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# The equations defining blowup algebras of height three Gorenstein ideals

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We find the defining equations of Rees rings of linearly presented height three Gorenstein ideals. To prove our main theorem we use local cohomology techniques to bound the maximum generator degree of the torsion submodule of symmetric powers in order to conclude that the defining equations of the Rees algebra and of the special fiber ring generate the same ideal in the symmetric algebra. We show that the ideal defining the special fiber ring is the unmixed part of the ideal generated by the maximal minors of a matrix of linear forms which is annihilated by a vector of indeterminates, and otherwise has maximal possible height. An important step in the proof is the calculation of the degree of the variety parametrized by the forms generating the height three Gorenstein ideal.

## 1. Introduction

This paper deals with the algebraic study of rings that arise in the process of blowing up a variety along a subvariety. These rings are the Rees ring and the special fiber ring of an ideal. Rees rings are also the bihomogeneous coordinate rings of graphs of rational maps between projective spaces, whereas the special fiber rings are the homogeneous coordinate rings of the images of such maps. More precisely, let $R$ be a standard graded polynomial ring over a field and $I$ be an ideal of $R$ minimally generated by forms $g_1, \dots, g_n$ of the same degree. These forms define a rational map $\Psi$ whose image is a variety $X$. The bihomogeneous coordinate ring of the graph of $\Psi$ is the Rees algebra of the ideal $I$, which is defined as the graded subalgebra $\mathcal{R}(I) = R[It]$ of the polynomial ring $R[t]$. This algebra contains, as direct summands, the ideal $I$ as well as the homogeneous coordinate ring of the variety $X$.

It is a fundamental problem to find the implicit defining equations of the Rees ring and thereby of the variety $X$. This problem has been studied extensively

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by commutative algebraists, algebraic geometers, and, most recently, by applied mathematicians in geometric modeling. A complete solution requires that one begins with a knowledge of the structure of the ideal $I$. There is a large body of work dealing with perfect ideals of height two, which are known to be ideals of maximal minors of an almost square matrix; see for instance [Herzog et al. 1983; Morey 1996; Morey and Ulrich 1996; Cox et al. 2008; Hong et al. 2008; Busé 2009; Kustin et al. 2011; Cortadellas Benítez and D'Andrea 2013; 2014; Nguyen 2014; Madsen 2015; Boswell and Mukundan 2016; Kustin et al. 2017a]. Much less is known for the "next cases", determinantal ideals of arbitrary size [Bruns et al. 2013; 2015] and Gorenstein ideals of height three [Morey 1996; Johnson 1997], the class of ideals that satisfy the Buchsbaum–Eisenbud structure theorem [Buchsbaum and Eisenbud 1977]. In the present paper we treat the case where $I$ is a height three Gorenstein ideal and the entries of a homogeneous presentation matrix of $I$ are linear forms or, more generally, generate a complete intersection ideal; see Theorem 9.1 and Remark 9.2. Height three Gorenstein ideals have been studied extensively, and there is a vast literature dealing with the various aspects of the subject; see for instance [Watanabe 1973; Buchsbaum and Eisenbud 1977; Elias and Iarrobino 1987; Boffi and Sánchez 1992; Kustin and Ulrich 1992; Iarrobino 1994; Aberbach et al. 1995; De Negri and Valla 1995; Diesel 1996; Geramita and Migliore 1997; Migliore and Peterson 1997; Kleppe and Miró-Roig 1998; Iarrobino and Kanev 1999; Polini and Ulrich 1999; Conca and Valla 2000; Hartshorne 2004; Boij et al. 2014; Kimura and Terai 2015].

Though it may be impossible to determine the implicit equations of Rees rings for general classes of ideals, one can still hope to bound the degrees of these equations, which is in turn an important step towards finding them. In the present paper we address this broader issue as well.

Traditionally one views the Rees algebra as a natural epimorphic image of the symmetric algebra $\mathrm{Sym}(I)$ of $I$ and one studies the kernel of this map,

$$0 \to \mathscr{A} \to \mathrm{Sym}(I) \to \mathscr{R}(I) \to 0.$$

The kernel $\mathscr{A}$ is the $R$-torsion submodule of $\mathrm{Sym}(I)$. The defining equations of the symmetric algebra can be easily read from the presentation matrix of the ideal $I$. Hence the simplest situation is when the symmetric algebra and the Rees algebra are isomorphic, in which case the ideal $I$ is said to be of *linear type*.

When the ideal $I$ is perfect of height two, or is Gorenstein of height three or, more generally, is in the linkage class of a complete intersection, then $I$ is of linear type if and only if the minimal number of generators of the ideal $I_{\mathfrak{p}}$, $\mu(I_{\mathfrak{p}})$, is at most $\dim R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \in V(I)$; see [Huneke 1982; Herzog et al. 1983]. If the condition $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ is only required for every nonmaximal prime ideal $\mathfrak{p} \in V(I)$, then we say that $I$ satisfies $G_d$, where $d$ is the dimension of the ring $R$. The condition $G_d$ can be interpreted in terms of the height of Fitting ideals, $i < \operatorname{ht}\operatorname{Fitt}_i(I)$, whenever $i < d$. The property $G_d$ is always satisfied if $R/I$ is 0-dimensional or if $I$ is a generic perfect ideal of height two or a generic Gorenstein ideal of height three. For ideals in the linkage class of a complete intersection the assumption $G_d$ means that $\mathcal{A}$ is annihilated by a power of the maximal homogeneous ideal $\mathfrak{m}$ of $R$, in other words $\mathcal{A}$ is the zeroth local cohomology $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I))$. One often requires the condition $G_d$ when studying Rees rings of ideals that are not necessarily of linear type.

To identify further implicit equations of the Rees ring, in addition to the equations defining the symmetric algebra, one uses the technique of *Jacobian dual*; see for instance [Vasconcelos 1991]. Let $x_1, \ldots, x_d$ be the variables of the polynomial ring $R$ and $T_1, \ldots, T_n$ be new variables. Let $\varphi$ be a minimal homogeneous presentation matrix of the forms $g_1, \ldots, g_n$ generating $I$. There exits a matrix $B$ with $d$ rows and entries in the polynomial ring $S = R[T_1, \ldots, T_n]$ such that the equality of row vectors

$$[T_1, \ldots, T_n] \cdot \varphi = [x_1, \ldots, x_d] \cdot B$$

holds. Choose $B$ so that its entries are linear in $T_1, \ldots, T_n$. The matrix $B$ is called a *Jacobian dual* of $\varphi$. If $\varphi$ is a matrix of linear forms, then the entries of $B$ can be taken from the ring $T = k[T_1, \ldots, T_n]$; this $B$ is uniquely determined by $\varphi$ and is called *the* Jacobian dual of $\varphi$. The choice of generators of $I$ gives rise to homogeneous epimorphisms of graded $R$-algebras

$$S \twoheadrightarrow \mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}(I).$$

We denote by $\mathcal{L}$ and $\mathcal{J}$ the ideals of $S$ defining $\mathrm{Sym}(I)$ and $\mathcal{R}(I)$, respectively. The ideal $\mathcal{L}$ is generated by the entries of the row vector $[T_1, \ldots, T_n]\cdot\varphi = [x_1, \ldots, x_d]\cdot B$. Cramer's rule shows that

$$\mathcal{L} + I_d(B) \subset \mathcal{J},$$

and if equality holds, then $\mathcal{J}$ or $\mathcal{R}(I)$ is said to have the *expected form*. The expected form is the next best possibility for the defining ideal of the Rees ring if $I$ is not of linear type. Linearly presented ideals whose Rees ring has the expected form are special cases of ideals of fiber type. An ideal $I$ is of *fiber type* if the defining ideal $\mathcal{J}$ of $\mathcal{R}(I)$ can be reconstructed from the defining ideal $I(X)$ of the image variety $X$ or, more precisely,

$$\mathcal{J} = \mathcal{L} + I(X)S.$$

It has been shown in [Morey and Ulrich 1996] that the Rees ring has the expected form if $I$ is a linearly presented height two perfect ideal satisfying $G_d$. In this situation the Rees algebra is Cohen–Macaulay. There has been a great deal of work investigating the defining ideal of Rees rings when $I$ is a height two perfect ideal that either fails to satisfy $G_d$ or is not linearly presented. In this case $\mathcal{R}(I)$ usually does not have the expected form and is not Cohen–Macaulay; see for instance [Cox et al. 2008; Hong et al. 2008; Busé 2009; Kustin et al. 2011; Cortadellas Benítez and D'Andrea 2013; 2014; Nguyen 2014; Boswell and Mukundan 2016; Madsen 2015; Kustin et al. 2017a].

Much less is known for height three Gorenstein ideals. In this case $n$ is odd and the presentation matrix $\varphi$ can be chosen to be alternating, according to the Buchsbaum–Eisenbud structure theorem [Buchsbaum and Eisenbud 1977]. The defining ideal $\mathcal{J}$ of the Rees ring has been determined, provided that $I$ satisfies $G_d$, $n = d+1$, and moreover $\varphi$ has linear entries [Morey 1996] or, more generally, $I_1(\varphi)$ is generated by the entries of a single row of $\varphi$ [Johnson 1997]. It turns out that $\mathcal{R}(I)$ does not have the expected form, but is Cohen–Macaulay under these hypotheses. On the other hand, this is the only case when the Rees ring is Cohen–Macaulay. In fact, the Rees ring of a height three Gorenstein ideal satisfying $G_d$ is Cohen–Macaulay if and only if either $n \le d$ or else $n = d+1$ and $I_1(\varphi)$ is generated by the entries of a single generalized row of $\varphi$ [Polini and Ulrich 1999]. Observe that if $n = d+1$ then $d$ is necessarily even. The "approximate resolutions" of the symmetric powers of $I$ worked out in [Kustin and Ulrich 1992] show that if $n \ge d+1$ and $d$ is even, then $\mathrm{Sym}(I)$ has $R$-torsion in too low a degree for $\mathcal{J}$ to have the expected form; see also [Morey 1996]. As it turns out, the alternating structure of the presentation matrix $\varphi$ is responsible for "unexpected" elements in $\mathcal{J}$. Let $\varphi$ be an alternating matrix of linear forms presenting a height three Gorenstein ideal, and let $B$ be its Jacobian dual. As in [Kustin and Ulrich 1992] one builds an alternating matrix from $\varphi$ and $B$,

$$\mathfrak{B} = \begin{bmatrix} \varphi & -B^{\mathrm{t}} \\ B & 0 \end{bmatrix}.$$

The submaximal Pfaffians of this matrix are easily seen to belong to $\mathcal{J}$. We regard the last of these Pfaffians as a polynomial in $T[x_1, \dots, x_d]$ and consider its content ideal $C(\varphi)$ in $T$. The main theorem of the present paper says that $C(\varphi)$, together with the expected equations, generates the ideal $\mathcal{J}$.

**Theorem 9.1.** *Let $I$ be a linearly presented height three Gorenstein ideal that satisfies $G_d$. Then the defining ideal of the Rees ring of $I$ is*

$$\mathcal{J} = \mathcal{L} + I_d(B)S + C(\varphi)S,$$

*and the defining ideal of the variety $X$ is*

$$I(X) = I_d(B) + C(\varphi),$$

*where $\varphi$ is a minimal homogeneous alternating presentation matrix for $I$. In particular, $I$ is of fiber type and, if $d$ is odd, then $C(\varphi) = 0$ and $\mathcal{R}(I)$ has the expected form.*

We now outline the method of proof of our main theorem. The proof has essentially four ingredients, each of which is of independent interest and applies in much greater generality than needed here. Some of these tools are developed in other articles [Kustin and Ulrich 1992; Kustin et al. 2015; 2016b; 2017b]. We describe the content of these articles to the extent necessary for the present paper.

The first step is to prove that any ideal $I$ satisfying the assumptions of Theorem 9.1 is of fiber type (Corollary 6.4). Once this is done, it suffices to determine $I(X)$. A crucial tool in this context is the standard bigrading of the symmetric algebra. If $\delta$ is the degree of the generators of $I$, then the $R$-module $I(\delta)$ is generated in degree zero and its symmetric algebra $\operatorname{Sym}(I(\delta))$ is naturally standard bigraded. As it turns out, the ideal $I$ is of fiber type if and only if the bihomogeneous ideal $\mathscr{A} = \mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}(I(\delta)))$ of the symmetric algebra is generated in degrees $(0, \star)$ or, equivalently, the $R$-modules $\mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}_q(I(\delta)))$ are generated in degree zero for all nonnegative integers $q$. This suggests the following general question: If $M$ is a finitely generated graded $R$-module, what are bounds for the generator degrees of the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$? This question is addressed in [Kustin et al. 2015]. There we consider *approximate resolutions*, which are homogeneous complexes of finitely generated graded $R$-modules $C_j$ of sufficiently high depth,

$$C_\bullet : \cdots \to C_1 \to C_0 \to 0,$$

such that $\mathrm{H}_0(C_\bullet) \cong M$ and $\mathrm{H}_j(C_\bullet)$ have sufficiently small dimension for all positive integers $j$. We prove, for instance, that $\mathrm{H}^0_{\mathfrak{m}}(M)$ is concentrated in degrees $\le b_0(C_d) - d$ and is generated in degrees $\le b_0(C_{d-1}) - d + 1$, where $b_0(C_j)$ denotes the largest generator degree of $C_j$. These theorems from [Kustin et al. 2015] are reproduced in the present paper as Corollary 5.4 and Theorem 5.6. Whereas similar results for the concentration degree of local cohomology modules have been established before to study Castelnuovo–Mumford regularity (see, e.g., [Gruson et al. 1983]), it appears that the sharper bound on the generation degree is new and more relevant for our purpose. Under the hypotheses of Theorem 9.1 the complexes $\mathscr{D}^q_\bullet$ from [Kustin and Ulrich 1992] are approximate resolutions of the symmetric powers $\operatorname{Sym}_q(I(\delta))$, and we deduce that indeed the $R$-modules $\mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}_q(I(\delta)))$ are generated in degree zero. This is done in Theorem 6.1 and Corollary 6.4.

In fact, Theorem 6.1 is considerably more general; it deals with height two perfect ideals and height three Gorenstein ideals that are not necessarily linearly presented. We record an explicit exponent $N$ so that $\mathfrak{m}^N \cdot \mathscr{A} = 0$ and we provide an explicit bound for the largest $x$-degree $p$ so that $\mathscr{A}_{(p,*)}$ contains a minimal bihomogeneous generator of $\mathscr{A}$. Theorem 6.1 is a significant generalization of part of [Kustin et al. 2017a], where the same results are established under fairly restrictive hypotheses.

With the hypotheses of Theorem 9.1 we know that $I_d(B) + C(\varphi) \subset I(X)$; see [Kustin et al. 2017b] or Corollary 8.8. To show that this inclusion is an equality, we first prove that the two ideals have the same height, which is $n-d$ if $d \leq n$. In fact, we show much more under weaker hypotheses: Assume $I$ is any ideal of $R$ generated by forms of degree $\delta$, $I$ is of linear type on the punctured spectrum, and a sufficiently high symmetric power $\mathrm{Sym}_q(I(\delta))$ has an approximate free resolution that is linear for the first $d$ steps, then up to radical, $\mathcal{J}$ has the expected form, in symbols,

$$\mathcal{J} = \sqrt{(\mathcal{L}, I_d(B))}$$

and, as a consequence,

$$I(X) = \sqrt{I_d(B')},$$

where $B'$ is the Jacobian dual of the largest submatrix of $\varphi$ that has linear entries; this is Theorem 7.2. We point out that even though our primary interest is the defining ideal $I(X)$ of the variety $X$, we need to consider the entire bigraded symmetric algebra of $I$ since our main assumption uses the first component of the bigrading, which is invisible to the homogeneous coordinate ring of $X$.

The third step in the proof of Theorem 9.1 is to show that the ideal $I_d(B)+C(\varphi)$ is unmixed, in the relevant case when $d<n$. In [Kustin et al. 2017b] we construct, more generally, complexes associated to any $d$ by $n$ matrix $B$ with linear entries in $T$ that is annihilated by a vector of indeterminates, $B \cdot \underline{T}^{\mathrm{t}} = 0$. The ideal of $d$ by $d$ minors of such a matrix cannot have generic height $n-d+1$. However, if the height is $n-d$, our complexes give resolutions of $I_d(B)$ and of $I_d(B)+C(\varphi)$, although these ideals fail to be perfect in general. Thus, we obtain resolutions in the context of Theorem 9.1 because we have seen in the previous step that $I_d(B)$ has height $n-d$. Using these resolutions we prove that the ideal $I_d(B)+C(\varphi)$ is unmixed and hence is the unmixed part of $I_d(B)$. In [Kustin et al. 2017b] we also compute the multiplicity of the rings defined by these ideals.

We have seen that the two ideals $I_d(B)+C(\varphi)$ and $I(X)$ are unmixed and have the same height. Thus, to conclude that the inclusion $I_d(B)+C(\varphi) \subset I(X)$ is an equality it suffices to prove that the rings defined by these ideals have the same multiplicity. Since the multiplicity of the first ring has been computed in [Kustin et al. 2017b], it remains to determine the multiplicity of the coordinate ring $A = T/I(X)$ or, equivalently, the degree of the variety $X$. To do so we observe that the rational map $\Psi$ is birational onto its image because the presentation matrix $\varphi$ is linear (Proposition 4.1). Therefore the degree of $X$ can be expressed in terms of the multiplicity of a ring defined by a certain residual intersection of $I$ (Proposition 4.2). Finally, the multiplicity of such residual intersections can be obtained from the resolutions in [Kustin and Ulrich 1992]; see [Kustin et al. 2016b] and Theorem 4.4. Once we prove in Theorem 9.1 that $I(X) = I_d(B)+C(\varphi)$, then in Corollary 9.3 we are able to harvest much information from the results of [Kustin et al. 2017b]. In particular, we learn the depth of the homogeneous coordinate ring $A$ of $X$, the entire Hilbert series of $A$, and the fact that $I(X)$ has a linear resolution when $d$ is odd.

## 2. Conventions and notation

**2.1.** If $\psi$ is a matrix, then $\psi^{\mathrm{t}}$ is the transpose of $\psi$. If $\psi$ is a matrix (or a homomorphism of finitely generated free $R$-modules), then $I_r(\psi)$ is the ideal generated by the $r \times r$ minors of $\psi$ (or any matrix representation of $\psi$).

**2.2.** We collect names for some of the invariants associated to a graded module. Let $R$ be a graded Noetherian ring and $M$ be a graded $R$-module. Define

$$\begin{aligned}\operatorname{topdeg} M &= \sup\{\, j \mid M_j \neq 0\},\\ \operatorname{indeg} M &= \inf\{\, j \mid M_j \neq 0\}, \text{ and}\\ b_0(M) &= \inf\Big\{p \mid R\Big(\bigoplus_{j\leq p} M_j\Big) = M\Big\}.\end{aligned}$$

If $R$ is nonnegatively graded, $R_0$ is local, $\mathfrak{m}$ is the maximal homogeneous ideal of $R$, $M$ is finitely generated, and $i$ is a nonnegative integer, then also define

$$a_i(M) = \operatorname{topdeg} \mathrm{H}^i_{\mathfrak{m}}(M) \quad \text{and} \quad b_i(M) = \operatorname{topdeg} \operatorname{Tor}_i^R(M, R/\mathfrak{m}).$$

Observe that both definitions of the maximal generator degree $b_0(M)$ give the same value. The expressions “topdeg”, “indeg”, and “$b_i$” are read “top degree”, “initial degree”, and “maximal $i$-th shift in a minimal homogeneous resolution”, respectively. If $M$ is the zero module, then

$$\operatorname{topdeg}(M) = b_0(M) = -\infty \quad \text{and} \quad \operatorname{indeg} M = \infty.$$

In general one has

$$a_i(M) < \infty \quad \text{and} \quad b_i(M) < \infty.$$

**2.3.** Recall the numerical functions of 2.2. Let $R$ be a nonnegatively graded Noetherian ring with $R_0$ local and $\mathfrak{m}$ be the maximal homogeneous ideal of $R$. The $a$-invariant of $R$ is defined to be

$$a(R) = a_{\dim R}(R).$$

Furthermore, if $\omega_R$ is the graded canonical module of $R$, then

$$a(R) = -\operatorname{indeg} \omega_R.$$

**2.4.** Let $R$ be a standard graded polynomial ring over a field $k$, $\mathfrak{m}$ be the maximal homogeneous ideal of $R$, and $M$ be a finitely generated graded $R$-module. Recall the numerical functions of 2.2. The Castelnuovo–Mumford *regularity* of $M$ is

$$\operatorname{reg} M = \sup\{a_i(M) + i\} = \sup\{b_i(M) - i\}.$$

**2.5.** Let $I$ be an ideal in a Noetherian ring. The *unmixed part* $I^{\text{unm}}$ of $I$ is the intersection of the primary components of $I$ that have height equal to the height of $I$.

**2.6.** Let $R$ be a Noetherian ring, $I$ be a proper ideal of $R$, and $M$ be a finitely generated nonzero $R$-module. We denote the projective dimension of the $R$-module $M$ by $\operatorname{pd}_R M$. The *grade* of $I$ is the length of a maximal regular sequence on $R$ which is contained in $I$. (If $R$ is Cohen–Macaulay, then the grade of $I$ is equal to the height of $I$.) The $R$-module $M$ is called *perfect* if the grade of the annihilator of $M$ (denoted $\operatorname{ann} M$) is equal to the projective dimension of $M$. (The inequality $\operatorname{grade} \operatorname{ann} M \le \operatorname{pd}_R M$ holds automatically.) The ideal $I$ in $R$ is called a *perfect ideal* if $R/I$ is a perfect $R$-module. A perfect ideal $I$ of grade $g$ is a *Gorenstein ideal* if $\operatorname{Ext}^g_R(R/I, R)$ is a cyclic $R$-module.

**2.7.** Denote by $\mu(M)$ the minimal number of generators of a finitely generated module $M$ over a local ring $R$. Recall from [Artin and Nagata 1972] that an ideal $I$ in a Noetherian ring $R$ satisfies the condition $G_s$ if $\mu(I_\mathfrak{p}) \le \dim R_\mathfrak{p}$ for each prime ideal $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \le s-1$. The condition $G_s$ can be interpreted in terms of the height of Fitting ideals. An ideal $I$ of positive height satisfies $G_s$ if and only if $i < \operatorname{ht} \operatorname{Fitt}_i(I)$ for $0 < i < s$.

The property $G_d$, for $d = \dim R$, is always satisfied if $R/I$ is 0-dimensional or if $I$ is a generic perfect ideal of grade two or a generic Gorenstein ideal of grade three.

**2.8.** Let $R$ be a Noetherian ring, $I$ be an ideal of height $g$, $K$ be a proper ideal, and $s$ be an integer with $g \le s$.

(a) The ideal $K$ is called an *$s$-residual intersection* of $I$ if there exists an $s$-generated ideal $\mathfrak{a} \subset I$ such that $K = \mathfrak{a} : I$ and $s \le \operatorname{ht} K$.

(b) The ideal $K$ is called a *geometric $s$-residual intersection* of $I$ if $K$ is an $s$-residual intersection of $I$ and if, in addition, $s+1 \le \operatorname{ht}(I+K)$.

(c) The ideal $I$ is said to be *weakly $s$-residually $S_2$* if for every $i$ with $g \le i \le s$ and every geometric $i$-residual intersection $K$ of $I$, $R/K$ is $S_2$.

**2.9.** If $I$ is an ideal of a ring $R$, then the *Rees ring* of $I$, denoted $\mathcal{R}(I)$, is the subring $R[It]$ of the polynomial ring $R[t]$. There is a natural epimorphism of $R$-algebras from the symmetric algebra of $I$, denoted $\operatorname{Sym}(I)$, onto $\mathcal{R}(I)$. If this map is an isomorphism, we say that the ideal $I$ is of *linear type*.

## 3. Defining equations of graphs and images of rational maps

**Data 3.1.** Let $k$ be a field and let $R = k[x_1, \ldots, x_d]$, $T = k[T_1, \ldots, T_n]$, and $S = R[T_1, \ldots, T_n]$ be polynomial rings. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the maximal homogeneous ideal of $R$, and $I$ a homogeneous ideal of $R$ minimally generated by forms $g_1, \ldots, g_n$ of the same positive degree $\delta$.

Such forms define a rational map

$$\Psi = [g_1 : \cdots : g_n] : \mathbb{P}_k^{d-1} \to \mathbb{P}_k^{n-1}$$

with base locus $V(I)$. We write $X$ for the closed image of $\Psi$, the variety parametrized by $\Psi$, and $A = T/I(X)$ for the homogeneous coordinate ring of this variety.

In this section we collect some general facts about rational maps and Rees rings. Data 3.1 is in effect throughout.

**3.2.** Recall the definition of the Rees ring $\mathcal{R}(I) = R[It] \subset R[t]$ from 2.9. We consider the epimorphism of $R$-algebras

$$\pi : S \twoheadrightarrow \mathcal{R}(I)$$

that sends $T_i$ to $g_i t$. Let $\mathcal{J}$ be the kernel of this map, which is the ideal defining the Rees ring.

We set $\deg x_i = (1, 0)$, $\deg T_i = (0, 1)$, and $\deg t = (-\delta, 1)$. Thus $S$ and the subring $\mathcal{R}(I)$ of $R[t]$ become standard bigraded $k$-algebras, $\pi$ is a bihomogeneous epimorphism, and $\mathcal{J}$ is a bihomogeneous ideal of $S$. If $M$ is a bigraded $S$-module and $p$ a fixed integer, we write

$$M_{(p,\star)} = \bigoplus_j M_{(p,j)} \quad \text{and} \quad M_{(>p,\star)} = \bigoplus_{i>p} M_{(i,\star)}.$$

**3.3.** The Rees algebra $\mathcal{R}(I)$ is the bihomogeneous coordinate ring of the graph of $\Psi$. In fact, the natural morphisms of projective varieties

$$\begin{tikzcd} \mathbb{P}_k^{d-1} \times \mathbb{P}_k^{n-1} \arrow[d] & \text{graph}\,\Psi \arrow[l, hook'] \arrow[d] \\ \mathbb{P}_k^{n-1} & X = \operatorname{im}\Psi \arrow[l, hook'] \end{tikzcd}$$

correspond to bihomogeneous homomorphisms of $k$-algebras,

$$\begin{tikzcd} S \arrow[r, two heads, "\pi"] & \mathcal{R}(I) \\ T \arrow[u, hook] \arrow[r, two heads, "\pi_{|T}"'] & A. \arrow[u, hook] \end{tikzcd}$$

Since $T = S_{(0,\star)}$, we see that

$$A = \mathcal{R}(I)_{(0,\star)} = k[g_1 t, \ldots, g_n t]$$

and

$$I(X) = \mathcal{J}_{(0,\star)} = \mathcal{J} \cap T. \tag{3.3.1}$$

We also observe that there is a not necessarily homogeneous isomorphism of $k$-algebras

$$k[g_1t, \dots, g_nt] \cong k[g_1, \dots, g_n] \subset R.$$

Moreover, the identification $\mathcal{R}(I)_{(0,\star)} \cong \mathcal{R}(I)/\mathcal{R}(I)_{(>0,\star)}$ gives a homogeneous isomorphism

$$A \cong \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I).$$

The latter ring is called the *special fiber ring* of $I$ and its dimension is the *analytic spread* of $I$, written $\ell(I)$. The analytic spread plays an important role in the study of reductions of ideals and satisfies the inequality

$$\ell(I) \le \min\{d, n\}. \tag{3.3.2}$$

Notice that $\ell(I) = \dim A = \dim X + 1$.

**3.4.** The graded $R$-module $I(\delta)$ is generated in degree zero and hence its symmetric algebra $\mathrm{Sym}(I(\delta))$ is a standard bigraded $k$-algebra. There is a natural bihomogeneous epimorphism of $k$-algebras

$$\mathrm{Sym}(I(\delta)) \twoheadrightarrow \mathcal{R}(I),$$

whose kernel we denote by $\mathcal{A}$. The epimorphism $\pi$ from 3.2 factors through this map and gives a bihomogeneous epimorphism of $k$-algebras

$$S \twoheadrightarrow \mathrm{Sym}(I(\delta)).$$

We write $\mathcal{L}$ for the kernel of this epimorphism. The advantage of $\mathcal{L}$ over $\mathcal{J}$ is that $\mathcal{L}$ can be easily described in terms of a presentation matrix of $I$; see 3.6. Notice that

$$\mathcal{A} = \mathcal{J}/\mathcal{L}.$$

The ideal $I$ is of linear type if any of the following equivalent conditions hold:

- $\mathcal{A} = 0$.
- $\mathcal{J} = \mathcal{L}$.
- $\mathcal{J}$ is generated in bidegrees $(\star, 1)$.

Notice that if $I$ is of linear type then necessarily $I(X) = 0$.

The ideal $I$ is of *fiber type* if any of the following equivalent conditions hold:

- $\mathcal{A} = I(X) \cdot \mathrm{Sym}(I(\delta))$.
- $\mathcal{A}$ is generated in bidegrees $(0, \star)$.
- $\mathcal{J} = \mathcal{L} + I(X) \cdot S$.
- $\mathcal{J}$ is generated in bidegrees $(\star, 1)$ and $(0, \star)$.

See (3.3.1) for the equivalence of these conditions.

**3.5.** The ideal $\mathscr{A}$ in the exact sequence

$$0 \to \mathscr{A} \to \mathrm{Sym}(I(\delta)) \to \mathscr{R}(I) \to 0$$

is the $R$-torsion of $\mathrm{Sym}(I(\delta))$. Hence $\mathscr{A}$ contains the zeroth local cohomology of $\mathrm{Sym}(I(\delta))$ as an $R$-module, which in turn contains the socle of $\mathrm{Sym}(I(\delta))$ as an $R$-module,

$$\mathscr{A} \supset \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta))) = 0 :_{\mathrm{Sym}(I(\delta))} \mathfrak{m}^\infty \supset 0 :_{\mathrm{Sym}(I(\delta))} \mathfrak{m}. \tag{3.5.1}$$

**3.6.** Consider the row vectors $\underline{x} = [x_1, \dots, x_d]$, $\underline{T} = [T_1, \dots, T_n]$, $\underline{g} = [g_1, \dots, g_n]$, and let $\varphi$ be a minimal homogeneous presentation matrix of $\underline{g}$. There exists a matrix $B$ with $d$ rows and entries in the polynomial ring $S$ such that the equality of row vectors

$$\underline{T} \cdot \varphi = \underline{x} \cdot B \tag{3.6.1}$$

holds. If $B$ is chosen so that its entries are linear in $T_1, \dots, T_n$, then $B$ is called a *Jacobian dual* of $\varphi$. We say that a Jacobian dual is *homogeneous* if the entries of any fixed column of $B$ are homogeneous of the same bidegree. Whenever $\varphi$ is a matrix of linear forms, then the entries of $B$ can be taken from the ring $k[T_1, \dots, T_n]$; this $B$ is uniquely determined by $\varphi$ and is called *the* Jacobian dual of $\varphi$; notice that the entries of $B$ are linear forms and that $B$ is indeed a Jacobian matrix of the $T$-algebra $\mathrm{Sym}(I(\delta))$.

The ideal $\mathscr{L}$ defining the symmetric algebra is equal to $I_1(\underline{T} \cdot \varphi)$, which coincides with $I_1(\underline{x} \cdot B)$. For the Rees algebra, the inclusions

$$\mathscr{L} + I_d(B) \subset \mathscr{L} :_S \mathfrak{m} \subset \mathscr{J} \tag{3.6.2}$$

obtain; see also [Vasconcelos 1991]. If the equality $\mathscr{L} + I_d(B) = \mathscr{J}$ holds, then $\mathscr{J}$ or $\mathscr{R}(I)$ is said to have the *expected form*. Notice that in this case the inclusions of (3.5.1) are equalities, in other words, $\mathscr{A}$ is the socle of $\mathrm{Sym}(I(\delta))$ as an $R$-module.

**3.7.** It is easier to determine when the first inclusion of (3.5.1) is an equality. Namely,

$$\mathscr{A} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta))) \iff I \text{ is of linear type on the punctured spectrum of } R. \tag{3.7.1}$$

The right hand side means that $\mathscr{A}_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ in $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$.

This leads to the question of when an ideal is of linear type. Thus let $J$ be an ideal of a local Cohen–Macaulay ring of dimension $d$. If $J$ is perfect of height two or Gorenstein of height three, then $J$ is in the linkage class of a complete intersection [Gaeta 1953; Watanabe 1973]. If $J$ is in the linkage class of a complete intersection, then $J$ is strongly Cohen–Macaulay [Huneke 1982, 1.11] and hence satisfies the sliding depth condition of [Herzog et al. 1985]. If $J$ satisfies the sliding depth condition and is $G_{d-2}$, then $J$ is weakly $(d-2)$-residually $S_2$ in the sense of

2.8.(c) [Herzog et al. 1985, 3.3]. In the presence of the weak $(d-2)$-residually $S_2$ condition it follows from [Chardin et al. 2001, 3.6(b)] that

$$J \text{ is of linear type} \iff J \text{ satisfies } G_{d+1};$$

in particular, since the residually $S_2$ condition localizes,

$$J \text{ is of linear type on the punctured spectrum} \iff J \text{ satisfies } G_d.$$

(Alternatively, one uses [Herzog et al. 1983].)

Combining these facts we see that if the ideal $I$ of Data 3.1 is perfect of height two or Gorenstein of height three then

$$I \text{ is of linear type} \iff I \text{ satisfies } G_d \text{ and } n \le d \tag{3.7.2}$$

and

$$\mathcal{A} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta))) \iff I \text{ satisfies } G_d. \tag{3.7.3}$$

## 4. The degree of the image of a rational map

This section deals with the dimension and the degree of the image of rational maps as in Data 3.1. Applied to the special case of linearly presented Gorenstein ideals of height three, this information will be an important ingredient in the proof of Theorem 9.1.

We first treat the question of when the rational map $\Psi$ is birational onto its image $X$ (Proposition 4.1). We then express the degree of $X$ in terms of the multiplicity of a ring defined by a residual intersection of the ideal $I$ defining the base locus of $\Psi$ (Proposition 4.2). The multiplicity of rings defined by residual intersections of linearly presented height three Gorenstein ideals was computed in [Kustin et al. 2016b], using the resolutions worked out in [Kustin and Ulrich 1992]. For completeness we state this result in Theorem 4.3. Finally, we combine Propositions 4.1, 4.2, and Theorem 4.3 in Theorem 4.4 to compute the degree of $X$ under the hypotheses of Theorem 9.1.

**Proposition 4.1.** *Adopt Data 3.1 and assume further that the ideal $I$ is linearly presented.*

(a) *If the ring $R/I$ has dimension zero, then the map $\Psi$ is biregular onto its image. In particular, $A$ has multiplicity $\delta^{d-1}$.*

(b) *If $A$ has dimension $d$, then the rational map $\Psi$ is birational onto its image.*

*Proof.* We may assume that the field $k$ is algebraically closed. Let $\varphi$ be a minimal presentation matrix of $\underline{g} = g_1, \dots, g_n$ with homogeneous linear entries in $R$ and let $P \in \mathbb{P}^{n-1}_k$ be a point in $X$. According to [Eisenbud and Ulrich 2008] the ideal $I_1(P\cdot\varphi):I^\infty$ defines the fiber $\Psi^{-1}(P)$ scheme theoretically. This ideal cannot be the unit ideal because the fiber is not empty. On the other hand, the ideal $I_1(P\cdot\varphi)$ is generated by linear forms, hence it is a prime ideal of $R$. Thus $I_1(P\cdot\varphi):I^\infty = I_1(P\cdot\varphi)$ is generated by linear forms.

(a) We consider the embedding $A \cong k[\underline{g}] \subset R$ which corresponds to the rational map $\Psi$ onto its image. If the ring $R/I = R/(\underline{g})$ has dimension zero, then $R$ is finitely generated as an $A$-module. It follows that the fiber $\Psi^{-1}(P)$ consists of finitely many points. Since this fiber is defined by a linear ideal, it consists of a single reduced point. This shows that $\Psi$ is biregular onto its image.

The claim about the multiplicity follows because, in particular, $\Psi$ is birational onto its image; see, for instance, [Kustin et al. 2016a, 5.8].

(b) Let $Q$ be a general point in $\mathbb{P}_k^{d-1}$ and $P$ be its image $\Psi(Q)$ in $X$. Since $A \cong k[\underline{g}]$ and $R$ have the same dimension, the fiber $\Psi^{-1}(P)$ consists of finitely many points. Therefore, again, $\Psi^{-1}(P)$ consists of a single reduced point, which means that $\Psi$ is birational onto its image. $\square$

**Proposition 4.2.** *Adopt Data 3.1. Further assume that the field $k$ is infinite, that $d \le n$, and that the ideal $I$ is weakly $(d-2)$-residually $S_2$ and satisfies $G_d$. Let $f_1,\dots,f_{d-1}$ be general $k$-linear combinations of the generators $g_1,\dots,g_n$ of $I$. Then the following statements hold*:

(a) *The ring $R/((f_1,\dots,f_{d-1}):I)$ is Cohen–Macaulay of dimension one.*

(b) *The ring $A$ has dimension $d$.*

(c) *The ring $A$ has multiplicity*

$$e(A) = \frac{1}{r}\cdot e\left(\frac{R}{(f_1,\dots,f_{d-1}):I}\right),$$

*where $r$ is the degree of the rational map $\Psi$.*

*Proof.* Let $f_1,\dots,f_d$ be general $k$-linear combinations of the generators $g_1,\dots,g_n$ of $I$. The $G_d$ property of the ideal $I$ implies that

$$d-1 \le \operatorname{ht}((f_1,\dots,f_{d-1}):I) \quad \text{and} \quad d \le \operatorname{ht}((f_1,\dots,f_{d-1}):I,\, f_d), \tag{4.2.1}$$

according to [Artin and Nagata 1972, 2.3] or [Ulrich 1994, 1.6(a)]. Furthermore the ideal $(f_1,\dots,f_{d-1}):I$ is proper because $I$ requires at least $d$ generators. Now 3.1 and 3.3(a) of [Chardin et al. 2001] imply that $(f_1,\dots,f_{d-1}):I$ is unmixed of height $d-1$, which proves (a).

To show (b) we suppose that $\dim A < d$. In this case the ring $k[g_1,\dots,g_n] \cong A$ is integral over the subring $k[f_1,\dots,f_{d-1}]$, and therefore the ideal $I$ is integral over the ideal $(f_1,\dots,f_{d-1})$. In 3.7 we have seen that, locally on the punctured spectrum, $I$ is of linear type and hence cannot be integral over a proper subideal. Thus $d \le \operatorname{ht}((f_1,\dots,f_{d-1}):I)$, which contradicts (a).

We now prove (c). Item (b) allows us to apply [Kustin et al. 2016a, 3.7], which gives

$$e(A) = \frac{1}{r} \cdot e\left(\frac{R}{(f_1, \dots, f_{d-1}) : I^\infty}\right).$$

On the other hand, item (a) and the inequality $d \le \operatorname{ht}((f_1, \dots, f_{d-1}) : I, f_d)$ imply that the element $f_d$ of $I$ is regular on $R/((f_1, \dots, f_{d-1}) : I)$. It follows that $(f_1, \dots, f_{d-1}) : I^\infty = (f_1, \dots, f_{d-1}) : I$. □

The following statement, which is used in the proof of Theorem 4.4, is part of [Kustin et al. 2016b, 1.2]. We do not need the entire $h$-vector of $\overline{R}$ in the present paper; but the entire $h$-vector is recorded in [Kustin et al. 2016b, 1.2].

**Theorem 4.3.** *Adopt Data 3.1. Further assume that the ideal $I$ is a linearly presented Gorenstein ideal of height three. Let $\mathfrak{a}$ be a subideal of $I$, minimally generated by $s$ homogeneous elements for some $s$ with $3 \le s$, let $\alpha$ be the minimal number of generators of $I/\mathfrak{a}$, $J$ be the ideal $\mathfrak{a} : I$, and $\overline{R} = R/J$. Assume that*

(1) *the homogeneous minimal generators of $\mathfrak{a}$ live in two degrees*: $\operatorname{indeg}(I)$ *and* $\operatorname{indeg}(I) + 1$,

(2) *the ideal $J$ is an $s$-residual intersection of $I$ (that is, $J \ne R$ and $s \le \operatorname{ht} J$).*

*Then the ring $\overline{R}$ has multiplicity*

$$e(\overline{R}) = \sum_{i=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{s+\alpha-2-2i}{s-1}.$$

*The latter is also equal to the number of monomials $\boldsymbol{m}$ of degree at most $\alpha - 1$ in $s$ variables with $(\deg \boldsymbol{m}) + \alpha$ odd.*

Theorem 4.4, the main result of this section, is where the dimension and the degree of the variety $X$ are computed.

**Theorem 4.4.** *Adopt Data 3.1. Further assume that $d \le n$, that $I$ is a linearly presented Gorenstein ideal of height three, and that $I$ satisfies $G_d$. Then $A$ has dimension $d$ and multiplicity*

$$e(A) = \sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2},$$

*which is equal to*

- *the number of monomials of even degree at most $n-d$ in $d-1$ variables, if $d$ is odd,*
- *the number of monomials of odd degree at most $n-d$ in $d-1$ variables, if $d$ is even.*

*Proof.* We may assume that the field $k$ is infinite. We first observe that the ideal $I$ satisfies the hypotheses of Proposition 4.2; see 3.7. According to Propositions 4.2.(b), 4.1.(b), and 4.2.(c), the dimension of $A$ is $d$ and the multiplicity of $A$ is

$$e(A) = e\left(\frac{R}{(f_1, \dots, f_{d-1}) : I}\right),$$

where $f_1, \dots, f_{d-1}$ are general $k$-linear combinations of homogeneous minimal generators of $I$.

Write $J = (f_1, \dots, f_{d-1}) : I$. This ideal is a $(d-1)$-residual intersection of $I$ by (4.2.1). Apply Theorem 4.3 with $s$ replaced by $d-1$ and $\alpha$ replaced by $n-d+1$ in order to obtain

$$e(A) = e(R/J) = \sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2}. \qquad \square$$

## 5. Degree bounds for local cohomology

An important step in the proof of Theorem 9.1 is to show that an ideal is of fiber type (see 3.4 for a definition) and to identify a power of the maximal ideal that annihilates the torsion of the symmetric algebra. In (3.7.3) and 3.4 we have seen that this amounts to finding degree bounds for local cohomology modules. Such degree bounds are given in [Kustin et al. 2015], we recall them here. These results are applied to the study of blowup algebras in Sections 6 and 7.

**Data 5.1.** Let $R$ be a nonnegatively graded Noetherian algebra over a local ring $R_0$ with $\dim R = d$, let $\mathfrak{m}$ be the maximal homogeneous ideal of $R$, let $M$ be a graded $R$-module, and let

$$C_\bullet : \ \cdots \to C_1 \to C_0 \to 0$$

be a homogeneous complex of finitely generated graded $R$-modules with $\mathrm{H}_0(C_\bullet) \cong M$.

In [Kustin et al. 2015] we find bounds on the degrees of interesting elements of the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ in terms of information about the ring $R$ and information that can be read from the complex $C_\bullet$. The ring $R$ need not be a polynomial ring, the complex $C_\bullet$ need not be finite, need not be acyclic, and need not consist of free modules, and the parameter $i$ need not be zero. Instead, we impose hypotheses on the Krull dimension of $\mathrm{H}_j(C_\bullet)$ and the depth of $C_j$ in order to make various local cohomology modules $\mathrm{H}^\ell_{\mathfrak{m}}(\mathrm{H}_j(C_\bullet))$ and $\mathrm{H}^\ell_{\mathfrak{m}}(C_j)$ vanish.

Recall that if $M$ and $N$ are modules over a ring, then $N$ is a *subquotient* of $M$ if $N$ is isomorphic to a submodule of a homomorphic image of $M$ or, equivalently, if $N$ is isomorphic to a homomorphic image of a submodule of $M$. Also recall the numerical functions of 2.2.

The next proposition is [Kustin et al. 2015, 3.6].

**Proposition 5.2.** *Adopt Data 5.1. Fix an integer $i$ with $0 \le i \le d$. Assume that*

(1) $j+i+1 \le \operatorname{depth} C_j$ *for all $j$ with* $0 \le j \le d-i-1$, *and*

(2) $\dim \mathrm{H}_j(C_\bullet) \le j+i$ *for all $j$ with* $1 \le j$.

*Then*

(a) $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is a graded subquotient of* $\mathrm{H}^d_{\mathfrak{m}}(C_{d-i})$, *and*

(b) $a_i(M) \le b_0(C_{d-i}) + a(R)$.

**Remark 5.3.** Typically, one applies Proposition 5.2 when the modules $C_j$ are maximal Cohen–Macaulay modules (for example, free modules over a Cohen–Macaulay ring) because, in this case, hypothesis (1) about depth $C_j$ is automatically satisfied. Similarly, if

$$\begin{aligned} \mathrm{H}_j(C_{\bullet\mathfrak{p}}) = 0 \text{ for all } j \text{ and } p \text{ with } 1 \le j \le d-i-1,\\ \mathfrak{p} \in \operatorname{Spec}(R), \text{ and } i+2 \le \dim R/\mathfrak{p}, \end{aligned} \tag{5.3.1}$$

then hypothesis (2) is satisfied.

We record an immediate consequence of Proposition 5.2; this is [Kustin et al. 2015, 3.8].

**Corollary 5.4.** *Adopt the hypotheses of Proposition 5.2, with $i = 0$. Then the following statements hold*:

(a) *If* $C_d = 0$, *then* $\mathrm{H}^0_{\mathfrak{m}}(M) = 0$.

(b) *If* $C_d \ne 0$, *then* $[\mathrm{H}^0_{\mathfrak{m}}(M)]_p = 0$ *for all $p$ with* $b_0(C_d) + a(R) < p$.

The next result, an obvious consequence of Corollary 5.4, is [Kustin et al. 2015, 3.9].

**Corollary 5.5.** *Adopt the hypotheses of Proposition 5.2, with $i = 0$. Assume further that $R = k[x_1, \dots, x_d]$ is a standard graded polynomial ring over a field and that the subcomplex*

$$C_d \to \cdots \to C_1 \to C_0 \to 0$$

*of $C_\bullet$ is a $q$-linear complex of free $R$-modules, for some integer $q$ (that is, $C_j \cong R(-j-q)^{\beta_j}$ for $0 \le j \le d$). Then $\mathrm{H}^0_{\mathfrak{m}}(M)$ is concentrated in degree $q$; that is, $[\mathrm{H}^0_{\mathfrak{m}}(M)]_p = 0$ for all $p$ with $p \ne q$.*

The next theorem is [Kustin et al. 2015, 4.8] and the main result of this section. It provides an upper bound for the generation degree of the zeroth local cohomology.

**Theorem 5.6.** *Let $R = k[x_1, \dots, x_d]$ be a standard graded polynomial ring over a field, with maximal homogeneous ideal $\mathfrak{m}$, let*

$$C_\bullet : \cdots \to C_2 \to C_1 \to C_0 \to 0$$

*be a homogeneous complex of finitely generated graded $R$-modules, and $M = \mathrm{H}_0(C_\bullet)$. Assume that*

- $\dim \mathrm{H}_j(C_\bullet) \le j$ *whenever* $1 \le j \le d-1$, *and*
- $\min\{d,\ j+2\} \le \operatorname{depth} C_j$ *whenever* $0 \le j \le d-1$.

*Then* $b_0(\mathrm{H}^0_{\mathfrak{m}}(M)) \le b_0(C_{d-1}) - d + 1$.

**Remark 5.7.** The hypotheses of Theorem 5.6 about depth and dimension are satisfied if the modules $C_j$ are free and condition (5.3.1) holds.

## 6. Bounding the degrees of defining equations of Rees rings

In this section we apply the local cohomology techniques of [Kustin et al. 2015] to bound the degrees of the defining equations of Rees rings. This style of argument was inspired by work of Eisenbud, Huneke, and Ulrich; see [Eisenbud et al. 2006].

To apply the results of [Kustin et al. 2015], most notably Corollary 5.4 and Theorem 5.6 of the present paper, we need "approximate resolutions" $C_\bullet$ of the symmetric powers of $I(\delta)$, for an ideal $I$ as in Data 3.1 that satisfies $G_d$. If $I$ is perfect of height two, we take $C_\bullet$ to be a homogeneous strand of a Koszul complex. The fact that this choice of $C_\bullet$ satisfies the hypotheses of Corollary 5.4 and Theorem 5.6 is shown in the proof of Theorem 6.1.(a). If $I$ is Gorenstein of height three, we take $C_\bullet$ to be one of the complexes $\mathfrak{D}^q_\bullet(\varphi)$ from [Kustin and Ulrich 1992]. In this paper these complexes are introduced in (6.1.4); they are shown to satisfy the hypotheses of Corollary 5.4 and Theorem 5.6 in Lemma 6.3.

In Theorem 6.1 we carefully record the degrees of the entries in a minimal homogeneous presentation matrix for $I$; but we do not insist that these entries be linear. For the ideal $\mathscr{A}$ of the symmetric algebra defined in 3.4, we give an explicit exponent $N$ so that $\mathfrak{m}^N \cdot \mathscr{A} = 0$ and we obtain an explicit bound for the largest $x$-degree $p$ so that $\mathscr{A}_{(p,*)} = \bigoplus_q \mathscr{A}_{(p,q)}$ contains a minimal bihomogeneous generator of $\mathscr{A}$.

Assertions (ai) and (aii) of Theorem 6.1, about perfect height two ideals, are significant generalizations of Corollaries 2.5(2) and 2.16(1), respectively, in [Kustin et al. 2017a], where the same results are obtained for $d = 2$ and $n = 3$. The starting point for [Kustin et al. 2017a] is the perfect pairing of Jouanolou [1996; 1997], which exists because the symmetric algebra of a three generated height two perfect ideal is a complete intersection. With the present hypotheses, the symmetric algebra is only a complete intersection on the punctured spectrum of $R$ and there is no perfect pairing analogous to the pairing of Jouanolou.

Theorem 6.1 is particularly interesting when the ideal $I$ is linearly presented. Indeed, assertions (aii) and (bii) say that $I$ is of fiber type. For height two perfect ideals this was already observed in [Morey and Ulrich 1996]. For height three Gorenstein ideals, this is a new result and a main ingredient in the proof of Theorem 9.1. The application of Theorem 6.1 to the situation where $I$ is a linearly presented height three Gorenstein ideal is recorded as Corollary 6.4.

**Theorem 6.1.** *Adopt Data 3.1 and recall the bihomogeneous ideal $\mathcal{A}$ of the symmetric algebra defined in 3.4. Assume further that $I$ satisfies the condition $G_d$.*

(a) *Assume that the ideal $I$ is perfect of height two and the column degrees of a homogeneous Hilbert–Burch matrix minimally presenting $I$ are $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{n-1}$. Then the following statements hold when $d \leq n-1$:*

  (i) *If $\sum_{i=1}^{d}(\epsilon_i - 1) < p$, then $\mathcal{A}_{(p,*)} = 0$. In particular,*

$$\mathfrak{m}^{1+\sum_{i=1}^{d}(\epsilon_i-1)}\mathcal{A} = 0.$$

  (ii) *If $\mathcal{A}_{(p,*)}$ contains a minimal bihomogeneous generator of $\mathcal{A}$, then*

$$p \leq \sum_{i=1}^{d-1}(\epsilon_i - 1).$$

(b) *Assume that the ideal $I$ is Gorenstein of height three and every entry of a homogeneous alternating matrix minimally presenting $I$ has degree $D$. Then the following statements hold:*

  (i) *If*

$$\begin{cases} d(D-1) < p & \text{when } d \text{ is odd,} \\ (d-1)(D-1) + \frac{1}{2}(n-d+1)D - 1 < p & \text{when } d \text{ is even,} \end{cases}$$

  *then $\mathcal{A}_{(p,*)} = 0$. In particular,*

$$\begin{cases} \mathfrak{m}^{d(D-1)+1}\mathcal{A} = 0 & \text{if } d \text{ is odd,} \\ \mathfrak{m}^{(d-1)(D-1)+(n-d+1)D/2}\mathcal{A} = 0 & \text{if } d \text{ is even.} \end{cases}$$

  (ii) *If $\mathcal{A}_{(p,*)}$ contains a minimal bihomogeneous generator of $\mathcal{A}$, then*

$$p \leq (d-1)(D-1).$$

**Remarks.** (a) We may safely assume that $d \leq n-1$ in Theorem 6.1 because otherwise $\mathcal{A} = 0$; see (3.7.2).

(b) The assumptions on the degrees of the entries in a presentation matrix do not impose any restriction: In (a), the inequalities $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{n-1}$ can be achieved by a permutation of the columns. In (b), all entries necessarily have the same degree, as can be seen from the symmetry of the minimal homogeneous $R$-resolution of $R/I$.

(c) Results similar to, but more complicated than, Theorem 6.1 can be obtained without the assumption that the ideal $I$ is generated by forms of the same degree.

(d) The degree $\delta$ of the forms $g_1, \dots, g_n$ in Data 3.1 is $\sum_{i=1}^{n-1} \epsilon_i$ in the setting of Theorem 6.1.(a) and is $\frac{1}{2}D(n-1)$ in Theorem 6.1.(b); see [Burch 1968] and [Buchsbaum and Eisenbud 1977], respectively.

*Proof.* We have seen in (3.7.3) that the ideal $\mathscr{A}$ of $\operatorname{Sym}(I(\delta))$ is equal to the local cohomology module $\mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}(I(\delta)))$. We apply Corollary 5.4 and Theorem 5.6 to learn about the $R$-modules

$$\mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}_q(I(\delta))) = \bigoplus_p \mathscr{A}_{(p,q)} = \mathscr{A}_{(*,q)}. \tag{6.1.1}$$

(a) Let $\varphi$ be a minimal homogeneous Hilbert–Burch matrix for $I$. In other words, $\varphi$ is an $n \times (n-1)$ matrix with homogeneous entries from $R$ such that

$$0 \to F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\underline{g}} I(\delta) \to 0$$

is a homogeneous exact sequence of $R$-modules, where $\underline{g} = [g_1, \dots, g_n]$ is the row vector of 3.6,

$$F_0 = R^n \quad \text{and} \quad F_1 = \bigoplus_{i=1}^{n-1} R(-\epsilon_i).$$

It induces a bihomogeneous presentation of the symmetric algebra

$$(\operatorname{Sym}(F_0) \otimes_R F_1)(0,-1) \xrightarrow{\underline{\ell}} \operatorname{Sym}(F_0) \to \operatorname{Sym}(I(\delta)) \to 0,$$

where $\operatorname{Sym}(F_0)$ is the standard bigraded polynomial ring $S$ of 3.2 and $\underline{\ell}$ is the row vector $[T_1, \dots, T_n] \cdot \varphi$ of 3.6. We consider the Koszul complex of $\underline{\ell}$,

$$\mathbb{K}_\bullet = \bigwedge_S^{\bullet}((S \otimes_R F_1)(0,-1)) = S \otimes_R \bigwedge_R^{\bullet}(F_1(0,-1)).$$

It is a bihomogeneous complex of free $S$-modules with $\mathrm{H}_0(\mathbb{K}_\bullet)$ equal to $\operatorname{Sym}(I(\delta))$.

The ideal $I$ is perfect of height two and satisfies $G_d$ (see 2.7). It follows that the sequence $\underline{\ell} = \ell_1, \dots, \ell_{n-1}$ is a regular sequence locally on the punctured spectrum of $R$ (see Observation 6.2); and therefore the Koszul complex $\mathbb{K}_\bullet$ is acyclic on the punctured spectrum of $R$. In particular, for each nonnegative integer $q$, the component of degree $(\star, q)$ of $\mathbb{K}_\bullet$ is a homogeneous complex of graded free $R$-modules, has zeroth homology equal to $\operatorname{Sym}_q(I(\delta))$, and is exact on the punctured spectrum of $R$. This component looks like

$$C^q_\bullet : 0 \to C^q_{n-1} \to C^q_{n-2} \to \cdots \to C^q_1 \to C^q_0 \to 0,$$

with

$$C^q_r = \operatorname{Sym}_{q-r}(F_0) \otimes_R \bigwedge^r F_1 \cong \bigoplus_{1 \le i_1 < \cdots < i_r \le n-1} R^{b_r}\Big(-\sum_{j=1}^r \epsilon_{i_j}\Big)$$

for

$$b_r = \operatorname{rank}(\operatorname{Sym}_{q-r}(F_0)) = \binom{q-r+n-1}{n-1}.$$

The assumption $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{n-1}$ yields that if $C_r^q \neq 0$ then

$$b_0(C_r^q) = \sum_{i=1}^{r} \epsilon_i. \tag{6.1.2}$$

Notice that if $C_r^q = 0$ then $b_0(C_r^q) = -\infty$.

Apply Corollary 5.4 (see also Remark 5.3) to the complex $C_\bullet^q$ to conclude that

$$[\mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}_q(I(\delta)))]_p = 0 \quad \text{for all } p \text{ with } b_0(C_d^q) + a(R) < p. \tag{6.1.3}$$

Recall that the index $q$ in (6.1.3) is arbitrary. Apply (6.1.1), (6.1.3), and (6.1.2) to conclude that

$$\mathcal{A}_{(p,*)} = 0 \quad \text{for all } p \text{ with } \sum_{i=1}^{d} \epsilon_i - d < p.$$

Assertion (ai) has been established.

Apply Theorem 5.6 (see also Remark 5.7) to the complex $C_\bullet^q$ to conclude that every minimal homogeneous generator $\alpha$ of $\mathrm{H}^0_{\mathfrak{m}}(\operatorname{Sym}_q(I(\delta)))$ satisfies

$$\deg \alpha \leq b_0(C_{d-1}^q) - d + 1.$$

Once again, (6.1.1) and (6.1.2) yield that

$$b_0(\mathcal{A}_{(*,q)}) \leq \sum_{i=1}^{d-1} \epsilon_i - d + 1.$$

The parameter $q$ remains arbitrary. Every minimal bihomogeneous generator of $\mathcal{A}$ is a minimal homogeneous generator of $\mathcal{A}_{(*,q)}$, for some $q$. We conclude that if $\mathcal{A}_{(p,q)}$ contains a minimal bihomogeneous generator of $\mathcal{A}$, then $p \leq \sum_{i=1}^{d-1} \epsilon_i - d + 1$, and this establishes assertion (aii).

(b) Let $\varphi$ be an $n \times n$ homogeneous alternating matrix which minimally presents $I$, and, for each nonnegative integer $q$, let

$$\mathcal{D}_\bullet^q(\varphi) : 0 \to \mathcal{D}_{n-1}^q \to \mathcal{D}_{n-2}^q \to \cdots \to \mathcal{D}_1^q \to \mathcal{D}_0^q \to 0$$

be the complex from [Kustin and Ulrich 1992, Definition 2.15 and Figure 4.7] that is associated to the alternating matrix $\varphi$. The zeroth homology of $\mathcal{D}_\bullet^q(\varphi)$ is $\operatorname{Sym}_q(I(\delta))$; see [Kustin and Ulrich 1992, 4.13.b]. The hypothesis that every entry of $\varphi$ has degree $D$ yields that

$$\mathfrak{D}_r^q = \begin{cases} K_{q-r,r} = R(-rD)^{\beta_r^q} & \text{if } r \le \min\{q, n-1\}, \\ Q_q = R\big(-(r-1)D - \frac{1}{2}(n-r+1)D\big) & \text{if } r = q+1 \le n-1,\ q \text{ odd}, \\ 0 & \text{if } r = q+1, q \text{ even}, \\ 0 & \text{if } \min\{q+2, n\} \le r, \end{cases} \tag{6.1.4}$$

for some nonzero Betti numbers $\beta_r^q$. Indeed, the graded $R$-module $\operatorname{Sym}_q(I(\delta))$ is generated in degree zero; moreover, each map $K_{a,b} \to K_{a+1,b-1}$ is linear in the entries of $\varphi$ (see [Kustin and Ulrich 1992, 2.15.d]) and the map $Q_q \to K_{0,q}$ may be represented by a column vector whose entries consist of the Pfaffians of the principal $(n-r+1) \times (n-r+1)$ alternating submatrices of $\varphi$ (see [Kustin and Ulrich 1992, 2.15.c and 2.15.f]).

Notice from (6.1.4) that

$$\text{if } \mathfrak{D}_r^q = Q_q, \text{ then } \mathfrak{D}_{r+1}^q = 0. \tag{6.1.5}$$

Notice also that

$$\max_{\{q \mid \mathfrak{D}_d^q \neq 0\}} b_0(\mathfrak{D}_d^q) \le \begin{cases} dD & \text{if } d \text{ is odd}, \\ (d-1)D + \frac{1}{2}(n-d+1)D & \text{if } d \text{ is even}, \end{cases} \tag{6.1.6}$$

$$\max_{\{q \mid \mathfrak{D}_d^q \neq 0\}} b_0(\mathfrak{D}_{d-1}^q) \le (d-1)D; \tag{6.1.7}$$

one uses (6.1.5) in order to see (6.1.7) because the condition $\mathfrak{D}_d^q \neq 0$ forces $\mathfrak{D}_{d-1}^q = K_{q-d+1,d-1}$.

By hypothesis the ideal $I$ satisfies the condition $G_d$; so $I$ satisfies the hypothesis (6.3.1) of Lemma 6.3 for $s = d-1$; and therefore, the complex $(\mathfrak{D}_\bullet^q(\varphi))_{\mathfrak{p}}$ is acyclic for every $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \le d-1$ and every nonnegative integer $q$. The hypotheses of Corollary 5.4 and Theorem 5.6 are satisfied by the complexes $\mathfrak{D}_\bullet^q(\varphi)$. We deduce that either $\mathfrak{D}_d^q$ and $\mathrm{H}_{\mathfrak{m}}^0(\operatorname{Sym}_q(I(\delta)))$ are both zero, or else $\mathfrak{D}_d^q \neq 0$, $[\mathrm{H}_{\mathfrak{m}}^0(\operatorname{Sym}_q(I(\delta)))]_p = 0$ for all $p$ with $b_0(\mathfrak{D}_d^q) + a(R) < p$, and every minimal homogeneous generator $\alpha$ of $\mathrm{H}_{\mathfrak{m}}^0(\operatorname{Sym}_q(I(\delta)))$ satisfies

$$\deg \alpha \le b_0(\mathfrak{D}_{d-1}^q) - d + 1.$$

In the case $\mathfrak{D}_d^q \neq 0$ we apply (6.1.1), (6.1.6), and (6.1.7) to conclude that

$$\mathcal{A}_{(p,q)} = 0 \quad \text{for} \quad \begin{cases} dD - d < p & \text{if } d \text{ is odd}, \\ (d-1)D + \frac{1}{2}(n-d+1)D - d < p & \text{if } d \text{ is even}, \end{cases}$$

and every minimal homogeneous generator $\alpha$ of $\mathcal{A}_{(*,q)}$ satisfies

$$\deg \alpha \le (d-1)D - d + 1.$$

Keep in mind that if $\alpha \in \mathscr{A}_{(p,q)}$ is a minimal bihomogeneous generator of $\mathscr{A}$, then $\alpha$ is also a minimal homogeneous generator of $\mathscr{A}_{(*,q)}$. Assertions (bi) and (bii) have been established. □

Two small facts were used in the proof of Theorem 6.1 to guarantee that the complexes $C^q_\bullet$ and $\mathscr{D}^q_\bullet(\varphi)$ satisfy the hypotheses of Corollary 5.4 and Theorem 5.6. We prove these facts now.

**Observation 6.2.** Let $R$ be a Cohen–Macaulay local ring and $I$ be an ideal of positive height. Assume that $\mu(I_\mathfrak{p}) \le \dim R_\mathfrak{p} + 1$ for each prime ideal $\mathfrak{p} \in V(I)$. If $\mathrm{Sym}(I) \cong P/J$ is any homogeneous presentation, where $P$ is a standard graded polynomial ring over $R$ in $n$ variables and $J$ is a homogeneous ideal of $P$, then $J$ has height $n-1$.

*Proof.* One has

$$\operatorname{ht} J = \dim P - \dim P/J = \dim R + n - \dim \mathrm{Sym}(I) = n-1,$$

where the first equality holds because $R$ is Cohen–Macaulay local and $J$ is homogeneous and the last equality follows from the dimension formula for symmetric algebras; see [Huneke and Rossi 1986, 2.6]. □

**Lemma 6.3.** *Let $R$ be a Cohen–Macaulay ring and $I$ be a perfect Gorenstein ideal of height three. Let $\varphi$ be an alternating presentation matrix for $I$, $\{\mathscr{D}^q_\bullet(\varphi)\}$ be the family of complexes in [Kustin and Ulrich 1992] which is associated to $\varphi$, and $s$ be an integer. If*

$$\mu(I_\mathfrak{p}) \le \dim R_\mathfrak{p} + 1 \quad \textit{for each prime ideal } \mathfrak{p} \in V(I) \textit{ with } \dim R_\mathfrak{p} \le s, \qquad (6.3.1)$$

*then the complex $(\mathscr{D}^q_\bullet(\varphi))_\mathfrak{p}$ is acyclic for every $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_\mathfrak{p} \le s$ and every nonnegative integer $q$.*

*Proof.* Let $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_\mathfrak{p} \le s$. Notice that $(\mathscr{D}^q_\bullet(\varphi))_\mathfrak{p} = \mathscr{D}^q_\bullet(\varphi_\mathfrak{p})$. This complex has length at most $n-1$, where $n$ is the size of the matrix $\varphi_\mathfrak{p}$ according to [Kustin and Ulrich 1992, 4.3.d]; see also (6.1.4). Moreover, by assumption (6.3.1), the matrix $\varphi_\mathfrak{p}$ satisfies condition $\mathrm{WMC}_1$, and therefore $\mathrm{WPC}_1$, see [Kustin and Ulrich 1992, 5.8, 5.9, 5.11.a]. Now [Kustin and Ulrich 1992, 6.2] shows that the complex $\mathscr{D}^q_\bullet(\varphi_\mathfrak{p})$ is acyclic. □

Theorem 6.1 is particularly interesting when $I$ is a linearly presented height three Gorenstein ideal. One quickly deduces that such ideals are of fiber type; see 3.4 for the definition. This is a new result and a main ingredient in the proof of Theorem 9.1.

**Corollary 6.4.** *Adopt the setting of Theorem 6.1.(b) with $D = 1$. Then the ideal $I$ is of fiber type. Furthermore, $\mathscr{A}$ is annihilated by $\mathfrak{m}$ when $d$ is odd and by $\mathfrak{m}^{(n-d+1)/2}$ when $d$ is even.*

*Proof.* Theorem 6.1.(bii) shows that the ideal $\mathscr{A}$ of $\mathrm{Sym}(I(\delta))$ is generated by $\mathscr{A}_{(0,\star)}$, which means that $I$ is of fiber type; see 3.4. The assertion about which power of $\mathfrak{m}$ annihilates $\mathscr{A}$ is explicitly stated in Theorem 6.1.(bi). □

## 7. The defining ideal of Rees rings up to radical

In this section we mainly work under the assumption that the ideal $I$ of Data 3.1 is of linear type on the punctured spectrum of $R$ and that a sufficiently high symmetric power $\mathrm{Sym}_t(I(\delta))$ has an approximate free resolution that is linear for the first $d$ steps. With these hypotheses alone we prove, somewhat surprisingly, that the defining ideal $\mathscr{J}$ of the Rees ring has the expected form up to radical; see 3.6 for the definition of expected form. We also describe the defining ideal $I(X)$ of the variety $X$ up to radical and we deduce that $I_d(B')$ has maximal possible height, where $B'$ is the Jacobian dual of the linear part of a minimal homogeneous presentation matrix of $I$; see 3.6 for the definition of Jacobian dual. This is done in Theorem 7.2. The computation of the height of $I_d(B')$ is relevant for the proof of Theorem 9.1 because it ensures that the complexes constructed in [Kustin et al. 2017b] are free resolutions of $I_d(B)$ and of $I_d(B)+C(\varphi)$, where $B=B'$ is the full Jacobian dual in this case and $C(\varphi)$ is the content ideal of Definition 8.1. A step towards the proof of Theorem 7.2 is Lemma 7.1, where we show that, up to radical, $\mathscr{A}$ is the socle of the symmetric algebra as an $R$-module and $I$ is of fiber type; see 3.4 for the definition of fiber type. We deduce this statement from the other parts of the same lemma, which say that even locally on the punctured spectrum of the polynomial ring $S$, the ideal $\mathscr{A}$ belongs to the socle and $I$ is of fiber type. These facts are proved using the methods of [Kustin et al. 2015], most notably Corollary 5.5 of the present paper.

The assumption that $I$ is of linear type on the punctured spectrum of $R$ is rather natural; see 3.7. It is satisfied for instance if $I$ is perfect of height two or Gorenstein of height three and $I$ satisfies $G_d$. If such an ideal is, in addition, linearly presented, then the hypothesis about an approximate resolution $C_\bullet$ of $\mathrm{Sym}_t(I(\delta))$ holds as well; see Remark 7.4.

The index $t$ of the symmetric power $\mathrm{Sym}_t(I(\delta))$ in Lemma 7.1 and Theorem 7.2 can be any integer greater than or equal to the relation type of $I$. The *relation type* of an ideal $I$ as in Data 3.1 is the largest integer $q$ such that $\mathscr{A}_{(*,q)}$ contains a minimal bihomogeneous generator of $\mathscr{A}$. We recall the bihomogeneous ideal $\mathscr{A}=\mathscr{J}/\mathscr{L}$ of $\mathrm{Sym}(I(\delta))$ as well as the bihomogeneous ideals $\mathscr{L}$ and $\mathscr{J}$ of $S$ that define the symmetric algebra and the Rees ring, respectively; see 3.2 and 3.4.

**Lemma 7.1.** *Adopt Data 3.1. Suppose further that $I$ is of linear type on the punctured spectrum of $R$ and that for some index $t$ greater than or equal to the relation type of $I$ there exists a homogeneous complex*

$$C_\bullet : \cdots \to C_1 \to C_0 \to 0$$

*of finitely generated graded free $R$-modules such that*

(1) $\mathrm{H}_0(C_\bullet) = \mathrm{Sym}_t(I(\delta))$,

(2) *the subcomplex* $C_d \to \cdots \to C_0 \to 0$ *of* $C_\bullet$ *is linear (that is* $C_j = R(-j)^{\beta_j}$ *for* $0 \le j \le d$*), and*

(3) $\dim \mathrm{H}_j(C_\bullet) \le j$ *for* $1 \le j \le d-1$.

*Let* $\mathfrak{M}$ *be the ideal* $(x_1, \dots, x_d, T_1, \dots, T_n)$ *of* $S$. *Then the following statements hold:*

(a) *On* $\mathrm{Spec}(S) \setminus \{\mathfrak{M}\}$, *the ideal* $\mathscr{A}$ *of* $\mathrm{Sym}(I(\delta))$ *is annihilated by* $\mathfrak{m}$; *that is,*

$$(\mathfrak{m}\mathscr{A})_{\mathfrak{P}} = 0 \quad \textit{for every } \mathfrak{P} \in \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}.$$

(b) *On* $\mathrm{Spec}(S) \setminus \{\mathfrak{M}\}$, $I$ *is of fiber type; that is,*

$$\mathscr{A}_{\mathfrak{P}} = I(X) \cdot \mathrm{Sym}(I(\delta))_{\mathfrak{P}} \quad \textit{for every } \mathfrak{P} \in \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}.$$

(c) *The ideal* $\mathscr{J}$ *of* $S$ *satisfies*

$$\mathscr{J} = \sqrt{\mathscr{L} :_S \mathfrak{m}} = \sqrt{(\mathscr{L}, I(X))}.$$

*Proof.* Recall that $\mathscr{A} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta)))$ since $I$ is of linear type on the punctured spectrum of $R$; see (3.7.1). In particular,

$$\mathscr{A}_{(\star,t)} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}_t(I(\delta))).$$

Apply Corollary 5.5 to the $R$-module $\mathrm{Sym}_t(I(\delta))$ to conclude that

$$\mathscr{A}_{(\star,t)} = \mathscr{A}_{(0,t)}.$$

Therefore

$$\mathscr{A}_{(\star,\ge t)} = \mathscr{A}_{(0,\ge t)}$$

because, by the definition of relation type, $\mathscr{A}_{(\star,t)}$ generates $\mathscr{A}_{(\star,\ge t)}$ as a module over $\mathrm{Sym}(I(\delta))$, hence over $\mathrm{Sym}(I(\delta))_{(0,\star)}$. It follows that $(T_1, \dots, T_n)^{\mathrm{t}}\mathscr{A} \subset \mathscr{A}_{(0,\ge t)}$, which gives

$$(T_1, \dots, T_n)^{\mathrm{t}}\mathscr{A}_{(>0,\star)} = 0.$$

Since, moreover, $\mathscr{A}_{(>0,\star)}$ vanishes locally on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$, we deduce that

$$\mathscr{A}_{(>0,\star)} = 0 \quad \text{locally on } \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}.$$

This completes the proof of (a) because $\mathfrak{m}\mathscr{A} \subset \mathscr{A}_{(>0,\star)}$, and of (b) because $\mathscr{A} = I(X) + \mathscr{A}_{(>0,\star)}$. As for (c), we recall the obvious inclusion $\mathscr{L} :_S \mathfrak{m} \subset \mathscr{J}$. Part (a) shows that if $\mathfrak{P} \in \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}$ contains $\mathscr{L} :_S \mathfrak{m}$ then $\mathfrak{P}$ contains $\mathscr{J}$. Since $\mathfrak{M}$ contains $\mathscr{J}$ anyway, we deduce that $\mathscr{J} \subset \sqrt{\mathscr{L} :_S \mathfrak{m}}$. The same argument, with part (a) replaced by (b), shows that $\mathscr{J} = \sqrt{(\mathscr{L}, I(X))}$. ☐

**Theorem 7.2.** *Retain all of the notation and hypotheses of Lemma 7.1. Let $\varphi$ be a minimal homogeneous presentation matrix of $g_1, \dots, g_n$, $\varphi'$ be the submatrix of $\varphi$ that consists of the linear columns of $\varphi$, $B$ be a homogeneous Jacobian dual of $\varphi$, and $B'$ be the Jacobian dual of $\varphi'$ in the sense of 3.6. Then the following statements hold*:

(a) *The ideal $I(X)$ of $T$ which defines the variety $X$ satisfies*

$$I(X) = \sqrt{I_d(B')}.$$

(b) $\operatorname{ht}(I_d(B')) = n - \dim A$.

(c) *The ideal $\mathcal{J}$ of $S$ which defines the Rees ring $\mathcal{R}(I)$ satisfies*

$$\mathcal{J} = \sqrt{(\mathcal{L}, I_d(B'))} = \sqrt{(\mathcal{L}, I_d(B))}.$$

*Proof.* Part (b) is an immediate consequence of (a). We prove items (a) and (c) simultaneously. We first notice that $B'$ is a submatrix of $B$, the ideal $I_d(B)$ of $S$ is bihomogeneous, the two ideals $(\mathfrak{m}, I_d(B))$ and $(\mathfrak{m}, I_d(B'))$ of $S$ are equal, and the two ideals $I_d(B) \cap T$ and $I_d(B')$ of $T$ are equal.

From Lemma 7.1.(c) we know that

$$\mathcal{J} = \sqrt{\mathcal{L} :_S \mathfrak{m}}.$$

The ideal $\mathcal{L} :_S \mathfrak{m}$ is the annihilator of the $S$-module $M = \mathfrak{m}S/\mathcal{L}$. The ideal $\mathfrak{m}S$ is generated by the entries of $\underline{x}$ and the ideal $\mathcal{L}$ is generated by the entries of $\underline{x} \cdot B$. It follows that $M$ is presented by $[\Pi \mid B]$, where $\Pi$ is a presentation matrix of $\underline{x}$ with entries in $\mathfrak{m}$. Therefore

$$\mathcal{L} :_S \mathfrak{m} = \operatorname{ann}_S M \subset \sqrt{\operatorname{Fitt}_0(M)} \subset \sqrt{(\mathfrak{m}, I_d(B))}.$$

It follows that

$$\mathcal{J} \subset \sqrt{(\mathfrak{m}, I_d(B))} = \sqrt{(\mathfrak{m}, I_d(B'))}.$$

Intersecting with $T$ we obtain

$$I(X) = \mathcal{J} \cap T \subset \sqrt{(\mathfrak{m}, I_d(B'))} \cap T = \sqrt{(\mathfrak{m}, I_d(B')) \cap T} = \sqrt{I_d(B')},$$

where the last two radicals are taken in the ring $T$. This proves part (a).

From Lemma 7.1.(c) we also know that

$$\mathcal{J} = \sqrt{(\mathcal{L}, I(X))},$$

and then by part (a)

$$\mathcal{J} = \sqrt{(\mathcal{L}, I_d(B'))}.$$

Finally, recall the inclusions $I_d(B') \subset I_d(B) \subset \mathcal{J}$, and the proof of part (c) is complete. $\square$

**Remark 7.3.** Retain the notation and the hypotheses of Theorem 7.2, and recall the definition of analytic spread from 3.3. If $d<n$ then $\ell(I)<n$; and if $\ell(I)<n$ then $\varphi'$ has at least $n-1$ columns.

*Proof.* Write $s$ for the number of columns of $\varphi'$, which is also the number of columns of $B'$. One has

$$n-d\le n-\ell(I)=n-\dim A=\operatorname{ht} I_d(B')\le \max\{s-d+1,0\},$$

where the first inequality holds by (3.3.2), the first equality follows from 3.3, the second equality is Theorem 7.2.(b), and the last inequality holds by the Eagon–Northcott bound on the height of determinantal ideals. These inequalities show that if $0<n-d$ then $0<n-\ell(I)$ and if $0<n-\ell(I)$ then $n-d\le s-d+1$. □

**Remark 7.4.** Adopt Data 3.1. Further assume that $I$ is perfect of height two or Gorenstein of height three, $I$ is linearly presented, and $I$ satisfies $G_d$. Then the hypotheses of Lemma 7.1 and Theorem 7.2 are satisfied.

*Proof.* We know from (3.7.1) and (3.7.3) that the ideal $I$ is of linear type on the punctured spectrum.

We now establish the existence of a complex $C_\bullet$. In the first case, when $I$ is perfect of height two, we can take $C_\bullet$ to be the complex $C_\bullet^t=\mathbb{K}_{\bullet(\star,t)}$ defined in the proof of Theorem 6.1.(a). This is a linear complex of finitely generated graded free $R$-modules which is acyclic on the punctured spectrum and has $\operatorname{Sym}_t(I(\delta))$ as zeroth homology. These facts are shown in the proof of Theorem 6.1.(a), most notably in (6.1.2).

In the second case, when $I$ is Gorenstein of height three, we take $C_\bullet$ to be the complex $\mathfrak{D}_\bullet^t(\varphi)$ used in the proof of Theorem 6.1.(b), with the additional restriction that $d\le t$. This is a complex of finitely generated graded free $R$-modules which has $\operatorname{Sym}_t(I(\delta))$ as zeroth homology. Moreover, this complex is acyclic on the punctured spectrum, see Lemma 6.3, and it is linear for the first $d$ steps, see (6.1.4). □

We end this section by recording a fact contained in the proof of Theorem 6.1.(b), that may well be of independent interest. Recall the functions $a_i$ and reg from 2.2 and 2.4.

**Proposition 7.5.** *Adopt Data 3.1. Further assume that $I$ is a linearly presented Gorenstein ideal of height three. Let $q$ be a nonnegative integer. If $\mu(I_\mathfrak{p})\le\dim R_\mathfrak{p}+1$ for each prime ideal $\mathfrak{p}\in V(I)$ with $\dim R_\mathfrak{p}\le d-2$, then*

$$a_i(\operatorname{Sym}_q(I))\le\begin{cases} q\delta-i & \text{if } q \text{ is even or } q\ne d-i-1,\\ q\delta+\frac{1}{2}(n-q)-1-i & \text{if } q \text{ is odd and } q=d-i-1.\end{cases}$$

*In particular,*

$$\begin{aligned} &\operatorname{reg}\operatorname{Sym}_q(I)=q\delta && \text{if } q \text{ is even or } d\le q,\\ &\operatorname{reg}\operatorname{Sym}_q(I)\le q\delta+\tfrac{1}{2}(n-q)-1 && \text{if } q \text{ is odd and } q\le d-1.\end{aligned}$$

*Proof.* Apply Proposition 5.2 with $M = \operatorname{Sym}_q(I(\delta))$ and $C_\bullet$ the complex $\mathfrak{D}^q_\bullet(\varphi)$ of (6.1.4). Lemma 6.3 guarantees that $\dim \mathrm{H}_j(\mathfrak{D}^q_\bullet(\varphi)) \le 1$ for every $j$ with $1 \le j$. Thus Proposition 5.2.(a) (see also Remark 5.3) gives that $\mathrm{H}^i_{\mathfrak{m}}(\operatorname{Sym}_q(I(\delta)))$ is a graded subquotient of $\mathrm{H}^d_{\mathfrak{m}}(\mathfrak{D}^q_{d-i})$ for $0 \le i \le d$. From (6.1.4) we see that either $\mathfrak{D}^q_{d-i} = 0$ or else

$$\mathfrak{D}^q_{d-i} \cong \begin{cases} R(-d+i)^{\beta^q_{d-i}} & \text{if } q \text{ is even or } q \ne d-i-1, \\ R\big(-d+i+1-\frac{1}{2}(n-q)\big) & \text{if } q \text{ is odd and } q = d-i-1. \end{cases}$$

Therefore the top degree of $\mathrm{H}^d_{\mathfrak{m}}(\mathfrak{D}^q_{d-i})$ is at most

$$\begin{aligned} d-i-d &= -i && \text{if } q \text{ is even or } q \ne d-i-1, \\ d-i-1+\tfrac{1}{2}(n-q)-d &= -i-1+\tfrac{1}{2}(n-q) && \text{if } q \text{ is odd and } q = d-i-1. \end{aligned}$$

Notice the second case cannot occur if $q$ is even or $d \le q$.

On the other hand, the fact that the $R$-module $\operatorname{Sym}_q(I(\delta))$ is generated in degree 0 implies that the regularity of $\operatorname{Sym}_q(I(\delta))$ is nonnegative. Now the assertions follow. $\square$

## 8. The candidate ideal $C(\varphi)$

In the present section we exhibit the candidate for the defining ideal of the variety $X$. Most of this material is taken from [Kustin et al. 2017b].

**Definition 8.1.** Let $k$ be a field and $n$ and $d$ be positive integers. For each $n \times n$ alternating matrix $\varphi$ with linear entries from the polynomial ring $R = k[x_1, \dots, x_d]$, define an ideal $C(\varphi)$ in the polynomial ring $T = k[T_1, \dots, T_n]$ as follows: Let $B$ be the $d \times n$ matrix with linear entries from $T$ such that the matrix equation

$$[T_1, \dots, T_n] \cdot \varphi = [x_1, \dots, x_d] \cdot B \tag{8.1.1}$$

holds. Consider the $(n+d) \times (n+d)$ alternating matrix

$$\mathfrak{B} = \begin{bmatrix} \varphi & -B^{\mathrm{t}} \\ B & 0 \end{bmatrix},$$

with entries in the polynomial ring $S = k[x_1, \dots, x_d, T_1, \dots, T_n]$. Let $F_i$ be $(-1)^{n+d-i}$ times the Pfaffian of $\mathfrak{B}$ with row and column $i$ removed. View each $F_i$ as a polynomial in $T[x_1, \dots, x_d]$ and let

$$C(\varphi) = c_T(F_{n+d})$$

be the content ideal of $F_{n+d}$ in $T$.

A description of $C(\varphi)$ which is almost coordinate-free is given in [Kustin et al. 2017b]. The description depends on the choice of a direct sum decomposition of the degree one component of the ring $T$ into $V_1 \oplus V_2$ where $V_1$ has dimension one.

