. The importance of this kind of module is that we have the following result (see [Savitt 2005, Corollary 2.10]).

Theorem 13.1. *The category of E-representations of $G_{\mathbb{Q}_p}$ which become semistable when restricted to G_F and the category of weakly admissible $(\varphi, N, F/\mathbb{Q}_p, E)$-modules are equivalent. Here $G_{\mathbb{Q}_p}$ (resp. G_F) is the absolute Galois group of $\mathbb{Q}_p$ (resp. F).*

Now I want to classify all two dimensional potentially semistable E-representations of $G_{\mathbb{Q}_p}$ that

- have Hodge–Tate weights (0, 1), and
- correspond to $D_{\mathrm{crys},\chi}$ if we forget about the filtration.

Proposition 13.2 [Savitt 2005, Proposition 2.18]. *Any such weakly admissible $(\varphi, N, F/\mathbb{Q}_p, E)$-module is of the form*

$$\mathrm{Fil}^n(D_F) = \begin{cases} D_F, & n \le 0, \\ (F \otimes_{\mathbb{Q}_p} E)\big((\varpi^{(p-1)i} \otimes a)\boldsymbol{e}_1 + (1 \otimes b)\boldsymbol{e}_2\big), & n = 1, \\ 0, & n \ge 2, \end{cases}$$

where $(a, b) \neq (0, 0) \in E^2$, and i, j are defined as follows: write $m = i + (p+1)j$ with $i \in \{1, \dots, p\}$ and $j \in \{0, \dots, p-2\}$.

We denote the filtered module in the above proposition by $D_{\chi,[a,b]}$. It's not hard to see

$$D_{\chi,[a,b]} \simeq D_{\chi^p,[bc_x/p,-a]} \quad \text{and} \quad D_{\chi,[a,b]} = D_{\chi,[ca,cb]}.$$

So we may assume $a = 1$ and $v_p(b) \ge 0$ (recall that c_x is defined in Proposition 12.8). We use $V_{\chi,[1,b]}$ to denote the Galois representation it corresponds to in Theorem 13.1.

Now suppose we have an element f in $D_F = F \otimes_{F_0} D_{\mathrm{crys},\chi}$. How do we check whether or not f is in $\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$ for a given b? First assume $f \in \mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$. Write $f = f_1 + f_2$, $f_1 \in (F \otimes_{\mathbb{Q}_p} E) \cdot \boldsymbol{e}_1$, $f_2 \in (F \otimes_{\mathbb{Q}_p} E) \cdot \boldsymbol{e}_2$. Then we must have

$$f_1 = \Big(\sum a_k \otimes b_k\Big)(\varpi^{(p-1)i} \otimes 1)\boldsymbol{e}_1, \quad f_2 = \Big(\sum a_k \otimes b_k\Big)(1 \otimes b)\boldsymbol{e}_2,$$

for some $a_k \in F,\ b_k \in E$. Notice that $g_\varphi \otimes \varphi$ is well-defined on $F \otimes_{F_0} D_{\mathrm{crys},\chi}$ since g_φ acts as Frobenius on F_0. Here g_φ is considered only acting on F, not on $D_{\mathrm{crys},\chi}$:

$$(g_\varphi \otimes \varphi)(f_1) = \left(\sum g_\varphi(a_k) \otimes b_k\right)(\varpi^{(p-1)i} \otimes 1)\boldsymbol{e}_2.$$

On the other hand, $g_\varphi(f_2) = (\Sigma g_\varphi(a_k) \otimes b_k)(1 \otimes b)\boldsymbol{e}_2$. Therefore,

$$(1 \otimes b)(g_\varphi \otimes \varphi)(f_1) = (\varpi^{(p-1)i} \otimes 1)g_\varphi(f_2).$$

A simple dimension counting shows that this condition is even sufficient. Hence:

Proposition 13.3. *Suppose $f \in F \otimes_{F_0} D_{\mathrm{crys},\chi}$. Write*

$$f = f_1 + f_2, \quad f_1 \in (F \otimes_{\mathbb{Q}_p} E) \cdot \boldsymbol{e}_1, \quad f_2 \in (F \otimes_{\mathbb{Q}_p} E) \cdot \boldsymbol{e}_2.$$

Then $f \in \mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$ if and only if

$$(1 \otimes b)(g_\varphi \otimes \varphi)(f_1) = (\varpi^{(p-1)i} \otimes 1)g_\varphi(f_2).$$

Remark 13.4. In practice, we will assume f is fixed by g_φ; then the condition above is simplified to $(1 \otimes b)(g_\varphi \otimes \varphi)(f_1) = (\varpi^{(p-1)i} \otimes 1) f_2$.

14. Construction of Banach space representations of $\mathbf{GL_2}(\mathbb{Q}_p)$

In this section, I want to construct some Banach space representations $B(\chi, [1, b])$ that should correspond to $V^\vee_{\chi,[1,b]}$ (up to a twist by some character) under the p-adic local Langlands correspondence.

First we define an integral structure ω^1 of $\Omega^1_{\Sigma_{1,F}}$, the sheaf of holomorphic differential forms, on $\widetilde{\Sigma_{1,O_F}}$ defined in Section 8. Recall that $\widetilde{\Sigma_{1,O_F}}$ is a formal model of $\Omega^1_{\Sigma_{1,F}}$ which is not semistable, but only has some mild singularities $(xy - \varpi^{p-1})$. From now on, I will do all computations on this formal model rather than the semistable model.

View $\Omega^1_{\Sigma_{1,F}}$ as a sheaf on $\widetilde{\Sigma_{1,O_F}}$. The coherent sheaf ω^1 will be a subsheaf of it. Recall that there is an open covering $\{\widetilde{\Sigma_{1,O_F,e,\xi}}\}_{e,\xi}$ of $\widetilde{\Sigma_{1,O_F}}$, where e takes value in the set of edges of the Bruhat–Tits tree and $\xi^{p-1} = -1$. Using the explicit description of Lemma 8.2, we define ω^1 on each $\widetilde{\Sigma_{1,O_F,e,\xi}}$ as the trivial line bundle with a basis $d\tilde{e}/\tilde{e} = -d\tilde{e}'/\tilde{e}'$ (recall that $\tilde{e} = e/\varpi$, $\tilde{e}' = e'/\varpi$). It's easy to see that this really defines a line bundle $\widetilde{\Sigma_{1,O_F,e,\xi}}$ which becomes $\Omega^1_{\Sigma_{1,F}}$ if we restrict this line bundle to the generic fiber.

Remark 14.1. We can do exactly the same thing on the semistable model $\widehat{\Sigma_{1,O_F}}$, but this won't give us any extra sections: the sections on $\widetilde{\Sigma_{1,O_F,e}}$ and $\widehat{\Sigma_{1,O_F,e}}$ will be the same. This can be checked locally around the singularities. So I can do all the computations on $\widetilde{\Sigma_{1,O_F,e}}$.

Remark 14.2. We note that ω^1 in fact has an "F_0-structure". In other words, we can define it on $\widehat{\Sigma_1}$. Using the explicit description in Corollary 7.7, locally on $\widehat{\Sigma_{1,e}}$, it is defined as the trivial line bundle generated by de/e. Notice that $de/e = d\tilde{e}/\tilde{e}$ since $e = \tilde{e}\varpi$. Hence its pull-back to $\widetilde{\Sigma_{1,O_F}}$ is ω^1.

Similarly, we can define the same thing on $\widetilde{\Sigma'_{1,O_F}}$, $\widetilde{\Sigma^{(0)}_{1,O_F}}$, which we still denote by ω^1, by abuse of notation. Now if we restrict ω^1 to the special fiber, it becomes the dualizing sheaf (over $\mathrm{Spec}\,\mathbb{F}_{p^2}$). So there is an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on it. In fact, $\mathrm{GL}_2(\mathbb{Q}_p)$ even acts on ω^1. This can be seen using the explicit description in Section 9. Also, it's clear from the definition that $O_D^\times$ and $\mathrm{Gal}(F/\mathbb{Q}_p)$ act on the global sections of ω^1.

Consider the following maps:

$$H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E \hookrightarrow H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}}) \otimes_{\mathbb{Q}_p} E \to H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E.$$

Both maps are $\mathrm{GL}_2(\mathbb{Q}_p)$, $O_D^\times$, $\mathrm{Gal}(F/\mathbb{Q}_p)$-equivariant. Take the χ-isotypic component, where $\chi \in \chi(E)$ (see Section 11). We get a map (use Corollary 12.10):

$$\begin{aligned} f_\chi : \big(H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E\big)^\chi &\to (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^\chi \\ &\simeq F \otimes_{F_0} D_{\mathrm{crys},\chi} \otimes_E \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_\chi. \end{aligned}$$

Now for each two dimensional Galois representation $V_{\chi,[1,b]}$ of $G_{\mathbb{Q}_p}$ defined in the previous section, we have a free $F \otimes_{\mathbb{Q}_p} E$-module $\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$ inside $F \otimes_{F_0} D_{\mathrm{crys},\chi}$. We note that $\mathrm{Gal}(F/\mathbb{Q}_p)$ acts on this $\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$. Define

$$\begin{aligned} M(\chi,[1,b]) &= \big(f_\chi^{-1}(\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]}) \otimes_E \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_\chi)\big)^{\mathrm{Gal}(F/\mathbb{Q}_p)} \\ &= f_\chi^{-1}((\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]}))^{\mathrm{Gal}(F/\mathbb{Q}_p)} \otimes_E \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_\chi. \end{aligned}$$

Schneider and Teitelbaum [2002] introduced a category $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(O_E)$ whose objects are all torsion-free and compact, Hausdorff linear-topological O_E-modules, and morphisms are all continuous O_E-linear maps. Our first result about $M(\chi,[1,b])$ is:

Proposition 14.3. *$M(\chi,[1,b])$ with the topology induced from*

$$H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}}) \otimes_{\mathbb{Q}_p} E$$

is an object in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(O_E)$.

Proof. I learned this argument from Proposition 4.2.1 of [Breuil 2004]. It is clear that $M(\chi,[1,b])$ is torsion free and Hausdorff. To prove compactness, we use Proposition 15.3(iii) of [Schneider 2002] (c-compactness is equivalent with compactness here since O_E is locally compact [Perez-Garcia and Schikhof 2010, Corollary 6.1.14]). Proposition 11.4 already shows that $H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}}) \otimes_{\mathbb{Q}_p} E$ is a reflexive Fréchet space, so it suffices to show $M(\chi,[1,b])$ is closed and bounded (see [Schneider 2002] for the definition of boundedness). In fact it's easy to see we only need to prove closedness and boundedness for $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)$ in $H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}})$. Recall the topology on $H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}})$ is defined in Section 11 by:

$$H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}}) = \varprojlim_n H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}}),$$

where $\{V_{n,F}\}_n$ is an admissible open covering of $\Sigma_{1,F}$ and each $V_{n,F}$ is affinoid and contained in $V_{n+1,F}$. Now $\{\widetilde{\Sigma_{1,O_F,e}}\}_e$ is another admissible open covering. Thus we have:

- Each $\widetilde{\Sigma_{1,O_F,e}}$ is contained in some $V_{n,F}$.
- Each $V_{n,F}$ is covered by finitely many generic fibers of $\widetilde{\Sigma_{1,O_F,e}}$.

Then closedness follows from the first claim above and boundedness follows from the second. □

Suppose M is an object in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(O_E)$, following [Schneider and Teitelbaum 2002], the E-vector space $M^d \overset{\mathrm{def}}{=} \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M, E)$ with the norm $\|f\| = \max_{m\in M}|f(m)|_E$ is a Banach space.

Definition 14.4. $B(\chi, [1, b]) \overset{\mathrm{def}}{=} (M(\chi, [1, b]))^d = \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M(\chi, [1, b]), E)$.

It's clear from the definition that this is a Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Remark 14.5. The relation between $B(\chi, [1, b])$ and the Banach representation $B(\pi, \mathcal{L})$ defined in the introduction (see Definitions 1.3 and 1.4) is as follows: Take $\pi = \mathrm{Ind}^{D^\times}_{O_D^\times \mathbb{Q}_p^\times} \chi$, where χ is viewed as a character of $O_D^\times \mathbb{Q}_p^\times$ trivial on p. Also $\mathrm{Fil}^1(D_{\chi,[1,b]} \otimes F)$ essentially gives a line "$\mathcal{L}_b$" in Definition 1.3 by taking $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariants. Then $B(\chi, [1, b]) = B(\pi, \mathcal{L}_b)$.

Back to the definition of $M(\chi, [1, b])$. By Remark 12.11, we can replace the induced representation by the dual representation of the compact induction. Also by Galois descent, we have $(\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]}))^{\mathrm{Gal}(F/\mathbb{Q}_p)} \simeq E$. Under these isomorphisms, f_χ induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map,

$$f_{\chi,[1,b]} : M(\chi, [1, b]) \to (\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_{\chi^{-1}})^\vee.$$

It is natural to ask whether such a map is injective or not. The answer is positive.

Proposition 14.6. *The composition*

$$H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1)^{\chi'} \hookrightarrow H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'} \to H^1_{\mathrm{dR}}(\Sigma_{1,F})^{\chi'},$$

for a character $\chi' \in \chi(F)$ such that $\chi' \neq \chi'^p$, is injective.

Proof. Since $\chi' \neq \chi'^p$, the kernel of the second map is $H^0(\Sigma_{1,F}, \mathcal{O}_{\Sigma_{1,F}})^{\chi'}$. Consider the intersection of $H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1)^{\chi'}$ and $H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}})^{\chi'}$ in $H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'}$. It can be viewed as a subset in $H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}})^{\chi'}$ and we denote it by H. On the other hand, we use J to denote the same set but viewed in $H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'}$. The induced topology on H and J can be different. Proposition 14.3 tells us that J is compact since $H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}})^{\chi'}$ is closed in $H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'}$ (Proposition 11.5). Clearly $\mathrm{SL}_2(\mathbb{Q}_p)$ preserves both H and J.

Let's recall some notation here: For each connected component of $\widetilde{\Sigma_{1,O_F}}$, the dual graph of its special fiber is the Bruhat–Tits tree (see Section 8), and $U_{s,\xi}$ is the tubular neighborhood of $\overline{U_{s,\xi}}$, the irreducible component indexed by (s,ξ) in the special fiber (see Definition 11.10).

Similar to what we did in the beginning of Section 12, we can prove $U_{s_0',\xi}$ is isomorphic with

$$\left\{z=(x,y)\in\mathbb{A}^2_F \,\middle|\, y^{p+1}=v_1w_1^{-1}\xi(x^p-x),\ |x-k|>p^{-1}, k=0,\dots,p-1,\ |x|<p\right\},$$

and its de Rham cohomology is of finite dimension. Since $U_{s_0',\xi}$ is a Stein space, $H^1_{\mathrm{dR}}(U_{s_0',\xi}) = H^0(U_{s_0',\xi}, \Omega^1)/H^0(U_{s_0',\xi}, \Omega^0)$ (we use Ω^i for $\Omega^i_{\Sigma_{1,F}}$ for simplicity).

Fix a ξ. Under the isomorphism above, we can write $U_{s_0',\xi} = \bigcup_{\rho<p} U_{s_0',\xi,\rho}$, where $U_{s_0',\xi,\rho} \subset U_{s_0',\xi}$ is defined by the same equation but with $|x-k| \geq \rho^{-1}$, $k=0,\dots,p-1$, $|x| \leq \rho$. Then for each $\rho < p$, $H^0(U_{s_0',\xi,\rho}, \Omega^i)$ is a Banach space, and we have $H^0(U_{s_0',\xi}, \Omega^i) = \varprojlim_{\rho\to p} H^0(U_{s_0',\xi,\rho}, \Omega^i)$. So $H^0(U_{s_0',\xi}, \Omega^i)$ is a Fréchet space.

Notice that $O_D^\times$ acts on $U_{s_0'}$, so $H^0(U_{s_0'}, \Omega^0)^{\chi'} \hookrightarrow H^0(U_{s_0'}, \Omega^1)^{\chi'}$ and the quotient is a finite dimensional space. Thus this inclusion has to be a closed embedding because both of them are Fréchet spaces.

Now consider the canonical maps $H^0(\Sigma_{1,F}, \Omega^k)^{\chi'} \to H^0(U_{s_0'}, \Omega^k)^{\chi'}$, $k=0,1$. They're clearly continuous and we denote the image of H and J by H_1 and J_1. Since J is compact, J_1 is compact. Hence H_1 is also compact in $H^0(\Sigma_{1,F}, \Omega^0)^{\chi'}$ because $H^0(U_{s_0'}, \Omega^0)^{\chi'} \hookrightarrow H^0(U_{s_0'}, \Omega^1)^{\chi'}$ is a closed embedding. We will show this cannot happen unless $H_1 = \{0\}$.

Suppose f is a nonzero rigid function in H. We will prove later that f is unbounded on $\Sigma_{1,F}$ (see the next lemma). For each $U_{s,\xi}$, the maximum principle implies that f must obtain its maximum on the boundary annuli which are the tubes of the singular points on the special fiber. Therefore f is unbounded on $\bigcup_{s'\text{ even}} U_{s',\xi}$. But we know $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on $\Sigma_{1,F}$ and acts transitively on the set of even vertices. Hence using the action of $\mathrm{SL}_2(\mathbb{Q}_p)$, we can get functions in H with arbitrary large norms when restricted to $U_{s_0',\xi}$ and H_1 cannot be compact. So there is no such f. ☐

Lemma 14.7. *Any globally bounded function on a connected component of* $\Sigma_{1,F}$ *must be a constant.*

Proof. Fix a connected component ξ. Suppose f is such a function. By multiplying f by some powers of ϖ, we may assume $f \in H^0(\widehat{\Sigma_{1,O_F}}_{,\xi}, \mathcal{O}_{\widehat{\Sigma_{1,O_F}}_{,\xi}})$. Recall that the special fiber is connected and each irreducible component is a complete curve.

Hence,

$$H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}/(\varpi)) = \mathbb{F}_{p^2}.$$

Using induction on n, we can prove $H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}/(\varpi^n)) = O_F/(\varpi^n)$. Here we use the fact that $\mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}$ is flat over the constant sheaf O_F. Now the lemma follows from

$$H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}) = \varprojlim_n H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}/(\varpi^n)) = O_F. \qquad \square$$

Remark 14.8. The proposition is also true if $\chi' \neq \chi'^p$. In this case, it is equivalent to the same result on the Drinfel'd upper half-plane. See Proposition 19 of [Teitelbaum 1993] for a proof.

So we have an injective $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map:

$$f_{\chi,[1,b]} : M(\chi, [1, b]) \to (\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee.$$

A simple consideration of the topology (see Remark 12.11) shows that this induces a map

$$\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}} \to B(\chi, [1, b]).$$

It is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and has to be injective if $B(\chi, [1, b])$ is nonzero since the left-hand side is an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. If $B(\chi, [1, b])$ is nonzero, or equivalently if $M(\chi, [1, b])$ is nonzero, we can define a lattice inside $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$:

$$\Theta(\chi, [1, b]) = \{X \in \text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}} \mid \langle X, f_{\chi,[1,b]}(Y)\rangle \in O_E, \forall Y \in M(\chi, [1, b])\},$$

where

$$\langle \cdot, \cdot \rangle : \text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}} \times (\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee \to O_E$$

denotes the canonical pairing. This is equivalent to the intersection of the unit ball of $B(\chi, [1, b])$ with $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$.

Proposition 14.9. *$B(\chi, [1, b])$ is the completion of $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ with respect to the lattice $\Theta(\chi, [1, b])$ if $M(\chi, [1, b]) \neq 0$.*

Proof. The argument of Proposition 4.3.5 of [Breuil 2004] works here. I would like to recall it here. By [Schneider and Teitelbaum 2002, Theorem 1.2.], it suffices to prove that the natural map $M(\chi, [1, b]) \to \mathrm{Hom}_{O_E}(\Theta(\chi, [1, b]), O_E)$ is a topological isomorphism. The topology on the right hand side is defined by pointwise convergence. Notice that $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ can be viewed as the continuous dual space of $(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee$ with the topology described in Remark 12.11 and $M(\chi, [1, b])$ is closed in $(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee$ since it's already compact. We can apply Corollary 13.5 of [Schneider 2002] and get the desired isomorphism. It's also clear from the definition that this is a topological isomorphism. □

So if we can show $M(\chi, [1, b])$ is nonzero and moreover admissible as defined in [Schneider and Teitelbaum 2002], we indeed get an admissible Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is a completion of the smooth representation $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$. This is the goal of the rest of the paper.

