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On Iwasawa theory, zeta elements for $\mathbb{G}_m$, and the equivariant Tamagawa number conjecture

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We develop an explicit "higher-rank" Iwasawa theory for zeta elements associated to the multiplicative group over abelian extensions of number fields. We show this theory leads to a concrete new strategy for proving special cases of the equivariant Tamagawa number conjecture and, as a first application of this approach, we prove new cases of the conjecture over natural families of abelian CM-extensions of totally real fields for which the relevant p-adic L-functions possess trivial zeroes.

1. Introduction

The "Tamagawa number conjecture" of Bloch and Kato [1990] concerns the special values of motivic L-functions and has had a pivotal influence on the development of arithmetic geometry.

Nevertheless, in any situation in which a semisimple algebra acts on a motive it is natural to search for an "equivariant" refinement of this conjecture that takes account, in some way, of the additional symmetries that arise in such cases.

The first such refinement was formulated by Kato [1993a; 1993b] (in the setting of abelian extensions of number fields, and modulo certain delicate sign ambiguities) by using determinant functors, and a definitive statement of the "equivariant Tamagawa number conjecture" (or eTNC for short in the remainder of this introduction) was subsequently given in [Burns and Flach 2001] by using virtual objects and relative algebraic K-theory.

It has since been shown that the eTNC specializes to give refined versions of most, if not all, of the important conjectures related to special values of motivic L-values that are studied in the literature and it is by now widely accepted that it

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provides a "universal" approach to the formulation of the strongest possible versions of such conjectures.

In this direction, we used the framework of the eTNC in our earlier article [Burns et al. 2016a] — hereafter abbreviated [BKS] — to develop a very general approach to the theory of abelian Stark conjectures that was principally concerned with the properties of canonical "zeta elements" and "Selmer groups" that one can naturally associate to the multiplicative group $\mathbb{G}_m$ over finite abelian extensions of number fields.

In this way we derived, amongst other things, several new and concrete results on the relevant case of the eTNC, the formulation, and in some interesting cases proof of precise conjectural families of fine integral congruence relations between Rubin–Stark elements of different ranks and detailed information on the Galois module structures of both ideal class groups and Selmer groups.

The purpose of the current article is now to develop an explicit Iwasawa theory for the zeta elements introduced in [BKS], to use this theory to derive a new approach to proving special cases of the eTNC, and finally to demonstrate the usefulness of this approach by using it to prove the conjecture in important new cases.

In the next two subsections we discuss briefly the main results that we obtain.

1A. ***Iwasawa main conjectures for general number fields.*** The first key aspect of our approach is the formulation of an explicit main conjecture of Iwasawa theory for abelian extensions of *general* number fields (we refer to this conjecture as a "higher-rank main conjecture" since the rank of any associated Euler system would in most cases be greater than one).

To give a little more detail, we fix a finite abelian extension K/k of general number fields and a $\mathbb{Z}_p$-extension k_∞ of k and set $K_\infty = Kk_\infty$. In this introduction, we suppose that k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, but this is only for simplicity.

Then our higher-rank main conjecture asserts the existence of an Iwasawa-theoretic zeta element that plays the role of p-adic L-functions for general number fields and has precisely prescribed interpolation properties in terms of the values at zero of the higher derivatives of abelian L-series. (For details, see Conjecture 3.1).

Modulo a natural hypothesis on μ-invariants, this conjecture can be reformulated in a more classical style as an equality between the characteristic ideals of a canonical Selmer module and of the quotient of a natural Rubin lattice of unit groups modulo the subgroup generated by the Rubin–Stark elements (see Conjecture 3.14 and Proposition 3.15). In this way it becomes clear that the higher-rank main conjecture extends classical main conjectures.

1B. ***Rubin–Stark congruences and the eTNC.*** It is also clear that the higher-rank main conjecture does not itself imply the validity of the p-part of the eTNC (as stated in Conjecture 2.3 below) and is much weaker than the type of main conjecture formulated by Fukaya and Kato [2006]. For example, if any p-adic place of k splits completely in K, then our conjectural Rubin–Stark element encodes no information at all concerning the L-values of characters of $\mathrm{Gal}(K/k)$.

To overcome this deficiency, we make a detailed Iwasawa-theoretic study of the fine congruence relations between Rubin–Stark elements of *differing ranks* that were independently formulated for finite abelian extensions in [Mazur and Rubin 2016] (where the congruences are referred to as a "refined class number formula for $\mathbb{G}_m$") and in [Sano 2014]. In this way we are led to conjecture a precise family of "Iwasawa-theoretic Rubin–Stark congruences" for K_∞/k which, roughly speaking, describes the link between the natural Rubin–Stark elements for K_∞/k and for K/k. (For full details see Conjectures 4.1 and 4.2).

To better understand the context of this conjectural family of congruences we prove in Theorem 4.9 that it constitutes a natural extension to general number fields of the "Gross–Stark conjecture" that was formulated in [Gross 1982] for CM extensions of totally real fields and has since been much studied in the literature.

We can now state one of the main results of the present article (for a more detailed statement see Theorem 5.2).

Theorem 1.1. *If each of the following conjectures is valid for K_∞/k, then the p-component of the eTNC* (see Conjecture 2.3) *is valid for every finite subextension of K_∞/k*:

- *The higher-rank Iwasawa main conjecture* (Conjecture 3.1).
- *The Iwawasa-theoretic Rubin–Stark congruences* (Conjecture 4.2).
- *Gross's finiteness conjecture* (see Remark 5.4).

An early indication of the usefulness of this result is that it quickly leads to a much simpler proof of the main results of [Burns and Greither 2003] and [Flach 2011], and also those of [Bley 2006], in which the eTNC is proved for abelian extensions over $\mathbb{Q}$ and certain abelian extensions over imaginary quadratic fields respectively (see Corollary 5.6 and Remark 5.10).

To describe an application giving new results we assume k is totally real and K is CM and consider the "minus component" $\mathrm{eTNC}(K/k)_p^-$ of the p-part of the eTNC for K/k (as formulated explicitly in Remark 2.4).

We write K^+ for the maximal totally real subfield of K and recall that if no p-adic place splits in K/K^+ and the Iwasawa-theoretic μ-invariant of K_∞/K vanishes, then $\mathrm{eTNC}(K/k)_p^-$ is already known to be valid (as far as we are aware, such a result was first implicitly discussed in the survey article by Flach [2004]).

However, by combining Theorems 1.1 and 4.9 with results on the Gross–Stark conjecture by Darmon, Dasgupta and Pollack [Dasgupta et al. 2011] and by Ventullo [2015], we can now prove the following concrete result (for a precise statement of which see Corollary 5.8).

Corollary 1.2. *Let K/k be a finite abelian extension of number fields such that K is CM and k is totally real. If p is any odd prime for which the Iwasawa-theoretic μ-invariant of K_∞/K vanishes and at most one p-adic place of k splits in K/K^+, then* $\mathrm{eTNC}(K/k)_p^-$ (*see Remark 2.4*) *is* (*unconditionally*) *valid.*

This result gives *the first verifications* of $\mathrm{eTNC}(K/k)_p^-$ in any case for which both $k \neq \mathbb{Q}$ and *the relevant p-adic L-series possess trivial zeroes*. For example, all of the hypotheses of Corollary 1.2 are satisfied by the concrete families of extensions described in Examples 5.9.

By combining Corollary 1.2 with [BKS, Corollary 1.14] we can also immediately deduce the following result concerning a refined version of the classical Brumer–Stark Conjecture. In this result we write $S_{\mathrm{ram}}(K/k)$ for the set of places of k that ramify in K and for any finite set of nonarchimedean places T of k we write $\mathrm{Cl}^T(K)$ for the ray class group of the ring of integers of K modulo the product of all places of K above T. We also use the equivariant L-series $\theta_{K/k,S_{\mathrm{ram}}(K/k),T}(s)$ defined in equation (1) of Section 2B below, and write $x \mapsto x^\#$ for the $\mathbb{Z}_p$-linear involution on $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$ that inverts elements of $\mathrm{Gal}(K/k)$.

Corollary 1.3. *Let K/k and p be as in Corollary 1.2 and set $G := \mathrm{Gal}(K/k)$. Then for any finite nonempty set of places T of k that is disjoint from $S_{\mathrm{ram}}(K/k)$ one has*

$$\theta_{K/k,S_{\mathrm{ram}}(K/k),T}(0)^\# \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Fitt}_{\mathbb{Z}[G]}\big(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}^T(K), \mathbb{Q}/\mathbb{Z})\big),$$

and hence also

$$\theta_{K/k,S_{\mathrm{ram}}(K/k),T}(0) \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(K)).$$

We note that the final assertion of this result gives the first verifications of the Brumer–Stark conjecture in a case for which the base field is not $\mathbb{Q}$ and the relevant p-adic L-series possess trivial zeroes. Thus the conclusion of this corollary unconditionally holds for the extensions in Examples 5.9.

Our methods also prove a natural equivariant "main conjecture" (see Theorem 3.16 and Corollary 3.17) involving the Selmer modules for $\mathbb{G}_m$ introduced in [BKS] and give a more straightforward proof of one of the main results of [Greither and Popescu 2015] (see Section 3E, especially Corollaries 3.18 and 3.20).

1C. *Further developments.* The ideas presented in this article extend naturally in at least two different directions.

Firstly, one can formulate a natural generalization of the theory discussed here in the context of arbitrary Tate motives. In this setting our theory is related to natural generalizations of both the notion of Rubin–Stark element and of the Rubin–Stark conjecture for special values of L-functions at any integer points. We can also formulate precise conjectural congruences between Rubin–Stark elements of differing "weights", and in this way obtain *p-adic families* of Rubin–Stark elements. For details see our recent paper [Burns et al. 2016b].

Secondly, using an approach developed in [Burns and Sano 2016], many of the constructions, conjectures and results discussed here extend naturally to the setting of noncommutative Iwasawa theory and can then be used to prove the same case of the eTNC that we consider here over natural families of nonabelian Galois extensions.

Note. After this article was submitted for publication we learned of the preprint [Dasgupta et al. 2016] by Dasgupta, Kakde and Ventullo, which gives a full proof of the Gross–Stark conjecture (as stated in Conjecture 4.7 below). Taking their result into account, one can now remove the hypothesis of the validity of (the relevant cases of) Conjecture 4.7 from the statement of Corollary 5.7 and, via Theorem 4.9, one obtains further strong evidence in support of the Iwasawa-theoretic Rubin–Stark Congruences that are formulated in Conjecture 4.2. This does not yet, however, allow one to extend the results of either Corollary 1.2 or Corollary 1.3 since, aside from certain special classes of fields discussed in Remark 5.4, Gross's finiteness conjecture is still (in the relevant cases) not known to be valid unless one assumes that all associated p-adic L-functions have at most one trivial zero.

1D. ***Notation.*** For the reader's convenience we collect here some basic notation.

For any (profinite) group G we write $\hat{G}$ for the group of homomorphisms $G \to \mathbb{C}^\times$ of finite order.

Let k be a number field. For a place v of k, the residue field of v is denoted by $\kappa(v)$ and we set $\mathrm{N}v := \#\kappa(v)$. We denote the set of places of k which lie above the infinite place ∞ of $\mathbb{Q}$ (resp. a prime number p) by $S_\infty(k)$ (resp. $S_p(k)$). For a Galois extension L/k, the set of places of k that ramify in L is denoted by $S_{\mathrm{ram}}(L/k)$. For any set Σ of places of k, we denote by Σ_L the set of places of L which lie above places in Σ.

Let L/k be an abelian extension with Galois group G. For a place v of k, the decomposition group at v in G is denoted by G_v. If v is unramified in L, the Frobenius automorphism at v is denoted by Fr_v.

Let E be either a field of characteristic 0 or $\mathbb{Z}_p$. For an abelian group A, we denote $E \otimes_{\mathbb{Z}} A$ by EA or A_E. For a $\mathbb{Z}_p$-module A and an extension field E of $\mathbb{Q}_p$, we also write EA or A_E for $E \otimes_{\mathbb{Z}_p} A$. (This abuse of notation would not make any confusion.) We use similar notation for complexes. For example, if C is a complex of abelian groups, then we denote $E \otimes^{\mathbb{L}}_{\mathbb{Z}} C$ by EC or C_E.

Let R be a commutative ring and M an R-module. The linear dual $\mathrm{Hom}_R(M, R)$ is denoted by M^*. If r and s are nonnegative integers with $r \le s$, then there is a canonical paring

$$\textstyle\bigwedge_R^s M \times \bigwedge_R^r \mathrm{Hom}_R(M, R) \to \bigwedge_R^{s-r} M$$

defined by

$$(a_1\wedge\cdots\wedge a_s,\varphi_1\wedge\cdots\wedge\varphi_r)\mapsto \sum_{\sigma\in\mathfrak{S}_{s,r}} \mathrm{sgn}(\sigma)\det(\varphi_i(a_{\sigma(j)}))_{1\le i,j\le r}a_{\sigma(r+1)}\wedge\cdots\wedge a_{\sigma(s)},$$

with $\mathfrak{S}_{s,r} := \{\sigma\in\mathfrak{S}_s \mid \sigma(1)<\cdots<\sigma(r) \text{ and } \sigma(r+1)<\cdots<\sigma(s)\}$. (See [BKS, Proposition 4.1].) We denote the image of (a,Φ) under the above pairing by $\Phi(a)$.

The total quotient ring of R is denoted by $Q(R)$.

2. Zeta elements for $\mathbb{G}_m$

In this section, we review the zeta elements for $\mathbb{G}_m$ that were introduced in [BKS].

2A. *The Rubin–Stark conjecture.* We review the formulation of the Rubin–Stark conjecture [Rubin 1996, Conjecture B′].

Let L/k be a finite abelian extension of number fields with Galois group G. Let S be a finite set of places of k which contains $S_\infty(k)\cup S_{\mathrm{ram}}(L/k)$. We fix a labeling $S=\{v_0,\dots,v_n\}$. Take $r\in\mathbb{Z}$ so that $v_1,\dots,v_r$ split completely in L. We put $V:=\{v_1,\dots,v_r\}$. For each place v of k, we fix a place w of L lying above v. In particular, for each i with $0\le i\le n$, we fix a place w_i of L lying above v_i. Such conventions are frequently used in this paper.

For $\chi\in\hat{G}$, let $L_{k,S}(\chi,s)$ denote the usual S-truncated L-function for χ. We put

$$r_{\chi,S} := \mathrm{ord}_{s=0} L_{k,S}(\chi,s).$$

Let $\mathcal{O}_{L,S}$ be the ring of S_L integers of L. For any set Σ of places of k, put $Y_{L,\Sigma} := \bigoplus_{w\in\Sigma_L}\mathbb{Z}w$, the free abelian group on Σ_L. We define

$$X_{L,\Sigma} := \left\{\sum_{w\in\Sigma_L} a_w w\in Y_{L,\Sigma} \;\middle|\; \sum_{w\in\Sigma_L} a_w = 0\right\}.$$

By Dirichlet's unit theorem, we know that the homomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{L,S}:\mathbb{R}\mathcal{O}_{L,S}^\times \xrightarrow{\sim} \mathbb{R}X_{L,S},\quad a\mapsto -\sum_{w\in S_L}\log|a|_w w,$$

is an isomorphism.

By [Tate 1984, Chapter I, Proposition 3.4] we know that

$$r_{\chi,S} = \dim_{\mathbb{C}}(e_\chi\mathbb{C}\mathcal{O}_{L,S}^\times) = \dim_{\mathbb{C}}(e_\chi\mathbb{C}X_{L,S}) = \begin{cases}\#\{v\in S\mid \chi(G_v)=1\} & \text{if } \chi\ne 1,\\ n\ (=\#S-1) & \text{if } \chi=1,\end{cases}$$

where $e_\chi := 1/\#G\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$. From this fact, we see that $r\le r_{\chi,S}$.

Let T be a finite set of places of k which is disjoint from S. The S-truncated T-modified L-function is defined by

$$L_{k,S,T}(\chi,s) := \Bigl(\prod_{v\in T}(1-\chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s})\Bigr)L_{k,S}(\chi,s).$$

The (S,T)-unit group of L is defined to be the kernel of $\mathcal{O}_{L,S}^\times \to \bigoplus_{w\in T_L}\kappa(w)^\times$. Note that $\mathcal{O}_{L,S,T}^\times$ is a subgroup of $\mathcal{O}_{L,S}^\times$ of finite index. We have

$$r \le r_{\chi,S} = \mathrm{ord}_{s=0}\, L_{k,S,T}(\chi,s) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{L,S,T}^\times).$$

We put

$$L_{k,S,T}^{(r)}(\chi,0) := \lim_{s\to 0} s^{-r} L_{k,S,T}(\chi,s).$$

We define the r-th order Stickelberger element by

$$\theta_{L/k,S,T}^{(r)} := \sum_{\chi\in\hat{G}} L_{k,S,T}^{(r)}(\chi^{-1},0)e_\chi \in \mathbb{R}[G].$$

The (r-th order) Rubin–Stark element

$$\epsilon_{L/k,S,T}^{V} \in \mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^\times$$

is defined to be the element which corresponds to

$$\theta_{L/k,S,T}^{(r)}\cdot(w_1-w_0)\wedge\cdots\wedge(w_r-w_0)\in \mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}X_{L,S}$$

under the isomorphism

$$\mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^\times \xrightarrow{\sim} \mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}X_{L,S}$$

induced by $\lambda_{L,S}$. We note that $\epsilon_{L/k,S,T}^{V}$ is independent of the choice of w_0 and v_0 (see [Sano 2015, Proposition 3.3]).

