Classifying tilting complexes over preprojective
algebras of Dynkin type

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We study tilting complexes over preprojective algebras of Dynkin type. We classify all tilting complexes by giving a bijection between tilting complexes and the braid group of the corresponding folded graph. In particular, we determine the derived equivalence class of the algebra. For the results, we develop the theory of silting-discrete triangulated categories and give a criterion for silting-discreteness.

1. Introduction

1A. *Background and motivation.* Derived categories are nowadays considered as a fundamental object in many branches of mathematics including representation theory and algebraic geometry. Among others, one of the most important problems is to understand their equivalences. Derived equivalences provide a lot of interesting connections between various different objects and they are also quite useful to study structures of the categories.

It is known that derived equivalences are controlled by tilting objects (complexes) [Rickard 1989; Keller 1994] and therefore these constructions have been extensively studied. As a tool for studying tilting objects, Keller and Vossieck [1988] introduced the notion of silting objects (Definition 2.1), which is a generalization of tilting

objects. After that, it was shown that their mutation properties are much better than tilting ones and they yield a nice combinatorial description [Aihara and Iyama 2012] (see Definition 2.3). Furthermore, silting objects have turned out to have deep connections with several important objects such as cluster tilting objects and t-structures, for example [Adachi et al. 2014; Buan et al. 2011; Koenig and Yang 2014; Brüstle and Yang 2013; Iyama et al. 2014; Qiu and Woolf 2014; Broomhead et al. 2016].

One of the aims of this paper is to give a further development of the mutation theory of silting objects. In particular, we study a criterion when a triangulated category is *silting-discrete* (Definition 2.2). A remarkable property of this class is that all silting objects are connected to each other by iterated mutation and this fact allows us to achieve a comprehensive understanding of the categories.

Another aim of the paper is, by applying this technique, to classify all tilting complexes of *preprojective algebras* of Dynkin type. Since preprojective algebras were introduced in [Gel'fand and Ponomarev 1979; Dlab and Ringel 1980; Baer et al. 1987], it turned out that they have fundamental importance in representation theory as well as algebraic and differential geometry. We refer to [Ringel 1998] for quiver representations, [Lusztig 1991; 2000; Kashiwara and Saito 1997] for quantum groups, [Auslander and Reiten 1996; Crawley-Boevey 2000] for Kleinian singularities, [Nakajima 1994; 1998; 2001] for quiver varieties, and [Geiß et al. 2006; 2011] for cluster algebras.

For the case of preprojective algebras of non-Dynkin type, its tilting theory has been extensively studied in [Buan et al. 2009; Iyama and Reiten 2008]. In particular, they show that certain ideals parametrized by the Coxeter group (see Theorem 4.1) give tilting modules over the preprojective algebra and this fact provides a method for studying the derived category. On the other hand, in the case of Dynkin type, they are no longer tilting modules. Moreover, there is no spherical objects in this case and a similar nice theory had never been observed. In this paper, via a new strategy, we succeed in classifying all tilting complexes as described below.

1B. *Our results.* To explain our results, we give the following set-up. Let Δ be a Dynkin graph and Λ the preprojective algebra of Δ.

First we study two-term tilting complexes of Λ. For this purpose, we use τ-*tilting theory*. Mizuno [2014] showed that the above ideals are support τ-tilting Λ-modules (Theorem 4.1). Then, combining the results of [Adachi et al. 2014], we obtain a bijection between two-term silting complexes of Λ and the Weyl group (Theorem 4.1). Moreover we analyze this connection in more detail and we can give a classification of two-term tilting complexes of Λ using the *folded graph* Δ^{f} of Δ (Definition 3.2) given by the following correspondences.

Δ	$\mathbb{A}_{2n-1}, \mathbb{A}_{2n}$	$\mathbb{D}_{2n}$	$\mathbb{D}_{2n+1}$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
Δ^{f}	$\mathbb{B}_n$	$\mathbb{D}_{2n}$	$\mathbb{B}_{2n}$	$\mathbb{F}_4$	$\mathbb{E}_7$	$\mathbb{E}_8$

Then our first result is summarized as follows.

Theorem 1.1 (Theorem 4.2). *Let W_{Δ^f} be the Weyl group of Δ^f and 2-tilt Λ the set of isomorphism classes of basic two-term tilting complexes of $\mathsf{K}^b(\mathrm{proj}\Lambda)$. Then we have a bijection*

$$W_{\Delta^f} \longleftrightarrow \text{2-tilt } \Lambda.$$

We remark that we can give not only a bijection but also an explicit description of all two-term tilting complexes (Theorem 4.1). On the other hand, we study an important relationship between two-term silting complexes and silting-discrete categories. More precisely, we give the following criterion of silting-discreteness (tilting-discreteness).

Theorem 1.2 (Theorem 2.4, Corollary 2.11). *Let A be a finite dimensional algebra (respectively, finite dimensional self-injective algebra). The following are equivalent.*

 (a) $\mathsf{K}^b(\mathrm{proj}A)$ *is silting-discrete (respectively, tilting-discrete).*

 (b) 2-silt$_P$ A *(respectively, 2-tilt$_P$ A) is a finite set for any silting (respectively, tilting) complex P.*

 (c) 2-silt$_P$ A *(respectively, 2-tilt$_P$ A) is a finite set for any silting (respectively, tilting) complex P which is given by iterated irreducible left silting (respectively, tilting) mutation from A.*

Here 2-silt$_P$ A (respectively, 2-tilt$_P$ A) denotes the subset of silting (respectively, tilting) objects T in $\mathsf{K}^b(\mathrm{proj}A)$ such that $P \geq T \geq P[1]$ (Definition 2.2). An advantage of this theorem is that we can understand the condition of all silting (respectively, tilting) objects by studying a certain special class of silting (respectively, tilting) objects. Then, we can apply Theorem 1.2 and obtain the following result.

Theorem 1.3 (Theorem 5.1, Proposition 5.4). *The endomorphism algebra of any irreducible left tilting mutation (Definition 2.3) of Λ is isomorphic to Λ. In particular, the condition* (b) *of Theorem 1.2 is satisfied and hence $\mathsf{K}^b(\mathrm{proj}\Lambda)$ is tilting-discrete.*

Then Theorem 1.3 implies that any tilting complexes are obtained from Λ by iterated irreducible mutation. As a consequence of this result, we determine the derived equivalence class of Λ as follows.

Corollary 1.4 (Theorem 5.1). *Any basic tilting complex T of Λ satisfies*

$$\mathrm{End}_{\mathsf{K}^b(\mathrm{proj}\Lambda)}(T) \cong \Lambda.$$

In particular, the derived and Morita equivalence classes coincide.

In fact, we give a more detailed description about tilting complexes. Indeed, using Theorem 1.1 and Corollary 1.4, we can show that irreducible tilting mutation

satisfy braid relations (Proposition 6.1), which provide a nice relationship between the braid group and tilting complexes (see [Brav and Thomas 2011; Seidel and Thomas 2001; Grant 2013; Khovanov and Seidel 2002]).

Recall that the braid group B_{Δ^f} is defined by generators a_i ($i \in \Delta_0^f$) with relations $(a_i a_j)^{m(i,j)} = 1$ for $i \neq j$ (see Section 3B for $m(i,j)$), that is, the difference with W_{Δ^f} is that we do not require the relations $a_i^2 = 1$ for $i \in \Delta_0^f$. We denote by $\boldsymbol{\mu}_i^+$ (respectively, $\boldsymbol{\mu}_i^-$) the irreducible left (respectively, right) tilting mutation associated with $i \in \Delta_0^f$.

Then we can define the map from the braid group to tilting complexes and it gives a classification of tilting complexes as follows.

Theorem 1.5 (Theorem 6.6). *Let B_{Δ^f} be the braid group of Δ^f and tilt Λ the set of isomorphism classes of basic tilting complexes of Λ. Then we have a bijection*

$$B_{\Delta^f} \to \text{tilt}\,\Lambda, \quad a = a_{i_1}^{\epsilon_{i_1}} \cdots a_{i_k}^{\epsilon_{i_k}} \mapsto \boldsymbol{\mu}_a(\Lambda) := \boldsymbol{\mu}_{i_1}^{\epsilon_{i_1}} \circ \cdots \circ \boldsymbol{\mu}_{i_k}^{\epsilon_{i_k}}(\Lambda).$$

We now describe the organization of this paper.

In Section 2, we deal with triangulated categories and study some properties of silting-discrete categories. In particular, we give a criterion of silting-discreteness. We also investigate a Bongartz-type lemma for silting objects. In Section 3, we recall definitions and some results related to preprojective algebras. In Section 4, we explain a connection between two-term silting complexes and the Weyl group. In particular, we characterize two-term tilting complexes in terms of the subgroup of the Weyl group and this observation is crucial in this paper. In Section 5, we show that preprojective algebras of Dynkin type are tilting-discrete. It implies that any tilting complex is obtained by iterated mutation from an arbitrary tilting complex. In Section 6, we show that there exists a map from the braid group to tilting complexes and we prove that it is a bijection.

Notation. Throughout this paper, let K be an algebraically closed field and $D := \text{Hom}_K(-, K)$. For a finite dimensional algebra Λ over K, we denote by $\text{mod}\,\Lambda$ the category of finitely generated right Λ-modules and by $\text{proj}\,\Lambda$ the category of finitely generated projective Λ-modules. We denote by $\mathsf{D}^b(\text{mod}\,\Lambda)$ the bounded derived category of $\text{mod}\,\Lambda$ and by $\mathsf{K}^b(\text{proj}\,\Lambda)$ the bounded homotopy category of $\text{proj}\,\Lambda$.

2. Silting-discrete triangulated categories

In this section, we study silting-discrete triangulated categories. In particular, we give a criterion for silting-discreteness. Moreover we apply this theory for tilting-discrete categories for self-injective algebras. We also study a relationship between silting-discrete categories and a Bongartz-type lemma.

Throughout this section, let $\mathcal{T}$ be a Krull–Schmidt triangulated category and assume that it satisfies the following property:

- For any object X of $\mathcal{T}$, the additive closure add X is functorially finite in $\mathcal{T}$.

