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Standard conjecture of Künneth type with torsion coefficients

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A. Venkatesh raised the following question, in the context of torsion automorphic forms: can the mod p analogue of Grothendieck's standard conjecture of Künneth type be true (especially for compact Shimura varieties)? In the first theorem of this article, by using a topological obstruction involving Bockstein, we show that the answer is in the negative and exhibit various counterexamples, including compact Shimura varieties.

It remains an open geometric question whether the conjecture can fail for varieties with torsion-free integral cohomology. Turning to the case of abelian varieties, we give upper bounds (in p) for possible failures, using endomorphisms, the Hodge–Lefschetz operators, and invariant theory.

The Schottky problem enters into consideration, and we find that, for the Jacobians of curves, the question of Venkatesh has an affirmative answer for every prime number p.

1. Introduction

Let X be a complex smooth projective variety of dimension n. Denote by $\mathrm{CH}^j(X)$ the Chow group of codimension-j cycles on X, and by

$$\mathrm{cl}^j_{X,\mathbb{Q}} : \mathrm{CH}^j(X) \to H^{2j}(X, \mathbb{Z}) \to H^{2j}(X, \mathbb{Q})$$

the cycle class map into the Betti cohomology of X with $\mathbb{Q}$-coefficients.

Poincaré duality lets $H^{2n}(X \times X, \mathbb{Q})$ act linearly on $H^*(X, \mathbb{Q})$ via correspondences: namely, $z \in H^{2n}(X \times X, \mathbb{Q})$ acts as

$$H^*(X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^*(X \times X, \mathbb{Q}) \xrightarrow{\cup z} H^{*+2n}(X \times X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_{2,*}} H^*(X, \mathbb{Q}),$$

where the duality is used in the definition of the Gysin map $\mathrm{pr}_{2,*}$.

For each integer i, we have the (rational) Künneth projector

$$\pi^i_{X,\mathbb{Q}} \in H^{2n}(X \times X, \mathbb{Q}),$$

which acts as 1 on $H^i(X, \mathbb{Q})$ and as 0 on $H^j(X, \mathbb{Q})$ for all $j \neq i$.

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Conjecture 1.1 (Grothendieck [Kleiman 1968]). *The Künneth projectors $\pi^i_{X,\mathbb{Q}}$ are algebraic*:

$$\pi^i_{X,\mathbb{Q}} \in \mathrm{Im}(\mathrm{cl}^n_{X\times X,\mathbb{Q}}) \otimes \mathbb{Q} \quad \textit{for all } i.$$

The conjecture has long been known for flag varieties, for abelian varieties, and in dimension ≤ 2.[1] For more recent progress on the conjecture, we also refer the reader to [Arapura 2006; Charles and Markman 2013; Tankeev 2011; 2015].

A. Venkatesh asked the present author whether the analogous statement could hold with $\mathbb{F}_p$-coefficients: we still have Poincaré duality for $\mathbb{F}_p$-coefficients, hence the action of $H^{2n}(X \times X, \mathbb{F}_p)$ on $H^*(X, \mathbb{F}_p)$, as well as the (mod p) Künneth projectors $\pi^i_{X,\mathbb{F}_p} \in H^{2n}(X \times X, \mathbb{F}_p)$. We also have the mod p cycle class map

$$\mathrm{cl}^j_{X,\mathbb{F}_p} : \mathrm{CH}^j(X)/p \to H^{2j}(X, \mathbb{F}_p).$$

Question 1.2 (Venkatesh). *Are the mod p Künneth projectors algebraic*? *That is*,

$$\pi^i_{X,\mathbb{F}_p} \in \mathrm{Im}(\mathrm{cl}^n_{X\times X,\mathbb{F}_p}) \quad \textit{for all } i?$$

Two aspects of the question got us interested. First, of course, the torsion invariants have long been of interest not only in algebraic topology, but also in algebraic geometry. For the latter, we mention just two of the most classic examples.

(A) Atiyah and Hirzebruch [1962] showed that the integral version of the Hodge conjecture is false in codimension ≥ 2, by creating torsion cohomology classes that are not annihilated by certain Steenrod operations. (Totaro [1997] later clarified this phenomenon in terms of complex cobordisms.)

(B) In response to Lüroth's question (in birational classification of varieties) Artin and Mumford [1972] constructed unirational but irrational threefolds, by exploiting nontrivial 2-torsion in the cohomology.

Secondly, with the recent progress in the theory of automorphic forms (including Arthur's conjectures), one has seen various results towards proving the Hodge, Tate, and standard conjectures for Shimura varieties; see among others [Bergeron et al. 2016; Morel and Suh 2016]. In particular, while the standard conjecture of Künneth type remains open, *the standard sign conjecture of Jannsen*, which (only) asks whether the even and odd idempotents

$$\pi^+_{X,\mathbb{Q}} = \sum_{2\mid i} \pi^i_{X,\mathbb{Q}} \quad \text{and} \quad \pi^-_{X,\mathbb{Q}} = \sum_{2\nmid i} \pi^i_{X,\mathbb{Q}}$$

are algebraic, has been proved for many Shimura varieties, which has Tannakian consequences for homological motives.

[1] The conjecture is also known for the ℓ-adic and crystalline cohomology theories over finite fields, thanks to Katz and Messing [1974], whose proof relies on Deligne's proof of the Weil conjectures.

Parallel to this development, there have also been striking progress of late in the theory of torsion automorphic forms (see among others [Scholze 2015; Treumann and Venkatesh 2016]) and some speculations on the extent to which one may have an analogue of Arthur's conjectures [Emerton and Gee 2015]. One wonders how far one could push the theory and obtain geometric results for Shimura varieties.

The first main goal of this article is to give a topological criterion for the question.

Theorem 1.3. (1) *If the integral cohomology $H^*(X, \mathbb{Z})$ has nontrivial p-torsion, then Venkatesh's question has a negative answer for X.*

(2) *For any integers $i > 0$ and $n \geq i+1$, there exists a projective smooth variety X of dimension n such that $\pi^i_{X,\mathbb{F}_p}$ is* not *in the image of the mod p cycle class map.*

This class of examples includes Godeaux–Serre varieties and Shimura varieties, as well as Enriques surfaces and the Artin–Mumford threefolds mentioned above. In the case of Shimura surfaces, we show that even (the analogue of) the *sign* conjecture with mod p coefficients can fail.

The next natural question, which is somewhat orthogonal in motivation to the original question, is if such failure can happen to varieties with *torsion-free* integral cohomology, such as hypersurfaces (in the direction of Griffiths–Harris conjectures [1985]) or abelian varieties. That is, whether the *integral* analogue of the standard conjecture of Künneth type can fail to hold. We define a natural measure of possible failure — Künneth defect — and note some first properties, but at the moment do not know the answer to the question.

In the final section, we look into the question in the case of abelian varieties and obtain upper bounds for the Künneth defects. Among the results we obtain is:

Theorem 1.4. *Venkatesh's question has an affirmative answer for all primes p, if X is the Jacobian of a curve.*

This leads us to put forth:

Conjecture 1.5. *The standard conjecture of Künneth type* fails *to hold integrally, for a very general principally polarized abelian variety X of dimension* ≥ 4.

A more ambitious conjecture would be: the integral version fails once X is outside the closure of the Schottky locus. More quantitatively, we relate this conjecture to the Prym–Tyurin theory; see Section 4.2 and Question 4.2.7 below.