Now assume that $\mathcal{O}_{L,S,T}^\times$ is $\mathbb{Z}$-free. Then, the Rubin–Stark conjecture (as formulated in [Rubin 1996, Conjecture B′]) predicts that the Rubin–Stark element $\epsilon_{L/k,S,T}^{V}$ lies in the $\mathbb{Z}[G]$-lattice obtained by setting

$$\bigcap_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^\times := \Bigl\{a\in \mathbb{Q}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^\times \;\Big|\; \Phi(a)\in\mathbb{Z}[G] \text{ for all } \Phi\in {\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S,T}^\times,\mathbb{Z}[G])\Bigr\}.$$

We stress, in particular, that in this context (and as used systematically in [BKS]) the notation $\bigcap_{\mathbb{Z}[G]}^{r}$ does not refer to an intersection.

In this paper, we consider the "p-part" of the Rubin–Stark conjecture for a fixed prime number p. We put

$$U_{L,S,T} := \mathbb{Z}_p\mathcal{O}_{L,S,T}^\times.$$

We also fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. From this, we regard

$$\epsilon^V_{L/k,S,T} \in \mathbb{C}_p {\textstyle\bigwedge}^r_{\mathbb{Z}_p[G]} U_{L,S,T}.$$

We define

$$\bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T} := \Bigl\{a \in \mathbb{Q}_p {\textstyle\bigwedge}^r_{\mathbb{Z}_p[G]} U_{L,S,T} \;\Big|\; \Phi(a) \in \mathbb{Z}_p[G] \text{ for all } \Phi \in {\textstyle\bigwedge}^r_{\mathbb{Z}_p[G]} \operatorname{Hom}_{\mathbb{Z}_p[G]}(U_{L,S,T}, \mathbb{Z}_p[G])\Bigr\}.$$

We easily see that there is a natural isomorphism $\mathbb{Z}_p \bigcap^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T} \simeq \bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T}$. We often denote $\bigwedge^r_{\mathbb{Z}_p[G]}$ and $\bigcap^r_{\mathbb{Z}_p[G]}$ simply by $\bigwedge^r$ and $\bigcap^r$ respectively.

We propose the "p-component version" of the Rubin–Stark conjecture as follows.

Conjecture 2.1 (RS$(L/k, S, T, V)_p$). *One has* $\epsilon^V_{L/k,S,T} \in \bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T}$.

Remark 2.2. Concerning known results on the Rubin–Stark conjecture, see [BKS, Remark 5.3], for example. Note that the Rubin–Stark conjecture is a consequence of the eTNC; this was first proved in [Burns 2007, Corollary 4.1], and then, in a much simpler way, in [BKS, Theorem 5.14].

2B. ***The eTNC for the untwisted Tate motive.*** We now review the formulation of the eTNC for the untwisted Tate motive.

Let L/k, G, S, T be as in the previous subsection. Fix a prime number p. We assume that $S_p(k) \subset S$. Consider the complex

$$C_{L,S} := R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p), \mathbb{Z}_p)[-2].$$

It is known that $C_{L,S}$ is a perfect complex of $\mathbb{Z}_p[G]$-modules, acyclic outside degrees zero and one. We have a canonical isomorphism

$$H^0(C_{L,S}) \simeq U_{L,S} \; (:= \mathbb{Z}_p \mathcal{O}^\times_{L,S}),$$

and a canonical exact sequence

$$0 \to A_S(L) \to H^1(C_{L,S}) \to \mathcal{X}_{L,S} \to 0,$$

where $A_S(L) := \mathbb{Z}_p \operatorname{Pic}(\mathcal{O}_{L,S})$ and $\mathcal{X}_{L,S} := \mathbb{Z}_p X_{L,S}$. The complex $C_{L,S}$ is identified with the p-completion of the complex obtained from the classical "Tate sequence" (if S is large enough), and also identified with $\mathbb{Z}_p R\Gamma((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$, where $R\Gamma((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ is the "Weil-étale cohomology complex" constructed in [BKS, §2.2] (see [Burns and Flach 1998, Proposition 3.3; Burns 2008, Proposition 3.5(e)]).

By a similar construction to [BKS, Proposition 2.4], we construct a canonical complex $C_{L,S,T}$ which lies in the distinguished triangle

$$C_{L,S,T} \to C_{L,S} \to \bigoplus_{w \in T_L} \mathbb{Z}_p \kappa(w)^\times[0].$$

(We can simply define $C_{L,S,T}$ by $\mathbb{Z}_p R\Gamma_T((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ in the terminology of [BKS].) We have

$$H^0(C_{L,S,T}) = U_{L,S,T}$$

and the exact sequence

$$0 \to A_S^T(L) \to H^1(C_{L,S,T}) \to \mathcal{X}_{L,S} \to 0,$$

where $A_S^T(L)$ is the p-part of the ray class group of $\mathcal{O}_{L,S}$ with modulus $\prod_{w \in T_L} w$.

We define the leading term of $L_{k,S,T}(\chi, s)$ at $s = 0$ by

$$L^*_{k,S,T}(\chi, 0) := \lim_{s \to 0} s^{-r_{\chi,S}} L_{k,S,T}(\chi, s).$$

The leading term at $s = 0$ of the equivariant L-function

$$\theta_{L/k,S,T}(s) := \sum_{\chi \in \hat{G}} L_{k,S,T}(\chi^{-1}, s) e_\chi \tag{1}$$

is defined by

$$\theta^*_{L/k,S,T}(0) := \sum_{\chi \in \hat{G}} L^*_{k,S,T}(\chi^{-1}, 0) e_\chi \in \mathbb{R}[G]^\times.$$

As in the previous subsection, we fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. We regard $\theta^*_{L/k,S,T}(0) \in \mathbb{C}_p[G]^\times$. The zeta element for $\mathbb{G}_m$

$$z_{L/k,S,T} \in \mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T})$$

is defined to be the element which corresponds to $\theta^*_{L/k,S,T}(0)$ under the isomorphism

$$\begin{aligned} \mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) &\simeq \mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S}) \\ &\xrightarrow{\sim} \mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S}) \otimes_{\mathbb{C}_p[G]} \mathrm{det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S}) \\ &\xrightarrow{\sim} \mathbb{C}_p[G], \end{aligned}$$

where the second isomorphism is induced by $\lambda_{L,S}$, and the last isomorphism is the evaluation map. Note that determinant modules must be regarded as graded invertible modules, but we omit the grading of any graded invertible modules as in [BKS].

The eTNC for the pair $(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ is formulated as follows.

Conjecture 2.3 (eTNC ($h^0(\mathrm{Spec}\, L)$, $\mathbb{Z}_p[G]$)). *One has*

$$\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = \det_{\mathbb{Z}_p[G]}(C_{L,S,T}).$$

Remark 2.4. When p is odd, k is totally real, and L is CM, we say that the minus part of the eTNC (which we denote by eTNC($h^0(\mathrm{Spec}\, L)$, $\mathbb{Z}_p[G]^-$)) is valid if we have the equality

$$e^-\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = e^-\det_{\mathbb{Z}_p[G]}(C_{L,S,T}),$$

where $e^- := (1-c)/2$ and $c \in G$ is the complex conjugation.

2C. ***The eTNC and Rubin–Stark elements.*** In this subsection, we interpret the eTNC using Rubin–Stark elements. The result in this subsection will be used in Section 5.

We continue to use the notation in the previous subsection. Take $\chi \in \hat{G}$, and suppose that $r_{\chi,S} < \#S$. Put $L_\chi := L^{\ker \chi}$ and $G_\chi := \mathrm{Gal}(L_\chi/k)$. Take $V_{\chi,S} \subset S$ so that all $v \in V_{\chi,S}$ split completely in L_χ (i.e., $\chi(G_v) = 1$) and $\#V_{\chi,S} = r_{\chi,S}$. Note that if $\chi \neq 1$, we have

$$V_{\chi,S} = \{v \in S \mid \chi(G_v) = 1\}.$$

Consider the Rubin–Stark element

$$\epsilon^{V_{\chi,S}}_{L_\chi/k,S,T} \in \mathbb{C}_p {\textstyle\bigwedge}^{r_{\chi,S}} U_{L_\chi,S,T}.$$

Note that a Rubin–Stark element depends on a fixed labeling of S, so in this case a labeling of S such that $S = \{v_0, \dots, v_n\}$ and $V_{\chi,S} = \{v_1, \dots, v_{r_{\chi,S}}\}$ is understood to be chosen.

For a set Σ of places of k and a finite extension F/k, put $\mathcal{Y}_{F,\Sigma} := \mathbb{Z}_p Y_{F,\Sigma} = \bigoplus_{w\in\Sigma_F} \mathbb{Z}_p w$ and $\mathcal{X}_{F,\Sigma} := \mathbb{Z}_p X_{F,\Sigma} = \ker(\mathcal{Y}_{F,\Sigma} \to \mathbb{Z}_p)$.

Then the natural surjection $\mathcal{X}_{L_\chi,S} \to \mathcal{Y}_{L_\chi,V_{\chi,S}}$ induces an injection

$$\mathcal{Y}^*_{L_\chi,V_{\chi,S}} \to \mathcal{X}^*_{L_\chi,S},$$

where $(\cdot)^* := \mathrm{Hom}_{\mathbb{Z}_p[G_\chi]}(\cdot, \mathbb{Z}_p[G_\chi])$. Since

$$\mathcal{Y}_{L_\chi,V_{\chi,S}} \simeq \mathbb{Z}_p[G_\chi]^{\oplus r_{\chi,S}}$$

and $\dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p \mathcal{X}_{L,S}) = r_{\chi,S}$, the above map induces an isomorphism

$$e_\chi \mathbb{C}_p \mathcal{Y}^*_{L_\chi,V_{\chi,S}} \xrightarrow{\sim} e_\chi \mathbb{C}_p \mathcal{X}^*_{L,S}.$$

From this, we have a canonical identification

$$\begin{aligned} &e_\chi \mathbb{C}_p\big({\textstyle\bigwedge}^{r_{\chi,S}} U_{L_\chi,S,T} \otimes {\textstyle\bigwedge}^{r_{\chi,S}} \mathcal{Y}^*_{L_\chi,V_{\chi,S}}\big) \\ &\qquad\qquad = e_\chi\big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big). \end{aligned}$$

Since $\{w_1, \dots, w_{r_{\chi,S}}\}$ is a basis of $\mathcal{Y}_{L_\chi, V_{\chi,S}}$, we have the (noncanonical) isomorphism

$$\bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \xrightarrow{\sim} \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \otimes \bigwedge^{r_{\chi,S}} \mathcal{Y}^*_{L_\chi,V_{\chi,S}}, \quad a \mapsto a \otimes w_1^* \wedge \cdots \wedge w^*_{r_{\chi,S}},$$

where w_i^* is the dual of w_i. Hence, we have the (noncanonical) isomorphism

$$e_\chi \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big).$$

Proposition 2.5. *Suppose $r_{\chi,S} < \#S$ for every $\chi \in \hat{G}$. A necessary and sufficient condition for* $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ *to hold is the existence of a $\mathbb{Z}_p[G]$-basis $\mathcal{L}_{L/k,S,T}$ of $\det_{\mathbb{Z}_p[G]}(C_{L,S,T})$ such that for every $\chi \in \hat{G}$ the image of $e_\chi \mathcal{L}_{L/k,S,T}$ under the isomorphism*

$$\begin{aligned} &e_\chi \mathbb{C}_p \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) \\ &\qquad \simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big) \simeq e_\chi \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \end{aligned}$$

coincides with $e_\chi \epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$.

Proof. By the definition of Rubin–Stark elements, the image of $e_\chi \epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$ under the isomorphism

$$\begin{aligned} e_\chi \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} &\simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big) \\ &\simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big) \\ &\simeq e_\chi \mathbb{C}_p[G] \end{aligned}$$

is equal to $e_\chi L^*_{k,S,T}(\chi^{-1}, 0)$. Necessity follows by putting $\mathcal{L}_{L/k,S,T} := z_{L/k,S,T}$. Sufficiency follows by noting that $\mathcal{L}_{L/k,S,T}$ must be equal to $z_{L/k,S,T}$. □

2D. ***The canonical projection maps.*** Let L/k, G, S, T, V, r be as in Section 2A. We put

$$e_r := \sum_{\chi \in \hat{G},\, r_{\chi,S} = r} e_\chi \in \mathbb{Q}[G].$$

As in Proposition 2.5, we construct the (noncanonical) isomorphism

$$e_r \mathbb{C}_p \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) \simeq e_r \mathbb{C}_p \bigwedge^r U_{L,S,T}.$$

In this subsection, we give an explicit description of the map

$$\begin{aligned} \pi^V_{L/k,S,T} : \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) &\xrightarrow{e_r \mathbb{C}_p \otimes} e_r \mathbb{C}_p \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) \\ &\qquad \simeq e_r \mathbb{C}_p \bigwedge^r U_{L,S,T} \subset \mathbb{C}_p \bigwedge^r U_{L,S,T}. \end{aligned}$$

This map is important since the image of the zeta element $z_{L/k,S,T}$ under this map is the Rubin–Stark element $\epsilon^V_{L/k,S,T}$.

Firstly, we choose a representative $\Pi \xrightarrow{\psi} \Pi$ of $C_{L,S,T}$, where the first term is placed in degree zero, such that Π is a free $\mathbb{Z}_p[G]$-module with basis $\{b_1, \dots, b_d\}$ (d is sufficiently large), and that the natural surjection

$$\Pi \to H^1(C_{L,S,T}) \to \mathcal{X}_{L,S}$$

sends b_i to $w_i - w_0$ for each i with $1 \le i \le r$. For the details of this construction, see [BKS, §5.4]. Note that the representative of $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ chosen there is of the form

$$P \to F,$$

where P is projective and F is free. By Swan's theorem [Curtis and Reiner 1981, (32.1)], we have an isomorphism $\mathbb{Z}_p P \simeq \mathbb{Z}_p F$. This shows that we can take the representative of $C_{L,S,T}$ as above.

We define $\psi_i \in \mathrm{Hom}_{\mathbb{Z}_p[G]}(\Pi, \mathbb{Z}_p[G])$ by

$$\psi_i := b_i^* \circ \psi,$$

where b_i^* is the dual of b_i. Note that $\bigwedge_{r<i\le d} \psi_i \in \bigwedge^{d-r} \mathrm{Hom}_{\mathbb{Z}_p[G]}(\Pi, \mathbb{Z}_p[G])$ defines the homomorphism

$$\bigwedge_{r<i\le d} \psi_i : \bigwedge^d \Pi \to \bigwedge^r \Pi$$

given as follows (see Notation):

$$\big(\bigwedge_{r<i\le d} \psi_i\big)(b_1 \wedge \cdots \wedge b_d) = \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}$$

Proposition 2.6. (i) *We have*

$$\bigcap^r U_{L,S,T} = \big(\mathbb{Q}_p \bigwedge^r U_{L,S,T}\big) \cap \bigwedge^r \Pi,$$

where we regard $U_{L,S,T} \subset \Pi$ via the natural inclusion

$$U_{L,S,T} = H^0(C_{L,S,T}) = \ker \psi \hookrightarrow \Pi.$$

(ii) *If we regard $\bigcap^r U_{L,S,T}$ as a subset of $\bigwedge^r \Pi$ by* (i), *then*

$$\mathrm{im}\big(\bigwedge_{r<i\le d} \psi_i : \bigwedge^d \Pi \to \bigwedge^r \Pi\big) \subset \bigcap^r U_{L,S,T}.$$

(iii) *The map*

$$\det_{\mathbb{Z}_p[G]}(C_{L,S,T}) = \bigwedge^d \Pi \otimes \bigwedge^d \Pi^* \to \bigcap^r U_{L,S,T}$$

given by

$$b_1 \wedge \cdots \wedge b_d \otimes b_1^* \wedge \cdots \wedge b_d^* \mapsto \big(\bigwedge_{r<i\le d} \psi_i\big)(b_1 \wedge \cdots \wedge b_d)$$

coincides with $(-1)^{r(d-r)}\pi^V_{L/k,S,T}$. *In particular, we have*

$$\pi^V_{L/k,S,T}(b_1\wedge\cdots\wedge b_d\otimes b_1^*\wedge\cdots\wedge b_d^*)$$
$$= (-1)^{r(d-r)}\sum_{\sigma\in\mathfrak{S}_{d,r}}\operatorname{sgn}(\sigma)\det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d}\,b_{\sigma(1)}\wedge\cdots\wedge b_{\sigma(r)}$$

and

$$\operatorname{im}\pi^V_{L/k,S,T}\subset\Big\{a\in\bigcap^r U_{L,S,T}\;\Big|\; e_r a = a\Big\}.$$

Proof. For (i), see [BKS, Lemma 4.7(ii)]. For (ii) and (iii), see [BKS, Lemma 4.3]. □

3. Higher rank Iwasawa theory

3A. ***Notation.*** We fix a prime number p. Let k be a number field, and K_∞/k a Galois extension such that $\mathcal{G}:=\mathrm{Gal}(K_\infty/k)\simeq\Delta\times\Gamma$, where Δ is a finite abelian group and $\Gamma\simeq\mathbb{Z}_p$. Set $\Lambda:=\mathbb{Z}_p[[\mathcal{G}]]$. Fix an isomorphism $\mathbb{C}\simeq\mathbb{C}_p$, and identify $\hat{\Delta}$ with $\mathrm{Hom}_{\mathbb{Z}}(\Delta,\overline{\mathbb{Q}_p}^{\times})$. For $\chi\in\hat{\Delta}$, put $\Lambda_\chi:=\mathbb{Z}_p[\operatorname{im}\chi][[\Gamma]]$. Note that the total quotient ring $Q(\Lambda)$ has the decomposition

$$Q(\Lambda)\simeq\bigoplus_{\chi\in\hat{\Delta}/\sim_{\mathbb{Q}_p}} Q(\Lambda_\chi),$$

where $\chi\sim_{\mathbb{Q}_p}\chi'$ if and only if there exists $\sigma\in G_{\mathbb{Q}_p}$ such that $\chi=\sigma\circ\chi'$.