For example, it is satisfied if $\mathcal{T}$ is the homotopy category of bounded complexes of finitely generated projective modules over a finite dimensional algebra, which is a main object in this paper.

2A. *Criteria for silting-discreteness.* Let us start with recalling the definition of silting objects [Aihara and Iyama 2012; Buan et al. 2011; Keller and Vossieck 1988].

Definition 2.1. (a) We say an object P in $\mathcal{T}$ is *presilting* (respectively, *pretilting*) if it satisfies $\operatorname{Hom}_{\mathcal{T}}(P, P[i]) = 0$ for any $i > 0$ (respectively, $i \neq 0$).

(b) We call an object P in $\mathcal{T}$ *silting* (respectively, *tilting*) if it is presilting (respectively, pretilting) and the smallest thick subcategory containing P is $\mathcal{T}$.

We denote by silt $\mathcal{T}$ (respectively, tilt $\mathcal{T}$) the set of isomorphism classes of basic silting objects (respectively, tilting objects) in $\mathcal{T}$.

It is known that the number of nonisomorphic indecomposable summands of a silting object does not depend on the choice of silting objects [Aihara and Iyama 2012, Corollary 2.28]. Moreover, for objects P and Q of $\mathcal{T}$, we write $P \geq Q$ if $\operatorname{Hom}_{\mathcal{T}}(P, Q[i]) = 0$ for any $i > 0$, which gives a partial order on silt $\mathcal{T}$ [Aihara and Iyama 2012, Theorem 2.11].

Then we give the definition of silting-discrete triangulated categories as follows.

Definition 2.2. (a) We call a triangulated category $\mathcal{T}$ *silting-discrete* if for any $P \in$ silt $\mathcal{T}$ and any $\ell > 0$, the set

$$\{T \in \text{silt}\,\mathcal{T} \mid P \geq T \geq P[\ell]\}$$

is finite. Note that the property of being silting-discrete does not depend on the choice of silting objects [Aihara 2013, Proposition 3.8]. Hence it is equivalent to say that, for a silting object $A \in \mathcal{T}$ and any $\ell > 0$, the set

$$\{T \in \text{silt}\,\mathcal{T} \mid A \geq T \geq A[\ell]\}$$

is finite. Similarly, we call $\mathcal{T}$ *tilting-discrete* if, for a tilting object $A \in \mathcal{T}$ and any $\ell > 0$, the set $\{T \in \text{tilt}\,\mathcal{T} \mid A \geq T \geq A[\ell]\}$ is finite.

(b) For a silting object P of $\mathcal{T}$, we denote by 2-silt$_P$ $\mathcal{T}$ the subset U of silt $\mathcal{T}$ such that $P \geq U \geq P[1]$. We call $\mathcal{T}$ *2-silting-finite* if 2-silt$_P$ $\mathcal{T}$ is a finite set for any silting object P of $\mathcal{T}$. Note that the finiteness of 2-silt$_P$ $\mathcal{T}$ depends on a silting object P in general. Similarly, we denote by 2-tilt$_P$ $\mathcal{T}$ the subset U of tilt $\mathcal{T}$ such that $P \geq U \geq P[1]$.

Moreover we recall mutation for silting objects [Aihara and Iyama 2012, Theorem 2.31].

Definition 2.3. Let P be a basic silting object of $\mathcal{T}$ and decompose it as $P = X \oplus M$. We take a triangle

$$X \xrightarrow{\ f\ } M' \longrightarrow Y \longrightarrow X[1]$$

with a minimal left (add M)-approximation f of X. Then $\mu_X^+(P) := Y \oplus M$ is again a silting object, and we call it the *left mutation* of P with respect to X. Dually, we define the right mutation $\mu_X^-(P)$.[1] Mutation will mean either left or right mutation. If X is indecomposable, then we say that mutation is *irreducible*. In this case, we have $P > \mu_X^+(P)$ and there is no silting object Q satisfying $P > Q > \mu_X^+(P)$ [Aihara and Iyama 2012, Theorem 2.35].

Moreover, if P and $\mu_X^+(P)$ are tilting objects, then we call it the (left) *tilting mutation*. In this case, if there exists no nontrivial direct summand X' of X such that $\mu_{X'}^+(T)$ is tilting, then we say that tilting mutation is *irreducible* ([Chan et al. 2015, Definition 5.3]).

We remark that all silting objects of a silting-discrete category are reachable by iterated irreducible mutation [Aihara 2013, Corollary 3.9].

Our first aim is to show the following theorem.

Theorem 2.4. *The following are equivalent.*

(a) $\mathcal{T}$ *is silting-discrete.*

(b) $\mathcal{T}$ *is 2-silting-finite.*

(c) *For a silting object $A \in \mathcal{T}$, 2-silt$_P$ $\mathcal{T}$ is a finite set for any silting object P which is given by iterated irreducible left mutation from A.*

We note that the theorem is different from [Qiu and Woolf 2014, Lemma 2.14], where the partial order is defined by a finite sequence of tilts; our partial order is valid for any silting objects.

Now we give some examples of silting-discrete categories.

Example 2.5. Let Λ be a finite dimensional algebra. Then $\mathsf{K}^b(\mathrm{proj}\Lambda)$ is silting-discrete if:

(a) Λ is a path algebra of Dynkin type, which immediately follows from the definition.

(b) Λ is a local algebra [Aihara and Iyama 2012, Corollary 2.43].

(c) Λ is a representation-finite symmetric algebra [Aihara 2013, Theorem 5.6], which is also tilting-discrete.

(d) Λ is a derived discrete algebra of finite global dimension [Broomhead et al. 2016, Proposition 6.8].

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[1]The convention of μ^+ and μ^- is different from [Mizuno 2014] in which the converse notation is used.

(e) Λ is a Brauer graph algebra whose Brauer graph contains at most one cycle of odd length and no cycle of even length [Adachi et al. 2015], which is also tilting-discrete.

For a proof of Theorem 2.4, we will introduce the following terminology.

Definition 2.6. We define a subset of silt $\mathcal{T}$

$$\nabla_A(T) := \{U \in \text{silt } \mathcal{T} \mid A \geq U \geq A[1] \text{ and } U \geq T\},$$

where A is a silting object and T is a presilting object in $\mathcal{T}$ satisfying $A \geq T$. Note that we have $T \geq A[\ell]$ for some $\ell \geq 0$ [Aihara and Iyama 2012, Proposition 2.4].

Moreover, we say that a silting object P is *minimal* in $\nabla_A(T)$ if it is a minimal element in the partially ordered set $\nabla_A(T)$.

To keep this notation, we will make the following assumption.

Assumption 2.7. In the rest of this section, we always assume that $\mathcal{T}$ admits a silting object A and a presilting object T satisfying $A \geq T$.

Then we give the following key proposition.

Proposition 2.8. *If a silting object P is minimal in $\nabla_A(T)$ and $T \geq A[\ell]$ for some $\ell > 0$, then we have $T \geq P[\ell - 1]$.*

For a proof, we recall the following proposition. See [Aihara and Iyama 2012, Proposition 2.23, 2.24, 2.36] and [Aihara 2013, Proposition 2.12].

Proposition 2.9. *Let P be a silting object of $\mathcal{T}$. Then the following hold.*

(a) *There exists $\ell \geq 0$ such that $P \geq T \geq P[\ell]$ if and only if there exist triangles*

$$T_1 \longrightarrow P_0 \xrightarrow{\ f_0\ } T_0 := T \longrightarrow T_1[1],$$

$$\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots$$

$$T_{\ell-1} \longrightarrow P_{\ell-2} \xrightarrow{\ f_{\ell-2}\ } T_{\ell-2} \longrightarrow T_{\ell-1}[1],$$

$$T_\ell \longrightarrow P_{\ell-1} \xrightarrow{\ f_{\ell-1}\ } T_{\ell-1} \longrightarrow P_\ell[1],$$

$$0 \longrightarrow P_\ell \xrightarrow{\ f_\ell\ } T_\ell \longrightarrow 0,$$

where f_i is a minimal right (add P)-approximation of T_i for $0 \leq i \leq \ell$.

(b) *In the situation of (a), if $\ell \neq 0$, then there is a nonzero direct summand $X \in \text{add}(P_\ell)$ such that the irreducible left mutation $\mu_X^+(P) \geq T$.*

Using Proposition 2.9, we give a proof of Proposition 2.8.

Proof of Proposition 2.8. Since P is minimal in $\nabla_A(T)$, we have

$$P \geq T \geq A[\ell] \geq P[\ell].$$

Then, by Proposition 2.9(a), there exist triangles

$$T_1 \longrightarrow P_0 \xrightarrow{\ f_0\ } T_0 := T \longrightarrow T_1[1],$$

$$\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots$$

$$T_{\ell-1} \longrightarrow P_{\ell-2} \xrightarrow{\ f_{\ell-2}\ } T_{\ell-2} \longrightarrow T_{\ell-1}[1],$$

$$T_\ell \longrightarrow P_{\ell-1} \xrightarrow{\ f_{\ell-1}\ } T_{\ell-1} \longrightarrow P_\ell[1],$$

$$0 \longrightarrow P_\ell \xrightarrow{\ f_\ell\ } T_\ell \longrightarrow 0,$$

where f_i is a minimal right (add P)-approximation of T_i for $0 \leq i \leq \ell$.

Similarly, since we have $P \geq A[1] \geq P[1]$, there is a triangle

$$Q_1 \longrightarrow Q_0 \xrightarrow{\ f\ } A[1] \longrightarrow Q_1[1], \qquad\qquad (2\text{-}1)$$

where f is a minimal right (add P)-approximation of $A[1]$ and $Q_1 \in$ add P.

(i) We show that P_ℓ belongs to add Q_1. First, we have $\mathrm{Hom}_{\mathcal{T}}(T, A[1+\ell]) = 0$ by the definition of $T \geq A[\ell]$. Hence it follows from [Aihara and Iyama 2012, Lemma 2.25] that $(\mathrm{add}\, P_\ell) \cap (\mathrm{add}\, Q_0) = 0$.

On the other hand, since $A[1]$ is a silting object, we find that $Q_0 \oplus Q_1$ is also a silting object by the sequence (2-1). From [Aihara and Iyama 2012, Theorem 2.18], it is observed that add $P = \mathrm{add}(Q_0 \oplus Q_1)$ and hence P_ℓ belongs to add Q_1.