Notation. For an abelian group M and an integer n, we use the notation

$$M[n] = \{m \in M : n \cdot m = 0\} \quad \text{and} \quad M/n = M/nM = M \otimes \mathbb{Z}/n.$$

If p is a prime number, we denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at (p), and write

$$M[p^\infty] = \bigcup_{n \geq 1} M[p^n].$$

2. Standard conjecture of Künneth type with torsion coefficients

2.1. ***Topological obstruction: Bockstein.*** The first obstruction to Question 1.2 that we will use in this section relies on the simple fact that the mod p cycle class map factors through the integral cohomology mod p:

$$\begin{array}{ccc} \mathrm{CH}^j(X) & \xrightarrow{\mathrm{cl}} & H^{2j}(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{CH}^j(X)/p & \xrightarrow{\mathrm{cl}_{\mathbb{F}_p}} & H^{2j}(X,\mathbb{F}_p) \end{array}$$

As such, we look at the obstruction to lifting $\pi^i_{X,\mathbb{F}_p}$ to an element in $H^{2n}(X\times X,\mathbb{Z})$. This naturally leads us to look at the version of the Bockstein homomorphism β

$$0\to H^j(X,\mathbb{Z})/p\to H^j(X,\mathbb{F}_p)\xrightarrow{\beta} H^{j+1}(X,\mathbb{Z})[p]\to 0, \tag{2.1.1}$$

which is the connecting homomorphism for the short exact sequence of sheaves

$$0\to\mathbb{Z}\xrightarrow{\times p}\mathbb{Z}\to\mathbb{F}_p\to 0.$$

Definition 2.1.1. Let X be a complex projective smooth variety and p a prime number. We define *the initial incidence of nontrivial p-torsion*, and denote it $i_p(X)$, as the smallest integer i such that the natural reduction map

$$H^i(X,\mathbb{Z})\to H^i(X,\mathbb{F}_p)$$

is *not* surjective, in other words, the Bockstein homomorphism

$$H^i(X,\mathbb{F}_p)\to H^{i+1}(X,\mathbb{Z})[p]$$

is nonzero. If no such i exists, we set $i_p(X)=\infty$.

Equivalently, in view of the exact sequence (2.1.1), we have $i_p(X)=j-1$, where j is the smallest integer such that $H^j(X,\mathbb{Z})$ has nontrivial p-torsion. We will show in Section 2.3 that, for any prime p and any integer $i\geq 1$, there exists a projective smooth variety X with $i_p(X)=i$.

Proposition 2.1.2. *If $i_p(X)<\infty$, then $i_p(X)\leq\dim X-1$.*

Proof. Let $n=\dim X$. By definition, we have

$$\dim_{\mathbb{Q}} H^{i_p(X)}(X,\mathbb{Q})<\dim_{\mathbb{F}_p} H^{i_p(X)}(X,\mathbb{F}_p),$$

so by Poincaré duality with field coefficients, we get $i_p(X)\leq n$. If $i_p(X)$ were equal to n, then we would have

$$\dim_{\mathbb{F}_p} H^j(X,\mathbb{F}_p)=\dim_{\mathbb{Q}} H^j(X,\mathbb{Q})$$

for all $j < n$, and hence also for all $j > n$, again by Poincaré duality. Then the jump

$$\dim_{\mathbb{F}_p} H^n(X, \mathbb{F}_p) > \dim_{\mathbb{Q}} H^n(X, \mathbb{Q})$$

in the middle degree would contradict the equality of the Euler characteristics of X with $\mathbb{F}_p$- and $\mathbb{Q}$-coefficients. □

Theorem 2.1.3. *Let X be a complex smooth projective variety of dimension n and p a prime number. Suppose that $i := i_p(X) < \infty$. Then the Bockstein of the three idempotents*

$$\pi^i_{X,\mathbb{F}_p}, \quad \pi^{2n-i}_{X,\mathbb{F}_p}, \quad \textit{and} \quad \pi^i_{X,\mathbb{F}_p} + \pi^{2n-i}_{X,\mathbb{F}_p}$$

are all nonzero, and as such cannot be the cohomology class of an algebraic cycle.

Proof. First we show that $\beta_{X\times X}(\pi^i_{X,\mathbb{F}_p}) \neq 0$, where $\beta_{X\times X}$ is the Bockstein for $X \times X$.

Lemma 2.1.4. *With the notation as above, the natural map*

$$H^{2n-i}(X, \mathbb{Z}) \to H^{2n-i}(X, \mathbb{F}_p)$$

is surjective.

Proof. Equivalently, we show that $H^{2n-i+1}(X, \mathbb{Z})$ is free of p-torsion. By Poincaré duality, we have

$$H^{2n-i+1}(X, \mathbb{Z}) \simeq H_{i-1}(X, \mathbb{Z})$$

and the universal coefficient theorem provides us with an exact sequence

$$0 \to \mathrm{Ext}^1(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to \mathrm{Hom}(H_i(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

Now if $H_{i-1}(X, \mathbb{Z})$ had nontrivial p-torsion, then so would $H^i(X, \mathbb{Z})$, which would contradict the alternate characterization of $i_p(X)$ given just after the definition. □

Step 1 (the p-primary torsion parts, Poincaré dual bases, and Pontryagin dual generators). Denote the p-primary torsion part by

$$T := H_i(X, \mathbb{Z})[p^\infty] \simeq H^{2n-i}(X, \mathbb{Z})[p^\infty]$$

(isomorphism via Poincaré duality with integral coefficients), hence giving us a natural exact sequence (thanks to Lemma 2.1.4)

$$0 \to T/p \to H^{2n-i}(X, \mathbb{F}_p) \to H^{2n-i}(X, \mathbb{Z}_{(p)})^{\mathrm{fr}}/p \to 0 \tag{a}$$

(where $(\,\cdot\,)^{\mathrm{fr}}$ denotes the maximal $\mathbb{Z}_{(p)}$-free quotient). Then choose generators

$$T = \bigoplus_{k=1}^{r} (\mathbb{Z}/p^{m_k})t_k.$$

By the universal coefficient theorem

$$0 \to \mathrm{Ext}^1(H_{2n-i-1}(X, \mathbb{Z}), \mathbb{Z}_{(p)}) \to H^{2n-i}(X, \mathbb{Z}_{(p)}) \to \mathrm{Hom}(H_{2n-i}(X, \mathbb{Z}), \mathbb{Z}_{(p)}) \to 0$$

we get a natural isomorphism

$$T \simeq \mathrm{Ext}^1(H_{2n-i-1}(X,\mathbb{Z}),\mathbb{Z}_{(p)}) \simeq \mathrm{Hom}(H^{i+1}(X,\mathbb{Z})[p^\infty],\mathbb{Q}/\mathbb{Z}),$$

in other words,

$$H^{i+1}(X,\mathbb{Z})[p^\infty] \simeq T^\vee,$$

the Pontryagin dual of T. Let $t_k^\vee \in T^\vee$ denote the Pontryagin dual generators:

$$t_k^\vee(t_j) = 0 \quad \text{if } j \neq k \quad \text{while } t_k^\vee(t_k) = 1/p^{m_k} + \mathbb{Z}.$$

It follows that the $u_k := p^{m_k-1}t_k^\vee$ form a basis of $T^\vee[p]$,

$$T^\vee[p] = \bigoplus_{k=1}^r \mathbb{F}_p u_k,$$

and the Bockstein short exact sequence reads

$$0 \to H^i(X,\mathbb{Z})/p \to H^i(X,\mathbb{F}_p) \xrightarrow{\beta} T^\vee[p] \to 0. \tag{b}$$

The exact sequences (a) and (b) are Poincaré dual to each other, via the universal coefficient theorem.

Denote the mod p Poincaré pairing by

$$\langle\,\cdot\,,\,\cdot\,\rangle : H^{2n-i}(X,\mathbb{F}_p) \times H^i(X,\mathbb{F}_p) \to \mathbb{F}_p.$$

We can then choose

(1) a preimage $\{\widetilde{u_k} \in H^i(X,\mathbb{F}_p)\}_k$ of $\{u_k\}_{k=1,\dots,r}$ under β,

(2) a lift $\{f_\ell \in H^{2n-i}(X,\mathbb{Z}_{(p)})\}_\ell$ of a $\mathbb{Z}_{(p)}$-basis of the free quotient $H^{2n-i}(X,\mathbb{Z}_{(p)})^{\mathrm{fr}}$, and

(3) a $\mathbb{Z}_{(p)}$-basis $\{f'_\ell\}$ of $H^i(X,\mathbb{Z}_{(p)})$ (note that this last group is free over $\mathbb{Z}_{(p)}$),

such that the $\mathbb{Z}_{(p)}$-bases $\{f_\ell\}$ and $\{f'_\ell\}$ are Poincaré dual, so that

$$\langle \overline{t_k}, \widetilde{u_j}\rangle = \delta_{jk}, \quad \langle \overline{t_k}, \overline{f'_\ell}\rangle = 0, \quad \text{and} \quad \langle \overline{f_\ell}, \overline{f'_m}\rangle = \delta_{\ell m}$$

(the second vanishing equation follows from the duality of the sequences (a) and (b)). Here by the notation $\bar{\xi}$ we mean the image of ξ in the cohomology with $\mathbb{F}_p$-coefficients.

