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Analytic functions on tubes of nonarchimedean analytic spaces

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Let k be a discretely valued nonarchimedean field. We give an explicit description of analytic functions whose norm is bounded by a given real number r on tubes of reduced k-analytic spaces associated to special formal schemes (including k-affinoid spaces as well as open polydiscs). As an application we study the connectedness of these tubes. In the discretely valued case, this generalizes a result of Siegfried Bosch. We use as a main tool a result of Aise Johan de Jong relating formal and analytic functions on special formal schemes and a generalization of de Jong's result which is proved in the joint appendix with Christian Kappen.

1. Introduction

Let us temporarily consider a nonarchimedean nontrivially valued field k. We work with k-analytic spaces, which were introduced by Vladimir Berkovich [1990; 1993] (and we always consider strictly k-affinoid and strictly k-analytic spaces). If X is a k-affinoid space, its ring of analytic functions $\mathcal{A}$ is a k-affinoid algebra which has nice algebraic properties. A k-analytic space is connected if and only if its ring of global analytic functions contains no nontrivial idempotents. In concrete situations, if X is a k-affinoid space, one can expect to use the nice algebraic properties of its k-affinoid algebra $\mathcal{A}$ to study the connectedness of X. For more general k-analytic spaces, it might be difficult to deal with their ring of global analytic functions. For instance, the ring of global analytic functions of the open unit disc is not Noetherian. The starting point of this work is to use generic fibres of special formal schemes to overcome this difficulty in certain situations.

A result of Siegfried Bosch. Our motivation is a generalization as well as a new proof of a result due to Siegfried Bosch [1977] in the discretely valued case. Let us consider a k-affinoid algebra $\mathcal{A}$ and let X be the associated k-affinoid space. Let x be a rigid point of X, $\mathfrak{m}_x$ the associated maximal ideal of $\mathcal{A}$ and $\tilde{x}$ the image of x under

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the reduction map red : $X \to \widetilde{X}$ (for the definition and properties of the reduction map, we refer to [Bosch et al. 1984, Section 7.1.5] for rigid spaces and to [Berkovich 1990, Section 2.4] for k-analytic spaces). Let $X_+(x) := \mathrm{red}^{-1}(\tilde{x})$. Following Pierre Berthelot's terminology [1996, Définitions 1.1.2], we call $X_+(x)$ the tube of $\tilde{x}$ in X. Bosch proves that if X is distinguished and equidimensional, $X_+(x)$ is connected. This connectedness result is a corollary of the main result of [Bosch 1977, Theorem 5.8], which asserts that if X is distinguished and equidimensional, then

$$\Gamma(X_+(x), \mathcal{O}_X^\circ) \simeq (\mathcal{A}^\circ)^{\wedge(t\cdot\mathcal{A}^\circ+\mathring{\mathfrak{m}}_x)} \simeq \varprojlim_n \mathcal{A}^\circ/(t\cdot\mathcal{A}^\circ+\mathring{\mathfrak{m}}_x)^n,$$

where $t \in k$ with $0 < |t| < 1$, $\mathcal{O}_X^\circ$ denotes the sheaf of analytic functions f such that $|f|_{\sup} \leq 1$, $\mathring{\mathfrak{m}}_x := \mathfrak{m}_x \cap \mathcal{A}^\circ$ and $^\wedge$ denotes the completion with respect to an ideal.

Analytic functions and formal functions. For the rest of the article, we assume that k is discretely valued, with nontrivial valuation. We denote its valuation ring by R and we fix a uniformizer π. Following [Berkovich 1996, Section 1], we say that an adic R-algebra A is a *special R-algebra* if it is isomorphic to a quotient of $R\langle T_1, \dots, T_m\rangle[\![S_1, \dots, S_n]\!]$ equipped with the $(\pi, S_1, \dots, S_n)$-adic topology. Let $\mathfrak{X} := \mathrm{Spf}(A)$ be its associated formal R-scheme. Following a construction due to Berthelot [1996, 0.2.6] for rigid spaces and extended to k-analytic spaces by Berkovich [1996, Section 1], one can associate to $\mathfrak{X}$ a k-analytic space denoted by $\mathfrak{X}_\eta$ called its generic fibre. For instance, if $A = R\langle T_1, \dots, T_m\rangle[\![S_1, \dots, S_n]\!]$, then $\mathfrak{X}_\eta \simeq E^m \times B^n$, where we denote by E^m the m-dimensional closed unit polydisc and by B^n the n-dimensional open unit polydisc. Following the terminology introduced by Christian Kappen [2010; 2012], we say that $\mathcal{A} := A \otimes_R k$ is a *semiaffinoid k-algebra*. Up to canonical isomorphism, the k-analytic space $\mathfrak{X}_\eta$ depends only on the semiaffinoid k-algebra $\mathcal{A}$ and we call it a *semiaffinoid k-analytic space* (this should not be confused with semiaffinoid k-spaces in [Kappen 2010; 2012]). So we can functorially associate to a semiaffinoid k-algebra a k-analytic space. If A is R-flat, one gets a natural injection $A \to \Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta}^\circ)$. When A is in addition normal, it was proven by A.J. de Jong [1995, Theorem 7.4.1] that

$$A \simeq \Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta}^\circ).$$

For the applications we have in mind, we need the following generalization, which was already stated without proof in [de Jong 1995, Remark 7.4.2].

Theorem 2.1. *Let A be a reduced special R-algebra which is R-flat and integrally closed in $A\otimes_R k$, and let X be the associated k-analytic space. Then $A \simeq \Gamma(X, \mathcal{O}_X^\circ)$.*

Let us mention that if A is a special R-algebra which is R-flat and integrally closed in $A \otimes_R k$, then A is automatically reduced (see the argument in [Kappen 2012, Remark 2.7]). Hence one can remove the assumption that A is reduced in the above theorem if necessary. Theorem 2.1 easily follows from the following result, which is proved in the appendix with Christian Kappen.

Theorem A.8. *Let $\mathcal{A}$ be a reduced semiaffinoid k-algebra, and let X be the associated k-analytic space. Then* $\mathcal{A} \simeq \{f \in \Gamma(X, \mathcal{O}_X) \mid |f|_{\sup} < \infty\}$.

Main result. Actually, the tube $X_+(x)$ is a semiaffinoid k-analytic space. Therefore, it seemed very natural to us to look for a generalization and a more direct proof of Bosch's result [1977, Theorem 5.8] using special formal R-schemes, semiaffinoid k-algebras and de Jong's result as well as its generalization (Theorem 2.1). This is the content of this article. Let us fix X a reduced semiaffinoid k-analytic space. Let $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X^\circ)$ and let

$$U := \{x \in X \mid |f_i(x)| < 1 \ \forall i = 1, \dots, n\}.$$

Theorem 3.1 and Proposition 5.13. *With the above notations,*

$$\Gamma(X, \mathcal{O}_X^\circ)^{\wedge(f_1,\dots,f_n)} \simeq \Gamma(U, \mathcal{O}_X^\circ). \tag{1}$$

More generally, for any positive real number r, $\Gamma(X, \mathcal{O}_X^{\leq r})^{\wedge(f_1,\dots,f_n)} \simeq \Gamma(U, \mathcal{O}_X^{\leq r})$, *where* $\mathcal{O}_X^{\leq r}$ *is the sheaf of analytic functions* f *such that* $|f| \leq r$.

In Section 4 we associate to a semiaffinoid k-analytic space X its *canonical reduction* $\widetilde{X}$, which is a $\tilde{k}$-scheme of finite type, and a canonical reduction map $\mathrm{red} : X \to \widetilde{X}$. If X is a k-affinoid space, then red coincides with the reduction map of [Berkovich 1990, Section 2.4]. We prove the following result:

Corollary 4.14. *Let* $Z \subset \widetilde{X}$ *be a connected Zariski closed subset. Then* $\mathrm{red}^{-1}(Z)$ *is connected.*

We want to stress that our results hold under the assumption that k is discretely valued, whereas this assumption is not made in [Bosch 1977]. We conjecture that Theorem 3.1 holds for any nontrivially valued nonarchimedean field k and for any reduced k-affinoid space (with $(f_1, \dots, f_n)$ replaced by $(t, f_1, \dots, f_n)$ for some $t \in k^*$ with $|t| < 1$). We do not see how this could be done using the techniques of [Bosch 1977]. We believe that quasiaffinoid k-algebras (which are a generalization of special R-algebras and semiaffinoid k-algebras to arbitrary nonarchimedean nontrivially valued fields [Lipshitz and Robinson 2000]) might be a good framework to tackle this. We also want to stress that our results are more general and the proofs simpler than in [Bosch 1977] regarding the following points.

– The proof we give of (1) is pretty short: one has to use Theorem 2.1 and a certain compatibility between integral closure and tensor product for excellent rings (whose use was suggested to us by Ofer Gabber).

– The explicit description of the rings $\Gamma(U, \mathcal{O}_X^\circ)$ given in Theorem 3.1 holds for any tube, whereas in [Bosch 1977] this was proved only for tubes over closed points.

– For a positive real number r, we extend Bosch's result to analytic functions f such that $|f|_{\sup} \leq r$ (Proposition 5.13).

– Unlike in [Bosch 1977], we do not assume that X is distinguished or equidimensional. Hence, Corollary 4.14 answers positively in the discretely valued case the question raised by Jérôme Poineau [2014, Remarque 2.9]. Let us point out that using Bosch's result, Antoine Ducros [2003, Lemma 3.1.2] proved that when k is algebraically closed and X is equidimensional and reduced, the tube of a connected Zariski closed subset of $\widetilde{X}$ in X is connected; afterwards Poineau [2014, Théorème 2.8] (still relying on Bosch's result) proved the same statement for any k and assuming only that X is equidimensional.

– Our results hold not only for k-affinoid spaces, but also for semiaffinoid k-analytic spaces.

Organization of the paper. In Section 2, we give general facts about semiaffinoid k-analytic spaces. In Section 3, we prove the main result of the article, Theorem 3.1. In Section 4, we define and study the canonical reduction of semiaffinoid k-analytic spaces, and apply it to study the connectedness of tubes. In Section 5, we prove Proposition 5.13, which is the graded version of Theorem 3.1. In Section 6, we grasp additional remarks about semiaffinoid k-analytic spaces. The joint appendix with Christian Kappen aims to prove Theorem A.8.

2. Semiaffinoid k-analytic spaces

Following [Kappen 2012, Definition 2.2], we say that a k-algebra $\mathcal{A}$ is a semiaffinoid k-algebra if it is of the form $A \otimes_R k$ for some special R-algebra A. Equivalently, a semiaffinoid k-algebra is a quotient of $R\langle T_1, \dots, T_m\rangle[\![S_1, \dots, S_n]\!] \otimes_R k$ for some given integers m and n. The category of semiaffinoid k-algebras is defined as the category whose objects are semiaffinoid k-algebras and whose morphisms are k-algebra morphisms. In particular, the category of k-affinoid algebras is a full subcategory of the category of semiaffinoid k-algebras. If $\mathcal{A}$ is a semiaffinoid k-algebra, a *special R-model* of $\mathcal{A}$ is an R-flat special R-algebra A such that there exists an isomorphism of k-algebras $A \otimes_R k \simeq \mathcal{A}$. If A is an R-flat special R-algebra, one has a natural inclusion $A \to \mathcal{A} := A \otimes_R k$ and through this inclusion, A is identified with a special R-model of $\mathcal{A}$. One can define a functor from the category of semiaffinoid k-algebras to the category of k-analytic spaces [Kappen 2012, Section 2C1]. If $\mathcal{A}$ is a semiaffinoid k-algebra, its associated k-analytic space X is called a semiaffinoid k-analytic space. For any special R-model A of $\mathcal{A}$, one has a natural isomorphism $X \simeq \mathrm{Spf}(A)_\eta$. If $f \in \mathcal{A}$, we set $|f|_{\sup} := \sup\{|f(x)| \mid x \in X\}$ and we define

$$\mathcal{A}^\circ = \{f \in \mathcal{A} \mid |f|_{\sup} \leq 1\}.$$

It is proved in the Appendix (Theorem A.8) that if the semiaffinoid k-algebra $\mathcal{A}$ is reduced, one has an isomorphism

$$\mathcal{A} \simeq \{f \in \Gamma(X, \mathcal{O}_X) \mid |f|_{\sup} < \infty\}.$$

We are particularly interested in the following consequence of this result.