(ii) We show that $T \geq P[\ell - 1]$. Suppose that $P_\ell \neq 0$. Then we can take a direct summand $X \neq 0$ of P_ℓ such that $\mu_X^+(P) \geq T$ from Proposition 2.9(b).

On the other hand, (i) implies that X belongs to add Q_1. Since $P \geq A[1] \geq P[1]$, by applying Proposition 2.9(b) to the sequence (2-1), we see that $\mu_X^+(P) \geq A[1]$. Thus, one gets a silting object $\mu_X^+(P)$ such that $P > \mu_X^+(P) \geq A[1]$ satisfying $\mu_X^+(P) \geq T$, which is a contradiction to the minimality of P. Therefore, we conclude that $P_\ell = 0$. Hence we get $T \geq P[\ell - 1]$ by Proposition 2.9(a). $\qquad\square$

On the other hand, we can easily check the following lemma.

Lemma 2.10. *Let A be a silting object. If 2-silt$_A$ $\mathcal{T}$ is a finite set, then there exists a minimal element in $\nabla_A(T)$.*

Then we give a proof of Theorem 2.4, which provides a criterion of silting-discreteness.

Proof of Theorem 2.4. It is obvious that the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold.

We show that the implication (c) $\Rightarrow$ (a) holds. Let T be a silting object such that $A \geq T \geq A[\ell]$ for some $\ell > 0$. Since 2-silt$_A$ $\mathcal{T}$ is a finite set, there exists a minimal object P in $\nabla_A(T)$. Hence we get $P \geq T \geq P[\ell-1]$ by Proposition 2.9.

Thus, one obtains

$$\{T \in \text{silt}\, \mathcal{T} \mid A \geq T \geq A[\ell]\} \subseteq \bigcup_{P \in \text{2-silt}_A\, \mathcal{T}} \{U \in \text{silt}\, \mathcal{T} \mid P \geq U \geq P[\ell-1]\}.$$

By [Aihara 2013, Theorem 3.5], the finiteness of 2-silt$_A$ $\mathcal{T}$ implies that P can be obtained from A by iterated irreducible left mutation. Therefore, our assumption yields that 2-silt$_P$ $\mathcal{T}$ is also a finite set. Repeating this argument leads to the assertion. $\square$

Moreover, using a statement analogous to Proposition 2.9 (see [Chan et al. 2015, Section 5]), we give a criterion for tilting-discreteness for self-injective algebras as follows.

Corollary 2.11. *Let Λ be a basic finite dimensional self-injective algebra and $\mathcal{T} := \mathsf{K}^b(\text{proj}\Lambda)$. Then the following are equivalent.*

(a) *$\mathcal{T}$ is tilting-discrete.*

(b) *$\mathcal{T}$ is 2-tilting-finite.*

(c) *2-tilt$_P$ $\mathcal{T}$ is a finite set for any tilting object P which is given by iterated irreducible left tilting mutation from Λ.*

Proof. It is obvious that the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold.

We show that the implication (c) $\Rightarrow$ (a) holds. Let T be a tilting object such that $\Lambda \geq T \geq \Lambda[\ell]$ for some $\ell > 0$. Since 2-tilt$_\Lambda$ $\mathcal{T}$ is a finite set, there exists a minimal tilting object P in $\nabla_\Lambda(T)$. Then, by [Chan et al. 2015, Proposition 5.10, Theorem 5.11], an argument the same as that of Proposition 2.9 works for tilting objects and irreducible tilting mutation. Hence we obtain Proposition 2.8 for tilting objects and one can get $P \geq T \geq P[\ell-1]$.

Thus, one obtains

$$\{T \in \text{tilt}\, \mathcal{T} \mid \Lambda \geq T \geq \Lambda[\ell]\} \subseteq \bigcup_{P \in \text{2-tilt}_\Lambda\, \mathcal{T}} \{U \in \text{tilt}\, \mathcal{T} \mid P \geq U \geq P[\ell-1]\}.$$

By [Chan et al. 2015, Theorem 5.11], the finiteness of 2-tilt$_\Lambda$ $\mathcal{T}$ implies that P can be obtained from Λ by iterated irreducible left tilting mutation. Therefore, our assumption yields that 2-tilt$_P$ $\mathcal{T}$ is also a finite set. Repeating this argument leads to the assertion. $\square$

Finally, as an application of Theorem 2.4, we show that silting-discrete categories satisfy a Bongartz-type lemma. For this purpose, we give the following definition.

Definition 2.12. We call a presilting object T in $\mathcal{T}$ *partial silting* if it is a direct summand of some silting object, that is, there exists an object T' such that $T \oplus T'$ is a silting object.

One of the important questions is if any presilting object is partial silting or not [Brüstle and Yang 2013, Question 3.13]. We will show that it has a positive answer in the case of silting-discrete categories.

Let us recall the following result.

Proposition 2.13 [Aihara 2013, Proposition 2.16]. *Let T be a presilting object in $\mathcal{T}$. If $A \geq T \geq A[1]$, then T is partial silting.*

Then we can improve Proposition 2.13 as follows.

Proposition 2.14. *Let T be a presilting object in $\mathcal{T}$ such that $A \geq T$. Assume that for any silting object B in $\mathcal{T}$ such that $A \geq B \geq T$, there exists a minimal object in $\nabla_B(T)$.*

Then there exists a silting object P in $\mathcal{T}$ satisfying $P \geq T \geq P[1]$. In particular, T is partial silting.

Proof. We can take $\ell \geq 0$ such that $A \geq T \geq A[\ell]$ by [Aihara and Iyama 2012, Proposition 2.4]. It is enough to show the statement for $\ell \geq 2$. Since there is a minimal silting object in $\nabla_A(T)$, which we denote by A_1, we have $A_1 \geq T \geq A_1[\ell-1]$ by Proposition 2.8. By our assumption, we can repeat this argument and we obtain a sequence

$$A = A_0 \geq A_1 \geq \cdots \geq A_{\ell-1} \geq T \geq A_{\ell-1}[1] \geq \cdots \geq A_1[\ell-1] \geq A[\ell],$$

where A_{i+1} is a minimal object in $\nabla_{A_i}(T)$ for $0 \leq i \leq \ell-2$. Thus, we get the desired silting object $P := A_{\ell-1}$.

The second assertion immediately follows from the first one and Proposition 2.13. $\qquad\square$

As a consequence, we obtain the following theorem.

Theorem 2.15. *If $\mathcal{T}$ is silting-discrete, then any presilting object is partial silting.*

Proof. Take a presilting object T in $\mathcal{T}$. If T is presilting, then so is $T[i]$ for any i. Hence we can assume that $A \geq T$. Then, by Theorem 2.4 and Lemma 2.10, $\mathcal{T}$ satisfies the assumption of Proposition 2.14 and hence we obtain the conclusion. $\square$

We remark that in [Broomhead et al. 2016, Secion 5] the authors also discuss the Bongartz completion using a different type of the partial order.

3. Basic properties of preprojective algebras of Dynkin type

In this section, we review some definitions and results we will use in the rest of this paper.

3A. *Preprojective algebras.* Let Q be a finite connected acyclic quiver. We denote by Q_0 vertices of Q and by Q_1 arrows of Q. We denote by $\overline{Q}$ the double quiver of Q, which is obtained by adding an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q_1. The *preprojective algebra* $\Lambda_Q = \Lambda$ associated to Q is the algebra $K\overline{Q}/I$, where I is the ideal in the path algebra $K\overline{Q}$ generated by the relation of the form

$$\sum_{a \in Q_1} (aa^* - a^*a).$$

We remark that Λ does not depend on the orientation of Q. Hence, for a graph Δ, we define the preprojective algebra by $\Lambda_\Delta = \Lambda_Q$, where Q is a quiver whose underlying graph is Δ. We denote by Δ_0 vertices of Δ.

Let Δ be a Dynkin graph (by Dynkin graph we always mean the one of type ADE). The preprojective algebra of Δ is finite dimensional and self-injective [Brenner et al. 2002, Theorem 4.8]. Without loss of generality, we may suppose that vertices are given as in Figure 1 and let e_i be the primitive idempotent of Λ associated with $i \in \Delta_0$. We denote the Nakayama permutation of Λ by $\iota : \Delta_0 \to \Delta_0$ (i.e., $D(\Lambda e_{\iota(i)}) \cong e_i \Lambda$). Then, one can check that we have $\iota = \mathrm{id}$ if Δ is type $\mathbb{D}_{2n}$, $\mathbb{E}_7$ and $\mathbb{E}_8$. Otherwise, we have $\iota^2 = \mathrm{id}$ and it is given as follows.

$$\iota(1) = 1 \text{ and } \iota(i) = i + n - 1 \text{ for } i \in \{2, \cdots, n\} \qquad \text{if } \mathbb{A}_{2n-1},$$
$$\iota(i) = i + n \text{ for } i \in \{1, \cdots, n\} \qquad \text{if } \mathbb{A}_{2n},$$
$$\iota(1) = n \text{ and } \iota(i) = i \text{ for } i \notin \{1, n\} \qquad \text{if } \mathbb{D}_{2n+1},$$
$$\iota(3) = 5, \iota(4) = 6 \text{ and } \iota(i) = i \text{ for } i \in \{1, 2\} \qquad \text{if } \mathbb{E}_6.$$

3B. *Weyl group.* Let Δ be a graph from Figure 1. The *Weyl group* W_Δ associated to Δ is defined by the generators s_i and relations $(s_i s_j)^{m(i,j)} = 1$, where

$$m(i, j) := \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if no edge between } i \text{ and } j \text{ in } \Delta, \\ 3 & \text{if there is an edge } i \text{ —— } j \text{ in } \Delta, \\ 4 & \text{if there is an edge } i \xrightarrow{4} j \text{ in } \Delta. \end{cases}$$

For $w \in W_\Delta$, we denote by $\ell(w)$ the length of w.