We use the following notation:

- $K:=K_\infty^\Gamma$ (so $\mathrm{Gal}(K/k)=\Delta$);
- $k_\infty:=K_\infty^\Delta$ (so k_∞/k is a $\mathbb{Z}_p$-extension with Galois group Γ);
- k_n: the n-th layer of k_∞/k;
- K_n: the n-th layer of K_∞/K;
- $\mathcal{G}_n:=\mathrm{Gal}(K_n/k)$.

For each character $\chi\in\hat{\mathcal{G}}$ we also set

- $L_\chi:=K_\infty^{\ker\chi}$;
- $L_{\chi,\infty}:=L_\chi\cdot k_\infty$;
- $L_{\chi,n}$: the n-th layer of $L_{\chi,\infty}/L_\chi$;
- $\mathcal{G}_\chi:=\mathrm{Gal}(L_{\chi,\infty}/k)$;
- $\mathcal{G}_{\chi,n}:=\mathrm{Gal}(L_{\chi,n}/k)$;
- $G_\chi:=\mathrm{Gal}(L_\chi/k)$;
- $\Gamma_\chi:=\mathrm{Gal}(L_{\chi,\infty}/L_\chi)$;
- $\Gamma_{\chi,n}:=\mathrm{Gal}(L_{\chi,n}/L_\chi)$;

- S: a finite set of places of k which contains $S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k) \cup S_p(k)$;
- T: a finite set of places of k which is disjoint from S;
- $V_\chi := \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$ (this is a proper subset of S);
- $r_\chi := \# V_\chi$.

For any intermediate field L of K_∞/k, we denote $\varprojlim_F U_{F,S,T}$ by $U_{L,S,T}$, where F runs over all intermediate fields of L/k that are finite over k and the inverse limit is taken with respect to the norm maps. Similarly, $C_{L,S,T}$ is the complex defined by the inverse limit of the complexes $C_{F,S,T}$ with respect to the natural transition maps, and $A_S^T(L)$ the inverse limit of the p-primary parts $A_S^T(F)$ of the T ray class groups of $\mathcal{O}_{F,S}$ with respect to the norm maps. We denote $\varprojlim_F \mathcal{Y}_{F,S}$ by $\mathcal{Y}_{L,S}$, where the inverse limit is taken with respect to the maps

$$\mathcal{Y}_{F',S} \to \mathcal{Y}_{F,S}, \quad w_{F'} \mapsto w_F,$$

where $F \subset F'$, $w_{F'} \in S_{F'}$, and $w_F \in S_F$ is the place lying under $w_{F'}$. We use similar notation for $\mathcal{X}_{L,S}$ etc.

3B. *Iwasawa main conjecture I.* In this section we formulate the main conjecture of Iwasawa theory for general number fields, which is a key to our study.

3B1. For any character χ in $\hat{\mathcal{G}}$ there is a natural composite homomorphism

$$\begin{aligned}\lambda_\chi : \det_\Lambda(C_{K_\infty,S,T}) &\to \det_{\mathbb{Z}_p[G_\chi]}(C_{L_\chi,S,T})\\ &\hookrightarrow \det_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p C_{L_\chi,S,T})\\ &\xrightarrow{\sim} \det_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p U_{L_\chi,S,T}) \otimes_{\mathbb{C}_p[G_\chi]} \det^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S})\\ &\xrightarrow{\sim} \det_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \otimes_{\mathbb{C}_p[G_\chi]} \det^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S})\\ &\simeq \mathbb{C}_p[G_\chi]\\ &\xrightarrow{\chi} \mathbb{C}_p,\end{aligned}$$

where the fourth map is induced by $\lambda_{L_\chi,S}$, the fifth map is the evaluation, and the last map is induced by χ.

We can now state our higher-rank main conjecture of Iwasawa theory in its first form.

Conjecture 3.1 (IMC(K_∞/k, S, T)). *There exists a Λ-basis $\mathcal{L}_{K_\infty/k,S,T}$ of the module* $\det_\Lambda(C_{K_\infty,S,T})$ *for which, at every $\chi \in \hat{\Delta}$ and every $\psi \in \hat{G}_\chi$ for which $r_{\psi,S} = r_\chi$ one has $\lambda_\psi(\mathcal{L}_{K_\infty/k,S,T}) = L^{(r_\chi)}_{k,S,T}(\psi^{-1}, 0)$.*

Remark 3.2. This conjecture is equivariant with respect to Δ. But it is important to note that this conjecture is much weaker than the (relevant case of the) equivariant Tamagawa number conjecture. For example, if k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, then for any ψ that is trivial on the decomposition group in $\mathcal{G}_\chi$ of any p-adic place of k one has $r_{\psi,S} > r_\chi$ and so there is no interpolation condition at ψ specified above. When $r_\chi = 0$, (the χ-component of) the element $\mathcal{L}_{K_\infty/k,S,T}$ is the p-adic L-function, and in the general case $r_\chi > 0$, it plays a role of p-adic L-functions. We will see in Section 3B2 that the interpolation condition characterizes $\mathcal{L}_{K_\infty/k,S,T}$ uniquely.

Remark 3.3. The explicit definition of the elements $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$ implies directly that the assertion of Conjecture 3.1 is valid if and only if there is a Λ-basis $\mathcal{L}_{K_\infty/k,S,T}$ of $\det_\Lambda(C_{K_\infty,S,T})$ for which, for every character $\chi \in \hat{\Delta}$ and every positive integer n, the image of $\mathcal{L}_{K_\infty/k,S,T}$ under the map

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} e_{r_\chi}\mathbb{C}_p{\textstyle\bigwedge}^{r_\chi} U_{L_{\chi,n},S,T}$$

is equal to $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$.

It is not difficult to see that the validity of Conjecture 3.1 is independent of T. We assume in the sequel that T contains two places of unequal residue characteristics and hence that each group $U_{L,S,T}$ is $\mathbb{Z}_p$-free.

3B2. For each character $\chi \in \hat{\Delta}$, there is a natural ring homomorphism

$$\mathbb{Z}_p[[\mathcal{G}_\chi]] = \mathbb{Z}_p[[G_\chi \times \Gamma]] \xrightarrow{\chi} \mathbb{Z}_p[\operatorname{im}\chi][[\Gamma]] = \Lambda_\chi \subset Q(\Lambda_\chi).$$

In the sequel we use this homomorphism to regard $Q(\Lambda_\chi)$ as a $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-algebra.

In the next result we describe an important connection between the element $\mathcal{L}_{K_\infty/k,S,T}$ that is predicted to exist by Conjecture 3.1 and the inverse limit (over n) of the Rubin–Stark elements $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$. This result shows, in particular, that the element $\mathcal{L}_{K_\infty/k,S,T}$ in Conjecture 3.1 is unique (if it exists).

In the sequel we set

$$\bigcap^{r_\chi} U_{L_{\chi,\infty},S,T} := \varprojlim_n \bigcap^{r_\chi} U_{L_{\chi,n},S,T},$$

where the inverse limit is taken with respect to the map

$$\bigcap^{r_\chi} U_{L_{\chi,m},S,T} \to \bigcap^{r_\chi} U_{L_{\chi,n},S,T}$$

induced by the norm map $U_{L_{\chi,m},S,T} \to U_{L_{\chi,n},S,T}$, where $n \le m$. Note that Rubin–Stark elements are norm compatible (see [Rubin 1996, Proposition 6.1; Sano 2014, Proposition 3.5]), so if we know that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all sufficiently large n, then we can define the element

$$\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T} := \varprojlim_n \epsilon^{V_\chi}_{L_{\chi,n}/k,S,T} \in \bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}.$$

Theorem 3.4. (i) *For each $\chi \in \hat{\Delta}$, the homomorphism*

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} {\bigcap}^{r_\chi} U_{L_{\chi,n},S,T}$$

(see Proposition 2.6(iii)) *induces an isomorphism of $\mathcal{Q}(\Lambda_\chi)$-modules*

$$\pi^{V_\chi}_{L_{\chi,\infty}/k,S,T} : \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda \mathcal{Q}(\Lambda_\chi) \simeq \Big({\bigcap}^{r_\chi} U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathcal{Q}(\Lambda_\chi).$$

(ii) *If Conjecture 3.1 is valid, then we have*

$$\pi^{V_\chi}_{L_{\chi,\infty}/k,S,T}(\mathcal{L}_{K_\infty/k,S,T}) = \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}.$$

(Note that in this case Conjecture RS$(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all n by Remark 3.3 and Proposition 2.6(iii).)

Proof. Since the module $A_S^T(K_\infty) \otimes_\Lambda \mathcal{Q}(\Lambda_\chi)$ vanishes, there are canonical isomorphisms

$$\begin{aligned}
&\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda \mathcal{Q}(\Lambda_\chi)\\
&\quad\simeq \det_{\mathcal{Q}(\Lambda_\chi)}(C_{K_\infty,S,T} \otimes_\Lambda \mathcal{Q}(\Lambda_\chi))\\
&\quad\simeq \det_{\mathcal{Q}(\Lambda_\chi)}(U_{K_\infty,S,T} \otimes_\Lambda \mathcal{Q}(\Lambda_\chi)) \otimes_{\mathcal{Q}(\Lambda_\chi)} \det^{-1}_{\mathcal{Q}(\Lambda_\chi)}(\mathcal{X}_{K_\infty,S} \otimes_\Lambda \mathcal{Q}(\Lambda_\chi)). \qquad (2)
\end{aligned}$$

It is also easy to check that there are natural isomorphisms

$$U_{K_\infty,S,T} \otimes_\Lambda \mathcal{Q}(\Lambda_\chi) \simeq U_{L_{\chi,\infty},S,T} \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathcal{Q}(\Lambda_\chi)$$

and

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda \mathcal{Q}(\Lambda_\chi) \simeq \mathcal{X}_{L_{\chi,\infty},S} \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathcal{Q}(\Lambda_\chi) \simeq \mathcal{Y}_{L_{\chi,\infty},V_\chi} \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathcal{Q}(\Lambda_\chi),$$

and that these are $\mathcal{Q}(\Lambda_\chi)$-vector spaces of dimension $r := r_\chi$ $(= \# V_\chi)$. The isomorphism (2) is therefore a canonical isomorphism of the form

$$\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda \mathcal{Q}(\Lambda_\chi) \simeq \big({\bigwedge}^r U_{L_{\chi,\infty},S,T} \otimes {\bigwedge}^r \mathcal{Y}^*_{L_{\chi,\infty},V_\chi}\big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathcal{Q}(\Lambda_\chi).$$

Composing this isomorphism with the map induced by the noncanonical isomorphism

$${\bigwedge}^r \mathcal{Y}^*_{L_{\chi,\infty},V_\chi} \xrightarrow{\sim} \mathbb{Z}_p[\![\mathcal{G}_\chi]\!], \quad w_1^* \wedge \cdots \wedge w_r^* \mapsto 1,$$

we have

$$\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda \mathcal{Q}(\Lambda_\chi) \simeq \big({\bigwedge}^r U_{L_{\chi,\infty},S,T}\big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathcal{Q}(\Lambda_\chi).$$

As in the proofs of Proposition 2.6(iii) and [BKS, Lemma 4.3], this isomorphism is induced by $\varprojlim_n \pi^{V_\chi}_{L_{\chi,n}/k,S,T}$. Now the isomorphism in claim (i) is thus obtained directly from Lemma 3.5 below.

Claim (ii) follows by noting that the image of $\mathcal{L}_{K_\infty/k,S,T}$ under the map

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} {\bigcap}^{r_\chi} U_{L_{\chi,n},S,T}$$

is equal to $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$. □

Lemma 3.5. *With notation as above, there is a canonical identification*

$$\Big(\bigcap^r U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} Q(\Lambda_\chi) = \Big(\bigwedge^r U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} Q(\Lambda_\chi).$$

Proof. Take a representative $\Pi_\infty \to \Pi_\infty$ of $C_{L_{\chi,\infty},S,T}$ as in Section 2D. Put $\Pi_n := \Pi_\infty \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} \mathbb{Z}_p[\mathcal{G}_{\chi,n}]$. We have (see Proposition 2.6(i))

$$\bigcap^r U_{L_{\chi,n},S,T} = \Big(\mathbb{Q}_p \bigwedge^r U_{L_{\chi,n},S,T}\Big) \cap \bigwedge^r \Pi_n,$$

and thus $\varprojlim_n \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]} U_{L_{\chi,n},S,T}$ can be regarded as a submodule of the free $\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]$-module $\varprojlim_n \bigwedge^r \Pi_n = \bigwedge^r \Pi_\infty$. For simplicity, we set $G_n := \mathcal{G}_{\chi,n}$, $G := \mathcal{G}_\chi$, $U_n := U_{L_{\chi,n},S,T}$, $U_\infty := U_{L_{\chi,\infty},S,T}$, and $Q := Q(\Lambda_\chi)$. We will show the equality

$$\Big(\big(\varprojlim_n \mathbb{Q}_p \bigwedge^r U_n\big) \cap \bigwedge^r \Pi_\infty\Big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q = \big(\bigwedge^r U_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q$$

of the submodules of $(\bigwedge^r \Pi_\infty) \otimes_{\mathbb{Z}_p[\![G]\!]} Q$.

It is easy to see that

$$\big(\bigwedge^r U_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q \subset \Big(\big(\varprojlim_n \mathbb{Q}_p \bigwedge^r U_n\big) \cap \bigwedge^r \Pi_\infty\Big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q.$$

Conversely, take $a \in (\varprojlim_n \mathbb{Q}_p \bigwedge^r U_n) \cap \bigwedge^r \Pi_\infty$ and set $M_n := \operatorname{coker}(U_n \to \Pi_n)$. Then we have

$$\varprojlim_n M_n \simeq \operatorname{coker}(U_\infty \to \Pi_\infty) =: M_\infty.$$

Since $\Pi_\infty \otimes_{\mathbb{Z}_p[\![G]\!]} Q \simeq (U_\infty \otimes_{\mathbb{Z}_p[\![G]\!]} Q) \oplus (M_\infty \otimes_{\mathbb{Z}_p[\![G]\!]} Q)$, we have the decomposition

$$\big(\bigwedge^r \Pi_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q \simeq \bigoplus_{i=0}^r \big(\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q.$$

Write

$$a = (a_i)_i \in \bigoplus_{i=0}^r \big(\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q.$$

It is sufficient to show that $a_i = 0$ for all $i > 0$. We may assume that

$$a_i \in \operatorname{im}\Big(\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty \to \big(\bigwedge^{r-i} U_\infty \otimes \bigwedge^i M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\Big)$$

for every i. Since $a \in \bigwedge^r \Pi_\infty$, we can also write $a = (a_{(n)})_n \in \varprojlim_n \bigwedge^r \Pi_n$. For each n, we have a decomposition

$$\mathbb{Q}_p \bigwedge^r \Pi_n \simeq \bigoplus_{i=0}^r \big(\mathbb{Q}_p \bigwedge^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p \bigwedge^i M_n\big),$$

and we write

$$a_{(n)} = (a_{(n),i})_i \in \bigoplus_{i=0}^{r} \Big(\mathbb{Q}_p {\textstyle\bigwedge}^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p {\textstyle\bigwedge}^{i} M_n \Big).$$

Since $a \in \varprojlim_n \mathbb{Q}_p \bigwedge^r U_n$, we must have $a_{(n),i} = 0$ for all $i > 0$. To prove $a_i = 0$ for all $i > 0$, it is sufficient to show that the natural map

$$\begin{aligned} \operatorname{im}\big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty \to \big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\big) \\ \to \varprojlim_n \big(\mathbb{Q}_p {\textstyle\bigwedge}^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p {\textstyle\bigwedge}^{i} M_n\big) \quad (3) \end{aligned}$$

is injective. Note that M_∞ is isomorphic to a submodule of Π_∞, since $M_\infty \simeq \ker(\Pi_\infty \to H^1(C_{L_{\chi,\infty},S,T}))$. Hence both U_∞ and M_∞ are embedded in Π_∞, and we have

$$\begin{aligned} &\ker\big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty \to \big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\big) \\ &\qquad = \ker\big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty \xrightarrow{\alpha} \big({\textstyle\bigwedge}^{r} (\Pi_\infty \oplus \Pi_\infty)\big) \otimes_{\mathbb{Z}_p[\![G]\!]} \Lambda_\chi\big). \end{aligned}$$

Set $\Lambda_{\chi,n} := \mathbb{Z}_p[\operatorname{im}\chi][\Gamma_{\chi,n}]$. The commutative diagram

$$\begin{tikzcd} {\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty \arrow[r, "\alpha"] \arrow[d, "\beta"'] & ({\textstyle\bigwedge}^{r} (\Pi_\infty \oplus \Pi_\infty)) \otimes_{\mathbb{Z}_p[\![G]\!]} \Lambda_\chi \arrow[d, "f"] \\ \varprojlim_n \mathbb{Q}_p (({\textstyle\bigwedge}^{r-i} U_n \otimes {\textstyle\bigwedge}^{i} M_n) \otimes_{\mathbb{Z}_p[G_n]} \Lambda_{\chi,n}) \arrow[r, "g"'] & \varprojlim_n \mathbb{Q}_p \big(({\textstyle\bigwedge}^{r} (\Pi_n \oplus \Pi_n)) \otimes_{\mathbb{Z}_p[G_n]} \Lambda_{\chi,n}\big) \end{tikzcd}$$

and the injectivity of f and g implies $\ker\alpha = \ker\beta$. Hence we have

$$\ker\big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty \to \big({\textstyle\bigwedge}^{r-i} U_\infty \otimes {\textstyle\bigwedge}^{i} M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\big) = \ker\alpha = \ker\beta.$$

This shows the injectivity of (3). □

Remark 3.6. Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all $\chi \in \hat{\Delta}$ and n. Using Theorem 3.4, we can define

$$\mathcal{L}_{K_\infty/k,S,T} \in \det\nolimits_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda) = \bigoplus_{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p}} (\det\nolimits_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi))$$

by $\mathcal{L}_{K_\infty/k,S,T} := (\pi^{V_\chi,-1}_{L_{\chi,\infty}/k,S,T}(\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}))_\chi$. Then Conjecture 3.1 is equivalent to

$$\Lambda \cdot \mathcal{L}_{K_\infty/k,S,T} = \det\nolimits_\Lambda(C_{K_\infty,S,T}).$$

3C. *Iwasawa main conjecture II.* In this subsection, we work under the following simplifying assumptions:

$(*)$ $\quad$ p is odd, and V_χ contains no finite places for every $\chi \in \hat{\Delta}$.

