

**SUPPLEMENT TO “MODULAR CURVES OF PRIME-POWER LEVEL WITH
INFINITELY MANY RATIONAL POINTS”**

Algebra and Number Theory **11:5**

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This section includes tables that list data for all the groups G , up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, from Theorem 1.1. The genus 0 groups are given in Tables 1, 2 and 3. The genus 1 groups are given in Table 4.

We now describe how to read Tables 1–4. Each row corresponds to a unique group G from Theorem 1.1 up to conjugacy; it is given a unique label of the form MZ^g-Nz , where M , N and g are integers and Z and z are letters. The integers N and g are the level and genus of G , respectively. Let Γ be the congruence subgroup consisting of matrices in $\mathrm{SL}_2(\mathbb{Z})$ whose image modulo N lies in the image of G modulo N . The integer g is also the genus of the Riemann surface obtained by taking the quotient of the complex upper-half plane by the action of Γ . The integer M is the level of Γ and Z is an uppercase letter that distinguishes Γ up to conjugacy in $\mathrm{GL}_2(\mathbb{Z})$; the prefix MZ^g for Γ matches the label used by Cummins and Pauli [*Experiment. Math.* **12** (2003), 243–255]. The letter z is chosen so that the label MZ^g-Nz distinguishes G up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$. In some of the tables, we also number the rows.

For each row of these tables, there is a positive integer N and a set of matrices S in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$; the corresponding open subgroup G of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ consists of the matrices $A \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ whose image in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ lies in the subgroup generated by S . The integer i in each row is the index $[\mathrm{GL}_2(\hat{\mathbb{Z}}) : G]$.

For each row of a table, we also have a list of minimal supergroups up to conjugacy (given by labels or numberings of other rows); more precisely, the groups G' satisfying $G \subsetneq G' \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$, up to conjugacy in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, for which there are no subgroups strictly between G and G' .

If G has genus 0, then X_G is isomorphic to $\mathbb{P}_{\mathbb{Q}}^1$; its function field is of the form $\mathbb{Q}(t)$. If G has genus 1, then the curve X_G is isomorphic to the elliptic curve in Table 5 given by the “curve” column of Table 4; its function field is of the form $\mathbb{Q}(x, y)$, where x and y are the given Weierstrass coordinates.

Now for simplicity, suppose that G has genus 0. The function F listed in the “map” column describes the morphism $X_G \rightarrow X_{G'}$ for one of the corresponding minimal supergroups G' of G (or the first minimal supergroup listed if only one map is given). More precisely, after appropriately conjugating G' satisfying $G \subsetneq G'$, the morphism $X_G \rightarrow X_{G'}$ is given by the rational function $F(t) \in \mathbb{Q}(t)$, i.e., the curves X_G and $X_{G'}$ have function fields $\mathbb{Q}(t)$ and $\mathbb{Q}(u)$, respectively, and are related by the equation $u = F(t)$. By composing these rational maps down to $X_{\mathrm{GL}_2(\hat{\mathbb{Z}})} = \mathbb{P}_{\mathbb{Q}}^1$ (i.e., to the group labeled 1A⁰-1a), we obtain from G a rational function

$$J(t) \in \mathbb{Q}(t);$$

in Section 5, we showed that $J(t)$ describes the morphism π_G from X_G to the j -line. If $J'(t) \in \mathbb{Q}(t)$ is the rational function arising from G' in the same manner, then we will have $J(t) = J'(F(t))$. The function $J(t)$ is independent of any choice of supergroups.

Similar remarks and conventions hold when G has genus 1.