Let Δ be a Dynkin graph, Λ the preprojective algebra and ι the Nakayama permutation of Λ. Then ι acts on an element of the Weyl group W_Δ by $\iota(w) := s_{\iota(i_1)} s_{\iota(i_2)} \cdots s_{\iota(i_k)}$ for $w = s_{i_1} s_{i_2} \cdots s_{i_k} \in W_\Delta$. We define the subgroup W_Δ^ι of W_Δ by

$$W_\Delta^\iota := \{w \in W \mid \iota(w) = w\}.$$

$\mathbb{A}_{2n-1}$: $n \relbar \cdots \relbar 2 \relbar 1 \relbar (n+1) \relbar \cdots \relbar (2n-1)$

$\mathbb{A}_{2n}$: $n \relbar \cdots \relbar 2 \relbar 1 \relbar (n+1) \relbar \cdots \relbar 2n$

$\mathbb{B}_n \ (n \geq 1)$: $1 \overset{4}{\relbar} 2 \relbar \cdots \relbar (n-1) \relbar n$

$\mathbb{D}_n \ (n \geq 4)$: $\begin{matrix} 1 \\ \\ n \end{matrix} \Big\rangle 2 \relbar 3 \relbar \cdots \relbar (n-1)$

$\mathbb{E}_n \ (n = 6, 7, 8)$: $\begin{matrix} & & 1 \\ & & | \\ 4 \relbar 3 & \relbar & 2 \relbar 5 \relbar \cdots \relbar n \end{matrix}$

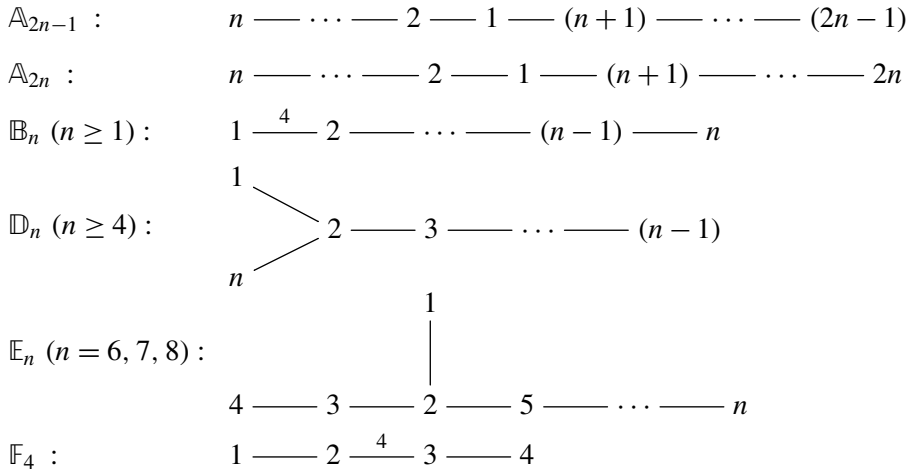$\mathbb{F}_4$: $1 \relbar 2 \overset{4}{\relbar} 3 \relbar 4$

Figure 1. Dynkin graphs with vertex labels.

Let w_0 be the longest element of W_Δ. Note that we have $w_0 w w_0 = \iota(w)$ for $w \in W_\Delta$ [Erdmann and Snashall 1998]. In particular we have $w_0 w = w w_0$ for any W_Δ^ι.

Theorem 3.1. *Let Δ be a Dynkin (ADE) graph whose vertices are given as in Figure 1 and W_Δ the Weyl group of Δ. Let Δ^{f} be a graph given by the following type.*

Δ	$\mathbb{A}_{2n-1}, \mathbb{A}_{2n}$	$\mathbb{D}_{2n}$	$\mathbb{D}_{2n+1}$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
Δ^{f}	$\mathbb{B}_n$	$\mathbb{D}_{2n}$	$\mathbb{B}_{2n}$	$\mathbb{F}_4$	$\mathbb{E}_7$	$\mathbb{E}_8$

Then we have $W_\Delta^\iota = \langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$, where

$$t_i := \begin{cases} s_i & \text{if } i = \iota(i) \text{ in } \Delta, \\ s_i s_{\iota(i)} s_i & \text{if there is an edge } i \relbar \iota(i) \text{ in } \Delta, \\ s_i s_{\iota(i)} & \text{if no edge between } i \text{ and } \iota(i) \text{ in } \Delta, \end{cases} \tag{T}$$

and W_Δ^ι is isomorphic to $W_{\Delta^{\mathrm{f}}}$.

Proof. This follows from the above property of the Nakayama permutation and [Carter 1989, Chapter 13]. □

Definition 3.2. We call the graph Δ^{f} given in Theorem 3.1 the *folded graph* of Δ.

Example 3.3. (a) Let Δ be a graph of type $\mathbb{A}_5$. Then one can check that W_Δ^ι is given by $\langle s_1, s_2 s_4, s_3 s_5 \rangle$ and this group is isomorphic to $W_{\Delta^{\mathrm{f}}}$, where Δ^{f} is a graph of type $\mathbb{B}_3$.

(b) Let Δ be a graph of type $\mathbb{A}_6$. Then one can check that W_Δ^ι is given by $\langle s_1 s_4 s_1, s_2 s_5, s_3 s_6 \rangle$ and this group is isomorphic to W_{Δ^f}, where Δ^f is a graph of type $\mathbb{B}_3$.

(c) Let Δ be a graph of type $\mathbb{D}_5$. Then one can check that W_Δ^ι is given by $\langle s_1 s_5, s_2, s_3, s_4 \rangle$ and this group is isomorphic to W_{Δ^f}, where Δ^f is a graph of type $\mathbb{B}_4$.

(d) Let Δ be a graph of type $\mathbb{E}_6$. Then one can check that W_Δ^ι is given by $\langle s_1, s_2, s_3 s_5, s_4 s_6 \rangle$ and this group is isomorphic to W_{Δ^f}, where Δ^f is a graph of type $\mathbb{F}_4$.

3C. *Support τ-tilting modules and two-term silting complexes.* In this subsection, we briefly recall the notion of support τ-tilting modules introduced in [Adachi et al. 2014], and its relationship with silting complexes. We refer to [Adachi et al. 2014; Iyama and Reiten 2014] for a background of support τ-tilting modules.

Let Λ be a finite dimensional algebra and we denote by τ the AR translation [Auslander et al. 1995].

Definition 3.4. (a) We call X in $\text{mod}\,\Lambda$ τ-*rigid* if $\text{Hom}_\Lambda(X, \tau X) = 0$.

(b) We call X in $\text{mod}\,\Lambda$ τ-*tilting* if X is τ-rigid and $|X| = |\Lambda|$, where $|X|$ denotes the number of nonisomorphic indecomposable direct summands of X.

(c) We call X in $\text{mod}\,\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e \rangle)$-module.

We can also describe these notions as pairs as follows.

(d) We call a pair (X, P) of $X \in \text{mod}\,\Lambda$ and $P \in \text{proj}\,\Lambda$ τ-*rigid* if X is τ-rigid and $\text{Hom}_\Lambda(P, X) = 0$.

(e) We call a τ-rigid pair (X, P) a *support τ-tilting* (respectively, *almost complete support τ-tilting*) pair if $|X| + |P| = |\Lambda|$ (respectively, $|X| + |P| = |\Lambda| - 1$).

We say that (X, P) is *basic* if X and P are basic, and we say that (X, P) is a *direct summand* of (X', P') if X is a direct summand of X' and P is a direct summand of P'. Note that a basic support τ-tilting module X determines a basic support τ-tilting pair (X, P) uniquely [Adachi et al. 2014, Proposition 2.3]. Hence we can identify basic support τ-tilting modules with basic support τ-tilting pairs. We denote by $s\tau$-tilt Λ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Finally we recall an important relationship between support τ-tilting modules and two-term silting complexes. We write silt $\Lambda := \text{silt}\,\mathsf{K}^\text{b}(\text{proj}\,\Lambda)$ and tilt $\Lambda := \text{tilt}\,\mathsf{K}^\text{b}(\text{proj}\,\Lambda)$ for simplicity. We denote by 2-silt Λ (respectively, 2-tilt Λ) the subset of silt Λ (respectively, tilt Λ) consisting of two-term (i.e., it is concentrated in the

degree 0 and -1) complexes. Note that a complex T is two-term if and only if $\Lambda \geq T \geq \Lambda[1]$.

Then we have the following nice correspondence.

Theorem 3.5 [Adachi et al. 2014, Theorem 3.2, Corollary 3.9]. *Let Λ be a finite dimensional algebra. There exists a bijection $\Psi : s\tau$-tilt $\Lambda \to$ 2-silt Λ,*

$$(X, P) \mapsto \Psi(X, P) := \begin{cases} \overset{-1}{P_X^1} \overset{f}{\longrightarrow} \overset{0}{P_X^0} \\ \qquad \oplus \qquad \in \mathsf{K}^{\mathsf{b}}(\mathrm{proj}\,\Lambda), \\ P \end{cases}$$

where $P_X^1 \overset{f}{\to} P_X^0 \to X \to 0$ is a minimal projective presentation of X. Moreover, it gives an isomorphism of the partially ordered sets between $s\tau$-tilt Λ and 2-silt Λ.

By the above correspondence, we can give a description of two-term silting complexes by calculating support τ-tilting modules, which is much simpler than calculations of two-term silting complexes.

4. Two-term tilting complexes and Weyl groups

In this section, we characterize 2-term tilting complexes in terms of the Weyl group. In particular, we provide a complete description of 2-term tilting complexes.

Throughout this section, let Δ be a Dynkin (ADE) graph with $\Delta_0 = \{1, \ldots, n\}$, Λ the preprojective algebra of Δ and $I_i := \Lambda(1 - e_i)\Lambda$, where e_i is the primitive idempotent of Λ associated with $i \in \Delta_0$. We denote by $\langle I_1, \ldots, I_n \rangle$ the set of ideals of Λ which can be written as

$$I_{i_1} I_{i_2} \cdots I_{i_k}$$

for some $k \geq 0$ and $i_1, \ldots, i_k \in \Delta_0$. Note that it has recently been understood that these ideals play an important role in several situations, for example [Iyama and Reiten 2008; Buan et al. 2009; Geiß et al. 2011; Oppermann et al. 2015; Baumann and Kamnitzer 2012; Baumann et al. 2014].

Then we use the following important results.