Theorem 2.1. *Let A be a reduced special R-algebra with associated k-analytic space X. Let us assume that A is R-flat and integrally closed in $A \otimes_R k$. Then $A \simeq \Gamma(X, \mathcal{O}_X^\circ)$.*

Proof. Let $\mathcal{A} = A \otimes_R k$. By definition, $\mathcal{A}$ is a reduced semiaffinoid k-algebra, so thanks to Theorem A.8, one has $\mathcal{A}^\circ \simeq \Gamma(X, \mathcal{O}_X^\circ)$. According to [Kappen 2012, Corollaries 2.10 and 2.11] $\mathcal{A}^\circ$ is a special R-algebra which contains A, and the inclusion $A \subset \mathcal{A}^\circ$ is integral. Since by assumption A is integrally closed in $\mathcal{A}$, A is also integrally closed in $\mathcal{A}^\circ$. So $A \simeq \mathcal{A}^\circ \simeq \Gamma(X, \mathcal{O}_X^\circ)$. □

Corollary 2.2. *Let $\mathcal{A}$ be a reduced semiaffinoid k-algebra with associated k-analytic space X.*

(i) *The R-algebra $\Gamma(X, \mathcal{O}_X^\circ)$ is a reduced special R-algebra.*

(ii) *If A is a reduced special R-algebra such that $A \otimes_R k \simeq \mathcal{A}$, then $\Gamma(X, \mathcal{O}_X^\circ)$ is isomorphic to the integral closure of A in $A \otimes_R k$.*

Proof. Let A be a reduced special R-model of $\mathcal{A}$, so that $A \otimes_R k \simeq \mathcal{A}$. Let A' be the integral closure of A in $\mathcal{A}$. Since A is excellent (see [Valabrega 1975; 1976]), A' is a finite A-algebra, so A' is a reduced special R-algebra. Since $A' \otimes_R k \simeq A \otimes_R k \simeq \mathcal{A}$, the k-analytic spaces attached to A and A' are both isomorphic to X. So thanks to Theorem 2.1, $\Gamma(X, \mathcal{O}_X^\circ) \simeq A'$, which is a special R-algebra. This proves (i) and (ii). □

Example 2.3. The two statements of Corollary 2.2 do not hold for nonreduced semiaffinoid k-algebras. For instance, if $\mathcal{A} = (R[\![T_1, T_2]\!] \otimes_R k)/(T_2^2)$ with associated k-analytic space X, then any element of $\Gamma(X, \mathcal{O}_X^\circ)$ is of the form $f(T_1) + g(T_1)T_2$, where $f(T_1) \in R[\![T_1]\!]$ and $g(T_1)$ is an arbitrary analytic function on the open unit disc. We can choose $g(T_1)$ such that $g(T_1) \notin R[\![T_1]\!] \otimes_R k$ (in mixed characteristic, one could take $g = \log$). This gives a counterexample to the statements (i) and (ii) of Corollary 2.2 (for (i), see [Kappen 2012, Remark 2.7]).

Corollary 2.4. *The functor*

$$\{\textit{semiaffinoid } k\textit{-algebras}\} \to \{k\textit{-analytic spaces}\}$$

is faithful and its restriction

$$\{\textit{reduced semiaffinoid } k\textit{-algebras}\} \to \{k\textit{-analytic spaces}\}$$

is fully faithful.

Proof. If $f : X \to Y$ is a morphism of k-analytic spaces induced by a morphism of semiaffinoid k-algebras $\mathcal{B} \to \mathcal{A}$ then f induces the diagram

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \Gamma(X, \mathcal{O}_X) \end{array}$$

whose vertical arrows are injective. This proves that the functor is faithful.

To prove that the restriction of the functor to reduced semiaffinoid k-algebras is full, let us fix a reduced semiaffinoid k-algebra $\mathcal{A}$ and let X be its associated k-analytic space. Let Y be the k-analytic space associated to another semiaffinoid k-algebra $\mathcal{B}$ and let us consider a presentation

$$\mathcal{B} = (R\langle T_1, \dots, T_m\rangle[\![S_1, \dots, S_n]\!] \otimes_R k)/I$$

giving rise to a closed immersion $Y \hookrightarrow E^m \times B^n$. If $f : X \to Y$ is a morphism of k-analytic spaces then using the composition

$$X \xrightarrow{f} Y \hookrightarrow E^m \times B^n,$$

one gets functions $f_1, \dots, f_m \in \Gamma(X, \mathcal{O}_X^\circ)$ and $f_{m+1}, \dots, f_{m+n} \in \Gamma(X, \mathcal{O}_X^\circ)$ with $|f_i(x)| < 1$ for all $x \in X$ and $i \in \{m+1, \dots, m+n\}$ such that the functions $f_1, \dots, f_{m+n}$ induce the morphism $X \to E^m \times B^n$. Thanks to Theorem A.8, $f_i \in \mathcal{A}^\circ$ for $i = 1, \dots, m+n$. Thanks to [Kappen 2012, Theorem 2.13] there is a unique morphism of semiaffinoid k-algebras $R\langle T_1, \dots, T_m\rangle[\![S_1, \dots, S_n]\!] \otimes_R k \to \mathcal{A}$ sending T_i to f_i for $i = 1, \dots, m$ and sending S_i to f_{m+i} for $i = 1, \dots, n$. It factorizes through $\mathcal{B}$ and gives the desired morphism of semiaffinoid k-algebras $\mathcal{B} \to \mathcal{A}$. □

Example 2.5. The functor from semiaffinoid k-algebras to k-analytic spaces is not fully faithful. Indeed, as in Example 2.3, consider $\mathcal{A} = (R[\![T_1, T_2]\!] \otimes_R k)/(T_2^2)$ with associated k-analytic space X. Let $g(T_1) \in k[\![T_1]\!]$ be a power series which converges on the open unit disc, but such that $g(T_1) \notin R[\![T_1]\!] \otimes_R k$. Then $T_2 g(T_1) \in \Gamma(X, \mathcal{O}_X^\circ)$ and hence it defines a morphism of k-analytic spaces $\phi : X \to \mathbb{A}_k^{1,\mathrm{an}}$ whose image is the origin of $\mathbb{A}_k^{1,\mathrm{an}}$. In particular it also defines a morphism $X \to E$, where E is the closed unit disc over k, which is a k-affinoid space (with affinoid algebra $\mathcal{B} = k\langle T\rangle$). But ϕ is not induced by a morphism of semiaffinoid k-algebras $\mathcal{B} \to \mathcal{A}$.

Remark 2.6. Theorem 2.1 was already stated in [de Jong 1995, Remark 7.4.2] without proof. Theorem A.8 is also stated in [Nicaise 2009, Lemma 2.14] but its proof is obtained as a corollary of [de Jong 1995, Remark 7.4.2]. Corollary 2.4 is also stated in [Kappen 2012, Remark 2.56] without proof.

Remark 2.7. Let $\mathcal{A}$ be a semiaffinoid k-algebra with associated k-analytic space X. Let $\mathcal{A}_{\mathrm{red}} := \mathcal{A}/(\mathrm{nil}\,\mathcal{A})$ be the reduced ring associated to $\mathcal{A}$. Then $\mathcal{A}_{\mathrm{red}}$ is a reduced semiaffinoid k-algebra. Let Y be the k-analytic space associated with $\mathcal{A}_{\mathrm{red}}$. Then $\mathcal{A} \to \mathcal{A}_{\mathrm{red}}$ induces a closed immersion of k-analytic spaces $Y \to X$ which is a bijection of sets. Let B be a special R-model of $\mathcal{A}_{\mathrm{red}}$. Since $B \subset B \otimes_R k \simeq \mathcal{A}_{\mathrm{red}}$, we deduce that B is reduced. Hence since $Y \simeq \mathrm{Spf}(B)_\eta$, by [de Jong 1995, Proposition 7.2.4(c)], Y is reduced. Hence $Y \simeq X_{\mathrm{red}}$.

3. Analytic functions and formal functions on tubes

I am very grateful to Ofer Gabber for having pointed out the reference to Proposition 6.14.4 of [EGA IV_2 1965], which greatly simplifies the proof of the following statement.

Theorem 3.1. *Let X be a k-analytic space associated with a reduced semiaffinoid k-algebra $\mathcal{A}$. Let $I \subset \Gamma(X, \mathcal{O}_X^\circ)$ be an ideal and let $U = \{x \in X \mid |f(x)| < 1 \; \forall f \in I\}$. Then*

$$\Gamma(X, \mathcal{O}_X^\circ)^{\wedge I} \simeq \Gamma(U, \mathcal{O}_X^\circ).$$

Proof. Let us set $A := \Gamma(X, \mathcal{O}_X^\circ)$. Thanks to Corollary 2.2(i), A is a reduced special R-algebra. Let us choose some functions $f_1, \dots, f_n \in A$ such that $(f_1, \dots, f_n) = I$. Then

$$A^{\wedge I} \simeq A[[\rho_1, \dots, \rho_n]]/(f_i - \rho_i)_{i=1,\dots,n}$$

is a special R-algebra. Let us set $\mathfrak{Y} = \mathrm{Spf}(A^{\wedge I})$. Then by [de Jong 1995, Lemma 7.2.5] (or see also [Lipshitz and Robinson 2000, Theorem 5.3.5 and Proposition 5.3.6]), the induced morphism of k-analytic spaces $\mathfrak{Y}_\eta \to X$ identifies $\mathfrak{Y}_\eta$ with U as an analytic domain of X. So U is the k-analytic space associated to the semiaffinoid k-algebra $\mathcal{B} := A^{\wedge I} \otimes_R k$ and by definition, $A^{\wedge I}$ is a special R-model of $\mathcal{B}$. Thanks to Theorem 2.1, it only remains to show that $A^{\wedge I}$ is integrally closed in $A^{\wedge I} \otimes_R k \simeq \mathcal{B}$.

Since A is an excellent ring (this follows from [Valabrega 1975, Proposition 7] when $\mathrm{char}(k) = p > 0$ and from [Valabrega 1976, Theorem 9] when $\mathrm{char}(k) = 0$), the morphism $\mathrm{Spec}(A^{\wedge I}) \to \mathrm{Spec}(A)$ is regular [EGA IV_2 1965, Scholie 7.8.3(v)], so in particular, $\mathrm{Spec}(A^{\wedge I}) \to \mathrm{Spec}(A)$ is a normal morphism (see [EGA IV_2 1965, Définition 6.8.1] for the definitions of normal and regular morphisms of schemes).

Since $A = \Gamma(X, \mathcal{O}_X^\circ)$, it follows that A is integrally closed in $A \otimes_R k$. So thanks to [EGA IV_2 1965, Proposition 6.14.4], $A^{\wedge I}$ is integrally closed in $A^{\wedge I} \otimes_R k$. Finally, by Theorem 2.1,

$$A^{\wedge I} \simeq \Gamma(U, \mathcal{O}_X^\circ). \qquad \square$$

4. Reduction and connectedness

Reduction. Let $\mathcal{A}$ be a semiaffinoid k-algebra and let X be its associated k-analytic space. We set

$$\mathcal{A}^\circ = \{f \in \mathcal{A} \mid |f|_{\sup} \le 1\}, \qquad \check{\mathcal{A}} = \{f \in \mathcal{A} \mid \forall x \in X\ |f(x)| < 1\},$$
$$\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} \mid |f|_{\sup} < 1\}, \qquad \widetilde{\mathcal{A}} = \mathcal{A}^\circ/\check{\mathcal{A}}, \qquad \widetilde{\mathcal{A}}^+ = \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}.$$

When $\mathcal{A}$ is a k-affinoid algebra, thanks to the maximum modulus principle [Bosch et al. 1984, Proposition 6.2.1.4], $\check{\mathcal{A}} = \mathcal{A}^{\circ\circ}$, $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}^+$ and in this case, $\widetilde{\mathcal{A}}$ corresponds to the reduction of $\mathcal{A}$ as defined in [Bosch et al. 1984, Section 6.3]. For a general semiaffinoid k-algebra, the maximum modulus principle does not hold[1] (consider for instance $S \in R[\![S]\!] \otimes_R k$), so in general, one has a strict inclusion $\mathcal{A}^{\circ\circ} \subset \check{\mathcal{A}}$ and $\widetilde{\mathcal{A}}$ is a strict quotient of $\widetilde{\mathcal{A}}^+$. If $\mathcal{A} = R\langle T_1, \dots, T_m\rangle[\![S_1, \dots, S_n]\!] \otimes_R k$, then $\widetilde{\mathcal{A}} = \tilde{k}[T_1, \dots, T_m]$ and $\widetilde{\mathcal{A}}^+ = \tilde{k}[T_1, \dots, T_m][\![S_1, \dots, S_n]\!]$.

For a k-analytic space X, recall that we defined the subsheaf $\mathcal{O}_X^\circ \subset \mathcal{O}_X$ of analytic functions f such that $|f(x)| \le 1$ for all x. Likewise, we denote by $\check{\mathcal{O}}_X \subset \mathcal{O}_X$ the subsheaf of analytic functions f such that $|f(x)| < 1$ for all x.