By modifying the f_ℓ by linear combinations of the t_k, which changes neither the fact that the f_ℓ lift a $\mathbb{Z}_{(p)}$-basis of the free quotient in condition (2) nor the mod p intersection numbers $\langle \overline{f_\ell}, \overline{f'_m}\rangle$, we may then assume in addition that

$$\langle \overline{f_\ell}, \widetilde{u_k}\rangle = 0,$$

that is, the bases

$$\{\bar{t}_1, \dots, \bar{t}_r; \bar{f}_1, \dots, \bar{f}_b\} \quad \text{and} \quad \{\widetilde{u}_1, \dots, \widetilde{u}_r; \bar{f}'_1, \dots, \bar{f}'_b\}$$

of $H^{2n-i}(X, \mathbb{F}_p)$ and of $H^i(X, \mathbb{F}_p)$, respectively, are Poincaré dual to each other (b stands for Betti).

It then follows, from the way the action of $H^{2n}(X \times X, \mathbb{F}_p)$ on $H^*(X, \mathbb{F}_p)$ via correspondence is constructed, that

$$\pi^i_{X,\mathbb{F}_p} = \sum_{k=1}^{r} \bar{t}_k \otimes \widetilde{u}_k + \sum_{\ell=1}^{b} \bar{f}_\ell \otimes \bar{f}'_\ell. \tag{$\star$}$$

Step 2 (integral Künneth theorem). Recall that the general Künneth theorem for integral coefficients says that we have a natural short exact sequence

$$0 \to \bigoplus_{a+b=m} H^a(X, \mathbb{Z}) \otimes H^b(Y, \mathbb{Z}) \to H^m(X \times Y, \mathbb{Z}) \to \\ \to \bigoplus_{a+b=m+1} \mathrm{Tor}_1(H^a(X, \mathbb{Z}), H^b(Y, \mathbb{Z})) \to 0$$

[Cartan and Eilenberg 1956, Chapter VI, Theorem 3.1]. Applied to our situation, $X = Y$, it gives us a natural *injection*

$$(H^{2n-i}(X, \mathbb{Z}) \otimes H^{i+1}(X, \mathbb{Z}))[p] \hookrightarrow H^{2n+1}(X \times X, \mathbb{Z})[p].$$

Thanks to ($\star$), the Bockstein of $\pi^i_{X,\mathbb{F}_p}$ belongs to this smaller subspace

$$\beta_{X\times X}(\pi^i_{X,\mathbb{F}_p}) = (-1)^{2n-i} \sum_{k=1}^{r} t_k \otimes u_k = (-1)^{2n-i} \sum_{k=1}^{r} p^{m_k - 1} t_k \otimes t_k^\vee;$$

here we are using the fact that the Bockstein homomorphism is a graded derivation on cohomology, that it annihilates elements obtained by reduction mod p, and that by construction $\beta(\widetilde{u}_k) = u_k$.

In fact, this belongs to a smaller subspace

$$(H^{2n-i}(X, \mathbb{Z}) \otimes H^{i+1}(X, \mathbb{Z}))[p] \supseteq (T \otimes T^\vee)[p],$$

which has basis

$$\{p^{\min(m_j, m_k)-1} t_i \otimes t_k^\vee\}, \quad \text{where } 1 \le j, k \le r,$$

because $(\mathbb{Z}/p^a) \otimes (\mathbb{Z}/p^b) \simeq \mathbb{Z}/p^a$ if $a \le b$. That is, up to sign, $\beta_{X\times X}(\pi^i_{X,\mathbb{F}_p})$ is the sum of the "diagonal" (that is, $j = k$) basis elements in $(T \otimes T^\vee)[p]$, and is nonzero because by assumption $T \neq 0$ and $r \ge 1$.

Step 3 (the other projectors). The proof for $\pi^{2n-i}_{X,\mathbb{F}_p}$ is essentially the same, with the two factors of $X \times X$ exchanging the roles. For the last idempotent, note that the Bocksteins of the two summands reside in distinct Künneth factors:

$$\beta_{X\times X}(\pi^{i}_{X,\mathbb{F}_p}) \in (H^{2n-i}(X,\mathbb{Z}) \otimes H^{i+1}(X,\mathbb{Z}))[p], \quad \text{while}$$
$$\beta_{X\times X}(\pi^{2n-i}_{X,\mathbb{F}_p}) \in (H^{i+1}(X,\mathbb{Z}) \otimes H^{2n-i}(X,\mathbb{Z}))[p]$$

($i+1$ and $2n-i$ have different parity). □

2.2. ***Shimura varieties with $i_p(X) = 1$.*** This section is a generalization of [Suh 2008, §3]. For the purpose of this article, it is enough to consider connected Shimura data $(G, \mathscr{X})$, where G is a simple algebraic group over $\mathbb{Q}$.

Proposition 2.2.1. *Assume that G is $\mathbb{Q}$-anisotropic so that the resulting Shimura varieties*

$$X_\Gamma = \Gamma \backslash \mathscr{X}$$

are compact, and that the congruence subgroups $\Gamma \subset G(\mathbb{Q})$ used in the definition satisfy the condition

$$\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma,\Gamma] < \infty. \tag{2.2.1}$$

Then for any prime number p, there exist torsion-free levels Γ such that $i_p(X_\Gamma) = 1$.

Remark 2.2.2. The condition (2.2.1) is satisfied whenever G has real rank at least 2 (see, e.g., [Margulis 1991, Theorem IV.4.9]) and also in the case of certain special unitary groups of real rank 1 [Rogawski 1990, Theorem 15.3.1]. (But it clearly rules out modular and Shimura curves as it must, as well as certain special unitary groups of real rank 1 [Kazhdan 1977].)

Proof. The condition (2.2.1) implies that

$$H^1(X_\Gamma, \mathbb{Z}) = 0.$$

On the other hand, for any fixed prime number p, one can always find lattices $\Gamma \subset G(\mathbb{Q})$ such that

$$H^1(X_\Gamma, \mathbb{F}_p) \neq 0;$$

the point is as follows. Let S be a finite set of prime numbers such that G has a good model over $\mathbb{Z}[S^{-1}]$, and let F be a number field over which G splits. Then by the Chebotarev density theorem, there are infinitely many prime numbers ℓ such that (i) $\ell \notin S$, (ii) ℓ splits completely in F, and (iii) $\ell \equiv 1 \pmod{p}$. Then, since $G \otimes \mathbb{F}_\ell$ contains a split torus of positive dimension, $G(\mathbb{F}_\ell)$ contains a nontrivial abelian p-subgroup, say $H \subseteq G(\mathbb{F}_\ell)$.

Now starting with a torsion-free level subgroup Γ, we can find a prime ℓ which is prime to Γ and satisfying the three conditions above. By first raising the level by a full mod ℓ level, and then lowering it by the abelian p-subgroup H, we arrive at a new level Γ' with

$$\pi_1(X_{\Gamma'})^{\mathrm{ab}}/p \neq 0, \quad \text{hence} \quad H^1(X_{\Gamma'}, \mathbb{F}_p) \neq 0.$$

Therefore, we get examples with $i_p(X_{\Gamma'}) = 1$. □

Remark 2.2.3. The groups H used in the proof are those p-subgroups that play a central role in the detection and the computation of the group cohomology of finite Lie groups, initiated by Quillen [1972]; see also [Adem and Milgram 2004] and references therein.