15. Computation of $(H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^{\chi,\mathrm{Gal}(F/\mathbb{Q}_p)}/p$

Our ultimate goal is to prove $M(\chi, [1, b])$ is nonzero and admissible. The method is by explicit computation of its mod p representation. First we review some notation defined in the previous sections that will be used frequently from now on.

Let $\chi \in \chi(E)$ be a character of $(O_D/\Pi)^\times$ such that $\chi^p \neq \chi$. Since we fix an embedding $\tau : F_0 \to E$, we may write $\chi = \tau \circ \chi'$, where χ' is a character of $(O_D/\Pi)^\times$ with values in $F_0^\times$. Then $\chi' = \chi_1^{-m}$, where χ_1 is one of the fundamental characters (Definition 3.2) and $m \in \{1, \dots, p^2 - 2\}$. Write $m = i + (p + 1) j$ with $i \in \{1, \dots, p\}$ and $j \in \{0, \dots, p - 2\}$. Finally, $g_\varphi \in \mathrm{Gal}(F/\mathbb{Q}_p)$ is the unique element that fixes ϖ and acts as Frobenius on F_0.

Also recall that for any integer n, we use $[n]$ to denote the unique integer in $\{0, 1, \dots, p^2-2\}$ congruent to n modulo p^2-1. For any O_{F_0}-module A, we denote $A \otimes_{O_{F_0},\tau} O_E$ by A_τ and $A \otimes_{O_{F_0},\bar{\tau}} O_E$ by $A_{\bar{\tau}}$.

Recall that

$$\widetilde{\Sigma^{(0)}_{1,O_F}} = \widetilde{\Sigma_{1,O_F}} \sqcup \widetilde{\Sigma'_{1,O_F}},$$

and g_φ interchanges $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$ and $(H^0(\widetilde{\Sigma'_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$. Hence a g_φ-invariant element in $(H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$ is determined by its $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$ component. By definition, $M(\chi, [1, b])$ is g_φ-invariant. Hence it suffices to work on $\widetilde{\Sigma_{1,O_F}}$. This means that we may identify $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)$ as the g_φ-invariant sections of $H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1)$. Hence there is a natural action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on it: this is nothing but $g_\varphi^{v_p(\det(g))} \circ g$.

Definition 15.1. For any $\chi \in \chi(E)$, $\chi' \in \chi(F_0)$, we define (see Section 8 for the definition of these formal schemes)

$$\begin{aligned} H^{(0),\chi,\mathbb{Q}_p} &= (H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1/p) \otimes_{\mathbb{Z}_p} O_E)^{\chi,\mathrm{Gal}(F/\mathbb{Q}_p)}, \\ H_*^{\chi,F_0} &= (H^0(\widetilde{\Sigma_{1,O_F,*}}, \omega^1/p) \otimes_{\mathbb{Z}_p} O_E)^{\chi,\mathrm{Gal}(F/F_0)}, \\ H_*^{\chi',F_0} &= H^0(\widetilde{\Sigma_{1,O_F,*}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)}, \\ H_{*,?}^{\chi',F_0} &= H_*^{\chi',F_0} \otimes_{O_{F_0},?} O_E = H^0(\widetilde{\Sigma_{1,O_F,*}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)} \otimes_{O_{F_0},?} O_E, \end{aligned}$$

where $*$ is either a vertex s or an edge e of the Bruhat–Tits tree or nothing, and $? = \tau, \bar{\tau}$.

It is clear from the definition that if $\chi = \tau \circ \chi'$, then

$$H_*^{\chi, F_0} \simeq H_{*,\tau}^{\chi', F_0} \oplus H_{*,\bar{\tau}}^{(\chi')^p, F_0}. \tag{45}$$

Also, the discussion above shows that we have a canonical isomorphism:

$$H^{(0),\chi,\mathbb{Q}_p} \simeq H^{\chi, F_0}.$$

Definition 15.2. For a vertex s in the Bruhat–Tits tree, we use $A(s)$ to denote the set of vertices adjacent to s.

Now fix $\xi^{p-1} = -1$. We can do all the computation on one ξ-component $\widetilde{\Sigma_{1,O_F,\xi}}$. This is because $O_D^\times$ acts transitively on all connected components.

The goal of this section is to compute $(H^0(\widetilde{\Sigma_{1,O_F}^{(0)}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^{\chi, \mathrm{Gal}(F/\mathbb{Q}_p)}/p$. The next lemma implies that this is nothing but $H^{(0),\chi,\mathbb{Q}_p}$.

Lemma 15.3. $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)/\varpi^n = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n)$.

Proof. Clearly there is an injection from the left-hand side to the right-hand side. Since we have

$$H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) = \varprojlim_n H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n),$$

we only need to prove the canonical map

$$H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n) \to H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^m), \quad n > m$$

is surjective. Notice that ω^1 is flat over the constant sheaf O_F. It suffices to prove $H^1(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n) = 0$ for all $n \in \mathbb{N}^+$. Do induction on n and use the flatness again. It turns out that it's enough to show $H^1(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi) = 0$. However, the construction of ω^1 tells us ω^1/ϖ is the dualizing sheaf on the special fiber. This means that if we restrict ω^1/ϖ to each irreducible component V of the special fiber, it is $\Omega_V^1(D_{\mathrm{sing}})$, where Ω_V^1 is the usual sheaf of differential forms on V, D_{sing} is the sum of singular points of D (considered in the whole special fiber) as a divisor. Also, we have the following exact sequence of sheaves:

$$0 \to \omega^1/\varpi \to \prod_V i_{V*}(\Omega_V^1(D_{\mathrm{sing}})) \to \prod_E i_{E*}(\mathbb{F}_{p^2}) \to 0,$$

where E (resp. V) runs through all singular points (resp. irreducible components) of the special fiber, and i_E (resp. i_V) is the corresponding inclusion. Take the long exact sequence of cohomologies of this sequence. H^0 of the third map is surjective since the dual graph of the special fiber of each connected component is a tree. H^1 of the middle term in the exact sequence above vanishes by Riemann–Roch. So we indeed get the vanishing of $H^1(\widetilde{\Sigma_{1,O_F}}, \omega^1)$. □

Hence we only need to compute

$$H^{(0),\chi,\mathbb{Q}_p} \simeq H^{\chi,F_0} \simeq H_{\tau}^{\chi',F_0} \oplus H_{\bar{\tau}}^{(\chi')^p,F_0}. \tag{46}$$

It's not hard to see that we have an injection:

$$H^{\chi',F_0} \hookrightarrow \prod_s H_s^{\chi',F_0},$$

where s takes values in the set of vertices of the Bruhat–Tits tree. Similarly, we have the same injection for $H^{(\chi')^p,F_0}$. Notice that by identifying the sections on $\widetilde{\Sigma_{1,O_F}}$ as the g_φ-invariant sections on $\widetilde{\Sigma^{(0)}_{1,O_F}}$, we have an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on

$$\prod_s (H_s^{\chi',F_0} \oplus H_s^{(\chi')^p,F_0})$$

(see the beginning of this section). Explicitly, g sends H_s^{χ',F_0} to H_{sg}^{χ',F_0} if $v_p(\det(g))$ ($g \in \mathrm{GL}_2(\mathbb{Q}_p)$) is even and to $H_{sg}^{(\chi')^p,F_0}$ if it is odd. From this description, we have an obvious $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism (recall s'_0 is the central vertex):

$$\prod_s (H_s^{\chi',F_0} \oplus H_s^{(\chi')^p,F_0}) \simeq \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H_{s'_0}^{\chi',F_0} \oplus \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H_{s'_0}^{(\chi')^p,F_0} \otimes_{\mathbb{F}_{p^2},\widetilde{\mathrm{Fr}}} \mathbb{F}_{p^2}.$$

The following lemma basically says that we may identify H_s^{χ',F_0} with sections of ω^1/ϖ on $\overline{U_s^0}$ introduced in Definition 11.10. Notice that ω^1/ϖ is the dualizing sheaf of the special fiber.

Lemma 15.4. *For each vertex s of the Bruhat–Tits tree, we have natural isomorphisms:*

$$\Psi_{s,\chi'} : H_s^{\chi',F_0} \xrightarrow{\sim} H^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1/\varpi)^{\chi'} = H^0(\overline{U_s^0}, \omega^1/\varpi)^{\chi'},$$
$$\Psi_{s,(\chi')^p} : H_s^{(\chi')^p,F_0} \xrightarrow{\sim} H^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1/\varpi)^{(\chi')^p} = H^0(\overline{U_s^0}, \omega^1/\varpi)^{(\chi')^p},$$

such that their product

$$\prod_s (\Psi_{s,\chi'}, \Psi_{s,(\chi')^p}) :$$
$$\prod_s H_s^{\chi',F_0} \oplus H_s^{(\chi')^p,F_0} \to \prod_s H^0(\overline{U_s^0}, \omega^1/\varpi)^{\chi'} \oplus \prod_s H^0(\overline{U_s^0}, \omega^1/\varpi)^{(\chi')^p}$$

is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. As usual, s takes its value in the set of vertices of Bruhat–Tits tree.

Proof. First let's see what happens when $s = s'_0$. Recall that we have a concrete description ((23), (24)) of $\widetilde{\Sigma_{1,O_F,s'_0,\xi}}$, $\widetilde{\Sigma_{1,O_F,s'_0}}$ from Section 8:

$$\widetilde{\Sigma_{1,O_F,s'_0,\xi}} \simeq \operatorname{Spf} O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p+1}+v_1 w_1^{-1}\xi\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big),$$
$$\widetilde{\Sigma_{1,O_F,s'_0}} \simeq \operatorname{Spf} O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p^2-1}-w_1^2\Big(\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big)^{p-1}\Big).$$

An element of $H^0(\widetilde{\Sigma_{1,O_F,s'_0}}, \omega^1)^{\chi'}$ is determined by its restriction to $\widetilde{\Sigma_{1,O_F,s'_0,\xi}}$. It's easy to see (using the results in Section 9) it must have the form

$$P(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}},$$

where $P(\eta) \in O_F[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$. Recall (Proposition 13.2) that $\chi' = \chi_1^{-m}$, and $m = i+(p+1)j$, $i \in \{1,\dots,p\}$, $j \in \{0,\dots,p-2\}$. It is $\operatorname{Gal}(F/F_0)$-invariant if and only if

$$P(\eta) = \varpi^{p^2-1-m}F_1(\eta),$$

where $F_1(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$. Similarly, a section of $H^0(\widetilde{\Sigma_{1,O_F,s'_0}}, \omega^1)^{(\chi')^p}$ fixed by $\operatorname{Gal}(F/F_0)$ must have the form

$$\varpi^{[-mp]}F_2(\eta)\tilde{e}^i\,\frac{d\tilde{e}}{\tilde{e}},$$

where $F_2(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$, and $[-mp]$ is defined in the beginning of this section.

Thus any element $\bar{F}$ of $H^0(\widetilde{\Sigma_{1,O_F,s'_0}}, \omega^1/p)^{\chi',\operatorname{Gal}(F/F_0)} = H_s^{\chi',F_0}$ can be written uniquely as

$$\varpi^{p^2-1-m}\bar{F}_1(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}$$

on ξ-components, where $\bar{F}_1(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(\eta^{p-1}-1)]$. Now define $\Psi_{s'_0,\chi'}(\bar{F}) = \bar{F}_1(\eta)\tilde{e}^{p+1-i}\,d\tilde{e}/\tilde{e}$. Equivalently, it is "multiplication" by $\varpi^{-(p^2-1-m)}$. It's trivial to see this is indeed an isomorphism. We can define $\Psi_{s'_0,(\chi')^p}$ in exactly the same way.

Note that $\Psi_{s'_0,\chi'}$, $\Psi_{s'_0,(\chi')^p}$ are $\operatorname{GL}_2(\mathbb{Z}_p)$-equivariant; we can extend both isomorphisms to any vertex s using the action of $\operatorname{GL}_2(\mathbb{Q}_p)$. Concretely, for an even vertex s', $\Psi_{s',\chi'}$ is "multiplication" by $\varpi^{-(p^2-1-m)}$ and $\Psi_{s',(\chi')^p}$ is "multiplication" by $\varpi^{-[-mp]}$. For an odd vertex s, $\Psi_{s,\chi'}$ is "multiplication" by $\varpi^{-[-mp]}$ and $\Psi_{s,(\chi')^p}$ is "multiplication" by $\varpi^{-(p^2-1-m)}$. □

By abuse of notation, I will identify $H^0(\overline{U_s^0}, \omega^1/\varpi)^{\chi'}$, $H^0(\overline{U_s^0}, \omega^1/\varpi)_\tau^{\chi'}$ with H_s^{χ',F_0}, $H_{s,\tau}^{\chi',F_0}$ via the isomorphisms in Lemma 15.4. Notice that ω^1/ϖ is the sheaf of differential forms on $\overline{U_s^0}$, thus we may view elements in H_s^{χ',F_0} as meromorphic differential forms on $\overline{U_s}$.

From now on, I would like to describe an element of H^{χ,F_0} via its image in $\prod_s H^{\chi',F_0}_{s,\tau} \oplus \prod_s H^{(\chi')^p,F_0}_{s,\bar\tau}$. In other words, using Lemma 15.4, any element $h=(h_1,h_2)$ in $H^{\chi,F_0} \simeq H^{\chi',F_0}_{\tau} \oplus H^{(\chi')^p,F_0}_{\bar\tau}$ corresponds to a family of meromorphic differential forms

$$\{(\omega_{s,\tau},\omega_{s,\bar\tau})\}_s,$$

where $\omega_{s,\tau}=h_1|_{\widetilde{\Sigma_{1,O_F,s}}} \in H^{\chi',F_0}_{s,\tau}$ and $\omega_{s,\bar\tau}=h_2|_{\widetilde{\Sigma_{1,O_F,s}}} \in H^{(\chi')^p,F_0}_{s,\bar\tau}$.

To further determine H^{χ,F_0}, we need to know when such a $\{(\omega_{s,\tau},\omega_{s,\bar\tau})\}_s$ comes from a global section. We will give a necessary condition in Proposition 15.8 and a sufficient condition in Proposition 15.11. To this end, it is crucial to understand the local structure of ω^1 on $\widetilde{\Sigma_{1,O_F,\xi}}$. Recall that $\widetilde{\Sigma_{1,O_F,\xi}}$ has an open covering $\{\widetilde{\Sigma_{1,O_F,e,\xi}}\}_e$ and an explicit description of $\widetilde{\Sigma_{1,O_F,e,\xi}}$ (Lemma 8.2) is:

$$\frac{\operatorname{Spf} O_F\Big[\eta,\zeta,\dfrac{1}{\eta^{p-1}-1},\dfrac{1}{\zeta^{p-1}-1},\tilde e,\tilde e'\Big]^{\widehat{}}}{\Big(\tilde e^{p+1}+v_1w_1^{-1}\xi\dfrac{\eta^p-\eta}{\zeta^{p-1}-1},\ \tilde e'^{p+1}+v_1^{-1}w_1\xi\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1},\ \tilde e\tilde e'-\varpi^{p-1}\xi\Big)}.$$

Note that e/ϖ, e'/ϖ in Lemma 8.2 is $\tilde e,\tilde e'$ here. Suppose $e=[s,s']$, where s' (resp. s) is an even (resp. odd) vertex and corresponds to η (resp. ζ). It's not too hard to see:

Lemma 15.5. *Any element h of $H^0(\widetilde{\Sigma_{1,O_F,[s,s']}},\omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$, when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$, can be written in the following form*:

$$h=\varpi^{p^2-1-m}f(\eta)\tilde e^{p+1-i}\,\frac{d\tilde e}{\tilde e}+\varpi^{[-mp]}g(\zeta)\tilde e'^{i}\,\frac{d\tilde e'}{\tilde e'},$$

where $f(\eta)\in O_{F_0}[\eta,1/(\eta^{p-1}-1)]^{\widehat{}},\ g(\zeta)\in O_{F_0}[\zeta,1/(\zeta^{p-1}-1)]^{\widehat{}}$.

Proof. It suffices to verify this after reducing modulo p. Equivalently, we need to show that any $h\in H^{\chi',F_0}_e$ has the form

$$\varpi^{p^2-1-m}f(\eta)\tilde e^{p+1-i}\,\frac{d\tilde e}{\tilde e}+\varpi^{[-mp]}g(\zeta)\tilde e'^{i}\,\frac{d\tilde e'}{\tilde e'},$$

where $f(\eta)\in\mathbb{F}_{p^2}[\eta,1/(1-\eta^{p-1})],\ g(\zeta)\in\mathbb{F}_{p^2}[\zeta,1/(1-\zeta^{p-1})]$ when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$.

Recall that ω^1 is free over $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$ with a basis $d\tilde e/\tilde e=-d\tilde e'/\tilde e'$ (see the beginning of the previous section). Hence any element h in $H^0(\widetilde{\Sigma_{1,O_F,[s,s'],\xi}},\omega^1/p)$ can be written as

$$\sum_{k=0}^{p} f_{1,k}(\eta,\zeta)\tilde e^k\,\frac{d\tilde e}{\tilde e}+\sum_{k=0}^{p} g_{1,k}(\eta,\zeta)\tilde e'^k\,\frac{d\tilde e'}{\tilde e'},$$

where $f_{1,k}(\eta,\zeta), g_{1,k}(\eta,\zeta) \in O_F/(p)\big[\eta,\zeta,1/(1-\eta^{p-1}),1/(1-\zeta^{p-1})\big]/(\eta\zeta)$. This is because using the explicit description of $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$ above, we see that $\tilde{e}^{p+1}$, $\tilde{e}'^{p+1}$, and $\tilde{e}\tilde{e}'$ can each be written as an element only containing η, ζ.

Using the results in Section 9, we see that such an element comes from an element in the χ'-isotypic component of $H^0(\widetilde{\Sigma_{1,O_F,[s,s']}}, \omega^1/p)$ if and only the coefficients of $\tilde{e}^k$ (resp. $\tilde{e}'^k$) are zero unless $k = p+1-i$ (resp. $k = i$). Hence we may write it as

$$h = f_{1,p+1-i}(\eta,\zeta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + g_{1,i}(\eta,\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}, \tag{47}$$

Next consider the action of $\mathrm{Gal}(F/F_0)$. Using the results in Section 9 once again, it's not hard to see that such an element comes from a Galois-invariant section if and only if

$$f_{1,p+1-i}(\eta,\zeta) = \varpi^{p^2-1-m} f_2(\eta,\zeta), \quad g_{1,i}(\eta,\zeta) = \varpi^{[-mp]} g_2(\eta,\zeta), \tag{48}$$

where $f_2(\eta,\zeta), g_2(\eta,\zeta) \in \mathbb{F}_{p^2}\big[\eta,\zeta,1/(1-\eta^{p-1}),1/(1-\zeta^{p-1})\big]/(\eta\zeta)$.

Now in order to prove the lemma, we need to "eliminate" the ζ in $f_2(\eta,\zeta)$ and η in $g_2(\eta,\zeta)$. We will prove this under the following assumption:

$$p^2 - 1 - m \geq [-mp].$$

Equivalently, this means $p^2-1-m = [-mp] + i(p-1)$. The other case is similar.

First we eliminate the η in $g_2(\eta,\zeta)$: We can write

$$g_2(\eta,\zeta) = f_3(\eta) + g_3(\zeta),$$

such that $g_3(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$ and $g_3(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$. This is because we can think $g_2(\eta,\zeta)$ as a regular function on a union of two irreducible smooth affine curves crossing transversally. Such a decomposition is obtained by restricting this function on each irreducible component (with some modification by some constants).