Lemma 4.1. *Let $\mathcal{A}$ be a semiaffinoid algebra and $\mathcal{A}_{\mathrm{red}}$ the associated reduced semiaffinoid k-algebra. Let X be the k-analytic space associated with $\mathcal{A}$. By Remark 2.7, the k-analytic space associated with $\mathcal{A}_{\mathrm{red}}$ can be identified with X_{red}. Then all the natural maps in the following commutative squares are isomorphisms of R-algebras*

$$\begin{array}{ccc}
\mathcal{A}^\circ/\check{\mathcal{A}} & \xrightarrow{\ \alpha\ } & (\mathcal{A}_{\mathrm{red}})^\circ/(\check{\mathcal{A}}_{\mathrm{red}}) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_{\mathrm{red}}} \\
\Gamma(X, \mathcal{O}_X^\circ)/\Gamma(X, \check{\mathcal{O}}_X) & \xrightarrow[\ \beta\ ]{} & \Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}^\circ)/\Gamma(X_{\mathrm{red}}, \check{\mathcal{O}}_{X_{\mathrm{red}}})
\end{array} \tag{2}$$

$$\begin{array}{ccc}
\mathcal{A}^\circ/\mathcal{A}^{\circ\circ} & \longrightarrow & (\mathcal{A}_{\mathrm{red}})^\circ/(\mathcal{A}_{\mathrm{red}})^{\circ\circ} \\
\downarrow & & \downarrow \\
\Gamma(X, \mathcal{O}_X^\circ)/\Gamma(X, \mathcal{O}_X)_{<1} & \longrightarrow & \Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}^\circ)/\Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})_{<1}
\end{array} \tag{3}$$

where $\Gamma(X, \mathcal{O}_X)_{<1} = \{f \in \Gamma(X, \mathcal{O}_X) \mid |f|_{\sup} < 1\}$, and similarly $\Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})_{<1}$.

Proof. Using Theorem A.8, we get isomorphisms $(\mathcal{A}_{\mathrm{red}})^\circ \simeq \Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}^\circ)$ and

$$\check{\mathcal{A}}_{\mathrm{red}} \simeq \Gamma(X_{\mathrm{red}}, \check{\mathcal{O}}_{X_{\mathrm{red}}}),$$

which imply ϕ_{red} is an isomorphism. To prove β is an isomorphism, we first remark that X is a quasi-Stein space in the sense of [Kiehl 1967, Definition 2.3]. Moreover, we have an isomorphism of ringed spaces $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}) \simeq (X, \mathcal{O}_X/(\operatorname{rad}\mathcal{O}_X))$, where $\operatorname{rad}\mathcal{O}_X$ is the nilradical of $\mathcal{O}_X$. Since the latter is a coherent $\mathcal{O}_X$-ideal sheaf by

[1] Actually, $\mathcal{A}$ is a k-affinoid algebra if and only if the maximum modulus principle holds [Lipshitz and Robinson 2000, Proposition 5.3.8].

[Bosch et al. 1984, Corollary 9.5.4], it follows from Theorem B for quasi-Stein spaces [Kiehl 1967, Satz 2.4.2] that $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$ is surjective. Hence $\Gamma(X, \mathcal{O}_X^\circ) \to \Gamma(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}^\circ)$ is also surjective, and so β is surjective. The injectivity of β follows easily from the fact that X and X_{red} are in natural bijection. So β is an isomorphism. Since $\mathcal{A}^\circ \to (\mathcal{A}_{\mathrm{red}})^\circ$ is surjective, we deduce similarly that α is an isomorphism. Since (2) commutes, we deduce that ϕ is also an isomorphism. The proof for (3) is analogous. □

Lemma 4.2. *Let $\mathcal{A}$ be a semiaffinoid k-algebra. Then $\widetilde{\mathcal{A}}$ is a reduced $\tilde{k}$-algebra of finite type and $\widetilde{\mathcal{A}}^+$ is a reduced $\tilde{k}$-algebra which is a quotient of some*

$$\tilde{k}[T_1, \dots, T_m][[S_1, \dots, S_n]].$$

Proof. By Lemma 4.1, $\widetilde{\mathcal{A}} \simeq \widetilde{\mathcal{A}_{\mathrm{red}}}$, so we can replace $\mathcal{A}$ by $\mathcal{A}_{\mathrm{red}}$ and assume that $\mathcal{A}$ is reduced. By [Kappen 2012, Corollary 2.11], under this assumption $\mathcal{A}^\circ$ is a special R-algebra, so there is an isomorphism

$$\mathcal{A}^\circ \simeq R\langle T_1, \dots, T_m\rangle[[S_1, \dots, S_n]]/I$$

for some ideal I of $R\langle T_1, \dots, T_m\rangle[[S_1, \dots, S_n]]$. The ideal $\check{\mathcal{A}}$ then contains the ideal generated by the image of $(\pi, S_1, \dots, S_n)$ modulo I. Hence $\widetilde{\mathcal{A}}$ is a quotient of $\tilde{k}[T_1, \dots, T_m]$, proving the first part of the lemma. Likewise $\mathcal{A}^{\circ\circ}$ contains the ideal (π), hence we get a surjective map

$$\tilde{k}[T_1, \dots, T_m][[S_1, \dots, S_n]] \to \widetilde{\mathcal{A}}^+. \qquad \square$$

Remark 4.3. If $\mathcal{A}$ is a reduced semiaffinoid k-algebra, then $\check{\mathcal{A}}$ is the biggest ideal of definition of the special R-algebra $\mathcal{A}^\circ$ (see [Kappen 2012, Remark 2.8]). Likewise, one can prove easily that $\mathcal{A}^{\circ\circ} = \mathrm{rad}(\pi)$.

4.4. Let A be an arbitrary special R-algebra and let $\mathfrak{X} := \mathrm{Spf}(A)$. We define $\mathfrak{X}_s := \mathrm{Spec}(A/J)$, where J is the biggest ideal of definition of A. Then there is a *specialization map* $\mathrm{sp}_{\mathfrak{X}} : \mathfrak{X}_\eta \to \mathfrak{X}_s$ which is defined in [de Jong 1995, 7.1.10] on the subset of rigid points $\mathfrak{X}^{\mathrm{rig}}$ and in general in [Berkovich 1996, §1, p. 371] (beware that in [Berkovich 1996] the map $\mathrm{sp}_{\mathfrak{X}}$ is called the reduction map).

Definition 4.5. Let X be a semiaffinoid k-analytic space coming from a semiaffinoid k-algebra $\mathcal{A}$. We set $\widetilde{X} = \mathrm{Spec}(\widetilde{\mathcal{A}})$. We call $\widetilde{X}$ the canonical reduction of X. According to Lemma 4.2, it is a reduced $\tilde{k}$-scheme of finite type. If Y is a semiaffinoid k-analytic space and $\varphi : X \to Y$ is a morphism of k-analytic spaces, then we get an associated morphism $\varphi^* : \Gamma(Y, \mathcal{O}_Y^\circ) \to \Gamma(X, \mathcal{O}_X^\circ)$. Hence by Lemma 4.1, we can functorially associate a morphism $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$. If φ comes from a morphism of semiaffinoid k-algebras $\psi : \mathcal{B} \to \mathcal{A}$, then $\tilde{\varphi}$ is induced by $\tilde{\psi} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{A}}$.

4.6. As in [Berkovich 1990, Section 2.4] one can define a canonical reduction map $\mathrm{red} : X \to \widetilde{X}$ in the following way. If $x \in X$, we obtain an associated map $\chi_x : \mathcal{A} \to \mathcal{H}(x)$, which gives rise to a map $\tilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$. Then

$$\mathrm{red} : X \to \widetilde{X}, \qquad x \mapsto \mathrm{Ker}(\tilde{\chi}_x).$$

To stress the dependence in X, we might write red_X instead of red. Similarly as in [Berkovich 1990, Corollary 2.4.3] one can check that the canonical reduction map red is anticontinuous, i.e., the inverse image of an open set is closed.

Remark 4.7. Identifying X_{red} with the k-analytic space associated with $\mathcal{A}_{\mathrm{red}}$ (see Remark 2.7), and using Lemma 4.1, we get a commutative diagram

$$\begin{array}{ccc} X_{\mathrm{red}} & \longrightarrow & X \\ {\scriptstyle \mathrm{red}_{X_{\mathrm{red}}}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}_X} \\ \widetilde{X_{\mathrm{red}}} & = & \widetilde{X} \end{array}$$

Remark 4.8. Let A be a special R-model of the semiaffinoid k-algebra $\mathcal{A}$, with associated k-analytic space X. Let J be the biggest ideal of definition A. The injection $A \to \mathcal{A}^\circ$ induces an injective morphism of $\tilde{k}$-algebras $\varphi : A/J \to \widetilde{\mathcal{A}}$, which induces a morphism $\iota : \widetilde{X} \to \mathfrak{X}_s$. If $\mathfrak{X} := \mathrm{Spf}(A)$ we get a commutative diagram

$$\begin{array}{ccccc} & & X \simeq \mathfrak{X}_\eta & & \\ & {\scriptstyle \mathrm{red}}\swarrow & & \searrow{\scriptstyle \mathrm{sp}_{\mathfrak{X}}} & \\ \widetilde{X} & & \xrightarrow[\iota]{\qquad} & & \mathfrak{X}_s \end{array}$$

If $\mathcal{A}$ is reduced, then $A := \mathcal{A}^\circ$ is a special R-model and in that case $J = \check{\mathcal{A}}$ (see Remark 4.3). Hence in that case, φ and ι are isomorphisms. In general, since $\widetilde{\mathcal{A}}$ is a finitely generated $\tilde{k}$-algebra, we can find $f_1, \dots, f_m \in \mathcal{A}^\circ$ such that $\widetilde{f_1}, \dots, \widetilde{f_m}$ generate $\widetilde{\mathcal{A}}$. By [Kappen 2012, Corollary 2.12], $A' := A[f_1, \dots, f_m] \subset \mathcal{A}^\circ$ is still a special R-model of $\mathcal{A}$, and by construction, if J' is the biggest ideal of definition of A', the map $\varphi : A'/J' \to \widetilde{\mathcal{A}}$ is surjective. Since φ is injective anyway, it is an isomorphism. In conclusion, if X is a semiaffinoid k-analytic space, one can always find a special R-scheme $\mathfrak{X} = \mathrm{Spf}(A)$ which is a model of X such that $\widetilde{X} \simeq \mathfrak{X}_s$ and such that red can be identified with $\mathrm{sp}_{\mathfrak{X}}$.

Corollary 4.9. *Let X be a semiaffinoid k-analytic space. Let Z be a Zariski closed subset of $\widetilde{X}$ and let $Y := \mathrm{red}^{-1}(Z)$. Then Y is a semiaffinoid k-analytic space and the naturally induced map $\widetilde{Y} \to \widetilde{X}$ is a closed immersion with image Z.*

Proof. Let $\mathcal{A}$ be the semiaffinoid k-algebra of X. Using Remark 4.7, we can replace $\mathcal{A}$ by $\mathcal{A}_{\mathrm{red}}$ and assume that $\mathcal{A}$ is reduced. Under this assumption, by [Kappen 2012, Corollary 2.11], $\mathcal{A}^\circ$ is a special R-algebra, and $X \simeq \mathrm{Spf}(\mathcal{A}^\circ)_\eta$. Let I be an ideal of $\mathcal{A}^\circ$ such that $Z = V(\widetilde{I})$, where $\widetilde{I} = \{\tilde{f} \mid f \in I\}$. Then as already seen in the proof of Theorem 3.1, $Y \simeq \mathrm{Spf}((\mathcal{A}^\circ)^{\wedge I})_\eta$, hence Y is the k-analytic space associated to the semiaffinoid k-algebra $\mathcal{B} := (\mathcal{A}^\circ)^{\wedge I} \otimes_R k$. Then according to Theorem 3.1, $\mathcal{B}^\circ = (\mathcal{A}^\circ)^{\wedge I}$. Since $\check{\mathcal{A}}$ is an ideal of definition of $\mathcal{A}^\circ$ [Kappen 2012, Remark 2.8], $\check{\mathcal{A}} + I$ is also an ideal of definition of $\mathcal{B}^\circ$. Since $\check{\mathcal{B}}$ is the biggest ideal of definition of $\mathcal{B}^\circ$ [Kappen 2012, Remark 2.8], it follows that

$$\widetilde{\mathcal{B}} = \mathcal{B}^\circ/\check{\mathcal{B}} = \big((\mathcal{A}^\circ)^{\wedge I}/(\check{\mathcal{A}} + I)\big)_{\mathrm{red}} = (\widetilde{\mathcal{A}}/I)_{\mathrm{red}}. \qquad \square$$

Remark 4.10. In general, when U is an affinoid domain of the k-affinoid space X, it is difficult to describe the induced canonical reduction map $\widetilde{U} \to \widetilde{X}$. However, it is proved in [Bosch et al. 1984, Proposition 7.2.6.3] that when U is the tube of a principal open subset of $\widetilde{X}$ (i.e., when $U = \mathrm{red}^{-1}(D(\tilde{f}))$ for some $f \in \mathcal{A}^\circ$), then $\widetilde{U} \to \widetilde{X}$ is an open immersion with image $D(\tilde{f})$. The above corollary is the counterpart of [Bosch et al. 1984, Proposition 7.2.6.3] for Zariski closed subsets of the canonical reduction.