Remark 2.2.4. This way we also get Shimura surfaces with $i_p(X) = 1$, and Theorem 2.1.3 in this case shows that even the analogue of the *sign* conjecture for $\mathbb{F}_p$-coefficients is false.

Remark 2.2.5. For PEL-type or quaternionic Shimura varieties, when p is a “good” prime (with respect to the Shimura data) not dividing the level Γ, then the Shimura varieties have good reduction at p, thanks to Kottwitz and Reimann. This way we also get nontrivial p-torsion in the *coherent* and *de Rham* cohomology of the integral model at p of the Shimura varieties. For details, see [Suh 2008, §3].

Remark 2.2.6 (Emerton). The resulting unliftable cohomology classes in $H^1(X, \mathbb{F}_p)$, and hence the nontrivial p-torsion classes in $H^2(X, \mathbb{Z})[p]$ via Bockstein homomorphism, are Eisenstein (in the sense that the associated Galois representations are completely reducible) [Emerton and Gee 2015, Remark 3.4.6]. The point is that, for any prime that is coprime to the level

$$\ker(\phi : \Gamma' \to \mathbb{F}_p),$$

the corresponding Hecke operator annihilates the classes. Indeed, already the pullback of the class is zero, and hence, the ensuing pushforward (under a different finite covering) is necessarily zero.

2.3. *Varieties with any prescribed $i_p(X) \geq 1$.*

Theorem 2.3.1. *Let p be any prime number, and $1 \leq i < n$ be two integers. Then there exists a complex projective smooth variety X of dimension n such that $i_p(X) = i$.*

Moreover, we may assume one of the following two conditions on X.

(1) *For any $n > i$, we can find X with ample canonical bundle.*

(2) *If $i \geq 3$ and $n \geq 2i - 1$, then we can find X that is rational.*

Our proof consists of two parts: group cohomology (the tools we use can be found, e.g., in [Adem and Milgram 2004]) and projective geometry.

Lemma 2.3.2. *For any prime number p, there exists a finite group G such that*

$$H^1(G, \mathbb{F}_p) = 0 \quad \textit{and} \quad \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) \geq 2.$$

It follows that

$$p \nmid |H^2(G, \mathbb{Z})| \quad \textit{and} \quad \dim_{\mathbb{F}_p} H^3(G, \mathbb{Z})[p] \geq 2.$$

Proof. Consider the 2-dimensional vector space $V = \mathbb{F}_p^{\oplus 2}$. For odd p, let $D := \mathbb{F}_p^\times$ be the units in $\mathbb{F}_p$, and let $\alpha \in D$ act on V as the diagonal matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

while for $p = 2$, let a chosen generator α of $D := \mathbb{Z}/3$ act on V as the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(One can regard V as $\mathbb{F}_4$ and D as $\mathbb{F}_4^\times$.) Let $V' = V$ be another copy of the same representation of D, and take the corresponding semidirect product

$$G := (V \oplus V') \rtimes D.$$

For p odd, the cohomology algebra of V (regarded as a finite group) with coefficients in $\mathbb{F}_p$ is the tensor product of a polynomial algebra and an exterior algebra [Adem and Milgram 2004, Corollary II.4.3]:

$$H^*(V, \mathbb{F}_p) \simeq \mathbb{F}_p[x_2, y_2] \otimes \bigwedge(e_1, f_1),$$

where the subscripts mark the degree. Then D acts as the identity character on the eigenspaces $\mathbb{F}_p x_2$ and $\mathbb{F}_p e_1$ and as the inverse character on $\mathbb{F}_p y_2$ and $\mathbb{F}_p f_1$. By the Künneth formula in group cohomology, we also have

$$H^*(V \oplus V', \mathbb{F}_p) \simeq \mathbb{F}_p[x_2, y_2, x_2', y_2'] \otimes \bigwedge(e_1, f_1, e_1', f_1'),$$

with a similar description of the action of D.

For $p = 2$, the cohomology algebra is a polynomial algebra [Adem and Milgram 2004, Theorem II.4.4]:

$$H^*(V, \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, y_1],$$

on which the chosen generator α of D acts as the matrix above, on the basis x_1, y_1. Similarly

$$H^*(V \oplus V', \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, y_1, x_1', y_1'].$$

Since D has order prime to p, the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{ab} = H^a(D, H^b(V \oplus V', \mathbb{F}_p)) \Rightarrow H^{a+b}(G, \mathbb{F}_p)$$

degenerates in E_2 and gives

$$H^*(G, \mathbb{F}_p) \simeq H^*(V \oplus V', \mathbb{F}_p)^D.$$

One can then show that there are no nonzero D-invariants of degree 1.

For p odd, all the D-invariants in $H^2(V \oplus V', \mathbb{F}_p)$ are in the second wedge product, on which the action of D is through the two characters, with multiplicity 2 each. It follows that the invariants have dimension 4 or 6, according to whether $p \geq 5$ or $p = 3$.

For $p = 2$, to compute the dimension of the D-invariants, first extend scalars to $\mathbb{F}_4$ so as to diagonalize the action of D. Then one can see that the invariants of degree 2 have dimension 4.

The last part of the lemma follows from the long exact sequence

$$\cdots \to H^i(G, \mathbb{Z}) \xrightarrow{\times p} H^i(G, \mathbb{Z}) \to H^i(G, \mathbb{F}_p) \to \cdots$$

and the fact that $H^i(G, \mathbb{Z})$ is finite for all $i > 0$. □

Proof of Theorem 2.3.1. Step 1 (the cases $i = 1, 2$). By applying the Godeaux–Serre construction [Serre 1958, §20] to the group $\mathbb{Z}/p$, we get a complete intersection Y of dimension $n \geq 2$ with a free action of $\mathbb{Z}/p$. Let X be the quotient of Y by $\mathbb{Z}/p$. Since Y is simply connected, X has fundamental group $\mathbb{Z}/p$, and we have

$$H^1(X, \mathbb{Z}) = 0 \quad \text{and} \quad H^1(X, \mathbb{F}_p) \simeq \mathbb{F}_p,$$

and we have $i_p(X) = 1$.

For $i = 2$, we apply the Godeaux–Serre construction to any group G satisfying the conditions of Lemma 2.3.2, to obtain a complete intersection Y of dimension $n \geq 3$ with a free G-action. Let $X = Y/G$. Because Y has dimension ≥ 3, the Lefschetz hyperplane theorem gives us

$$H^1(Y, \mathbb{Z}) = 0 \quad \text{and} \quad H^2(Y, \mathbb{Z}) \simeq \mathbb{Z},$$

on which G acts trivially. Then the Serre–Hochschild spectral sequence

$$E_2^{ab} = H^a(G, H^b(Y, \mathbb{Z})) \Rightarrow H^{a+b}(X, \mathbb{Z})$$

gives us $H^1(X, \mathbb{Z}) = 0$ and then an exact sequence

$$0 \to H^2(G, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \mathbb{Z} \to H^3(G, \mathbb{Z}).$$

It follows that there is a noncanonical isomorphism

$$H^2(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus H^2(G, \mathbb{Z}),$$

hence

$$\dim_{\mathbb{F}_p} H^2(X, \mathbb{Z}) \otimes \mathbb{F}_p = 1.$$

In parallel, the spectral sequence applied to the mod p cohomology gives us $H^1(X, \mathbb{F}_p) = 0$ and

$$0 \to H^2(G, \mathbb{F}_p) \to H^2(X, \mathbb{F}_p) \to \mathbb{F}_p \to H^3(G, \mathbb{F}_p),$$

which gives the dimension count

$$\dim_{\mathbb{F}_p} H^2(X, \mathbb{F}_p) \geq 2.$$

Therefore, $i_p(X) = 2$.

For analysis of the cohomology of Godeaux–Serre varieties, see also [Atiyah and Hirzebruch 1962, p. 42].