#	label	i	N	generators	map	sup
0	1A ⁰ -1a	1	1			
1	3A ⁰ -3a	3	3	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	t^3	0
2	3B ⁰ -3a	4	3	$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$	$(t+3)^3(t+27)/t$	0
3	3C ⁰ -3a	6	3	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$(t-9)(t+3)/t$	1
4	3D ⁰ -3a	12	3	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$729/(t^3-27)$	2
					$-27(t-3)/(t^2+3t+9)$	3
5	9A ⁰ -9a	9	9	$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	t^3+9t-6	1
6	9B ⁰ -9a	12	9	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$t(t^2+9t+27)$	2
7	9C ⁰ -9a	12	9	$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	t^3	2
8	9D ⁰ -9a	18	9	$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$-27/t^3$	3
					$(t^2-3)/t$	5
9	9E ⁰ -9a	18	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix}$	$-9(t^3+3t^2-9t-3)/(8t^3)$	1
10	9F ⁰ -9a	27	9	$\begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 0 & 5 \end{pmatrix}$	$\frac{3^7(t^2-1)^3(t^6+3t^5+6t^4+t^3-3t^2+12t+16)^3(2t^3+3t^2-3t-5)}{(t^3-3t-1)^9}$	0
11	9G ⁰ -9a	27	9	$\begin{pmatrix} 0 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 0 & 4 \end{pmatrix}$	$\frac{(t^3-9t-12)(9-3t^3)(5t^3+18t^2+18t+3)}{(t^3+3t^2-3)^3}$	1
12	9H ⁰ -9a	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$	$3(t^3+9)/t^3$	4
					$3t/(2t^2-3t+6)$	9
13	9H ⁰ -9b	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$3(t^3+9t^2-9t-9)/(t^3-9t^2-9t+9)$	4
14	9H ⁰ -9c	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix}$	$-6(t^3-9t)/(t^3+9t^2-9t-9)$	4
					$-(t^2+3)/(t^2+8t+3)$	10
15	9I ⁰ -9a	36	9	$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$	$-6(t^3-9t)/(t^3-3t^2-9t+3)$	6
16	9I ⁰ -9b	36	9	$\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 3 & 5 \end{pmatrix}$	$-3(t^3+9t^2-9t-9)/(t^3+3t^2-9t-3)$	6
17	9I ⁰ -9c	36	9	$\begin{pmatrix} 2 & 2 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$	$(t^3-6t^2+3t+1)/(t^2-t)$	6
18	9J ⁰ -9a	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$	$(t^3-3t+1)/(t^2-t)$	7
19	9J ⁰ -9b	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$-18(t^2-1)/(t^3-3t^2-9t+3)$	7
20	9J ⁰ -9c	36	9	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$	$3(t^3+3t^2-9t-3)/(t^3-3t^2-9t+3)$	7
21	27A ⁰ -27a	36	27	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 9 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$	t^3	6

TABLE 1. Genus zero groups of 3-power level.