Lemma 4.11. *Let A be an R-flat special R-algebra and let $\mathfrak{X} = \mathrm{Spf}(A)$. Then the specialization map $\mathrm{sp}_{\mathfrak{X}} : \mathfrak{X}_\eta \to \mathfrak{X}_s$ is surjective.*

Proof. This proof is strongly inspired by the proof of Lemme 1.2 in [Poineau 2008]. Let $\tilde{x} \in \mathfrak{X}_s$, and let r be the transcendence degree of the field extension $\tilde{k}(\tilde{x})/\tilde{k}$. Let K be the completion of $k(U_1, \dots, U_r)$ with respect to the Gauss norm. Then K is a discretely valued nonarchimedean field extension of k, and $\widetilde{K} \simeq \tilde{k}(U_1, \dots, U_r)$. Hence $\tilde{x}$ is the image of a closed point $\tilde{y} \in \mathfrak{X}_s \times_{\mathrm{Spec}(\tilde{k})} \mathrm{Spec}(\widetilde{K})$ with respect to the canonical map $\mathfrak{X}_s \times_{\mathrm{Spec}(\tilde{k})} \mathrm{Spec}(\widetilde{K}) \to \mathfrak{X}_s$.

Using base change extension induced by the inclusion $R \to K^\circ$ (see [de Jong 1995, 7.2.6]),

$$\mathfrak{X}' := \mathfrak{X} \times_{\mathrm{Spf}(R)} \mathrm{Spf}(K^\circ)$$

is a special formal scheme over K° of the form $\mathfrak{X}' = \mathrm{Spf}(A')$, where $A' := A \widehat{\otimes}_R K^\circ$. Let J be the biggest ideal of definition of A. Then JA' is an ideal of definition of A' and $J' := \mathrm{rad}(JA')$ is the biggest ideal of definition of A'. There is a well-defined morphism of $\tilde{k}$-algebras $\alpha : A/J \otimes_{\tilde{k}} \widetilde{K} \to A'/J'$ defined by $\tilde{a} \otimes \tilde{\lambda} \mapsto \widetilde{a \otimes \lambda}$ for $a \in A$, $\lambda \in K^\circ$ and where $\tilde{\cdot}$ stands for the various residue maps. We claim that α induces an isomorphism

$$(A/J \otimes_{\tilde{k}} \widetilde{K})_{\mathrm{red}} \simeq A'/J'. \tag{4}$$

Indeed by construction α is surjective, so it remains to prove that $\ker \alpha$ is the nilradical of $A/J \otimes_{\tilde{k}} \widetilde{K}$. There is also a well-defined surjective map

$$\beta : A' \to A/J \otimes_{\tilde{k}} \widetilde{K}, \qquad a \otimes \lambda \mapsto \tilde{a} \otimes \tilde{\lambda},$$

and by construction $\alpha \circ \beta : A' \to A'/J'$ is the quotient map. So for some $a_i \in A$ and $\lambda_i \in K^\circ$, let $\tilde{z} = \sum_i \widetilde{a_i} \otimes \widetilde{\lambda_i} \in \ker \alpha$ and $z := \sum_i a_i \otimes \lambda_i \in A'$. Then $\beta(z) = \tilde{z}$, hence $z \in \ker \alpha \circ \beta$, hence $z \in J'$. Since $J' := \mathrm{rad}(JA')$, we have $z^n \in JA'$ for some $n \in \mathbb{N}^*$. Hence $\beta(z^n) = 0$ and $\tilde{z}^n = 0$, thus proving (4). We get a natural composite morphism

$$\iota : (\mathfrak{X}')_s \xrightarrow{\iota_1} \mathfrak{X}_s \times_{\mathrm{Spec}(\tilde{k})} \mathrm{Spec}(\widetilde{K}) \xrightarrow{\iota_2} \mathfrak{X}_s,$$

where ι_1 is induced by (4) and hence is bijective, and ι_2 is the canonical map. Thus we can identify $\tilde{y}$ with a closed point of $(\mathfrak{X}')_s$. We also get a commutative diagram of sets

$$\begin{array}{ccc} (\mathfrak{X}')_\eta & \xrightarrow{\alpha} & (\mathfrak{X})_\eta \\ {\scriptstyle \mathrm{sp}_{\mathfrak{X}'}}\downarrow & & \downarrow{\scriptstyle \mathrm{sp}_{\mathfrak{X}}} \\ (\mathfrak{X}')_s & \xrightarrow[\iota]{} & \mathfrak{X}_s \end{array}$$

Thanks to [Kappen 2012, Lemma 2.3 and Remark 2.5] we know that $\mathrm{sp}_{\mathfrak{X}'}$ induces a surjective map from the set of rigid points $(\mathfrak{X}')^{\mathrm{rig}}$ to the set of closed points of $(\mathfrak{X}')_s$. Hence we can find $y \in (\mathfrak{X}')_\eta$ such that $\mathrm{sp}_{\mathfrak{X}'}(y) = \tilde{y}$. In conclusion, if $x := \alpha(y)$ we get $\mathrm{sp}_{\mathfrak{X}}(x) = \tilde{x}$, proving the surjectivity of $\mathrm{sp}_{\mathfrak{X}}$. □

Corollary 4.12. *Let X be a semiaffinoid k-analytic space. Then the canonical reduction map* red : $X \to \widetilde{X}$ *is surjective.*

Proof. This follows from Lemma 4.11 and Remark 4.8. □

Connected components. In this subsection, we consider a semiaffinoid k-algebra $\mathcal{A}$ with associated k-analytic space X. It follows from Theorem A.8 and Remark 2.7 that $\mathrm{Spec}(\mathcal{A})$ is connected if and only if X is connected. Indeed by Remark 2.7, we can assume that $\mathcal{A}$ is reduced, and we conclude since the connected components of X are in correspondence with the set of idempotents of $\Gamma(X, \mathcal{O}_X)$, which themselves are equal to the set of idempotents of $\{f \in \Gamma(X, \mathcal{O}_X) \mid |f| < \infty\} = \mathcal{A}$. From the Noetherianity of $\mathcal{A}$ it follows that $\mathcal{A}$ can be uniquely decomposed as

$$\mathcal{A} \simeq \mathcal{A}_1 \times \cdots \times \mathcal{A}_n, \tag{5}$$

where each $\mathcal{A}_i$ is a semiaffinoid k-algebra such that $\mathrm{Spec}(\mathcal{A}_i)$ is connected. If we denote by X_i the k-analytic space associated to $\mathcal{A}_i$ it follows that

$$X = X_1 \amalg \cdots \amalg X_n \tag{6}$$

is the decomposition of X in connected components. These remarks easily imply the following.

Lemma 4.13. *A semiaffinoid k-analytic space X is connected if and only if $\widetilde{X}$ is connected.*

Proof. The equations (5) and (6) imply that $\widetilde{X} \simeq \coprod \widetilde{X}_i$. So if X is not connected, $\widetilde{X}$ is also not connected. Conversely, since $\mathrm{red} : X \to \widetilde{X}$ is surjective (Corollary 4.12) and anticontinuous, if one has a decomposition $\widetilde{X} = U_1 \amalg U_2$ in two nonempty closed-open sets, then $X = \mathrm{red}^{-1}(U_1) \amalg \mathrm{red}^{-1}(U_2)$ is a decomposition of X in nonempty closed-open sets. □

Corollary 4.14. *Let X be a semiaffinoid k-analytic space. Let Z be a Zariski closed subset of $\widetilde{X}$ and let $Y := \mathrm{red}^{-1}(Z)$. Then Y is connected if and only if Z is connected.*

Proof. This follows from Corollary 4.9 and from Lemma 4.13. □

Using Theorem 2.1, one checks that if A is a reduced special R-algebra which is integrally closed in $A \otimes_R k$, then there is a one-to-one correspondence between the connected components of $\mathrm{Spec}(A)$ and the connected components of $\mathrm{Spf}(A)_\eta$.

Example 4.15. In general, if A is a reduced special R-algebra with associated k-analytic space X, the connected components of $\mathrm{Spec}(A)$ do not coincide with the connected components of X, as the example $A = \mathbb{Z}_p\langle T\rangle/(T^2 + pT)$ shows.

5. Analytic functions and formal functions on tubes: a graded version

In this section, we fix $\mathcal{A}$ a reduced semiaffinoid k-algebra. Let us recall that this implies that $\mathcal{A}^\circ$ is a special R-algebra [Kappen 2012, Corollary 2.11].

5.1. For $r \in \mathbb{R}_+^*$ we set

$$\mathcal{A}_r^\circ = \{f \in \mathcal{A} \mid |f|_{\sup} \le r\}, \qquad \mathcal{A}_r^{\circ\circ} = \{f \in \mathcal{A} \mid |f|_{\sup} < r\}, \qquad \widetilde{\mathcal{A}}_r^+ = \mathcal{A}_r^\circ / \mathcal{A}_r^{\circ\circ}.$$

By definition, $\mathcal{A}_1^\circ = \mathcal{A}^\circ$, $\mathcal{A}_1^{\circ\circ} = \mathcal{A}^{\circ\circ}$ and $\widetilde{\mathcal{A}}_1^+ = \widetilde{\mathcal{A}}^+$. We also set

$$\rho(\mathcal{A}) = \{|f|_{\sup} \mid f \in \mathcal{A},\ f \neq 0\} \subset \mathbb{R}_+^*.$$

By [Kappen 2010, Proposition 1.2.5.9], $\rho(\mathcal{A}) \subset \sqrt{|k^*|}$. We denote by G the subgroup of $\sqrt{|k^*|}$ generated by $\rho(\mathcal{A})$.

Lemma 5.2. *Let $r \in \mathbb{R}_+^*$. Then $\mathcal{A}_r^\circ$, $\mathcal{A}_r^{\circ\circ}$ and $\widetilde{\mathcal{A}}_r^+$ are finitely generated $\mathcal{A}^\circ$-modules.*

Proof. Let us pick some $\lambda \in k^*$ such that $|\lambda| r \le 1$. Then

$$\mathcal{A}_r^\circ \to \mathcal{A}^\circ{}_{|\lambda| r}, \qquad f \mapsto \lambda f$$

is an isomorphism of $\mathcal{A}^\circ$-modules. So, replacing r by $|\lambda| r$, we can assume that $r \le 1$. Then $\mathcal{A}_r^\circ$ is an ideal of $\mathcal{A}^\circ$, hence it is a finitely generated $\mathcal{A}^\circ$-module because $\mathcal{A}^\circ$ is Noetherian. We conclude since $\mathcal{A}_r^{\circ\circ}$ is a submodule, and $\widetilde{\mathcal{A}}_r^+$ a quotient, of $\mathcal{A}_r^\circ$. □

Lemma 5.3. *The index $[G : |k^*|]$ is finite. As a consequence, $G \simeq \mathbb{Z}$.*

Proof. Let $\kappa := \sup\{\lambda \mid \lambda < 1 \text{ and } \lambda \in \rho(\mathcal{A})\}$. We claim that κ is actually a maximum, that is to say, there exists $\lambda < 1$ with $\lambda \in \rho(\mathcal{A})$ such that $\kappa = \lambda$ (this implies in particular that $\kappa < 1$). Indeed, if κ was not a maximum, we could find an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n < 1$ in $\rho(\mathcal{A})$. Then $I_n = \{f \in \mathcal{A} \mid |f|_{\sup} \leq \lambda_n\}$ would be an infinite increasing sequence of ideals of $\mathcal{A}^\circ$, which is Noetherian. This would be a contradiction. Hence, $\kappa < 1$ and $\kappa \in \rho(\mathcal{A})$.