Step 2 (the case $i \geq 3$). Here we use the trick of blowing up to increase i_p. The main point is:

Lemma 2.3.3. *Let X be a closed smooth subvariety of a complex projective smooth variety Y of codimension $c \geq 2$, and let p be a prime number. Then the blow-up Y' of Y along X satisfies*

$$i_p(Y') = \min\{i_p(X) + 2, i_p(Y)\}.$$

In particular, if $H^*(Y, \mathbb{Z})$ is free of p-torsion (e.g., if Y is a projective space), then $i_p(Y') = i_p(X) + 2$.

Proof. This follows from the isomorphism

$$H^j(Y', \mathbb{Z}) \simeq H^j(Y, \mathbb{Z}) \oplus \bigoplus_{k=1}^{c-1} H^{j-2k}(X, \mathbb{Z}).$$

(Use the Leray spectral sequence and the computation of the cohomology of a projective bundle [Katz 1973, Théorème 2.2].) □

Now we are ready to prove Theorem 2.3.1. Start with a variety $X = X_0$ from Step 1, and repeat $\lceil i/2 - 1 \rceil$ times the procedure of embedding X_k in a projective space (whose dimension can be chosen to be $\leq 2\dim(X_k) + 1$), then taking the blow-up X_{k+1} of the projective space along X_k.

The resulting variety will in general have a large dimension. However, we can cut down the dimension without changing $i_p(X)$, by using the Lefschetz hyperplane theorem, as long as the desired dimension n is at least $i_p(X) + 1$.

To get a rational example, apply the previous procedure to get X of dimension $i-1$ and $i_p(X) = i - 2$. Embed it in the projective space of dimension $n \geq 2(i-1)+1$ and blow up the projective space along it. □

Remark 2.3.4. The numerical condition in (2) for rational examples is sharp in the sense that, for $i = 2$ and $n = 2i - 1 = 3$, there is no rational projective smooth threefold with $i_p(X) = 2$. This is the crucial observation of Artin and Mumford [1972] mentioned in the introduction.

3. Integral Künneth defect

Let X be a complex projective smooth variety of dimension n with torsion-free $H^*(X, \mathbb{Z})$. The Künneth theorem for integral coefficients says in this case

$$H^i(X \times X, \mathbb{Z}) \simeq \bigoplus_{j+k=i} H^i(X, \mathbb{Z}) \otimes H^j(X, \mathbb{Z}).$$

We begin by noting that the Künneth idempotent $\pi_X^i \in H^{2n}(X \times X, \mathbb{Q})$ in fact belongs to $H^{2n}(X \times X, \mathbb{Z})$, that in the integral cohomology group, it (if nonzero) is not divisible by any prime number, and that its reduction mod p gives the mod p Künneth idempotents $\pi^i_{X,\mathbb{F}_p}$.

Definition 3.1. The (*integral*) *Künneth defect* is defined as the index

$$\kappa_i = \kappa_{i,X} := [\mathbb{Z}\pi_X^i : \mathbb{Z}\pi_X^i \cap \mathrm{Im}(\mathrm{cl}^n_{X\times X}(\mathrm{CH}^n(X \times X)))].$$

Thus, for X with torsion-free cohomology, Grothendieck's standard conjecture of Künneth type becomes the assertion $\kappa_i < \infty$ for all i, while the integral analogue amounts to $\kappa_i = 1$, and the mod p analogue amounts to $p \nmid \kappa_i$.

Proposition 3.2. *Let X and Y be nonempty complex projective smooth varieties with torsion-free integral cohomology, and let $n = \dim X$.*

(1) $\kappa_{0,X} = \kappa_{2n,X} = 1$.

(2) $\kappa_{i,X} = \kappa_{2n-i,X}$ *for all i.*

(3) *Suppose that for some i and some integer m, we have $\kappa_{j,X} \mid m$ for all $j \neq i, 2n-i$. Then $\kappa_{i,X}$ and $\kappa_{2n-i,X}$ also divide m.*

(4) *Suppose that $f : Y \to X$ is a generically finite covering of degree d. Then $\kappa_{i,X} \mid d\kappa_{i,Y}$.*

(5a) *For the product, we have the "convolution" formula*

$$\kappa_{\ell,X\times Y} \quad \textit{divides} \quad \mathrm{lcm}_{i+j=\ell}(\kappa_{i,X} \cdot \kappa_{j,Y}).$$

(5b) *Conversely, both $\kappa_{i,X}$ and $\kappa_{i,Y}$ divide $\kappa_{i,X\times Y}$.*

(6) *Let $\mathscr{X}$ be a complex projective smooth family of varieties with torsion-free cohomology over a (base) connected complex variety S. If $\kappa_{i,\mathscr{X}_s} \mid m$ for a very general fiber $\mathscr{X}_s$, then $\kappa_{i,\mathscr{X}_t} \mid m$ for any fiber $\mathscr{X}_t$.*

Proof. The fibers $\{*\} \times X$ and $X \times \{*\}$ represent π_X^0 and π_X^{2n}, respectively, hence (1). The $\mathbb{Z}/2$ symmetry on $X \times X$ gives (2). The diagonal Δ_X represents the sum of all π_X^i, from which we get (3).

The projection formula tells us that $(f \times f)_*(\pi_Y^i) = d\pi_X^i$: for any $x \in H^*(X, \mathbb{Q})$,

$$\begin{aligned}\mathrm{pr}_{X,2,*}(\mathrm{pr}_{X,1}^*(x) \cup (f \times f)_*(\pi_Y^i)) &= \mathrm{pr}_{X,2,*}(f \times f)_*((f \times f)^* \mathrm{pr}_{X,1}^*(x) \cup \pi_Y^i) \\ &= f_* \mathrm{pr}_{Y,2,*}(\mathrm{pr}_{Y,1}^*(f^*(x)) \cup \pi_Y^i) \\ &= f_* f^*(x_i) = \deg(f)(x_i),\end{aligned}$$

where x_i is the degree i component of x, hence (4).

Part (5a) follows from the Künneth formula and the fact that

$$\pi_{X \times Y}^{\ell} = \sum_{i+j=\ell} \pi_X^i \otimes \pi_Y^j$$

in $H^*(X \times Y \times X \times Y, \mathbb{Q}) \simeq H^*(X \times X, \mathbb{Q}) \otimes H^*(Y \times Y, \mathbb{Q})$ (up to sign à la Koszul).

For part (5b), note that, for any point $*$ on Y, we have

$$\mathrm{pr}_{1,3,*}(\pi_{X \times Y}^i \cup [X \times \{*\} \times X \times Y]) = \pi_X^i$$

as elements of $H^{2\dim X}(X \times X, \mathbb{Q})$, where $\mathrm{pr}_{1,3} : X \times Y \times X \times Y \to X \times X$ is the projection onto the product of the first and third factors.

For (6), use the fact that the Hilbert scheme of $\mathscr{X} \times_S \mathscr{X}/S$ is the countable union of proper irreducible schemes $\{\mathscr{H}_n\}_{n=1,2,\dots}$ over S. Subtract from S all the images of those $\mathscr{H}_n$ which map onto proper subvarieties of S, and call the complement U. If $\kappa_{i,\mathscr{X}_s} \mid m$ for one $s \in U$, then by construction the algebraic cycle representing $m \cdot \pi_X^i$ specializes to any $t \in S$ in a flat family. □

It follows that the integral analogue of the standard conjecture of Künneth type is true for surfaces with $H^1(X, \mathbb{Z}) = 0 = H^3(X, \mathbb{Z})$, e.g., K3 surfaces and complete intersection surfaces.

Proposition 3.3. *Let X be a smooth complete intersection of hypersurfaces of degrees $d_1, \dots, d_c$ in $\mathbb{P}^{n+c}$. Then $\kappa_i = 1$ for all i odd, and $\kappa_i \mid d_1 \cdots d_c$ for all i.*

Proof. By Proposition 3.2(2)–(3), we may prove the statements just for $i < n$. For such i, by the Lefschetz hyperplane theorem, $H^i(X, \mathbb{Z})$ is zero if i is odd, and is generated by the cup-power $D^{\cup i/2}$ if i is even, where D is a hyperplane section of X. Since $D^{\cup n} = d_1 \cdots d_c$, the cycle

$$\mathrm{pr}_1^*(D^{\cup n-i/2}) \cup \mathrm{pr}_2^*(D^{\cup i/2})$$

represents $d_1 \cdots d_c \pi_X^i$ for $i < n$ even. □

Question 3.4. *Does there exist a projective smooth variety X with torsion-free cohomology such that $\kappa_{i,X} > 1$ for some i?*

In this regard, we mention:[2]

[2]The theorem cited is stated only for $n = 2$, but the argument carries over to general n without much difficulty.