Notice that $f_3(0)$ makes sense here. Replacing $f_3(\eta)$ with $f_3(\eta) - f_3(0)$, we may assume

$$f_3(\eta) = \eta f_4(\eta),$$

where $f_4(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$. Now in $\mathcal{O}_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}}$, we have

$$\eta = C\tilde{e}^{p+1}, \quad \text{where } C = -v_1 w_1 \xi^{-1}\frac{\zeta^{p-1}-1}{\eta^{p-1}-1}.$$

Plug this into (47) and use (48):

$$
\begin{aligned}
h &= \varpi^{p^2-1-m} f_2(\eta,\zeta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}(\eta f_4(\eta)+g_3(\zeta))\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'} \\
&= \varpi^{p^2-1-m} f_2(\eta,\zeta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]} g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'} \\
&\qquad + f_4(\eta)\varpi^{[-mp]} C\tilde{e}^{p+1}\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.
\end{aligned}
$$

Since $\tilde{e}\tilde{e}' = \varpi^{p-1}\xi$, the last term in the above equation is

$$
\begin{aligned}
f_4(\eta)C\varpi^{[-mp]}\varpi^{i(p-1)}\xi^i\tilde{e}^{p+1-i}\,\frac{d\tilde{e}'}{\tilde{e}'} &= f_4(\eta)C\varpi^{p^2-1-m}\xi^i\tilde{e}^{p+1-i}\,\frac{d\tilde{e}'}{\tilde{e}'} \\
&= -\varpi^{p^2-1-m}Cf_4(\eta)\xi^i\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}},
\end{aligned}
$$

by our assumption. In other words,

$$
h = \varpi^{p^2-1-m}\big(f_2(\eta,\zeta) - Cf_4(\eta)\xi^i\big)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.
$$

Hence in (47), we may assume

$$
g_{1,i}(\eta,\zeta) = \varpi^{[-mp]}g_3(\zeta), \quad \text{where } g_3(\zeta)\in\mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})].
$$

Now we are going to eliminate the ζ in $f_2(\eta,\zeta)$. As before, write $f_2(\eta,\zeta) = f_5(\eta) + \zeta g_5(\zeta)$ and notice that in $\mathcal{O}_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}}$, we can write $\zeta = C'\tilde{e}'^{p+1}$. Plug this into (47):

$$
\begin{aligned}
h &= \varpi^{p^2-1-m}(f_5(\eta)+\zeta g_5(\zeta))\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'} \\
&= \varpi^{p^2-1-m} f_5(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} g_5(\zeta)C'\tilde{e}'^{p+1}\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} \\
&\qquad + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.
\end{aligned}
$$

Here comes the difference between this case and the former case. The middle term actually vanishes:

$$
\begin{aligned}
\varpi^{p^2-1-m}g_5(\zeta)C'\tilde{e}'^{p+1}\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} &= \varpi^{p^2-1-m}g_5(\zeta)C'\varpi^{(p+1-i)(p-1)}\xi^{p+1-i}\tilde{e}'^{i}\,\frac{d\tilde{e}}{\tilde{e}} \\
&= 0,
\end{aligned}
$$

since $\varpi^{p^2-1-m+(p+1-i)(p-1)} = \varpi^{[-mp]+(p+1)(p-1)} = -p\cdot\varpi^{-[mp]} = 0$ by our assumption. Hence we may write

$$
h = \varpi^{p^2-1-m} f_5(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'},
$$

which is exactly what we want. □

Now suppose $h\in H_e^{\chi',F_0}$. We may assume it has the form

$$
\varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'},
$$

where $f(\eta)\in\mathbb{F}_{p^2}[\eta,1/(1-\eta^{p-1})]$, $g(\zeta)\in\mathbb{F}_{p^2}[\zeta,1/(1-\zeta^{p-1})]$ when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$. What's its restriction to $\widetilde{\Sigma_{1,O_F,s',\xi}}$? Algebraically, this means that we replace ζ by $p/\eta=0$ and $\tilde{e}'$ by $\varpi^{p-1}\xi/\tilde{e}$. So we have (notice that $d\tilde{e}/\tilde{e}=-d\tilde{e}'/\tilde{e}'$):

$$h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}=\varpi^{p^2-1-m}f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}-\varpi^{[-mp]}g(0)\varpi^{i(p-1)}\xi^i\tilde{e}^{-i}\,\frac{d\tilde{e}}{\tilde{e}}.$$

We make the following assumption in the rest of this section:

$$p^2-1-m\geq[-mp]. \tag{49}$$

Equivalently, this means $p^2-1-m=[-mp]+i(p-1)$.

On $\widetilde{\Sigma_{1,O_F,s',\xi}}$, we have

$$\tilde{e}^{-i}=\frac{\tilde{e}^{p+1-i}}{\tilde{e}^{p+1}}=-\frac{(p/\eta)^{p-1}-1}{v_1w_1^{-1}\xi(\eta^p-\eta)}\tilde{e}^{p+1-i}\equiv\frac{1}{v_1w_1^{-1}\xi(\eta^p-\eta)}\tilde{e}^{p+1-i}\pmod{p}.$$

Hence,

$$\begin{aligned}h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}&=\varpi^{p^2-1-m}f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}\\&\qquad\qquad-\varpi^{p^2-1-m}g(0)\xi^i\frac{1}{v_1w_1^{-1}\xi(\eta^p-\eta)}\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}\\&=\varpi^{p^2-1-m}\Big(f(\eta)+g(0)\xi^{i-1}v_1^{-1}w_1\Big(\frac{1}{\eta}-\frac{\eta^{p-2}}{\eta^{p-1}-1}\Big)\Big)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}.\end{aligned}\tag{50}$$

Write $F(\eta)=f(\eta)-g(0)\xi^{i-1}v_1^{-1}w_1\frac{\eta^{p-2}}{\eta^{p-1}-1}$, $C_1=g(0)\xi^{i-1}v_1^{-1}w_1$.

Lemma 15.6. *Under the assumption $p^2-1-m\geq[-mp]$,*

$$h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}=\varpi^{p^2-1-m}C_1\frac{\tilde{e}^{p+1-i}}{\eta}\,\frac{d\tilde{e}}{\tilde{e}}+\varpi^{p^2-1-m}F(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}, \tag{51}$$

where $F(\eta)\in\mathbb{F}_{p^2}[\eta,1/(1-\eta^{p-1})]$, $C_1\in\mathbb{F}_{p^2}$.

Now if we view $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ as a differential form on $\overline{U^0_{s',\xi}}$, or what's the same, a meromorphic differential form on $\overline{U_{s',\xi}}$ with poles at the singular points ($\overline{U_{s',\xi}}$ is viewed as a subvariety in the special fiber of $\widetilde{\Sigma_{1,O_F,\xi}}$), the order of the pole at the intersection point of $\overline{U_{s',\xi}}$ and $\overline{U_{s,\xi}}$ must be $i+1$ (if there is a pole) since $1/\eta$ has order $p+1$ at this point ($\eta=\tilde{e}=0$) and $\tilde{e}$ is a uniformizer of this point.

Now restrict h to $\widetilde{\Sigma_{1,O_F,s,\xi}}$. This time we replace η by $p/\zeta=0$ and $\tilde{e}$ by $\varpi^{p-1}\xi/\tilde{e}'$.

$$\begin{aligned}h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}&=-\varpi^{p^2-1-m}f(0)\varpi^{(p+1-i)(p-1)}\xi^{p+1-i}\tilde{e}'^{-(p+1-i)}\,\frac{d\tilde{e}'}{\tilde{e}'}\\&\qquad\qquad+\varpi^{[-mp]}g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}\\&=\varpi^{[-mp]}g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.\end{aligned}\tag{52}$$

The first term is zero since $\varpi^{p^2-1-m+(p+1-i)(p-1)} = -p\cdot\varpi^{-[mp]}$ by our assumption.

Lemma 15.7. *Under the assumption* $p^2-1-m \geq [-mp]$,

$$h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} = \varpi^{[-mp]} g(\zeta)\tilde{e}'^i \frac{d\tilde{e}'}{\tilde{e}'},$$

where $g(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$.

Thus if we view $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ as a meromorphic differential form on $\overline{U_{s,\xi}}$, it is holomorphic at the intersection point of $\overline{U_{s,\xi}}$ and $\overline{U_{s',\xi}}$. In summary,

Proposition 15.8. *Assume* $p^2-1-m \geq [-mp]$. *Under the identification in Lemma 15.4, an element* h *of* $H^{\chi',F_0} = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)}$ *has the following description*:

(1) *If* s *is odd, then* $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ *is a holomorphic differential form on* $\overline{U_{s,\xi}}$.

(2) *If* s' *is even, then* $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ *can have poles at the intersection points of* $\overline{U_{s',\xi}}$ *with adjacent components. If there are poles, their order must be* $i+1$. *Moreover, as an element of the space of meromorphic differential forms on* $\overline{U_{s',\xi}}$ *modulo holomorphic differential forms,* $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ *is uniquely determined by the restriction of* h *to the components adjacent to* s'. *In other words,* $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ *is holomorphic on* $\overline{U_{s',\xi}}$ *if the restriction of* h *to the components adjacent to* s' *is zero.*

Proof. The first part is a direct consequence of Lemma 15.7. The assertion for the order of poles follows from Lemma 15.6. As for the last assertion, using the notation before Lemma 15.6, we know that the pole of $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ at the intersection point of $\overline{U_{s',\xi}}$ and $\overline{U_{s,\xi}}$ is determined by $g(0)$ (in fact this pole is given by

$$g(0)\xi^{i-1}v_1^{-1}w_1\frac{\tilde{e}^{p+1-i}}{\eta}\frac{d\tilde{e}}{\tilde{e}}$$

modulo holomorphic terms). However, $g(0)$ is indeed determined by $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ since $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} = \varpi^{[-mp]}g(\zeta)\tilde{e}'^i\, d\tilde{e}'/\tilde{e}'$. □

Remark 15.9. Under the assumption $p^2-1-m \geq [-mp]$, we have a similar description for elements in $H^{(\chi')^p,F_0}$ while interchanging the descriptions for odd and even vertices. This is obvious if one uses the action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

If we assume $p^2-1-m \leq [-mp]$, an element h of H^{χ',F_0} has the following similar description:

(1) If s' is even, then $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ is a holomorphic differential form on $\overline{U_{s',\xi}}$.

(2) If s is odd, then $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ can have poles at the intersection points of $\overline{U_{s,\xi}}$ with adjacent components. The order of these poles, if they exist, must be $p+2-i$. Moreover, $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ is holomorphic on $\overline{U_{s,\xi}}$ if the restriction of h to the components adjacent to s is zero.

To get a converse result, we need one more lemma to see when we can glue sections on $\widetilde{\Sigma_{1,O_F,s}}$, $\widetilde{\Sigma_{1,O_F,s'}}$ to a section on $\widetilde{\Sigma_{1,O_F,[s,s']}}$.

Lemma 15.10. *Assume $p^2-1-m \geq [-mp]$, and s' is an even vertex and $s \in A(s')$. Given $h_{s'} \in H_{s'}^{\chi',F_0}$, $h_s \in H_s^{\chi',F_0}$ such that they have the forms in Lemmas 15.6 and 15.7* (under the explicit description in Lemma 8.2):

$$h_{s'}|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} = \varpi^{p^2-1-m} C_1 \frac{\tilde{e}^{p+1-i}}{\eta} \frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} F(\eta) \tilde{e}^{p+1-i} \frac{d\tilde{e}}{\tilde{e}}, \tag{53}$$

$$h_s|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} = \varpi^{[-mp]} g(\zeta) \tilde{e}'^i \frac{d\tilde{e}'}{\tilde{e}'}, \tag{54}$$

where $F(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$, $C_1 \in \mathbb{F}_{p^2}$, $g(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$. Moreover assume

$$C_1 = g(0)\xi^{i-1} v_1^{-1} w_1. \tag{55}$$

Then we can find a (*unique*) *section $h \in H_{[s,s']}^{\chi',F_0}$ such that*

$$h|_{\widetilde{\Sigma_{1,O_F,s'}}} = h_{s'}, \quad h|_{\widetilde{\Sigma_{1,O_F,s}}} = h_s.$$

Proof. It is direct to see that the following section h_ξ on $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$ can be extended to an element in $H_{[s,s']}^{\chi',F_0}$ and satisfies all the conditions:

$$h_\xi = \varpi^{p^2-1-m}\Big(F(\eta) + C_1 \frac{\eta^{p-2}}{\eta^{p-1}-1}\Big) \tilde{e}^{p+1-i} \frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]} g(\zeta) \tilde{e}'^i \frac{d\tilde{e}'}{\tilde{e}'}. \qquad \square$$

Proposition 15.11. *Assume $p^2-1-m \geq [-mp]$.*

(1) *Given $h_s \in H_s^{\chi',F_0}$ for each odd vertex s that corresponds to a holomorphic differential form on $\overline{U_{s,\xi}}$, we can find an element h in H^{χ',F_0} such that for any odd vertices s,*

$$h|_{\widetilde{\Sigma_{1,O_F,s}}} = h_s.$$

(2) *Moreover, we have the following freedom of choosing h: given $f_{s'} \in H_{s'}^{\chi',F_0}$ for each even vertex s' that corresponds to a holomorphic differential form on $\overline{U_{s',\xi}}$, we may find a* (*unique*) *element f in H^{χ',F_0} such that*

$$f|_{\widetilde{\Sigma_{1,O_F,s'}}} = f_{s'} \quad \text{for any even vertices } s',$$

$$f|_{\widetilde{\Sigma_{1,O_F,s}}} = 0 \quad \text{for any odd vertices } s.$$

Proof. Both are local questions. The second part is a direct consequence of Lemma 15.10: For any even vertex s' and $s \in A(s')$, applying Lemma 15.10 with $h_s = 0$, $h_{s'} = f_{s'}$ (in this case, $C_1 = 0$), we can glue to a section on $\widetilde{\Sigma_{1,O_F,[s,s']}}$ whose restriction to $\widetilde{\Sigma_{1,O_F,s'}}$ (resp. $\widetilde{\Sigma_{1,O_F,s}}$) is $f_{s'}$ (resp. zero). Hence we can glue to a global section on $\widetilde{\Sigma_{1,O_F}}$.

As for the first part, our strategy is similar. For any even vertex s', we will find a section $h_{s'} \in H_{s'}^{\chi', F_0}$ such that for any vertex $s \in A(s')$, we can use Lemma 15.10 to glue $h_s, h_{s'}$ to a section on $\widetilde{\Sigma_{1,O_F,[s,s']}}$ and obtain a global section on $\widetilde{\Sigma_{1,O_F}}$.

By Lemma 15.4, we may identify elements in $H_{s'}^{\chi', F_0}$ with differential forms on $\overline{U_{s'}^0}$. Since $(O_D/\Pi)^\times \simeq \mathbb{F}_{p^2}^\times$ acts transitively on the connected components of $\overline{U_{s'}^0}$, it is easy to see $\mu_{p+1}(\mathbb{F}_{p^2}) = \{a \in \mathbb{F}_{p^2} \mid a^{p+1} = 1\}$ fixes $\overline{U_{s',\xi}^0}$. As we noted in the proof of Lemma 12.9,

$$H^0(\overline{U_{s'}^0}, \omega^1/\varpi)^{\chi'} \simeq H^0(\overline{U_{s',\xi}^0}, \omega^1/\varpi)^{\mathrm{Id}^{-i}},$$

where we view $\mathrm{Id} : \mu_{p+1}(\mathbb{F}_{p^2}) \to \mathbb{F}_{p^2}^\times$ as a character of $\mu_{p+1}(\mathbb{F}_{p^2})$, and Id^{-i} is its $(-i)$-th power. We denote the intersection point of $\overline{U_{s',\xi}}$ with $\overline{U_{s,\xi}}$, by P_s for $s \in A(s')$.

Now using Lemma 15.10, finding such an $h_{s'} \in H_{s'}^{\chi', F_0}$ is equivalent to finding a meromorphic differential form $\omega_{s'} \in H^0(\overline{U_{s',\xi}^0}, \omega^1/\varpi)^{\mathrm{Id}^{-i}}$ such that:

- It can only have poles at P_s, $s \in A(s')$ with order at most $i+1$ (in fact, it has to be $i+1$ if there is a pole, by considering the action of $\mu_{p+1}(\mathbb{F}_{p^2})$).
- The "leading coefficient" of the pole at P_s is prescribed by h_s for all $s \in A(s')$.

More precisely, using the explicit description in Lemma 8.2, the first condition allows us to write $\omega_{s'}$ into the form (53). Also our condition in the proposition allows us to write h_s into the form (54). Then C_1 in (53) is the leading coefficient in this case and we want it to satisfy (55).

The existence of such a meromorphic differential form follows from:

Lemma 15.12. *Let C be a smooth geometrically connected curve over $\mathbb{F}_{p^2}$ and $\{P_k\}_k$ be a nonempty finite subset of $C(\mathbb{F}_{p^2})$. Then for $n \geq 2$, the restriction map*

$$H^0(C, \Omega_C^1(nD)) \to \bigoplus_k H^0(P_k, \Omega_C^1(nD)|_{P_k})$$

is surjective, where D is the divisor $\sum_k P_k$.

Assume this lemma for the moment. In our case, let $C = \overline{U_{s',\xi}}$, $\{P_k\} = \{P_s\}$ and $n = i+1$. The prescribed leading coefficients become a family of elements $c_s \in H^0(P_s, \Omega_C^1(nD)|_{P_s})$, $s \in A(s')$. Notice that the uniformizer for P_s is either $\tilde{e}$ or $\tilde{e}/\eta$, hence $\mu_{p+1}(\mathbb{F}_{p^2})$ acts on

$$H^0\left(P_s, \Omega_C^1\left(\sum_k (i+1)P_k\right)\Big|_{P_s}\right) = H^0(P_s, \Omega_C^1((i+1)D)|_{P_s})$$

via Id^{-i}. So taking the Id^{-i}-isotypic component of the map in the lemma (which remains surjective since $p+1$ is coprime to p), we may find an element in $H^0\big(\overline{U_{s',\xi}}, \Omega^1\big(\sum_s (i+1)P_s\big)\big)^{\mathrm{Id}^{-i}}$ having the correct leading coefficient at each P_s and that's exactly what we want. $\square$

Proof of Lemma 15.12. Consider the following short exact sequence of sheaves:

$$0 \to \Omega^1_C((n-1)D) \to \Omega^1_C(nD) \to \bigoplus_k \Omega^1_C(nD)|_{P_k} \to 0.$$

It suffices to show $H^1(C, \Omega^1_C((n-1)D))$ vanishes. However by Serre duality, this space is dual to $H^0(C, \mathcal{O}_C(-(n-1)D))$, which is zero since we assume $n \geq 2$. □

Now, we can prove the main proposition of this section.

Proposition 15.13. *Assume* $p^2 - 1 - m \geq [-mp]$. *There exists a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant short exact sequence*:

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau} \to H^{(0),\chi,\mathbb{Q}_p} \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}} \to 0.$$

Proof. Write $\prod_s H^{\chi,F_0}_s = \prod_s (H^{\chi',F_0}_{s,\tau} \oplus H^{(\chi')^p,F_0}_{s,\bar{\tau}})$, where as usual, s runs over the vertices of the Bruhat–Tits tree. Define

$$H_1 = \prod_{s' \text{ even}} H^{\chi',F_0}_{s',\tau} \oplus \prod_{s \text{ odd}} H^{(\chi')^p,F_0}_{s,\bar{\tau}},$$

$$H_2 = \prod_{s \text{ odd}} H^{\chi',F_0}_{s,\tau} \oplus \prod_{s' \text{ even}} H^{(\chi')^p,F_0}_{s',\bar{\tau}}.$$

Notice that $\mathrm{GL}_2(\mathbb{Q}_p)$ actually acts on H_1, H_2. Then we have a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant (split) short exact sequence:

$$0 \to H_1 \to \prod_s H^{\chi,F_0}_s \to H_2 \to 0.$$

Recall that we have an injection of $H^{(0),\chi,\mathbb{Q}_p} \simeq H^{\chi,F_0}$ into $\prod_s H^{\chi,F_0}_s$. So this short exact sequence induces another short exact sequence:

$$0 \to K \to H^{\chi,F_0} \to C \to 0.$$

It remains to determine K, and C.

Let f be an element of H^{χ,F_0}. We will write $f = f_\tau + f_{\bar{\tau}}$ under the decomposition $H^{\chi,F_0} \simeq H^{\chi',F_0}_{\tau} \oplus H^{(\chi')^p,F_0}_{\bar{\tau}}$ (see (46)).

