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The sixth moment of automorphic L-functions

Vorrapan Chandee and Xiannan Li

We consider the L-functions $L(s, f)$, where f is an eigenform for the congruence subgroup $\Gamma_1(q)$. We prove an asymptotic formula for the sixth moment of this family of automorphic L-functions.

1. Introduction

Moments of L-functions are of great interest to analytic number theorists. For instance, for $\zeta(s)$ denoting the Riemann zeta function and

$$I_k(T) := \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt,$$

asymptotic formulae were proven for $k = 1$ by Hardy and Littlewood and for $k = 2$ by Ingham; see [Titchmarsh 1986, Chapter VII]. This work is closely related to zero density results and the distribution of primes in short intervals. More recently, moments of other families of L-functions have been studied for their numerous applications, including nonvanishing and subconvexity results. In many applications, it is important to develop technology which can understand such moments for larger k.

The behavior of moments for larger k remain mysterious. However, recently there has been great progress in our understanding. First, good heuristics and conjectures on the behavior of $I_k(T)$ have appeared in the literature. To be precise, a folklore conjecture states that

$$I_k(T) \sim c_k T (\log T)^{k^2}$$

for constants c_k depending on k, but the values of c_k were unknown for general k until the work of Keating and Snaith [2000], which related these moments to circular unitary ensembles and provided precise conjectures for c_k. The choice of group is consistent with the Katz–Sarnak philosophy [Katz and Sarnak 1999], which indicates that the symmetry group associated to this family should be unitary.

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Based on heuristics for shifted divisor sums, Conrey and Ghosh [1998] derived a conjecture in the case $k=3$ and Conrey and Gonek [2001] derived a conjecture in the case $k=4$. In particular, the conjecture for the sixth moment is

$$I_3(T) \sim 42a_3 \frac{T(\log T)^9}{9!}$$

for some arithmetic factor a_3. Further conjectures, including lower order terms and cases of other symmetry groups, are available from the work of Conrey, Farmer, Keating, Rubinstein and Snaith [Conrey et al. 2005] as well as from the work of Diaconu, Goldfeld and Hoffstein [Diaconu et al. 2003].

In support of these conjectures, lower bounds of the right order of magnitude are available due to Rudnick and Soundararajan [2005], while good upper bounds of the right order of magnitude are available, conditionally on the Riemann hypothesis, due to Soundararajan [2009] and later improved by Harper [2013].

Despite this, verifications of the moment conjectures for high moments remain elusive. Typically, even going slightly beyong the fourth moment to obtain a twisted fourth moment is quite difficult, and there are few families for which this is known.

Quite recently, Conrey, Iwaniec and Soundararajan [Conrey et al. 2012] derived an asymptotic formula for the sixth moment of Dirichlet L-functions with a power saving error term. Instead of fixing the modulus q and only averaging over primitive characters $\chi \pmod q$, they also averaged over the modulus $q \le Q$, giving them a larger family of size Q^2. Further, they included a short average on the critical line. In particular, they showed that

$$\sum_{q\le Q} \sideset{}{^*}\sum_{\chi \pmod q} \int_{-\infty}^{\infty} \left|L\left(\tfrac12+it,\chi\right)\right|^6 \left|\Gamma\left(\tfrac14+\tfrac12 it\right)\right|^6 dt \\ \sim 42b_3 \frac{Q^2(\log Q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac14+\tfrac12 it\right)\right|^6 dt$$

for some constant b_3. This is consistent with the analogous conjecture for the Riemann zeta function above.

The authors of this paper subsequently derived an asymptotic formula for the eight moment of this family of L-functions, conditionally on the generalized Riemann hypothesis [Chandee and Li 2014], namely

$$\sum_{q\le Q} \sideset{}{^*}\sum_{\chi \pmod q} \int_{-\infty}^{\infty} \left|L\left(\tfrac12+it,\chi\right)\right|^8 \left|\Gamma\left(\tfrac14+\tfrac12 it\right)\right|^8 dt \\ \sim 24024b_4 \frac{Q^2(\log Q)^{16}}{16!} \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac14+\tfrac12 it\right)\right|^8 dt$$

for some constant b_4.

In this paper, we study a family of L-functions attached to automorphic forms on GL(2). To be more precise, let $S_k(\Gamma_0(q),\chi)$ be the space of cusp forms of weight

$k \geq 2$ for the group $\Gamma_0(q)$ and the nebentypus character $\chi \pmod q$, where

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1,\ c \equiv 0 \pmod q \right\}.$$

Also, let $S_k(\Gamma_1(q))$ be the space of holomorphic cusp forms for the group

$$\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1,\ c \equiv 0 \pmod q,\ a \equiv d \equiv 1 \pmod q \right\}.$$

Note that $S_k(\Gamma_1(q))$ is a Hilbert space with the Petersson's inner product

$$\langle f, g \rangle = \int_{\Gamma_1(q)\backslash\mathbb{H}} f(z)\bar{g}(z) y^{k-2}\, dx\, dy,$$

and

$$S_k(\Gamma_1(q)) = \bigoplus_{\chi \pmod q} S_k(\Gamma_0(q), \chi).$$

Let $\mathcal{H}_\chi \subset S_k(\Gamma_0(q), \chi)$ be an orthogonal basis of $S_k(\Gamma_0(q), \chi)$ consisting of Hecke cusp forms, normalized so that the first Fourier coefficient is 1. For each $f \in \mathcal{H}_\chi$, we let $L(f, s)$ be the L-function associated to f, defined for $\mathrm{Re}(s) > 1$ as

$$L(f, s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}, \tag{1-1}$$

where $\{\lambda_f(n)\}$ are the Hecke eigenvalues of f. With our normalization, $\lambda_f(1) = 1$. In general, the Hecke eigenvalues satisfy the Hecke relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right), \tag{1-2}$$

for all $m, n \geq 1$. We define the completed L-function as

$$\Lambda\left(f, \tfrac{1}{2} + s\right) = \left(\frac{q}{4\pi^2}\right)^{s/2} \Gamma\left(s + \tfrac{1}{2}k\right) L\left(f, \tfrac{1}{2} + s\right), \tag{1-3}$$

which satisfies the functional equation

$$\Lambda\left(f, \tfrac{1}{2} + s\right) = i^k \bar{\eta}_f \Lambda\left(\bar{f}, \tfrac{1}{2} - s\right),$$

where $|\eta_f| = 1$ when f is a newform.

Suppose for each $f \in \mathcal{H}_\chi$ we have an associated number α_f. Then we define the harmonic average of α_f over $\mathcal{H}_\chi$ to be

$$\sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \alpha_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_\chi} \frac{\alpha_f}{\|f\|^2}.$$

Note that when the first coefficient $\lambda_f(1) = 1$, $\|f\|^2$ is essentially the value of a certain L-function at 1, and so on average, $\|f\|^2$ is constant. As in other works, it is possible to remove the weighting by $\|f\|^2$ through what is now a standard argument.

We shall be interested in moments of the form

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sum_{f\in\mathcal{H}_\chi}^{h} \big|L\big(f, \tfrac{1}{2}\big)\big|^{2k}.$$

We note that the size of the family is around q^2. For prime level, η_f can be expressed in terms of Gauss sums, and in particular we expect η_f to equidistribute on the circle as f varies over an orthogonal basis of $S_k(\Gamma_1(q))$. Thus, we expect our family of L-functions to be unitary.

In this paper, we prove an asymptotic formula for the sixth moment; this will be the first time that the sixth moment of a family of L-functions over GL(2) has been understood. Following [Conrey et al. 2005], we have the following conjecture for the sixth moment of our family. We refer the reader to Appendix A for a brief derivation of the arithmetic factor in the conjecture.

Conjecture 1.1. Let q be a prime number. As $q \to \infty$, we have

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sum_{f\in\mathcal{H}_\chi}^{h} \big|L\big(f, \tfrac{1}{2}\big)\big|^{6} \sim 42\mathscr{C}_3\Big(1-\frac{1}{q}\Big)^4\Big(1+\frac{4}{q}+\frac{1}{q^2}\Big)C_q^{-1}\frac{(\log q)^9}{9!},$$

where

$$\begin{aligned} \mathscr{C}_3 &:= \prod_p C_p, \\ C_p &:= \Big(1+\frac{4}{p}+\frac{7}{p^2}-\frac{2}{p^3}+\frac{9}{p^4}-\frac{16}{p^5}+\frac{1}{p^6}-\frac{4}{p^7}\Big)\Big(1+\frac{1}{p}\Big)^{-4}. \end{aligned} \tag{1-4}$$

Iwaniec and Xiaoqing Li [2007] proved a large sieve result for this family, and Djankovic [2011] used their result to prove, for an odd integer $k \ge 3$ and prime q, that

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \sum_{f\in\mathcal{H}_\chi}^{h} \big|L\big(f, \tfrac{1}{2}\big)\big|^{6} \ll q^{\varepsilon}.$$

as $q \to \infty$. In this paper, we shall prove the following.

Theorem 1.2. *Let q be a prime and $k \ge 5$ be odd. Then, as $q \to \infty$, letting $\mathscr{C}_3$ and C_p be as defined in* (1-4), *we have*

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \sum_{f\in\mathcal{H}_\chi}^{h} \int_{-\infty}^{\infty} \big|\Lambda\big(f, \tfrac{1}{2}+it\big)\big|^6\,dt \\ \sim 42\mathscr{C}_3\Big(1-\frac{1}{q}\Big)^4\Big(1+\frac{4}{q}+\frac{1}{q^2}\Big)C_q^{-1}\frac{(\log q)^9}{9!}\int_{-\infty}^{\infty}\big|\Gamma\big(\tfrac{1}{2}k+it\big)\big|^6\,dt.$$

In fact, we are able to prove this with an error term of $q^{-1/4}$, as opposed to the $q^{-1/10}$ error term in [Conrey et al. 2012]. The reason behind this superior error term is explained in the outline in Section 1A. In future work, we hope to extend our attention to the eighth moment.

The assumption that k is odd implies that all $f \in \mathcal{H}_\chi$ are newforms. This is for convenience only and is not difficult to remove. Indeed, when k is even, all $f \in \mathcal{H}_\chi$ are newforms except possibly when χ is the principal character and f is induced by a cusp form of full level. We avoid this case for the sake of brevity. Similarly, the assumption that $k \geq 5$ simplifies parts of the calculation; it is possible to prove Theorem 1.2 for smaller k.

Since the Γ function decays rapidly on vertical lines, the average over t is fairly short. It is included for the same reason as in [Conrey et al. 2012; Chandee and Li 2014]: it allows us to avoid certain unbalanced sums in the computation of the moment. Although this appears to be a small technical change in the main statement, evaluating such moments without the short integration over t is a significant challenge. Our theorem follows from the more general Theorem 2.5 for shifted moments in Section 2.

1A. ***Outline of the paper.*** To help orient the reader, we provide a sketch of the proof, and introduce the various sections of the paper. After applying the approximate functional equation developed in Section 3, the main object to be understood is roughly of the form

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \sum_{m,n \asymp q^{3/2}} \frac{\sigma_3(m)\sigma_3(n)\lambda_f(n)\overline{\lambda_f}(m)}{\sqrt{mn}}.$$

In fact, since the coefficients $\lambda_f(n)$ are not completely multiplicative, the expression is significantly more complicated for the purpose of extracting main terms.

Applying Petersson's formula for the average over $f \in \mathcal{H}_\chi$ leads to diagonal terms $m = n$ which are evaluated fairly easily in Section 4A, as well as off-diagonal terms which involve sums of the form

$$\sum_{m,n \asymp q^{3/2}} \frac{\sigma_3(m)\sigma_3(n)}{\sqrt{mn}} \frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sum_c S_\chi(m,n;cq) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right),$$

where $S_\chi(m,n;cq)$ is the Kloosterman sum defined in Lemma 2.3, and $J_{k-1}(x)$ is the J-Bessel function of order $k-1$.

Let us focus on the transition region for the Bessel function where $c \asymp q^{1/2}$, so that the conductor is a priori of size $qc \asymp q^{3/2}$. It is here that the addition average over

$\chi \pmod q$ comes into play. To be more precise, to understand the exponential sum

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=(-1)^k}} S_\chi(m,n;cq),$$

it suffices to understand

$$\sideset{}{^*}\sum_{\substack{a \ (\mathrm{mod}\ cq) \\ a\equiv 1 \ (\mathrm{mod}\ q)}} \mathrm{e}\Big(\frac{am+\bar{a}n}{cq}\Big),$$

which, assuming that $(c,q)=1$ and using the Chinese remainder theorem and reciprocity, is

$$\mathrm{e}\Big(\frac{m+n}{cq}\Big) \sideset{}{^*}\sum_{a \ (\mathrm{mod}\ c)} \mathrm{e}\Big(\frac{\bar{q}(a-1)m+\bar{q}(\bar{a}-1)n}{c}\Big).$$

Note that $\mathrm{e}(x)=\exp(2\pi ix)$. The factor $\mathrm{e}\big(\frac{m+n}{cq}\big)$ has small derivatives and may be treated as a smooth function, while the conductor of the rest of the exponential sum has decreased to $c \asymp q^{1/2}$. The details of these calculations are in Section 5.

This phenomenon of the drop in conductor appears in other examples. In the case of the sixth moment of Dirichlet L-functions in [Conrey et al. 2012], it occurs when replacing q with the complementary divisor $(m-n)/q \asymp q^{1/2}$. It is quite interesting that the same drop in conductor occurs by seemingly very different mechanisms. However, note that when the complementary divisor is small, the ordered pair (m,n) is forced to be in a narrow region. That this does not occur in our case is one of the reasons behind the superior error term in our result.

After the conductor drop, we apply Voronoi summation to the sum over m and n in Section 6. We need a version of Voronoi summation including shifts. The proof of this is essentially the same as the proof of the standard Voronoi summation formula for $\sigma_3(n)$ by Ivić [1997]. We state the result required in Appendix B.

After applying Voronoi, it is easy to guess which terms should contribute to the main terms and which terms should be error terms. The main terms are described in Proposition 6.1 and the error terms are bounded in Proposition 6.2. Essentially, we expect the main terms to be a sum of products of 9 factors of ζ, the same as the diagonal contribution but with permutations in the shifts, as in Theorem 2.5. This is by no means immediately visible from the expression in Proposition 6.1. Indeed, it takes some effort to see that we get the right number of ζ factors. Along the way, we use, among other things, a calculation of Iwaniec and Xiaoqing Li in [Iwaniec and Li 2007]. This is done in Section 7. In order to finish the verifications, we need to check that the local factors of two expressions agree. The details here are standard but intricate, and are provided in Appendix A.

Finally, the error terms from Voronoi summation are bounded in Section 8. Here, one needs to show that the dual sums from Voronoi summation are essentially quite short, which is related to the reduction in conductor from cq to c earlier.

2. Notation and the shifted sixth moment

We begin with some notation. Let $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_3)$. For a complex number s, we write $\boldsymbol{\alpha} + s := (\alpha_1 + s, \alpha_2 + s, \alpha_3 + s)$. We define

$$\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \frac{1}{2}\sum_{j=1}^{3}(\alpha_j - \beta_j), \tag{2-1}$$

$$G(s; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \prod_{j=1}^{3} \Gamma\big(s + \tfrac{1}{2}(k-1) + \alpha_j\big)\Gamma\big(s + \tfrac{1}{2}(k-1) - \beta_j\big), \tag{2-2}$$

and

$$\Lambda(f, s; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \prod_{j=1}^{3} \Lambda(f, s + \alpha_j)\Lambda(\bar{f}, s - \beta_j). \tag{2-3}$$

Note that we have

$$\begin{aligned}\Lambda(f; \boldsymbol{\alpha}, \boldsymbol{\beta}) &= \Lambda\big(f, \tfrac{1}{2}; \boldsymbol{\alpha}, \boldsymbol{\beta}\big)\\ &= \Big(\frac{q}{4\pi^2}\Big)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} G\big(\tfrac{1}{2}; \boldsymbol{\alpha}, \boldsymbol{\beta}\big)\prod_{j=1}^{3} L\big(f, \tfrac{1}{2} + \alpha_j\big)L\big(\bar{f}, \tfrac{1}{2} - \beta_j\big).\end{aligned} \tag{2-4}$$

We define the shifted k-divisor function by

$$\sigma_k(n; \alpha_1, \dots, \alpha_k) = \sum_{n_1 n_2 \cdots n_k = n} n_1^{-\alpha_1} n_2^{-\alpha_2} \cdots n_k^{-\alpha_k}. \tag{2-5}$$

Let

$$\mathscr{B}(a, b; \boldsymbol{\alpha}) := \frac{\mu(a)\sigma_3(b; \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1)}{a^{\alpha_1 + \alpha_2 + \alpha_3}}. \tag{2-6}$$

Next we need the following lemmas, which help us generate the conjecture of the sixth moment, namely

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi(-1) = (-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \Lambda(f; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{2-7}$$

Lemma 2.1. *We have*

$$\sigma_2(n_1 n_2; \alpha_1, \alpha_2) = \sum_{d \mid (n_1, n_2)} \mu(d) d^{-\alpha_1 - \alpha_2} \sigma_2\Big(\frac{n_1}{d}; \alpha_1, \alpha_2\Big)\sigma_2\Big(\frac{n_2}{d}; \alpha_1, \alpha_2\Big).$$

Proof. Since both sides are multiplicative functions, it is enough to prove the lemma when n_1n_2 is a prime power. We set $n_1 = p^a$ and $n_2 = p^b$, where $1 \le a \le b$. Then

$$\begin{aligned}\sigma_2(p^a p^b; \alpha_1, \alpha_2) &= \sum_{0\le k\le a} \mu(p^k) p^{-k(\alpha_1+\alpha_2)} \sigma_2(p^{a-k}; \alpha_1, \alpha_2)\sigma_2(p^{b-k}; \alpha_1, \alpha_2)\\ &= \sigma_2(p^a; \alpha_1, \alpha_2)\sigma_2(p^b; \alpha_1, \alpha_2)\\ &\qquad - p^{-(\alpha_1+\alpha_2)}\sigma_2(p^{a-1}; \alpha_1, \alpha_2)\sigma_2(p^{b-1}; \alpha_1, \alpha_2). \qquad (2\text{-}8)\end{aligned}$$

On the other hand,

$$\sigma_2(p^n; \alpha_1, \alpha_2) = \sum_{\ell=0}^{n} p^{-\ell\alpha_1} p^{-(n-\ell)\alpha_2} = p^{-n\alpha_2} \frac{1 - 1/p^{(n+1)(\alpha_1-\alpha_2)}}{1 - 1/p^{\alpha_1-\alpha_2}},$$

and the lemma follows by substituting the above formula into (2-8). □

We write the product of L-functions in terms of Dirichlet series in the next lemma.

Lemma 2.2. *Let $L(f, w)$ be an L-function in $\mathcal{H}_\chi$. For* $\mathrm{Re}(s + \alpha_i) > 1$, *we have*

$$L(f, s+\alpha_1)L(f, s+\alpha_2)L(f, s+\alpha_3) = \sum_{a,b\ge 1}\sum \frac{\chi(ab)\mathscr{B}(a, b; \boldsymbol{\alpha})}{(ab)^{2s}} \sum_{n\ge 1} \frac{\lambda_f(an)\sigma_3(n; \boldsymbol{\alpha})}{(an)^s}.$$

Proof. From the Hecke relation (1-2) and Lemma 2.1, we have

$$\begin{aligned}&L(f, s+\alpha_1)L(f, s+\alpha_2)L(f, s+\alpha_3)\\ &= \sum_{n_1\ge 1} \frac{\lambda_f(n_1)}{n_1^{s+\alpha_1}} \sum_{d\ge 1} \frac{\chi(d)}{d^{2s+\alpha_2+\alpha_3}} \sum_{j\ge 1} \frac{\lambda_f(j)\sigma_2(j; \alpha_2, \alpha_3)}{j^s}\\ &= \sum_{d\ge 1} \frac{\chi(d)}{d^{2s+\alpha_2+\alpha_3}} \sum_{n_1, j\ge 1}\sum \frac{\sigma_2(j; \alpha_2, \alpha_3)}{n_1^{s+\alpha_1} j^s} \sum_{e|(n_1, j)} \chi(e)\lambda_f\Big(\frac{jn_1}{e^2}\Big)\\ &= \sum_{a\ge 1} \frac{\mu(a)\chi(a)}{a^{2s+\alpha_1+\alpha_2+\alpha_3}} \sum_{d,e\ge 1}\sum \frac{\chi(de)\sigma_2(e; \alpha_2, \alpha_3)}{(de)^{2s} d^{\alpha_2+\alpha_3} e^{\alpha_1}}\\ &\qquad\qquad \times \sum_{j,n_1\ge 1}\sum \frac{\lambda_f(n_1 ja)\sigma_2(j; \alpha_2, \alpha_3)}{(ajn_1)^s n_1^{\alpha_1}}\\ &= \sum_{a\ge 1} \frac{\mu(a)\chi(a)}{a^{2s+\alpha_1+\alpha_2+\alpha_3}} \sum_{b\ge 1} \frac{\chi(b)\sigma_3(b; \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_1)}{b^{2s}}\\ &\qquad\qquad \times \sum_{n\ge 1} \frac{\lambda_f(an)\sigma_3(n; \alpha_1, \alpha_2, \alpha_3)}{(an)^s}. \qquad (2\text{-}9)\end{aligned}$$

This completes the lemma. □

Lemma 2.3. *The orthogonality relation for Dirichlet characters is*

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \chi(m)\overline{\chi}(n) = \begin{cases} 1 & \text{if } m \equiv n \pmod q,\ (mn,q)=1, \\ (-1)^k & \text{if } m \equiv -n \pmod q,\ (mn,q)=1, \\ 0 & \text{otherwise.} \end{cases} \tag{2-10}$$

Petersson's formula gives

$$\sum_{f\in\mathcal{H}_\chi}^{h} \bar{\lambda}_f(m)\lambda_f(n) = \delta_{m=n} + \sigma_\chi(m,n), \tag{2-11}$$

where

$$\begin{aligned}\sigma_\chi(m,n) &= 2\pi i^{-k} \sum_{c\equiv 0 \pmod q} c^{-1} S_\chi(m,n;c) J_{k-1}\Big(\frac{4\pi}{c}\sqrt{mn}\Big) \\ &= 2\pi i^{-k} \sum_{c=1}^{\infty} (cq)^{-1} S_\chi(m,n;cq) J_{k-1}\Big(\frac{4\pi}{cq}\sqrt{mn}\Big),\end{aligned}$$

and S_χ is the Kloosterman sum defined by

$$S_\chi(m,n;cq) = \sum_{a\bar{a}\equiv 1 \pmod{cq}} \chi(a)\, \mathrm{e}\Big(\frac{am+\bar{a}n}{cq}\Big).$$