Theorem 3.5 [Soulé and Voisin 2005, Theorem 3]. *Let X be a projective smooth variety of dimension n, and let m be an integer prime to* 6. *Then for a very general degree-m^3 hypersurface Y in $\mathbb{P}^4$ and any irreducible subvariety Z of $X \times Y$ of codimension* 3, *the Künneth* (2, 4) *component*

$$[Z]^{(2,4)} \in H^2(X, \mathbb{Z}) \otimes H^4(Y, \mathbb{Z})$$

is divisible by m in the group.

While this theorem is relevant to our discussion, it does not quite answer our question, since one is not allowed to take $X = Y$. The nature of the proof requires that X must precede Y: the "very general" condition on Y is formulated in terms of the Hilbert scheme of $X \times Y$.

4. Abelian varieties

We now turn to investigating Question 3.4 in the case of abelian varieties.

Perhaps a few words are in order to put our results in relation to what is known in the literature. The standard conjecture of Künneth type with $\mathbb{Q}$-coefficients has long been known in the case of abelian varieties, treated as early as in [Kleiman 1968]. Part of what follows can be seen as an improvement of it with $\mathbb{Z}$-coefficients. (However, the use of higher Abel–Jacobi maps to produce requisite integral algebraic cycles seems to be new in this context.)

A remarkable, arithmetic development in a rather different direction has been seen in the consideration of the *rational* equivalence of algebraic cycles on abelian varieties (but still with $\mathbb{Q}$-coefficients), a refinement of the *homological* equivalence. (The standard conjecture, as stated, only concerns the homological equivalence.) Here a great deal of striking results have been obtained by use of the Fourier transformation. See among others [Beauville 1986; 2010; Künnemann 1993].[3]

4.1. ***Endomorphisms and Lieberman's trick.*** First we note that Poincaré duality is valid for compact complex tori, with or without polarization. As the cohomology is also torsion-free, we can still speak of the Künneth defects (Section 3).

Definition 4.1.1. Let $g \geq 1$, $0 < i < 2g$, and $a \neq -1, 0, 1$ be integers. Define the polynomial

$$P_{i,g,a}(T) := \prod_{0<j<2g \text{ and } j \neq i} \frac{T - a^j}{a^i - a^j} \in \mathbb{Q}[T],$$

[3] An explicit formula is given in [Künnemann 1993] for the projectors up to rational equivalence with $\mathbb{Q}$-coefficients. However, the denominators required there are no better than the ones given in Section 4.1 below, and not as sharp as the ones in Section 4.2 (where a principal polarization is assumed in addition).

and let $d_{i,g,a}$ be the denominator (up to sign) of $P_{i,g,a}$:

$$d_{i,g,a} = \prod_{0<j<2g \text{ and } j\neq i} (a^i - a^j).$$

Proposition 4.1.2. *Let X be a compact complex torus of dimension $g \geq 1$. For any integers $a \neq -1, 0, 1$ and $0 < i < 2g$, we have*

$$\kappa_{i,X} \mid d_{i,g,a}.$$

In particular, no prime $p \geq 2g$ divides any $\kappa_{i,X}$.

Proof. The pullback by the multiplication map $[a] : X \to X$ acts as a^j on $H^j(X, \mathbb{Z})$. The polynomial $P_{i,g,a}$ was constructed so that

$$P_{i,g,a}([a]^*) = a_{0,i}\pi_X^0 + \pi_X^i + a_{2g,i}\pi_X^{2g} \quad \text{on } H^*(X, \mathbb{Q}),$$

where $a_{0,i}$ and $a_{2g,i}$ are rational numbers whose denominators divide $d_{i,g,a}$. Because π_X^0 and π_X^{2g} are integrally algebraic (Proposition 3.2(1)), we get the first estimate by multiplying the equation through by $d_{i,g,a}$.

For the second statement, choose $a \neq 0, \pm 1$ to be a primitive root mod p, so that $a, \dots, a^{2g-1}$ are distinct mod p and $p \nmid d_{i,g,a}$. □

4.2. ***Invariant theory and Jacobians.*** From now on, we consider a principally polarized[4] abelian variety (X, L) of dimension $g \geq 1$.

Recall the following fact from the classical invariant theory. Let V be a $2g$-dimensional vector space over a field k of characteristic zero endowed with a nondegenerate alternating pairing $\langle \cdot, \cdot \rangle$, and let V_1 and V_2 be two copies of V. Then the $\mathbf{Sp}_{2g}$-invariants in the exterior algebra of the dual $(V_1 \oplus V_2)^\vee$ (on which $\mathbf{Sp}_{2g}$ acts diagonally and then dually) are generated by those in degree 2:

$$\Big(\bigwedge (V_1 \oplus V_2)^\vee\Big)^{\mathbf{Sp}_{2g}} = k\langle E_1, E_2, M\rangle,$$

where E_1 and E_2 are the pairings $\langle \cdot, \cdot \rangle$ through V_1 and V_2, respectively, and M is the pairing

$$M(u, w) := E(u_1, w_2) + E(u_2, w_2) = E(m(u), m(w)) - E_1(u, w) - E_2(u, w),$$

where $u = u_1 + u_2$ and $w = w_1 + w_2$ with $u_i, w_i \in V_i$ and $m : V_1 \oplus V_2 \to V$ is the sum map.

[4] The calculation of bounds that follows, when implemented for a general polarization L, would always involve the degree of L in the denominator. In view of Proposition 3.2(4), we do not lose much by first passing to an isogenous abelian variety with a principal polarization.

This has the following geometric consequence [Milne 1999]. The Chern class $E \in H^2(X, \mathbb{Z})$ of the principal polarization L can be regarded as a perfect alternating form on $H_1(X, \mathbb{Z})$. Then we have

$$H^*(X \times X, \mathbb{Q})^{\mathbf{Sp}_{2g}} = \mathbb{Q}\langle E_1, E_2, M\rangle,$$

where $E_i = \mathrm{pr}_i^*(E)$ and

$$M = \mu^* E - E_1 - E_2, \tag{4.2.1}$$

where $\mu : X \times X \to X$ is the group law. Since E_1, E_2, and M have Künneth types $(2, 0)$, $(0, 2)$, and $(1, 1)$, respectively, it follows that there exist constants $\gamma_{i,g}(a, b, c) \in \mathbb{Q}$ such that the Künneth projectors can be expressed as

$$\pi_X^i = \sum_{(a,b,c):2a+b=2g-i \text{ and } b+2c=i} \gamma_{i,g}(a, b, c) E_1^a M^b E_2^c.$$

Theorem 4.2.1 [Scholl 1994, §5.9]. *For all* $0 \le i \le 2g$, *we have*

$$\gamma_{i,g}(a, b, c) = \frac{(-1)^i}{a!\, b!\, c!}.$$

(We note in passing that in [loc. cit.] Scholl proves Künneth-type decomposition even for *rational* equivalence (with $\mathbb{Q}$-coefficients).)

For each index $1 \le i < g$, the sum ranges over the following triples (a, b, c):

$$(g-i, i, 0), (g-i+1, i-2, 1), (g-i+2, i-4, 2), \ldots, (g-i+\lfloor i/2\rfloor, 0 \text{ or } 1, \lfloor i/2\rfloor).$$

Since E_1, E_2, and M are integral algebraic cycles, we have:

Corollary 4.2.2. *Let* $1 \le i < g$. *If a prime* p *divides* $\kappa_{i,X}$, $p > \max(i, g-i+\lfloor i/2\rfloor)$. *In particular, the mod* p *analogue of the standard conjecture of Künneth type is true for all* $p \ge g$.