Theorem 4.1. (a) *There exists a bijection $W_\Delta \to \langle I_1, \ldots, I_n \rangle$, which is given by*
  $$w \mapsto I_w = I_{i_1} I_{i_2} \cdots I_{i_k} \text{ for any reduced expression } w = s_{i_1} \cdots s_{i_k}.$$

(b) *There exist bijections*

$$W_\Delta \to s\tau\text{-tilt}\,\Lambda \; \to 2\text{-silt}\,\Lambda,$$

$$w \mapsto (I_w, P_w) \mapsto S_w := \Psi(I_w, P_w).$$

(c) *The Weyl group W_Δ acts transitively and faithfully on 2-silt Λ by*

$$s_i \cdot (S_w) := \mu_i(S_w) \cong S_{s_i w},$$

where μ_i is the silting mutation associated with $i \in \Delta_0$.

Proof. (a) This follows from [Mizuno 2014, Theorem 2.14; Buan et al. 2009, III.1.9].

(b) This follows from [Mizuno 2014, Theorem 2.21] and Theorem 3.5.

(c) By [Mizuno 2014, Theorem 2.16], W_Δ acts transitively and faithfully on sτ-tilt Λ by mutation of support τ-tilting pairs (see [Adachi et al. 2014, Theorems 2.18, 2.28] for mutation of support τ-tilting pairs). On the other hand, [Adachi et al. 2014, Corollary 3.9] implies that the bijection (b) gives the compatibility of mutation of support τ-tilting pairs and two-term silting complexes. Hence we get the conclusion.

$\square$

Now, the aim of this section is to show the following result.

Theorem 4.2. *Let Δ be a Dynkin graph, Λ the preprojective algebra of Δ and ι the Nakayama permutation of Λ.*

(a) *Let ν be the Nakayama functor of Λ. Then $\nu(I_w) \cong I_w$ if and only if $\iota(w) = w$.*

(b) *We have a bijection*

$$W_\Delta^\iota \to 2\text{-tilt } \Lambda, \quad w \mapsto S_w.$$

(c) *Let Δ^{f} be the folded graph of Δ (Definition 3.2) and define $\langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$ by (T) of Theorem 3.1. Then $\langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$ acts transitively and faithfully on 2-tilt Λ.*

For a proof, we recall the notion of g-vectors of support τ-tilting modules. See [Mizuno 2014, Section 3; Adachi et al. 2014, Section 5] for details.

Let $K_0(\mathrm{proj}\Lambda)$ be the Grothendieck group of the additive category $\mathrm{proj}\Lambda$, which is isomorphic to the free abelian group $\mathbb{Z}^n$, and we identify the set of isomorphism classes of projective Λ-modules with the canonical basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$.

For a Λ-module X, take a minimal projective presentation

$$P_X^1 \longrightarrow P_X^0 \longrightarrow X \longrightarrow 0$$

and let $g(X) = (g_1(X), \cdots, g_n(X))^t := [P_X^0] - [P_X^1] \in \mathbb{Z}^n$. Then, for any $w \in W_\Delta$ and $i \in \Delta_0$, we define a *g-vector* by

$$\mathbb{Z}^n \ni g^i(w) = \begin{cases} g(e_i I_w) & \text{if } e_i I_w \neq 0, \\ -e_{\iota(i)} & \text{if } e_i I_w = 0. \end{cases}$$

Then we define a *g-matrix* of a support τ-tilting Λ-module I_w by

$$g(w) := (g^1(w), \ldots, g^n(w)) \in \mathrm{GL}_n(\mathbb{Z}).$$

Note that the g-vectors form a basis of $\mathbb{Z}^n$ [Adachi et al. 2014, Theorem 5.1].

On the other hand, we define a matrix $M_\iota := (e_{\iota(1)}, \ldots, e_{\iota(n)}) \in \mathrm{GL}_n(\mathbb{Z})$ and, for $X \in \mathrm{GL}_n(\mathbb{Z})$, we define

$$\iota(X) := M_\iota \cdot X \cdot M_\iota.$$

Clearly the left multiplication (respectively, right multiplication) of M_ι to X gives a permutation of X from j-th to $\iota(j)$-th rows (respectively, columns) for any $j \in \Delta_0$ and $M_\iota^2 = \mathrm{id}$.

Moreover, we recall the following definition (see [Mizuno 2014, Definition 3.5]).

Definition 4.3 [Björner and Brenti 2005]. The *contragradient* $r : W_\Delta \to \mathrm{GL}_n(\mathbb{Z})$ *of the geometric representation* is defined by

$$r(s_i)(e_j) = r_i(e_j) = \begin{cases} e_j, & i \neq j, \\ -e_i + \sum_{k - i} e_k, & i = j, \end{cases}$$

where the sum is taken over all edges of i in Δ. We regard r_i as a matrix of $\mathrm{GL}_n(\mathbb{Z})$ and this extends to a group homomorphism.

Lemma 4.4. *For any $i \in \Delta_0$, we have*

$$\iota(r_i) = r_{\iota(i)}.$$

Proof. Since the left multiplication (respectively, right multiplication) of M_ι gives a permutation of rows (respectively, columns) from j-th to $\iota(j)$-th for any $j \in \Delta_0$, this follows from the definition of r_i and $r_{\iota(i)}$. $\square$

Lemma 4.5. *For any $w \in W_\Delta$, we have*

$$\iota(g(w)) = g(\iota(w)).$$

Proof. Let $w = s_{i_1} \cdots s_{i_k}$ be an expression of w. Then, by [Mizuno 2014, Proposition 3.6], we conclude

$$g(w) = r_{i_k} \cdots r_{i_1}.$$

Hence we have

$$\begin{aligned} \iota(g(w)) &= M_\iota(r_{i_k} \ldots r_{i_1}) M_\iota \\ &= (M_\iota r_{i_k} M_\iota) \cdots (M_\iota r_{i_1} M_\iota) \quad (M_\iota^2 = \mathrm{id}) \\ &= r_{\iota(i_k)} \ldots r_{\iota(i_1)} \quad\quad\quad\ \ (\text{Lemma 4.4}) \\ &= g(\iota(w)). \end{aligned}$$ $\square$

Moreover, we give the following lemma.

Lemma 4.6. *Let $w \in W_\Delta$.*

(a) *$\nu(I_w)$ is also a support τ-tilting Λ-module. In particular, there exists some $w' \in W_\Delta$ such that $\nu(I_w) \cong I_{w'}$.*

(b) *For the above w', we have*

$$g(w') = \iota(g(w)).$$

Proof. (a) Let (I_w, P_w) be a basic support τ-tilting pair of Λ, where P_w is the corresponding projective Λ-module. By Theorem 3.5, we have the two-term silting complex in $\mathsf{K}^{\mathsf{b}}(\mathrm{proj}\Lambda)$ by

$$S_w := (P^1_{I_w} \xrightarrow{f} P^0_{I_w}) \oplus P_w[1] \in \mathsf{K}^{\mathsf{b}}(\mathrm{proj}\Lambda),$$

where

$$P^1_{I_w} \xrightarrow{\;f\;} P^0_{I_w} \longrightarrow I_w \longrightarrow 0$$

is a minimal projective presentation of I_w.

Then $\nu(S_w) = (\nu(P^1_{I_w}) \to \nu(P^0_{I_w})) \oplus \nu(P_w)[1] \in \mathsf{K}^{\mathsf{b}}(\mathrm{proj}\Lambda)$ is clearly a two-term silting complex. Hence, by Theorem 3.5, $(\nu(I_w), \nu(P_w))$ is also a basic support τ-tilting pair of Λ. Thus, by Theorem 4.1, there exists $w' \in W_\Delta$ such that $\nu(I_w) \cong I_{w'}$.

(b) Take $i \in \Delta_0$. First assume that $e_i I_w \neq 0$ and take a minimal projective presentation of $e_i I_w$

$$P^1 \to P^0 \to e_i I_w \to 0.$$

By applying ν to this sequence, we have

$$\nu(P^1) \to \nu(P^0) \to \nu(e_i I_w) \to 0.$$

Because $[\nu(e_j \Lambda)] = [e_{\iota(j)}\Lambda] = M_\iota[e_j\Lambda]$ for any $j \in \Delta_0$, we have

$$[(\nu(P^0)] - [(\nu(P^1)] = M_\iota([P^0] - [P^1]) = M_\iota(g^i(w)).$$

Then, since we have $\nu(e_i I_w) \cong e_{\iota(i)} I_{w'}$, we obtain $g^{\iota(i)}(w') = M_\iota(g^i(w))$.

Next assume that $e_i I_w = 0$. Then we have $g^i(w) = -e_{\iota(i)}$ by the definition. Because $\nu(e_j\Lambda) \cong e_{\iota(j)}\Lambda$ for any $j \in \Delta_0$, we obtain $g^{\iota(i)}(w') = -e_i = M_\iota(g^i(w))$.

Consequently, we have

$$
\begin{aligned}
g(w') &= (g^1(w'), \ldots, g^n(w'))\\
&= (g^{\iota(1)}(w'), \ldots, g^{\iota(n)}(w')) \cdot M_\iota\\
&= (M_\iota(g^1(w)), \ldots, M_\iota(g^n(w))) \cdot M_\iota\\
&= M_\iota \cdot (g^1(w), \ldots, g^n(w)) \cdot M_\iota\\
&= \iota(g(w)).
\end{aligned}
$$

This finishes the proof. $\square$

For the proof of Theorem 4.2 we recall the following nice property.

Theorem 4.7 [Adachi et al. 2014, Theorem 5.5]. *The map $X \to g(X)$ induces an injection from the set of isomorphism classes of τ-rigid pairs for Λ to $K_0(\mathrm{proj}\Lambda)$.*

Proof of Theorem 4.2. (a) We have the following equivalent conditions

$$\nu(I_w) \cong I_w \Leftrightarrow \iota(g(w)) = g(w) \quad \text{(Lemma 4.6 and Theorem 4.7)}$$
$$\Leftrightarrow g(\iota(w)) = g(w) \quad \text{(Lemma 4.5)}$$
$$\Leftrightarrow I_{\iota(w)} \cong I_w \quad \text{(Theorem 4.7)}$$
$$\Leftrightarrow \iota(w) = w \quad \text{(Theorem 4.1)}.$$

Thus we get the desired result.