From Lemma 2.2, we have that

$$\begin{aligned}\prod_{i=1}^{3} L(f,s+\alpha_i)L(f,s-\beta_i) &= \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\ge 1} \frac{\chi(a_1b_1)\overline{\chi}(a_2b_2)\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{(a_1b_1a_2b_2)^{2s}} \\ &\qquad \times \sum\sum_{n,m\ge 1} \frac{\lambda_f(a_1n)\overline{\lambda_f}(a_2m)\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{(a_1na_2m)^s}. \end{aligned} \tag{2-12}$$

By the orthogonality relation of Dirichlet characters and Petersson's formula in Lemma 2.3, a naive guess might be that the main contribution comes from the diagonal terms $a_1b_1 = a_2b_2$ and $a_1n = a_2m$, where $(a_ib_i, q) = 1$, which is

$$\mathcal{C}(s,\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum\sum\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2,m,n\ge 1 \\ a_1n=a_2m \\ a_1b_1=a_2b_2 \\ (a_i,q)=(b_j,q)=1}} \frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{(a_1b_1)^{2s}} \frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{(a_2b_2)^{2s}} \frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{(a_1n)^s(a_2m)^s} \tag{2-13}$$

for $\mathrm{Re}(s)$ large enough. This can be written as the Euler product

$$\mathcal{C}(s,\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_p \mathcal{C}_p(s,\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where for $p \neq q$,

$$\mathcal{C}_p(s,\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum\sum\sum\sum\sum\sum_{\substack{r_1,t_1,r_2,t_2,u_1,u_2\geq 0\\ r_1+u_1=r_2+u_2\\ r_1+t_1=r_2+t_2}} \frac{\mathscr{B}(p^{r_1},p^{t_1};\boldsymbol{\alpha})}{p^{2s(r_1+t_1)}}\frac{\mathscr{B}(p^{r_2},p^{t_2};-\boldsymbol{\beta})}{p^{2s(r_2+t_2)}} \times \frac{\sigma_3(p^{u_1};\boldsymbol{\alpha})\sigma_3(p^{u_2};-\boldsymbol{\beta})}{p^{s(r_1+u_1)}p^{s(r_2+u_2)}}, \tag{2-14}$$

and for $p = q$,

$$\mathcal{C}_q(s,\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum_{u\geq 0}\frac{\sigma_3(q^u;\boldsymbol{\alpha})\sigma_3(q^u;-\boldsymbol{\beta})}{q^{2us}}. \tag{2-15}$$

Next, for $\zeta_p(w) := (1-1/p^w)^{-1}$, let

$$\begin{aligned}\mathcal{Z}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta}) &:= \prod_{i=1}^{3}\prod_{j=1}^{3}\zeta_p(2s+\alpha_i-\beta_j),\\ \mathcal{Z}(s;\boldsymbol{\alpha},\boldsymbol{\beta}) &:= \prod_{i=1}^{3}\prod_{j=1}^{3}\zeta(2s+\alpha_i-\beta_j).\end{aligned} \tag{2-16}$$

and

$$\mathcal{A}(s;\boldsymbol{\alpha},\boldsymbol{\beta}) := \mathcal{C}(s;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{Z}(s;\boldsymbol{\alpha},\boldsymbol{\beta})^{-1} = \prod_p \mathcal{C}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{Z}_p(s;\boldsymbol{\alpha},\boldsymbol{\beta})^{-1}. \tag{2-17}$$

We define

$$\mathcal{M}(q;\boldsymbol{\alpha},\boldsymbol{\beta}) := \left(\frac{q}{4\pi^2}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} G\big(\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)\mathcal{A}\mathcal{Z}\big(\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big). \tag{2-18}$$

When $\mathrm{Re}(\alpha_i), \mathrm{Re}(\beta_i) \ll 1/\log q$, the term $\mathcal{A}\big(\frac{1}{2},\boldsymbol{\alpha},\boldsymbol{\beta}\big)$ is absolutely convergent. Now, let S_j be the permutation group of j variables. Based on the analysis of the diagonal contribution, we expect $\mathcal{M}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ to be a part of the average in (2-7), and we also notice that the expression $\mathcal{M}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ is fixed by the action of $S_3\times S_3$. Since we expect our final answer to be symmetric under the full group S_6, we sum over the cosets $S_6/(S_3\times S_3)$. In fact, the method of Conrey, Farmer, Keating, Rubinstein and Snaith [Conrey et al. 2005] gives the following conjecture for the average of $\Lambda(f;\boldsymbol{\alpha},\boldsymbol{\beta})$.

Conjecture 2.4. Assume that $\boldsymbol{\alpha},\boldsymbol{\beta}$ satisfy

$$\begin{aligned}\mathrm{Re}(\alpha_i), \mathrm{Re}(\beta_i) &\ll \frac{1}{\log q},\\ \mathrm{Im}\,(\alpha_i), \mathrm{Im}\,(\beta_i) &\ll q^{1-\varepsilon}.\end{aligned}$$

We have

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{H}_\chi} \Lambda(f;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\pi\in S_6/(S_3\times S_3)} \mathcal{M}(q;\pi(\boldsymbol{\alpha},\boldsymbol{\beta}))(1+O(q^{-1/2+\varepsilon})),$$

where we define $\pi(\boldsymbol{\alpha},\boldsymbol{\beta}) = \pi(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)$ for $\pi\in S_{2k}$, where π acts on the $2k$-tuple $(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)$ as usual.

We also write $\pi(\boldsymbol{\alpha},\boldsymbol{\beta}) = (\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))$ by an abuse of notation, where $\pi(\boldsymbol{\alpha},\boldsymbol{\beta})$ is as above. Our main goal is to find an asymptotic formula for

$$\mathcal{M}_6(q) := \frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{H}_\chi} \int_{-\infty}^{\infty} \Lambda(f;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)\,dt, \tag{2-19}$$

and we will prove the following result.

Theorem 2.5. *Let q be prime and $k\ge 5$ be odd. For $\alpha_i,\beta_j \ll \frac{1}{\log q}$, we have that*

$$\mathcal{M}_6(q) = \int_{-\infty}^{\infty} \sum_{\pi\in S_6/(S_3\times S_3)} \mathcal{M}(q,\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it)\,dt + O(q^{-1/4+\varepsilon}).$$

We note that as the shifts go to 0, the main term of this moment is of the size $(\log q)^9$, and we derive Theorem 1.2. We refer the reader to [Conrey et al. 2005] for the details of this type of calculation.

3. Approximate functional equation

In this section, we prove an approximate functional equation for the product of L-functions. Let

$$H(s;\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{j=1}^{3}\prod_{\ell=1}^{3}\Big(s^2-\Big(\frac{\alpha_j-\beta_\ell}{2}\Big)^2\Big)^3,$$

and define, for any $\xi>0$,

$$W(\xi;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{2\pi i}\int_{(1)} G\big(\tfrac12+s;\boldsymbol{\alpha},\boldsymbol{\beta}\big)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\xi^{-s}\,\frac{ds}{s}.$$

Moreover, let $\Lambda_0(f,\boldsymbol{\alpha},\boldsymbol{\beta})$ be

$$\Big(\frac{q}{4\pi^2}\Big)^{\delta(\alpha,\beta)} \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\ge1} \frac{\chi(a_1b_1)\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1b_1}\,\frac{\overline{\chi}(a_2b_2)\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2b_2}$$
$$\times \sum\sum_{n,m\ge1} \frac{\lambda_f(a_1n)\sigma_3(n;\boldsymbol{\alpha})}{(a_1n)^{1/2}}\,\frac{\overline{\lambda_f}(a_2m)\sigma_3(m;-\boldsymbol{\beta})}{(a_2m)^{1/2}}$$
$$\times W\Big(\frac{(2\pi)^6a_1^3b_1^2a_2^3b_2^2nm}{q^3};\boldsymbol{\alpha},\boldsymbol{\beta}\Big). \tag{3-1}$$

Lemma 3.1. *We have*

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\Lambda(f;\boldsymbol{\alpha},\boldsymbol{\beta}) = \Lambda_0(f,\boldsymbol{\alpha},\boldsymbol{\beta}) + \Lambda_0(f,\boldsymbol{\beta},\boldsymbol{\alpha}).$$

Proof. We consider

$$I := \frac{1}{2\pi i}\int_{(1)} \Lambda\big(f, s+\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\,\frac{ds}{s}.$$

Moving the contour integral to (-1), we obtain that

$$\begin{aligned} I &= \Lambda(f;\boldsymbol{\alpha},\boldsymbol{\beta})H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) + \frac{1}{2\pi i}\int_{(-1)} \Lambda\big(f, s+\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\,\frac{ds}{s} \\ &= \Lambda(f;\boldsymbol{\alpha},\boldsymbol{\beta})H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) - \frac{1}{2\pi i}\int_{(1)} \Lambda\big(f, -s+\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)H(-s;\boldsymbol{\alpha},\boldsymbol{\beta})\,\frac{ds}{s}. \end{aligned}$$

By the functional equation, we have $\Lambda\big(f, s+\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big) = \Lambda\big(f, -s+\tfrac{1}{2};\boldsymbol{\beta},\boldsymbol{\alpha}\big)$. Moreover, H is an even function, and $H(s;\boldsymbol{\alpha},\boldsymbol{\beta}) = H(s;\boldsymbol{\beta},\boldsymbol{\alpha})$. Therefore,

$$\begin{aligned} \Lambda(f;\boldsymbol{\alpha},\boldsymbol{\beta})H(0;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{2\pi i}\int_{(1)} \Lambda\big(f, s+\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\,\frac{ds}{s} \\ + \frac{1}{2\pi i}\int_{(1)} \Lambda\big(f, s+\tfrac{1}{2};\boldsymbol{\beta},\boldsymbol{\alpha}\big)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\,\frac{ds}{s}. \end{aligned}$$

The lemma follows after writing Λ as a product of L-functions and Gamma functions and using Lemma 2.2. □

Next, we let

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi,\eta;\mu) = \Big(\frac{\mu}{4\pi^2}\Big)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\int_{-\infty}^{\infty}\Big(\frac{\eta}{\xi}\Big)^{it} W\bigg(\frac{\xi\eta(4\pi^2)^3}{\mu^3};\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\bigg)\,dt,$$

and

$$\begin{aligned} \Lambda_1(f;\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum\sum\sum\sum_{a_1,b_1,a_2,b_2\ge 1} \frac{\chi(a_1b_1)\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1b_1}\,\frac{\overline{\chi}(a_2b_2)\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2b_2} \\ \times \sum\sum_{n,m\ge 1} \frac{\lambda_f(a_1n)\sigma_3(n;\boldsymbol{\alpha})}{(a_1n)^{1/2}}\,\frac{\overline{\lambda_f}(a_2m)\sigma_3(m;-\boldsymbol{\beta})}{(a_2m)^{1/2}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n, a_2^3b_2^2m;q). \end{aligned} \tag{3-2}$$

Lemma 3.2. *With notation as above, we have*

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\int_{-\infty}^{\infty}\Lambda(f;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)\,dt = \Lambda_1(f;\boldsymbol{\alpha},\boldsymbol{\beta}) + \Lambda_1(f;\boldsymbol{\beta},\boldsymbol{\alpha}).$$

The proof follows easily from Lemma 3.1.

Remark. The integration over t is added so that the main contribution comes from when $a_1^3b_1^2n \ll q^{3/2+\varepsilon}$ and $a_2^3b_2^2m \ll q^{3/2+\varepsilon}$, as we will see from Lemma 3.3 below. Without the integration over t, the ranges of a_i, b_j, m, n that we need to consider satisfy the weaker condition $a_1^3b_1^2na_2^3b_2^2m \ll q^{3+\varepsilon}$, and the proof presented here does not extend to this range.

Lemma 3.3. *If ξ or $\eta \gg q^{3/2+\varepsilon}$, then for any $A > 1$, we have*

$$V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi, \eta; q) \ll q^{-A},$$

where the implied constant depends on ε and A.

Proof. From the definition of W and V and a change of variables ($s + it = w$, $s - it = z$), we can write $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\xi, \eta; \mu)$ as

$$\left(\frac{q}{4\pi^2}\right)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \frac{4\pi}{(2\pi i)^2} \int_{(0)} \prod_{j=1}^{3} \Gamma\big(w + \tfrac{1}{2}(k-1) + \alpha_j\big)\left(\frac{q^{3/2}}{(4\pi^2)^{3/2}\xi}\right)^{w}$$
$$\times \int_{(1)} \prod_{j=1}^{3} \Gamma\big(z + \tfrac{1}{2}(k-1) - \beta_j\big)\left(\frac{q^{3/2}}{(4\pi^2)^{3/2}\eta}\right)^{z} H\big(\tfrac{1}{2}(z+w); \boldsymbol{\alpha}, \boldsymbol{\beta}\big) \frac{dz\,dw}{z+w}.$$

When $\xi \gg q^{3/2+\varepsilon}$, we move the contour integral over w to the far right, and similarly, when $\eta \gg q^{3/2+\varepsilon}$, we move the contour integral over z to the far right. The lemma then follows. □

4. Setup for the proof of Theorem 2.5 and diagonal terms

From Lemma 3.2, we have that for $\alpha_i, \beta_i \ll 1/\log q$,

$$H(0; \boldsymbol{\alpha}, \boldsymbol{\beta})\mathcal{M}_6(q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{H}_\chi} \{\Lambda_1(f; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \Lambda_1(f; \boldsymbol{\beta}, \boldsymbol{\alpha})\}.$$

Therefore, to evaluate $\mathcal{M}_6(q)$, it is sufficient to compute asymptotically

$$\mathscr{M}_1(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \frac{2}{\phi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f\in\mathcal{H}_\chi} \Lambda_1(f; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4-1}$$

Applying Petersson's formula, we obtain

$$\sideset{}{^h}\sum_{f\in\mathcal{H}_\chi} \bar{\lambda}_f(a_2m)\lambda_f(a_1n) = \delta_{a_2m=a_1n} + \sigma_\chi(a_2m, a_1n),$$

where $\sigma_\chi(a_2m, a_1n)$ is defined as in Lemma 2.3. We then write

$$\mathscr{M}_1(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathscr{D}(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathscr{K}(q; \boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{4-2}$$

where $\mathscr{D}(q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is the diagonal contribution from $\delta_{a_2m=a_1n}$, and $\mathscr{K}(q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is the contribution from $\sigma_\chi(a_2m, a_1n)$.

In Section 4A, we show that the term $\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ contributes one of the twenty terms in Conjecture 2.4, namely the term corresponding to $\mathcal{M}(q;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)$. Moreover, $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ gives another nine terms in the conjecture, namely those transpositions in $S_6/S_3\times S_3$ which switch α_i and β_j for a fixed $i,j=1,2,3$. We explicitly work out one of these terms in Proposition 6.1. Similarly, $\mathscr{D}(q;\boldsymbol{\beta},\boldsymbol{\alpha})$ gives rise to the term corresponding to $\mathcal{M}(q;\boldsymbol{\beta}+it,\boldsymbol{\alpha}+it)$, and the last nine expressions arise from $\mathscr{K}(q;\boldsymbol{\beta},\boldsymbol{\alpha})$.

4A. ***Evaluating the diagonal terms $\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$.*** Recall that

$$\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})=\frac{2}{\phi(q)}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi(-1)=(-1)^k}}\sum\sum\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2,m,n\geq 1\\ a_1n=a_2m}}\chi(a_1b_1)\overline{\chi}(a_2b_2)\\ \times\frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2b_2}\frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{(a_1n)^{1/2}(a_2m)^{1/2}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n,a_2^3b_2^2m;q).$$

We compute the diagonal contribution in the following lemma.

Lemma 4.1. *With the same notation as above, we have*

$$\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})=H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\int_{-\infty}^{\infty}\mathcal{M}(q;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)\,dt+O(q^{-3/4+\varepsilon}).$$

Proof. We apply the orthogonality relation for Dirichlet characters in (2-10) and obtain that for $(a_ib_i,q)=1$,

$$\frac{2}{\phi(q)}\sum_{\substack{\chi \ (\mathrm{mod}\ q)\\ \chi(-1)=(-1)^k}}\chi(a_1b_1)\overline{\chi}(a_2b_2)=\begin{cases}1 & \text{if } a_1b_1\equiv a_2b_2 \ (\mathrm{mod}\ q),\\ (-1)^k & \text{if } a_1b_1\equiv -a_2b_2 \ (\mathrm{mod}\ q),\\ 0 & \text{otherwise.}\end{cases}$$

When $a_1b_1\geq q/4$ or $a_2b_2\geq q/4$, we have that $a_1^3b_1^2n\geq q^2/16$ or $a_2^3b_2^2m\geq q^2/16$. From Lemma 3.3, $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n,a_2^3b_2^2m;q)\ll q^{-A}$ in that range, so the contribution from these terms is negligible. Hence the main contribution from $\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ comes from the terms with $a_1b_1=a_2b_2$ when $(a_ib_i,q)=1$, and

$$\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})=\sum\sum\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2,m,n\geq 1\\ a_1n=a_2m\\ a_1b_1=a_2b_2\\ (a_ib_i,q)=1}}\frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2b_2}\\ \times\frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{(a_1n)^{1/2}(a_2m)^{1/2}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n,a_2^3b_2^2m;q)+O(q^{-A}).$$

Since $a_1b_1=a_2b_2$ and $a_1n=a_2m$, $a_1^3b_1^2n=a_2^3b_2^2m$. Therefore, $\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ can be written as

$$\frac{1}{2\pi i}\Big(\frac{q}{4\pi^2}\Big)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\int_{-\infty}^{\infty}\int_{(1)} G\big(\tfrac{1}{2}+s;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\big) \times H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\Big(\frac{q}{4\pi^2}\Big)^{3s}\mathcal{AZ}\big(\tfrac{1}{2}+s,\boldsymbol{\alpha},\boldsymbol{\beta}\big)\,\frac{ds}{s}\,dt,$$

where we have used equations (2-13)–(2-17).

Note that $\mathcal{A}(s;\boldsymbol{\alpha},\boldsymbol{\beta})$ is absolutely convergent when $\mathrm{Re}(s)>\frac{1}{4}+\varepsilon$. Furthermore, the pole at $s=-\frac{1}{2}(\alpha_i-\beta_j)$ from the zeta factor $\mathcal{Z}\big(\frac{1}{2}+s;\boldsymbol{\alpha},\boldsymbol{\beta}\big)$ is canceled by the zero at the same point from $H(s;\boldsymbol{\alpha},\boldsymbol{\beta})$. Thus, in the region $\mathrm{Re}(s)>-\frac{1}{4}+\varepsilon$, the integrand is analytic except for a simple pole at $s=0$. Moving the line of integration to $\mathrm{Re}(s)=-\frac{1}{4}+\varepsilon$, we obtain that $\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ is

$$\Big(\frac{q}{4\pi^2}\Big)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\int_{-\infty}^{\infty} G\big(\tfrac{1}{2};\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\big)H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{AZ}\big(\tfrac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)\,dt+O(q^{-3/4+\varepsilon}).$$

The lemma now follows from (2-18) and $\mathcal{AZ}\big(\frac{1}{2};\boldsymbol{\alpha},\boldsymbol{\beta}\big)=\mathcal{AZ}\big(\frac{1}{2};\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\big)$. □

5. Setup for the off-diagonal terms $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$

Define $\mathcal{K}f=i^{-k}f+i^k\bar{f}$. If g is a real function, then $g\mathcal{K}f=\mathcal{K}(gf)$. Applying the orthogonality relation for χ from (2-10) to $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$, we obtain that

$$\begin{aligned}\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta}) &= 2\pi\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\ge 1\\(a_1a_2b_1b_2,q)=1}}\frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2b_2}\\ &\quad\times\sum\sum_{n,m\ge 1}\frac{\sigma_3(n;\boldsymbol{\alpha})}{(a_1n)^{1/2}}\frac{\sigma_3(m;-\boldsymbol{\beta})}{(a_2m)^{1/2}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n,a_2^3b_2^2m;q)\\ &\quad\times\sum_{c=1}^{\infty}\frac{1}{cq}J_{k-1}\Big(\frac{4\pi}{cq}\sqrt{a_2ma_1n}\Big)\mathcal{K}\sideset{}{^*}\sum_{\substack{a \ (\mathrm{mod}\ cq)\\ a\equiv\overline{a_1b_1}a_2b_2\ (\mathrm{mod}\ q)}}\mathrm{e}\Big(\frac{aa_2m+\bar{a}a_1n}{cq}\Big),\end{aligned}$$

where $\sum^*$ denotes a sum over reduced residues. Let f be a smooth partition of unity such that

$$\sideset{}{^d}\sum_{M} f\Big(\frac{m}{M}\Big)=1,$$

where f is supported in $\big[\frac{1}{2},3\big]$ and $\sum^d_M$ denotes a dyadic sum over $M=2^k$, $k\ge 0$.

Rearranging the sum, we have

$$\begin{aligned}\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta}) &= \frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\ge 1\\(a_1a_2b_1b_2,q)=1}}\frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1^{3/2}b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2^{3/2}b_2}\\ &\quad\times\sideset{}{^d}\sum_{M}\sideset{}{^d}\sum_{N} S(\boldsymbol{a},\boldsymbol{b},M,N;\boldsymbol{\alpha},\boldsymbol{\beta}),\end{aligned}$$

where

$$S(\boldsymbol{a},\boldsymbol{b},M,N;\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum_{c=1}^{\infty}\frac{1}{c}\sum\sum_{m,n\geq 1}\frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{n^{1/2}m^{1/2}}\mathcal{F}(\boldsymbol{a},\boldsymbol{b},m,n,c),$$

$$\mathcal{F}(\boldsymbol{a},\boldsymbol{b},m,n,c) := \mathcal{G}(\boldsymbol{a},\boldsymbol{b},m,n,c) \times \mathcal{K}\sum^{*}_{\substack{a \ (\mathrm{mod}\ cq)\\ a\equiv \overline{a_1b_1}a_2b_2 \ (\mathrm{mod}\ q)}} \mathrm{e}\Big(\frac{aa_2m+\bar{a}a_1n}{cq}\Big)J_{k-1}\Big(\frac{4\pi}{cq}\sqrt{a_2ma_1n}\Big), \tag{5-1}$$

$$\mathcal{G}(\boldsymbol{a},\boldsymbol{b},m,n,c) := V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n, a_2^3b_2^2m; q)f\Big(\frac{m}{M}\Big)f\Big(\frac{n}{N}\Big). \tag{5-2}$$

As described in the outline of the paper, we now endeavor to compute $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$.