(The almost coordinate-free description of $C(\varphi)$ depends on the choice of direct sum decomposition; however every decomposition gives rise to the same ideal $C(\varphi)$; see also Proposition 8.4.(b) below).

For the convenience of the reader, we provide direct proofs of the more immediate properties of the ideal $C(\varphi)$; most notably we describe a generating set of this ideal in Proposition 8.5. The more difficult results about depth, unmixedness, Hilbert series, and resolutions, however, are merely restated from [Kustin et al. 2017b]; see Theorem 8.7.

**Lemma 8.2.** *Retain the setting of Definition 8.1. There exists a polynomial $h$ in $S$ so that the equality of row vectors*

$$[F_1, \dots, F_{n+d}] = h \cdot [T_1, \dots, T_n, -x_1, \dots, -x_d]$$

*obtains.*

*Proof.* We may assume that the row vector $\underline{F} = [F_1, \dots, F_{n+d}]$ is not zero. Then $\mathfrak{B}$ has rank at least $n+d-1$. The alternating property of $\varphi$ and equality (8.1.1) imply

$$0 = \underline{T} \cdot \varphi \cdot \underline{T}^{\mathrm{t}} = \underline{x} \cdot B \cdot \underline{T}^{\mathrm{t}}.$$

Since the matrix $B$ has only entries in the polynomial ring $T$, we conclude that

$$B \cdot \underline{T}^{\mathrm{t}} = 0 \quad \text{and} \quad \underline{T} \cdot B^{\mathrm{t}} = 0.$$

Hence

$$[T_1, \dots, T_n, -x_1, \dots, -x_d] \cdot \mathfrak{B} = 0.$$

Since

$$[F_1, \dots, F_{n+d}] \cdot \mathfrak{B} = 0$$

and $\mathfrak{B}$ has almost maximal rank, there exists an element $h$ in the quotient field of $S$ such that

$$[F_1, \dots, F_{n+d}] = h \cdot [T_1, \dots, T_n, -x_1, \dots, -x_d].$$

The element $h$ is necessarily in $S$ because the ideal $(T_1, \dots, T_n, x_1, \dots, x_d)$ of $S$ has grade at least two. $\square$

Lemma 8.2 shows that either $h$ and the submaximal Pfaffians of $\mathfrak{B}$ all vanish, or else these elements are all nonzero. In the latter case, $n+d$ has to be odd. Moreover, regarding $S$ as standard bigraded with $\deg x_j = (1, 0)$ and $\deg T_i = (0, 1)$, we see that $\mathfrak{B}$ is the matrix of a bihomogeneous linear map. Thus the Pfaffians of this matrix are bihomogeneous, hence $h$ is bihomogeneous, and computing bidegrees one sees that

$$\deg h = \left(\frac{n-d-1}{2}, d-1\right).$$

In particular, if $h \neq 0$ then $d < n$. The next remark is now immediate.

**Remark 8.3.** Retain the setting of Definition 8.1.

(a) The ideal $C(\varphi)$ of $T$ is generated by homogeneous forms of degree $d-1$.

(b) If $n \leq d$ or $n+d$ is even, then $C(\varphi) = 0$.

Proposition 8.4 shows that the ideal $C(\varphi)$ can be defined using any submaximal Pfaffian of $\mathfrak{B}$; it also relates $C(\varphi)$ to the socle modulo the ideal $I_d(B)$ of $T$.

**Proposition 8.4.** *Retain the setting of Definition 8.1.*

(a) $c_T(F_i) = T_i \cdot c_T(F_{n+j}) = T_i \cdot C(\varphi)$ *for every* $1 \leq i \leq n$ *and* $1 \leq j \leq d$.

(b) $c_T(F_{n+j}) = C(\varphi)$ *for every* $1 \leq j \leq d$.

(c) $C(\varphi) \subset I_d(B) :_T (T_1, \ldots, T_n)$; *in other words, the image of* $C(\varphi)$ *is contained in the socle of the standard graded k-algebra* $T/I_d(B)$.

*Proof.* Lemma 8.2 shows that

$$\begin{aligned} c_T(F_i) &= c_T(h \cdot T_i) = T_i \cdot c_T(h) \quad && \text{for } 1 \leq i \leq n, \\ c_T(F_{n+j}) &= c_T(h \cdot x_j) = c_T(h) && \text{for } 1 \leq j \leq d. \end{aligned}$$

This proves (a). Part (b) is an immediate consequence of (a).

Finally, the definition of $F_i$ shows that in the range $1 \leq i \leq n$,

$$F_i \in I_d(B) \cdot S, \quad \text{and therefore} \quad c_T(F_i) \subset I_d(B).$$

Now apply (a) to deduce (c). □

In light of Remark 8.3.(b) we now assume that $d < n$ and $n+d$ is odd. Fix an integer $i$ with $1 \leq i \leq n$. Let $J = \{j_1, \ldots, j_d\}$ be a subset of $\{1, \ldots, n\} \setminus \{i\}$ with $j_1 < \cdots < j_s < i < j_{s+1} < \cdots < j_d$. Write $|J| = j_1 + \cdots + j_d - d + s$. Let $\mathrm{Pf}_J(\varphi_i)$ be the Pfaffian of the matrix obtained from $\varphi$ by deleting rows and columns $i, j_1, \ldots, j_d$ and denote by $\Delta_J(B)$ the determinant of the $d \times d$ matrix consisting of columns $j_1, \ldots, j_d$ of $B$.

Let $k[y_1, \ldots, y_d]$ be another polynomial ring in $d$ variables that acts on the polynomial ring $T[x_1, \ldots, x_d]$ by contraction and let $\circ$ denote the contraction operation.

For $M$ a monomial in $k[y_1, \ldots, y_d]$ of degree $\frac{1}{2}(n-d-1)$ we define

$$f_M = \frac{1}{T_i} \sum_J (-1)^{|J|} (M \circ \mathrm{Pf}_J(\varphi_i)) \cdot \Delta_J(B), \tag{8.4.1}$$

where $J$ ranges over all subsets of $\{1, \ldots, n\} \setminus \{i\}$ of cardinality $d$. Notice that, since $\mathrm{Pf}_J(\varphi_i) \in k[x_1, \ldots, x_d]$ is a homogeneous polynomial of degree $\frac{1}{2}(n-d-1)$, the element $M \circ \mathrm{Pf}_J(\varphi_i) \in k$ is simply the coefficient in this polynomial of the monomial in $k[x_1, \dots, x_d]$ corresponding to $M$.

Similarly, fix an integer $j$ with $1 \le j \le d$. Let $J = \{j_1, \dots, j_{d-1}\}$ be a subset of $\{1, \dots, n\}$ and set $\|J\| = j_1 + \cdots + j_{d-1}$. Write $\mathrm{Pf}_J(\varphi)$ for the Pfaffian of the matrix obtained from $\varphi$ by deleting rows and columns $j_1, \dots, j_{d-1}$ and denote by $\Delta_J(B_j)$ the determinant of the matrix consisting of columns $j_1, \dots, j_{d-1}$ of $B$ with row $j$ removed. Let $M \in k[y_1, \dots, y_d]$ be a monomial of degree $\frac{1}{2}(n-d+1)$ that is divisible by $y_j$. We define

$$h_M = \sum_J (-1)^{\|J\|}(M \circ \mathrm{Pf}_J(\varphi)) \cdot \Delta_J(B_j), \tag{8.4.2}$$

where $J$ ranges over all subsets of $\{1, \dots, n\}$ of cardinality $d-1$. Notice that $M \circ \mathrm{Pf}_J(\varphi) \in k$ is the coefficient in $\mathrm{Pf}_J(\varphi)$ of the monomial corresponding to $M$.

**Proposition 8.5.** *Adopt the setting of Definition 8.1 with $d < n$ and $n+d$ odd. The ideal $C(\varphi)$ of $T$ is generated by the elements $f_M$ of (8.4.1) for a fixed $i$, where $M$ ranges over all monomials in $k[y_1, \dots, y_d]$ of degree $\frac{1}{2}(n-d-1)$; it is also generated by the elements $h_M$ of (8.4.2) for a fixed $j$, where $M$ ranges over all monomials of degree $\frac{1}{2}(n-d+1)$ that are divisible by $y_j$.*

*Proof.* We prove the first claim. From Proposition 8.4.(a) we know that $C(\varphi) = 1/T_i \cdot c_T(F_i)$, for $1 \le i \le n$. Expanding the Pfaffian $F_i$ by maximal minors of $B$, one obtains

$$F_i = \pm \sum_J (-1)^{|J|} \mathrm{Pf}_J(\varphi_i) \cdot \Delta_J(B),$$

where, again, $J$ ranges over all subsets of $\{1, \dots, n\} \setminus \{i\}$ of cardinality $d$; see for instance [Iarrobino and Kanev 1999, Lemma B.1]. Regarded as polynomials in $T[x_1, \dots, x_d]$, the elements $\Delta_J(B)$ are constants and $F_i$ is homogeneous of degree $\frac{1}{2}(n-d-1)$. Therefore, the content ideal $c_T(F_i)$ is generated by the elements

$$M \circ F_i = \pm \sum_J (-1)^{|J|}(M \circ \mathrm{Pf}_J(\varphi_i)) \cdot \Delta_J(B),$$

where $M$ ranges over all monomials in $k[y_1, \dots, y_d]$ of degree $\frac{1}{2}(n-d-1)$.

To prove the second claim we recall that $C(\varphi) = c_T(F_{n+j})$, for $1 \le j \le d$; see Proposition 8.4.(b). Now expand the Pfaffian $F_{n+j}$ along the last $d-1$ rows and use the fact that the monomials in the support of $F_{n+j} \in T[x_1, \dots, x_d]$ have degree $\frac{1}{2}(n-d+1)$ and are divisible by $x_j$; see Lemma 8.2 and the discussion following it. $\square$

We illustrate the propositions above with an application to the ideal $I_d(B)+C(\varphi)$ in the case $n = d+1$; see [Johnson 1997, 2.10] for a similar result.

**Example 8.6.** Adopt the setting of Definition 8.1 with $n=d+1$. Write $\Delta_i$ for the maximal minor of $B$ obtained by deleting column $i$. The first part of Proposition 8.5 (or Proposition 8.4.(a) and the fact that $F_i=\pm\Delta_i$) show that $C(\varphi)$ is the principal ideal generated by $\Delta_i/T_i$, for any $i$ with $1\le i\le n$. Therefore $I_d(B)+C(\varphi)$ is the principal ideal generated by $\Delta_i/T_i$.

The next theorem is [Kustin et al. 2017b, 8.3].

**Theorem 8.7.** *Adopt the setting of Definition 8.1 with* $3\le d<n$ *and* $n$ *odd. If* $n-d\le \operatorname{ht} I_d(B)$, *then the following statements hold*:

(a) *The ring* $T/(I_d(B)+C(\varphi))$ *is Cohen–Macaulay on the punctured spectrum and*

$$\operatorname{depth}(T/(I_d(B)+C(\varphi)))=\begin{cases}1 & \text{if } d \text{ is odd},\\ 2 & \text{if } d \text{ is even and } d+3\le n,\\ n-1 & \text{if } d+1=n.\end{cases}$$

(b) *The unmixed part of* $I_d(B)$ *is equal to* $I_d(B)^{\mathrm{unm}}=I_d(B)+C(\varphi)$.

(c) *If* $d$ *is odd, then the ideal* $I_d(B)$ *is unmixed*; *furthermore*, $I_d(B)$ *has a linear free resolution and* $\operatorname{reg}(T/I_d(B))=d-1$.

(d) *The Hilbert series of* $T/(I_d(B)+C(\varphi))$ *is*

$$\frac{1}{(1-z)^d}\Bigg(\sum_{\ell=0}^{d-2}\binom{\ell+n-d-1}{n-d-1}z^{\ell}+\sum_{\ell=0}^{n-d-2}(-1)^{\ell+d+1}\binom{\ell+d-1}{d-1}z^{\ell+2d-n}\Bigg)$$
$$+(-1)^d\sum_{j\le\lceil\frac{n-d-3}{2}\rceil}\binom{j+d-1}{d-1}z^{2j+2d-n}.$$

(e) *The multiplicity of* $T/(I_d(B)+C(\varphi))$ *is*

$$\sum_{i=0}^{\lfloor\frac{n-d}{2}\rfloor}\binom{n-2-2i}{d-2}.$$

**Remark.** The conclusions of assertion (c) do not hold when $d$ is even: the ideal $I_d(B)$ can be mixed and the minimal homogeneous resolution of $I_d(B)^{\mathrm{unm}}$ may not be linear; see, for example, [Kustin et al. 2017b, (4.4.5)].

**Corollary 8.8.** *Adopt Data 3.1. Further assume that* $I$ *is a linearly presented Gorenstein ideal of height three. Let* $\varphi$ *be a minimal homogeneous alternating presentation matrix for* $g_1,\dots,g_n$, $B$ *be the Jacobian dual of* $\varphi$ *in the sense of 3.6, and* $C(\varphi)$ *be the content ideal of Definition 8.1. Then*

$$I_d(B)+C(\varphi)\subset I_d(B):_T(T_1,\dots,T_n)\subset I(X).$$

*In particular*, $I_d(B)+C(\varphi)$ *is contained in the defining ideal* $\mathcal{J}$ *of the Rees ring of* $I$.

*Proof.* The first containment follows from Proposition 8.4.(c). Since $I$ is linearly presented, $I_d(B)$ is an ideal of $T$ and therefore $I_d(B) \subset I(X)$ according to (3.6.2) and (3.3.1). The second containment follows because $I(X)$ is a prime ideal containing $I_d(B)$ but not $(T_1, \dots, T_n)$. □

## 9. Defining equations of blowup algebras of linearly presented height three Gorenstein ideals

In this section we assemble the proof of the main result of the paper, Theorem 9.1. We also provide a more detailed version of the theorem than the one stated in the Introduction; in particular, we express the relevant defining ideals as socles and iterated socles.

**Theorem 9.1.** *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field $k$, $I$ be a linearly presented Gorenstein ideal of height three which is minimally generated by homogeneous forms $g_1, \dots, g_n$, $\varphi$ be a minimal homogeneous alternating presentation matrix for $g_1, \dots, g_n$, $B$ be the Jacobian dual of $\varphi$ in the sense of 3.6, $C(\varphi)$ be the ideal of $T = k[T_1, \dots, T_n]$ given in Definition 8.1, and $\mathcal{L}$ be the ideal of $S = R[T_1, \dots, T_n]$ which defines* Sym$(I)$ *as described in 3.6. If $I$ satisfies $G_d$, then the following statements hold*:

(a) *The ideal of $S$ defining the Rees ring of $I$ is*
$$\mathcal{J} = \mathcal{L} + I_d(B)S + C(\varphi)S.$$
*If $d$ is odd, then $C(\varphi)$ is zero and $\mathcal{J}$ has the expected form.*

(b) *The ideal of $T$ defining the variety $X$ parametrized by $g_1, \dots, g_n$ is*
$$I(X) = I_d(B) + C(\varphi).$$

(c) *The two ideals $\mathcal{J}$ and $\mathcal{L} :_S (x_1, \dots, x_d)(T_1, \dots, T_n)$ of $S$ are equal.*

(d) *The three ideals $I(X)$, $I_d(B) :_T (T_1, \dots, T_n)$, and $I_d(B)^{\mathrm{unm}}$ of $T$ are equal.*

*Proof.* From Corollary 8.8 we have the containment
$$I_d(B) + C(\varphi) \subset I(X). \tag{9.1.1}$$
If $n \le d$, then $\mathcal{J} = \mathcal{L}$ by (3.7.2) and hence $I(X) = 0$ by (3.3.1), which also gives $I_d(B) + C(\varphi) = 0$. Therefore we may assume that $d < n$.

Corollary 6.4 guarantees that the ideal $I$ is of fiber type, which means that $\mathcal{J} = \mathcal{L} + I(X)S$; see 3.4. Thus (a) follows from part (b) (and Remark 8.3.b).

According to Theorem 4.4 the homogeneous coordinate ring $A = T/I(X)$ of $X$ has dimension $d$. Now Remark 7.4 and Theorem 7.2.(a) show that
$$\mathrm{ht}\, I_d(B) = \mathrm{ht}\, I(X) = n - d. \tag{9.1.2}$$

The hypothesis of Theorem 8.7 is satisfied; thus assertions (b) and (e) of Theorem 8.7 yield

$$I_d(B)^{\mathrm{unm}} = I_d(B) + C(\varphi), \tag{9.1.3}$$

and the multiplicity of $T/(I_d(B)+C(\varphi))$ is

$$\sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2}.$$

By Theorem 4.4, the coordinate ring $A = T/I(X)$ has the same multiplicity, by (9.1.1) and (9.1.2), the rings $T/(I_d(B)+C(\varphi))$ and $T/I(X)$ have the same dimension, and by (9.1.3) both rings are unmixed. Hence the containment $I_d(B)+C(\varphi) \subset I(X)$ of (9.1.1) is an equality, and (b) is also established.

Part (d) follows from Corollary 8.8, part (b), and (9.1.3). For the proof of (c) notice that parts (a), (b), and (d) give

$$\mathcal{J} = \mathcal{L} + (I_d(B) :_T (T_1, \dots, T_n))S.$$

Therefore,

$$\mathcal{J} \subset (\mathcal{L} + I_d(B)S) :_S (T_1, \dots, T_n).$$

Since $\mathcal{L} + I_d(B)S \subset \mathcal{L} :_S (x_1, \dots, x_d)$ according to (3.6.2), we obtain

$$\mathcal{J} \subset (\mathcal{L} :_S (x_1, \dots, x_d)) :_S (T_1, \dots, T_n) = \mathcal{L} :_S (x_1, \dots, x_d)(T_1, \dots, T_n) \subset \mathcal{J}. \quad \square$$

**Remark 9.2.** Adopt Data 3.1. The assumption in Theorem 9.1 that $I$ is linearly presented can be weakened to the condition that the entries of a minimal homogeneous presentation matrix $\varphi$ of $I$ generate a complete intersection ideal.

Indeed, let $\varphi$ be a minimal homogeneous alternating presentation matrix of $g_1, \dots, g_n$. By the symmetry of the minimal homogeneous $R$-resolution of $R/I$ all entries of $\varphi$ have the same degree $D$. Let $y_1, \dots, y_s$ be a regular sequence of homogeneous forms of degree $D$ that generate $I_1(\varphi)$. These forms are algebraically independent over $k$, and $R = k[x_1, \dots, x_d]$ is flat over the polynomial ring $k[y_1, \dots, y_s]$. The entries of $\varphi$ are forms of degree $D$ and are $R$-linear combinations of $y_1, \dots, y_s$; hence these entries are linear forms in the ring $k[y_1, \dots, y_s]$. Since $g_1, \dots, g_n$ are signed submaximal Pfaffians of $\varphi$ according to [Buchsbaum and Eisenbud 1977], these elements also belong to the ring $k[y_1, \dots, y_s]$, and by flat descent they satisfy the assumptions of Theorem 9.1 as elements of this ring. Now apply Theorem 9.1. Flat base change then gives the statements about the Rees ring over $R$, with $y_1, \dots, y_s$ in place of $x_1, \dots, x_d$; the statements about the ideal $I(X)$ are independent of the ambient ring.

The paper [Kustin et al. 2017b] was written with the intention of understanding the ideals $I_d(B)^{\mathrm{unm}}$ in order to determine the equations defining the Rees algebra and the special fiber ring of height three Gorenstein ideals, the ultimate goal of the present paper. In particular, in [Kustin et al. 2017b] the ideals $I_d(B)^{\text{unm}}$ have been resolved. Now that we have proven that $I_d(B)^{\text{unm}}$ defines the special fiber ring, we are able to harvest much information from the results of [Kustin et al. 2017b].

**Corollary 9.3.** *Adopt the notation and hypotheses of Theorem 9.1. The following statements hold*:

(a) *The variety $X$ is Cohen–Macaulay.*

(b) *If $n \le d+1$, then the homogeneous coordinate ring $A$ of $X$ is Cohen–Macaulay. Otherwise*
$$\operatorname{depth} A = \begin{cases} 1 & \textit{if } d \textit{ is odd},\\ 2 & \textit{if } d \textit{ is even}.\end{cases}$$

(c) *If $d$ is odd and $d < n$, then $I(X)$ has a linear free resolution and* $\operatorname{reg}(A) = d-1$.

(d) *If $d < n$, then the Hilbert series of $A$ is*
$$\frac{1}{(1-z)^d}\left(\sum_{\ell=0}^{d-2}\binom{\ell+n-d-1}{n-d-1}z^\ell + \sum_{\ell=0}^{n-d-2}(-1)^{\ell+d+1}\binom{\ell+d-1}{d-1}z^{\ell+2d-n}\right) + (-1)^d \sum_{j\le\lceil\frac{n-d-3}{2}\rceil}\binom{j+d-1}{d-1}z^{2j+2d-n}.$$

*Proof.* We may assume that $d < n$ since otherwise $I(X) = 0$; see (3.7.2) and (3.3.1). From Theorem 9.1.(b) we know that $I(X) = I_d(B) + C(\varphi)$, where $C(\varphi) = 0$ if $d$ is odd. Thus the assertions are items (a), (c), and (d) of Theorem 8.7, which applies due to (9.1.2). □

**Remark.** The minimal homogeneous resolution of the ideal $I(X)$ may not be linear when $d$ is even; see [Kustin et al. 2017b, (4.4.5)].

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# On Iwasawa theory, zeta elements for $\mathbb{G}_m$, and the equivariant Tamagawa number conjecture

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We develop an explicit "higher-rank" Iwasawa theory for zeta elements associated to the multiplicative group over abelian extensions of number fields. We show this theory leads to a concrete new strategy for proving special cases of the equivariant Tamagawa number conjecture and, as a first application of this approach, we prove new cases of the conjecture over natural families of abelian CM-extensions of totally real fields for which the relevant $p$-adic $L$-functions possess trivial zeroes.

## 1. Introduction

The "Tamagawa number conjecture" of Bloch and Kato [1990] concerns the special values of motivic $L$-functions and has had a pivotal influence on the development of arithmetic geometry.

Nevertheless, in any situation in which a semisimple algebra acts on a motive it is natural to search for an "equivariant" refinement of this conjecture that takes account, in some way, of the additional symmetries that arise in such cases.

The first such refinement was formulated by Kato [1993a; 1993b] (in the setting of abelian extensions of number fields, and modulo certain delicate sign ambiguities) by using determinant functors, and a definitive statement of the "equivariant Tamagawa number conjecture" (or eTNC for short in the remainder of this introduction) was subsequently given in [Burns and Flach 2001] by using virtual objects and relative algebraic $K$-theory.

It has since been shown that the eTNC specializes to give refined versions of most, if not all, of the important conjectures related to special values of motivic $L$-values that are studied in the literature and it is by now widely accepted that it

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provides a "universal" approach to the formulation of the strongest possible versions of such conjectures.

In this direction, we used the framework of the eTNC in our earlier article [Burns et al. 2016a] — hereafter abbreviated [BKS] — to develop a very general approach to the theory of abelian Stark conjectures that was principally concerned with the properties of canonical "zeta elements" and "Selmer groups" that one can naturally associate to the multiplicative group $\mathbb{G}_m$ over finite abelian extensions of number fields.

In this way we derived, amongst other things, several new and concrete results on the relevant case of the eTNC, the formulation, and in some interesting cases proof of precise conjectural families of fine integral congruence relations between Rubin–Stark elements of different ranks and detailed information on the Galois module structures of both ideal class groups and Selmer groups.

The purpose of the current article is now to develop an explicit Iwasawa theory for the zeta elements introduced in [BKS], to use this theory to derive a new approach to proving special cases of the eTNC, and finally to demonstrate the usefulness of this approach by using it to prove the conjecture in important new cases.

In the next two subsections we discuss briefly the main results that we obtain.

**1A.** ***Iwasawa main conjectures for general number fields.*** The first key aspect of our approach is the formulation of an explicit main conjecture of Iwasawa theory for abelian extensions of *general* number fields (we refer to this conjecture as a "higher-rank main conjecture" since the rank of any associated Euler system would in most cases be greater than one).

To give a little more detail, we fix a finite abelian extension $K/k$ of general number fields and a $\mathbb{Z}_p$-extension $k_\infty$ of $k$ and set $K_\infty = Kk_\infty$. In this introduction, we suppose that $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, but this is only for simplicity.

Then our higher-rank main conjecture asserts the existence of an Iwasawa-theoretic zeta element that plays the role of $p$-adic $L$-functions for general number fields and has precisely prescribed interpolation properties in terms of the values at zero of the higher derivatives of abelian $L$-series. (For details, see Conjecture 3.1).

Modulo a natural hypothesis on $\mu$-invariants, this conjecture can be reformulated in a more classical style as an equality between the characteristic ideals of a canonical Selmer module and of the quotient of a natural Rubin lattice of unit groups modulo the subgroup generated by the Rubin–Stark elements (see Conjecture 3.14 and Proposition 3.15). In this way it becomes clear that the higher-rank main conjecture extends classical main conjectures.

**1B.** ***Rubin–Stark congruences and the eTNC.*** It is also clear that the higher-rank main conjecture does not itself imply the validity of the $p$-part of the eTNC (as stated in Conjecture 2.3 below) and is much weaker than the type of main conjecture formulated by Fukaya and Kato [2006]. For example, if any $p$-adic place of $k$ splits completely in $K$, then our conjectural Rubin–Stark element encodes no information at all concerning the $L$-values of characters of $\mathrm{Gal}(K/k)$.

To overcome this deficiency, we make a detailed Iwasawa-theoretic study of the fine congruence relations between Rubin–Stark elements of *differing ranks* that were independently formulated for finite abelian extensions in [Mazur and Rubin 2016] (where the congruences are referred to as a "refined class number formula for $\mathbb{G}_m$") and in [Sano 2014]. In this way we are led to conjecture a precise family of "Iwasawa-theoretic Rubin–Stark congruences" for $K_\infty/k$ which, roughly speaking, describes the link between the natural Rubin–Stark elements for $K_\infty/k$ and for $K/k$. (For full details see Conjectures 4.1 and 4.2).

To better understand the context of this conjectural family of congruences we prove in Theorem 4.9 that it constitutes a natural extension to general number fields of the "Gross–Stark conjecture" that was formulated in [Gross 1982] for CM extensions of totally real fields and has since been much studied in the literature.

We can now state one of the main results of the present article (for a more detailed statement see Theorem 5.2).

**Theorem 1.1.** *If each of the following conjectures is valid for $K_\infty/k$, then the $p$-component of the eTNC* (see Conjecture 2.3) *is valid for every finite subextension of $K_\infty/k$:*

- *The higher-rank Iwasawa main conjecture* (Conjecture 3.1).
- *The Iwawasa-theoretic Rubin–Stark congruences* (Conjecture 4.2).
- *Gross's finiteness conjecture* (see Remark 5.4).

An early indication of the usefulness of this result is that it quickly leads to a much simpler proof of the main results of [Burns and Greither 2003] and [Flach 2011], and also those of [Bley 2006], in which the eTNC is proved for abelian extensions over $\mathbb{Q}$ and certain abelian extensions over imaginary quadratic fields respectively (see Corollary 5.6 and Remark 5.10).

To describe an application giving new results we assume $k$ is totally real and $K$ is CM and consider the "minus component" $\mathrm{eTNC}(K/k)_p^-$ of the $p$-part of the eTNC for $K/k$ (as formulated explicitly in Remark 2.4).

We write $K^+$ for the maximal totally real subfield of $K$ and recall that if no $p$-adic place splits in $K/K^+$ and the Iwasawa-theoretic $\mu$-invariant of $K_\infty/K$ vanishes, then $\mathrm{eTNC}(K/k)_p^-$ is already known to be valid (as far as we are aware, such a result was first implicitly discussed in the survey article by Flach [2004]).

However, by combining Theorems 1.1 and 4.9 with results on the Gross–Stark conjecture by Darmon, Dasgupta and Pollack [Dasgupta et al. 2011] and by Ventullo [2015], we can now prove the following concrete result (for a precise statement of which see Corollary 5.8).

**Corollary 1.2.** *Let $K/k$ be a finite abelian extension of number fields such that $K$ is CM and $k$ is totally real. If $p$ is any odd prime for which the Iwasawa-theoretic $\mu$-invariant of $K_\infty/K$ vanishes and at most one $p$-adic place of $k$ splits in $K/K^+$, then* $\mathrm{eTNC}(K/k)_p^-$ (*see Remark 2.4*) *is* (*unconditionally*) *valid.*

This result gives *the first verifications* of $\mathrm{eTNC}(K/k)_p^-$ in any case for which both $k \neq \mathbb{Q}$ and *the relevant $p$-adic $L$-series possess trivial zeroes*. For example, all of the hypotheses of Corollary 1.2 are satisfied by the concrete families of extensions described in Examples 5.9.

By combining Corollary 1.2 with [BKS, Corollary 1.14] we can also immediately deduce the following result concerning a refined version of the classical Brumer–Stark Conjecture. In this result we write $S_{\mathrm{ram}}(K/k)$ for the set of places of $k$ that ramify in $K$ and for any finite set of nonarchimedean places $T$ of $k$ we write $\mathrm{Cl}^T(K)$ for the ray class group of the ring of integers of $K$ modulo the product of all places of $K$ above $T$. We also use the equivariant $L$-series $\theta_{K/k,S_{\mathrm{ram}}(K/k),T}(s)$ defined in equation (1) of Section 2B below, and write $x \mapsto x^\#$ for the $\mathbb{Z}_p$-linear involution on $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$ that inverts elements of $\mathrm{Gal}(K/k)$.

**Corollary 1.3.** *Let $K/k$ and $p$ be as in Corollary 1.2 and set $G := \mathrm{Gal}(K/k)$. Then for any finite nonempty set of places $T$ of $k$ that is disjoint from $S_{\mathrm{ram}}(K/k)$ one has*

$$\theta_{K/k,S_{\mathrm{ram}}(K/k),T}(0)^\# \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Fitt}_{\mathbb{Z}[G]}\big(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}^T(K), \mathbb{Q}/\mathbb{Z})\big),$$

*and hence also*

$$\theta_{K/k,S_{\mathrm{ram}}(K/k),T}(0) \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(K)).$$

We note that the final assertion of this result gives the first verifications of the Brumer–Stark conjecture in a case for which the base field is not $\mathbb{Q}$ and the relevant $p$-adic $L$-series possess trivial zeroes. Thus the conclusion of this corollary unconditionally holds for the extensions in Examples 5.9.

Our methods also prove a natural equivariant "main conjecture" (see Theorem 3.16 and Corollary 3.17) involving the Selmer modules for $\mathbb{G}_m$ introduced in [BKS] and give a more straightforward proof of one of the main results of [Greither and Popescu 2015] (see Section 3E, especially Corollaries 3.18 and 3.20).

**1C. *Further developments.*** The ideas presented in this article extend naturally in at least two different directions.

Firstly, one can formulate a natural generalization of the theory discussed here in the context of arbitrary Tate motives. In this setting our theory is related to natural generalizations of both the notion of Rubin–Stark element and of the Rubin–Stark conjecture for special values of $L$-functions at any integer points. We can also formulate precise conjectural congruences between Rubin–Stark elements of differing “weights”, and in this way obtain *p-adic families* of Rubin–Stark elements. For details see our recent paper [Burns et al. 2016b].

Secondly, using an approach developed in [Burns and Sano 2016], many of the constructions, conjectures and results discussed here extend naturally to the setting of noncommutative Iwasawa theory and can then be used to prove the same case of the eTNC that we consider here over natural families of nonabelian Galois extensions.

***Note.*** After this article was submitted for publication we learned of the preprint [Dasgupta et al. 2016] by Dasgupta, Kakde and Ventullo, which gives a full proof of the Gross–Stark conjecture (as stated in Conjecture 4.7 below). Taking their result into account, one can now remove the hypothesis of the validity of (the relevant cases of) Conjecture 4.7 from the statement of Corollary 5.7 and, via Theorem 4.9, one obtains further strong evidence in support of the Iwasawa-theoretic Rubin–Stark Congruences that are formulated in Conjecture 4.2. This does not yet, however, allow one to extend the results of either Corollary 1.2 or Corollary 1.3 since, aside from certain special classes of fields discussed in Remark 5.4, Gross’s finiteness conjecture is still (in the relevant cases) not known to be valid unless one assumes that all associated $p$-adic $L$-functions have at most one trivial zero.

**1D.** ***Notation.*** For the reader’s convenience we collect here some basic notation.

For any (profinite) group $G$ we write $\hat{G}$ for the group of homomorphisms $G \to \mathbb{C}^\times$ of finite order.

Let $k$ be a number field. For a place $v$ of $k$, the residue field of $v$ is denoted by $\kappa(v)$ and we set $\mathrm{N}v := \#\kappa(v)$. We denote the set of places of $k$ which lie above the infinite place $\infty$ of $\mathbb{Q}$ (resp. a prime number $p$) by $S_\infty(k)$ (resp. $S_p(k)$). For a Galois extension $L/k$, the set of places of $k$ that ramify in $L$ is denoted by $S_{\mathrm{ram}}(L/k)$. For any set $\Sigma$ of places of $k$, we denote by $\Sigma_L$ the set of places of $L$ which lie above places in $\Sigma$.

Let $L/k$ be an abelian extension with Galois group $G$. For a place $v$ of $k$, the decomposition group at $v$ in $G$ is denoted by $G_v$. If $v$ is unramified in $L$, the Frobenius automorphism at $v$ is denoted by $\mathrm{Fr}_v$.

Let $E$ be either a field of characteristic 0 or $\mathbb{Z}_p$. For an abelian group $A$, we denote $E \otimes_{\mathbb{Z}} A$ by $EA$ or $A_E$. For a $\mathbb{Z}_p$-module $A$ and an extension field $E$ of $\mathbb{Q}_p$, we also write $EA$ or $A_E$ for $E \otimes_{\mathbb{Z}_p} A$. (This abuse of notation would not make any confusion.) We use similar notation for complexes. For example, if $C$ is a complex of abelian groups, then we denote $E \otimes^{\mathbb{L}}_{\mathbb{Z}} C$ by $EC$ or $C_E$.

Let $R$ be a commutative ring and $M$ an $R$-module. The linear dual $\mathrm{Hom}_R(M, R)$ is denoted by $M^*$. If $r$ and $s$ are nonnegative integers with $r \le s$, then there is a canonical paring

$$\textstyle\bigwedge_R^s M \times \bigwedge_R^r \mathrm{Hom}_R(M, R) \to \bigwedge_R^{s-r} M$$

defined by

$$(a_1\wedge\cdots\wedge a_s,\varphi_1\wedge\cdots\wedge\varphi_r)\mapsto\sum_{\sigma\in\mathfrak{S}_{s,r}}\operatorname{sgn}(\sigma)\det(\varphi_i(a_{\sigma(j)}))_{1\le i,j\le r}a_{\sigma(r+1)}\wedge\cdots\wedge a_{\sigma(s)},$$

with $\mathfrak{S}_{s,r}:=\{\sigma\in\mathfrak{S}_s\mid\sigma(1)<\cdots<\sigma(r)\text{ and }\sigma(r+1)<\cdots<\sigma(s)\}$. (See [BKS, Proposition 4.1].) We denote the image of $(a,\Phi)$ under the above pairing by $\Phi(a)$.

The total quotient ring of $R$ is denoted by $Q(R)$.

## 2. Zeta elements for $\mathbb{G}_m$

In this section, we review the zeta elements for $\mathbb{G}_m$ that were introduced in [BKS].

**2A.** ***The Rubin–Stark conjecture.*** We review the formulation of the Rubin–Stark conjecture [Rubin 1996, Conjecture B′].

Let $L/k$ be a finite abelian extension of number fields with Galois group $G$. Let $S$ be a finite set of places of $k$ which contains $S_\infty(k)\cup S_{\mathrm{ram}}(L/k)$. We fix a labeling $S=\{v_0,\dots,v_n\}$. Take $r\in\mathbb{Z}$ so that $v_1,\dots,v_r$ split completely in $L$. We put $V:=\{v_1,\dots,v_r\}$. For each place $v$ of $k$, we fix a place $w$ of $L$ lying above $v$. In particular, for each $i$ with $0\le i\le n$, we fix a place $w_i$ of $L$ lying above $v_i$. Such conventions are frequently used in this paper.

For $\chi\in\hat{G}$, let $L_{k,S}(\chi,s)$ denote the usual $S$-truncated $L$-function for $\chi$. We put

$$r_{\chi,S}:=\operatorname{ord}_{s=0}L_{k,S}(\chi,s).$$

Let $\mathcal{O}_{L,S}$ be the ring of $S_L$ integers of $L$. For any set $\Sigma$ of places of $k$, put $Y_{L,\Sigma}:=\bigoplus_{w\in\Sigma_L}\mathbb{Z}w$, the free abelian group on $\Sigma_L$. We define

$$X_{L,\Sigma}:=\left\{\sum_{w\in\Sigma_L}a_ww\in Y_{L,\Sigma}\;\middle|\;\sum_{w\in\Sigma_L}a_w=0\right\}.$$

By Dirichlet's unit theorem, we know that the homomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{L,S}:\mathbb{R}\mathcal{O}_{L,S}^\times\xrightarrow{\sim}\mathbb{R}X_{L,S},\quad a\mapsto-\sum_{w\in S_L}\log|a|_ww,$$

is an isomorphism.

By [Tate 1984, Chapter I, Proposition 3.4] we know that

$$r_{\chi,S}=\dim_{\mathbb{C}}(e_\chi\mathbb{C}\mathcal{O}_{L,S}^\times)=\dim_{\mathbb{C}}(e_\chi\mathbb{C}X_{L,S})=\begin{cases}\#\{v\in S\mid\chi(G_v)=1\} & \text{if }\chi\ne1,\\ n\ (=\#S-1) & \text{if }\chi=1,\end{cases}$$

where $e_\chi:=1/\#G\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1}$. From this fact, we see that $r\le r_{\chi,S}$.

Let $T$ be a finite set of places of $k$ which is disjoint from $S$. The $S$-truncated $T$-modified $L$-function is defined by

$$L_{k,S,T}(\chi,s) := \left(\prod_{v\in T}(1-\chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s})\right)L_{k,S}(\chi,s).$$

The $(S,T)$-unit group of $L$ is defined to be the kernel of $\mathcal{O}_{L,S}^{\times} \to \bigoplus_{w\in T_L}\kappa(w)^{\times}$. Note that $\mathcal{O}_{L,S,T}^{\times}$ is a subgroup of $\mathcal{O}_{L,S}^{\times}$ of finite index. We have

$$r \le r_{\chi,S} = \mathrm{ord}_{s=0}\, L_{k,S,T}(\chi,s) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{L,S,T}^{\times}).$$

We put

$$L_{k,S,T}^{(r)}(\chi,0) := \lim_{s\to 0} s^{-r} L_{k,S,T}(\chi,s).$$

We define the $r$-th order Stickelberger element by

$$\theta_{L/k,S,T}^{(r)} := \sum_{\chi\in\hat{G}} L_{k,S,T}^{(r)}(\chi^{-1},0)e_\chi \in \mathbb{R}[G].$$

The ($r$-th order) Rubin–Stark element

$$\epsilon_{L/k,S,T}^{V} \in \mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^{\times}$$

is defined to be the element which corresponds to

$$\theta_{L/k,S,T}^{(r)}\cdot(w_1-w_0)\wedge\cdots\wedge(w_r-w_0) \in \mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}X_{L,S}$$

under the isomorphism

$$\mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^{\times} \xrightarrow{\sim} \mathbb{R}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}X_{L,S}$$

induced by $\lambda_{L,S}$. We note that $\epsilon_{L/k,S,T}^{V}$ is independent of the choice of $w_0$ and $v_0$ (see [Sano 2015, Proposition 3.3]).

Now assume that $\mathcal{O}_{L,S,T}^{\times}$ is $\mathbb{Z}$-free. Then, the Rubin–Stark conjecture (as formulated in [Rubin 1996, Conjecture B′]) predicts that the Rubin–Stark element $\epsilon_{L/k,S,T}^{V}$ lies in the $\mathbb{Z}[G]$-lattice obtained by setting

$$\bigcap_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^{\times} := \Big\{a\in\mathbb{Q}{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathcal{O}_{L,S,T}^{\times} \;\Big|\; \Phi(a)\in\mathbb{Z}[G] \text{ for all } \Phi\in{\textstyle\bigwedge}_{\mathbb{Z}[G]}^{r}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S,T}^{\times},\mathbb{Z}[G])\Big\}.$$

We stress, in particular, that in this context (and as used systematically in [BKS]) the notation $\bigcap_{\mathbb{Z}[G]}^{r}$ does not refer to an intersection.

In this paper, we consider the "$p$-part" of the Rubin–Stark conjecture for a fixed prime number $p$. We put

$$U_{L,S,T} := \mathbb{Z}_p\mathcal{O}_{L,S,T}^{\times}.$$

We also fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. From this, we regard

$$\epsilon^V_{L/k,S,T} \in \mathbb{C}_p {\bigwedge}^r_{\mathbb{Z}_p[G]} U_{L,S,T}.$$

We define

$$\bigcap_{\mathbb{Z}_p[G]}^{r} U_{L,S,T} := \big\{ a \in \mathbb{Q}_p {\bigwedge}^r_{\mathbb{Z}_p[G]} U_{L,S,T} \mid \\ \Phi(a) \in \mathbb{Z}_p[G] \text{ for all } \Phi \in {\bigwedge}^r_{\mathbb{Z}_p[G]} \operatorname{Hom}_{\mathbb{Z}_p[G]}(U_{L,S,T}, \mathbb{Z}_p[G]) \big\}.$$

We easily see that there is a natural isomorphism $\mathbb{Z}_p \bigcap^r_{\mathbb{Z}[G]} \mathcal{O}^\times_{L,S,T} \simeq \bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T}$. We often denote ${\bigwedge}^r_{\mathbb{Z}_p[G]}$ and $\bigcap^r_{\mathbb{Z}_p[G]}$ simply by ${\bigwedge}^r$ and $\bigcap^r$ respectively.

We propose the "$p$-component version" of the Rubin–Stark conjecture as follows.

**Conjecture 2.1** ($\mathrm{RS}(L/k, S, T, V)_p$). *One has* $\epsilon^V_{L/k,S,T} \in \bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T}$.

**Remark 2.2.** Concerning known results on the Rubin–Stark conjecture, see [BKS, Remark 5.3], for example. Note that the Rubin–Stark conjecture is a consequence of the eTNC; this was first proved in [Burns 2007, Corollary 4.1], and then, in a much simpler way, in [BKS, Theorem 5.14].

**2B.** ***The eTNC for the untwisted Tate motive.*** We now review the formulation of the eTNC for the untwisted Tate motive.

Let $L/k$, $G$, $S$, $T$ be as in the previous subsection. Fix a prime number $p$. We assume that $S_p(k) \subset S$. Consider the complex

$$C_{L,S} := R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p), \mathbb{Z}_p)[-2].$$

It is known that $C_{L,S}$ is a perfect complex of $\mathbb{Z}_p[G]$-modules, acyclic outside degrees zero and one. We have a canonical isomorphism

$$H^0(C_{L,S}) \simeq U_{L,S} \; (:= \mathbb{Z}_p \mathcal{O}^\times_{L,S}),$$

and a canonical exact sequence

$$0 \to A_S(L) \to H^1(C_{L,S}) \to \mathcal{X}_{L,S} \to 0,$$

where $A_S(L) := \mathbb{Z}_p \operatorname{Pic}(\mathcal{O}_{L,S})$ and $\mathcal{X}_{L,S} := \mathbb{Z}_p X_{L,S}$. The complex $C_{L,S}$ is identified with the $p$-completion of the complex obtained from the classical "Tate sequence" (if $S$ is large enough), and also identified with $\mathbb{Z}_p R\Gamma((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$, where $R\Gamma((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ is the "Weil-étale cohomology complex" constructed in [BKS, §2.2] (see [Burns and Flach 1998, Proposition 3.3; Burns 2008, Proposition 3.5(e)]).

By a similar construction to [BKS, Proposition 2.4], we construct a canonical complex $C_{L,S,T}$ which lies in the distinguished triangle

$$C_{L,S,T} \to C_{L,S} \to \bigoplus_{w \in T_L} \mathbb{Z}_p \kappa(w)^\times[0].$$

(We can simply define $C_{L,S,T}$ by $\mathbb{Z}_p R\Gamma_T((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ in the terminology of [BKS].) We have

$$H^0(C_{L,S,T}) = U_{L,S,T}$$

and the exact sequence

$$0 \to A_S^T(L) \to H^1(C_{L,S,T}) \to \mathcal{X}_{L,S} \to 0,$$

where $A_S^T(L)$ is the $p$-part of the ray class group of $\mathcal{O}_{L,S}$ with modulus $\prod_{w \in T_L} w$.

We define the leading term of $L_{k,S,T}(\chi, s)$ at $s = 0$ by

$$L_{k,S,T}^*(\chi, 0) := \lim_{s \to 0} s^{-r_{\chi,S}} L_{k,S,T}(\chi, s).$$

The leading term at $s = 0$ of the equivariant $L$-function

$$\theta_{L/k,S,T}(s) := \sum_{\chi \in \hat{G}} L_{k,S,T}(\chi^{-1}, s) e_\chi \tag{1}$$

is defined by

$$\theta_{L/k,S,T}^*(0) := \sum_{\chi \in \hat{G}} L_{k,S,T}^*(\chi^{-1}, 0) e_\chi \in \mathbb{R}[G]^\times.$$

As in the previous subsection, we fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. We regard $\theta_{L/k,S,T}^*(0) \in \mathbb{C}_p[G]^\times$. The zeta element for $\mathbb{G}_m$

$$z_{L/k,S,T} \in \mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T})$$

is defined to be the element which corresponds to $\theta_{L/k,S,T}^*(0)$ under the isomorphism

$$\begin{aligned}
\mathbb{C}_p \mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) &\simeq \mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \mathrm{det}_{\mathbb{C}_p[G]}^{-1}(\mathbb{C}_p \mathcal{X}_{L,S}) \\
&\xrightarrow{\sim} \mathrm{det}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S}) \otimes_{\mathbb{C}_p[G]} \mathrm{det}_{\mathbb{C}_p[G]}^{-1}(\mathbb{C}_p \mathcal{X}_{L,S}) \\
&\xrightarrow{\sim} \mathbb{C}_p[G],
\end{aligned}$$

where the second isomorphism is induced by $\lambda_{L,S}$, and the last isomorphism is the evaluation map. Note that determinant modules must be regarded as graded invertible modules, but we omit the grading of any graded invertible modules as in [BKS].

The eTNC for the pair $(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ is formulated as follows.

**Conjecture 2.3** (eTNC ($h^0(\operatorname{Spec} L), \mathbb{Z}_p[G]$)). *One has*

$$\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = \det_{\mathbb{Z}_p[G]}(C_{L,S,T}).$$

**Remark 2.4.** When $p$ is odd, $k$ is totally real, and $L$ is CM, we say that the minus part of the eTNC (which we denote by eTNC($h^0(\operatorname{Spec} L), \mathbb{Z}_p[G]^-$)) is valid if we have the equality

$$e^-\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = e^-\det_{\mathbb{Z}_p[G]}(C_{L,S,T}),$$

where $e^- := (1-c)/2$ and $c \in G$ is the complex conjugation.

**2C.** ***The eTNC and Rubin–Stark elements.*** In this subsection, we interpret the eTNC using Rubin–Stark elements. The result in this subsection will be used in Section 5.

We continue to use the notation in the previous subsection. Take $\chi \in \hat{G}$, and suppose that $r_{\chi,S} < \#S$. Put $L_\chi := L^{\ker \chi}$ and $G_\chi := \operatorname{Gal}(L_\chi/k)$. Take $V_{\chi,S} \subset S$ so that all $v \in V_{\chi,S}$ split completely in $L_\chi$ (i.e., $\chi(G_v) = 1$) and $\#V_{\chi,S} = r_{\chi,S}$. Note that if $\chi \neq 1$, we have

$$V_{\chi,S} = \{v \in S \mid \chi(G_v) = 1\}.$$

Consider the Rubin–Stark element

$$\epsilon^{V_{\chi,S}}_{L_\chi/k,S,T} \in \mathbb{C}_p \textstyle\bigwedge^{r_{\chi,S}} U_{L_\chi,S,T}.$$

Note that a Rubin–Stark element depends on a fixed labeling of $S$, so in this case a labeling of $S$ such that $S = \{v_0, \ldots, v_n\}$ and $V_{\chi,S} = \{v_1, \ldots, v_{r_{\chi,S}}\}$ is understood to be chosen.

For a set $\Sigma$ of places of $k$ and a finite extension $F/k$, put $\mathcal{Y}_{F,\Sigma} := \mathbb{Z}_p Y_{F,\Sigma} = \bigoplus_{w \in \Sigma_F} \mathbb{Z}_p w$ and $\mathcal{X}_{F,\Sigma} := \mathbb{Z}_p X_{F,\Sigma} = \ker(\mathcal{Y}_{F,\Sigma} \to \mathbb{Z}_p)$.

Then the natural surjection $\mathcal{X}_{L_\chi,S} \to \mathcal{Y}_{L_\chi,V_{\chi,S}}$ induces an injection

$$\mathcal{Y}^*_{L_\chi,V_{\chi,S}} \to \mathcal{X}^*_{L_\chi,S},$$

where $(\cdot)^* := \operatorname{Hom}_{\mathbb{Z}_p[G_\chi]}(\cdot, \mathbb{Z}_p[G_\chi])$. Since

$$\mathcal{Y}_{L_\chi,V_{\chi,S}} \simeq \mathbb{Z}_p[G_\chi]^{\oplus r_{\chi,S}}$$

and $\dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p \mathcal{X}_{L,S}) = r_{\chi,S}$, the above map induces an isomorphism

$$e_\chi \mathbb{C}_p \mathcal{Y}^*_{L_\chi,V_{\chi,S}} \xrightarrow{\sim} e_\chi \mathbb{C}_p \mathcal{X}^*_{L,S}.$$

From this, we have a canonical identification

$$\begin{aligned} &e_\chi \mathbb{C}_p \big(\textstyle\bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \otimes \textstyle\bigwedge^{r_{\chi,S}} \mathcal{Y}^*_{L_\chi,V_{\chi,S}}\big) \\ &\qquad\qquad = e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big). \end{aligned}$$

Since $\{w_1, \dots, w_{r_{\chi,S}}\}$ is a basis of $\mathcal{Y}_{L_\chi, V_{\chi,S}}$, we have the (noncanonical) isomorphism

$$\bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \xrightarrow{\sim} \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \otimes \bigwedge^{r_{\chi,S}} \mathcal{Y}^*_{L_\chi, V_{\chi,S}}, \quad a \mapsto a \otimes w_1^* \wedge \cdots \wedge w^*_{r_{\chi,S}},$$

where $w_i^*$ is the dual of $w_i$. Hence, we have the (noncanonical) isomorphism

$$e_\chi \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big).$$

**Proposition 2.5.** *Suppose $r_{\chi,S} < \#S$ for every $\chi \in \hat{G}$. A necessary and sufficient condition for* eTNC$(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ *to hold is the existence of a $\mathbb{Z}_p[G]$-basis $\mathcal{L}_{L/k,S,T}$ of* $\det_{\mathbb{Z}_p[G]}(C_{L,S,T})$ *such that for every $\chi \in \hat{G}$ the image of $e_\chi \mathcal{L}_{L/k,S,T}$ under the isomorphism*

$$\begin{aligned} &e_\chi \mathbb{C}_p \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) \\ &\qquad \simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big) \simeq e_\chi \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} \end{aligned}$$

*coincides with $e_\chi \epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$.*

*Proof.* By the definition of Rubin–Stark elements, the image of $e_\chi \epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$ under the isomorphism

$$\begin{aligned} e_\chi \mathbb{C}_p \bigwedge^{r_{\chi,S}} U_{L_\chi,S,T} &\simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big) \\ &\simeq e_\chi \big(\det_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S}) \otimes_{\mathbb{C}_p[G]} \det^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p \mathcal{X}_{L,S})\big) \\ &\simeq e_\chi \mathbb{C}_p[G] \end{aligned}$$

is equal to $e_\chi L^*_{k,S,T}(\chi^{-1}, 0)$. Necessity follows by putting $\mathcal{L}_{L/k,S,T} := z_{L/k,S,T}$. Sufficiency follows by noting that $\mathcal{L}_{L/k,S,T}$ must be equal to $z_{L/k,S,T}$. □

**2D.** ***The canonical projection maps.*** Let $L/k$, $G$, $S$, $T$, $V$, $r$ be as in Section 2A. We put

$$e_r := \sum_{\chi \in \hat{G},\, r_{\chi,S} = r} e_\chi \in \mathbb{Q}[G].$$

As in Proposition 2.5, we construct the (noncanonical) isomorphism

$$e_r \mathbb{C}_p \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) \simeq e_r \mathbb{C}_p \bigwedge^r U_{L,S,T}.$$

In this subsection, we give an explicit description of the map

$$\begin{aligned} \pi^V_{L/k,S,T} : \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) &\xrightarrow{e_r \mathbb{C}_p \otimes} e_r \mathbb{C}_p \det_{\mathbb{Z}_p[G]}(C_{L,S,T}) \\ &\qquad \simeq e_r \mathbb{C}_p \bigwedge^r U_{L,S,T} \subset \mathbb{C}_p \bigwedge^r U_{L,S,T}. \end{aligned}$$

This map is important since the image of the zeta element $z_{L/k,S,T}$ under this map is the Rubin–Stark element $\epsilon^V_{L/k,S,T}$.

Firstly, we choose a representative $\Pi \xrightarrow{\psi} \Pi$ of $C_{L,S,T}$, where the first term is placed in degree zero, such that $\Pi$ is a free $\mathbb{Z}_p[G]$-module with basis $\{b_1, \ldots, b_d\}$ ($d$ is sufficiently large), and that the natural surjection

$$\Pi \to H^1(C_{L,S,T}) \to \mathcal{X}_{L,S}$$

sends $b_i$ to $w_i - w_0$ for each $i$ with $1 \le i \le r$. For the details of this construction, see [BKS, §5.4]. Note that the representative of $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ chosen there is of the form

$$P \to F,$$

where $P$ is projective and $F$ is free. By Swan's theorem [Curtis and Reiner 1981, (32.1)], we have an isomorphism $\mathbb{Z}_p P \simeq \mathbb{Z}_p F$. This shows that we can take the representative of $C_{L,S,T}$ as above.

We define $\psi_i \in \mathrm{Hom}_{\mathbb{Z}_p[G]}(\Pi, \mathbb{Z}_p[G])$ by

$$\psi_i := b_i^* \circ \psi,$$

where $b_i^*$ is the dual of $b_i$. Note that $\bigwedge_{r<i\le d} \psi_i \in \bigwedge^{d-r} \mathrm{Hom}_{\mathbb{Z}_p[G]}(\Pi, \mathbb{Z}_p[G])$ defines the homomorphism

$$\bigwedge\nolimits_{r<i\le d} \psi_i : \bigwedge\nolimits^{d} \Pi \to \bigwedge\nolimits^{r} \Pi$$

given as follows (see Notation):

$$\Big(\bigwedge\nolimits_{r<i\le d} \psi_i\Big)(b_1 \wedge \cdots \wedge b_d) = \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d} b_{\sigma(1)} \wedge \cdots \wedge b_{\sigma(r)}$$

**Proposition 2.6.** (i) *We have*

$$\bigcap^{r} U_{L,S,T} = \Big(\mathbb{Q}_p \bigwedge\nolimits^{r} U_{L,S,T}\Big) \cap \bigwedge\nolimits^{r} \Pi,$$

*where we regard $U_{L,S,T} \subset \Pi$ via the natural inclusion*

$$U_{L,S,T} = H^0(C_{L,S,T}) = \ker \psi \hookrightarrow \Pi.$$

(ii) *If we regard $\bigcap^r U_{L,S,T}$ as a subset of $\bigwedge^r \Pi$ by (i), then*

$$\mathrm{im}\Big(\bigwedge\nolimits_{r<i\le d} \psi_i : \bigwedge\nolimits^{d} \Pi \to \bigwedge\nolimits^{r} \Pi\Big) \subset \bigcap^{r} U_{L,S,T}.$$

(iii) *The map*

$$\det\nolimits_{\mathbb{Z}_p[G]}(C_{L,S,T}) = \bigwedge\nolimits^{d} \Pi \otimes \bigwedge\nolimits^{d} \Pi^* \to \bigcap^{r} U_{L,S,T}$$

*given by*

$$b_1 \wedge \cdots \wedge b_d \otimes b_1^* \wedge \cdots \wedge b_d^* \mapsto \Big(\bigwedge\nolimits_{r<i\le d} \psi_i\Big)(b_1 \wedge \cdots \wedge b_d)$$

*coincides with* $(-1)^{r(d-r)}\pi^V_{L/k,S,T}$. *In particular, we have*

$$\pi^V_{L/k,S,T}(b_1\wedge\cdots\wedge b_d\otimes b_1^*\wedge\cdots\wedge b_d^*) = (-1)^{r(d-r)}\sum_{\sigma\in\mathfrak{S}_{d,r}}\operatorname{sgn}(\sigma)\det(\psi_i(b_{\sigma(j)}))_{r<i,j\le d}b_{\sigma(1)}\wedge\cdots\wedge b_{\sigma(r)}$$

*and*

$$\operatorname{im}\pi^V_{L/k,S,T}\subset\Big\{a\in\bigcap^r U_{L,S,T}\ \Big|\ e_r a=a\Big\}.$$

*Proof.* For (i), see [BKS, Lemma 4.7(ii)]. For (ii) and (iii), see [BKS, Lemma 4.3]. □

## 3. Higher rank Iwasawa theory

**3A.** ***Notation.*** We fix a prime number $p$. Let $k$ be a number field, and $K_\infty/k$ a Galois extension such that $\mathcal{G}:=\operatorname{Gal}(K_\infty/k)\simeq\Delta\times\Gamma$, where $\Delta$ is a finite abelian group and $\Gamma\simeq\mathbb{Z}_p$. Set $\Lambda:=\mathbb{Z}_p[[\mathcal{G}]]$. Fix an isomorphism $\mathbb{C}\simeq\mathbb{C}_p$, and identify $\hat{\Delta}$ with $\operatorname{Hom}_{\mathbb{Z}}(\Delta,\overline{\mathbb{Q}_p}^\times)$. For $\chi\in\hat{\Delta}$, put $\Lambda_\chi:=\mathbb{Z}_p[\operatorname{im}\chi][[\Gamma]]$. Note that the total quotient ring $Q(\Lambda)$ has the decomposition

$$Q(\Lambda)\simeq\bigoplus_{\chi\in\hat{\Delta}/\sim_{\mathbb{Q}_p}}Q(\Lambda_\chi),$$

where $\chi\sim_{\mathbb{Q}_p}\chi'$ if and only if there exists $\sigma\in G_{\mathbb{Q}_p}$ such that $\chi=\sigma\circ\chi'$.

We use the following notation:

- $K:=K_\infty^\Gamma$ (so $\operatorname{Gal}(K/k)=\Delta$);
- $k_\infty:=K_\infty^\Delta$ (so $k_\infty/k$ is a $\mathbb{Z}_p$-extension with Galois group $\Gamma$);
- $k_n$: the $n$-th layer of $k_\infty/k$;
- $K_n$: the $n$-th layer of $K_\infty/K$;
- $\mathcal{G}_n:=\operatorname{Gal}(K_n/k)$.

For each character $\chi\in\hat{\mathcal{G}}$ we also set

- $L_\chi:=K_\infty^{\ker\chi}$;
- $L_{\chi,\infty}:=L_\chi\cdot k_\infty$;
- $L_{\chi,n}$: the $n$-th layer of $L_{\chi,\infty}/L_\chi$;
- $\mathcal{G}_\chi:=\operatorname{Gal}(L_{\chi,\infty}/k)$;
- $\mathcal{G}_{\chi,n}:=\operatorname{Gal}(L_{\chi,n}/k)$;
- $G_\chi:=\operatorname{Gal}(L_\chi/k)$;
- $\Gamma_\chi:=\operatorname{Gal}(L_{\chi,\infty}/L_\chi)$;
- $\Gamma_{\chi,n}:=\operatorname{Gal}(L_{\chi,n}/L_\chi)$;

- $S$: a finite set of places of $k$ which contains $S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k) \cup S_p(k)$;
- $T$: a finite set of places of $k$ which is disjoint from $S$;
- $V_\chi := \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$ (this is a proper subset of $S$);
- $r_\chi := \# V_\chi$.

For any intermediate field $L$ of $K_\infty/k$, we denote $\varprojlim_F U_{F,S,T}$ by $U_{L,S,T}$, where $F$ runs over all intermediate fields of $L/k$ that are finite over $k$ and the inverse limit is taken with respect to the norm maps. Similarly, $C_{L,S,T}$ is the complex defined by the inverse limit of the complexes $C_{F,S,T}$ with respect to the natural transition maps, and $A_S^T(L)$ the inverse limit of the $p$-primary parts $A_S^T(F)$ of the $T$ ray class groups of $\mathcal{O}_{F,S}$ with respect to the norm maps. We denote $\varprojlim_F \mathcal{Y}_{F,S}$ by $\mathcal{Y}_{L,S}$, where the inverse limit is taken with respect to the maps

$$\mathcal{Y}_{F',S} \to \mathcal{Y}_{F,S}, \quad w_{F'} \mapsto w_F,$$

where $F \subset F'$, $w_{F'} \in S_{F'}$, and $w_F \in S_F$ is the place lying under $w_{F'}$. We use similar notation for $\mathcal{X}_{L,S}$ etc.

**3B. *Iwasawa main conjecture I.*** In this section we formulate the main conjecture of Iwasawa theory for general number fields, which is a key to our study.

**3B1.** For any character $\chi$ in $\hat{\mathcal{G}}$ there is a natural composite homomorphism

$$\begin{aligned}
\lambda_\chi : \det\nolimits_\Lambda(C_{K_\infty,S,T}) &\to \det\nolimits_{\mathbb{Z}_p[G_\chi]}(C_{L_\chi,S,T}) \\
&\hookrightarrow \det\nolimits_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p C_{L_\chi,S,T}) \\
&\xrightarrow{\sim} \det\nolimits_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p U_{L_\chi,S,T}) \otimes_{\mathbb{C}_p[G_\chi]} \det\nolimits^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \\
&\xrightarrow{\sim} \det\nolimits_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \otimes_{\mathbb{C}_p[G_\chi]} \det\nolimits^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p \mathcal{X}_{L_\chi,S}) \\
&\simeq \mathbb{C}_p[G_\chi] \\
&\xrightarrow{\chi} \mathbb{C}_p,
\end{aligned}$$

where the fourth map is induced by $\lambda_{L_\chi,S}$, the fifth map is the evaluation, and the last map is induced by $\chi$.

We can now state our higher-rank main conjecture of Iwasawa theory in its first form.

**Conjecture 3.1** (IMC($K_\infty/k$, $S$, $T$)). *There exists a $\Lambda$-basis $\mathcal{L}_{K_\infty/k,S,T}$ of the module* $\det_\Lambda(C_{K_\infty,S,T})$ *for which, at every $\chi \in \hat{\Delta}$ and every $\psi \in \hat{G}_\chi$ for which $r_{\psi,S} = r_\chi$ one has $\lambda_\psi(\mathcal{L}_{K_\infty/k,S,T}) = L_{k,S,T}^{(r_\chi)}(\psi^{-1}, 0)$.*

**Remark 3.2.** This conjecture is equivariant with respect to $\Delta$. But it is important to note that this conjecture is much weaker than the (relevant case of the) equivariant Tamagawa number conjecture. For example, if $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, then for any $\psi$ that is trivial on the decomposition group in $\mathcal{G}_\chi$ of any $p$-adic place of $k$ one has $r_{\psi,S} > r_\chi$ and so there is no interpolation condition at $\psi$ specified above. When $r_\chi = 0$, (the $\chi$-component of) the element $\mathcal{L}_{K_\infty/k,S,T}$ is the $p$-adic $L$-function, and in the general case $r_\chi > 0$, it plays a role of $p$-adic $L$-functions. We will see in Section 3B2 that the interpolation condition characterizes $\mathcal{L}_{K_\infty/k,S,T}$ uniquely.

**Remark 3.3.** The explicit definition of the elements $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$ implies directly that the assertion of Conjecture 3.1 is valid if and only if there is a $\Lambda$-basis $\mathcal{L}_{K_\infty/k,S,T}$ of $\det_\Lambda(C_{K_\infty,S,T})$ for which, for every character $\chi \in \hat{\Delta}$ and every positive integer $n$, the image of $\mathcal{L}_{K_\infty/k,S,T}$ under the map

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} e_{r_\chi}\mathbb{C}_p{\textstyle\bigwedge}^{r_\chi} U_{L_{\chi,n},S,T}$$

is equal to $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$.

It is not difficult to see that the validity of Conjecture 3.1 is independent of $T$. We assume in the sequel that $T$ contains two places of unequal residue characteristics and hence that each group $U_{L,S,T}$ is $\mathbb{Z}_p$-free.

**3B2.** For each character $\chi \in \hat{\Delta}$, there is a natural ring homomorphism

$$\mathbb{Z}_p[[\mathcal{G}_\chi]] = \mathbb{Z}_p[[G_\chi \times \Gamma]] \xrightarrow{\chi} \mathbb{Z}_p[\operatorname{im}\chi][[\Gamma]] = \Lambda_\chi \subset Q(\Lambda_\chi).$$

In the sequel we use this homomorphism to regard $Q(\Lambda_\chi)$ as a $\mathbb{Z}_p[[\mathcal{G}_\chi]]$-algebra.

In the next result we describe an important connection between the element $\mathcal{L}_{K_\infty/k,S,T}$ that is predicted to exist by Conjecture 3.1 and the inverse limit (over $n$) of the Rubin–Stark elements $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$. This result shows, in particular, that the element $\mathcal{L}_{K_\infty/k,S,T}$ in Conjecture 3.1 is unique (if it exists).

In the sequel we set

$$\bigcap^{r_\chi} U_{L_{\chi,\infty},S,T} := \varprojlim_n \bigcap^{r_\chi} U_{L_{\chi,n},S,T},$$

where the inverse limit is taken with respect to the map

$$\bigcap^{r_\chi} U_{L_{\chi,m},S,T} \to \bigcap^{r_\chi} U_{L_{\chi,n},S,T}$$

induced by the norm map $U_{L_{\chi,m},S,T} \to U_{L_{\chi,n},S,T}$, where $n \le m$. Note that Rubin–Stark elements are norm compatible (see [Rubin 1996, Proposition 6.1; Sano 2014, Proposition 3.5]), so if we know that Conjecture RS$(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all sufficiently large $n$, then we can define the element

$$\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T} := \varprojlim_n \epsilon^{V_\chi}_{L_{\chi,n}/k,S,T} \in \bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}.$$

**Theorem 3.4.** (i) *For each $\chi \in \hat{\Delta}$, the homomorphism*

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} {\bigcap}^{r_\chi} U_{L_{\chi,n},S,T}$$

(see Proposition 2.6(iii)) *induces an isomorphism of $Q(\Lambda_\chi)$-modules*

$$\pi^{V_\chi}_{L_{\chi,\infty}/k,S,T} : \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \simeq \Big({\bigcap}^{r_\chi} U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

(ii) *If Conjecture 3.1 is valid, then we have*

$$\pi^{V_\chi}_{L_{\chi,\infty}/k,S,T}(\mathcal{L}_{K_\infty/k,S,T}) = \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}.$$

(Note that in this case Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all $n$ by Remark 3.3 and Proposition 2.6(iii).)

*Proof.* Since the module $A^T_S(K_\infty) \otimes_\Lambda Q(\Lambda_\chi)$ vanishes, there are canonical isomorphisms

$$\begin{aligned}
&\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi)\\
&\quad \simeq \det_{Q(\Lambda_\chi)}(C_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\chi))\\
&\quad \simeq \det_{Q(\Lambda_\chi)}(U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\chi)) \otimes_{Q(\Lambda_\chi)} \det^{-1}_{Q(\Lambda_\chi)}(\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_\chi)). \qquad (2)
\end{aligned}$$

It is also easy to check that there are natural isomorphisms

$$U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\chi) \simeq U_{L_{\chi,\infty},S,T} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi)$$

and

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_\chi) \simeq \mathcal{X}_{L_{\chi,\infty},S} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi) \simeq \mathcal{Y}_{L_{\chi,\infty},V_\chi} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi),$$

and that these are $Q(\Lambda_\chi)$-vector spaces of dimension $r := r_\chi$ $(= \#V_\chi)$. The isomorphism (2) is therefore a canonical isomorphism of the form

$$\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \simeq \Big({\bigwedge}^r U_{L_{\chi,\infty},S,T} \otimes {\bigwedge}^r \mathcal{Y}^*_{L_{\chi,\infty},V_\chi}\Big) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

Composing this isomorphism with the map induced by the noncanonical isomorphism

$${\bigwedge}^r \mathcal{Y}^*_{L_{\chi,\infty},V_\chi} \xrightarrow{\sim} \mathbb{Z}_p[[\mathcal{G}_\chi]], \quad w_1^* \wedge \cdots \wedge w_r^* \mapsto 1,$$

we have

$$\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi) \simeq \Big({\bigwedge}^r U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi).$$

As in the proofs of Proposition 2.6(iii) and [BKS, Lemma 4.3], this isomorphism is induced by $\varprojlim_n \pi^{V_\chi}_{L_{\chi,n}/k,S,T}$. Now the isomorphism in claim (i) is thus obtained directly from Lemma 3.5 below.

Claim (ii) follows by noting that the image of $\mathcal{L}_{K_\infty/k,S,T}$ under the map

$$\det_\Lambda(C_{K_\infty,S,T}) \to \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n}/k,S,T}} {\bigcap}^{r_\chi} U_{L_{\chi,n},S,T}$$

is equal to $\epsilon^{V_\chi}_{L_{\chi,n}/k,S,T}$. □

**Lemma 3.5.** *With notation as above, there is a canonical identification*

$$\Big(\bigcap^r U_{L_{\chi,\infty},S,T}\Big)\otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} Q(\Lambda_\chi)=\Big(\bigwedge^r U_{L_{\chi,\infty},S,T}\Big)\otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} Q(\Lambda_\chi).$$

*Proof.* Take a representative $\Pi_\infty\to\Pi_\infty$ of $C_{L_{\chi,\infty},S,T}$ as in Section 2D. Put $\Pi_n:=\Pi_\infty\otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]}\mathbb{Z}_p[\mathcal{G}_{\chi,n}]$. We have (see Proposition 2.6(i))

$$\bigcap^r U_{L_{\chi,n},S,T}=\Big(\mathbb{Q}_p\bigwedge^r U_{L_{\chi,n},S,T}\Big)\cap\bigwedge^r\Pi_n,$$

and thus $\varprojlim_n\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}U_{L_{\chi,n},S,T}$ can be regarded as a submodule of the free $\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]$-module $\varprojlim_n\bigwedge^r\Pi_n=\bigwedge^r\Pi_\infty$. For simplicity, we set $G_n:=\mathcal{G}_{\chi,n}$, $G:=\mathcal{G}_\chi$, $U_n:=U_{L_{\chi,n},S,T}$, $U_\infty:=U_{L_{\chi,\infty},S,T}$, and $Q:=Q(\Lambda_\chi)$. We will show the equality

$$\Big(\big(\varprojlim_n\mathbb{Q}_p\bigwedge^r U_n\big)\cap\bigwedge^r\Pi_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q=\Big(\bigwedge^r U_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q$$

of the submodules of $(\bigwedge^r\Pi_\infty)\otimes_{\mathbb{Z}_p[\![G]\!]}Q$.

It is easy to see that

$$\Big(\bigwedge^r U_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q\subset\Big(\big(\varprojlim_n\mathbb{Q}_p\bigwedge^r U_n\big)\cap\bigwedge^r\Pi_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q.$$

Conversely, take $a\in(\varprojlim_n\mathbb{Q}_p\bigwedge^r U_n)\cap\bigwedge^r\Pi_\infty$ and set $M_n:=\operatorname{coker}(U_n\to\Pi_n)$. Then we have

$$\varprojlim_n M_n\simeq\operatorname{coker}(U_\infty\to\Pi_\infty)=:M_\infty.$$

Since $\Pi_\infty\otimes_{\mathbb{Z}_p[\![G]\!]}Q\simeq(U_\infty\otimes_{\mathbb{Z}_p[\![G]\!]}Q)\oplus(M_\infty\otimes_{\mathbb{Z}_p[\![G]\!]}Q)$, we have the decomposition

$$\Big(\bigwedge^r\Pi_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q\simeq\bigoplus_{i=0}^r\Big(\bigwedge^{r-i}U_\infty\otimes\bigwedge^i M_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q.$$

Write

$$a=(a_i)_i\in\bigoplus_{i=0}^r\Big(\bigwedge^{r-i}U_\infty\otimes\bigwedge^i M_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q.$$

It is sufficient to show that $a_i=0$ for all $i>0$. We may assume that

$$a_i\in\operatorname{im}\Big(\bigwedge^{r-i}U_\infty\otimes\bigwedge^i M_\infty\to\Big(\bigwedge^{r-i}U_\infty\otimes\bigwedge^i M_\infty\Big)\otimes_{\mathbb{Z}_p[\![G]\!]}Q\Big)$$

for every $i$. Since $a\in\bigwedge^r\Pi_\infty$, we can also write $a=(a_{(n)})_n\in\varprojlim_n\bigwedge^r\Pi_n$. For each $n$, we have a decomposition

$$\mathbb{Q}_p\bigwedge^r\Pi_n\simeq\bigoplus_{i=0}^r\Big(\mathbb{Q}_p\bigwedge^{r-i}U_n\otimes_{\mathbb{Q}_p[G_n]}\mathbb{Q}_p\bigwedge^i M_n\Big),$$

and we write

$$a_{(n)} = (a_{(n),i})_i \in \bigoplus_{i=0}^{r} \Big(\mathbb{Q}_p {\bigwedge}^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p {\bigwedge}^i M_n\Big).$$

Since $a \in \varprojlim_n \mathbb{Q}_p {\bigwedge}^r U_n$, we must have $a_{(n),i} = 0$ for all $i > 0$. To prove $a_i = 0$ for all $i > 0$, it is sufficient to show that the natural map

$$\begin{aligned} \operatorname{im}\Big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty \to \big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\Big) \\ \to \varprojlim_n \Big(\mathbb{Q}_p {\bigwedge}^{r-i} U_n \otimes_{\mathbb{Q}_p[G_n]} \mathbb{Q}_p {\bigwedge}^i M_n\Big) \end{aligned} \tag{3}$$

is injective. Note that $M_\infty$ is isomorphic to a submodule of $\Pi_\infty$, since $M_\infty \simeq \ker(\Pi_\infty \to H^1(C_{L_{\chi,\infty},S,T}))$. Hence both $U_\infty$ and $M_\infty$ are embedded in $\Pi_\infty$, and we have

$$\begin{aligned} &\ker\Big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty \to \big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\Big) \\ &\qquad = \ker\Big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty \xrightarrow{\alpha} \big({\bigwedge}^{r} (\Pi_\infty \oplus \Pi_\infty)\big) \otimes_{\mathbb{Z}_p[\![G]\!]} \Lambda_\chi\Big). \end{aligned}$$

Set $\Lambda_{\chi,n} := \mathbb{Z}_p[\operatorname{im}\chi][\Gamma_{\chi,n}]$. The commutative diagram

$$\begin{tikzcd} {\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty \arrow[r, "\alpha"] \arrow[d, "\beta"] & ({\bigwedge}^{r} (\Pi_\infty \oplus \Pi_\infty)) \otimes_{\mathbb{Z}_p[\![G]\!]} \Lambda_\chi \arrow[d, "f"] \\ \varprojlim_n \mathbb{Q}_p(({\bigwedge}^{r-i} U_n \otimes {\bigwedge}^i M_n) \otimes_{\mathbb{Z}_p[G_n]} \Lambda_{\chi,n}) \arrow[r, "g"'] & \varprojlim_n \mathbb{Q}_p\big(({\bigwedge}^{r} (\Pi_n \oplus \Pi_n)) \otimes_{\mathbb{Z}_p[G_n]} \Lambda_{\chi,n}\big) \end{tikzcd}$$

and the injectivity of $f$ and $g$ implies $\ker\alpha = \ker\beta$. Hence we have

$$\ker\Big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty \to \big({\bigwedge}^{r-i} U_\infty \otimes {\bigwedge}^i M_\infty\big) \otimes_{\mathbb{Z}_p[\![G]\!]} Q\Big) = \ker\alpha = \ker\beta.$$

This shows the injectivity of (3). ☐

**Remark 3.6.** Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all $\chi \in \hat{\Delta}$ and $n$. Using Theorem 3.4, we can define

$$\mathcal{L}_{K_\infty/k,S,T} \in \det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda) = \bigoplus_{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p}} (\det_\Lambda(C_{K_\infty,S,T}) \otimes_\Lambda Q(\Lambda_\chi))$$

by $\mathcal{L}_{K_\infty/k,S,T} := (\pi^{V_\chi,-1}_{L_{\chi,\infty}/k,S,T}(\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}))_\chi$. Then Conjecture 3.1 is equivalent to

$$\Lambda \cdot \mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T}).$$

**3C.** ***Iwasawa main conjecture II.*** In this subsection, we work under the following simplifying assumptions:

(∗) $p$ is odd, and $V_\chi$ contains no finite places for every $\chi \in \hat{\Delta}$.

We note that the second assumption here is satisfied whenever $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension.

**3C1.** We start by quickly reviewing some basic facts concerning the height-one prime ideals of $\Lambda$.

We say that a height-one prime ideal $\mathfrak{p}$ of $\Lambda$ is *regular* if $p \notin \mathfrak{p}$, and *singular* if $p \in \mathfrak{p}$.

If $\mathfrak{p}$ is regular, then $\Lambda_{\mathfrak{p}}$ is identified with the localization of $\Lambda[1/p]$ at $\mathfrak{p}\Lambda[1/p]$. Since we have the decomposition

$$\Lambda\left[\frac{1}{p}\right] = \bigoplus_{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p}} \Lambda_\chi\left[\frac{1}{p}\right],$$

we have $Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda_{\chi_{\mathfrak{p}}})$ for some $\chi_{\mathfrak{p}} \in \hat{\Delta}/\sim_{\mathbb{Q}_p}$. Since $\Lambda_{\chi_{\mathfrak{p}}}[1/p]$ is a regular local ring, $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring.

Next, suppose that $\mathfrak{p}$ is a singular prime. We have the decomposition

$$\Lambda = \bigoplus_{\chi \in \hat{\Delta}'/\sim_{\mathbb{Q}_p}} \mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][\![\Gamma]\!],$$

where $\Delta_p$ is the Sylow $p$-subgroup of $\Delta$, and $\Delta'$ is the unique subgroup of $\Delta$ which is isomorphic to $\Delta/\Delta_p$. From this, we see that $\Lambda_{\mathfrak{p}}$ is identified with the localization of some $\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][\![\Gamma]\!]$ at $\mathfrak{p}\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][\![\Gamma]\!]$. By [Burns and Greither 2003, Lemma 6.2(i)], we have

$$\mathfrak{p}\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p][\![\Gamma]\!] = (\sqrt{p\mathbb{Z}_p[\operatorname{im}\chi][\Delta_p]}),$$

where we denote the radical of an ideal $I$ by $\sqrt{I}$. This shows that there is a one-to-one correspondence between the set of all singular primes of $\Lambda$ and the set $\hat{\Delta}'/\sim_{\mathbb{Q}_p}$. We denote by $\chi_{\mathfrak{p}} \in \hat{\Delta}'/\sim_{\mathbb{Q}_p}$ the character corresponding to $\mathfrak{p}$. The next lemma shows that

$$Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\substack{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p} \\ \chi|_{\Delta'} = \chi_{\mathfrak{p}}}} Q(\Lambda_\chi).$$

**Lemma 3.7.** *Let $E/\mathbb{Q}_p$ be a finite unramified extension, and $\mathcal{O}$ its ring of integers. Let $P$ be a finite abelian group whose order is a power of $p$. Put $\Lambda := \mathcal{O}[P][\![\Gamma]\!]$ and $\mathfrak{p} := \sqrt{p\mathcal{O}[P]}\Lambda$ ($\mathfrak{p}$ is the unique singular prime of $\Lambda$). Then we have*

$$Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda) = \bigoplus_{\chi \in \hat{P}/\sim_E} Q(\mathcal{O}[\operatorname{im}\chi][\![\Gamma]\!]).$$

*Proof.* Since $Q(\Lambda_{\mathfrak{p}}) = Q(\Lambda_{\mathfrak{p}}[1/p])$ and $\Lambda_{\mathfrak{p}}[1/p] = \bigoplus_{\chi \in \hat{P}/\sim_E} e_\chi \Lambda_{\mathfrak{p}}[1/p]$, where $e_\chi := \sum_{\chi' \sim_E \chi} e_{\chi'}$, we have

$$Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\chi \in \hat{P}/\sim_E} Q\Big(e_\chi \Lambda_{\mathfrak{p}}\Big[\frac{1}{p}\Big]\Big).$$

For $\chi \in \hat{P}/\sim_E$, put $\mathfrak{q}_\chi := \ker(\Lambda \xrightarrow{\chi} \mathcal{O}[\operatorname{im}\chi][[\Gamma]])$. We can easily see that $\sqrt{p\mathcal{O}[P]} = (p, I_{\mathcal{O}}(P))$, where $I_{\mathcal{O}}(P)$ is the kernel of the augmentation map $\mathcal{O}[P] \to \mathcal{O}$. From this, we also see that

$$\sqrt{p\mathcal{O}[P]} = \ker(\mathcal{O}[P] \xrightarrow{\chi} \mathcal{O}[\operatorname{im}\chi] \to \mathcal{O}[\operatorname{im}\chi]/\pi_\chi\mathcal{O}[\operatorname{im}\chi] \simeq \mathcal{O}/p\mathcal{O})$$

holds for any $\chi \in \hat{P}/\sim_E$, where $\pi_\chi \in \mathcal{O}[\operatorname{im}\chi]$ is a uniformizer. This shows that $\mathfrak{q}_\chi \subset \mathfrak{p}$. Hence, we know that $\Lambda_{\mathfrak{q}_\chi}$ is the localization of $\Lambda_{\mathfrak{p}}[1/p]$ at $\mathfrak{q}_\chi\Lambda_{\mathfrak{p}}[1/p]$. One can check that $\Lambda_{\mathfrak{q}_\chi} = Q(e_\chi\Lambda_{\mathfrak{p}}[1/p])$. Since we have $\Lambda_{\mathfrak{q}_\chi} = Q(\mathcal{O}[\operatorname{im}\chi][[\Gamma]])$, the lemma follows. □

For a height-one prime ideal $\mathfrak{p}$ of $\Lambda$, define a subset $\Upsilon_{\mathfrak{p}} \subset \hat{\Delta}/\sim_{\mathbb{Q}_p}$ by

$$\Upsilon_{\mathfrak{p}} := \begin{cases} \{\chi_{\mathfrak{p}}\} & \text{if } \mathfrak{p} \text{ is regular,} \\ \{\chi \in \hat{\Delta}/\sim_{\mathbb{Q}_p} \mid \chi|_{\Delta'} = \chi_{\mathfrak{p}}\} & \text{if } \mathfrak{p} \text{ is singular.} \end{cases}$$

The above argument shows that $Q(\Lambda_{\mathfrak{p}}) = \bigoplus_{\chi\in\Upsilon_{\mathfrak{p}}} Q(\Lambda_\chi)$.

To end this subsection we recall a useful result concerning $\mu$-invariants, whose proof is in [Flach 2004, Lemma 5.6].

**Lemma 3.8.** *Let $M$ be a finitely generated torsion $\Lambda$-module. Let $\mathfrak{p}$ be a singular prime of $\Lambda$. Then the following are equivalent*:

(i) *The $\mu$-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_{\mathfrak{p}}}M$ vanishes.*

(ii) *For any $\chi \in \Upsilon_{\mathfrak{p}}$, the $\mu$-invariant of the $\mathbb{Z}_p[\operatorname{im}\chi][[\Gamma]]$-module $M \otimes_{\mathbb{Z}_p[\Delta']} \mathbb{Z}_p[\operatorname{im}\chi]$ vanishes.*

(iii) $M_{\mathfrak{p}} = 0$.