We note that the second assumption here is satisfied whenever k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension.

3C1. We start by quickly reviewing some basic facts concerning the height-one prime ideals of Λ.

We say that a height-one prime ideal $\mathfrak{p}$ of Λ is *regular* if $p \notin \mathfrak{p}$, and *singular* if $p \in \mathfrak{p}$.

If $\mathfrak{p}$ is regular, then $\Lambda_{\mathfrak{p}}$ is identified with the localization of $\Lambda[1/p]$ at $\mathfrak{p}\Lambda[1/p]$. Since we have the decomposition

$$\Lambda\left[\frac{1}{p}\right] = \bigoplus_{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p}} \Lambda_\chi\left[\frac{1}{p}\right],$$

we have $Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda_{\chi_{\mathfrak{p}}})$ for some $\chi_{\mathfrak{p}} \in \hat{\Delta}/\sim_{\mathbb{Q}_p}$. Since $\Lambda_{\chi_{\mathfrak{p}}}[1/p]$ is a regular local ring, $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring.

Next, suppose that $\mathfrak{p}$ is a singular prime. We have the decomposition

$$\Lambda = \bigoplus_{\chi \in \hat{\Delta}'/\sim_{\mathbb{Q}_p}} \mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]],$$

where Δ_p is the Sylow p-subgroup of Δ, and Δ' is the unique subgroup of Δ which is isomorphic to Δ/Δ_p. From this, we see that $\Lambda_{\mathfrak{p}}$ is identified with the localization of some $\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]]$ at $\mathfrak{p}\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]]$. By [Burns and Greither 2003, Lemma 6.2(i)], we have

$$\mathfrak{p}\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][[\Gamma]] = (\sqrt{p\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p]}),$$

where we denote the radical of an ideal I by $\sqrt{I}$. This shows that there is a one-to-one correspondence between the set of all singular primes of Λ and the set $\hat{\Delta}'/\sim_{\mathbb{Q}_p}$. We denote by $\chi_{\mathfrak{p}} \in \hat{\Delta}'/\sim_{\mathbb{Q}_p}$ the character corresponding to $\mathfrak{p}$. The next lemma shows that

$$Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\substack{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p} \\ \chi|_{\Delta'} = \chi_{\mathfrak{p}}}} Q(\Lambda_\chi).$$

Lemma 3.7. *Let $E/\mathbb{Q}_p$ be a finite unramified extension, and $\mathcal{O}$ its ring of integers. Let P be a finite abelian group whose order is a power of p. Put $\Lambda := \mathcal{O}[P][[\Gamma]]$ and $\mathfrak{p} := \sqrt{p\mathcal{O}[P]}\Lambda$ ($\mathfrak{p}$ is the unique singular prime of Λ). Then we have*

$$Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda) = \bigoplus_{\chi \in \hat{P}/\sim_E} Q(\mathcal{O}[\operatorname{im}\chi][[\Gamma]]).$$

Proof. Since $Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda_{\mathfrak{p}}[1/p])$ and $\Lambda_{\mathfrak{p}}[1/p] = \bigoplus_{\chi \in \hat{P}/\sim_E} e_\chi \Lambda_{\mathfrak{p}}[1/p]$, where $e_\chi := \sum_{\chi' \sim_E \chi} e_{\chi'}$, we have

$$Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\chi \in \hat{P}/\sim_E} Q\Big(e_\chi \Lambda_{\mathfrak{p}}\left[\frac{1}{p}\right]\Big).$$

For $\chi \in \hat{P}/{\sim_E}$, put $\mathfrak{q}_\chi := \ker(\Lambda \xrightarrow{\chi} \mathcal{O}[\mathrm{im}\,\chi][[\Gamma]])$. We can easily see that $\sqrt{p\mathcal{O}[P]} = (p, I_{\mathcal{O}}(P))$, where $I_{\mathcal{O}}(P)$ is the kernel of the augmentation map $\mathcal{O}[P] \to \mathcal{O}$. From this, we also see that

$$\sqrt{p\mathcal{O}[P]} = \ker(\mathcal{O}[P] \xrightarrow{\chi} \mathcal{O}[\mathrm{im}\,\chi] \to \mathcal{O}[\mathrm{im}\,\chi]/\pi_\chi \mathcal{O}[\mathrm{im}\,\chi] \simeq \mathcal{O}/p\mathcal{O})$$

holds for any $\chi \in \hat{P}/{\sim_E}$, where $\pi_\chi \in \mathcal{O}[\mathrm{im}\,\chi]$ is a uniformizer. This shows that $\mathfrak{q}_\chi \subset \mathfrak{p}$. Hence, we know that $\Lambda_{\mathfrak{q}_\chi}$ is the localization of $\Lambda_{\mathfrak{p}}[1/p]$ at $\mathfrak{q}_\chi \Lambda_{\mathfrak{p}}[1/p]$. One can check that $\Lambda_{\mathfrak{q}_\chi} = Q(e_\chi \Lambda_{\mathfrak{p}}[1/p])$. Since we have $\Lambda_{\mathfrak{q}_\chi} = Q(\mathcal{O}[\mathrm{im}\,\chi][[\Gamma]])$, the lemma follows. □

For a height-one prime ideal $\mathfrak{p}$ of Λ, define a subset $\Upsilon_{\mathfrak{p}} \subset \hat{\Delta}/{\sim_{\mathbb{Q}_p}}$ by

$$\Upsilon_{\mathfrak{p}} := \begin{cases} \{\chi_{\mathfrak{p}}\} & \text{if } \mathfrak{p} \text{ is regular}, \\ \{\chi \in \hat{\Delta}/{\sim_{\mathbb{Q}_p}} \mid \chi|_{\Delta'} = \chi_{\mathfrak{p}}\} & \text{if } \mathfrak{p} \text{ is singular}. \end{cases}$$

The above argument shows that $Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\chi \in \Upsilon_{\mathfrak{p}}} Q(\Lambda_\chi)$.

To end this subsection we recall a useful result concerning μ-invariants, whose proof is in [Flach 2004, Lemma 5.6].

Lemma 3.8. *Let M be a finitely generated torsion Λ-module. Let $\mathfrak{p}$ be a singular prime of Λ. Then the following are equivalent*:

(i) *The μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_{\mathfrak{p}}} M$ vanishes.*

(ii) *For any $\chi \in \Upsilon_{\mathfrak{p}}$, the μ-invariant of the $\mathbb{Z}_p[\mathrm{im}\,\chi][[\Gamma]]$-module $M \otimes_{\mathbb{Z}_p[\Delta']} \mathbb{Z}_p[\mathrm{im}\,\chi]$ vanishes.*

(iii) $M_{\mathfrak{p}} = 0$.

3C2. In the rest of this section we assume the condition $(*)$ from the beginning of Section 3C.

Lemma 3.9. *Let $\mathfrak{p}$ be a singular prime of Λ. Then V_χ is independent of $\chi \in \Upsilon_{\mathfrak{p}}$. In particular, for any $\chi \in \Upsilon_{\mathfrak{p}}$, the $Q(\Lambda_{\mathfrak{p}})$-module $U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$ is free of rank r_χ.*

Proof. It is sufficient to show that $V_\chi = V_{\chi_{\mathfrak{p}}}$ for any $\chi \in \Upsilon_{\mathfrak{p}}$. Note that the extension degree $[L_{\chi,\infty} : L_{\chi_{\mathfrak{p}},\infty}] = [L_\chi : L_{\chi_{\mathfrak{p}}}]$ is a power of p. Since p is odd by the assumption $(*)$, we see that an infinite place of k which splits completely in $L_{\chi_{\mathfrak{p}},\infty}$ also splits completely in $L_{\chi,\infty}$. By the assumption $(*)$, we know every place in $V_{\chi_{\mathfrak{p}}}$ is infinite. Hence we have $V_\chi = V_{\chi_{\mathfrak{p}}}$. □

The above result motivates us, for any height-one prime ideal $\mathfrak{p}$ of Λ, to define $V_{\mathfrak{p}} := V_\chi$ and $r_{\mathfrak{p}} := r_\chi$ by choosing some $\chi \in \Upsilon_{\mathfrak{p}}$.

Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ holds for all $\chi \in \hat{\Delta}$ and n. We then define the "$\mathfrak{p}$-part" of the Rubin–Stark element

$$\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} \in \left(\bigwedge\nolimits^{r_{\mathfrak{p}}} U_{K_\infty,S,T}\right) \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$$

as the image of

$$(\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T})_{\chi\in\Upsilon_{\mathfrak{p}}} \in \bigoplus_{\chi\in\Upsilon_{\mathfrak{p}}} \bigcap^{r_{\mathfrak{p}}} U_{L_{\chi,\infty},S,T}$$

under the natural map

$$\bigoplus_{\chi\in\Upsilon_{\mathfrak{p}}} \bigcap^{r_{\mathfrak{p}}} U_{L_{\chi,\infty},S,T} \to \bigoplus_{\chi\in\Upsilon_{\mathfrak{p}}} \Big(\bigcap^{r_{\mathfrak{p}}} U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} Q(\Lambda_\chi) = \big(\textstyle\bigwedge^{r_{\mathfrak{p}}} U_{K_\infty,S,T}\big) \otimes_\Lambda Q(\Lambda_{\mathfrak{p}}).$$

(See Lemma 3.5.)

Lemma 3.10. *Let $\mathfrak{p}$ be a height-one prime ideal of Λ. When $\mathfrak{p}$ is singular, assume that the μ-invariant of $e_{\chi_{\mathfrak{p}}} A^T_S(K_\infty)$ (as $\mathbb{Z}_p[\![\Gamma]\!]$-module) vanishes.*

- (i) *The $\Lambda_{\mathfrak{p}}$-module $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is free of rank $r_{\mathfrak{p}}$.*
- (ii) *If Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for every χ in $\hat{\Delta}$ and every natural number n, then there is an inclusion*

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} \subset \big(\textstyle\bigwedge^{r_{\mathfrak{p}}}_\Lambda U_{K_\infty,S,T}\big)_{\mathfrak{p}}.$$

Proof. As in the proof of Lemma 3.5, we choose a representative $\psi_\infty : \Pi_\infty \to \Pi_\infty$ of $C_{K_\infty,S,T}$. We have the exact sequence

$$0 \to U_{K_\infty,S,T} \to \Pi_\infty \xrightarrow{\psi_\infty} \Pi_\infty \to H^1(C_{K_\infty,S,T}) \to 0. \tag{4}$$

If $\mathfrak{p}$ is regular, then $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring and the exact sequence (4) implies that the $\Lambda_{\mathfrak{p}}$-modules $(U_{K_\infty,S,T})_{\mathfrak{p}}$ and $\mathrm{im}(\psi_\infty)_{\mathfrak{p}}$ are free. Since $U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$ is isomorphic to $\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$, we also know that the rank of $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is $r_{\mathfrak{p}}$.

Suppose next that $\mathfrak{p}$ is singular. Since the μ-invariant of $e_{\chi_{\mathfrak{p}}} \mathcal{X}_{K_\infty,S\setminus V_{\mathfrak{p}}}$ vanishes, we apply Lemma 3.8 to deduce that $(\mathcal{X}_{K_\infty,S})_{\mathfrak{p}} = (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$. In a similar way, the assumption that the μ-invariant of $e_{\chi_{\mathfrak{p}}} A^T_S(K_\infty)$ vanishes implies that $A^T_S(K_\infty)_{\mathfrak{p}} = 0$. Hence we have $H^1(C_{K_\infty,S,T})_{\mathfrak{p}} = (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$. By assumption $(*)$, we know that $\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}}$ is projective as a Λ-module. This implies that $H^1(C_{K_\infty,S,T})_{\mathfrak{p}} = (\mathcal{Y}_{K_\infty,V_{\mathfrak{p}}})_{\mathfrak{p}}$ is a free $\Lambda_{\mathfrak{p}}$-module of rank $r_{\mathfrak{p}}$. By choosing splittings of the sequence (4), we then easily deduce that the $\Lambda_{\mathfrak{p}}$-modules $(U_{K_\infty,S,T})_{\mathfrak{p}}$ and $\mathrm{im}(\psi_\infty)_{\mathfrak{p}}$ are free and that the rank of $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is equal to $r_{\mathfrak{p}}$.

At this stage we have proved that, for any height-one prime ideal $\mathfrak{p}$ of Λ, the $\Lambda_{\mathfrak{p}}$-module $(U_{K_\infty,S,T})_{\mathfrak{p}}$ is both free of rank $r_{\mathfrak{p}}$ (as required to prove claim (i)) and also a direct summand of $(\Pi_\infty)_{\mathfrak{p}}$, and hence that

$$\big(\textstyle\bigwedge^{r_{\mathfrak{p}}}_\Lambda U_{K_\infty,S,T}\big)_{\mathfrak{p}} = \big(\textstyle\bigwedge^{r_{\mathfrak{p}}}_\Lambda U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})\big) \cap \big(\textstyle\bigwedge^{r_{\mathfrak{p}}}_\Lambda \Pi_\infty\big)_{\mathfrak{p}}. \tag{5}$$

Now we make the stated assumption concerning the validity of the p-part of the Rubin–Stark conjecture. This implies, by the proof of Theorem 3.4(i), that for each $\mathfrak{p}$ the element $\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}$ lies in both $(\bigwedge^{r_\mathfrak{p}}_\Lambda \Pi_\infty)_\mathfrak{p}$ and

$$\bigoplus_{\chi\in\Upsilon_\mathfrak{p}} \left(\bigwedge\nolimits^{r_\chi}_\Lambda U_{K_\infty,S,T}\right)\otimes_\Lambda Q(\Lambda_\chi) = \left(\bigwedge\nolimits^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\right)\otimes_\Lambda Q(\Lambda_\mathfrak{p}),$$

and hence, by (5) that it belongs to $(\bigwedge^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T})_\mathfrak{p}$, as required to prove claim (ii). □

We can now decompose Conjecture 3.1 into the statements for $\mathfrak{p}$ components.

Proposition 3.11. *Assume that Conjecture* $\mathrm{RS}(L_{\chi,n}/k,\, S,\, T,\, V_\chi)_p$ *holds for all characters* χ *in* $\hat{\Delta}$ *and all sufficiently large* n *and that for each character* χ *in* $\hat{\Delta}'/{\sim_{\mathbb{Q}_p}}$ *the* μ*-invariant of the* $\mathbb{Z}_p[[\Gamma]]$*-module* $e_\chi A^T_S(K_\infty)$ *vanishes. Then Conjecture 3.1 holds if and only if*

$$\Lambda_\mathfrak{p}\cdot\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}^{r_\mathfrak{p}}_\Lambda(H^1(C_{K_\infty,S,T}))_\mathfrak{p}\cdot\left(\bigwedge\nolimits^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\right)_\mathfrak{p} \tag{6}$$

for every height-one prime ideal $\mathfrak{p}$ *of* Λ.

Remark 3.12. At every height-one prime ideal $\mathfrak{p}$ there is an equality

$$\mathrm{Fitt}^{r_\mathfrak{p}}_\Lambda(H^1(C_{K_\infty,S,T}))_\mathfrak{p} = \mathrm{Fitt}^0_\Lambda(A^T_S(K_\infty))_\mathfrak{p}\mathrm{Fitt}^0_\Lambda(\mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}})_\mathfrak{p}\ .$$

If $\mathfrak{p}$ is regular, then $\Lambda_\mathfrak{p}$ is a discrete valuation ring and this equality follows directly from the exact sequence

$$0\to A^T_S(K_\infty)\to H^1(C_{K_\infty,S,T})\to\mathcal{X}_{K_\infty,S}\to 0.$$

If $\mathfrak{p}$ is singular, then the equality is valid since the result of Lemma 3.8 implies $(\mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}})_\mathfrak{p}$ vanishes and so $H^1(C_{K_\infty,S,T})_\mathfrak{p}$ is isomorphic to the direct sum $A^T_S(K_\infty)_\mathfrak{p}\oplus(\mathcal{Y}_{K_\infty,V_\mathfrak{p}})_\mathfrak{p}$.