We write

$$\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathscr{K}_M(q;\boldsymbol{\alpha},\boldsymbol{\beta}) + \mathscr{K}_E(q;\boldsymbol{\alpha},\boldsymbol{\beta}), \tag{5-3}$$

letting $\mathscr{K}_M(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ represent the contribution from the sum over $c < C$, where $C = \sqrt{a_1a_2MN}/q^{2/3}$, and $\mathscr{K}_E(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ the rest. We show that the contribution from $\mathscr{K}_E(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ is small in Section 5A. This is possible by the decay of the Bessel functions, and such a truncation bounds the size of the conductor inside the exponential sum.

For $\mathscr{K}_M(q;\boldsymbol{\alpha},\boldsymbol{\beta})$, we start by reducing the conductor inside the exponential sum from cq to c in Section 5B. This step takes advantage of the average over $\chi \pmod q$.

Before showing each step, we provide properties of Bessel functions that will be used later.

Lemma 5.1. *We have*

$$J_{k-1}(2\pi x) = \frac{1}{\pi\sqrt{x}}\big\{W(2\pi x)\,\mathrm{e}\big(x-\tfrac{1}{4}k+\tfrac{1}{8}\big) + \overline{W}(2\pi x)\,\mathrm{e}\big(-x+\tfrac{1}{4}k-\tfrac{1}{8}\big)\big\}, \tag{5-4}$$

where $W^{(j)}(x) \ll_{j,k} x^{-j}$. Moreover,

$$J_{k-1}(2x) = \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{x^{2\ell+k-1}}{\ell!(\ell+k-1)!}, \tag{5-5}$$

and

$$J_{k-1}(x) \ll \min(x^{-1/2}, x^{k-1}). \tag{5-6}$$

Finally, when calculating the main terms of $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$, we use the fact that for $\alpha,\beta,\gamma > 0$,

$$\mathcal{K}\int_0^{\infty}\mathrm{e}((\alpha+\beta)x+\gamma x^{-1})J_{k-1}\big(4\pi\sqrt{\alpha\beta x}\big)\,\frac{dx}{x} = 2\pi\, J_{k-1}\big(4\pi\sqrt{\alpha\gamma}\big)J_{k-1}\big(4\pi\sqrt{\beta\gamma}\big), \tag{5-7}$$

and the integration is 0 *if $\alpha,\beta > 0$ and $\gamma \leq 0$.*

These results are standard. We refer the reader to [Watson 1944] for the first three claims, and to [Oberhettinger 1972] for the last claim.

5A. ***Truncating the sum over c.*** In this section we show that we can truncate the sum over c in $S(\boldsymbol{a}, \boldsymbol{b}, M, N; \boldsymbol{\alpha}, \boldsymbol{\beta})$ with small error contribution.

Proposition 5.2. *Let $C = \sqrt{a_1 a_2 MN}/q^{2/3}$, $k \geq 5$, and $\mathcal{F}(\boldsymbol{a}, \boldsymbol{b}, m, n, c)$ be defined as in* (5-1). *Further, let*

$$\mathscr{K}_E(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{2\pi}{q} \sum_{\substack{a_1, b_1, a_2, b_2 \geq 1 \\ (a_1 a_2 b_1 b_2, q) = 1}} \sum\sum\sum \frac{\mathscr{B}(a_1, b_1; \boldsymbol{\alpha})}{a_1^{3/2} b_1} \frac{\mathscr{B}(a_2, b_2; -\boldsymbol{\beta})}{a_2^{3/2} b_2} \times \sum_M^d \sum_N^d S_E(\boldsymbol{a}, \boldsymbol{b}, M, N; \boldsymbol{\alpha}, \boldsymbol{\beta}),$$

where

$$S_E(\boldsymbol{a}, \boldsymbol{b}, M, N; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{c \geq C} \frac{1}{c} \sum_{m, n \geq 1} \sum \frac{\sigma_3(n; \boldsymbol{\alpha}) \sigma_3(m; -\boldsymbol{\beta})}{n^{1/2} m^{1/2}} \mathcal{F}(\boldsymbol{a}, \boldsymbol{b}, m, n, c).$$

Then $\mathscr{K}_E(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll q^{-5/12+\varepsilon}$.

Proof. Note that the contribution of terms when $a_1^3 b_1^2 N$ or $a_2^3 b_2^2 M \gg q^{3/2+\varepsilon}$ is $\ll_{\varepsilon, A} q^{-A}$, due to the fast decay rate of $\mathcal{G}(\boldsymbol{a}, \boldsymbol{b}, m, n, c)$ defined in (5-2). Thus we discount such terms in the rest of the proof. For $k \geq 5$, we let

$$\begin{aligned} \tilde{\Delta}(m, n) &= \tilde{\Delta}(m, n, \boldsymbol{a}, \boldsymbol{b}; C) \\ &= \sum_{c \geq C} \frac{1}{c} \mathcal{K} \sideset{}{^*}\sum_{\substack{a \ (\mathrm{mod}\ cq) \\ a \equiv \overline{a_1 b_1} a_2 b_2 \ (\mathrm{mod}\ q)}} \mathrm{e}\Big(\frac{a a_2 m + \bar{a} a_1 n}{cq}\Big) J\Big(\frac{4\pi \sqrt{a_1 a_2 mn}}{cq}\Big) \\ &= \tilde{\Delta}_1(m, n) + \tilde{\Delta}_2(m, n), \end{aligned}$$

where

$$\tilde{\Delta}_1(m, n) := \sum_{\substack{c \geq C \\ (c, q) = 1}} \frac{1}{c} \mathcal{K} \sideset{}{^*}\sum_{\substack{a \ (\mathrm{mod}\ cq) \\ a \equiv \overline{a_1 b_1} a_2 b_2 \ (\mathrm{mod}\ q)}} \mathrm{e}\Big(\frac{a a_2 m + \bar{a} a_1 n}{cq}\Big) J\Big(\frac{4\pi \sqrt{a_1 a_2 mn}}{cq}\Big),$$

and $\tilde{\Delta}_2(m, n)$ is the sum of the terms where $(c, q) > 1$. Now for $(c, q) = 1$, the Weil bound gives

$$\sideset{}{^*}\sum_{\substack{a \ (\mathrm{mod}\ cq) \\ a \equiv \overline{a_1 b_1} a_2 b_2 \ (\mathrm{mod}\ q)}} \mathrm{e}\Big(\frac{a a_2 m + \bar{a} a_1 n}{cq}\Big) \ll c^{1/2+\varepsilon} \sqrt{(a_1 m, a_2 n, c)},$$

and from the bound in (5-6), we have

$$J\Big(\frac{4\pi \sqrt{a_1 a_2 mn}}{cq}\Big) \ll \Big(\frac{\sqrt{a_1 a_2 mn}}{cq}\Big)^{k-1}.$$

When $a_1^3b_1^2N$ and $a_2^3b_2^2M \ll q^{3/2+\varepsilon}$, we obtain that for $k \geq 5$,

$$\sum_{m,n} \frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{\sqrt{mn}} \tilde{\Delta}_1(m,n)\mathcal{G}(\boldsymbol{a},\boldsymbol{b},m,n,c) \ll q^{\varepsilon}M^{k/2}N^{k/2}\sum_{c\geq C}\frac{1}{c}c^{1/2+\varepsilon}\biggl(\frac{\sqrt{a_1a_2}}{cq}\biggr)^{k-1} \ll q^{7/12+\varepsilon}.$$

In the above, we have used that $\max(a_1^3b_1^2N, a_2^3b_2^2M) \ll q^{3/2+\varepsilon}$. Then summing over a_1, b_1, a_2, b_2 gives the desired bound.

Now, for $(c,q) > 1$, we use the bound

$$\sum_{\substack{a \ (\mathrm{mod}\ cq) \\ a\equiv \overline{a_1b_1}a_2b_2 \ (\mathrm{mod}\ q)}}^{*} \mathrm{e}\biggl(\frac{aa_2m+\bar{a}a_1n}{cq}\biggr) \ll (cq)^{1/2+\varepsilon}\sqrt{(a_1m, a_2n, cq)}.$$

Hence, for $q \mid c$ and $k \geq 5$, we obtain

$$\sum_{m,n} \frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{\sqrt{mn}} \tilde{\Delta}_2(m,n)\mathcal{G}(\boldsymbol{a},\boldsymbol{b},m,n,c) \ll q^{\varepsilon}M^{k/2}N^{k/2}\sum_{\substack{c\geq C \\ q\mid c}}\frac{1}{c}(qc)^{1/2}\biggl(\frac{\sqrt{a_1a_2}}{cq}\biggr)^{k-1} \ll q^{1/12+\varepsilon}.$$

Then summing over a_1, b_1, a_2, b_2 gives the desired bound. □

From this proposition, we are left to consider only

$$\mathscr{K}_M(q;\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1 \\ (a_1a_2b_1b_2,q)=1}} \frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1^{3/2}b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2^{3/2}b_2} \times \sum_M^d \sum_N^d S_M(\boldsymbol{a},\boldsymbol{b},M,N;\boldsymbol{\alpha},\boldsymbol{\beta}), \tag{5-8}$$

where

$$S_M(\boldsymbol{a},\boldsymbol{b},M,N;\boldsymbol{\alpha},\boldsymbol{\beta}) := \sum_{c<C}\frac{1}{c}\sum_{m,n\geq 1}\frac{\sigma_3(n;\boldsymbol{\alpha})\sigma_3(m;-\boldsymbol{\beta})}{n^{1/2}m^{1/2}}\mathcal{F}(\boldsymbol{a},\boldsymbol{b},m,n,c). \tag{5-9}$$

5B. ***Treatment of the exponential sum.*** Next, we reduce the conductor in the exponential sum in $\mathcal{F}(\boldsymbol{a},\boldsymbol{b},m,n,c)$ before applying Voronoi summation.

Lemma 5.3 (treatment of the exponential sum). *Assume that* $(c,q) = 1$ *and let*

$$Y := \sum_{\substack{a \ (\mathrm{mod}\ cq) \\ a\equiv \overline{a_1b_1}a_2b_2 \ (\mathrm{mod}\ q)}}^{*} \mathrm{e}\biggl(\frac{au+\bar{a}v}{cq}\biggr).$$

Then we have

$$Y=\mathrm{e}\left(\frac{(a_2b_2)^2u+(a_1b_1)^2v}{cqa_1b_1a_2b_2}\right)\sum_{x \pmod c}^{*}\mathrm{e}\left(\frac{\bar{q}(a_1b_1x-a_2b_2)u}{a_1b_1c}\right)\mathrm{e}\left(\frac{\bar{q}(a_2b_2\bar{x}-a_1b_1)v}{a_2b_2c}\right).$$

Proof. By the Chinese remainder theorem, for each $a \pmod{cq}$, there exist unique $x \pmod c$ and $y \pmod q$ such that

$$a = xq\bar{q} + yc\bar{c}, \tag{5-10}$$

where $\bar{q}$ denotes the inverse of $q \pmod c$, and $\bar{c}$ denotes the inverse of $c \pmod q$. Using (5-10) and the reciprocity relation

$$\frac{\bar{a}}{b}+\frac{\bar{b}}{a}\equiv\frac{1}{ab} \pmod 1,$$

where $(a, b) = 1$, $\bar{a}$ is the inverse of $a \pmod b$, and $\bar{b}$ is the inverse of $b \pmod a$, we obtain that

$$Y=\mathrm{e}\left(\frac{\overline{a_1b_1}a_2b_2\bar{c}u+a_1b_1\overline{a_2b_2}\bar{c}v}{q}\right)\sum_{x \pmod c}^{*}\mathrm{e}\left(\frac{x\bar{q}u+\bar{x}\bar{q}v}{c}\right).$$

Thus

$$Y=\mathrm{e}\left(\frac{(a_2b_2)^2u+(a_1b_1)^2v}{cqa_1b_1a_2b_2}\right)\sum_{x \pmod c}^{*}\mathrm{e}\left(\frac{x\bar{q}u+\bar{x}\bar{q}v}{c}\right)\mathrm{e}\left(\frac{-\bar{q}a_2b_2u}{ca_1b_1}\right)\mathrm{e}\left(\frac{-\bar{q}a_1b_1v}{ca_2b_2}\right),$$

and the lemma follows. □

Note that when $c < C < q$, we automatically have $(c, q) = 1$. The point of this lemma is that we may treat

$$\mathrm{e}\left(\frac{(a_2b_2)^2u+(a_1b_1)^2v}{cqa_1b_1a_2b_2}\right)$$

as a smooth function with small derivatives, while the other exponentials have conductor at most $ca_ib_i \le q^{1+\varepsilon}$ after truncation. It should be noted, however, that we are most concerned with the contribution from the transition region of the Bessel function, where the conductor ca_ib_i should be thought of as around size $q^{1/2}$.

6. Applying Voronoi summation

To calculate $\mathscr{K}_M(q; \boldsymbol{\alpha}, \boldsymbol{\beta})$ from Section 5A, we first evaluate $S_M(\boldsymbol{a}, \boldsymbol{b}, M, N; \boldsymbol{\alpha}, \boldsymbol{\beta})$, defined in (5-9). We write

$$S_M(\boldsymbol{a}, \boldsymbol{b}, M, N; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{c<C}\sum_{x \pmod c}^{*}\frac{1}{c}(\mathcal{S}^{+}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c, x)+\mathcal{S}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c, x)),$$

in which

$$\mathcal{S}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x) = i^{\mp k}\sum_{m\geq 1}\frac{\sigma_3(m;-\boldsymbol{\beta})}{m^{1/2}} f\Big(\frac{m}{M}\Big)\,\mathrm{e}\Big(\pm\frac{a_2^2b_2m}{cqa_1b_1}\Big)\,\mathrm{e}\Big(\pm\frac{\bar{q}(a_1b_1x-a_2b_2)a_2m}{a_1b_1c}\Big)$$
$$\times\sum_{n\geq 1}\sigma_3(n;\boldsymbol{\alpha})F^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n,c)\,\mathrm{e}\Big(\pm\frac{\bar{q}(a_2b_2\bar{x}-a_1b_1)a_1n}{a_2b_2c}\Big),$$

with

$$F^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(m,n,c) = \frac{1}{n^{1/2}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2n, a_2^3b_2^2m; q)J_{k-1}\Big(\frac{4\pi}{cq}\sqrt{a_2ma_1n}\Big)\,\mathrm{e}\Big(\pm\frac{a_1^2b_1n}{cqa_2b_2}\Big)f\Big(\frac{n}{N}\Big).$$

Let

$$\frac{\lambda_1}{\eta_1} = \frac{\bar{q}(a_2b_2\bar{x}-a_1b_1)a_1}{a_2b_2c} \tag{6-1}$$

and

$$\frac{\lambda_2}{\eta_2} = \frac{\bar{q}(a_1b_1x-a_2b_2)a_2}{a_1b_1c}, \tag{6-2}$$

where $(\lambda_1,\eta_1)=(\lambda_2,\eta_2)=1$. Moreover, define

$$\mathcal{V}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c;\boldsymbol{a},\boldsymbol{b},y,z) = \frac{1}{y^{1/2}}\frac{1}{z^{1/2}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2y, a_2^3b_2^2z; q)J_{k-1}\bigg(\frac{4\pi\sqrt{a_2za_1y}}{cq}\bigg)$$
$$\times i^{\mp k}\,\mathrm{e}\Big(\pm\frac{a_1^2b_1y}{cqa_2b_2}\pm\frac{a_2^2b_2z}{cqa_1b_1}\Big)f\Big(\frac{y}{N}\Big)f\Big(\frac{z}{M}\Big).$$

We then apply Voronoi summation as in Theorem B.1 to the sum over n,m and obtain that $\mathcal{S}^{+}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)+\mathcal{S}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)$ is

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\operatorname*{Res}_{s_1=1-\alpha_i}\operatorname*{Res}_{s_2=1+\beta_j}(\mathcal{T}^{+}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2)+\mathcal{T}^{-}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2))$$
$$+\sum_{i=1}^{8}(\mathcal{T}^{+}_{i,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)+\mathcal{T}^{-}_{i,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)),$$

where in the region of absolute convergence,

$$\mathcal{T}^{\pm}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2) := \mathcal{F}^{\pm}_{M}(c;\boldsymbol{\alpha},\boldsymbol{\beta})D_3\Big(s_1,\pm\frac{\lambda_1}{\eta_1},\boldsymbol{\alpha}\Big)D_3\Big(s_2,\pm\frac{\lambda_2}{\eta_2},-\boldsymbol{\beta}\Big), \tag{6-3}$$
$$\mathcal{F}^{\pm}_{M}(c;\boldsymbol{\alpha},\boldsymbol{\beta}) := \mathcal{F}^{\pm}_{M}(c,s_1,s_2;\boldsymbol{\alpha},\boldsymbol{\beta})$$
$$:= \int_0^\infty\int_0^\infty y^{s_1-1}z^{s_2-1}\mathcal{V}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c;\boldsymbol{a},\boldsymbol{b},y,z)\,dy\,dz,$$

$$\mathcal{T}^{\pm}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x) := \frac{\pi^{3/2+\alpha_1+\alpha_2+\alpha_3}}{\eta_1^{3+\alpha_1+\alpha_2+\alpha_3}} \sum_{i=1}^{3} \operatorname*{Res}_{s=1+\beta_i} D_3\Big(s_2, \pm\frac{\lambda_2}{\eta_2}, -\boldsymbol{\beta}\Big) \times \sum_{n=1}^{\infty} A_3\Big(n, \pm\frac{\lambda_1}{\eta_1}, \boldsymbol{\alpha}\Big)\mathcal{F}_1^{\pm}(c,n;\boldsymbol{\alpha},\boldsymbol{\beta}), \quad (6\text{-}4)$$

$$\mathcal{F}_1^{\pm}(c,n;\boldsymbol{\alpha},\boldsymbol{\beta}) := \mathcal{F}_1^{\pm}(c,n,s;\boldsymbol{\alpha},\boldsymbol{\beta}) := \int_0^\infty\int_0^\infty z^{s-1}\mathcal{V}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c;\boldsymbol{a},\boldsymbol{b},y,z)U_3\Big(\frac{\pi^3 ny}{\eta_1^3};\boldsymbol{\alpha}\Big)\,dy\,dz,$$

and $\mathcal{T}^{\pm}_{i,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)$ is defined similarly for $i=2,3,4$. Further,

$$\mathcal{T}^{\pm}_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x) := \frac{\pi^{3+\alpha_1-\beta_1+\alpha_2-\beta_2+\alpha_3-\beta_3}}{\eta_1^{3+\alpha_1+\alpha_2+\alpha_3}\eta_2^{3-\beta_1-\beta_2-\beta_3}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_3\Big(n,\pm\frac{\lambda_1}{\eta_1},\boldsymbol{\alpha}\Big) \times A_3\Big(m,\pm\frac{\lambda_2}{\eta_2},-\boldsymbol{\beta}\Big)\mathcal{F}_5^{\pm}(c,n,m;\boldsymbol{\alpha},\boldsymbol{\beta}), \quad (6\text{-}5)$$

$$\mathcal{F}_5^{\pm}(c,n,m;\boldsymbol{\alpha},\boldsymbol{\beta}) := \int_0^\infty\int_0^\infty \mathcal{V}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c;\boldsymbol{a},\boldsymbol{b},y,z) \times U_3\Big(\frac{\pi^3 mz}{\eta_2^3};-\boldsymbol{\beta}\Big)U_3\Big(\frac{\pi^3 ny}{\eta_1^3};\boldsymbol{\alpha}\Big)\,dy\,dz,$$

and $\mathcal{T}^{\pm}_{i,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)$ is defined similarly for $i=6,7,8$.

As mentioned in Section 4, there are nine terms from $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$. In particular, we will show that these terms arise from

$$\sum_{c<C}\ \sideset{}{^*}\sum_{x \ (\mathrm{mod}\ c)} \frac{1}{c}(\mathcal{T}^{+}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2)+\mathcal{T}^{-}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2)),$$

and in fact each term comes from the residues at $s_1=1-\alpha_i$ and $s_2=1+\beta_j$ for $i=1,2,3$. We state the contribution from the residues $s_1=1-\alpha_1$ and $s_2=1+\beta_1$ in Proposition 6.1 below, and prove it in Section 7. By symmetry, the analogous result holds for the other residues. Then, we will show that the rest of $\mathcal{T}^{\pm}_{i,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)$ are negligible in Section 8 as stated in Proposition 6.2.

Proposition 6.1. *Let*

$$R_{\alpha_1,\beta_1} := \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\ge 1\\(a_1a_2b_1b_2,q)=1}} \frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1^{3/2}b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2^{3/2}b_2} \times \sideset{}{^d}\sum_{M}\ \sideset{}{^d}\sum_{N}\ \sum_{c<C}\ \sideset{}{^*}\sum_{x\ (\mathrm{mod}\ c)} \frac{1}{c}\operatorname*{Res}_{s_1=1-\alpha_1}\ \operatorname*{Res}_{s_2=1+\beta_1}(\mathcal{T}^{+}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2) + \mathcal{T}^{-}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2)).$$

Then we have

$$R_{\alpha_1,\beta_1} = H(0; \boldsymbol{\alpha}, \boldsymbol{\beta}) \int_{-\infty}^{\infty} \mathcal{M}(q; \pi(\boldsymbol{\alpha}) + it, \pi(\boldsymbol{\beta}) + it)\, dt + O(q^{-1/2+\varepsilon}),$$

where $(\pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})) = (\beta_1, \alpha_2, \alpha_3; \alpha_1, \beta_2, \beta_3)$.

Proposition 6.2. *For* $i = 1, \dots, 8$, *define*

$$\mathcal{E}_i^{\pm}(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2 \ge 1 \\ (a_1a_2b_1b_2,q)=1}} \frac{\mathscr{B}(a_1, b_1; \boldsymbol{\alpha})}{a_1^{3/2} b_1} \frac{\mathscr{B}(a_2, b_2; -\boldsymbol{\beta})}{a_2^{3/2} b_2} \times \sum_M^d \sum_N^d \sum_{c<C} \sideset{}{^*}\sum_{x \pmod c} \frac{\mathcal{T}^{\pm}_{i,\boldsymbol{\alpha},\boldsymbol{\beta}}(c, x)}{c}.$$

Then

$$\mathcal{E}_i^{\pm}(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll q^{-1/4+\varepsilon}.$$

We will prove this proposition in Section 8.

7. Proof of Proposition 6.1

We begin by collecting some lemmas which will be used in this section.