Since $\kappa \in \rho(\mathcal{A})$, we can write $\kappa = |\pi|^{a/b}$ with a and b relatively prime and $b > 0$. One can easily prove using Bézout's theorem that $a = 1$. Hence $\kappa = |\pi|^{1/b}$. Likewise, using again Bézout's theorem, one can prove that if $|\pi|^{\alpha/\beta} \in \rho(\mathcal{A})$ with $\beta > 0$ and α and β coprime, then $\beta \leq b$. Hence, if $v := \operatorname{lcm}(1, 2, \ldots, b)$, one has $\rho(\mathcal{A}) \subset |\pi|^{\mathbb{Z}/v}$. □

Remark 5.4. If $\|\cdot\|$ is a k-Banach algebra norm on $\mathcal{A}$, according to [Kappen 2010, Lemma 1.2.5.8], for any $f \in \mathcal{A}$, $|f|_{\sup} = \lim_{n\in\mathbb{N}} \sqrt[n]{\|f^n\|}$. The notation $\rho(\mathcal{A})$ that we have introduced is compatible with [Berkovich 1990, Section 1.3], where the spectral radius of an element of a Banach algebra $\mathcal{A}$ is defined to be $\rho(f) = \lim_{n\in\mathbb{N}} \sqrt[n]{\|f^n\|}$. Let us recall from [Kappen 2010, Lemma 1.2.5.4] that a surjective morphism of semiaffinoid k-algebras

$$R\langle T_1, \ldots, T_m\rangle[\![S_1, \ldots, S_n]\!] \otimes_R k \to \mathcal{A}$$

induces a k-Banach norm on $\mathcal{A}$ by taking the residue seminorm of the Gauss norm on

$$R\langle T_1, \ldots, T_m\rangle[\![S_1, \ldots, S_n]\!] \otimes_R k.$$

Example 5.5. In general, $\rho(\mathcal{A})$ is not a subgroup, and not even a monoid. For instance if $\mathcal{A} = \mathbb{Q}_p(\sqrt{p}) \times \mathbb{Q}_p(\sqrt[3]{p})$ then $\rho(\mathcal{A}) = p^{\mathbb{Z}/2} \cup p^{\mathbb{Z}/3}$, which is not a monoid.

The following definition is inspired by [Temkin 2004, Section 3].

Definition 5.6. We define the *total reduction ring of* $\mathcal{A}$ as

$$\widetilde{\mathcal{A}}^+_{\text{tot}} := \bigoplus_{r\in G} \widetilde{\mathcal{A}}^+_r.$$

Corollary 5.7. *The total reduction ring* $\widetilde{\mathcal{A}}^+_{\text{tot}}$ *of* $\mathcal{A}$ *is excellent and reduced.*

Proof. By Lemma 5.3, $[G : |k^*|]$ is finite, so we can introduce $n := [G : |k^*|] \in \mathbb{N}$. Let $r_1, \ldots, r_n \in G$ be some representatives of the classes of $G/|k^*|$. For each $i \in \{1, \ldots, n\}$, by Lemma 5.2, we can find a finite family $(f_{i,j})_{j\in J}$ which is a finite set of generators of the $\mathcal{A}^\circ$-module $\widetilde{\mathcal{A}}^+_{r_i}$. Let us denote by $\tilde{\pi}$ the image of π in $\widetilde{\mathcal{A}}^+_{1/|\pi|}$, and likewise by $\widetilde{(1/\pi)}$ the image of $1/\pi$ in $\widetilde{\mathcal{A}}^+_{1/|\pi|}$. We get that

$$\bigoplus_{r\in G} \widetilde{\mathcal{A}}^+_r = \widetilde{\mathcal{A}}^+\big[\tilde{\pi}, \widetilde{\big(\tfrac{1}{\pi}\big)}, f_{i,j}\big].$$

Hence $\widetilde{\mathcal{A}}^+_{\mathrm{tot}}$ is a finitely generated $\widetilde{\mathcal{A}}^+$-algebra. Since $\widetilde{\mathcal{A}}^+$ is excellent, it follows that $\widetilde{\mathcal{A}}^+_{\mathrm{tot}}$ is also excellent. For the reducedness, consider a nonzero element $\tilde{f} \in \widetilde{\mathcal{A}}^+_r$ for some $f \in \mathcal{A}$ with $|f|_{\sup} = r$. Then for any integer $k > 0$, the associated element $\widetilde{f^k} = \tilde{f}^k \in \widetilde{\mathcal{A}}^+_{r^k}$ is also nonzero since $|f^k|_{\sup} = |f|^k_{\sup} = r^k$. □

Remark 5.8. Let $r \in \rho(\mathcal{A})$. Since $\rho(\mathcal{A})$ is discrete in $\mathbb{R}^*_+$ (see Lemma 5.3), there exists a real number $s \in \rho(\mathcal{A})$ which is the biggest element of $\rho(\mathcal{A})$ such that $s < r$. It follows that $\mathcal{A}^{\circ\circ}_r = \mathcal{A}^\circ_s$.

We fix I an ideal of $\mathcal{A}^\circ$. If M is an $\mathcal{A}^\circ$-module, we denote by $M^{\wedge I}$ the completion of M with respect to the I-adic topology. So

$$M^{\wedge I} \simeq \varprojlim_n M/I^n M.$$

Lemma 5.9. *Let $r \in \rho(\mathcal{A})$. There is a short exact sequence of $\mathcal{A}^\circ$-modules*

$$0 \to (\mathcal{A}^{\circ\circ}_r)^{\wedge I} \to (\mathcal{A}^\circ_r)^{\wedge I} \to (\widetilde{\mathcal{A}}^+_r)^{\wedge I} \to 0. \tag{7}$$

Proof. By definition of $\widetilde{\mathcal{A}}^+_r$, there is a short exact sequence

$$0 \to \mathcal{A}^{\circ\circ}_r \to \mathcal{A}^\circ_r \to \widetilde{\mathcal{A}}^+_r \to 0 \tag{8}$$

of finitely generated $\mathcal{A}^\circ$-modules. So the I-adic completion of (8) remains exact; see [Matsumura 1989, Theorems 8.7 and 8.8]. □

5.10. Let I be an ideal of $\mathcal{A}^\circ$ and let us set $U = \{x \in X \mid |f(x)| < 1 \ \forall f \in I\}$. Let us denote by $\mathcal{B}$ its associated semiaffinoid k-algebra, which can be defined as

$$\mathcal{B} = (\mathcal{A}^\circ[\![S_1, \dots, S_n]\!]/(f_i - S_i)_{i=1,\dots,n}) \otimes_R k,$$

where $I = (f_1, \dots, f_n)$. According to Theorem 3.1, one has $\mathcal{B}^\circ \simeq (\mathcal{A}^\circ)^{\wedge I}$. We denote by Y the k-analytic space associated with $\mathcal{B}$. If $g \in \mathcal{B}$, set $|g|_{\sup} = \sup_{y \in Y} |g(y)|$.

Remark 5.11. Let $r \in |k^*|$ and let $\lambda \in k$ with $|\lambda| = r^{-1}$. Multiplication by λ induces an isomorphism of $\mathcal{A}^\circ$-modules $\mathcal{A}^\circ_r \xrightarrow{\times\lambda} \mathcal{A}^\circ$. Completing with respect to I one gets an isomorphism of $(\mathcal{A}^\circ)^{\wedge I}$-modules $(\mathcal{A}^\circ_r)^{\wedge I} \simeq (\mathcal{A}^\circ)^{\wedge I}$. Finally, using Theorem 3.1 we get an isomorphism of $(\mathcal{A}^\circ)^{\wedge I}$-modules $\mathcal{B}^\circ_r \simeq (\mathcal{A}^\circ_r)^{\wedge I}$ obtained as the composition

$$(\mathcal{A}^\circ_r)^{\wedge I} \xrightarrow{\times\lambda} (\mathcal{A}^\circ)^{\wedge I} \to \mathcal{B}^\circ \xrightarrow{\times\lambda^{-1}} \mathcal{B}^\circ_r.$$

More generally, if $r \in \rho(\mathcal{A})$, we can find $s \in |k^*|$ with $r \leq s$, leading to an inclusion of $\mathcal{A}^\circ$-modules $\mathcal{A}^\circ_r \to \mathcal{A}^\circ_s$. Then completing with respect to I we get an inclusion of $(\mathcal{A}^\circ)^{\wedge I}$-modules $(\mathcal{A}^\circ_r)^{\wedge I} \to (\mathcal{A}^\circ_s)^{\wedge I}$. Using the above identifications, we can assimilate $(\mathcal{A}^\circ_r)^{\wedge I}$ as a $\mathcal{B}^\circ$-submodule of $\mathcal{B}^\circ_s$, hence as a $\mathcal{B}^\circ$-submodule of $\mathcal{B}$.

Lemma 5.12. *Let $r \in |k^*|$.*

(i) *Let $g \in \mathcal{B}_r^\circ \simeq (\mathcal{A}_r^\circ)^{\wedge I}$ (see Remark 5.11). Let $\tilde{g} \in (\widetilde{\mathcal{A}}_r^+)^{\wedge I}$ be the image of g by the reduction map of the short exact sequence (7) of Lemma 5.9. Then $\tilde{g} = 0$ if and only if $|g|_{\sup} < r$. Equivalently, $\tilde{g} \neq 0$ if and only if $|g|_{\sup} = r$.*

(ii) *There is a natural isomorphism $\mathcal{B}_r^{\circ\circ} \simeq (\mathcal{A}_r^{\circ\circ})^{\wedge I}$.*

Proof. Using the same arguments as in Remark 5.11, we can assume that $r = 1$. We then consider the short exact sequence of Lemma 5.9 for $r = 1$:

$$0 \to (\mathcal{A}^{\circ\circ})^{\wedge I} \to (\mathcal{A}^\circ)^{\wedge I} \to (\widetilde{\mathcal{A}}^+)^{\wedge I} \to 0.$$

Let us then consider $g \in \mathcal{B}^\circ \simeq (\mathcal{A}^\circ)^{\wedge I}$ and let us assume that $\tilde{g} = 0$. This implies that $g \in (\mathcal{A}^{\circ\circ})^{\wedge I}$. By Remark 5.8, there exists $s < 1$ such that $(\mathcal{A}^{\circ\circ})^{\wedge I} = (\mathcal{A}_s^\circ)^{\wedge I}$ for some $s < 1$. It follows that $|g|_{\sup} \leq s < 1$. Conversely, let us assume that $|g|_{\sup} < 1$. There exists an integer $d \in \mathbb{N}$ such that $|g^d|_{\sup} \leq |\pi|$. Hence, according to Remark 5.11, $g^d \in \mathcal{B}_{|\pi|}^\circ \simeq (A_{|\pi|}^\circ)^{\wedge I} \subset (\mathcal{A}^{\circ\circ})^{\wedge I}$. This proves (i), and (ii) follows from (i). □

We can now generalize Theorem 3.1 to an arbitrary $r \in \rho(\mathcal{A})$.

Proposition 5.13. *We use the notations of 5.10.*

(i) *There is an inclusion $\rho(\mathcal{B}) \subset \rho(\mathcal{A})$.*

(ii) *Let $r \in \rho(\mathcal{A})$. There are isomorphisms $\mathcal{B}_r^\circ \simeq (\mathcal{A}_r^\circ)^{\wedge I}$, $\mathcal{B}_r^{\circ\circ} \simeq (\mathcal{A}_r^{\circ\circ})^{\wedge I}$ and $\widetilde{\mathcal{B}}_r^+ \simeq (\widetilde{\mathcal{A}}_r^+)^{\wedge I}$.*

(iii) *There is a natural isomorphism $\widetilde{\mathcal{B}}_{\mathrm{tot}}^+ \simeq \bigoplus_{r \in G} (\widetilde{\mathcal{A}}_r^+)^{\wedge I}$.*

Proof. Let $g \in \mathcal{B}$ be a nonzero element. Then there exists $s \in \rho(\mathcal{A})$ such that $g \in \mathcal{B}_s^\circ$. Since $\rho(\mathcal{A})$ is discrete by Lemma 5.3, we can then define the smallest element $r \in \rho(\mathcal{A})$ such that $g \in \mathcal{B}_r^\circ$. By Remark 5.11, for each $r \in \rho(\mathcal{A})$, we can naturally identify $(\mathcal{A}_r^\circ)^{\wedge I}$ with a $\mathcal{B}^\circ$-submodule of $\mathcal{B}$, and under these identifications, $\mathcal{B} = \bigcup_{r \in \rho(\mathcal{A})} (\mathcal{A}_r^\circ)^{\wedge I}$. Since $\rho(\mathcal{A})$ is discrete, we can then define the smallest element $r \in \rho(\mathcal{A})$ such that $g \in (\mathcal{A}_r^\circ)^{\wedge I}$. We then consider the short exact sequence (7):

$$0 \to (\mathcal{A}_r^{\circ\circ})^{\wedge I} \to (\mathcal{A}_r^\circ)^{\wedge I} \to (\widetilde{\mathcal{A}}_r^+)^{\wedge I} \to 0.$$

The minimality of r and Remark 5.8 imply that $g \notin (\mathcal{A}_r^{\circ\circ})^{\wedge I}$. It follows that $\tilde{g} \neq 0$, where $\tilde{g} \in (\widetilde{\mathcal{A}}_r^+)^{\wedge I}$ denotes the reduction of g in $(\widetilde{\mathcal{A}}_r^+)^{\wedge I}$. Thanks to Lemma 5.3 we can pick some $d \in \mathbb{N}^*$ such that $r^d \in |k^*|$. Then $g^d \in (\mathcal{A}_{r^d}^\circ)^{\wedge I}$. Since $(\widetilde{\mathcal{A}}_{\mathrm{tot}}^+)$ is reduced and excellent (Corollary 5.7), it follows that $(\widetilde{\mathcal{A}}_{\mathrm{tot}}^+)^{\wedge I}$ is also reduced. Hence $\widetilde{g^d} = \tilde{g}^d \neq 0$ in $(\widetilde{\mathcal{A}}_{r^d}^+)^{\wedge I}$. Since $r^d \in |k^*|$, Lemma 5.12 implies that $|g^d|_{\sup} = r^d$. So $|g|_{\sup} = r$. This proves (i) as well as (ii) and (iii). □

We also obtain the following generalization of [Bosch and Lütkebohmert 1985, Lemma 2.1].