(b) A silting complex S_w is a tilting complex if and only if $\nu(S_w) \cong S_w$ (see [Aihara 2013, Appendix]). Hence (a) implies that it is equivalent to say that $\iota(w) = w$. This proves our claim.

(c) By (b) and Theorem 3.1, the action of Theorem 4.1 induces the action of $\langle t_i \mid i \in \Delta_0^{\mathrm{f}} \rangle$ on 2-tilt Λ.   □

Example 4.8. Let Δ be a graph of type $\mathbb{A}_3$ and Λ the preprojective algebra of Δ. Then the *support τ-tilting quiver* of Λ [Adachi et al. 2014, Definition 2.29] is given in Figure 2.

The framed modules indicate ν-*stable* modules [Mizuno 2015] (i.e., $I_w \cong \nu(I_w)$), which is equivalent to say that $\iota(w) = w$. Hence Theorems 3.1 and 4.2 imply that these modules are in bijection with the elements of the subgroup $W_\Delta^\iota = \langle s_1 s_3, s_2 \rangle$ and this group is isomorphic to the Weyl group of type $\mathbb{B}_2$.

5. Preprojective algebras are tilting-discrete

In this section, we show that preprojective algebras of Dynkin type are tilting-discrete. It implies that all tilting complexes are connected to each other by successive tilting mutation [Chan et al. 2015, Theorem 5.14; Aihara 2013, Theorem 3.5]. From this result, we can determine the derived equivalence class of the algebra.

Throughout this section, let Δ be a Dynkin graph with $\Delta_0 = \{1, \ldots, n\}$, Λ the preprojective algebra of Δ, e_i the primitive idempotent of Λ associated with $i \in \Delta_0$ and Δ^{f} the folded graph of Δ. We also keep the notation of previous sections.

The aim of this section is to show the following theorem.

Theorem 5.1. *Let Λ be a preprojective algebra of Dynkin type.*

 (a) $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ *is tilting-discrete.*

 (b) *Any basic tilting complex T of Λ satisfies $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)}(T) \cong \Lambda$. In particular, the derived equivalence class coincides with the Morita equivalence class.*

Notation. Let $\widetilde{\Delta}$ be an extended Dynkin graph obtained from Δ by adding a vertex 0 (i.e., $\widetilde{\Delta}_0 = \{0\} \cup \Delta_0$) with the associated arrows. Since

$$W_\Delta = \langle s_1, \ldots, s_n \rangle \subset W_{\widetilde{\Delta}} = \langle s_1, \ldots, s_n, s_0 \rangle,$$
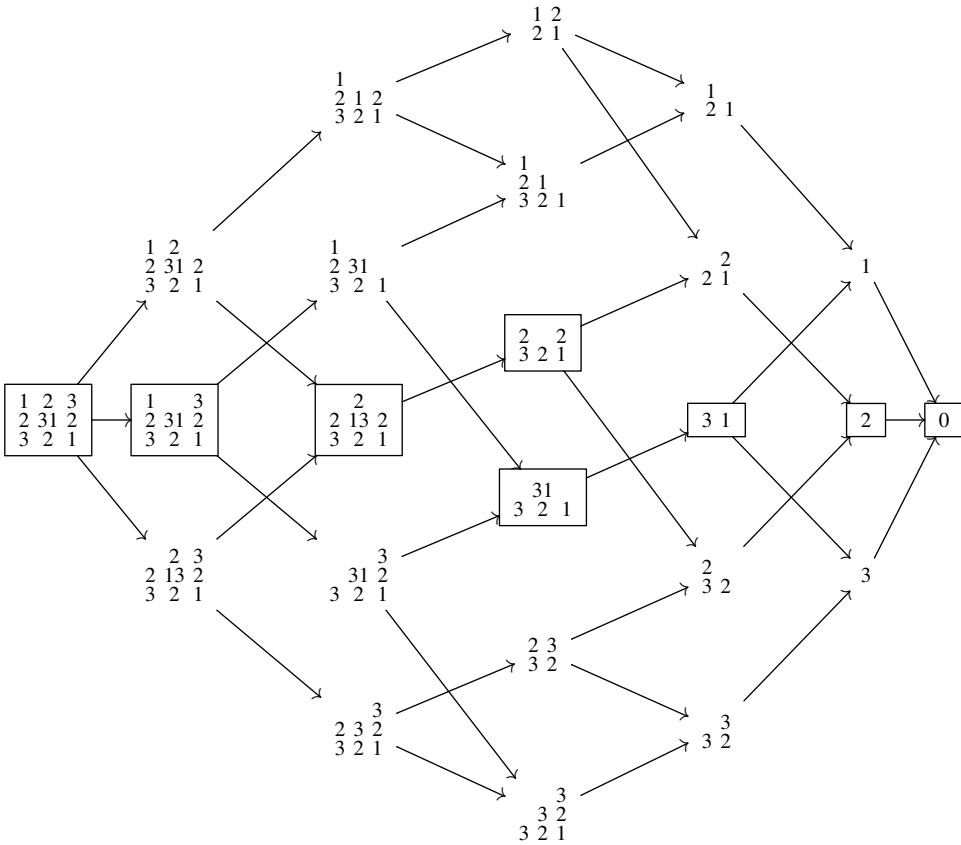
Figure 2. Diagram for Example 4.8.

we can regard elements of W_Δ as those of $W_{\widetilde{\Delta}}$. We denote by $\widetilde{\Lambda}$ the $\mathfrak{m}$-adic completion of the preprojective algebra of $\widetilde{\Delta}$, where $\mathfrak{m}$ is the ideal generated by all arrows. It implies that the Krull–Schmidt theorem holds for finitely generated projective $\widetilde{\Lambda}$-modules. Moreover we denote $\tilde{I}_i := \widetilde{\Lambda}(1 - e_i)\widetilde{\Lambda}$, where e_i is the primitive idempotent of $\widetilde{\Lambda}$ associated with $i \in \widetilde{\Delta}_0$.

Recall that, by Theorem 4.1, we have a bijection between $W_{\widetilde{\Delta}}$ and $\langle \tilde{I}_1, \ldots, \tilde{I}_n, \tilde{I}_0 \rangle$ [Buan et al. 2009, III.1.9] and hence for each element $w \in W_{\widetilde{\Delta}}$, we can define $\tilde{I}_w := \tilde{I}_{i_1} \cdots \tilde{I}_{i_k}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression. Furthermore, it is shown that $\tilde{I}_w$ is a tilting $\widetilde{\Lambda}$-module [Buan et al. 2009, Theorem III.1.6].

Note that if $i \neq 0 \in \widetilde{\Delta}_0$, then we have

$$\Lambda = \widetilde{\Lambda}/\langle e_0 \rangle \quad \text{and} \quad I_i = \tilde{I}_i/\langle e_0 \rangle.$$

In particular, for $w \in W_\Delta$, we have $\tilde{I}_w/\langle e_0 \rangle = I_w$ and hence $\widetilde{\Lambda}/\tilde{I}_w \cong \Lambda/I_w$.

Recall that we can describe the two-term silting complex of $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ by

$$S_w := \begin{cases} P^1_{I_w} \xrightarrow{f} P^0_{I_w} \\ \qquad\quad \oplus \\ P_w \end{cases}$$

where $P^1_{I_w} \xrightarrow{f} P^0_{I_w} \to I_w \to 0$ is a minimal projective presentation of I_w.

Then we show that $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda$ gives a two-term silting complex S_w.

Proposition 5.2. *For $w \in W_\Delta$, $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda$ is isomorphic to S_w in $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$.*

Proof. Since $\tilde{I}_w$ is a tilting $\tilde{\Lambda}$-module, we have a minimal projective resolution

$$0 \longrightarrow \tilde{P}_1 \xrightarrow{g} \tilde{P}_0 \longrightarrow \tilde{I}_w \longrightarrow 0.$$

By applying the functor $- \otimes_{\tilde{\Lambda}} \Lambda$, we have the following exact sequence [Mizuno 2014, Proposition 3.2]

$$0 \to \nu^{-1}(\Lambda/I_w) \to \tilde{P}_1 \otimes_{\tilde{\Lambda}} \Lambda \xrightarrow{g \otimes \Lambda} \tilde{P}_0 \otimes_{\tilde{\Lambda}} \Lambda \to \tilde{I}_w \otimes_{\tilde{\Lambda}} \Lambda \to 0.$$

Because we have an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\tilde{\Lambda})$

$$\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda \cong \left(\cdots \to 0 \to \tilde{P}_1 \otimes_{\tilde{\Lambda}} \Lambda \xrightarrow{g \otimes \Lambda} \tilde{P}_0 \otimes_{\tilde{\Lambda}} \Lambda \to 0 \to \cdots \right),$$

one can check that $\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \Lambda$ is isomorphic to S_w (Theorem 3.5). $\qquad\square$

For $w \in W_\Delta$, we denote the inclusion by $\boldsymbol{i} : \tilde{I}_w \hookrightarrow \tilde{\Lambda}$. Then we show the following lemma.