7A. *Preliminary lemmas.*

Lemma 7.1. *Let* $(a, \ell) = 1$. *We have*

$$f(c, \ell) := \sideset{}{^*}\sum_{\substack{x \pmod{c\ell} \\ x \equiv a \pmod{\ell}}} 1 = c \prod_{\substack{p \mid c \\ p \nmid \ell}} \Big(1 - \frac{1}{p}\Big) = \phi(c) \prod_{p \mid (\ell, c)} \Big(1 - \frac{1}{p}\Big)^{-1}.$$

Proof. We first prove that if $(m, n\ell) = 1$, then

$$f(mn, \ell) = f(n, \ell)\phi(m). \tag{7-1}$$

For all x satisfying $(x, mn\ell) = 1$, we can write $x = um\bar{m} + vn\ell\overline{n\ell}$ such that $m\bar{m} \equiv 1 \pmod{n\ell}$, $n\ell\overline{n\ell} \equiv 1 \pmod m$, and $(u, n\ell) = (v, m) = 1$. Moreover, $x \equiv a \pmod \ell$ if and only if $u \equiv a \pmod \ell$. By the Chinese remainder theorem,

$$f(mn, \ell) = \sideset{}{^*}\sum_{\substack{x \pmod{mn\ell} \\ x \equiv a \pmod{\ell}}} 1 = \sideset{}{^*}\sum_{\substack{u \pmod{n\ell} \\ u \equiv a \pmod{\ell}}} \sideset{}{^*}\sum_{v \pmod m} 1 = f(n, \ell)\phi(m).$$

Let $c = c_1c_2$, where all prime factors of c_1 also divide ℓ, and $(c_2, \ell) = 1$. From (7-1), we have that $f(c, \ell) = f(c_1, \ell)\phi(c_2)$.

Now let x be any residue modulo $c_1\ell$ with $x \equiv a \pmod \ell$. Then $(x, c_1\ell) = 1$ since $(a, \ell) = 1$. Thus all such x can be uniquely written as $x = a + k\ell$, where $k = 0, \dots, c_1 - 1$, so $f(c_1, \ell) = c_1$. We then have $f(c, \ell) = c_1\phi(c_2)$, and the statement follows from the identity $\phi(c_2) = c_2 \prod_{p \mid c_2}(1 - 1/p)$. □

Lemma 7.2. *Let α, β, y, z be nonnegative real numbers satisfying $\alpha y, \beta z \ll q^2$ and define*

$$T = T(y, z, \alpha, \beta) := \sum_{\delta=1}^{\infty} \frac{1}{\delta} J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{\delta}\right)\mathcal{K}\,\mathrm{e}\left(\frac{\alpha y}{\delta} + \frac{\beta z}{\delta}\right).$$

Further, let $L = q^{100}$ and w be a smooth function on $\mathbb{R}^+$ with $w(x) = 1$ if $0 \le x \le 1$, and $w(x) = 0$ if $x > 2$. Then for any $A > 0$, we have

$$T = 2\pi \sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\left(4\pi\sqrt{\alpha y \ell}\right) J_{k-1}\left(4\pi\sqrt{\beta z \ell}\right) - 2\pi \int_0^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\left(4\pi\sqrt{\alpha y \ell}\right) J_{k-1}\left(4\pi\sqrt{\beta z \ell}\right) d\ell + O_A(q^{-A}).$$

Proof. We follow the arguments of Section 3 of [Iwaniec and Li 2007] to evaluate T. Let $\eta(s)$ be a smooth function on $\mathbb{R}^+$ with $\eta(s) = 0$ if $0 \le s < \frac{1}{4}$, $0 \le \eta(s) \le 1$ if $\frac{1}{4} \le s \le \frac{1}{2}$, and $\eta(s) = 1$ if $s > \frac{1}{2}$. We then obtain that

$$T = \mathcal{K} \sum_{\delta} \frac{\eta(\delta)}{\delta} J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{\delta}\right) \mathrm{e}\left(\frac{\alpha y}{\delta} + \frac{\beta z}{\delta}\right).$$

After inserting this smooth function we apply Poisson summation to obtain that

$$T = \sum_{\ell} \hat{F}(\ell) := \sum_{\ell} \mathcal{K} \int_0^{\infty} \frac{\eta(u)}{u} \mathrm{e}\left(\ell u + \frac{\alpha y}{u} + \frac{\beta z}{u}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) du.$$

By (5-4), we can write the integral above in terms of two integrals with the phase

$$\ell u + \frac{\alpha y \pm \sqrt{\alpha\beta yz} + \beta z}{u}.$$

If $|\ell| > L$, the factor ℓu dominates. Then integrating by parts A times, we have that

$$\int_0^{\infty} \frac{\eta(u)}{u} \mathrm{e}\left(\ell u + \frac{\alpha y}{u} + \frac{\beta z}{u}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) du \ll q^{-A}.$$

Therefore,

$$T = \sum_{\ell} \hat{F}(\ell) w\left(\frac{|\ell|}{L}\right) + O(q^{-A}).$$

Now, we write $\sum_{\ell} \hat{F}(\ell) w(|\ell|/L) = T_1 - T_2$, where

$$T_1 := \sum_{\ell} w\left(\frac{|\ell|}{L}\right) \mathcal{K} \int_0^{\infty} \frac{1}{u} \mathrm{e}\left(\ell u + \frac{\alpha y}{u} + \frac{\beta z}{u}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) du,$$

and

$$T_2 := \sum_{\ell} w\left(\frac{|\ell|}{L}\right)\mathcal{K}\int_0^\infty \frac{1-\eta(u)}{u}\,\mathrm{e}\left(\ell u + \frac{\alpha y}{u} + \frac{\beta z}{u}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) du.$$

We use (5-7) to evaluate T_1 and obtain

$$T_1 = 2\pi\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\big(4\pi\sqrt{\alpha y\ell}\big) J_{k-1}\big(4\pi\sqrt{\beta z\ell}\big). \tag{7-2}$$

For T_2, we note that $\xi(u) = 1-\eta(u) = 1$ if $0 < s < \frac{1}{4}$, $0 \le \xi(u) \le 1$ if $\frac{1}{4} \le s \le \frac{1}{2}$, and $\xi(u) = 0$ if $s > \frac{1}{2}$. Interchanging the sum over ℓ and the integration over u and applying the Poisson summation formula, we have

$$T_2 = \mathcal{K}\int_0^\infty \frac{\xi(u)}{u}\,\mathrm{e}\left(\frac{\alpha y}{\delta} + \frac{\beta z}{\delta}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) \sum_{\ell} L\hat{w}(L(\ell+u))\,du.$$

Since $\hat{w}(y) \ll (1+|y|)^{-A}$, the main contribution comes from $\ell = 0$ and $0 \le u < \frac{1}{4}$. Therefore

$$\begin{aligned} T_2 &= \mathcal{K}\int_0^\infty \frac{\xi(u)}{u}\,\mathrm{e}\left(\frac{\alpha y}{\delta} + \frac{\beta z}{\delta}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) L\hat{w}(Lu)\,du + O(q^{-A}) \\ &= \mathcal{K}\int_0^\infty \frac{1}{u}\,\mathrm{e}\left(\frac{\alpha y}{\delta} + \frac{\beta z}{\delta}\right) J_{k-1}\left(\frac{4\pi\sqrt{\alpha\beta yz}}{u}\right) L\hat{w}(Lu)\,du + O(q^{-A}) \\ &= 2\pi\int_0^\infty w\left(\frac{\ell}{L}\right) J_{k-1}\big(4\pi\sqrt{\alpha y\ell}\big) J_{k-1}\big(4\pi\sqrt{\beta z\ell}\big)\,d\ell + O(q^{-A}), \end{aligned} \tag{7-3}$$

where the last equality comes from Plancherel's formula and (5-7). □

The next lemma deals with the sum and the integral involving ℓ.

Lemma 7.3. *Let w be a smooth function on $\mathbb{R}^+$ with $w(x) = 1$ if $0 \le x \le 1$, and $w(x) = 0$ if $x > 2$. Also let γ be a complex number with $\operatorname{Re}\gamma \ll 1/\log q$, $\operatorname{Re}\gamma < 0$, and $L = q^{100}$. Then*

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right)\frac{1}{\ell^{1+\gamma}} - \int_0^\infty w\left(\frac{\ell}{L}\right)\frac{1}{\ell^{1+\gamma}}\,d\ell = \zeta(1+\gamma) + O(q^{-20}).$$

Proof. Let $\tilde{w}(z)$ be the Mellin transform of w, defined by

$$\tilde{w}(z) = \int_0^\infty w(t)\frac{t^z}{t}\,dt.$$

From the definition, $\tilde{w}(z)$ is analytic for $\operatorname{Re} z > 0$, and integration by parts gives

$$\tilde{w}(z) = -\frac{1}{z}\int_0^\infty w'(t)t^z\,dt,$$

so $\tilde{w}(z)$ can be analytically continued to $\operatorname{Re} z > -1$ except at $z = 0$, where it has a simple pole with residue $w(0) = 1$. For $\sigma > \min\{0, -\operatorname{Re}\gamma\}$, we have

$$\sum_{\ell=1}^{\infty} w\Big(\frac{\ell}{L}\Big)\frac{1}{\ell^{1+\gamma}} = \sum_{\ell=1}^{\infty} \frac{1}{2\pi i}\int_{(\sigma)} \tilde{w}(z)\Big(\frac{\ell}{L}\Big)^{-z}\frac{1}{\ell^{1+\gamma}}\,dz$$
$$= \frac{1}{2\pi i}\int_{(\sigma)} \tilde{w}(z)L^{z}\zeta(1+\gamma+z)\,dz. \tag{7-4}$$

Shifting the contour to $\operatorname{Re}(z) = -\frac{1}{4}$, we have that (7-4) is

$$\zeta(1+\gamma) + \tilde{w}(-\gamma)L^{-\gamma} + O(q^{-20}).$$

The lemma follows from noting that

$$\tilde{w}(-\gamma)L^{-\gamma} = \int_0^{\infty} w\Big(\frac{\ell}{L}\Big)\frac{1}{\ell^{1+\gamma}}\,d\ell. \qquad \square$$

7B. ***Calculation of residues.*** In this section, we calculate

$$\operatorname*{Res}_{s_1=1-\alpha_1}\ \operatorname*{Res}_{s_2=1+\beta_1}\big(\mathcal{T}^{+}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2) + \mathcal{T}^{-}_{M,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x,s_1,s_2)\big).$$

To do this, we essentially need to consider

$$\operatorname*{Res}_{s_1=1-\alpha_1} D_3\Big(s_1, \pm\frac{\lambda_1}{\eta_1}, \boldsymbol{\alpha}\Big)y^{s_1-1},$$

where λ_1/η_1 is defined in (6-1). Let $(a_1b_1, a_2b_2) = \lambda$, $a_1b_1 = u_1\lambda$, $a_2b_2 = u_2\lambda$, where $(u_1, u_2) = 1$. Note that $(u_1x - u_2, c) = (u_2\bar{x} - u_1, c) = \delta$. Hence

$$\delta_1 := ((a_2b_2\bar{x} - a_1b_1)a_1, a_2b_2c) = \lambda((u_2\bar{x} - u_1)a_1, u_2c) = \lambda\delta(a_1, u_2c/\delta),$$

and $\lambda_1 = \bar{q}(a_2b_2\bar{x} - a_1b_1)a_1/\delta_1$ and $\eta_1 = a_2b_2c/\delta_1$.

By (B-4), we obtain that

$$\mathcal{R}_1\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big) := \operatorname*{Res}_{s_1=1-\alpha_1} D_3\Big(s_1, \pm\frac{\lambda_1}{\eta_1}, \boldsymbol{\alpha}\Big)$$
$$= \frac{1}{\eta_1^{2-2\alpha_1+\alpha_2+\alpha_3}} \sum_{\substack{1\le a_1,a_2\le\eta_1\\ \eta_1\mid a_1a_2}}\sum \zeta\Big(1-\alpha_1+\alpha_2, \frac{a_2}{\eta_1}\Big)\zeta\Big(1-\alpha_1+\alpha_3, \frac{a_3}{\eta_1}\Big).$$

Hence

$$\operatorname*{Res}_{s_1=1-\alpha_1} D_3\Big(s_1, \pm\frac{\lambda_1}{\eta_1}, \boldsymbol{\alpha}\Big)y^{s_1-1} = \mathcal{R}_1\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big)y^{-\alpha}.$$

Similarly, we let $\mathcal{R}_2(c, \boldsymbol{a}, \boldsymbol{b}) := \operatorname{Res}_{s_2=1+\beta_1} D_3(s_2, \pm\lambda_2/\eta_2, -\boldsymbol{\beta})$. Then

$$\mathcal{R}_2\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big) = \frac{1}{\eta_2^{2+2\beta_1-\beta_2-\beta_3}} \sum_{\substack{1\le a_1,a_2\le\eta_2\\ \eta_2\mid a_1a_2}}\sum \zeta\Big(1+\beta_1-\beta_2, \frac{a_2}{\eta_2}\Big)\zeta\Big(1+\beta_1-\beta_3, \frac{a_3}{\eta_2}\Big),$$

and

$$\operatorname*{Res}_{s_2=1+\beta_1} D_3\Big(s_1, \pm\frac{\lambda_2}{\eta_2}, \boldsymbol{\alpha}\Big) z^{s_2-1} = \mathcal{R}_2\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big) z^{\beta}.$$

7C. *Computing R_{α_1,β_1}*. From the previous section, R_{α_1,β_1} can be written as

$$R_{\alpha_1,\beta_1} = \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\ge 1\\ (a_1a_2b_1b_2,q)=1}} \frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1^{3/2}b_1} \frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2^{3/2}b_2} \times \sum_M^d \sum_N^d \mathcal{A}(\boldsymbol{a},\boldsymbol{b},M,N) + O(q^{-1/2+\varepsilon}),$$

where $\mathcal{A}(\boldsymbol{a},\boldsymbol{b},M,N)$ is defined as

$$\sum_{c=1}^{\infty} \frac{\mathcal{F}(c)}{c} \sideset{}{^*}\sum_{x \ (\mathrm{mod}\ c)} \mathcal{R}_1\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big)\mathcal{R}_2\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big), \tag{7-5}$$

and

$$\begin{aligned}\mathcal{F}(c) &:= \mathcal{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(c,\boldsymbol{a},\boldsymbol{b})\\ &:= \int_0^\infty\int_0^\infty \frac{1}{y^{1/2+\alpha_1}} \frac{1}{z^{1/2-\beta_1}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2y, a_2^3b_2^2z; q) J_{k-1}\Big(\frac{4\pi\sqrt{a_2ya_1z}}{cq}\Big)\\ &\qquad \times f\Big(\frac{y}{N}\Big) f\Big(\frac{z}{M}\Big) \mathcal{K}\, \mathrm{e}\Big(\frac{a_1^2b_1y}{cqa_2b_2} + \frac{a_2^2b_2z}{cqa_1b_1}\Big)\, dy\, dz.\end{aligned}$$

We remark that we can extend the sum over c to all positive integers in a similar manner as in the truncation argument in Proposition 5.2. Now, we let

$$\frac{1}{c^2}\mathcal{G}(c,\boldsymbol{a},\boldsymbol{b}) := \mathcal{R}_1(c,\boldsymbol{a},\boldsymbol{b})\mathcal{R}_2(c,\boldsymbol{a},\boldsymbol{b}),$$

so that we can write the sum over c in (7-5) as

$$\begin{aligned}&\sum_{c=1}^{\infty} \frac{\mathcal{F}(c)}{c} \sum_{\delta\mid c} \sideset{}{^*}\sum_{\substack{x \ (\mathrm{mod}\ c)\\ (u_1x-u_2,c)=\delta}} \frac{1}{(c/\delta)^2} \mathcal{G}\Big(\frac{c}{\delta}, \boldsymbol{a}, \boldsymbol{b}\Big)\\ &\qquad = \sum_{\delta=1}^{\infty} \frac{1}{\delta} \sum_{c=1}^{\infty} \frac{\mathcal{F}(c\delta)}{c} \sideset{}{^*}\sum_{\substack{x \ (\mathrm{mod}\ c\delta)\\ (u_1x-u_2,c\delta)=\delta}} \frac{\mathcal{G}(c,\boldsymbol{a},\boldsymbol{b})}{c^2}\\ &\qquad = \sum_{\delta=1}^{\infty} \frac{1}{\delta} \sum_{c=1}^{\infty} \frac{\mathcal{F}(c\delta)}{c} \sideset{}{^*}\sum_{x \ (\mathrm{mod}\ c\delta)} \sum_{b\mid((u_1x-u_2)/\delta,c)} \frac{\mu(b)\mathcal{G}(c,\boldsymbol{a},\boldsymbol{b})}{c^2}\\ &\qquad = \sum_{\substack{\delta\ge 1\\ (\delta,u_1u_2)=1}} \frac{1}{\delta} \sum_{\substack{b\ge 1\\ (b,u_1u_2)=1}} \frac{\mu(b)}{b^3} \sum_{c\ge 1} \frac{\mathcal{G}(cb,\boldsymbol{a},\boldsymbol{b})\mathcal{F}(cb\delta)}{c^3} \sideset{}{^*}\sum_{\substack{x \ (\mathrm{mod}\ c\delta b)\\ x\equiv u_2\bar{u}_1 \ (\mathrm{mod}\ b\delta)}} 1,\end{aligned}$$

where the sum over x is 0 if $(u_1u_2, b\delta) \neq 1$ since $(u_1, u_2) = 1$. Applying Lemma 7.1 to the sum over x, we then obtain that

$$\begin{aligned}
&\mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, M, N) \\
&\quad = \sum_{\substack{\delta\geq 1\\ (\delta,u_1u_2)=1}} \frac{1}{\delta} \sum_{\substack{b\geq 1\\ (b,u_1u_2)=1}} \frac{\mu(b)}{b^3} \sum_{c\geq 1} \frac{1}{c^2} \prod_{\substack{p\mid c\\ p\nmid b\delta}} \Big(1-\frac{1}{p}\Big) \mathcal{F}(cb\delta)\mathcal{G}(cb, \boldsymbol{a}, \boldsymbol{b}) \\
&\quad = \sum_{\substack{h\geq 1\\ h\mid u_1u_2}} \frac{\mu(h)}{h} \sum_{\delta\geq 1} \frac{1}{\delta} \sum_{\substack{b\geq 1\\ (b,u_1u_2)=1}} \frac{\mu(b)}{b^3} \sum_{c\geq 1} \frac{1}{c^2} \prod_{\substack{p\mid c\\ p\nmid bh\delta}} \Big(1-\frac{1}{p}\Big) \mathcal{F}(cbh\delta)\mathcal{G}(cb, \boldsymbol{a}, \boldsymbol{b}) \\
&\quad = \sum_{\substack{h\geq 1\\ h\mid u_1u_2}} \frac{\mu(h)}{h} \sum_{\substack{b\geq 1\\ (b,u_1u_2)=1}} \frac{\mu(b)}{b^3} \sum_{c\geq 1} \frac{\mathcal{G}(cb, \boldsymbol{a}, \boldsymbol{b})}{c^2} \\
&\qquad\qquad \times \sum_{\gamma\mid c} \prod_{\substack{p\mid c\\ p\nmid bh\gamma}} \Big(1-\frac{1}{p}\Big) \sum_{\delta\geq 1} \frac{1}{\delta} \mathcal{F}(cbh\delta) \sum_{g\mid (c/\gamma, \delta/\gamma)} \mu(g) \\
&\quad = \sum\nolimits^{\#} \mathscr{G}_{\boldsymbol{a},\boldsymbol{b}}(1; h, b, c, \gamma, g) \sum_{\delta\geq 1} \frac{1}{\delta} \mathcal{F}(cbhg\gamma\delta),
\end{aligned}$$

where

$$\begin{aligned}
&\sum\nolimits^{\#} \mathscr{G}_{\boldsymbol{a},\boldsymbol{b}}(s; h, b, c, \gamma, g) \\
&= \sum_{\substack{h\geq 1\\ h\mid u_1u_2}} \frac{\mu(h)}{h^s} \sum_{\substack{b\geq 1\\ (b,u_1u_2)=1}} \frac{\mu(b)}{b^{2+s}} \sum_{c\geq 1} \frac{\mathcal{G}(cb, \boldsymbol{a}, \boldsymbol{b})}{c^{1+s}} \sum_{\gamma\mid c} \frac{1}{\gamma^s} \sum_{g\mid (c/\gamma)} \frac{\mu(g)}{g^s} \prod_{\substack{p\mid c\\ p\nmid bh\gamma}} \Big(1-\frac{1}{p}\Big). \qquad (7\text{-}6)
\end{aligned}$$

Next, applying Lemma 7.2 to the sum over δ and summing $\sum_M^d \sum_N^d$, we have that

$$\begin{aligned}
&\sum_M^d \sum_N^d \mathcal{A}(\boldsymbol{a}, \boldsymbol{b}, M, N) \\
&\quad = 2\pi \frac{1}{2\pi i} \Big(\frac{q}{4\pi^2}\Big)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})} \int_{-\infty}^{\infty} \int_{(1)} \sum\nolimits^{\#} \mathscr{G}_{\boldsymbol{a},\boldsymbol{b}}(1; h, b, c, \gamma, g) \\
&\qquad \times \Big\{ \sum_{\ell=1}^{\infty} w\Big(\frac{\ell}{L}\Big) - \int_0^{\infty} w\Big(\frac{\ell}{L}\Big)\, d\ell \Big\} G\Big(\frac{1}{2}+s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\Big) H(s; \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
&\qquad \times \bigg(\frac{a_2^3 b_2^2}{a_1^3 b_1^2}\bigg)^{it} (a_1^3 b_1^2 a_2^3 b_2^2)^{-s} \frac{q^{3s}}{(4\pi^2)^{3s}} \\
&\qquad \times \int_0^{\infty}\int_0^{\infty} \frac{y^{-1/2-\alpha_1}}{y^{s+it}} \frac{z^{-1/2+\beta_1}}{z^{s-it}} J_{k-1}\Bigg(4\pi \sqrt{\frac{a_1^2 b_1 y\ell}{a_2 b_2 cbhg\gamma q}}\Bigg) \\
&\qquad \times J_{k-1}\Bigg(4\pi \sqrt{\frac{a_2^2 b_2 z\ell}{a_1 b_1 cbhg\gamma q}}\Bigg)\, dy\, dz\, \frac{ds}{s}\, dt.
\end{aligned}$$