Remark 3.13. If the prime $\mathfrak{p}$ is singular, then $(\mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}})_\mathfrak{p}$ vanishes and

$$\mathrm{Fitt}^0_\Lambda(A^T_S(K_\infty))_\mathfrak{p} = \Lambda_\mathfrak{p}$$

if the μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_\mathfrak{p}}A^T_S(K_\infty)$ vanishes (see Lemma 3.8). Thus, in this case, for any such $\mathfrak{p}$ the equality (6) is equivalent to

$$\Lambda_\mathfrak{p}\cdot\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \left(\bigwedge\nolimits^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\right)_\mathfrak{p}\ .$$

Thus, we know that by Lemma 3.10(ii) the validity of the p-part of the Rubin–Stark conjecture already gives strong evidence of the above equality.

Proof. Since $\det_\Lambda(C_{K_\infty,S,T})$ is an invertible Λ-module the equality $\Lambda\cdot\mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T})$ in Conjecture 3.1 is valid if and only if at every height-one prime ideal $\mathfrak{p}$ of Λ one has

$$\Lambda_\mathfrak{p}\cdot\mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T})_\mathfrak{p} \tag{7}$$

(see [Burns and Greither 2003, Lemma 6.1]).

If $\mathfrak{p}$ is regular, then one easily sees that this equality is valid if and only if the equality

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}^{r_{\mathfrak{p}}}_{\Lambda}(H^1(C_{K_\infty,S,T})) \cdot \big({\textstyle\bigwedge}^{r_{\mathfrak{p}}}_{\Lambda} U_{K_\infty,S,T}\big)_{\mathfrak{p}}$$

is valid, by using Theorem 3.4(ii).

If $\mathfrak{p}$ is singular, then the assumed vanishing of the μ-invariants and the argument in the proof of Lemma 3.10(i) together show that the $\Lambda_{\mathfrak{p}}$-modules $(U_{K_\infty,S,T})_{\mathfrak{p}}$ and $H^1(C_{K_\infty,S,T})_{\mathfrak{p}}$ are both free of rank $r_{\mathfrak{p}}$. Noting this, we see that (7) holds if and only if

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \big({\textstyle\bigwedge}^{r_{\mathfrak{p}}}_{\Lambda} U_{K_\infty,S,T}\big)_{\mathfrak{p}},$$

and so in this case the claimed result follows from Remark 3.13. □

3C3. In [BKS] we defined canonical Selmer modules $\mathcal{S}_{S,T}(\mathbb{G}_{m/F})$ and $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/F})$ for $\mathbb{G}_m$ over number fields F that are of finite degree over $\mathbb{Q}$. For any intermediate field L of K_∞/k, we now set

$$\mathcal{S}_{p,S,T}(\mathbb{G}_{m/L}) := \varprojlim_F \mathcal{S}_{S,T}(\mathbb{G}_{m/F}) \otimes \mathbb{Z}_p, \quad \mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L}) := \varprojlim_F \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/F}) \otimes \mathbb{Z}_p,$$

where in both limits F runs over all finite extensions of k in L and the transition morphisms are the natural corestriction maps.

We note in particular that, by its very definition, $\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})$ coincides with $H^1(C_{L,S,T})$. In addition, this definition implies that for any subset V of S comprising places that split completely in L the kernel of the natural (composite) projection map

$$\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})_V := \ker(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L}) \to \mathcal{X}_{L,S} \to \mathcal{Y}_{L,V})$$

lies in a canonical exact sequence of the form

$$0 \to A^T_S(L) \to \mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})_V \to \mathcal{X}_{L,S\setminus V} \to 0. \tag{8}$$

We now interpret our Iwasawa main conjecture in terms of classical characteristic ideals.

Conjecture 3.14 (IMC(K_∞/k, S, T) II). *Assume Conjecture* RS$(L_{\chi,n}/k, S, T, V_\chi)_p$ *holds for all* $\chi \in \hat{\Delta}$ *and all nonnegative integers* n *where* $L_{\chi,n}$, Δ, *etc. are defined in Section 3. Then for any* $\chi \in \hat{\Delta}$ *there are equalities*

$$\begin{aligned}\mathrm{char}_{\Lambda_\chi}\Big(\Big(\bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}/\langle \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}\rangle\Big)^{\chi}\Big) &\\ = \mathrm{char}_{\Lambda_\chi}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L_{\chi,\infty}})^{\chi}_{V_\chi}) &\\ = \mathrm{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^{\chi})\mathrm{char}_{\Lambda_\chi}((\mathcal{X}_{L_{\chi,\infty},S\setminus V_\chi})^{\chi}).&\end{aligned} \tag{9}$$

Here, for any $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-module M we write M^χ for the Λ_χ-module $M \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Z}_p[\operatorname{im}\chi]$ and $\operatorname{char}_{\Lambda_\chi}(M^\chi)$ for its characteristic ideal in Λ_χ. In addition, the second displayed equality is a direct consequence of the appropriate case of the exact sequence (8).

Proposition 3.15. *Assume that Conjecture* $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ *is valid for all characters χ in $\hat{\Delta}$ and all n and that for each character $\chi \in \hat{\Delta}'/\sim_{\mathbb{Q}_p}$ the μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_\chi A_S^T(K_\infty)$ vanishes. Then Conjecture 3.1 is equivalent to Conjecture 3.14.*

Proof. Note that by our assumption $\mu = 0$ we have $(\bigcap^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p} = (\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}$ for any height-one prime $\mathfrak{p}$, using (5). Thus, the equality (6) implies the equality (9) for any χ.

On the other hand, for a height-one regular prime $\mathfrak{p}$, we can regard $\mathfrak{p}$ to be a prime of Λ_χ for some χ, so the equality (9) implies the equality (6). For a singular prime $\mathfrak{p}$, by Lemma 3.8, (9) for any χ implies $(\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}/\langle \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}\rangle = 0$, thus the equality (6) by Remark 3.13.

The proposition therefore follows from Proposition 3.11. □

3D. ***The case of CM-fields.*** Concerning the minus components for CM-extensions, we can prove our equivariant main conjecture using the usual main conjecture proved by Wiles.

Theorem 3.16. *Suppose that p is odd, k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and K is CM. If the μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension K_∞/K vanishes, then the minus part of Conjecture 3.1 is valid for $(K_\infty/k, S, T)$.*

Proof. In fact, for an odd character χ, one has $r_\chi = 0$ and the Rubin–Stark elements are Stickelberger elements. Therefore, $\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}$ is the p-adic L-function of Deligne–Ribet.

We shall prove the equality (9) in Conjecture 3.14 for each odd $\chi \in \hat{\Delta}$. We fix such a character χ, and may take $K = L_\chi$ and $S = S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k) \cup S_p(k)$. Let S'_p be the set of p-adic primes which split completely in K. If $v \in S \setminus V_\chi$ is prime to p, it is ramified in $L_\chi = K$, so we have $\operatorname{char}_{\Lambda_\chi}(\mathcal{X}^\chi_{L_{\chi,\infty},S\setminus V_\chi}) = \operatorname{char}_{\Lambda_\chi}(\mathcal{Y}^\chi_{L_{\chi,\infty},S'_p})$. Let $A^T(L_{\chi,\infty})$ be the inverse limit of the p-component of the T-ray class group of the full integer ring of $L_{\chi,n}$. By sending the prime w above v in S'_p to the class of w, we obtain a homomorphism $\mathcal{Y}^\chi_{L_{\chi,\infty},S'_p} \to A^T(L_{\chi,\infty})^\chi$, which is known to be injective. Since the sequence

$$\mathcal{Y}^\chi_{L_{\chi,\infty},S} \to A^T(L_{\chi,\infty})^\chi \to A^T_S(L_{\chi,\infty})^\chi \to 0$$

is exact and the kernel of $\mathcal{Y}^\chi_{L_{\chi,\infty},S} \to \mathcal{Y}^\chi_{L_{\chi,\infty},S'_p}$ is finite, we have

$$\operatorname{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^\chi)\operatorname{char}_{\Lambda_\chi}((\mathcal{Y}_{L_{\chi,\infty},S})^\chi) = \operatorname{char}_{\Lambda_\chi}(A^T(L_{\chi,\infty})^\chi).$$

Therefore, by noting $\chi \neq 1$, the equality (9) in Conjecture 3.14 becomes

$$\mathrm{char}_{\Lambda_\chi}(A^T(L_{\chi,\infty})^\chi) = \theta^\chi_{L_{\chi,\infty}/k,S,T}(0)\Lambda_\chi,$$

where $\theta^\chi_{L_{\chi,\infty}/k,S,T}(0)$ is the χ-component of $\epsilon^\varnothing_{L_{\chi,\infty}/k,S,T}$, which is the Stickelberger element in this case. This equality is nothing but the usual main conjecture proved in [Wiles 1990], so we have proved this theorem. □

3E. *Consequences for number fields of finite degree.* Let p, k, k_∞, and K be as in Theorem 3.16. We shall describe unconditional equivariant results on the Galois module structure of Selmer modules for K, which follow from the validity of Theorem 3.16.

To do this we set $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ and for any Λ-module M we denote by M^- the minus part consisting of elements on which the complex conjugation acts as -1 (namely, $M^- = e^- M$). We note, in particular, that $\theta_{K_\infty/k,S,T}(0)$ belongs to Λ^-.

We also write $x \mapsto x^\#$ for the $\mathbb{Z}_p$-linear involutions of both Λ and the group rings $\mathbb{Z}_p[G]$ for finite quotients G of $\mathrm{Gal}(K_\infty/k)$ which is induced by inverting elements of $\mathrm{Gal}(K_\infty/k)$.

Corollary 3.17. *If the p-adic μ-invariant of K_∞/K vanishes, then*

$$\mathrm{Fitt}_{\Lambda^-}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/K_\infty})^-) = \Lambda \cdot \theta_{K_\infty/k,S,T}(0)$$

and

$$\mathrm{Fitt}_{\Lambda^-}(\mathcal{S}_{p,S,T}(\mathbb{G}_{m/K_\infty})^-) = \Lambda \cdot \theta_{K_\infty/k,S,T}(0)^\#.$$

Proof. Since $r_\chi = 0$ for any odd character χ, the first displayed equality is equivalent to Conjecture 3.1 in this case and is therefore valid as a consequence of Theorem 3.16.

The second displayed equality is then obtained directly by applying the general result of [BKS, Lemma 2.8] to the first equality. □

Corollary 3.18. *Let L be an intermediate CM-field of K_∞/k which is finite over k, and set $G := \mathrm{Gal}(L/k)$. If the p-adic μ-invariant of K_∞/K vanishes, then there are equalities*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})^-) = \mathbb{Z}_p[G] \cdot \theta_{L/k,S,T}(0)$$

and

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathcal{S}_{p,S,T}(\mathbb{G}_{m/L})^-) = \mathbb{Z}_p[G] \cdot \theta_{L/k,S,T}(0)^\#.$$

Proof. This follows by combining Corollary 3.17 with the general result of Lemma 3.19 below and standard properties of Fitting ideals. □

Lemma 3.19. *Suppose that L/k is a Galois extension of finite number fields with Galois group G. Then there are natural isomorphisms*

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})_G \xrightarrow{\sim} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/k}) \quad \textit{and} \quad \mathcal{S}_{S,T}(\mathbb{G}_{m/L})_G \xrightarrow{\sim} \mathcal{S}_{S,T}(\mathbb{G}_{m/k}).$$

Proof. The "Weil-étale cohomology complex" $R\Gamma_T((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ is perfect and so there exist projective $\mathbb{Z}[G]$-modules P_1 and P_2, and a homomorphism of $\mathbb{Z}[G]$-modules $P_1 \to P_2$ whose cokernel identifies with $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})$ and is such that the cokernel of the induced map $P_1^G \to P_2^G$ identifies with $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/k})$ (see [BKS, §5.4]).

The first isomorphism is then obtained by noting that the norm map induces an isomorphism of modules $(P_2)_G \xrightarrow{\sim} P_2^G$.

The second claimed isomorphism can also be obtained in a similar way, noting that $\mathcal{S}_{S,T}(\mathbb{G}_{m/L})$ is obtained as the cohomology in the highest (nonzero) degree of a perfect complex (see [BKS, Proposition 2.4]). □

We write $\mathcal{O}_L$ for the ring of integers of L and $\mathrm{Cl}^T(L)$ for the ray class group of $\mathcal{O}_L$ with modulus $\prod_{w \in T_L} w$. We denote the Sylow p-subgroup of $\mathrm{Cl}^T(L)$ by $A^T(L)$ and write $(A^T(L)^-)^\vee$ for the Pontrjagin dual of the minus part of $A^T(L)$.

The next corollary of Theorem 3.16 that we record coincides with one of the main results of [Greither and Popescu 2015].

Corollary 3.20. *Let L be an intermediate CM-field of K_∞/k which is finite over k, and set $G := \mathrm{Gal}(L/k)$. If the p-adic μ-invariant for K_∞/K vanishes, then*

$$\theta_{L/k,S,T}(0)^{\#} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A^T(L)^-)^\vee).$$

Proof. The canonical exact sequence

$$0 \to \mathrm{Cl}^T(L)^\vee \to \mathcal{S}_{S_\infty(k),T}(\mathbb{G}_{m/L}) \to \mathrm{Hom}(\mathcal{O}_L^\times, \mathbb{Z}) \to 0$$

from [BKS, Proposition 2.2] implies that the natural map $\mathcal{S}_{p,S_\infty(k),T}(\mathbb{G}_{m/L})^- \simeq (A^T(L)^-)^\vee$ is bijective.

In addition, from [BKS, Proposition 2.4(ii)], we know that the canonical homomorphism $\mathcal{S}_{S,T}(\mathbb{G}_{m/L}) \to \mathcal{S}_{S_\infty(k),T}(\mathbb{G}_{m/L})$ is surjective.

The claim therefore follows directly from the second equality in Corollary 3.18. □

Remark 3.21. (i) Our derivation of the equality in Corollary 3.20 differs from that given in [Greither and Popescu 2015] in that we avoid any use of Galois modules related to 1-motives. Instead, we used the theory of Selmer modules $\mathcal{S}_{S,T}(\mathbb{G}_{m/L})$ introduced in [BKS].

(ii) The Brumer–Stark conjecture predicts $\theta_{L/k,S_{\mathrm{ram}}(L/k),T}(0)$ belongs to the annihilator $\mathrm{Ann}_{\mathbb{Z}_p[G]^-}(A^T(L))$ and if no p-adic place of L^+ splits in L, then Corollary 3.20 implies a stronger version of this conjecture.

(iii) We have assumed throughout Section 3 that S contains all p-adic places of k and so the Stickelberger element $\theta_{L/k,S,T}(0)$ that occurs in Corollary 3.20 is, in general, imprimitive. In particular, if any p-adic place of k splits completely in L, then $\theta_{L/k,S,T}(0)$ vanishes and the assertion of Corollary 3.20 is trivially valid. However, by applying Corollary 1.2 and [BKS, Corollary 1.14] in this context, one can now also obtain results such as Corollary 1.3.

4. Iwasawa-theoretic Rubin–Stark congruences

In this section, we formulate an Iwasawa-theoretic version of the conjecture proposed in [Mazur and Rubin 2016] and [Sano 2014] (see also [BKS, Conjecture 5.4]). This conjecture is a natural generalization of the Gross–Stark conjecture [Gross 1982], and plays a key role in the descent argument that we present in the next section.

We maintain the notation of the previous section.

4A. ***Statement of the congruences.*** We first recall the formulation of the conjecture of Mazur, Rubin and of the third author.

Take a character $\chi \in \hat{\mathcal{G}}$. Take a proper subset $V' \subset S$ so that all $v \in V'$ splits completely in L_χ (i.e., $\chi(G_v) = 1$) and that $V_\chi \subset V'$. Put $r' := \#V'$. We recall the formulation of the conjecture of Mazur and Rubin and of the third author for $(L_{\chi,n}/L_\chi/k, S, T, V_\chi, V')$. For simplicity, put

- $L_n := L_{\chi,n}$;
- $L := L_\chi$;
- $\mathcal{G}_n := \mathcal{G}_{\chi,n} = \mathrm{Gal}(L_{\chi,n}/k)$;
- $G := G_\chi = \mathrm{Gal}(L_\chi/k)$;
- $\Gamma_n := \Gamma_{\chi,n} = \mathrm{Gal}(L_{\chi,n}/L_\chi)$;
- $V := V_\chi = \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$;
- $r := r_\chi = \#V_\chi$.