Lemma 5.3. *Let w_0 be the longest element of W_Δ. For $w \in W^\iota_\Delta$, we have isomorphisms $p : \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0} \to \tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w$ and $q : \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \to \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w$, which make the following diagram commutative*

$$
\begin{array}{ccc}
\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0} & \xrightarrow{\mathrm{id} \otimes \boldsymbol{i}} & \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \\
\cong \downarrow p & & \cong \downarrow q \\
\tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w & \xrightarrow{\boldsymbol{i} \otimes \mathrm{id}} & \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w
\end{array}
$$

Proof. Because $\ell(w_0 w^{-1}) + \ell(w) = \ell(w_0)$, [Buan et al. 2009, Propositions II.1.5(a), II.1.10] gives the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{I}_{w_0} & \xrightarrow{\quad \boldsymbol{i} \quad} & \tilde{\Lambda} \\
\cong \uparrow & & \cong \downarrow \\
\tilde{I}_{w_0 w^{-1}} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w & \xrightarrow{\boldsymbol{i} \otimes \boldsymbol{i}} & \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda}
\end{array}
$$

Hence we have

$$
\begin{array}{ccc}
\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0} & \xrightarrow{\text{id} \otimes \boldsymbol{i}} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \xrightarrow{\boldsymbol{i} \otimes \text{id}} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \\
\cong \big\uparrow & \qquad\qquad\qquad \cong \big\downarrow \\
\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w_0 w^{-1}} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w & \xrightarrow{\quad \boldsymbol{i} \otimes \boldsymbol{i} \otimes \boldsymbol{i} \quad} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda}
\end{array}
$$

Since $w \in W^l_\Delta$, we have $w_0 w = w w_0$ (Section 3B) and hence $\tilde{I}_{w_0 w^{-1}} = \tilde{I}_{w^{-1} w_0}$. Then similarly we have the following commutative diagram

$$
\begin{array}{ccc}
\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w^{-1} w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w & \xrightarrow{\quad \boldsymbol{i} \otimes \boldsymbol{i} \otimes \boldsymbol{i} \quad} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \\
\cong \big\downarrow & \qquad\qquad\qquad \cong \big\uparrow \\
\tilde{I}_{w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w & \xrightarrow{\boldsymbol{i} \otimes \text{id}} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \xrightarrow{\text{id} \otimes \boldsymbol{i}} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda}
\end{array}
$$

Moreover we have the following commutative diagram

$$
\begin{array}{ccc}
\tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} & \xrightarrow{\quad \boldsymbol{i} \otimes \text{id} \quad} & \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \\
\cong \big\downarrow & & \cong \big\downarrow \\
\tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} & \xrightarrow{\text{id} \otimes \boldsymbol{i} \otimes \text{id}} & \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda} \\
\cong \big\uparrow & & \cong \big\uparrow \\
\tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w & \xrightarrow{\quad \text{id} \otimes \boldsymbol{i} \quad} & \tilde{\Lambda} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{\Lambda}
\end{array}
$$

Put $L := \tilde{I}_w \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_{w^{-1} w_0} \otimes^{\boldsymbol{L}}_{\tilde{\Lambda}} \tilde{I}_w$. Consider a morphism $u : L \to \tilde{I}_w$ and the triangle

$$
\cdots \longrightarrow \tilde{\Lambda}/\tilde{I}_w[-1] \longrightarrow \tilde{I}_w \xrightarrow{\ \boldsymbol{i}\ } \tilde{\Lambda} \longrightarrow \tilde{\Lambda}/\tilde{I}_w \longrightarrow \cdots.
$$

If $\boldsymbol{i} \circ u = 0$, then there exists a map $v : L \to \tilde{\Lambda}/\tilde{I}_w[-1]$ which makes commutative the diagram

$$
\begin{array}{ccc}
& & L \\
& \swarrow^{v} & \big\downarrow^{u} \\
\cdots \longrightarrow \tilde{\Lambda}/\tilde{I}_w[-1] \longrightarrow & \tilde{I}_w & \xrightarrow{\ \boldsymbol{i}\ } \tilde{\Lambda} \longrightarrow \tilde{\Lambda}/\tilde{I}_w \longrightarrow \cdots
\end{array}
$$

Because $H^i(L) = 0$ for any $i > 0$, we get $v = 0$ and hence $u = 0$. Thus the above diagrams provide the required morphisms. □

From the above results, we have the following nice consequence.

Proposition 5.4. *For any $w \in W_\Delta^\iota$, we have an isomorphism*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(\tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \Lambda) \cong \Lambda.$$

In particular, the endomorphism algebra of any basic two-term tilting complex is isomorphic to Λ.

Proof. Let w_0 be the longest element of W_Δ. Since $\tilde{I}_{w_0} = \langle e_0 \rangle$, we have the following exact sequence

$$0 \longrightarrow \tilde{I}_{w_0} \longrightarrow \tilde{\Lambda} \longrightarrow \Lambda \longrightarrow 0.$$

Then applying the functor $\tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} -$ to the exact sequence, we have the triangle

$$\tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_{w_0} \longrightarrow \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{\Lambda} \longrightarrow \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \Lambda \longrightarrow \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_{w_0}[1].$$

Similarly, applying the functor $- \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w$ to the first exact sequence, we have the triangle

$$\tilde{I}_{w_0} \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w \longrightarrow \tilde{\Lambda} \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w \longrightarrow \Lambda \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w \longrightarrow \tilde{I}_{w_0} \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w[1].$$

By Lemma 5.3, we have the following commutative diagram

$$
\begin{array}{ccccccc}
\tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_{w_0} & \longrightarrow & \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{\Lambda} & \longrightarrow & \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \Lambda & \longrightarrow & \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_{w_0}[1] \\
\cong \downarrow p & & \cong \downarrow q & & \downarrow r & & \downarrow \\
\tilde{I}_{w_0} \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w & \longrightarrow & \tilde{\Lambda} \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w & \longrightarrow & \Lambda \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w & \longrightarrow & \tilde{I}_{w_0} \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w[1]
\end{array}
$$

and the isomorphism r. Because $\tilde{I}_w$ is a tilting module [Buan et al. 2009, Theorem III.1.6] and we have $\tilde{\Lambda} \cong \mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \tilde{I}_w)$ [Buan et al. 2009, Proposition II.1.4], we obtain

$$\mathbf{R}\mathrm{Hom}_\Lambda(\tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \Lambda, \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \Lambda) \cong \mathbf{R}\mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \tilde{I}_w \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \Lambda)$$
$$\cong \mathbf{R}\mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \Lambda \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{I}_w)$$
$$\cong \Lambda \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \mathbf{R}\mathrm{Hom}_{\tilde{\Lambda}}(\tilde{I}_w, \tilde{I}_w)$$
$$\cong \Lambda \otimes_{\tilde{\Lambda}}^{\boldsymbol{L}} \tilde{\Lambda}$$
$$\cong \Lambda.$$

Then by taking the 0th part, we get the assertion. The second statement immediately follows from the first one, Theorem 4.2 and Proposition 5.2. $\qquad\square$

Corollary 5.5. *Let T be a tilting complex which is given by iterated irreducible left tilting mutation from Λ. Then we have*

$$\operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(T) \cong \Lambda.$$

Proof. Let $T = \boldsymbol{\mu}^{+}_{(\ell)} \circ \cdots \circ \boldsymbol{\mu}^{+}_{(1)}(\Lambda)$, where $\boldsymbol{\mu}$ denotes irreducible left tilting mutation. We proceed by induction on ℓ. Assume that, for $T' = \boldsymbol{\mu}^{+}_{(\ell-1)} \circ \cdots \circ \boldsymbol{\mu}^{+}_{(1)}(\Lambda)$, we have $\operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(T') \cong \Lambda$. Then we have an equivalence $F : \mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda) \to \mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)$ such that $F(T') \cong \Lambda$ [Rickard 1989]. Therefore we have $\operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(\boldsymbol{\mu}^{+}_{(\ell)}(T')) \cong \operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(\boldsymbol{\mu}^{+}_{(\ell)}(\Lambda))$ and hence it is isomorphic to Λ by Proposition 5.4. $\qquad\square$

Now we are ready to give a proof of Theorem 5.1.

Proof of Theorem 5.1. (a) We will check the condition (c) of Corollary 2.11.

Recall that 2-tilt$_T$ $\Lambda := \{U \in \operatorname{tilt}\Lambda \mid T \geq U \geq T[1]\}$. We denote by $\sharp$ 2-tilt$_T$ Λ the number of 2-tilt$_T$ Λ.

By Theorem 4.2, the set 2-tilt$_\Lambda$ $\Lambda =$ 2-tilt Λ is finite. Let T be a tilting complex which is given by iterated irreducible left tilting mutation from Λ. Then we have $\operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(T) \cong \Lambda$ from Corollary 5.5. Therefore, we have an equivalence

$$F : \mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda) \to \mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)$$

such that $F(T) \cong \Lambda$ and hence we get

$$\sharp\bigl\{U \in \operatorname{tilt}\Lambda \,\bigm|\, T \geq U \geq T[1]\bigr\} = \sharp\bigl\{F(U) \in \operatorname{tilt}\Lambda \,\bigm|\, \Lambda \geq F(U) \geq \Lambda[1]\bigr\}.$$

Thus it is also finite and we obtain the statement.

(b) Let T be a basic tilting complex such that $\Lambda \geq T$. Since Λ is tilting-discrete, T is obtained by iterated irreducible left tilting mutation from Λ [Chan et al. 2015, Theorem 5.11; Aihara 2013, Theorem 3.5]. Thus the statement follows from Corollary 5.5. Because for any tilting complex T, we have $\Lambda \geq T[\ell]$ for some ℓ [Aihara and Iyama 2012, Proposition 2.4] and $\operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(T) \cong \operatorname{End}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)}(T[\ell])$, we get the conclusion from the above argument. $\qquad\square$

6. Tilting complexes and braid groups

In this section, we show that irreducible mutation satisfy the braid relations and we give a bijection from the elements of the braid group to the set of tilting complexes.

We keep the notation of previous sections.

Define $W^\iota_\Delta = \langle t_i \mid i \in \Delta^{\mathsf{f}}_0 \rangle$ by (T) of Theorem 3.1. By Theorems 4.1 and 4.2, we have $S_{t_i} = \boldsymbol{\mu}^{+}_i(\Lambda)$ $(i \in \Delta^{\mathsf{f}}_0)$ in $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)$, where $\boldsymbol{\mu}^{+}_i$ is given as a composition of

left silting mutation as follows

$$\boldsymbol{\mu}_i^+ := \begin{cases} \mu_i^+ & \text{if } i = \iota(i) \text{ in } \Delta, \\ \mu_i^+ \circ \mu_{\iota(i)}^+ \circ \mu_i^+ & \text{if there is an edge } i \,\text{---}\, \iota(i) \text{ in } \Delta, \\ \mu_i^+ \circ \mu_{\iota(i)}^+ & \text{if there is no edge between } i \text{ and } \iota(i) \text{ in } \Delta. \end{cases}$$

Moreover, we let

$$e_{t_i} := \begin{cases} e_i & \text{if } i = \iota(i) \text{ in } \Delta, \\ e_i + e_{\iota(i)} & \text{if } i \neq \iota(i) \text{ in } \Delta. \end{cases}$$

Then, it is easy to check that $\boldsymbol{\mu}_i^+(\Lambda) = \mu_{(e_{t_i}\Lambda)}^+(\Lambda)$ and hence we have

$$S_{t_i} = \begin{cases} e_{t_i}\Lambda \overset{f}{\xrightarrow{}} R_{t_i} \\ \qquad\quad \oplus \\ \qquad (1 - e_{t_i})\Lambda \end{cases} \in \mathsf{K}^\mathsf{b}(\mathsf{proj}\,\Lambda),$$

where f is a minimal left $(\mathsf{add}((1 - e_{t_i})\Lambda))$-approximation.