Corollary 5.14. *Let X be a k-affinoid space, and let $Z \subset \widetilde{X}$ be a Zariski closed subset. Then $\widehat{\widetilde{X}_{/Z}}$, the formal completion of $\widetilde{X}$ along Z, depends intrinsically on the k-analytic space* $\mathrm{red}^{-1}(Z)$.

Proof. Let us denote by $\mathcal{B}$ the semiaffinoid k-algebra of $\mathrm{red}^{-1}(Z)$. The space $\widehat{\widetilde{X}_{/Z}}$ is the formal scheme associated to the adic $\tilde{k}$-algebra $(\widetilde{\mathcal{A}}^{+})^{\wedge I}$. Thanks to the short exact sequence (7) for $r = 1$,

$$(\widetilde{\mathcal{A}}^{+})^{\wedge I} \simeq \mathcal{A}^{\circ\wedge I}/\mathcal{A}^{\circ\circ\wedge I}.$$

Thanks to Theorem 3.1 and Lemma 5.12(ii), one gets that

$$(\widetilde{\mathcal{A}}^{+})^{\wedge I} \simeq \mathcal{B}^{\circ}/\mathcal{B}^{\circ\circ} = \widetilde{\mathcal{B}}^{+},$$

which depends intrinsically on $\mathrm{red}^{-1}(Z)$ since $\mathcal{B}$ does. □

Remark 5.15. More generally, if X is the k-analytic space associated to the reduced semiaffinoid k-algebra $\mathcal{A}$, then $\widetilde{\mathcal{A}}^{+}$ is naturally an adic algebra with an ideal of definition given by $\check{\mathcal{A}}$. This adic algebra is isomorphic to a quotient of $\tilde{k}[T_i][[S_j]]$. Let us set $\mathcal{X} := \mathrm{Spf}(\widetilde{\mathcal{A}}^{+})$. Then $|\mathcal{X}| = \widetilde{X}$. Let Z be a Zariski closed subset of $\widetilde{X}$, Y the k-analytic space defined by $Y = \mathrm{red}^{-1}(Z)$ and $\mathcal{B}$ its associated semiaffinoid k-algebra. One shows similarly that the inclusion of analytic domains $Y \to X$ induces an isomorphism $\widehat{\mathcal{X}_{/Z}} \simeq \mathrm{Spf}(\widetilde{\mathcal{B}}^{+})$, where we denote by $\widehat{\mathcal{X}_{/Z}}$ the completion of $\mathcal{X}$ along Z. In particular, $\widehat{\mathcal{X}_{/Z}}$ depends intrinsically on the k-analytic space $\mathrm{red}^{-1}(Z)$.

6. Additional remarks

Finite morphisms.

Proposition 6.1. *Let $\varphi : \mathcal{B} \to \mathcal{A}$ be a finite morphism of semiaffinoid k-algebras, with $\mathcal{A}$ reduced. Then $\varphi^{\circ} : \mathcal{B}^{\circ} \to \mathcal{A}^{\circ}$ is finite.*

Proof. Since $\mathcal{B}^{\circ} \to (\mathcal{B}_{\mathrm{red}})^{\circ}$ is finite, we can also assume that $\mathcal{B}$ is reduced. Let $f_1, \dots, f_n$ be elements of $\mathcal{A}$ such that

$$\mathcal{A} = \mathcal{B}[f_1, \dots, f_n]. \tag{9}$$

Each $f \in \{f_1, \dots, f_n\}$ satisfies a unitary polynomial equation with coefficients in $\mathcal{B}$ of the form

$$f^d + b_{d-1} f^{d-1} + \cdots + b_1 f + b_0 = 0. \tag{10}$$

Then for $m \in \mathbb{N}^*$, multiplying by π^{md}, (10) becomes

$$(\pi^m f)^d + \pi^m b_{d-1}(\pi^m f)^{d-1} + \cdots + \pi^{m(d-1)} b_1 (\pi^m f) + \pi^{md} b_0 = 0.$$

But for m big enough, all the coefficients $\pi^m b_{d-1}, \dots, \pi^{m(d-1)} b_1, \pi^{md} b_0$ appearing in the above equation belong to $\mathcal{B}^{\circ}$. Hence, for m big enough, $\pi^m f_i$ satisfies a unitary polynomial equation with coefficients in $\mathcal{B}^\circ$. So replacing each f_i by $\pi^m f_i$ (which will not change (9)), we can assume that the f_i belong to $\mathcal{A}^\circ$ and are integral over $\mathcal{B}^\circ$. So, thanks to (9), $\mathcal{B}^\circ[f_1, \dots, f_n]$ is a special R-model of $\mathcal{A}$. According to [Kappen 2012, Corollary 2.10], $\mathcal{A}^\circ$ is finite over $\mathcal{B}^\circ[f_1, \dots, f_n]$, and hence also over $\mathcal{B}^\circ$. □

We conjecture that for an arbitrary nonarchimedean nontrivially valued field k, a similar statement holds for quasiaffinoid k-algebras (see [Lipshitz and Robinson 2000, Remark 2.1.8] for the definition of a quasiaffinoid k-algebra).

Corollary 6.2. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a finite morphism of semiaffinoid k-algebras. The associated morphisms $\tilde{\varphi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$, $\tilde{\varphi}^+ : \widetilde{\mathcal{A}}^+ \to \widetilde{\mathcal{B}}^+$, $\tilde{\varphi}_{\mathrm{tot}} : \widetilde{\mathcal{A}}^+_{\mathrm{tot}} \to \widetilde{\mathcal{B}}^+_{\mathrm{tot}}$ are finite.*

Proof. Since the above morphisms do not change if one replaces $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}_{\mathrm{red}}$ and $\mathcal{B}_{\mathrm{red}}$, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are reduced. The first two points then follow from Proposition 6.1. To prove that $\tilde{\varphi}_{\mathrm{tot}}$ is finite, one has to use that $\tilde{\varphi}^+$ is finite, and then argue as in the proof of Corollary 5.7. □

The nonaffine case.

Lemma 6.3. *Let A be a reduced special R-algebra which is integrally closed in the semiaffinoid k-algebra $A \otimes_R k$. Let $\mathfrak{U}$ be a formal open affine subset of $\mathfrak{X} = \mathrm{Spf}(A)$. Then $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ is also a reduced special R-algebra which is integrally closed in the semiaffinoid k-algebra $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes_R k$ and $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \simeq \Gamma(\mathrm{sp}_{\mathfrak{X}}^{-1}(\mathfrak{U}), \mathcal{O}^\circ_{\mathfrak{X}_\eta})$.*

Proof. We first assume that $\mathfrak{U}$ is a principal formal open subset of the form $\mathfrak{U} = \mathfrak{D}(f)$ for some $f \in A$. Let J be an ideal of definition of A. Then $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \simeq A_{\{f\}}$, where $A_{\{f\}} \simeq \widehat{A_f}$ is the completion of the localization A_f with respect to the ideal JA_f. The composition morphism $A \to A_f \to \widehat{A_f}$ is regular. Indeed, $A \to A_f$ is regular since it is a localization, and $A_f \to \widehat{A_f}$ is regular since it is the completion of an excellent ring. It follows that the morphism $A \to \widehat{A_f}$ is regular since regular morphisms are stable under composition [EGA IV$_2$ 1965, Proposition 6.8.3]. Using [EGA IV$_2$ 1965, Proposition 6.14.4], we conclude that $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \simeq \widehat{A_f}$ is integrally closed in $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes_R k$.

Let us now assume that $\mathfrak{U}$ is an arbitrary formal open affine subset of $\mathfrak{X}$. It follows from [de Jong 1995, §7, p. 74–75] that $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ is a special R-algebra. Moreover, we can cover $\mathfrak{U}$ by some principal formal open subsets of the form $\mathfrak{D}(f_i)$ for some $f_i \in A$. Let us now consider an element $g \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes_R k$ which is integral over $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$. This means that g satisfies an equation

$$g^d + b_{d-1} g^{d-1} + \cdots + b_1 g + b_0 = 0 \tag{11}$$

for some $b_j \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$. But for each principal formal open subset $\mathfrak{D}(f_i)$, we can restrict (11) to $\Gamma(\mathfrak{D}(f_i), \mathcal{O}_{\mathfrak{X}}) \otimes_R k$. We then get that $g|_{\mathfrak{D}(f_i)} \in \Gamma(\mathfrak{D}(f_i), \mathcal{O}_{\mathfrak{X}}) \otimes_R k$ is integral over $\Gamma(\mathfrak{D}(f_i), \mathcal{O}_{\mathfrak{X}})$, so by the first part of the proof, $g|_{\mathfrak{D}(f_i)} \in \Gamma(\mathfrak{D}(f_i), \mathcal{O}_{\mathfrak{X}})$.

Since the $\mathfrak{D}(f_i)$ form a covering of $\mathfrak{U}$, we deduce that $g \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$, which proves that $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ is integrally closed in $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes_R k$. Finally, the equality $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \simeq \Gamma(\mathrm{sp}_{\mathfrak{X}}^{-1}(\mathfrak{U}), \mathcal{O}^{\circ}_{\mathfrak{X}_\eta})$ now follows from Theorem 2.1. □

The following result extends [Bosch et al. 1984, Proposition 7.2.6.3] from affinoid to semiaffinoid k-algebras.

Corollary 6.4. *Let X be a semiaffinoid k-analytic space. Let $f \in \Gamma(X, \mathcal{O}^{\circ}_X)$ and let $Y = \{x \in X \mid |f(x)| = 1\}$. Then Y is a semiaffinoid k-analytic space and the associated map $\widetilde{Y} \to \widetilde{X}$ is the Zariski open embedding of the principal open subset $D(\tilde{f}) \subset \widetilde{X}$.*

Proof. It follows from general properties of semiaffinoid k-analytic spaces that the inclusion $Y \to X$ is induced by a morphism of semiaffinoid k-algebras (see [de Jong 1995, Proposition 7.2.1(a)]). Using Remark 4.7, we can easily assume that X and Y are reduced. Let $\mathcal{A}$ be the reduced semiaffinoid k-algebra associated with X. By [Kappen 2012, Corollary 2.11], $\mathcal{A}^{\circ}$ is a special R-algebra. Let $\mathfrak{X} := \mathrm{Spf}(\mathcal{A}^{\circ})$. Let $\mathfrak{D}(f) \subset \mathfrak{X}$ be the principal formal open subset associated with f. Hence $Y = \mathrm{sp}_{\mathfrak{X}}^{-1}(\mathfrak{D}(f))$. By Lemma 6.3, we have $\Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}) \simeq \mathcal{B}^{\circ}$, where $\mathcal{B}$ is the semiaffinoid k-algebra associated with Y. Let $\mathfrak{J}$ be the biggest ideal sheaf of definition of $\mathfrak{X}$ (see [EGA I 1960, Proposition 10.5.4] for the definition and the properties of $\mathfrak{J}$). By [EGA I 1960, 10.5.2] there is an isomorphism of schemes $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}) \simeq \mathrm{Spec}(\widetilde{\mathcal{A}}) = \widetilde{X}$. Moreover, by [EGA I 1960, Corollaire 10.5.5], $\mathfrak{J}|_{\mathfrak{U}}$ is also the biggest ideal sheaf of definition of the formal scheme $\mathfrak{U}$. Hence $\Gamma(\mathfrak{U}, \mathfrak{J})$ is the biggest ideal of definition of the adic ring $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \simeq \mathcal{B}^{\circ}$, and as a consequence $\Gamma(\mathfrak{U}, \mathfrak{J}) \simeq \check{\mathcal{B}}$ by [Kappen 2012, Remark 2.8]. Hence we can conclude, since

$$\widetilde{\mathcal{B}} = \mathcal{B}^{\circ}/\check{\mathcal{B}} \simeq \Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}) \simeq \widetilde{\mathcal{A}}[\tilde{f}^{-1}].$$ □