label	i	N	generators	map	supergroup
1A ⁰ -1a	1	1			
5A ⁰ -5a	5	5	$(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$t^3(t^2 + 5t + 40)$	1A ⁰ -1a
5B ⁰ -5a	6	5	$(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$(t^2 + 10t + 5)^3 / t$	1A ⁰ -1a
5C ⁰ -5a	10	5	$(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 \\ 2 & 0 \end{smallmatrix}), (\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix})$	$8000t^3(t+1)(t^2-5t+10)^3 / (t^2-5)^5$	1A ⁰ -1a
5D ⁰ -5a	12	5	$(\begin{smallmatrix} 4 & 0 \\ 1 & 4 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$125t / (t^2 - 11t - 1)$	5B ⁰ -5a
5D ⁰ -5b	12	5	$(\begin{smallmatrix} 4 & 0 \\ 1 & 4 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^2 - 11t - 1) / t$	5B ⁰ -5a
5E ⁰ -5a	15	5	$(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 2 & 3 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 2 & 2 \end{smallmatrix})$	$(t+5)(t^2-5) / (t^2+5t+5)$	5A ⁰ -5a
5G ⁰ -5a	30	5	$(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix})$	$125 / (t(t^4 + 5t^3 + 15t^2 + 25t + 25))$ $(t^2 + 5) / t$	5E ⁰ -5a 5B ⁰ -5a
5G ⁰ -5b	30	5	$(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}), (\begin{smallmatrix} 2 & 1 \\ 3 & 3 \end{smallmatrix})$	$-t(t^2 + 5t + 10) / (t^3 + 5t^2 + 10t + 10)$ $-5(t^2 + 4t + 5) / (t^2 + 5t + 5)$	5C ⁰ -5a 5E ⁰ -5a
5H ⁰ -5a	60	5	$(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$-1 / t^5$ $\frac{-(t^4 - 2t^3 + 4t^2 - 3t + 1)}{t(t^4 + 3t^3 + 4t^2 + 2t + 1)}$ $5t / (t^2 - t - 1)$	5D ⁰ -5a 5D ⁰ -5b 5G ⁰ -5a
25A ⁰ -25a	30	25	$(\begin{smallmatrix} 2 & 2 \\ 0 & 13 \end{smallmatrix}), (\begin{smallmatrix} 4 & 1 \\ 3 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 3 \\ 0 & 6 \end{smallmatrix})$	$(t-1)(t^4 + t^3 + 6t^2 + 6t + 11)$	5B ⁰ -5a
25B ⁰ -25a	60	25	$(\begin{smallmatrix} 9 & 10 \\ 0 & 14 \end{smallmatrix}), (\begin{smallmatrix} 0 & 7 \\ 7 & 2 \end{smallmatrix}), (\begin{smallmatrix} 2 & 8 \\ 0 & 1 \end{smallmatrix})$	$-t^5$	5D ⁰ -5b
25B ⁰ -25b	60	25	$(\begin{smallmatrix} 9 & 10 \\ 0 & 14 \end{smallmatrix}), (\begin{smallmatrix} 0 & 7 \\ 7 & 2 \end{smallmatrix}), (\begin{smallmatrix} 4 & 1 \\ 0 & 7 \end{smallmatrix})$	$(1 - t^2) / t$ $\frac{-(t^4 - 2t^3 + 4t^2 - 3t + 1)}{t(t^4 + 3t^3 + 4t^2 + 2t + 1)}$ $(t^2 + 4t - 1) / (t^2 - t - 1)$	25A ⁰ -25a 5D ⁰ -5a 25A ⁰ -25a
7B ⁰ -7a	8	7	$(\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}), (\begin{smallmatrix} 3 & 0 \\ 1 & 5 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix})$	$(t^2 + 5t + 1)^3(t^2 + 13t + 49) / t$	1A ⁰ -1a
7D ⁰ -7a	21	7	$(\begin{smallmatrix} 2 & 3 \\ 0 & 3 \end{smallmatrix}), (\begin{smallmatrix} 2 & 4 \\ 4 & 5 \end{smallmatrix}), (\begin{smallmatrix} 3 & 1 \\ 0 & 4 \end{smallmatrix})$	$\frac{(2t-1)^3(t^2-t+2)^3(2t^2+5t+4)^3(5t^2+2t-4)^3}{(t^3+2t^2-t-1)^7}$	1A ⁰ -1a
7E ⁰ -7a	24	7	$(\begin{smallmatrix} 6 & 0 \\ 1 & 6 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix})$	$49(t^2 - t) / (t^3 - 8t^2 + 5t + 1)$	7B ⁰ -7a
7E ⁰ -7b	24	7	$(\begin{smallmatrix} 6 & 0 \\ 1 & 6 \end{smallmatrix}), (\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^3 - 8t^2 + 5t + 1) / (t^2 - t)$	7B ⁰ -7a
7E ⁰ -7c	24	7	$(\begin{smallmatrix} 6 & 0 \\ 1 & 6 \end{smallmatrix}), (\begin{smallmatrix} 3 & 0 \\ 0 & 4 \end{smallmatrix})$	$-7(t^3 - 2t^2 - t + 1) / (t^3 - t^2 - 2t + 1)$	7B ⁰ -7a
7F ⁰ -7a	28	7	$(\begin{smallmatrix} 3 & 1 \\ 4 & 4 \end{smallmatrix}), (\begin{smallmatrix} 4 & 4 \\ 1 & 3 \end{smallmatrix}), (\begin{smallmatrix} 3 & 4 \\ 0 & 4 \end{smallmatrix})$	$\frac{t(t+1)^3(t^2-5t+1)^3(t^2-5t+8)^3(t^4-5t^3+8t^2-7t+7)^3}{(t^3-4t^2+3t+1)^7}$	1A ⁰ -1a
13A ⁰ -13a	14	13	$(\begin{smallmatrix} 2 & 0 \\ 0 & 7 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$(t^2 + 5t + 13)(t^4 + 7t^3 + 20t^2 + 19t + 1)^3 / t$	1A ⁰ -1a
13B ⁰ -13a	28	13	$(\begin{smallmatrix} 3 & 0 \\ 0 & 9 \end{smallmatrix}), (\begin{smallmatrix} 4 & 0 \\ 1 & 10 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$13t / (t^2 - 3t - 1)$	13A ⁰ -13a
13B ⁰ -13b	28	13	$(\begin{smallmatrix} 3 & 0 \\ 0 & 9 \end{smallmatrix}), (\begin{smallmatrix} 4 & 0 \\ 1 & 10 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^2 - 3t - 1) / t$	13A ⁰ -13a
13C ⁰ -13a	42	13	$(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})$	$13(t^2 - t) / (t^3 - 4t^2 + t + 1)$	13A ⁰ -13a
13C ⁰ -13b	42	13	$(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})$	$(t^3 - 4t^2 + t + 1) / (t^2 - t)$	13A ⁰ -13a
13C ⁰ -13c	42	13	$(\begin{smallmatrix} 5 & 0 \\ 0 & 8 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix})$	$-(5t^3 - 7t^2 - 8t + 5) / (t^3 - 4t^2 + t + 1)$	13A ⁰ -13a

TABLE 2. Genus zero groups of ℓ -power level for primes $\ell > 3$.

$$J(x, y) := \frac{(f_1 f_2 f_3 f_4)^3}{f_5^2 f_6^{11}},$$

$$f_1 = x^2 + 3x - 6,$$

$$f_2 = 11(x^2 - 5)y + (2x^4 + 23x^3 - 72x^2 - 28x + 127),$$

$$f_3 = 6y + 11x - 19,$$

$$f_4 = 22(x - 2)y + (5x^3 + 17x^2 - 112x + 120),$$

$$f_5 = 11y + (2x^2 + 17x - 34), \quad f_6 = (x - 4)y - (5x - 9).$$