**3C2.** In the rest of this section we assume the condition $(*)$ from the beginning of Section 3C.

**Lemma 3.9.** *Let $\mathfrak{p}$ be a singular prime of $\Lambda$. Then $V_\chi$ is independent of $\chi \in \Upsilon_{\mathfrak{p}}$. In particular, for any $\chi \in \Upsilon_{\mathfrak{p}}$, the $Q(\Lambda_{\mathfrak{p}})$-module $U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$ is free of rank $r_\chi$.*

*Proof.* It is sufficient to show that $V_\chi = V_{\chi_{\mathfrak{p}}}$ for any $\chi \in \Upsilon_{\mathfrak{p}}$. Note that the extension degree $[L_{\chi,\infty} : L_{\chi_{\mathfrak{p}},\infty}] = [L_\chi : L_{\chi_{\mathfrak{p}}}]$ is a power of $p$. Since $p$ is odd by the assumption $(*)$, we see that an infinite place of $k$ which splits completely in $L_{\chi_{\mathfrak{p}},\infty}$ also splits completely in $L_{\chi,\infty}$. By the assumption $(*)$, we know every place in $V_{\chi_{\mathfrak{p}}}$ is infinite. Hence we have $V_\chi = V_{\chi_{\mathfrak{p}}}$. □

The above result motivates us, for any height-one prime ideal $\mathfrak{p}$ of $\Lambda$, to define $V_{\mathfrak{p}} := V_\chi$ and $r_{\mathfrak{p}} := r_\chi$ by choosing some $\chi \in \Upsilon_{\mathfrak{p}}$.

Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ holds for all $\chi \in \hat{\Delta}$ and $n$. We then define the "$\mathfrak{p}$-part" of the Rubin–Stark element

$$\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} \in \left(\bigwedge\nolimits^{r_{\mathfrak{p}}} U_{K_\infty,S,T}\right) \otimes_\Lambda Q(\Lambda_{\mathfrak{p}})$$

as the image of

$$(\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T})_{\chi\in\Upsilon_\mathfrak{p}} \in \bigoplus_{\chi\in\Upsilon_\mathfrak{p}} \bigcap^{r_\mathfrak{p}} U_{L_{\chi,\infty},S,T}$$

under the natural map

$$\bigoplus_{\chi\in\Upsilon_\mathfrak{p}} \bigcap^{r_\mathfrak{p}} U_{L_{\chi,\infty},S,T} \to \bigoplus_{\chi\in\Upsilon_\mathfrak{p}} \Big(\bigcap^{r_\mathfrak{p}} U_{L_{\chi,\infty},S,T}\Big) \otimes_{\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]} Q(\Lambda_\chi) = \big(\textstyle\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T}\big) \otimes_\Lambda Q(\Lambda_\mathfrak{p}).$$

(See Lemma 3.5.)

**Lemma 3.10.** *Let $\mathfrak{p}$ be a height-one prime ideal of $\Lambda$. When $\mathfrak{p}$ is singular, assume that the $\mu$-invariant of $e_{\chi_\mathfrak{p}} A^T_S(K_\infty)$ (as $\mathbb{Z}_p[\![\Gamma]\!]$-module) vanishes.*

(i) *The $\Lambda_\mathfrak{p}$-module $(U_{K_\infty,S,T})_\mathfrak{p}$ is free of rank $r_\mathfrak{p}$.*

(ii) *If Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for every $\chi$ in $\hat{\Delta}$ and every natural number $n$, then there is an inclusion*

$$\Lambda_\mathfrak{p} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} \subset \big(\textstyle\bigwedge^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\big)_\mathfrak{p}.$$

*Proof.* As in the proof of Lemma 3.5, we choose a representative $\psi_\infty : \Pi_\infty \to \Pi_\infty$ of $C_{K_\infty,S,T}$. We have the exact sequence

$$0 \to U_{K_\infty,S,T} \to \Pi_\infty \xrightarrow{\psi_\infty} \Pi_\infty \to H^1(C_{K_\infty,S,T}) \to 0. \tag{4}$$

If $\mathfrak{p}$ is regular, then $\Lambda_\mathfrak{p}$ is a discrete valuation ring and the exact sequence (4) implies that the $\Lambda_\mathfrak{p}$-modules $(U_{K_\infty,S,T})_\mathfrak{p}$ and $\mathrm{im}(\psi_\infty)_\mathfrak{p}$ are free. Since $U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\mathfrak{p})$ is isomorphic to $\mathcal{Y}_{K_\infty,V_\mathfrak{p}} \otimes_\Lambda Q(\Lambda_\mathfrak{p})$, we also know that the rank of $(U_{K_\infty,S,T})_\mathfrak{p}$ is $r_\mathfrak{p}$.

Suppose next that $\mathfrak{p}$ is singular. Since the $\mu$-invariant of $e_{\chi_\mathfrak{p}} \mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}}$ vanishes, we apply Lemma 3.8 to deduce that $(\mathcal{X}_{K_\infty,S})_\mathfrak{p} = (\mathcal{Y}_{K_\infty,V_\mathfrak{p}})_\mathfrak{p}$. In a similar way, the assumption that the $\mu$-invariant of $e_{\chi_\mathfrak{p}} A^T_S(K_\infty)$ vanishes implies that $A^T_S(K_\infty)_\mathfrak{p} = 0$. Hence we have $H^1(C_{K_\infty,S,T})_\mathfrak{p} = (\mathcal{Y}_{K_\infty,V_\mathfrak{p}})_\mathfrak{p}$. By assumption $(*)$, we know that $\mathcal{Y}_{K_\infty,V_\mathfrak{p}}$ is projective as a $\Lambda$-module. This implies that $H^1(C_{K_\infty,S,T})_\mathfrak{p} = (\mathcal{Y}_{K_\infty,V_\mathfrak{p}})_\mathfrak{p}$ is a free $\Lambda_\mathfrak{p}$-module of rank $r_\mathfrak{p}$. By choosing splittings of the sequence (4), we then easily deduce that the $\Lambda_\mathfrak{p}$-modules $(U_{K_\infty,S,T})_\mathfrak{p}$ and $\mathrm{im}(\psi_\infty)_\mathfrak{p}$ are free and that the rank of $(U_{K_\infty,S,T})_\mathfrak{p}$ is equal to $r_\mathfrak{p}$.

At this stage we have proved that, for any height-one prime ideal $\mathfrak{p}$ of $\Lambda$, the $\Lambda_\mathfrak{p}$-module $(U_{K_\infty,S,T})_\mathfrak{p}$ is both free of rank $r_\mathfrak{p}$ (as required to prove claim (i)) and also a direct summand of $(\Pi_\infty)_\mathfrak{p}$, and hence that

$$\big(\textstyle\bigwedge^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T}\big)_\mathfrak{p} = \big(\textstyle\bigwedge^{r_\mathfrak{p}}_\Lambda U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_\mathfrak{p})\big) \cap \big(\textstyle\bigwedge^{r_\mathfrak{p}}_\Lambda \Pi_\infty\big)_\mathfrak{p}. \tag{5}$$

Now we make the stated assumption concerning the validity of the $p$-part of the Rubin–Stark conjecture. This implies, by the proof of Theorem 3.4(i), that for each $\mathfrak{p}$ the element $\epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}$ lies in both $(\bigwedge_\Lambda^{r_\mathfrak{p}} \Pi_\infty)_\mathfrak{p}$ and

$$\bigoplus_{\chi\in\Upsilon_\mathfrak{p}} \left(\bigwedge\nolimits_\Lambda^{r_\chi} U_{K_\infty,S,T}\right) \otimes_\Lambda Q(\Lambda_\chi) = \left(\bigwedge\nolimits_\Lambda^{r_\mathfrak{p}} U_{K_\infty,S,T}\right) \otimes_\Lambda Q(\Lambda_\mathfrak{p}),$$

and hence, by (5) that it belongs to $(\bigwedge_\Lambda^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}$, as required to prove claim (ii). □

We can now decompose Conjecture 3.1 into the statements for $\mathfrak{p}$ components.

**Proposition 3.11.** *Assume that Conjecture* $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ *holds for all characters* $\chi$ *in* $\hat{\Delta}$ *and all sufficiently large* $n$ *and that for each character* $\chi$ *in* $\hat{\Delta}'/{\sim_{\mathbb{Q}_p}}$ *the* $\mu$*-invariant of the* $\mathbb{Z}_p[[\Gamma]]$*-module* $e_\chi A_S^T(K_\infty)$ *vanishes. Then Conjecture 3.1 holds if and only if*

$$\Lambda_\mathfrak{p} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}_\Lambda^{r_\mathfrak{p}}(H^1(C_{K_\infty,S,T}))_\mathfrak{p} \cdot \left(\bigwedge\nolimits_\Lambda^{r_\mathfrak{p}} U_{K_\infty,S,T}\right)_\mathfrak{p} \tag{6}$$

*for every height-one prime ideal* $\mathfrak{p}$ *of* $\Lambda$.

**Remark 3.12.** At every height-one prime ideal $\mathfrak{p}$ there is an equality

$$\mathrm{Fitt}_\Lambda^{r_\mathfrak{p}}(H^1(C_{K_\infty,S,T}))_\mathfrak{p} = \mathrm{Fitt}_\Lambda^0(A_S^T(K_\infty))_\mathfrak{p}\mathrm{Fitt}_\Lambda^0(\mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}})_\mathfrak{p}\ .$$

If $\mathfrak{p}$ is regular, then $\Lambda_\mathfrak{p}$ is a discrete valuation ring and this equality follows directly from the exact sequence

$$0 \to A_S^T(K_\infty) \to H^1(C_{K_\infty,S,T}) \to \mathcal{X}_{K_\infty,S} \to 0.$$

If $\mathfrak{p}$ is singular, then the equality is valid since the result of Lemma 3.8 implies $(\mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}})_\mathfrak{p}$ vanishes and so $H^1(C_{K_\infty,S,T})_\mathfrak{p}$ is isomorphic to the direct sum $A_S^T(K_\infty)_\mathfrak{p} \oplus (\mathcal{Y}_{K_\infty,V_\mathfrak{p}})_\mathfrak{p}$.

**Remark 3.13.** If the prime $\mathfrak{p}$ is singular, then $(\mathcal{X}_{K_\infty,S\setminus V_\mathfrak{p}})_\mathfrak{p}$ vanishes and

$$\mathrm{Fitt}_\Lambda^0(A_S^T(K_\infty))_\mathfrak{p} = \Lambda_\mathfrak{p}$$

if the $\mu$-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_\mathfrak{p}} A_S^T(K_\infty)$ vanishes (see Lemma 3.8). Thus, in this case, for any such $\mathfrak{p}$ the equality (6) is equivalent to

$$\Lambda_\mathfrak{p} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \left(\bigwedge\nolimits_\Lambda^{r_\mathfrak{p}} U_{K_\infty,S,T}\right)_\mathfrak{p}\ .$$

Thus, we know that by Lemma 3.10(ii) the validity of the $p$-part of the Rubin–Stark conjecture already gives strong evidence of the above equality.

*Proof.* Since $\det_\Lambda(C_{K_\infty,S,T})$ is an invertible $\Lambda$-module the equality $\Lambda \cdot \mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T})$ in Conjecture 3.1 is valid if and only if at every height-one prime ideal $\mathfrak{p}$ of $\Lambda$ one has

$$\Lambda_\mathfrak{p} \cdot \mathcal{L}_{K_\infty/k,S,T} = \det_\Lambda(C_{K_\infty,S,T})_\mathfrak{p} \tag{7}$$

(see [Burns and Greither 2003, Lemma 6.1]).

If $\mathfrak{p}$ is regular, then one easily sees that this equality is valid if and only if the equality

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \mathrm{Fitt}^{r_{\mathfrak{p}}}_{\Lambda}(H^1(C_{K_\infty,S,T})) \cdot \big(\textstyle\bigwedge^{r_{\mathfrak{p}}}_{\Lambda} U_{K_\infty,S,T}\big)_{\mathfrak{p}}$$

is valid, by using Theorem 3.4(ii).

If $\mathfrak{p}$ is singular, then the assumed vanishing of the $\mu$-invariants and the argument in the proof of Lemma 3.10(i) together show that the $\Lambda_{\mathfrak{p}}$-modules $(U_{K_\infty,S,T})_{\mathfrak{p}}$ and $H^1(C_{K_\infty,S,T})_{\mathfrak{p}}$ are both free of rank $r_{\mathfrak{p}}$. Noting this, we see that (7) holds if and only if

$$\Lambda_{\mathfrak{p}} \cdot \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T} = \big(\textstyle\bigwedge^{r_{\mathfrak{p}}}_{\Lambda} U_{K_\infty,S,T}\big)_{\mathfrak{p}},$$

and so in this case the claimed result follows from Remark 3.13. □

**3C3.** In [BKS] we defined canonical Selmer modules $\mathcal{S}_{S,T}(\mathbb{G}_{m/F})$ and $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/F})$ for $\mathbb{G}_m$ over number fields $F$ that are of finite degree over $\mathbb{Q}$. For any intermediate field $L$ of $K_\infty/k$, we now set

$$\mathcal{S}_{p,S,T}(\mathbb{G}_{m/L}) := \varprojlim_{F} \mathcal{S}_{S,T}(\mathbb{G}_{m/F}) \otimes \mathbb{Z}_p, \quad \mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L}) := \varprojlim_{F} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/F}) \otimes \mathbb{Z}_p,$$

where in both limits $F$ runs over all finite extensions of $k$ in $L$ and the transition morphisms are the natural corestriction maps.

We note in particular that, by its very definition, $\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})$ coincides with $H^1(C_{L,S,T})$. In addition, this definition implies that for any subset $V$ of $S$ comprising places that split completely in $L$ the kernel of the natural (composite) projection map

$$\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})_V := \ker(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L}) \to \mathcal{X}_{L,S} \to \mathcal{Y}_{L,V})$$

lies in a canonical exact sequence of the form

$$0 \to A^T_S(L) \to \mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})_V \to \mathcal{X}_{L,S\setminus V} \to 0. \tag{8}$$

We now interpret our Iwasawa main conjecture in terms of classical characteristic ideals.

**Conjecture 3.14** (IMC($K_\infty/k$, $S$, $T$) II). *Assume Conjecture* RS($L_{\chi,n}/k$, $S$, $T$, $V_\chi$)$_p$ *holds for all $\chi \in \hat{\Delta}$ and all nonnegative integers $n$ where $L_{\chi,n}$, $\Delta$, etc. are defined in Section 3. Then for any $\chi \in \hat{\Delta}$ there are equalities*

$$\begin{aligned}\mathrm{char}_{\Lambda_\chi}\Big(\Big(\bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}/\langle \epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}\rangle\Big)^{\chi}\Big) \\ &= \mathrm{char}_{\Lambda_\chi}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L_{\chi,\infty}})^{\chi}_{V_\chi}) \\ &= \mathrm{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^{\chi})\mathrm{char}_{\Lambda_\chi}((\mathcal{X}_{L_{\chi,\infty},S\setminus V_\chi})^{\chi}).\end{aligned} \tag{9}$$

*Here, for any $\mathbb{Z}_p[\![\mathcal{G}_\chi]\!]$-module $M$ we write $M^\chi$ for the $\Lambda_\chi$-module $M \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Z}_p[\operatorname{im}\chi]$ and $\operatorname{char}_{\Lambda_\chi}(M^\chi)$ for its characteristic ideal in $\Lambda_\chi$. In addition, the second displayed equality is a direct consequence of the appropriate case of the exact sequence (8).*

**Proposition 3.15.** *Assume that Conjecture $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ is valid for all characters $\chi$ in $\hat{\Delta}$ and all $n$ and that for each character $\chi \in \hat{\Delta}'/\sim_{\mathbb{Q}_p}$ the $\mu$-invariant of the $\mathbb{Z}_p[\![\Gamma]\!]$-module $e_\chi A_S^T(K_\infty)$ vanishes. Then Conjecture 3.1 is equivalent to Conjecture 3.14.*

*Proof.* Note that by our assumption $\mu = 0$ we have $(\bigcap^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p} = (\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}$ for any height-one prime $\mathfrak{p}$, using (5). Thus, the equality (6) implies the equality (9) for any $\chi$.

On the other hand, for a height-one regular prime $\mathfrak{p}$, we can regard $\mathfrak{p}$ to be a prime of $\Lambda_\chi$ for some $\chi$, so the equality (9) implies the equality (6). For a singular prime $\mathfrak{p}$, by Lemma 3.8, (9) for any $\chi$ implies $(\bigwedge^{r_\mathfrak{p}} U_{K_\infty,S,T})_\mathfrak{p}/\langle \epsilon^{\mathfrak{p}}_{K_\infty/k,S,T}\rangle = 0$, thus the equality (6) by Remark 3.13.

The proposition therefore follows from Proposition 3.11. □

**3D. *The case of CM-fields.*** Concerning the minus components for CM-extensions, we can prove our equivariant main conjecture using the usual main conjecture proved by Wiles.

**Theorem 3.16.** *Suppose that $p$ is odd, $k$ is totally real, $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, and $K$ is CM. If the $\mu$-invariant of the cyclotomic $\mathbb{Z}_p$-extension $K_\infty/K$ vanishes, then the minus part of Conjecture 3.1 is valid for $(K_\infty/k, S, T)$.*

*Proof.* In fact, for an odd character $\chi$, one has $r_\chi = 0$ and the Rubin–Stark elements are Stickelberger elements. Therefore, $\epsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}$ is the $p$-adic $L$-function of Deligne–Ribet.

We shall prove the equality (9) in Conjecture 3.14 for each odd $\chi \in \hat{\Delta}$. We fix such a character $\chi$, and may take $K = L_\chi$ and $S = S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k) \cup S_p(k)$. Let $S'_p$ be the set of $p$-adic primes which split completely in $K$. If $v \in S \setminus V_\chi$ is prime to $p$, it is ramified in $L_\chi = K$, so we have $\operatorname{char}_{\Lambda_\chi}(\mathcal{X}^\chi_{L_{\chi,\infty},S\setminus V_\chi}) = \operatorname{char}_{\Lambda_\chi}(\mathcal{Y}^\chi_{L_{\chi,\infty},S'_p})$. Let $A^T(L_{\chi,\infty})$ be the inverse limit of the $p$-component of the $T$-ray class group of the full integer ring of $L_{\chi,n}$. By sending the prime $w$ above $v$ in $S'_p$ to the class of $w$, we obtain a homomorphism $\mathcal{Y}^\chi_{L_{\chi,\infty},S'_p} \to A^T(L_{\chi,\infty})^\chi$, which is known to be injective. Since the sequence

$$\mathcal{Y}^\chi_{L_{\chi,\infty},S} \to A^T(L_{\chi,\infty})^\chi \to A^T_S(L_{\chi,\infty})^\chi \to 0$$

is exact and the kernel of $\mathcal{Y}^\chi_{L_{\chi,\infty},S} \to \mathcal{Y}^\chi_{L_{\chi,\infty},S'_p}$ is finite, we have

$$\operatorname{char}_{\Lambda_\chi}(A^T_S(L_{\chi,\infty})^\chi)\operatorname{char}_{\Lambda_\chi}((\mathcal{Y}_{L_{\chi,\infty},S})^\chi) = \operatorname{char}_{\Lambda_\chi}(A^T(L_{\chi,\infty})^\chi).$$

Therefore, by noting $\chi \neq 1$, the equality (9) in Conjecture 3.14 becomes

$$\mathrm{char}_{\Lambda_\chi}(A^T(L_{\chi,\infty})^\chi) = \theta^\chi_{L_{\chi,\infty}/k,S,T}(0)\Lambda_\chi,$$

where $\theta^\chi_{L_{\chi,\infty}/k,S,T}(0)$ is the $\chi$-component of $\epsilon^\varnothing_{L_{\chi,\infty}/k,S,T}$, which is the Stickelberger element in this case. This equality is nothing but the usual main conjecture proved in [Wiles 1990], so we have proved this theorem. □

**3E. *Consequences for number fields of finite degree.*** Let $p$, $k$, $k_\infty$, and $K$ be as in Theorem 3.16. We shall describe unconditional equivariant results on the Galois module structure of Selmer modules for $K$, which follow from the validity of Theorem 3.16.

To do this we set $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ and for any $\Lambda$-module $M$ we denote by $M^-$ the minus part consisting of elements on which the complex conjugation acts as $-1$ (namely, $M^- = e^- M$). We note, in particular, that $\theta_{K_\infty/k,S,T}(0)$ belongs to $\Lambda^-$.

We also write $x \mapsto x^\#$ for the $\mathbb{Z}_p$-linear involutions of both $\Lambda$ and the group rings $\mathbb{Z}_p[G]$ for finite quotients $G$ of $\mathrm{Gal}(K_\infty/k)$ which is induced by inverting elements of $\mathrm{Gal}(K_\infty/k)$.

**Corollary 3.17.** *If the $p$-adic $\mu$-invariant of $K_\infty/K$ vanishes, then*

$$\mathrm{Fitt}_{\Lambda^-}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/K_\infty})^-) = \Lambda \cdot \theta_{K_\infty/k,S,T}(0)$$

*and*

$$\mathrm{Fitt}_{\Lambda^-}(\mathcal{S}_{p,S,T}(\mathbb{G}_{m/K_\infty})^-) = \Lambda \cdot \theta_{K_\infty/k,S,T}(0)^\#.$$

*Proof.* Since $r_\chi = 0$ for any odd character $\chi$, the first displayed equality is equivalent to Conjecture 3.1 in this case and is therefore valid as a consequence of Theorem 3.16.

The second displayed equality is then obtained directly by applying the general result of [BKS, Lemma 2.8] to the first equality. □

**Corollary 3.18.** *Let $L$ be an intermediate CM-field of $K_\infty/k$ which is finite over $k$, and set $G := \mathrm{Gal}(L/k)$. If the $p$-adic $\mu$-invariant of $K_\infty/K$ vanishes, then there are equalities*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathcal{S}^{\mathrm{tr}}_{p,S,T}(\mathbb{G}_{m/L})^-) = \mathbb{Z}_p[G] \cdot \theta_{L/k,S,T}(0)$$

*and*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathcal{S}_{p,S,T}(\mathbb{G}_{m/L})^-) = \mathbb{Z}_p[G] \cdot \theta_{L/k,S,T}(0)^\#.$$

*Proof.* This follows by combining Corollary 3.17 with the general result of Lemma 3.19 below and standard properties of Fitting ideals. □

**Lemma 3.19.** *Suppose that $L/k$ is a Galois extension of finite number fields with Galois group $G$. Then there are natural isomorphisms*

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})_G \xrightarrow{\sim} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/k}) \quad \textit{and} \quad \mathcal{S}_{S,T}(\mathbb{G}_{m/L})_G \xrightarrow{\sim} \mathcal{S}_{S,T}(\mathbb{G}_{m/k}).$$

*Proof.* The "Weil-étale cohomology complex" $R\Gamma_T((\mathcal{O}_{L,S})_{\mathcal{W}}, \mathbb{G}_m)$ is perfect and so there exist projective $\mathbb{Z}[G]$-modules $P_1$ and $P_2$, and a homomorphism of $\mathbb{Z}[G]$-modules $P_1 \to P_2$ whose cokernel identifies with $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/L})$ and is such that the cokernel of the induced map $P_1^G \to P_2^G$ identifies with $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbb{G}_{m/k})$ (see [BKS, §5.4]).

The first isomorphism is then obtained by noting that the norm map induces an isomorphism of modules $(P_2)_G \xrightarrow{\sim} P_2^G$.

The second claimed isomorphism can also be obtained in a similar way, noting that $\mathcal{S}_{S,T}(\mathbb{G}_{m/L})$ is obtained as the cohomology in the highest (nonzero) degree of a perfect complex (see [BKS, Proposition 2.4]). □

We write $\mathcal{O}_L$ for the ring of integers of $L$ and $\mathrm{Cl}^T(L)$ for the ray class group of $\mathcal{O}_L$ with modulus $\prod_{w\in T_L} w$. We denote the Sylow $p$-subgroup of $\mathrm{Cl}^T(L)$ by $A^T(L)$ and write $(A^T(L)^-)^\vee$ for the Pontrjagin dual of the minus part of $A^T(L)$.

The next corollary of Theorem 3.16 that we record coincides with one of the main results of [Greither and Popescu 2015].

**Corollary 3.20.** *Let $L$ be an intermediate CM-field of $K_\infty/k$ which is finite over $k$, and set $G := \mathrm{Gal}(L/k)$. If the $p$-adic $\mu$-invariant for $K_\infty/K$ vanishes, then*

$$\theta_{L/k,S,T}(0)^\# \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A^T(L)^-)^\vee).$$

*Proof.* The canonical exact sequence

$$0 \to \mathrm{Cl}^T(L)^\vee \to \mathcal{S}_{S_\infty(k),T}(\mathbb{G}_{m/L}) \to \mathrm{Hom}(\mathcal{O}_L^\times, \mathbb{Z}) \to 0$$

from [BKS, Proposition 2.2] implies that the natural map $\mathcal{S}_{p,S_\infty(k),T}(\mathbb{G}_{m/L})^- \simeq (A^T(L)^-)^\vee$ is bijective.

In addition, from [BKS, Proposition 2.4(ii)], we know that the canonical homomorphism $\mathcal{S}_{S,T}(\mathbb{G}_{m/L}) \to \mathcal{S}_{S_\infty(k),T}(\mathbb{G}_{m/L})$ is surjective.

The claim therefore follows directly from the second equality in Corollary 3.18. □

**Remark 3.21.** (i) Our derivation of the equality in Corollary 3.20 differs from that given in [Greither and Popescu 2015] in that we avoid any use of Galois modules related to 1-motives. Instead, we used the theory of Selmer modules $\mathcal{S}_{S,T}(\mathbb{G}_{m/L})$ introduced in [BKS].

(ii) The Brumer–Stark conjecture predicts $\theta_{L/k,S_{\mathrm{ram}}(L/k),T}(0)$ belongs to the annihilator $\mathrm{Ann}_{\mathbb{Z}_p[G]^-}(A^T(L))$ and if no $p$-adic place of $L^+$ splits in $L$, then Corollary 3.20 implies a stronger version of this conjecture.

(iii) We have assumed throughout Section 3 that $S$ contains all $p$-adic places of $k$ and so the Stickelberger element $\theta_{L/k,S,T}(0)$ that occurs in Corollary 3.20 is, in general, imprimitive. In particular, if any $p$-adic place of $k$ splits completely in $L$, then $\theta_{L/k,S,T}(0)$ vanishes and the assertion of Corollary 3.20 is trivially valid. However, by applying Corollary 1.2 and [BKS, Corollary 1.14] in this context, one can now also obtain results such as Corollary 1.3.

## 4. Iwasawa-theoretic Rubin–Stark congruences

In this section, we formulate an Iwasawa-theoretic version of the conjecture proposed in [Mazur and Rubin 2016] and [Sano 2014] (see also [BKS, Conjecture 5.4]). This conjecture is a natural generalization of the Gross–Stark conjecture [Gross 1982], and plays a key role in the descent argument that we present in the next section.

We maintain the notation of the previous section.

**4A.** ***Statement of the congruences.*** We first recall the formulation of the conjecture of Mazur, Rubin and of the third author.

Take a character $\chi \in \hat{\mathcal{G}}$. Take a proper subset $V' \subset S$ so that all $v \in V'$ splits completely in $L_\chi$ (i.e., $\chi(G_v) = 1$) and that $V_\chi \subset V'$. Put $r' := \#V'$. We recall the formulation of the conjecture of Mazur and Rubin and of the third author for $(L_{\chi,n}/L_\chi/k, S, T, V_\chi, V')$. For simplicity, put

- $L_n := L_{\chi,n}$;
- $L := L_\chi$;
- $\mathcal{G}_n := \mathcal{G}_{\chi,n} = \mathrm{Gal}(L_{\chi,n}/k)$;
- $G := G_\chi = \mathrm{Gal}(L_\chi/k)$;
- $\Gamma_n := \Gamma_{\chi,n} = \mathrm{Gal}(L_{\chi,n}/L_\chi)$;
- $V := V_\chi = \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}$;
- $r := r_\chi = \#V_\chi$.

Put $e := r' - r$. Let $I(\Gamma_n)$ denote the augmentation ideal of $\mathbb{Z}_p[\Gamma_n]$. It is shown in [Sano 2014, Lemma 2.11] that there exists a canonical injection

$$\bigcap^r U_{L,S,T} \hookrightarrow \bigcap^r U_{L_n,S,T},$$

which induces the injection

$$\nu_n : \Big(\bigcap^r U_{L,S,T}\Big) \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1} \hookrightarrow \Big(\bigcap^r U_{L_n,S,T}\Big) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

Note that this injection does not coincide with the map induced by the inclusion

$U_{L,S,T} \hookrightarrow U_{L_n,S,T}$, and we have

$$\nu_n(\mathrm{N}^r_{L_n/L}(a)) = \mathrm{N}_{L_n/L}\, a$$

for all $a \in \bigcap^r U_{L_n,S,T}$ (see [Sano 2014, Remark 2.12]). For an explicit description of the map $\nu_n$, see [Mazur and Rubin 2016, Lemma 4.9; Sano 2015, Remark 4.2].

Let $I_n$ be the kernel of the natural map $\mathbb{Z}_p[\mathcal{G}_n] \to \mathbb{Z}_p[G]$. For $v \in V' \setminus V$, let $\mathrm{rec}_w : L^\times \to \Gamma_n$ denote the local reciprocity map at $w$ (recall that $w$ is the fixed place lying above $v$). Define

$$\mathrm{Rec}_w := \sum_{\sigma \in G} (\mathrm{rec}_w(\sigma(\cdot)) - 1)\sigma^{-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(L^\times, I_n/I_n^2).$$

It is shown in [Sano 2014, Proposition 2.7] that $\bigwedge_{v\in V'\setminus V} \mathrm{Rec}_w$ induces a homomorphism

$$\mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1}.$$

Finally, define

$$\mathcal{N}_n : \bigcap^{r} U_{L_n,S,T} \to \bigcap^{r} U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}$$

by

$$\mathcal{N}_n(a) := \sum_{\sigma \in \Gamma_n} \sigma a \otimes \sigma^{-1}.$$

We now state the formulation of [Sano 2014, Conjecture 3] (or [Mazur and Rubin 2016, Conjecture 5.2]).

**Conjecture 4.1** (MRS$(L_n/L/k, S, T, V, V')_p$). *Assume Conjectures* RS$(L_n/k, S, T, V)_p$ *and* RS$(L/k, S, T, V')_p$. *Then*

$$\mathcal{N}_n(\epsilon^V_{L_n/k,S,T}) = (-1)^{re}\nu_n(\mathrm{Rec}_n(\epsilon^{V'}_{L/k,S,T})) \ \ \textit{in} \ \ \bigcap^{r} U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

(Note that the sign in the right-hand side depends on the labeling of $S$. We follow the convention in [BKS, §5.3].)

Note that [BKS, Conjecture MRS$(K/L/k, S, T, V, V')$] is slightly stronger than the above conjecture (see [BKS, Remark 5.7]).

We shall next give an Iwasawa theoretic version of the above conjecture. Note that, since the inverse limit $\varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$ is isomorphic to $\mathbb{Z}_p$, the map

$$\varprojlim_n \mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$$

uniquely extends to give a $\mathbb{C}_p$-linear map

$$\mathbb{C}_p\bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p\big(\bigwedge^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}\big),$$

which we denote by $\mathrm{Rec}_\infty$.

**Conjecture 4.2** (MRS($K_\infty/k, S, T, \chi, V'$)). *Assume that Conjecture* RS($L_n/k, S, T, V)_p$ *is valid for all $n$. Then, there exists a (unique)*

$$\kappa = (\kappa_n)_n \in \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e / I(\Gamma_n)^{e+1}$$

*such that* $\nu_n(\kappa_n) = \mathcal{N}_n(\epsilon^V_{L_n/k,S,T})$ *for all $n$ and that*

$$e_\chi \kappa = (-1)^{re} e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}) \text{ in } \mathbb{C}_p\big(\textstyle\bigwedge^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e / I(\Gamma_n)^{e+1}\big).$$

**Remark 4.3.** Clearly the validity of Conjecture MRS($L_n/L/k, S, T, V, V')_p$ for all $n$ implies the validity of MRS($K_\infty/k, S, T, \chi, V'$). A significant advantage of the above formulation of Conjecture MRS($K_\infty/k, S, T, \chi, V'$) is that we do not need to assume that Conjecture RS($L/k, S, T, V')_p$ is valid.

**Proposition 4.4.** (i) MRS($K_\infty/k, S, T, \chi, V'$) *is valid if* $V = V'$.

(ii) MRS($K_\infty/k, S, T, \chi, V'$) *implies* MRS($K_\infty/k, S, T, \chi, V''$) *if* $V \subset V'' \subset V'$.

(iii) *Suppose that* $\chi(G_v) = 1$ *for all* $v \in S$ *and* $\#V' = \#S - 1$. *Then, for any* $V'' \subset S$ *with* $V \subset V''$ *and* $\#V'' = \#S - 1$, MRS($K_\infty/k, S, T, \chi, V'$) *and* MRS($K_\infty/k, S, T, \chi, V''$) *are equivalent.*

(iv) MRS($K_\infty/k, S \setminus \{v\}, T, \chi, V' \setminus \{v\}$) *implies* MRS($K_\infty/k, S, T, \chi, V'$) *if* $v \in V' \setminus V$ *is a finite place which is unramified in* $L_\infty$.

(v) *If* $\#V' \neq \#S - 1$ *and* $v \in S \setminus V'$ *is a finite place which is unramified in* $L_\infty$, *then* MRS($K_\infty/k, S \setminus \{v\}, T, \chi, V'$) *implies* MRS($K_\infty/k, S, T, \chi, V'$).

*Proof.* Claim (i) follows from the "norm relation" of Rubin–Stark elements; see [Sano 2014, Remark 3.9; Mazur and Rubin 2016, Proposition 5.7]. Claim (ii) follows from [Sano 2014, Proposition 3.12]. Claim (iii) follows from [Sano 2015, Lemma 5.1]. Claim (iv) follows from the proof of [Sano 2014, Proposition 3.13]. Claim (v) follows by noting $\epsilon^V_{L_n/k,S,T} = (1 - \mathrm{Fr}_v^{-1})\epsilon^V_{L_n/k,S\setminus\{v\},T}$ and $\epsilon^{V'}_{L/k,S,T} = (1 - \mathrm{Fr}_v^{-1})\epsilon^{V'}_{L/k,S\setminus\{v\},T}$. □

**Corollary 4.5.** *If every place $v$ in $V' \setminus V$ is both nonarchimedean and unramified in $L_\infty$, then* MRS($K_\infty/k, S, T, \chi, V'$) *is valid.*

*Proof.* By Proposition 4.4(iv), we may assume $V = V'$. By Proposition 4.4(i), we know that MRS($K_\infty/k, S, T, \chi, V'$) is valid in this case. □

Consider the following condition:

NTZ($K_\infty/k, \chi$) $\quad \chi(G_\mathfrak{p}) \neq 1$ for all $\mathfrak{p} \in S_p(k)$ which ramify in $L_{\chi,\infty}$.

This condition is usually called "no trivial zeros".

**Corollary 4.6.** *If $\chi$ satisfies* $\mathrm{NTZ}(K_\infty/k, \chi)$, *then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ *is valid.*

*Proof.* In this case we see that every $v \in V' \setminus V$ is finite and unramified in $L_\infty$. □

**4B.** ***Connection to the Gross–Stark conjecture.*** In this subsection we help set the context for Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ by showing that it specializes to recover the Gross–Stark conjecture (as stated in Conjecture 4.7 below).

To do this we assume throughout that $k$ is totally real, $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, and $\chi$ is totally odd. We also set $V' := \{v \in S \mid \chi(G_v) = 1\}$ (and note that this is a proper subset of $S$ since $\chi$ is totally odd) and we assume that every $v \in V'$ lies above $p$ (noting that this assumption is not restrictive as a consequence of Proposition 4.4(iv)).

We shall now show that this case of $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is equivalent to the Gross–Stark conjecture.

First, we note that in this case $V$ is empty (that is, $r = 0$) and so one knows that Conjecture $\mathrm{RS}(L_n/k, S, T, V)_p$ is valid for all $n$ (by [Rubin 1996, Theorem 3.3]). In fact, one has $\epsilon^V_{L_n/k,S,T} = \theta_{L_n/k,S,T}(0) \in \mathbb{Z}_p[\mathcal{G}_n]$ and, by [Mazur and Rubin 2016, Proposition 5.4], the assertion of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is equivalent to the following claims:

$$\theta_{L_n/k,S,T}(0) \in I_n^{r'} \tag{10}$$

for all $n$ and

$$e_\chi \theta_{L_\infty/k,S,T}(0) = e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}) \text{ in } \mathbb{C}_p[G] \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^{r'}/I(\Gamma_n)^{r'+1}, \tag{11}$$

where we set

$$\theta_{L_\infty/k,S,T}(0) := \varprojlim_n \theta_{L_n/k,S,T}(0) \in \varprojlim_n I_n^{r'}/I_n^{r'+1} \simeq \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^{r'}/I(\Gamma_n)^{r'+1}.$$

We also note that the validity of (10) follows as a consequence of our Iwasawa main conjecture (Conjecture 3.1) by using Proposition 2.6(iii) and the result of [BKS, Lemma 5.20] (see the argument in Section 5C).

To study (11) we set $\chi_1 := \chi|_\Delta \in \hat{\Delta}$ and regard (as we may) the product $\chi_2 := \chi\chi_1^{-1}$ as a character of $\Gamma = \mathrm{Gal}(k_\infty/k)$.

Note that $\mathrm{Gal}(L_\infty/k) = G_{\chi_1} \times \Gamma_{\chi_1}$. Fix a topological generator $\gamma \in \Gamma_{\chi_1}$, and identify $\mathbb{Z}_p[\mathrm{im}(\chi_1)][[\Gamma_{\chi_1}]]$ with the ring of power series $\mathbb{Z}_p[\mathrm{im}(\chi_1)][[t]]$ via the correspondence $\gamma = 1 + t$.

We then define $g^{\chi_1}_{L_\infty/k,S,T}(t)$ to be the image of $\theta_{L_\infty/k,S,T}(0)$ under the map

$$\mathbb{Z}_p[[\mathrm{Gal}(L_\infty/k)]] = \mathbb{Z}_p[G_{\chi_1}][[\Gamma_{\chi_1}]] \to \mathbb{Z}_p[\mathrm{im}(\chi_1)][[\Gamma_{\chi_1}]] = \mathbb{Z}_p[\mathrm{im}(\chi_1)][[t]]$$

induced by $\chi_1$. We recall that the $p$-adic $L$-function of Deligne–Ribet is defined by

$$L_{k,S,T,p}(\chi^{-1}\omega, s) := g^{\chi_1}_{L_\infty/k,S,T}(\chi_2(\gamma)\chi_{\mathrm{cyc}}(\gamma)^s - 1),$$

where $\chi_{\mathrm{cyc}}$ is the cyclotomic character; one can show this to be independent of the choice of $\gamma$.

The validity of (10) implies an inequality

$$\operatorname{ord}_{s=0} L_{k,S,T,p}(\chi^{-1}\omega, s) \geq r'. \tag{12}$$

It is known that (12) is a consequence of the Iwasawa main conjecture (in the sense of [Wiles 1990]), which is itself known to be valid when $p$ is odd. In addition, Spiess [2014] proved that (12) is valid, including the case $p = 2$, by using Shintani cocycles. In all cases, therefore, we can define

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) := \lim_{s\to 0} s^{-r'} L_{k,S,T,p}(\chi^{-1}\omega, s) \in \mathbb{C}_p.$$

For $v \in V'$, define

$$\operatorname{Log}_w : L^\times \to \mathbb{Z}_p[G]$$

by $\operatorname{Log}_w(a) := -\sum_{\sigma\in G} \log_p(\mathrm{N}_{L_w/\mathbb{Q}_p}(\sigma a))\sigma^{-1}$, where $\log_p : \mathbb{Q}_p^\times \to \mathbb{Z}_p$ is Iwasawa's logarithm (in the sense that $\log_p(p) = 0$). We set

$$\operatorname{Log}_{V'} := \textstyle\bigwedge_{v\in V'} \operatorname{Log}_w : \mathbb{C}_p \textstyle\bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p[G].$$

We shall denote the map $\mathbb{C}_p[G] \to \mathbb{C}_p$ induced by $\chi$ also by $\chi$.

For $v \in V'$, we define

$$\operatorname{Ord}_w : L^\times \to \mathbb{Z}[G]$$

by $\operatorname{Ord}_w(a) := \sum_{\sigma\in G} \operatorname{ord}_w(\sigma a)\sigma^{-1}$, and set

$$\operatorname{Ord}_{V'} := \textstyle\bigwedge_{v\in V'} \operatorname{Ord}_w : \mathbb{C}_p \textstyle\bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p[G].$$

On the $\chi$-component, $\operatorname{Ord}_{V'}$ induces an isomorphism

$$\chi \circ \operatorname{Ord}_{V'} : e_\chi \mathbb{C}_p \textstyle\bigwedge^{r'} U_{L,S,T} \xrightarrow{\sim} \mathbb{C}_p.$$

Taking a nonzero element $x \in e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T}$, we define the $\mathcal{L}$-invariant by

$$\mathcal{L}(\chi) := \frac{\chi(\operatorname{Log}_{V'}(x))}{\chi(\operatorname{Ord}_{V'}(x))} \in \mathbb{C}_p.$$

Since $e_\chi \mathbb{C}_p \bigwedge^{r'} U_{L,S,T}$ is a one-dimensional $\mathbb{C}_p$-vector space, we see that $\mathcal{L}(\chi)$ does not depend on the choice of $x$.

Then the Gross–Stark conjecture is stated as follows.

**Conjecture 4.7** (GS$(L/k, S, T, \chi)$). *One has*

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) = \mathcal{L}(\chi) L_{k,S\setminus V',T}(\chi^{-1}, 0).$$

**Remark 4.8.** This formulation constitutes a natural higher-rank generalization of the form of the Gross–Stark conjecture stated in [Dasgupta et al. 2011, Conjecture 1].

Letting $x = e_\chi \epsilon^{V'}_{L/k,S,T}$, we obtain

$$\chi(\mathrm{Log}_{V'}(\epsilon^{V'}_{L/k,S,T})) = \mathcal{L}(\chi) L_{k,S\setminus V',T}(\chi^{-1}, 0).$$

Thus we see that Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to the equality

$$L^{(r')}_{k,S,T,p}(\chi^{-1}\omega, 0) = \chi(\mathrm{Log}_{V'}(\epsilon^{V'}_{L/k,S,T})).$$

Concerning the relation between $\mathrm{Rec}_\infty$ and $\mathrm{Log}_{V'}$, we note the fact

$$\chi_{\mathrm{cyc}}(\mathrm{rec}_w(a)) = \mathrm{N}_{L_w/\mathbb{Q}_p}(a)^{-1},$$

where $v \in V'$ and $a \in L^\times$.

Given this fact, it is straightforward to check (under the validity of (10)) that Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to (11).

At this stage we have therefore proved the following result.

**Theorem 4.9.** *Suppose that $k$ is totally real, $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, and $\chi$ is totally odd. Set $V' := \{v \in S \mid \chi(G_v) = 1\}$ and assume that every $v \in V'$ lies above $p$. Assume also that (10) is valid. Then Conjecture $\mathrm{GS}(L/k, S, T, \chi)$ is equivalent to Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$.*

**4C.** ***A proof in the case $k = \mathbb{Q}$.*** In [BKS, Corollary 1.2] the known validity of the eTNC for Tate motives over abelian fields is used to prove that Conjecture $\mathrm{MRS}(K/L/k, S, T, V, V')$ is valid in the case $k = \mathbb{Q}$.

In this subsection, we shall give a much simpler proof of the latter result which uses only Theorem 4.9, the known validity of the Gross–Stark conjecture over abelian fields and a classical result from [Solomon 1992].

We note that for any $\chi$ and $n$ the Rubin–Stark conjecture is known to be true for $(L_{\chi,n}/\mathbb{Q}, S, T, V_\chi)$. In fact, in this setting the Rubin–Stark element is given by a cyclotomic unit when $r_\chi = 1$ and by the Stickelberger element when $r_\chi = 0$ (see [Popescu 2011, §4.2 and Example 3.2.10], for example).

**Theorem 4.10.** *Suppose that $k = \mathbb{Q}$. Then, $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')$ is valid.*

*Proof.* By Proposition 4.4(ii), we may assume that $V'$ is maximal, namely,

$$r' = \min\{\#\{v \in S \mid \chi(G_v) = 1\}, \#S - 1\}.$$

By Corollary 4.6, we may assume that $\chi(p) = 1$.

Suppose first that $\chi$ is odd. Since Conjecture $\mathrm{GS}(L/\mathbb{Q}, S, T, \chi)$ is valid (see [Gross 1982, §4]), Conjecture $\mathrm{MRS}(K_\infty/\mathbb{Q}, S, T, \chi, V')$ follows from Theorem 4.9.

Suppose next that $\chi = 1$. In this case we have $r' = \#S - 1$. We may assume $p \notin V'$ by Proposition 4.4(iii). In this case every $v \in V' \setminus V$ is unramified in $L_\infty$. Hence, the theorem follows from Corollary 4.5.

Finally, suppose that $\chi \neq 1$ is even. By Proposition 4.4(iv) and (v), we may assume $S = \{\infty, p\} \cup S_{\mathrm{ram}}(L/\mathbb{Q})$ and $V' = \{\infty, p\}$. We label $S = \{v_0, v_1, \dots\}$ so that $v_1 = \infty$ and $v_2 = p$.

Fix a topological generator $\gamma$ of $\Gamma = \mathrm{Gal}(L_\infty/L)$. Then we construct an element $\kappa(L, \gamma) \in \varprojlim_n L^\times/(L^\times)^{p^n}$ as follows. Note that $\mathrm{N}_{L_n/L}(\epsilon^V_{L_n/\mathbb{Q},S,T})$ vanishes since $\chi(p) = 1$. So we can take $\beta_n \in L_n^\times$ such that $\beta_n^{\gamma-1} = \epsilon^V_{L_n/\mathbb{Q},S,T}$ (Hilbert's Theorem 90). Define

$$\kappa_n := \mathrm{N}_{L_n/L}(\beta_n) \in L^\times/(L^\times)^{p^n}.$$

This element is independent of the choice of $\beta_n$, and for any $m > n$ the natural map

$$L^\times/(L^\times)^{p^m} \to L^\times/(L^\times)^{p^n}$$

sends $\kappa_m$ to $\kappa_n$. We define

$$\kappa(L, \gamma) := (\kappa_n)_n \in \varprojlim_n L^\times/(L^\times)^{p^n}.$$

Then, by [Solomon 1992, Proposition 2.3(i)], we know that

$$\kappa(L, \gamma) \in \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L\left[\frac{1}{p}\right]^\times \hookrightarrow \varprojlim_n L^\times/(L^\times)^{p^n}.$$

Fix a prime $\mathfrak{p}$ of $L$ lying above $p$. Define

$$\mathrm{Ord}_{\mathfrak{p}} : L^\times \to \mathbb{Z}_p[G]$$

by $\mathrm{Ord}_{\mathfrak{p}}(a) := \sum_{\sigma \in G} \mathrm{ord}_{\mathfrak{p}}(\sigma a)\sigma^{-1}$. Similarly, define

$$\mathrm{Log}_{\mathfrak{p}} : L^\times \to \mathbb{Z}_p[G]$$

by $\mathrm{Log}_{\mathfrak{p}}(a) := -\sum_{\sigma \in G} \log_p(\iota_{\mathfrak{p}}(\sigma a))\sigma^{-1}$, where $\iota_{\mathfrak{p}} : L \hookrightarrow L_{\mathfrak{p}} = \mathbb{Q}_p$ is the natural embedding.

Then by [Solomon 1992, Theorem 2.1 and Remark 2.4], one deduces

$$\mathrm{Ord}_{\mathfrak{p}}(\kappa(L, \gamma)) = -\frac{1}{\log_p(\chi_{\mathrm{cyc}}(\gamma))}\mathrm{Log}_{\mathfrak{p}}(\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}).$$

From this, we have

$$\mathrm{Ord}_{\mathfrak{p}}(\kappa(L, \gamma)) \otimes (\gamma - 1) = -\mathrm{Rec}_{\mathfrak{p}}(\epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}) \text{ in } \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2, \quad (13)$$

where $I(\Gamma)$ is the augmentation ideal of $\mathbb{Z}_p[[\Gamma]]$.

We know that $e_\chi \mathbb{C}_p U_{L,S}$ is a two-dimensional $\mathbb{C}_p$-vector space. Lemma 4.11 below shows that $\{e_\chi \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}, e_\chi \kappa(L,\gamma)\}$ is a $\mathbb{C}_p$-basis of this space. For simplicity, set $\epsilon^V_L := \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}$. Note that the isomorphism

$$\mathrm{Ord}_{\mathfrak{p}} : e_\chi \mathbb{C}_p {\textstyle\bigwedge}^2 U_{L,S} \xrightarrow{\sim} e_\chi \mathbb{C}_p U_L$$

sends $e_\chi \epsilon^V_L \wedge \kappa(L,\gamma)$ to $-\chi(\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)))e_\chi \epsilon^V_L$. Since we have

$$\mathrm{Ord}_{\mathfrak{p}}(e_\chi \epsilon^{V'}_{L/\mathbb{Q},S,T}) = -e_\chi \epsilon^V_L$$

(see [Rubin 1996, Proposition 5.2; Sano 2014, Proposition 3.6]), we have

$$e_\chi \epsilon^{V'}_{L/\mathbb{Q},S,T} = -\chi(\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)))^{-1} e_\chi \epsilon^V_L \wedge \kappa(L,\gamma).$$

Hence we have

$$\begin{aligned}\mathrm{Rec}_{\mathfrak{p}}(e_\chi \epsilon^{V'}_{L/\mathbb{Q},S,T}) &= \chi(\mathrm{Ord}_{\mathfrak{p}}(\kappa(L,\gamma)))^{-1} e_\chi \kappa(L,\gamma)\cdot \mathrm{Rec}_{\mathfrak{p}}(\epsilon^V_L)\\ &= -e_\chi \kappa(L,\gamma)\otimes(\gamma-1),\end{aligned}$$

where the first equality follows by noting that $\mathrm{Rec}_{\mathfrak{p}}(\kappa(L,\gamma)) = 0$ (since $\kappa(L,\gamma)$ lies in the universal norm by definition), and the second by (13).

Now, noting that

$$\nu_n : U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)/I(\Gamma_n)^2 \hookrightarrow U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^2$$

is induced by the inclusion map $L \hookrightarrow L_n$, and that

$$\mathcal{N}_n(\epsilon^V_{L_n/\mathbb{Q},S,T}) = \kappa_n \otimes (\gamma-1),$$

it is easy to see that the element $\kappa := \kappa(L,\gamma)\otimes(\gamma-1)$ has the properties in the statement of Conjecture MRS($K_\infty/\mathbb{Q}, S, T, \chi, V'$). This completes the proof. □

**Lemma 4.11.** *Assume that $k = \mathbb{Q}$ and $\chi \neq 1$ is even such that $\chi(p) = 1$. Assume also that $S = \{\infty, p\} \cup S_{\mathrm{ram}}(L/\mathbb{Q})$. Then, $\{e_\chi \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}, e_\chi \kappa(L,\gamma)\}$ is a $\mathbb{C}_p$-basis of $e_\chi \mathbb{C}_p U_{L,S}$.*

*Proof.* This result follows from [Solomon 1994, Remark 4.4]. But we sketch another proof, essentially given in [Flach 2004].

In the next section, we define the "Bockstein map"

$$\beta : e_\chi \mathbb{C}_p U_{L,S} \to e_\chi \mathbb{C}_p(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2).$$

We see that $\beta$ is injective on $e_\chi \mathbb{C}_p U_L$, and that $\ker \beta \simeq U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p$ where we put $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$ and $\mathbb{C}_p$ is regarded as a $\Lambda$-algebra via $\chi$. Hence we have

$$e_\chi \mathbb{C}_p U_{L,S} = e_\chi \mathbb{C}_p U_L \oplus (U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p).$$

Since $e_\chi \epsilon^V_{L/\mathbb{Q},S\setminus\{p\},T}$ is nonzero, this is a basis of $e_\chi \mathbb{C}_p U_{L,S\setminus\{p\}} = e_\chi \mathbb{C}_p U_L$. We prove that $e_\chi \kappa(L,\gamma)$ is a basis of $U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p$.

By using the exact sequence $0 \to U_{L_\infty,S} \xrightarrow{\gamma-1} U_{L_\infty,S} \to U_{L,S}$, we see that there exists a unique element $\alpha \in U_{L_\infty,S}$ such that $(\gamma-1)\alpha = \epsilon^V_{L_\infty/\mathbb{Q},S,T}$. By the cyclotomic Iwasawa main conjecture over $\mathbb{Q}$, we see that $\alpha$ is a basis of $U_{L_\infty,S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi}$, where $\mathfrak{p}_\chi := \ker(\chi : \Lambda \to \mathbb{C}_p)$. The image of $\alpha$ under the map

$$U_{L_\infty,S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi} \xrightarrow{\chi} U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p \hookrightarrow e_\chi \mathbb{C}_p U_{L,S}$$

is equal to $e_\chi \kappa(L,\gamma)$. □

## 5. A strategy for proving the eTNC

**5A.** ***Statement of the main result and applications.*** In the sequel we fix an intermediate field $L$ of $K_\infty/k$ which is finite over $k$ and set $G := \mathrm{Gal}(L/k)$. In this section we always assume the following conditions to be satisfied:

(R) For every $\chi \in \hat{G}$, one has $r_{\chi,S} < \#S$.

(S) No finite place of $k$ splits completely in $k_\infty$.

**Remark 5.1.** Before proceeding we note that the condition (R) is very mild since it is automatically satisfied when the class number of $k$ is equal to one and, for any $k$, is satisfied when $S$ is large enough. We also note that the condition (S) is satisfied when, for example, $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension.

The following result is one of the main results of this article and, as we will see, it provides an effective strategy for proving the special case of the eTNC that we are considering here.

**Theorem 5.2.** *Assume the following conditions*:

(hIMC) *The main conjecture* $\mathrm{IMC}(K_\infty/k, S, T)$ (Conjecture 3.1) *is valid.*

(F) *For every $\chi$ in $\hat{G}$, the module of $\Gamma_\chi$-coinvariants of $A_S^T(L_{\chi,\infty})$ is finite.*

(MRS) *For every $\chi$ in $\hat{G}$, Conjecture* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'_\chi)$ (Conjecture 4.2) *is valid for a maximal set $V'_\chi$, so that*

$$\#V'_\chi = \min\{\#\{v \in S \mid \chi(G_v) = 1\}, \#S - 1\}.$$

*Then, the conjecture* $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ (*Conjecture 2.3*) *is valid.*

**Remark 5.3.** We note that the set $V'_\chi$ in condition (MRS) is not uniquely determined when every place $v$ in $S$ satisfies $\chi(G_v) = 1$, but that the validity of Conjecture $\mathrm{MRS}(K_\infty/k, S, T, \chi, V'_\chi)$ is independent of the choice of $V'_\chi$ (by Proposition 4.4(iii)).

**Remark 5.4.** One checks easily that the condition (F) is equivalent to the finiteness of the module of $\Gamma_\chi$-coinvariants of $A_S(L_{\chi,\infty})$. Hence, taking account of [1991, Theorem 1.14], the condition (F) can be regarded as a natural generalization of

Conjecture 1.15 of [Gross 1982]. We also note here that this conjecture of Gross was asserted in a special setting as Conjecture 2.2 in [Coates and Lichtenbaum 1973]. In particular, we recall that the condition (F) is satisfied in each of the following cases:

- $L$ is abelian over $\mathbb{Q}$ (this is due to Greenberg [1973]).
- $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension and $L$ has unique $p$-adic place (in this case "$\delta_L = 0$" holds obviously; see [Kolster 1991]).
- $L$ is totally real and the Leopoldt conjecture is valid for $L$ at $p$ (see [Kolster 1991, Corollary 1.3]).

**Remark 5.5.** The condition (MRS) is satisfied for $\chi$ in $\hat{G}$ when the condition $\mathrm{NTZ}(K_\infty/k, \chi)$ is satisfied (see Corollary 4.6).

As an immediate corollary of Theorem 5.2, we obtain a new proof of a theorem that was first proved in [Burns and Greither 2003] for $p$ odd and in [Flach 2011] for $p = 2$.

**Corollary 5.6.** *If $k = \mathbb{Q}$, then the conjecture* $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G])$ *is valid.*

*Proof.* As we mentioned above, the conditions (R), (S) and (F) are all satisfied in this case. In addition, the condition (hIMC) is a direct consequence of the classical Iwasawa main conjecture solved by Mazur and Wiles (see [Burns and Greither 2003; Flach 2011]) and the condition (MRS) is satisfied by Theorem 4.10. □

We also obtain a result over totally real fields.

**Corollary 5.7.** *Suppose that $p$ is odd, $k$ is totally real, $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, and $K$ is CM. Assume that (F) is satisfied, that the $\mu$-invariant of $K_\infty/K$ vanishes, and that for every odd character $\chi \in \hat{G}$, Conjecture* $\mathrm{GS}(L_\chi/k, S, T, \chi)$ *is valid. Then, Conjecture* $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]^-)$ *is valid.*

*Proof.* Fix $S$ so that the condition (R) is satisfied. Then the minus-part of condition (hIMC) is satisfied by Theorem 3.16 and the minus part of condition (MRS) by Theorem 4.9. □

When at most one $p$-adic place $\mathfrak{p}$ of $k$ satisfies $\chi(G_\mathfrak{p}) = 1$, the validity of Conjecture $\mathrm{GS}(L_\chi/k, S, T, \chi)$ was proved by Dasgupta, Darmon and Pollack [Dasgupta et al. 2011] under certain assumptions, including Leopoldt's conjecture. Those assumptions were removed in [Ventullo 2015], so Conjecture $\mathrm{GS}(L_\chi/k, S, T, \chi)$ is unconditionally valid in this case (see also the note on p. 1531). Condition (F) is then valid too, by the argument of [Gross 1982, Proposition 2.13]. Hence we get:

**Corollary 5.8.** *Suppose that $p$ is odd, $k$ is totally real, $k_\infty/k$ is the cyclotomic $\mathbb{Z}_p$-extension, and $K$ is CM. Assume that the $\mu$-invariant of $K_\infty/K$ vanishes, and that for each odd character $\chi \in \hat{G}$ there is at most one $p$-adic place $\mathfrak{p}$ of $k$ which satisfies $\chi(G_\mathfrak{p}) = 1$. Then, Conjecture* $\mathrm{eTNC}(h^0(\mathrm{Spec}\, L), \mathbb{Z}_p[G]^-)$ *is valid.*

**Examples 5.9.** It is not difficult to find many concrete families of examples satisfying the hypotheses of Corollary 5.8 and hence to deduce the unconditional validity of $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G]^-)$ in some new and interesting cases. In particular, we shall now describe several families of examples in which the extension $k/\mathbb{Q}$ is not abelian (noting that if $L/\mathbb{Q}$ is abelian and $k \subset L$, then $\mathrm{eTNC}(h^0(\operatorname{Spec} L), \mathbb{Z}_p[G])$ is already known to be valid).

(i) The case $p = 3$. As a simple example, we consider the case that $k/\mathbb{Q}$ is a $S_3$-extension. To do this we fix an irreducible cubic polynomial $f(x)$ in $\mathbb{Z}[x]$ with discriminant $27d$ where $d$ is strictly positive and congruent to 2 modulo 3. (For example, one can take $f(x)$ to be $x^3-6x-3$, $x^3-15x-3$, etc.) The minimal splitting field $k$ of $f(x)$ over $\mathbb{Q}$ is then totally real (since $27d > 0$) and an $S_3$-extension of $\mathbb{Q}$ (since $27d$ is not a square). Also, since the discriminant of $f(x)$ is divisible by 27 but not 81, the prime 3 is totally ramified in $k$. Now set $p := 3$ and $K := k(\mu_p) = k(\sqrt{-p}) = k(\sqrt{-d})$. Then the prime above $p$ splits in $K/k$ because $-d \equiv 1 \pmod 3$. In addition, as $K/\mathbb{Q}(\sqrt{d}, \sqrt{-p})$ is a cyclic cubic extension, the $\mu$-invariant of $K_\infty/K$ vanishes and so the extension $K/k$ satisfies all the conditions of Corollary 5.8 (with $p = 3$).

(ii) The case $p > 3$. In this case one can construct a suitable field $K$ in the following way. Fix a primitive $p$-th root of unity $\zeta$, an integer $i$ such that $1 \le i \le (p-3)/2$, and an integer $b$ which is prime to $p$, and then set

$$a := \frac{1+b(\zeta-1)^{2i+1}}{1+b(\zeta^{-1}-1)^{2i+1}}.$$

Write $\mathrm{ord}_\pi$ for the normalized additive valuation of $\mathbb{Q}(\mu_p)$ associated to the prime element $\pi = \zeta - 1$. Then, since $\mathrm{ord}_\pi(a-1) = 2i+1 < p$, $(\pi)$ is totally ramified in $\mathbb{Q}(\mu_p, \sqrt[p]{a})/\mathbb{Q}(\mu_p)$. Also, since $c(a) = a^{-1}$ where $c$ is the complex conjugation, $\mathbb{Q}(\mu_p, \sqrt[p]{a})$ is the composite of a cyclic extension of $\mathbb{Q}(\mu_p)^+$ of degree $p$ and $\mathbb{Q}(\mu_p)$. This shows that $\mathbb{Q}(\mu_p, \sqrt[p]{a})$ is a CM-field and, since $1 < 2i+1 < p$, the extension $\mathbb{Q}(\mu_p, \sqrt[p]{a})^+/\mathbb{Q}$ is nonabelian. We now take a negative integer $-d$ which is a quadratic residue modulo $p$, let $K$ denote the CM-field $\mathbb{Q}(\mu_p, \sqrt[p]{a}, \sqrt{-d})$ and set $k := K^+$. Then $p$ is totally ramified in $k/\mathbb{Q}$ and the $p$-adic prime of $k$ splits in $K$. In addition, $k/\mathbb{Q}$ is not abelian and the $\mu$-invariant of $K_\infty/K$ vanishes since $K/\mathbb{Q}(\mu_p, \sqrt{-d})$ is cyclic of degree $p$. This shows that the extension $K/k$ satisfies all of the hypotheses of Corollary 5.8.

(iii) In cases (i) and (ii) above, $p$ is totally ramified in the extension $k_\infty/\mathbb{Q}$ and so Corollary 5.8 implies that $\mathrm{eTNC}(h^0(\operatorname{Spec} K_n), \mathbb{Z}_p[G]^-)$ is valid for any nonnegative integer $n$. In addition, if $F$ is any real abelian field of degree prime to $[k:\mathbb{Q}]$ in which $p$ is totally ramified, the minus component of the $p$-part of eTNC for $FK_n/k$ holds for any nonnegative integer $n$.

**Remark 5.10.** By using similar methods to the proofs of the above corollaries it is also possible to deduce the main result of [Bley 2006] as a consequence of Theorem 5.2. In this case $k$ is imaginary quadratic, the validity of (hIMC) can be derived from [Rubin 1991] (as explained in [Bley 2006]), and the conjecture (MRS) from the result in [Bley 2004], which is itself an analogue of Solomon's theorem [1992] for elliptic units, by using the same argument as Theorem 4.10.

**5B.** ***A computation of Bockstein maps.*** Fix a character $\chi \in \hat{G}$ and set

$$\begin{aligned} L_n &:= L_{\chi,n}, \\ L &:= L_\chi, \\ V &:= V_\chi = \{v \in S \mid v \text{ splits completely in } L_{\chi,\infty}\}, \\ r &:= r_\chi = \# V_\chi, \\ V' &:= V'_\chi \qquad \text{(as in (MRS) in Theorem 5.2)}, \\ r' &:= r_{\chi,S} = \# V', \\ e &:= r' - r. \end{aligned}$$

As in Section 4A, we label $S = \{v_0, v_1, \dots\}$ so that $V = \{v_1, \dots, v_r\}$ and $V' = \{v_1, \dots, v_{r'}\}$, and fix a place $w$ lying above each $v \in S$. Also, as in Section 2D, it will be useful to fix a representative $\Pi_{K_\infty} \to \Pi_{K_\infty}$ of $C_{K_\infty,S,T}$ where the first term is placed in degree zero, and $\Pi_{K_\infty}$ is a free $\Lambda$-module with basis $\{b_1, \dots, b_d\}$. This representative is chosen so that the natural surjection

$$\Pi_{K_\infty} \to H^1(C_{K_\infty,S,T}) \to \mathcal{X}_{K_\infty,S}$$

sends $b_i$ to $w_i - w_0$ for every $i$ with $1 \le i \le r'$.

We define a height-one regular prime ideal of $\Lambda$ by setting

$$\mathfrak{p} := \ker(\Lambda \xrightarrow{\chi} \mathbb{Q}_p(\chi) := \mathbb{Q}_p(\operatorname{im}\chi)).$$

Then the localization $R := \Lambda_{\mathfrak{p}}$ is a discrete valuation ring and we write $P$ for its maximal ideal. We see that $\chi$ induces an isomorphism

$$E := R/P \xrightarrow{\sim} \mathbb{Q}_p(\chi).$$

We set $C := C_{K_\infty,S,T} \otimes_\Lambda R$ and $\Pi := \Pi_{K_\infty} \otimes_\Lambda R$.

**Lemma 5.11.** *Let $\gamma$ be a topological generator of $\Gamma = \mathrm{Gal}(K_\infty/K)$. Let $n$ be an integer which satisfies $\gamma^{p^n} \in \mathrm{Gal}(K_\infty/L)$. Then $\gamma^{p^n} - 1$ is a uniformizer of $R$.*

*Proof.* Regard $\chi \in \hat{\mathcal{G}}$, and put $\chi_1 := \chi|_\Delta \in \hat{\Delta}$. We identify $R$ with the localization of $\Lambda_{\chi_1}[1/p] = \mathbb{Z}_p[\operatorname{im}\chi_1][[\Gamma]][1/p]$ at $\mathfrak{q} := \ker(\Lambda_{\chi_1}[1/p] \xrightarrow{\chi|_\Gamma} \mathbb{Q}_p(\chi))$.

Then the lemma follows by noting the localization of $\Lambda_{\chi_1}[1/p]/(\gamma^{p^n} - 1) = \mathbb{Z}_p[\operatorname{im}\chi_1][\Gamma_n][1/p]$ at $\mathfrak{q}$ is identified with $\mathbb{Q}_p(\chi)$. $\square$

**Lemma 5.12.** *Assume that the condition (F) is satisfied.*

(i) *$H^0(C)$ is isomorphic to $U_{K_\infty,S,T} \otimes_\Lambda R$, and $R$-free of rank $r$.*

(ii) *$H^1(C)$ is isomorphic to $\mathcal{X}_{K_\infty,S} \otimes_\Lambda R$.*

(iii) *The maximal $R$-torsion submodule $H^1(C)_{\mathrm{tors}}$ of $H^1(C)$ is isomorphic to $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$, and annihilated by $P$. (So $H^1(C)_{\mathrm{tors}}$ is an $E$-vector space.)*

(iv) *$H^1(C)_{\mathrm{tf}} := H^1(C)/H^1(C)_{\mathrm{tors}}$ is isomorphic to $\mathcal{Y}_{K_\infty,V} \otimes_\Lambda R$ and is therefore $R$-free of rank $r$.*

(v) *$\dim_E(H^1(C)_{\mathrm{tors}}) = e$.*

*Proof.* Since $U_{K_\infty,S,T} \otimes_\Lambda R = H^0(C)$ is regarded as a submodule of $\Pi$, we see that $U_{K_\infty,S,T} \otimes_\Lambda R$ is $R$-free. Put $\chi_1 := \chi|_\Delta \in \hat{\Delta}$. Note that $L_\infty := L_{\chi,\infty} = L_{\chi_1,\infty}$, and that the quotient field of $R$ is $Q(\Lambda_{\chi_1})$. As in the proof of Theorem 3.4, we have

$$U_{K_\infty,S,T} \otimes_\Lambda Q(\Lambda_{\chi_1}) \simeq \mathcal{Y}_{L_\infty,V} \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_{\chi_1}).$$

These are $r$-dimensional $Q(\Lambda_{\chi_1})$-vector spaces. This proves (i).

To prove (ii), it is sufficient to show that $A_S^T(K_\infty) \otimes_\Lambda R = 0$. Fix a topological generator $\gamma$ of $\Gamma$, and regard $\mathbb{Z}_p[[\Gamma]]$ as the ring of power series $\mathbb{Z}_p[[t]]$ via the identification $\gamma = 1 + t$. Let $f$ be the characteristic polynomial of the $\mathbb{Z}_p[[t]]$-module $A_S^T(L_\infty)$. By Lemma 5.11, for sufficiently large $n$, $\gamma^{p^n} - 1$ is a uniformizer of $R$. On the other hand, by the assumption (F), we see that $f$ is prime to $\gamma^{p^n} - 1$. This implies (ii).

We prove (iii). Proving that $H^1(C)_{\mathrm{tors}}$ is isomorphic to $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$, it is sufficient to show that

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda Q(\Lambda_{\chi_1}) \simeq \mathcal{Y}_{K_\infty,V} \otimes_\Lambda Q(\Lambda_{\chi_1}),$$

by (ii). This was shown in the proof of Theorem 3.4. We prove that $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R$ is annihilated by $P$. Note that $\mathcal{X}_{K_\infty,S\setminus V} \otimes_\Lambda R = \mathcal{X}_{K_\infty,S\setminus(V\cup S_\infty)} \otimes_\Lambda R$, since the complex conjugation $c$ at $v \in S_\infty \setminus (V \cap S_\infty)$ is nontrivial in $G_{\chi_1}$, and hence $c - 1 \in R^\times$. Hence, it is sufficient to show that, for every $v \in S \setminus (V \cup S_\infty)$, there exists $\sigma \in G_v \cap \Gamma$ such that $\sigma - 1$ is a uniformizer of $R$, where $G_v \subset \mathcal{G}$ is the decomposition group at a place of $K_\infty$ lying above $v$. Thanks to the assumption (S), we find such $\sigma$ by Lemma 5.11.

The assertion (iv) is immediate from the above argument.

The assertion (v) follows from (iii), (iv), and the fact that

$$\mathcal{X}_{K_\infty,S} \otimes_\Lambda E \simeq \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{X}_{L,S} \simeq e_\chi \mathbb{Q}_p(\chi)\mathcal{Y}_{L,V'}$$

is an $r'$-dimensional $E$-vector space. □

In the following for any $R$-module $M$ we often denote $M \otimes_R E$ by $M_E$. Also, we assume that (F) is satisfied.

**Definition 5.13.** The "Bockstein map" is the homomorphism

$$\beta : H^0(C_E) \to H^1(C \otimes_R P) = H^1(C) \otimes_R P \to H^1(C_E) \otimes_E P/P^2$$

induced by the natural exact triangle $C \otimes_R P \to C \to C_E$.

Note that there are canonical isomorphisms

$$H^0(C_E) \simeq U_{L,S,T} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi) U_{L,S,T},$$
$$H^1(C_E) \simeq \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p[G_\chi]} \mathbb{Q}_p(\chi) \simeq e_\chi \mathbb{Q}_p(\chi) \mathcal{X}_{L,S} \simeq e_\chi \mathbb{Q}_p(\chi) \mathcal{Y}_{L,V'},$$

where $\mathbb{Q}_p(\chi)$ is regarded as a $\mathbb{Z}_p[G_\chi]$-algebra via $\chi$. Note also that $P$ is generated by $\gamma^{p^n} - 1$ with sufficiently large $n$, where $\gamma$ is a fixed topological generator of $\Gamma$ (see Lemma 5.11). There is a canonical isomorphism

$$I(\Gamma_\chi)/I(\Gamma_\chi)^2 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\chi) \simeq P/P^2,$$

where $I(\Gamma_\chi)$ denotes the augmentation ideal of $\mathbb{Z}_p[[\Gamma_\chi]]$ (note that $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\Gamma_\chi = \mathrm{Gal}(L_\infty/L)$). Thus, the Bockstein map is regarded as the map

$$\begin{aligned}\beta : e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} &\to e_\chi \mathbb{Q}_p(\chi)(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \\ &\simeq e_\chi \mathbb{Q}_p(\chi)(\mathcal{Y}_{L,V'} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2).\end{aligned}$$

**Proposition 5.14.** *The Bockstein map $\beta$ is induced by the map*

$$U_{L,S,T} \to \mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2$$

*given by* $a \mapsto \sum_{w \in S_L} w \otimes (\mathrm{rec}_w(a) - 1)$.

*Proof.* The proof is the same as for [Flach 2004, Lemma 5.8] and we sketch the proof therein.

Take $n$ so that the image of $\gamma^{p^n} \in \mathrm{Gal}(K_\infty/L)$ in $\mathrm{Gal}(L_\infty/L) = \Gamma_\chi$ is a generator. We regard $\gamma^{p^n} \in \Gamma_\chi$. Define $\theta \in H^1(L, \mathbb{Z}_p) = \mathrm{Hom}(G_L, \mathbb{Z}_p)$ by $\gamma^{p^n} \mapsto 1$. Define

$$\beta' : e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} \to e_\chi \mathbb{Q}_p(\chi)(\mathcal{X}_{L,S} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \xrightarrow{\sim} e_\chi \mathbb{Q}_p(\chi) \mathcal{X}_{L,S}$$

by $\beta(a) = \beta'(a) \otimes (\gamma^{p^n} - 1)$. Then, $\beta'$ is induced by the cup product

$$\cdot \cup \theta : \mathbb{Q}_p U_{L,S} \simeq H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \to H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p \mathcal{X}_{L,S \setminus S_\infty}.$$

By class field theory we see that $\beta$ is induced by the map

$$a \mapsto \sum_{w \in S_L \setminus S_\infty(L)} w \otimes (\mathrm{rec}_w(a) - 1).$$

Since $\mathrm{rec}_w(a) = 1 \in \Gamma_\chi$ for all $w \in S_\infty(L)$, the proposition follows. □

**Proposition 5.15.** *We have canonical isomorphisms*

$$\ker \beta \simeq H^0(C)_E \quad \textit{and} \quad \mathrm{coker}\, \beta \simeq H^1(C)_{\mathrm{tf}} \otimes_R P/P^2.$$

*Proof.* Let $\delta$ be the boundary map $H^0(C_E) \to H^1(C \otimes_R P) = H^1(C) \otimes_R P$. We have

$$\ker \delta \simeq \operatorname{coker}(H^0(C \otimes_R P) \to H^0(C)) = H^0(C)_E$$

and

$$\operatorname{im} \delta = \ker(H^1(C) \otimes_R P \to H^1(C)) = H^1(C)[P] \otimes_R P,$$

where $H^1(C)[P]$ is the submodule of $H^1(C)$ which is annihilated by $P$. By Lemma 5.12(iii), we know $H^1(C)[P] = H^1(C)_{\mathrm{tors}}$. Hence, the natural map

$$H^1(C) \otimes_R P \to H^1(C) \otimes_R P/P^2 \simeq H^1(C)_E \otimes_E P/P^2 \simeq H^1(C_E) \otimes_E P/P^2$$

is injective on $H^1(C)_{\mathrm{tors}} \otimes_R P$. From this we see that $\ker \beta \simeq H^0(C)_E$. We also have

$$\operatorname{coker} \beta \simeq \operatorname{coker}(H^1(C)_{\mathrm{tors}} \otimes_R P \to H^1(C) \otimes_R P/P^2) \simeq H^1(C)_{\mathrm{tf}} \otimes_R P/P^2.$$

Hence we have completed the proof. □

By Lemma 5.12, we see that there are canonical isomorphisms

$$\begin{aligned} H^0(C)_E &\simeq U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi), \\ H^1(C)_E &\simeq \mathcal{X}_{K_\infty,S} \otimes_\Lambda \mathbb{Q}_p(\chi), \\ H^1(C)_{\mathrm{tf},E} &\simeq \mathcal{Y}_{K_\infty,V} \otimes_\Lambda \mathbb{Q}_p(\chi). \end{aligned}$$

Hence, by Proposition 5.15, we have the exact sequence

$$\begin{aligned} 0 \to U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi) &\to e_\chi \mathbb{Q}_p(\chi) U_{L,S,T} \\ &\xrightarrow{\beta} e_\chi \mathbb{Q}_p(\chi)(\mathcal{Y}_{L,V'} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)/I(\Gamma_\chi)^2) \to \mathcal{Y}_{K_\infty,V} \otimes_\Lambda P/P^2 \to 0. \end{aligned}$$

This induces an isomorphism

$$\begin{aligned} \tilde{\beta} : e_\chi \mathbb{Q}_p(\chi)\big({\textstyle\bigwedge}^{r'} U_{L,S,T} \otimes {\textstyle\bigwedge}^{r'} \mathcal{Y}^*_{L,V'}\big) \\ \xrightarrow{\sim} {\textstyle\bigwedge}^{r} (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes {\textstyle\bigwedge}^{r} (\mathcal{Y}^*_{K_\infty,V} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1}. \end{aligned}$$

We have isomorphisms

$$\begin{aligned} {\textstyle\bigwedge}^{r'} \mathcal{Y}^*_{L,V'} &\xrightarrow{\sim} \mathbb{Z}_p[G_\chi], \quad w_1^* \wedge \cdots \wedge w_{r'}^* \mapsto 1, \\ {\textstyle\bigwedge}^{r} (\mathcal{Y}^*_{K_\infty,V} \otimes_\Lambda \mathbb{Q}_p(\chi)) &\xrightarrow{\sim} \mathbb{Q}_p(\chi), \qquad w_1^* \wedge \cdots \wedge w_r^* \mapsto 1. \end{aligned}$$

By these isomorphisms, we see that $\tilde{\beta}$ induces an isomorphism

$$e_\chi \mathbb{Q}_p(\chi) {\textstyle\bigwedge}^{r'} U_{L,S,T} \xrightarrow{\sim} {\textstyle\bigwedge}^{r} (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1},$$

which we denote also by $\tilde{\beta}$. Note that we have a natural injection

$${\textstyle\bigwedge}^{r} (U_{K_\infty,S,T} \otimes_\Lambda \mathbb{Q}_p(\chi)) \otimes P^e/P^{e+1} \hookrightarrow e_\chi \mathbb{Q}_p(\chi)\big({\textstyle\bigwedge}^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e/I(\Gamma_\chi)^{e+1}\big).$$

Composing this with $\tilde{\beta}$, we have an injection

$$\tilde{\beta} : e_\chi \mathbb{Q}_p(\chi) {\textstyle\bigwedge}^{r'} U_{L,S,T} \hookrightarrow e_\chi \mathbb{Q}_p(\chi) \big( {\textstyle\bigwedge}^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e / I(\Gamma_\chi)^{e+1} \big).$$

By Proposition 5.14, we obtain the following.

**Proposition 5.16.** *Let*

$$\mathrm{Rec}_\infty : \mathbb{C}_p {\textstyle\bigwedge}^{r'} U_{L,S,T} \to \mathbb{C}_p \big( {\textstyle\bigwedge}^{r} U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_\chi)^e / I(\Gamma_\chi)^{e+1} \big)$$

*be the map defined in Section 4A. Then we have*

$$(-1)^{re} e_\chi \mathrm{Rec}_\infty = \tilde{\beta}.$$

*In particular, $e_\chi \mathrm{Rec}_\infty$ is injective.*

**5C.** ***The proof of the main result.*** In this section we prove Theorem 5.2.

We start with an important technical observation. Let $\Pi_n$ denote the free $\mathbb{Z}_p[\mathcal{G}_{\chi,n}]$-module $\Pi_{K_\infty} \otimes_\Lambda \mathbb{Z}_p[\mathcal{G}_{\chi,n}]$, and $I(\Gamma_{\chi,n})$ denote the augmentation ideal of $\mathbb{Z}_p[\Gamma_{\chi,n}]$.

We recall from [BKS, Lemma 5.20] that the image of

$$\pi^V_{L_n/k,S,T} : \det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_n,S,T}) \to {\textstyle\bigwedge}^r \Pi_n$$

is contained in $I(\Gamma_{\chi,n})^e \cdot {\textstyle\bigwedge}^r \Pi_n$ (see Proposition 2.6(iii)) and also from [BKS, Proposition 4.17] that $\nu_n^{-1} \circ \mathcal{N}_n$ induces the map

$$I(\Gamma_{\chi,n})^e \cdot {\textstyle\bigwedge}^r \Pi_n \to {\textstyle\bigwedge}^r \Pi_0 \otimes_{\mathbb{Z}_p} I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1}.$$

**Lemma 5.17.** *There exists a commutative diagram*

$$\begin{tikzcd}
\det_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_n,S,T}) \arrow[r] \arrow[d, "\pi^V_{L_n/k,S,T}"'] & \det_{\mathbb{Z}_p[G_\chi]}(C_{L,S,T}) \arrow[d, "\pi^{V'}_{L/k,S,T}"] \\
I(\Gamma_{\chi,n})^e \cdot {\textstyle\bigwedge}^r \Pi_n \arrow[d, "\nu_n^{-1} \circ \mathcal{N}_n"'] & {\textstyle\bigcap}^{r'} U_{L,S,T} \arrow[d, "(-1)^{re}\mathrm{Rec}_n"] \\
{\textstyle\bigwedge}^r \Pi_0 \otimes_{\mathbb{Z}_p} I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1} & {\textstyle\bigcap}^{r} U_{L,S,T} \otimes_{\mathbb{Z}} I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1}. \arrow[l, hook', "\supset"']
\end{tikzcd}$$

*Proof.* This follows from Proposition 2.6(iii) and [BKS, Lemma 5.22]. $\square$

For any intermediate field $F$ of $K_\infty/k$, we denote by $\mathcal{L}_{F/k,S,T}$ the image of the (conjectured) element $\mathcal{L}_{K_\infty/k,S,T}$ of $\det_\Lambda(C_{K_\infty,S,T})$ under the isomorphism

$$\mathbb{Z}_p[[\mathrm{Gal}(F/k)]] \otimes_\Lambda \det_\Lambda(C_{K_\infty,S,T}) \simeq \det_{\mathbb{Z}_p[[\mathrm{Gal}(F/k)]]}(C_{F,S,T}).$$

Note that we have

$$\pi^V_{L_n/k,S,T}(\mathcal{L}_{L_n/k,S,T}) = \epsilon^V_{L_n/k,S,T}.$$

Hence, Lemma 5.17 implies that

$$(-1)^{re}\mathrm{Rec}_n(\pi^{V'}_{L/k,S,T}(\mathcal{L}_{L/k,S,T})) = \nu_n^{-1} \circ \mathcal{N}_n(\epsilon^{V}_{L_n/k,S,T}) =: \kappa_n.$$

We set

$$\kappa := (\kappa_n)_n \in \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_{\chi,n})^e / I(\Gamma_{\chi,n})^{e+1}.$$

Then the validity of Conjecture MRS($K_\infty/k$, $S$, $T$, $\chi$, $V'$) implies that

$$e_\chi \kappa = (-1)^{re} e_\chi \mathrm{Rec}_\infty(\epsilon^{V'}_{L/k,S,T}).$$

In addition, by Proposition 5.16, we know that $e_\chi \mathrm{Rec}_\infty$ is injective, and so

$$\pi^{V'}_{L/k,S,T}(e_\chi \mathcal{L}_{L/k,S,T}) = e_\chi \epsilon^{V'}_{L/k,S,T}.$$

Hence, by Proposition 2.5, we see that eTNC($h^0(\operatorname{Spec} L)$, $\mathbb{Z}_p[G]$) is valid, as claimed.