Lemma 6.5. *Let $\mathfrak{X}$ be a special formal scheme over R. The following are equivalent.*

(i) *Any formal open affine subscheme of $\mathfrak{X}$ is isomorphic to* $\mathrm{Spf}(A)$, *where A is a reduced special R-algebra integrally closed in $A \otimes_R k$.*

(ii) *There exists a covering by formal open affine subschemes $\mathfrak{U}_i = \mathrm{Spf}(A_i)$, where for each i, A_i is a reduced special R-algebra which is integrally closed in $A_i \otimes_R k$.*

Proof. That (i) implies (ii) is clear. To prove the converse implication, let $\mathfrak{U}$ be a formal open affine subset of $\mathfrak{X}$. Then $\mathfrak{U}$ is covered by finitely many $\mathfrak{U} \cap \mathfrak{U}_i$. For each i we can find a finite covering $\{\mathfrak{U}_{i,j}\}$ of $\mathfrak{U} \cap \mathfrak{U}_i$ by formal open affine subsets. Thanks to Lemma 6.3, all the $\mathfrak{U}_{i,j}$ satisfy the expected property. We are then reduced to the situation where $\mathfrak{X}$ is affine and is covered by finitely many formal open affine $\mathfrak{U}_i$'s which all satisfy the expected property. Let us then consider a function f in the integral closure of $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$ in $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}) \otimes_R k$. Then for all i, $f|_{\mathfrak{U}_i}$ is in the integral closure of $\Gamma(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}})$ in $\Gamma(\mathfrak{U}_i \mathcal{O}_{\mathfrak{X}}) \otimes_R k$, which is by assumption $\Gamma(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}})$. So $f \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})$. □

Appendix: Bounded functions on reduced semiaffinoid *k*-spaces

by Christian Kappen and Florent Martin

In this appendix, we assume that k is a nontrivially discretely valued nonarchimedean field, and we let R denote its valuation ring. We fix a reduced semiaffinoid k-algebra $\mathcal{A}$, and we let X denote the associated rigid analytic k-space.

Projective limits. Let us start with a reminder on derived functors of projective limits of abelian groups. Let

$$\cdots \xrightarrow{\sigma_3} G_2 \xrightarrow{\sigma_2} G_1 \xrightarrow{\sigma_1} G_0$$

be a projective system of abelian groups indexed by $\mathbb{N}$, and let

$$\varphi : \prod_{n\in\mathbb{N}} G_n \to \prod_{n\in\mathbb{N}} G_n, \qquad (a_n)_n \mapsto (a_n - \sigma_{n+1}(a_{n+1}))_n.$$

Then $\ker\varphi \simeq \varprojlim G_n$. Let us denote by $\varprojlim^i$ the i-th derived functor of $\varprojlim$. According to [Weibel 1994, Corollary 3.5.4], one has the following descriptions:

$$\begin{aligned} \varprojlim{}^1 G_n &\simeq \operatorname{coker}\varphi, \\ \varprojlim{}^i G_n &= 0 \qquad \text{for } i > 1. \end{aligned}$$

A flatness result. Let us fix a presentation $\mathcal{A} \simeq (k \otimes_R R\langle T_1, \dots, T_m\rangle[[S_1, \dots, S_n]])/I$ and let us equip $\mathcal{A}$ with the associated k-Banach algebra norm $\|\cdot\|$ as in [Kappen 2010, Section 1.2.5]. For a real number $\varepsilon \in \sqrt{|k^*|}$ such that $0 < \varepsilon < 1$, we set

$$X_\varepsilon := \{x \in X \mid |S_i(x)| \le \varepsilon,\ i = 1, \dots, n\}.$$

Then X_ε is an affinoid k-space (depending on the chosen presentation of $\mathcal{A}$), and we let $\mathcal{A}_\varepsilon$ denote the associated affinoid k-algebra. It comes with a natural presentation

$$\mathcal{A}_\varepsilon \simeq k\langle T_1, \dots, T_m, \varepsilon^{-1} S_1, \dots, \varepsilon^{-1} S_n\rangle/I$$

and with an associated k-Banach algebra norm $\|\cdot\|_\varepsilon$ such that if $\varepsilon \le \varepsilon'$, the restriction morphism $\mathcal{A}_{\varepsilon'} \to \mathcal{A}_\varepsilon$ is contractive, that is to say, for $f \in \mathcal{A}_{\varepsilon'}$ we have $\|f\|_\varepsilon \le \|f\|_{\varepsilon'}$.

Let us now fix an increasing sequence of positive real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 1$ and such that for all $n \in \mathbb{N}$, we have that $\varepsilon_n \in \sqrt{|k^*|}$. We set $\mathcal{A}_n := \mathcal{A}_{\varepsilon_n}$. We denote by X_n the k-affinoid space associated to $\mathcal{A}_n$. For $n \in \mathbb{N}$ we denote by $\tau_n : \mathcal{A} \to \mathcal{A}_n$ the associated canonical map, and for $m \ge n \in \mathbb{N}$ we denote by $\sigma_{m,n} : \mathcal{A}_m \to \mathcal{A}_n$ the restriction morphism. By the above remark, each $\sigma_{m,n}$ is a contractive morphism with respect to the norms $\|\cdot\|_{\varepsilon_m}$ and $\|\cdot\|_{\varepsilon_n}$. The sequence

$(\mathcal{A}_n)_{n\in\mathbb{N}}$ and the restriction morphisms form a projective system of abelian groups. The next lemma is inspired by [Bosch 1977, Satz 2.1], and the proof is verbatim the same as in [Bosch 1977]. For the convenience of the reader, we recall it.

Lemma A.1. *In the above setting*, $\varprojlim^1 \mathcal{A}_n = 0$.

Proof. We have to show that the map

$$\varphi : \prod_{n\in\mathbb{N}} \mathcal{A}_n \to \prod_{n\in\mathbb{N}} \mathcal{A}_n, \qquad (a_n)_n \mapsto (a_n - \sigma_{n+1,n}(a_{n+1}))_n,$$

is surjective. So let us consider a sequence $(g_n) \in \prod_{n\in\mathbb{N}} \mathcal{A}_n$, and let us find a sequence $(f_n) \in \prod_{n\in\mathbb{N}} \mathcal{A}_n$ satisfying the conditions

$$g_n = f_n - \sigma_{n+1,n}(f_{n+1}) \quad \forall n \in \mathbb{N}. \tag{12}$$

Step 1. Let us first assume that for all $n \in \mathbb{N}$, g_n lies in the image of the restriction map $\tau_n : \mathcal{A} \to \mathcal{A}_n$. For each n, let us then choose $G_n \in \mathcal{A}$ such that $g_n = \tau_n(G_n)$. We define inductively a sequence $F_n \in \mathcal{A}$ via

$$\begin{cases} F_0 := 0, \\ F_{n+1} := F_n - G_n \quad \text{for } n \geq 0. \end{cases}$$

Setting $f_n := \tau_n(F_n)$, we obtain a solution $(f_n)_n$ of (12).

Step 2. Let us now pick some arbitrary $(g_n) \in \prod_n \mathcal{A}_n$. For each $n \in \mathbb{N}$, the image of $\mathcal{A}$ in $\mathcal{A}_n$ is dense with respect to the topology induced by $\|\cdot\|_{\varepsilon_n}$. Hence for all $n \in \mathbb{N}$, there exists $h_n \in \mathcal{A}$ such that $\|g_n - \tau_n(h_n)\|_{\varepsilon_n} \leq 2^{-n}$. For $n \in \mathbb{N}$, we have $g_n = \tau_n(h_n) + (g_n - \tau_n(h_n))$. By Step 1, there exists $(H_n)_{n\in\mathbb{N}} \in \prod_n \mathcal{A}_n$ such that $\varphi((H_n)_{n\in\mathbb{N}}) = (\tau_n(h_n))_{n\in\mathbb{N}}$. Hence it remains to prove that $(g_n - \tau_n(h_n))_{n\in\mathbb{N}} \in \operatorname{im}(\varphi)$. Replacing g_n by $g_n - \tau_n(h_n)$, we can thus assume that

$$\|g_n\|_{\varepsilon_n} \leq 2^{-n} \quad \forall n \in \mathbb{N}.$$

Since the morphisms $\sigma_{m,n}$ are contractive, for each $m \geq n$ we have $\|\sigma_{m,n}(g_m)\|_{\varepsilon_n} \leq \|g_m\|_{\varepsilon_m} \leq 2^{-m}$. Hence, for each $n \in \mathbb{N}$, since $\mathcal{A}_n$ is a k-Banach algebra, it makes sense to define

$$f_n := \sum_{m\geq n} \sigma_{m,n}(g_m).$$

Finally, we have

$$\begin{aligned} f_n - \sigma_{n+1,n}(f_{n+1}) &= \sum_{m\geq n} \sigma_{m,n}(g_m) - \sigma_{n+1,n}\Bigg(\sum_{m\geq n+1} \sigma_{m,n+1}(g_m)\Bigg) \\ &= \sum_{m\geq n} \sigma_{m,n}(g_m) - \sum_{m\geq n+1} \sigma_{m,n}(g_m) = g_n, \end{aligned}$$

which proves that $\varphi((f_n)_n) = (g_n)$. □

For any $\mathcal{A}$-module M, we set $M_n := M \otimes_{\mathcal{A}} \mathcal{A}_n$.

Definition A.2. We let Θ denote the functor

$$\Theta : \{\text{finitely generated } \mathcal{A}\text{-modules}\} \to \{\Gamma(X, \mathcal{O}_X)\text{-modules}\}, \qquad M \mapsto \varprojlim_n M_n.$$

The statement and the proof of the following result are again copied almost verbatim from [Bosch 1977] (see however Remark A.4 below).

Lemma A.3. *The functor Θ has the following properties.*

(i) *The functor Θ is exact.*

(ii) *For any finitely generated $\mathcal{A}$-module M, there is a natural isomorphism*

$$\tau : M \otimes_{\mathcal{A}} \Gamma(X, \mathcal{O}_X) \simeq \Theta(M).$$

Proof. Let us first show that Θ is exact. By the local theory of uniformly rigid spaces as developed in [Kappen 2010], the rings $\mathcal{A}_n$ are flat over $\mathcal{A}$. Hence, it suffices to show that $\varprojlim^1 M_n$ vanishes for all finitely generated $\mathcal{A}$-modules M. Thus, let M be a finitely generated $\mathcal{A}$-module, and let $F \to M \to 0$ be a finite presentation of M. Since the higher derivatives of the projective limit functor vanish, $\varprojlim^1 F_n$ maps onto $\varprojlim^1 M_n$. Since $\varprojlim^1$ commutes with finite direct sums, Lemma A.1 shows that $\varprojlim^1 F_n = 0$. The claim follows. Let us now prove the second statement. Since $\Gamma(X, \mathcal{O}_X)$ is naturally isomorphic to $\varprojlim \mathcal{A}_n$, one has a natural morphism

$$\tau : M \otimes_{\mathcal{A}} \Gamma(X, \mathcal{O}_X) \to \varprojlim M_n.$$

By definition, τ is an isomorphism when $M = \mathcal{A}$, and more generally τ is an isomorphism when M is finite and free. In general, since $\mathcal{A}$ is Noetherian, and M is finitely generated, there is an exact sequence

$$F_1 \to F_2 \to M \to 0,$$

where F_1 and F_2 are finite free $\mathcal{A}$-modules. Using exactness of Θ, one obtains the exact diagram

$$\begin{array}{ccccccccc}
\longrightarrow & F_2 \otimes_{\mathcal{A}} \Gamma(X, \mathcal{O}_X) & \longrightarrow & F_1 \otimes_{\mathcal{A}} \Gamma(X, \mathcal{O}_X) & \longrightarrow & M \otimes_{\mathcal{A}} \Gamma(X, \mathcal{O}_X) & \longrightarrow & 0 \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \tau & & \\
\longrightarrow & \Theta(F_2) & \longrightarrow & \Theta(F_1) & \longrightarrow & \Theta(M) & \longrightarrow & 0
\end{array}$$

and it follows that τ is an isomorphism. ☐

Remark A.4. The statements of Lemma A.3 do not hold for general $\mathcal{A}$-modules. Likewise, Satz 2.1 and Korollar 2.2 from [Bosch 1977] do not hold for general A-modules either, although this is not explicitly mentioned there. Indeed, in Example A.5 below, we give a counterexample to Satz 2.1 and Korollar 2.2 of

[Bosch 1977] involving modules which are not finitely generated. We want to stress that the results of [Bosch 1977, Section 2] are not affected by this observation: using the notations of [Bosch 1977], a correct replacement of [Bosch 1977, Satz 2.1 and Korollar 2.2] is to say that the functor θ is exact on the category of finitely generated $A\langle\zeta\rangle$-modules and that τ is an isomorphism for finitely generated $A\langle\zeta\rangle$-modules.