Put $e := r' - r$. Let $I(\Gamma_n)$ denote the augmentation ideal of $\mathbb{Z}_p[\Gamma_n]$. It is shown in [Sano 2014, Lemma 2.11] that there exists a canonical injection

$$\bigcap^r U_{L,S,T} \hookrightarrow \bigcap^r U_{L_n,S,T},$$

which induces the injection

$$\nu_n : \Big(\bigcap^r U_{L,S,T}\Big) \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1} \hookrightarrow \Big(\bigcap^r U_{L_n,S,T}\Big) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

Note that this injection does not coincide with the map induced by the inclusion

$U_{L,S,T} \hookrightarrow U_{L_n,S,T}$, and we have

$$\nu_n(\mathrm{N}^r_{L_n/L}(a)) = \mathrm{N}_{L_n/L}\, a$$

for all $a \in \bigcap^r U_{L_n,S,T}$ (see [Sano 2014, Remark 2.12]). For an explicit description of the map ν_n, see [Mazur and Rubin 2016, Lemma 4.9; Sano 2015, Remark 4.2].

Let I_n be the kernel of the natural map $\mathbb{Z}_p[\mathcal{G}_n] \to \mathbb{Z}_p[G]$. For $v \in V' \setminus V$, let $\mathrm{rec}_w : L^\times \to \Gamma_n$ denote the local reciprocity map at w (recall that w is the fixed place lying above v). Define

$$\mathrm{Rec}_w := \sum_{\sigma \in G} (\mathrm{rec}_w(\sigma(\cdot)) - 1)\sigma^{-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(L^\times, I_n/I_n^2).$$

It is shown in [Sano 2014, Proposition 2.7] that $\bigwedge_{v \in V' \setminus V} \mathrm{Rec}_w$ induces a homomorphism

$$\mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e / I(\Gamma_n)^{e+1}.$$

Finally, define

$$\mathcal{N}_n : \bigcap^{r} U_{L_n,S,T} \to \bigcap^{r} U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}$$

by

$$\mathcal{N}_n(a) := \sum_{\sigma \in \Gamma_n} \sigma a \otimes \sigma^{-1}.$$

We now state the formulation of [Sano 2014, Conjecture 3] (or [Mazur and Rubin 2016, Conjecture 5.2]).

Conjecture 4.1 ($\mathrm{MRS}(L_n/L/k, S, T, V, V')_p$). *Assume Conjectures* $\mathrm{RS}(L_n/k, S, T, V)_p$ *and* $\mathrm{RS}(L/k, S, T, V')_p$. *Then*

$$\mathcal{N}_n(\epsilon^V_{L_n/k,S,T}) = (-1)^{re} \nu_n(\mathrm{Rec}_n(\epsilon^{V'}_{L/k,S,T})) \ \ in \ \bigcap^{r} U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

(Note that the sign in the right-hand side depends on the labeling of S. We follow the convention in [BKS, §5.3].)

Note that [BKS, Conjecture $\mathrm{MRS}(K/L/k, S, T, V, V')$] is slightly stronger than the above conjecture (see [BKS, Remark 5.7]).

We shall next give an Iwasawa theoretic version of the above conjecture. Note that, since the inverse limit $\varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$ is isomorphic to $\mathbb{Z}_p$, the map

$$\varprojlim_n \mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$$

uniquely extends to give a $\mathbb{C}_p$-linear map

$$\mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p\big(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}\big),$$

which we denote by Rec_∞.

Conjecture 4.2 ($\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$). *Assume that Conjecture* $\mathrm{RS}(L_n/k, S, T, V)_p$ *is valid for all n. Then, there exists a (unique)*

$$\kappa = (\kappa_n)_n \in \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e / I(\Gamma_n)^{e+1}$$

such that $\nu_n(\kappa_n) = \mathcal{N}_n(\epsilon^V_{L_n/k,S,T})$ for all n and that

$$e_\chi \kappa = (-1)^{re} e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}) \text{ in } \mathbb{C}_p\Big(\bigwedge^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e / I(\Gamma_n)^{e+1}\Big).$$

Remark 4.3. Clearly the validity of Conjecture $\mathrm{MRS}(L_n/L/k, S, T, V, V')_p$ for all n implies the validity of $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$. A significant advantage of the above formulation of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is that we do not need to assume that Conjecture $\mathrm{RS}(L/k, S, T, V')_p$ is valid.

Proposition 4.4. (i) $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid if* $V = V'$.

(ii) $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *implies* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'')$ *if* $V \subset V'' \subset V'$.

(iii) *Suppose that $\chi(G_v) = 1$ for all $v \in S$ and $\#V' = \#S - 1$. Then, for any $V'' \subset S$ with $V \subset V''$ and $\#V'' = \#S - 1$,* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *and* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'')$ *are equivalent.*

(iv) $\mathrm{MRS}(K_\infty/k, S \setminus \{v\}, T, \chi, V' \setminus \{v\})$ *implies* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *if $v \in V' \setminus V$ is a finite place which is unramified in L_∞.*

(v) *If $\#V' \neq \#S - 1$ and $v \in S \setminus V'$ is a finite place which is unramified in L_∞, then* $\mathrm{MRS}(K_\infty/k, S \setminus \{v\}, T, \chi, V')$ *implies* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$.

Proof. Claim (i) follows from the "norm relation" of Rubin–Stark elements; see [Sano 2014, Remark 3.9; Mazur and Rubin 2016, Proposition 5.7]. Claim (ii) follows from [Sano 2014, Proposition 3.12]. Claim (iii) follows from [Sano 2015, Lemma 5.1]. Claim (iv) follows from the proof of [Sano 2014, Proposition 3.13]. Claim (v) follows by noting $\epsilon^V_{L_n/k,S,T} = (1 - \mathrm{Fr}_v^{-1})\epsilon^V_{L_n/k,S\setminus\{v\},T}$ and $\epsilon^{V'}_{L/k,S,T} = (1 - \mathrm{Fr}_v^{-1})\epsilon^{V'}_{L/k,S\setminus\{v\},T}$. □

Corollary 4.5. *If every place v in $V' \setminus V$ is both nonarchimedean and unramified in L_∞, then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*

Proof. By Proposition 4.4(iv), we may assume $V = V'$. By Proposition 4.4(i), we know that $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is valid in this case. □

Consider the following condition:

$$\mathrm{NTZ}(K_\infty/k, \chi) \quad \chi(G_{\mathfrak{p}}) \neq 1 \text{ for all } \mathfrak{p} \in S_p(k) \text{ which ramify in } L_{\chi,\infty}.$$

This condition is usually called "no trivial zeros".

Corollary 4.6. *If χ satisfies* $\mathrm{NTZ}(K_\infty/k,\chi)$, *then* $\mathrm{MRS}(K_\infty/k,S,T,\chi,V')$ *is valid.*

Proof. In this case we see that every $v\in V'\setminus V$ is finite and unramified in L_∞. $\square$

4B. ***Connection to the Gross–Stark conjecture.*** In this subsection we help set the context for Conjecture $\mathrm{MRS}(K_\infty/k,S,T,\chi,V')$ by showing that it specializes to recover the Gross–Stark conjecture (as stated in Conjecture 4.7 below).

To do this we assume throughout that k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and χ is totally odd. We also set $V':=\{v\in S\mid \chi(G_v)=1\}$ (and note that this is a proper subset of S since χ is totally odd) and we assume that every $v\in V'$ lies above p (noting that this assumption is not restrictive as a consequence of Proposition 4.4(iv)).

We shall now show that this case of $\mathrm{MRS}(K_\infty/k,S,T,\chi,V')$ is equivalent to the Gross–Stark conjecture.

First, we note that in this case V is empty (that is, $r=0$) and so one knows that Conjecture $\mathrm{RS}(L_n/k,S,T,V)_p$ is valid for all n (by [Rubin 1996, Theorem 3.3]). In fact, one has $\epsilon^V_{L_n/k,S,T}=\theta_{L_n/k,S,T}(0)\in\mathbb{Z}_p[\mathcal{G}_n]$ and, by [Mazur and Rubin 2016, Proposition 5.4], the assertion of Conjecture $\mathrm{MRS}(K_\infty/k,S,T,\chi,V')$ is equivalent to the following claims:

$$\theta_{L_n/k,S,T}(0)\in I_n^{r'} \tag{10}$$

for all n and

$$e_\chi\theta_{L_\infty/k,S,T}(0)=e_\chi\mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T})\text{ in }\mathbb{C}_p[G]\otimes_{\mathbb{Z}_p}\varprojlim_n I(\Gamma_n)^{r'}/I(\Gamma_n)^{r'+1}, \tag{11}$$

where we set

$$\theta_{L_\infty/k,S,T}(0):=\varprojlim_n\theta_{L_n/k,S,T}(0)\in\varprojlim_n I_n^{r'}/I_n^{r'+1}\simeq\mathbb{Z}_p[G]\otimes_{\mathbb{Z}_p}\varprojlim_n I(\Gamma_n)^{r'}/I(\Gamma_n)^{r'+1}.$$

We also note that the validity of (10) follows as a consequence of our Iwasawa main conjecture (Conjecture 3.1) by using Proposition 2.6(iii) and the result of [BKS, Lemma 5.20] (see the argument in Section 5C).

To study (11) we set $\chi_1:=\chi|_\Delta\in\hat{\Delta}$ and regard (as we may) the product $\chi_2:=\chi\chi_1^{-1}$ as a character of $\Gamma=\mathrm{Gal}(k_\infty/k)$.

Note that $\mathrm{Gal}(L_\infty/k)=G_{\chi_1}\times\Gamma_{\chi_1}$. Fix a topological generator $\gamma\in\Gamma_{\chi_1}$, and identify $\mathbb{Z}_p[\mathrm{im}(\chi_1)][\![\Gamma_{\chi_1}]\!]$ with the ring of power series $\mathbb{Z}_p[\mathrm{im}(\chi_1)][\![t]\!]$ via the correspondence $\gamma=1+t$.

We then define $g^{\chi_1}_{L_\infty/k,S,T}(t)$ to be the image of $\theta_{L_\infty/k,S,T}(0)$ under the map

$$\mathbb{Z}_p[\![\mathrm{Gal}(L_\infty/k)]\!]=\mathbb{Z}_p[G_{\chi_1}][\![\Gamma_{\chi_1}]\!]\to\mathbb{Z}_p[\mathrm{im}(\chi_1)][\![\Gamma_{\chi_1}]\!]=\mathbb{Z}_p[\mathrm{im}(\chi_1)][\![t]\!]$$

induced by χ_1. We recall that the p-adic L-function of Deligne–Ribet is defined by

$$L_{k,S,T,p}(\chi^{-1}\omega,s):=g^{\chi_1}_{L_\infty/k,S,T}(\chi_2(\gamma)\chi_{\mathrm{cyc}}(\gamma)^s-1),$$

where χ_{cyc} is the cyclotomic character; one can show this to be independent of the choice of γ.

The validity of (10) implies an inequality

$$\mathrm{ord}_{s=0} L_{k,S,T,p}(\chi^{-1}\omega, s) \geq r'. \tag{12}$$

It is known that (12) is a consequence of the Iwasawa main conjecture (in the sense of [Wiles 1990]), which is itself known to be valid when p is odd. In addition, Spiess [2014] proved that (12) is valid, including the case $p=2$, by using Shintani cocycles. In all cases, therefore, we can define

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) := \lim_{s\to 0} s^{-r'} L_{k,S,T,p}(\chi^{-1}\omega, s) \in \mathbb{C}_p.$$

For $v \in V'$, define

$$\mathrm{Log}_w : L^{\times} \to \mathbb{Z}_p[G]$$

by $\mathrm{Log}_w(a) := -\sum_{\sigma\in G} \log_p(\mathrm{N}_{L_w/\mathbb{Q}_p}(\sigma a))\sigma^{-1}$, where $\log_p : \mathbb{Q}_p^{\times} \to \mathbb{Z}_p$ is Iwasawa's logarithm (in the sense that $\log_p(p)=0$). We set

$$\mathrm{Log}_{V'} := \textstyle\bigwedge_{v\in V'} \mathrm{Log}_w : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p[G].$$

We shall denote the map $\mathbb{C}_p[G] \to \mathbb{C}_p$ induced by χ also by χ.

For $v \in V'$, we define

$$\mathrm{Ord}_w : L^{\times} \to \mathbb{Z}[G]$$

by $\mathrm{Ord}_w(a) := \sum_{\sigma\in G} \mathrm{ord}_w(\sigma a)\sigma^{-1}$, and set

$$\mathrm{Ord}_{V'} := \textstyle\bigwedge_{v\in V'} \mathrm{Ord}_w : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p[G].$$

On the χ-component, $\mathrm{Ord}_{V'}$ induces an isomorphism

$$\chi \circ \mathrm{Ord}_{V'} : e_\chi \mathbb{C}_p \textstyle\bigwedge^{r'} U_{L,S,T} \xrightarrow{\sim} \mathbb{C}_p.$$

Taking a nonzero element $x \in e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T}$, we define the $\mathcal{L}$-invariant by

$$\mathcal{L}(\chi) := \frac{\chi(\mathrm{Log}_{V'}(x))}{\chi(\mathrm{Ord}_{V'}(x))} \in \mathbb{C}_p.$$

Since $e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T}$ is a one-dimensional $\mathbb{C}_p$-vector space, we see that $\mathcal{L}(\chi)$ does not depend on the choice of x.

Then the Gross–Stark conjecture is stated as follows.

Conjecture 4.7 (GS(L/k, S, T, χ)). *One has*

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) = \mathcal{L}(\chi) L_{k,S\setminus V',T}(\chi^{-1}, 0).$$

Remark 4.8. This formulation constitutes a natural higher-rank generalization of the form of the Gross–Stark conjecture stated in [Dasgupta et al. 2011, Conjecture 1].

Letting $x = e_\chi \epsilon^{V'}_{L/k,S,T}$, we obtain

$$\chi(\mathrm{Log}_{V'}(\epsilon^{V'}_{L/k,S,T})) = \mathcal{L}(\chi) L_{k,S\setminus V',T}(\chi^{-1}, 0).$$

Thus we see that Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to the equality

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) = \chi(\mathrm{Log}_{V'}(\epsilon^{V'}_{L/k,S,T})).$$

Concerning the relation between Rec_∞ and $\mathrm{Log}_{V'}$, we note the fact

$$\chi_{\mathrm{cyc}}(\mathrm{rec}_w(a)) = \mathrm{N}_{L_w/\mathbb{Q}_p}(a)^{-1},$$

where $v \in V'$ and $a \in L^\times$.

Given this fact, it is straightforward to check (under the validity of (10)) that Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to (11).

At this stage we have therefore proved the following result.

Theorem 4.9. *Suppose that k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and χ is totally odd. Set $V' := \{v \in S \mid \chi(G_v) = 1\}$ and assume that every $v \in V'$ lies above p. Assume also that* (10) *is valid. Then Conjecture* $\mathrm{GS}(L/k, S, T, \chi)$ *is equivalent to Conjecture* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$.

4C. ***A proof in the case $k = \mathbb{Q}$.*** In [BKS, Corollary 1.2] the known validity of the eTNC for Tate motives over abelian fields is used to prove that Conjecture $\mathrm{MRS}(K/L/k, S, T, V, V')$ is valid in the case $k = \mathbb{Q}$.

In this subsection, we shall give a much simpler proof of the latter result which uses only Theorem 4.9, the known validity of the Gross–Stark conjecture over abelian fields and a classical result from [Solomon 1992].

We note that for any χ and n the Rubin–Stark conjecture is known to be true for $(L_{\chi,n}/\mathbb{Q}, S, T, V_\chi)$. In fact, in this setting the Rubin–Stark element is given by a cyclotomic unit when $r_\chi = 1$ and by the Stickelberger element when $r_\chi = 0$ (see [Popescu 2011, §4.2 and Example 3.2.10], for example).

Theorem 4.10. *Suppose that $k = \mathbb{Q}$. Then,* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*

Proof. By Proposition 4.4(ii), we may assume that V' is maximal, namely,

$$r' = \min\{\#\{v \in S \mid \chi(G_v) = 1\}, \#S - 1\}.$$

By Corollary 4.6, we may assume that $\chi(p) = 1$.

Suppose first that χ is odd. Since Conjecture $\mathrm{GS}(L/\mathbb{Q}, S, T, \chi)$ is valid (see [Gross 1982, §4]), Conjecture $\mathrm{MRS}(K_\infty/\mathbb{Q}, S, T, \chi, V')$ follows from Theorem 4.9.

Suppose next that $\chi = 1$. In this case we have $r' = \#S - 1$. We may assume $p \notin V'$ by Proposition 4.4(iii). In this case every $v \in V' \setminus V$ is unramified in L_∞. Hence, the theorem follows from Corollary 4.5.

Finally, suppose that $\chi \neq 1$ is even. By Proposition 4.4(iv) and (v), we may assume $S = \{\infty, p\} \cup S_{\mathrm{ram}}(L/\mathbb{Q})$ and $V' = \{\infty, p\}$. We label $S = \{v_0, v_1, \ldots\}$ so that $v_1 = \infty$ and $v_2 = p$.