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# Standard conjecture of Künneth type with torsion coefficients

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A. Venkatesh raised the following question, in the context of torsion automorphic forms: can the mod $p$ analogue of Grothendieck's standard conjecture of Künneth type be true (especially for compact Shimura varieties)? In the first theorem of this article, by using a topological obstruction involving Bockstein, we show that the answer is in the negative and exhibit various counterexamples, including compact Shimura varieties.

It remains an open geometric question whether the conjecture can fail for varieties with torsion-free integral cohomology. Turning to the case of abelian varieties, we give upper bounds (in $p$) for possible failures, using endomorphisms, the Hodge–Lefschetz operators, and invariant theory.

The Schottky problem enters into consideration, and we find that, for the Jacobians of curves, the question of Venkatesh has an affirmative answer for every prime number $p$.

## 1. Introduction

Let $X$ be a complex smooth projective variety of dimension $n$. Denote by $\mathrm{CH}^j(X)$ the Chow group of codimension-$j$ cycles on $X$, and by

$$\mathrm{cl}^j_{X,\mathbb{Q}} : \mathrm{CH}^j(X) \to H^{2j}(X, \mathbb{Z}) \to H^{2j}(X, \mathbb{Q})$$

the cycle class map into the Betti cohomology of $X$ with $\mathbb{Q}$-coefficients.

Poincaré duality lets $H^{2n}(X \times X, \mathbb{Q})$ act linearly on $H^*(X, \mathbb{Q})$ via correspondences: namely, $z \in H^{2n}(X \times X, \mathbb{Q})$ acts as

$$H^*(X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^*(X \times X, \mathbb{Q}) \xrightarrow{\cup z} H^{*+2n}(X \times X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_{2,*}} H^*(X, \mathbb{Q}),$$

where the duality is used in the definition of the Gysin map $\mathrm{pr}_{2,*}$.

For each integer $i$, we have the (rational) Künneth projector

$$\pi^i_{X,\mathbb{Q}} \in H^{2n}(X \times X, \mathbb{Q}),$$

which acts as 1 on $H^i(X, \mathbb{Q})$ and as 0 on $H^j(X, \mathbb{Q})$ for all $j \neq i$.

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**Conjecture 1.1** (Grothendieck [Kleiman 1968]). *The Künneth projectors $\pi^i_{X,\mathbb{Q}}$ are algebraic*:

$$\pi^i_{X,\mathbb{Q}} \in \mathrm{Im}(\mathrm{cl}^n_{X\times X,\mathbb{Q}}) \otimes \mathbb{Q} \quad \textit{for all } i.$$

The conjecture has long been known for flag varieties, for abelian varieties, and in dimension $\leq 2$.[1] For more recent progress on the conjecture, we also refer the reader to [Arapura 2006; Charles and Markman 2013; Tankeev 2011; 2015].

A. Venkatesh asked the present author whether the analogous statement could hold with $\mathbb{F}_p$-coefficients: we still have Poincaré duality for $\mathbb{F}_p$-coefficients, hence the action of $H^{2n}(X \times X, \mathbb{F}_p)$ on $H^*(X, \mathbb{F}_p)$, as well as the (mod $p$) Künneth projectors $\pi^i_{X,\mathbb{F}_p} \in H^{2n}(X \times X, \mathbb{F}_p)$. We also have the mod $p$ cycle class map

$$\mathrm{cl}^j_{X,\mathbb{F}_p} : \mathrm{CH}^j(X)/p \to H^{2j}(X, \mathbb{F}_p).$$

**Question 1.2** (Venkatesh). *Are the mod p Künneth projectors algebraic*? *That is*,

$$\pi^i_{X,\mathbb{F}_p} \in \mathrm{Im}(\mathrm{cl}^n_{X\times X,\mathbb{F}_p}) \quad \textit{for all } i?$$

Two aspects of the question got us interested. First, of course, the torsion invariants have long been of interest not only in algebraic topology, but also in algebraic geometry. For the latter, we mention just two of the most classic examples.

(A) Atiyah and Hirzebruch [1962] showed that the integral version of the Hodge conjecture is false in codimension $\geq 2$, by creating torsion cohomology classes that are not annihilated by certain Steenrod operations. (Totaro [1997] later clarified this phenomenon in terms of complex cobordisms.)

(B) In response to Lüroth's question (in birational classification of varieties) Artin and Mumford [1972] constructed unirational but irrational threefolds, by exploiting nontrivial 2-torsion in the cohomology.

Secondly, with the recent progress in the theory of automorphic forms (including Arthur's conjectures), one has seen various results towards proving the Hodge, Tate, and standard conjectures for Shimura varieties; see among others [Bergeron et al. 2016; Morel and Suh 2016]. In particular, while the standard conjecture of Künneth type remains open, *the standard sign conjecture of Jannsen*, which (only) asks whether the even and odd idempotents

$$\pi^+_{X,\mathbb{Q}} = \sum_{2\mid i} \pi^i_{X,\mathbb{Q}} \quad \text{and} \quad \pi^-_{X,\mathbb{Q}} = \sum_{2\nmid i} \pi^i_{X,\mathbb{Q}}$$

are algebraic, has been proved for many Shimura varieties, which has Tannakian consequences for homological motives.

[1] The conjecture is also known for the $\ell$-adic and crystalline cohomology theories over finite fields, thanks to Katz and Messing [1974], whose proof relies on Deligne's proof of the Weil conjectures.

Parallel to this development, there have also been striking progress of late in the theory of torsion automorphic forms (see among others [Scholze 2015; Treumann and Venkatesh 2016]) and some speculations on the extent to which one may have an analogue of Arthur's conjectures [Emerton and Gee 2015]. One wonders how far one could push the theory and obtain geometric results for Shimura varieties.

The first main goal of this article is to give a topological criterion for the question.

**Theorem 1.3.** (1) *If the integral cohomology $H^*(X, \mathbb{Z})$ has nontrivial $p$-torsion, then Venkatesh's question has a negative answer for $X$.*

(2) *For any integers $i > 0$ and $n \geq i+1$, there exists a projective smooth variety $X$ of dimension $n$ such that $\pi^i_{X,\mathbb{F}_p}$ is* not *in the image of the mod $p$ cycle class map.*

This class of examples includes Godeaux–Serre varieties and Shimura varieties, as well as Enriques surfaces and the Artin–Mumford threefolds mentioned above. In the case of Shimura surfaces, we show that even (the analogue of) the *sign* conjecture with mod $p$ coefficients can fail.

The next natural question, which is somewhat orthogonal in motivation to the original question, is if such failure can happen to varieties with *torsion-free* integral cohomology, such as hypersurfaces (in the direction of Griffiths–Harris conjectures [1985]) or abelian varieties. That is, whether the *integral* analogue of the standard conjecture of Künneth type can fail to hold. We define a natural measure of possible failure — Künneth defect — and note some first properties, but at the moment do not know the answer to the question.

In the final section, we look into the question in the case of abelian varieties and obtain upper bounds for the Künneth defects. Among the results we obtain is:

**Theorem 1.4.** *Venkatesh's question has an affirmative answer for all primes $p$, if $X$ is the Jacobian of a curve.*

This leads us to put forth:

**Conjecture 1.5.** *The standard conjecture of Künneth type* fails *to hold integrally, for a very general principally polarized abelian variety $X$ of dimension $\geq 4$.*

A more ambitious conjecture would be: the integral version fails once $X$ is outside the closure of the Schottky locus. More quantitatively, we relate this conjecture to the Prym–Tyurin theory; see Section 4.2 and Question 4.2.7 below.

***Notation.*** For an abelian group $M$ and an integer $n$, we use the notation

$$M[n] = \{m \in M : n \cdot m = 0\} \quad \text{and} \quad M/n = M/nM = M \otimes \mathbb{Z}/n.$$

If $p$ is a prime number, we denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at $(p)$, and write

$$M[p^\infty] = \bigcup_{n \geq 1} M[p^n].$$

## 2. Standard conjecture of Künneth type with torsion coefficients

**2.1.** ***Topological obstruction: Bockstein.*** The first obstruction to Question 1.2 that we will use in this section relies on the simple fact that the mod $p$ cycle class map factors through the integral cohomology mod $p$:

$$\begin{tikzcd} \mathrm{CH}^j(X) \arrow[r, "\mathrm{cl}"] \arrow[d] & H^{2j}(X, \mathbb{Z}) \arrow[d] \\ \mathrm{CH}^j(X)/p \arrow[r, "\mathrm{cl}_{\mathbb{F}_p}"] & H^{2j}(X, \mathbb{F}_p) \end{tikzcd}$$

As such, we look at the obstruction to lifting $\pi^i_{X,\mathbb{F}_p}$ to an element in $H^{2n}(X \times X, \mathbb{Z})$. This naturally leads us to look at the version of the Bockstein homomorphism $\beta$

$$0 \to H^j(X, \mathbb{Z})/p \to H^j(X, \mathbb{F}_p) \xrightarrow{\beta} H^{j+1}(X, \mathbb{Z})[p] \to 0, \tag{2.1.1}$$

which is the connecting homomorphism for the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{F}_p \to 0.$$

**Definition 2.1.1.** Let $X$ be a complex projective smooth variety and $p$ a prime number. We define *the initial incidence of nontrivial $p$-torsion*, and denote it $i_p(X)$, as the smallest integer $i$ such that the natural reduction map

$$H^i(X, \mathbb{Z}) \to H^i(X, \mathbb{F}_p)$$

is *not* surjective, in other words, the Bockstein homomorphism

$$H^i(X, \mathbb{F}_p) \to H^{i+1}(X, \mathbb{Z})[p]$$

is nonzero. If no such $i$ exists, we set $i_p(X) = \infty$.

Equivalently, in view of the exact sequence (2.1.1), we have $i_p(X) = j - 1$, where $j$ is the smallest integer such that $H^j(X, \mathbb{Z})$ has nontrivial $p$-torsion. We will show in Section 2.3 that, for any prime $p$ and any integer $i \geq 1$, there exists a projective smooth variety $X$ with $i_p(X) = i$.

**Proposition 2.1.2.** *If $i_p(X) < \infty$, then $i_p(X) \leq \dim X - 1$.*

*Proof.* Let $n = \dim X$. By definition, we have

$$\dim_{\mathbb{Q}} H^{i_p(X)}(X, \mathbb{Q}) < \dim_{\mathbb{F}_p} H^{i_p(X)}(X, \mathbb{F}_p),$$

so by Poincaré duality with field coefficients, we get $i_p(X) \leq n$. If $i_p(X)$ were equal to $n$, then we would have

$$\dim_{\mathbb{F}_p} H^j(X, \mathbb{F}_p) = \dim_{\mathbb{Q}} H^j(X, \mathbb{Q})$$

for all $j < n$, and hence also for all $j > n$, again by Poincaré duality. Then the jump

$$\dim_{\mathbb{F}_p} H^n(X, \mathbb{F}_p) > \dim_{\mathbb{Q}} H^n(X, \mathbb{Q})$$

in the middle degree would contradict the equality of the Euler characteristics of $X$ with $\mathbb{F}_p$- and $\mathbb{Q}$-coefficients. $\square$

**Theorem 2.1.3.** *Let $X$ be a complex smooth projective variety of dimension $n$ and $p$ a prime number. Suppose that $i := i_p(X) < \infty$. Then the Bockstein of the three idempotents*

$$\pi^i_{X,\mathbb{F}_p}, \quad \pi^{2n-i}_{X,\mathbb{F}_p}, \quad \textit{and} \quad \pi^i_{X,\mathbb{F}_p} + \pi^{2n-i}_{X,\mathbb{F}_p}$$

*are all nonzero, and as such cannot be the cohomology class of an algebraic cycle.*

*Proof.* First we show that $\beta_{X\times X}(\pi^i_{X,\mathbb{F}_p}) \neq 0$, where $\beta_{X\times X}$ is the Bockstein for $X \times X$.

**Lemma 2.1.4.** *With the notation as above, the natural map*

$$H^{2n-i}(X, \mathbb{Z}) \to H^{2n-i}(X, \mathbb{F}_p)$$

*is surjective.*

*Proof.* Equivalently, we show that $H^{2n-i+1}(X, \mathbb{Z})$ is free of $p$-torsion. By Poincaré duality, we have

$$H^{2n-i+1}(X, \mathbb{Z}) \simeq H_{i-1}(X, \mathbb{Z})$$

and the universal coefficient theorem provides us with an exact sequence

$$0 \to \operatorname{Ext}^1(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to \operatorname{Hom}(H_i(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

Now if $H_{i-1}(X, \mathbb{Z})$ had nontrivial $p$-torsion, then so would $H^i(X, \mathbb{Z})$, which would contradict the alternate characterization of $i_p(X)$ given just after the definition. $\square$

*Step 1* (the $p$-primary torsion parts, Poincaré dual bases, and Pontryagin dual generators). Denote the $p$-primary torsion part by

$$T := H_i(X, \mathbb{Z})[p^\infty] \simeq H^{2n-i}(X, \mathbb{Z})[p^\infty]$$

(isomorphism via Poincaré duality with integral coefficients), hence giving us a natural exact sequence (thanks to Lemma 2.1.4)

$$0 \to T/p \to H^{2n-i}(X, \mathbb{F}_p) \to H^{2n-i}(X, \mathbb{Z}_{(p)})^{\mathrm{fr}}/p \to 0 \tag{a}$$

(where $(\,\cdot\,)^{\mathrm{fr}}$ denotes the maximal $\mathbb{Z}_{(p)}$-free quotient). Then choose generators

$$T = \bigoplus_{k=1}^{r} (\mathbb{Z}/p^{m_k}) t_k.$$

By the universal coefficient theorem

$$0 \to \operatorname{Ext}^1(H_{2n-i-1}(X,\mathbb{Z}), \mathbb{Z}_{(p)}) \to H^{2n-i}(X, \mathbb{Z}_{(p)}) \to \operatorname{Hom}(H_{2n-i}(X,\mathbb{Z}), \mathbb{Z}_{(p)}) \to 0$$

we get a natural isomorphism

$$T \simeq \mathrm{Ext}^1(H_{2n-i-1}(X,\mathbb{Z}),\mathbb{Z}_{(p)}) \simeq \mathrm{Hom}(H^{i+1}(X,\mathbb{Z})[p^\infty],\mathbb{Q}/\mathbb{Z}),$$

in other words,

$$H^{i+1}(X,\mathbb{Z})[p^\infty] \simeq T^\vee,$$

the Pontryagin dual of $T$. Let $t_k^\vee \in T^\vee$ denote the Pontryagin dual generators:

$$t_k^\vee(t_j)=0 \quad \text{if } j\neq k \quad \text{while } t_k^\vee(t_k)=1/p^{m_k}+\mathbb{Z}.$$

It follows that the $u_k := p^{m_k-1}t_k^\vee$ form a basis of $T^\vee[p]$,

$$T^\vee[p]=\bigoplus_{k=1}^r \mathbb{F}_p u_k,$$

and the Bockstein short exact sequence reads

$$0\to H^i(X,\mathbb{Z})/p \to H^i(X,\mathbb{F}_p) \xrightarrow{\beta} T^\vee[p]\to 0. \tag{b}$$

The exact sequences (a) and (b) are Poincaré dual to each other, via the universal coefficient theorem.

Denote the mod $p$ Poincaré pairing by

$$\langle\cdot,\cdot\rangle : H^{2n-i}(X,\mathbb{F}_p)\times H^i(X,\mathbb{F}_p)\to \mathbb{F}_p.$$

We can then choose

(1) a preimage $\{\widetilde{u_k}\in H^i(X,\mathbb{F}_p)\}_k$ of $\{u_k\}_{k=1,\dots,r}$ under $\beta$,

(2) a lift $\{f_\ell \in H^{2n-i}(X,\mathbb{Z}_{(p)})\}_\ell$ of a $\mathbb{Z}_{(p)}$-basis of the free quotient $H^{2n-i}(X,\mathbb{Z}_{(p)})^{\mathrm{fr}}$, and

(3) a $\mathbb{Z}_{(p)}$-basis $\{f'_\ell\}$ of $H^i(X,\mathbb{Z}_{(p)})$ (note that this last group is free over $\mathbb{Z}_{(p)}$),

such that the $\mathbb{Z}_{(p)}$-bases $\{f_\ell\}$ and $\{f'_\ell\}$ are Poincaré dual, so that

$$\langle \overline{t_k},\widetilde{u_j}\rangle=\delta_{jk},\quad \langle \overline{t_k},\overline{f'_\ell}\rangle=0,\quad \text{and}\quad \langle \overline{f_\ell},\overline{f'_m}\rangle=\delta_{\ell m}$$

(the second vanishing equation follows from the duality of the sequences (a) and (b)). Here by the notation $\bar{\xi}$ we mean the image of $\xi$ in the cohomology with $\mathbb{F}_p$-coefficients.

By modifying the $f_\ell$ by linear combinations of the $t_k$, which changes neither the fact that the $f_\ell$ lift a $\mathbb{Z}_{(p)}$-basis of the free quotient in condition (2) nor the mod $p$ intersection numbers $\langle \overline{f_\ell},\overline{f'_m}\rangle$, we may then assume in addition that

$$\langle \overline{f_\ell},\widetilde{u_k}\rangle=0,$$

that is, the bases

$$\{\bar{t}_1, \dots, \bar{t}_r;\ \bar{f}_1, \dots, \bar{f}_b\} \quad \text{and} \quad \{\widetilde{u}_1, \dots, \widetilde{u}_r;\ \bar{f}'_1, \dots, \bar{f}'_b\}$$

of $H^{2n-i}(X, \mathbb{F}_p)$ and of $H^i(X, \mathbb{F}_p)$, respectively, are Poincaré dual to each other ($b$ stands for Betti).

It then follows, from the way the action of $H^{2n}(X \times X, \mathbb{F}_p)$ on $H^*(X, \mathbb{F}_p)$ via correspondence is constructed, that

$$\pi^i_{X,\mathbb{F}_p} = \sum_{k=1}^{r} \bar{t}_k \otimes \widetilde{u}_k + \sum_{\ell=1}^{b} \bar{f}_\ell \otimes \bar{f}'_\ell. \tag{$\star$}$$

*Step 2* (integral Künneth theorem). Recall that the general Künneth theorem for integral coefficients says that we have a natural short exact sequence

$$0 \to \bigoplus_{a+b=m} H^a(X, \mathbb{Z}) \otimes H^b(Y, \mathbb{Z}) \to H^m(X \times Y, \mathbb{Z}) \to \\ \to \bigoplus_{a+b=m+1} \mathrm{Tor}_1(H^a(X, \mathbb{Z}), H^b(Y, \mathbb{Z})) \to 0$$

[Cartan and Eilenberg 1956, Chapter VI, Theorem 3.1]. Applied to our situation, $X = Y$, it gives us a natural *injection*

$$(H^{2n-i}(X, \mathbb{Z}) \otimes H^{i+1}(X, \mathbb{Z}))[p] \hookrightarrow H^{2n+1}(X \times X, \mathbb{Z})[p].$$

Thanks to ($\star$), the Bockstein of $\pi^i_{X,\mathbb{F}_p}$ belongs to this smaller subspace

$$\beta_{X\times X}(\pi^i_{X,\mathbb{F}_p}) = (-1)^{2n-i} \sum_{k=1}^{r} t_k \otimes u_k = (-1)^{2n-i} \sum_{k=1}^{r} p^{m_k - 1} t_k \otimes t_k^\vee;$$

here we are using the fact that the Bockstein homomorphism is a graded derivation on cohomology, that it annihilates elements obtained by reduction mod $p$, and that by construction $\beta(\widetilde{u}_k) = u_k$.

In fact, this belongs to a smaller subspace

$$(H^{2n-i}(X, \mathbb{Z}) \otimes H^{i+1}(X, \mathbb{Z}))[p] \supseteq (T \otimes T^\vee)[p],$$

which has basis

$$\{p^{\min(m_j, m_k)-1} t_i \otimes t_k^\vee\}, \quad \text{where } 1 \le j, k \le r,$$

because $(\mathbb{Z}/p^a) \otimes (\mathbb{Z}/p^b) \simeq \mathbb{Z}/p^a$ if $a \le b$. That is, up to sign, $\beta_{X\times X}(\pi^i_{X,\mathbb{F}_p})$ is the sum of the "diagonal" (that is, $j = k$) basis elements in $(T \otimes T^\vee)[p]$, and is nonzero because by assumption $T \neq 0$ and $r \ge 1$.

*Step 3* (the other projectors). The proof for $\pi^{2n-i}_{X,\mathbb{F}_p}$ is essentially the same, with the two factors of $X \times X$ exchanging the roles. For the last idempotent, note that the Bocksteins of the two summands reside in distinct Künneth factors:

$$\beta_{X\times X}(\pi^{i}_{X,\mathbb{F}_p}) \in (H^{2n-i}(X,\mathbb{Z}) \otimes H^{i+1}(X,\mathbb{Z}))[p], \quad \text{while}$$
$$\beta_{X\times X}(\pi^{2n-i}_{X,\mathbb{F}_p}) \in (H^{i+1}(X,\mathbb{Z}) \otimes H^{2n-i}(X,\mathbb{Z}))[p]$$

($i+1$ and $2n-i$ have different parity). □

**2.2.** ***Shimura varieties with $i_p(X) = 1$.*** This section is a generalization of [Suh 2008, §3]. For the purpose of this article, it is enough to consider connected Shimura data $(G, \mathscr{X})$, where $G$ is a simple algebraic group over $\mathbb{Q}$.

**Proposition 2.2.1.** *Assume that $G$ is $\mathbb{Q}$-anisotropic so that the resulting Shimura varieties*

$$X_\Gamma = \Gamma \backslash \mathscr{X}$$

*are compact, and that the congruence subgroups $\Gamma \subset G(\mathbb{Q})$ used in the definition satisfy the condition*

$$\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma] < \infty. \tag{2.2.1}$$

*Then for any prime number $p$, there exist torsion-free levels $\Gamma$ such that $i_p(X_\Gamma) = 1$.*

**Remark 2.2.2.** The condition (2.2.1) is satisfied whenever $G$ has real rank at least 2 (see, e.g., [Margulis 1991, Theorem IV.4.9]) and also in the case of certain special unitary groups of real rank 1 [Rogawski 1990, Theorem 15.3.1]. (But it clearly rules out modular and Shimura curves as it must, as well as certain special unitary groups of real rank 1 [Kazhdan 1977].)

*Proof.* The condition (2.2.1) implies that

$$H^1(X_\Gamma, \mathbb{Z}) = 0.$$

On the other hand, for any fixed prime number $p$, one can always find lattices $\Gamma \subset G(\mathbb{Q})$ such that

$$H^1(X_\Gamma, \mathbb{F}_p) \neq 0;$$

the point is as follows. Let $S$ be a finite set of prime numbers such that $G$ has a good model over $\mathbb{Z}[S^{-1}]$, and let $F$ be a number field over which $G$ splits. Then by the Chebotarev density theorem, there are infinitely many prime numbers $\ell$ such that (i) $\ell \notin S$, (ii) $\ell$ splits completely in $F$, and (iii) $\ell \equiv 1 \pmod{p}$. Then, since $G \otimes \mathbb{F}_\ell$ contains a split torus of positive dimension, $G(\mathbb{F}_\ell)$ contains a nontrivial abelian $p$-subgroup, say $H \subseteq G(\mathbb{F}_\ell)$.

Now starting with a torsion-free level subgroup $\Gamma$, we can find a prime $\ell$ which is prime to $\Gamma$ and satisfying the three conditions above. By first raising the level by a full mod $\ell$ level, and then lowering it by the abelian $p$-subgroup $H$, we arrive at a new level $\Gamma'$ with

$$\pi_1(X_{\Gamma'})^{\mathrm{ab}}/p \neq 0, \quad \text{hence} \quad H^1(X_{\Gamma'}, \mathbb{F}_p) \neq 0.$$

Therefore, we get examples with $i_p(X_{\Gamma'}) = 1$. □

**Remark 2.2.3.** The groups $H$ used in the proof are those $p$-subgroups that play a central role in the detection and the computation of the group cohomology of finite Lie groups, initiated by Quillen [1972]; see also [Adem and Milgram 2004] and references therein.

**Remark 2.2.4.** This way we also get Shimura surfaces with $i_p(X) = 1$, and Theorem 2.1.3 in this case shows that even the analogue of the *sign* conjecture for $\mathbb{F}_p$-coefficients is false.

**Remark 2.2.5.** For PEL-type or quaternionic Shimura varieties, when $p$ is a "good" prime (with respect to the Shimura data) not dividing the level $\Gamma$, then the Shimura varieties have good reduction at $p$, thanks to Kottwitz and Reimann. This way we also get nontrivial $p$-torsion in the *coherent* and *de Rham* cohomology of the integral model at $p$ of the Shimura varieties. For details, see [Suh 2008, §3].

**Remark 2.2.6** (Emerton). The resulting unliftable cohomology classes in $H^1(X, \mathbb{F}_p)$, and hence the nontrivial $p$-torsion classes in $H^2(X, \mathbb{Z})[p]$ via Bockstein homomorphism, are Eisenstein (in the sense that the associated Galois representations are completely reducible) [Emerton and Gee 2015, Remark 3.4.6]. The point is that, for any prime that is coprime to the level

$$\ker(\phi : \Gamma' \to \mathbb{F}_p),$$

the corresponding Hecke operator annihilates the classes. Indeed, already the pullback of the class is zero, and hence, the ensuing pushforward (under a different finite covering) is necessarily zero.

### 2.3. *Varieties with any prescribed $i_p(X) \geq 1$.*

**Theorem 2.3.1.** *Let $p$ be any prime number, and $1 \leq i < n$ be two integers. Then there exists a complex projective smooth variety $X$ of dimension $n$ such that $i_p(X) = i$.*

*Moreover, we may assume one of the following two conditions on $X$.*

(1) *For any $n > i$, we can find $X$ with ample canonical bundle.*

(2) *If $i \geq 3$ and $n \geq 2i - 1$, then we can find $X$ that is rational.*

Our proof consists of two parts: group cohomology (the tools we use can be found, e.g., in [Adem and Milgram 2004]) and projective geometry.

**Lemma 2.3.2.** *For any prime number $p$, there exists a finite group $G$ such that*

$$H^1(G, \mathbb{F}_p) = 0 \quad \textit{and} \quad \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) \geq 2.$$

*It follows that*

$$p \nmid |H^2(G, \mathbb{Z})| \quad \textit{and} \quad \dim_{\mathbb{F}_p} H^3(G, \mathbb{Z})[p] \geq 2.$$

*Proof.* Consider the 2-dimensional vector space $V = \mathbb{F}_p^{\oplus 2}$. For odd $p$, let $D := \mathbb{F}_p^\times$ be the units in $\mathbb{F}_p$, and let $\alpha \in D$ act on $V$ as the diagonal matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

while for $p = 2$, let a chosen generator $\alpha$ of $D := \mathbb{Z}/3$ act on $V$ as the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(One can regard $V$ as $\mathbb{F}_4$ and $D$ as $\mathbb{F}_4^\times$.) Let $V' = V$ be another copy of the same representation of $D$, and take the corresponding semidirect product

$$G := (V \oplus V') \rtimes D.$$

For $p$ odd, the cohomology algebra of $V$ (regarded as a finite group) with coefficients in $\mathbb{F}_p$ is the tensor product of a polynomial algebra and an exterior algebra [Adem and Milgram 2004, Corollary II.4.3]:

$$H^*(V, \mathbb{F}_p) \simeq \mathbb{F}_p[x_2, y_2] \otimes \bigwedge(e_1, f_1),$$

where the subscripts mark the degree. Then $D$ acts as the identity character on the eigenspaces $\mathbb{F}_p x_2$ and $\mathbb{F}_p e_1$ and as the inverse character on $\mathbb{F}_p y_2$ and $\mathbb{F}_p f_1$. By the Künneth formula in group cohomology, we also have

$$H^*(V \oplus V', \mathbb{F}_p) \simeq \mathbb{F}_p[x_2, y_2, x_2', y_2'] \otimes \bigwedge(e_1, f_1, e_1', f_1'),$$

with a similar description of the action of $D$.

For $p = 2$, the cohomology algebra is a polynomial algebra [Adem and Milgram 2004, Theorem II.4.4]:

$$H^*(V, \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, y_1],$$

on which the chosen generator $\alpha$ of $D$ acts as the matrix above, on the basis $x_1, y_1$. Similarly

$$H^*(V \oplus V', \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, y_1, x_1', y_1'].$$

Since $D$ has order prime to $p$, the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{ab} = H^a(D, H^b(V \oplus V', \mathbb{F}_p)) \Rightarrow H^{a+b}(G, \mathbb{F}_p)$$

degenerates in $E_2$ and gives

$$H^*(G, \mathbb{F}_p) \simeq H^*(V \oplus V', \mathbb{F}_p)^D.$$

One can then show that there are no nonzero $D$-invariants of degree 1.

For $p$ odd, all the $D$-invariants in $H^2(V \oplus V', \mathbb{F}_p)$ are in the second wedge product, on which the action of $D$ is through the two characters, with multiplicity 2 each. It follows that the invariants have dimension 4 or 6, according to whether $p \geq 5$ or $p = 3$.

For $p = 2$, to compute the dimension of the $D$-invariants, first extend scalars to $\mathbb{F}_4$ so as to diagonalize the action of $D$. Then one can see that the invariants of degree 2 have dimension 4.

The last part of the lemma follows from the long exact sequence

$$\cdots \to H^i(G, \mathbb{Z}) \xrightarrow{\times p} H^i(G, \mathbb{Z}) \to H^i(G, \mathbb{F}_p) \to \cdots$$

and the fact that $H^i(G, \mathbb{Z})$ is finite for all $i > 0$. □

*Proof of Theorem 2.3.1*. *Step 1* (the cases $i = 1, 2$). By applying the Godeaux–Serre construction [Serre 1958, §20] to the group $\mathbb{Z}/p$, we get a complete intersection $Y$ of dimension $n \geq 2$ with a free action of $\mathbb{Z}/p$. Let $X$ be the quotient of $Y$ by $\mathbb{Z}/p$. Since $Y$ is simply connected, $X$ has fundamental group $\mathbb{Z}/p$, and we have

$$H^1(X, \mathbb{Z}) = 0 \quad \text{and} \quad H^1(X, \mathbb{F}_p) \simeq \mathbb{F}_p,$$

and we have $i_p(X) = 1$.

For $i = 2$, we apply the Godeaux–Serre construction to any group $G$ satisfying the conditions of Lemma 2.3.2, to obtain a complete intersection $Y$ of dimension $n \geq 3$ with a free $G$-action. Let $X = Y/G$. Because $Y$ has dimension $\geq 3$, the Lefschetz hyperplane theorem gives us

$$H^1(Y, \mathbb{Z}) = 0 \quad \text{and} \quad H^2(Y, \mathbb{Z}) \simeq \mathbb{Z},$$

on which $G$ acts trivially. Then the Serre–Hochschild spectral sequence

$$E_2^{ab} = H^a(G, H^b(Y, \mathbb{Z})) \Rightarrow H^{a+b}(X, \mathbb{Z})$$

gives us $H^1(X, \mathbb{Z}) = 0$ and then an exact sequence

$$0 \to H^2(G, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \mathbb{Z} \to H^3(G, \mathbb{Z}).$$

It follows that there is a noncanonical isomorphism

$$H^2(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus H^2(G, \mathbb{Z}),$$

hence

$$\dim_{\mathbb{F}_p} H^2(X, \mathbb{Z}) \otimes \mathbb{F}_p = 1.$$

In parallel, the spectral sequence applied to the mod $p$ cohomology gives us $H^1(X, \mathbb{F}_p) = 0$ and

$$0 \to H^2(G, \mathbb{F}_p) \to H^2(X, \mathbb{F}_p) \to \mathbb{F}_p \to H^3(G, \mathbb{F}_p),$$

which gives the dimension count

$$\dim_{\mathbb{F}_p} H^2(X, \mathbb{F}_p) \geq 2.$$

Therefore, $i_p(X) = 2$.

For analysis of the cohomology of Godeaux–Serre varieties, see also [Atiyah and Hirzebruch 1962, p. 42].

*Step 2* (the case $i \geq 3$). Here we use the trick of blowing up to increase $i_p$. The main point is:

**Lemma 2.3.3.** *Let $X$ be a closed smooth subvariety of a complex projective smooth variety $Y$ of codimension $c \geq 2$, and let $p$ be a prime number. Then the blow-up $Y'$ of $Y$ along $X$ satisfies*

$$i_p(Y') = \min\{i_p(X) + 2, i_p(Y)\}.$$

*In particular, if $H^*(Y, \mathbb{Z})$ is free of $p$-torsion (e.g., if $Y$ is a projective space), then $i_p(Y') = i_p(X) + 2$.*

*Proof.* This follows from the isomorphism

$$H^j(Y', \mathbb{Z}) \simeq H^j(Y, \mathbb{Z}) \oplus \bigoplus_{k=1}^{c-1} H^{j-2k}(X, \mathbb{Z}).$$

(Use the Leray spectral sequence and the computation of the cohomology of a projective bundle [Katz 1973, Théorème 2.2].) □

Now we are ready to prove Theorem 2.3.1. Start with a variety $X = X_0$ from Step 1, and repeat $\lceil i/2 - 1 \rceil$ times the procedure of embedding $X_k$ in a projective space (whose dimension can be chosen to be $\leq 2\dim(X_k) + 1$), then taking the blow-up $X_{k+1}$ of the projective space along $X_k$.

The resulting variety will in general have a large dimension. However, we can cut down the dimension without changing $i_p(X)$, by using the Lefschetz hyperplane theorem, as long as the desired dimension $n$ is at least $i_p(X) + 1$.

To get a rational example, apply the previous procedure to get $X$ of dimension $i-1$ and $i_p(X) = i - 2$. Embed it in the projective space of dimension $n \geq 2(i-1)+1$ and blow up the projective space along it. □

**Remark 2.3.4.** The numerical condition in (2) for rational examples is sharp in the sense that, for $i = 2$ and $n = 2i - 1 = 3$, there is no rational projective smooth threefold with $i_p(X) = 2$. This is the crucial observation of Artin and Mumford [1972] mentioned in the introduction.

## 3. Integral Künneth defect

Let $X$ be a complex projective smooth variety of dimension $n$ with torsion-free $H^*(X, \mathbb{Z})$. The Künneth theorem for integral coefficients says in this case

$$H^i(X \times X, \mathbb{Z}) \simeq \bigoplus_{j+k=i} H^i(X, \mathbb{Z}) \otimes H^j(X, \mathbb{Z}).$$

We begin by noting that the Künneth idempotent $\pi_X^i \in H^{2n}(X \times X, \mathbb{Q})$ in fact belongs to $H^{2n}(X \times X, \mathbb{Z})$, that in the integral cohomology group, it (if nonzero) is not divisible by any prime number, and that its reduction mod $p$ gives the mod $p$ Künneth idempotents $\pi^i_{X,\mathbb{F}_p}$.

**Definition 3.1.** The (*integral*) *Künneth defect* is defined as the index

$$\kappa_i = \kappa_{i,X} := [\mathbb{Z}\pi_X^i : \mathbb{Z}\pi_X^i \cap \mathrm{Im}(\mathrm{cl}^n_{X\times X}(\mathrm{CH}^n(X \times X)))].$$

Thus, for $X$ with torsion-free cohomology, Grothendieck's standard conjecture of Künneth type becomes the assertion $\kappa_i < \infty$ for all $i$, while the integral analogue amounts to $\kappa_i = 1$, and the mod $p$ analogue amounts to $p \nmid \kappa_i$.

**Proposition 3.2.** *Let $X$ and $Y$ be nonempty complex projective smooth varieties with torsion-free integral cohomology, and let $n = \dim X$.*

(1) $\kappa_{0,X} = \kappa_{2n,X} = 1$.

(2) $\kappa_{i,X} = \kappa_{2n-i,X}$ *for all $i$.*

(3) *Suppose that for some $i$ and some integer $m$, we have $\kappa_{j,X} \mid m$ for all $j \neq i, 2n-i$. Then $\kappa_{i,X}$ and $\kappa_{2n-i,X}$ also divide $m$.*

(4) *Suppose that $f : Y \to X$ is a generically finite covering of degree $d$. Then $\kappa_{i,X} \mid d\kappa_{i,Y}$.*

(5a) *For the product, we have the "convolution" formula*

$$\kappa_{\ell,X\times Y} \quad \textit{divides} \quad \mathrm{lcm}_{i+j=\ell}(\kappa_{i,X} \cdot \kappa_{j,Y}).$$

(5b) *Conversely, both $\kappa_{i,X}$ and $\kappa_{i,Y}$ divide $\kappa_{i,X\times Y}$.*

(6) *Let $\mathscr{X}$ be a complex projective smooth family of varieties with torsion-free cohomology over a (base) connected complex variety $S$. If $\kappa_{i,\mathscr{X}_s} \mid m$ for a very general fiber $\mathscr{X}_s$, then $\kappa_{i,\mathscr{X}_t} \mid m$ for any fiber $\mathscr{X}_t$.*

*Proof.* The fibers $\{*\} \times X$ and $X \times \{*\}$ represent $\pi_X^0$ and $\pi_X^{2n}$, respectively, hence (1). The $\mathbb{Z}/2$ symmetry on $X \times X$ gives (2). The diagonal $\Delta_X$ represents the sum of all $\pi_X^i$, from which we get (3).

The projection formula tells us that $(f\times f)_*(\pi_Y^i)=d\pi_X^i$: for any $x\in H^*(X,\mathbb{Q})$,

$$\begin{aligned}\mathrm{pr}_{X,2,*}(\mathrm{pr}_{X,1}^*(x)\cup(f\times f)_*(\pi_Y^i)) &= \mathrm{pr}_{X,2,*}(f\times f)_*((f\times f)^*\,\mathrm{pr}_{X,1}^*(x)\cup\pi_Y^i)\\ &= f_*\,\mathrm{pr}_{Y,2,*}(\mathrm{pr}_{Y,1}^*(f^*(x))\cup\pi_Y^i)\\ &= f_*f^*(x_i)=\deg(f)(x_i),\end{aligned}$$

where $x_i$ is the degree $i$ component of $x$, hence (4).

Part (5a) follows from the Künneth formula and the fact that

$$\pi_{X\times Y}^{\ell}=\sum_{i+j=\ell}\pi_X^i\otimes\pi_Y^j$$

in $H^*(X\times Y\times X\times Y,\mathbb{Q})\simeq H^*(X\times X,\mathbb{Q})\otimes H^*(Y\times Y,\mathbb{Q})$ (up to sign à la Koszul).

For part (5b), note that, for any point $*$ on $Y$, we have

$$\mathrm{pr}_{1,3,*}(\pi_{X\times Y}^i\cup[X\times\{*\}\times X\times Y])=\pi_X^i$$

as elements of $H^{2\dim X}(X\times X,\mathbb{Q})$, where $\mathrm{pr}_{1,3}:X\times Y\times X\times Y\to X\times X$ is the projection onto the product of the first and third factors.

For (6), use the fact that the Hilbert scheme of $\mathscr{X}\times_S\mathscr{X}/S$ is the countable union of proper irreducible schemes $\{\mathscr{H}_n\}_{n=1,2,\dots}$ over $S$. Subtract from $S$ all the images of those $\mathscr{H}_n$ which map onto proper subvarieties of $S$, and call the complement $U$. If $\kappa_{i,\mathscr{X}_s}\mid m$ for one $s\in U$, then by construction the algebraic cycle representing $m\cdot\pi_X^i$ specializes to any $t\in S$ in a flat family. □

It follows that the integral analogue of the standard conjecture of Künneth type is true for surfaces with $H^1(X,\mathbb{Z})=0=H^3(X,\mathbb{Z})$, e.g., K3 surfaces and complete intersection surfaces.

**Proposition 3.3.** *Let $X$ be a smooth complete intersection of hypersurfaces of degrees $d_1,\dots,d_c$ in $\mathbb{P}^{n+c}$. Then $\kappa_i=1$ for all $i$ odd, and $\kappa_i\mid d_1\cdots d_c$ for all $i$.*

*Proof.* By Proposition 3.2(2)–(3), we may prove the statements just for $i<n$. For such $i$, by the Lefschetz hyperplane theorem, $H^i(X,\mathbb{Z})$ is zero if $i$ is odd, and is generated by the cup-power $D^{\cup i/2}$ if $i$ is even, where $D$ is a hyperplane section of $X$. Since $D^{\cup n}=d_1\cdots d_c$, the cycle

$$\mathrm{pr}_1^*(D^{\cup n-i/2})\cup\mathrm{pr}_2^*(D^{\cup i/2})$$

represents $d_1\cdots d_c\pi_X^i$ for $i<n$ even. □

**Question 3.4.** *Does there exist a projective smooth variety $X$ with torsion-free cohomology such that $\kappa_{i,X}>1$ for some $i$?*

In this regard, we mention:[2]

[2]The theorem cited is stated only for $n=2$, but the argument carries over to general $n$ without much difficulty.

**Theorem 3.5** [Soulé and Voisin 2005, Theorem 3]. *Let $X$ be a projective smooth variety of dimension $n$, and let $m$ be an integer prime to* 6. *Then for a very general degree-$m^3$ hypersurface $Y$ in $\mathbb{P}^4$ and any irreducible subvariety $Z$ of $X \times Y$ of codimension* 3, *the Künneth* (2, 4) *component*

$$[Z]^{(2,4)} \in H^2(X, \mathbb{Z}) \otimes H^4(Y, \mathbb{Z})$$

*is divisible by $m$ in the group.*

While this theorem is relevant to our discussion, it does not quite answer our question, since one is not allowed to take $X = Y$. The nature of the proof requires that $X$ must precede $Y$: the "very general" condition on $Y$ is formulated in terms of the Hilbert scheme of $X \times Y$.

## 4. Abelian varieties

We now turn to investigating Question 3.4 in the case of abelian varieties.

Perhaps a few words are in order to put our results in relation to what is known in the literature. The standard conjecture of Künneth type with $\mathbb{Q}$-coefficients has long been known in the case of abelian varieties, treated as early as in [Kleiman 1968]. Part of what follows can be seen as an improvement of it with $\mathbb{Z}$-coefficients. (However, the use of higher Abel–Jacobi maps to produce requisite integral algebraic cycles seems to be new in this context.)

A remarkable, arithmetic development in a rather different direction has been seen in the consideration of the *rational* equivalence of algebraic cycles on abelian varieties (but still with $\mathbb{Q}$-coefficients), a refinement of the *homological* equivalence. (The standard conjecture, as stated, only concerns the homological equivalence.) Here a great deal of striking results have been obtained by use of the Fourier transformation. See among others [Beauville 1986; 2010; Künnemann 1993].[3]

**4.1.** ***Endomorphisms and Lieberman's trick.*** First we note that Poincaré duality is valid for compact complex tori, with or without polarization. As the cohomology is also torsion-free, we can still speak of the Künneth defects (Section 3).

**Definition 4.1.1.** Let $g \geq 1$, $0 < i < 2g$, and $a \neq -1, 0, 1$ be integers. Define the polynomial

$$P_{i,g,a}(T) := \prod_{0<j<2g \text{ and } j \neq i} \frac{T - a^j}{a^i - a^j} \in \mathbb{Q}[T],$$

[3] An explicit formula is given in [Künnemann 1993] for the projectors up to rational equivalence with $\mathbb{Q}$-coefficients. However, the denominators required there are no better than the ones given in Section 4.1 below, and not as sharp as the ones in Section 4.2 (where a principal polarization is assumed in addition).

and let $d_{i,g,a}$ be the denominator (up to sign) of $P_{i,g,a}$:

$$d_{i,g,a} = \prod_{0<j<2g \text{ and } j\neq i} (a^i - a^j).$$

**Proposition 4.1.2.** *Let $X$ be a compact complex torus of dimension $g \geq 1$. For any integers $a \neq -1, 0, 1$ and $0 < i < 2g$, we have*

$$\kappa_{i,X} \mid d_{i,g,a}.$$

*In particular, no prime $p \geq 2g$ divides any $\kappa_{i,X}$.*

*Proof.* The pullback by the multiplication map $[a] : X \to X$ acts as $a^j$ on $H^j(X, \mathbb{Z})$. The polynomial $P_{i,g,a}$ was constructed so that

$$P_{i,g,a}([a]^*) = a_{0,i}\pi_X^0 + \pi_X^i + a_{2g,i}\pi_X^{2g} \quad \text{on } H^*(X, \mathbb{Q}),$$

where $a_{0,i}$ and $a_{2g,i}$ are rational numbers whose denominators divide $d_{i,g,a}$. Because $\pi_X^0$ and $\pi_X^{2g}$ are integrally algebraic (Proposition 3.2(1)), we get the first estimate by multiplying the equation through by $d_{i,g,a}$.

For the second statement, choose $a \neq 0, \pm 1$ to be a primitive root mod $p$, so that $a, \ldots, a^{2g-1}$ are distinct mod $p$ and $p \nmid d_{i,g,a}$. $\square$

**4.2.** ***Invariant theory and Jacobians.*** From now on, we consider a principally polarized[4] abelian variety $(X, L)$ of dimension $g \geq 1$.

Recall the following fact from the classical invariant theory. Let $V$ be a $2g$-dimensional vector space over a field $k$ of characteristic zero endowed with a nondegenerate alternating pairing $\langle \cdot, \cdot \rangle$, and let $V_1$ and $V_2$ be two copies of $V$. Then the $\mathbf{Sp}_{2g}$-invariants in the exterior algebra of the dual $(V_1 \oplus V_2)^\vee$ (on which $\mathbf{Sp}_{2g}$ acts diagonally and then dually) are generated by those in degree 2:

$$\left(\bigwedge (V_1 \oplus V_2)^\vee\right)^{\mathbf{Sp}_{2g}} = k\langle E_1, E_2, M\rangle,$$

where $E_1$ and $E_2$ are the pairings $\langle \cdot, \cdot \rangle$ through $V_1$ and $V_2$, respectively, and $M$ is the pairing

$$M(u, w) := E(u_1, w_2) + E(u_2, w_2) = E(m(u), m(w)) - E_1(u, w) - E_2(u, w),$$

where $u = u_1 + u_2$ and $w = w_1 + w_2$ with $u_i, w_i \in V_i$ and $m : V_1 \oplus V_2 \to V$ is the sum map.

[4]The calculation of bounds that follows, when implemented for a general polarization $L$, would always involve the degree of $L$ in the denominator. In view of Proposition 3.2(4), we do not lose much by first passing to an isogenous abelian variety with a principal polarization.

This has the following geometric consequence [Milne 1999]. The Chern class $E \in H^2(X, \mathbb{Z})$ of the principal polarization $L$ can be regarded as a perfect alternating form on $H_1(X, \mathbb{Z})$. Then we have

$$H^*(X \times X, \mathbb{Q})^{\mathbf{Sp}_{2g}} = \mathbb{Q}\langle E_1, E_2, M\rangle,$$

where $E_i = \mathrm{pr}_i^*(E)$ and

$$M = \mu^* E - E_1 - E_2, \tag{4.2.1}$$

where $\mu : X \times X \to X$ is the group law. Since $E_1$, $E_2$, and $M$ have Künneth types $(2, 0)$, $(0, 2)$, and $(1, 1)$, respectively, it follows that there exist constants $\gamma_{i,g}(a, b, c) \in \mathbb{Q}$ such that the Künneth projectors can be expressed as

$$\pi_X^i = \sum_{(a,b,c):2a+b=2g-i \text{ and } b+2c=i} \gamma_{i,g}(a, b, c) E_1^a M^b E_2^c.$$

**Theorem 4.2.1** [Scholl 1994, §5.9]. *For all* $0 \le i \le 2g$, *we have*

$$\gamma_{i,g}(a, b, c) = \frac{(-1)^i}{a!\, b!\, c!}.$$

(We note in passing that in [loc. cit.] Scholl proves Künneth-type decomposition even for *rational* equivalence (with $\mathbb{Q}$-coefficients).)

For each index $1 \le i < g$, the sum ranges over the following triples $(a, b, c)$:

$$(g-i, i, 0), (g-i+1, i-2, 1), (g-i+2, i-4, 2), \dots, (g-i+\lfloor i/2\rfloor, 0 \text{ or } 1, \lfloor i/2\rfloor).$$

Since $E_1$, $E_2$, and $M$ are integral algebraic cycles, we have:

**Corollary 4.2.2.** *Let* $1 \le i < g$. *If a prime* $p$ *divides* $\kappa_{i,X}$, $p > \max(i, g-i+\lfloor i/2\rfloor)$. *In particular, the mod* $p$ *analogue of the standard conjecture of Künneth type is true for all* $p \ge g$.

With a fixed $g$, as $i$ ranges from 1 to $g$, the corollary gives the best bound around $i \approx 2g/3$, when $i$ surpasses $g - i/2$.

For Jacobians we can do much better. Let $C$ be a projective smooth curve, $J = \mathrm{Jac}(C)$ its Jacobian, $c \in C$ any base point,

$$\alpha_n : C^{(n)} = \mathrm{Sym}^n C \to J$$

the (higher) Abel–Jacobi map for $n = 0, \dots, g$, and

$$w^{[n]} := \alpha_{g-n,*}[C^{(g-n)}] \in H^{2n}(J, \mathbb{Z})$$

the cohomology class (which is independent of $c$).

**Theorem 4.2.3.** *Let $C$, $J=\mathrm{Jac}(C)$, $\mu$, and $w$ be as above. Then we have*

$$\pi_J^i=(-1)^i \sum_{(a,b,c):2a+b=2g-i \text{ and } b+2c=i} \mathrm{pr}_1^*(w^{[a]})\,\mathrm{pr}_2^*(w^{[c]}) \times \sum_{d+e+f=b} (-1)^{d+f}\,\mathrm{pr}_1^*(w^{[d]})\mu^*(w^{[e]})\,\mathrm{pr}_2^*(w^{[f]})$$

*for all $i$, and for both $\mathbb{Q}$- and $\mathbb{F}_p$-coefficients.*

*In particular, the mod $p$ and integral analogues of the standard conjecture of Künneth type are true for $J$.*

*Proof.* The key point is the celebrated formula of Poincaré [Birkenhake and Lange 2004, §11.2.1]

$$n!\,w^{[n]}=[c_1(L)]^n \quad \text{in } H^{2n}(J,\mathbb{Z})$$

(hence the divided power notation). This allows us to write

$$\frac{E_1^a}{a!}=\mathrm{pr}_1^*(w^{[a]}) \quad \text{and} \quad \frac{E_2^c}{c!}=\mathrm{pr}_2^*(w^{[c]}).$$

As for the second term, apply the trinomial theorem (in the divided power form)[5] to the defining equation (4.2.1):

$$\begin{aligned} M^{[b]}=M^b/b! &= (-\mathrm{pr}_1^*(E)+\mu^*(E)-\mathrm{pr}_2^*(E))^{[b]} \\ &= \sum_{d+e+f=b} (-\mathrm{pr}_1^*(E))^{[d]}(\mu^*(E))^{[e]}(-\mathrm{pr}_2^*(E))^{[f]} \\ &= \sum_{d+e+f=b} (-1)^{d+f}\,\mathrm{pr}_1^*(w^{[d]})\mu^*(w^{[e]})\,\mathrm{pr}_2^*(w^{[f]}). \end{aligned}$$ □

**Corollary 4.2.4.** *The torsion and integral versions of the standard conjecture of Künneth type is true for all principally polarized abelian varieties of dimension $\le 3$.*

*Proof.* Via the Torelli map, a general PPAV of dimension $g\le 3$ is the Jacobian of a curve of genus $g$. □

This is the motivation for Conjecture 1.5 in the introduction.

*Prym–Tyurin theory and curves on abelian varieties.* Recall [Birkenhake and Lange 2004, §12.3] that, to a double covering $f: C'\to C$ of curves that is either unramified or ramified at exactly 2 points, one attaches a principally polarized abelian variety $P(f)$ (the Prym variety of $f$), in such a way that $P(f)\times \mathrm{Jac}(C)$ is isogenous to the Jacobian $\mathrm{Jac}(C')$ via an isogeny of exponent 2.

**Corollary 4.2.5.** *For all primes $p\ge 3$, the mod $p$ analogue of the standard conjecture of Künneth type is true for the Prym variety $P(f)$.*

[5]Namely, $(\alpha+\beta+\gamma)^n/n!=\sum_{i+j+k=n}(\alpha^i/i!)(\beta^j/j!)(\gamma^k/k!)$.

*Proof.* By Theorem 4.2.3, the analogue is true for $\mathrm{Jac}(C')$, hence for $P(f) \times \mathrm{Jac}(C)$ by Proposition 3.2(4). Then the analogue, for odd $p$, follows for $P(f)$ by Proposition 3.2(5b). □

**Corollary 4.2.6.** *For every principally polarized abelian variety $X$ of dimension $g \leq 5$, the mod $p$ analogue of the standard conjecture of Künneth type is true for all primes $p \geq 3$.*

*Proof.* A general PPAV of genus $g \leq 5$ is the Prym variety of a suitable double cover. □

More generally, following Prym and Tyurin [loc. cit.], one studies a general principally polarized abelian variety by embedding it into the Jacobian of some curve, usually of high genus, with some exponent, usually $\geq 2$. In this way, the validity of the integral analogue of the standard conjecture of Künneth type (or its failure) is related to the existence of low-genus curves on abelian varieties (or lack thereof).

We thus quantify Conjecture 1.5 into:

**Question 4.2.7.** *For any integer $g \geq 1$, let $\kappa(g)$ denote the least common multiple of $\kappa_{i,X}$ as $i$ ranges over the interval $[0, 2g]$, where $X$ is a very general principally polarized abelian variety of dimension $g$.*

*How does $\kappa(g)$ vary with $g$? In particular, is $\kappa(g) > 1$ and is $\kappa(g) > 2$ when $g > 3$ and $g > 5$, respectively?*

**4.3.** ***Hodge and Lefschetz operators.*** In this section, we consider operators closely related to the Künneth projectors, and find the denominators required in their definition. As in the previous section, we focus on principal polarizations. We recall a set of operators in the Hodge–Lefschetz theory (see, e.g., [Wells 2008, §V.3] or [Kleiman 1968]). First, cup product with $[L]$ gives

$$L : H^*(X, \mathbb{Q}) \to H^{*+2}(X, \mathbb{Q}),$$

which is defined by an integral algebraic correspondence and preserves the integral cohomology. One then defines (see the references above) the operator, with $\mathbb{Q}$-coefficients,

$$\Lambda : H^*(X, \mathbb{Q}) \to H^{*-2}(X, \mathbb{Q}).$$

Finally, the Pontryagin product is also defined by an integral algebraic correspondence

$$\begin{aligned} H^i(X, \mathbb{Z}) \otimes H^j(X, \mathbb{Z}) &\to H^{i+j-2g}(X, \mathbb{Z}), \\ x \otimes y &\mapsto x \star y := \mu_*(\mathrm{pr}_1^*(x) \otimes \mathrm{pr}_2^*(y)), \end{aligned}$$

where $\mu : X \times X \to X$ is the group law.[6] This allows us to recover the Lefschetz $\Lambda$-operator:[7]

**Proposition 4.3.1.** *For any* $x \in H^*(X, \mathbb{Q})$, *we have*

$$\Lambda(x) = \frac{1}{g!}([L]^{g-1}) \star x.$$

*In particular, $\Lambda$ is defined by an algebraic correspondence with coefficients in $\mathbb{Z}[1/g!]$ and preserves $H^*(X, \mathbb{Z}[1/g!])$.*

*Moreover, if $X$ is the Jacobian of a curve or the Prym variety associated with a double covering, then $\Lambda$ is defined by a correspondence with coefficients in $\mathbb{Z}[1/g]$ or $\mathbb{Z}[1/2g]$, respectively.*

*Proof.* For the proof of the first formula, see [Birkenhake and Lange 2004, Proposition 4.11.3]. The latter statement follows from the fact that, in case $X$ is a Jacobian or Prym, $[L]^{g-1}/(g-1)!$ or $2[L]^{g-1}/(g-1)!$, respectively, is represented by an integral algebraic cycle [Birkenhake and Lange 2004, §11.2.1 and Criterion 12.2.2]. □

Now that we have $L$ and $\Lambda$, we can apply the representation theory of $\mathfrak{sl}_2$ on cohomology (Jacobson–Morozov); see, e.g., [Wells 2008, §V.3]. With the usual notation

$$B := [\Lambda, L] = \Lambda L - L\Lambda = \sum_{i=0}^{2g} (g-i)\pi^i,$$

the operators correspond to the following matrices in $\mathfrak{sl}_2$:

$$\Lambda \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad B \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

**Definition 4.3.2** (integral Lefschetz algebra). We denote by $\mathfrak{L}$ the $\mathbb{Z}$-subalgebra of linear operators on $H^*(X, \mathbb{Q})$ generated by the five operators

$$L, \quad \Lambda, \quad w = [-1]^*, \quad \pi^0 = [\{0\} \times X], \quad \text{and} \quad \pi^{2g} = [X \times \{0\}].$$

(Recall that $[-1]^*$ acts as 1 on $H^{\text{even}}$ and as $-1$ on $H^{\text{odd}}$.)

It follows from Proposition 4.3.1 that any element of the localization $\mathfrak{L}[1/g!]$ is defined by an algebraic correspondence with coefficients in $\mathbb{Z}[1/g!]$.

Recall that $x \in H^i(X, \mathbb{Q})$ is called *primitive* if $\Lambda(x) = 0$. For any $x \in H^i(X, \mathbb{Q})$, we have the primitive decomposition

$$x = \sum_{k \geq i_0} L^k x_k, \quad x_k \in H^{i-2k}(X, \mathbb{Q}) \text{ primitive}, \tag{4.3.1}$$

[6] A natural way to index is to use homology, via Poincaré duality $H^i(X, \mathbb{Z}) \simeq H_{2g-i}(X, \mathbb{Z})$. If $x \in H_i(X, \mathbb{Z})$ and $y \in H_j(X, \mathbb{Z})$, then $x \star y \in H_{i+j}(X, \mathbb{Z})$.

[7] This operator is named $\Lambda^c$ in [Kleiman 1968], in which $\Lambda$ is reserved for something else.

where $i_0 = \max(i-g, 0)$. In terms of (4.3.1), the Lefschetz operator on $H^*(X, \mathbb{Q})$ is

$$\Lambda(x) = \sum_{k \geq i_1} k(g-i+k+1) L^{k-1} x_k \quad \text{for } x \in H^i(X, \mathbb{Q}),$$

where $i_1 = \max(i-g, 1)$; see the formula for "$\Lambda^c$" in [Kleiman 1968]. We can then define[8]

$$\Lambda'(x) = \sum_{k \geq i_1} L^{k-1} x_k \quad \text{for } x \in H^i(X, \mathbb{Q}).$$

The Hodge $*$-operator on $H^*(X, \mathbb{Q})$ in Kleiman's convention[9] is given by

$$*_K(x) = \sum_{k \geq i_0} (-1)^{(i-2k)(i-2k+1)/2} L^{g-i+k} x_k \quad \text{for } x \in H^i(X, \mathbb{Q}).$$

**Proposition 4.3.3.** *Let $j$ and $k$ be two integers in $[0, g]$. Then $\mathfrak{L}[1/2(g!)]$ contains the following 4 operators on $H^*(X, \mathbb{Q})$: the Künneth projectors $\pi^j$ and $\pi^{2g-j}$ and the primitive part extractors*

$$p^{j,k}(x) := \begin{cases} x_k & \text{if } x \in H^j(X, \mathbb{Q}), \\ 0 & \text{if } x \in H^i(X, \mathbb{Q}) \text{ and } i \neq j \end{cases}$$

*and $p^{2g-j,k}$ defined similarly ($x_k$ on $H^{2g-j}$ and 0 in other degrees).*

*Proof.* We use a nested induction, ascending in $j$ outside and descending in $k$ inside. We first note that $\pi^j$ and $\pi^{2g-j}$ are generated by $p^{j,k}$ and $L$ and by $p^{2g-j,k}$ and $L$, respectively, with integer coefficients, so for any fixed $j$, it is enough to prove the statement for $p^{j,k}$ and $p^{2g-j,k}$.

The structure constants that appear below, and need to be inverted, are given by:

**Lemma 4.3.4.** *Let $0 \neq x \in H^i(X, \mathbb{Q})$ be primitive. Then*

$$\Lambda^{g-i} L^{g-i}(x) = ((g-i)!)^2 x.$$

The point is that the irreducible subrepresentation of $\mathfrak{sl}_2(\mathbb{Q})$ generated by $x$ has weight $g-i$. See, e.g., [Wells 2008, §V.3].

In the induction base $j = 0$, the elements $\pi^0 = p^{0,0}$ and $\pi^{2g}$ are already in $\mathfrak{L}$ by definition. From the fact that $L^g : H^0(X, \mathbb{Z}[1/g!]) \to H^{2g}(X, \mathbb{Z}[1/g!])$ is an isomorphism, we see from Lemma 4.3.4 that

$$p^{2g,g} = \frac{1}{(g!)} \Lambda^g \circ \pi^{2g}.$$

---

[8] In the next line is Kleiman's definition of "$\Lambda$".

[9] This again differs from the notation of Wells [2008]; among other things, the latter acts on $H^*(X, \mathbb{C})$ but not necessarily on $H^*(X, \mathbb{Q})$.

Now suppose that, for some $j_0$, we have proven the statement for all $j < j_0$ and for all $k$. Let $x \in H^*(X, \mathbb{Q})$, and let $x^{(i)}$ denote its component in $H^i(X, \mathbb{Q})$. Applying to $x$ the composite

$$\Lambda^{g-\lfloor j_0/2\rfloor} \circ L^{g-\lfloor j_0-2\rfloor} \circ \left(1 - \sum_{j=0}^{j_0-1} \pi^j\right),$$

and then

$$\frac{1 \pm [-1]^*}{2}$$

according to whether $j_0$ is even or odd, we are left with the homogeneous component

$$\Lambda^{g-\lfloor j_0/2\rfloor} L^{g-\lfloor j_0/2\rfloor} x^{(j_0)}.$$

From the primitive decomposition,

$$x^{(j_0)} = \sum_k L^k x_k^{(j_0)}, \qquad x_k^{(j_0)} \in H^{j_0-2k} \text{ primitive},$$

what we have is

$$\Lambda^{g-\lfloor j_0/2\rfloor} L^{g-\lfloor j_0/2\rfloor} x^{(j_0)} = \sum_k \Lambda^{g-\lfloor j_0/2\rfloor} L^{g-\lfloor j_0/2\rfloor+k} x_k^{(j_0)}.$$

Now we use the descending induction in $k$, with the case $k = g+1$ being trivially true. Suppose that we have shown that $p^{j_0,k}$ is in $\mathfrak{L}[1/2g!]$ for all $k > k_0$. It follows then that the previous operator *curtailed* in degrees $k \leq k_0$

$$x \mapsto \sum_{k \leq k_0} \Lambda^{g-\lfloor j_0/2\rfloor} L^{g-\lfloor j_0/2\rfloor+k} x_k^{(j_0)}$$

on $H^*$ is in $\mathfrak{L}[1/2g!]$. Applying $\Lambda^{k_0}$ will then annihilate all the terms with $k < k_0$ (again, the point is that $x_k^{(j_0)}$, when nonzero, generates a subrepresentation of weight $g-(i-2k)$), and we obtain

$$x \mapsto \Lambda^{g-\lfloor j_0/2\rfloor+k_0} L^{g-\lfloor j_0/2\rfloor+k_0} x_k^{(j_0)} = ((g - \lfloor j_0/2\rfloor + k_0)!)^2 x_{k_0}^{(j_0)},$$

by Lemma 4.3.4, and we have shown that

$$((g - \lfloor j_0/2\rfloor + k_0)!)^2 \cdot p^{j_0,k_0}$$

is in $\mathfrak{L}[1/2g!]$, hence also $p^{j_0,k_0}$. Go down in $k$ inside, then up in $j$ outside.

The proof for $p^{2g-j,k}$ is similar, and we omit the details. □

We summarize the calculations.

**Theorem 4.3.5.** *Let $(X, L)$ be a principally polarized abelian variety of dimension $g$. The Lefschetz operator $\Lambda$ is defined by an algebraic correspondence with coefficients in $\mathbb{Z}[1/g!]$, even with coefficients in $\mathbb{Z}[1/g]$ or $\mathbb{Z}[1/2g]$ in case $X$ is* *a Jacobian or a Prym, respectively. The Künneth projectors $\pi_X^i$, the Hodge star operator (à la Kleiman) $*_K$, and the primitive part extractors $p^{j,k}$ are all defined by algebraic correspondences with coefficients in $\mathbb{Z}[1/2(g!)]$.*

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# New cubic fourfolds with odd-degree unirational parametrizations

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We prove that the moduli space of cubic fourfolds $\mathcal{C}$ contains a divisor $\mathcal{C}_{42}$ whose general member has a unirational parametrization of degree 13. This result follows from a thorough study of the Hilbert scheme of rational scrolls and an explicit construction of examples. We also show that $\mathcal{C}_{42}$ is uniruled.



## Introduction

Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$. We say that $X$ has a *degree-$\varrho$ unirational parametrization* if there is a dominant rational map $\rho : \mathbb{P}^n \dashrightarrow X$ with $\deg \rho = \varrho$. Such a parametrization implies that the smallest positive integer $N$ which allows the rational equivalence

$$N\Delta_X \equiv N\{x \times X\} + Z \quad \text{in } \mathrm{CH}^n(X \times X) \tag{0-1}$$

would divide $\varrho$, where $x \in X$ and $Z$ is a cycle supported on $X \times Y$ for some divisor $Y \subset X$. The relation (0-1) for arbitrary integer $N$ is called a *decomposition of the diagonal* of $X$, and it is called an *integral* decomposition of the diagonal if $N = 1$. (See [Bloch and Srinivas 1983] and also [Voisin 2014, Chapter 3].)

This paper studies the unirationality of *cubic fourfolds*, i.e., smooth cubic hypersurfaces in $\mathbb{P}^5$ over $\mathbb{C}$. Let $\mathrm{Hdg}^4(X, \mathbb{Z}) := H^4(X, \mathbb{Z}) \cap H^2(\Omega^2_X)$ be the group of

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integral Hodge classes of degree 4 for a cubic fourfold $X$. In the coarse moduli space of cubic fourfolds $\mathcal{C}$, the Noether–Lefschetz locus

$$\{X \in \mathcal{C} : \mathrm{rk}(\mathrm{Hdg}^4(X, \mathbb{Z})) \geq 2\}$$

is a countably infinite union of irreducible divisors $\mathcal{C}_d$ indexed by $d \geq 8$ and $d \equiv 0, 2 \pmod 6$. Here $\mathcal{C}_d$ consists of the *special cubic fourfolds* which admit a rank-2 saturated sublattice of discriminant $d$ in $\mathrm{Hdg}^4(X, \mathbb{Z})$ [Hassett 2000]. Because the integral Hodge conjecture is valid for cubic fourfolds [Voisin 2013, Theorem 1.4], $X \in \mathcal{C}$ is special if and only if there is an algebraic surface $S \subset X$ not homologous to a complete intersection.

Voisin [2017, Theorem 5.6] proves that a special cubic fourfold of discriminant $d \equiv 2 \pmod 4$ admits an integral decomposition of the diagonal. Because every cubic fourfold has a unirational parametrization of degree 2 [Harris 1992, Example 18.19], it is natural to ask whether they have odd-degree unirational parametrizations.

For a general $X \in \mathcal{C}_d$ with $d = 14, 18, 26, 30, 38$, the examples constructed by Nuer [2016] combined with an algorithm by Hassett [2016, Proposition 38] support the expectation. In this paper, we improve the list by solving the case $d = 42$.

**Theorem 0.1.** *A generic $X \in \mathcal{C}_{42}$ has a degree-13 unirational parametrization.*

Recall that a variety $Y$ is *uniruled* if there is a variety $Z$ and a dominant rational map $Z \times \mathbb{P}^1 \dashrightarrow Y$ which doesn't factor through the projection to $Z$. As a byproduct of the proof of Theorem 0.1, we also prove:

**Theorem 0.2.** *$\mathcal{C}_{42}$ is uniruled.*

***Strategy of proof.*** When $d = 2(n^2 + n + 1)$ with $n \geq 2$ and $X \in \mathcal{C}_d$ is general, the Fano variety of lines $F_1(X)$ is isomorphic to the Hilbert scheme of two points $\Sigma^{[2]}$, where $\Sigma$ is a K3 surface polarized by a primitive ample line bundle of degree $d$ [Hassett 2000, Theorem 6.1.4].

The isomorphism $F_1(X) \cong \Sigma^{[2]}$ implies $X$ contains a family of two-dimensional rational scrolls parametrized by $\Sigma$. Indeed, the divisor $\Delta \subset \Sigma^{[2]}$ parametrizing the nonreduced subschemes can be naturally identified as the projectivization of the tangent bundle of $\Sigma$. Each fiber of this $\mathbb{P}^1$-bundle induces a smooth rational curve in $F_1(X)$ through the isomorphism and hence corresponds to a rational scroll in $X$.

Let $S \subset X$ be one such scroll. Since $S$ is rational, its symmetric square $W = \mathrm{Sym}^2 S$ is also rational. A generic element $s_1 + s_2 \in W$ spans a line $l(s_1, s_2)$ not contained in $X$, so there is a rational map

$$\rho : W \dashrightarrow X,$$
$$s_1 + s_2 \mapsto x$$

where $l(s_1, s_2) \cap X = \{s_1, s_2, x\}$. By [Hassett 2016, Proposition 38], this map becomes a unirational parametrization if $S$ has isolated singularities. Moreover, its degree is odd as long as $4 \nmid d$.

Discriminant $d = 42$ corresponds to the case $n = 4$ above. Note that $4 \nmid d = 42$. Thus, a generic $X \in \mathcal{C}_{42}$ admits an odd-degree unirational parametrization once we prove:

**Theorem 0.3.** *A generic $X \in \mathcal{C}_{42}$ contains a degree-9 rational scroll $S$ which has 8 double points and is smooth otherwise.*

Here a *double point* means a *nonnormal ordinary double point*. It's a point where the surface has two branches that meet transversally.

The idea in proving Theorem 0.3 is as follows.

Degree-9 scrolls in $\mathbb{P}^5$ form a component $\mathcal{H}_9$ in the associated Hilbert scheme. Let $\mathcal{H}_9^8 \subset \mathcal{H}_9$ parametrize scrolls with 8 isolated singularities. By definition (see Section 4) an element $\bar{S} \in \mathcal{H}_9^8$ is nonreduced. We use $S$ to denote its underlying variety.

Let $U_{42} \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the locus of special cubic fourfolds with discriminant 42. Consider the incidence variety

$$\mathcal{Z} = \{(\bar{S}, X) \in \mathcal{H}_9^8 \times U_{42} : S \subset X\}.$$

Then there is a diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ \mathcal{H}_9^8 & & U_{42} \arrow[r] & \mathcal{C}_{42} \end{tikzcd}$$

Theorem 0.3 is proved by showing that $p_2$ is dominant. Two main ingredients in the proof are

- constructing an explicit example and
- estimating the dimension of the Hilbert scheme parametrizing singular scrolls.

Section 1 provides an introduction of rational scrolls and the basic properties required in the proof. We construct an example in Section 2 and then prove the main results in Section 3. The general description about the Hilbert schemes $\mathcal{H}_9^8 \subset \mathcal{H}_9$ and the estimate of the dimensions are left to Section 4.

Throughout the paper we will frequently deal with the rational map

$$\Lambda_Q : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^N$$

defined as the projection from some $(D-N)$-plane $Q$. Here $D$ and $N$ are positive integers such that $D+1 \geq N \geq 3$. We will assume $D \geq N \geq 5$ when we are studying singular scrolls.

## 1. Preliminary: rational scrolls

We provide a brief review of rational scrolls and introduce necessary terminologies and lemmas in this section.

**1A.** ***Hirzebruch surfaces.*** Let $m$ be a nonnegative integer, and let $\mathscr{E}$ be a rank-2 locally free sheaf on $\mathbb{P}^1$ isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$. The Hirzebruch surface $\mathbb{F}_m$ is defined to be the associated projective space bundle $\mathbb{P}(\mathscr{E})$.

Let $f$ be the divisor class of a fiber, and let $g$ be the divisor class of a section, i.e., the divisor class associated with Serre's twisting sheaf $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. The Picard group of $\mathbb{F}_m$ is freely generated by $f$ and $g$, and the intersection pairing is given by

|  | $f$ | $g$ |
|---|---|---|
| $f$ | 0 | 1 |
| $g$ | 1 | $m$ |

The canonical divisor is $K_{\mathbb{F}_m} = -2g + (m-2)f$.

Let $a$ and $b$ be two integers, and let $h = ag + bf$ be a divisor on $\mathbb{F}_m$. The ampleness and the very ampleness for $h$ are equivalent on $\mathbb{F}_m$, and it happens if and only if $a > 0$ and $b > 0$ [Hartshorne 1977, Chapter V, §2.18].

**Lemma 1.1.** *Suppose the divisor $ag + bf$ is ample. We have*

$$h^0(\mathbb{F}_m, ag+bf) = (a+1)(\tfrac{1}{2}am + b + 1),$$
$$h^i(\mathbb{F}_m, ag+bf) = 0 \quad \textit{for all } i > 0.$$

These formulas appear in several places in the literature with slightly different details depending on the contexts, for example [Laface 2002, Proposition 2.3; Ballico et al. 2004, p. 543; Coskun 2006, Lemma 2.6]. It can be proved by induction on the integers $a$ and $b$ or by applying the projection formula to the bundle map $\pi : \mathbb{F}_m \to \mathbb{P}^1$.

**1B.** ***Deformations of Hirzebruch surfaces.*** $\mathbb{F}_m$ admits a deformation to $\mathbb{F}_{m-2k}$ for all $m > 2k \geq 0$. More precisely, there exists a holomorphic family $\tau : \mathcal{F} \to \mathbb{C}$ such that $\mathcal{F}_0 \cong \mathbb{F}_m$ and $\mathcal{F}_t \cong \mathbb{F}_{m-2k}$ for $t \neq 0$. The family can be written down explicitly by the equation

$$\mathcal{F} = \{x_0^m y_1 - x_1^m y_2 + t x_0^{m-k} x_1^k y_0 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}, \tag{1-1}$$

where $([x_0, x_1], [y_0, y_1, y_2], t)$ is the coordinate of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}$ [Barth et al. 1984, p. 205].

Generally, $\mathbb{F}_m$ admits an analytic versal deformation with a base manifold of dimension $h^1(\mathbb{F}_m, T_{\mathbb{F}_m})$ by the following lemma.

**Lemma 1.2** [Seiler 1992, Lemma 1 and Theorem 4]. *There is a natural isomorphism $H^1(\mathbb{F}_m, T_{\mathbb{F}_m}) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m))$. We also have $H^2(\mathbb{F}_m, T_{\mathbb{F}_m}) = 0$.*

Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ be the underlying locally free sheaf of $\mathbb{F}_m$. It is straightforward to compute that $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E}) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m))$, so there is a natural isomorphism

$$H^1(\mathbb{F}_m, T_{\mathbb{F}_m}) \cong \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E}), \tag{1-2}$$

by Lemma 1.2. The elements of the group $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E})$ are in one-to-one correspondence with the deformations of $\mathcal{E}$ over the dual numbers $D_t \cong \mathbb{C}[t]/(t^2)$ [Hartshorne 2010, Theorem 2.7]. Thus, (1-2) says that the infinitesimal deformation of $\mathbb{F}_m$ can be identified with the infinitesimal deformation of its underlying locally free sheaf.

Every element in $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(m), \mathcal{O}_{\mathbb{P}^1})$ is represented by a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m-k) \to \mathcal{O}_{\mathbb{P}^1}(m) \to 0$$

for some $k$ satisfying $m > 2k \geq 0$. By tracking the construction of the correspondence in [Hartshorne 2010, Theorem 2.7], the above sequence corresponds to a coherent sheaf $\mathscr{E}$ on $\mathbb{P}^1 \times D_t$, flat over $D_t$, such that $\mathscr{E}_0 \cong \mathcal{E}$ and $\mathscr{E}_t \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m-k)$ for $t \neq 0$. So it induces a flat family $\mathcal{F}$ of Hirzebruch surfaces over $D_t$ such that $\mathcal{F}_0 \cong \mathbb{F}_m$ and $\mathcal{F}_t \cong \mathbb{F}_{m-2k}$ for $t \neq 0$.

**1C. *Rational normal scrolls.*** Let $u$ and $v$ be positive integers with $u \leq v$, and let $N = u + v + 1$. Let $P_1$ and $P_2$ be complementary linear subspaces of dimensions $u$ and $v$ in $\mathbb{P}^N$. Choose rational normal curves $C_1 \subset P_1$ and $C_2 \subset P_2$ and an isomorphism $\varphi : C_1 \to C_2$. Then the union of the lines $\bigcup_{p \in C_1} \overline{p\varphi(p)}$ forms a smooth surface $S_{u,v}$ called a *rational normal scroll of type* $(u, v)$. The line $\overline{p\varphi(p)}$ is called a *ruling*. When $u < v$, we call the curve $C_1 \subset S_{u,v}$ the *directrix* of $S_{u,v}$.

A rational normal scroll of type $(u, v)$ is uniquely determined up to projective isomorphism. In particular, each $S_{u,v}$ is projectively equivalent to the one given by the parametric equation

$$\begin{aligned} &\mathbb{C}^2 \to \mathbb{P}^N, \\ &(s, t) \mapsto (1, s, \dots, s^u, t, st, \dots, s^v t). \end{aligned} \tag{1-3}$$

One can check by this expression that a hyperplane section of $S_{u,v}$ which doesn't contain a ruling is a rational normal curve of degree $u + v$. It easily follows that $S_{u,v}$ has degree $D = u + v$.

The rulings of $S_{u,v}$ form a rational curve in $\mathbb{G}(1, N)$ the Grassmannian of lines in $\mathbb{P}^N$. By using (1-3), we can parametrize this curve as

$$\begin{aligned} &\mathbb{C} \to \mathbb{G}(1, N), \\ &s \mapsto \begin{pmatrix} 1 & s & \cdots & s^u & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & s & \cdots & s^v \end{pmatrix}, \end{aligned} \tag{1-4}$$

where the matrix on the right represents the line spanned by the row vectors.

The embedding $S_{u,v} \subset \mathbb{P}^N$ can be seen as the Hirzebruch surface $\mathbb{F}_{v-u}$ embedded in $\mathbb{P}^N$ through the complete linear system $|g+uf|$. Conversely, every nondegenerate, irreducible, and smooth surface of degree $D$ in $\mathbb{P}^{D+1}$ isomorphic to $\mathbb{F}_{v-u}$ must be $S_{u,v}$ [Griffiths and Harris 1978, p. 522–525].

It's not hard to compute that $H^1(T_{\mathbb{P}^N}|_{\mathbb{F}_{u-v}}) = 0$ under the above embedding. Combining this with the rigidity result, it implies that every abstract deformation of $\mathbb{F}_{v-u}$ can be lifted to an embedded deformation as a family of rational normal scrolls in $\mathbb{P}^N$ [Hartshorne 2010, Remark 20.2.1]. We conclude this as the following lemma.

**Lemma 1.3.** *For $m > 2k \geq 0$, let $\mathcal{F}$ be an abstract deformation of Hirzebruch surfaces such that $\mathcal{F}_0 \cong \mathbb{F}_m$ and $\mathcal{F}_t \cong \mathbb{F}_{m-2k}$ for $t \neq 0$. Then $\mathcal{F}$ can be realized as an embedded deformation $\mathcal{S}$ in $\mathbb{P}^{D+1}$ with $\mathcal{S}_0 \cong S_{u,v}$ and $\mathcal{S}_t \cong S_{u+k,v-k}$ for $t \neq 0$, where $D = u + v$, and $u \leq v$ are any positive integers satisfying $v - u = m$.*

## 1D. *Rational scrolls.*

**Definition 1.4.** We call a surface $S \subset \mathbb{P}^N$ a rational scroll (or a scroll) of type $(u, m+u)$ if it is the image of a Hirzebruch surface $\mathbb{F}_m$ through a birational morphism defined by an $N$-dimensional subsystem $\mathfrak{d} \subset |g+uf|$ for some $u > 0$.

Equivalently, $S \subset \mathbb{P}^N$ is a rational scroll of type $(u, v)$ either if it is a rational normal scroll $S_{u,v}$ or if it is the projection image of $S_{u,v} \subset \mathbb{P}^{D+1}$ from a $(D-N)$-plane disjoint from $S_{u,v}$. Here $D = u+v$ is the degree of $S_{u,v}$ as well as the degree of $S$. In the latter case, we also call a line on $S$ a ruling if its preimage is a ruling on $S_{u,v}$.

The following lemma computes the cohomology groups of the normal bundle for an arbitrary embedding of a Hirzebruch surface into a projective space.

**Lemma 1.5.** *Let $\iota : \mathbb{F}_m \hookrightarrow \mathbb{P}^N$ be an embedding with image $S$ and $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_S(h)$, where $h = ag + bf$ with $a > 0$ and $b > 0$. Let $N_{S/\mathbb{P}^N}$ be the normal bundle of $S$ in $\mathbb{P}^N$; then*

$$h^0(S, N_{S/\mathbb{P}^N}) = (N+1)(a+1)(\tfrac{1}{2}am + b + 1) - 7$$

*and $h^i(S, N_{S/\mathbb{P}^N}) = 0$ for all $i > 0$. Especially, if $S$ is a smooth scroll of degree $D$, then the formula for $h^0$ reduces to*

$$h^0(S, N_{S/\mathbb{P}^N}) = (N+1)(D+2) - 7.$$

*Proof.* The short exact sequence

$$0 \to T_S \to T_{\mathbb{P}^N}|_S \to N_{S/\mathbb{P}^N} \to 0 \tag{1-5}$$

has the long exact sequence

$$\begin{aligned}
0 &\to H^0(S, T_S) \to H^0(S, T_{\mathbb{P}^N}|_S) \to H^0(S, N_{S/\mathbb{P}^N}) \\
&\to H^1(S, T_S) \to H^1(S, T_{\mathbb{P}^N}|_S) \to H^1(S, N_{S/\mathbb{P}^N}) \\
&\to H^2(S, T_S) \to H^2(S, T_{\mathbb{P}^N}|_S) \to H^2(S, N_{S/\mathbb{P}^N}) \to 0.
\end{aligned}$$

In order to calculate the dimensions in the right, we need the dimensions in the first two columns.

For the middle column, we can restrict the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1)^{\oplus(N+1)} \to T_{\mathbb{P}^N} \to 0$$

to $S$ and obtain

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(h)^{\oplus(N+1)} \to T_{\mathbb{P}^N}|_S \to 0.^{1}$$

Lemma 1.1 confirms that $h^i(S, \mathcal{O}_S(h)) = 0$ for $i > 0$, so we have

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(h))^{\oplus(N+1)} \to H^0(S, T_{\mathbb{P}^N}|_S) \to 0$$

from the associated long exact sequence while the other terms are all vanishing. It follows that

$$\begin{aligned}
h^0(S, T_{\mathbb{P}^N}|_S) &= (N+1)h^0(S, \mathcal{O}_S(h)) - h^0(S, \mathcal{O}_S) \\
&\overset{1.1}{=} (N+1)(a+1)(\tfrac{1}{2}am + b + 1) - 1.
\end{aligned}$$

For the first column, one can use the Hirzebruch–Riemann–Roch formula to compute that $\chi(T_S) = 6$. We also have $h^2(S, T_S) = 0$ by Lemma 1.2. Thus, $h^0(S, T_S) - h^1(S, T_S) = \chi(T_S) = 6$.