The integration over y and z can be evaluated by Equation 707.14 in [Gradshteyn and Ryzhik 2007], which is

$$\int_0^\infty v^\mu (vk)^{1/2} J_\nu(vk)\,dv = 2^{\mu+1/2}k^{-\mu-1}\frac{\Gamma(\mu/2+\nu/2+3/4)}{\Gamma(\nu/2-\mu/2+1/4)},$$

for $-\operatorname{Re}\nu-\frac{3}{2}<\operatorname{Re}\mu<0$. Then we apply Lemma 7.3 to the sum and the integration over ℓ. Therefore, after summing over a_1, a_2, b_1, b_2, we obtain that the main term of R_{α_1,β_1} is

$$\frac{1}{2\pi i}\Big(\frac{q}{4\pi^2}\Big)^{\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}\int_{-\infty}^{\infty}\int_{(\varepsilon)} G_{\alpha_1,\beta_1}\Big(\frac{1}{2}+s;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it\Big)H(s;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{M}_{\alpha_1,\beta_1}(s) \times\zeta(1-\alpha_1+\beta_1-2s)\frac{q^s}{(4\pi^2)^s}\frac{ds}{s}\,dt, \quad (7\text{-}7)$$

where $(\pi(\alpha),\pi(\beta))=(\beta_1,\alpha_2,\alpha_3;\alpha_1,\beta_2,\beta_3)$,

$$G_{\alpha_i,\beta_j}\Big(\frac{1}{2}+s;\boldsymbol{\alpha},\boldsymbol{\beta}\Big) = \Gamma\Big(\frac{k}{2}-s-\alpha_i\Big)\Gamma\Big(\frac{k}{2}-s+\beta_j\Big)\prod_{\ell\neq i}\Gamma\Big(\frac{k}{2}+s+\alpha_\ell\Big)\prod_{\ell\neq j}\Gamma\Big(\frac{k}{2}+s-\beta_\ell\Big),$$

and

$$\mathcal{M}_{\alpha_1,\beta_1}(s) := \sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\geq 1\\ (a_1a_2b_1b_2,q)=1}} \frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1^{2-2\alpha_1-\beta_1+2s}b_1^{1-\alpha_1-\beta_1+2s}}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2^{2+\alpha_1+2\beta_1+2s}b_2^{1+\alpha_1+\beta_1+2s}} \times\sum\nolimits^{\#}\mathscr{G}_{\boldsymbol{a},\boldsymbol{b}}(2s+\alpha_1-\beta_1;h,b,c,\gamma,g). \quad (7\text{-}8)$$

Now, in the ensuing discussion, we temporarily assume $\operatorname{Re}(\alpha_1)<\operatorname{Re}(\alpha_2),\operatorname{Re}(\alpha_3)$ and $\operatorname{Re}(\beta_1)>\operatorname{Re}(\beta_2),\operatorname{Re}(\beta_3)$. In this region,

$$\mathcal{R}_1\Big(\frac{c}{\delta},\boldsymbol{a},\boldsymbol{b}\Big)=\sum\sum_{\substack{n_2,n_3\geq 1\\ \frac{u_2c}{\delta(a_1,u_2c/\delta)}\mid n_2n_3}}\frac{1}{n_2^{1+\alpha_2-\alpha_1}n_3^{1+\alpha_3-\alpha_1}}$$

and

$$\operatorname*{Res}_{s_1=1-\alpha_i} D_3\Big(s_1,\pm\frac{\lambda_1}{\eta_1},\boldsymbol{\alpha}\Big)y^{s_1-1}=\mathcal{R}_1\Big(\frac{c}{\delta},\boldsymbol{a},\boldsymbol{b}\Big)y^{-\alpha_1} = y^{-\alpha_1}\sum\sum_{\substack{n_2,n_3\geq 1\\ \frac{u_2c}{\delta(a_1,u_2c/\delta)}\mid n_2n_3}}\frac{1}{n_2^{1+\alpha_2-\alpha_1}n_3^{1+\alpha_3-\alpha_1}}.$$

Similarly,

$$\operatorname*{Res}_{s_2=1+\beta_1} D_3\Big(s_2,\pm\frac{\lambda_2}{\eta_2},-\boldsymbol{\beta}\Big)z^{s_2-1}=\mathcal{R}_2\Big(\frac{c}{\delta},\boldsymbol{a},\boldsymbol{b}\Big)z^{\beta_1} = z^{\beta_1}\sum\sum_{\substack{m_2,m_3\geq 1\\ \frac{u_1c}{\delta(a_2,u_1c/\delta)}\mid m_2m_3}}\frac{1}{m_2^{1+\beta_1-\beta_2}m_3^{1+\beta_1-\beta_3}}.$$

Thus,

$$\frac{1}{c^2}\mathcal{G}(c,\boldsymbol{a},\boldsymbol{b})$$
$$= \sum_{\substack{n_2,n_3\ge 1\\ u_2c/(a_1,u_2c)\,|\,n_2n_3}}\sum \frac{1}{n_2^{1+\alpha_2-\alpha_1}n_3^{1+\alpha_3-\alpha_1}} \sum_{\substack{m_2,m_3\ge 1\\ u_1c/(a_2,u_1c)\,|\,m_2m_3}}\sum \frac{1}{m_2^{1+\beta_1-\beta_2}m_3^{1+\beta_1-\beta_3}}$$
$$= \frac{(a_1,u_2c)(a_2,u_1c)}{u_1u_2c^2}\sum_{n=1}^{\infty}\frac{\sigma_2(u_2cn/(a_1,u_2c);\alpha_2-\alpha_1,\alpha_3-\alpha_1)}{n}$$
$$\times\sum_{m=1}^{\infty}\frac{\sigma_2(u_1cm/(a_2,u_1c);\beta_1-\beta_2,\beta_1-\beta_3)}{m}. \tag{7-9}$$

From this, we may then check that

$$\mathcal{M}_{\alpha_1,\beta_1}(s)=\prod_{j=2}^{3}\zeta(1+2s+\alpha_j-\beta_1)\zeta(1+2s+\alpha_1-\beta_j)\mathcal{J}_{\alpha_1,\beta_1}(s), \tag{7-10}$$

where $\mathcal{J}_{\alpha_1,\beta_1}$ is absolutely convergent in the region $\mathrm{Re}(s)=-\frac{1}{4}+\varepsilon$. Although we have a priori only verified (7-10) for the region $\mathrm{Re}(\alpha_1)<\mathrm{Re}(\alpha_2),\mathrm{Re}(\alpha_3)$ and $\mathrm{Re}(\beta_1)>\mathrm{Re}(\beta_2),\mathrm{Re}(\beta_3)$, we see that (7-10) must hold for all values of α_i,β_j by analytic continuation.

We note that the pole of $\zeta(1-\alpha_1+\beta_1-2s)$ at $s=\frac{1}{2}(\alpha_1-\beta_1)$ and the poles of $\zeta(1+2s+\alpha_i-\beta_j)$ at $s=\frac{1}{2}(\alpha_i-\beta_j)$ cancel with the zeros at the same point from $H(s;\boldsymbol{\alpha},\boldsymbol{\beta})$. Thus, the integrand in (7-7) has only a simple pole at $s=0$ and is analytic for all values of s with $\mathrm{Re}\,s>-\frac{1}{4}+\epsilon$. Moving the line of integration to $\mathrm{Re}(s)=-\frac{1}{4}+\varepsilon$, we then obtain the main term

$$\zeta(1-\alpha_1+\beta_1)\mathcal{M}_{\alpha_1,\beta_1}(0)\biggl\{\Bigl(\frac{q}{4\pi^2}\Bigr)^{\delta(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))}H(0;\boldsymbol{\alpha},\boldsymbol{\beta})$$
$$\times\int_{-\infty}^{\infty}G\Bigl(\frac{1}{2};\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it\Bigr)\,dt\biggr\},$$

with negligible error term. To finish the proof of Proposition 6.1, we will show that the local factor at prime p of the Euler product of $\zeta(1-\alpha_1+\beta_1)\mathcal{M}_{\alpha_1,\beta_1}(0)$ is the same as the one in $\mathcal{AZ}\bigl(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\bigr)$ defined in (2-16) and (2-17). The details of this are in Appendix A.

8. Proof of Proposition 6.2

To prove the proposition, it suffices to show that $\mathcal{E}_i^{+}(q;\boldsymbol{\alpha},\boldsymbol{\beta})\ll q^{-1/4+\varepsilon}$ for $i=1$ and $i=5$, since the proofs of upper bounds for other terms are similar. We start with a lemma that will be used in the proof.

Lemma 8.1. *Let λ, η be integers such that $(\lambda, \eta) = 1$ and $\lambda, \eta \ll q^A$, where A is a fixed constant. Moreover, for $i, j = 1, 2, 3$, let $\alpha_i \ll 1/\log q$ and $|\alpha_i - \alpha_j| \gg 1/q^{\varepsilon_1}$ when $i \neq j$. Then for $\varepsilon > \varepsilon_1$,*

$$\operatorname*{Res}_{s=1-\alpha_i} D_3\Big(s, \frac{\lambda}{\eta}, \boldsymbol{\alpha}\Big) \ll \frac{q^\varepsilon}{\eta},$$

where $D_3(s, \lambda/\eta, \boldsymbol{\alpha})$ is defined in (B-4).

Proof. By symmetry, it suffices to prove the statement for the residue at $1 - \alpha_1$. For $\mathrm{Re}(s) > 1 + \mathrm{Re}(\alpha_1 - \alpha_j)$, where $j = 2, 3$, let

$$\begin{aligned} D(s) &:= \sum_{\substack{m=1\\ \eta \mid m}}^{\infty} \frac{\sigma_2(m; \alpha_2 - \alpha_1, \alpha_3 - \alpha_1)}{m^s} \\ &= \frac{1}{\eta^s} \sum_{d \mid \eta} \frac{\mu(d)\sigma_2(\eta/d; \alpha_2 - \alpha_1, \alpha_3 - \alpha_1)}{d^{s+\alpha_2+\alpha_3-2\alpha_1}} \zeta(s + \alpha_2 - \alpha_1)\zeta(s + \alpha_3 - \alpha_1), \end{aligned}$$

where we have used Lemma 2.1 to derive the last line. Now, $D(s)$ can be continued analytically to the whole complex plane except for poles at $s = 1 + \alpha_1 - \alpha_j$ for $j = 2, 3$. Moreover, $D(1) \ll q^\varepsilon/\eta$.

For $i = 1, 2, 3$, and $\mathrm{Re}(s + \alpha_i) > 1$, the sum in the lemma can be rewritten as

$$\frac{1}{\eta^{3s+\alpha_1+\alpha_2+\alpha_3}} \sum\sum\sum_{r_1, r_2, r_3 \pmod{\eta}} \mathrm{e}\Big(\frac{\lambda r_1 r_2 r_3}{\eta}\Big) \zeta\Big(s + \alpha_1; \frac{r_1}{\eta}\Big) \zeta\Big(s + \alpha_2; \frac{r_2}{\eta}\Big) \zeta\Big(s + \alpha_3; \frac{r_3}{\eta}\Big).$$

This sum can be analytically continued to the whole complex plane except for poles at $s = 1 - \alpha_i$ for $i = 1, 2, 3$. After some arrangement, the contribution of the residue at $s = 1 - \alpha_1$ is

$$\frac{1}{\eta^{2+\alpha_2+\alpha_3-2\alpha_1}} \sum\sum_{\substack{r_2, r_3 \pmod{\eta}\\ \eta \mid r_2 r_3}} \zeta\Big(1 + \alpha_2 - \alpha_1; \frac{r_2}{\eta}\Big) \zeta\Big(1 + \alpha_3 - \alpha_1; \frac{r_3}{\eta}\Big) = D(1),$$

and the lemma follows. □

8A. *Bounding $\mathcal{E}_1^+(q; \boldsymbol{\alpha}, \boldsymbol{\beta})$*. With the same notation as in Section 7 and $\mathscr{B}$ defined as in (2-6), we recall that

$$\begin{aligned} \mathcal{E}^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(q; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{2\pi}{q} \sum\sum\sum\sum_{\substack{a_1, b_1, a_2, b_2 \geq 1\\ (a_1 a_2 b_1 b_2, q) = 1}} \frac{\mathscr{B}(a_1, b_1; \boldsymbol{\alpha})}{a_1^{3/2} b_1} \frac{\mathscr{B}(a_2, b_2; -\boldsymbol{\beta})}{a_2^{3/2} b_2} \\ \times \sum_M^d \sum_N^d E^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a}, \boldsymbol{b}, M, N), \end{aligned}$$

where

$$E^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N) := \sum_{c<C} \sideset{}{^*}\sum_{x \pmod{\delta c}} \frac{\mathcal{T}^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)}{c}$$
$$= \sum_{\delta<C} \frac{1}{\delta} \sum_{c<C/\delta} \sideset{}{^*}\sum_{\substack{x \pmod{\delta c}\\ (u_1x-u_2,c\delta)=\delta}} \frac{\mathcal{T}^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(c\delta,x)}{c},$$

$$\mathcal{T}^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(c\delta,x) := \frac{\pi^{3/2+\alpha_1+\alpha_2+\alpha_3}}{\eta_1^{3+\alpha_1+\alpha_2+\alpha_3}} \sum_{i=1}^{3} \operatorname*{Res}_{s=1+\beta_i} D_3\Big(s, \frac{\lambda_2}{\eta_2}, -\boldsymbol{\beta}\Big) \times \sum_{n=1}^{\infty} A_3\Big(n, \frac{\lambda_1}{\eta_1}, \boldsymbol{\alpha}\Big) \mathcal{F}_1^+(c\delta, n; \boldsymbol{\alpha}, \boldsymbol{\beta}),$$

for

$$\lambda_1 = \frac{\bar{q}(u_2\bar{x}-u_1)a_1}{\delta(a_1,u_2c)}, \qquad \eta_1 = \frac{u_2c}{(a_1,u_2c)}, \qquad u_i = \frac{a_ib_i}{(a_1b_1,a_2b_2)},$$
$$\lambda_2 = \frac{\bar{q}(u_1x-u_2)a_2}{\delta(a_2,u_1c)}, \qquad \eta_2 = \frac{u_1c}{(a_2,u_1c)},$$

and

$$\mathcal{F}_1^+(c\delta,n;\boldsymbol{\alpha},\boldsymbol{\beta}) = \int_0^\infty \int_0^\infty \frac{1}{y^{1/2}} \frac{z^{s-1}}{z^{1/2}} V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2y, a_2^3b_2^2z; q) J_{k-1}\Big(\frac{4\pi\sqrt{a_2ya_1z}}{c\delta q}\Big) f\Big(\frac{y}{N}\Big) f\Big(\frac{z}{M}\Big) \times i^{-k}\, \mathrm{e}\Big(\frac{a_1^2b_1y}{c\delta q a_2b_2} + \frac{a_2^2b_2z}{c\delta q a_1b_1}\Big) U_3\Big(\frac{\pi^3 ny}{\eta_1^3}; \boldsymbol{\alpha}\Big)\, dy\, dz.$$

We first note that the contribution from the terms $a_1^3b_1^2y \gg q^{3/2+\varepsilon}$ or $a_2^3b_2^2z \gg q^{3/2+\varepsilon}$ can be bounded by q^{-A} for any A due to the factor $V_{\alpha,\beta}(a_1^3b_1^2y, a_2^3b_2^2z; q)$. So from now on we assume $a_1^3b_1^2N \ll q^{3/2+\varepsilon}$ and $a_2^3b_2^2M \ll q^{3/2+\varepsilon}$.

Moreover, the dyadic sum over M and N contains only $\ll \log^2 q$ terms, so it suffices to prove that

$$E^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N) \ll a_1^{1/2} q^{3/4+\varepsilon} \tag{8-1}$$

for fixed $\boldsymbol{a}$, $\boldsymbol{b}$, M, N satisfying $a_1^3b_1^2N \ll q^{3/2+\varepsilon}$ and $a_2^3b_2^2M \ll q^{3/2+\varepsilon}$. On a first reading, the reader may set $a_1 = a_2 = b_1 = b_2 = 1$ as this simplifies the notation without substantially changing the calculation.

We now write

$$E^+_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N) = H_1 + H_2, \tag{8-2}$$

where H_1 is the contribution from the sum over $n \le \eta_1^3 q^\varepsilon/N$, and H_2 is the rest.

8A1. *Bounding H_1.* By (B-6), $U_3(\pi^3 ny/\eta_1^3) \ll q^\varepsilon$ when $n \ll \eta_1^3 q^\varepsilon/N$. This and (5-6) gives us that

$$\mathcal{F}_1^+(c\delta, n; \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll M^{1/2}N^{1/2}q^\varepsilon \min\biggl\{\biggl(\frac{\sqrt{a_1a_2MN}}{c\delta q}\biggr)^{-1/2}, \biggl(\frac{\sqrt{a_1a_2MN}}{c\delta q}\biggr)^{k-1}\biggr\}.$$

Then, from Lemma 8.1, Lemma B.2, (B-1), and the fact that $(a_2, u_1c) \le a_2$ and $1/(a_1, u_2c) \le 1$, we obtain that H_1 is bounded by

$$\begin{aligned} M^{1/2}N^{1/2}q^\varepsilon &\sum_{\delta<C}\sum_{c<C/\delta}\frac{1}{\eta_1^3\eta_2} \\ &\times \sum_{n\ll\eta_1^3q^\varepsilon/N}\sum_{h|\eta_1,\, h|n^2}\eta_1^{3/2}\sqrt{h}\min\biggl\{\biggl(\frac{\sqrt{a_1a_2MN}}{c\delta q}\biggr)^{-1/2}, \biggl(\frac{\sqrt{a_1a_2MN}}{c\delta q}\biggr)^{k-1}\biggr\} \\ &\ll M^{5/4}N^{1/4}q^\varepsilon\frac{a_1^{3/4}a_2^{7/4}u_2^{3/2}}{u_1q^{3/2}} \ll q^{3/4+\varepsilon}, \end{aligned}$$

as desired. In the above, we have used $(a_1b_1, a_2b_2) \ge 1$, $a_1^3b_1^2N \ll q^{3/2+\varepsilon}$, and $a_2^3b_2^2M \ll q^{3/2+\varepsilon}$.

8A2. *Bounding H_2.* We start by rewriting $\mathcal{F}_1^+(c\delta, n; \boldsymbol{\alpha}, \boldsymbol{\beta})$ as

$$\int_{-\infty}^{\infty}\int_{(1/\log q)} V_1(s,t)\int_0^\infty z^{\beta_1}F_{s-it}\Bigl(\frac{z}{M}\Bigr)\,\mathrm{e}\Bigl(\frac{a_2^2b_2z}{c\delta q a_1b_1}\Bigr)\mathcal{I}(n,z)\,dz\,\frac{ds}{s}\,dt,$$

where $V_1(s,t) := V_1(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\alpha}, \boldsymbol{\beta}, s, t, M, N)$ is defined as

$$\begin{aligned} \frac{1}{2\pi i}\Bigl(\frac{q}{4\pi^2}\Bigr)^{\delta(\boldsymbol{\alpha},\boldsymbol{\beta})}\biggl(\frac{a_2^3b_2^2}{a_1^3b_1^2}\biggr)^{it} & G\bigl(\tfrac12+s; \boldsymbol{\alpha}+it, \boldsymbol{\beta}+it\bigr)H(s;\boldsymbol{\alpha},\boldsymbol{\beta}) \\ &\times\Bigl(\frac{q}{4\pi^2}\Bigr)^{3s}M^{-1/2-s+it}N^{-1/2-s-it}, \end{aligned} \tag{8-3}$$

and $\mathcal{I}(n,z) := \mathcal{I}_{\boldsymbol{\alpha}}(\boldsymbol{a}, \boldsymbol{b}, N, n, z, c, \delta)$ is defined as

$$\int_0^\infty F_{s+it}\Bigl(\frac{y}{N}\Bigr)\,\mathrm{e}\Bigl(\frac{a_1^2b_1y}{c\delta q a_2b_2}\Bigr)J_{k-1}\Bigl(\frac{4\pi\sqrt{a_2ya_1z}}{c\delta q}\Bigr)U_3\Bigl(\frac{\pi^3ny}{\eta_1^3};\boldsymbol{\alpha}\Bigr)\,dy, \tag{8-4}$$

and $F_v(x) := f(x)/x^{1/2+v}$. Note that the j-th derivative, $F_{s\pm it}^{(j)}(y/N) = O(|t|^jN^{-j})$.

Note that the trivial bound for $\mathcal{F}_1^+(c\delta, n; \boldsymbol{\alpha}, \boldsymbol{\beta})$ is

$$\mathcal{F}_1^+(c\delta, n; \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll \sqrt{MN}(qn)^\varepsilon. \tag{8-5}$$

There are two cases to consider: $c > \dfrac{8\pi\sqrt{a_1a_2MN}}{\delta q}$ and $c \le \dfrac{8\pi\sqrt{a_1a_2MN}}{\delta q}$.

Case 1. $$c > \frac{8\pi\sqrt{a_1a_2MN}}{\delta q}.$$

By (5-5), (B-7), and $\pi^3 ny/\eta_1^3 \gg q^\varepsilon$, we can rewrite $\mathcal{I}(n,z)$ as

$$\sum_{j=1}^{K}\Big(\frac{\pi^3 ny}{\eta_1^3}\Big)^{(\beta_1+\beta_2+\beta_3)/3}\Big(\frac{\eta_1}{\pi n^{1/3}N^{1/3}}\Big)^j \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!(\ell+k-1)!}$$
$$\times \int_0^\infty \mathscr{F}_j(y,z,\ell)\,\mathrm{e}\Big(\frac{a_1^2 b_1 y}{c\delta q a_2 b_2}\Big)\Big\{c_j\,\mathrm{e}\Big(\frac{3n^{1/3}y^{1/3}}{\eta_1}\Big)+d_j\,\mathrm{e}\Big(-\frac{3n^{1/3}y^{1/3}}{\eta_1}\Big)\Big\}\,dy$$
$$+\,O(q^{-\varepsilon(K+1)}),$$

where c_j, d_j are some constants, and

$$\mathscr{F}_j(y,z,\ell) = F_{s+it}\Big(\frac{y}{N}\Big)\Big(\frac{2\pi\sqrt{a_2 y a_1 z}}{c\delta q}\Big)^{2\ell+k-1}\Big(\frac{y}{N}\Big)^{-j/3}$$

is supported on $y \in [N, 2N]$. Moreover,

$$\frac{\partial^i \mathscr{F}_j(y,z,\ell)}{\partial y^i} \ll \frac{1}{2^{2\ell}}\frac{|t|^i}{N^i} \quad \text{and} \quad \mathscr{F}_j(y,z,\ell) \ll 1.$$

Thus, picking K large enough so that $q^{-\varepsilon(K+1)}$ is negligible, it suffices to bound integrals of the form

$$\int_0^\infty \mathscr{F}_j(y,z,\ell)\,\mathrm{e}(\theta_z(y,n))\,dy,$$

where

$$\theta_z(y,n) = \pm\frac{3n^{1/3}y^{1/3}}{\eta_1} + By \quad \text{and} \quad B = \frac{a_1^2 b_1}{c\delta q a_2 b_2}.$$

Taking the derivative of $\theta_z(y,n)$ with respect to y, we have that

$$\theta_z'(y,n) = B \pm \frac{n^{1/3}}{y^{2/3}\eta_1}.$$

When $n \ge 64(B\eta_1)^3 N^2$ or $n \le \frac{1}{4}(B\eta_1)^3 N^2$, since $n \gg \eta_1^3 q^\varepsilon/N$ we have

$$|\theta_z'(y,n)| \gg \frac{n^{1/3}}{y^{2/3}\eta_1} \gg \frac{q^\varepsilon}{N}.$$

Thus, integrating by parts many times shows that the contribution from these terms is negligible. Therefore we only consider the contribution from when $\frac{1}{4}(B\eta_1)^3 N^2 \le n \le 64(B\eta_1)^3 N^2$. Note however that

$$(B\eta_1)^3 N^2 \ll \frac{(a_1^2 b_1)^3 N^2}{\delta^3 q^3} \ll \frac{q^\varepsilon}{\delta^3},$$

and that there are no terms of this form unless $N \gg q^{3/2}/(a_1^2 b_1)^{3/2}$ and $\delta \ll q^\varepsilon$. From (8-5), trivially $\mathcal{F}_1^+(c\delta, n; \boldsymbol{\alpha}, \boldsymbol{\beta}) = O(M^{1/2}N^{1/2}(nq)^\varepsilon)$. Hence the contribution to H_2 from these terms is bounded by

$$M^{1/2}N^{1/2}q^{\varepsilon}\sum_{\delta<q^{\varepsilon}}\sum_{c\gg\sqrt{a_1a_2MN}/\delta q}\frac{1}{\eta_1^{3/2}\eta_2}\sum_{h|\eta_1}\sqrt{h}\sum_{\substack{n\ll q^{\varepsilon}\\ h^2|n}}\frac{\eta_1}{n^{1/3}N^{1/3}}$$
$$\ll a_1^{5/4}b_1^{1/2}a_2^{3/4}M^{1/4}N^{1/4}q^{\varepsilon}\ll a_1^{1/2}q^{3/4+\varepsilon},$$

similar to before.