Suppose f is in K. This means for any odd vertex s and even vertex s',

$$f_\tau|_{\widetilde{\Sigma_{1,O_F,s}}} = 0 \quad \text{and} \quad f_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s'}}} = 0.$$

By the second part of Proposition 15.8, we know that $f_\tau|_{\widetilde{\Sigma_{1,O_F,s'}}}$ corresponds to a holomorphic differential form on $\overline{U_{s',\xi}}$ for any even vertex s' (tensored with O_E). However the second part of Proposition 15.11 indicates that $f_\tau|_{\widetilde{\Sigma_{1,O_F,s'}}}$ can be any holomorphic differential form inside $H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_{s'}}})^{\chi'}_{\tau}$. Similarly $f_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s}}}$ can be any holomorphic differential form inside $H^0(\overline{U_s}, \Omega^1_{\overline{U_s}})^{\chi'}_{\tau}$, where s is an odd vertex. This certainly implies that

$$K \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau}.$$

By the first part of Proposition 15.8, we know that C is inside

$$\prod_{s\colon \text{odd}} H^0(\overline{U_s}, \Omega^1_{\overline{U_s}})^{\chi'}_{\tau} \oplus \prod_{s'\colon \text{even}} H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_{s'}}})^{(\chi')^p}_{\bar{\tau}},$$

as a subset of H_2. However the first part of Proposition 15.11 tells us that in fact C is equal to this set. Clearly this is nothing but $\operatorname{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}_{\bar{\tau}}$. □

Remark 15.14. See the beginning of the paper for the notation here: Under the isomorphism (27), an element of $H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{\chi'}$ must have the form $f(\eta)\tilde{e}^{p+1-i}\, d\tilde{e}/\tilde{e}$ on $\overline{U_{s'_0,\xi}}$, where $f(\eta)$ is a polynomial of η of degree at most $i-2$. Using the results in Section 9, it's not hard to construct a $\mathrm{GL}_2(\mathbb{F}_p)$-equivariant isomorphism:

$$H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{\chi'} \to (\mathrm{Sym}^{i-2}\, \mathbb{F}^2_{p^2}) \otimes \det{}^{j+1}, \tag{56}$$

$$\eta^r \tilde{e}^{p+1-i} \frac{d\tilde{e}}{\tilde{e}} \mapsto x^r y^{i-2-r}, \tag{57}$$

where $\mathrm{Sym}^{i-2}\, \mathbb{F}^2_{p^2}$ is the $(i-2)$-th symmetric power of the natural representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the canonical basis of $\mathbb{F}^2_{p^2}$.

Similarly, we can identify $H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}$ with $(\mathrm{Sym}^{p-1-i}\, \mathbb{F}^2_{p^2}) \otimes \det^{i+j}$. Then we can rewrite the exact sequence in Proposition 15.13 as

$$0 \to \sigma_{i-2}(j+1) \to H^{(0),\chi,\mathbb{Q}_p} \to \sigma_{p-1-i}(i+j) \to 0.$$

Remark 15.15. If we assume $p^2-1-m \le [-mp]$, then we have the exact sequence of the opposite direction:

$$0 \to \operatorname{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}_{\bar{\tau}} \to H^{(0),\chi,\mathbb{Q}_p} \to \operatorname{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{\chi'}_{\tau} \to 0.$$

16. Computation of $M(\chi, [1, b])/p$, I: $2 \le i \le p-1$

In this section, we compute $M(\chi, [1, b])/p$ as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ when $i \in \{2, \dots, p-1\}$. The strategy is as follows. We first identify the crystalline cohomology with the de Rham cohomology of some formal scheme. Then H^{χ,F_0} will map to some meromorphic differential forms on this formal scheme. Now any cohomology class of the de Rham cohomology can be expressed using 1-hypercocycles and any meromorphic differential form can be naturally viewed as a 1-hypercocycle. The question becomes how to write this 1-hypercocycle into some "good form". This will be done by explicit calculations. We keep the notation from the last section.

Consider the composite of the following maps, which we denote by ι,

$$H^{\chi',F_0} \to H^0(\Sigma_{1,F}, \Omega^1)^{\chi'} \to H^1_{\mathrm{dR}}(\Sigma_{1,F})^{\chi'} \simeq \prod_s H^1_{\mathrm{dR}}(U_s)^{\chi'} \simeq \prod_s H^1_{\mathrm{crys}}(\overline{U_s}/F_0)^{\chi'} \otimes_{F_0} F.$$

See Sections 11 and 12 for the notation. Our first result is about the image of ι. We denote the first crystalline cohomology of $\overline{U_s}$ (over Spec $\mathbb{F}_{p^2}$) by $H^1_{\text{crys}}(\overline{U_s}/O_{F_0})$. It is not hard to see that this is a lattice inside $H^1_{\text{crys}}(\overline{U_s}/F_0) = H^1_{\text{crys}}(\overline{U_s}/O_{F_0}) \otimes_{O_{F_0}} F_0$.

Proposition 16.1. $$\iota(H^{\chi',F_0}) \subset \prod_s H^1_{\text{crys}}(\overline{U_s}/O_{F_0})^{\chi'} \otimes_{O_{F_0}} O_F.$$

Proof. We only deal with the even case, that is to say, for an even vertex s', we will prove that the image of H^{χ',F_0} in $H^1_{\text{crys}}(\overline{U_{s'}}/F_0)^{\chi'} \otimes_{F_0} F$ is actually inside $H^1_{\text{crys}}(\overline{U_{s'}}/O_{F_0}) \otimes_{O_{F_0}} O_F$. The odd case is similar.

First we recall some results from Section 12; see the discussion below Lemma 12.1. We constructed an isomorphism $\psi_{s',\xi} : U_{s',\xi} \to F_{0,\xi}$ (recall that $U_{s',\xi}$ is the tubular neighborhood of $\overline{U_{s',\xi}}$ in $\Sigma_{1,F}$), where

$$F_{0,\xi} \overset{\text{def}}{=} \Big\{(x,y) \in \mathbb{A}^2_F \,\Big|\, y^{p+1} = v_1 w_1^{-1} \xi (x^p - x),\ |x-k| > p^{-1/(p-1)}, k = 0, \dots, p-1, \\ |x| < p^{1/(p-1)}\Big\}.$$

Cleary $F_{0,\xi}$ is an open set in a projective curve $D_{0,\xi}$ in $\mathbb{P}^2_F$ defined by $y^{p+1} = v_1 w_1^{-1} \xi (x^p - x)$. The curve $D_{0,\xi}$ has an obvious formal model $\widehat{D_{0,O_{F_0},\xi}}$ over O_{F_0}. Its special fiber can be canonically identified with $\overline{U_{s',\xi}}$. Hence we can identify $H^1_{\text{crys}}(\overline{U_{s',\xi}}/O_{F_0})$ with $H^1_{\text{dR}}(\widehat{D_{0,O_{F_0},\xi}})$.

Definition 16.2. For $s \in A(s')$, let $V_{s,\xi}$ be the affine open formal subscheme of $\widehat{D_{0,O_{F_0},\xi}}$ whose underlying space is the union of $\overline{U^0_{s',\xi}}$ and the intersection point of $\overline{U_{s',\xi}}$ and $\overline{U_{s,\xi}}$. Also we define $V_{c,\xi} = \bigcap_{s_v \in A(s')} V_{s_v,\xi}$ (it is equal to $V_{s_1,\xi} \cap V_{s_2,\xi}$ for any $s_1 \neq s_2 \in A(s')$).

Hence $\mathcal{C} = \{V_{s,\xi}\}_{s \in A(s')}$ is an open covering of $\widehat{D_{0,O_{F_0},\xi}}$. Any element in $H^1_{\text{dR}}(\widehat{D_{0,O_{F_0},\xi}})$ can be represented as a 1-hypercocycle $(\{\omega_s\}_{s \in A(s')}, \{f_{s_1,s_2}\}_{s_1,s_2 \in A(s)})$, where $\omega_s \in H^0(V_{s,\xi}, \Omega^1_{V_{s,\xi}})$, and $f_{s_1,s_2} \in H^0(V_{s_1,\xi} \cap V_{s_2,\xi}, \mathcal{O}_{V_{s_1,\xi} \cap V_{s_2,\xi}})$, such that

$$df_{s_1,s_2} = \omega_{s_1}|_{V_{s_1} \cap V_{s_2}} - \omega_{s_2}|_{V_{s_1} \cap V_{s_2}}.$$

Two 1-hypercocycles $(\{\omega_s\}, \{f_{s_1,s_2}\})$, $(\{\omega'_s\}, \{f'_{s_1,s_2}\})$ represent the same cohomology class if and only there exists a family of functions $\{g_s\}_{s \in A(s)}$, $g_s \in H^0(V_{s,\xi}, \mathcal{O}_{V_{s,\xi}})$, such that

$$\omega_s - \omega'_s = dg_s, \quad f_{s_1,s_2} - f'_{s_1,s_2} = g_{s_1}|_{V_{s_1} \cap V_{s_2}} - g_{s_2}|_{V_{s_1} \cap V_{s_2}}.$$

Given a differential form ω on $F_{0,\xi}$, we view it as a cohomology class in $H^1_{\text{dR}}(F_{0,\xi})$. How do we relate it, as above, with a 1-hypercocycle in

$$H^1_{\text{dR}}(D_{0,\xi}) = H^1_{\text{dR}}(\widehat{D_{0,O_{F_0},\xi}}) \otimes_{F_0} F?$$

Definition 16.3. Since the generic fiber of $\widehat{D_{0,O_{F_0},\xi}}$ becomes $D_{0,\xi}$ when tensored with F, the generic fiber of $V_{s,\xi}$ corresponds to an open rigid subspace of $D_{0,\xi}$, which we denote by $W_{s,\xi}$. We also define $Z_{s,\xi} = W_{s,\xi} \cap F_{0,\xi}$.

If ω is in the χ'-isotypic component and $\chi' \neq \chi'^p$, we will see later that we can find a rigid analytic function f_s on $Z_{s,\xi}$ for each $s \in A(s')$ such that $\omega|_{Z_{s,\xi}} - df_s$ can be extended to a holomorphic differential form ω_s on $W_{s,\xi}$. Define

$$f_{s_1,s_2} = f_{s_2}|_{W_{s_1,\xi}\cap W_{s_2,\xi}} - f_{s_1}|_{W_{s_1,\xi}\cap W_{s_2,\xi}}. \tag{58}$$

Then $(\{\omega_s\}, \{f_{s_1,s_2}\})$ is an element in $H^1_{\mathrm{dR}}(D_{0,\xi})$, whose image in $H^1_{\mathrm{dR}}(F_{0,\xi})$ is ω.

Roughly speaking, what we did above is to "remove" the poles of ω so that ω can be extended to a hypercocycle on $D_{0,\xi}$.

Now apply the above abstract discussion to our situation. Let s' be an even vertex and $s \in A(s')$. Then, under the isomorphism in Lemma 8.2,

$$W_{s,\xi} = \big\{(x, y) \in D_{0,\xi} \bigm| |x-k| = 1, k = 1, \dots, p-1,\ |x| \le 1\big\},$$
$$Z_{s,\xi} = \big\{(x, y) \in W_{s,\xi} \bigm| |x| > p^{-1/(p-1)}\big\}.$$

Recall that in Lemma 15.5, we showed that a section ω of $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi'}$ has the following form when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$:

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]} g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}, \tag{59}$$

where $f(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]^{\widehat{}}$, $g(\zeta) \in O_{F_0}[\zeta, 1/(\zeta^{p-1}-1)]^{\widehat{}}$.

Hence if we restrict it to $\widetilde{\Sigma_{1,O_F,s',\xi}}$ (replace ζ by p/η and $\tilde{e}'$ by $\varpi^{p-1}\xi/\tilde{e}$):

$$\omega|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} - \varpi^{[-mp]} g\Big(\frac{p}{\eta}\Big)\varpi^{i(p-1)}\xi^i\tilde{e}^{-i}\,\frac{d\tilde{e}}{\tilde{e}}, \tag{60}$$

where

$$f(\eta) \in O_{F_0}\Big[\eta, \frac{1}{\eta^{p-1}-1}\Big]^{\widehat{}},\quad g\Big(\frac{p}{\eta}\Big) \in O_{F_0}\Big[\frac{p}{\eta}, \frac{1}{(p/\eta)^{p-1}-1}\Big]^{\widehat{}} \subset O_{F_0}\Big[\Big[\frac{p}{\eta}\Big]\Big].$$

Notice that the restriction of $\psi_{0,\xi}$ to the generic fiber of $\widetilde{\Sigma_{1,O_F,s',\xi}}$ has the form

$$x \mapsto \eta, \quad y \mapsto \tilde{e}(1-(p/\eta)^{p-1})^{1/(p+1)}. \tag{61}$$

Lemma 16.4. *Under the isomorphism $\psi_{0,\xi}$, the 1-form ω has the following form on $Z_{s,\xi}$*

$$\varpi^{p^2-1-m}\Big(F(x)y^{p+1-i} + G\Big(\frac{p}{x}\Big)y^{-i}\Big)\frac{dy}{y}, \tag{62}$$

where $F(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^{\widehat{}}$ and $G(p/x) = \sum_{n=0}^{+\infty} a_n(p/x)^n$ with $a_n \in O_{F_0}$ for all n. Moreover, using (60):

(1) $f(x) \equiv F(x) \bmod pO_{F_0}[x, 1/(x^{p-1}-1)]^{\widehat{}}$.

(2) $a_0 \equiv -\xi^i g(0) \bmod pO_{F_0}$ *if* $p^2-1-m \ge [-mp]$ *and* $a_0 \equiv 0 \bmod pO_{F_0}$ *otherwise. When* $i = p$, $a_0/p \equiv -\xi^i g(0) \bmod pO_{F_0}$.

Assume this lemma for the moment. Hence we can write ω on $Z_{s,\xi}$ as

$$\varpi^{p^2-1-m}\Big(F(x)y^{p+1-i}+G\Big(\frac{p}{x}\Big)y^{-i}\Big)\,\frac{dy}{y},$$

where $F(x)\in O_{F_0}[x,1/(x^{p-1}-1)]\widehat{\ }$, $G(p/x)=\sum_{n=0}^{+\infty}a_n(p/x)^n\in O_{F_0}[\![p/x]\!]$. Certainly $F(x)y^{p+1-i}\,dy/y$ extends to $W_{s,\xi}$, so we only need to "remove" the poles of the other term (essentially the pole at $x=y=0$). On $Z^0_{s,\xi}\overset{\text{def}}{=}\{(x,y)\in Z_{s,\xi}\mid |x|<1\}$, we can write

$$x=\sum_{n=1}^{+\infty}c_n y^{(p+1)n},$$

where $c_n\in O_{F_0}$, $c_1=v_1^{-1}w_1\xi^{-1}\in O_{F_0}^{\times}$. Thus a simple computation shows:

Lemma 16.5. *On $Z^0_{s,\xi}$,*

$$\sum_{n=0}^{+\infty}a_n\Big(\frac{p}{x}\Big)^n y^{-i}\frac{dy}{y}=\sum_{n=-\infty}^{+\infty}b_n y^{-n(p+1)-i-1}\,dy,$$

where $b_n\in O_{F_0}$, $\forall n\in\mathbb{Z}$ and for $n\geq 0$, $v_p(b_n)\geq n$. Moreover $b_0\equiv a_0 \bmod p$.

Now define

$$f_s=\varpi^{p^2-1-m}\sum_{n=0}^{+\infty}\frac{b_n}{-n(p+1)-i}y^{-n(p+1)-i}. \tag{63}$$

It can be viewed as a rigid analytic function on $Z_{s,\xi}$. Also, it is clear from the above computation that $\omega-df_s$ can be extended to a holomorphic differential form ω_s on $W_{s,\xi}$. Do the same thing for each $s\in A(s')$; we can define ω_s, f_{s_1,s_2} as explained before. Then $(\{\omega_s\},\{f_{s_1,s_2}\})$ is the 1-hypercocycle in $H^1_{\mathrm{dR}}(D_{0,\xi})\simeq H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}},_\xi)\otimes_{O_{F_0}}F$ that represents ω.

Notice that for $i\in\{1,\ldots,p-1\}$, $v_p(b_n/(-n(p+1)-i))\geq 0$ since $v_p(b_n)\geq n$. When $i=p$, $b_0\equiv a_0\equiv 0 \bmod p$ since we are in the case $p^2-1-m\leq[-mp]$. We still have $v_p(b_n/(-n(p+1)-i))\geq 0$. In fact, equality only can happen when $n=0$. Therefore all the coefficients appearing in ω_s, f_{s_1,s_2} will be integral. In other words,

$$(\{\omega_s\},\{f_{s_1,s_2}\})\in H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}},_\xi)\otimes_{O_{F_0}}O_F. \qquad \square$$

Proof of Lemma 16.4. We only give a sketch of the computations here. Using the notation in (60), it suffices to deal with the case $g(p/\eta)=0$ and $f(\eta)=0$ separately.

(1) Assume $g(p/\eta)=0$. Plug (61) into (60). A direct computation shows that ω has the form

$$\varpi^{p^2-1-m}f(x)\Big(1+\Big(\frac{p}{x}\Big)^{p-1}G_1(x)\Big)y^{p+1-i}\,\frac{dy}{y},$$

where $G_1(x)\in O_{F_0}\Big[x,\frac{1}{x^{p-1}-1}\Big]\widehat{\ }\Big[\!\Big[\frac{p}{x}\Big]\!\Big]$.

Let $G_2(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^\wedge[\![p/x]\!]$ be

$$G_2(x) = v_1 w_1^{-1}\xi(x^{p-1}-1)G_1(x)f(x)\Big(\frac{p}{x}\Big)^{p-2}.$$

Clearly we can decompose $G_2(x)$ as

$$G_2(x) = F_3(x) + G_3(\frac{p}{x}),$$

where $F_3(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^\wedge$, $G_3(p/x) \in O_{F_0}[\![p/x]\!]$. Replacing $F_3(x)$ by $F_3(x) - F_3(0)$, we may assume $F_3(x) \in x O_{F_0}[x, 1/(x^{p-1}-1)]^\wedge$. Since there is a $(p/x)^{p-2}$ in the definition of $G_2(x)$, it is easy to see (for example, expand $G_2(x)$ as an element in $F_0[\![x, 1/x]\!]$) that the constant term of $G_3(p/x)$ is divisible by p (in fact, by p^{p-2}). Recall that we assume p is odd; hence at least 3.

Now $\varpi^{-(p^2-1-m)}\omega$ can be written as

$$f(x)y^{p+1-i}\frac{dy}{y} + p\Big(\frac{F_3(x)}{x}\Big)\Big(\frac{1}{(x^{p-1}-1)v_1 w_1^{-1}\xi}\Big)y^{p+1-i}\frac{dy}{y} + p\frac{G_3(p/x)}{(x^p-x)v_1 w_1^{-1}\xi}y^{p+1-i}\frac{dy}{y}.$$

Notice that $y^{p+1} = v_1 w_1^{-1}\xi(x^p - x)$. The last term is nothing but

$$pG_3\Big(\frac{p}{x}\Big)y^{-i}\frac{dy}{y}.$$

Now let

$$F(x) = f(x) + p\Big(\frac{F_3(x)}{x}\Big)\Big(\frac{1}{(x^{p-1}-1)v_1 w_1^{-1}\xi}\Big), \quad G\Big(\frac{p}{x}\Big) = pG_3\Big(\frac{p}{x}\Big).$$

It is clear they satisfy all the conditions in the lemma. So we're done in this case.

(2) Assume $f(\eta) = 0$. When $p^2 - 1 - m \geq [-mp]$, we can write $\varpi^{-(p^2-1-m)}\omega$ as

$$-g\Big(\frac{p}{x}\Big)\xi^i\Big(1 + \Big(\frac{p}{x}\Big)^{p-1}H(x)\Big)y^{-i}\frac{dy}{y},$$

where $H(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^\wedge[\![p/x]\!]$. Make the decomposition

$$-g\Big(\frac{p}{x}\Big)\xi^i H(x)\Big(\frac{p}{x}\Big)^{p-2} = F_1(x)x^2 + Ax + H_1\Big(\frac{p}{x}\Big),$$

where $F_1(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^\wedge$, $A \in pO_{F_0}$, $H_1(p/x) \in O_{F_0}[\![p/x]\!]$. Notice that A is divisible by p since there is a $(p/x)^{p-2}$ in the expression. Then $\varpi^{-(p^2-1-m)}\omega$ is

$$-g\Big(\frac{p}{x}\Big)\xi^i y^{-i}\frac{dy}{y} + pxF_1(x)y^{-i}\frac{dy}{y} + pAy^{-i}\frac{dy}{y} + \frac{p}{x}H_1\Big(\frac{p}{x}\Big)y^{-i}\frac{dy}{y}.$$

Using $y^{p+1} = v_1 w_1^{-1}\xi(x^p - x)$, the second term is

$$pF_1(x)\frac{1}{v_1 w_1^{-1}\xi(x^{p-1}-1)}y^{p+1-i}\frac{dy}{y}.$$

It's easy to see the following $F(x)$, $G(p/x)$ actually work:

$$F(x) = pF_1(x)\frac{1}{v_1 w_1^{-1}\xi(x^{p-1}-1)}, \quad G\Big(\frac{p}{x}\Big) = -g\Big(\frac{p}{x}\Big)\xi^i + pA + \frac{p}{x}H_1\Big(\frac{p}{x}\Big).$$

When $p^2 - 1 - m \geq [-mp]$ does not hold, then

$$\omega = -\varpi^{[-mp]}g\Big(\frac{p}{\eta}\Big)\varpi^{i(p-1)}\xi^i\tilde{e}^{-i}\frac{d\tilde{e}}{\tilde{e}} = p\varpi^{p^2-1-m}g\Big(\frac{p}{\eta}\Big)\xi\tilde{e}^{-i}\frac{d\tilde{e}}{\tilde{e}}.$$

Repeat the previous argument and it's direct to see the claim in the lemma is true. □

In the previous proposition, we showed how to turn a differential form $\omega \in H^{\chi',F_0}$, when restricted to $U_{s'}$, into a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ inside the de Rham cohomology $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}}) \otimes_{O_{F_0}} O_F$ (via the isomorphism $\psi_{s',\xi}$). It is crucial to understand the mod p properties of this hypercocycle. Essentially, we need to understand f_s in (63) modulo p (recall that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$; see (58)).