Example A.5. We use the notations of Satz 2.1 and Korollar 2.2 of [Bosch 1977], and we consider $A = k$. Moreover, we assume that $\zeta = (\zeta_1)$; that is, ζ is made of only one variable. So θ is the functor sending a $k\langle\zeta\rangle$-module M to the $k\langle\langle\zeta\rangle\rangle$-module

$$\theta(M) = \varprojlim_{n\in\mathbb{N}} M \otimes_{k\langle\zeta\rangle} k\langle\varepsilon_n^{-1}\zeta\rangle,$$

and for each M, $\tau = \tau_M$ is the natural map

$$\tau_M : M \otimes_{k\langle\zeta\rangle} k\langle\langle\zeta\rangle\rangle \to \theta(M).$$

Let us write $\mathcal{A} = k\langle\zeta\rangle$ and $\mathcal{A}_j = k\langle\varepsilon_j^{-1}\zeta\rangle$, and let us consider the $\mathcal{A}$-modules

$$M' = \bigoplus_{j\in\mathbb{N}} \mathcal{A}, \qquad M = \bigoplus_{j\in\mathbb{N}} \mathcal{A}_j, \qquad M'' = \bigoplus_{j\in\mathbb{N}} (\mathcal{A}_j/\mathcal{A}),$$

which form a natural short exact sequence of $\mathcal{A}$-modules

$$0 \to M' \to M \to M'' \to 0.$$

We denote be $(e_j)_{j\in\mathbb{N}}$ the canonical basis of M. For simplicity of notation, we also denote by $(e_j)_{j\in\mathbb{N}}$ the canonical bases of M' and M''. We claim that both the natural morphism $\tau_{M''}$ for M'' and the induced map $\theta(M) \to \theta(M'')$ are not surjective, contrary to the statements of Satz 2.1 and Korollar 2.2 of [Bosch 1977]. To this end, let us choose, for each $j \in \mathbb{N}$, a function $f_j \in \mathcal{A}$ such that f_j is invertible in $\mathcal{A}$ and such that for any $m > j$, f_j is not invertible in $\mathcal{A}_m$, by picking an element $t \in k$ as well as some positive integers a, b such that

$$\varepsilon_{j+1} \geq |t|^{a/b} > \varepsilon_j$$

and by setting $f_j := \zeta^b - t^a \in \mathcal{A} = k\langle\zeta\rangle$. Let us now consider the element $g \in \prod_n M''_n$ which is defined by giving, for each n, the element

$$g_n := \sum_{j=0}^{n-1} [f_j^{-1} \otimes 1] \cdot e_j$$

of the $\mathcal{A}_n$-module

$$M''_n = \bigoplus_{j\in\mathbb{N}} (\mathcal{A}_j/\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}_n \cong \bigoplus_{j\in\mathbb{N}} (A_j \otimes_{\mathcal{A}} \mathcal{A}_n)/\mathcal{A}_n,$$

where $[\cdot]$ denotes the formation of the residue class and where we have used flatness of $\mathcal{A}_n$ over $\mathcal{A}$ to establish the above isomorphism. Then

$$g \in \varprojlim_{n \in \mathbb{N}} M''_n,$$

because in M''_n, we have that $[f_n^{-1} \otimes 1] = [1 \otimes f_n^{-1}] = 0$. Let us now consider the natural map

$$\tau_{M''} : M'' \otimes_{\mathcal{A}} \varprojlim_n \mathcal{A}_n \to \varprojlim_n M''_n.$$

For each element h in the image of this map, there exists a j_0 such that for all $j > j_0$, the j-th component of h_n is zero for all n. On the other hand, for each n, all of the summands $[f_j^{-1} \otimes 1] \cdot e_j$ with $j < n$ defining g_n are nonzero. Indeed, if there was an element $h \in \mathcal{A}_n$ with

$$f_j^{-1} \otimes 1 = 1 \otimes h \quad \text{in } \mathcal{A}_j \otimes_{\mathcal{A}} \mathcal{A}_n,$$

then the same equality would hold in the completed tensor product, which is $\mathcal{A}_n$, and f_j^{-1} would thus extend to $\mathcal{A}_n$, which is not the case. We have shown that $g \notin \operatorname{im} \tau_{M''}$ and thus established our claim that $\tau_{M''}$ is not surjective. The same statement regarding the structure of g shows our second claim, namely that g does not lie in the image of the natural map $\theta(M) \to \theta(M'')$. Indeed, it suffices to remark that

$$\begin{aligned}\theta(M) &= \varprojlim_n \Big(\bigoplus_j (\mathcal{A}_j \otimes_{\mathcal{A}} \mathcal{A}_n) \Big) = \varinjlim_i \varprojlim_n \Big(\bigoplus_{j \le i} (\mathcal{A}_j \otimes_{\mathcal{A}} \mathcal{A}_n) \Big) \\ &= \bigoplus_j \Big(\varprojlim_n (\mathcal{A}_j \otimes_{\mathcal{A}} \mathcal{A}_n) \Big),\end{aligned}$$

which follows from the fact that the transition maps $\mathcal{A}_{n+1} \to \mathcal{A}_n$ are injective and that the $\mathcal{A}_j$ are flat over $\mathcal{A}$.

Proposition A.6. *The $\mathcal{A}$-module $\Gamma(X, \mathcal{O}_X)$ is faithfully flat.*

Proof. Flatness follows from Lemma A.3. For faithfully flatness, let us consider a maximal ideal $\mathfrak{m}$ of $\mathcal{A}$. By the Nullstellensatz for semiaffinoid k-algebras, $\mathfrak{m}$ corresponds to a rigid point x of X. Since the X_n cover X, for n big enough one has $x \in X_n$. Hence $\mathfrak{m}' := \{f \in \Gamma(X, \mathcal{O}_X) \mid f(x) = 0\}$ is a maximal ideal of $\Gamma(X, \mathcal{O}_X)$ such that $\mathfrak{m}' \cap \mathcal{A} = \mathfrak{m}$. □

Lemma A.7. *The following are equivalent.*

(i) *The semiaffinoid k-algebra $\mathcal{A}$ is normal.*

(ii) *The special R-algebra $\mathcal{A}^\circ$ of power-bounded functions in $\mathcal{A}$ is normal.*

Proof. Let us first remark that $\mathrm{Quot}(\mathcal{A}) = \mathrm{Quot}(\mathcal{A}^\circ)$. Let us first prove that (i) $\Rightarrow$ (ii). Let $f \in \mathrm{Quot}(\mathcal{A}^\circ)$ be an element which satisfies an equation $f^n + \sum_{i=0}^{n-1} a_i f^i = 0$ with $a_i \in \mathcal{A}^\circ$. Then $f \in \mathcal{A}$ by (i), and since the a_i are in $\mathcal{A}^\circ$, it follows that $f \in \mathcal{A}^\circ$. Let us now prove that (ii) $\Rightarrow$ (i). Let $f \in \mathrm{Quot}(\mathcal{A})$ be an element which satisfies an equation $f^n + \sum_{i=0}^{n-1} a_i f^i = 0$ with $a_i \in \mathcal{A}$. Then there exists an integer m such that for all i, $\pi^m a_i \in \mathcal{A}^\circ$. Using the same argument as in the proof of Proposition 6.1, it follows that $\pi^m f$ satisfies a unitary equation with coefficients in $\mathcal{A}^\circ$, hence $\pi^m f \in \mathcal{A}^\circ$ by (ii), hence $f \in \mathcal{A}$. □

Theorem A.8. *Let $\mathcal{A}$ be a reduced semiaffinoid k-algebra, and X its associated rigid analytic k-space. Then $\mathcal{A} \simeq \{f \in \Gamma(X, \mathcal{O}_X) \mid |f|_{\sup} < \infty\}$.*

Proof. If $\mathcal{A}$ is normal, then $\mathcal{A}^\circ$ is a normal special R-algebra by Lemma A.7, and [de Jong 1995, Theorem 7.4.1] shows that $\mathcal{A}^\circ \simeq \Gamma(X, \mathcal{O}_X^\circ)$. The theorem follows in that case. In general, let $\mathcal{B}$ denote the normalization of $\mathcal{A}$. According to [Valabrega 1975; 1976], $\mathcal{B}$ is a semiaffinoid k-algebra. Let X' denote the rigid analytic k-space associated to $\mathcal{B}$ and $p : X' \to X$ the induced morphism, and let us consider the induced commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \Gamma(X, \mathcal{O}_X) & \xrightarrow[p^*]{} & \Gamma(X', \mathcal{O}_{X'}) \end{array}$$

Let us first observe that

$$\Gamma(X', \mathcal{O}_{X'}) \simeq \Gamma(X, \mathcal{O}_X) \otimes_{\mathcal{A}} \mathcal{B}.$$

Indeed, since $\mathcal{A} \to \mathcal{B}$ is finite, for each n the morphism $\mathcal{A}_n \to \mathcal{A}_n \otimes_{\mathcal{A}} \mathcal{B}$ is also finite. Since $\mathcal{A}_n$ is a k-affinoid algebra, it follows that $\mathcal{A}_n \otimes_{\mathcal{A}} \mathcal{B}$ is also a k-affinoid algebra [Bosch et al. 1984, Proposition 6.1.1.6]. If X'_n denotes the k-affinoid space associated to $\mathcal{A}_n \otimes_{\mathcal{A}} \mathcal{B}$, then X'_n is an affinoid domain of X' and the X'_n cover X'. It follows that

$$\Gamma(X', \mathcal{O}_{X'}) \simeq \varprojlim_{n \in \mathbb{N}} \Gamma(X'_n, \mathcal{O}_{X'}) \simeq \varprojlim_{n \in \mathbb{N}} (\mathcal{A}_n \otimes_{\mathcal{A}} \mathcal{B}) \simeq \Gamma(X, \mathcal{O}_X) \otimes_{\mathcal{A}} \mathcal{B},$$

where the second equality follows from the fact that X'_n is a k-affinoid space, and the third equality follows from Lemma A.3(ii).

Let now $f \in \Gamma(X, \mathcal{O}_X)$ be a bounded function. Then $p^*(f) \in \Gamma(X', \mathcal{O}_{X'}) \simeq \Gamma(X, \mathcal{O}_X) \otimes_{\mathcal{A}} \mathcal{B}$ is also bounded on X', and according to what we have shown in the first part of the proof, $p^*(f)$ comes from an element $b \in \mathcal{B}$.

Finally, $\mathcal{A} \subset \mathcal{B}$ is a sub-$\mathcal{A}$-module of $\mathcal{B}$ because $\mathcal{A}$ is reduced. Since $\Gamma(X, \mathcal{O}_X)$ is flat over $\mathcal{A}$ (Proposition A.6), it follows that $\Gamma(X, \mathcal{O}_X)$ is a submodule of

$\Gamma(X', \mathcal{O}_{X'}) \simeq \Gamma(X, \mathcal{O}_X) \otimes_{\mathcal{A}} \mathcal{B}$. Since $\Gamma(X, \mathcal{O}_X)$ is even faithfully flat over $\mathcal{A}$, we conclude that

$$\Gamma(X, \mathcal{O}_X) \cap \mathcal{B} = \mathcal{A}.$$

Indeed, this follows from [Bourbaki 1998, Section I.3.5, Proposition 10(ii)], which asserts that if $C \to C'$ is a faithfully flat ring morphism, if N is a C-module and if $N' \subset N$ is a sub-C-module, then $(N' \otimes_C C') \cap N = N'$. Since $f \in \Gamma(X, \mathcal{O}_X) \cap \mathcal{B}$, it follows that $f \in \mathcal{A}$. □

Example A.9. Theorem A.8 does not hold if we do not assume the semiaffinoid k-algebra $\mathcal{A}$ to be reduced. For instance, as in Example 2.3, let

$$\mathcal{A} := R[\![T_1, T_2]\!]/(T_2^2) \otimes_R k,$$

and let X be the associated rigid k-space. Let $f(T_1) \in k[\![T_1]\!]$ be a formal power series which converges on the open unit disc, but such that $f(T_1) \notin R[\![T_1]\!] \otimes_R k$. Then $f(T_1)T_2 \in \Gamma(X, \mathcal{O}_X^\circ) \setminus \mathcal{A}$.

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