Case 2. $$c\le\frac{8\pi\sqrt{a_1a_2MN}}{\delta q}.$$

By (5-4), we write $\mathcal{I}(n,z)$ as

$$\int_0^\infty\Bigl\{R^+(y,z)+R^-(y,z)\Bigr\}\frac{\sqrt{c\delta q}}{\pi(a_1a_2yz)^{1/4}}U_3\Bigl(\frac{\pi^3ny}{\eta_1^3};\boldsymbol{\alpha}\Bigr)\,\mathrm{e}\Bigl(\frac{a_1^2b_1y}{c\delta qa_2b_2}\Bigr)\,dy,$$

where

$$R^{\pm}(y,z):=F_{s+it}\Bigl(\frac{y}{N}\Bigr)W^{\pm}\biggl(\frac{4\pi\sqrt{a_1a_2yz}}{c\delta q}\biggr)\,\mathrm{e}\biggl(\pm\biggl(\frac{2\sqrt{a_1a_2yz}}{c\delta q}-\frac{k}{4}+\frac{1}{8}\biggr)\biggr),$$

and $W^+=W$, $W^-=\overline{W}$.

Similar to Case 1, we explicitly write $U_3(\pi^3ny/\eta_1^3;\boldsymbol{\alpha})$ as in (B-7) so it suffices to bound

$$\sum_{j=1}^{K}\Bigl(\frac{\pi^3ny}{\eta_1^3}\Bigr)^{(\beta_1+\beta_2+\beta_3)/3}\Bigl(\frac{\eta_1}{\pi n^{1/3}N^{1/3}}\Bigr)^j\frac{1}{N^{1/4}}\frac{\sqrt{c\delta q}}{\pi(a_1a_2z)^{1/4}}$$
$$\times\int_0^\infty\mathscr{H}_j^{\pm}(y,z)\biggl\{c_j\,\mathrm{e}\biggl(\frac{3n^{1/3}y^{1/3}}{\eta_1}\biggr)+d_j\,\mathrm{e}\biggl(-\frac{3n^{1/3}y^{1/3}}{\eta_1}\biggr)\biggr\}\,\mathrm{e}\Bigl(\frac{a_1^2b_1y}{c\delta qa_2b_2}\Bigr)$$
$$\times\,\mathrm{e}\biggl(\pm\biggl(\frac{2\sqrt{a_1a_2yz}}{c\delta q}-\frac{k}{4}+\frac{1}{8}\biggr)\biggr)\,dy+O\bigl(q^{-\varepsilon(K+1)}\bigr),$$

where

$$\mathscr{H}_j^{\pm}(y,z)=\frac{F_{s+it}(y/N)W^{\pm}(4\pi\sqrt{a_1a_2yz}/c\delta q)}{(y/N)^{j/3+1/4}}$$

is supported on $y\in[N,2N]$. Note that

$$\frac{\partial^{(i)}\mathscr{H}_j^{\pm}(y,z)}{\partial y^i}\ll_{j,i}\frac{|t|^i}{N^i}\quad\text{and}\quad\mathscr{H}_j^{\pm}(y,z)\ll 1.$$

Thus, the integration over y is of the form

$$\int_0^\infty\mathscr{H}_j^{\pm}(y,z)\,\mathrm{e}\bigl(g_z(y,n)\bigr)\,dy,$$

where

$$g_z(y,n)=\pm\frac{3n^{1/3}y^{1/3}}{\eta_1}\pm\Big(2A\sqrt{y}+\frac{k}{4}-\frac{1}{8}\Big)+By,\quad A=\frac{\sqrt{a_1a_2z}}{c\delta q},\quad B=\frac{a_1^2b_1}{c\delta q a_2 b_2}.$$

Differentiating $g_z(y,n)$ with respect to y, we have

$$g_z'(y,n)=\pm\frac{n^{1/3}}{y^{2/3}\eta_1}\pm\frac{A}{y^{1/2}}+B.$$

When $a_2^{3/2}b_2M^{1/2}\geq 4a_1^{3/2}b_1N^{1/2}$, it follows that

$$\frac{A}{y^{1/2}}\geq\frac{A}{y^{1/2}}-B\geq\frac{1}{2}\frac{A}{y^{1/2}}\quad\text{and}\quad\frac{3}{2}\frac{A}{y^{1/2}}\geq\frac{A}{y^{1/2}}+B\geq\frac{A}{y^{1/2}}.$$

Therefore

$$\frac{1}{2}\frac{A}{y^{1/2}}\leq\left|\pm\frac{A}{y^{1/2}}+B\right|\leq\frac{3}{2}\frac{A}{y^{1/2}}.$$

When $n\geq 54(A\eta_1)^3N^{1/2}$ or $n\leq\frac{1}{64}(A\eta_1)^3N^{1/2}$, since $n\gg\eta_1^3q^\varepsilon/N$ we have that

$$|g_z'(y,n)|\gg\frac{n^{1/3}}{y^{2/3}\eta_1}\gg\frac{q^\varepsilon}{N}.$$

Integrating by parts many times shows that these terms are negligible. We then consider only the terms where $\frac{1}{64}(A\eta_1)^3N^{1/2}\leq n\leq 54(A\eta_1)^3N^{1/2}$. Note however that

$$(A\eta_1)^3N^{1/2}\ll\left(\frac{\sqrt{a_1a_2M}u_2c}{c\delta q}\right)^3N^{1/2}\ll\left(\frac{\sqrt{a_1}}{\delta q^{1/4}}\right)^3N^{1/2}\ll\frac{q^\varepsilon}{\delta^3},$$

and that the left side is only $\gg 1$ if $N\gg q^{3/2}/a_1^3$ and $\delta\ll q^\varepsilon$.

By (8-5), the contribution of $\mathcal{F}_1^+(c\delta,n;\boldsymbol{\alpha},\boldsymbol{\beta})$ to the terms in this range is $O(M^{1/2}N^{1/2}(nq)^\varepsilon)$. So the contribution to H_2 from these terms is bounded by

$$M^{1/2}N^{1/2}q^\varepsilon\sum_{\delta<q^\varepsilon}\sum_{c\ll\sqrt{a_1a_2MN}/(\delta q)}\frac{1}{\eta_1^{3/2}\eta_2}\sum_{h|\eta_1}\sqrt{h}\sum_{\substack{n\ll q^\varepsilon\\ h^2|n}}\Big(\frac{\eta_1}{n^{1/3}N^{1/3}}\Big)\frac{\sqrt{c\delta q}}{(a_1a_2MN)^{1/4}}$$
$$\ll a_1^{5/4}b_1^{1/2}a_2^{3/4}M^{1/4}N^{1/4}q^\varepsilon\ll a_1^{1/2}q^{3/4+\varepsilon},$$

which suffices.

When $4a_2^{3/2}b_2M^{1/2}\leq a_1^{3/2}b_1N^{1/2}$, we have that $\frac{1}{2}B<B-A/y^{1/2}<B$ and $\frac{3}{2}B>Ay^{1/2}+B>B$. By the same arguments as in Case 1, the range of n that should be considered is of the size $(B\eta_1)^3N^2$ and give a contribution to H_2 bounded by $a_1^{1/2}q^{3/4+\varepsilon}$.

When $\frac{1}{4}a_1^{3/2}b_1N^{1/2}<a_2^{3/2}b_2M^{1/2}<4a_1^{3/2}b_1N^{1/2}$, we have that $A/y^{1/2}\asymp B$, and so the range of n that should be considered is of the size $(A\eta_1)^3N^{1/2}$ by the same arguments as above. Hence the contribution from these terms to H_2 is $O\big(a_1^{1/4}q^{3/4+\varepsilon}\big)$.

This completes the proof of Proposition 6.2 for $\mathcal{E}_1^+(q;\boldsymbol{\alpha},\boldsymbol{\beta})$. The same proof applies to bound $\mathcal{E}_i^{\pm}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ for $i=2,3,4$.

8B. ***Bounding*** $\mathcal{E}_5^+(q;\boldsymbol{\alpha},\boldsymbol{\beta})$**.** We first recall that

$$\mathcal{E}_5^+(q;\boldsymbol{\alpha},\boldsymbol{\beta})=\frac{2\pi}{q}\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2\ge 1\\(a_1a_2b_1b_2,q)=1}}\frac{\mathscr{B}(a_1,b_1;\boldsymbol{\alpha})}{a_1^{3/2}b_1}\frac{\mathscr{B}(a_2,b_2;-\boldsymbol{\beta})}{a_2^{3/2}b_2}\times\sum_M^d\sum_N^d E^+_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N),$$

where

$$\begin{aligned}E^+_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N)&:=\sum_{c<C}\ \sideset{}{^*}\sum_{x\ (\mathrm{mod}\ \delta c)}\frac{\mathcal{T}^+_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(c,x)}{c}\\&=\sum_{\delta<C}\frac{1}{\delta}\sum_{c<C/\delta}\ \sideset{}{^*}\sum_{\substack{x\ (\mathrm{mod}\ \delta c)\\(u_1x-u_2,c\delta)=\delta}}\frac{\mathcal{T}^+_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(c\delta,x)}{c},\end{aligned}$$

$$\mathcal{T}^+_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(c\delta,x):=\frac{\pi^{3+\alpha_1-\beta_1+\alpha_2-\beta_2+\alpha_3-\beta_3}}{\eta_1^{3+\alpha_1+\alpha_2+\alpha_3}\eta_2^{3-\beta_1-\beta_2-\beta_3}}\sum\sum_{n,m\ge 1}A_3\Big(n,\frac{\lambda_1}{\eta_1},\boldsymbol{\alpha}\Big)\times A_3\Big(m,\frac{\lambda_2}{\eta_2},-\boldsymbol{\beta}\Big)\mathcal{F}_5^+(c\delta,n,m;\boldsymbol{\alpha},\boldsymbol{\beta});$$

for

$$\lambda_1=\frac{\bar{q}(u_2\bar{x}-u_1)a_1}{\delta(a_1,u_2c)},\quad \eta_1=\frac{u_2c}{(a_1,u_2c)},\qquad \lambda_2=\frac{\bar{q}(u_1x-u_2)a_2}{\delta(a_2,u_1c)},\quad \eta_2=\frac{u_1c}{(a_2,u_1c)},\qquad u_i=\frac{a_ib_i}{(a_1b_1,a_2b_2)},$$

and

$$\mathcal{F}_5^+(c\delta,n,m,\boldsymbol{\alpha},\boldsymbol{\beta})=\int_0^\infty\int_0^\infty\frac{1}{y^{1/2}}\frac{1}{z^{1/2}}V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3b_1^2y,a_2^3b_2^2z;q)\times J_{k-1}\Big(\frac{4\pi\sqrt{a_2ya_1z}}{c\delta q}\Big)f\Big(\frac{y}{N}\Big)f\Big(\frac{z}{M}\Big)i^{-k}\,\mathrm{e}\Big(\frac{a_1^2b_1y}{c\delta qa_2b_2}+\frac{a_2^2b_2z}{c\delta qa_1b_1}\Big)\times U_3\Big(\frac{\pi^3ny}{\eta_1^3};\boldsymbol{\alpha}\Big)U_3\Big(\frac{\pi^3mz}{\eta_2^3};-\boldsymbol{\beta}\Big)\,dy\,dz.$$

The proofs in this section are very similar to the ones in the previous section. Previously, we had one sum over n and now we have a double sum over m and n which can be treated in a similar manner. To be precise, we begin by dividing

$E^{+}_{5,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N)$ into $\sum_{i=1}^{4} E^{+}_{5,i}(\boldsymbol{a},\boldsymbol{b},M,N)$, where

$$E^{+}_{5,i}(\boldsymbol{a},\boldsymbol{b},M,N) := E^{+}_{5,i,\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{a},\boldsymbol{b},M,N)$$

is the contribution from case (i) below:

(1) $n \ll \dfrac{\eta_1^3 q^{\varepsilon}}{N}$ and $m \ll \dfrac{\eta_2^3 q^{\varepsilon}}{M}$;

(2) $n \gg \dfrac{\eta_1^3 q^{\varepsilon}}{N}$ and $m \ll \dfrac{\eta_2^3 q^{\varepsilon}}{M}$;

(3) $n \ll \dfrac{\eta_1^3 q^{\varepsilon}}{N}$ and $m \gg \dfrac{\eta_2^3 q^{\varepsilon}}{M}$;

(4) $n \gg \dfrac{\eta_1^3 q^{\varepsilon}}{N}$ and $m \gg \dfrac{\eta_2^3 q^{\varepsilon}}{M}$.

By symmetry, the treatment for cases (2) and (3) is the same, so we show only the second case.

Similar to Section 8A, the contribution from the terms $a_1^3 b_1^2 y \gg q^{3/2+\varepsilon}$ or $a_2^3 b_2^2 z \gg q^{3/2+\varepsilon}$ can be bounded by q^{-A} due to the factor $V_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_1^3 b_1^2 y, a_2^3 b_2^2 z; q)$. Thus it suffices to prove that

$$E^{+}_{5,i}(\boldsymbol{a},\boldsymbol{b},M,N) \ll a_1^{1/2} a_2^{1/2} q^{3/4+\varepsilon}, \tag{8-6}$$

for fixed $\boldsymbol{a}, \boldsymbol{b}, M, N$ satisfying

$$a_1^3 b_1^2 N \ll q^{3/2+\varepsilon} \quad \text{and} \quad a_2^3 b_2^2 M \ll q^{3/2+\varepsilon}.$$

In fact, we prove the stronger bound $E^{+}_{5,i}(\boldsymbol{a},\boldsymbol{b},M,N) \ll a_1^{1/2} a_2^{1/2} q^{1/2+\varepsilon}$.

8B1. *Bounding* $E^{+}_{5,1}(\boldsymbol{a},\boldsymbol{b},M,N)$. For this case, we have $U_3(\pi^3 ny/\eta_1^3; \boldsymbol{\alpha}) \ll q^{\varepsilon}$ and $U_3(\pi^3 mz/\eta_2^3; -\boldsymbol{\beta}) \ll q^{\varepsilon}$ by (B-6). Similar to the arguments in Section 8A1, from Lemma 8.1, Lemma B.2, and (5-6), we have that for $k \geq 5$,

$$\begin{aligned} &E^{+}_{5,1}(\boldsymbol{a},\boldsymbol{b},M,N) \\ &\quad \ll M^{1/2} N^{1/2} q^{\varepsilon} \sum_{\delta<C} \sum_{c<C/\delta} \frac{1}{\eta_1^3 \eta_2^3} \sum_{n \ll \eta_1^3 q^{\varepsilon}/N} \sum_{h_1 \mid \eta_1,\, h_1 \mid n^2} \eta_1^{3/2} \sqrt{h_1} \\ &\qquad \times \sum_{m \ll \eta_2^3 q^{\varepsilon}/M} \sum_{h_2 \mid \eta_2,\, h_2 \mid m^2} \eta_2^{3/2} \sqrt{h_2} \min\left\{ \left(\frac{\sqrt{a_1 a_2 MN}}{c\delta q} \right)^{-1/2}, \left(\frac{\sqrt{a_1 a_2 MN}}{c\delta q} \right)^{k-1} \right\} \end{aligned}$$

$$\ll M^{-1/2}N^{-1/2}q^{\varepsilon}\sum_{\delta<C}\Bigg\{\sum_{\sqrt{a_1a_2MN}/(q\delta)\ll c<C/\delta}\eta_1^{3/2}\eta_2^{3/2}\Big(\frac{\sqrt{a_1a_2MN}}{c\delta q}\Big)^{k-1} + \sum_{c\ll\sqrt{a_1a_2MN}/(q\delta)}\eta_1^{3/2}\eta_2^{3/2}\Big(\frac{\sqrt{a_1a_2MN}}{c\delta q}\Big)^{-1/2}\Bigg\}$$

$$\ll M^{3/2}N^{3/2}q^{\varepsilon}\frac{a_1^2a_2^2u_1^{3/2}u_2^{3/2}}{q^4}\ll q^{1/2+\varepsilon}.$$

8B2. *Bounding* $E^+_{5,2}(\boldsymbol{a},\boldsymbol{b},M,N)$. We can write $\mathcal{F}^+_5(c\delta,n,m;\boldsymbol{\alpha},\boldsymbol{\beta})$ as

$$\int_{-\infty}^{\infty}\int_{(1/\log q)}V_1(s,t)\int_0^{\infty}F_{s-it}\Big(\frac{z}{M}\Big)\,\mathrm{e}\Big(\frac{a_2^2b_2z}{c\delta qa_1b_1}\Big)U_3\Big(\frac{\pi^3mz}{\eta_2^3};-\boldsymbol{\beta}\Big)\mathcal{I}(n,z)\,dz\,\frac{ds}{s}\,dt,$$

where $V_1(s,t)$ and $\mathcal{I}(n,z)$ are defined as in (8-3) and (8-4), respectively, and $F_v(x)=1/x^{1/2+v}f(x)$. Note that $F^{(j)}_{s\pm it}(y/N)\ll|t|^jN^{-j}$.

The integration over z can be bounded trivially, and the sum over m, h_2 can be treated in the same way as in Section 8B1. For the integration over y, we argue as in Cases 1 and 2 of Section 8A2 and obtain that $E^+_{5,2}(\boldsymbol{a},\boldsymbol{b},M,N)\ll a_1^{1/2}q^{1/2+\varepsilon}$.

8B3. *Bounding* $E^+_{5,4}(\boldsymbol{a},\boldsymbol{b},M,N)$. We split into two cases as follows.

Case 1.
$$c>\frac{8\pi\sqrt{a_1a_2MN}}{\delta q}.$$

We use (5-5) and (B-7), and the integral that we consider is of the form

$$\int_0^{\infty}\int_0^{\infty}G(y,z)\,\mathrm{e}\Big(\frac{a_1^2b_1y}{c\delta qa_2b_2}+\frac{a_2^2b_2z}{c\delta qa_1b_1}\pm\frac{3n^{1/3}y^{1/3}}{\eta_1}\pm\frac{3m^{1/3}z^{1/3}}{\eta_2}\Big)\,dy\,dz,$$

where $\partial^j\partial^iG(y,z)/\partial y^j\partial z^i\ll 1/(N^jM^i)$, $G(x,y)\ll 1$, and it is supported in $[N,2N]\times[M,2M]$. Therefore, the integration over y, z above is $O(MN)$.

By the same arguments as Case 1 of Section 8A2, it is sufficient to consider when $c_1(B_1\eta_1)^3N^2\ll n\ll c_2(B_1\eta_1)^3N^2$ and $c_1(B_2\eta_2)^3M^2\ll m\ll c_2(B_2\eta_2)^3M^2$, where c_1, c_2 are some constants, $B_1=a_1^2b_1/(c\delta qa_2b_2)$, and $B_2=a_2^2b_2/(c\delta qa_1b_1)$, since the terms outside these ranges give negligible contribution from integration by parts many times. By the same arguments as in Section 8A,

$$(B\eta_1)^3N^2\ll\frac{(a_1^2b_1)^3N^2}{\delta^3q^3}\ll\frac{q^{\varepsilon}}{\delta^3}\quad\text{and}\quad(B\eta_2)^3M^2\ll\frac{(a_2^2b_2)^3M^2}{\delta^3q^3}\ll\frac{q^{\varepsilon}}{\delta^3}.$$

So there are no terms of this form unless $N\gg q^{3/2}/(a_1^2b_1)^{3/2}$, $M\gg q^{3/2}/(a_2^2b_2)^{3/2}$, and $\delta\ll q^{\varepsilon}$. We then obtain that the contribution from these terms to $E^+_{5,4}(\boldsymbol{a},\boldsymbol{b},M,N)$

is bounded by

$$M^{1/2}N^{1/2}q^{\varepsilon}\sum_{\delta<q^{\varepsilon}}\sum_{c\gg\sqrt{a_1a_2MN}/(\delta q)}\frac{1}{\eta_1^{3/2}\eta_2^{3/2}}\sum_{h_1|\eta_1}\sqrt{h_1}\sum_{\substack{n\ll q^{\varepsilon}\\ h_1^2|n}}\sum_{h_2|\eta_1}\sqrt{h_2}\sum_{\substack{m\ll q^{\varepsilon}\\ h_2^2|m}}1 \ll a_1^{1/2}a_2^{1/2}q^{1/2+\varepsilon}.$$

Case 2. $$c\le\frac{8\pi\sqrt{a_1a_2MN}}{\delta q}.$$

For this case, we use (5-4), and the integral that we consider is of the form

$$\frac{\eta_1\eta_2}{m^{1/3}n^{1/3}M^{1/3}N^{1/3}}\frac{\sqrt{c\delta q}}{M^{1/4}N^{1/4}(a_1a_2)^{1/4}}\int_0^{\infty}\int_0^{\infty}\mathcal{H}(y,z)\,\mathrm{e}(g(y,z,n,m))\,dy\,dz,$$

where
$$\frac{\partial^j\partial^i\mathcal{H}(y,z)}{\partial y^j\partial z^i}\ll 1/(N^jM^i),$$

$\mathcal{H}(x,y)$ is supported in $[N,2N]\times[M,2M]$, and

$$g(y,z,n,m)=\frac{a_1^2b_1y}{c\delta qa_2b_2}+\frac{a_2^2b_2z}{c\delta qa_1b_1}\pm\frac{3n^{1/3}y^{1/3}}{\eta_1}\pm\frac{3m^{1/3}z^{1/3}}{\eta_2}\pm\frac{2\sqrt{a_1a_2yz}}{c\delta q}.$$

We note that the integration over y,z above is $O(MN)$. Hence we obtain that

$$\frac{\partial g(y,z,n,m)}{\partial y}=B_1\pm\frac{n^{1/3}}{y^{2/3}\eta_1}\pm\frac{A_1}{y^{1/2}},$$
$$\frac{\partial g(y,z,n,m)}{\partial z}=B_2\pm\frac{m^{1/3}}{z^{2/3}\eta_2}\pm\frac{A_2}{z^{1/2}},$$

where $A_1=\sqrt{a_1a_2z}/(c\delta q)$ and $A_2=\sqrt{a_1a_2y}/(c\delta q)$. We divide into three cases to consider.