Fix an even vertex s' and $s \in A(s')$. Recall that $V_{c,\xi} = \bigcap_{s_v \in A(s')} V_{s_v,\xi}$. It is clear from our definition that $\varpi^{-(p^2-1-m)} f_s \in H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$.

Lemma 16.6. (Using notation from the proof of Proposition 16.1.)

(1) *When $i = p$,*

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{b_0 y^{-p}}{-p} \equiv \frac{a_0 y^{-p}}{-p} \equiv \xi^p g(0) y^{-p} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

(2) *When $i \in \{1, \dots, p-1\}$ and $p^2 - 1 - m \leq [-mp]$,*

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

(3) *When $p^2 - 1 - m \geq [-mp]$, we have*

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{b_0 y^{-i}}{-i} \equiv \frac{a_0 y^{-i}}{-i} \equiv \frac{\xi^i g(0) y^{-i}}{i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$

except the case $i = p-1$ and the case $p = 3$, $i = 1$. I claim that in these exceptional cases, we can find another 1-hypercocycle $(\{\omega'_{s_v}\}, \{f'_{s_1,s_2}\})$ in the same cohomology class as $(\{\omega_{s_v}\}, \{f_{s_1,s_2}\})$ such that we can write $f'_{s_1,s_2} = f'_{s_2} - f'_{s_1}$ for any $s_1, s_2 \in A(s')$ and

$$\varpi^{-(p^2-1-m)} f'_s \equiv \frac{b_0 y^{-i}}{-i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$
$$f'_{s_v} = f_s \quad \textit{for any } s_v \neq s.$$

Proof of Lemma 16.6. Everything is clear by Lemma 16.5 and the definition of f_s in (63), except for the exceptional cases. First we assume $i = p-1$, then

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{b_0 y^{-(p-1)}}{-(p-1)} + \frac{b_1 y^{-2p}}{-2p} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

This makes sense since $v_p(b_1) \geq 1$. Now define $g_{s_v} \in H^0(V_{s_v,\xi}, \mathcal{O}_{V_{s_v,\xi}}),\ s_v \in A(s')$ as:

$$g_{s_v} = \begin{cases} -\dfrac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \dfrac{(x^{2p-4} - 2x^{p-3})y^2}{(x^{p-1}-1)^2} & \text{if } s_v = s, \\ -\dfrac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \dfrac{y^2}{x^2} & \text{if } s_v \neq s. \end{cases}$$

Hence define $\omega'_{s_v} = \omega_{s_v} + \varpi^{p^2-1-m} dg_{s_v},\ f'_{s_1,s_2} = f_{s_1,s_2} + \varpi^{p^2-1-m}(g_{s_1} - g_{s_2})$; the hypercocycle $(\{\omega_{s_v}\}, \{f_{s_1,s_2}\})$ and $(\{\omega'_{s_v}\}, \{f'_{s_1,s_2}\})$ are in the same cohomology class. A simple computation shows the following identity in $H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$:

$$-\frac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \frac{(x^{2p-4} - 2x^{p-3})y^2}{(x^{p-1}-1)^2} + \frac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \frac{y^2}{x^2} = \frac{b_1 y^{-2p}}{-2p}.$$

Thus if we define $f'_s = f_s - b_1 y^{-2p}/(-2p),\ f'_{s_v} = f_{s_v},\ s_v \neq s$, they satisfy

$$f'_{s_1,s_2} = f'_{s_2} - f'_{s_1},$$

and clearly have the property we want.

The case $i = 1,\ p = 3$ can be done by the same method. This time

$$\begin{aligned} \varpi^{-(p^2-1-m)} f_s &\equiv b_0 y^{-1} + \frac{b_2}{-9} y^{-9} \\ &\equiv b_0 y^{-1} + \frac{b_2}{-9v_1^3 w_1^{-3}\xi^3}\left(\frac{x^3y^3}{(x^2-1)^3} - \frac{y^3}{x^3}\right) \mod 3H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}). \end{aligned}$$

We can define g_{s_v} similarly. I omit the details here. □

Remark 16.7. In the odd case, things are similar. We only restrict ourselves to the case $p^2 - 1 - m \geq [-mp]$. Let s be an odd vertex. We also have $\psi_{s,\xi}$ (see the beginning of Section 12). Let ω be an element of H^{χ',F_0}. Similarly to Lemma 16.4, ω has the form (using (52)):

$$\varpi^{[-mp]}\left(F(x)y^i + pG\left(\frac{p}{x}\right)y^{-p-1+i}\right)\frac{dy}{y}, \tag{64}$$

where $F(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]\widehat{\ }$, $G(p/x) = \sum_{n=0}^{+\infty} a_n(p/x)^n,\ a_n \in O_{F_0}\ \forall n$. All the above arguments work here and we can define a 1-hypercocycle $(\{\omega_{s'}\}, \{f_{s'_1,s'_2}\})$ that represents ω. Notice that there is a "p" in front of $G(p/x)$ in (64). Thus when

$2 \le i \le p$ (resp. $i = 1$),

$$\varpi^{-[-mp]} f_{s'} \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}) \text{ (resp. } H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})),$$
$$\varpi^{-[-mp]} f_{s'_1,s'_2} \in pH^0(V_{s'_1,\xi} \cap V_{s'_2,\xi}, \mathcal{O}_{V_{s'_1,\xi} \cap V_{s'_2,\xi}}) \text{ (resp. } H^0(V_{s'_1,\xi} \cap V_{s'_2,\xi}, \mathcal{O}_{V_{s'_1,\xi} \cap V_{s'_2,\xi}})).$$

Here $V_{s',\xi},\ s' \in A(s)$ is defined similarly.

Before stating the main result of this section, we still need to do some extra work. Most results here can be found in [Haastert and Jantzen 1990]. Since $\widehat{D_{0,O_{F_0},\xi}}$ is a curve in $\mathbb{P}^2$, the Hodge–de Rham spectral sequence gives us the following exact sequence:

$$0 \to H^0(\widehat{D_{0,O_{F_0},\xi}}, \Omega^1_{\widehat{D_{0,O_{F_0},\xi}}}) \to H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}}) \to H^1(\widehat{D_{0,O_{F_0},\xi}}, \mathcal{O}_{\widehat{D_{0,O_{F_0},\xi}}}) \to 0. \quad (65)$$

And each group in this exact sequence is a finite free O_{F_0}-module. If we use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent a cohomology class in $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}})$, then every element in $H^0(\widehat{D_{0,O_{F_0},\xi}}, \Omega^1_{\widehat{D_{0,O_{F_0},\xi}}})$ can be identified as the hypercocycle with all $f_{s_1,s_2} = 0$. And the map to $H^1(\widehat{D_{0,O_{F_0},\xi}}, \mathcal{O}_{\widehat{D_{0,O_{F_0},\xi}}})$ is just mapping the hypercocycle to $\{f_{s_1,s_2}\}$, which is considered as a 1-cocycle. Similarly, we have

$$0 \to H^0(\overline{U_{s'_0,\xi}}, \Omega^1_{\overline{U_{s'_0,\xi}}}) \to H^1_{\mathrm{dR}}(\overline{U_{s'_0,\xi}}) \to H^1(\overline{U_{s'_0,\xi}}, \mathcal{O}_{\overline{U_{s'_0,\xi}}}) \to 0,$$

which can be identified with the reduction mod p of the previous exact sequence.

Recall that the de Rham cohomology of $\widehat{D_{0,O_{F_0},\xi}}$ can be identified as the crystalline cohomology of $\overline{U_{s'_0,\xi}}$. It is equipped with a Frobenius operator φ. It is important to understand the relationship between φ and the above exact sequence. Denote $\bigcup_{\xi^{p-1}=-1} \widehat{D_{0,O_{F_0},\xi}}$ by $\widehat{D_{0,O_{F_0}}}$.

Lemma 16.8. *Under the isomorphism between $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}})$ and $H^1_{\mathrm{crys}}(\overline{U_{s'_0,\xi}}/O_{F_0})$,*

(1) $\varphi(H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}) \subset H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}$.

(2) $\varphi(H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}) \subset H^0(\widehat{D_{0,O_{F_0}}}, \Omega^1_{\widehat{D_{0,O_{F_0}}}})^{(\chi')^p} + pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}$.

(3) *The above inclusion is in fact an equality and φ induces an isomorphism between $H^1(\overline{U_{s'_0}}, \mathcal{O}_{\overline{U_{s'_0}}})^{\chi'}$ and $H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}$.*

Proof. See Section 3 of [Haastert and Jantzen 1990], especially Proposition 3.5. Although our curve is slightly different from the curve in that paper, all arguments in their paper work here. □

Remark 16.9. A variant of Lemma 16.8 is that φ induces an isomorphism

$$H^0(\widehat{D_{0,O_{F_0}}}, \Omega^1_{\widehat{D_{0,O_{F_0}}}})^{\chi'} + pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'} \xrightarrow{\sim} pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}.$$

This follows from the fact that φ^2 is a scalar c_x on these spaces and $v_p(c_x) = 1$; see Proposition 12.8. A direct corollary is that φ induces an isomorphism between

$$\big(H^0(\widehat{D_{0,O_{F_0}}}, \Omega^1_{\widehat{D_{0,O_{F_0}}}})^{\chi'} + pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}\big)/pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}$$

and

$$pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}/\big(pH^0(\widehat{D_{0,O_{F_0}}}, \Omega^1_{\widehat{D_{0,O_{F_0}}}})^{(\chi')^p} + p^2H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}\big),$$

which can be viewed as an isomorphism $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'} \xrightarrow{\sim} H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}$.

In fact, we can write down the isomorphism between $H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}$ and $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}$ explicitly (Lemma 16.13 below). Some notation here: as before (see (27)), we may identify $\overline{U_{s_0',\xi}}$ with the projective curve defined by $\tilde{e}^{p+1} = v_1 w_1^{-1}\xi(\eta^p - \eta)$ and the singular points of $\overline{U_{s_0'}}$ (considered in the special fiber of $\widetilde{\Sigma_{1,O_F,\xi}}$) are those points with $\tilde{e} = 0$.

Definition 16.10. We write $A(s_0') = \{s_0, \dots, s_{p-1}, s_\infty\}$, where for $k = 0, \dots, p-1$, s_k is the vertex that corresponds to $\eta = k$, $\tilde{e} = 0$ in $\overline{U_{s_0',\xi}}$ and s_∞ corresponds to the point $\eta = \infty$, $\tilde{e} = 0$ (equivalently, if we use projective coordinates $[\eta, \tilde{e}, 1]$, then this point is $[1, 0, 0]$).

Definition 16.11. Let V_0 be the open set of $\overline{U_{s_0',\xi}}$ that is the complement of the point $\eta = \infty$, $\tilde{e} = 0$. We also define V_∞ as the complement of $\eta = \tilde{e} = 0$.

Using the notation from Definition 16.2, it is clear that set theoretically, V_0 is the union of $V_{s_0,\xi}, \dots, V_{s_{p-1},\xi}$ and V_∞ is the union of $V_{s_1,\xi}, \dots, V_{s_{p-1},\xi}, V_{s_\infty,\xi}$. By abuse of notation, we also view V_0, V_∞ as open affine formal subschemes of $\widehat{D_{0,O_{F_0},\xi}}$.

Notice that V_0, V_∞ is an open covering of $\widehat{D_{0,O_{F_0},\xi}}$. Hence every cohomology class of $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}})$ can be represented by a 1-hypercocycle $(\omega_0, \omega_\infty, f_{0,\infty})$ as before. Every element of $H^1(\overline{U_{s_0',\xi}}, \mathcal{O}_{\overline{U_{s_0',\xi}}})$ can be represented by an element in $H^0(V_0 \cap V_\infty, \mathcal{O}_{V_0\cap V_\infty})$, viewed as a 1-cocycle. The next lemma is easy to see.

Lemma 16.12. *$H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}$ has a basis, when restricted to $\overline{U_{s_0',\xi}}$, given by*

$$\frac{\tilde{e}^{p+1-i}}{\eta^k}, \quad k = 1, \dots, p-i.$$

If $i = p$, then $H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{\chi'} = 0$.

Hence we may view $\tilde{e}^{p+1-i}/\eta^k$ as an element in $H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}$. Then, as a 1-hypercocycle, $\varphi(\tilde{e}^{p+1-i}/\eta^k)$ is $(0, 0, \tilde{e}^{(p+1-i)p}/\eta^{pk})$. A direct computation shows:

Lemma 16.13. *$\varphi(\tilde{e}^{p+1-i}/\eta^k)$ is the same as the holomorphic differential form*

$$(v_1 w_1^{-1}\xi)^{p-i}(-1)^{p-i-k}k\binom{p-i}{k}\eta^{p-i-k}\,\tilde{e}^{i-1}\,d\tilde{e}.$$

Remark 16.14. We will need to translate a 1-cocycle inside $H^1(\overline{U_{s'_0,\xi}}, \mathcal{O}_{\overline{U_{s'_0,\xi}}})$ using the open covering $\{V_{s,\xi}\}_{s\in A(s'_0)}$ to a 1-cocycle using the open covering $\{V_0, V_\infty\}$. This is done as follows. If we start with a 1-cocycle $\{f'_{s,s''}\}$, we can find another 1-cocycle $\{f_{s,s''}\}$ that represents the same cohomology class and f_{s_0,s_∞} can be extended to a section in $H^0(V_0\cap V_\infty, \mathcal{O}_{V_0\cap V_\infty})$. Then f_{s_0,s_∞} can be viewed as a 1-cocycle of the covering $\{V_0, V_\infty\}$. In fact, this is just what we want.

Example 16.15. Let's compute one example here. Consider the 1-cocycle $\{f'_{s,s''}\}$:

$$f'_{s,s''} = f'_{s''} - f'_s, \quad \text{where } f'_{s_0} = \tilde{e}^{-i}, \ f'_s = 0 \text{ for } s \neq s_0.$$

Then clearly f'_{s_0,s_∞} has poles on $V_0\cap V_\infty$. But we can modify this cocycle a little bit: define

$$g_{s_0} = \frac{\eta^{p-2}\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi(\eta^{p-1}-1)} \in H^0(V_{s_0,\xi}, \mathcal{O}_{V_{s_0,\xi}}), \quad g_s = 0 \text{ for } s \neq s_0,$$

and let

$$f_{s,s''} = f'_{s,s''} - g_{s''} + g_s.$$

Then $\{f_{s,s''}\}$ and $\{f'_{s,s''}\}$ represent the same cohomology class. Moreover,

$$f_{s_0,s_\infty} = f'_{s_0,s_\infty} + g_{s_0} = -f_{s_0} + g_{s_0} = -\tilde{e}^{-i} + \frac{\eta^{p-2}\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi(\eta^{p-1}-1)} = \frac{\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi\eta}$$

(using $\tilde{e}^{p+1} = v_1 w_1^{-1}\xi(\eta^p - \eta)$) clearly extends to $V_0\cap V_\infty$. Hence,

$$\frac{\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi\eta},$$

viewed as a 1-cocycle of the covering $\{V_0, V_\infty\}$, represents the same cohomology class as $\{f'_{s.s''}\}$.

A combination of Remark 16.7 and the Lemma 16.8 gives:

Lemma 16.16. *Assume $p^2-1-m \geq [-mp]$ and $i \neq 1$. Let s be an odd vertex and $\omega \in H^{\chi',F_0}$.*

(1) *Using the method in the proof of Proposition 16.1, we may view $\varpi^{-[-mp]}\omega$ as a cohomology class inside $H^1_{\text{crys}}(\overline{U}_s/O_{F_0})^{\chi'}$. Then*

$$\varphi(\varpi^{-[-mp]}\omega) \in pH^1_{\text{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p},$$

or equivalently (using Remark 16.9),

$$\varpi^{-[-mp]}\omega \in \varphi(H^1_{\text{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p}).$$

(2) *In fact, Proposition 15.8 shows that* $\varpi^{-[-mp]}\omega$ *modulo* p *is a holomorphic differential form inside*

$$H^0(\overline{U}_s, \Omega^1_{\overline{U}_s}) = \varphi(H^1_{\mathrm{dR}}(\overline{U}_s)),$$

which is nothing but $\varpi^{-[-mp]}\omega$ *considered as a cohomology class in* $H^1_{\mathrm{dR}}(\overline{U}_s)$. *In particular, if*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1),$$

then the cohomology class of $\varpi^{-[-mp]}\omega$ *is inside* $pH^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})$.

Proof. Following Remark 16.7, let $(\{\omega_{s'}\}, \{f_{s'_1,s'_2}\})$ be the 1-hypercocycle that represents ω in $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0)^{\chi'}$ (by identifying the crystalline cohomology of $\overline{U}_s$ with the de Rham cohomology of $\widehat{D_{0,O_{F_0}}}$). Since

$$\varpi^{-[-mp]} f_{s'_1,s'_2} \in pH^0(V_{s'_1,\xi} \cap V_{s'_2,\xi}, \mathcal{O}_{V_{s'_1,\xi}\cap V_{s'_2,\xi}}),$$

all these $\varpi^{-[-mp]} f_{s'_1,s'_2}$ vanish if we reduce modulo p. This means that the image of $\varpi^{-[-mp]}\omega$ in $H^1_{\mathrm{dR}}(\overline{U}_s)$ actually lies inside $H^0(\overline{U}_s, \Omega^1_{\overline{U}_s})$. Now our first claim is a direct consequence of Lemma 16.8. Referring again to Remark 16.7, the rest of the lemma follows from

$$\varpi^{-[-mp]} f_{s'} \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

Thus when we restrict everything to the special fiber of $V_{c,\xi}$ (equivalently, $\overline{U^0_{s,\xi}}$),

$$\varpi^{[-mp]}\omega_{s'} = \varpi^{-[-mp]}\omega - df_{s'} \equiv \varpi^{-[-mp]}\omega \mod pH^0(V_{c,\xi}, \Omega^1_{V_{c,\xi}}).$$

This indicates that the cohomology class of $\varpi^{-[-mp]}\omega$ is just the 1-form $\varpi^{-[-mp]}\omega$ after reducing modulo p. □

Remark 16.17. Using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, it's not hard to see that if we replace s by an even vertex s' and $\omega \in H^{\chi',F_0}$ by $\omega \in H^{(\chi')^p,F_0}$, we have a similar result:

$$\varpi^{-[-mp]}\omega \in \varphi(H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}),$$

and exactly the same statement for the second part.

Similarly, by combining Lemmas 16.6 and 16.8, we obtain:

Lemma 16.18. *Let* $\omega \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$ *and* s' *be an even vertex. Assume*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)} \tag{66}$$

for any $s \in A(s')$.

(1) *The image of* $\varpi^{-(p^2-1-m)}\omega$ *in* $H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}$ *is actually inside*

$$\varphi(H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}).$$

Equivalently, if we view $\varpi^{-(p^2-1-m)}\omega$ *as an element inside* $H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}$,

$$\varphi(\varpi^{-(p^2-1-m)}\omega) \in pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}.$$

(2) *Assume* $i \neq p$. *Proposition 15.8 shows that in this case* $\varpi^{-(p^2-1-m)}\omega$ *modulo* p *is a holomorphic differential form inside*

$$H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_s}}) = \varphi(H^1_{\mathrm{dR}}(\overline{U_{s'}})),$$

and we may identify it with the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ *in* $H^1_{\mathrm{dR}}(\overline{U_{s'}})$. *In particular, if*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1),$$

then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ *is inside* $pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})$.

(3) *Assume* $i = p$. *We have a slightly weaker result: assume*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1); \tag{67}$$

then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ *is inside* $pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})$.