With a fixed g, as i ranges from 1 to g, the corollary gives the best bound around $i \approx 2g/3$, when i surpasses $g - i/2$.

For Jacobians we can do much better. Let C be a projective smooth curve, $J = \mathrm{Jac}(C)$ its Jacobian, $c \in C$ any base point,

$$\alpha_n : C^{(n)} = \mathrm{Sym}^n C \to J$$

the (higher) Abel–Jacobi map for $n = 0, \ldots, g$, and

$$w^{[n]} := \alpha_{g-n,*}[C^{(g-n)}] \in H^{2n}(J, \mathbb{Z})$$

the cohomology class (which is independent of c).

Theorem 4.2.3. *Let C, $J = \mathrm{Jac}(C)$, μ, and w be as above. Then we have*

$$\pi_J^i = (-1)^i \sum_{(a,b,c):2a+b=2g-i \text{ and } b+2c=i} \mathrm{pr}_1^*(w^{[a]})\,\mathrm{pr}_2^*(w^{[c]}) \times \sum_{d+e+f=b} (-1)^{d+f}\,\mathrm{pr}_1^*(w^{[d]})\mu^*(w^{[e]})\,\mathrm{pr}_2^*(w^{[f]})$$

for all i, and for both $\mathbb{Q}$- and $\mathbb{F}_p$-coefficients.

In particular, the mod p and integral analogues of the standard conjecture of Künneth type are true for J.

Proof. The key point is the celebrated formula of Poincaré [Birkenhake and Lange 2004, §11.2.1]

$$n!\,w^{[n]} = [c_1(L)]^n \quad \text{in } H^{2n}(J, \mathbb{Z})$$

(hence the divided power notation). This allows us to write

$$\frac{E_1^a}{a!} = \mathrm{pr}_1^*(w^{[a]}) \quad \text{and} \quad \frac{E_2^c}{c!} = \mathrm{pr}_2^*(w^{[c]}).$$

As for the second term, apply the trinomial theorem (in the divided power form)[5] to the defining equation (4.2.1):

$$\begin{aligned} M^{[b]} = M^b/b! &= (-\mathrm{pr}_1^*(E) + \mu^*(E) - \mathrm{pr}_2^*(E))^{[b]} \\ &= \sum_{d+e+f=b} (-\mathrm{pr}_1^*(E))^{[d]}(\mu^*(E))^{[e]}(-\mathrm{pr}_2^*(E))^{[f]} \\ &= \sum_{d+e+f=b} (-1)^{d+f}\,\mathrm{pr}_1^*(w^{[d]})\mu^*(w^{[e]})\,\mathrm{pr}_2^*(w^{[f]}). \qquad \square \end{aligned}$$

Corollary 4.2.4. *The torsion and integral versions of the standard conjecture of Künneth type is true for all principally polarized abelian varieties of dimension ≤ 3.*

Proof. Via the Torelli map, a general PPAV of dimension $g \leq 3$ is the Jacobian of a curve of genus g. □

This is the motivation for Conjecture 1.5 in the introduction.

Prym–Tyurin theory and curves on abelian varieties. Recall [Birkenhake and Lange 2004, §12.3] that, to a double covering $f : C' \to C$ of curves that is either unramified or ramified at exactly 2 points, one attaches a principally polarized abelian variety $P(f)$ (the Prym variety of f), in such a way that $P(f) \times \mathrm{Jac}(C)$ is isogenous to the Jacobian $\mathrm{Jac}(C')$ via an isogeny of exponent 2.

Corollary 4.2.5. *For all primes $p \geq 3$, the mod p analogue of the standard conjecture of Künneth type is true for the Prym variety $P(f)$.*

[5] Namely, $(\alpha+\beta+\gamma)^n/n! = \sum_{i+j+k=n} (\alpha^i/i!)(\beta^j/j!)(\gamma^k/k!)$.

Proof. By Theorem 4.2.3, the analogue is true for $\mathrm{Jac}(C')$, hence for $P(f) \times \mathrm{Jac}(C)$ by Proposition 3.2(4). Then the analogue, for odd p, follows for $P(f)$ by Proposition 3.2(5b). $\square$

Corollary 4.2.6. *For every principally polarized abelian variety X of dimension $g \leq 5$, the mod p analogue of the standard conjecture of Künneth type is true for all primes $p \geq 3$.*

Proof. A general PPAV of genus $g \leq 5$ is the Prym variety of a suitable double cover. $\square$

More generally, following Prym and Tyurin [loc. cit.], one studies a general principally polarized abelian variety by embedding it into the Jacobian of some curve, usually of high genus, with some exponent, usually ≥ 2. In this way, the validity of the integral analogue of the standard conjecture of Künneth type (or its failure) is related to the existence of low-genus curves on abelian varieties (or lack thereof).

We thus quantify Conjecture 1.5 into:

Question 4.2.7. *For any integer $g \geq 1$, let $\kappa(g)$ denote the least common multiple of $\kappa_{i,X}$ as i ranges over the interval $[0, 2g]$, where X is a very general principally polarized abelian variety of dimension g.*

How does $\kappa(g)$ vary with g? In particular, is $\kappa(g) > 1$ and is $\kappa(g) > 2$ when $g > 3$ and $g > 5$, respectively?

4.3. *Hodge and Lefschetz operators.* In this section, we consider operators closely related to the Künneth projectors, and find the denominators required in their definition. As in the previous section, we focus on principal polarizations. We recall a set of operators in the Hodge–Lefschetz theory (see, e.g., [Wells 2008, §V.3] or [Kleiman 1968]). First, cup product with $[L]$ gives

$$L : H^*(X, \mathbb{Q}) \to H^{*+2}(X, \mathbb{Q}),$$

which is defined by an integral algebraic correspondence and preserves the integral cohomology. One then defines (see the references above) the operator, with $\mathbb{Q}$-coefficients,

$$\Lambda : H^*(X, \mathbb{Q}) \to H^{*-2}(X, \mathbb{Q}).$$

Finally, the Pontryagin product is also defined by an integral algebraic correspondence

$$H^i(X, \mathbb{Z}) \otimes H^j(X, \mathbb{Z}) \to H^{i+j-2g}(X, \mathbb{Z}),$$
$$x \otimes y \mapsto x \star y := \mu_*(\mathrm{pr}_1^*(x) \otimes \mathrm{pr}_2^*(y)),$$

where $\mu : X \times X \to X$ is the group law.[6] This allows us to recover the Lefschetz Λ-operator:[7]

Proposition 4.3.1. *For any $x \in H^*(X, \mathbb{Q})$, we have*

$$\Lambda(x) = \frac{1}{g!}([L]^{g-1}) \star x.$$

In particular, Λ is defined by an algebraic correspondence with coefficients in $\mathbb{Z}[1/g!]$ and preserves $H^*(X, \mathbb{Z}[1/g!])$.

Moreover, if X is the Jacobian of a curve or the Prym variety associated with a double covering, then Λ is defined by a correspondence with coefficients in $\mathbb{Z}[1/g]$ or $\mathbb{Z}[1/2g]$, respectively.

Proof. For the proof of the first formula, see [Birkenhake and Lange 2004, Proposition 4.11.3]. The latter statement follows from the fact that, in case X is a Jacobian or Prym, $[L]^{g-1}/(g-1)!$ or $2[L]^{g-1}/(g-1)!$, respectively, is represented by an integral algebraic cycle [Birkenhake and Lange 2004, §11.2.1 and Criterion 12.2.2]. □

Now that we have L and Λ, we can apply the representation theory of $\mathfrak{sl}_2$ on cohomology (Jacobson–Morozov); see, e.g., [Wells 2008, §V.3]. With the usual notation

$$B := [\Lambda, L] = \Lambda L - L\Lambda = \sum_{i=0}^{2g} (g-i)\pi^i,$$

the operators correspond to the following matrices in $\mathfrak{sl}_2$:

$$\Lambda \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad B \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition 4.3.2 (integral Lefschetz algebra). We denote by $\mathfrak{L}$ the $\mathbb{Z}$-subalgebra of linear operators on $H^*(X, \mathbb{Q})$ generated by the five operators

$$L, \quad \Lambda, \quad w = [-1]^*, \quad \pi^0 = [\{0\} \times X], \quad \text{and} \quad \pi^{2g} = [X \times \{0\}].$$

(Recall that $[-1]^*$ acts as 1 on H^{even} and as -1 on H^{odd}.)