Collecting the above results, the long exact sequence for (1-5) now becomes

$$\begin{aligned}
0 &\to H^0(S, T_S) \to H^0(S, T_{\mathbb{P}^N}|_S) \to H^0(S, N_{S/\mathbb{P}^N}) \\
&\to H^1(S, T_S) \to 0 \qquad\qquad\quad \to H^1(S, N_{S/\mathbb{P}^N}) \\
&\to 0 \qquad\qquad \to 0 \qquad\qquad\quad \to H^2(S, N_{S/\mathbb{P}^N}) \to 0.
\end{aligned}$$

Therefore, we have $h^i(S, N_{S/\mathbb{P}^N}) = 0$ for all $i > 0$, and

$$\begin{aligned}
h^0(S, N_{S/\mathbb{P}^N}) &= h^0(S, T_{\mathbb{P}^N}|_S) - \chi(T_S) \\
&= (N+1)(a+1)(\tfrac{1}{2}am + b + 1) - 7.
\end{aligned}$$

When $S$ is a rational scroll, we have $h = g + bf$. Then the formula is obtained by inserting $a = 0$ and $D = h^2 = m + 2b$ into the equation. □

[1] $\mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_S, \mathscr{F}) = 0$ for all $i > 0$ and locally free sheaf $\mathscr{F}$, so the Euler exact sequence keeps exact after the restriction.

**1E. *Isolated singularities on rational scrolls.*** The singularities on a rational scroll are all caused from projection by definition, so we assume $D \geq N$. We also assume $N \geq 5$.

Let $S \subset \mathbb{P}^N$ be a rational scroll under $S_{u,v} \subset \mathbb{P}^{D+1}$, and let $q : S_{u,v} \to S$ be the projection. A point $p \in S$ is singular if and only if one of the following situations occurs:

- There are two distinct rulings $l, l' \subset S_{u,v}$ such that $p \in q(l) \cap q(l')$.
- There is a ruling $l \subset S_{u,v}$ such that $p \in q(l)$ and the map $q$ is ramified at $l$.

Suppose that $S$ has isolated singularities, i.e., the singular locus of $S$ has dimension 0. Then each singular point is set-theoretically the intersection of two or more rulings. Let $m$ be the number of the ruling which passes through any of the singular points. Then the number of singularities on $S$ is counted as $\binom{m}{2}$.

Note that $S_{u,v}$ is cut out by quadrics, so every secant line intersects $S_{u,v}$ in exactly two points transversally. Let $T(S_{u,v}) \subset S(S_{u,v})$ be the tangent and the secant varieties of $S_{u,v}$, respectively. Then every $x \in S(S_{u,v}) - T(S_{u,v})$ belongs to one of the two conditions:

(1) The point $x$ lies on one and only one secant line.

(2) The point $x$ lies on two secant lines. Let $Z_2 \subset S(S_{u,v})$ denote the union of such points.

**Lemma 1.6.** *The subset $Z_2 \neq \varnothing$ if and only if $u = 2$. In this situation, $Z_2 \cong \mathbb{P}^2$ and each $x \in Z_2$ lies on infinitely many secant lines.*

*Proof.* We retain the notation used in constructing $S_{u,v}$ throughout the proof.

Let $x \in Z_2$ be any point. First we claim that the intersection points of $S_{u,v}$ with the union of the two secants described in (2) lie on four distinct rulings.

The intersection points don't lie on two rulings because any two distinct rulings are linearly independent [Harris 1992, Exercise 8.21]. If they lie on three rulings, then the projection to $P_2$ would be a trisecant line of $C_2$. But this is impossible because $C_2$ has degree $v \geq \lceil D/2 \rceil \geq 2$. Hence, the claim holds.

The claim admits a rational normal curve $C \subset S_{u,v}$ (either sectional or residual) of degree $\geq u$ passing through the four intersection points [Harris 1992, Example 8.17]. This imposes a nontrivial linear relation on four distinct points on $C$, which forces $C$ to be either a line or a conic. If $C$ is a line, then $C$ coincides with the two secant lines, which is impossible. Hence, $C$ must be a conic.

It follows that $u \leq \deg C \leq 2$. If $u = 1$, then the conic $C$ would dominate $C_2$ through the projection from $P_1$. However, this cannot happen since $C_2$ has degree $v = D - u \geq 4$. Hence, $u = 2$. In this condition, $C$ can only be the directrix since $u < v$. It follows that the $Z_2$ coincides with the 2-plane spanned by $C$ and each point of $Z_2$ lies on infinitely many secants.

Conversely, $u = 2$ implies that $Z_2$ contains the 2-plane spanned by $C_2$. By the same argument above, they coincide and every $x \in Z_2$ lies on infinitely many secants. □

Assume that $S$ is the projection of $S_{u,v}$ from a $(D-N)$-plane $Q \subset \mathbb{P}^{D+1}$.

**Corollary 1.7.** *The scroll $S$ is singular along $r$ points if and only if $Q$ intersects $S(S_{u,v})$ in $r$ points away from $T(S_{u,v}) \cup Z_2$.*

*Proof.* Assume that $S$ has isolated singularities. Recall that the number $r$ counts the number of the pair $(l, l')$ of distinct rulings on $S_{u,v}$ such that $q(l)$ intersects $q(l')$ in one point. (Different pairs might intersect in the same point.) It then counts the number of the unique line joining $l$, $l'$, and $Q$. By Lemma 1.6, each $x \in S(S_{u,v})$ away from $T(S_{u,v}) \cup Z_2$ lies on a unique secant. Thus, it is the same as the number of the intersection between $Q$ and $S(S_{u,v})$ away from $T(S_{u,v}) \cup Z_2$. □

In the end we provide a criterion for $S$ to have isolated singularities when $u = 1$. This is going to be used in proving Proposition 2.1.

**Proposition 1.8.** *Assume $u = 1$. If $Q \cap T(S_{1,v}) = \varnothing$, $S$ has isolated singularities.*

*Proof.* If $Q$ intersects $S(S_{1,v})$ in points, then the proposition follows by Corollary 1.7.

Assume $Q \cap S(S_{1,v})$ contains a curve $\Gamma$. We are going to show that $\Gamma$ intersects $T(S_{1,v})$ nontrivially, which then contradicts our hypothesis.

Let $f$ be the fiber class of $S_{1,v}$. Then the linear system $|f|$ parametrizes the rulings of $S_{1,v}$. For distinct $l, l' \in |f|$, the linear span of $l$ and $l'$ is a 3-space $P_{l,l'} \subset S(S_{1,v})$. Consider the incidence correspondence

$$\mathbb{S} = \{(x, l + l') \in \mathbb{P}^{D+1} \times |2f| : x \in P_{l,l'}\}.$$

Observe that $\mathbb{S}$ is a $\mathbb{P}^3$-bundle over $|2f| \cong \mathbb{P}^2$ via the second projection

$$p_2 : \mathbb{S} \to |2f|.$$

On the other hand, the image of $\mathbb{S}$ under the first projection

$$p_1 : \mathbb{S} \to \mathbb{P}^{D+1}$$

is the secant variety $S(S_{1,v})$. Consider the diagonal

$$\Delta := \{2l : l \in |f|\} \subset |2f|.$$

It's easy to see that the tangent variety $T(S_{1,v}) \subset S(S_{1,v})$ is the image of $p_2^{-1}(\Delta)$ via the first projection.

If $\Gamma \not\subset P_{l,l'}$ for all $l + l'$, then the curve $p_1^{-1}(\Gamma)$ is mapped to a curve in $|2f|$ which intersects $\Delta$ nontrivially. It follows that $\Gamma \cap T(S_{u,v}) \neq \varnothing$.

Suppose $\Gamma \subset P_{l,l'}$ for some $l+l'$. The directrix $C_1$ is a line in $P_{l,l'}$ by hypothesis, so we have

$$T(S_{u,v}) \cap P_{l,l'} = P \cup P'$$

where $P$ and $P'$ are the 2-planes spanned by $C_1$, and $l$ and $l'$, respectively. So $\Gamma$ and $T(S_{u,v})$ have a nontrivial intersection in $P_{l,l'}$. $\square$

## 2. Construction of singular scrolls in $\mathbb{P}^5$

This section provides a construction of singular scrolls in $\mathbb{P}^5$ of type $(1, v)$ with isolated singularities. The construction actually relates the existence of the singular scrolls to the solvability of a four-square equation as follows.

**Proposition 2.1.** *Assume $v \geq 4$. There exists a rational scroll in $\mathbb{P}^5$ of type $(1, v)$ with isolated singularities which has at least $r$ singularities if there are four odd integers $a \geq b \geq c \geq d > 0$ satisfying*

(1) $8r+4 = a^2+b^2+c^2+d^2$ *and*

(2) $a+b+c \leq 2v-3$.

We use the construction to produce an explicit example which can be manipulated by a computer algebra system. With the help of a computer, we prove:

**Proposition 2.2.** *There is a degree-9 rational scroll $S \subset \mathbb{P}^5$ which has eight isolated singularities and is smooth otherwise such that*

(1) $h^0(\mathbb{P}^5, \mathcal{I}_S(3)) = 6$, *where $\mathcal{I}_S$ is the ideal sheaf of $S$ in $\mathbb{P}^5$*,

(2) *$S$ is contained in a smooth cubic fourfold $X$*,

(3) *$S$ deforms in $X$ to the first order as a two-dimensional family*, *and*

(4) *$S$ is also contained in a singular cubic fourfold $Y$*.

We introduce the construction first and prove Proposition 2.2 in the end. Recall that, with a fixed rational normal scroll $S_{1,v} \subset \mathbb{P}^{D+1}$, every degree-$D$ scroll $S \subset \mathbb{P}^5$ of type $(1, v)$ is the projection of $S_{1,v}$ from $P^{\perp}$ for some $P \in \mathbb{G}(5, D+1)$.

**2A.** ***Plane k-chains.*** Let $k$ be a positive integer. It can be proved by induction that $k$ distinct lines in a projective space intersect in at most $\binom{k}{2}$ points counted with multiplicity, and the maximal number is attained exactly when the $k$ lines span a 2-plane.

**Definition 2.3.** Let $k \geq 1$ be an integer. We call the union of $k$ distinct lines which span a 2-plane a plane $k$-chain. Let $W \subset \mathbb{P}^N$ be the union of a finite number of lines. A plane $k$-chain in $W$ is called maximal if it is not a subset of a plane $k'$-chain in $W$ for some $k' > k$.

Let $S \subset \mathbb{P}^5$ be a singular scroll with isolated singularities. There's a subset $W \subset S$ consisting of a finite number of rulings defined by

$$W = \bigcup l : l \text{ is a ruling passing through a singular point on } S. \tag{2-1}$$

By Zorn's lemma, $W$ can be expressed as

$$W = \bigcup_{i=1}^{n} K_i : K_i \text{ is a maximal plane } k_i\text{-chain with } k_i \geq 2.$$

If two plane $k$-chains share more than one line, then they must lie on the same 2-plane. In particular, both of them cannot be maximal. Therefore, for distinct maximal plane $k$-chains $K_i$ and $K_j$ in $W$, we have either $K_i \cap K_j = \varnothing$ or $K_i \cap K_j = l$ a single ruling. It follows that the number of singularities on $S$ equals $\sum_{i=1}^{n} \binom{k_i}{2}$ since a plane $k$-chain contributes $\binom{k}{2}$ singularities.

Let $l_1, \ldots, l_k \subset S_{1,v}$ be $k$ distinct rulings which span a subspace $P_{l_1,\ldots,l_k} \subset \mathbb{P}^{D+1}$. The images of the rulings form a plane $k$-chain on $S$ through projection if and only if $P_{l_1,\ldots,l_k}$ is projected onto a 2-plane in $\mathbb{P}^5$. Parametrize the rulings as in (1-4) with $l_j = l_j(s_j)$, $j = 1, \ldots, k$. Then $P_{l_1,\ldots,l_k}$ is spanned by the row vectors of the $(k+2) \times (D+2)$ matrix

$$P(s_1, \ldots, s_k) = \begin{pmatrix} 1 & s_1 & 0 & 0 & \cdots & 0 \\ 1 & s_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & s_1 & \cdots & s_1^v \\ & & \vdots & & & \\ 0 & 0 & 1 & s_k & \cdots & s_k^v \end{pmatrix}. \tag{2-2}$$

The projection $S_{1,v} \to S$ is restricted from a linear map

$$\Lambda : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^5.$$

Suppose $\Lambda$ is represented by a $(D+2) \times 6$ matrix

$$\Lambda = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{pmatrix},$$

where $v_1, \ldots, v_6$ are vectors in $\mathbb{P}^{D+1}$. Then $P_{l_1,\ldots,l_k}$ is projected onto a 2-plane if and only if the $(k+2) \times 6$ matrix

$$P(s_1, \ldots, s_k) \cdot \Lambda$$

has rank 3.

**2B.** ***Control the number of singularities.*** Let $r$ be a nonnegative integer. We introduce a method to find a projection $\Lambda$ which maps $S_{1,v}$ to a singular scroll $S$ with isolated singularities. The method allows us to control the number of singularities such that it is bounded below by $r$. For simplicity, we consider only the cases when the configuration $W \subset S$ defined in (2-1) consists of four disjoint maximal plane $k$-chains.

We start by picking distinct rulings on $S_{u,v}$ and produce four matrices $P_1$, $P_2$, $P_3$, and $P_4$ as in (2-2). Suppose $P_i$ consists of $k_i$ rulings. Note that $P_i$ contribute $\binom{k_i}{2}$ singularities if its rulings are mapped to a plane $k_i$-chain. Thus, we also assume that

$$r = \binom{k_1}{2} + \binom{k_2}{2} + \binom{k_3}{2} + \binom{k_4}{2}. \tag{2-3}$$

Here we allow $k_i = 1$, which means that $P_i$ consists of a single ruling and thus contributes no singularity.

Consider $\Lambda = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{pmatrix}$ as an unknown. Let $P$ be the 5-plane spanned by $v_1, \dots, v_6$. We are going to construct $\Lambda$ satisfying

(1) $\mathrm{rk}(P_i \cdot \Lambda) = 3$, $i = 1, 2, 3, 4$, and

(2) $P^{\perp} \cap T(S_{1,v}) = \varnothing$.

Note that (1) makes the number of isolated singularities $\geq r$, while (2) confirms that no curve singularity occurs. We divide the construction into two steps.

*Step 1* (find $v_1$, $v_2$, $v_3$, and $v_4$ to satisfy (1)). Consider each $P_i$ as a linear map by multiplication from the left. We are trying to find independent vectors $v_1$, $v_2$, $v_3$, and $v_4$ such that for each $P_i$ three of them are in the kernel while the remaining one isn't. The four vectors arranged in this way contribute exactly one rank to each $P_i \cdot \Lambda$. In the next step, $v_5$ and $v_6$ will be general vectors in $\mathbb{P}^{D+1}$ satisfying some open conditions. This contributes two additional ranks to each $P_i \cdot \Lambda$, which makes (1) true.

Under the standard parametrization for $S_{1,v} \subset \mathbb{P}^{D+1}$, the underlying vector space of $\mathbb{P}^{D+1}$ can be decomposed as $A \oplus B$ with $A$ representing the first two coordinates and $B$ representing the last $v+1$ coordinates. With this decomposition, the matrix $P$ in (2-2) can be decomposed into two Vandermonde matrices

$$P^A = \begin{pmatrix} 1 & s_1 \\ 1 & s_2 \end{pmatrix} \quad \text{and} \quad P^B = \begin{pmatrix} 1 & s_1 & \cdots & s_1^v \\ & \vdots & & \\ 1 & s_k & \cdots & s_k^v \end{pmatrix}.$$

Note that $\ker P = \ker P^B$. So we can search for the vectors from $\ker P^B$.

In our situation, we have four matrices $P_1^B$, $P_2^B$, $P_3^B$, and $P_4^B$ which have four kernels $\ker P_1^B$, $\ker P_2^B$, $\ker P_3^B$, and $\ker P_4^B$, respectively. By the assumption $k_i \leq v$ and the fact that a Vandermonde matrix has full rank, each $\ker P_i^B$ is a codimension-$k_i$ subspace of $B$.

Now we want to pick $v_1, \dots, v_4$ from $B$ such that each $\ker P_i^B$ contains exactly three of the four vectors; i.e., we want

$$|\ker P_i^B \cap \{v_1, v_2, v_3, v_4\}| = 3 \quad \text{for } i = 1, 2, 3, 4. \tag{2-4}$$

One way to satisfy (2-4) is to pick $v_i$ from

$$\Big(\bigcap_{j\neq i}\ker P_j^B\Big)-\ker P_i^B \quad \text{for } i=1,2,3,4. \tag{2-5}$$

The sets in (2-5) are nonempty if and only if

$$\dim(\ker P_\alpha^B\cap\ker P_\beta^B\cap\ker P_\gamma^B)\geq 1$$

for all distinct $\alpha,\beta,\gamma\in\{1,2,3,4\}$. This is equivalent to

$$k_\alpha+k_\beta+k_\gamma\leq v \quad \text{for distinct } \alpha,\beta,\gamma\in\{1,2,3,4\}. \tag{2-6}$$

So we have to include (2-6) as one of our assumptions.

*Step 2* (adjust $v_1,\dots,v_4$ and then pick $v_5$ and $v_6$ to satisfy (2)).

**Lemma 2.4.** *Let $v_i^\perp$ be the hyperplane in $\mathbb{P}^{D+1}$ orthogonal to $v_i$. The four vectors $v_1,\dots,v_4$ can be chosen generally such that $\bigcap_{i=1}^4 v_i^\perp$ intersects $T(S_{1,v})$ only in the directrix of $S_{1,v}$.*

*Proof.* Parametrize the rational normal curve $C=S_{1,v}\cap\mathbb{P}(B)$ by

$$\theta(s)=(0,0,1,s,\dots,s^v).$$

Then the standard parametrization (1-3) can be written as

$$(1,s,0,\dots,0)+t\theta(s).$$

Let $a$ and $b$ be the parameters for the tangent plane over each point. Then the tangent variety $T(S_{1,v})$ has the parametric equation

$$\begin{aligned}(1,s,0,\dots,0)+t\theta(s)+a\left[(0,1,0,\dots,0)+t\frac{d\theta}{ds}(s)\right]+b\theta(s)\\ =(1,s+a,0,\dots,0)+(t+b)\theta(s)+ta\frac{d\theta}{ds}(s).\end{aligned}$$

Each point on $T(S_{1,v})$ lying in $\bigcap_{i=1}^4 v_i^\perp$ is a common zero of the equations

$$(t+b)(\theta(s)\cdot v_i)+ta\left(\frac{d\theta}{ds}(s)\cdot v_i\right)=0 \quad \text{for } i=1,2,3,4. \tag{2-7}$$

By considering $(t+b)$ and $ta$ as variables, (2-7) becomes a system of linear equations given by the matrix

$$\begin{pmatrix}\theta(s)\cdot v_1 & \theta(s)\cdot v_2 & \theta(s)\cdot v_3 & \theta(s)\cdot v_4\\ \theta'(s)\cdot v_1 & \theta'(s)\cdot v_2 & \theta'(s)\cdot v_3 & \theta'(s)\cdot v_4\end{pmatrix}.$$

The matrix fails to be of full rank exactly when $s$ admits the existence of $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$, such that

$$(\alpha\theta(s) + \beta\theta'(s)) \cdot v_i = 0 \quad \text{for } i = 1, 2, 3, 4. \tag{2-8}$$

Note that (2-8) has a solution if and only if $\bigcap_{i=1}^{4} v_i^{\perp}$ and the tangent variety $T(C)$ of $C$ intersect each other.

One can choose $v_2$, $v_3$, and $v_4$ in general from (2-5) so that $\bigcap_{i=2}^{4} v_i^{\perp}$ is disjoint from $C$. This forces $\bigcap_{i=2}^{4} v_i^{\perp}$ to intersect $T(C)$ in either the empty set or points. By the properties of a rational normal curve, the hyperplane orthogonal to a point on $C$ contains no invariant subspace when one perturbs the point. Hence, after necessary perturbation of the chosen rulings, one can choose $v_1$ from (2-5) such that $\left(\bigcap_{i=1}^{4} v_i^{\perp}\right) \cap T(C) = \varnothing$. As a result, the equations in (2-7) become independent, so the solutions are $t = b = 0$, or $a = 0$ and $t = -b$. Both solutions form the directrix of $S_{1,v}$. □

With the above adjustment, we can pick $v_5$ and $v_6$ in general in $\mathbb{P}^{D+1}$ so that the $(D-5)$-plane $Q = v_1^{\perp} \cap \cdots \cap v_6^{\perp}$ has no intersection with $T(S_{1,v})$. Note that the projection defined by $\Lambda$ is the same as the projection from $Q$. By Proposition 1.8, this projection produces a rational scroll with isolated singularities.

**Proposition 2.5.** *There exists a rational scroll in $\mathbb{P}^5$ of type $(1, v)$ with isolated singularities which has at least $r$ singularities if there are four positive integers $k_1 \geq k_2 \geq k_3 \geq k_4$ satisfying (2-3) and (2-6):*

$$r = \binom{k_1}{2} + \binom{k_2}{2} + \binom{k_2}{2} + \binom{k_2}{2} \quad \textit{and} \quad k_1 + k_2 + k_3 \leq v.$$

Proposition 2.1 is obtained by expanding the binomial coefficients followed by a change of variables.

**2C.** ***Proof of Proposition 2.2.*** In the following we exhibit an explicit example which can be manipulated by a computer algebra system over characteristic 0. The main program used in our work is Singular [Decker et al. 2015].

Consider $\mathbb{P}^{10}$ with homogeneous coordinate $\boldsymbol{x} = (x_0, \ldots, x_{10})$. We define the rational normal scroll $S_{1,8}$ by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_1 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{pmatrix}.$$

In order to project $S_{1,8}$ onto a rational scroll whose singular locus is zero-dimensional and consists of at least eight singular points, we use the method introduced previously to construct a projection

$$\Lambda = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 & 120 & -34 & -203 & 91 & 70 & -56 & 13 & -1 \\ 0 & 0 & 2880 & 5184 & -2372 & -2196 & 633 & 261 & -63 & -9 & 2 \\ 0 & 0 & 0 & 480 & 304 & -510 & -339 & 30 & 36 & 0 & -1 \\ 0 & 0 & 0 & 144 & 36 & -196 & -49 & 56 & 14 & -4 & -1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Let $z = (z_0, \dots, z_5)$ be the coordinate for $\mathbb{P}^5$. Then the projection $\mathbb{P}^{10} \dashrightarrow \mathbb{P}^5$ defined by $\Lambda$ can be explicitly written as

$$z = x \cdot \Lambda.$$

Let $S$ be the image of $S_{1,8}$ under the projection. Due to the limit of the author's computer, we check that $S$ has eight singularities and is smooth otherwise over the finite field of order 31. On the other hand, the double point formula implies that $S$ has eight double points if the singular locus is isolated. Hence, the singularity of $S$ consists of eight double points over characteristic 0 as required.

The generators of the ideal of $S$ contain six cubics, so property (1) is confirmed. Properties (2) and (4) can be easily checked by examining the linear combinations of those cubics.

The final step is to verify property (3). Let $X \subset \mathbb{P}^5$ be a smooth cubic containing $S$. Let $F_1(S)$ and $F_1(X)$ denote the Fano variety of lines on $S$ and $X$, respectively. Then it is equivalent to show that $F_1(S)$ deforms in $F_1(X)$ to the first order with dimension 2.

Let $\mathbb{G}(1,5)$ be the grassmannian of lines in $\mathbb{P}^5$. Every element $\boldsymbol{b} \in \mathbb{G}(1,5)$ is parametrized by a $2 \times 6$ matrix

$$\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix} = \begin{pmatrix} b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{20} & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{pmatrix} \tag{2-9}$$

where $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are two vectors which span the line $\boldsymbol{b}$.

Let $P_X = P_X(z)$ be the homogeneous polynomial defining $X$. Let $V$ be the six-dimensional linear space underlying $\mathbb{P}^5$. Consider $P_X$ as a symmetric function defined on $V \oplus V \oplus V$. Then $F_1(X) \subset \mathbb{G}(1,5)$ is cut out by the four equations

$$P_X(\boldsymbol{b}_1, \boldsymbol{b}_1, \boldsymbol{b}_1), \quad P_X(\boldsymbol{b}_1, \boldsymbol{b}_1, \boldsymbol{b}_2), \quad P_X(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_2), \quad P_X(\boldsymbol{b}_2, \boldsymbol{b}_2, \boldsymbol{b}_2). \tag{2-10}$$

Consider the Fano variety of lines on $S_{1,8}$ as a rational curve $\mathbb{P}^1 \subset \mathbb{G}(1,10)$ parametrized by

$$Q = \begin{pmatrix} r & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^8 & r^7 s & r^6 s^2 & r^5 s^3 & r^4 s^4 & r^3 s^5 & r^2 s^6 & r s^7 & s^8 \end{pmatrix}$$

where $(r, s)$ is the homogeneous coordinate for $\mathbb{P}^1$. Then $F_1(S) \subset \mathbb{G}(1, 5)$ is defined by the parametric equation

$$R = Q \cdot \Lambda.$$

Now consider a $2 \times 6$ matrix $dR$ whose first row consists of arbitrary linear forms on $\mathbb{P}^1$ while the second row consists of arbitrary 8-forms. The coefficients of those forms introduce $2 \cdot 6 + 9 \cdot 6 = 66$ variables $c_1, \dots, c_{66}$. Then an abstract first-order deformation of $F_1(S)$ in $\mathbb{G}(1, 5)$ is given by

$$R + dR.$$

Inserting $R+dR$ into (2-10) gives us four polynomials in $r$ and $s$ with coefficients in $c_1, \dots, c_{66}$. The linear parts of the coefficients form a system of linear equations in $c_1, \dots, c_{66}$ whose associated matrix has rank 53. Then the first-order deformation of $F_1(S)$ in $F_1(X)$ appears as solutions of the system.

In addition to the 53 constraints contributed by the above linear equations, we also have

- four constraints from the $\mathrm{GL}(2)$ action on the coordinates (2-9),
- three constraints from the automorphism group of $\mathbb{P}^1$, and
- four constraints from rescaling the four equations (2-10).

So $F_1(S)$ deforms in $F_1(X)$ to the first order with dimension $66-53-4-3-4=2$.

## 3. Special cubic fourfolds of discriminant 42

This section proves that a generic special cubic fourfold $X \in \mathcal{C}_{42}$ has a unirational parametrization of odd degree and also that $\mathcal{C}_{42}$ is uniruled. We also provide a discussion in the end talking about the difficulty of generalizing our method to higher discriminants.

**3A.** ***The space of singular scrolls.*** The Zariski closure of the locus for degree-9 scrolls forms a component $\mathcal{H}_9$ in the associated Hilbert scheme. Let $\mathcal{H}_9^8 \subset \mathcal{H}_9$ be the closure of the locus parametrizing scrolls with eight isolated singularities. By Proposition 2.1 and Theorem 4.2 we have the estimate:

**Corollary 3.1.** *$\mathcal{H}_9^8$ has codimension at most 8 in $\mathcal{H}_9$.*

Note that $\mathcal{H}_9^8$ parametrizes nonreduced schemes by definition. In the following, we use an overline to specify an element $\bar{S} \in \mathcal{H}_9^8$ and denote by $S$ its underlying reduced subscheme.

Let $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the locus parametrizing smooth cubic fourfolds. Define

$$\mathcal{Z} = \{(\bar{S}, X) \in \mathcal{H}_9^8 \times U : S \subset X\}.$$

By Proposition 2.2 there exists $(\bar{S}, X) \in \mathcal{Z}$ such that $S$ has isolated singularities and $X$ is smooth.

The right projection $p_2 : \mathcal{Z} \to U$ factors through $U_{42}$, the preimage of $\mathcal{C}_{42}$ in $U$. Indeed, by definition $S$ is the image of a rational normal scroll $F \subset \mathbb{P}^{10}$ through a projection. Let $\epsilon : F \to X$ be the composition of the projection followed by the inclusion into $X$. Let $[S]_X^2$ be the self-intersection of $S$ in $X$. Then the number of singularities $D_{S \subset X} = 8$ on $S$ satisfies the double point formula [Fulton 1998, Theorem 9.3]:

$$D_{S\subset X} = \tfrac{1}{2}\big([S]_X^2 - \epsilon^* c_2(T_X) + c_1(T_F)\cdot \epsilon^* c_1(T_X) - c_1(T_F)^2 + c_2(T_F)\big).$$

By using this formula one can get $[S]_X^2 = 41$. Let $h_X$ be the hyperplane class of $X$. Then the intersection table for $X$ is

| | $h_X^2$ | $S$ |
|---|---|---|
| $h_X^2$ | 3 | 9 |
| $S$ | 9 | 41 |

So $X$ has discriminant $3 \cdot 41 - 9^2 = 42$.

**3B.** ***Odd-degree unirational parametrizations.***

**Theorem 3.2.** *Consider the diagram*

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
\mathcal{H}_9^8 & & U_{42} \arrow[r] & \mathcal{C}_{42}
\end{tikzcd}$$

(1) *$\mathcal{Z}$ dominates $U_{42}$. Therefore, a general $X \in \mathcal{C}_{42}$ contains a degree-9 rational scroll with eight isolated singularities and that is smooth otherwise.*

(2) *$\mathcal{C}_{42}$ is uniruled.*

*Proof.* Let $(\bar{S}, X) \in \mathcal{Z}$ be a pair satisfying Proposition 2.2. Then

$$h^0(\mathbb{P}^5, \mathcal{I}_S(3)) = 6.$$

On the other hand, the short exact sequence

$$0 \to \mathcal{I}_S(3) \to \mathcal{O}_{\mathbb{P}^5}(3) \to \mathcal{O}_S(3) \to 0$$

implies that

$$h^0(\mathbb{P}^5, \mathcal{I}_S(3)) \geq h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) - h^0(S, \mathcal{O}_S(3)). \tag{3-1}$$

Let $F \subset \mathbb{P}^{10}$ be the preimage scroll of $S$. Then $H^0(S, \mathcal{O}_S(3))$ consists of the sections in $H^0(F, \mathcal{O}_F(3))$ which cannot distinguish the preimage of a singular point. We have $h^0(F, \mathcal{O}_F(3)) = 58$ by Lemma 1.1, so $h^0(S, \mathcal{O}_S(3)) = 58 - 8 = 50$. So the right-hand side of (3-1) equals $56 - 50 = 6$. Thus, $h^0(\mathbb{P}^5, \mathcal{I}_S(3))$ attains a minimum.

The left projection $p_1 : \mathcal{Z} \to \mathcal{H}_9^8$ has fiber $\mathbb{P}H^0(\mathbb{P}^5, \mathcal{I}_S(3))$ over all $\bar{S} \in \mathcal{H}_9^8$. Because the fiber dimension is an upper-semicontinuous function, there is an open subset $V \subset \mathcal{H}_9^8$ containing $\bar{S}$ such that $\mathcal{Z}$ is a $\mathbb{P}^5$-bundle over $V$. We have $\dim \mathcal{H}_9 = 59$ by Proposition 4.1. Hence, $\dim \mathcal{H}_9^8 \geq 59 - 8 = 51$ by Theorem 4.2. Thus, $\mathcal{Z}$ has dimension at least $51 + 5 = 56$ in a neighborhood of $(\bar{S}, X)$.

By Proposition 2.2(3), $\mathcal{Z}$ has fiber dimension at most 2 over an open subset of $p_2(\mathcal{Z})$ which contains $X$. Hence, $p_2(\mathcal{Z})$ has dimension at least $56 - 2 = 54$ in a neighborhood of $X$. On the other hand, $U_{42}$ is an irreducible divisor in $U$. In particular, $U_{42}$ has dimension 54. So $\mathcal{Z}$ must dominate $U_{42}$.

Next we prove the uniruledness of $\mathcal{C}_{42}$.

We already know that $\mathcal{Z}$ has an open dense subset $\mathcal{Z}^\circ$ isomorphic to a $\mathbb{P}^5$-bundle over $V \subset \mathcal{H}_9^8$. If we can prove that the composition $\mathcal{Z}^\circ \xrightarrow{p_2} U_{42} \to \mathcal{C}_{42}$ does not factor through this bundle map, then the proof is done.

Let $(\bar{S}, X) \in \mathcal{Z}^\circ$ be the pair as before. By Proposition 2.2, $S$ is also contained in a singular cubic $Y$. Assume that the map $\mathcal{Z}^\circ \to \mathcal{C}_{42}$ does factor through the bundle map instead. Then all of the cubics in $\mathbb{P}H^0(\mathbb{P}^5, \mathcal{I}_S(3))$ would be in the same $\mathbb{P}\mathrm{GL}(6)$-orbit. In particular, the smooth cubic $X$ and the singular cubic $Y$ would be isomorphic, but this is impossible. □

**Proposition 3.3** [Hassett 2016, Proposition 38; Hassett and Tschinkel 2001, Proposition 7.4]. *Let $X$ be a cubic fourfold and $S \subset X$ be a rational surface. Suppose $S$ has isolated singularities and smooth normalization, with invariants $D = \deg S$, section genus $g_H$, and self-intersection $\langle S, S\rangle_X$. If*

$$\varrho = \varrho(S, X) := \frac{D(D-2)}{2} + (2 - 2g_H) - \frac{\langle S, S\rangle_X}{2} > 0, \tag{3-2}$$

*then $X$ admits a unirational parametrization $\rho : \mathbb{P}^4 \dashrightarrow X$ of degree $\varrho$.*

**Corollary 3.4.** *A general $X \in \mathcal{C}_{42}$ has a unirational parametrization of degree* 13.

*Proof.* By Theorem 3.2(1), a general cubic fourfold $X \in \mathcal{C}_{42}$ contains a degree-9 scroll $S$ having eight isolated singularities, with $\langle S, S\rangle_X = 41$ and $g_H = 0$. Thus, $\varrho = \frac{1}{2}(9 \cdot 7) + 2 - \frac{41}{2} = 13$ by Proposition 3.3. □

**3C.** ***Problems in higher discriminants.*** Let $\Delta \subset \Sigma^{[2]}$ denote the divisor parametrizing nonreduced subschemes. Recall that it is a $\mathbb{P}^1$-bundle over $\Sigma$. Its fibers correspond to smooth rational curves of degree $2n+1$ in $F_1(X)$, where the polarization on $F_1(X)$ is induced from $\mathbb{G}(1, 5)$. Each rational scroll $S \subset X$ induced by these rational curves has the intersection product

| | $h_X^2$ | $S$ |
|---|---|---|
| $h_X^2$ | $3$ | $2n+1$ |
| $S$ | $2n+1$ | $2n^2+2n+1$ |

where $h_X$ is the hyperplane section class of $X$. One can compute by the double point formula that $S$ has $n(n-2)$ singularities provided that they are all isolated [Hassett and Tschinkel 2001, Proposition 7.2].

In order to obtain an odd-degree unirational parametrization for a generic member of $\mathcal{C}_d$ by Proposition 3.3, we need the existence of a degree-$(2n+1)$ scroll $S \subset \mathbb{P}^5$ with isolated singularities which has $n(n-2)$ singularities and is contained in a cubic fourfold $X$. We also need an estimate of the dimension of the associated Hilbert scheme $\mathcal{H}_{2n+1}^{n(n-2)}$ which contains $S$.

Section 2 builds up a method to find such $S$, but the existence of a cubic fourfold $X$ containing $S$ requires examination with a computer. This works well with $n=4$ because in this case such a generic $S$ is contained in a cubic hypersurface. However, the same phenomenon may fail when $n \geq 5$. Indeed, the Hilbert scheme $\mathcal{H}_{2n+1}^{n(n-2)}$ of degree-$(2n+1)$ scrolls with $n(n-2)$ singularities satisfies $\dim \mathcal{H}_{2n+1}^{n(n-2)} \geq -n^2+14n+11$ by Theorem 4.2 and Proposition 4.1. When $5 \leq n \leq 8$, $\dim \mathcal{H}_{2n+1}^{n(n-2)} \geq 55$, the dimension of cubic hypersurfaces in $\mathbb{P}^5$, so a generic $S \in \mathcal{H}_{2n+1}^{n(n-2)}$ is not in a cubic fourfold. We don't know what happens when $n \geq 9$, but working in this range involves tedious trial and error.

**Question.** Assume $n \geq 2$. Let $S \subset \mathbb{P}^5$ be a degree-$(2n+1)$ rational scroll which has $n(n-2)$ isolated singularities and is smooth otherwise. When is $S$ contained in a cubic fourfold?

## 4. The Hilbert scheme of rational scrolls

Let $N \geq 3$ be an integer. The Hilbert polynomial $P_S$ for a degree-$D$ smooth surface $S \subset \mathbb{P}^N$ has the form

$$P_S(x) = \tfrac{1}{2}Dx^2 + (\tfrac{1}{2}D + 1 - \pi)x + 1 + p_a,$$

where $\pi$ is the genus of a generic hyperplane section and $p_a$ is the arithmetic genus of $S$ [Hartshorne 1977, Chapter V, Exercise 1.2].

We are interested in the case when $S$ is a rational scroll. In this case $\pi = p_a = 0$, so

$$P_S(x) = \tfrac{1}{2}Dx^2 + (\tfrac{1}{2}D + 1)x + 1.$$

Every smooth surface sharing the same Hilbert polynomial has $\pi = 0$ and $p_a = 0$ also and thus is rational. We denote by $\mathrm{Hilb}_{P_S}(\mathbb{P}^N)$ the Hilbert scheme of subschemes in $\mathbb{P}^N$ with Hilbert polynomial $P_S$.

The closure of the locus parametrizing degree-$D$ scrolls forms a component $\mathcal{H}_D \subset \mathrm{Hilb}_{P_S}(\mathbb{P}^N)$. We study this space by stratifying it according to the types of the scrolls. Recall that, by fixing a rational normal scroll $S_{u,v} \subset \mathbb{P}^{D+1}$ where $D = u + v$, a rational scroll $S \subset \mathbb{P}^N$ of type $(u, v)$ is either $S_{u,v}$ itself or the image of $S_{u,v}$ projected from a disjoint $(D - N)$-plane. We define $\mathcal{H}_{u,v} \subset \mathcal{H}_D$ as the closure of the subset consisting of smooth rational scrolls of type $(u, v)$. In this section, we will first show:

**Proposition 4.1.** *Assume $D + 1 \geq N \geq 3$.*

(1) *$\mathcal{H}_D$ is generically smooth of dimension $(N + 1)(D + 2) - 7$.*

(2) *$\mathcal{H}_{u,v}$ is unirational of dimension $(D + 2)N + 2u - 4 - \delta_{u,v}$,*

*where $\delta_{u,v}$ is the Kronecker delta. We also have*

(3) *$\mathcal{H}_{u,v} \subset \mathcal{H}_{u+k,v-k}$ for $0 \leq 2k < v - u$, and $\mathcal{H}_{\lfloor D/2\rfloor,\lceil D/2\rceil} = \mathcal{H}_D$.*

When $D + 1 = N$, a generic element of $\mathcal{H}_{u,v}$ is projectively equivalent to a fixed rational normal scroll $S_{u,v} \subset \mathbb{P}^{D+1}$. In this case $\mathcal{H}_{u,v}$ is birational to $\mathbb{P}\mathrm{GL}(D + 2)$ quotient by the stabilizer of $S_{u,v}$.

When $D \geq N$, a generic element in $\mathcal{H}_{u,v}$ is the projection of $S_{u,v}$ from a $(D - N)$-plane. Note that $\mathcal{H}_{u,v}$ also records the scrolls equipped with embedded points along their singular loci. Such an element occurs when the $(D - N)$-plane contacts the secant variety of $S_{u,v}$. We denote by $\mathcal{H}^r_{u,v} \subset \mathcal{H}_{u,v}$ the closure of the subset parametrizing the schemes such that the singular locus of each of the underlying varieties consists of $\geq r$ isolated singularities. Let $\mathcal{H}^r_D \subset \mathcal{H}_D$ denote the union of $\mathcal{H}^r_{u,v}$ through all possible types.

The main goal of this section is to prove:

**Theorem 4.2.** *Assume $D \geq N \geq 5$, and assume the existence of a degree-$D$ rational scroll with isolated singularities in $\mathbb{P}^N$ which has at least $r$ singularities. Suppose $rN \leq (D + 2)^2 - 1$; then $\mathcal{H}^r_D$ has codimension at most $r(N - 4)$ in $\mathcal{H}_D$. Especially when $r = 1$, $\mathcal{H}^1_D$ is unirational of codimension exactly $N - 4$.*

**4A.** ***The component of rational scrolls.*** Here we give a general picture of the component $\mathcal{H}_D$ and also prove Proposition 4.1. Note that Proposition 4.1(1) follows immediately from Lemma 1.5.

As mentioned before, $\mathcal{H}_{u,v}$ is birational to $\mathbb{P}\mathrm{GL}(D + 2)$ when $D + 1 = N$. In order to study the case of $D \geq N$, we introduce the *projective Stiefel variety*.

**Definition 4.3.** Let $V_{N+1}(\mathbb{C}^{D+2}) = \mathrm{GL}(D+2)/\mathrm{GL}(D-N+1)$ be the homogeneous space of $(N + 1)$-frames in $\mathbb{C}^{D+2}$. The group $\mathbb{C}^*$ acts on $V_{N+1}(\mathbb{C}^{D+2})$ by rescaling, which induces a geometric quotient $\mathbb{V}(N, D + 1)$ that we call a projective Stiefel variety.

$\mathbb{V}(N, D+1)$ has a fiber structure over $\mathbb{G}(N, D+1)$:

$$\begin{tikzcd} \mathbb{P}\mathrm{GL}(N+1) \arrow[r, hook] & \mathbb{V}(N, D+1) \arrow[d, "p"] \\ & \mathbb{G}(N, D+1) \end{tikzcd}$$

An element $\Lambda \in \mathbb{V}(N, D+1)$ over $P \in \mathbb{G}(N, D+1)$ can be expressed as a $(D+2)\times(N+1)$ matrix

$$\Lambda = \begin{pmatrix} v_1 & v_2 & \cdots & v_{N+1} \end{pmatrix}_{(D+2)\times(N+1)}$$

up to rescaling, where $v_1, \ldots, v_{N+1}$ are column vectors which form a basis of the underlying vector space of $P$. In particular, each $\Lambda \in \mathbb{V}(N, D+1)$ naturally defines a projection $\cdot\Lambda : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^N$ by multiplying the coordinates from the right.

Let $S_{u,v} \subset \mathbb{P}^{D+1}$ be the rational normal scroll given by the standard parametrization (1-3). When $D \geq N$, every rational scroll in $\mathcal{H}_{u,v}$ is the image of $S_{u,v}$ under the projection defined by some $\Lambda \in \mathbb{V}(N, D+1)$. So there is a dominant rational map

$$\begin{aligned} \pi = \pi(S_{u,v}) : \mathbb{V}(N, D+1) &\dashrightarrow \mathcal{H}_{u,v}, \\ \Lambda &\mapsto S_{u,v}\cdot\Lambda, \end{aligned} \tag{4-1}$$

where $S_{u,v}\cdot\Lambda$ is the rational scroll given by the parametric equation

$$\begin{aligned} \mathbb{C}^2 &\to \mathbb{P}^N, \\ (s,t) &\mapsto (1, s, \ldots, s^u, t, st, \ldots, s^v t)\cdot\Lambda. \end{aligned}$$

*Proof of Proposition 4.1(2).* Both $\mathbb{P}\mathrm{GL}(D+2)$ and $\mathbb{V}(N, D+1)$ are rational quasiprojective varieties, so $\mathcal{H}_{u,v}$ is unirational either when $D+1 = N$ or $D \geq N$ by the above construction. The formula for the dimension of $\mathcal{H}_{u,v}$ holds by [Coskun 2006, Lemma 2.6]. □

*Proof of Proposition 4.1(3).* By Lemma 1.3, there exists an embedded deformation $\mathcal{S}$ in $\mathbb{P}^{D+1}$ over the dual numbers $D_t = \mathbb{C}[t]/(t^2)$ with $\mathcal{S}_0 \cong S_{u,v}$ and $\mathcal{S}_t \cong S_{u+k,v-k}$ for $t \neq 0$. For every rational scroll $S \in \mathcal{H}_{u,v}$, we can find a $\Lambda \in \mathbb{V}(N, D+1)$ such that $S = S_{u,v}\cdot\Lambda$. Then $\mathcal{S}\cdot\Lambda$ defines an infinitesimal deformation of $S$ to a rational scroll of type $(u+k, v-k)$, which forces the inclusion $\mathcal{H}_{u,v} \subset \mathcal{H}_{u+k,v-k}$ to hold.

When $(u, v) = (\lfloor D/2\rfloor, \lceil D/2\rceil)$, i.e., when $u = v$ or $u = v-1$, we have $\dim \mathcal{H}_D = \dim \mathcal{H}_{u,v} = (N+1)(D+2) - 7$ by Proposition 4.1(1)–(2). Because $\mathcal{H}_D = \bigcup_{u+v=D} \mathcal{H}_{u,v}$, we must have $\mathcal{H}_{\lfloor D/2\rfloor,\lceil D/2\rceil} = \mathcal{H}_D$. □

**4B. *Projections that produce one singularity.*** We are ready to study the locus in $\mathcal{H}_D$ which parametrizes singular scrolls. Assume $D \geq N \geq 5$. Let us start from studying the projections that produce one singularity.

*Notations and facts.* Let $K$ and $L$ be any linear subspaces of $\mathbb{P}^{D+1}$.

(1) We use the same symbol to denote a projective space and its underlying vector space. The dimension always means the projective dimension.

(2) Assume $K \subset L$; we write $K^{\perp L}$ for the orthogonal complement of $K$ in $L$. When $L = \mathbb{P}^{D+1}$, we write $K^{\perp}$ instead of $K^{\perp \mathbb{P}^{D+1}}$.

(3) $K + L$ means the space spanned by $K$ and $L$. We write it as $K \oplus L$ if $K \cap L = \{0\}$ and write it as $K \oplus_{\perp} L$ if $K$ and $L$ are orthogonal to each other.

The two relations

$$(K \cap L)^{\perp} = K^{\perp} + L^{\perp}, \tag{4-2}$$

$$(K \cap L)^{\perp K} = (K \cap L)^{\perp} \cap K \tag{4-3}$$

can be derived by linear algebra.

**Definition 4.4.** Let $l$ and $l'$ be a pair of distinct rulings on $S_{u,v}$, and let $P_{l,l'}$ be the 3-plane spanned by them. We define $\sigma(l, l')$ to be a subvariety of $\mathbb{G}(N, D+1)$ by

$$\sigma(l, l') = \{P \in \mathbb{G}(N, D+1) : \dim(P \cap P_{l,l'}^{\perp}) \geq N - 3\}.$$

**Lemma 4.5.** *Let $p : \mathbb{V}(N, D+1) \to \mathbb{G}(N, D+1)$ be the bundle map. Then $p^{-1}(\sigma(l, l')) \subset \mathbb{V}(N, D+1)$ consists of the projections which produce singularities by making $l$ and $l'$ intersect.*

*Proof.* Let $P \in \mathbb{G}(N, D+1)$ and $\Lambda \in p^{-1}(P)$ be arbitrary. The target space of the projection map $\cdot\,\Lambda$ is actually $P$. Let $L \subset \mathbb{P}^{D+1}$ be any linear subspace; then the image $L \cdot \Lambda$ is identical to $(P^{\perp} + L) \cap P$. On the other hand, (4-2) and (4-3) imply that $(P \cap L^{\perp})^{\perp P} = (P \cap L^{\perp})^{\perp} \cap P = (P^{\perp} + L) \cap P$. Therefore,

$$\begin{aligned} N - 1 = \dim P - 1 &= \dim(P \cap L^{\perp}) + \dim(P \cap L^{\perp})^{\perp P} \\ &= \dim(P \cap L^{\perp}) + \dim((P^{\perp} + L) \cap P) = \dim(P \cap L^{\perp}) + \dim(L \cdot \Lambda). \end{aligned}$$

With $L = P_{l,l'}$, the equation implies that

$$\dim(P \cap P_{l,l'}^{\perp}) \geq N - 3 \iff \dim(P_{l,l'} \cdot \Lambda) \leq 2.$$

It follows that

$$p^{-1}(\sigma(l, l')) = \{\Lambda \in \mathbb{V}(N, D+1) : \dim(P_{l,l'} \cdot \Lambda) \leq 2\}.$$

The image $P_{l,l'} \cdot \Lambda \subset \mathbb{P}^N$ lies in a plane if and only if $l$ and $l'$ intersect each other after the projection $\cdot\,\Lambda : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^N$. As a consequence, every $\Lambda \in p^{-1}(\sigma(l, l'))$ defines a projection which produces a singularity by making $l$ and $l'$ intersect. $\square$

**4C.** ***The geometry of the variety $\sigma(l, l')$.*** The properties of the singular scroll locus in which we are interested are the unirationality and the dimension. As a preliminary, we describe here the geometry of the variety $\sigma(l, l')$, which implies immediately the rationality of $\sigma(l, l')$ and also allows us to find its dimension easily.

Instead of studying $\sigma(l, l')$ alone, the geometry would be more apparent if we consider generally the linear subspaces in $\mathbb{P}^{D+1}$ which satisfy a certain intersectional condition. Fix a $(D-3)$-plane $L \subset \mathbb{P}^{D+1}$. For every $j \geq 0$, we define

$$\sigma_j(L) = \{P \in \mathbb{G}(N, D+1) : \dim(P \cap L) \geq N - 4 + j\}. \tag{4-4}$$

For example, $\sigma_0(L) = \mathbb{G}(N, D+1)$, and $\sigma_1(P_{l,l'}^{\perp}) = \sigma(l, l')$. Note that $P \subset L$ or $L \subset P$ if $j \geq \min(4, D-N+1)$ in (4-4), so we have

$$\begin{aligned} \sigma_j(L) &\supsetneq \sigma_{j+1}(L) && \text{if } 0 \leq j < \min(4, D-N+1), \\ \sigma_j(L) &= \sigma_{j+1}(L) && \text{if } \quad\ j \geq \min(4, D-N+1). \end{aligned}$$

Define $\sigma_j^{\circ}(L) = \{P \in \mathbb{G}(N, D+1) : \dim(P \cap L) = N - 4 + j\}$; then

$$\begin{aligned} \sigma_j^{\circ}(L) &= \sigma_j(L) - \sigma_{j+1}(L) && \text{if } 0 \leq j < \min(4, D-N+1), \\ \sigma_j^{\circ}(L) &= \sigma_j(L) && \text{if } \quad\ j = \min(4, D-N+1). \end{aligned}$$

**Lemma 4.6.** *Assume that* $1 \leq j < \min(4, D-N+1)$; *then* $\sigma_j(L)$ *is singular along* $\sigma_{j+1}(L)$ *and is smooth otherwise. The singularity can be resolved by a* $\mathbb{G}(3-j, D-N+4-j)$*-bundle over* $\mathbb{G}(N-4+j, D-3)$. *Especially*, $\sigma_j(L)$ *is rational with codimension* $j(N-3+j)$ *in* $\mathbb{G}(N, D+1)$.

*Proof.* We define $\boldsymbol{G}_j(L)$ to be the fiber bundle

$$\begin{tikzcd} \mathbb{G}(3-j, D-N+4-j) \arrow[r, hook] & \boldsymbol{G}_j(L) \arrow[d] \\ & \mathbb{G}(N-4+j, L) \end{tikzcd}$$

by taking $\mathbb{G}(3-j, Q^{\perp})$ as the fiber over $Q \in \mathbb{G}(N-4+j, L)$. Apparently $\boldsymbol{G}_j(L)$ is smooth and rational. We denote an element of $\boldsymbol{G}_j(L)$ as $(Q, R)$, where $Q$ belongs to the base and $R$ belongs to the fiber over $Q$.

In the following, we will construct a birational morphism from $\boldsymbol{G}_j(L)$ to $\sigma_j(L)$, which determines the rationality and the codimension immediately. Then we will study the singular locus by analyzing the tangent cone to $\sigma_j(L)$ at a point on $\sigma_{j+1}(L)$.

*Step 1* (a birational morphism from $\boldsymbol{G}_j(L)$ to $\sigma_j(L)$). Every $P \in \sigma_j^{\circ}(L)$ can be decomposed as $P = (P \cap L) \oplus_{\perp} (P \cap L)^{\perp P}$. Because $P \cap L \in \mathbb{G}(N-4+j, L)$

and $(P \cap L)^{\perp P}$ is a $(3-j)$-plane in $(P \cap L)^{\perp}$, this induces a morphism

$$\begin{aligned} \iota : \sigma_j^{\circ}(L) &\to \boldsymbol{G}_j(L), \\ P &\mapsto (P \cap L, (P \cap L)^{\perp P}). \end{aligned}$$

On the other hand, $Q \oplus_{\perp} R \in \sigma_j(L)$ for every $(Q, R) \in \boldsymbol{G}_j(L)$ since $\dim(Q \cap L) = N - 4 + j$ by definition. Thus, there is a morphism

$$\begin{aligned} \epsilon : \boldsymbol{G}_j(L) &\to \sigma_j(L), \\ (Q, R) &\mapsto Q \oplus_{\perp} R. \end{aligned} \tag{4-5}$$

Clearly, the composition $\epsilon \circ \iota$ is the same as the inclusion $\sigma_j^{\circ}(L) \subset \sigma_j(L)$. Therefore, $\epsilon$ is a birational morphism.

The smoothness and rationality of $\boldsymbol{G}_j(L)$ implies that $\sigma_j^{\circ}(L)$ is smooth and that $\sigma_j(L)$ is rational. Moreover,

$$\begin{aligned} \dim \sigma_j(L) &= \dim \boldsymbol{G}_j(L) \\ &= (4-j)(D-N+1) + (N-3+j)(D-N+1-j) \\ &= (N+1)(D-N+1) - j(N-3+j) \\ &= \dim \mathbb{G}(N, D+1) - j(N-3+j). \end{aligned}$$

Hence, $\sigma_j(L)$ has codimension $j(N-3+j)$ in $\mathbb{G}(N, D+1)$.

*Step 2* (the tangent cones to $\sigma_j(L)$). Choose any $P \in \sigma_j(L)$, and fix a $\phi \in T_P\mathbb{G}(N, D+1) \cong \operatorname{Hom}(P, P^{\perp})$. Let $T_P\sigma_j(L)$ be the tangent cone to $\sigma_j(L)$ at $P$. By definition, $\phi \in T_P\sigma_j(L)$ if and only if the condition $\dim(P \cap L) \geq N - 4 + j$ is kept when $P$ moves infinitesimally in the direction of $\phi$, which is equivalent to the condition that $P \cap L$ has a subspace $Q$ of dimension $N-4+j$ such that $\phi(Q) \subset L$.

Consider the decomposition

$$P^{\perp} = (P^{\perp} \cap L) \oplus_{\perp} (P^{\perp} \cap L)^{\perp P^{\perp}}.$$

Define

$$\Gamma : \operatorname{Hom}(P, P^{\perp}) \to \operatorname{Hom}(P \cap L, (P^{\perp} \cap L)^{\perp P^{\perp}})$$

to be the composition of the restriction to $P \cap L$ followed by the right projection of the above decomposition.

For any subspace $Q \subset P \cap L$, $\phi(Q) \subset L$ if and only if $\phi(Q) \subset P^{\perp} \cap L$, if and only if $Q \subset \ker \Gamma(\phi)$. So $L$ has a subspace $Q$ of dimension $N - 4 + j$ such that $\phi(Q) \subset L$ if and only if the (projective) dimension of $\ker \Gamma(\phi)$ is at least $N-4+j$. Therefore,

$$T_P\sigma_j(L) = \{\phi \in \operatorname{Hom}(P, P^{\perp}) : \dim(\ker \Gamma(\phi)) \geq N - 4 + j\}. \tag{4-6}$$

Note that $\sigma_j(L)$ is the disjoint union of $\sigma^\circ_{j+k}(L)$ for all $k$ satisfying

$$0 \le k \le \min(4, D-N+1) - j.$$

Assume $P \in \sigma^\circ_{j+k}(L)$, i.e., $\dim(P \cap L) = N-4+j+k$; then (4-6) is equivalent to

$$T_P\sigma_j(L) = \{\phi \in \operatorname{Hom}(P, P^\perp) : \operatorname{rk}\Gamma(\phi) \le k\}. \tag{4-7}$$

When $k=0$, the constraint becomes $\operatorname{rk}\Gamma(\phi)=0$, so $T_P\sigma_j(L) = \ker\Gamma$ is a vector space. This reflects the fact that $\sigma_j(L)$ is smooth on $\sigma^\circ_j(L)$ for all $j$. On the other hand, from the inequality

$$\dim(P\cap L) + \dim(P^\perp \cap L) \le \dim(L) - 1,$$

we get

$$\begin{aligned}\dim(P^\perp \cap L) &\le \dim(L) - \dim(P\cap L) - 1\\ &= (D-3) - (N-4+j+k) - 1 = D-N-j-k.\end{aligned}$$

It follows that

$$\begin{aligned}\dim((P^\perp \cap L)^{\perp P^\perp}) &= \dim(P^\perp) - \dim(P^\perp\cap L) - 1\\ &\ge (D-N) - (D-N-j-k) - 1 = j+k-1.\end{aligned}$$

So $\dim((P^\perp \cap L)^{\perp P^\perp}) \ge j+k-1 \ge k$ once $k \ge 1$. Under this condition, the linear combination of members of rank $k$ in $\operatorname{Hom}(P\cap L, (P^\perp\cap L)^{\perp P^\perp})$ can have rank exceeding $k$. So $T_P\sigma_j(L)$ cannot be a vector space; thus, $P$ is a singularity of $\sigma_j(L)$. □

Recall that $\sigma(l,l') = \sigma_1(P^\perp_{l,l'})$, so Lemma 4.6 implies:

**Corollary 4.7.** *$\sigma(l,l')$ is rational with codimension $N-2$ in $\mathbb{G}(N, D+1)$.*

**4D.** ***Families of the projections.*** The singularities we have studied are those produced from the intersection of a fixed pair of distinct rulings. Now we are going to make use of the variety $\sigma(l,l')$ to control multiple singularities.

Let $\mathbb{P}^{1[2]}$ be the Hilbert scheme of two points on $\mathbb{P}^1$ and $U \subset \mathbb{P}^{1[2]}$ be the open subset parametrizing reduced subschemes. On the rational normal scroll $S_{u,v}$, the set of $r$ pairs of distinct rulings

$$\{(l_1+l'_1, \dots, l_r+l'_r) : l_i \ne l'_i \text{ for all } i\}$$

is parametrized by $U^{\times r}$.

Let $\Sigma_r$ be a subset of $U^{\times r} \times \mathbb{G}(N, D+1)$ defined by

$$\Sigma_r = \left\{(l_1+l'_1, \dots, l_r+l'_r, P) \in U^{\times r} \times \mathbb{G}(N, D+1) : P \in \bigcap_{i=1}^{r} \sigma(l_i, l'_i)\right\}.$$

Let $p_1$ be the left projection and $p_2$ the right projection. Then there is a diagram

$$\begin{array}{ccc} \bigcap_{i=1}^{r} \sigma(l_i, l_i') & \subset & \Sigma_r \xrightarrow{p_2} \mathbb{G}(N, D+1) \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ (l_1 + l_1', \ldots, l_r + l_r') & \in & U^{\times r} \end{array}$$

By Lemma 4.5, the image $p_2(\Sigma_r)$ consists of the $N$-planes such that the projections to them produce at least $r$ singularities. By the diagram above and Corollary 4.7, the codimension of $\Sigma_r$ in $U^{\times r} \times \mathbb{G}(N, D+1)$ is at most $r(N-4)$. When $r = 1$, $\Sigma_1$ is rational with codimension exactly $N-4$.

Our goal is to compute the dimension of $p_2(\Sigma_r)$, so we care about whether $p_2$ is generically finite onto its image or not. It turns out that the following condition is sufficient (see Lemma 4.9):

> There exists a rational scroll with isolated singularities $S \subset \mathbb{P}^N$ of type $(u, v)$ which has at least $r$ singularities. (4-8)

By considering $S$ as the projection of $S_{u,v}$ from $P^\perp$ for some $N$-plane $P$, we can apply Corollary 1.7 to translate (4-8) into the equivalent statement:

> There exists an $N$-plane $P$ such that $P^\perp$ intersects $S(S_{u,v})$ in $\geq r$ points away from $T(S_{u,v}) \cup Z_2$. (4-8′)

**Proposition 4.8.** *Equation (4-8) holds for $r \leq D - N + 1$.*

*Proof.* By [Catalano-Johnson 1996, Proposition 2.2; Harris 1992, Example 19.10], $\deg(S(S_{u,v})) = \binom{D-2}{2}$. Since $\dim(S(S_{u,v})) = 5$ and $T(S_{u,v}) \cup Z_2$ forms a proper closed subvariety of $S(S_{u,v})$, we can use Bertini's theorem to choose a $(D-4)$-plane $R$ which intersects $S(S_{u,v})$ in $\binom{D-2}{2}$ points outside $T(S_{u,v}) \cup Z_2$. It is easy to check that $\binom{D-2}{2} \geq D - N + 1$. Thus, we can choose $D - N + 1$ of the intersection points to span a $(D - N)$-plane $Q \subset R$. Then $P = Q^\perp$ satisfies the hypothesis. □

Unfortunately, Proposition 4.8 doesn't cover the case $D = 9$, $N = 5$, and $r = 8$ in our proof of the unirationality of discriminant-42 cubic fourfolds. In the following, we estimate the dimension of $p_2(\Sigma_r)$ under the assumption (4-8) and leave the construction of examples to Section 2.

**Lemma 4.9.** *Suppose (4-8) holds. Then $p_2(\Sigma_r)$ has codimension $\leq r(N-4)$ in $\mathbb{G}(N, D+1)$. When $r = 1$, $p_2(\Sigma_1)$ is rational of codimension exactly $N-4$.*

*Proof.* Let $l$ and $l'$ be distinct rulings on $S_{u,v}$. We write $P_{l,l'}$ for the 3-plane spanned by them. Note that $S(S_{u,v}) = \overline{\bigcup_{l \neq l'} P_{l,l'}}$. Let $P$ be an $N$-plane satisfying (4-8′). Then there exists $r$ pairs of distinct rulings $(l_1, l_1'), \ldots, (l_r, l_r')$ such that $P^\perp$ and $P_{l_i, l_i'}$ intersect in exactly one point for each pair $(l_i, l_i')$. This implies that $\dim(P^\perp + P_{l_i,l_i'}) \leq D - N + 3$ for all $i$, which is equivalent to $\dim(P \cap P_{l_i,l_i'}^\perp) \geq N - 3$ for all $i$ by (4-2). Hence, $P \in \bigcap_{i=1}^{r} \sigma(l_i, l_i')$, i.e., $P$ belongs to the image of $p_2$.

Suppose $P^\perp$ intersects $S(S_{u,v})$ in $m$ points; then the $(l_1+l_1', \dots, l_r+l_r')$ in the preimage of $P$ is unique up to the choices of $r$ from $m$ pairs, the reordering of the $r$ pairs, and the transpositions of the rulings in a pair. Hence, $p_2$ is generically finite with $\deg p_2 = \binom{m}{r}\cdot r!$. In particular, $\Sigma_r$ and $p_2(\Sigma_r)$ are equidimensional.

From $\dim(S(S_{u,v})) = 5$ and our assumption that $N \geq 5$, we are able to choose a $(D-N)$-plane which intersects $S(S_{u,v})$ in any one and exactly one point. Therefore, we can find $P$ so that $P^\perp$ intersects $S(S_{u,v})$ in one point outside $T(S_{u,v}) \cup Z_2$. This provides an example of (4-8) for $m=r=1$. It follows that $p_2$ has degree 1, and the image is rational since $\sigma(l_1, l_1')$ is rational by Corollary 4.7. □

## 4E. *Proof of Theorem 4.2.*

**Lemma 4.10.** *Assume $D \geq N \geq 5$, and assume the existence of a degree-$D$ rational scroll $S \subset \mathbb{P}^N$ with isolated singularities which has at least $r$ singularities. Then $\mathcal{H}^r_{u,v}$ has codimension at most $r(N-4)$ in $\mathcal{H}_{u,v}$. For $r=1$, $\mathcal{H}^1_{u,v}$ is unirational of codimension exactly $N-4$.*

*Proof.* We have the diagram

$$\begin{tikzcd} & \mathbb{V}(N, D+1) \arrow[r, dashed, "\pi"] \arrow[d, "p"] & \mathcal{H}_{u,v} \\ \Sigma_r \arrow[r, "p_2"] & \mathbb{G}(N, D+1) & \end{tikzcd}$$

By definition, $\mathcal{H}^r_{u,v} = \pi(p^{-1}(p_2(\Sigma_r)))$.

By Lemma 4.9, $p_2(\Sigma_1)$ is rational, which implies that $\mathcal{H}^1_{u,v}$ is unirational.

It's clear that $p_2(\Sigma_r)$ and $p^{-1}(p_2(\Sigma_r))$ have the same codimension. On the other hand, $p^{-1}(p_2(\Sigma_r))$ and $\pi^{-1}(\pi(p^{-1}(p_2(\Sigma_r))))$ have the same dimension since both contain an open dense subset consisting of the projections which generate $r$ singularities, so the codimension of $p^{-1}(p_2(\Sigma_r))$ is the same as its image through $\pi$. Therefore, $p_2(\Sigma_r)$ and $\mathcal{H}^r_{u,v}$ have the same codimension in their own ambient spaces, and the results follows from Lemma 4.9. □

Lemma 4.10 is the special case of Theorem 4.2 when restricting to the locus of a particular type on the Hilbert scheme. The next lemma shows that a general $S \in \mathcal{H}^r_D$ deforms equisingularly between different types under the assumption $rN \leq (D+2)^2 - 1$. Hence, the dimension estimate made by Lemma 4.10 can be extended regardless of the types.

**Lemma 4.11.** *Assume (4-8) and $rN \leq (D+2)^2 - 1$; then $\mathcal{H}^r_{u,v} = \mathcal{H}_{u,v} \cap \mathcal{H}^r_{u+k,v-k}$ for $0 \leq 2k < v-u$.*

*Proof.* It is trivial that $\mathcal{H}^r_{u,v} \supset \mathcal{H}_{u,v} \cap \mathcal{H}^r_{u+k,v-k}$. To prove that $\mathcal{H}^r_{u,v} \subset \mathcal{H}_{u,v} \cap \mathcal{H}^r_{u+k,v-k}$, it is sufficient to show that a generic element in $\mathcal{H}^r_{u,v}$ deforms equisingularly to an element in $\mathcal{H}_{u+k,v-k}$.

If $(u, v) = (\lfloor D/2\rfloor, \lceil D/2\rceil)$, then there is nothing to prove, so we assume $(u, v) \neq (\lfloor D/2\rfloor, \lceil D/2\rceil)$. The elements satisfying (4-8) form an open dense subset of $\mathcal{H}^r_{u,v}$. Let $S \in \mathcal{H}^r_{u,v}$ be one of them, and assume $S$ is the image of $F \cong S_{u,v} \subset \mathbb{P}^{D+1}$ projected from some $(D-N)$-plane $Q$. By hypothesis, $F$ has $r$ secants $\gamma_1, \ldots, \gamma_r$ incident to $Q$. Assume $\gamma_j \cap F = \{x_j, y_j\}$ for $j = 1, \ldots, r$.

$H^1(T_{\mathbb{P}^{D+1}}|_F) = 0$ by Lemma 1.3, so the short exact sequence $0 \to T_F \to T_{\mathbb{P}^{D+1}}|_F \to N_{F/\mathbb{P}^{D+1}} \to 0$ induces the exact sequence

$$0 \to H^0(T_F) \to H^0(T_{\mathbb{P}^{D+1}}|_F) \to H^0(N_{F/\mathbb{P}^{D+1}}) \to H^1(T_F) \to 0.$$

By Lemma 1.2, $h^1(F, T_F) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(u-v)) = v - u - 1$, the same as the codimension of $\mathcal{H}_{u,v}$ in $\mathcal{H}_D$; thus, a deformation normal to $\mathcal{H}_{u,v}$ is induced from an element in $H^1(T_F)$. In order to prove that the deformation is equisingular, it is sufficient to prove that, for all $\mathcal{F} \in H^1(T_F)$ and its lift $\mathcal{S} \in H^0(N_{F/\mathbb{P}^{D+1}})$, there exists $\alpha \in H^0(T_{\mathbb{P}^{D+1}}|_F)$ such that the vectors $\mathcal{S}(x_j) + \alpha(x_j) \in T_{\mathbb{P}^{D+1},x_j}$ and $\mathcal{S}(y_j) + \alpha(y_j) \in T_{\mathbb{P}^{D+1},y_j}$ keep $\gamma_j$ contact with $Q$ for $j = 1, \ldots, r$, so that $\mathcal{S} + \alpha$ is a lift of $\mathcal{F}$ representing an embedded deformation which preserves the incidence of the $r$ secants to $Q$.

Note that, for arbitrary $p \in \mathbb{P}^{D+1}$, the tangent space $T_{\mathbb{P}^{D+1},p} \cong \mathrm{Hom}(p, p^\perp) \cong p^\perp$ can be considered as a subspace of $\mathbb{P}^{D+1}$. We identify a point in $\mathbb{P}^{D+1}$ with its underlying vector. Let $\gamma = \gamma_j$ for some $j$, and let $\{x, y\} = \gamma \cap S_{u,v}$ with $x = (x_1, \ldots, x_{D+1})$ and $y = (y_1, \ldots, y_{D+1})$. The condition that $\mathcal{S}(x) + \alpha(x)$ and $\mathcal{S}(y) + \alpha(y)$ keep $\gamma$ contact with $Q$ is equivalent to the condition that the set of vectors consisting of $x + \mathcal{S}(x) + \alpha(x)$, $y + \mathcal{S}(y) + \alpha(y)$, and the basis of $Q$ is not independent.

One can compute that $h^0(T_{\mathbb{P}^{D+1}}|_F) = (D+2)^2 - 1$ by the Euler exact sequence $0 \to \mathcal{O}_F \to \mathcal{O}_F(1)^{\oplus(D+2)} \to T_{\mathbb{P}^{D+1}}|_F \to 0$ and Lemma 1.1. Suppose $H^0(T_{\mathbb{P}^{D+1}}|_F)$ has basis $e_1, \ldots, e_{(D+2)^2-1}$; we write the evaluation of $e_i$ at $p$ as $e_i(p) = (e_i(p)_1, \ldots, e_i(p)_{D+1})$. Let $\alpha = \sum_{i\geq 1} \alpha_i e_i$, $\mathcal{S}(x) = \sum_{i\geq 1} c_i e_i(x)$, and $\mathcal{S}(y) = \sum_{i\geq 1} d_i e_i(y)$; also write $Q = (q_{i,j})$ as a $(D-N+1)\times(D+2)$ matrix. Then the dependence condition is equivalent to the condition that the $(D-N+3)\times(D+2)$ matrix

$$A_\gamma = \begin{pmatrix} \alpha_0 x + \alpha_0 \mathcal{S}(x) + \alpha(x) \\ \alpha_0 y + \alpha_0 \mathcal{S}(y) + \alpha(y) \\ Q \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_0 x_0 + \sum(\alpha_0 c_i + \alpha_i) e_i(x)_0 & \cdots & \alpha_0 x_{D+1} + \sum(\alpha_0 c_i + \alpha_i) e_i(x)_{D+1} \\ \alpha_0 y_0 + \sum(\alpha_0 d_i + \alpha_i) e_i(y)_0 & \cdots & \alpha_0 y_{D+1} + \sum(\alpha_0 d_i + \alpha_i) e_i(y)_{D+1} \\ q_{0,0} & \cdots & q_{0,D+1} \\ \vdots & & \vdots \\ q_{D-N,0} & \cdots & q_{D-N,D+1} \end{pmatrix}$$

has rank at most $D-N+2$. Here we homogenize the first two rows by $\alpha_0$, so that the matrix defines a morphism

$$\boldsymbol{A}_\gamma : \mathbb{P}(\mathbb{C} \oplus H^0(T_{\mathbb{P}^{D+1}}|_F)) \cong \mathbb{P}^{(D+2)^2-1} \to \mathbb{P}^{(D-N+3)(D+2)-1},$$
$$(\alpha_0, \dots, \alpha_{(D+2)^2-1}) \mapsto A_\gamma.$$

Let $M_{D-N+2} \subset \mathbb{P}^{(D-N+3)(D+2)-1}$ be the determinantal variety of matrices of rank at most $D-N+2$. Then $\boldsymbol{A}_\gamma^{-1}(M_{D-N+2}) \subset \mathbb{P}(\mathbb{C} \oplus H^0(T_{\mathbb{P}^{D+1}}|_F))$ is an irreducible and nondegenerate subvariety of codimension $N$, whose locus outside $\alpha_0 = 0$ parametrizes those $\alpha \in H^0(T_{\mathbb{P}^{D+1}}|_F)$ such that $\mathcal{S}+\alpha$ preserves the incidence between $\gamma$ and $Q$.

It follows that the intersection $\bigcap_{j=1}^r \boldsymbol{A}_{\gamma_j}^{-1}(M_{D-N+2})$ is nonempty by the hypothesis $rN \leq (D+2)^2 - 1$. Moreover, it is not contained in the hyperplane $\alpha_0 = 0$ for a generic $S \in \mathcal{H}_{u,v}^r$. Indeed, if this doesn't hold, then the limit case $\gamma_1 = \cdots = \gamma_r$ should also be inside the hyperplane $\alpha_0 = 0$. However, the intersection in that case is a multiple of a nondegenerate variety, a contradiction. As a result, for a generic $S \in \mathcal{H}_{u,v}^r$ we can find $\alpha$ from $\bigcap_{j=1}^r \boldsymbol{A}_{\gamma_j}^{-1}(M_{D-N+2})$ which lies on $\{\alpha_0 = 1\} = H^0(T_{\mathbb{P}^{D+1}}|_F)$, so that $\mathcal{S}+\alpha$ preserves the incidence condition between $\gamma_1, \dots, \gamma_r$ and $Q$. □

Now we are ready to finish the proof of Theorem 4.2.

Note that $\mathcal{H}_D^r = \bigcup_{u+v=D} \mathcal{H}_{u,v}^r$. By Lemma 4.11

$$\bigcup_{u+v=D} \mathcal{H}_{u,v}^r = \bigcup_{u+v=D} (\mathcal{H}_{u,v} \cap \mathcal{H}_{\lfloor D/2 \rfloor, \lceil D/2 \rceil}^r) = \mathcal{H}_D \cap \mathcal{H}_{\lfloor D/2 \rfloor, \lceil D/2 \rceil}^r = \mathcal{H}_{\lfloor D/2 \rfloor, \lceil D/2 \rceil}^r.$$

Therefore, $\mathcal{H}_D^r = \mathcal{H}_{\lfloor D/2 \rfloor, \lceil D/2 \rceil}^r$, and the result follows from Lemma 4.10 with $(u, v) = (\lfloor D/2 \rfloor, \lceil D/2 \rceil)$.

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# Quantitative equidistribution of Galois orbits of small points in the $N$-dimensional torus

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We present a quantitative version of Bilu's theorem on the limit distribution of Galois orbits of sequences of points of small height in the $N$-dimensional algebraic torus. Our result gives, for a given point, an explicit bound for the discrepancy between its Galois orbit and the uniform distribution on the compact subtorus, in terms of the height and the generalized degree of the point.

## 1. Introduction

One of the first results concerning the distribution of Galois orbits of points of small height in algebraic varieties is due to Bilu [1997]. It establishes that the Galois orbits of strict sequences of points of small Weil height in an algebraic torus tend to the uniform distribution around the unit polycircle.

Let us introduce some notation before giving the precise formulation of this result. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ together with an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. By $\mathbb{C}^\times$ and $\overline{\mathbb{Q}}^\times$ we denote the multiplicative groups of $\mathbb{C}$ and $\overline{\mathbb{Q}}$, respectively. Let $N \geq 1$; the *Galois orbit* of a point in $(\overline{\mathbb{Q}}^\times)^N$ is its orbit under the action of the *absolute Galois group*, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For a finite set $T \subset (\mathbb{C}^\times)^N$, the discrete probability measure on $(\mathbb{C}^\times)^N$ associated to it is given by

$$\mu_T = \frac{1}{\#T} \sum_{\alpha \in T} \delta_\alpha,$$

where $\#T$ denotes the cardinality of $T$ and $\delta_\alpha$ the Dirac delta measure on $(\mathbb{C}^\times)^N$ supported on $\alpha$. The *unit polycircle* $(S^1)^N$ is the set of points $(z_1, \dots, z_n) \in \mathbb{C}^N$ such that $|z_1| = \cdots = |z_N| = 1$. It is a compact subgroup of $(\mathbb{C}^\times)^N$. We denote by $\lambda_{(S^1)^N}$ the Haar probability measure of $(S^1)^N$, considered as a measure on $(\mathbb{C}^\times)^N$.

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A sequence $(\mu_k)_{k\geq 1}$ of probability measures on $(\mathbb{C}^\times)^N$ *converges weakly* to a probability measure $\mu$ on $(\mathbb{C}^\times)^N$ if, for every compactly supported continuous function $F : (\mathbb{C}^\times)^N \to \mathbb{R}$,

$$\lim_{k\to\infty} \int_{(\mathbb{C}^\times)^N} F\, d\mu_k = \int_{(\mathbb{C}^\times)^N} F\, d\mu.$$

Let $\xi \in \overline{\mathbb{Q}}^\times$ and $f_\xi \in \mathbb{Z}[x]$ be the minimal polynomial of $\xi$ over the integers. The *Weil height* of $\xi$ is defined as

$$\mathrm{h}(\xi) = \frac{m(f_\xi)}{\deg(\xi)},$$

where $m(f_\xi)$ is the *(logarithmic) Mahler measure* of $f_\xi$, given by

$$m(f_\xi) = \frac{1}{2\pi} \int_0^{2\pi} \log|f_\xi(e^{i\theta})|\, d\theta,$$

and $\deg(\xi) = [\mathbb{Q}(\xi) : \mathbb{Q}]$ is the degree of the point $\xi$.

This notion of height extends to $(\overline{\mathbb{Q}}^\times)^N$ as follows:

$$\mathrm{h}(\boldsymbol{\xi}) = \mathrm{h}(\xi_1) + \cdots + \mathrm{h}(\xi_N), \quad \text{for every } \boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in (\overline{\mathbb{Q}}^\times)^N. \tag{1-1}$$

A sequence $(\boldsymbol{\xi}_k)_{k\geq 1}$ in $(\overline{\mathbb{Q}}^\times)^N$ is *strict* if, for every proper algebraic subgroup $Y \subset (\overline{\mathbb{Q}}^\times)^N$, the cardinality of the set $\{k : \boldsymbol{\xi}_k \in Y\}$ is finite.

**Theorem 1.1** [Bilu 1997, Theorem 1.1]. *Let $(\boldsymbol{\xi}_k)_{k\geq 1}$ be a strict sequence in $(\overline{\mathbb{Q}}^\times)^N$ such that $\lim_{k\to\infty} \mathrm{h}(\boldsymbol{\xi}_k) = 0$. Then*

$$\lim_{k\to\infty} \mu_{S_k} = \lambda_{(S^1)^N},$$

*where $\mu_{S_k}$ is the discrete probability measure associated to the Galois orbit $S_k$ of $\boldsymbol{\xi}_k$.*

This result was inspired by a previous work of Szpiro, Ullmo and Zhang [Szpiro et al. 1997] on the equidistribution of points of small Néron–Tate height in abelian varieties. It was originally motivated by Bogomolov's conjecture, solved in [Ullmo 1998; Zhang 1998]. The results of Szpiro, Ullmo and Zhang and of Bilu were largely generalized to other heights and places [Rumely 1999; Baker and Hsia 2005; Favre and Rivera-Letelier 2006; Baker and Rumely 2006; Chambert-Loir 2006; Yuan 2008; Gubler 2008; Berman and Boucksom 2010; Chen 2011; Burgos Gil et al. 2015]. In particular, these results established the equidistribution of Galois orbits of sequences of small points for all places of $\mathbb{Q}$ and heights associated to algebraic dynamical systems. Moreover, this equidistribution phenomenon holds for the bigger set of test functions with logarithmic singularities along divisors with minimal height; see [Chambert-Loir and Thuillier 2009].

As a general fact, these equidistribution theorems are formulated in a qualitative way, in the sense that no information is provided on the rate of convergence towards the equidistribution. An exception is [Favre and Rivera-Letelier 2006], where a bound for this rate of convergence is given for a large class of heights of points in the projective line and all places of $\mathbb{Q}$. Independently, Petsche [2005] gave a quantitative version of Bilu's result for the case of dimension one.

In this paper, we present a quantitative version of Theorem 1.1 for the general $N$-dimensional case. In particular, we provide a bound for the integral of a suitable test function with respect to the signed measure defined by the difference of the discrete probability measure associated to the Galois orbit of a point in $(\overline{\mathbb{Q}}^{\times})^N$ and the measure $\lambda_{(S^1)^N}$. This bound is given in terms of the height of the point, a higher dimensional generalization of the notion of the degree of an algebraic number, and a constant depending only on the test function.

To state our main result properly, let us introduce further definitions and notations. For every $\boldsymbol{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$, consider the monomial map

$$\begin{aligned} \chi^{\boldsymbol{n}} : (\overline{\mathbb{Q}}^{\times})^N &\to \overline{\mathbb{Q}}^{\times} \\ \boldsymbol{z} = (z_1, \dots, z_N) &\mapsto \chi^{\boldsymbol{n}}(\boldsymbol{z}) = z_1^{n_1} \cdots z_N^{n_N}. \end{aligned}$$

We define the *generalized degree* of a point $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$ by

$$\mathscr{D}(\boldsymbol{\xi}) = \min_{\boldsymbol{n} \neq \boldsymbol{0}} \bigl\{ \|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \bigr\}, \tag{1-2}$$

where $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$ is the degree of the point $\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) \in \overline{\mathbb{Q}}^{\times}$ and $\|\cdot\|_1$ is the 1-norm on $\mathbb{C}^N$. For a particular choice of $\boldsymbol{\xi}$, the generalized degree can be computed with a finite number of operations; see Remark 2.7.

Let us write $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and identify $\mathbb{T}^N \times \mathbb{R}^N$ with $(\mathbb{C}^{\times})^N$ via the logarithmic-polar coordinate change of variables

$$(\boldsymbol{\theta}, \boldsymbol{u}) = ((\theta_1, \dots, \theta_N), (u_1, \dots, u_N)) \mapsto (e^{2\pi i \theta_1 + u_1}, \dots, e^{2\pi i \theta_N + u_N}).$$

On $\mathbb{T}^N \times \mathbb{R}^N \simeq (\mathbb{C}^{\times})^N$, consider the translation invariant distance, defined as

$$\mathrm{d}((\boldsymbol{\theta}, \boldsymbol{u}), (\boldsymbol{\theta}', \boldsymbol{u}')) = \left( \sum_{l=1}^{N} \mathrm{d}_{\mathrm{ang}}(\theta_l, \theta_l')^2 + |u_l - u_l'|^2 \right)^{\frac{1}{2}},$$

where $\mathrm{d}_{\mathrm{ang}}(\theta_l, \theta_l')$ is the Euclidean distance in $S^1$ between $e^{2\pi i \theta_l}$ and $e^{2\pi i \theta_l'}$, divided by $2\pi$.

A function $F : (\mathbb{C}^{\times})^N \to \mathbb{R}$ belongs to the set of test functions $\mathcal{F}$ if it satisfies:

(i) $F$ is a Lipschitz function with respect to the distance d;

(ii) the restriction $F_0 = F|_{(S^1)^N}$ is in $\mathscr{C}^{N+1}((S^1)^N, \mathbb{R})$.

The set $\mathcal{F}$ contains all compactly supported functions in $\mathscr{C}^{N+1}((\mathbb{C}^\times)^N, \mathbb{R})$.

The following is the main result of this paper.

**Theorem 1.2.** *There is a constant* $C \leq 64$ *such that, for every* $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ *with* $\mathrm{h}(\boldsymbol{\xi}) \leq 1$ *and every* $F \in \mathcal{F}$,

$$\left| \int_{(\mathbb{C}^\times)^N} F\, d\mu_S - \int_{(\mathbb{C}^\times)^N} F\, d\lambda_{(S^1)^N} \right| \leq c(F) \left( 4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}},$$

*where* $S$ *is the Galois orbit of* $\boldsymbol{\xi}$, $\mu_S$ *the discrete probability measure associated to it and* $c(F)$ *a positive constant depending only on* $F$.