Case 2.1: $a_2^{3/2}b_2M^{1/2}\ge 4a_1^{3/2}b_1N^{1/2}$. For this case, we have that

$$\left|\frac{A_1}{y^{1/2}}\pm B_1\right|\asymp\frac{A_1}{y^{1/2}}\quad\text{and}\quad\left|\frac{A_2}{z^{1/2}}\pm B_2\right|\asymp B_2.$$

Similar to Case 2 of Section 8A2, we consider the ranges $n\asymp(A_1\eta_1)^3N^{1/2}$ and $m\asymp(B_2\eta_2)^3M^2$. By the same arguments as in Section 8A, we note that

$$(A\eta_1^3)N^{1/2}\ll\left(\frac{\sqrt{a_1}}{\delta q^{1/2}}\right)^3N^{1/2}\ll\frac{q^{\varepsilon}}{\delta^3}\quad\text{and}\quad(B\eta_2)^3M^2\ll\frac{(a_2^2b_2)^3M^2}{\delta^3q^3}\ll\frac{q^{\varepsilon}}{\delta^3},$$

and there are no terms of this form unless $N\gg q^{3/2}/a_1^3$, $M\gg q^{3/2}/(a_2^2b_2)^{3/2}$, and $\delta\ll q^{\varepsilon}$. Hence the contribution from these terms to $E_{5,4}^{+}(\boldsymbol{a},\boldsymbol{b},M,N)$ is

$$M^{1/6}N^{1/6}\sum_{\delta<q^{\varepsilon}}\sum_{c\ll\sqrt{a_1a_2MN}/(\delta q)}\frac{1}{\eta_1^{1/2}\eta_2^{1/2}}\sum_{h_1|\eta_1}\sqrt{h_1}\sum_{\substack{n\ll q^{\varepsilon}\\ h_1^2|n}}\frac{1}{n^{1/3}}\sum_{h_2|\eta_1}\sqrt{h_2}\sum_{m\ll q^{\varepsilon}h_2^2|m}\frac{1}{m^{1/3}} \ll a_1^{1/2}a_2^{1/2}q^{1/2+\varepsilon}.$$

Case 2.2: $a_1^{3/2}b_1N^{1/2}\ge 4a_2^{3/2}b_2M^{1/2}$. For this case, we do the same calculation as in Case 2.1 and obtain that the contribution is also $O(a_1^{1/2}a_2^{1/2}q^{1/2+\varepsilon})$.

Case 2.3: $\frac{1}{4}a_1^{3/2}b_1N^{1/2}<a_2^{3/2}b_2M^{1/2}<4a_1^{3/2}b_1N^{1/2}$. For this case, we have that

$$\frac{A_1}{y^{1/2}}\asymp B_1 \quad\text{and}\quad \frac{A_2}{z^{1/2}}\asymp B_2.$$

By similar arguments to Case 2 of Section 8A2, we can focus on the ranges $n\asymp(A_1\eta_1)^3N^{1/2}$ and $m\asymp(A_1\eta_1)^3N^{1/2}$. The contribution from these terms to $E^{+}_{5,4}(\boldsymbol{a},\boldsymbol{b},M,N)$ is then $\ll a_1^{1/2}a_2^{1/2}q^{1/2+\varepsilon}$.

9. Conclusion of the proof of Theorem 2.5

Recall that from Section 4 and (4-1), we want to evaluate

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{M}_6(q)=\mathscr{M}_1(q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathscr{M}_1(q;\boldsymbol{\beta},\boldsymbol{\alpha}).$$

By (4-2), we see that $\mathscr{M}_1(q;\boldsymbol{\alpha},\boldsymbol{\beta})=\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$, and in Lemma 4.1, we showed that

$$\mathscr{D}(q;\boldsymbol{\alpha},\boldsymbol{\beta})=H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\int_{-\infty}^{\infty}\mathcal{M}(q;\boldsymbol{\alpha}+it,\boldsymbol{\beta}+it)\,dt+O(q^{-3/4+\varepsilon}),$$

which is one of the twenty main terms of the asymptotic formula. Then we decomposed $\mathscr{K}(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ as $\mathscr{K}_M(q;\boldsymbol{\alpha},\boldsymbol{\beta})+\mathscr{K}_E(q;\boldsymbol{\alpha},\boldsymbol{\beta})$. We proved in Section 5A that $\mathscr{K}_E(q;\boldsymbol{\alpha},\boldsymbol{\beta})\ll q^{-1/2+\varepsilon}$, and then using the Voronoi summation formula, we extracted another nine main terms of the asymptotic formula from $\mathscr{K}_M(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ with an error term $O(q^{-1/4+\varepsilon})$ (see Propositions 6.1 and 6.2, Section 7, Section 8, and Appendix A). As briefly discussed in Section 4, those terms correspond to $\mathcal{M}(q;\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it)$, where π is the transposition (α_i,β_j) for $i=1,2,3$ in $S_6/S_3\times S_3$. Hence $\mathscr{M}_1(q;\boldsymbol{\alpha},\boldsymbol{\beta})$ gives ten main terms the desired asymptotic formula, and similarly the remaining ten terms come from $\mathscr{M}_1(q;\boldsymbol{\beta},\boldsymbol{\alpha})$.

Therefore, combining everything together, we have that

$$H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\mathcal{M}_6(q) = H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\int_{-\infty}^{\infty}\sum_{\pi\in S_6/S_3\times S_3}\mathcal{M}(q;\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it)\,dt+O(q^{-1/4+\varepsilon}).$$

If $|\alpha_i-\beta_j|\gg q^{-\varepsilon}$ for all $1\le i,\, j\le 3$, then $H(0;\boldsymbol{\alpha},\boldsymbol{\beta})\gg q^{-\varepsilon}$ and we immediately get

$$\mathcal{M}_6(q)=\int_{-\infty}^{\infty}\sum_{\pi\in S_6/S_3\times S_3}\mathcal{M}(q;\pi(\boldsymbol{\alpha})+it,\pi(\boldsymbol{\beta})+it)\,dt+O(q^{-1/4+\varepsilon}).$$

However, since all expressions above, including the term bounded by $O(q^{-1/2+\varepsilon})$, are analytic in the α_i and β_j, we see that this in fact holds in general.

Appendix A: Comparing the main term of R_{α_1,β_1} and $\mathcal{M}(q;\pi(\alpha),\pi(\beta))$

To finish the proof of Proposition 6.1, we show that the local factor at prime p of the Euler product of $\zeta(1-\alpha_1+\beta_1)\mathcal{M}_{\alpha_1,\beta_1}(0)$ is the same as the one in $\mathcal{AZ}\big(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\big)$, where

$$(\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))=(\beta_1,\alpha_1,\alpha_2;\alpha_1,\beta_2,\beta_3),$$

and $\mathcal{M}_{\alpha_1,\beta_1}(s)$ is defined as in (7-10). To simplify the presentation, we work within the ring of formal Dirichlet series, so that we need not worry about convergence issues in this section. Indeed, if we show that $\zeta(1-\alpha_1+\beta_1)\mathcal{M}_{\alpha_1,\beta_1}(0)$ is the same as $\mathcal{AZ}\big(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\big)$ as formal series, then they must have the same region of absolute convergence. Thus, as analytic functions, they agree on the region of absolute convergence, and so must be the same by analytic continuation. Note that we have already verified that there is a nonempty open region of absolute convergence at the end of Section 7.

For notational convenience, $\boldsymbol{\alpha}_{2,3}=(\alpha_2,\alpha_3)$, and $-\boldsymbol{\beta}_{2,3}=(-\beta_2,-\beta_3)$ in this section.

A1. ***Euler product at prime p of $\mathcal{AZ}\big(\frac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\big)$.*** We start by rearranging the sums in $\mathcal{AZ}(s;\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))$ by the same method as in (2-9). When

$$\operatorname{Re}(s+\beta_1+\alpha_2+\alpha_3),\ \operatorname{Re}(s-\alpha_1-\beta_2-\beta_3)>1,$$

we recall that from (2-13) and (2-17), $\mathcal{AZ}(s;\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta}))$ is

$$\sum\sum\sum\sum\sum\sum_{\substack{a_1,b_1,a_2,b_2,m,n\ge1\\ a_1n=a_2m\\ a_1b_1=a_2b_2\\ (a_i,q)=(b_j,q)=1}}\frac{\mathscr{B}(a_1,b_1;\pi(\boldsymbol{\alpha}))}{(a_1b_1)^{2s}}\frac{\mathscr{B}(a_2,b_2;-\pi(\boldsymbol{\beta}))}{(a_2b_2)^{2s}}\times\frac{\sigma_3(n;\pi(\boldsymbol{\alpha}))\sigma_3(m;-\pi(\boldsymbol{\beta}))}{(a_1n)^s(a_2m)^s}.$$

Using Lemma 2.1, the proof of (2-9), and the fact that

$$\sigma_3(a;\alpha_1,\alpha_2,\alpha_3)=\sum_{df=a}d^{-\alpha_1}\sigma_2(f;\alpha_2,\alpha_3),\tag{A-1}$$

we see after a change of variables that

$$\mathcal{AZ}\big(\tfrac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\big) = \zeta(1-\alpha_1+\beta_1)\sum\sum\sum\sum_{\substack{d_1,d_2,e_1,e_2\ge 1\\ d_1e_1=d_2e_2\\ (d_ie_i,q)=1}} \frac{1}{d_1^{1+\alpha_2+\alpha_3}d_2^{1-\beta_2-\beta_3}}\frac{1}{e_1^{1+\beta_1}e_2^{1-\alpha_1}}\mathcal{J}(e_1,e_2), \tag{A-2}$$

where

$$\mathcal{J}(e_1,e_2)=\sum\sum_{j_1,j_2\ge 1}\frac{\sigma_2(j_1e_1;\boldsymbol{\alpha}_{2,3})\sigma_2(j_2e_2;-\boldsymbol{\beta}_{2,3})(j_1,j_2)^{1-\alpha_1+\beta_1}}{j_1^{1-\alpha_1}j_2^{1+\beta_1}}. \tag{A-3}$$

Because both $\zeta(1-\alpha_1+\beta_1)\mathcal{M}_{\alpha_1,\beta_1}(0)$ and $\mathcal{AZ}\big(\tfrac{1}{2};\pi(\boldsymbol{\alpha}),\pi(\boldsymbol{\beta})\big)$ have the factor $\zeta(1-\alpha_1+\beta_1)$, it suffices to consider only the local factor at prime p of the sum over d_i, e_i in (A-2). For $p\neq q$, this is

$$\sum\sum\sum\sum_{\substack{\delta_1,\delta_2,\epsilon_1,\epsilon_2\ge 0\\ \delta_1+\epsilon_1=\delta_2+\epsilon_2}}\frac{1}{p^{D+\epsilon_1+\epsilon_2+\epsilon_1\beta_1-\epsilon_2\alpha_1}}\sum_{k\ge 0}\frac{\mathcal{J}_p(\epsilon_1,\epsilon_2,k)}{p^k}, \tag{A-4}$$

where $p^{\delta_i}\parallel d_i$, $p^{\epsilon_i}\parallel e_i$, $p^{\iota_i}\parallel j_i$,

$$D:=\delta_1+\delta_2+\delta_1(\alpha_2+\alpha_3)-\delta_2(\beta_2-\beta_3), \tag{A-5}$$

and

$$\begin{aligned}\mathcal{J}_p(\epsilon_1,\epsilon_2,k) :=\ & \sigma_2(p^{k+\epsilon_1};\boldsymbol{\alpha}_{2,3})\sigma_2(p^{k+\epsilon_2};-\boldsymbol{\beta}_{2,3}) \\ &+\sum_{0\le\iota_1<k}\frac{\sigma_2(p^{\iota_1+\epsilon_1};\boldsymbol{\alpha}_{2,3})\sigma_2(p^{k+\epsilon_2};-\boldsymbol{\beta}_{2,3})}{p^{\beta_1(k-\iota_1)}} \\ &+\sum_{0\le\iota_2<k}\frac{\sigma_2(p^{k+\epsilon_1};\boldsymbol{\alpha}_{2,3})\sigma_2(p^{\iota_2+\epsilon_2};-\boldsymbol{\beta}_{2,3})}{p^{-\alpha_1(k-\iota_2)}}.\end{aligned} \tag{A-6}$$

For $p=q$, we have that $\delta_i=\epsilon_i=0$, and the local factor at p is

$$\sum_{k\ge 0}\frac{\mathcal{J}_p(0,0,k)}{p^k}. \tag{A-7}$$

We also comment here that when $\alpha_i=\beta_i=0$ for $i=1,2,3$, using $\sigma_2(p^k)=k+1$ in (A-4), (A-6), (A-7) and some straightforward calculation, we derive that the local factor at $p\neq q$ of $\mathcal{AZ}\big(\tfrac{1}{2};0,0\big)$ is

$$\Big(1-\frac{1}{p}\Big)^{-9}C_p,$$

where C_p is defined in (1-4), and the local factor at q is

$$\Big(1-\frac{1}{q}\Big)^{-5}\Big(1+\frac{4}{q}+\frac{1}{q^2}\Big).$$

This explains the presence of the arithmetic factor in our Conjecture 1.1, as in the work [Conrey et al. 2005].

A2. ***The Euler product at p of $\mathcal{M}_{\alpha_1,\beta_1}(0)$.*** By the definition of $\mathscr{B}(a, b; \alpha_1, \alpha_2, \alpha_3)$ in (2-6), $\mathcal{G}(c, \boldsymbol{a}, \boldsymbol{b})$ in (7-9), $\sum^{\#} \mathscr{G}_{\boldsymbol{a},\boldsymbol{b}}(s; h, b, c, \gamma, g)$ in (7-6), (A-1), and a change of variables, we obtain that $\mathcal{M}_{\alpha_1,\beta_1}(0)$ can be rewritten as

$$\sum\sum\sum\sum_{\substack{d_1,f_1,d_2,f_2\geq 1\\(d_if_i,q)=1}} \frac{1}{d_1^{1-\alpha_1-\beta_1+\alpha_2+\alpha_3} d_2^{1+\alpha_1+\beta_1-\beta_2-\beta_3}} \frac{1}{f_1^{1-\beta_1} f_2^{1+\alpha_1}}$$
$$\times \sum_{\substack{h\geq 1\\ h|u_1u_2}} \frac{\mu(h)}{h^{\alpha_1-\beta_1}} \sum_{\substack{b\geq 1\\(b,u_1u_2)=1}} \frac{\mu(b)}{b^{\alpha_1-\beta_1}} \sum_{c\geq 1} \frac{c}{c^{\alpha_1-\beta_1}} \sum_{\gamma|c} \frac{1}{\gamma^{\alpha_1-\beta_1}} \sum_{g|(c/\gamma)} \frac{\mu(g)}{g^{\alpha_1-\beta_1}} \prod_{\substack{p|c\\ p\nmid bh\gamma}} \Big(1-\frac{1}{p}\Big)$$
$$\times \sum_{n\geq 1}\sum_{a_1|f_1} \frac{\mu(a_1)}{a_1^{\alpha_2+\alpha_3}} \bigg(\frac{(a_1, u_2cb)}{a_1u_2cbn}\bigg)^{1-\alpha_1} \sigma_2\bigg(\frac{u_2cbn}{(a_1,u_2cb)}; \boldsymbol{\alpha}_{2,3}\bigg) \sigma_2\bigg(\frac{f_1}{a_1}; \boldsymbol{\alpha}_{2,3}\bigg)$$
$$\times \sum_{m\geq 1}\sum_{a_2|f_2} \frac{\mu(a_2)}{a_2^{-\beta_2-\beta_3}} \bigg(\frac{(a_2, u_1cb)}{a_2u_1cbm}\bigg)^{1+\beta_1} \sigma_2\bigg(\frac{u_1cbm}{(a_2,u_1cb)}; -\boldsymbol{\beta}_{2,3}\bigg) \sigma_2\bigg(\frac{f_2}{a_2}; -\boldsymbol{\beta}_{2,3}\bigg).$$

In Section 7B, $u_i = a_ib_i/(a_1b_1, a_2b_2)$, but after changing variables, we write that $u_i = f_id_i/(f_1d_1, f_2d_2)$. By comparing Euler products, we can show that

$$\sum_{n\geq 1}\sum_{a_1|f_1} \frac{\mu(a_1)}{a_1^{\alpha_2+\alpha_3}} \bigg(\frac{(a_1, u_2cb)}{a_1u_2cbn}\bigg)^{1-\alpha_1} \sigma_2\bigg(\frac{u_2cbn}{(a_1,u_2cb)}; \boldsymbol{\alpha}_{2,3}\bigg) \sigma_2\bigg(\frac{f_1}{a_1}; \boldsymbol{\alpha}_{2,3}\bigg) = \sum_{n'\geq 1} \frac{\sigma_2(f_1u_2cbn'; \boldsymbol{\alpha}_{2,3})}{(u_2cbn')^{1-\alpha_1}}.$$

We also have a similar expression for the sum over m and a_2. Hence we can write $\mathcal{M}_{\alpha_1,\beta_1}(0)$ as

$$\sum\sum\sum\sum_{\substack{d_1,f_1,d_2,f_2\geq 1\\(d_if_i,q)=1}} \frac{1}{d_1^{1-\alpha_1-\beta_1+\alpha_2+\alpha_3} d_2^{1+\alpha_1+\beta_1-\beta_2-\beta_3}} \frac{1}{f_1^{1-\beta_1} u_2^{1-\alpha_1} f_2^{1+\alpha_1} u_1^{1+\beta_1}}$$
$$\times \sum_{\substack{h\geq 1\\ h|u_1u_2}} \frac{\mu(h)}{h^{\alpha_1-\beta_1}} \sum_{\substack{b\\(b,u_1u_2)=1}} \frac{\mu(b)}{b^2} \sum_{c} \frac{1}{c} \sum_{\gamma|c} \frac{1}{\gamma^{\alpha_1-\beta_1}} \sum_{g|(c/\gamma)} \frac{\mu(g)}{g^{\alpha_1-\beta_1}} \prod_{\substack{p|c\\ p\nmid bh\gamma}} \Big(1-\frac{1}{p}\Big)$$
$$\times \sum\sum_{n,m\geq 1} \frac{\sigma_2(f_1u_2cbn; \boldsymbol{\alpha}_{2,3})}{n^{1-\alpha_1}} \frac{\sigma_2(f_2u_1cbm; -\boldsymbol{\beta}_{2,3})}{m^{1+\beta_1}}. \tag{A-8}$$

We note here that $d_1f_1u_2 = d_2f_2u_1$ by the definition of u_1, u_2. Next, we consider the local factor at $p \neq q$ of (A-8), which is of the form

$$\sum\sum\sum\sum_{\substack{\delta_1,\delta_2,\xi_1,\xi_2\geq 0\\ \delta_1+\ell_1=\delta_2+\ell_2}} \frac{1}{p^{D'(\ell_1,\ell_2)+\ell_2\beta_1-\ell_1\alpha_1-(\xi_1+\xi_2)(\beta_1-\alpha_1)}}\mathscr{L}_p(\delta_1,\delta_2,\xi_1,\xi_2), \tag{A-9}$$

where $p^{\delta_i} \parallel d_i$, $p^{\xi_i} \parallel f_i$, $p^{\upsilon_i} \parallel u_i$, $\ell_1 = \xi_1 + \upsilon_2$, $\ell_2 = \xi_2 + \upsilon_1$, $\min\{\upsilon_1, \upsilon_2\} = 0$, $D'(\ell_1, \ell_2) = D + (\delta_2 - \delta_1)(\alpha_1 + \beta_1) + \ell_1 + \ell_2$, and D is defined in (A-5). We examine $\mathscr{L}_p(d_1, d_2, f_1, f_2)$ below, but before that analysis, we need the following two lemmas.

Lemma A.1. *The contribution to the local factor at p from*

$$\sum_{\gamma|c}\frac{1}{\gamma^{\alpha_1-\beta_1}}\sum_{g|(c/\gamma)}\frac{\mu(g)}{g^{\alpha_1-\beta_1}}\prod_{\substack{p|c\\ p\nmid bh\gamma}}\Big(1-\frac{1}{p}\Big)$$

is 1 *if $p \nmid c$ or $p \mid bh$. Otherwise, it is*

$$1-\frac{1}{p}+\frac{1}{p^{1+\alpha_1-\beta_1}}.$$

Proof. For $p \nmid c$ or $p \mid bh$, the contribution to the local factor is 1 because

$$\sum_{\gamma|c}\frac{1}{\gamma^{\alpha_1-\beta_1}}\sum_{g|(c/\gamma)}\frac{\mu(g)}{g^{\alpha_1-\beta_1}}=\sum_{a|c}\frac{1}{a^{\alpha_1-\beta_1}}\sum_{g|a}\mu(g)=1.$$

Now suppose $p \mid c$ and $p \nmid bh$. Below we write $p^{c_p} \parallel c$ and $p^{\gamma_p} \parallel \gamma$. Then the contribution to the local factor at p is

$$\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)\Big(1-\frac{1}{p}\Big)+\sum_{1\leq\gamma_p<c_p}\frac{1}{p^{\gamma_p(\alpha_1-\beta_1)}}\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)+\frac{1}{p^{c_p(\alpha_1-\beta_1)}}$$
$$=1-\frac{1}{p}+\frac{1}{p^{1+\alpha_1-\beta_1}}. \qquad \square$$

Lemma A.2. *Let*

$$\mathscr{P}(\ell_1,\ell_2):=\sum\sum\sum_{c_p\geq 0,\ n_p,m_p\geq 0}\frac{1}{p^{c_p}}\frac{\sigma_2(p^{\ell_1+n_p+c_p};\boldsymbol{\alpha}_{2,3})}{p^{n_p(1-\alpha_1)}}\frac{\sigma_2(p^{\ell_2+m_p+c};-\boldsymbol{\beta}_{2,3})}{p^{m_p(1+\beta_1)}}. \tag{A-10}$$

Then

$$\mathscr{P}(\ell_1,\ell_2)=\sum_{k\geq 0}\frac{\mathcal{J}_p(\ell_1,\ell_2,k)}{p^k}+\frac{p^{\alpha_1-\beta_1}}{p^2}\mathscr{P}(\ell_1+1,\ell_2+1),$$

where $\mathcal{J}_p(\ell_1,\ell_2,k)$ is defined as in (A-6).