It follows from Proposition 4.3.1 that any element of the localization $\mathfrak{L}[1/g!]$ is defined by an algebraic correspondence with coefficients in $\mathbb{Z}[1/g!]$.

Recall that $x \in H^i(X, \mathbb{Q})$ is called *primitive* if $\Lambda(x) = 0$. For any $x \in H^i(X, \mathbb{Q})$, we have the primitive decomposition

$$x = \sum_{k \geq i_0} L^k x_k, \quad x_k \in H^{i-2k}(X, \mathbb{Q}) \text{ primitive}, \tag{4.3.1}$$

[6] A natural way to index is to use homology, via Poincaré duality $H^i(X, \mathbb{Z}) \simeq H_{2g-i}(X, \mathbb{Z})$. If $x \in H_i(X, \mathbb{Z})$ and $y \in H_j(X, \mathbb{Z})$, then $x \star y \in H_{i+j}(X, \mathbb{Z})$.

[7] This operator is named Λ^c in [Kleiman 1968], in which Λ is reserved for something else.

where $i_0 = \max(i-g, 0)$. In terms of (4.3.1), the Lefschetz operator on $H^*(X, \mathbb{Q})$ is

$$\Lambda(x) = \sum_{k \geq i_1} k(g-i+k+1)L^{k-1}x_k \quad \text{for } x \in H^i(X, \mathbb{Q}),$$

where $i_1 = \max(i-g, 1)$; see the formula for "Λ^c" in [Kleiman 1968]. We can then define[8]

$$\Lambda'(x) = \sum_{k \geq i_1} L^{k-1}x_k \quad \text{for } x \in H^i(X, \mathbb{Q}).$$

The Hodge $*$-operator on $H^*(X, \mathbb{Q})$ in Kleiman's convention[9] is given by

$$*_K(x) = \sum_{k \geq i_0} (-1)^{(i-2k)(i-2k+1)/2} L^{g-i+k}x_k \quad \text{for } x \in H^i(X, \mathbb{Q}).$$

Proposition 4.3.3. *Let j and k be two integers in* $[0, g]$. *Then* $\mathfrak{L}[1/2(g!)]$ *contains the following* 4 *operators on* $H^*(X, \mathbb{Q})$: *the Künneth projectors π^j and π^{2g-j} and the primitive part extractors*

$$p^{j,k}(x) := \begin{cases} x_k & \text{if } x \in H^j(X, \mathbb{Q}), \\ 0 & \text{if } x \in H^i(X, \mathbb{Q}) \text{ and } i \neq j \end{cases}$$

and $p^{2g-j,k}$ defined similarly (x_k on H^{2g-j} and 0 *in other degrees).*

Proof. We use a nested induction, ascending in j outside and descending in k inside. We first note that π^j and π^{2g-j} are generated by $p^{j,k}$ and L and by $p^{2g-j,k}$ and L, respectively, with integer coefficients, so for any fixed j, it is enough to prove the statement for $p^{j,k}$ and $p^{2g-j,k}$.

The structure constants that appear below, and need to be inverted, are given by:

Lemma 4.3.4. *Let $0 \neq x \in H^i(X, \mathbb{Q})$ be primitive. Then*

$$\Lambda^{g-i}L^{g-i}(x) = ((g-i)!)^2 x.$$

The point is that the irreducible subrepresentation of $\mathfrak{sl}_2(\mathbb{Q})$ generated by x has weight $g-i$. See, e.g., [Wells 2008, §V.3].

In the induction base $j = 0$, the elements $\pi^0 = p^{0,0}$ and π^{2g} are already in $\mathfrak{L}$ by definition. From the fact that $L^g : H^0(X, \mathbb{Z}[1/g!]) \to H^{2g}(X, \mathbb{Z}[1/g!])$ is an isomorphism, we see from Lemma 4.3.4 that

$$p^{2g,g} = \frac{1}{(g!)}\Lambda^g \circ \pi^{2g}.$$

[8] In the next line is Kleiman's definition of "Λ".

[9] This again differs from the notation of Wells [2008]; among other things, the latter acts on $H^*(X, \mathbb{C})$ but not necessarily on $H^*(X, \mathbb{Q})$.

Now suppose that, for some j_0, we have proven the statement for all $j < j_0$ and for all k. Let $x \in H^*(X, \mathbb{Q})$, and let $x^{(i)}$ denote its component in $H^i(X, \mathbb{Q})$. Applying to x the composite

$$\Lambda^{g-\lfloor j_0/2 \rfloor} \circ L^{g-\lfloor j_0-2 \rfloor} \circ \left(1 - \sum_{j=0}^{j_0-1} \pi^j\right),$$

and then

$$\frac{1 \pm [-1]^*}{2}$$

according to whether j_0 is even or odd, we are left with the homogeneous component

$$\Lambda^{g-\lfloor j_0/2 \rfloor} L^{g-\lfloor j_0/2 \rfloor} x^{(j_0)}.$$

From the primitive decomposition,

$$x^{(j_0)} = \sum_k L^k x_k^{(j_0)}, \quad x_k^{(j_0)} \in H^{j_0-2k} \text{ primitive},$$

what we have is

$$\Lambda^{g-\lfloor j_0/2 \rfloor} L^{g-\lfloor j_0/2 \rfloor} x^{(j_0)} = \sum_k \Lambda^{g-\lfloor j_0/2 \rfloor} L^{g-\lfloor j_0/2 \rfloor + k} x_k^{(j_0)}.$$

Now we use the descending induction in k, with the case $k = g + 1$ being trivially true. Suppose that we have shown that $p^{j_0,k}$ is in $\mathfrak{L}[1/2g!]$ for all $k > k_0$. It follows then that the previous operator *curtailed* in degrees $k \leq k_0$

$$x \mapsto \sum_{k \leq k_0} \Lambda^{g-\lfloor j_0/2 \rfloor} L^{g-\lfloor j_0/2 \rfloor + k} x_k^{(j_0)}$$

on H^* is in $\mathfrak{L}[1/2g!]$. Applying Λ^{k_0} will then annihilate all the terms with $k < k_0$ (again, the point is that $x_k^{(j_0)}$, when nonzero, generates a subrepresentation of weight $g - (i - 2k)$), and we obtain

$$x \mapsto \Lambda^{g-\lfloor j_0/2 \rfloor + k_0} L^{g-\lfloor j_0/2 \rfloor + k_0} x_k^{(j_0)} = ((g - \lfloor j_0/2 \rfloor + k_0)!)^2 x_{k_0}^{(j_0)},$$

by Lemma 4.3.4, and we have shown that

$$((g - \lfloor j_0/2 \rfloor + k_0)!)^2 \cdot p^{j_0,k_0}$$

is in $\mathfrak{L}[1/2g!]$, hence also p^{j_0,k_0}. Go down in k inside, then up in j outside.

The proof for $p^{2g-j,k}$ is similar, and we omit the details. □

We summarize the calculations.

Theorem 4.3.5. *Let (X, L) be a principally polarized abelian variety of dimension g. The Lefschetz operator Λ is defined by an algebraic correspondence with coefficients in $\mathbb{Z}[1/g!]$, even with coefficients in $\mathbb{Z}[1/g]$ or $\mathbb{Z}[1/2g]$ in case X is* *a Jacobian or a Prym, respectively. The Künneth projectors* π_X^i*, the Hodge star operator (à la Kleiman)* $*_K$*, and the primitive part extractors* $p^{j,k}$ *are all defined by algebraic correspondences with coefficients in* $\mathbb{Z}[1/2(g!)]$.

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