Proof. First we prove the first two parts. If $i = p$, we know that (Lemma 16.12)

$$H^1(\overline{U_{s'}}, \mathcal{O}_{\overline{U_{s'}}})^{\chi'} = H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_{s'}}})^{(\chi')^p} = 0.$$

Hence Lemma 16.8 tells us that

$$\varphi(H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}) = H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}.$$

So in this case, the first part is trivially true.

Now assume $i \neq p$. We need to use some results from Lemma 16.6; see the notation there. We can represent $\omega|_{\widetilde{\Sigma_{1,O_F,s'}}}$ as a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ and for $s \in A(s')$, there exists

$$f_s \in \varpi^{p^2-1-m} H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$

such that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$. Hence, as in the proof of Lemma 16.16, it suffices to prove

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

If $p^2-1-m \leq [-mp]$ and $i \neq p$, this already follows from the second part of Lemma 16.6. So we only need to treat the case $p^2-1-m \geq [-mp]$. Then the desired result follows from our condition that $\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$.

More precisely, using the notation from the proof of Proposition 16.1 (especially (59)), we can write

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g(\zeta)\tilde{e}'^{i}\frac{d\tilde{e}'}{\tilde{e}'}. \tag{68}$$

Notice that Lemma 16.6 tells us that

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{\xi^i g(0) y^{-i}}{i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

It suffices to show $g(0) \in pO_{F_0}$. But by Lemma 15.7,

$$g(\zeta) \in pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]^{\wedge},$$

since we assume $\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$. So we're done for the first two parts.

As for the last claim, we keep using the notation $(\{\omega_s\}, \{f_{s_1,s_2}\})$ and $\{f_s\}$ as above. Notice that we already assume $\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1)$. Hence if we can show

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}), \tag{69}$$

then we would know that both ω_s and f_{s_1,s_2} are divisible by p. Thus it suffices to prove (69).

But using the notation (68) above and Lemma 16.6, which says that

$$\varpi^{-(p^2-1-m)} f_s \equiv \xi^p g(0) y^{-p} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$

we only need to show $g(0)$ is divisible by p. But by our assumption (67),

$$f(\eta) \in pO_{F_0}\Big[\eta, \frac{1}{\eta^{p-1}-1}\Big]^{\wedge}.$$

Since we also assume $\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)$,

$$g(\zeta) \in pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]^{\wedge}.$$

See the computations around Lemma 15.6. Therefore $g(0) \in pO_{F_0}$. □

Remark 16.19. Using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we can get a variant of the previous lemma. Let $\omega \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{(\chi')^p,\mathrm{Gal}(F/F_0)}$ and s be an odd vertex. Assume

$$\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1)^{(\chi')^p,\mathrm{Gal}(F/F_0)}$$

for any $s' \in A(s)$. Then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ in $H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}$ is inside $\varphi(H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{\chi'})$. And we have a similar result for the last two parts: if we assume

$$\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1),$$

then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ is inside $pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})$.

Remark 16.20. When $i = p$, we will see in Section 17 that the second part of the lemma is actually still true (Lemma 17.12).

Now let's recall the construction of $M(\chi, [1, b])$ in Section 14. First we write

$$\prod_s (F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_s}/F_0) \otimes_{\mathbb{Q}_p} E)^\chi = F_1 \oplus F_2,$$

where

$$\begin{aligned} F_1 &\stackrel{\text{def}}{=} \prod_{s' \text{ even}} F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_{s'}}/F_0)^{\chi'}_{\tau} \oplus \prod_{s \text{ odd}} F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_s}/F_0)^{(\chi')^p}_{\bar{\tau}}, \\ F_2 &\stackrel{\text{def}}{=} \prod_{s' \text{ even}} F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_{s'}}/F_0)^{(\chi')^p}_{\bar{\tau}} \oplus \prod_{s \text{ odd}} F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_s}/F_0)^{\chi'}_{\tau}. \end{aligned} \tag{70}$$

It is clear from Lemma 16.18 that $g_\varphi \otimes \varphi \otimes \text{Id}_E$ sends F_1 to F_2. Let f be an element of $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi, \text{Gal}(F/F_0)}$. By Proposition 14.6, we have an injective map $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi, \text{Gal}(F/F_0)}$ into $\prod_s \big(F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_s}/F_0) \otimes_{\mathbb{Q}_p} E\big)^\chi$. Let (f_1, f_2) be the decomposition of the image of f into $F_1 \oplus F_2$. Then:

Lemma 16.21. $M(\chi, [1, b]) =$

$$\big\{f \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi, \text{Gal}(F/F_0)} \,\big|\, (1 \otimes b)(g_\varphi \otimes \varphi \otimes \text{Id}_E)(f_1) = (\varpi^{(p-1)i} \otimes 1) f_2\big\}.$$

Here $1 \otimes b$ and $\varpi^{(p-1)i} \otimes 1$ are viewed as elements in $F \otimes_{\mathbb{Q}_p} E$.

Proof. Considering

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi, \text{Gal}(F/F_0)} = (H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes O_E)^{\chi, \text{Gal}(F/\mathbb{Q}_p)}, \tag{71}$$

the lemma follows from Proposition 13.3 and the remark below it. □

Thus we can rewrite $M(\chi, [1, b])$ as the kernel of θ_b, which is defined as the composite of the following maps:

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi, \text{Gal}(F/F_0)} \to \prod_s \big(F \otimes_{F_0} H^1_{\text{crys}}(\overline{U_s}/F_0) \otimes_{\mathbb{Q}_p} E\big)^\chi \xrightarrow{L_b} F_2, \tag{72}$$

where $L_b : F_1 \oplus F_2 \to F_2$ is defined as

$$(f_1, f_2) \mapsto -(1 \otimes b)(g_\varphi \otimes \varphi \otimes \text{Id}_E)(f_1) + (\varpi^{(p-1)i} \otimes 1) f_2.$$

To understand the image of θ_b, we introduce:

Definition 16.22.

$$J_1 \stackrel{\text{def}}{=} \prod_{s' \text{ even}} H^1_{\text{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\tau} \oplus \prod_{s \text{ odd}} H^1_{\text{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}_{\bar{\tau}} \subset F_1, \tag{73}$$

$$J_2 \stackrel{\text{def}}{=} (\varpi^{p^2-1-m} g_\varphi \otimes \varphi \otimes \text{Id}_{O_E})(J_1) \subset F_2. \tag{74}$$

Lemma 16.23. *Under the assumption $p^2-1-m \geq [-mp]$ and $2 \leq i \leq p-1$, we have*

$$\theta_b\big((H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big) \subset J_2.$$

Remark 16.24. In the next section we show that the lemma also holds for $i = 1, p$.

Proof. Let ω be an element in $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$. Write $\omega = (\omega_1, \omega_2)$ as the decomposition into $F_1 \oplus F_2$. By Proposition 16.1, we have $\omega_1 \in \varpi^{p^2-1-m} J_1$. Hence

$$L_b((\omega_1, 0)) = -(g_\varphi \otimes \varphi \otimes b(\omega_1)) \in J_2.$$

It remains to prove that $L_b((0, \omega_2)) \in J_2$, or equivalently, $(\varpi^{(p-1)i} \otimes 1)\omega_2 \in J_2$. Using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we only need to check this for one odd vertex. In other words, it suffices to show

$$(\varpi^{(p-1)i} \otimes 1)\omega_{2,s} \in (\varpi^{p^2-1-m} g_\varphi \otimes \varphi \otimes \mathrm{Id}_{O_E})(H^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p}_{\tilde{\tau}}),$$

where s is an odd vertex and $\omega_{2,s}$ is the image of ω inside $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0)^{\chi'}_{\tau}$. But this is nothing but the first part of Lemma 16.16. $\square$

By abuse of notation, we use θ_b to denote the map

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2.$$

Also we use $\bar{\theta}_b$ to denote the modulo p map of θ_b, that is:

$$\bar{\theta}_b : H^{\chi,F_0} = (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E/p)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2/p.$$

Recall that we have an exact sequence (Proposition 15.13):

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{U_{s_0'}})^{\chi'}_{\tau} \to H^{\chi,F_0} \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{U_{s_0'}})^{(\chi')^p}_{\tilde{\tau}} \to 0.$$

As for J_2/p, it's obvious that

$$J_2/p \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'}_{\tau})/p.$$

Using Lemma 16.8, the filtration

$$p\varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'}) \subset pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p} \subset \varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'})$$

induces the following exact sequence:

$$0 \to H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p} \to \varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'})/p \to H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p} \to 0.$$

Another way to see this is that J_2/p is canonically isomorphic with J_1/p, and J_1/p has the usual exact sequence for de Rham cohomology. In other words, we have:

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\tilde{\tau}} \to J_2/p \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\tilde{\tau}} \to 0.$$

Lemma 16.25. *Assume* $p^2-1-m \geq [-mp]$ *and* $i \in \{2, \dots, p-1\}$. *Then* $\bar{\theta}_b$ *induces the following commutative diagram*:

$$\begin{array}{ccccc}
\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_\tau & \longrightarrow & H^{\chi,F_0} & \longrightarrow & \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau} \\
\downarrow{\scriptstyle \bar\theta_{b,1}} & & \downarrow{\scriptstyle \bar\theta_b} & & \downarrow{\scriptstyle \bar\theta_{b,2}} \\
\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau} & \longrightarrow & J_2/p & \longrightarrow & \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}
\end{array}$$

Proof. Let ω be an element of $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ whose mod p reduction lies in $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_\tau$. We need to show that

$$\theta_b(\omega) \in \varpi^{p^2-1-m} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar\tau} \subset J_2.$$

Write $\omega = \omega_\tau + \omega_{\bar\tau}$ as in the decomposition

$$\big(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes_{\mathbb{Q}_p} O_E\big)^{\chi,\mathrm{Gal}(F/F_0)} = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}_\tau \oplus H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{(\chi')^p,\mathrm{Gal}(F/F_0)}_{\bar\tau}.$$

It is clear from the construction in the proof of Proposition 15.13 that ω is in H_1 modulo p (see the notation there). This means that

$$\omega_\tau|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi'}_\tau \quad \text{and} \quad \omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1)^{(\chi')^p}_{\bar\tau} \tag{75}$$

for any odd vertex s and even vertex s'. Then, by Lemma 16.16, we know that the image of $\varpi^{-[-mp]}\omega_\tau$ in $H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{\chi'}_\tau$ actually lies in $pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{\chi'}_\tau$ for any odd vertex s. Similarly the image of $\varpi^{-[-mp]}\omega_{\bar\tau}$ will be in $pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\bar\tau}$ for any even vertex s'. Let $\omega = (\omega_1, \omega_2)$ be the decomposition of ω into $F_1 \oplus F_2$. Then the discussion before indicates that

$$(\varpi^{(p-1)i} \otimes 1)\omega_2 \in \varpi^{p^2-1-m} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar\tau} \subset J_2.$$

It remains to prove that

$$(g_\varphi \otimes \varphi \otimes \mathrm{Id}_E)(\omega_1) \in \varpi^{p^2-1-m} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar\tau} \subset J_2.$$

This follows from Lemma 16.18 (the condition in that lemma is satisfied since we have (75)). □

Now we can state the main theorem of this paper.

Theorem 16.26. *The maps* $\bar\theta_{b,1}, \bar\theta_{b,2}$ *are surjective. More precisely, if we identify* $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}$ *with* $(\mathrm{Sym}^{p-1-i}(O_E/p)^2) \otimes \det^{i+j}$ (see Remark 15.14), *and identify*

$$H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_\tau \simeq H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}$$

with $(\mathrm{Sym}^{i-2}(O_E/p)^2)\otimes\det^{j+1}$, *where the isomorphism is induced by* φ (see Remark 16.9), *then* $\bar\theta_{b,1},\bar\theta_{b,2}$ *are given by*

$$\bar\theta_{b,1}:\sigma_{i-2}(j+1)\to\sigma_{i-2}(j+1),\qquad X\mapsto -bX+((-1)^{j+1}\tau(w_1^i))T(X),$$
$$\bar\theta_{b,2}:\sigma_{p-1-i}(i+j)\to\sigma_{p-1-i}(i+j),\quad X\mapsto X-((-1)^{j+1}\tau(w_1^{-i})b)T(X),$$

where T *is the Hecke operator* (defined in [Breuil 2007]). *See the beginning of the paper for its definition.*

We list some direct consequences of this theorem.

Corollary 16.27. *$\bar\theta_b$ is surjective.*

Corollary 16.28. $\theta_b:(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\to J_2$ *is surjective and we have the following exact sequence*:

$$0\to M(\chi,[1,b])\to(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\to J_2\to 0. \tag{76}$$

Applying the functor $M\mapsto M^d=\mathrm{Hom}^{\mathrm{cont}}_{O_E}(M,E)$ *defined in Section 14, we get*

$$0\to J_2^d\to\big((H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big)^d\to B(\chi,[1,b])\to 0. \tag{77}$$

Notice that the kernel and the middle term of this exact sequence do not depend on b. In fact, the unitary representation J_2^d *is the completion of* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}}$ *with respect to the lattice* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho^o_{\chi^{-1}}$, *where* $\rho^o_{\chi^{-1}}\subset\rho_{\chi^{-1}}$ *is an* O_E*-lattice. It is the universal unitary completion of* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}}$.

Proof. Recall that $B(\chi,[1,b])=(M(\chi,[1,b]))^d$ defined in Section 14. The surjectivity of θ_b follows from the surjectivity of $\bar\theta_b$ and the fact that J_2 and $(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ are p-adically complete. The explicit description of J_2^d follows from the obvious isomorphism between J_2 and J_1, which is clearly isomorphic to $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'^{-1}}_{\tau}$. It is easy to verify that it satisfies the universal property. □

Corollary 16.29. *Under the assumption* $p^2-1-m\ge[-mp]$, $i\in\{2,\dots,p-1\}$, *as a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$,

$$0\to\big\{X\in\sigma_{i-2}(j+1)\mid c(\chi,b)X=T(X)\big\}\to M(\chi,[1,b])/p$$
$$\to\big\{X\in\sigma_{p-1-i}(i+j)\mid X=c(\chi,b)T(X)\big\}\to 0,$$

where $c(\chi,b)=(-1)^{j+1}\tau(w_1^{-i})b\in O_E/p$. *Thus* $B(\chi,[1,b])$ *is nonzero and admissible.*

Remark 16.30. If we assume $p^2-1-m\le[-mp]$, $i\in\{2,\dots,p-1\}$, the same proof will yield a similar exact sequence:

$$0\to\big\{X\in\sigma_{p-1-i}(i+j)\mid X=c(\chi,b)T(X)\big\}\to M(\chi,[1,b])/p$$
$$\to\big\{X\in\sigma_{i-2}(j+1)\mid c(\chi,b)X=T(X)\big\}\to 0.$$

Proof of Theorem 16.26. First we introduce some notation.

Definition 16.31. Let $\omega \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$. Then

(1) $\omega_\tau + \omega_{\bar\tau}$ will be the decomposition of ω in

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi} = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi'}_{\tau} \oplus H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{(\chi')^p}_{\bar\tau}.$$

We will use $\omega_{\tau,s,\xi}$ (resp. $\omega_{\bar\tau,s,\xi}$) to denote the restriction of ω_τ (resp. $\omega_{\bar\tau}$) to $\widetilde{\Sigma_{1,O_F,s,\xi}}$, where s is a vertex of the Bruhat–Tits tree and $\xi^{p-1} = -1$.

(2) $\omega = \omega_1 + \omega_2$ will be its decomposition into $F_1 \oplus F_2$ (see (70)). For an even vertex s' and odd vertex s, we define $\omega_{1,s'}$ and $\omega_{1,s}$ as the images of ω inside $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_{s'}}/F_0)^{\chi'}_{\tau}$ and $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_s}/F_0)^{(\chi')^p}_{\bar\tau}$, respectively. It is clear that $\omega_{2,s'}$, $\omega_{2,s}$ can be defined similarly. We also use $\omega_{1,s,\xi}$ to denote the image of ω in $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_{s,\xi}}/F_0)^{(\chi')^p}_{\bar\tau}$ and define $\omega_{1,s',\xi}$, $\omega_{2,s,\xi}$, and $\omega_{2,s',\xi}$ similarly.

In fact, Proposition 16.1 tells us that for an even vertex s' and odd vertex s,

$$\omega_{1,s'} \in \varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\tau}, \quad \omega_{1,s} \in \varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}_{\bar\tau},$$

and

$$\omega_{2,s'} \in \varpi^{[-mp]} H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\bar\tau}, \quad \omega_{2,s} \in \varpi^{[-mp]} H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{\chi'}_{\tau}).$$

Now we start to prove the surjectivity of

$$\bar\theta_{b,2} : \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau} \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}.$$

Consider $[\mathrm{Id}, x^k y^{p-1-i-k}]$ in (See the beginning of the paper for the notation here)

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}(\mathrm{Sym}^{p-1-i}(O_E/p)^2) \otimes \det{}^{i+j} \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}.$$

Let $\bar\omega \in H^{\chi,F_0}$ be a lift of $[\mathrm{Id}, x^k y^{p-1-i-k}]$ in the first row of Lemma 16.25 and let $\omega \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ be a lift of $\bar\omega$.

It is clear that we may assume $\omega_\tau = 0$. Then our choice of ω implies:

Lemma 16.32. *Under the identification in (27),*

$$\omega_{\bar\tau,s_0',\xi} \equiv \varpi^{[-mp]}\eta^k \tilde e^i \frac{d\tilde e}{\tilde e} \mod pH^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}}, \omega^1)_{\bar\tau}, \tag{78}$$

$$\omega_{\bar\tau,s',\xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s,\xi}}, \omega^1)_{\bar\tau} \quad \textit{for any even vertex } s' \neq s_0'. \tag{79}$$

Using this and Remark 16.17, we know that for any even vertex $s' \neq s_0$,

$$\varpi^{(p-1)i}\omega_{2,s'} \in \varpi^{p^2-1-m} pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\bar\tau},$$

and considered as elements in $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}}) \subset H^1_{\mathrm{dR}}(\overline{U_{s_0'}})$,

$$\varpi^{-[-mp]}\omega_{2,s_0',\xi} \equiv \eta^k \tilde{e}^i \,\frac{d\tilde{e}}{\tilde{e}} \mod pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar{\tau}}.$$

Similarly, Remark 16.19 tells us that for any odd vertex $s \notin A(s_0')$,

$$\varphi(\varpi^{-(p^2-1-m)}\omega_{2,s}) \in pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}_{\bar{\tau}}.$$

Hence it is clear from the definition of θ_b that we have:

Lemma 16.33. $\bar{\theta}_{b,2}([\mathrm{Id}, x^k y^{p-1-i-k}]) = [\mathrm{Id}, v_{s_0'}] + \sum_{s\in A(s_0')} [g_s^{-1}, v_s],$

where g_s is a chosen representative in the coset defined by s. Recall that we identify the set of vertices of the Bruhat–Tits tree with $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \backslash \mathrm{GL}_2(\mathbb{Q}_p)$.

Since $\omega_{1,s_0'} = 0$, it follows from (78) that $v_{s_0'} = x^k y^{p-1-i-k}$.

To determine other terms, we recall some results in Section 7. Recall that s_0 is the vertex that corresponds to $\eta = \tilde{e} = 0$. As a coset, it corresponds to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \cdot w$, where

$$w = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}. \tag{80}$$

Then $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$ is isomorphic to

$$\mathrm{Spf}\, \frac{O_F\Big[\eta, \zeta, \dfrac{1}{\eta^{p-1}-1}, \dfrac{1}{\zeta^{p-1}-1}, \tilde{e}, \tilde{e}'\Big]^\wedge}{\Big(\tilde{e}^{p+1} + v_1 w_1^{-1}\xi \dfrac{\eta^p-\eta}{\zeta^{p-1}-1}, \tilde{e}'^{p+1} + v_1^{-1} w_1 \xi \dfrac{\zeta^p-\zeta}{\eta^{p-1}-1}, \tilde{e}\tilde{e}' - \varpi^{p-1}\xi\Big)}$$

in such a way that the following lemma is true.