Thus $\boldsymbol{\mu}_i^+$ is an irreducible left tilting mutation of Λ and any irreducible left tilting mutation of Λ is given as $\boldsymbol{\mu}_i^+$ for some $i \in \Delta_0^\mathsf{f}$. Dually, we define $\boldsymbol{\mu}_i^-$ so that $\boldsymbol{\mu}_i^- \circ \boldsymbol{\mu}_i^+ = \mathrm{id}$ [Aihara and Iyama 2012, Proposition 2.33].

Let F_{Δ^f} be the free group generated by a_i ($i \in \Delta_0^\mathsf{f}$). Then we define the map

$$F_{\Delta^\mathsf{f}} \to \mathsf{tilt}\,\Lambda,$$

$$a = a_{i_1}^{\epsilon_{i_1}} \cdots a_{i_k}^{\epsilon_{i_k}} \mapsto \boldsymbol{\mu}_a(\Lambda) := \boldsymbol{\mu}_{i_1}^{\epsilon_{i_1}} \circ \cdots \circ \boldsymbol{\mu}_{i_k}^{\epsilon_{i_k}}(\Lambda).$$

Then we give the following proposition.

Proposition 6.1. *For any $a \in F_{\Delta^\mathsf{f}}$, we let $T := \boldsymbol{\mu}_a(\Lambda)$. Then we have the following braid relations in $\mathsf{D}^\mathsf{b}(\mathsf{mod}\,\Lambda)$:*

$$\boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+(T) \cong \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+(T) \qquad\qquad \textit{if } \nexists \textit{ an edge between } i \textit{ and } j \textit{ in } \Delta^\mathsf{f},$$

$$\boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+(T) \cong \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+(T) \qquad\qquad \textit{if } \exists \textit{ an edge } i \,\text{---}\, j \textit{ in } \Delta^\mathsf{f},$$

$$\boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+(T) \cong \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+ \circ \boldsymbol{\mu}_j^+ \circ \boldsymbol{\mu}_i^+(T) \quad \textit{if } \exists \textit{ an edge } i \xrightarrow{\; 4 \;} j \textit{ in } \Delta^\mathsf{f}.$$

Proof. By Theorem 4.2, the assertion holds for $T = \Lambda$. Moreover, by Theorem 5.1, T satisfies $\mathrm{End}_{\mathsf{K}^\mathsf{b}(\mathsf{proj}\,\Lambda)}(T) \cong \Lambda$ and hence we have an equivalence $F : \mathsf{K}^\mathsf{b}(\mathsf{proj}\,\Lambda) \to \mathsf{K}^\mathsf{b}(\mathsf{proj}\,\Lambda)$ such that $F(T) \cong \Lambda$. Since mutation is preserved by an equivalence, the assertion holds for T. $\qquad\square$

Now we recall the following definition.

Definition 6.2. The braid group B_{Δ^f} is defined by generators a_i ($i \in \Delta_0^\mathsf{f}$) and relations $(a_i a_j)^{m(i,j)} = 1$ for $i \neq j$ (i.e., the difference with W_{Δ^f} is that we do not require the relations $a_i^2 = 1$ for $i \in \Delta_0^\mathsf{f}$). Moreover we denote the positive braid monoid by $B_{\Delta^\mathsf{f}}^+$.

As a consequence of the above results, we have the following proposition.

Proposition 6.3. *There is a map*

$$B_{\Delta^f} \to \mathrm{tilt}\,\Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda).$$

Moreover, it is surjective.

Proof. The first statement follows from Proposition 6.1. Since Λ is tilting-discrete, any tilting complex can be obtained from Λ by iterated irreducible tilting mutation [Chan et al. 2015, Theorem 5.11; Aihara and Iyama 2012, Theorem 3.5]. Thus the map is surjective. $\qquad\square$

Finally, we will show that the map of Proposition 6.3 is injective.

Recall that $T > \boldsymbol{\mu}_a(T)$ for any $a \in B_{\Delta^f}^+$ (Definition 2.3). Then we have the following result.

Lemma 6.4. *The map*

$$B_{\Delta^f}^+ \to \mathrm{tilt}\,\Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda)$$

is injective.

Proof. We denote by $\ell(a)$ the length of $a \in B_{\Delta^f}^+$, that is, the number of elements of the expression a. We show by induction on the length of $B_{\Delta^f}^+$. Take $b, c \in B_{\Delta^f}^+$ such that $\boldsymbol{\mu}_b(\Lambda) \cong \boldsymbol{\mu}_c(\Lambda)$ in $\mathsf{D}^b(\mathrm{mod}\,\Lambda)$. Without loss of generality, we can assume that $\ell(b) \le \ell(c)$.

If $\ell(b) = 0$, (or equivalently, $b = \mathrm{id}$), then $\boldsymbol{\mu}_b(\Lambda) = \Lambda$. Then we have $c = \mathrm{id}$ because otherwise $\Lambda > \boldsymbol{\mu}_c(\Lambda)$.

Next assume that $\ell(b) > 0$ and the statement holds for any element if the length is less than $\ell(b)$. We write $b = b'a_i$ and $c = c'a_j$ for some $b', c' \in B_{\Delta^f}^+$ and $i, j \in \Delta_0^f$. If $i = j$, then $\boldsymbol{\mu}_{b'}(\Lambda) \cong \boldsymbol{\mu}_{c'}(\Lambda)$ and the induction hypothesis implies that $b' = c'$ and hence $b = c$.

Hence assume that $i \ne j$. Then we define

$$a_{i,j} := \begin{cases} a_i a_j & \text{if no edge between } i \text{ and } j \text{ in } \Delta^f, \\ a_i a_j a_i & \text{if there is an edge } i \underline{\quad} j \text{ in } \Delta^f, \\ a_i a_j a_i a_j & \text{if there is an edge } i \overset{4}{\underline{\quad}} j \text{ in } \Delta^f. \end{cases}$$

Then $\boldsymbol{\mu}_{a_{i,j}}(\Lambda)$ is a meet of $\boldsymbol{\mu}_{a_i}(\Lambda)$ and $\boldsymbol{\mu}_{a_j}(\Lambda)$ by Theorem 4.2, [Mizuno 2014, Theorem 2.30] and [Adachi et al. 2014, Corollary 3.9]. Therefore we get $\boldsymbol{\mu}_{a_{i,j}}(\Lambda) \ge \boldsymbol{\mu}_b(\Lambda)$ since $\boldsymbol{\mu}_{a_i}(\Lambda) \ge \boldsymbol{\mu}_b(\Lambda)$ and $\boldsymbol{\mu}_{a_j}(\Lambda) \ge \boldsymbol{\mu}_c(\Lambda) \cong \boldsymbol{\mu}_b(\Lambda)$.

Because Λ is tilting-discrete and $\Lambda > \boldsymbol{\mu}_{a_{i,j}}(\Lambda)$, there exists $d \in B_{\Delta^f}^+$ such that $\boldsymbol{\mu}_d(\boldsymbol{\mu}_{a_{i,j}}(\Lambda)) = \boldsymbol{\mu}_{da_{i,j}}(\Lambda) \cong \boldsymbol{\mu}_b(\Lambda)$. Then we have $\boldsymbol{\mu}_{da_{i,j}a_i^{-1}}(\Lambda) \cong \boldsymbol{\mu}_{b'}(\Lambda)$. Since we have $da_{i,j}a_i^{-1} \in B_{\Delta^f}^+$, the induction hypothesis implies that $da_{i,j}a_i^{-1} = b'$ and hence $da_{i,j} = b$. Similarly, we have $\boldsymbol{\mu}_{da_{i,j}a_j^{-1}}(\Lambda) \cong \boldsymbol{\mu}_{c'}(\Lambda)$ and we get $da_{i,j}a_j^{-1} = c'$. Therefore, we get $b = da_{i,j} = c'a_j = c$ and the assertion holds. $\qquad\square$

As an immediate consequence, we obtain the following result (cf. [Brav and Thomas 2011, Lemma 2.3]).

Proposition 6.5. *The map*

$$B_{\Delta^f} \to \text{tilt}\,\Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda)$$

is injective.

Proof. It is enough to show that $\boldsymbol{\mu}_a(\Lambda) \cong \Lambda$ in $\mathsf{D}^b(\text{mod}\,\Lambda)$ implies $a = \text{id}$. In fact, $\boldsymbol{\mu}_a(\Lambda) \cong \boldsymbol{\mu}_{a'}(\Lambda)$ implies $\boldsymbol{\mu}_{aa'^{-1}}(\Lambda) \cong \Lambda$. Then if $aa'^{-1} = \text{id}$, then we get $a = a'$.

It is well-known that any element $a \in B_{\Delta^f}$ is given by $a = b^{-1}c$ for some $b, c \in B_{\Delta^f}^+$ [Kassel and Turaev 2008, Section 6.6]. Hence, $\boldsymbol{\mu}_a(\Lambda) \cong \Lambda$ is equivalent to saying that $\boldsymbol{\mu}_{b^{-1}c}(\Lambda) \cong \Lambda$. Then we have $\boldsymbol{\mu}_b(\Lambda) \cong \boldsymbol{\mu}_c(\Lambda)$ and Lemma 6.4 implies $b = c$. Thus we get the assertion. $\square$

Consequently, we obtain the following conclusion.

Theorem 6.6. *There is a bijection*

$$B_{\Delta^f} \to \text{tilt}\,\Lambda, \quad a \mapsto \boldsymbol{\mu}_a(\Lambda).$$

Proof. The statement follows from Propositions 6.3 and 6.5. $\square$

Acknowledgements

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