For every test function $F \in \mathcal{F}$, the function $F_0$, its Fourier transform $\widehat{F}_0$, all the first order partial derivatives of $F_0$, and their corresponding Fourier transforms are integrable with respect to a Haar measure (Theorem A.1). In logarithmic-polar coordinates $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$. Then, as shown in the proof of Theorem 1.2, the constant $c(F)$ can be bounded by

$$c(F) \leq 2\,\mathrm{Lip}(F) + 16 \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial \theta_l}} \right\|_{\mathrm{L}^1},$$

where $\mathrm{Lip}(F)$ is the Lipschitz constant of $F$ with respect to the distance d of $(\mathbb{C}^\times)^N$ and where $\|\cdot\|_{\mathrm{L}^1}$ stands for the $\mathrm{L}^1$-norm of a function on the locally compact abelian group $\mathbb{Z}^N$ with respect to the standard Haar measure.

Our main theorem is a quantitative version of Bilu's result. Indeed, if we consider a strict sequence $(\boldsymbol{\xi}_k)_{k\geq 1}$ in $(\overline{\mathbb{Q}}^\times)^N$ such that $\mathrm{h}(\boldsymbol{\xi}_k) \to 0$ as $k \to \infty$, we necessarily have that $\mathscr{D}(\boldsymbol{\xi}_k) \to \infty$ as $k \to \infty$ (Lemma 2.8). Hence, for every function $F \in \mathcal{F}$, Theorem 1.2 implies that

$$\lim_{k\to\infty} \int_{(\mathbb{C}^\times)^N} F\, d\mu_{S_k} = \int_{(\mathbb{C}^\times)^N} F\, d\lambda_{(S^1)^N},$$

where $\mu_{S_k}$ is the discrete probability measure associated to the Galois orbit $S_k$ of $\boldsymbol{\xi}_k$. Since $\mathcal{F}$ contains a dense subset of the set of compactly supported continuous functions on $(\mathbb{C}^\times)^N$, we deduce Theorem 1.1.

The rate of convergence in Theorem 1.2 has the expected exponent $\frac{1}{2}$, as in [Favre and Rivera-Letelier 2006], see also Theorem 3.1. On the other hand, one could ask if, for the general $N$-dimensional case, the constant $c(F)$ might be bounded by the Lipschitz constant of the test function, as in their paper.

The idea of the proof of our result is to reduce the problem, via monomial maps, to the one-dimensional situation as it was done in [Bilu 1997; D'Andrea et al. 2014]. In this setting, we apply Favre and Rivera-Letelier's result (Theorem 3.1). Then, we lift the obtained quantitative control to the $N$-dimensional torus by applying the Fourier inversion formula and a study of the Fourier–Stieltjes transform of the discrete probability measure associated to the orbit of the point.

This paper is structured as follows. Section 2 contains preliminary theory and general results on Fourier analysis, measures on the Riemann sphere, Galois invariant sets, and the generalized degree. In Section 3, we give the proof of Theorem 1.2, which is divided in several propositions and lemmas. At the end of the paper there are two appendices, the first one studies the set of test functions $\mathcal{F}$, and the second the Lipschitz constant of an auxiliary function used in Section 3.

## 2. Preliminaries

**2.1.** ***Fourier analysis.*** In this section we review basic concepts of Fourier analysis on $\mathbb{T}^N$. We refer the reader to [Rudin 1962] for the proof of the stated results.

Let $p \geq 1$. Given a function $H : \mathbb{T}^N \to \mathbb{C}$, its $\mathrm{L}^p$*-norm* is defined by

$$\|H\|_{\mathrm{L}^p} = \left(\int_{\mathbb{T}^N} |H(\boldsymbol{\theta})|^p \, d\boldsymbol{\theta}\right)^{\frac{1}{p}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

We say that $H \in \mathrm{L}^p(\mathbb{T}^N)$ if this norm is finite. In particular, the function $H$ is *Haar-integrable* if it lies in $\mathrm{L}^1(\mathbb{T}^N)$. Similarly, for a function $G : \mathbb{Z}^N \to \mathbb{C}$, its $\mathrm{L}^p$*-norm* is defined by

$$\|G\|_{\mathrm{L}^p} = \left(\sum_{\boldsymbol{n} \in \mathbb{Z}^N} |G(\boldsymbol{n})|^p\right)^{\frac{1}{p}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

and we say that $G \in \mathrm{L}^p(\mathbb{Z}^N)$ if this norm is finite. Also, $G$ is *Haar-integrable* if it lies in $\mathrm{L}^1(\mathbb{Z}^N)$.

Let $H : \mathbb{T}^N \to \mathbb{C}$ be Haar-integrable. Its Fourier transform is the function $\widehat{H} : \mathbb{Z}^N \to \mathbb{C}$, defined as

$$\widehat{H}(\boldsymbol{n}) = \int_{\mathbb{T}^N} H(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}} \, d\boldsymbol{\theta},$$

where

$$\boldsymbol{n} \cdot \boldsymbol{\theta} = (n_1, \dots, n_N) \cdot (\theta_1, \dots, \theta_N) = n_1\theta_1 + \cdots + n_N\theta_N.$$

If $\widehat{H}$ is also Haar-integrable, the *Fourier inversion formula* states that

$$H(\boldsymbol{\theta}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \widehat{H}(\boldsymbol{n}) e^{2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}}.$$

For $H \in (\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{T}^N)$, *Plancherel's theorem* states that $\widehat{H} \in \mathrm{L}^2(\mathbb{Z}^N)$ and

$$\|\widehat{H}\|_{\mathrm{L}^2} = \|H\|_{\mathrm{L}^2}.$$

For every finite and regular positive measure $\lambda$ on $\mathbb{T}^N$, its *Fourier–Stieltjes transform* is the function $\widehat{\lambda}:\mathbb{Z}^N\to\mathbb{C}$ given by

$$\widehat{\lambda}(\boldsymbol{n})=\int_{\mathbb{T}^N} e^{-2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}\,d\lambda(\boldsymbol{\theta}).$$

We now establish some auxiliary results that will be useful for the proof of Theorem 1.2.

**Lemma 2.1.** *Let $H:\mathbb{T}^N\to\mathbb{C}$ be a Haar-integrable function such that its Fourier transform $\widehat{H}$ is also Haar-integrable. For any finite regular measure $\lambda$ on $\mathbb{T}^N$ we have that $H$ is integrable with respect to $\lambda$ and $\widehat{H}\widehat{\lambda}$ is Haar-integrable. Moreover,*

$$\int_{\mathbb{T}^N} H\,d\lambda=\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}.$$

*Proof.* Let $\lambda$ be a finite regular measure on $\mathbb{T}^N$. Its Fourier–Stieltjes transform is the function $\widehat{\lambda}:\mathbb{Z}^N\to\mathbb{C}$ given by

$$\widehat{\lambda}(\boldsymbol{n})=\int_{\mathbb{T}^N} e^{-2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}\,d\lambda(\boldsymbol{\theta}).$$

Since both $H$ and $\widehat{H}$ are Haar-integrable, we apply the Fourier inversion formula that, together with Fubini's theorem, leads to

$$\begin{aligned}\int_{\mathbb{T}^N} H d\lambda&=\int_{\mathbb{T}^N}\Big(\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{H}(\boldsymbol{n})e^{2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}\Big)\,d\lambda(\boldsymbol{\theta})\\&=\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{H}(\boldsymbol{n})\Big(\int_{\mathbb{T}^N} e^{2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}\,d\lambda(\boldsymbol{\theta})\Big)\\&=\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})},\end{aligned}$$

this equality containing the fact that $H$ is integrable with respect to $\lambda$ and that $\widehat{H}\widehat{\lambda}$ is Haar-integrable. $\square$

**Lemma 2.2.** *Let $H:\mathbb{T}^N\to\mathbb{C}$ be a Haar-integrable function such that $\widehat{H}$ is also Haar-integrable, and let $\lambda$ be a finite regular measure on $\mathbb{T}^N$. Then*

$$\int_{\mathbb{T}^N} H\,d\lambda-\int_{\mathbb{T}^N} H\,d\lambda_{(S^1)^N}=\widehat{H}(\boldsymbol{0})\big(\overline{\widehat{\lambda}(\boldsymbol{0})}-1\big)+\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}.$$

*Proof.* Since $\lambda_{(S^1)^N}$ is the Haar probability measure of $\mathbb{T}^N$, for any $\boldsymbol{n}\in\mathbb{Z}^N$,

$$\widehat{\lambda}_{(S^1)^N}(\boldsymbol{n})=\int_{\mathbb{T}^N} e^{-2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}\,d\boldsymbol{\theta}=\begin{cases}1 & \text{if } \boldsymbol{n}=\boldsymbol{0},\\ 0 & \text{otherwise.}\end{cases}$$

Hence, by Lemma 2.1 we obtain

$$\int_{(\mathbb{C}^\times)^N} H\, d\lambda_{(S^1)^N} = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}_{(S^1)^N}(\boldsymbol{n})} = \widehat{H}(\boldsymbol{0}).$$

Then we have

$$\int_{\mathbb{T}^N} H d\lambda - \int_{\mathbb{T}^N} H d\lambda_{(S^1)^N}$$
$$= \Big(\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}\Big) - \widehat{H}(\boldsymbol{0}) = \widehat{H}(\boldsymbol{0})\big(\overline{\widehat{\lambda}(\boldsymbol{0})} - 1\big) + \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \widehat{H}(\boldsymbol{n})\overline{\widehat{\lambda}(\boldsymbol{n})}. \qquad \square$$

**2.2.** ***Galois invariant sets.*** In this section we work with Galois invariant sets and study their height. For further details on basic Galois theory we refer to [Lang 2002], and on heights of points to [Bombieri and Gubler 2006].

Let $\xi \in \overline{\mathbb{Q}}^\times$ and $f_\xi \in \mathbb{Z}[x]$ be the minimal polynomial of $\xi$ over the integers. Recall that the Weil height of $\xi$ is defined as

$$\mathrm{h}(\xi) = \frac{m(f_\xi)}{\deg(\xi)},$$

where $m(f_\xi)$ is the Mahler measure of $f_\xi$, given by

$$m(f_\xi) = \frac{1}{2\pi}\int_0^{2\pi} \log|f_\xi(e^{i\theta})|\, d\theta,$$

and $\deg(\xi) = [\mathbb{Q}(\xi) : \mathbb{Q}]$ is the degree of the point $\xi$. This notion of height coincides with that in [Bombieri and Gubler 2006, §1.5], which is defined using local decompositions.

Let $T \subset (\overline{\mathbb{Q}}^\times)^N$ be a finite Galois-invariant set, its height is defined as

$$\mathrm{h}(T) = \sum_{\boldsymbol{\alpha}\in T} \mathrm{h}(\boldsymbol{\alpha}),$$

where $\mathrm{h}(\alpha)$ is the height of $\alpha \in (\overline{\mathbb{Q}}^\times)^N$ as in (1-1). In particular, since the height of two Galois conjugate points coincide, if $T \subset (\overline{\mathbb{Q}}^\times)^N$ is a Galois orbit of cardinality $D$, then

$$\mathrm{h}(T) = D\,\mathrm{h}(\boldsymbol{\alpha}),$$

for any $\boldsymbol{\alpha} \in T$.

**Lemma 2.3.** *Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ in $(\overline{\mathbb{Q}}^\times)^N$, $S$ its Galois orbit, and set $D = \#S$. Then*

(1) $D = [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}]$,

(2) $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$ *divides $D$ for every $\boldsymbol{n} \in \mathbb{Z}^N$.*

*Proof.* If a group $G$ acts on a finite set $S$ transitively, then for any $x \in S$ the index of the stabilizer $G_x$ is equal to $\#S$, because the cosets $G/G_x$ stay in a natural one-to-one correspondence with the element of $S$. Applying this to $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $x = \boldsymbol{\xi}$, and $S$ the Galois orbit of $\boldsymbol{\xi}$, we find $[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\boldsymbol{\xi}))] = D$, which by Galois theory implies that $[\mathbb{Q}(\boldsymbol{\xi}) : \mathbb{Q}] = D$, proving the first statement. The second statement is immediate because $\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) \in \mathbb{Q}(\boldsymbol{\xi})$. $\square$

**Lemma 2.4.** *Let* $\xi \in \overline{\mathbb{Q}}^{\times}$, $d = \deg(\xi)$, *and* $S$ *its Galois orbit. Then*

$$\frac{1}{d}\sum_{\alpha\in S} \bigl|\log|\alpha|\bigr| \le 2\,\mathrm{h}(\xi).$$

*Proof.* We have

$$\begin{aligned}\frac{1}{d}\sum_{\alpha\in S} \bigl|\log|\alpha|\bigr| &= \frac{1}{d}\sum_{\alpha\in S} \max\{-\log|\alpha|, \log|\alpha|\}\\ &= \frac{1}{d}\sum_{\alpha\in S} \log\max\Bigl\{\frac{1}{|\alpha|}, |\alpha|\Bigr\}\\ &= \frac{1}{d}\sum_{\alpha\in S} \bigl(\log\max\{1, |\alpha|^2\} - \log|\alpha|\bigr).\end{aligned}$$

Let $P_\xi(x) = a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of $\xi$ over $\mathbb{Z}$. Since $S$ is the Galois orbit of $\xi$,

$$P_\xi(x) = a_d \prod_{\alpha\in S} (x-\alpha) \quad \text{and} \quad a_0 = (-1)^d a_d \prod_{\alpha\in S} \alpha.$$

Since $|a_0|$ is a nonzero positive integer, we obtain

$$\begin{aligned}\frac{1}{d}\sum_{\alpha\in S} (\log\max\{1, |\alpha|^2\} - \log|\alpha|) &= \frac{1}{d}\sum_{\alpha\in S} \log\max\{1, |\alpha|^2\} + \log\frac{|a_d|}{|a_0|}\\ &\le \frac{1}{d}\sum_{\alpha\in S} \log\max\{1, |\alpha|^2\} + \log|a_d|\\ &\le 2\Bigl(\frac{1}{d}\sum_{\alpha\in S} \log\max\{1, |\alpha|\} + \log|a_d|\Bigr)\\ &= 2\,\mathrm{h}(\xi),\end{aligned}$$

where the last equality is given by Jensen's formula for the Mahler measure [Bombieri and Gubler 2006, Proposition 1.6.5]. $\square$

**Lemma 2.5.** *Let* $\boldsymbol{\xi}_1 \in (\overline{\mathbb{Q}}^\times)^N$ *and consider its Galois orbit* $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_D\}$, *where* $\boldsymbol{\xi}_j = (\xi_{j,1}, \dots, \xi_{j,N})$ *for every* $j = 1, \dots, D$. *Then*

$$\frac{1}{D}\sum_{l=1}^{N}\sum_{j=1}^{D}\bigl|\log|\xi_{j,l}|\bigr| \le 2\,\mathrm{h}(\boldsymbol{\xi}_1).$$

*Proof.* For every $l = 1, \dots, N$, the elements $\xi_{j,l}$ and $\xi_{k,l}$ are conjugates. Denote by $S_l$ the Galois orbit of $\xi_{1,l}$. By Lemma 2.3, we have that $\#S_l = \deg(\xi_{1,l})$ divides $D$. That is, there is a positive integer $k_l$ such that $D = \deg(\xi_{1,l})k_l$, where $k_l$ is exactly the number of times each element of the orbit is repeated in $\{\xi_{1,l}, \dots, \xi_{D,l}\}$. Thus

$$\begin{aligned}\frac{1}{D}\sum_{l=1}^{N}\sum_{j=1}^{D}\bigl|\log|\xi_{j,l}|\bigr| &= \sum_{l=1}^{N}\frac{1}{k_l\deg(\xi_{1,l})}\sum_{j=1}^{D}\bigl|\log|\xi_{j,l}|\bigr| \\ &= \sum_{l=1}^{N}\frac{1}{\deg(\xi_{1,l})}\sum_{\alpha\in S_l}\bigl|\log|\alpha|\bigr| \\ &\le \sum_{l=1}^{N} 2\,\mathrm{h}(\xi_{1,l}) = 2\,\mathrm{h}(\boldsymbol{\xi}_1),\end{aligned}$$

where the inequality follows from Lemma 2.4. ☐

**Lemma 2.6.** *Let* $S \subset \overline{\mathbb{Q}}^\times$ *be a Galois-invariant set of cardinality* $D$. *For every* $0 < \delta < 1$,

$$\#S_\delta < 2\Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\mathrm{h}(S),$$

*where* $S_\delta = \Bigl\{\alpha \in S : \bigl|\log|\alpha|\bigr| > \log\frac{1}{\delta}\Bigr\}$.

*Proof.* Write $S$ as a finite disjoint union of Galois orbits

$$S = S_1 \sqcup \cdots \sqcup S_m.$$

By definition, for any $\alpha \in S_\delta$,

$$1 < \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\bigl|\log|\alpha|\bigr|.$$

Hence,

$$\begin{aligned}\#S_\delta &< \sum_{\alpha\in S_\delta}\Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\bigl|\log|\alpha|\bigr| \le \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\sum_{\alpha\in S}\bigl|\log|\alpha|\bigr| \\ &= \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\sum_{l=1}^{m}\sum_{\alpha\in S_l}\bigl|\log|\alpha|\bigr| \le \Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\sum_{l=1}^{m}2\,\mathrm{h}(S_l) \\ &= 2\Bigl(\log\frac{1}{\delta}\Bigr)^{-1}\mathrm{h}(S),\end{aligned}$$

where the last inequality holds by Lemma 2.4. □

**2.3.** ***The generalized degree.*** We now study the notion of the generalized degree of a point in the algebraic torus defined in (1-2). First of all, let us see that in dimension one, it coincides with the notion of the degree of the algebraic number. Let $\xi \in \overline{\mathbb{Q}}^\times$; then

$$\mathscr{D}(\xi) = \min_{n \neq 0}\{|n| \deg(\xi^n)\}.$$

For every nonzero integer $n$, let $Q_n(x)$ be the minimal polynomial of $\xi^{|n|}$ over $\mathbb{Z}$, which is of degree $\deg(\xi^{|n|}) = \deg(\xi^n)$. By setting $R_n(x) = Q_n(x^{|n|}) \in \mathbb{Z}[x]$ we obtain that $R_n(\xi) = 0$ and this implies that

$$\deg(\xi) \leq \deg(R_n(x)) = |n| \deg(\xi^n).$$

Hence, $\mathscr{D}(\xi) = \deg(\xi)$.

**Remark 2.7.** For $N \geq 1$ and every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ in $(\overline{\mathbb{Q}}^\times)^N$,

$$\mathscr{D}(\boldsymbol{\xi}) \leq \min\{\deg(\xi_1), \dots, \deg(\xi_N)\}.$$

This holds since

$$\{\deg(\xi_1), \dots, \deg(\xi_N)\} \subset \{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) : \boldsymbol{n} \neq \boldsymbol{0}\}.$$

Thus, for a particular choice of $\boldsymbol{\xi}$, the generalized degree can be computed after a finite number of steps by considering all $\boldsymbol{n} \neq \boldsymbol{0}$ such that

$$\|\boldsymbol{n}\|_1 \leq \min\{\deg(\xi_1), \dots, \deg(\xi_N)\}.$$

For $N = 1$, a strict sequence $(\xi_k)_{k \geq 1}$ in $\overline{\mathbb{Q}}^\times$ such that $\lim_{k\to\infty} \mathrm{h}(\xi_k) = 0$ shows that $\lim_{k\to\infty} \deg(\xi_k) = \infty$. Indeed, to the contrary suppose there is some $c > 0$ such that $\deg(\xi_k) \leq c$ for every $k \geq 0$. By Northcott's theorem [Bombieri and Gubler 2006, Theorem 1.6.8], there are only finitely many elements with bounded degree and height. Hence, there is some $\alpha \in \overline{\mathbb{Q}}^\times$ such that $\xi_k = \alpha$ for infinitely many values of $k$. Since $\mathrm{h}(\xi_k)$ tends to 0 as $k$ goes to infinity, by Kronecker's theorem [op. cit., Theorem 1.5.9] we necessarily have $\mathrm{h}(\alpha) = 0$, which implies that $\alpha$ is a root of unity. In particular, there is an infinite subsequence of $(\xi_k)_{k \geq 1}$ contained in a proper algebraic subgroup of $\overline{\mathbb{Q}}^\times$ which is not possible by the assumption that the sequence is strict.

The following lemma is a generalization of this fact to higher dimensions.

**Lemma 2.8.** *Let $(\boldsymbol{\xi}_k)_{k \geq 1}$ be a strict sequence in $(\overline{\mathbb{Q}}^\times)^N$ such that*

$$\lim_{k\to\infty} \mathrm{h}(\boldsymbol{\xi}_k) = 0.$$

*Then*

$$\lim_{k\to\infty} \mathscr{D}(\boldsymbol{\xi}_k) = \infty.$$

*Proof.* Since the sequence $(\boldsymbol{\xi}_k)_{k\geq 0}$ is strict, the sequence $(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k))_{k\geq 0}$ is a strict sequence in $\overline{\mathbb{Q}}^\times$ for every $\boldsymbol{n} \neq \boldsymbol{0}$.

Write $\boldsymbol{\xi}_k = (\xi_{k,1}, \ldots, \xi_{k,N})$ and let $\boldsymbol{n} = (n_1, \ldots, n_N) \neq \boldsymbol{0}$. Then

$$\begin{aligned} \mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k)) &= \mathrm{h}(\xi_{k,1}^{n_1} \cdots \xi_{k,N}^{n_N}) \\ &\leq \mathrm{h}(\xi_{k,1}^{n_1}) + \ldots + \mathrm{h}(\xi_{k,N}^{n_N}) \\ &= |n_1|\,\mathrm{h}(\xi_{k,1}) + \cdots + |n_N|\,\mathrm{h}(\xi_{k,N}) \\ &\leq \|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}_k) \xrightarrow{k\to\infty} 0, \end{aligned}$$

where the first inequality follows from [Bombieri and Gubler 2006, §1.5.14].

Thus, as we just saw, for every $\boldsymbol{n} \neq \boldsymbol{0}$

$$\lim_{k\to\infty} \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k)) = \infty.$$

Finally, by Remark 2.7, for every $k \geq 0$ there is $\boldsymbol{n}_k \neq \boldsymbol{0}$ with bounded 1-norm such that $\mathscr{D}(\boldsymbol{\xi}_k) = \|\boldsymbol{n}_k\|_1 \deg(\chi^{\boldsymbol{n}_k}(\boldsymbol{\xi}_k))$. Hence

$$\lim_{k\to\infty} \mathscr{D}(\boldsymbol{\xi}_k) = \infty,$$

completing the proof. □

## 3. Proof of the main result

In this section we give the proof of Theorem 1.2. As we mentioned in the introduction, we do so by using Fourier analysis techniques and reducing the problem, via projections, to the one-dimensional case, where the result follows from [Favre and Rivera-Letelier 2006, Corollary 1.4].

Before stating this result, we give the definition of the spherical distance on the Riemann sphere. Let us identify the projective complex line with the unit sphere $S^2$ of $\mathbb{R}^3$. Let $S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$ be the stereographic projection, where we identify the equator of $S^2$ with the set $\{z \in \mathbb{C} : |z| = 1\}$. Composing it with the standard inclusion $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ gives a map $S^2 \setminus \{(0,0,1)\} \to \mathbb{P}^1(\mathbb{C}) \setminus \{(0:1)\}$, that we extend to a homeomorphism $\rho : S^2 \to \mathbb{P}^1(\mathbb{C})$ by setting $\rho(0,0,1) = (0:1)$. The spherical distance $\mathrm{d}_{\mathrm{sph}}$ on $\mathbb{P}^1(\mathbb{C})$ is given by the length of the arc on $S^2$ under this identification and extended to $\mathbb{P}^1(\mathbb{C})^N$ for $\boldsymbol{p} = (p_1, \ldots, p_N)$ and $\boldsymbol{p}' = (p'_1, \ldots, p'_N)$ as

$$\mathrm{d}_{\mathrm{sph}}(\boldsymbol{p}, \boldsymbol{p}') = \left(\sum_{l=1}^{N} \mathrm{d}_{\mathrm{sph}}(p_l, p'_l)^2\right)^{\frac{1}{2}}.$$

A function $f : \mathbb{P}^1(\mathbb{C})^N \to \mathbb{C}$ is a *Lipschitz function* with respect to the distance

$\mathrm{d_{sph}}$ if there is a constant $K \geq 0$ such that

$$|f(\boldsymbol{p}) - f(\boldsymbol{p}')| \leq K\, \mathrm{d_{sph}}(\boldsymbol{p}, \boldsymbol{p}') \quad \text{for every } \boldsymbol{p}, \boldsymbol{p}' \in \mathbb{P}^1(\mathbb{C})^N. \tag{3-1}$$

If $f$ is a Lipschitz function with respect to the spherical distance, then its *Lipschitz constant* $\mathrm{Lip_{sph}}(f)$ is the smallest $K \geq 0$ such that (3-1) holds.

We now state the result of Favre and Rivera-Letelier together with the explicit constants computed in the Ph.D. thesis of Narváez-Clauss [2016, Theorem II].

**Theorem 3.1.** *There is a positive constant $C_0 \leq 15$ such that for every $\mathscr{C}^1$-function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ and every $\xi \in \overline{\mathbb{Q}}^\times$*

$$\left|\int_{\mathbb{P}^1(\mathbb{C})} f\, d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f\, d\lambda_{S^1}\right| \\ \leq \mathrm{Lip_{sph}}(f)\left(\frac{\pi}{\deg(\xi)} + \left(4\,\mathrm{h}(\xi) + C_0 \frac{\log(\deg(\xi)+1)}{\deg(\xi)}\right)^{\frac{1}{2}}\right),$$

*where $S$ is the Galois orbit of $\xi$, $\mu_S$ is the discrete probability measure associated to it, and $\mathrm{Lip_{sph}}$ stands for the Lipschitz constant with respect to the spherical distance on the Riemann sphere.*

*In particular, if $\mathrm{h}(\xi) \leq 1$, then*

$$\left|\int_{\mathbb{P}^1(\mathbb{C})} f\, d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f\, d\lambda_{S^1}\right| \leq \mathrm{Lip_{sph}}(f)\left(4\,\mathrm{h}(\xi) + C \frac{\log(\deg(\xi)+1)}{\deg(\xi)}\right)^{\frac{1}{2}},$$

*for $C \leq 64$.*

The proof of this result relies on the interpretation of the height of a point in terms of the potential theory over the complex projective line. Given $\xi \in \overline{\mathbb{Q}}^\times$, it can be shown that the mutual energy of the signed measure $\mu_S - \lambda_{S^1}$ is bounded above by twice the height of the point. Since this signed measure is not regular enough, Favre and Rivera-Letelier consider a regularization such that it has vanishing total mass and its trace measure has continuous potential. This allows them to apply a Cauchy–Schwartz type inequality to the integral of the function with respect to the regularized measure. Together with the study of the integral of the function with respect to the difference of the measure and its regularization, this leads to their result. The constant in [Narváez-Clauss 2016] is made explicit by considering a specific regularization of the measure, which is done by convolution with a specific *mollifier*.

Consider the projection

$$\begin{aligned} \pi : \mathbb{T}^N \times \mathbb{R}^N &\to \mathbb{T}^N, \\ (\boldsymbol{\theta}, \boldsymbol{u}) &\mapsto \boldsymbol{\theta}. \end{aligned}$$

Under the natural identifications

$$(\mathbb{C}^\times)^N \to \mathbb{T}^N \times \mathbb{R}^N,$$
$$(z_1,\ldots,z_N) \mapsto \left(\left(\frac{\arg(z_1)}{2\pi},\ldots,\frac{\arg(z_N)}{2\pi}\right),(\log|z_1|,\ldots,\log|z_N|)\right)$$

and

$$(S^1)^N \to \mathbb{T}^N,$$
$$(z_1,\ldots,z_N) \mapsto \left(\frac{\arg(z_1)}{2\pi},\ldots,\frac{\arg(z_N)}{2\pi}\right),$$

the map $\pi$ can be rewritten as

$$(\mathbb{C}^\times)^N \to (S^1)^N,$$
$$(z_1,\ldots,z_N) \mapsto \left(\frac{z_1}{|z_1|},\ldots,\frac{z_N}{|z_N|}\right).$$

Let $\boldsymbol{\xi} \in (\mathbb{Q}^\times)^N$, $S$ its Galois orbit, and $\mu_S$ the discrete probability measure associated to it. If $F : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{C}$ is integrable with respect to the measure $\lambda_{(S^1)^N}$, then

$$\begin{aligned}\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\lambda_{(S^1)^N}\right| \\ \le \left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N} F_0\,d\nu_S\right| + \left|\int_{\mathbb{T}^N} F_0\,d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\lambda_{(S^1)^N}\right|, \quad (3\text{-}2)\end{aligned}$$

where $F_0 : \mathbb{T}^N \to \mathbb{R}$ is defined by $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta},\mathbf{0})$, and the measure $\nu_S$ is the pushforward of the measure $\mu_S$, which is given by

$$\nu_S = \pi_*\mu_S = \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}\delta_{\boldsymbol{\alpha}/|\boldsymbol{\alpha}|}. \qquad (3\text{-}3)$$

Using (3-2), we are able to divide the proof of the main result into two parts. The following proposition corresponds to the first one.

**Proposition 3.2.** *Let $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ and $S$ its Galois orbit. Let $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ be a Lipschitz function with respect to the distance* d *and such that it is integrable with respect to $\lambda_{(S^1)^N}$. Then*

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S - \int_{\mathbb{T}^N} F_0\,d\nu_S\right| \le 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}),$$

*where $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta},\mathbf{0})$ and* $\mathrm{Lip}(F)$ *is the Lipschitz constant of $F$.*

*Proof.* With the above notation, we have

$$\begin{aligned}
\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S-\int_{\mathbb{T}^N} F_0\,d\nu_S\right| \\
&= \left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S-\int_{\mathbb{T}^N\times\mathbb{R}^N} (F\circ\pi)\,d\mu_S\right| \\
&= \left|\int_{(\mathbb{C}^\times)^N}\left(F(z_1,\dots,z_N)-F\left(\frac{z_1}{|z_1|},\dots,\frac{z_N}{|z_N|}\right)\right)d\mu_S(z_1,\dots,z_N)\right| \\
&\le \frac{1}{\#S}\sum_{(\alpha_1,\dots,\alpha_N)\in S}\left|F(\alpha_1,\dots,\alpha_N)-F\left(\frac{\alpha_1}{|\alpha_1|},\dots,\frac{\alpha_N}{|\alpha_N|}\right)\right| \\
&\le \frac{1}{\#S}\operatorname{Lip}(F)\sum_{(\alpha_1,\dots,\alpha_N)\in S}\mathrm{d}\left((\alpha_1,\dots,\alpha_N),\left(\frac{\alpha_1}{|\alpha_1|},\dots,\frac{\alpha_N}{|\alpha_N|}\right)\right),
\end{aligned}$$

where the last inequality is given by the fact that $F$ is a Lipschitz function with respect to the distance d of $(\mathbb{C}^\times)^N$. By the definition of this distance,

$$\mathrm{d}\left((\alpha_1,\dots,\alpha_N),\left(\frac{\alpha_1}{|\alpha_1|},\dots,\frac{\alpha_N}{|\alpha_N|}\right)\right)=\left(\sum_{l=1}^N\big|\log|\alpha_l|\big|^2\right)^{\frac{1}{2}}\le\sum_{l=1}^N\big|\log|\alpha_l|\big|.$$

Hence, by Lemma 2.5, we conclude

$$\begin{aligned}
\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\mu_S-\int_{\mathbb{T}^N} F_0\,d\nu_S\right| &\le \frac{1}{\#S}\operatorname{Lip}(F)\sum_{(\alpha_1,\dots,\alpha_N)\in S}\sum_{l=1}^N\big|\log|\alpha_l|\big| \\
&\le 2\operatorname{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}). \qquad \square
\end{aligned}$$

Let us study now the second summand in (3-2). First of all we observe that, since the measure $\lambda_{(S^1)^N}$ is supported on $\mathbb{T}^N\times\{\mathbf{0}\}$, we can reduce the problem to the compact torus $(S^1)^N$. Indeed, with the notation as in (3-2), we have

$$\left|\int_{\mathbb{T}^N} F_0\,d\nu_S-\int_{\mathbb{T}^N\times\mathbb{R}^N} F\,d\lambda_{(S^1)^N}\right|=\left|\int_{\mathbb{T}^N} F_0\,d\nu_S-\int_{\mathbb{T}^N} F_0\,d\lambda_{(S^1)^N}\right|,$$

where $\nu_S$ is given by (3-3).

If $F_0:\mathbb{T}^N\to\mathbb{R}$ is Haar-integrable and such that its Fourier transform $\widehat{F_0}$ is also Haar-integrable, by Lemma 2.2,

$$\int_{\mathbb{T}^N} F_0\,d\nu_S-\int_{\mathbb{T}^N} F_0\,d\lambda_{(S^1)^N}=\widehat{F_0}(\mathbf{0})\big(\overline{\widehat{\nu_S}(\mathbf{0})}-1\big)+\sum_{\boldsymbol{n}\neq\mathbf{0}}\widehat{F_0}(\boldsymbol{n})\overline{\widehat{\nu_S}(\boldsymbol{n})},$$

where the Fourier–Stieltjes transform of $\nu_S$ is

$$\widehat{\nu}_S(\boldsymbol{n}) = \int_{\mathbb{T}^N} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}}\, d\nu_S(\boldsymbol{\theta}) = \frac{1}{\# S} \sum_{(\alpha_1,\dots,\alpha_N)\in S} e^{-i\boldsymbol{n}\cdot(\arg(\alpha_1),\dots,\arg(\alpha_N))}, \tag{3-4}$$

for every $\boldsymbol{n} \in \mathbb{Z}^N$. In particular, $\widehat{\nu}_S(\boldsymbol{0}) = 1$.

We obtain the following lemma.

**Lemma 3.3.** *Let $F_0 : \mathbb{T}^N \to \mathbb{R}$ be Haar-integrable and such that its Fourier transform is also Haar-integrable. With the notation as above,*

$$\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N} F_0\, d\lambda_{(S^1)^N} = \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \widehat{F}_0(\boldsymbol{n})\overline{\widehat{\nu}_S(\boldsymbol{n})}.$$

We now study the Fourier–Stieltjes transform of the measure $\nu_S = \pi_*\mu_S$.

**Proposition 3.4.** *There is a constant $C \le 64$ such that, for every $\boldsymbol{n} \neq \boldsymbol{0}$ and every $0 < \delta < 1$, if $\mathrm{h}(\boldsymbol{\xi}) \le 1$,*

$$|\widehat{\nu}_S(\boldsymbol{n})| \le \frac{-2}{\log\delta}\|\boldsymbol{n}\|_1\, \mathrm{h}(\boldsymbol{\xi}) + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1 \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

*Proof.* Let $\boldsymbol{n} \neq \boldsymbol{0}$ and let $S_{\boldsymbol{n}}$ be the Galois orbit of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$. By Lemma 2.3, there is an integer $l_{\boldsymbol{n}}$ such that $\# S = l_{\boldsymbol{n}} \# S_{\boldsymbol{n}}$ and we know that every element $\alpha \in S_{\boldsymbol{n}}$ is repeated $l_{\boldsymbol{n}}$ times in $\{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in S\}$. Hence, by (3-4), we obtain

$$\overline{\widehat{\nu}_S(\boldsymbol{n})} = \frac{1}{\# S} \sum_{(\alpha_1,\dots,\alpha_N)\in S} e^{i\boldsymbol{n}\cdot(\arg(\alpha_1),\dots,\arg(\alpha_N))} = \frac{1}{\# S}\sum_{\boldsymbol{\alpha}\in S} \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|} = \frac{1}{\# S_{\boldsymbol{n}}}\sum_{\alpha\in S_{\boldsymbol{n}}} \frac{\alpha}{|\alpha|}.$$

For $0 < \delta < 1$, consider the function $f_\delta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}$ given by

$$f_\delta(0:1) = 0 \quad \text{and} \quad f_\delta(1:z) = \rho_\delta(|z|)\frac{z}{|z|} \quad \text{for any } z \in \mathbb{C},$$

where the function $\rho_\delta : \mathbb{R} \to [0,1]$ is given by

$$\rho_\delta(r) = \begin{cases} 0 & \text{if } r < \delta/2, \\ (5\delta - 4r)(\delta - 2r)^2/\delta^3 & \text{if } \delta/2 \le r \le \delta, \\ 1 & \text{if } \delta < r < 1/\delta, \\ (-2+\delta r)^2(-1+2\delta r) & \text{if } 1/\delta \le r \le 2/\delta, \\ 0 & \text{if } r > 2/\delta. \end{cases}$$

In Lemma B.1, we prove that $f_\delta$ is a $\mathscr{C}^1$-function such that, if we write $f_\delta = u_\delta + i v_\delta$,

$$\mathrm{Lip}_{\mathrm{sph}}(u_\delta), \mathrm{Lip}_{\mathrm{sph}}(v_\delta) \le \frac{2\sqrt{2}(\delta^2+9)}{\delta^3},$$

where $\mathrm{Lip}_{\mathrm{sph}}$ stands for the Lipschitz constant with respect to the spherical distance on the Riemann sphere.

For every $\boldsymbol{n} \neq \boldsymbol{0}$,

$$\begin{aligned}\left|\overline{\widehat{\nu}_S(\boldsymbol{n})} - \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}^{D} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}))\right| &= \left|\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}\frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|} - \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|}\right| \\ &= \left|\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}\frac{\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})}{|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|}\big(1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\big)\right| \\ &\leq \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}\big|1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\big|. \end{aligned} \tag{3-5}$$

Let us define, for every $\boldsymbol{n} \neq \boldsymbol{0}$ and $0 < \delta < 1$, the set

$$J_{\boldsymbol{n},\delta} = \Big\{\boldsymbol{\alpha} \in S : \delta \leq |\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})| \leq \frac{1}{\delta}\Big\}.$$

If $\boldsymbol{\alpha} \in J_{\boldsymbol{n},\delta}$, then $\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|) = 1$, and otherwise $0 \leq \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|) < 1$. Hence,

$$\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S}\big|1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|)\big| = \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\notin J_{\boldsymbol{n},\delta}} 1-\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})|) \leq \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\notin J_{\boldsymbol{n},\delta}} 1. \tag{3-6}$$

Set

$$S_{\boldsymbol{n},\delta} = \Big\{\alpha \in S_{\boldsymbol{n}} : \big|\log|\boldsymbol{\alpha}|\big| > \log\frac{1}{\delta}\Big\},$$

then we obtain

$$\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\notin J_{\boldsymbol{n},\delta}} 1 = \frac{1}{\#S_{\boldsymbol{n}}}\sum_{\alpha\in S_{\boldsymbol{n},\delta}} 1 \leq 2\Big(\log\frac{1}{\delta}\Big)^{-1}\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})), \tag{3-7}$$

where the last inequality is given by Lemma 2.6.

As we saw in the proof of Lemma 2.8, for $\boldsymbol{n} \neq \boldsymbol{0}$ we have

$$\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \leq \|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}).$$

Thus, putting this together with (3-5), (3-6) and (3-7) we deduce that

$$\left|\overline{\widehat{\nu}_S(\boldsymbol{n})} - \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}))\right| \leq 2\Big(\log\frac{1}{\delta}\Big)^{-1}\|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}). \tag{3-8}$$

On the other hand,

$$\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha})) = \frac{1}{l_{\boldsymbol{n}}\#S_{\boldsymbol{n}}}\sum_{\alpha\in S_{\boldsymbol{n}}} l_{\boldsymbol{n}} f_\delta(1:\alpha) = \int_{\mathbb{P}^1(\mathbb{C})} f_\delta\, d\mu_{S_{\boldsymbol{n}}}, \tag{3-9}$$

where $\mu_{S_{\boldsymbol{n}}}$ is the discrete probability measure on $\mathbb{P}^1(\mathbb{C})$ associated to the Galois orbit $S_{\boldsymbol{n}}$ of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$.

Since $\lambda_{S^1}$ is the measure on $\mathbb{P}^1(\mathbb{C})$ supported on the unit circle, where it coincides with the Haar probability measure and, by definition, $f_\delta(1:z) = z$ if $|z| = 1$,

$$\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\lambda_{S^1} = \int_{\mathbb{C}^\times} z \, d\lambda_{S^1}(z) = 0.$$

By Theorem 3.1, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}} \right| \\
&\quad = \left| \int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\lambda_{S^1} \right| \\
&\quad \le \left| \int_{\mathbb{P}^1(\mathbb{C})} u_\delta \, d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} u_\delta \, d\lambda_{S^1} \right| \\
&\qquad\quad + \left| \int_{\mathbb{P}^1(\mathbb{C})} v_\delta \, d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} v_\delta \, d\lambda_{S^1} \right| \\
&\quad \le (\mathrm{Lip}_{\mathrm{sph}}(u_\delta) + \mathrm{Lip}_{\mathrm{sph}}(v_\delta)) \\
&\qquad\quad \times \left( \frac{\pi}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left( 4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0 \frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}} \right) \\
&\quad \le \frac{4\sqrt{2}(\delta^2 + 9)}{\delta^3} \\
&\qquad\quad \times \left( \frac{\pi}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left( 4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0 \frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}} \right),
\end{aligned} \tag{3-10}$$

where $C_0 \le 15$.

Since $\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \le \|\boldsymbol{n}\|_1 \,\mathrm{h}(\boldsymbol{\xi})$,

$$\begin{aligned}
&\left( 4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0 \frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}} \\
&\qquad \le \left( 4\|\boldsymbol{n}\|_1 \,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\|\boldsymbol{n}\|_1 \log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}} \\
&\qquad \le \sqrt{\|\boldsymbol{n}\|_1} \left( 4\,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}} \\
&\qquad \le \|\boldsymbol{n}\|_1 \left( 4\,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, this together with (3-9) and (3-10) gives

$$\begin{aligned}
\left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}}\right| &\\
&\le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\Biggl(\frac{\pi}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\\
&\qquad + \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C_0 \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}\Biggr)\\
&\le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}, \qquad (3\text{-}11)
\end{aligned}$$

with $C \le 64$.

The function $\log(x+1)/x$ is monotonically decreasing for $x \ge 1$. We deduce that, for every $\boldsymbol{n} \ne \boldsymbol{0}$,

$$\frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \le \frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}.$$

Together with (3-11), this implies that, for every $\boldsymbol{n} \ne \boldsymbol{0}$,

$$\left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta \, d\mu_{S_{\boldsymbol{n}}}\right| \le \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

By using this inequality and (3-5) we deduce that

$$\begin{aligned}
\left|\widehat{\nu_S}(\boldsymbol{n})\right| &\le \left|\widehat{\nu_S}(\boldsymbol{n}) - \frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}))\right| + \left|\frac{1}{\#S}\sum_{\boldsymbol{\alpha}\in S} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\alpha}))\right|\\
&\le \frac{-2}{\log\delta}\|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}) + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}},
\end{aligned}$$

proving the proposition. □

The following proposition bounds the second summand in the inequality (3-2).

**Proposition 3.5.** *There is a constant $C \le 64$ such that, for every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ with $\mathrm{h}(\boldsymbol{\xi}) \le 1$, every $0 < \delta < 1$ and every $F \in \mathcal{F}$,*

$$\begin{aligned}
\left|\int_{\mathbb{T}^N} F_0 \, d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F \, d\lambda_{(S^1)^N}\right| &\\
&\le \frac{1}{2\pi}\left(\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{l=1}^{N}\left\|\widehat{\frac{\partial F_0}{\partial \theta_l}}\right\|_{\mathrm{L}^1},
\end{aligned}$$

*where $S$ is the Galois orbit of $\boldsymbol{\xi}$, $\mu_S$ the discrete probability measure associated to it, $\mathscr{D}(\boldsymbol{\xi})$ the generalized degree of $\boldsymbol{\xi}$, and $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \boldsymbol{0})$.*

*Proof.* In Appendix A we prove that, given $F \in \mathcal{F}$, the function $F_0$ is Haar-integrable as well as its Fourier transform $\widehat{F}_0$. Thus, by Lemma 3.3 and Proposition 3.4,

$$\begin{aligned}
&\left|\int_{\mathbb{T}^N} F_0 \, d\nu_S - \int_{\mathbb{T}^N \times \mathbb{R}^N} F \, d\lambda_{(S^1)^N}\right| \\
&\quad = \left|\int_{\mathbb{T}^N} F_0 \, d\nu_S - \int_{\mathbb{T}^N} F_0 \, d\lambda_{(S^1)^N}\right| \\
&\quad = \left|\sum_{\boldsymbol{n} \neq \boldsymbol{0}} \widehat{F}_0(\boldsymbol{n}) \overline{\widehat{\nu}_S(\boldsymbol{n})}\right| \leq \sum_{\boldsymbol{n} \neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})| |\widehat{\nu}_S(\boldsymbol{n})| \\
&\quad \leq \sum_{\boldsymbol{n} \neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})| \Bigg( \frac{-2}{\log \delta} \|\boldsymbol{n}\|_1 \, \mathrm{h}(\boldsymbol{\xi}) \\
&\qquad\qquad + \frac{4\sqrt{2}(\delta^2 + 9)}{\delta^3} \|\boldsymbol{n}\|_1 \left( 4 \, \mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}} \Bigg) \\
&\quad \leq \left( \frac{-2}{\log \delta} + \frac{4\sqrt{2}(\delta^2 + 9)}{\delta^3} \right) \left( 4 \, \mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}} \sum_{\boldsymbol{n} \neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})| \|\boldsymbol{n}\|_1,
\end{aligned}$$

where the last inequality is given by the fact that $\mathrm{h}(\boldsymbol{\xi}) \leq 1$.

By Lemma A.2, for every $l = 1, \dots, N$,

$$\widehat{\frac{\partial F_0}{\partial \theta_l}}(\boldsymbol{n}) = 2\pi i n_l \widehat{F}_0(\boldsymbol{n}).$$

Hence, we obtain

$$\begin{aligned}
\sum_{\boldsymbol{n} \neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})| \|\boldsymbol{n}\|_1 &= \frac{1}{2\pi} \sum_{l=1}^{N} \sum_{\boldsymbol{n} \neq \boldsymbol{0}} |\widehat{F}_0(\boldsymbol{n})| \cdot |2\pi n_l| \\
&= \frac{1}{2\pi} \sum_{l=1}^{N} \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \left| \widehat{\frac{\partial F_0}{\partial \theta_l}}(\boldsymbol{n}) \right| \\
&= \frac{1}{2\pi} \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial \theta_l}} \right\|_{\mathrm{L}^1}.
\end{aligned}$$

Finally, we conclude:

$$\begin{aligned}
&\left|\int_{\mathbb{T}^N} F_0 \, d\pi_* \mu_S - \int_{\mathbb{T}^N \times \mathbb{R}^N} F \, d\lambda_{(S^1)^N}\right| \\
&\quad \leq \left( \frac{-2}{\log \delta} + \frac{4\sqrt{2}(\delta^2 + 9)}{\delta^3} \right) \left( 4 \, \mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}} \frac{1}{2\pi} \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial \theta_l}} \right\|_{\mathrm{L}^1} \qquad \square
\end{aligned}$$

*Proof of Theorem 1.2*. Let $F \in \mathcal{F}$ and set $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$. By Theorem A.1, the function $F_0$ and its Fourier transform $\widehat{F}_0$ are Haar-integrable and thus, as shown in (3-2),

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\ \leq \left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N} F_0\, d\nu_S\right| + \left|\int_{\mathbb{T}^N} F_0\, d\nu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right|.$$

Since the function $F$ is Lipschitz with respect to the distance d, by Propositions 3.2 and 3.5, there is a constant $C \leq 64$ such that

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\ \leq 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}) \\ + \frac{1}{2\pi}\left(\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial\theta_l}} \right\|_{\mathrm{L}^1}.$$

By Theorem A.1, the Fourier transforms of the first order partial derivatives of $F_0$ are Haar-integrable and so this bound is finite.

We search numerically for the minimum of the function

$$\frac{-2}{\log\delta} + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3},$$

for $0 < \delta < 1$, and we obtain the value 94.9591, attained at $\delta \approx 0.9071$. Hence, since $\mathrm{h}(\boldsymbol{\xi}) \leq 1$ and $94.9591/2\pi < 16$,

$$\left|\int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\mu_S - \int_{\mathbb{T}^N\times\mathbb{R}^N} F\, d\lambda_{(S^1)^N}\right| \\ \leq 2\,\mathrm{Lip}(F)\,\mathrm{h}(\boldsymbol{\xi}) + 16\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial\theta_l}} \right\|_{\mathrm{L}^1} \\ \leq \left(2\,\mathrm{Lip}(F) + 16\sum_{l=1}^{N} \left\| \widehat{\frac{\partial F_0}{\partial\theta_l}} \right\|_{\mathrm{L}^1}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C\,\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

□

**Remark 3.6.** The functions in $\mathcal{F}$ are functions with logarithmic singularities along toric divisors in a toric compactification of $(\overline{\mathbb{Q}}^\times)^N$. The qualitative equidistribution with respect to this set of test functions is given by [Chambert-Loir and Thuillier 2009, Théorème 1.2].

## Appendix A: The set of test functions

In this appendix, we show that the test functions in $\mathcal{F}$, when restricted to the unit polycircle $(S^1)^N$, are Haar-integrable as well as their Fourier transforms. We also prove the Haar-integrability of all their first order partial derivatives and their corresponding Fourier transforms.

Recall the definition of the test functions. The set $\mathcal{F}$ is given by all real-valued functions $F$ satisfying

(i) $F$ is Lipschitz with respect to the distance d on $(\mathbb{C}^\times)^N$,

(ii) $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$ is in $\mathscr{C}^{N+1}(\mathbb{T}^N, \mathbb{R})$.

The main theorem of this section is this:

**Theorem A.1.** *For any $F \in \mathcal{F}$, the function $F_0(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \mathbf{0})$ has these properties*:

(i) *$F_0$ is Haar-integrable.*

(ii) *$\widehat{F}_0$ is Haar-integrable.*

(iii) *$\partial F_0/\partial\theta_l$ is Haar-integrable for every $l = 1, \dots, N$.*

(iv) *$\widehat{\partial F_0/\partial\theta_l}$ is Haar-integrable for every $l = 1, \dots, N$.*

Before proving this result, let us consider a technical lemma. For every function $H : \mathbb{T}^N \to \mathbb{R}$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \{0, 1\}^N$, we will use the notation

$$\frac{\partial^{|\boldsymbol{\alpha}|} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}(\boldsymbol{\theta}) = \frac{\partial^{\alpha_1 + \dots + \alpha_N} H}{\partial \theta_1^{\alpha_1} \cdots \theta_N^{\alpha_N}}(\boldsymbol{\theta}),$$

whenever it makes sense.

**Lemma A.2.** *Let $H : \mathbb{T}^N \to \mathbb{R}$ of class $\mathscr{C}^{N+1}$ be such that*

$$\frac{\partial^{|\boldsymbol{\alpha}|} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}} \in \mathrm{L}^1(\mathbb{T}^N) \quad \textit{and} \quad \frac{\partial^{|\boldsymbol{\alpha}|+1} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \theta_l} \in \mathrm{L}^1(\mathbb{T}^N),$$

*for $\boldsymbol{\alpha} \in \{0, 1\}^N$ and $l = 1, \dots, N$. Then,*

$$\widehat{\frac{\partial^{|\boldsymbol{\alpha}|} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n}) = \prod_{k=1}^{N} (2\pi i n_k)^{\alpha_k} \widehat{H}(\boldsymbol{n}) \quad \textit{and} \quad \widehat{\frac{\partial^{|\boldsymbol{\alpha}|+1} H}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \theta_l}} = (2\pi i n_l) \prod_{k=1}^{N} (2\pi i n_k)^{\alpha_k} \widehat{H}(\boldsymbol{n}).$$

*Proof.* This lemma is proved by recursively applying the following calculation:

$$\begin{aligned}
\widehat{\frac{\partial F_0}{\partial \theta_1}}(\boldsymbol{n}) &= \int_{\mathbb{T}^N} \frac{\partial F_0}{\partial \theta_1}(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} \, d\boldsymbol{\theta} \\
&= \int_{\mathbb{T}^{N-1}} \left( \int_{\mathbb{T}} \frac{\partial F_0}{\partial \theta_1}(\boldsymbol{\theta}) e^{-2\pi i n_1 \theta_1} \, d\theta_1 \right) e^{-2\pi i \sum_{j\neq 1} n_j \theta_j} \, d\theta_2 \cdots d\theta_N \\
&= \int_{\mathbb{T}^{N-1}} \left( 2\pi i n_1 \int_{\mathbb{T}} F_0(\boldsymbol{\theta}) e^{-2\pi i n_1 \theta_1} \, d\theta_1 \right) e^{-2\pi i \sum_{j\neq 1} n_j \theta_j} \, d\theta_2 \cdots d\theta_N \\
&= (2\pi i n_1) \int_{\mathbb{T}^N} F_0(\boldsymbol{\theta}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} \, d\boldsymbol{\theta} \\
&= (2\pi i n_1) \widehat{F}_0(\boldsymbol{n}). \qquad \square
\end{aligned}$$

*Proof of Theorem A.1*. By definition, for every $F \in \mathcal{F}$, the function $F_0$ is of class $N+1$. This is, all its partial derivatives up to order $N+1$ are continuous and, since they are defined on a compact space, they are bounded. Hence, for every $\boldsymbol{\alpha} \in \{0,1\}^N$ and every $l = 1, \dots, N$,

$$\frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}, \frac{\partial^{|\boldsymbol{\alpha}|+1} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \theta_l} \in (\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{T}^N).$$

In particular, we obtain parts (i) and (iii).

Let us prove (ii), we have to see that

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^N} |\widehat{F}_0(\boldsymbol{n})| < \infty.$$

To do so, we will divide the sum over all $\boldsymbol{n} \in \mathbb{Z}^N$ in several subsets. Let $\alpha \in \{0,1\}$ and set

$$W(\alpha) = \begin{cases} \mathbf{0} & \text{if } \alpha = 0, \\ \mathbb{Z}^N \setminus \{\mathbf{0}\} & \text{if } \alpha = 1. \end{cases}$$

For $\boldsymbol{\alpha} \in \{0,1\}^N$, set also

$$\boldsymbol{W}(\boldsymbol{\alpha}) = W(\alpha_1) \times \cdots \times W(\alpha_N).$$

Hence,

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^N} |\widehat{F}_0(\boldsymbol{n})| = \sum_{\boldsymbol{\alpha}\in\{0,1\}^N} \sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})} |\widehat{F}_0(\boldsymbol{n})|.$$

For $\boldsymbol{\alpha} \in \{0,1\}^N$,

$$\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})} |\widehat{F}_0(\boldsymbol{n})| = \sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})} \prod_{k:\alpha_k\neq 0} (2\pi n_k)^{-1} \left| \widehat{\frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n}) \right|.$$

We saw that $\partial^{|\boldsymbol{\alpha}|} F_0 / \partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \in (\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{T}^N)$ and so, by Plancherel's theorem,

$$\left\| \widehat{\frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}}} \right\|_{\mathrm{L}^2(\mathbb{Z}^N)} = \left\| \frac{\partial^{|\boldsymbol{\alpha}|} F_0}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}}} \right\|_{\mathrm{L}^2(\mathbb{T}^N)}.$$

Using the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}&\Big(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in W(\boldsymbol{\alpha})}|\widehat{F}_0(\boldsymbol{n})|\Big)^2\\&\quad=\Big(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in W(\boldsymbol{\alpha})}\prod_{k:\alpha_k\neq 0}(2\pi n_k)^{-1}\Big|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|}F_0}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n})\Big|\Big)^2\\&\quad\leq\Big(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in W(\boldsymbol{\alpha})}\prod_{k:\alpha_k\neq 0}\frac{1}{4\pi^2n_k^2}\Big)\Big(\sum_{\boldsymbol{\alpha}\in\{0,1\}^N}\sum_{\boldsymbol{n}\in W(\boldsymbol{\alpha})}\Big|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|}F_0}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}}}(\boldsymbol{n})\Big|^2\Big)<\infty.\end{aligned}$$

Part (iv) of the theorem is proved by applying the same argument to the function $\partial F_0/\partial\theta_l$ for every $l=1,\dots,N$. □

## Appendix B: Bounds for the Lipschitz constant of the function $f_\delta$

In this appendix, we give a bound for the Lipschitz constant with respect to the spherical distance of the function $f_\delta:\mathbb{P}^1(\mathbb{C})\to\mathbb{C}$ defined by

$$f_\delta(0:1)=0 \quad\text{and}\quad f_\delta(1:z)=\rho_\delta(|z|)\frac{z}{|z|}\quad\text{for any } z\in\mathbb{C},$$

where $\rho_\delta:\mathbb{R}\to[0,1]$, with $0<\delta<1$, is given by

$$\rho_\delta(r)=\begin{cases}0 & \text{if } r<\delta/2,\\ (5\delta-4r)(\delta-2r)^2/\delta^3 & \text{if } \delta/2\leq r\leq\delta,\\ 1 & \text{if } \delta<r<1/\delta,\\ (-2+\delta r)^2(-1+2\delta r) & \text{if } 1/\delta\leq r\leq 2/\delta,\\ 0 & \text{if } r>2/\delta.\end{cases}$$

First we prove that $f_\delta\in\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}),\mathbb{C})$. Afterwards, we will study the Lipschitz constant of its real and imaginary parts. Let us define the usual charts in $\mathbb{P}^1(\mathbb{C})$,

$$U_0:=\{(z_0:z_1)\in\mathbb{P}^1(\mathbb{C}):z_0\neq 0\}\quad\text{and}\quad U_1:=\{(z_0:z_1)\in\mathbb{P}^1(\mathbb{C}):z_1\neq 0\}.$$

It is easy to see that the function $f_\delta$ is compactly supported on $U_0\cap U_1$. In fact, we have that

$$\operatorname{supp}(f_\delta)=\Big\{(1:z):\frac{\delta}{2}\leq|z|\leq\frac{2}{\delta}\Big\}.$$

For this reason, to prove that $f_\delta$ is in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}),\mathbb{C})$, it is enough to prove that the function $\rho_\delta(|z|)z/|z|$ is of class $\mathscr{C}^1$ in a neighborhood of the set $\{z:\delta/2\leq|z|\leq 2/\delta\}$.

The piecewise-defined function $\rho_\delta$ is continuous, as well as its derivative, which is given by

$$\rho_\delta'(r) = \begin{cases} -24(\delta-2r)(\delta-r)/\delta^3 & \text{if } \delta/2 \le r \le \delta, \\ 6\delta(-2+\delta r)(-1+\delta r) & \text{if } 1/\delta \le r \le 2/\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, since $|z|$ and $z/|z|$ are smooth on $\mathbb{C}^\times$, we conclude that $\rho_\delta(|z|)z/|z|$ is of class $\mathscr{C}^1$.

**Lemma B.1.** *Let $f_\delta$ be defined as above, and set $f_\delta = u_\delta + i v_\delta$. Then,*

$$\mathrm{Lip}_{\mathrm{sph}}(u_\delta), \mathrm{Lip}_{\mathrm{sph}}(v_\delta) \le 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.$$

The spherical distance $\mathrm{d}_{\mathrm{sph}}$ on $\mathbb{P}^1(\mathbb{C})$ can be computed by

$$\mathrm{d}_{\mathrm{sph}}(p,p') := 2\arccos \frac{\left|z_0\overline{z_0'} + z_1\overline{z_1'}\right|}{\sqrt{|z_0|^2+|z_1|^2}\sqrt{|z_0'|^2+|z_1'|^2}},$$

for $p=(z_0:z_1)$ and $p'=(z_0':z_1')$ in $\mathbb{P}^1(\mathbb{C})$.

To simplify the computations, we will work with an equivalent distance, the *chordal distance* $\mathrm{d}_{\mathrm{ch}}$ on $\mathbb{P}^1(\mathbb{C})$, which is given by the length of the chord joining two points of $S^2$. For $p=(z_0:z_1)$ and $p'=(z_0':z_1')$ in $\mathbb{P}^1(\mathbb{C})$, we have

$$\mathrm{d}_{\mathrm{ch}}(p,p') := \frac{2|z_0z_1' - z_1z_0'|}{\sqrt{|z_0|^2+|z_1|^2}\sqrt{|z_0'|^2+|z_1'|^2}}.$$

These distances can be compared as follows:

**Lemma B.2.** *For every $p,p' \in \mathbb{P}^1(\mathbb{C})$,*

$$\frac{2}{\pi}\,\mathrm{d}_{\mathrm{sph}}(p,p') \le \mathrm{d}_{\mathrm{ch}}(p,p') \le \mathrm{d}_{\mathrm{sph}}(p,p').$$

*Proof.* We work on the sphere using the stereographic projection. Since the chordal distance $\mathrm{d}_{\mathrm{ch}}$ between two points in the sphere is the length of the chord joining them and the spherical distance $\mathrm{d}_{\mathrm{sph}}$ is the angle between the vectors both points define, we have

$$\mathrm{d}_{\mathrm{ch}}(p,p') = 2\sin\frac{\mathrm{d}_{\mathrm{sph}}(p,p')}{2}, \quad \text{for every } p,p' \in \mathbb{P}^1(\mathbb{C}).$$

For any pair of points, we have $\mathrm{d}_{\mathrm{sph}}(p,p') \le \pi$ so we deduce

$$\mathrm{d}_{\mathrm{ch}}(p,p') \le \mathrm{d}_{\mathrm{sph}}(p,p').$$

Now, let $\beta > 0$ be such that $\beta\,\mathrm{d}_{\mathrm{sph}}(p,p') \le \mathrm{d}_{\mathrm{ch}}(p,p')$ for all $p,p' \in \mathbb{P}^1(\mathbb{C})$. This is equivalent to $\beta x \le 2\sin\frac{x}{2}$ for every $0 \le x \le \pi$. By the convexity of the function $2\sin\frac{x}{2}$, we deduce that the optimal value is $\beta = \frac{2}{\pi}$. □

*Proof of Lemma B.1*. Let us compute now a bound for the Lipschitz constants, with respect to the spherical distance, of the $u_\delta$ and $v_\delta$. To do so, we choose coordinates $(x, y)$ in $\mathbb{R}^2 \cong \mathbb{C}$. Let

$$\tilde{u}_\delta(x, y) := u_\delta(1 : x + iy) = \frac{\rho_\delta(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} x,$$

and

$$\tilde{v}_\delta(x, y) := v_\delta(1 : x + iy) = \frac{\rho_\delta(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} y.$$

Since the computations are symmetric for both the real and imaginary parts of $f_\delta$, it is enough to study the Lipschitz constant of one of them. To simplify these computations, we will study the Lipschitz constant with respect to the chordal distance in the Riemann sphere and conclude by applying the comparison between the chordal and spherical distances.

First of all, recall that the chordal distance restricted to the open subset $U_0 \subset \mathbb{P}^1(\mathbb{C})$ is given by

$$\mathrm{d}_{\mathrm{ch}}\big((1 : x_0 + iy_0), (1 : x_1 + iy_1)\big) = \frac{2\|(x_0, y_0) - (x_1, y_1)\|}{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}},$$

where $\|\cdot\|$ denotes the Euclidean metric on $\mathbb{R}^2$ and $m(x, y) = \sqrt{x^2 + y^2}$. Now, since the function $u_\delta$ is supported on $U_0$,

$$\begin{aligned}\sup_{z_0, z_1 \in \mathbb{C}} &\frac{|u_\delta(1 : z_0) - u_\delta(1 : z_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : z_0), (1 : z_1))} \\ &= \sup_{\substack{(x_0, y_0) \in \mathbb{R}^2 \\ (x_1, y_1) \in \mathbb{R}^2}} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\|(x_0, y_0) - (x_1, y_1)\|} \frac{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}}{2}.\end{aligned}$$

We consider different cases.

<u>Case 1</u>. If $(x_0, y_0), (x_1, y_1) \notin D(0, 2/\delta)$, we trivially obtain

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d}_{\mathrm{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} = 0.$$

<u>Case 2</u>. Suppose $(x_0, y_0), (x_1, y_1) \in D(0, 2/\delta)$. For $t \in [0, 1]$, consider the function

$$g(t) = \tilde{u}_\delta((1 - t)(x_0, y_0) + t(x_1, y_1)).$$

By the mean value theorem, there is some $c \in (0, 1)$ such that $g(1) - g(0) = g'(c)$. Applying the chain rule, we obtain

$$\tilde{u}_\delta(x_1, y_1) - \tilde{u}_\delta(x_0, y_0) = \nabla\tilde{u}_\delta((1 - c)(x_0, y_0) + c(x_1, y_1)) \cdot (x_1 - x_0, y_1 - y_0).$$

Hence, we deduce

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\|(x_0,y_0)-(x_1,y_1)\|} \le \sup_{(x,y)\in D(0,\frac{2}{\delta})} \|\nabla \tilde{u}_\delta(x,y)\|. \tag{B-1}$$

Let us study the gradient of $\tilde{u}_\delta$. For every $(x,y)\in\mathbb{R}^2$,

$$\frac{\partial \tilde{u}_\delta}{\partial x}(x,y) = \left(\frac{x}{m(x,y)}\right)^2 \rho'_\delta(m(x,y)) + \left(\frac{y}{m(x,y)}\right)^2 \frac{\rho_\delta(m(x,y))}{m(x,y)}$$

and

$$\frac{\partial \tilde{u}_\delta}{\partial y}(x,y) = \frac{xy}{m(x,y)^2}\left(\rho'_\delta(m(x,y)) - \frac{\rho_\delta(m(x,y))}{m(x,y)}\right).$$

Without loss of generality, we restrict ourselves to the situation where $(x,y)$ satisfies $\delta/2 \le m(x,y) \le 2/\delta$, since otherwise both partial derivatives would vanish. It can be easily shown that $|\rho'_\delta(r)| \le 3/\delta$ for every $r\ge 0$. This, together with the fact that $0\le\rho_\delta\le 1,\ x\le m(x,y),\ y\le m(x,y),$ and $m(x,y)\ge\delta/2$, leads to

$$\left|\frac{\partial \tilde{u}_\delta}{\partial x}(x,y)\right|, \left|\frac{\partial \tilde{u}_\delta}{\partial y}(x,y)\right| \le \frac{4}{\delta}.$$

We then conclude that $\|\nabla\tilde{u}_\delta(x,y)\| \le 4\sqrt{2}/\delta$. for any $(x,y)\in\mathbb{R}^2$.

On the other hand, given $(x_0,y_0),(x_1,y_1)\in D(0,2/\delta)$ we have that

$$\frac{\sqrt{1+m(x_0,y_0)^2}\sqrt{1+m(x_1,y_1)^2}}{2} \le \frac{\delta^2+4}{2\delta^2}.$$

Therefore, we obtain

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))} \le 2\sqrt{2}\,\frac{\delta^2+4}{\delta^3}.$$

<u>Case 3</u>. Suppose now that $(x_0,y_0)\in D(0,2/\delta)$ and $(x_1,y_1)\in D(0,3/\delta)\setminus D(0,2/\delta)$. As we did in the previous case, we can deduce that

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\|(x_0,y_0)-(x_1,y_1)\|} \le \frac{4\sqrt{2}}{\delta} \quad\text{and}\quad \frac{\sqrt{1+m(x_0,y_0)^2}\sqrt{1+m(x_1,y_1)^2}}{2} \le \frac{\delta^2+9}{2\delta^2}.$$

and

$$\frac{\sqrt{1+m(x_0,y_0)^2}\sqrt{1+m(x_1,y_1)^2}}{2} \le \frac{\delta^2+9}{2\delta^2}.$$

Hence,

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))} \le 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.$$

Case 4. Finally suppose that $(x_0, y_0) \in D(0, 2/\delta)$ and $(x_1, y_1) \notin D(0, 3/\delta)$. In this situation, we have $\tilde{u}_\delta(x_1, y_1) = 0$ and

$$|\tilde{u}_\delta(x_1, y_1)| = |\rho_\delta(m(x_0, y_0))| \frac{|x_0|}{m(x_0, y_0)} \leq 1.$$

Since

$$\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1)) \geq \mathrm{d_{ch}}\Big(\Big(1 : \frac{2}{\delta}\Big), \Big(1 : \frac{3}{\delta}\Big)\Big) = \frac{2\delta}{\sqrt{(\delta^2+9)(\delta^2+4)}},$$

we conclude

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} \leq \frac{2\delta}{\sqrt{(\delta^2 + 9)(\delta^2 + 4)}} 2\delta.$$

Having studied all these cases, we deduce that

$$\sup_{\substack{(x_0,y_0)\in\mathbb{R}^2\\(x_1,y_1)\in\mathbb{R}^2}} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} \leq 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.$$

As we mentioned above, we were looking for a bound of the Lipschitz constant of $u_\delta$ with respect to the spherical distance. By Lemma B.2, we know that $\mathrm{d_{sph}}(p, p') \geq \mathrm{d_{ch}}(p, p')$ for any pair of points $p, p' \in \mathbb{P}^1(\mathbb{C})$ and we obtain

$$\begin{aligned}\mathrm{Lip_{sph}}(u_\delta) &= \sup_{p,p'\in\mathbb{P}^1(\mathbb{C})} \frac{|u_\delta(p) - u_\delta(p')|}{\mathrm{d_{sph}}(p, p')}\\ &\leq \sup_{\substack{(x_0,y_0)\in\mathbb{R}^2\\(x_1,y_1)\in\mathbb{R}^2}} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))}\\ &\leq 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}.\end{aligned}$$

Analogously, we deduce that $\mathrm{Lip_{sph}}(v_\delta) \leq 2\sqrt{2}\,\frac{\delta^2+9}{\delta^3}$. □

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# Rational curves on smooth hypersurfaces of low degree

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We establish the dimension and irreducibility of the moduli space of rational curves (of fixed degree) on arbitrary smooth hypersurfaces of sufficiently low degree. A spreading out argument reduces the problem to hypersurfaces defined over finite fields of large cardinality, which can then be tackled using a function field version of the Hardy-Littlewood circle method, in which particular care is taken to ensure uniformity in the size of the underlying finite field.



## 1. Introduction

The geometry of a variety is intimately linked to the geometry of the space of rational curves on it. Given a projective variety $X$ defined over $\mathbb{C}$, a natural object to study is the moduli space of rational curves on $X$. There are many results in the literature establishing the irreducibility of such mapping spaces, but most statements are only proved for generic $X$. Following a strategy of Ellenberg and Venkatesh, we shall use tools from analytic number theory to prove such a result for all smooth hypersurfaces of sufficiently low degree.

Let $X \subset \mathbb{P}^n$ be a smooth Fano hypersurface of degree $d$ defined over $\mathbb{C}$, with $n \geqslant 3$. For each positive integer $e$, the Kontsevich moduli space $\bar{\mathcal{M}}_{0,0}(X, e)$ is a compactification of the space $\mathcal{M}_{0,0}(X, e)$ of morphisms of degree $e$ from $\mathbb{P}^1$ to $X$,

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up to isomorphism. According to Kollár [1996, Theorem II.1.2/3], any irreducible component of $\bar{\mathcal{M}}_{0,0}(X, e)$ has dimension at least

$$\bar{\mu} = (n+1-d)e + n - 4. \tag{1-1}$$

Work of Harris, Roth and Starr [Harris et al. 2004] shows that $\bar{\mathcal{M}}_{0,0}(X, e)$ is an irreducible, local complete intersection scheme of dimension $\bar{\mu}$, provided that $X$ is general and $d < \frac{1}{2}(n+1)$. The restriction on $d$ has since been weakened to $d < \frac{2}{3}(n+1)$ by Beheshti and Kumar [2013] (assuming that $n \geqslant 23$), and then to $d \leqslant n-2$ by Riedl and Yang [2016].

In the setting $d = 3$ of cubic hypersurfaces it is possible to obtain results for all smooth hypersurfaces in the family. Thus Coskun and Starr [2009] have shown that $\bar{\mathcal{M}}_{0,0}(X, e)$ is irreducible and of dimension $\bar{\mu}$ for any smooth cubic hypersurface $X \subset \mathbb{P}^n$ over $\mathbb{C}$, provided that $n > 4$. (If $n = 4$ then $\bar{\mathcal{M}}_{0,0}(X, e)$ has two irreducible components of the expected dimension $\bar{\mu} = 2e$.)

At the expense of a much stronger condition on the degree, our main result establishes the irreducibility and dimension of the space $\mathcal{M}_{0,0}(X, e)$, for an arbitrary smooth hypersurface $X \subset \mathbb{P}^n$ over $\mathbb{C}$. Let

$$n_0(d) = 2^{d-1}(5d-4). \tag{1-2}$$

We shall prove the following statement.

**Theorem 1.1.** *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geqslant 3$ defined over $\mathbb{C}$, with $n \geqslant n_0(d)$. Then for each $e \geqslant 1$ the space $\mathcal{M}_{0,0}(X, e)$ is irreducible and of the expected dimension.*

The example of Fermat hypersurfaces, discussed in [op. cit., §1], shows that the analogous result for $\bar{\mathcal{M}}_{0,0}(X, e)$ is false when $d > 3$ and $e$ is large enough. When $e = 1$ we have $\bar{\mathcal{M}}_{0,0}(X, 1) = \mathcal{M}_{0,0}(X, 1) = F_1(X)$, where $F_1(X)$ is the Fano scheme of lines on $X$. It has been conjectured, independently by Debarre and de Jong, that $\dim F_1(X) = 2n - d - 3$ for any smooth Fano hypersurface $X \subset \mathbb{P}^n$ of degree $d$. Beheshti [2014] has confirmed this for $d \leqslant 8$. Taking $e = 1$ in Theorem 1.1, we conclude that $\dim F_1(X) = 2n - d - 3$ for any $d \geqslant 3$, provided that $n \geqslant n_0(d)$.

Our proof of Theorem 1.1 ultimately relies on techniques from analytic number theory. The first step is "spreading out", in the sense of Grothendieck [EGA IV$_3$ 1966, §10.4.11] (compare [Serre 2009]), which will take us to the analogous problem for smooth hypersurfaces defined over the algebraic closure of a finite field. Passing to a finite field $\mathbb{F}_q$ of sufficiently large cardinality, for a smooth degree $d$ hypersurface $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$, the cardinality of $\mathbb{F}_q$-points on $\mathcal{M}_{0,0}(X, e)$ can be related to the number of $\mathbb{F}_q(t)$-points on $X$ of degree $e$. We shall access the latter quantity through a function field version of the Hardy–Littlewood circle method. A comparison with the estimate of Lang and Weil [1954] then allows us to make deductions about the irreducibility and dimension of $\mathcal{M}_{0,0}(X, e)$.

The idea of using the circle method to study the moduli space of rational curves on varieties is due to Ellenberg and Venkatesh. The traditional setting for the circle method is a fixed finite field $\mathbb{F}_q$, with the goal being to understand the $\mathbb{F}_q(t)$-points on $X$ of degree $e$, as $e \to \infty$. This is the point of view taken in [Lee 2013; 2011] on a $\mathbb{F}_q(t)$-version of Birch's work on systems of forms in many variables. In contrast to this, we will be required to handle any fixed $e \geqslant 1$, as $q \to \infty$. Pugin developed an "algebraic circle method" in his Ph.D. thesis [2011] to study the spaces $\mathcal{M}_{0,0}(X, e)$, when $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ is the diagonal cubic hypersurface

$$a_0x_0^3 + \cdots + a_nx_n^3 = 0, \quad \text{for } a_0, \ldots, a_n \in \mathbb{F}_q^*.$$

Assuming that $n \geqslant 12$ and $\mathrm{char}(\mathbb{F}_q) > 3$, he succeeds in showing that the space $\mathcal{M}_{0,0}(X, e)$ is irreducible and of the expected dimension. Our work, on the other hand, applies to arbitrary smooth hypersurfaces of sufficiently low degree, which are defined over the complex numbers. Finally, our investigation bears comparison with work of Bourqui [2012; 2013]. He has also investigated the moduli space of curves on varieties using counting arguments. In place of the circle method, however, Bourqui draws on the theory of universal torsors.

## 2. Spreading out

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$, defined by a homogeneous polynomial

$$F(x_0, \ldots, x_n) = \sum_{\substack{\boldsymbol{i} \in \mathbb{Z}^{n+1}_{\geqslant 0} \\ i_0 + \cdots + i_n = d}} c_{\boldsymbol{i}} x_0^{i_0} \ldots x_n^{i_n},$$

with coefficients $c_{\boldsymbol{i}} \in \mathbb{C}$. Rather than working with $\mathcal{M}_{0,0}(X, e)$, it will suffice to study the naive space $\mathrm{Mor}_e(\mathbb{P}^1, X)$ of actual maps $\mathbb{P}^1 \to X$ of degree $e$. The expected dimension of $\mathrm{Mor}_e(\mathbb{P}^1, X)$ is $\mu = \bar{\mu} + 3$, where $\bar{\mu}$ is given by (1-1), since $\mathbb{P}^1$ has automorphism group of dimension 3. We now recall the construction of $\mathrm{Mor}_e(\mathbb{P}^1, X)$.

Let $G_e$ be the set of all homogeneous polynomials in $u, v$ of degree $e \geqslant 1$, with coefficients in $\mathbb{C}$. A *rational curve* of degree $e$ on $X$ is a nonconstant morphism $f : \mathbb{P}^1_{\mathbb{C}} \to X$ of degree $e$. It is given by

$$f = (f_0(u, v), \ldots, f_n(u, v)),$$

with $f_0, \ldots, f_n \in G_e$, with no nonconstant common factor in $\mathbb{C}[u, v]$, such that $F(f_0(u, v), \ldots, f_n(u, v))$ vanishes identically. We may regard $f$ as a point in the space $\mathbb{P}_{\mathbb{C}}^{(n+1)(e+1)-1}$. The morphisms of degree $e$ on $X$ are parametrised by $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$, which is an open subvariety of $\mathbb{P}_{\mathbb{C}}^{(n+1)(e+1)-1}$ cut out by a system of

$de+1$ equations of degree $d$. In this way we obtain the expected dimension

$$(n+1)(e+1)-1-(de+1)=(n+1-d)e+n-1=\mu,$$

of $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$. It follows from [Kollár 1996, Theorem II.1.2] that all irreducible components of $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ have dimension at least $\mu$. In order to establish Theorem 1.1 it will therefore suffice to show that $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ is irreducible, with $\dim \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X) \leqslant \mu$, provided that $n \geqslant n_0(d)$.

The complement to $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ in its closure is the set of $(f_0, \dots, f_n)$ with a common zero. We can obtain explicit equations for $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ by noting that $f_0, \dots, f_n$ have a common zero if and only if the resultant $\mathrm{Res}(\sum_i \lambda_i f_i, \sum_j \mu_j f_j)$ is identically zero as a polynomial in $\lambda_i, \mu_j$. It is clear that both $X$ and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ are defined by equations with coefficients belonging to the finitely generated $\mathbb{Z}$-algebra $\Lambda = \mathbb{Z}[c_{\boldsymbol{i}}]$, obtained by adjoining the coefficients of $F$ to $\mathbb{Z}$. In this way we may view $X$ and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ as schemes over $\Lambda$, with structure morphisms $X \to \operatorname{Spec} \Lambda$ and

$$\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X) \to \operatorname{Spec} \Lambda.$$

By Chevalley's upper semicontinuity theorem [EGA IV$_3$ 1966, Theorem 13.1.3], there exists a nonempty open set $U$ of $\operatorname{Spec} \Lambda$ such that

$$\dim \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X) \leqslant \dim \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$$

for any closed point $\mathfrak{m} \in U$. Here $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$ denotes the fibre above $\mathfrak{m}$, which is obtained via the base change $\operatorname{Spec} \Lambda/\mathfrak{m} \to \operatorname{Spec} \Lambda$. Likewise, since integrality is an open condition, the space $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ will be irreducible if $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$ is.

Choose a maximal ideal $\mathfrak{m}$ in $U$. The quotient $\Lambda/\mathfrak{m}$ is a finite field by arithmetic weak Nullstellensatz. By enlarging $\Lambda$, we may assume that it contains $1/d!$. In particular, it follows that $\mathrm{char}(\Lambda/\mathfrak{m}) = p$, say, with $p > d$, since any prime less than or equal to $d$ is invertible in $\Lambda$. The quasiprojective varieties $X_{\mathfrak{m}}$ and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$ are defined over $\overline{\mathbb{F}}_p$, being given explicitly by reducing modulo $\mathfrak{m}$ the coefficients of the original system of defining equations. By further enlarging $\Lambda$, if necessary, we may assume that $X_{\mathfrak{m}}$ is smooth. There exists a finite field $\mathbb{F}_{q_0}$ such that $X_{\mathfrak{m}}$ and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X_{\mathbb{C}})_{\mathfrak{m}}$ are both defined over $\mathbb{F}_{q_0}$. In view of the Lang–Weil estimate, Theorem 1.1 is a direct consequence of the following result, together with the fact that $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X_{\mathbb{C}})_{\mathfrak{m}}$ is nonempty in the cases under consideration.

**Theorem 2.1.** *Let $n \geqslant n_0(d)$ and let $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a smooth hypersurface of degree $d \geqslant 3$ defined over a finite field $\mathbb{F}_q$, with $\mathrm{char}(\mathbb{F}_q) > d$. Then for each $e \geqslant 1$,*

$$\lim_{\ell \to \infty} q^{-\ell\mu} \# \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{F}_q}, X)(\mathbb{F}_{q^\ell}) \leqslant 1.$$

## 3. The Hardy–Littlewood circle method

We now initiate the proof of Theorem 2.1. We henceforth redefine $q^\ell$ to be $q$ and we replace $n$ by $n-1$ in the statement of the theorem. In particular the expected dimension is now $\mu=(n-d)e+n-2$. Our proof of Theorem 2.1 is based on a version of the Hardy–Littlewood circle method for the function field $K=\mathbb{F}_q(t)$, always under the assumption that $\operatorname{char}(\mathbb{F}_q)>d$. The main input for this comes from [Lee 2013; 2011], combined with our own recent contribution to the subject, in the setting of cubic forms [Browning and Vishe 2015].

We begin by laying down some basic notation and terminology. To begin with, for any real number $R$ we set $\hat{R}=q^R$. Let $\mathbb{O}=\mathbb{F}_q[t]$ be the ring of integers of $K$ and let $\Omega$ be the set of places of $K$. These correspond to either monic irreducible polynomials $\varpi$ in $\mathbb{O}$, which we call the *finite primes*, or the *prime at infinity* $t^{-1}$ which we usually denote by $\infty$. The associated absolute value $|\cdot|_v$ is either $|\cdot|_\varpi$ for some prime $\varpi\in\mathbb{O}$ or $|\cdot|$, according to whether $v$ is a finite or infinite place, respectively. These are given by

$$|a/b|_\varpi=q^{-(\deg\varpi)\operatorname{ord}_\varpi(a/b)}\quad\text{and}\quad |a/b|=q^{\deg a-\deg b},$$

for any $a/b\in K^*$. We extend these definitions to $K$ by taking $|0|_\varpi=|0|=0$.

For $v\in\Omega$ we let $K_v$ denote the completion of $K$ at $v$ with respect to $|\cdot|_v$. We may identify $K_\infty$ with the set

$$\mathbb{F}_q((1/t))=\left\{\sum_{i\leqslant N}a_it^i:a_i\in\mathbb{F}_q\text{ and }N\in\mathbb{Z}\right\}.$$

We can extend the absolute value at the infinite place to $K_\infty$ to get a nonarchimedean absolute value $|\cdot|:K_\infty\to\mathbb{R}_{\geqslant 0}$ given by $|\alpha|=q^{\operatorname{ord}\alpha}$, where $\operatorname{ord}\alpha$ is the largest $i\in\mathbb{Z}$ such that $a_i\neq 0$ in the representation $\alpha=\sum_{i\leqslant N}a_it^i$. In this context we adopt the convention $\operatorname{ord}0=-\infty$ and $|0|=0$. We extend this to vectors by setting $|\boldsymbol{x}|=\max_{1\leqslant i\leqslant n}|x_i|$, for any $\boldsymbol{x}\in K_\infty^n$.