Lemma 16.34. *The action of w sends $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$ to*

$$\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi^p}} = \widetilde{\Sigma_{1,O_F,[s_0',s_0],-\xi}}.$$

Explicitly, it is given by (see Corollary 7.7; recall that $\tilde{e} = e/\varpi$):

$$\eta \mapsto -\zeta, \quad \zeta \mapsto -\eta, \quad \tilde{e} \mapsto v_1\tilde{e}', \quad \tilde{e}' \mapsto v_1^{-1}\tilde{e}.$$

Now we come back to our situation. Using Lemma 15.5, the restriction of $\omega_{\bar{\tau}}$ to $\widetilde{\Sigma_{1,O_F,[s_0',s_0]}}$ can be written as

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}} = \varpi^{[-mp]} f(\eta)\tilde{e}^i \,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} g(\zeta)\tilde{e}'^{p+1-i}\,\frac{d\tilde{e}'}{\tilde{e}'},$$

where $f(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$, $g(\zeta) \in O_{F_0}[\zeta, 1/(\zeta^{p-1}-1)]\widehat{\ }$. Since ω is in the $(\chi')^p$-isotypic component, we must have (using results in Section 9):

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],-\xi}}} = \varpi^{[-mp]} f(\eta)\tilde{e}^i(-1)^{-(i+j)}\frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} g(\zeta)\tilde{e}'^{p+1-i}(-1)^{-(j+1)}\frac{d\tilde{e}'}{\tilde{e}'}.$$

By our construction of ω,

$$\omega_{\bar{\tau},s_0',\xi} = \omega_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}} \equiv \varpi^{[-mp]}\eta^k\tilde{e}^i\frac{d\tilde{e}}{\tilde{e}} \mod pH^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}}, \omega^1).$$

Hence:

Lemma 16.35. $$f(\eta) \equiv \eta^k \mod pO_{F_0}\Big[\eta, \frac{1}{\eta^{p-1}-1}\Big]\widehat{\ }.$$

I would like do all the computations on the central component, so we define

$$h_{s_0} = (w^{-1})^*(\omega_{\bar{\tau}}) \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)_\tau. \tag{81}$$

Then (notice that w maps the $(-\xi)$-component to the ξ-component) a direct consequence of Lemma 16.34 is:

Lemma 16.36. *$h_{s_0}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}} = (w^{-1})^*(\omega_{\bar{\tau},s_0,-\xi})$ has the form*

$$\varpi^{p^2-1-m}\tilde{g}(-\eta)\tilde{e}^{p+1-i}(-v_1^{-1})^{p+1-i}(-1)^{j+1}\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}\tilde{f}(-\zeta)\tilde{e}'^i(-v_1)^i(-1)^{i+j}\frac{d\tilde{e}'}{\tilde{e}'},$$

where $\tilde{f}(-\zeta) = \widetilde{\mathrm{Fr}}(f(-\zeta))$, applying Frobenius operator on the coefficients, and $\tilde{g}(-\eta)$ is defined similarly.

In fact, by Lemma 16.35, we know that

$$\tilde{f}(-\zeta) \equiv (-\zeta)^k \mod pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]\widehat{\ }.$$

We need to compute the cohomology class of $\varpi^{-(p^2-1-m)}h_{s_0}$ in $H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_\tau$ (modulo p). Following the strategy in the proof of Proposition 16.1 (see the notation there), we may use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent h_{s_0}. Also recall that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$ (all considered as elements in $\varpi^{p^2-1-m}H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$). By definition of $\bar{\theta}_{b,2}$, we only need to know the image of $\varphi(\varpi^{-(p^2-1-m)}h_{s_0})$ in

$$H^1_{\mathrm{dR}}(\overline{U_{s_0'}})_{\bar{\tau}} = H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_{\bar{\tau}}/pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_{\bar{\tau}}.$$

Hence Lemma 16.8 tells us that we only need to know the image of $\varpi^{-(p^2-1-m)}h_{s_0}$ inside

$$H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})_\tau.$$

In other words, we are only concerned with the mod p properties of f_s.

Since w interchanges s_0' and s_0, $w(s) \neq s_0'$ for any odd vertex $s \neq s_0$. Then it follows from Lemma 16.32 that for any $s \in A(s_0')$, $s \neq s_0$,

$$h_{s_0}|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)_\tau.$$

Therefore the proof of Lemma 16.18 implies that for any $s \in A(s_0')$, $s \neq s_0$,

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})_\tau.$$

Moreover, Lemma 16.6 tells us that (compare Lemma 16.36 with (59) and notice that $g(\zeta)$ there is $\tilde{f}(-\zeta)(-v_1)^i(-1)^{i+j}$ here)

$$\varpi^{-(p^2-1-m)} f_{s_0} \equiv \frac{\xi^i \tilde{f}(0)(-v_1)^i(-1)^{i+j} y^{-i}}{i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})_\tau.$$

Recall that the identification of $\overline{U_{s_0',\xi}}$ and the special fiber of $\widehat{D_{0,O_{F_0},\xi}}$ is given by

$$x \mapsto \eta, \quad y \mapsto \tilde{e}.$$

Lemma 16.37. *The image of* $\varpi^{-(p^2-1-m)} h_{s_0}$ *in* $H^1(\overline{U_{s_0',\xi}}, \mathcal{O}_{\overline{U_{s_0',\xi}}})_\tau$ *is the following* 1*-cocycle* $\{f'_{s,s''}\}$ *if we use the open covering* $\{V_{s,\xi}\}_s$:

$$f'_{s,s''} = f'_{s''} - f'_s, \quad \text{where } f'_{s_0} = \xi^i \tilde{f}(0)(-v_1)^i(-1)^{i+j} i^{-1} \tilde{e}^{-i}, \quad f'_s = 0 \text{ for } s \neq s_0.$$

Now we want to write this cohomology class as a 1-cocycle $f_{0,\infty}$ of the open covering $\{V_0, V_\infty\}$ (Definition 16.11). But this is already computed in Example 16.15:

Lemma 16.38. *The image of* $\varpi^{-(p^2-1-m)} h_{s_0}$ *in* $H^1(\overline{U_{s_0',\xi}}, \mathcal{O}_{\overline{U_{s_0',\xi}}})_\tau$ *is the following* 1*-cocycle* $\{f_{0,\infty}\}$ *if we use the open covering* $\{V_0, V_\infty\}$:

$$f_{0,\infty} = \tilde{f}(0)(-1)^j i^{-1} w_1 (v_1 \xi)^{i-1} \frac{\tilde{e}^{p+1-i}}{\eta}.$$

Thanks to Lemma 16.13, a simple computation shows:

Lemma 16.39. *The image of* $\varphi(\varpi^{-(p^2-1-m)} h_{s_0})$ *in* $H^1_{\mathrm{dR}}(\overline{U_{s_0',\xi}})_{\bar\tau}$ *is*

$$\varphi\Big(\tilde{f}(0)(-1)^j i^{-1} w_1 (v_1 \xi)^{i-1} \frac{\tilde{e}^{p+1-i}}{\eta}\Big)$$
$$= f(0)(-1)^{i+j+1} w_1^i \eta^{p-1-i} \tilde{e}^i \frac{d\tilde{e}}{\tilde{e}} \in H^0(\overline{U_{s_0',\xi}}, \Omega^1_{\overline{U_{s_0',\xi}}})_{\bar\tau}.$$

Recall that in the isomorphism

$$H^0(\overline{U_{s_0'}}, \Omega^1_{U_{s_0'}})^{(\chi')^p} \to (\mathrm{Sym}^{p-1-i}\, \mathbb{F}_{p^2}^2) \otimes \det{}^{i+j},$$

$\eta^{p-1-i}\tilde{e}^i\, d\tilde{e}/\tilde{e}$ is identified with x^{p-1-i}. By Lemma 16.35, $f(0) = 1$ if $k = 0$ and $f(0) = 0$ otherwise. Hence, considering the definition of $\bar{\theta}_{b,2}$, Lemma 16.39 implies:

Lemma 16.40. $$[w, v_{[w^{-1}]}] = \begin{cases} [w, (-1)^{j+1}\tau(w_1^{-i})bx^{p-1-i}] & \textit{if } k = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

Now we compute the same term of $T([\mathrm{Id}, x^k y^{p-1-i-k}])$ (see the beginning of the paper for the notation here):

$$[w, \varphi_r(w^{-1})(x^k y^{p-1-i-k})] = \left[w, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(x^k y^{p-1-i-k})\right],$$

which is nonzero if and only if $k = 0$. When $k = 0$,

$$\begin{aligned} [w, \varphi_r(w^{-1})(y^{p-1-i})] &= \left[w, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(y^{p-1-i})\right] \\ &= \left[w, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(y^{p-1-i})\right] = [w, x^{p-1-i}]. \end{aligned}$$

Lemma 16.41. $$T([\mathrm{Id}, x^k y^{p-1-i-k}]) = \begin{cases} [w, x^{p-1-i}] + \textit{other terms}, & k = 0, \\ [w, 0] + \textit{other terms}, & k \neq 0. \end{cases}$$

Since $\mathrm{GL}_2(\mathbb{Z}_p)$ acts transitively on $A(s_0')$, the above computation implies

$$\begin{aligned} &\bar{\theta}_{b,2}([\mathrm{Id}, x^k y^{p-1-i-k}]) \\ &\qquad = [\mathrm{Id}, x^k y^{p-1-i-k}] - ((-1)^{j+1}\tau(w_1^{-i})b)T([\mathrm{Id}, x^k y^{p-1-i-k}]). \end{aligned}$$

Notice that $\bar{\theta}_{b,2}$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. Therefore,

$$\bar{\theta}_{b,2} = \mathrm{Id} - ((-1)^{j+1}\tau(w_1^{-i})b)T.$$

As for $\bar{\theta}_{b,1}$, the computation is almost the same. I omit the details here. □

17. Computation of $M(\chi, [1, b])/p$, II: $i = 1, p$

In this section, we deal with the case $i = 1, p$. We keep the notation used in the last two sections. Now Proposition 15.13 becomes:

Proposition 17.1. (1) *If $i = 1$, there exists a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism*

$$H^{\chi, F_0} \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})_{\tau}^{(\chi')^p}.$$

(2) *If $i = p$,*

$$H^{\chi, F_0} \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})_{\tau}^{\chi'}.$$

Proof. Notice that when $i = 1$, $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'} = 0$. So everything follows from Proposition 15.13 and Remark 15.15. □

In fact, we can see the above isomorphisms in the following way. If $i=p$, for any $\bar{h}\in H^{\chi,F_0}$, the restriction of $\bar{h}_\tau$ (resp. $\bar{h}_{\bar{\tau}}$) to $\widetilde{\Sigma_{1,O_F,s'}}$ (resp. $\widetilde{\Sigma_{1,O_F,s}}$) for an odd (resp. even) vertex s' (resp. s) corresponds to a holomorphic differential form on $\overline{U_{s'}}$ (resp. $\overline{U_s}$) under the isomorphism in Lemma 15.4. Hence we can define the above map. The case $i=1$ is similar.

As we promised earlier, we have:

Lemma 17.2. *Assume $i=1$ or p. Then*

$$\theta_b\big((H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big)\subset J_2.$$

Proof. See (73), (74) for the definitions of J_1, J_2. First we assume $i=p$. Then by Lemma 16.8, we have

$$\varphi(H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'})=pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}.$$

Thus we may identify $J_2=(\varpi^{p^2-1-m}\otimes\varphi\otimes\mathrm{Id}_{O_E})(J_1)$ with (recall F_2 is an $F\otimes_{\mathbb{Q}_p}E$-module)

$$(\varpi^{p^2-1-m}\otimes 1)\Bigg(\prod_{s'\text{ even}}pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\bar{\tau}}\oplus\prod_{s\text{ odd}}pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\tau}\Bigg)\subset F_2.$$

Let $\omega\in(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ and $\omega=\omega_1+\omega_2$ be the decomposition of ω into $F_1\oplus F_2$. By definition $\varphi(\omega_1)\in J_2$. Since

$$[-mp]+i(p-1)=(p^2-1)+p^2-1-m$$

in this case, Proposition 16.1 implies that $\varpi^{i(p-1)}\omega_2\in J_2$. Hence $\theta_b(\omega)\in J_2$.

Now assume $i=1$; then $\varphi(H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'})=pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}$. Hence

$$J_2=\varpi^{p^2-1-m}\Bigg(\prod_{s'\text{ even}}H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\bar{\tau}}\oplus\prod_{s\text{ odd}}H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\tau}\Bigg).$$

So the lemma follows directly from Proposition 16.1. □

Let $\bar{\theta}_b:H^{\chi,F_0}\to J_2/p$ be the mod p map of θ_b. It is clear that

$$J_2/p\simeq\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^1_{\mathrm{dR}}(\overline{U_{s_0'}})^{(\chi')^p}_{\bar{\tau}}\simeq\begin{cases}\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}, & i=p,\\ \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}, & i=1.\end{cases}$$

We can now state our main results of this section.

Theorem 17.3. *$\bar{\theta}_b$ is surjective. More precisely*:

(1) *Assume $i=p$. If we consider the following isomorphism induced by φ* (Remark 16.9):

$$H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau}\simeq H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}},$$

and use Remark 15.14 to make the identification

$$H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau} \simeq (\mathrm{Sym}^{p-2}(O_E/p)^2) \otimes \det{}^{j+1},$$

then $\bar{\theta}_b$ *is given by*

$$\begin{aligned}\bar{\theta}_b : \sigma_{p-2}(j+1) &\to \sigma_{p-2}(j+1),\\ X &\mapsto -bX + (-1)^{j+1}\tau(w_1^p)T(X) - bT^2(X).\end{aligned}$$

(2) *Assume* $i = 1$. *If we use Remark 15.14 to make the identification*

$$H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}} \simeq (\mathrm{Sym}^{p-2}(O_E/p)^2) \otimes \det{}^{j+1},$$

then $\bar{\theta}_b$ *is given by*

$$\begin{aligned}\bar{\theta}_b : \sigma_{p-2}(j+1) &\to \sigma_{p-2}(j+1)\\ X &\mapsto X + (-1)^{j+1}b\tau(w_1^{-1})T(X) + T^2(X).\end{aligned}$$

Just like the previous section, we list some corollaries first.

Corollary 17.4. $\bar{\theta}_b$ *is surjective.*

Corollary 17.5. $\theta_b : (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2$ *is surjective and we have the following exact sequence*:

$$0 \to M(\chi, [1, b]) \to (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2 \to 0. \tag{82}$$

Applying the functor $M \mapsto M^d = \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M, E)$, *we get*

$$0 \to J_2^d \to \big((H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big)^d \to B(\chi, [1, b]) \to 0. \tag{83}$$

The kernel and the middle term of this exact sequence are independent of b. The kernel J_2^d *is the completion of* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}}$ *with respect to the lattice* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho^o_{\chi^{-1}}$, *where* $\rho^o_{\chi^{-1}} \subset \rho_{\chi^{-1}}$ *is an* O_E*-lattice. It is the universal unitary completion of* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}}$.

Corollary 17.6. *Assume* $i = p$. *As a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$,

$$M(\chi, [1, b])/p \simeq \big\{X \in \sigma_{p-2}(j+1) \;\big|\; -bX + (-1)^{j+1}\tau(w_1^p)T(X) - bT^2(X) = 0\big\}.$$

When $i = 1$,

$$M(\chi, [1, b])/p \simeq \big\{X \in \sigma_{p-2}(j+1) \;\big|\; X + (-1)^{j+1}b\tau(w_1^{-1})T(X) + T^2(X) = 0\big\}.$$

Thus in any case, $B(\chi, [1, b])$ *is nonzero and admissible.*

Proof of Theorem 17.3. We only deal with the case $i = 1$. The case where $i = p$ can be treated in almost the same way.

Consider $[\mathrm{Id}, x^k y^{p-2-k}]$ as an element in

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{Sym}^{p-2}(O_E/p)^2)\otimes \det^{j+1} \simeq \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U}_{s_0'}, \Omega^1_{\overline{U_{s_0'}}})_{\bar\tau}^{(\chi')^p}.$$

Let $\omega \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ be a lift of $[\mathrm{Id}, x^k y^{p-2-k}]$. As before, we may assume $\omega_\tau = 0$. It is clear from our construction that for any even vertex $s' \neq s_0'$,

$$\omega_{\bar\tau,s',\xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s',\xi}}, \omega^1)_{\bar\tau}. \tag{84}$$

Hence for any odd vertex $s \notin A(s_0')$,

$$\omega_{\bar\tau,s,\xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s,\xi}}, \omega^1)_{\bar\tau}. \tag{85}$$

This follows from Remark 15.9 and the fact $H^0(\overline{U}_s, \Omega^1_{\overline{U_s}})^{(\chi')^p} = 0$. Thus using Lemma 16.18 and Remark 16.19, we know that $\bar\theta_b([\mathrm{Id}, x^k y^{p-2-k}])$ must be of the following form:

Lemma 17.7. $\bar\theta_b([\mathrm{Id}, x^k y^{p-2-k}]) = [\mathrm{Id}, u_{s_0'}] + \sum_{s\in A(s_0')} [g_s^{-1}, u_s] + \sum_{s'\in A^2(s_0')} [g_{s'}^{-1}, u_{s'}],$

where $A^2(s_0') = \{s' \in A(s) \mid s \in A(s_0'), s' \neq s_0'\}$.

First we compute $[\mathrm{Id}, u_{s_0'}]$. It suffices to compute the image of $\varpi^{-[-pm]}\omega_{\bar\tau}$ in

$$H^1_{\mathrm{dR}}(\overline{U_{s_0'}})_{\bar\tau}^{(\chi')^p} = H^0(\overline{U}_{s_0'}, \Omega^1_{\overline{U_{s_0'}}})_{\bar\tau}^{(\chi')^p}.$$

As before, on $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$, we can write (use a variant of Lemma 15.5 and notice that $\omega_{\bar\tau}$ is in the $(\chi')^p$-isotypic component)

$$\omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}} = \varpi^{[-mp]} f(\eta)\tilde e \frac{d\tilde e}{\tilde e} + \varpi^{p^2-1-m} g(\zeta)\tilde e'^p \frac{d\tilde e'}{\tilde e'},$$

where $f(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]^\wedge$, $g(\zeta) \in O_{F_0}[\zeta, 1/(\zeta^{p-1}-1)]^\wedge$. As usual, we identify $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$ with

$$\mathrm{Spf}\,\frac{O_F\Big[\eta,\zeta,\frac{1}{\eta^{p-1}-1},\frac{1}{\zeta^{p-1}-1},\tilde e,\tilde e'\Big]^\wedge}{\Big(\tilde e^{p+1}+v_1w_1^{-1}\xi\frac{\eta^p-\eta}{\zeta^{p-1}-1},\ \tilde e'^{p+1}+v_1^{-1}w_1\xi\frac{\zeta^p-\zeta}{\eta^{p-1}-1},\ \tilde e\tilde e'-\varpi^{p-1}\xi\Big)}.$$

Our choice of ω implies:

Lemma 17.8. $$f(\eta) \equiv \eta^k \mod pO_{F_0}\Big[\eta, \frac{1}{\eta^{p-1}-1}\Big]^\wedge.$$

Now restricted to $\widetilde{\Sigma_{1,O_F,s_0,\xi}}$,

$$\omega_{\tau,s_0,\xi} = \omega_\tau|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}} = \varpi^{p^2-1-m}\Big(-\xi\tilde e'^{-2} f\Big(\frac{p}{\zeta}\Big)\,d\tilde e' + g(\zeta)\tilde e'^{p-1}\,d\tilde e'\Big).$$

By (84), we know that for any $s' \in A(s_0)$ that is not s_0',

$$\omega_{\bar\tau,s',\xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s',\xi}}, \omega^1)_{\bar\tau}.$$

Then Remark 15.9 implies that the reduction of $\varpi^{-(p^2-1-m)}\omega_{\bar{\tau},s_0,\xi}$ modulo p, as a meromorphic differential form on $\overline{U_{s_0,\xi}}$, can only have poles at $\zeta=\tilde{e}'=0$. Here we identify $\overline{U_{s_0,\xi}}$ with the projective curve in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ defined by

$$\tilde{e}'^{p+1}=v^{-1}w_1\xi(\zeta^p-\zeta).$$

Therefore the only possible pole must come from $-\xi f(p/\zeta)\,d\tilde{e}'/\tilde{e}'^2$. Notice that by Lemma 17.8, this term is nonzero modulo p if and only if $k=0$. Thus when $k\neq 0$, the reduction of $\omega_{\bar{\tau},s_0,\xi}$ is a holomorphic differential form on $\overline{U_{s_0,\xi}}$. But

$$H^0(\overline{U_{s_0}},\Omega^1_{\overline{U_{s_0}}})^{(\chi')^p}=0,$$

hence $g(\zeta)$ has to be zero modulo p in this case. Therefore we have proved:

Lemma 17.9. *If $k\neq 0$, then $g(\zeta)\in pO_{F_0}[\zeta,1/(\zeta^{p-1}-1)]^\wedge$, and*

$$\omega_{\bar{\tau},s_0,\xi}\in pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}},\omega^1)_{\bar{\tau}}.$$

When $k=0$. Rewrite

$$\begin{aligned}\tilde{e}'^{-2}f\Big(\frac{p}{\zeta}\Big)&\equiv\frac{1}{\tilde{e}'^2}=\frac{\tilde{e}'^{p-1}}{\tilde{e}'^{p+1}}\equiv\frac{\tilde{e}'^{p-1}}{v_1^{-1}w_1\xi(\zeta^p-\zeta)}\\&\equiv-\frac{\tilde{e}'^{p-1}}{v_1^{-1}w_1\xi\zeta}+\frac{\tilde{e}'^{p-1}\zeta^{p-2}}{v_1^{-1}w_1\xi(\zeta^{p-1}-1)}\\&\qquad\Big(\mathrm{mod}\ pO_{F_0}\Big[\tilde{e}',\zeta,\frac{1}{\zeta^p-\zeta}\Big]^\wedge\Big/\Big(\tilde{e}'^{p+1}+v_1^{-1}w_1\xi\frac{\zeta^p-\zeta}{(p/\zeta)^{p-1}-1}\Big)\Big).\end{aligned}$$

Thus

$$\begin{aligned}&\varpi^{-(p^2-1-m)}\omega_{\bar{\tau},s_0,\xi}\\&\equiv\frac{\tilde{e}'^{p-1}\,d\tilde{e}'}{v_1^{-1}w_1\zeta}+\Big(-\frac{\tilde{e}'^{p-1}\zeta^{p-2}}{v_1^{-1}w_1(\zeta^{p-1}-1)}+g(\zeta)\tilde{e}'^{p-1}\Big)\,d\tilde{e}'\quad\mathrm{mod}\ pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}},\omega^1)_{\bar{\tau}}.\end{aligned}$$

Notice that the first term, $\tilde{e}'^{p-1}\,d\tilde{e}'/(v_1^{-1}w_1\zeta)$, only has a pole at $\tilde{e}'=\zeta=0$ and the second term is holomorphic at this point. Therefore the second term (modulo p) is a holomorphic differential form on $\overline{U_{s_0}}$, which has to be zero since it is in $H^0(\overline{U_{s_0}},\Omega^1_{\overline{U_{s_0}}})^{(\chi')^p}=0$. Hence:

Lemma 17.10. *When $k=0$,*

$$\begin{aligned}\omega_{\bar{\tau},s_0,\xi}&\equiv\varpi^{p^2-1-m}\frac{\tilde{e}'^{p-1}d\tilde{e}'}{v_1^{-1}w_1\zeta}\quad\mathrm{mod}\ pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}},\omega^1)_{\bar{\tau}}\\g(\zeta)&\equiv\frac{\zeta^{p-2}}{v_1^{-1}w_1(\zeta^{p-1}-1)}\quad\mathrm{mod}\ pO_{F_0}\Big[\zeta,\frac{1}{\zeta^{p-1}-1}\Big]^\wedge.\end{aligned}$$

A direct corollary of Lemma 17.9 and Lemma 17.10 is:

Lemma 17.11. *For any k, we always have $g(0) \in pO_{F_0}$.*

Now we try to compute the image of ω inside $\varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_{\bar\tau} \pmod p$. As we did in the previous section, we can use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent this cohomology class. Moreover, there exists $\{f_s\}_{s\in A(s_0')}$, where $f_s \in \varpi^{p^2-1-m} H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$ such that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$ and $\omega_s = \omega - df_s$. See the proof of Proposition 16.1 for the notation here.

From Lemma 17.11, we know that $g(0)$ is divisible by p. Therefore Lemma 16.6 tells us that

$$\varpi^{-(p^2-1-m)} f_{s_0} \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

Using the action of $\mathrm{GL}_2(\mathbb{Z}_p)$, it is easy to see that the above inclusion is also true for other vertex $s \in A(s_0')$. Hence all f_{s_1,s_2} are divisible by p and all ω_s are congruent to ω modulo p. This certainly implies that the image of $\varpi^{-(p^2-1-m)}\omega$ in $H^1_{\mathrm{dR}}(\overline{U_{s_0'}})_{\bar\tau}^{(\chi')^p} = H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})_{\bar\tau}^{(\chi')^p}$ is

$$\varpi^{-(p^2-1-m)}\omega \equiv \eta^k \, d\tilde{e},$$

considered as a differential form using Lemma 15.4. In other words:

Lemma 17.12. $$u_{s_0'} = x^k y^{p-2-k}.$$

Next we compute u_{s_0}. As we did in the previous section, we define

$$h'_{s_0} = (w_1^{-1})^*(\omega_{\bar\tau}) \in H^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}}, \omega^1)^{\chi'}_{\tau}. \tag{86}$$

Hence Lemma 16.36 tells us that

$$\begin{aligned} &h'_{s_0}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}} \\ &\quad = \varpi^{p^2-1-m}\tilde{g}(-\eta)\tilde{e}^p(-v_1^{-1})^p(-1)^{j+1}\frac{d\tilde{e}}{\tilde{e}} - \varpi^{[-mp]}\tilde{f}(-\zeta)\tilde{e}'v_1(-1)^{j+1}\frac{d\tilde{e}'}{\tilde{e}'}, \end{aligned}$$

where $\tilde{f}(-\zeta) = \widetilde{\mathrm{Fr}}(f(-\zeta))$, and $\tilde{g}(-\eta)$ is defined similarly.

We need to compute the image of $\varpi^{-(p^2-1-m)}h'_{s_0}$ in

$$H^1_{\mathrm{dR}}(\overline{U_{s_0'}})^{\chi'}_{\tau} = H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}_{\tau}.$$

Now the argument becomes exactly the same as the proof of Theorem 16.26: By abuse of notation, we use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent the cohomology class of $h'_{s_0} \in \varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'}_{\tau}$. Also there exists $\{f_s\}$ such that $f_{s_2,s_1} = f_{s_2} - f_{s_1}$. By (84) and Lemma 16.18, we know that all f_s are divisible by p for $s \neq s_0$. As for f_{s_0}, we can compute it using Lemma 16.6 and Lemma 17.8. We omit all the details here but just refer to the arguments from Lemma 16.36 to Lemma 16.38 in the proof of Theorem 16.26.

Lemma 17.13. $u_{s_0} = u_{[w^{-1}]} = \begin{cases} (-1)^{j+1} b\tau(w_1^{-1}) x^{p-2}, & k=0, \\ 0, & k \neq 0. \end{cases}$

Finally we come to the case $s' \in A^2(s_0')$, which does not exist when $i \in \{2, \dots, p-1\}$.

Definition 17.14. We define $s_0'' \in A(s_0)$ as the vertex that corresponds to the coset

$$\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}^{-1} \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \backslash \mathrm{GL}_2(\mathbb{Q}_p).$$

When $k \neq 0$, Lemma 17.9 tells us that $\omega_{\bar\tau, s_0, \xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}}, \omega^1)_{\bar\tau}$. Therefore by Lemma 16.18, the cohomology class of $\varpi^{-[-mp]}\omega_{\bar\tau}$ in $H^1_{\mathrm{crys}}(\overline{U_{s_0''}}/O_{F_0})_{\bar\tau}$ is inside $pH^1_{\mathrm{crys}}(\overline{U_{s_0''}}/O_{F_0})_{\bar\tau}$.

Lemma 17.15. *When $k \neq 0$, $u_{s_0''} = 0$.*

So we assume $k = 0$ from now on.

Notice that

$$\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}^{-1}.$$

Hence the (right) action of $\left(\begin{smallmatrix} 1 & 1/p \\ 0 & 1 \end{smallmatrix}\right)$ fixes the vertex s_0 and sends s_0'' to s_0'. This clearly implies that $\left(\begin{smallmatrix} 1 & 1/p \\ 0 & 1 \end{smallmatrix}\right)$ sends the edge $[s_0'', s_0]$ to $[s_0', s_0]$. In other words, we get an isomorphism

$$\Psi_{s_0', s_0''} : \widetilde{\Sigma_{1,O_F,[s_0'',s_0]}} \xrightarrow{\sim} \widetilde{\Sigma_{1,O_F,[s_0',s_0]}}.$$

Restrict $\Psi_{s_0',s_0''}$ to $\widetilde{\Sigma_{1,O_F,s_0}}$, we thus get an automorphism of $\widetilde{\Sigma_{1,O_F,s_0}}$. As usual, we identify $\widetilde{\Sigma_{1,O_F,s_0,\xi}}$ with

$$\mathrm{Spf}\, O_F\Big[\zeta, \tilde e', \frac{1}{\zeta^p - \zeta}\Big] \Big/ \Big(\tilde e'^{p+1} + v_1^{-1} w_1 \xi \frac{\zeta^p - \zeta}{(p/\zeta)^{p-1} - 1}\Big)^{\widehat{}}.$$

To see $\Psi_{s_0',s_0''}$ explicitly on it, we use $\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ to send $\widetilde{\Sigma_{1,O_F,s_0,-\xi}}$ to $\widetilde{\Sigma_{1,O_F,s_0',\xi}}$ and then apply the results in Section 9. An easy computation shows:

Lemma 17.16. $\Psi_{s_0',s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$ *is*

$$\zeta \mapsto \zeta + 1, \quad \tilde e' \mapsto \tilde e' \bmod pO_F\Big[\zeta, \tilde e', \frac{1}{\zeta^p - \zeta}\Big] \Big/ \Big(\tilde e'^{p+1} + v_1^{-1} w_1 \xi \frac{\zeta^p - \zeta}{(p/\zeta)^{p-1} - 1}\Big)^{\widehat{}}.$$

Now consider

$$h_{s_0''} \stackrel{\text{def}}{=} \left(\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}^{-1}\right)^*(\omega_{\bar\tau}) \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)_{\bar\tau}^{(\chi')^p}. \tag{87}$$

On $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$, it can be written as

$$h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}} = \varpi^{[-mp]} f_1(\eta)\tilde e \frac{d\tilde e}{\tilde e} + \varpi^{p^2-1-m} g_1(\zeta)\tilde e'^p \frac{d\tilde e'}{\tilde e'}.$$

By our construction (see (84)), $\omega_{\bar\tau,s_0'',\xi}\in pH^0(\widetilde{\Sigma_{1,O_F,s_0'',\xi}},\omega^1)_{\bar\tau}$. Hence,

$$h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}}\in pH^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}},\omega^1)_{\bar\tau}.$$

This implies:

Lemma 17.17. $$f_1(\eta)\in pO_{F_0}\Big[\eta,\frac{1}{\eta^{p-1}-1}\Big]^{\widehat{}}.$$

Restrict $h_{s_0''}$ to $\widetilde{\Sigma_{1,O_F,s_0,\xi}}$. Then we have

$$h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}\equiv\varpi^{p^2-1-m}g_1(\zeta)\tilde e'^p\frac{d\tilde e'}{\tilde e'}\mod pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}},\omega^1)_{\bar\tau}.$$

By definition,

$$h_{s_0''}=\left(\begin{pmatrix}1&1/p\\0&1\end{pmatrix}^{-1}\right)^*(\omega_{\bar\tau}).$$

Hence $\Psi_{s_0',s_0''}$ maps $\omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$ to $h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$. Thanks to Lemma 17.16, we can write down this map explicitly (after reducing modulo p). Recall that an explicit expression of $\omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$ is given in Lemma 17.10. Thus a simple computation gives:

Lemma 17.18. *When $k=0$,*

$$g_1(\zeta)\equiv\frac{1}{v_1^{-1}w_1(\zeta-1)}\mod O_{F_0}\Big[\zeta,\frac{1}{\zeta^{p-1}-1}\Big]^{\widehat{}}.$$

With this lemma in hand, we can compute the image of $\varpi^{-(p^2-1-m)}h_{s_0''}$ in $H^1_{\mathrm{dR}}(\overline{U_{s_0'}})^{(\chi')^p}_{\bar\tau}=H^0(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}$. We note that $h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s}}}\in H^0(\widetilde{\Sigma_{1,O_F,s}},\omega^1)_{\bar\tau}$ for any $s\in A(s_0')$ that is not s_0. So the computation is exactly the same as the case where we computed u_{s_0}. I omit the details here. The final result is:

Lemma 17.19. *When $k=0$, $u_{s_0''}=-x^{p-2}$.*

We need to compute the same term of $T^2([\mathrm{Id},x^ky^{p-2-k}])$. Assume $k=0$. When $k\neq0$, it's easy to see this term is zero. We already computed that

$$T([\mathrm{Id},y^{p-2}])=\left[\begin{pmatrix}0&-1\\p&0\end{pmatrix},x^{p-2}\right]+\text{other terms}.$$

Since

$$\begin{pmatrix}1&1/p\\0&1\end{pmatrix}=\begin{pmatrix}0&-1\\p&0\end{pmatrix}\begin{pmatrix}0&p^{-1}\\-1&-p^{-1}\end{pmatrix},$$

by definition we have,

$$T\left(\left[\begin{pmatrix}0&-1\\p&0\end{pmatrix},x^{p-2}\right]\right)=\left[\begin{pmatrix}1&1/p\\0&1\end{pmatrix},\varphi_{p-2}\left(\begin{pmatrix}0&p^{-1}\\-1&p^{-1}\end{pmatrix}^{-1}\right)(x^{p-2})\right].$$

Write

$$\begin{pmatrix} 0 & p^{-1} \\ -1 & -p^{-1} \end{pmatrix}^{-1} = p\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \varphi_{p-2}\left(\begin{pmatrix} 0 & p^{-1} \\ -1 & p^{-1} \end{pmatrix}^{-1}\right)(x^{p-2}) &= \varphi_{p-2}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)((x+y)^{p-2}) \\ &= \varphi_{p-2}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)(y^{p-2}) \\ &= -x^{p-2}. \end{aligned}$$

Hence:

Lemma 17.20. $T^2([\mathrm{Id}, y^{p-2}]) = \begin{cases} \left[\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}, -x^{p-2}\right] + \textit{other terms}, & k = 0, \\ \left[\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}, 0\right] + \textit{other terms}, & k \neq 0. \end{cases}$

Combining the results of Lemmas 17.12, 17.13, 17.15, and 17.19 together with Lemmas 16.41 and 17.20:

$$\bar{\theta}_b(X) = X + (-1)^{j+1} b\tau(w_1^{-1})T(X) + T^2(X). \qquad \square$$

18. A conjecture on $B(\chi, [1, b])$

In the previous two sections, we have proved the admissibility of $B(\chi, [1, b])$ and explicitly compute its residue representation (see Corollaries 16.29 and 17.6, and Remark 16.30). Recall that for each data $(\chi, [1, b])$, we associate a two dimensional Galois representation $V_{\chi,[1,b]}$ (Proposition 13.2) and prove that $B(\chi, [1, b])$ is a completion of the smooth representation $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ with respect to the lattice $\Theta(\chi, [1, b])$ (Proposition 14.9). Up to some twist, this smooth representation, via the classical local Langlands correspondence for GL_2, corresponds to the Weil–Deligne representation associated to $V_{\chi,[1,b]}^\vee$ in [Fontaine 1994]. It is natural to make the following:

Conjecture 18.1. *Up to a twist of some character*, $B(\chi, [1, b])$ *is isomorphic to* $\Pi(V_{\chi,[1,b]}^\vee)$ *as a Banach space representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$, *where* $\Pi(V_{\chi,[1,b]}^\vee)$ *is defined via the* p*-adic local Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Colmez 2010; Colmez et al. 2014]).

The evidence for this conjecture is that we can verify it modulo ϖ_E, the uniformizer of E, namely:

Theorem 18.2. *Via the semisimple modulo-*p *Langlands correspondence defined by Breuil* ([2003] or [2007]), *up to a twist by some character and semisimplification,*

$\Theta(\chi, [1, b])/\varpi_E$ corresponds to the residue representation of $V^\vee_{\chi,[1,b]}$ with respect to some lattice inside.

Proof. The residue representation of $V_{\chi,[1,b]}$ is computed in Theorem 6.12 of [Savitt 2005]. I almost follow his notation except that his w there is my u_x here. $\Theta(\chi, [1, b])/\varpi_E\Theta(\chi, [1, b])$ is computed in Corollaries 16.29 and 17.6, and Remark 16.30. A direct computation shows that they indeed match via Breuil's dictionary. I omit the details here. □

Remark 18.3. There is some duality involved in the conjecture. The reason is that we are using de Rham cohomology rather than its dual, de Rham cohomology with compact support.

Remark 18.4. It seems that this conjecture follows from the work of Dospinescu and Le Bras [2015] by taking the universal unitary completion in their construction. The interested reader is referred to their paper.

List of symbols

Section 2	Ω; $\widehat{\Omega}$; $\widehat{\Omega}_e$; $\widehat{\Omega}_s$; s_0'; X_n; $\mathcal{X}_n$; Σ_n; T_0; T_1
Section 3	χ_1; χ_2; $\mathcal{L}_i$; c_i; d_i
Section 5	λ_1; $\lambda_{\mathcal{L}_1}$; $\widetilde{\lambda_{\mathcal{L}_1}}$; $\widehat{\Sigma_1^{\mathrm{nr}}}$ (Proposition 5.7)
Section 7	$\widehat{\Sigma^{\mathrm{nr}}_{1,e}}$; $\widehat{\Sigma^{\mathrm{nr}}_{1,s}}$; w_1; v_1, $\widehat{\Sigma}_1$ (Proposition 7.5)
Section 8	$\widehat{\Sigma}_1'$; F_0; F, ϖ (Definition 8.1); $\widehat{\Sigma_{1,O_F}}$; $\widehat{\Sigma_{1,O_F,s}}$; $\widetilde{\Sigma_{1,O_F}}$; $\widetilde{\Sigma_{1,O_F,s}}$; $\widetilde{\Sigma_{1,O_F,s,\xi}}$; $\widehat{\Sigma^{(0)}_{1,O_F}}$, ..., g_φ (Remark 8.5)
Section 9	$\tilde{\omega}_2$ (Definition 9.3)
Section 10	$\Sigma_{1,F}$; $\Sigma^{(0)}_{1,F}$; U_n
Section 11	$\Omega^i_{\Sigma_{1,F}}$; $H^i_{\mathrm{dR}}(\Sigma_{1,F})$; U_e, U_s (Definition 11.7); (s, ξ) (Definition 11.9); $\overline{U_s}$, $\overline{U_{s,\xi}}$, $\overline{U^0_{s,\xi}}$, $\overline{U^0_s}$, $U_{s,\xi}$ (Definition 11.10)
Section 12	$F_{1,\xi}$; $D_{1,\xi}$; $\psi_{s,\xi}$; $\widehat{D_{1,O_{F_0},\xi}}$; ρ_χ (Definition 12.6); $D_{\mathrm{crys},\chi}$, m, c_x (Proposition 12.8)
Section 13	i, j, $D_{\chi,[a,b]}$, $V_{\chi,[1,b]}$ (Proposition 13.2)
Section 14	ω^1; $M(\chi, [1, b])$; $B(\chi, [1, b])$
Section 15	$H^{(0),\chi,\mathbb{Q}_p}$, H^{χ,F_0}_*, H^{χ',F_0}_*, $H^{\chi',F_0}_{*,?}$ (Definition 15.1); $A(s)$ (Definition 15.2)
Section 16	$F_{0,\xi}$; $D_{0,\xi}$; $\psi_{s',\xi}$; $\widehat{D_{0,O_{F_0},\xi}}$; $V_{s,\xi}$, $V_{c,\xi}$ (Definition 16.2); $W_{s,\xi}$, $Z_{s,\xi}$ (Definition 16.3); ω_s; f_{s_1,s_2}, f_s (58); $A(s_0')$ (Definition 16.10); V_0, V_∞ (Definition 16.11); F_1, F_2 (70); θ_b (72); J_1, J_2 (Definition 16.22); $\bar{\theta}_b$; $\bar{\theta}_{b,1}$, $\bar{\theta}_{b,2}$ (Lemma 16.25); v_s (Lemma 16.33); h_{s_0} (81);

	$\omega_\tau, \omega_{\bar\tau}, \omega_{\tau,s,\xi}, \omega_1, \omega_2, \omega_{1,s}, \omega_{1,s,\xi}, \dots$ (Definition 16.31)
Section 17	$u_{s_0'}, u_{s_0}, u_{s_0''}$ (Lemma 17.7); s_0'' (Definition 17.14); h'_{s_0} (86); $h_{s_0''}$ (87)

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