Proof. We have

$$\begin{aligned}\mathscr{P}(\ell_1,\ell_2) &= \sum_{c_p\ge 0}\frac{\sigma_2(p^{\ell_1+c_p};\boldsymbol{\alpha}_{2,3})\sigma_2(p^{\ell_2+c_p};-\boldsymbol{\beta}_{2,3})}{p^{c_p}}\\ &\quad+\sum_{c_p\ge 0,}\sum_{n_p\ge 1}\frac{\sigma_2(p^{\ell_1+n_p+c_p};\boldsymbol{\alpha}_{2,3})\sigma_2(p^{\ell_2+c_p};-\boldsymbol{\beta}_{2,3})}{p^{c_p+n_p-n_p\alpha_1}}\\ &\quad+\sum_{c_p\ge 0,}\sum_{m_p\ge 1}\frac{\sigma_2(p^{\ell_1+c_p};\boldsymbol{\alpha}_{2,3})\sigma_2(p^{\ell_2+m_p+c_p};-\boldsymbol{\beta}_{2,3})}{p^{c_p+m_p+m_p\beta_1}}\\ &\quad+\frac{p^{\alpha_1-\beta_1}}{p^2}\mathscr{P}(\ell_1+1,\ell_2+1)\\ &= \sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1,\ell_2,k)}{p^k}+\frac{p^{\alpha_1-\beta_1}}{p^2}\mathscr{P}(\ell_1+1,\ell_2+1),\end{aligned}$$

after some arrangement. □

Now we examine $\mathscr{L}_p(\delta_1,\delta_2,\xi_1,\xi_2)$, which we separate into two cases below.

Case 1: $\boldsymbol{\delta_1+\xi_1=\delta_2+\xi_2}$. For this case, we have $\upsilon_1=\upsilon_2=0$, so $\xi_i=\ell_i$. Hence $u_1u_2=1$ and $p\nmid h$. From Lemma A.1, we then obtain that

$$\begin{aligned}&\mathscr{L}_p(\delta_1,\delta_2,\xi_1,\xi_2)\\ &\quad=\mathscr{P}(\ell_1,\ell_2)+\frac{1}{p^2}(p^{\beta_1-\alpha_1}-1)\mathscr{P}(\ell_1+1,\ell_2+1)-\frac{1}{p^2}\mathscr{P}(\ell_1+1,\ell_2+1).\end{aligned}$$

From Lemma A.2, $\mathscr{L}_p(d_1,d_2,f_1,f_2)$ can be written as

$$\begin{aligned}&\sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1,\ell_2,k)}{p^k}+\frac{1}{p^2}(p^{\alpha_1-\beta_1}-2+p^{\beta_1-\alpha_1})\mathscr{P}(\ell_1+1,\ell_2+1)\\ &\quad=\sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1,\ell_2,k)}{p^k}+\frac{1}{p^2}(p^{\alpha_1-\beta_1}-2+p^{\beta_1-\alpha_1})\sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1+1,\ell_2+1,k)}{p^k}\\ &\qquad+\frac{p^{\alpha_1-\beta_1}}{p^4}(p^{\alpha_1-\beta_1}-2+p^{\beta_1-\alpha_1})\mathscr{P}(\ell_1+2,\ell_2+2)\\ &\quad=\sum_{k\ge 0}\frac{1}{p^k}\Big\{\mathcal{J}_p(\ell_1,\ell_2,k)+(p^{\alpha_1-\beta_1}-2+p^{\beta_1-\alpha_1})\\ &\qquad\times\sum_{1\le m_p\le\lfloor k/2\rfloor}p^{(m_p-1)(\alpha_1-\beta_1)}\mathcal{J}_p(\ell_1+m_p,\ell_2+m_p,k-2m_p)\Big\}.\end{aligned}$$

Case 2: $\boldsymbol{\delta_1+\xi_1\neq\delta_2+\xi_2}$. For this case, $\upsilon_1+\upsilon_2\ge 1$. So $p\mid u_1u_2$, and $b_p=0$, where $p^{b_p}\parallel b$. By Lemma A.1 and A.2, we have

$$\begin{aligned}
&\mathscr{L}_p(\delta_1,\delta_2,\xi_1,\xi_2)\\
&\quad=\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)\mathscr{P}(\ell_1,\ell_2)-\frac{1}{p^2}\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)\mathscr{P}(\ell_1+1,\ell_2+1)\\
&\quad=\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)\Big\{\sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1,\ell_2,k)}{p^k}+\frac{1}{p^2}(p^{\alpha_1-\beta_1}-1)\mathscr{P}(\ell_1+1,\ell_2+1)\Big\}\\
&\quad=\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)\Big\{\sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1,\ell_2,k)}{p^k}+\frac{1}{p^2}(p^{\alpha_1-\beta_1}-1)\sum_{k\ge 0}\frac{\mathcal{J}_p(\ell_1+1,\ell_2+1,k)}{p^k}\\
&\qquad\qquad+\frac{p^{\alpha_1-\beta_1}}{p^4}(p^{\alpha_1-\beta_1}-1)\mathscr{P}(\ell_1+2,\ell_2+2)\Big\}\\
&\quad=\Big(1-\frac{1}{p^{\alpha_1-\beta_1}}\Big)\sum_{k\ge 0}\frac{1}{p^k}\Big\{\mathcal{J}_p(\ell_1,\ell_2,k)\\
&\qquad+(p^{\alpha_1-\beta_1}-1)\sum_{1\le m_p\le\lfloor k/2\rfloor}p^{(m_p-1)(\alpha_1-\beta_1)}\mathcal{J}_p(\ell_1+m_p,\ell_2+m_p,k-2m_p)\Big\}.
\end{aligned}$$

From both cases, we obtain that (A-9) is, say,

$$\begin{aligned}
&\sum\sum\sum\sum_{\substack{\delta_1,\delta_2,\ell_1,\ell_2\ge 0\\ \delta_1+\ell_1=\delta_2+\ell_2}}\frac{1}{p^{D'(\ell_1,\ell_2)-\ell_1\beta_1+\ell_2\alpha_1}}\mathscr{L}_p(\delta_1,\delta_2,\ell_1,\ell_2)\\
&\qquad+\sum\sum\sum\sum_{\substack{\delta_1,\delta_2,\ell_1,\ell_2\ge 0\\ \delta_1+\ell_1=\delta_2+\ell_2}}\Big(\sum_{\substack{0\le\xi_2<\ell_2\\ \xi_1=\ell_1}}\frac{\mathscr{L}_p(\delta_1,\delta_2,\xi_1,\xi_2)}{p^{(\ell_2-\ell_1)\beta_1-\xi_2(\beta_1-\alpha_1)}}\\
&\qquad+\sum_{\substack{0\le\xi_1<\ell_1\\ \xi_2=\ell_2}}\frac{\mathscr{L}_p(\delta_1,\delta_2,\xi_1,\xi_2)}{p^{(\ell_2-\ell_1)\alpha_1-\xi_1(\beta_1-\alpha_1)}}\Big)\frac{1}{p^{D'(\ell_1,\ell_2)}}=:\sum\sum_{\delta_1,\delta_2\ge 0}\mathcal{S}_p(\delta_1,\delta_2). \qquad \text{(A-11)}
\end{aligned}$$

For fixed δ_1,δ_2, where $\delta_2\ge\delta_1$, we rearrange the term $\mathcal{S}_p(\delta_1,\delta_2)$ and obtain that

$$\begin{aligned}
&S_p(\delta_1,\delta_2)\\
&\ =\sum\sum_{\substack{\epsilon_1,\epsilon_2\ge 0\\ \delta_1+\epsilon_1=\delta_2+\epsilon_2}}\sum_{k\ge 0}\frac{\mathcal{J}_p(\epsilon_1,\epsilon_2,k)}{p^{D'(\epsilon_1,\epsilon_2)+k}}\Big\{\frac{1}{p^{(\epsilon_2-\epsilon_1)\alpha_1}}+\frac{1}{p^{(\epsilon_2-\epsilon_1)\beta_1}}-\frac{1}{p^{-\epsilon_1\beta_1+\epsilon_2\alpha_1}}\\
&\qquad+(p^{\alpha_1-\beta_1}-2+p^{\beta_1-\alpha_1})\sum_{1\le\ell_2\le\epsilon_2}\frac{p^{(\ell_2-1)(\alpha_1-\beta_1)}}{p^{-(\epsilon_1-\ell_2)\beta_1+(\epsilon_2-\ell_2)\alpha_1}}\\
&\qquad+(p^{\alpha_1-\beta_1}-1)\sum_{1\le\ell_2\le\epsilon_2}p^{(\ell_2-1)(\alpha_1-\beta_1)}\Big(\frac{1}{p^{(\epsilon_2-\epsilon_1)\alpha_1}}+\frac{1}{p^{(\epsilon_2-\epsilon_1)\beta_1}}-\frac{2p^{\ell_2(\alpha_1-\beta_1)}}{p^{-\epsilon_1\beta_1+\epsilon_2\alpha_1}}\Big)\Big\}\\
&\ =\sum\sum_{\substack{\epsilon_1,\epsilon_2\ge 0\\ \delta_1+\epsilon_1=\delta_2+\epsilon_2}}\sum_{k\ge 0}\frac{\mathcal{J}_p(\epsilon_1,\epsilon_2,k)}{p^{D+\epsilon_1+\epsilon_2+k+\epsilon_1\beta_1-\epsilon_2\alpha_1}}. \qquad \text{(A-12)}
\end{aligned}$$

By a similar calculation, $\mathcal{S}_p(\delta_1, \delta_2)$ yields the same value for $\delta_1 > \delta_2$. In summary, from (A-9), (A-11), and (A-12), the local factor at p of $\mathcal{M}_{\alpha_1,\beta_1}$ is

$$\sum_{\substack{\epsilon_1,\epsilon_2\geq 0\\ \delta_1+\epsilon_1=\delta_2+\epsilon_2}}\sum \sum_{k\geq 0} \frac{\mathcal{J}_p(\epsilon_1, \epsilon_2, k)}{p^{D+\epsilon_1+\epsilon_2+k+\epsilon_1\beta_1-\epsilon_2\alpha_1}},$$

which is the same as the local factor at p of $\mathcal{AZ}\big(\frac{1}{2}; \pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})\big)$ in (A-4), as desired.

For $p = q$, we use similar arguments, with $\delta_i = \epsilon_i = 0$, so that the Euler factor is

$$\sum_{k\geq 0} \frac{\mathcal{J}_p(0, 0, k)}{p^k},$$

which is the same as the local factor at q of $\mathcal{AZ}\big(\frac{1}{2}; \pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})\big)$ in (A-7). This completes the proof of Proposition 6.1.

Appendix B. Voronoi summation

In this section, we state the Voronoi summation formula for the shifted k-divisor function defined in (2-5). The proof of this formula is essentially the same as the proof by Ivić of the Voronoi summation formula for the k-divisor function, so we state the results and refer the reader to [Ivić 1997] for detailed proofs.

Let ω be a smooth compactly supported function. For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, let $\sigma_k(n; \boldsymbol{\alpha}) = \sigma_k(n; \alpha_1, \dots, \alpha_k)$. Define

$$S\Big(\frac{a}{c}, \boldsymbol{\alpha}\Big) = \sum_{n=1}^{\infty} \sigma_k(n; \boldsymbol{\alpha})\, \mathrm{e}\Big(\frac{an}{c}\Big)\omega(n),$$

where $(a, c) = 1$, and the Mellin transform of ω,

$$\tilde{\omega}(s) = \int_0^{\infty} \omega(x)x^{s-1}\, dx.$$

Since ω was chosen to be from the Schwarz class, $\tilde{\omega}$ is entire and decays rapidly on vertical lines. We have the Mellin inversion formula

$$\omega(n) = \frac{1}{2\pi i}\int_{(c)} \tilde{\omega}(s)\, \frac{ds}{n^s},$$

where c is any vertical line. Let

$$A_k\Big(m, \frac{a}{c}, \boldsymbol{\alpha}\Big) = \frac{1}{2} \sum_{\substack{m_1,\dots,m_k\geq 1\\ m_1\cdots m_k=m}} m_1^{\alpha_1}\cdots m_k^{\alpha_k}$$
$$\times \sum_{a_1 \pmod c} \cdots \sum_{a_k \pmod c} \Big\{\mathrm{e}\Big(\frac{aa_1\cdots a_k + \boldsymbol{a}\cdot\boldsymbol{m}}{c}\Big) + \mathrm{e}\Big(\frac{-aa_1\cdots a_k + \boldsymbol{a}\cdot\boldsymbol{m}}{c}\Big)\Big\}, \quad \text{(B-1)}$$

and

$$B_k\Big(m,\frac{a}{c},\boldsymbol{\alpha}\Big)=\frac{1}{2}\sum_{\substack{m_1,\dots,m_k\ge 1\\ m_1\cdots m_k=m}} m_1^{\alpha_1}\cdots m_k^{\alpha_k}$$
$$\times\sum_{a_1 \pmod c}\cdots\sum_{a_k \pmod c}\Big\{\mathrm{e}\Big(\frac{aa_1\cdots a_k+\boldsymbol{a}\cdot\boldsymbol{m}}{c}\Big)-\mathrm{e}\Big(\frac{-aa_1\cdots a_k+\boldsymbol{a}\cdot\boldsymbol{m}}{c}\Big)\Big\}. \tag{B-2}$$

Moreover, we define

$$G_k(s,n,\boldsymbol{\alpha})=\frac{c^{ks}}{\pi^{ks}n^s}\bigg(\prod_{\ell=1}^{k}\frac{\Gamma\big(\frac{1}{2}(s-\alpha_\ell)\big)}{\Gamma\big(\frac{1}{2}(1-s+\alpha_\ell)\big)}\bigg),$$
$$H_k(s,n,\boldsymbol{\alpha})=\frac{c^{ks}}{\pi^{ks}n^s}\bigg(\prod_{\ell=1}^{k}\frac{\Gamma\big(\frac{1}{2}(1+s-\alpha_\ell)\big)}{\Gamma\big(\frac{1}{2}(2-s+\alpha_\ell)\big)}\bigg),$$

and for $0<\sigma<\frac{1}{2}-\frac{1}{k}+\frac{1}{k}\sum_{\ell=1}^{k}\mathrm{Re}(\alpha_\ell)$, we let

$$\begin{aligned}U_k(x;\boldsymbol{\alpha})&=\frac{1}{2\pi i}\int_{(\sigma)}\prod_{\ell=1}^{k}\frac{\Gamma\big(\frac{1}{2}(s-\alpha_\ell)\big)}{\Gamma\big(\frac{1}{2}(1-s+\alpha_\ell)\big)}\frac{ds}{x^s},\\ V_k(x;\boldsymbol{\alpha})&=\frac{1}{2\pi i}\int_{(\sigma)}\prod_{\ell=1}^{k}\frac{\Gamma\big(\frac{1}{2}(1+s-\alpha_\ell)\big)}{\Gamma\big(\frac{1}{2}(2-s+\alpha_\ell)\big)}\frac{ds}{x^s}.\end{aligned} \tag{B-3}$$

We note that by Stirling's formula, both integrals for U_k and V_k are absolutely convergent.

Finally, we define the Dirichlet series to be

$$D_k\Big(s,\frac{a}{c},\boldsymbol{\alpha}\Big)=\sum_n\frac{\sigma_k(n;\boldsymbol{\alpha})e(an/c)}{n^s},$$

which converges absolutely for $\mathrm{Re}\, s>1$. We have that

$$\begin{aligned}D_k\Big(s,\frac{a}{c},\boldsymbol{\alpha}\Big)&=\sum_{n_1,\dots,n_k}\frac{e(an_1n_2\cdots n_k/c)}{n_1^{s+\alpha_1}\cdots n_k^{s+\alpha_k}}\\ &=\frac{1}{c^{ks+\alpha_1+\cdots+\alpha_k}}\sum_{a_1,\dots,a_k \pmod c} e\Big(\frac{aa_1\cdots a_k}{c}\Big)\prod_{j=1}^{k}\zeta\Big(s+\alpha_j,\frac{a_j}{c}\Big),\end{aligned} \tag{B-4}$$

where $\zeta(s,a/c)$ is the Hurwitz zeta function defined for $\mathrm{Re}\, s>1$ as

$$\zeta\Big(s,\frac{a}{c}\Big)=\sum_{n=0}^{\infty}\frac{1}{(n+a/c)^s}. \tag{B-5}$$

The Hurwitz zeta function may be analytically continued to all of $\mathbb{C}$ except for a simple pole at $s=1$. Therefore, $D_k(s, a/c)$ can be analytically continued to all of $\mathbb{C}$ except for a simple pole at $s = 1-\alpha_j$ for $j = 1, \dots, k$.

Theorem B.1. *With notation as above and* $(a, c) = 1$ *and*

$$\alpha_i \ll \frac{1}{1000k \log(|c|+100)},$$

we have

$$S\Big(\frac{a}{c}, \boldsymbol{\alpha}\Big) = \sum_{\ell=1}^{k} \operatorname*{Res}_{s=1-\alpha_\ell} \tilde{\omega}(s) D_k\big(s, \frac{a}{c}, \boldsymbol{\alpha}\big)$$
$$+ \frac{\pi^{k/2+\alpha_1+\cdots+\alpha_k}}{c^{k+\alpha_1+\cdots+\alpha_k}} \sum_{n=1}^{\infty} A_k\Big(n, \frac{a}{c}, \boldsymbol{\alpha}\Big) \int_0^\infty \omega(x) U_k\Big(\frac{\pi^k n x}{c^k}; \boldsymbol{\alpha}\Big)\, dx$$
$$+ i^{3k} \frac{\pi^{k/2+\alpha_1+\cdots+\alpha_k}}{c^{k+\alpha_1+\cdots+\alpha_k}} \sum_{n=1}^{\infty} B_k\Big(n, \frac{a}{c}, \boldsymbol{\alpha}\Big) \int_0^\infty \omega(x) V_k\Big(\frac{\pi^k n x}{c^k}; \boldsymbol{\alpha}\Big)\, dx.$$

We refer the reader to the proof of Theorem 2 in [Ivić 1997] for details. Next, we collect properties of $A_3(n, a/c, \boldsymbol{\alpha})$, $B_3(n, a/c, \boldsymbol{\alpha})$, $U_3(x; \boldsymbol{\alpha})$, and $V_3(x; \boldsymbol{\alpha})$. These are useful for bounding error terms of $\mathcal{M}_6(q)$.

Lemma B.2. *Let* a, n, γ *be integers such that* $(a, \gamma) = 1$. *Moreover*, $A_3(n, a/c, \boldsymbol{\alpha})$ *and* $B_3(n, a/c, \boldsymbol{\alpha})$ *are defined as in* (B-1) *and* (B-2). *Then*

$$\sum_{n_1 n_2 n_3 = n} n_1^{\alpha_1} n_2^{\alpha_2} n_3^{\alpha_3} \sum\sum\sum_{r_1, r_2, r_3 \pmod{\gamma}} \mathrm{e}\Big(\frac{a r_1 r_2 r_3 + r_1 n_1 + r_2 n_2 + r_3 n_3}{\gamma}\Big)$$
$$= \gamma \sum_{h \mid \gamma,\, h^2 \mid n} h \Delta(n, h, \gamma) S\Big(\frac{n}{h^2}, -\bar{a}, \frac{\gamma}{h}\Big),$$

where $\Delta(n, h, \gamma)$ *is a divisor function satisfying* $\Delta(n, h, \gamma) \ll (\gamma n)^{\varepsilon}$. *Moreover*,

$$A_3\Big(n, \frac{a}{\gamma}, \boldsymbol{\alpha}\Big) \ll (\gamma n)^{\varepsilon} \gamma^{3/2} \sum_{h \mid \gamma,\, h^2 \mid n} \sqrt{h},$$
$$B_3\Big(n, \frac{a}{\gamma}, \boldsymbol{\alpha}\Big) \ll (\gamma n)^{\varepsilon} \gamma^{3/2} \sum_{h \mid \gamma,\, h^2 \mid n} \sqrt{h}.$$

The proof of this lemma can be found in equations (8.7)–(8.9) in [Ivić 1997].

Lemma B.3. *If* $U(x; \boldsymbol{\alpha}) := U_3(x; \boldsymbol{\alpha})$ *and* $V(x; \boldsymbol{\alpha}) := V_3(x; \boldsymbol{\alpha})$, *as defined in* (B-3), *and*

$$\alpha_i \ll \frac{1}{1000k \log(|c|+100)},$$

then for any $0 < \varepsilon < \frac{1}{6}$ *and* $x > 0$, *we have*

$$U(x;\boldsymbol{\alpha}) \ll x^{\varepsilon}, \qquad V(x;\boldsymbol{\alpha}) \ll x^{\varepsilon}. \tag{B-6}$$

Moreover for any fixed integer $K \geq 1$ *and* $x \geq x_0 > 0$,

$$U(x;\boldsymbol{\alpha}) = \sum_{j=1}^{K} \frac{c_j \cos(6x^{1/3}) + d_j \sin(6x^{1/3})}{x^{j/3+(\alpha_1+\alpha_2+\alpha_3)/3}} + O\Big(\frac{1}{x^{(K+1)/3+(\alpha_1+\alpha_2+\alpha_3)/3}}\Big), \tag{B-7}$$

$$V(x;\boldsymbol{\alpha}) = \sum_{j=1}^{K} \frac{e_j \cos(6x^{1/3}) + f_j \sin(6x^{1/3})}{x^{j/3+(\alpha_1+\alpha_2+\alpha_3)/3}} + O\Big(\frac{1}{x^{(K+1)/3+(\alpha_1+\alpha_2+\alpha_3)/3}}\Big), \tag{B-8}$$

with suitable constants $c_j, \ldots, f_j$, *and* $c_1 = 0$, $d_1 = -2/\sqrt{3\pi}$, $e_1 = -2/\sqrt{3\pi}$, $f_1 = 0$.

The proof of this lemma is a minor modification of the proof of Lemma 3 in [Ivić 1997].

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