Next, we put

$$\mathbb{T}=\{\alpha\in K_\infty:|\alpha|<1\}=\left\{\sum_{i\leqslant -1}a_it^i:\text{for }a_i\in\mathbb{F}_q\right\}.$$

Since $\mathbb{T}$ is a locally compact additive subgroup of $K_\infty$ it possesses a unique Haar measure $\mathrm{d}\alpha$, which is normalised so that $\int_{\mathbb{T}}\mathrm{d}\alpha=1$. We can extend $\mathrm{d}\alpha$ to a (unique) translation-invariant measure on $K_\infty$, in such a way that

$$\int_{\{\alpha\in K_\infty:|\alpha|<\hat{N}\}}\mathrm{d}\alpha=\hat{N},$$

for any $N\in\mathbb{Z}_{>0}$. These measures also extend to $\mathbb{T}^n$ and $K_\infty^n$, for any $n\in\mathbb{Z}_{>0}$. There is a nontrivial additive character $e_q : \mathbb{F}_q \to \mathbb{C}^*$ defined for each $a \in \mathbb{F}_q$ by taking $e_q(a) = \exp(2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)/p)$. This character yields a nontrivial (unitary) additive character $\psi : K_\infty \to \mathbb{C}^*$ by defining $\psi(\alpha) = e_q(a_{-1})$ for any $\alpha = \sum_{i \leqslant N} a_i t^i$ in $K_\infty$.

Let $F \in \mathbb{F}_q[\boldsymbol{x}]$ be a nonsingular form of degree $d \geqslant 3$, with $\boldsymbol{x} = (x_1, \dots, x_n)$. We may express this polynomial as

$$F(\boldsymbol{x}) = \sum_{i_1,\dots,i_d=1}^{n} c_{i_1,\dots,i_d} x_{i_1} \cdots x_{i_d},$$

with coefficients $c_{i_1,\dots,i_d} \in \mathbb{F}_q$. In particular $F$ and the discriminant $\Delta_F$ are nonzero, or equivalently, $\max_{\boldsymbol{i}} |c_{\boldsymbol{i}}| = 1$ and $|\Delta_F| = 1$. We will make frequent use of these facts in what follows. Associated to $F$ are the multilinear forms

$$\Psi_i(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(d-1)}) = \sum_{i_1,\dots,i_{d-1}=1}^{n} c_{i_1,\dots,i_{d-1},i}\, x_{i_1}^{(1)} \cdots x_{i_{d-1}}^{(d-1)}, \tag{3-1}$$

for $1 \leqslant i \leqslant n$.

To establish Theorem 2.1 we work with the naive space

$$M_e = \bigl\{\boldsymbol{x} = (x_1, \dots, x_n) \in G_e(\mathbb{F}_q)^n \setminus \{\boldsymbol{0}\} : F(\boldsymbol{x}) = 0\bigr\},$$

where $G_e(\mathbb{F}_q)$ is the set of binary forms of degree $e$ with coefficients in $\mathbb{F}_q$. Thus $M_e$ corresponds to the $\mathbb{F}_q$-points on the affine cone of $\operatorname{Mor}_e(\mathbb{P}^1_{\mathbb{F}_q}, X)$, where we drop the condition that $x_1, \dots, x_n$ share no common factor. Let us set

$$\hat{\mu} = \mu + 1 = (n-d)e + n - 1 = (e+1)n - de - 1. \tag{3-2}$$

It will clearly suffice to show that

$$\lim_{q \to \infty} q^{-\hat{\mu}} \# M_e \leqslant 1, \tag{3-3}$$

for $n > n_0(d)$, where $n_0(d)$ is given by (1-2). We proceed by relating $\# M_e$ to the counting function that lies at the heart of our earlier investigation [Browning and Vishe 2015].

Let $w : K_\infty^n \to \{0, 1\}$ be given by $w(\boldsymbol{x}) = \prod_{1 \leqslant i \leqslant n} w_\infty(x_i)$, where

$$w_\infty(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Putting $P = t^{e+1}$, we then have $\# M_e \leqslant N(P)$, where

$$N(P) = \sum_{\substack{\boldsymbol{x} \in \mathcal{O}^n \\ F(\boldsymbol{x})=0}} w(\boldsymbol{x}/P).$$

It follows from [Browning and Vishe 2015, Equation (4.1)] that for any $Q \geqslant 1$,

$$N(P) = \sum_{\substack{r \in \mathbb{O} \\ |r| \leqslant \hat{Q} \\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \int_{|\theta|<|r|^{-1}\hat{Q}^{-1}} S\Big(\frac{a}{r}+\theta\Big)\,\mathrm{d}\theta, \tag{3-4}$$

where $\sum^*$ means that the sum is taken over residue classes $|a| < |r|$ for which $\gcd(a, r) = 1$, and where

$$S(\alpha) = \sum_{\boldsymbol{x} \in \mathbb{O}^n} \psi(\alpha F(\boldsymbol{x}))\, w(\boldsymbol{x}/P), \tag{3-5}$$

for any $\alpha \in \mathbb{T}$. We will work with the choice $Q = d(e+1)/2$, so that $\hat{Q} = |P|^{d/2}$.

The *major arcs* for our problem are given by $r = 1$ and $|\theta| < |P|^{-d}q^{d-1}$. We let the *minor arcs* be everything else: i.e., those $\alpha = a/r + \theta$ appearing in (3-4) for which either $|r| > q$, or else $r = 1$ and $|\theta| \geqslant |P|^{-d}q^{d-1}$. The contribution $N_{\text{major}}(P)$ from the major arcs is easy to deal with. Indeed, for $|\theta| < |P|^{-d}q^{d-1}$ and $|\boldsymbol{x}| < |P|$ we have $|\theta F(\boldsymbol{x})| < |P|^{-d}q^{d-1}q^{de} = q^{-1}$, whence $\psi(\theta F(\boldsymbol{x})) = 1$. Thus $S(\alpha) = |P|^n$, for $\alpha = \theta$ belonging to the major arcs, whence

$$N_{\text{major}}(P) = |P|^n \int_{|\theta|<|P|^{-d}q^{d-1}} \mathrm{d}\theta = |P|^{n-d}q^{d-1} = q^{\hat{\mu}}.$$

In order to prove (3-3), it therefore remains to show that

$$\lim_{q\to\infty} q^{-\hat{\mu}} N_{\text{minor}}(P) = 0, \tag{3-6}$$

for $n > n_0(d)$, where $N_{\text{minor}}(P)$ is the overall contribution to (3-4) from the minor arcs. This will complete the proof of Theorem 2.1.

## 4. Geometry of numbers in function fields

The purpose of this section is to record a technical result about lattice point counting over $K_\infty$. A *lattice* in $K_\infty^N$ is a set of points of the form $\boldsymbol{x} = \Lambda \boldsymbol{u}$, where $\Lambda$ is an $N \times N$ matrix over $K_\infty$ and $\boldsymbol{u}$ runs over elements of $\mathbb{O}^N$. By an abuse of notation we will also denote the set of such points by $\Lambda$. Given a lattice $M$, the *adjoint lattice* $\Lambda$ is defined to satisfy $\Lambda^T M = I_N$, where $I_N$ is the $N \times N$ identity matrix.

Let $\gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in $K_\infty$. Given any positive integer $m$, we define the special lattice

$$M_m = \begin{pmatrix} t^{-m} I_n & 0 \\ t^m \gamma & t^m I_n \end{pmatrix},$$

with corresponding adjoint lattice

$$\Lambda_m = \begin{pmatrix} t^m I_n & -t^m \gamma \\ 0 & t^{-m} I_n \end{pmatrix}.$$

Let $\hat{R}_1, \dots, \hat{R}_{2n}$ denote the successive minima of the lattice corresponding to $M_m$. For any vector $\boldsymbol{x} \in K_\infty^{2n}$ let $\boldsymbol{x}_1 = (x_1, \dots, x_n)$ and $\boldsymbol{x}_2 = (x_{n+1}, \dots, x_{2n})$. We claim that $M_m$ and $\Lambda_m$ can be identified with one another. Now $M_m$ is the set of points $\boldsymbol{x} = M_m \boldsymbol{u}$ where $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ runs over elements of $\mathbb{O}^{2n}$. Likewise, $\Lambda_m$ is the set of points $\boldsymbol{y} = \Lambda_m \boldsymbol{v}$ where $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ runs over elements of $\mathbb{O}^{2n}$. We can therefore identify $M_m$ with $\Lambda_m$ through the process of changing the sign of $\boldsymbol{v}_2$, then the sign of $\boldsymbol{y}_2$, then switching $\boldsymbol{v}_1$ with $\boldsymbol{v}_2$, and finally interchanging $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. It now follows from [Lee 2013, Lemma 3.3.6] (see also [Lee 2011, Lemma B.6]) that

$$R_\nu + R_{2n-\nu+1} = 0, \tag{4-1}$$

for $1 \leqslant \nu \leqslant n$. An important step in the proof of [Lee 2013, Lemma 3.3.6] (see also [Lee 2011, Lemma B.6]) is a nonarchimedean version of Gram–Schmidt orthogonalisation, which is used without reference in the proof of [Lee 2013, Lemma 3.3.3] (see also [Lee 2011, Lemma B.3]). This deficit is remedied by appealing to recent work of Usher and Zhang [2016, Theorem 2.16].

For any $Z \in \mathbb{R}$ and any lattice $\Gamma$ we define the counting function

$$\Gamma(Z) = \#\{\boldsymbol{x} \in \Gamma : |\boldsymbol{x}| < \hat{Z}\}.$$

Note that $\Gamma(Z) = \Gamma(\lceil Z \rceil)$ for any $Z \in \mathbb{R}$. We proceed to establish the following inequality.

**Lemma 4.1.** *Let $m, Z_1, Z_2 \in \mathbb{Z}$ such that $Z_1 \leqslant Z_2 \leqslant 0$. Then*

$$\frac{M_m(Z_1)}{M_m(Z_2)} \geqslant \left(\frac{\hat{Z}_1}{\hat{Z}_2}\right)^n.$$

*Proof.* Let $1 \leqslant \mu, \nu \leqslant 2n$ be such that $R_\mu < Z_1 \leqslant R_{\mu+1}$ and $R_\nu < Z_2 \leqslant R_{\nu+1}$. Since $R_j$ is a nondecreasing sequence which satisfies $R_j + R_{2n-j+1} = 0$, by (4-1), we must have $0 \leqslant R_{n+1}$, whence in fact $\mu \leqslant \nu \leqslant n$. It follows from [Lee 2013, Lemma 3.3.5] (see also [Lee 2011, Lemma B.5]) that

$$\frac{M_m(Z_1)}{M_m(Z_2)} = \begin{cases} 1 & \text{if } Z_1, Z_2 < R_1, \\ \left(\prod_{j=1}^{\nu} \hat{R}_j / \hat{Z}_1\right)(\hat{Z}_1/\hat{Z}_2)^\nu & \text{if } Z_1 < R_1 \leqslant Z_2, \\ \left(\prod_{j=\mu+1}^{\nu} \hat{R}_j / \hat{Z}_1\right)(\hat{Z}_1/\hat{Z}_2)^\nu & \text{if } R_1 \leqslant Z_1 \leqslant Z_2, \end{cases}$$

The statement of the lemma is now obvious. □

As above, let $\gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in $K_\infty$. For $1 \leqslant i \leqslant n$ we introduce the linear forms

$$L_i(u_1, \dots, u_n) = \sum_{j=1}^{n} \gamma_{ij} u_j.$$

Next, for given real numbers $a, Z$, we let $N(a, Z)$ denote the number of vectors $(u_1, \dots, u_{2n}) \in \mathbb{O}^{2n}$ such that

$$|u_j| < \hat{a}\hat{Z} \quad \text{and} \quad |L_j(u_1, \dots, u_n) + u_{j+n}| < \frac{\hat{Z}}{\hat{a}} \quad \text{for } 1 \leqslant j \leqslant n.$$

If we put $m = \lfloor a \rfloor$, then it is clear that

$$M_m(Z - \{a\}) \leqslant N(a, Z) \leqslant M_m(Z + \{a\}),$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of $a$. The following result is a direct consequence of Lemma 4.1.

**Lemma 4.2.** *Let $a, Z_1, Z_2 \in \mathbb{R}$ with $Z_1 \leqslant Z_2 \leqslant 0$. Then*

$$\frac{N(a, Z_1)}{N(a, Z_2)} \geqslant \hat{K}^n,$$

*where $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$.*

## 5. Weyl differencing

In everything that follows we shall assume that $\mathrm{char}(\mathbb{F}_q) > d$ and we will allow all our implied constants to depend at most on $d$ and $n$. This section is concerned with a careful analysis of the exponential sum (3-5), using the function field version of Weyl differencing that was worked out by Lee [2013; 2011]. Our task is to make the dependence on $q$ completely explicit and it turns out that gaining satisfactory control requires considerable care. Since we are concerned with hypersurfaces one needs to take $R = 1$ in Lee's results.

For any $\beta = \sum_{i \leqslant N} b_i t^i \in K_\infty$, we let $\|\beta\| = \left|\sum_{i \leqslant -1} b_i t^i\right|$. Recalling the definition (3-1) of the multilinear forms associated to $F$, we let

$$N(\alpha) = \#\left\{ \underline{\boldsymbol{u}} \in \mathbb{O}^{(d-1)n} : \begin{array}{c} |\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_{d-1}| < |P| \\ \|\alpha \Psi_i(\underline{\boldsymbol{u}})\| < |P|^{-1} \ (\forall i \leqslant n) \end{array} \right\}, \tag{5-1}$$

where $\underline{\boldsymbol{u}} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_{d-1})$. We begin with an application of [Lee 2013, Corollary 4.3.2] (see also [Lee 2011, Corollary 3.3]), which leads to the inequality

$$|S(\alpha)|^{2^{d-1}} \leqslant |P|^{(2^{d-1}-d+1)n} N(\alpha), \tag{5-2}$$

for any $\alpha \in \mathbb{T}$.

The next stage in the analysis of $S(\alpha)$ is a multiple application of the function field analogue of Davenport's "shrinking lemma", as proved in [Lee 2013, Lemma 4.3.3] (see also [Lee 2011, Lemma 3.4]), ultimately leading to [Lee 2013, Lemma 4.3.4] (see also [Lee 2011, Lemma 3.5]). Unfortunately the implied constant in these estimates is allowed to depend on $q$ and so we must work harder to control it. Let

$$N_\eta(\alpha) = \#\left\{\underline{\boldsymbol{u}} \in \mathbb{O}^{(d-1)n} : \begin{array}{c} |\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_{d-1}| < |P|^\eta \\ \|\alpha\Psi_i(\underline{\boldsymbol{u}})\| < |P|^{-d+(d-1)\eta}\ (\forall i \leqslant n) \end{array}\right\},$$

for any parameter $\eta \in [0, 1]$. Recalling that $P = t^{e+1}$, we shall prove the following uniform version of [Lee 2013, Lemma 4.3.4] (see also [Lee 2011, Lemma 3.5]).

**Lemma 5.1.** *Let $\alpha \in \mathbb{T}$ and suppose that $\eta \in [0, 1)$ is chosen so that*

$$\frac{(e+1)(\eta+1)}{2} \in \mathbb{Z}. \tag{5-3}$$

*Then we have $N(\alpha) \leqslant |P|^{(n-\eta n)(d-1)} N_\eta(\alpha)$. In particular,*

$$|S(\alpha)|^{2^{d-1}} \leqslant \frac{|P|^{2^{d-1}n}}{|P|^{\eta(d-1)n}} N_\eta(\alpha).$$

*Proof.* In view of (5-1) and (5-2), the final part follows from the first part. For each $v \in \{0, \dots, d-1\}$, define $N^{(v)}(\alpha)$ to be the number of $\underline{\boldsymbol{u}} \in \mathbb{O}^{(d-1)n}$ such that

$$|\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_v| < |P|^\eta \quad \text{and} \quad |\boldsymbol{u}_{v+1}|, \dots, |\boldsymbol{u}_{d-1}| < |P| \tag{5-4}$$

and $\|\alpha\Psi_i(\underline{\boldsymbol{u}})\| < |P|^{-v-1+v\eta}$, for $1 \leqslant i \leqslant n$. Thus we have $N^{(0)}(\alpha) = N(\alpha)$ and $N^{(d-1)}(\alpha) = N_\eta(\alpha)$. It will suffice to show that

$$N^{(v)}(\alpha) \geqslant |P|^{-n+\eta n} N^{(v-1)}(\alpha), \tag{5-5}$$

for each $v \in \{1, \dots, d-1\}$.

Fix a choice of $v$, together with $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{v-1}, \boldsymbol{u}_{v+1}, \dots, \boldsymbol{u}_{d-1} \in \mathbb{O}^n$ such that (5-4) holds. For each $1 \leqslant i \leqslant n$ we consider the linear form

$$L_i(\boldsymbol{u}) = \alpha\Psi_i(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{v-1}, \boldsymbol{u}, \boldsymbol{u}_{v+1}, \dots, \boldsymbol{u}_{d-1}) = \sum_{j=1}^n \gamma_{ij} u_j,$$

say, for a suitable symmetric $n \times n$ matrix $\gamma = (\gamma_{ij})$, with entries in $K_\infty$. Given real numbers $a$ and $Z$, define $N(a, Z)$ to be the number of vectors $(u_1, \dots, u_{2n})$ in $\mathbb{O}^{2n}$ satisfying

$$|u_j| < \widehat{Z+a} \quad \text{and} \quad |L_j(u_1, \dots, u_n) - u_{j+n}| < \widehat{Z-a} \quad \text{for } 1 \leqslant j \leqslant n.$$

We are interested in estimating the number of $\boldsymbol{u} \in \mathbb{O}^n$ such that $|\boldsymbol{u}| < |P|^\eta$ and $\|L_i(\boldsymbol{u})\| < |P|^{-v-1+v\eta}$, for $1 \leqslant i \leqslant n$, in terms of the number of $\boldsymbol{u} \in \mathbb{O}^n$ such that

$|\boldsymbol{u}| < |P|$ and $\|L_i(\boldsymbol{u})\| < |P|^{-v+(v-1)\eta}$, for $1 \leqslant i \leqslant n$. That is, we wish to compare $N(a, Z_1)$ with $N(a, Z_2)$, where

$$\hat{a} = |P|^{(v+1-(v-1)\eta)/2}, \quad \hat{Z}_1 = |P|^{(v+1)(\eta-1)/2}, \quad \text{and} \quad \hat{Z}_2 = |P|^{(v-1)(\eta-1)/2}.$$

Note that $\hat{a}\hat{Z}_1 = |P|^\eta$ and $\hat{a}\hat{Z}_2 = |P|$. Moreover, our hypothesis (5-3) implies that

$$a = \frac{(e+1)(v+1)}{2} - \frac{(v-1)(e+1)\eta}{2} = v(e+1) - \frac{(v-1)(e+1)(\eta+1)}{2} \in \mathbb{Z}.$$

Similarly, (5-3) implies that $Z_1, Z_2 \in \mathbb{Z}$. It now follows from Lemma 4.2 that

$$\frac{N(a, Z_1)}{N(a, Z_2)} \geqslant \big(\widehat{Z_1 - Z_2}\big)^n = |P|^{-n+\eta n},$$

which thereby completes the proof of (5-5). □

Lemma 5.1 doesn't allow us to handle the case $e = 1$ of lines. To circumvent this difficulty we shall invoke a simpler version of the shrinking lemma, as follows.

**Lemma 5.2.** *Let $\alpha \in \mathbb{T}$ and let $v \in \{1, \dots, d\}$. Then we have*

$$|S(\alpha)|^{2^{d-1}} \leqslant |P|^{(2^{d-1}-d+1)n} q^{e(v-1)n} M^{(v)}(\alpha),$$

*where $M^{(v)}(\alpha)$ is the number of $\underline{\boldsymbol{u}} \in \mathbb{O}^{(d-1)n}$ such that*

$$|\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_{v-1}| < q \quad \textit{and} \quad |\boldsymbol{u}_v|, \dots, |\boldsymbol{u}_{d-1}| < |P|$$

*and $\|\alpha\Psi_i(\underline{\boldsymbol{u}})\| < |P|^{-1}$ for $1 \leqslant i \leqslant n$.*

*Proof.* Noting that $N(\alpha) = M^{(1)}(\alpha)$, it follows from (5-2) that it will be enough to prove that $M^{(v-1)}(\alpha) \leqslant q^{en} M^{(v)}(\alpha)$ for $2 \leqslant v \leqslant d$. The proof follows that of Lemma 5.1 and so we shall be brief. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{v-1}, \boldsymbol{u}_{v+1}, \dots, \boldsymbol{u}_{d-1} \in \mathbb{O}^n$ be vectors satisfying

$$|\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_{v-1}| < q \quad \text{and} \quad |\boldsymbol{u}_{v+1}|, \dots, |\boldsymbol{u}_{d-1}| < |P|.$$

Let $\gamma$ and $N(a, Z)$ be as in the proof of Lemma 5.1, corresponding to this choice. Lemma 4.2 clearly implies that

$$\frac{N(e+1, -e)}{N(e+1, 0)} \geqslant q^{-en}.$$

However, $N(e+1, -e)$ denotes the number of $\boldsymbol{u} \in \mathbb{O}^n$ such that $|\boldsymbol{u}| < q$ and $\|L_i(\boldsymbol{u})\| < q^{-2e-1}$, for $1 \leqslant i \leqslant n$. The lemma follows on noting that $q^{-2e-1} < q^{-e-1} = |P|^{-1}$. □

The next step is an application of the function field analogue of Heath-Brown's Diophantine approximation lemma, as worked out in [Lee 2013, Lemma 4.3.5] (see also [Lee 2011, Lemma 3.6]). Let $\alpha = a/r + \theta$, where $a/r \in K$ and $\theta \in \mathbb{T}$. Note that the maximum absolute value of the coefficients of each multilinear form $\Psi_j$

is 1. We shall apply those lemmas with $\hat{M}=|P|^{(d-1)\eta}$ and $\hat{Y}=|P|^{d-(d-1)\eta}$. We want a maximal choice of $\eta\geqslant 0$ such that

$$|P|^{(d-1)\eta}<\min\left\{|P|^{d-1},\frac{1}{|r\theta|},\frac{|P|^d}{|r|}\right\}$$

and

$$|P|^{(d-1)\eta}\leqslant |r|\max\bigl\{1,|P^d\theta|\bigr\}.$$

This leads to the constraint $(e+1)\eta\leqslant\Gamma$, where

$$\Gamma=\frac{1}{d-1}\operatorname{ord}\left(\min\left\{\frac{|P|^{d-1}}{q},\frac{1}{q|r\theta|},\frac{|P|^d}{q|r|},|r|\max\bigl\{1,|P^d\theta|\bigr\}\right\}\right),\tag{5-6}$$

in which we abuse notation and denote by ord the integer exponent of $q$ that appears. For $i\in\{0,1\}$, let $[\Gamma]_i$ denote the largest nonnegative integer not exceeding $\Gamma$, which is congruent to $i$ modulo 2. We then choose $\eta$ via

$$(e+1)\eta=\begin{cases}[\Gamma]_0 & \text{if } 2\nmid e,\\ [\Gamma]_1 & \text{if } 2\mid e.\end{cases}\tag{5-7}$$

One notes that $(e+1)\eta\leqslant\Gamma$ and (5-3) is satisfied.

It now follows from [Lee 2013, Lemma 4.3.5] (see also [Lee 2011, Lemma 3.6]) that $N_\eta(\alpha)\leqslant U_\eta$, where $U_\eta$ denotes the number of $\underline{\boldsymbol{u}}\in\mathbb{O}^{(d-1)n}$ such that

$$|\boldsymbol{u}_1|,\dots,|\boldsymbol{u}_{d-1}|<|P|^\eta\quad\text{and}\quad \Psi_i(\underline{\boldsymbol{u}})=0\quad\text{for } 1\leqslant i\leqslant n.$$

A standard calculation, which we recall here for completeness, now shows that the latter system of equations defines an affine variety $V\subset\mathbb{A}^{(d-1)n}$ of dimension at most $(d-2)n$. To see this, we note that the intersection of $V$ with the diagonal $\Delta=\{\underline{\boldsymbol{u}}\in\mathbb{A}^{(d-1)n}:\boldsymbol{u}_1=\cdots=\boldsymbol{u}_{d-1}\}$ is contained in the singular locus of $F$ and so has affine dimension 0. The claim follows on noting that $0=\dim(V\cap\Delta)\geqslant\dim V+\dim\Delta-(d-1)n=\dim V-(d-2)n$.

We now apply [Browning and Vishe 2015, Lemma 2.8]. Since $|P|^\eta=q^{(e+1)\eta}$, with $(e+1)\eta\in\mathbb{Z}$, this directly yields the existence of a positive constant $c_{d,n}$, independent of $q$, such that $U_\eta\leqslant c_{d,n}|P|^{\eta(d-2)n}$. Inserting this into Lemma 5.1, we therefore arrive at the following conclusion.

**Lemma 5.3.** *Let $L=2^{-d+1}n$, let $a/r\in K$ and let $\theta\in\mathbb{T}$. Let $\eta$ be given by (5-7). Then there exists a constant $c_{d,n}>0$, independent of $q$, such that*

$$|S(a/r+\theta)|\leqslant c_{d,n}|P|^{n-L\eta}.$$

It turns out that this estimate is inefficient when $|r|$ is small. Let

$$\kappa=\begin{cases}1 & \text{if } 2\nmid e,\\ 0 & \text{if } 2\mid e.\end{cases}\tag{5-8}$$

It will also be advantageous to consider the effect of taking $(e+1)\eta = 1+\kappa$, instead of (5-7). Since

$$\frac{(e+1)(\eta+1)}{2} = 1 + \frac{e+\kappa}{2} \in \mathbb{Z},$$

it follows from Lemma 5.1 that

$$|S(\alpha)| \leqslant \frac{|P|^n \mathcal{N}^{2^{-d+1}}}{q^{(1+\kappa)(d-1)L}}, \tag{5-9}$$

where

$$\mathcal{N} = \#\left\{ \underline{\boldsymbol{u}} \in \mathbb{O}^{(d-1)n} : \begin{array}{c} |\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_{d-1}| \leqslant q^{\kappa} \\ \|\alpha \Psi_i(\underline{\boldsymbol{u}})\| < q^{\kappa(d-1)-de-1} \ (\forall i \leqslant n) \end{array} \right\}, \tag{5-10}$$

Supposing that $\alpha = a/r + \theta$ for $a/r \in K$ and $\theta \in \mathbb{T}$, our argument now bifurcates according to the degree of $r$.

**Lemma 5.4** ($\deg(r) \geqslant 1$). *Let $L = 2^{-d+1}n$, let $a/r \in K$, and let $\theta \in \mathbb{T}$. Assume that*

(i) $e \geqslant 1$, $q \leqslant |r| < q^{de+1-\kappa(d-1)}$ *and* $|r\theta| < q^{-\kappa(d-1)}$; *or*

(ii) $e = 1$, $q^2 \leqslant |r| \leqslant q^d$, *and* $|r\theta| \leqslant q^{-d}$.

*Then there exists a constant $c'_{d,n} > 0$, independent of $q$, such that*

$$|S(a/r+\theta)| \leqslant c'_{d,n} |P|^n q^{-L}.$$

*Proof.* To deal with case (i) we apply [Lee 2013, Lemma 4.3.5] (see also [Lee 2011, Lemma 3.6]) with $Y = de + 1 - \kappa(d-1)$ and $M = \kappa(d-1) + \frac{1}{2}$. Our hypotheses ensure that $|r| < \hat{Y}$ and $|r\theta| < \hat{M}^{-1}$. Thus it follows that $\Psi_i(\underline{\boldsymbol{u}}) \equiv 0 \bmod r$ in (5-10), for all $i \leqslant n$. In particular we have $\mathcal{N} = 0$ unless $\kappa = 1$, which we now assume.

Pick a prime $\varpi \mid r$ with $|\varpi| \geqslant q$. If $|\varpi| \leqslant q^2$ we may break into residue classes modulo $\varpi$, finding that

$$\mathcal{N} \leqslant \sum_{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-1}} \#\big\{ |\boldsymbol{u}_1|, \dots, |\boldsymbol{u}_{d-1}| \leqslant q : \boldsymbol{u}_i \equiv \boldsymbol{v}_i \bmod \varpi \ (\text{for } 1 \leqslant i \leqslant d-1) \big\},$$

where the sum is over all $\underline{\boldsymbol{v}} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-1}) \in \mathbb{F}_{\varpi}^{(d-1)n}$ such that $\Psi_i(\underline{\boldsymbol{v}}) = 0$, for all $i \leqslant n$, over $\mathbb{F}_{\varpi}$. The inner cardinality is $O((q^2/|\varpi|)^{(d-1)n})$, with an implied constant that is independent of $q$. We may use the Lang–Weil estimate to deduce that the outer sum is $O(|\varpi|^{(d-2)n})$, again with an implied constant that depends at most on $d$ and $n$. Hence we get the overall contribution

$$\mathcal{N} \ll \frac{q^{2(d-1)n}}{|\varpi|^n} \leqslant q^{2(d-1)n-n}.$$

Alternatively, if $|\varpi| > q^2$, we may assume that the system of equations $\Psi_i = 0$, for $i \leqslant n$, has dimension $(d-2)n$ over $\mathbb{F}_{\varpi}$. We now appeal to an argument of

Browning and Heath-Brown [2009, Lemma 4]. Using induction on the dimension, as in the proof of [op. cit., Equation (3.7)], we easily conclude that

$$\mathcal{N} \ll (q^2)^{(d-2)n} \leqslant q^{2(d-1)n-2n},$$

for an implied constant that only depends on $d$ and $n$. Recalling that $\kappa = 1$, the first part of the lemma now follows on substituting these bounds into (5-9).

We now consider case (ii), in which $e = 1$, $q^2 \leqslant |r| \leqslant q^d$, and $|r\theta| \leqslant q^{-d}$. Let $|a/r| = q^{-\alpha}$ for $1 \leqslant \alpha \leqslant d$. Let $v \in \{1, \ldots, d\}$ be such that $d - v - \alpha = -1$. Then an application of Lemma 5.2 yields

$$\begin{aligned}|S(\alpha)|^{2^{d-1}} &\leqslant |P|^{(2^{d-1}-d+1)n} q^{(v-1)n} M^{(v)}(\alpha)\\ &= |P|^{2^{d-1}n} q^{(-2d+1+v)n} M^{(v)}(\alpha).\end{aligned}$$

Let $\underline{\boldsymbol{u}} \in \mathbb{O}^{n(d-1)}$ be counted by $M^{(v)}(\alpha)$. Since $|\theta| \leqslant q^{-d-2}$, it follows that $|\theta\Psi_i(\underline{\boldsymbol{u}})| \leqslant q^{-d-2} \cdot q^{d-v} = q^{-2-v} \leqslant q^{-3}$, for $1 \leqslant i \leqslant n$. Similarly, for $1 \leqslant i \leqslant n$, we have $|(a/r)\Psi_i(\underline{\boldsymbol{u}})| \leqslant q^{-\alpha} \cdot q^{d-v} = q^{-1}$. If we write $\boldsymbol{u}_j = \boldsymbol{u}_j' + t\boldsymbol{u}_j''$, for $v \leqslant j \leqslant d$, where $\boldsymbol{u}_j', \boldsymbol{u}_j'' \in \mathbb{F}_q^n$, then the coefficient of $t^{-1}$ in the $t$-expansion of $(a/r)\Psi_i(\underline{\boldsymbol{u}})$ is equal to $\Psi_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{v-1}, \boldsymbol{u}_v'', \ldots, \boldsymbol{u}_{d-1}'')$. The condition $\|\alpha\Psi_i(\underline{\boldsymbol{u}})\| < |P|^{-1}$ in $M^{(v)}(\alpha)$ implies that this coefficient must necessarily vanish, whence $M^{(v)}(\alpha)$ is at most the number of $\boldsymbol{u}_1, \ldots \boldsymbol{u}_{v-1}, \boldsymbol{u}_v', \ldots, \boldsymbol{u}_{d-1}', \boldsymbol{u}_v'', \ldots, \boldsymbol{u}_{d-1}'' \in \mathbb{F}_q^n$ for which $\Psi_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{v-1}, \boldsymbol{u}_v'', \ldots, \boldsymbol{u}_{d-1}'') = 0$, for $1 \leqslant i \leqslant n$. Thus

$$M^{(v)}(\alpha) \ll q^{(d-v)n} \cdot q^{(d-2)n} = q^{(2d-v-2)n},$$

by the Lang–Weil estimate, which implies the statement of the lemma. □

**Lemma 5.5** ($\deg(r) = 0$). *Let $L = 2^{-d+1}n$ and let $\theta \in \mathbb{T}$. Assume that*

$$q^{-de-1} \leqslant |\theta| \leqslant q^{-1-\kappa(d-1)}.$$

*Then there exists a constant $c_{d,n}'' > 0$, independent of $q$, such that*

$$|S(\theta)| \leqslant c_{d,n}'' |P|^n q^{-L}.$$

*Proof.* The upper bound assumed of $|\theta|$ implies that $|\theta\Psi_i(\underline{\boldsymbol{u}})| \leqslant q^{-1}$ in (5-10), for $1 \leqslant i \leqslant n$. Hence $\|\theta\Psi_i(\underline{\boldsymbol{u}})\| = |\theta\Psi_i(\underline{\boldsymbol{u}})|$ for $1 \leqslant i \leqslant n$. Since $\alpha = \theta$ and $|\theta| \geqslant q^{-de-1}$, it follows that the condition $\|\alpha\Psi_i(\underline{\boldsymbol{u}})\| < q^{\kappa(d-1)-de-1}$ is equivalent to $|\Psi_i(\underline{\boldsymbol{u}})| < q^{\kappa(d-1)}$. If $\kappa = 0$ then it follows from (5-10) that

$$\mathcal{N} = \#\big\{\underline{\boldsymbol{u}} \in \mathbb{F}_q^{(d-1)n} : \Psi_i(\underline{\boldsymbol{u}}) = 0\ (\forall i \leqslant n)\big\} \ll q^{(d-2)n},$$

by the Lang–Weil estimate. If, on the other hand, $\kappa = 1$ then we write $\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}' + t\underline{\boldsymbol{u}}''$ in $\mathcal{N}$, under which transformation $|\Psi_i(\underline{\boldsymbol{u}})| < q^{d-1}$ is equivalent to $\Psi_i(\underline{\boldsymbol{u}}'') = 0$, for $i \leqslant n$. Applying the Lang–Weil estimate to this system of equations, we therefore deduce that $\mathcal{N} = O(q^{(1+\kappa)(d-1)n-n})$ for $\kappa \in \{0, 1\}$. An application of (5-9) now completes the proof of the lemma. $\square$

## 6. The contribution from the minor arcs

We assume that $d \geqslant 3$ throughout this section. Our goal is to prove (3-6) for all $e \geqslant 1$, provided that $n > n_0(d)$, where $n_0(d)$ is given by (1-2). The overall contribution to (3-4) from $|\theta| < q^{-3de}$ is easily seen to be negligible. Hence we may redefine the minor arcs to incorporate the condition $|\theta| \geqslant q^{-3de}$. For $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$, let $E(\alpha, \beta)$ denote the overall contribution to $N_{\text{minor}}(P)$, from values of $a, r, \theta$ for which $|r| = q^\alpha$ and $|\theta| = q^{-\beta}$. The contribution is empty unless

$$0 \leqslant \alpha \leqslant \frac{d(e+1)}{2} \quad \text{and} \quad \alpha + \frac{d(e+1)}{2} \leqslant \beta \leqslant 3de, \tag{6-1}$$

with $\beta \leqslant de + 1$ if $\alpha = 0$. Since there are only finitely many choices of $\alpha, \beta$, in order to prove (3-6), it will suffice to show that

$$\lim_{q\to\infty} q^{-\hat{\mu}} E(\alpha, \beta) = 0,$$

for each pair $(\alpha, \beta)$ under consideration, assuming that $n > n_0(d)$. To begin with, summing trivially over $a$, we have

$$E(\alpha, \beta) \leqslant q^{2\alpha-\beta+1} \max_{\substack{a,r,\theta \\ |a|<|r|=q^\alpha \\ |\theta|=q^{-\beta}}} |S(a/r + \theta)|. \tag{6-2}$$

We start by dealing with generic values of $\alpha$ and $\beta$. Lemma 5.3 implies that

$$E(\alpha, \beta) \leqslant c_{d,n} q^{2\alpha-\beta+1+(e+1)n-L(e+1)\eta},$$

where $L = 2^{-d+1}n$. Recalling (3-2), the definition of $\hat{\mu}$, the exponent of $q$ is $\hat{\mu} - \hat{\nu}$, with

$$\begin{aligned} \hat{\nu} &= \{(n-d)e + n - 1\} - \{2\alpha - \beta + 1 + (e+1)n - L(e+1)\eta\} \\ &= L(e+1)\eta + \beta - de - 2\alpha - 2. \end{aligned} \tag{6-3}$$

For the choice of $\eta$ in (5-7), and $n > n_0(d)$, we want to determine when $\hat{\nu} > 0$. Returning to (5-6), we now see that

$$\Gamma = \frac{1}{d-1} \min\bigl\{(e+1)(d-1) - 1,\ \beta - \alpha - 1,\ (e+1)d - \alpha - 1,\ \alpha + M\bigr\},$$

where $M = \max\{0, (e+1)d - \beta\}$. The remainder of the argument is a case by case analysis. When $[\Gamma] \leqslant 1$ we shall return to (6-2), and argue differently based instead on Lemmas 5.4 and 5.5.

***Case 1:* $\boldsymbol{\alpha \geqslant 2(d-1)}$ *and* $\boldsymbol{\beta \geqslant (e+1)d+1}$.** In this case $M=0$. Using (6-1), one finds that

$$\Gamma = \frac{1}{d-1} \times \begin{cases} \alpha & \text{if } \alpha < \frac{d(e+1)}{2}, \\ \alpha - 1 & \text{if } \alpha = \frac{d(e+1)}{2}. \end{cases}$$

Let $\iota \in \{0, 1\}$. We write $\alpha - \iota = k(d-1) + \ell$, for $k \in \mathbb{Z}_{\geqslant 0}$ and $\ell \in \{0, \dots, d-2\}$. Then (5-7) implies that $(e+1)\eta = k - \delta$, where

$$\delta = \begin{cases} 0 & \text{if } k \not\equiv e \mod 2, \\ 1 & \text{if } k \equiv e \mod 2. \end{cases} \tag{6-4}$$

We claim that the assumption $\alpha \geqslant 2(d-1)$ implies that $k \geqslant 2$, or else $k=1$ and $\delta = 0$. This is obvious when $\alpha < \frac{d(e+1)}{2}$. Suppose that $k=1$ and $\alpha = \frac{d(e+1)}{2}$. Then $\iota = 0$ and $\ell = d-2$, whence $\alpha = 2(d-1) = \frac{d(e+1)}{2}$. Since $d \geqslant 3$, this equation has no solutions in odd integers $e$. Thus $\delta = 0$.

Recalling (6-3) and substituting for $\alpha$, we find that

$$\begin{aligned} \hat{\nu} &= L(k-\delta) + \beta - de - 2 - 2\iota - 2k(d-1) - 2\ell \\ &= (L - 2(d-1))k - \delta L + \beta - de - 2 - 2\iota - 2\ell \\ &\geqslant (L-2(d-1))k - \delta L - d + 3 - 2\iota, \end{aligned}$$

since $\beta \geqslant (e+1)d+1$ and $\ell \leqslant d-2$. Taking $3 - 2\iota \geqslant 0$, we have therefore shown that $\hat{\nu} \geqslant \hat{\nu}_0$, with

$$\hat{\nu}_0 = (L - 2(d-1))k - \delta L - d.$$

If $k \geqslant 2$, then we take $\delta \leqslant 1$ to conclude that

$$\hat{\nu}_0 \geqslant (2-\delta)L - 4(d-1) - d \geqslant L - 5d + 4.$$

Thus $\hat{\nu}_0 > 0$ if $n > n_0(d)$. Alternatively, if $k = 1$ then we must have $\delta = 0$. It follows that

$$\hat{\nu}_0 = L - 2(d-1) - d = L - 3d + 2,$$

whence $\hat{\nu}_0 > 0$ if $n > n_0(d)$, since $n_0(d) \geqslant 2^{d-1} \cdot (3d-2)$ in (1-2).

***Case 2:* $\boldsymbol{\alpha + de - d + 2 > \beta}$ *and* $\boldsymbol{\beta \leqslant (e+1)d}$.** In this case $M = (e+1)d - \beta$. It follows from (6-1) that

$$\Gamma = \frac{1}{d-1} \times \begin{cases} \alpha + (e+1)d - \beta & \text{if } \beta > 2\alpha, \\ \alpha + (e+1)d - \beta - 1 & \text{if } \beta \leqslant 2\alpha. \end{cases}$$

We proceed as before. Thus for $\iota \in \{0, 1\}$, we write

$$\alpha + (e+1)d - \beta - \iota = k(d-1) + \ell, \tag{6-5}$$

with $k \in \mathbb{Z}_{\geqslant 0}$ and $\ell \in \{0, \dots, d-2\}$. Then (5-7) implies that $(e+1)\eta = k - \delta$, where $\delta$ is given by (6-4). If $k \geqslant 2$ then (6-3) yields

$$\begin{aligned}\hat{\nu} &= L(k-\delta) - \beta + de - 2 - 2\iota - 2k(d-1) - 2\ell + 2d \\ &= (L - 2(d-1))k - \delta L - \beta + de - 2 - 2\iota - 2\ell + 2d \\ &\geqslant L - 4d + 4 - \beta + de,\end{aligned}$$

since $\delta, \iota \leqslant 1$ and $\ell \leqslant d-2$. But $\beta \leqslant (e+1)d$, and so it follows that $\hat{\nu} \geqslant L - 5d + 4$, which is positive if $n > n_0(d)$. Suppose that $k \leqslant 1$. Then, on taking $\iota \leqslant 1$ and $\ell \leqslant d-2$ in (6-5), we must have that

$$\alpha + de - d + 2 \leqslant \beta,$$

which contradicts the hypothesis.

***Case 3: $\alpha \leqslant 2(d-1)$ and $\beta \geqslant (e+1)d + 1$.*** In this case we return to (6-2), and we recall the definition (5-8) of $\kappa$. Suppose first that $\alpha = 0$. It follows from Lemma 5.5 that $S(a/r + \theta) \ll |P|^n q^{-L}$ if

$$1 + \kappa(d-1) \leqslant \beta \leqslant de + 1.$$

The upper bound $\beta \leqslant de + 1$ follows from the definition of the minor arcs when $\alpha = 0$. Moreover, the lower bound holds, since for $e \geqslant 1$ it follows from (6-1) that $\beta \geqslant d \geqslant 1 + \kappa(d-1)$. Recalling (3-2), we conclude that

$$E(\alpha, \beta) \ll q^{-\beta+1+(e+1)n-L} = q^{\hat{\mu}-\hat{\nu}},$$

with $\hat{\nu} = L + \beta - de - 2 \geqslant L > 0$, which is satisfactory.

Suppose next that $\alpha \geqslant 1$. Then $S(a/r + \theta) \ll |P|^n q^{-L}$, by Lemma 5.4, provided that

$$e \geqslant 1, \quad 1 \leqslant \alpha < de + 1 - \kappa(d-1), \quad \text{and} \quad \alpha - \beta < -\kappa(d-1), \tag{6-6}$$

or

$$e = 1, \quad 2 \leqslant \alpha \leqslant d, \quad \text{and} \quad \alpha - \beta \leqslant -d. \tag{6-7}$$

In view of (6-1), it is easily seen that $\alpha - \beta < -(d-1) \leqslant -\kappa(d-1)$. Next, we claim that $2d - 2 < de + 1 - \kappa(d-1)$ for any $e \geqslant 2$. This is enough to confirm (6-6), since $\alpha \leqslant 2(d-1)$. The claim is obvious when $\kappa = 1$ and $e \geqslant 3$. On the other hand, if $\kappa = 0$ then $e \geqslant 2$ and it is clear that $2d - 2 \leqslant 2d + 1 \leqslant de + 1$. Next, suppose that $e = 1$, so that $\kappa = 1$. If $\alpha = 1$ then we are plainly in the situation covered by (6-6). If $\alpha \geqslant 2$, on the other hand, then (6-1) implies that $\alpha \leqslant d$ and $\alpha - \beta \leqslant -d$, so that we are in the case covered by (6-7). It follows that

$$E(\alpha, \beta) \ll q^{2\alpha-\beta+1+(e+1)n-L} = q^{\hat{\mu}-\hat{\nu}},$$

with
$$\hat{\nu} = L+\beta-de-2-2\alpha \geqslant L+d-1-2\alpha \\ \geqslant L-3d+3,$$
since $\alpha \leqslant 2(d-1)$ and $\beta \geqslant (e+1)d+1$. This is positive for $n > n_0(d)$.

***Case 4:*** $\boldsymbol{\alpha+de-d+2 \leqslant \beta}$ ***and*** $\boldsymbol{\beta \leqslant (e+1)d}$***.*** We begin as in the previous case. If $\alpha=0$, the same argument goes through, leading to $E(\alpha,\beta) \ll q^{\hat{\mu}-\hat{\nu}}$, with $\hat{\nu} = L+\beta-de-2 \geqslant L-d$. This is certainly positive for $n > n_0(d)$. Suppose next that $\alpha \geqslant 1$. Then $S(a/r+\theta) \ll |P|^n q^{-L}$, by Lemma 5.4, provided that (6-6) or (6-7) hold. Note that
$$\alpha \leqslant \beta-de+d-2 \leqslant 2d-2 < de+1-\kappa(d-1),$$
for any $e \geqslant 2$, by the calculation in the previous case. Likewise, the previous argument shows that we are covered by (6-6) or (6-7) when $e=1$. Thus we find that $E(\alpha,\beta) \ll q^{\hat{\mu}-\hat{\nu}}$, with
$$\hat{\nu} = L+\beta-de-2-2\alpha \geqslant L-d-\alpha \\ \geqslant L-3d+2,$$
since $\alpha \leqslant 2(d-1)$. This is also positive for $n > n_0(d)$.

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# Thick tensor ideals of right bounded derived categories

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Let $R$ be a commutative noetherian ring. Denote by $\mathsf{D}^-(R)$ the derived category of cochain complexes $X$ of finitely generated $R$-modules with $\mathsf{H}^i(X) = 0$ for $i \gg 0$. Then $\mathsf{D}^-(R)$ has the structure of a tensor triangulated category with tensor product $\cdot \otimes^{\boldsymbol{L}}_R \cdot$ and unit object $R$. In this paper, we study thick tensor ideals of $\mathsf{D}^-(R)$, i.e., thick subcategories closed under the tensor action by each object in $\mathsf{D}^-(R)$, and investigate the Balmer spectrum $\operatorname{Spc} \mathsf{D}^-(R)$ of $\mathsf{D}^-(R)$, i.e., the set of prime thick tensor ideals of $\mathsf{D}^-(R)$. First, we give a complete classification of the thick tensor ideals of $\mathsf{D}^-(R)$ generated by bounded complexes, establishing a generalized version of the Hopkins–Neeman smash nilpotence theorem. Then, we define a pair of maps between the Balmer spectrum $\operatorname{Spc} \mathsf{D}^-(R)$ and the Zariski spectrum Spec $R$, and study their topological properties. After that, we compare several classes of thick tensor ideals of $\mathsf{D}^-(R)$, relating them to specialization-closed subsets of Spec $R$ and Thomason subsets of $\operatorname{Spc} \mathsf{D}^-(R)$, and construct a counterexample to a conjecture of Balmer. Finally, we explore thick tensor ideals of $\mathsf{D}^-(R)$ in the case where $R$ is a discrete valuation ring.



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## Introduction

*Tensor triangular geometry* is a theory established by Balmer at the beginning of this century. Let $(\mathscr{T}, \otimes, \mathbf{1})$ be an (essentially small) *tensor triangulated category*, that is, a triangulated category $\mathscr{T}$ equipped with symmetric tensor product $\otimes$ and unit object $\mathbf{1}$. One can then define *prime* thick tensor ideals of $\mathscr{T}$, which behave similarly to prime ideals of commutative rings. The *Balmer spectrum* $\mathtt{Spc}\,\mathscr{T}$ of $\mathscr{T}$ is defined as the set of prime thick tensor ideals of $\mathscr{T}$. This set has the structure of a topological space. Tensor triangular geometry studies Balmer spectra and develops commutative-algebraic and algebrogeometric observations on them. Tensor triangular geometry is related to a lot of areas of mathematics, including commutative/noncommutative algebra, commutative/noncommutative algebraic geometry, stable homotopy theory, modular representation theory, motivic theory, noncommutative topology, and symplectic geometry. Understandably, tensor triangular geometry has been attracting a great deal of attention, and Balmer [2010b] gave an invited lecture at the 2010 International Congress of Mathematicians.

By virtue of a landmark theorem due to Balmer [2005], the radical thick tensor ideals of $\mathscr{T}$ correspond to the Thomason subsets of the Balmer spectrum $\mathtt{Spc}\,\mathscr{T}$ of $\mathscr{T}$. It is thus a main subject in tensor triangular geometry to determine/describe the Balmer spectrum of a given tensor triangulated category. Such studies have been done for these thirty years considerably widely; one can find ones at least in stable homotopy theory [Balmer and Sanders 2017; Devinatz et al. 1988; Hopkins and Smith 1998], commutative algebra [Hopkins 1987; Neeman 1992; Takahashi 2010], algebraic geometry [Balmer 2002; Stevenson 2014b; Thomason 1997], modular representation theory [Balmer 2016; Benson et al. 1997; 2011; Friedlander and Pevtsova 2007], and motivic theory [Dell'Ambrogio and Tabuada 2012; Peter 2013].

Let $R$ be a commutative noetherian ring. Denote by $\mathsf{D}^-(R)$ the right bounded derived category of finitely generated $R$-modules, namely, the derived category of (cochain) complexes $X$ of finitely generated $R$-modules such that $\mathsf{H}^i(X) = 0$ for all $i \gg 0$. Then $(\mathsf{D}^-(R), \otimes_R^{\boldsymbol{L}}, R)$ is a tensor triangulated category. The main purpose of this paper is to investigate thick tensor ideals of the tensor triangulated category $\mathsf{D}^-(R)$, analyzing the structure of the Balmer spectrum $\mathtt{Spc}\,\mathsf{D}^-(R)$ of $\mathsf{D}^-(R)$.

Here, we should remark that results in the literature which we can apply for our purpose are quite limited. For example, many people have been studying the Balmer spectra of tensor triangulated categories which arise as the compact objects of compactly generated tensor triangulated categories, but our tensor triangulated category $\mathsf{D}^-(R)$ does not arise in this way. Also, there are various results on the Balmer spectrum of a rigid tensor triangulated category, but again they do not apply to our case because $\mathsf{D}^-(R)$ is not rigid (nor even closed); see Remark 1.3. Furthermore, several properties have been found for tensor triangulated categories which are generated by their unit object as a thick subcategory, but $\mathsf{D}^-(R)$ does not satisfy this property. Thus, the only existing results that are available and useful for our goal are basically general fundamental results given in [Balmer 2005], and we need to start with establishing basic tools by ourselves.

From now on, let us explain the main results of this paper. First of all, recall that an object $X$ of a triangulated category $\mathcal{T}$ is *compact* if the natural morphism

$$\bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_{\mathcal{T}}(M, N_\lambda) \to \mathrm{Hom}_{\mathcal{T}}\Big(M, \bigoplus_{\lambda\in\Lambda} N_\lambda\Big)$$

is an isomorphism for every family $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$ with $\bigoplus_{\lambda\in\Lambda} N_\lambda \in \mathcal{T}$. Furthermore, $X$ is *cocompact* if the natural morphism

$$\bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_{\mathcal{T}}(N_\lambda, M) \to \mathrm{Hom}_{\mathcal{T}}\Big(\prod_{\lambda\in\Lambda} N_\lambda, M\Big)$$

is an isomorphism for every family $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$ with $\prod_{\lambda\in\Lambda} N_\lambda \in \mathcal{T}$. A thick tensor ideal of $\mathsf{D}^-(R)$ is called *compactly generated* or *cocompactly generated* if it is generated by compact or cocompact objects, respectively, of $\mathsf{D}^-(R)$ as a thick tensor ideal. For a subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$ we denote by $\mathrm{Supp}\,\mathcal{X}$ the union of the supports of complexes in $\mathcal{X}$, and for a subset $S$ of $\mathrm{Spec}\,R$ we denote by $\langle S\rangle$ the thick tensor ideal of $\mathsf{D}^-(R)$ generated by $R/\mathfrak{p}$ with $\mathfrak{p}\in S$. We shall prove the following theorem.

**Theorem A** (Proposition 2.1, Theorem 2.12, and Corollary 2.16). *The compact or cocompact objects of* $\mathsf{D}^-(R)$ *are the perfect or bounded complexes*, *respectively*; *hence*, *all compactly generated thick tensor ideals are cocompactly generated. The assignments* $\mathcal{X}\mapsto \mathrm{Supp}\,\mathcal{X}$ *and* $\langle W\rangle \leftarrow\!\shortmid W$ *make mutually inverse bijections*

$$\{\textit{Cocompactly generated thick } \otimes\textit{-ideals of } \mathsf{D}^-(R)\} \rightleftarrows \{\textit{Specialization-closed subsets of } \mathrm{Spec}\,R\}.$$

*Consequently*, *all cocompactly generated thick tensor ideals of* $\mathsf{D}^-(R)$ *are compactly generated.*

The core of this theorem is constituted by the classification of the *cocompactly* generated thick tensor ideals of $\mathsf{D}^-(R)$, which is obtained by establishment of a *generalized smash nilpotence theorem*, extending the classical smash nilpotence theorem due to Hopkins [1987] and Neeman [1992] for the homotopy category of perfect complexes. In view of Theorem A, we may simply call $\mathcal{X}$ *compact* if $\mathcal{X}$ is compactly generated and/or cocompactly generated. We should remark that in general we have

$$\langle W\rangle \neq \mathrm{Supp}^{-1} W,$$

where $\operatorname{Supp}^{-1} W$ consists of the complexes whose supports are contained in $W$. Thus, we call a thick tensor ideal of $\mathsf{D}^{-}(R)$ *tame* if it has the form $\operatorname{Supp}^{-1} W$ for some specialization-closed subset $W$ of $\operatorname{Spec} R$.

Next, we relate the Balmer spectrum $\operatorname{Spc} \mathsf{D}^{-}(R)$ of $\mathsf{D}^{-}(R)$ to the Zariski spectrum $\operatorname{Spec} R$ of $R$, i.e., the set of prime ideals of $R$. More precisely, we introduce a pair of order-reversing maps

$$\mathcal{S} : \operatorname{Spec} R \rightleftarrows \operatorname{Spc} \mathsf{D}^{-}(R) : \mathfrak{s}$$

and investigate their topological properties. These maps are defined as follows. Let $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathcal{P} \in \operatorname{Spc} \mathsf{D}^{-}(R)$. Then $\mathcal{S}(\mathfrak{p})$ consists of the complexes $X \in \mathsf{D}^{-}(R)$ with $X_{\mathfrak{p}} = 0$, and $\mathfrak{s}(\mathcal{P})$ is the unique maximal element of ideals $I$ of $R$ with $R/I \notin \mathcal{P}$ with respect to the inclusion relation. Our main result in this direction is the following theorem. Denote by ${}^{\mathrm{t}}\operatorname{Spc} \mathsf{D}^{-}(R)$ the set of tame prime thick tensor ideals of $\mathsf{D}^{-}(R)$, and by $\operatorname{Mx} \mathsf{D}^{-}(R)$ and $\operatorname{Mn} \mathsf{D}^{-}(R)$ the maximal and minimal elements, respectively, of $\operatorname{Spc} \mathsf{D}^{-}(R)$ with respect to the inclusion relation. For each full subcategory $\mathcal{X}$ of $\mathsf{D}^{-}(R)$, let $\mathcal{X}^{\mathsf{tame}}$ stand for the smallest tame thick tensor ideal of $\mathsf{D}^{-}(R)$ containing $\mathcal{X}$.

**Theorem B** (Theorems 3.9, 4.5, 4.7, 4.12, and 4.14 and Corollary 3.14). *The following statements hold.*

(1) $\mathfrak{s} \cdot \mathcal{S} = 1$ *and* $\mathcal{S} \cdot \mathfrak{s} = \operatorname{Supp}^{-1} \operatorname{Supp} = (\,\cdot\,)^{\mathsf{tame}}$. *In particular*, $\dim(\operatorname{Spc} \mathsf{D}^{-}(R)) \geqslant \dim R$.

(2) *The image of* $\mathcal{S}$ *coincides with* ${}^{\mathrm{t}}\operatorname{Spc} \mathsf{D}^{-}(R)$, *and it is dense in* $\operatorname{Spc} \mathsf{D}^{-}(R)$.

(3) *The map* $\mathfrak{s}$ *is continuous, and its restriction* $\mathfrak{s}' : {}^{\mathrm{t}}\operatorname{Spc} \mathsf{D}^{-}(R) \to \operatorname{Spec} R$ *is a continuous bijection.*

(4) *The map* $\mathcal{S}' : \operatorname{Spec} R \to {}^{\mathrm{t}}\operatorname{Spc} \mathsf{D}^{-}(R)$ *induced by* $\mathcal{S}$ *is an open and closed bijection.*

(5) *The map* $\operatorname{Min} R \to \operatorname{Mx} \mathsf{D}^{-}(R)$ *induced by* $\mathcal{S}$ *is a homeomorphism.*

(6) *The map* $\operatorname{Max} R \to \operatorname{Mn} \mathsf{D}^{-}(R)$ *induced by* $\mathcal{S}$ *is a homeomorphism if* $R$ *is semilocal.*

(7) $\mathcal{S}$ *is continuous* $\Longleftrightarrow$ $\mathcal{S}'$ *is homeomorphic* $\Longleftrightarrow$ $\mathfrak{s}'$ *is homeomorphic* $\Longleftrightarrow$ $\operatorname{Spec} R$ *is finite.*

The celebrated classification theorem due to Balmer [2005] asserts that taking the Balmer support $\operatorname{Spp}$ makes a one-to-one correspondence between the set **Rad** of radical thick tensor ideals of $\mathsf{D}^{-}(R)$ and the set **Thom** of Thomason subsets of $\operatorname{Spc} \mathsf{D}^{-}(R)$:

$$\operatorname{Spp} : \textbf{Rad} \rightleftarrows \textbf{Thom} : \operatorname{Spp}^{-1}.$$

Our next goal is to complete this one-to-one correspondence to the following commutative diagram, giving complete classifications of compact and tame thick tensor ideals of $\mathsf{D}^-(R)$. Denote by **Cpt** and **Tame** the sets of compact and tame thick tensor ideals of $\mathsf{D}^-(R)$, respectively, and by **Spcl**(Spec) and **Spcl**($^{\mathtt{t}}\mathtt{Spc}$) the sets of specialization-closed subsets of Spec $R$ and $^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, respectively.

**Theorem C** (Theorems 5.13 and 5.20). *There is a diagram*

$$\begin{tikzcd}
\mathbf{Rad} \arrow[r, shift left, "\mathsf{Spp}"] \arrow[d, shift left, "(\,\cdot\,)_{\mathsf{cpt}}"] & \mathbf{Thom} \arrow[l, shift left, "\mathsf{Spp}^{-1}"] \arrow[d, shift left, "\mathscr{S}^{-1}"] \arrow[dr, shift left, "(\,\cdot\,)_{\mathsf{spcl}}"] & \\
\mathbf{Cpt} \arrow[u, shift left, "(\,\cdot\,)^{\mathsf{rad}}"] \arrow[r, shift left, "\mathrm{Supp}"] \arrow[dr, shift left, "(\,\cdot\,)^{\mathsf{tame}}"] & \mathbf{Spcl}(\mathrm{Spec}) \arrow[l, shift left, "\langle\,\cdot\,\rangle"] \arrow[u, shift left, "\overline{\mathscr{S}}"] \arrow[r, shift left, "\mathscr{S}"] \arrow[d, shift left, "\mathrm{Supp}^{-1}"] & \mathbf{Spcl}(^{\mathtt{t}}\mathtt{Spc}) \arrow[l, shift left, "\mathfrak{s}"] \arrow[ul, shift left, "(\,\cdot\,)^{\mathsf{spcl}}"] \arrow[dl, shift left, "\mathsf{Sp}^{-1}"] \\
& \mathbf{Tame} \arrow[ul, shift left, "(\,\cdot\,)_{\mathsf{cpt}}"] \arrow[u, shift left, "\mathrm{Supp}"] \arrow[ur, shift left, "\mathsf{Sp}"] &
\end{tikzcd}$$

*where the pairs of maps* $A=((\,\cdot\,)^{\mathsf{rad}},(\,\cdot\,)_{\mathsf{cpt}})$, $B=(\overline{\mathscr{S}},\mathscr{S}^{-1})$, *and* $C=((\,\cdot\,)^{\mathsf{spcl}},(\,\cdot\,)_{\mathsf{spcl}})$ *are section-retraction pairs* (*as sets*), *and all the other pairs consist of mutually inverse bijections. The diagram with the sections and retractions*, *respectively*, *and bijections is commutative.*

We do not give here the definitions of the maps appearing above (we do this in Section 5); what we want to emphasize now is that those maps are given explicitly.

Moreover, we prove that some/any of the three section-retraction pairs $A$, $B$, $C$ in the above theorem are bijections if and only if $R$ is artinian, which is incorporated into the following theorem.

**Theorem D** (Theorem 6.5). *The following are equivalent.*

(1) *$R$ is artinian.*

(2) *Every thick tensor ideal of $\mathsf{D}^-(R)$ is compact*, *tame*, *and radical.*

(3) *Every radical thick tensor ideal of $\mathsf{D}^-(R)$ is tame.*

(4) *The pair of maps* $(\mathscr{S},\mathfrak{s})$ *consists of mutually inverse homeomorphisms.*

(5) *Some/all of the maps $\mathscr{S}$, $\mathfrak{s}$ are bijective.*

(6) *Some/all of the pairs $A$, $B$, $C$ consist of mutually inverse bijections.*

This theorem says that in the case of artinian rings everything is clear. An essential role is played in the proof of this theorem by a certain complex in $\mathsf{D}^-(R)$ constructed from shifted Koszul complexes.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. Balmer [2010a] constructs a continuous map

$$\rho^\bullet_{\mathcal{T}} : \operatorname{Spc} \mathcal{T} \to \operatorname{Spec}^{\mathrm{h}} \mathrm{R}^\bullet_{\mathcal{T}},$$

where $\mathrm{R}^\bullet_{\mathcal{T}} = \operatorname{Hom}_{\mathcal{T}}(\mathbf{1}, \Sigma^\bullet \mathbf{1})$ is a graded-commutative ring. Balmer [2010b] conjectures that the map $\rho^\bullet_{\mathcal{T}}$ is (locally) injective when $\mathcal{T}$ is an *algebraic* triangulated category, that is, a triangulated category arisen as the stable category of a Frobenius exact category. Our $\mathsf{D}^-(R)$ is evidently an algebraic triangulated category, but does not satisfy this conjecture under a quite mild assumption:

**Theorem E** (Corollary 6.10). *Assume that* $\dim R > 0$ *and that $R$ is either a domain or a local ring. Then the map $\rho^\bullet_{\mathsf{D}^-(R)}$ is not locally injective. Hence, Balmer's conjecture does not hold for $\mathsf{D}^-(R)$.*

In fact, the assumption of the theorem gives an element $x \in R$ with $\operatorname{ht}(x) > 0$. Then we can find a nontame prime thick tensor ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$ associated with $x$ at which $\rho^\bullet_{\mathsf{D}^-(R)}$ is not locally injective.

Finally, we explore thick tensor ideals of $\mathsf{D}^-(R)$ in the case where $R$ is a discrete valuation ring, because this should be the simplest unclear case, now that everything is clarified by Theorem D in the case of artinian rings. We show the following theorem, which says that even if $R$ is such a good ring, the structure of the Balmer spectrum of $\mathsf{D}^-(R)$ is rather complicated. (Here, $\ell\ell(\cdot)$ stands for the Loewy length.)

**Theorem F** (Propositions 7.7 and 7.17 and Theorems 7.11 and 7.14). *Let $(R, xR)$ be a discrete valuation ring, and let $n \geqslant 0$ be an integer. Let $\mathcal{P}_n$ be the full subcategory of $\mathsf{D}^-(R)$ consisting of complexes $X$ with finite length homologies such that there exists an integer $t \geqslant 0$ with $\ell\ell(\mathsf{H}^{-i} X) \leqslant t i^n$ for all $i \gg 0$. Then*:

(1) *$\mathcal{P}_n$ coincides with the smallest thick tensor ideal of $\mathsf{D}^-(R)$ containing the complex*

$$\bigoplus_{i>0} (R/x^{i^n} R)[i] = (\cdots \xrightarrow{0} R/x^{3^n} R \xrightarrow{0} R/x^{2^n} R \xrightarrow{0} R/x^{1^n} R \to 0).$$

(2) *$\mathcal{P}_n$ is a prime thick tensor ideal of $\mathsf{D}^-(R)$ which is not tame. If $n \geqslant 1$, then $\mathcal{P}_n$ is not compact.*

(3) *One has $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots$. Hence, $\operatorname{Spc} \mathsf{D}^-(R)$ has infinite Krull dimension.*

The paper is organized as follows. Section 1 is devoted to giving several basic definitions and studying fundamental properties that are used in later sections. In Section 2, we study compactly and cocompactly generated thick tensor ideals of $\mathsf{D}^-(R)$, and classify them completely. The generalized smash nilpotence theorem and Theorem A are proved in this section. In Section 3, we define the maps $\mathcal{S}$ and $\mathfrak{s}$ between $\operatorname{Spec} R$ and $\operatorname{Spc} \mathsf{D}^-(R)$, and prove part of Theorem B. In

Section 4, we study topological properties of the maps $\mathcal{S}$, $\mathfrak{s}$ and the Balmer spectrum $\mathtt{Spc}\,\mathsf{D}^-(R)$. We complete in this section the proof of Theorem B. In Section 5, we compare compact, tame, and radical thick tensor ideals of $\mathsf{D}^-(R)$, relating them to specialization-closed subsets of Spec $R$ and ${}^{\mathrm{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ and Thomason subsets of $\mathtt{Spc}\,\mathsf{D}^-(R)$. Theorem C is proved in this section. In Section 6, we consider when the section-retraction pairs in Theorem C are one-to-one correspondences, and deal with the conjecture of Balmer for $\mathsf{D}^-(R)$. We show Theorems D and E in this section. The final Section 7 concentrates on investigation of the case of discrete valuation rings. Several properties that are specific to this case are found out, and Theorem F is proved in this section.

## 1. Fundamental materials

In this section, we give several basic definitions and study fundamental properties, which will be used in later sections. We begin with our convention.

**Convention 1.1.** Throughout the paper, unless otherwise specified, $R$ is a commutative noetherian ring, and all subcategories are nonempty and full. We put $I^0 = R$ and $x^0 = 1$ for an ideal $I$ of $R$ and an element $x \in R$. We denote by Spec $R$, Max $R$, and Min $R$ the set of prime, maximal prime, and minimal prime ideals of $R$, respectively. For an ideal $I$ of $R$, we denote by $\mathrm{V}(I)$ the set of prime ideals of $R$ containing $I$, and set $\mathrm{D}(I) = \mathrm{V}(I)^{\complement} = \operatorname{Spec} R \setminus \mathrm{V}(I)$. When $I$ is generated by a single element $x$, we simply write $\mathrm{V}(x)$ and $\mathrm{D}(x)$. For a prime ideal $\mathfrak{p}$ of $R$, the residue field of $R_{\mathfrak{p}}$ is denoted by $\kappa(\mathfrak{p})$, i.e., $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. For a sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of $R$, the *Koszul complex* of $R$ with respect to $\boldsymbol{x}$ is denoted by $\mathrm{K}(\boldsymbol{x}, R)$. For an additive category $\mathscr{C}$ we denote by $\mathbf{0}$ the *zero subcategory* of $\mathscr{C}$, that is, the full subcategory consisting of objects isomorphic to the zero object. For objects $X, Y$ of $\mathscr{C}$, we mean by $X \lessdot Y$ (or $Y \gtrdot X$) that $X$ is a direct summand of $Y$ in $\mathscr{C}$. We often omit subscripts, superscripts, and parentheses, if there is no danger of confusion.

Let $\mathcal{T}$ be a triangulated category. A *thick* subcategory of $\mathcal{T}$ is by definition a triangulated subcategory closed under direct summands; in other words, it is a subcategory closed under direct summands, shifts, and cones. For a subcategory $\mathscr{X}$ of $\mathcal{T}$ we denote by $\mathtt{thick}\,\mathscr{X}$ the *thick closure* of $\mathscr{X}$, that is, the smallest thick subcategory of $\mathcal{T}$ containing $\mathscr{X}$.

Now we recall the definitions of a tensor triangulated category and a thick tensor ideal.

**Definition 1.2.** (1) We say that $(\mathcal{T}, \otimes, \mathbf{1})$ is a *tensor triangulated category* if $\mathcal{T}$ is a triangulated category equipped with a symmetric monoidal structure which is compatible with the triangulated structure of $\mathcal{T}$; see [Hovey et al. 1997, Appendix A] for the precise definition. In particular, $\cdot \otimes \cdot$ is exact in each variable.

(2) Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. A subcategory $\mathcal{X}$ of $\mathcal{T}$ is said to be a *thick tensor ideal* provided that $\mathcal{X}$ is a thick subcategory of $\mathcal{T}$ and for any $T \in \mathcal{T}$ and $X \in \mathcal{X}$ one has $T \otimes X \in \mathcal{X}$. We often abbreviate "tensor ideal" to "$\otimes$-ideal". For a subcategory $\mathcal{C}$ of $\mathcal{T}$, we define the *thick $\otimes$-ideal closure* of $\mathcal{C}$ to be the smallest thick $\otimes$-ideal of $\mathcal{T}$ containing $\mathcal{C}$, and denote it by $\mathsf{thick}^{\otimes} \mathcal{C}$.

We denote by $\mathsf{D}^-(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$ the derived categories of (cochain) complexes $X$ of finitely generated $R$-modules with $\mathsf{H}^i(X) = 0$ for all $i \gg 0$ and $|i| \gg 0$, respectively. We denote by $\mathsf{D}^-_{\mathsf{fl}}(R)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$ the subcategories of $\mathsf{D}^-(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$, respectively, consisting of complexes $X$ whose homologies have finite length as $R$-modules. By $\mathsf{K}^-(R)$ and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ we denote the homotopy categories of complexes $P$ of finitely generated projective $R$-modules with $P^i = 0$ for all $i \gg 0$ and $|i| \gg 0$, respectively. By $\mathsf{K}^{-,\mathsf{b}}(R)$ we denote the subcategory of $\mathsf{K}^-(R)$ consisting of complexes $P$ with $\mathsf{H}^i(P) = 0$ for all $i \ll 0$. Note that there are chains

$$\begin{aligned} \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) \subseteq \mathsf{D}^{\mathsf{b}}(R) &\subseteq \mathsf{D}^-(R), \\ \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) \subseteq \mathsf{D}^-_{\mathsf{fl}}(R) &\subseteq \mathsf{D}^-(R), \\ \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R) \subseteq \mathsf{K}^{-,\mathsf{b}}(R) &\subseteq \mathsf{K}^-(R) \end{aligned}$$

of thick subcategories and triangle equivalences

$$\mathsf{D}^-(R) \cong \mathsf{K}^-(R), \qquad \mathsf{D}^{\mathsf{b}}(R) \cong \mathsf{K}^{-,\mathsf{b}}(R).$$

We will often identify $\mathsf{D}^-(R)$, $\mathsf{D}^{\mathsf{b}}(R)$ with $\mathsf{K}^-(R)$, $\mathsf{K}^{-,\mathsf{b}}(R)$, respectively, via these equivalences. Note that $(\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R), \otimes_R, R)$ and $(\mathsf{D}^-(R), \otimes^{\boldsymbol{L}}_R, R)$ are essentially small tensor triangulated categories. (In general, if $\mathcal{C}$ is an essentially small additive category, then so is the category of complexes of objects in $\mathcal{C}$, and so is the homotopy category.)

**Remark 1.3.** The tensor triangulated category $\mathsf{D}^-(R)$ is never rigid. More strongly, it is never closed. In fact, assume there is a functor $F : \mathsf{D}^-(R) \times \mathsf{D}^-(R) \to \mathsf{D}^-(R)$ such that $\mathrm{Hom}_{\mathsf{D}^-(R)}(X \otimes^{\boldsymbol{L}}_R Y, Z) \cong \mathrm{Hom}_{\mathsf{D}^-(R)}(Y, F(X, Z))$ for all $X, Y, Z \in \mathsf{D}^-(R)$. We have $\mathrm{Hom}_{\mathsf{D}^-(R)}(X \otimes^{\boldsymbol{L}}_R Y, Z) = \mathrm{Hom}_{\mathsf{D}(R)}(X \otimes^{\boldsymbol{L}}_R Y, Z) \cong \mathrm{Hom}_{\mathsf{D}(R)}(Y, \boldsymbol{R}\mathrm{Hom}_R(X, Z))$, where $\mathsf{D}(R)$ is the unbounded derived category of $R$-modules. Letting $Y = R[-i]$ for $i \in \mathbb{Z}$, we obtain $\mathsf{H}^i(F(X, Z)) \cong \mathrm{Ext}^i_R(X, Z)$. Since $F(X, Z)$ is in $\mathsf{D}^-(R)$, we have $\mathsf{H}^i(F(X, Z)) = 0$ for $i \gg 0$. Hence, $\mathrm{Ext}^{\gg 0}_R(X, Z) = 0$ for all $X, Z \in \mathsf{D}^-(R)$. This is a contradiction.

Here we compute some thick closures and thick $\otimes$-ideal closures.

**Proposition 1.4.** *There are equalities*:

(1) $\mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} R = \mathsf{D}^-(R)$.

(2) $\mathsf{thick}_{\mathsf{D}^-(R)} R = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)} R = \mathsf{thick}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)} R = \mathsf{thick}^{\otimes}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)} R = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$.

(3) $\mathsf{thick}_{\mathsf{D}^-(R)}\, k = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\, k = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$, *if $R$ is local with residue field $k$.*

*Proof.* The following hold in general and are easy to check.

(a) Let $\mathcal{T}$ be a triangulated category, $\mathcal{U}$ be a thick subcategory, and $U \in \mathcal{U}$. Then $\mathsf{thick}_{\mathcal{U}}\, U = \mathsf{thick}_{\mathcal{T}}\, U$.

(b) Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category. Then $\mathsf{thick}^{\otimes}\, \mathbf{1} = \mathcal{T}$.

The assertion is shown by these two statements. $\square$

From now on, we deal with the supports of objects and subcategories of $\mathsf{D}^-(R)$. Recall that the *support* of an $R$-module $M$ is defined as the set of prime ideals $\mathfrak{p}$ of $R$ such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is nonzero, which is denoted by $\operatorname{Supp}_R M$.

**Proposition 1.5.** *Let $X$ be a complex in $\mathsf{D}^-(R)$. Then the following three sets are equal*:

(1) $\bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R \mathsf{H}^i(X)$,

(2) $\{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \not\cong 0$ *in* $\mathsf{D}^-(R_{\mathfrak{p}})\}$, *and*

(3) $\{\mathfrak{p} \in \operatorname{Spec} R \mid \kappa(\mathfrak{p}) \otimes^{\boldsymbol{L}}_R X \not\cong 0$ *in* $\mathsf{D}^-(R_{\mathfrak{p}})\}$.

*Proof.* It is clear that the first and second sets coincide. For a prime ideal $\mathfrak{p}$ of $R$ one has $\kappa(\mathfrak{p}) \otimes^{\boldsymbol{L}}_R X \cong \kappa(\mathfrak{p}) \otimes^{\boldsymbol{L}}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}$. It is seen by [Christensen 2000, Corollary A.4.16] that the second and third sets coincide. $\square$

**Definition 1.6.** The set in Proposition 1.5 is called the *support* of $X$ and denoted by $\operatorname{Supp}_R X$. For a subcategory $\mathcal{C}$ of $\mathsf{D}^-(R)$, we set $\operatorname{Supp} \mathcal{C} = \bigcup_{C \in \mathcal{C}} \operatorname{Supp} C$, and call this the *support* of $\mathcal{C}$. For a subset $S$ of $\operatorname{Spec} R$, we denote by $\operatorname{Supp}^{-1} S$ the subcategory of $\mathsf{D}^-(R)$ consisting of complexes whose supports are contained in $S$.

**Remark 1.7.** The fact that the second and third sets in Proposition 1.5 coincide will often play an important role in this paper. Note that these two sets are different if $X$ is a complex outside $\mathsf{D}^-(R)$. For example, let $(R, \mathfrak{m}, k)$ be a local ring of positive Krull dimension. Take any nonmaximal prime ideal $P$, and let $X$ be the injective hull $\mathrm{E}(R/P)$ of the $R$-module $R/P$. Then $k \otimes^{\boldsymbol{L}}_R X = 0$, while $X_{\mathfrak{m}} \neq 0$.

**Remark 1.8.** For $X \in \mathsf{D}^-(R)$ one has $\operatorname{Supp} X = \varnothing$ if and only if $X = 0$. In other words, it holds that $\operatorname{Supp}^{-1} \varnothing = \mathbf{0}$. (If we define the support of $X$ as the third set in Proposition 1.5, then the assumption that $X$ belongs to $\mathsf{D}^-(R)$ is essential, as the example given in Remark 1.7 shows.)

In the following lemma and proposition, we state several basic properties of Supp and $\operatorname{Supp}^{-1}$ defined above. Both results will often be used later.

**Lemma 1.9.** *The following statements hold.*

(1) $\operatorname{Supp}(X[n]) = \operatorname{Supp} X$ *for all* $X \in \mathsf{D}^-(R)$ *and* $n \in \mathbb{Z}$.

(2) *If $X$ is a direct summand of $Y$ in* $\mathsf{D}^-(R)$, *then* $\operatorname{Supp} X \subseteq \operatorname{Supp} Y$.

(3) *If* $X \to Y \to Z \to X[1]$ *is an exact triangle in* $\mathsf{D}^-(R)$, *then* $\operatorname{Supp} A \subseteq \operatorname{Supp} B \cup \operatorname{Supp} C$ *for all* $\{A, B, C\} = \{X, Y, Z\}$.

(4) $\operatorname{Supp}(X \otimes_R^{\boldsymbol{L}} Y) = \operatorname{Supp} X \cap \operatorname{Supp} Y$ *for all* $X, Y \in \mathsf{D}^-(R)$.

*Proof.* The assertions (1), (2), and (3) are straightforward by definition. For each prime ideal $\mathfrak{p}$ of $R$ there is an isomorphism $(X \otimes_R^{\boldsymbol{L}} Y)_\mathfrak{p} \cong X_\mathfrak{p} \otimes_{R_\mathfrak{p}}^{\boldsymbol{L}} Y_\mathfrak{p}$. Hence, $(X \otimes_R^{\boldsymbol{L}} Y)_\mathfrak{p} = 0$ if and only if either $X_\mathfrak{p} = 0$ or $Y_\mathfrak{p} = 0$ by [Christensen 2000, Corollary A.4.16]. This shows the assertion (4). □

Let $X$ be a topological space. A subset $A$ of $X$ is called *specialization-closed* provided that for each point $a \in A$ the closure $\overline{\{a\}}$ of $\{a\}$ in $X$ is contained in $A$. Hence, a subset $S$ of $\operatorname{Spec} R$ is specialization-closed if and only if for each $\mathfrak{p} \in S$ one has $\mathrm{V}(\mathfrak{p}) \subseteq S$. Note that $A$ is specialization-closed if and only if $A$ is a (possibly infinite) union of closed subsets of $X$. Therefore, a union of specialization-closed subsets is again specialization-closed, and thus, one can define the largest specialization-closed subset $A_{\mathsf{spcl}}$ of $X$ contained in $A$, which will be called the $\mathsf{spcl}$-*closure* of $A$ in Section 5.

**Proposition 1.10.** (1) *Let* $S$ *be a subset of* $\operatorname{Spec} R$. *Then there are equalities* $\operatorname{Supp}^{-1} S = \operatorname{Supp}^{-1}(S_{\mathsf{spcl}})$ *and* $\operatorname{Supp}(\operatorname{Supp}^{-1} S) = S_{\mathsf{spcl}}$. *Moreover*, $\operatorname{Supp}^{-1} S$ *is a thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$.

(2) *Let* $\mathcal{X}$ *be any subcategory of* $\mathsf{D}^-(R)$. *Then* $\operatorname{Supp} \mathcal{X}$ *is a specialization-closed subset of* $\operatorname{Spec} R$, *and one has* $\operatorname{Supp} \mathcal{X} = \operatorname{Supp}(\mathsf{thick}^{\otimes} \mathcal{X})$.

(3) *It holds that* $\mathsf{D}^-_{\mathsf{fl}}(R) = \operatorname{Supp}^{-1}(\operatorname{Max} R)$. *In particular*, $\mathsf{D}^-_{\mathsf{fl}}(R)$ *is a thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$.

*Proof.* (1) We put $W = S_{\mathsf{spcl}}$. Let $X$ be a complex in $\mathsf{D}^-(R)$. Since $\operatorname{Supp} X$ is specialization-closed, it is contained in $S$ if and only if it is contained in $W$. Hence, $\operatorname{Supp}^{-1} S = \operatorname{Supp}^{-1} W$. Evidently, $W$ contains $\operatorname{Supp}(\operatorname{Supp}^{-1} W)$, while we have $\mathfrak{p} \in \operatorname{Supp} R/\mathfrak{p} = \mathrm{V}(\mathfrak{p}) \subseteq W$ for $\mathfrak{p} \in W$. Hence, $\operatorname{Supp}(\operatorname{Supp}^{-1} W) = W$, and thus, $\operatorname{Supp}(\operatorname{Supp}^{-1} S) = W$. It is seen from Lemma 1.9 that $\operatorname{Supp}^{-1} S$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$.

(2) We have $\operatorname{Supp} \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp} X = \bigcup_{X \in \mathcal{X}} \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} \mathsf{H}^i X$ by Proposition 1.5. Since $\mathsf{H}^i X$ is a finitely generated $R$-module, $\operatorname{Supp} \mathsf{H}^i X$ is closed. Hence, $\operatorname{Supp} \mathcal{X}$ is specialization-closed. A prime ideal $\mathfrak{p}$ of $R$ is not in $\operatorname{Supp} \mathcal{X}$ if and only if $\mathcal{X}$ is contained in $\operatorname{Supp}^{-1}(\{\mathfrak{p}\}^\complement)$, if and only if $\mathsf{thick}^{\otimes} \mathcal{X}$ is contained in $\operatorname{Supp}^{-1}(\{\mathfrak{p}\}^\complement)$, if and only if $\mathfrak{p}$ does not belong to $\operatorname{Supp}(\mathsf{thick}^{\otimes} \mathcal{X})$. It follows from (1) that $\operatorname{Supp}^{-1}(\{\mathfrak{p}\}^\complement)$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$, which shows the second equivalence. The other two equivalences are obvious.

(3) The equality is straightforward, and the last assertion is shown by (1). □

## 2. Classification of compact thick tensor ideals

In this section, we prove a generalized version of the smash nilpotence theorem due to Hopkins [1987] and Neeman [1992], and using this we give a complete classification of cocompact thick tensor ideals of $\mathsf{D}^-(R)$.