Fix a topological generator γ of $\Gamma = \mathrm{Gal}(L_\infty/L)$. Then we construct an element $\kappa(L, \gamma) \in \varprojlim_n L^\times/(L^\times)^{p^n}$ as follows. Note that $\mathrm{N}_{L_n/L}(\epsilon^V_{L_n/\mathbb{Q},S,T})$ vanishes since $\chi(p) = 1$. So we can take $\beta_n \in L_n^\times$ such that $\beta_n^{\gamma-1} = \epsilon^V_{L_n/\mathbb{Q},S,T}$ (Hilbert's Theorem 90). Define

$$\kappa_n := \mathrm{N}_{L_n/L}(\beta_n) \in L^\times/(L^\times)^{p^n}.$$

This element is independent of the choice of β_n, and for any $m > n$ the natural map

$$L^\times/(L^\times)^{p^m} \to L^\times/(L^\times)^{p^n}$$

sends κ_m to κ_n. We define

$$\kappa(L, \gamma) := (\kappa_n)_n \in \varprojlim_n L^\times/(L^\times)^{p^n}.$$

Then, by [Solomon 1992, Proposition 2.3(i)], we know that

$$\kappa(L, \gamma) \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L\left[\frac{1}{p}\right]^\times \hookrightarrow \varprojlim_n L^\times/(L^\times)^{p^n}.$$

Fix a prime $\mathfrak{p}$ of L lying above p. Define

$$\mathrm{Ord}_{\mathfrak{p}} : L^\times \to \mathbb{Z}_p[G]$$

by $\mathrm{Ord}_{\mathfrak{p}}(a) := \sum_{\sigma \in G} \mathrm{ord}_{\mathfrak{p}}(\sigma a)\sigma^{-1}$. Similarly, define

$$\mathrm{Log}_{\mathfrak{p}} : L^\times \to \mathbb{Z}_p[G]$$

by $\mathrm{Log}_{\mathfrak{p}}(a) := -\sum_{\sigma \in G} \log_p(\iota_{\mathfrak{p}}(\sigma a))\sigma^{-1}$, where $\iota_{\mathfrak{p}} : L \hookrightarrow L_{\mathfrak{p}} = \mathbb{Q}_p$ is the natural embedding.

Then by [Solomon 1992, Theorem 2.1 and Remark 2.4], one deduces

$$\mathrm{Ord}_{\mathfrak{p}}(\kappa(L, \gamma)) = -\frac{1}{\log_p(\chi_{\mathrm{cyc}}(\gamma))}\mathrm{Log}_{\mathfrak{p}}(\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}).$$

From this, we have

$$\mathrm{Ord}_{\mathfrak{p}}(\kappa(L, \gamma)) \otimes (\gamma - 1) = -\mathrm{Rec}_{\mathfrak{p}}(\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}) \text{ in } \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2, \quad (13)$$

where $I(\Gamma)$ is the augmentation ideal of $\mathbb{Z}_p[[\Gamma]]$.

We know that $e_\chi \mathbb{C}_p U_{L,S}$ is a two-dimensional $\mathbb{C}_p$-vector space. Lemma 4.11 below shows that $\{e_\chi \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}, e_\chi \kappa(L,\gamma)\}$ is a $\mathbb{C}_p$-basis of this space. For simplicity, set $\epsilon^V_L := \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}$. Note that the isomorphism

$$\mathrm{Ord}_\mathfrak{p} : e_\chi \mathbb{C}_p \bigwedge^2 U_{L,S} \xrightarrow{\sim} e_\chi \mathbb{C}_p U_L$$

sends $e_\chi \epsilon^V_L \wedge \kappa(L,\gamma)$ to $-\chi(\mathrm{Ord}_\mathfrak{p}(\kappa(L,\gamma)))e_\chi \epsilon^V_L$. Since we have

$$\mathrm{Ord}_\mathfrak{p}(e_\chi \epsilon^{V'}_{L/\mathbb{Q},S,T}) = -e_\chi \epsilon^V_L$$

(see [Rubin 1996, Proposition 5.2; Sano 2014, Proposition 3.6]), we have

$$e_\chi \epsilon^{V'}_{L/\mathbb{Q},S,T} = -\chi(\mathrm{Ord}_\mathfrak{p}(\kappa(L,\gamma)))^{-1} e_\chi \epsilon^V_L \wedge \kappa(L,\gamma).$$

Hence we have

$$\begin{aligned}\mathrm{Rec}_\mathfrak{p}(e_\chi \epsilon^{V'}_{L/\mathbb{Q},S,T}) &= \chi(\mathrm{Ord}_\mathfrak{p}(\kappa(L,\gamma)))^{-1} e_\chi \kappa(L,\gamma) \cdot \mathrm{Rec}_\mathfrak{p}(\epsilon^V_L)\\ &= -e_\chi \kappa(L,\gamma) \otimes (\gamma - 1),\end{aligned}$$

where the first equality follows by noting that $\mathrm{Rec}_\mathfrak{p}(\kappa(L,\gamma)) = 0$ (since $\kappa(L,\gamma)$ lies in the universal norm by definition), and the second by (13).

Now, noting that

$$\nu_n : U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)/I(\Gamma_n)^2 \hookrightarrow U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^2$$

is induced by the inclusion map $L \hookrightarrow L_n$, and that

$$\mathcal{N}_n(\epsilon^V_{L_n/\mathbb{Q},S,T}) = \kappa_n \otimes (\gamma - 1),$$

it is easy to see that the element $\kappa := \kappa(L,\gamma) \otimes (\gamma - 1)$ has the properties in the statement of Conjecture MRS($K_\infty/\mathbb{Q}$, S, T, χ, V'). This completes the proof. □

Lemma 4.11. *Assume that $k = \mathbb{Q}$ and $\chi \neq 1$ is even such that $\chi(p) = 1$. Assume also that $S = \{\infty, p\} \cup S_{\mathrm{ram}}(L/\mathbb{Q})$. Then, $\{e_\chi \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}, e_\chi \kappa(L,\gamma)\}$ is a $\mathbb{C}_p$-basis of $e_\chi \mathbb{C}_p U_{L,S}$.*

Proof. This result follows from [Solomon 1994, Remark 4.4]. But we sketch another proof, essentially given in [Flach 2004].

In the next section, we define the "Bockstein map"

$$\beta : e_\chi \mathbb{C}_p U_{L,S} \to e_\chi \mathbb{C}_p(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2).$$

We see that β is injective on $e_\chi \mathbb{C}_p U_L$, and that $\ker \beta \simeq U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p$ where we put $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$ and $\mathbb{C}_p$ is regarded as a Λ-algebra via χ. Hence we have

$$e_\chi \mathbb{C}_p U_{L,S} = e_\chi \mathbb{C}_p U_L \oplus (U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p).$$

Since $e_\chi \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}$ is nonzero, this is a basis of $e_\chi \mathbb{C}_p U_{L,S\setminus\{p\}} = e_\chi \mathbb{C}_p U_L$. We prove that $e_\chi \kappa(L,\gamma)$ is a basis of $U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p$.

By using the exact sequence $0 \to U_{L_\infty,S} \xrightarrow{\gamma-1} U_{L_\infty,S} \to U_{L,S}$, we see that there exists a unique element $\alpha \in U_{L_\infty,S}$ such that $(\gamma-1)\alpha = \epsilon^V_{L_\infty/\mathbb{Q},S,T}$. By the cyclotomic Iwasawa main conjecture over $\mathbb{Q}$, we see that α is a basis of $U_{L_\infty,S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi}$, where $\mathfrak{p}_\chi := \ker(\chi : \Lambda \to \mathbb{C}_p)$. The image of α under the map

$$U_{L_\infty,S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi} \xrightarrow{\chi} U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p \hookrightarrow e_\chi \mathbb{C}_p U_{L,S}$$

is equal to $e_\chi \kappa(L,\gamma)$. □

5. A strategy for proving the eTNC

5A. *Statement of the main result and applications.* In the sequel we fix an intermediate field L of K_∞/k which is finite over k and set $G := \mathrm{Gal}(L/k)$. In this section we always assume the following conditions to be satisfied:

(R) For every $\chi \in \hat{G}$, one has $r_{\chi,S} < \#S$.

(S) No finite place of k splits completely in k_∞.

Remark 5.1. Before proceeding we note that the condition (R) is very mild since it is automatically satisfied when the class number of k is equal to one and, for any k, is satisfied when S is large enough. We also note that the condition (S) is satisfied when, for example, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension.

The following result is one of the main results of this article and, as we will see, it provides an effective strategy for proving the special case of the eTNC that we are considering here.

Theorem 5.2. *Assume the following conditions*:

(hIMC) *The main conjecture* $\mathrm{IMC}(K_\infty/k, S, T)$ (Conjecture 3.1) *is valid.*

(F) *For every χ in $\hat{G}$, the module of Γ_χ-coinvariants of $A^T_S(L_{\chi,\infty})$ is finite.*

(MRS) *For every χ in $\hat{G}$, Conjecture* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'_\chi)$ (Conjecture 4.2) *is valid for a maximal set V'_χ, so that*

$$\#V'_\chi = \min\{\#\{v \in S \mid \chi(G_v) = 1\}, \#S - 1\}.$$

Then, the conjecture $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ (*Conjecture 2.3*) *is valid.*

Remark 5.3. We note that the set V'_χ in condition (MRS) is not uniquely determined when every place v in S satisfies $\chi(G_v) = 1$, but that the validity of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'_\chi)$ is independent of the choice of V'_χ (by Proposition 4.4(iii)).

Remark 5.4. One checks easily that the condition (F) is equivalent to the finiteness of the module of Γ_χ-coinvariants of $A_S(L_{\chi,\infty})$. Hence, taking account of [1991, Theorem 1.14], the condition (F) can be regarded as a natural generalization of

Conjecture 1.15 of [Gross 1982]. We also note here that this conjecture of Gross was asserted in a special setting as Conjecture 2.2 in [Coates and Lichtenbaum 1973]. In particular, we recall that the condition (F) is satisfied in each of the following cases:

- L is abelian over $\mathbb{Q}$ (this is due to Greenberg [1973]).
- k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension and L has unique p-adic place (in this case "$\delta_L = 0$" holds obviously; see [Kolster 1991]).
- L is totally real and the Leopoldt conjecture is valid for L at p (see [Kolster 1991, Corollary 1.3]).

Remark 5.5. The condition (MRS) is satisfied for χ in $\hat{G}$ when the condition $\mathrm{NTZ}(K_\infty/k, \chi)$ is satisfied (see Corollary 4.6).

As an immediate corollary of Theorem 5.2, we obtain a new proof of a theorem that was first proved in [Burns and Greither 2003] for p odd and in [Flach 2011] for $p = 2$.

Corollary 5.6. *If $k = \mathbb{Q}$, then the conjecture* $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G])$ *is valid.*

Proof. As we mentioned above, the conditions (R), (S) and (F) are all satisfied in this case. In addition, the condition (hIMC) is a direct consequence of the classical Iwasawa main conjecture solved by Mazur and Wiles (see [Burns and Greither 2003; Flach 2011]) and the condition (MRS) is satisfied by Theorem 4.10. □

We also obtain a result over totally real fields.

Corollary 5.7. *Suppose that p is odd, k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and K is CM. Assume that (F) is satisfied, that the μ-invariant of K_∞/K vanishes, and that for every odd character $\chi \in \hat{G}$, Conjecture* $\mathrm{GS}(L_\chi/k, S, T, \chi)$ *is valid. Then, Conjecture* $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G]^-)$ *is valid.*

Proof. Fix S so that the condition (R) is satisfied. Then the minus-part of condition (hIMC) is satisfied by Theorem 3.16 and the minus part of condition (MRS) by Theorem 4.9. □

When at most one p-adic place $\mathfrak{p}$ of k satisfies $\chi(G_\mathfrak{p}) = 1$, the validity of Conjecture $\mathrm{GS}(L_\chi/k, S, T, \chi)$ was proved by Dasgupta, Darmon and Pollack [Dasgupta et al. 2011] under certain assumptions, including Leopoldt's conjecture. Those assumptions were removed in [Ventullo 2015], so Conjecture $\mathrm{GS}(L_\chi/k, S, T, \chi)$ is unconditionally valid in this case (see also the note on p. 1531). Condition (F) is then valid too, by the argument of [Gross 1982, Proposition 2.13]. Hence we get:

Corollary 5.8. *Suppose that p is odd, k is totally real, k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension, and K is CM. Assume that the μ-invariant of K_∞/K vanishes, and that for each odd character $\chi \in \hat{G}$ there is at most one p-adic place $\mathfrak{p}$ of k which satisfies $\chi(G_\mathfrak{p}) = 1$. Then, Conjecture* $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G]^-)$ *is valid.*

Examples 5.9. It is not difficult to find many concrete families of examples satisfying the hypotheses of Corollary 5.8 and hence to deduce the unconditional validity of $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]^-)$ in some new and interesting cases. In particular, we shall now describe several families of examples in which the extension $k/\mathbb{Q}$ is not abelian (noting that if $L/\mathbb{Q}$ is abelian and $k \subset L$, then $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ is already known to be valid).

(i) The case $p = 3$. As a simple example, we consider the case that $k/\mathbb{Q}$ is a S_3-extension. To do this we fix an irreducible cubic polynomial $f(x)$ in $\mathbb{Z}[x]$ with discriminant $27d$ where d is strictly positive and congruent to 2 modulo 3. (For example, one can take $f(x)$ to be $x^3 - 6x - 3$, $x^3 - 15x - 3$, etc.) The minimal splitting field k of $f(x)$ over $\mathbb{Q}$ is then totally real (since $27d > 0$) and an S_3-extension of $\mathbb{Q}$ (since $27d$ is not a square). Also, since the discriminant of $f(x)$ is divisible by 27 but not 81, the prime 3 is totally ramified in k. Now set $p := 3$ and $K := k(\mu_p) = k(\sqrt{-p}) = k(\sqrt{-d})$. Then the prime above p splits in K/k because $-d \equiv 1 \pmod 3$. In addition, as $K/\mathbb{Q}(\sqrt{d}, \sqrt{-p})$ is a cyclic cubic extension, the μ-invariant of K_∞/K vanishes and so the extension K/k satisfies all the conditions of Corollary 5.8 (with $p = 3$).

(ii) The case $p > 3$. In this case one can construct a suitable field K in the following way. Fix a primitive p-th root of unity ζ, an integer i such that $1 \le i \le (p-3)/2$, and an integer b which is prime to p, and then set

$$a := \frac{1 + b(\zeta - 1)^{2i+1}}{1 + b(\zeta^{-1} - 1)^{2i+1}}.$$

Write ord_π for the normalized additive valuation of $\mathbb{Q}(\mu_p)$ associated to the prime element $\pi = \zeta - 1$. Then, since $\mathrm{ord}_\pi(a - 1) = 2i + 1 < p$, (π) is totally ramified in $\mathbb{Q}(\mu_p, \sqrt[p]{a})/\mathbb{Q}(\mu_p)$. Also, since $c(a) = a^{-1}$ where c is the complex conjugation, $\mathbb{Q}(\mu_p, \sqrt[p]{a})$ is the composite of a cyclic extension of $\mathbb{Q}(\mu_p)^+$ of degree p and $\mathbb{Q}(\mu_p)$. This shows that $\mathbb{Q}(\mu_p, \sqrt[p]{a})$ is a CM-field and, since $1 < 2i + 1 < p$, the extension $\mathbb{Q}(\mu_p, \sqrt[p]{a})^+/\mathbb{Q}$ is nonabelian. We now take a negative integer $-d$ which is a quadratic residue modulo p, let K denote the CM-field $\mathbb{Q}(\mu_p, \sqrt[p]{a}, \sqrt{-d})$ and set $k := K^+$. Then p is totally ramified in $k/\mathbb{Q}$ and the p-adic prime of k splits in K. In addition, $k/\mathbb{Q}$ is not abelian and the μ-invariant of K_∞/K vanishes since $K/\mathbb{Q}(\mu_p, \sqrt{-d})$ is cyclic of degree p. This shows that the extension K/k satisfies all of the hypotheses of Corollary 5.8.

(iii) In cases (i) and (ii) above, p is totally ramified in the extension $k_\infty/\mathbb{Q}$ and so Corollary 5.8 implies that $\mathrm{eTNC}(h^0(\mathrm{Spec}\, K_n), \mathbb{Z}_p[G]^-)$ is valid for any nonnegative integer n. In addition, if F is any real abelian field of degree prime to $[k : \mathbb{Q}]$ in which p is totally ramified, the minus component of the p-part of eTNC for FK_n/k holds for any nonnegative integer n.

Remark 5.10. By using similar methods to the proofs of the above corollaries it is also possible to deduce the main result of [Bley 2006] as a consequence of Theorem 5.2. In this case k is imaginary quadratic, the validity of (hIMC) can be derived from [Rubin 1991] (as explained in [Bley 2006]), and the conjecture (MRS) from the result in [Bley 2004], which is itself an analogue of Solomon's theorem [1992] for elliptic units, by using the same argument as Theorem 4.10.

5B. ***A computation of Bockstein maps.*** Fix a character $\chi \in \hat{G}$ and set

$$\begin{aligned} L_n &:= L_{\chi,n}, \\ L &:= L_\chi, \\ V &:= V_\chi = \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}, \\ r &:= r_\chi = \# V_\chi, \\ V' &:= V'_\chi \qquad \text{(as in (MRS) in Theorem 5.2)}, \\ r' &:= r_{\chi,S} = \# V', \\ e &:= r' - r. \end{aligned}$$

As in Section 4A, we label $S = \{v_0, v_1, \ldots\}$ so that $V = \{v_1, \ldots, v_r\}$ and $V' = \{v_1, \ldots, v_{r'}\}$, and fix a place w lying above each $v \in S$. Also, as in Section 2D, it will be useful to fix a representative $\Pi_{K_\infty} \to \Pi_{K_\infty}$ of $C_{K_\infty,S,T}$ where the first term is placed in degree zero, and Π_{K_∞} is a free Λ-module with basis $\{b_1, \ldots, b_d\}$. This representative is chosen so that the natural surjection

$$\Pi_{K_\infty} \to H^1(C_{K_\infty,S,T}) \to \mathcal{X}_{K_\infty,S}$$

sends b_i to $w_i - w_0$ for every i with $1 \le i \le r'$.

We define a height-one regular prime ideal of Λ by setting

$$\mathfrak{p} := \ker(\Lambda \xrightarrow{\chi} \mathbb{Q}_p(\chi) := \mathbb{Q}_p(\operatorname{im}\chi)).$$

Then the localization $R := \Lambda_{\mathfrak{p}}$ is a discrete valuation ring and we write P for its maximal ideal. We see that χ induces an isomorphism

$$E := R/P \xrightarrow{\sim} \mathbb{Q}_p(\chi).$$

We set $C := C_{K_\infty,S,T} \otimes_\Lambda R$ and $\Pi := \Pi_{K_\infty} \otimes_\Lambda R$.

Lemma 5.11. *Let γ be a topological generator of $\Gamma = \mathrm{Gal}(K_\infty/K)$. Let n be an integer which satisfies $\gamma^{p^n} \in \mathrm{Gal}(K_\infty/L)$. Then $\gamma^{p^n} - 1$ is a uniformizer of R.*

Proof. Regard $\chi \in \hat{\mathcal{G}}$, and put $\chi_1 := \chi|_\Delta \in \hat{\Delta}$. We identify R with the localization of $\Lambda_{\chi_1}[1/p] = \mathbb{Z}_p[\operatorname{im}\chi_1][[\Gamma]][1/p]$ at $\mathfrak{q} := \ker(\Lambda_{\chi_1}[1/p] \xrightarrow{\chi|_\Gamma} \mathbb{Q}_p(\chi))$.

Then the lemma follows by noting the localization of $\Lambda_{\chi_1}[1/p]/(\gamma^{p^n} - 1) = \mathbb{Z}_p[\operatorname{im}\chi_1][\Gamma_n][1/p]$ at $\mathfrak{q}$ is identified with $\mathbb{Q}_p(\chi)$. □

Lemma 5.12. *Assume that the condition* (F) *is satisfied.*

(i) *$H^0(C)$ is isomorphic to $U_{K_\infty,S,T} \otimes_\Lambda R$, and R-free of rank r.*

(ii) *$H^1(C)$ is isomorphic to $\mathcal{X}_{K_\infty,S} \otimes_\Lambda R$.*

(iii) *The maximal R-torsion submodule $H^1(C)_{\mathrm{tors}}$ of $H^1(C)$ is isomorphic to $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$, and annihilated by P. (So $H^1(C)_{\mathrm{tors}}$ is an E-vector space.)*

(iv) *$H^1(C)_{\mathrm{tf}} := H^1(C)/H^1(C)_{\mathrm{tors}}$ is isomorphic to $\mathcal{Y}_{K_\infty,V} \otimes_\Lambda R$ and is therefore R-free of rank r.*

(v) $\dim_E(H^1(C)_{\mathrm{tors}}) = e$.

Proof. Since $U_{K_\infty,S,T} \otimes_\Lambda R = H^0(C)$ is regarded as a submodule of Π, we see that $U_{K_\infty,S,T} \otimes_\Lambda R$ is R-free. Put $\chi_1 := \chi|_\Delta \in \hat{\Delta}$. Note that $L_\infty := L_{\chi,\infty} = L_{\chi_1,\infty}$, and that the quotient field of R is $Q(\Lambda_{\chi_1})$. As in the proof of Theorem 3.4, we have

$$U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\chi_1}) \simeq \mathcal{Y}_{L_\infty,V} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_{\chi_1}).$$

These are r-dimensional $Q(\Lambda_{\chi_1})$-vector spaces. This proves (i).

To prove (ii), it is sufficient to show that $A_S^T(K_\infty) \otimes_\Lambda R = 0$. Fix a topological generator γ of Γ, and regard $\mathbb{Z}_p[[\Gamma]]$ as the ring of power series $\mathbb{Z}_p[[t]]$ via the identification $\gamma = 1 + t$. Let f be the characteristic polynomial of the $\mathbb{Z}_p[[t]]$-module $A_S^T(L_\infty)$. By Lemma 5.11, for sufficiently large n, $\gamma^{p^n} - 1$ is a uniformizer of R. On the other hand, by the assumption (F), we see that f is prime to $\gamma^{p^n} - 1$. This implies (ii).

We prove (iii). Proving that $H^1(C)_{\mathrm{tors}}$ is isomorphic to $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$, it is sufficient to show that

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_{\chi_1}) \simeq \mathcal{Y}_{K_\infty,V} \otimes_\Lambda Q(\Lambda_{\chi_1}),$$

by (ii). This was shown in the proof of Theorem 3.4. We prove that $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$ is annihilated by P. Note that $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R = \mathcal{X}_{K_\infty,S\setminus(V\cup S_\infty)} \otimes_\Lambda R$, since the complex conjugation c at $v \in S_\infty \setminus (V \cap S_\infty)$ is nontrivial in G_{χ_1}, and hence $c - 1 \in R^\times$. Hence, it is sufficient to show that, for every $v \in S \setminus (V \cup S_\infty)$, there exists $\sigma \in G_v \cap \Gamma$ such that $\sigma - 1$ is a uniformizer of R, where $G_v \subset \mathcal{G}$ is the decomposition group at a place of K_∞ lying above v. Thanks to the assumption (S), we find such σ by Lemma 5.11.

The assertion (iv) is immediate from the above argument.

The assertion (v) follows from (iii), (iv), and the fact that

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda E \simeq \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S} \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{Y}_{L,V'}$$

is an r'-dimensional E-vector space. □

In the following for any R-module M we often denote $M \otimes_R E$ by M_E. Also, we assume that (F) is satisfied.

Definition 5.13. The "Bockstein map" is the homomorphism

$$\beta : H^0(C_E) \to H^1(C \otimes_R P) = H^1(C) \otimes_R P \to H^1(C_E) \otimes_E P/P^2$$

induced by the natural exact triangle $C \otimes_R P \to C \to C_E$.

Note that there are canonical isomorphisms

$$H^0(C_E) \simeq U_{L,S,T} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi) U_{L,S,T},$$
$$H^1(C_E) \simeq \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S} \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{Y}_{L,V'},$$

where $\mathbb{Q}_p(\chi)$ is regarded as a $\mathbb{Z}_p[G_\chi]$-algebra via χ. Note also that P is generated by $\gamma^{p^n} - 1$ with sufficiently large n, where γ is a fixed topological generator of Γ (see Lemma 5.11). There is a canonical isomorphism

$$I(\Gamma_\chi)/I(\Gamma_\chi)^2 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\chi) \simeq P/P^2,$$

where $I(\Gamma_\chi)$ denotes the augmentation ideal of $\mathbb{Z}_p[\![\Gamma_\chi]\!]$ (note that $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\Gamma_\chi = \mathrm{Gal}(L_\infty/L)$). Thus, the Bockstein map is regarded as the map

$$\begin{aligned}\beta : e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} &\to e_\chi \mathbb{Q}_p(\chi)(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2)\\ &\simeq e_\chi \mathbb{Q}_p(\chi)(\mathcal{Y}_{L,V'} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2).\end{aligned}$$

Proposition 5.14. *The Bockstein map β is induced by the map*

$$U_{L,S,T} \to \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2$$

given by $a \mapsto \sum_{w \in S_L} w \otimes (\mathrm{rec}_w(a) - 1)$.

Proof. The proof is the same as for [Flach 2004, Lemma 5.8] and we sketch the proof therein.

Take n so that the image of $\gamma^{p^n} \in \mathrm{Gal}(K_\infty/L)$ in $\mathrm{Gal}(L_\infty/L) = \Gamma_\chi$ is a generator. We regard $\gamma^{p^n} \in \Gamma_\chi$. Define $\theta \in H^1(L, \mathbb{Z}_p) = \mathrm{Hom}(G_L, \mathbb{Z}_p)$ by $\gamma^{p^n} \mapsto 1$. Define

$$\beta' : e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} \to e_\chi \mathbb{Q}_p(\chi)(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \xrightarrow{\sim} e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S}$$

by $\beta(a) = \beta'(a) \otimes (\gamma^{p^n} - 1)$. Then, β' is induced by the cup product

$$\cdot \cup \theta : \mathbb{Q}_p U_{L,S} \simeq H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \to H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p \mathcal{X}_{L,S \setminus S_\infty}.$$

By class field theory we see that β is induced by the map

$$a \mapsto \sum_{w \in S_L \setminus S_\infty(L)} w \otimes (\mathrm{rec}_w(a) - 1).$$

Since $\mathrm{rec}_w(a) = 1 \in \Gamma_\chi$ for all $w \in S_\infty(L)$, the proposition follows. □

Proposition 5.15. *We have canonical isomorphisms*

$$\ker \beta \simeq H^0(C)_E \quad \textit{and} \quad \mathrm{coker}\, \beta \simeq H^1(C)_{\mathrm{tf}} \otimes_R P/P^2.$$

Proof. Let δ be the boundary map $H^0(C_E) \to H^1(C \otimes_R P) = H^1(C) \otimes_R P$. We have

$$\ker \delta \simeq \operatorname{coker}(H^0(C \otimes_R P) \to H^0(C)) = H^0(C)_E$$

and

$$\operatorname{im} \delta = \ker(H^1(C) \otimes_R P \to H^1(C)) = H^1(C)[P] \otimes_R P,$$

where $H^1(C)[P]$ is the submodule of $H^1(C)$ which is annihilated by P. By Lemma 5.12(iii), we know $H^1(C)[P] = H^1(C)_{\mathrm{tors}}$. Hence, the natural map

$$H^1(C) \otimes_R P \to H^1(C) \otimes_R P/P^2 \simeq H^1(C)_E \otimes_E P/P^2 \simeq H^1(C_E) \otimes_E P/P^2$$

is injective on $H^1(C)_{\mathrm{tors}} \otimes_R P$. From this we see that $\ker \beta \simeq H^0(C)_E$. We also have

$$\operatorname{coker} \beta \simeq \operatorname{coker}(H^1(C)_{\mathrm{tors}} \otimes_R P \to H^1(C) \otimes_R P/P^2) \simeq H^1(C)_{\mathrm{tf}} \otimes_R P/P^2.$$

Hence we have completed the proof. $\square$

By Lemma 5.12, we see that there are canonical isomorphisms

$$\begin{aligned} H^0(C)_E &\simeq U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi), \\ H^1(C)_E &\simeq \mathcal{X}_{K_\infty,S} \otimes_\Lambda \mathbb{Q}_p(\chi), \\ H^1(C)_{\mathrm{tf},E} &\simeq \mathcal{Y}_{K_\infty,V} \otimes_\Lambda \mathbb{Q}_p(\chi). \end{aligned}$$

Hence, by Proposition 5.15, we have the exact sequence

$$\begin{aligned} 0 \to U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi) &\to e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} \\ &\xrightarrow{\beta} e_\chi \mathbb{Q}_p(\chi)(\mathcal{Y}_{L,V'} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \to \mathcal{Y}_{K_\infty,V} \otimes_\Lambda P/P^2 \to 0. \end{aligned}$$

This induces an isomorphism

$$\begin{aligned} \tilde{\beta} : e_\chi \mathbb{Q}_p(\chi)\big(\textstyle\bigwedge^{r'} U_{L,S,T} \otimes \bigwedge^{r'} \mathcal{Y}_{L,V'}^*\big) \\ \xrightarrow{\sim} \textstyle\bigwedge^{r} (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes \bigwedge^{r} (\mathcal{Y}_{K_\infty,V}^* \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1}. \end{aligned}$$

We have isomorphisms

$$\begin{aligned} \textstyle\bigwedge^{r'} \mathcal{Y}_{L,V'}^* &\xrightarrow{\sim} \mathbb{Z}_p[G_\chi], \quad w_1^* \wedge \cdots \wedge w_{r'}^* \mapsto 1, \\ \textstyle\bigwedge^{r} (\mathcal{Y}_{K_\infty,V}^* \otimes_\Lambda \mathbb{Q}_p(\chi)) &\xrightarrow{\sim} \mathbb{Q}_p(\chi), \qquad w_1^* \wedge \cdots \wedge w_r^* \mapsto 1. \end{aligned}$$

By these isomorphisms, we see that $\tilde{\beta}$ induces an isomorphism

$$e_\chi \mathbb{Q}_p(\chi) \textstyle\bigwedge^{r'} U_{L,S,T} \xrightarrow{\sim} \bigwedge^{r} (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1},$$

which we denote also by $\tilde{\beta}$. Note that we have a natural injection

$$\textstyle\bigwedge^{r} (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1} \hookrightarrow e_\chi \mathbb{Q}_p(\chi)\big(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e/I(\Gamma_\chi)^{e+1}\big).$$

Composing this with $\tilde{\beta}$, we have an injection

$$\tilde{\beta} : e_\chi \mathbb{Q}_p(\chi) \bigwedge^{r'} U_{L,S,T} \hookrightarrow e_\chi \mathbb{Q}_p(\chi)\big(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e / I(\Gamma_\chi)^{e+1}\big).$$

By Proposition 5.14, we obtain the following.

Proposition 5.16. *Let*

$$\mathrm{Rec}_\infty : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p\big(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e / I(\Gamma_\chi)^{e+1}\big)$$

be the map defined in Section 4A. Then we have

$$(-1)^{re} e_\chi \mathrm{Rec}_\infty = \tilde{\beta}.$$

In particular, $e_\chi \mathrm{Rec}_\infty$ *is injective.*

5C. ***The proof of the main result.*** In this section we prove Theorem 5.2.

We start with an important technical observation. Let Π_n denote the free $\mathbb{Z}_p[\mathcal{G}_{\chi,n}]$-module $\Pi_{K_\infty} \otimes_\Lambda \mathbb{Z}_p[\mathcal{G}_{\chi,n}]$, and $I(\Gamma_{\chi,n})$ denote the augmentation ideal of $\mathbb{Z}_p[\Gamma_{\chi,n}]$.

We recall from [BKS, Lemma 5.20] that the image of

$$\pi^V_{L_n/k,S,T} : \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_n,S,T}) \to \bigwedge^r \Pi_n$$

is contained in $I(\Gamma_{\chi,n})^e \cdot \bigwedge^r \Pi_n$ (see Proposition 2.6(iii)) and also from [BKS, Proposition 4.17] that $\nu_n^{-1} \circ \mathcal{N}_n$ induces the map

$$I(\Gamma_{\chi,n})^e \cdot \bigwedge^r \Pi_n \to \bigwedge^r \Pi_0 \otimes_{\mathbb{Z}_p} I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1}.$$

Lemma 5.17. *There exists a commutative diagram*

$$\begin{tikzcd}
\det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_n,S,T}) \arrow[rr] \arrow[d, "\pi^V_{L_n/k,S,T}"'] & & \det_{\mathbb{Z}_p[G_\chi]}(C_{L,S,T}) \arrow[d, "\pi^{V'}_{L/k,S,T}"] \\
I(\Gamma_{\chi,n})^e \cdot \bigwedge^r \Pi_n \arrow[d, "\nu_n^{-1} \circ \mathcal{N}_n"'] & & \bigcap^{r'} U_{L,S,T} \arrow[d, "(-1)^{re}\mathrm{Rec}_n"] \\
\bigwedge^r \Pi_0 \otimes_{\mathbb{Z}_p} I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1} & & \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}} I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1}. \arrow[ll, "\supset"']
\end{tikzcd}$$

Proof. This follows from Proposition 2.6(iii) and [BKS, Lemma 5.22]. □

For any intermediate field F of K_∞/k, we denote by $\mathcal{L}_{F/k,S,T}$ the image of the (conjectured) element $\mathcal{L}_{K_\infty/k,S,T}$ of $\det_\Lambda(C_{K_\infty,S,T})$ under the isomorphism

$$\mathbb{Z}_p[\![\mathrm{Gal}(F/k)]\!] \otimes_\Lambda \det_\Lambda(C_{K_\infty,S,T}) \simeq \det_{\mathbb{Z}_p[\![\mathrm{Gal}(F/k)]\!]}(C_{F,S,T}).$$

Note that we have

$$\pi^V_{L_n/k,S,T}(\mathcal{L}_{L_n/k,S,T}) = \epsilon^V_{L_n/k,S,T}.$$

Hence, Lemma 5.17 implies that

$$(-1)^{re}\mathrm{Rec}_n(\pi^{V'}_{L/k,S,T}(\mathcal{L}_{L/k,S,T})) = \nu_n^{-1}\circ\mathcal{N}_n(\epsilon^{V}_{L_n/k,S,T}) =: \kappa_n.$$

We set

$$\kappa := (\kappa_n)_n \in \bigcap^{r} U_{L,S,T}\otimes_{\mathbb{Z}_p}\varprojlim_n I(\Gamma_{\chi,n})^e/I(\Gamma_{\chi,n})^{e+1}.$$

Then the validity of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ implies that

$$e_\chi\kappa = (-1)^{re}e_\chi\mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}).$$

In addition, by Proposition 5.16, we know that $e_\chi\mathrm{Rec}_\infty$ is injective, and so

$$\pi^{V'}_{L/k,S,T}(e_\chi\mathcal{L}_{L/k,S,T}) = e_\chi\epsilon^{V'}_{L/k,S,T}.$$

Hence, by Proposition 2.5, we see that $\mathrm{eTNC}(h^0(\mathrm{Spec}\,L), \mathbb{Z}_p[G])$ is valid, as claimed.

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