We begin with recalling the definitions of compact and cocompact objects. Let $\mathcal{T}$ be a triangulated category. We say that an object $M \in \mathcal{T}$ is *compact* if the natural morphism

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(M, N_\lambda) \to \mathrm{Hom}_{\mathcal{T}}\Bigl(M, \bigoplus_{\lambda\in\Lambda} N_\lambda\Bigr)$$

is an isomorphism for every family $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$ with $\bigoplus_{\lambda\in\Lambda} N_\lambda \in \mathcal{T}$. Furthermore, we say that an object $M \in \mathcal{T}$ is *cocompact* if the natural morphism

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}(N_\lambda, M) \to \mathrm{Hom}_{\mathcal{T}}\Bigl(\prod_{\lambda\in\Lambda} N_\lambda, M\Bigr)$$

is an isomorphism for every family $\{N_\lambda\}_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$ with $\prod_{\lambda\in\Lambda} N_\lambda \in \mathcal{T}$. We denote by $\mathcal{T}^{\mathsf{c}}$ and $\mathcal{T}^{\mathsf{cc}}$ the subcategories of $\mathcal{T}$ consisting of compact and cocompact objects, respectively. For $\mathcal{T} = \mathsf{D}^-(R)$ we have explicit descriptions of the compact objects and cocompact objects:

**Proposition 2.1.** *One has* $\mathsf{D}^-(R)^{\mathsf{c}} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ *and* $\mathsf{D}^-(R)^{\mathsf{cc}} = \mathsf{D}^{\mathsf{b}}(R)$.

*Proof.* The second statement follows from [Stevenson 2014a, Theorem 18]. The first one can be shown in the same way as of the fact that the compact objects of the unbounded derived category of all $R$-modules coincide with $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. For the convenience of the reader, we give a proof.

First of all, $R$ is compact since each homology functor $\mathsf{H}^i$ commutes with direct sums. Since the compact objects form a thick subcategory, $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R) \subseteq \mathsf{D}^-(R)^{\mathsf{c}}$. Next, let $X \in \mathsf{D}^-(R)$ be a compact object. Replacing $X$ with its projective resolution, we may assume $X \in \mathsf{K}^-(R)$. Consider the chain map

$$\begin{array}{ccc}
X & = & (\cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots) \\
\downarrow{\scriptstyle f_n} & & \qquad\qquad\quad \downarrow \qquad\qquad \downarrow{\scriptstyle f_n^n} \qquad\qquad \downarrow \\
C^n[-n] & = & (\cdots \longrightarrow 0 \longrightarrow C^n \longrightarrow 0 \longrightarrow \cdots)
\end{array}$$

where $C^n$ is the cokernel of $d^{n-1}$, and $f_n^n : X^n \to C^n$ is a natural surjection. Put $Y = \bigoplus_{n\in\mathbb{Z}} C^n[-n]$. A chain map $f : X \to Y$ is induced by $\{f_n\}_{n\in\mathbb{Z}}$. As $X \in \mathsf{K}^-(R)$ is compact in $\mathsf{D}^-(R)$, we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{K}}(X, Y) &\cong \mathrm{Hom}_{\mathsf{D}^-(R)}(X, Y) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}^-(R)}(X, C^n[-n]) \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{K}}(X, C^n[-n]),
\end{aligned}$$

where $\mathcal{K}$ is the homotopy category of $R$-modules. The composition of these isomorphisms sends $f$ to $(f_n)_{n\in\mathbb{Z}}$, which implies that there exists $t \in \mathbb{Z}$ such that $f_n = 0$ in $\mathcal{K}$ for all $n \leqslant t$. Hence, there is an $R$-linear map $g : X^{n+1} \to C^n$ such that $g \circ d^n = f_n^n$. Let $\overline{d^n} : C^n \to X^{n+1}$ be the map induced by $d^n$. We have $g\overline{d^n} f_n^n = gd^n = f_n^n$, and obtain $g\overline{d^n} = 1$ as $f_n^n$ is a surjection. Thus, $C^n$ is a direct summand of $X^{n+1}$, and thereby projective. Also, $\mathsf{H}^n X$ is isomorphic to the kernel of $\overline{d^n}$, which vanishes since $\overline{d^n}$ is a split monomorphism. Consequently, the truncated complex $X' := (0 \to C^t \xrightarrow{\overline{d^t}} X^{t+1} \xrightarrow{d^{t+1}} X^{t+2} \xrightarrow{d^{t+2}} \cdots)$, which is quasi-isomorphic to $X$, is in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. We now conclude that $X$ belongs to $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. □

Next, we give the definitions of the annihilators of morphisms and objects in $\mathsf{D}^-(R)$.

**Definition 2.2.** (1) Let $f : X \to Y$ be a morphism in $\mathsf{D}^-(R)$. We define the *annihilator* of $f$ as the set of elements $a \in R$ such that $af = 0$ in $\mathsf{D}^-(R)$, and denote it by $\operatorname{Ann}_R(f)$. This is an ideal of $R$.

(2) The *annihilator* of an object $X \in \mathsf{D}^-(R)$ is defined as the annihilator of the identity morphism $\mathrm{id}_X$, and denoted by $\operatorname{Ann}_R(X)$. This is the set of elements $a \in R$ such that $(X \xrightarrow{a} X) = 0$ in $\mathsf{D}^-(R)$.

Here are some properties of annihilators.

**Proposition 2.3.** (1) *Let $f : X \to Y$ be a morphism in $\mathsf{D}^-(R)$ and $\mathfrak{p}$ a prime ideal of $R$.*

(a) *The ideal $\operatorname{Ann}_R(f)$ is the kernel of the map $\eta_f : R \to \operatorname{Hom}_{\mathsf{D}^-(R)}(X, Y)$ given by $a \mapsto af$.*

(b) *If the natural map $\tau_{X,Y,\mathfrak{p}} : \operatorname{Hom}_{\mathsf{D}^-(R)}(X, Y)_\mathfrak{p} \to \operatorname{Hom}_{\mathsf{D}^-(R_\mathfrak{p})}(X_\mathfrak{p}, Y_\mathfrak{p})$ is an isomorphism, then there is an equality $\operatorname{Ann}_R(f)_\mathfrak{p} = \operatorname{Ann}_{R_\mathfrak{p}}(f_\mathfrak{p})$.*

(2) *For any $X \in \mathsf{D}^-(R)$ one has $\mathrm{V}(\operatorname{Ann} X) \supseteq \operatorname{Supp} X$. The equality holds if $\tau_{X,X,\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \operatorname{Spec} R$. In particular, for $X \in \mathsf{D}^{\mathsf{b}}(R)$ one has $\mathrm{V}(\operatorname{Ann} X) = \operatorname{Supp} X$.*

(3) *Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements of $R$. Then it holds that $\operatorname{Ann} \mathrm{K}(\boldsymbol{x}, R) = \boldsymbol{x}R$. In particular, there is an equality $\operatorname{Supp} \mathrm{K}(\boldsymbol{x}, R) = \mathrm{V}(\boldsymbol{x})$, and $\mathrm{K}(\boldsymbol{x}, R)$ belongs to $\operatorname{Supp}^{-1} \mathrm{V}(\boldsymbol{x})$.*

*Proof.* (1) The assertion (a) is obvious, while (b) follows from (a) and the commutative diagram

$$\begin{tikzcd} R_\mathfrak{p} \arrow[r, "(\eta_f)_\mathfrak{p}"] \arrow[d, equal] & \operatorname{Hom}_{\mathsf{D}^-(R)}(X, Y)_\mathfrak{p} \arrow[d, "\tau_{X,Y,\mathfrak{p}}", "\cong"'] \\ R_\mathfrak{p} \arrow[r, "\eta_{f_\mathfrak{p}}"] & \operatorname{Hom}_{\mathsf{D}^-(R_\mathfrak{p})}(X_\mathfrak{p}, Y_\mathfrak{p}) \end{tikzcd}$$

(2) The first assertion is easy to show. Suppose that $\tau_{X,X,\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \operatorname{Spec} R$. By (1) one has $(\operatorname{Ann}_R X)_\mathfrak{p} = \operatorname{Ann}_{R_\mathfrak{p}} X_\mathfrak{p}$. We have $X_\mathfrak{p} \neq 0$ if and only if $(\operatorname{Ann}_R X)_\mathfrak{p} \neq R_\mathfrak{p}$, if and only if $\mathfrak{p} \in \mathrm{V}(\operatorname{Ann}_R X)$. This shows $\mathrm{V}(\operatorname{Ann}_R X) = \operatorname{Supp}_R X$. As for the last assertion, use [Avramov and Foxby 1991, Lemma 5.2(b)].

(3) The second statement follows from the first one and (2). Therefore, it suffices to show the equality $\operatorname{Ann} \mathrm{K}(\boldsymbol{x}, R) = \boldsymbol{x}R$. It follows from [Bruns and Herzog 1998, Proposition 1.6.5] that $\operatorname{Ann} \mathrm{K}(\boldsymbol{x}, R)$ contains $\boldsymbol{x}R$. Conversely, pick $a \in \operatorname{Ann} \mathrm{K}(\boldsymbol{x}, R)$. Then the multiplication map $a : \mathrm{K}(\boldsymbol{x}, R) \to \mathrm{K}(\boldsymbol{x}, R)$ is null-homotopic, and there is a homotopy $\{s_i : \mathrm{K}_{i-1}(\boldsymbol{x}, R) \to \mathrm{K}_i(\boldsymbol{x}, R)\}$ from $a$ to 0. In particular, we have $a = d_1 s_1$, where $d_1$ is the first differential of $\mathrm{K}(\boldsymbol{x}, R)$. Writing $d_1 = (x_1, \dots, x_n) : R^n \to R$ and $s_1 = {}^t(a_1, \dots, a_n) : R \to R^n$, we get $a = (x_1, \dots, x_n)\, {}^t(a_1, \dots, a_n) = a_1 x_1 + \cdots + a_n x_n \in \boldsymbol{x}R$. Consequently, we obtain $\operatorname{Ann} \mathrm{K}(\boldsymbol{x}, R) = \boldsymbol{x}R$. □

To state our next results, we need to introduce some notation.

**Definition 2.4.** Let $\mathcal{T}$ be a triangulated category.

(1) For two subcategories $\mathcal{C}_1, \mathcal{C}_2$ of $\mathcal{T}$, we denote by $\mathcal{C}_1 * \mathcal{C}_2$ the subcategory of $\mathcal{T}$ consisting of objects $M$ such that there is an exact triangle $C_1 \to M \to C_2 \rightsquigarrow$ with $C_i \in \mathcal{C}_i$ for $i = 1, 2$.

(2) For a subcategory $\mathcal{C}$ of $\mathcal{T}$, we denote by $\mathsf{add}^\Sigma \mathcal{C}$ the smallest subcategory of $\mathcal{T}$ that contains $\mathcal{C}$ and is closed under finite direct sums, direct summands, and shifts. Inductively we define $\mathsf{thick}^1_\mathcal{T}(\mathcal{C}) = \mathsf{add}^\Sigma \mathcal{C}$ and $\mathsf{thick}^r_\mathcal{T}(\mathcal{C}) = \mathsf{add}^\Sigma(\mathsf{thick}^{r-1}_\mathcal{T}(\mathcal{C}) * \mathsf{add}^\Sigma \mathcal{C})$ for $r > 1$. This is sometimes called the $r$-th *thickening* of $\mathcal{C}$. When $\mathcal{C}$ consists of a single object $X$, we simply denote it by $\mathsf{thick}^r_\mathcal{T}(X)$.

(3) For a morphism $f : X \to Y$ in $\mathcal{T}$ and an integer $n \geqslant 1$, we denote by $f^{\otimes n}$ the $n$-fold tensor product $\underbrace{f \otimes \cdots \otimes f}_{n}$. By $f^{\otimes n}$ we mean the morphism $\underbrace{f \otimes^{\boldsymbol{L}}_R \cdots \otimes^{\boldsymbol{L}}_R f}_{n}$ for $\mathcal{T} = \mathsf{D}^-(R)$.

We establish two lemmas, which will be used to show the generalized smash nilpotence theorem. The first one concerns general tensor triangulated categories, while the second one is specific to our $\mathsf{D}^-(R)$.

**Lemma 2.5.** *Let $\mathcal{T}$ be a tensor triangulated category.*

(1) *Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{T}$. Let $f : M \to M'$ and $g : N \to N'$ be morphisms in $\mathcal{T}$. If $f \otimes \mathcal{X} = 0$ and $g \otimes \mathcal{Y} = 0$, then $f \otimes g \otimes (\mathcal{X} * \mathcal{Y}) = 0$.*

(2) *Let $\phi : A \to B$ be a morphism in $\mathcal{T}$, and let $C$ be an object of $\mathcal{T}$. If $\phi \otimes C = 0$, then $\phi^{\otimes n} \otimes \mathsf{thick}^n_\mathcal{T}(C) = 0$ for all integers $n > 0$.*

*Proof.* As (2) is shown by induction on $n$ and (1), let us show (1). Let $X \to E \to Y \rightsquigarrow$ be an exact triangle in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then $f \otimes X = 0$ and $g \otimes Y = 0$ by assumption. There is a diagram

$$\begin{tikzcd}
M \otimes N \otimes X \arrow[r] \arrow[d, "M\otimes g\otimes X"'] & M \otimes N \otimes E \arrow[r] \arrow[d, "M\otimes g\otimes E"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes N \otimes Y \arrow[r, squiggly] \arrow[d, "M\otimes g\otimes Y"', "0"] & {} \\
M \otimes N' \otimes X \arrow[r] \arrow[d, "f\otimes N'\otimes X"', "0"] \arrow[dr, "0"'] \arrow[dr, phantom, "\circlearrowleft", shift left=3] & M \otimes N' \otimes E \arrow[r] \arrow[d, "f\otimes N'\otimes E"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes N' \otimes Y \arrow[r, squiggly] \arrow[d, "f\otimes N'\otimes Y"'] \arrow[dl, dotted, "h"] & {} \\
M' \otimes N' \otimes X \arrow[r] & M' \otimes N' \otimes E \arrow[r] & M' \otimes N' \otimes Y \arrow[r, squiggly] & {}
\end{tikzcd}$$

in $\mathcal{T}$ whose rows are exact triangles, and we obtain a morphism $h$ as in it. It is observed from this diagram that $f \otimes g \otimes E = (f \otimes N' \otimes E) \circ (M \otimes g \otimes E)$ is a zero morphism. $\square$

**Lemma 2.6.** (1) *Let $f : X \to Y$ be a morphism in $\mathsf{D}^-(R)$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of $R$. If $f \otimes_R^{\boldsymbol{L}} R/(\boldsymbol{x}) = 0$ in $\mathsf{D}^-(R)$, then $f^{\otimes 2^n} \otimes_R^{\boldsymbol{L}} \mathrm{K}(\boldsymbol{x}, R) = 0$ in $\mathsf{D}^-(R)$.*

(2) *Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of $R$, and let $e > 0$ be an integer. Then $\mathrm{K}(\boldsymbol{x}^e, R)$ belongs to $\mathsf{thick}^{ne}_{\mathsf{K}^-(R)}(\mathrm{K}(\boldsymbol{x}, R))$, where $\boldsymbol{x}^e = x_1^e, \dots, x_n^e$.*

*Proof.* (1) We use induction on $n$. Let $n = 1$ and set $x = x_1$. There are exact sequences $0 \to (0 : x) \to R \to (x) \to 0$ and $0 \to (x) \to R \to R/(x) \to 0$. Applying the octahedral axiom to $(R \to (x) \to R) = (R \xrightarrow{x} R)$ gives an exact triangle $(0 : x)[1] \to \mathrm{K}(x, R) \to R/(x) \rightsquigarrow$ in $\mathsf{D}^-(R)$. We have $f \otimes_R^{\boldsymbol{L}} R/(x) = 0$, and $f \otimes_R^{\boldsymbol{L}} (0 : x)[1] = (f \otimes_R^{\boldsymbol{L}} R/(x)) \otimes_{R/(x)}^{\boldsymbol{L}} (0 : x)[1] = 0$. Lemma 2.5(1) yields $f^{\otimes 2} \otimes_R^{\boldsymbol{L}} \mathrm{K}(x, R) = 0$.

Let $n \geqslant 2$. We have $0 = f \otimes_R^{\boldsymbol{L}} R/(\boldsymbol{x}) = (f \otimes_R^{\boldsymbol{L}} R/(x_1)) \otimes_{R/(x_1)}^{\boldsymbol{L}} R/(\boldsymbol{x})$. The induction hypothesis gives

$$\begin{aligned} 0 = (f \otimes_R^{\boldsymbol{L}} R/(x_1))^{\otimes 2^{n-1}} \otimes_{R/(x_1)}^{\boldsymbol{L}} \mathrm{K}(x_2, \dots, x_n, R/(x_1)) \\ = (f^{\otimes 2^{n-1}} \otimes_R^{\boldsymbol{L}} \mathrm{K}(x_2, \dots, x_n, R)) \otimes_R^{\boldsymbol{L}} R/(x_1). \end{aligned}$$

The induction basis shows $0 = (f^{\otimes 2^{n-1}} \otimes_R^{\boldsymbol{L}} \mathrm{K}(x_2, \dots, x_n, R))^{\otimes 2} \otimes_R^{\boldsymbol{L}} \mathrm{K}(x_1, R) = f^{\otimes 2^n} \otimes_R^{\boldsymbol{L}} \mathrm{K}(x_2, \dots, x_n, \boldsymbol{x}, R)$. Note that $\mathrm{K}(\boldsymbol{x}, R)$ is a direct summand of $\mathrm{K}(x_2, \dots, x_n, \boldsymbol{x}, R)$ [Bruns and Herzog 1998, Proposition 1.6.21]. We thus obtain the desired equality $f^{\otimes 2^n} \otimes_R^{\boldsymbol{L}} \mathrm{K}(\boldsymbol{x}, R) = 0$.

(2) Again, we use induction on $n$. Consider the case $n = 1$. Put $x = x_1$. Applying the octahedral axiom to $(R \xrightarrow{x^{e-1}} R \xrightarrow{x} R) = (R \xrightarrow{x^e} R)$, we get an exact triangle $\mathrm{K}(x^{e-1}, R) \to \mathrm{K}(x^e, R) \to \mathrm{K}(x, R) \rightsquigarrow$. Induction on $e$ shows $\mathrm{K}(x^e, R) \in \mathsf{thick}^e\, \mathrm{K}(x, R)$. Let $n \geqslant 2$. By the induction hypothesis, $\mathrm{K}(x_1^e, \dots, x_{n-1}^e, R)$ belongs to $\mathsf{thick}^{(n-1)e}\, \mathrm{K}(x_1, \dots, x_{n-1}, R)$. Applying the exact functor $\cdot \otimes \mathrm{K}(x_n^e, R)$, we see

that $\mathrm{K}(\boldsymbol{x}^e, R)$ belongs to $\mathsf{thick}^{(n-1)e}\,\mathrm{K}(x_1, \dots, x_{n-1}, x_n^e, R)$. Applying the exact functor $\mathrm{K}(x_1, \dots, x_{n-1}, R) \otimes \cdot$ to the containment $\mathrm{K}(x_n^e, R) \in \mathsf{thick}^e\,\mathrm{K}(x_n, R)$ gives rise to $\mathrm{K}(x_1, \dots, x_{n-1}, x_n^e, R) \in \mathsf{thick}^e\,\mathrm{K}(\boldsymbol{x}, R)$. Therefore, $\mathrm{K}(\boldsymbol{x}^e, R)$ belongs to $\mathsf{thick}^{ne}\,\mathrm{K}(\boldsymbol{x}, R)$. □

We now achieve the goal of generalizing the Hopkins–Neeman smash nilpotence theorem.

**Theorem 2.7** (generalized smash nilpotence). *Let $f : X \to Y$ be a morphism in $\mathsf{K}^-(R)$ with $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. Suppose that $f \otimes \kappa(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Then $f^{\otimes t} = 0$ for some $t > 0$.*

*Proof.* We have an ascending chain $\operatorname{Ann}_R(f) \subseteq \operatorname{Ann}_R(f^{\otimes 2}) \subseteq \operatorname{Ann}_R(f^{\otimes 3}) \subseteq \cdots$ of ideals of $R$. Since $R$ is noetherian, there is an integer $c$ such that $\operatorname{Ann}_R(f^{\otimes c}) = \operatorname{Ann}_R(f^{\otimes i})$ for all $i > c$. Replacing $f$ by $f^{\otimes c}$, we may assume that $\operatorname{Ann}_R(f) = \operatorname{Ann}_R(f^{\otimes i})$ for all $i > 0$. Note that $\operatorname{Ann}_R(f) = R$ if and only if $f = 0$.

We assume $\operatorname{Ann}_R(f) \neq R$, and shall derive a contradiction. Take a minimal prime ideal $\mathfrak{p}$ of $\operatorname{Ann}_R(f)$. Then localization at $\mathfrak{p}$ reduces to the following situation:

> $(R, \mathfrak{m}, k)$ is a local ring, $\operatorname{Ann}_R(f)$ is an $\mathfrak{m}$-primary ideal, $f \otimes_R k = 0$, and $\operatorname{Ann}_R(f) = \operatorname{Ann}_R(f^{\otimes i})$ for all $i > 0$.

Indeed, since $Y$ is in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$, it follows from [Avramov and Foxby 1991, Lemma 5.2(b)] that the map $\tau_{X,Y,\mathfrak{p}}$ is an isomorphism, and Proposition 2.3(1) yields $\operatorname{Ann}_{R_\mathfrak{p}}(f_\mathfrak{p}) = \operatorname{Ann}_R(f)_\mathfrak{p}$, which is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal of $R_\mathfrak{p}$. Also, we have $\operatorname{Ann}_{R_\mathfrak{p}}(f_\mathfrak{p}) = \operatorname{Ann}_R(f)_\mathfrak{p} = \operatorname{Ann}_R(f^{\otimes i})_\mathfrak{p} = \operatorname{Ann}_{R_\mathfrak{p}}((f^{\otimes i})_\mathfrak{p}) = \operatorname{Ann}_{R_\mathfrak{p}}((f_\mathfrak{p})^{\otimes i})$ for all $i > 0$. Furthermore, it holds that $f_\mathfrak{p} \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p}) = f \otimes_R \kappa(\mathfrak{p}) = 0$ by the assumption of the theorem.

For each nonnegative integer $n$, consider the following two statements.

- $F(n)$**:** Let $(R, \mathfrak{m}, k)$ be a reduced local ring with $\dim R \leqslant n$. Let $f : X \to Y$ be a morphism in $\mathsf{K}^-(R)$ with $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. If $\operatorname{Ann}_R(f)$ is $\mathfrak{m}$-primary and $f \otimes_R k = 0$, then $f^{\otimes t} = 0$ for some $t > 0$.
- $G(n)$**:** Let $(R, \mathfrak{m}, k)$ be a local ring with $\dim R \leqslant n$. Let $f : X \to Y$ be a morphism in $\mathsf{K}^-(R)$ with $Y \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. If $\operatorname{Ann}_R(f)$ is $\mathfrak{m}$-primary and $f \otimes_R k = 0$, then $f^{\otimes t} = 0$ for some $t > 0$.

If the statement $G(n)$ holds true for all $n \geqslant 0$, we have $\operatorname{Ann}_R(f) = \operatorname{Ann}_R(f^{\otimes t}) = R$, which gives a desired contradiction. Note that the statement $F(0)$ always holds true since a 0-dimensional reduced local ring is a field. It is thus enough to show the implications $F(n) \Longrightarrow G(n) \Longrightarrow F(n+1)$.

$F(n) \Longrightarrow G(n)$. We consider the reduced ring $R_{\mathrm{red}} = R/\operatorname{nil} R$, where $\operatorname{nil} R$ stands for the nilradical of $R$. The ideal $\operatorname{Ann}_{R_{\mathrm{red}}}(f \otimes_R R_{\mathrm{red}})$ of $R_{\mathrm{red}}$ is $\mathfrak{m}R_{\mathrm{red}}$-primary since it contains $(\operatorname{Ann}_R f)R_{\mathrm{red}}$. We have $(f \otimes_R R_{\mathrm{red}}) \otimes_{R_{\mathrm{red}}} k = f \otimes_R k = 0$. Thus, $R_{\mathrm{red}}$

and $f \otimes_R R_{\mathrm{red}}$ satisfy the assumption $F(n)$, and we find an integer $t > 0$ such that $f^{\otimes t} \otimes_R R_{\mathrm{red}} = (f \otimes_R R_{\mathrm{red}})^{\otimes t} = 0$. Using Lemma 2.6(1), we get $f^{\otimes tu} \otimes_R \mathrm{K}(\boldsymbol{x}, R) = 0$, where $\boldsymbol{x} = x_1, \dots, x_n$ is a system of generators of nil $R$ and $u = 2^n$. Choose an integer $e > 0$ such that $x_i^e = 0$ for all $1 \leqslant i \leqslant n$. Then $R$ is a direct summand of $\mathrm{K}(\boldsymbol{x}^e, R)$ by [Bruns and Herzog 1998, Proposition 1.6.21], whence $R$ is in $\mathsf{thick}^{ne}\, K(\boldsymbol{x}, R)$ by Lemma 2.6(2). Finally, Lemma 2.5(2) gives rise to the equality $f^{\otimes netu} = 0$.

$G(n) \Longrightarrow F(n+1)$. We may assume $\dim R = n + 1 > 0$. Since $R$ is reduced and $\mathrm{Ann}_R(f)$ is $\mathfrak{m}$-primary, we can choose an $R$-regular element $x \in \mathrm{Ann}_R(f)$. Then the local ring $R/(x)$ has dimension $n$, the ideal $\mathrm{Ann}_{R/(x)}(f \otimes_R R/(x))$ of $R/(x)$ is $\mathfrak{m}/(x)$-primary, and $(f \otimes_R R/(x)) \otimes_{R/(x)} k = 0$. Hence, $R/(x)$ and $f \otimes_R R/(x)$ satisfy the assumption of $G(n)$, and there is an integer $t > 0$ such that $(f \otimes_R R/(x))^{\otimes t} = 0$. The short exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ induces an exact triangle $R/(x)[-1] \to R \xrightarrow{x} R \rightsquigarrow$ in $\mathsf{D}^-(R)$. Tensoring $Y$ with this gives an exact triangle $Y \otimes_R R/(x)[-1] \xrightarrow{g} Y \xrightarrow{x} Y \rightsquigarrow$ in $\mathsf{D}^-(R)$. As $xf = 0$, there is a morphism $h : X \to Y \otimes_R R/(x)[-1]$ with $f = gh$. Now $f^{\otimes t+1}$ is decomposed as

$$X^{\otimes t+1} \xrightarrow{h \otimes X^{\otimes t}} (Y \otimes_R R/(x)[-1]) \otimes_R X^{\otimes t}$$
$$\xrightarrow{(Y \otimes R/(x)[-1]) \otimes f^{\otimes t}} (Y \otimes_R R/(x)[-1]) \otimes_R Y^{\otimes t} \xrightarrow{g \otimes Y^{\otimes t}} Y^{\otimes t+1}.$$

The middle morphism is identified with $Y[-1] \otimes_R (f \otimes_R R/(x))^{\otimes t}$, which is zero. Thus, $f^{\otimes t+1} = 0$. □

**Remark 2.8.** (1) Theorem 2.7 extends the smash nilpotence theorem due to Hopkins [1987, Theorem 10] and Neeman [1992, Theorem 1.1], where $X$ is also assumed to belong to $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$, so that $f : X \to Y$ is a morphism in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$. Under this assumption one can reduce to the case where $X = R$, which plays a key role in the proof of the original Hopkins–Neeman smash nilpotence theorem.

(2) The proof of Theorem 2.7 has a similar frame to that of the original Hopkins–Neeman smash nilpotence theorem, but we should notice that various delicate modifications are actually made there. Indeed, Proposition 2.3 and Lemmas 2.5 and 2.6 are all established to prove Theorem 2.7, which are not necessary to prove the original smash nilpotence theorem.

(3) The assumption in Theorem 2.7 that $Y$ belongs to $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ is used only to have $\mathrm{Ann}_{R_{\mathfrak{p}}}(f_{\mathfrak{p}}) = \mathrm{Ann}_R(f)_{\mathfrak{p}}$.

Our next goal is to classify cocompactly generated thick tensor ideals of $\mathsf{D}^-(R)$. To this end, we begin with deducing the following proposition concerning generation of thick tensor ideals of $\mathsf{D}^-(R)$, which will play an essential role throughout the rest of the paper.

**Proposition 2.9.** *Let $X$ be an object of $\mathsf{D}^-(R)$, and let $\mathcal{Y}$ be a subcategory of $\mathsf{D}^-(R)$. If $\mathrm{V}(\operatorname{Ann} X) \subseteq \operatorname{Supp} \mathcal{Y}$, then $X \in \mathsf{thick}^{\otimes} \mathcal{Y}$.*

*Proof.* Clearly, we may assume $X \neq 0$. We prove the proposition by replacing $\mathsf{D}^-(R)$ with $\mathsf{K}^-(R)$. There are a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of $R$ such that $\mathrm{V}(\operatorname{Ann} X) = \bigcup_{i=1}^n \mathrm{V}(\mathfrak{p}_i)$. Since each $\mathfrak{p}_i$ is in the support of $\mathcal{Y}$, we find an object $Y_i \in \mathcal{Y}$ with $\mathfrak{p}_i \in \operatorname{Supp} Y_i$. All $\mathfrak{p}_i$ are in the support of $Y := Y_1 \oplus \cdots \oplus Y_n \in \mathsf{K}^-(R)$. Choose an integer $t$ with $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \bigcup_{i>t} \operatorname{Supp} \mathsf{H}^i(Y)$, and let $Y' = (\cdots \to 0 \to 0 \to Y^t \to Y^{t+1} \to \cdots) \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ be the truncated complex of $Y$. Then $\mathrm{V}(\operatorname{Ann} X)$ is contained in $\operatorname{Supp} Y'$. Let $f : Y' \to Y$ be the natural morphism, and let $\phi : R \to \operatorname{Hom}_R(Y', Y)$ be the composition of the homothety morphism $R \to \operatorname{Hom}_R(Y, Y)$ and $\operatorname{Hom}_R(f, Y) : \operatorname{Hom}_R(Y, Y) \to \operatorname{Hom}_R(Y', Y)$. There is an exact triangle $Z \xrightarrow{\psi} R \xrightarrow{\phi} \operatorname{Hom}_R(Y', Y) \rightsquigarrow$ in $\mathsf{K}^-(R)$. We establish two claims.

**Claim 1.** *Let $\Phi : R \to C$ be a nonzero morphism in $\mathsf{K}^-(R)$. If $R$ is a field, then $\Phi$ is a split monomorphism.*

*Proof.* Since $C$ is isomorphic to $\mathsf{H}(C)$ in $\mathsf{K}^-(R)$, we may assume that the differentials of $C$ are zero. As $z := \Phi^0(1)$ is nonzero, we can construct a chain map $\Psi : C \to R$ with $\Psi^0(z) = 1$ and $\Psi^i = 0$ for all $i \neq 0$. It then holds that $\Psi\Phi = 1$. □

**Claim 2.** *The morphism $\phi \otimes_R \kappa(\mathfrak{p})$ in $\mathsf{K}^-(\kappa(\mathfrak{p}))$ is a split monomorphism for each $\mathfrak{p} \in \mathrm{V}(\operatorname{Ann} X)$.*

*Proof.* Set $S = \bigcup_{i>t} \operatorname{Supp} \mathsf{H}^i(Y)$; note that this contains $\mathrm{V}(\operatorname{Ann} X)$. We prove the stronger statement that $\phi \otimes \kappa(\mathfrak{p})$ is a split monomorphism for each $\mathfrak{p} \in S$. Since $Y'$ is a perfect complex, there are natural isomorphisms $\operatorname{Hom}_R(Y', Y) \otimes \kappa(\mathfrak{p}) \cong \operatorname{Hom}_R(Y', Y \otimes \kappa(\mathfrak{p})) \cong \operatorname{Hom}_{\kappa(\mathfrak{p})}(Y' \otimes \kappa(\mathfrak{p}), Y \otimes \kappa(\mathfrak{p}))$, which says that $\phi \otimes \kappa(\mathfrak{p})$ is identified with the natural morphism $\kappa(\mathfrak{p}) \to \operatorname{Hom}_{\kappa(\mathfrak{p})}(Y' \otimes \kappa(\mathfrak{p}), Y \otimes \kappa(\mathfrak{p}))$. This induces a map $\mathsf{H}^0(\phi \otimes \kappa(\mathfrak{p})) : \kappa(\mathfrak{p}) \to \operatorname{Hom}_{\mathsf{K}^-(\kappa(\mathfrak{p}))}(Y' \otimes \kappa(\mathfrak{p}), Y \otimes \kappa(\mathfrak{p}))$, sending 1 to $f \otimes \kappa(\mathfrak{p})$. If $f \otimes \kappa(\mathfrak{p}) = 0$ in $\mathsf{K}^-(\kappa(\mathfrak{p}))$, then we see that $\mathsf{H}^{>t}(Y \otimes \kappa(\mathfrak{p})) = 0$, contradicting the fact that $\mathfrak{p} \in S$. Thus, $\mathsf{H}^0(\phi \otimes \kappa(\mathfrak{p}))$ is nonzero, and so is $\phi \otimes \kappa(\mathfrak{p})$. Applying Claim 1 completes the proof. □

Claim 2 implies $\psi \otimes_R \kappa(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{V}(\operatorname{Ann} X)$. Using Theorem 2.7 for the morphism $\psi \otimes_R (R/\operatorname{Ann} X)$ in $\mathsf{K}^-(R/\operatorname{Ann} X)$, we have $\psi^{\otimes m} \otimes_R (R/\operatorname{Ann} X) = 0$ for some $m > 0$. Lemma 2.6(1) shows

$$0 = \psi^{\otimes u} \otimes_R \mathrm{K}(\boldsymbol{x}, R) : Z^{\otimes u} \otimes \mathrm{K}(\boldsymbol{x}, R) \to \mathrm{K}(\boldsymbol{x}, R), \tag{2.9.1}$$

where $\boldsymbol{x} = x_1, \dots, x_r$ is a system of generators of the ideal $\operatorname{Ann} X$, and $u = 2^r m$.

For each $i > 0$, let $W_i$ be the cone of the morphism $\psi^{\otimes i} : Z^{\otimes i} \to R$. Applying the octahedral axiom to the composition $\psi \circ (\psi^{\otimes i} \otimes Z) = \psi^{\otimes i+1}$, we get an exact triangle $W_i \otimes Z \to W_{i+1} \to W_1 \rightsquigarrow$ in $\mathsf{K}^-(R)$. As $W_1 \cong \operatorname{Hom}_R(Y', Y)$ and $Y' \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$, we see that $W_1$ is in $\mathsf{thick}\, Y$. Using the triangle, we inductively observe that $W_i$

belongs to $\mathsf{thick}^{\otimes} Y$ for all $i > 0$, and so does $W_u \otimes \mathrm{K}(\boldsymbol{x}, R)$. It follows from (2.9.1) that $\mathrm{K}(\boldsymbol{x}, R)$ is a direct summand of $W_u \otimes \mathrm{K}(\boldsymbol{x}, R)$, and therefore, $\mathrm{K}(\boldsymbol{x}, R)$ belongs to $\mathsf{thick}^{\otimes} Y$.

There is an exact triangle $R \xrightarrow{x_i} R \to \mathrm{K}(x_i, R) \rightsquigarrow$ in $\mathsf{K}^{-}(R)$ for each $1 \leqslant i \leqslant r$. Tensoring $X$ with this and using the fact that each $x_i$ kills $X$, we see that $X$ is a direct summand of $X \otimes \mathrm{K}(\boldsymbol{x}, R)$. Consequently, $X$ belongs to $\mathsf{thick}^{\otimes} Y$. By construction $Y$ is in $\mathsf{thick}\, \mathcal{Y}$, and hence, $X$ belongs to $\mathsf{thick}^{\otimes} \mathcal{Y}$. □

**Remark 2.10.** (1) Proposition 2.9 extends Neeman's result [1992, Lemma 1.2], where both $X$ and $\mathcal{Y}$ are contained in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ (and $\mathcal{Y}$ is assumed to consist of a single object).

(2) Proposition 2.9 is no longer true if we replace $\mathrm{V}(\mathrm{Ann}\, X)$ with $\mathrm{Supp}\, X$, or if we replace $\mathrm{Supp}\, \mathcal{Y}$ with $\mathrm{V}(\mathrm{Ann}\, \mathcal{Y})$. This will be explained in Remarks 6.7(1) and 7.15.

The following result is a consequence of Proposition 2.9, which will often be used later.

**Corollary 2.11.** *Let $\mathcal{X}$ be a thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Let $I$ be an ideal of $R$ and $\boldsymbol{x} = x_1, \ldots, x_n$ a system of generators of $I$. Then there are equivalences*

$$\mathrm{V}(I) \subseteq \mathrm{Supp}\, \mathcal{X} \iff R/I \in \mathcal{X} \iff \mathrm{K}(\boldsymbol{x}, R) \in \mathcal{X}.$$

*Proof.* By Proposition 2.3(3), we have that $\mathrm{Supp}\, R/I = \mathrm{V}(\mathrm{Ann}\, R/I) = \mathrm{V}(I) = \mathrm{V}(\mathrm{Ann}\, \mathrm{K}(\boldsymbol{x}, R)) = \mathrm{Supp}\, \mathrm{K}(\boldsymbol{x}, R)$. The assertion is shown by combining this with Proposition 2.9. □

Now we can give a complete classification of the cocompactly generated thick tensor ideals of $\mathsf{D}^{-}(R)$, using Proposition 2.9. For each subset $S$ of $\mathrm{Spec}\, R$, we set $\langle S \rangle = \mathsf{thick}^{\otimes}\{R/\mathfrak{p} \mid \mathfrak{p} \in S\}$.

**Theorem 2.12.** *The assignments $\mathcal{X} \mapsto \mathrm{Supp}\, \mathcal{X}$ and $\langle W \rangle \leftarrow\!\shortmid W$ make mutually inverse bijections*

$$\{\textit{Cocompactly generated thick } \otimes\textit{-ideals of } \mathsf{D}^{-}(R)\} \rightleftarrows \{\textit{Specialization-closed subsets of } \mathrm{Spec}\, R\}.$$

*Proof.* Proposition 1.10(2) shows that the map $\mathcal{X} \mapsto \mathrm{Supp}\, \mathcal{X}$ is well defined and that for a specialization-closed subset $W$ of $\mathrm{Spec}\, R$ the equality $W = \mathrm{Supp}\langle W \rangle$ holds. It remains to show that for any cocompactly generated thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^{-}(R)$ one has $\mathcal{X} = \langle \mathrm{Supp}\, \mathcal{X} \rangle$. Proposition 2.9 implies that $\mathcal{X}$ contains $\langle \mathrm{Supp}\, \mathcal{X} \rangle$. Since $\mathcal{X}$ is cocompactly generated, there is a subcategory $\mathcal{C}$ of $\mathsf{D}^{\mathsf{b}}(R)$ with $\mathcal{X} = \mathsf{thick}^{\otimes} \mathcal{C}$ by Proposition 2.1. Thus, it suffices to prove that each $M \in \mathcal{C}$ belongs to $\langle \mathrm{Supp}\, \mathcal{X} \rangle$. The complex $M$ belongs to $\mathsf{thick}\, \mathsf{H}(M)$ as $M \in \mathsf{D}^{\mathsf{b}}(R)$, and the finitely generated module $\mathsf{H}(M)$ has a finite filtration each of whose subquotients has the form $R/\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Supp} \mathsf{H}(M)$. Hence, $M$ is in $\langle \operatorname{Supp} M \rangle$, and we are done. □

Let us give several applications of our Theorem 2.12.

**Corollary 2.13.** (1) *Let $\mathscr{C}$ be a subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. Then $\mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} \mathscr{C}$ consists of the complexes $X \in \mathsf{D}^-(R)$ with $\mathrm{V}(\operatorname{Ann} X) \subseteq \operatorname{Supp} \mathscr{C}$. In particular, those complexes $X$ form a thick $\otimes$-ideal of $\mathsf{D}^-(R)$.*

(2) *Let $I$ be an ideal of $R$. Then $\mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)}(R/I)$ consists of the complexes $X \in \mathsf{D}^-(R)$ with $I \subseteq \sqrt{\operatorname{Ann} X}$.*

(3) *Let $W$ be a specialization-closed subset of $\operatorname{Spec} R$. Then $\langle W \rangle$ consists of the complexes $X \in \mathsf{D}^-(R)$ such that $\mathrm{V}(\operatorname{Ann} X) \subseteq W$.*

(4) *Let $\mathscr{X}, \mathscr{Y}$ be thick subcategories in $\mathsf{D}^{\mathsf{b}}(R)$. Then $\mathsf{thick}^{\otimes} \mathscr{X} = \mathsf{thick}^{\otimes} \mathscr{Y}$ if and only if $\operatorname{Supp} \mathscr{X} = \operatorname{Supp} \mathscr{Y}$.*

*Proof.* (1) Let $\mathscr{X}$ be the subcategory of $\mathsf{D}^-(R)$ consisting of objects $X \in \mathsf{D}^-(R)$ with $\mathrm{V}(\operatorname{Ann} X) \subseteq \operatorname{Supp} \mathscr{C}$. Proposition 2.9 says that $\mathsf{thick}^{\otimes} \mathscr{C}$ contains $\mathscr{X}$. Propositions 1.10(2) and 2.1 and Theorem 2.12 yield $\mathsf{thick}^{\otimes} \mathscr{C} = \langle \operatorname{Supp}(\mathsf{thick}^{\otimes} \mathscr{C}) \rangle = \langle \operatorname{Supp} \mathscr{C} \rangle$. For each $\mathfrak{p} \in \operatorname{Supp} \mathscr{C}$, the set $\mathrm{V}(\operatorname{Ann} R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ is contained in $\operatorname{Supp} \mathscr{C}$, whence $R/\mathfrak{p}$ is in $\mathscr{X}$. Hence, $\mathsf{thick}^{\otimes} \mathscr{C}$ is contained in $\mathscr{X}$, and we get the equality $\mathsf{thick}^{\otimes} \mathscr{C} = \mathscr{X}$.

(2) Applying (1) to $\mathscr{C} = \{R/I\}$, we immediately obtain the assertion.

(3) Setting $\mathscr{C} = \{R/\mathfrak{p} \mid \mathfrak{p} \in W\} \subseteq \mathsf{D}^{\mathsf{b}}(R)$, we have $\operatorname{Supp} \mathscr{C} = W$. The assertion follows from (1).

(4) Let $\mathscr{C}$ be either $\mathscr{X}$ or $\mathscr{Y}$. By Proposition 2.1 the thick $\otimes$-ideal $\mathsf{thick}^{\otimes} \mathscr{C}$ is cocompactly generated, and $\operatorname{Supp}(\mathsf{thick}^{\otimes} \mathscr{C}) = \operatorname{Supp} \mathscr{C}$ by Proposition 1.10(2). The assertion now follows from Theorem 2.12. □

We obtain the following one-to-one correspondence by combining our Theorem 2.12 with the celebrated Hopkins–Neeman classification theorem [Neeman 1992, Theorem 1.5].

**Corollary 2.14.** *The assignments $\mathscr{X} \mapsto \mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ and $\mathsf{thick}^{\otimes} \mathscr{Y} \leftarrow\!\shortmid \mathscr{Y}$ make mutually inverse bijections*

$$\{\textit{Cocompactly generated thick } \otimes\textit{-ideals of } \mathsf{D}^-(R)\} \rightleftarrows \{\textit{Thick subcategories of } \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)\}.$$

*In particular, all cocompactly generated thick $\otimes$-ideals of $\mathsf{D}^-(R)$ are compactly generated.*

*Proof.* It is directly verified and follows from Proposition 2.1 that the assignments $\mathscr{X} \mapsto \mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, R)$ and $\mathsf{thick}^{\otimes} \mathscr{Y} \leftarrow\!\shortmid \mathscr{Y}$, respectively, make well defined maps. It follows from [Neeman 1992, Theorem 1.5] that

(#) the assignments $\mathscr{X} \mapsto \operatorname{Supp}\mathscr{X}$ and $W \mapsto \operatorname{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)}(W) := \operatorname{Supp}^{-1} W \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$ make mutually inverse bijections between the thick subcategories of $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$ and the specialization-closed subsets of $\operatorname{Spec} R$.

In view of Theorem 2.12 and (#), we only have to show that

(a) $\operatorname{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)}(\operatorname{Supp}\mathscr{X}) = \mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$ for any cocompactly generated thick $\otimes$-ideal $\mathscr{X}$ of $\mathsf{D}^{-}(R)$, and

(b) $\langle \operatorname{Supp}\mathscr{Y}\rangle = \operatorname{thick}^{\otimes}\mathscr{Y}$ for any thick subcategory $\mathscr{Y}$ of $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$.

Using Propositions 2.1 and 1.10(2), we see that $\langle \operatorname{Supp}\mathscr{Y}\rangle$ and $\operatorname{thick}^{\otimes}\mathscr{Y}$ are cocompactly generated thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$ whose supports are equal to $\operatorname{Supp}\mathscr{Y}$. Now Theorem 2.12 shows the statement (b).

Clearly, $\operatorname{Supp}(\mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R))$ is contained in $\operatorname{Supp}\mathscr{X}$. Take a prime ideal $\mathfrak{p} \in \operatorname{Supp}\mathscr{X}$, and let $\boldsymbol{x}$ be a system of generators of $\mathfrak{p}$. Then $\mathrm{V}(\operatorname{Ann}\mathrm{K}(\boldsymbol{x}, R)) = \operatorname{Supp}\mathrm{K}(\boldsymbol{x}, R) = \mathrm{V}(\mathfrak{p}) \subseteq \operatorname{Supp}\mathscr{X}$ by Proposition 2.3(3), and $\mathrm{K}(\boldsymbol{x}, R) \in \mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$ by Proposition 2.9. It follows that $\mathfrak{p} \in \operatorname{Supp}\mathrm{K}(\boldsymbol{x}, R) \subseteq \operatorname{Supp}(\mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R))$. Thus, we get $\operatorname{Supp}(\mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)) = \operatorname{Supp}\mathscr{X}$, and obtain $\operatorname{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)}(\operatorname{Supp}\mathscr{X}) = \operatorname{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)}(\operatorname{Supp}(\mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R))) = \mathscr{X} \cap \mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$, where the last equality is shown by (#). Now the statement (a) is proved. □

**Remark 2.15.** Corollary 2.14 in particular gives a classification of the *compactly* generated thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$. This itself can also be deduced as follows. Let $\mathscr{X}, \mathscr{Y}$ be thick subcategories of $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$ with $\operatorname{Supp}(\operatorname{thick}^{\otimes}\mathscr{X}) = \operatorname{Supp}(\operatorname{thick}^{\otimes}\mathscr{Y})$. Then $\operatorname{Supp}\mathscr{X} = \operatorname{Supp}\mathscr{Y}$ by Proposition 1.10(2), and the Hopkins–Neeman theorem yields $\mathscr{X} = \mathscr{Y}$. Hence, $\operatorname{thick}^{\otimes}\mathscr{X} = \operatorname{thick}^{\otimes}\mathscr{Y}$.

The essential benefit that Corollary 2.14 produces is the classification of the *cocompactly* generated thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$. This should not follow from the Hopkins–Neeman theorem or other known results, but requires the arguments established in this section so far (especially, the generalized smash nilpotence Theorem 2.7). A *compactly* generated thick tensor ideal of $\mathsf{D}^{-}(R)$ is clearly *cocompactly* generated by Proposition 2.1, but the converse (shown in Corollary 2.14) should be rather nontrivial.

In view of Corollary 2.14 and Proposition 2.1, we obtain the following result and definition.

**Corollary 2.16.** *The following four conditions are equivalent for a thick $\otimes$-ideal $\mathscr{X}$ of $\mathsf{D}^{-}(R)$.*

- *$\mathscr{X}$ is compactly generated.*
- *$\mathscr{X}$ is cocompactly generated.*
- *$\mathscr{X}$ is generated by objects in $\mathsf{K}^{\mathsf{b}}(\operatorname{proj} R)$.*
- *$\mathscr{X}$ is generated by objects in $\mathsf{D}^{\mathsf{b}}(R)$.*

**Definition 2.17.** Let $\mathcal{X}$ be a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. We say that $\mathcal{X}$ is *compact* if it satisfies one (hence all) of the equivalent conditions in Corollary 4.19.

Next, for two thick $\otimes$-ideals $\mathcal{X}, \mathcal{Y}$ of $\mathsf{D}^-(R)$ we define the thick $\otimes$-ideals $\mathcal{X} \wedge \mathcal{Y}$ and $\mathcal{X} \vee \mathcal{Y}$ by

$$\mathcal{X} \wedge \mathcal{Y} = \mathsf{thick}^{\otimes}\{X \otimes_R^{\boldsymbol{L}} Y \mid X \in \mathcal{X},\ Y \in \mathcal{Y}\}, \qquad \mathcal{X} \vee \mathcal{Y} = \mathsf{thick}^{\otimes}(\mathcal{X} \cup \mathcal{Y}).$$

These two operations yield a lattice structure in the compact thick $\otimes$-ideals of $\mathsf{D}^-(R)$:

**Proposition 2.18.** (1) *Let $A$ and $B$ be specialization-closed subsets of* Spec $R$. *One then has equalities*

$$\langle A \rangle \wedge \langle B \rangle = \langle A \cap B \rangle, \qquad \langle A \rangle \vee \langle B \rangle = \langle A \cup B \rangle.$$

(2) *The set of compact thick $\otimes$-ideals of* $\mathsf{D}^-(R)$ *is a lattice with meet $\wedge$ and join $\vee$.*

*Proof.* (1) It is evident that the second equality holds. Let us show the first one.

We claim that for two subcategories $\mathcal{M}, \mathcal{N}$ of $\mathsf{D}^-(R)$ it holds that

$$(\mathsf{thick}^{\otimes} \mathcal{M}) \wedge (\mathsf{thick}^{\otimes} \mathcal{N}) = \mathsf{thick}^{\otimes}\{M \otimes_R^{\boldsymbol{L}} N \mid M \in \mathcal{M},\ N \in \mathcal{N}\}.$$

In fact, clearly $(\mathsf{thick}^{\otimes} \mathcal{M}) \wedge (\mathsf{thick}^{\otimes} \mathcal{N})$ contains $\mathcal{C} := \mathsf{thick}^{\otimes}\{M \otimes_R^{\boldsymbol{L}} N \mid M \in \mathcal{M}, N \in \mathcal{N}\}$. For each $N \in \mathcal{N}$, the subcategory of $\mathsf{D}^-(R)$ consisting of objects $X$ with $X \otimes_R^{\boldsymbol{L}} N \in \mathcal{C}$ is a thick $\otimes$-ideal containing $\mathcal{M}$, so contains $\mathsf{thick}^{\otimes} \mathcal{M}$. Let $X$ be an object in $\mathsf{thick}^{\otimes} \mathcal{M}$. Then $X \otimes_R^{\boldsymbol{L}} N$ belongs to $\mathcal{C}$ for all $N \in \mathcal{N}$. The subcategory of $\mathsf{D}^-(R)$ consisting of objects $Y$ with $X \otimes_R^{\boldsymbol{L}} Y \in \mathcal{C}$ is a thick $\otimes$-ideal containing $\mathcal{N}$, so contains $\mathsf{thick}^{\otimes} \mathcal{N}$. Hence, $X \otimes_R^{\boldsymbol{L}} Y$ is in $\mathcal{C}$ for all $X \in \mathsf{thick}^{\otimes} \mathcal{M}$ and $Y \in \mathsf{thick}^{\otimes} \mathcal{N}$, and the claim follows.

Using the claim, we see that $\langle A \rangle \wedge \langle B \rangle = \mathsf{thick}^{\otimes}\{R/\mathfrak{p} \otimes_R^{\boldsymbol{L}} R/\mathfrak{q} \mid \mathfrak{p} \in A,\ \mathfrak{q} \in B\}$. Therefore,

$$\begin{aligned}\operatorname{Supp}(\langle A \rangle \wedge \langle B \rangle) &= \operatorname{Supp}\{R/\mathfrak{p} \otimes_R^{\boldsymbol{L}} R/\mathfrak{q} \mid \mathfrak{p} \in A,\ \mathfrak{q} \in B\} \\ &= \bigcup_{\mathfrak{p} \in A,\, \mathfrak{q} \in B} \operatorname{Supp}(R/\mathfrak{p} \otimes_R^{\boldsymbol{L}} R/\mathfrak{q}) = \bigcup_{\mathfrak{p} \in A,\, \mathfrak{q} \in B} (\mathrm{V}(\mathfrak{p}) \cap \mathrm{V}(\mathfrak{q})) \\ &= A \cap B = \operatorname{Supp}\langle A \cap B \rangle\end{aligned}$$

by Proposition 1.10(2), Lemma 1.9(4), and the assumption we made that $A, B$ are specialization-closed. Theorem 2.12 implies that $\langle A \rangle \wedge \langle B \rangle = \langle A \cap B \rangle$.

(2) Let $\mathcal{X}, \mathcal{Y}$ be compact thick $\otimes$-ideals of $\mathsf{D}^-(R)$. Theorem 2.12 implies that $\mathcal{X} = \langle \operatorname{Supp} \mathcal{X} \rangle$ and $\mathcal{Y} = \langle \operatorname{Supp} \mathcal{Y} \rangle$, and $\operatorname{Supp} \mathcal{X}$ and $\operatorname{Supp} \mathcal{Y}$ are specialization-closed. It follows from (1) that $\mathcal{X} \wedge \mathcal{Y} = \langle \operatorname{Supp} \mathcal{X} \cap \operatorname{Supp} \mathcal{Y} \rangle$ and $\mathcal{X} \vee \mathcal{Y} = \langle \operatorname{Supp} \mathcal{X} \cup \operatorname{Supp} \mathcal{Y} \rangle$, which are compact. It is seen by definition that any thick $\otimes$-ideal containing both $\mathcal{X}$ and $\mathcal{Y}$ contains $\mathcal{X} \vee \mathcal{Y}$. Let $\mathcal{Z}$ be a compact thick $\otimes$-ideal contained in both $\mathcal{X}$ and $\mathcal{Y}$. By Theorem 2.12 again we get $\mathcal{Z} = \langle \operatorname{Supp} \mathcal{Z} \rangle$. Since $\operatorname{Supp} \mathcal{Z}$ is contained in

$\operatorname{Supp}\mathcal{X} \cap \operatorname{Supp}\mathcal{Y}$, we have that $\mathcal{Z}$ is contained in $\mathcal{X} \wedge \mathcal{Y}$. These arguments prove the assertion. □

Note that the specialization-closed subsets of $\operatorname{Spec} R$ form a lattice with meet $\cap$ and join $\cup$. As an immediate consequence of this fact and Proposition 2.18(2), we obtain a refinement of Theorem 2.12:

**Theorem 2.19.** *The assignments $\mathcal{X} \mapsto \operatorname{Supp}\mathcal{X}$ and $\langle W \rangle \leftarrow\!\shortmid W$ induce a lattice isomorphism*

$$\{\textit{Compact thick } \otimes\textit{-ideals of } \mathsf{D}^-(R)\} \cong \{\textit{Specialization-closed subsets of } \operatorname{Spec} R\}.$$

Restricting to the artinian case, we get a complete classification of thick tensor ideals of $\mathsf{D}^-(R)$.

**Corollary 2.20.** *Let $R$ be an artinian ring. Then the following statements are true.*

(1) *All the thick $\otimes$-ideals of $\mathsf{D}^-(R)$ are compact.*

(2) *The assignments $\mathcal{X} \mapsto \operatorname{Supp}\mathcal{X}$ and $\langle S \rangle \leftarrow\!\shortmid S$ induce a lattice isomorphism*

$$\{\textit{Thick } \otimes\textit{-ideals of } \mathsf{D}^-(R)\} \cong \{\textit{Subsets of } \operatorname{Spec} R\}.$$

*Proof.* (1) Take any thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$. We want to show $\mathcal{X} = \langle \operatorname{Supp}\mathcal{X} \rangle$. Corollary 2.11 implies that $\mathcal{X}$ contains $\langle \operatorname{Supp}\mathcal{X} \rangle$. To show the opposite inclusion, we may assume that $\mathcal{X}$ consists of a single object $X$. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s, \mathfrak{m}_{s+1}, \dots, \mathfrak{m}_n$ be the maximal ideals of $R$ with $\operatorname{Supp} X = \{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$. Find an integer $t > 0$ with $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)^t = 0$. The Chinese remainder theorem yields an isomorphism $R \cong R/\mathfrak{m}_1^t \oplus \cdots \oplus R/\mathfrak{m}_n^t$ of $R$-modules. Tensoring $X$, we obtain an isomorphism $X \cong (X \otimes_R^{\boldsymbol{L}} R/\mathfrak{m}_1^t) \oplus \cdots \oplus (X \otimes_R^{\boldsymbol{L}} R/\mathfrak{m}_n^t)$. Lemma 1.9(4) gives $\operatorname{Supp}(X \otimes_R^{\boldsymbol{L}} R/\mathfrak{m}_i^t) = \operatorname{Supp} X \cap \{\mathfrak{m}_i\}$, which is an empty set for $s + 1 \leqslant i \leqslant n$. For such an $i$ we have $X \otimes_R^{\boldsymbol{L}} R/\mathfrak{m}_i^t = 0$ by Remark 1.8, and get $X \cong (X \otimes_R^{\boldsymbol{L}} R/\mathfrak{m}_1^t) \oplus \cdots \oplus (X \otimes_R^{\boldsymbol{L}} R/\mathfrak{m}_s^t)$. It follows that $X$ is in $\operatorname{thick}^{\otimes}\{R/\mathfrak{m}_1^t, \dots, R/\mathfrak{m}_s^t\}$, which is the same as $\langle \operatorname{Supp} X \rangle$ by Corollary 2.13.

(2) Since all prime ideals of $R$ are maximal, every subset of $\operatorname{Spec} R$ is specialization-closed. (A more general statement will be given in Lemma 4.6.) The assertion follows from (1) and Theorem 2.19. □

## 3. Correspondence between the Balmer and Zariski spectra

In this section, we construct a pair of maps between the Balmer spectrum $\operatorname{Spc} \mathsf{D}^-(R)$ and the Zariski spectrum $\operatorname{Spec} R$, which will play a crucial role in later sections. First of all, let us recall the definitions of a prime thick tensor ideal of a tensor triangulated category and its Balmer spectrum.

**Definition 3.1.** Let $\mathcal{T}$ be an essentially small tensor triangulated category. A thick $\otimes$-ideal $\mathcal{P}$ of $\mathcal{T}$ is called *prime* provided that $\mathcal{P} \neq \mathcal{T}$ and if $X \otimes Y$ is in $\mathcal{P}$, then so is either $X$ or $Y$. The set of prime thick $\otimes$-ideals of $\mathcal{T}$ is denoted by $\mathtt{Spc}\,\mathcal{T}$ and called the *Balmer spectrum* of $\mathcal{T}$.

Here is an example of a prime thick tensor ideal of $\mathsf{D}^-(R)$.

**Example 3.2.** When $R$ is local, the zero subcategory $\mathbf{0}$ of $\mathsf{D}^-(R)$ is a prime thick $\otimes$-ideal. In fact, it is easy to verify that $\mathbf{0}$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. (This also follows from Remark 1.8 and Proposition 1.10(1).) If $X, Y$ are objects of $\mathsf{D}^-(R)$ with $X \otimes_R^{\boldsymbol{L}} Y = 0$, then either $X = 0$ or $Y = 0$ by Lemma 1.9(4).

Now we introduce the following notation.

**Notation 3.3.** For a prime ideal $\mathfrak{p}$ of $R$, we denote by $\mathcal{S}(\mathfrak{p})$ the subcategory of $\mathsf{D}^-(R)$ consisting of complexes $X$ with $X_\mathfrak{p} \cong 0$ in $\mathsf{D}^-(R_\mathfrak{p})$.

The subcategory $\mathcal{S}(\mathfrak{p})$ is always a prime thick tensor ideal:

**Proposition 3.4.** *Let $\mathfrak{p}$ be a prime ideal of $R$. Then $\mathcal{S}(\mathfrak{p})$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ satisfying*

$$\operatorname{Supp} \mathcal{S}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

*Proof.* Since $\mathcal{S}(\mathfrak{p})$ does not contain $R$, it is not equal to $\mathsf{D}^-(R)$. Note that $\mathcal{S}(\mathfrak{p}) = \operatorname{Supp}^{-1}(\{\mathfrak{p}\}^\complement)$. Using Lemma 1.9(4) and Proposition 1.10(1), we observe that $\mathcal{S}(\mathfrak{p})$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$.

Fix a prime ideal $\mathfrak{q}$ of $R$. If $\mathfrak{q}$ is in $\operatorname{Supp} \mathcal{S}(\mathfrak{p})$, then there is a complex $X \in \mathcal{S}(\mathfrak{p})$ with $\mathfrak{q} \in \operatorname{Supp} X$, and it follows that $X_\mathfrak{p} = 0 \neq X_\mathfrak{q}$. If $\mathfrak{q}$ is contained in $\mathfrak{p}$, then we have $X_\mathfrak{q} = (X_\mathfrak{p})_\mathfrak{q}$ and get a contradiction. Therefore, $\mathfrak{q}$ is not contained in $\mathfrak{p}$. Conversely, assume this. Take a system of generators $\boldsymbol{x} = x_1, \ldots, x_n$ of $\mathfrak{q}$, and put $K = \mathrm{K}(\boldsymbol{x}, R)$. Then we have $K_\mathfrak{q} \neq 0 = K_\mathfrak{p}$ by Proposition 2.3(3). Hence, $K$ belongs to $\mathcal{S}(\mathfrak{p})$ and $\mathfrak{q}$ is in $\operatorname{Supp} K$, which implies $\mathfrak{q} \in \operatorname{Supp} \mathcal{S}(\mathfrak{p})$. We thus obtain the equality in the proposition. $\square$

As an easy consequence of the above proposition, we get another example of a prime thick tensor ideal.

**Corollary 3.5.** *Let $R$ be an integral domain of dimension one. It then holds that $\mathsf{D}^-_{\mathsf{fl}}(R) = \mathcal{S}((0))$, where $(0)$ stands for the zero ideal of $R$. Hence, $\mathsf{D}^-_{\mathsf{fl}}(R)$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$.*

*Proof.* For a complex $X \in \mathsf{D}^-(R)$ it holds that

$$\begin{aligned} X \in \mathsf{D}^-_{\mathsf{fl}}(R) &\iff \ell(\mathsf{H}^i X) < \infty \text{ for all } i \\ &\iff \mathsf{H}^i X_{(0)} = 0 \text{ for all } i \iff X_{(0)} = 0 \iff X \in \mathcal{S}((0)), \end{aligned}$$

where the second equivalence follows from the fact that $\operatorname{Spec} R = \{(0)\} \cup \operatorname{Max} R$. This shows $\mathsf{D}^-_{\mathsf{fl}}(R) = \mathcal{S}((0))$. Proposition 3.4 implies that $\mathcal{S}((0))$ is prime, which gives the last statement of the corollary. □

**Remark 3.6.** Corollary 3.5 is no longer valid if we remove the assumption that $R$ is an integral domain. More precisely, the assertion of the corollary is not true even if $R$ is reduced. In fact, consider the ring $R = k[\![x, y]\!]/(xy)$, where $k$ is a field. Then $R$ is a 1-dimensional reduced local ring. It is observed by Proposition 2.3(3) that the Koszul complexes $\mathrm{K}(x, R)$, $\mathrm{K}(y, R)$ are outside $\mathsf{D}^-_{\mathsf{fl}}(R)$, while the complex $\mathrm{K}(x, R) \otimes^{\boldsymbol{L}}_R \mathrm{K}(y, R) = \mathrm{K}(x, y, R)$ is in $\mathsf{D}^-_{\mathsf{fl}}(R)$. This shows that $\mathsf{D}^-_{\mathsf{fl}}(R)$ is not prime.

We have constructed from each prime ideal $\mathfrak{p}$ of $R$ the prime thick tensor ideal $\mathcal{S}(\mathfrak{p})$ of $\mathsf{D}^-(R)$. Now we are concerned with the opposite direction; that is, we construct from a prime thick tensor ideal of $\mathsf{D}^-(R)$ a prime ideal of $R$, which is done in the following proposition.

**Proposition 3.7.** *Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Let $K$ be the set of ideals $I$ of $R$ such that $\mathrm{V}(I)$ is not contained in $\operatorname{Supp} \mathcal{P}$. Then $K$ has the maximum element $P$ with respect to the inclusion relation, and $P$ is a prime ideal of $R$.*

*Proof.* We claim that for ideals $I$, $J$ of $R$, if $\operatorname{Supp} \mathcal{P}$ contains $\mathrm{V}(I + J)$, then it contains either $\mathrm{V}(I)$ or $\mathrm{V}(J)$. Indeed, let $\boldsymbol{x} = x_1, \dots, x_a$ and $\boldsymbol{y} = y_1, \dots, y_b$ be systems of generators of $I$ and $J$, respectively. Corollary 2.11 yields that $\mathrm{K}(\boldsymbol{x}, \boldsymbol{y}, R)$ is in $\mathcal{P}$. There is an isomorphism $\mathrm{K}(\boldsymbol{x}, R) \otimes^{\boldsymbol{L}}_R \mathrm{K}(\boldsymbol{y}, R) \cong \mathrm{K}(\boldsymbol{x}, \boldsymbol{y}, R)$ of complexes, whence $\mathrm{K}(\boldsymbol{x}, R) \otimes^{\boldsymbol{L}}_R \mathrm{K}(\boldsymbol{y}, R)$ belongs to $\mathcal{P}$. Since $\mathcal{P}$ is prime, it contains either $\mathrm{K}(\boldsymbol{x}, R)$ or $\mathrm{K}(\boldsymbol{y}, R)$. Thus, $\operatorname{Supp} \mathcal{P}$ contains either $\mathrm{V}(I)$ or $\mathrm{V}(J)$ by Corollary 2.11 again.

The claim says that $K$ is closed under sums of ideals of $R$. Taking into account that $R$ is noetherian, we see that $K$ has the maximum element $P$ with respect to the inclusion relation. There is a filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = R/P$ of submodules of the $R$-module $R/P$ such that for every $1 \leqslant i \leqslant t$ one has $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ with some $\mathfrak{p}_i \in \operatorname{Supp}_R R/P$, whence each $\mathfrak{p}_i$ contains $P$. Suppose that $P$ is not a prime ideal of $R$. Then the $\mathfrak{p}_i$ strictly contain $P$, and the maximality of $P$ shows that $\operatorname{Supp} \mathcal{P}$ contains $\mathrm{V}(\mathfrak{p}_i)$. There is an equality $\operatorname{Supp}_R R/P = \bigcup_{i=1}^t \operatorname{Supp} R/\mathfrak{p}_i$, or equivalently, $\mathrm{V}(P) = \bigcup_{i=1}^t \mathrm{V}(\mathfrak{p}_i)$. It follows that $\operatorname{Supp} \mathcal{P}$ contains $\mathrm{V}(P)$, which is a contradiction. Consequently, $P$ is a prime ideal of $R$. □

Thus, we have got two maps in the mutually inverse directions, between $\operatorname{Spec} R$ and $\operatorname{Spc} \mathsf{D}^-(R)$:

**Notation 3.8.** Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. With the notation of Proposition 3.7, we set $\mathbb{I}(\mathcal{P}) = K$ and $\mathfrak{s}(\mathcal{P}) = P$. In view of Proposition 3.4, we obtain a pair of maps

$$\mathcal{S} : \operatorname{Spec} R \rightleftarrows \operatorname{Spc} \mathsf{D}^-(R) : \mathfrak{s}$$

given by $\mathfrak{p} \mapsto \mathcal{S}(\mathfrak{p})$ and $\mathcal{P} \mapsto \mathfrak{s}(\mathcal{P})$ for $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathcal{P} \in \operatorname{Spc} \mathsf{D}^-(R)$.

Now we compare the maps $\mathcal{S}$, $\mathfrak{s}$, and for this recall two basic definitions from set theory. Let $f : A \to B$ be a map of partially ordered sets. We say that $f$ is *order-reversing* if $x \leqslant y$ implies $f(x) \geqslant f(y)$ for all $x, y \in A$. Also, we call $f$ an *order antiembedding* if $x \leqslant y$ is equivalent to $f(x) \geqslant f(y)$ for all $x, y \in A$. Note that any order antiembedding is an injection. We regard $\operatorname{Spec} R$ and $\mathsf{Spc}\,\mathsf{D}^-(R)$ as partially ordered sets with respect to the inclusion relations. The following theorem is the main result of this section.

**Theorem 3.9.** *The maps* $\mathcal{S} : \operatorname{Spec} R \rightleftarrows \mathsf{Spc}\,\mathsf{D}^-(R) : \mathfrak{s}$ *are order-reversing, and satisfy*

$$\mathfrak{s} \cdot \mathcal{S} = 1, \qquad \mathcal{S} \cdot \mathfrak{s} = \operatorname{Supp}^{-1} \operatorname{Supp}.$$

*Hence, $\mathcal{S}$ is an order antiembedding.*

*Proof.* Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of $R$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Then Proposition 3.4 shows that $\mathfrak{q}$ is not in $\operatorname{Supp} \mathcal{S}(\mathfrak{p})$. Hence, $X_{\mathfrak{q}} = 0$ for all $X \in \mathcal{S}(\mathfrak{p})$, which means that $\mathcal{S}(\mathfrak{p})$ is contained in $\mathcal{S}(\mathfrak{q})$. On the other hand, let $\mathcal{P}, \mathcal{Q}$ be prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ with $\mathcal{P} \subseteq \mathcal{Q}$. Then $\operatorname{Supp} \mathcal{P}$ is contained in $\operatorname{Supp} \mathcal{Q}$, and we see from the definition of $\mathfrak{s}$ that $\mathfrak{s}(\mathcal{P})$ contains $\mathfrak{s}(\mathcal{Q})$. Therefore, the maps $\mathcal{S}, \mathfrak{s}$ are order-reversing.

Fix a prime ideal $\mathfrak{p}$ of $R$. Then $\mathfrak{s}(\mathcal{S}(\mathfrak{p}))$ is the maximum element of $\mathbb{I}(\mathcal{S}(\mathfrak{p}))$, which consists of ideals $I$ with $\mathrm{V}(I) \nsubseteq \operatorname{Supp} \mathcal{S}(\mathfrak{p})$. This is equivalent to saying that $I \subseteq \mathfrak{p}$ by Proposition 3.4. Hence, $\mathfrak{s}(\mathcal{S}(\mathfrak{p})) = \mathfrak{p}$.

Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Note that a prime ideal $\mathfrak{p}$ of $R$ belongs to $\mathbb{I}(\mathcal{P})$ if and only if $\mathfrak{p}$ is not in $\operatorname{Supp} \mathcal{P}$. Let $X \in \mathsf{D}^-(R)$ be a complex with $X_{\mathfrak{s}(\mathcal{P})} = 0$. If $\mathfrak{p}$ is a prime ideal of $R$ with $X_{\mathfrak{p}} \neq 0$, then $\mathfrak{p}$ is not contained in $\mathfrak{s}(\mathcal{P})$, and $\mathfrak{p}$ must not belong to $\mathbb{I}(\mathcal{P})$, which means $\mathfrak{p} \in \operatorname{Supp} \mathcal{P}$. Therefore, $\operatorname{Supp} X$ is contained in $\operatorname{Supp} \mathcal{P}$, and we obtain $\mathcal{S}(\mathfrak{s}(\mathcal{P})) \subseteq \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{P}$. Conversely, let $X \in \mathsf{D}^-(R)$ be a complex with $\operatorname{Supp} X \subseteq \operatorname{Supp} \mathcal{P}$. Since $\mathfrak{s}(\mathcal{P})$ is in $\mathbb{I}(\mathcal{P})$, it does not belong to $\operatorname{Supp} \mathcal{P}$. Hence, $\mathfrak{s}(\mathcal{P})$ is not in $\operatorname{Supp} X$, which means $X_{\mathfrak{s}(\mathcal{P})} = 0$. We thus conclude that $\mathcal{S}(\mathfrak{s}(\mathcal{P})) = \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{P}$.

The last assertion is shown by using the equality $\mathfrak{p} = \mathfrak{s}(\mathcal{S}(\mathfrak{p}))$ for all prime ideals $\mathfrak{p}$ of $R$. □

The above theorem gives rise to several corollaries, which will often be used later. The rest of this section is devoted to stating and proving them.

**Corollary 3.10.** *Let $\mathfrak{p}$ be a prime ideal of $R$, and let $\mathcal{P}$ a prime thick $\otimes$-ideal of* $\mathsf{D}^-(R)$. *It holds that*

$$\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P}) \iff R/\mathfrak{p} \notin \mathcal{P} \iff \mathfrak{p} \notin \operatorname{Supp} \mathcal{P} \iff \mathcal{P} \subseteq \mathcal{S}(\mathfrak{p}).$$

*In particular, $\mathfrak{s}(\mathcal{P})$ is the maximum element of* $(\operatorname{Supp} \mathcal{P})^{\complement}$ *with respect to the inclusion relation.*

*Proof.* The second equivalence follows from Corollary 2.11, while the third one is trivial. If $\mathfrak{p} \notin \operatorname{Supp} \mathcal{P}$, then $\mathfrak{p} \subseteq \mathfrak{s}(\mathcal{P})$. If this is the case, then $\mathcal{S}(\mathfrak{p}) \supseteq \mathcal{S}(\mathfrak{s}(\mathcal{P})) = \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{P} \supseteq \mathcal{P}$ by Theorem 3.9. □

**Corollary 3.11.** *For two prime thick $\otimes$-ideals $\mathcal{P}, \mathcal{Q}$ of $\mathsf{D}^-(R)$ one has*

$$\mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}(\mathcal{Q}) \iff \operatorname{Supp} \mathcal{P} \supseteq \operatorname{Supp} \mathcal{Q}, \qquad \mathfrak{s}(\mathcal{P}) = \mathfrak{s}(\mathcal{Q}) \iff \operatorname{Supp} \mathcal{P} = \operatorname{Supp} \mathcal{Q}.$$

*Proof.* Theorem 3.9 and Proposition 1.10(1) yield the first equivalence, which implies the second one. □

Here we introduce two notions of thick tensor ideals, which will play main roles in the rest of this paper.

**Definition 3.12.** (1) For a thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ we denote by $\sqrt{\mathcal{X}}$ the *radical* of $\mathcal{X}$, that is, the subcategory of $\mathsf{D}^-(R)$ consisting of objects $M$ such that the $n$-fold tensor product $M \otimes_R^{\boldsymbol{L}} \cdots \otimes_R^{\boldsymbol{L}} M$ belongs to $\mathcal{X}$ for some $n \geqslant 1$.

(2) A thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ is called *radical* if $\mathcal{X} = \sqrt{\mathcal{X}}$. Any prime thick $\otimes$-ideal is radical.

(3) A thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ is called *tame* if one can write $\mathcal{X} = \operatorname{Supp}^{-1} S$ for some subset $S$ of $\operatorname{Spec} R$. The set of tame prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ is denoted by ${}^{\mathtt{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$.

**Remark 3.13.** For each subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$ the following are equivalent:

(1) $\mathcal{X}$ is a tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$,

(2) $\mathcal{X} = \operatorname{Supp}^{-1} S$ for some subset $S$ of $\operatorname{Spec} R$, and

(3) $\mathcal{X} = \operatorname{Supp}^{-1} W$ for some specialization-closed subset $W$ of $\operatorname{Spec} R$.

This is a direct consequence of Proposition 1.10(1).

The following corollary of Theorem 3.9 gives an explicit description of tame prime thick tensor ideals.

**Corollary 3.14.** ${}^{\mathtt{t}}\mathtt{Spc}\, \mathsf{D}^-(R) = \operatorname{Im} \mathcal{S} = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}$.

*Proof.* For a prime ideal $\mathfrak{p}$ of $R$, we have $\mathcal{S}(\mathfrak{p}) = \mathcal{S}\mathfrak{s}\mathcal{S}(\mathfrak{p}) = \operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{S}(\mathfrak{p}))$ by Theorem 3.9, which shows that the prime thick $\otimes$-ideal $\mathcal{S}(\mathfrak{p})$ of $\mathsf{D}^-(R)$ is tame. On the other hand, let $\mathcal{P}$ be a tame prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Using Theorem 3.9 and Proposition 1.10, we get $\mathcal{S}(\mathfrak{s}(\mathcal{P})) = \operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{P}) = \mathcal{P}$. □

Here is one more application of Theorem 3.9, giving a criterion for a thick tensor ideal to be prime.

**Corollary 3.15.** *Let $W$ be a specialization-closed subset of* $\operatorname{Spec} R$. *The following are equivalent*:

(1) *the tame thick $\otimes$-ideal* $\operatorname{Supp}^{-1} W$ *of* $\mathsf{D}^-(R)$ *is prime,*

(2) *there exists a prime ideal $\mathfrak{p}$ of $R$ such that* $W = \operatorname{Supp} \mathcal{S}(\mathfrak{p})$,

(3) *there exists a prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$ such that $W = \operatorname{Supp}\mathcal{P}$, and*

(4) *the set $W^{\complement}$ has a unique maximal element with respect to the inclusion relation.*

*Proof.* (1) $\Longrightarrow$ (2). By Corollary 3.10, the complement of $W = \operatorname{Supp}(\operatorname{Supp}^{-1} W)$ (see Proposition 1.10(2)) has the maximum element $\mathfrak{p} := \mathfrak{s}(\operatorname{Supp}^{-1} W)$. Using Theorem 3.9, we obtain $W = \operatorname{Supp}\mathcal{S}(\mathfrak{p})$.

(2) $\Longrightarrow$ (3). Take $\mathcal{P} = \mathcal{S}(\mathfrak{p})$, which is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ by Proposition 3.4.

(3) $\Longrightarrow$ (4). This implication follows from Corollary 3.10.

(4) $\Longrightarrow$ (1). Let $\mathfrak{p}$ be a unique maximal element of $W^{\complement}$. We claim that there is an equality $W = \operatorname{Supp}\mathcal{S}(\mathfrak{p})$. Indeed, $\operatorname{Supp}\mathcal{S}(\mathfrak{p})$ consists of the prime ideals $\mathfrak{q}$ of $R$ not contained in $\mathfrak{p}$ by Proposition 3.4. Now fix a prime ideal $\mathfrak{q}$ of $R$. Suppose that $\mathfrak{q}$ is in $W$. If $\mathfrak{q}$ is contained in $\mathfrak{p}$, then $\mathfrak{p}$ belongs to $W$ as $W$ is specialization-closed. This contradicts the choice of $\mathfrak{p}$, whence $\mathfrak{q}$ belongs to $\operatorname{Supp}\mathcal{S}(\mathfrak{p})$. Conversely, if $\mathfrak{q}$ is not in $W$, then $\mathfrak{q}$ is in $W^{\complement}$, and the choice of $\mathfrak{p}$ shows that $\mathfrak{q}$ is contained in $\mathfrak{p}$. Thus, the claim follows. Applying Theorem 3.9, we obtain $\operatorname{Supp}^{-1} W = \mathcal{S}(\mathfrak{p})$ and this is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. $\square$

## 4. Topological structures of the Balmer spectrum

In this section, we study various topological properties of the maps $\mathcal{S}$, $\mathfrak{s}$ defined in the previous section, and explore the structure of the Balmer spectrum $\operatorname{Spc}\mathsf{D}^-(R)$ as a topological space. We begin with recalling the definition of the topology which the Balmer spectrum possesses.

**Definition 4.1.** Let $\mathcal{T}$ be an essentially small tensor triangulated category.

(1) For an object $X \in \mathcal{T}$ the *Balmer support* of $X$, denoted by $\operatorname{Spp} X$, is defined as the set of prime thick $\otimes$-ideals of $\mathcal{T}$ not containing $X$. We set $\mathrm{U}(X) = (\operatorname{Spp} X)^{\complement} = \operatorname{Spc}\mathcal{T} \setminus \operatorname{Spp} X$.

(2) The set $\operatorname{Spc}\mathcal{T}$ is a topological space with open basis $\{\mathrm{U}(X) \mid X \in \mathsf{D}^-(R)\}$ [Balmer 2005, Definition 2.1].

Therefore, $\operatorname{Spc}\mathsf{D}^-(R)$ is a topological space. We regard ${}^{\mathrm{t}}\operatorname{Spc}\mathsf{D}^-(R)$ as a subspace of $\operatorname{Spc}\mathsf{D}^-(R)$ by the relative topology.

We first consider a direct sum decomposition of the Balmer spectrum.

**Proposition 4.2.** *There is a direct sum decomposition of sets*

$$\operatorname{Spc}\mathsf{D}^-(R) = \coprod_{\mathfrak{p}\in\operatorname{Spec} R} \mathfrak{s}^{-1}(\mathfrak{p}),$$

*where* $\mathfrak{s}^{-1}(\mathfrak{p}) := \{\mathcal{P} \in \operatorname{Spc}\mathsf{D}^-(R) \mid \mathfrak{s}(\mathcal{P}) = \mathfrak{p}\} = \{\mathcal{P} \in \operatorname{Spc}\mathsf{D}^-(R) \mid \operatorname{Supp}\mathcal{P} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}\}$.

*Proof.* Theorem 3.9 says that the map $\mathfrak{s}$ is surjective. Using this, we easily get the direct sum decomposition. Applying Theorem 3.9, Corollary 3.11, and Proposition 3.4, we observe that for any $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathcal{P} \in \operatorname{Spc} \mathsf{D}^-(R)$ one has $\mathfrak{s}(\mathcal{P}) = \mathfrak{p}$ if and only if $\operatorname{Supp} \mathcal{P} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}$. ☐

Next we investigate the dimension of the Balmer spectrum. The (*Krull*) *dimension* of a topological space $X$, denoted by $\dim X$, is defined to be the supremum of integers $n \geqslant 0$ such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ of nonempty irreducible closed subsets of $X$. (Recall that a subset of $X$ is called *irreducible* if it cannot be written as a union of two nonempty proper closed subsets.)

**Proposition 4.3.** (1) *Let $\mathcal{T}$ be an essentially small $\otimes$-triangulated category. The dimension of $\operatorname{Spc} \mathcal{T}$ is equal to the supremum of integers $n \geqslant 0$ such that there is a chain $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n$ in $\operatorname{Spc} \mathcal{T}$.*

(2) *There is an inequality*

$$\dim(\operatorname{Spc} \mathsf{D}^-(R)) \geqslant \dim R.$$

*Proof.* Applying [Balmer 2005, Propositions 2.9 and 2.18] shows (1), while (2) follows from (1) and Theorem 3.9. ☐

**Remark 4.4.** We will see that the inequality in Proposition 4.3(2) sometimes becomes an equality, and sometimes becomes a strict inequality. See Corollaries 4.16 and 7.13 and Theorem 7.11.

Let $\mathcal{P}, \mathcal{Q}$ be prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$. We write $\mathcal{P} \sim \mathcal{Q}$ if $\operatorname{Supp} \mathcal{P} = \operatorname{Supp} \mathcal{Q}$. Then $\sim$ defines an equivalence relation on $\operatorname{Spc} \mathsf{D}^-(R)$. We denote by $\operatorname{Spc} \mathsf{D}^-(R)/\operatorname{Supp}$ the quotient topological space of $\operatorname{Spc} \mathsf{D}^-(R)$ by the equivalence relation $\sim$, so that a subset $U$ of $\operatorname{Spc} \mathsf{D}^-(R)/\operatorname{Supp}$ is open if and only if $\pi^{-1}(U)$ is open in $\operatorname{Spc} \mathsf{D}^-(R)$, where $\pi : \operatorname{Spc} \mathsf{D}^-(R) \to \operatorname{Spc} \mathsf{D}^-(R)/\operatorname{Supp}$ stands for the canonical surjection. By definition, $\pi$ is a continuous map. Denote by $\theta : {}^{\mathsf{t}}\operatorname{Spc} \mathsf{D}^-(R) \to \operatorname{Spc} \mathsf{D}^-(R)$ the inclusion map, which is continuous. Now we can state our first main result in this section.

**Theorem 4.5.** (1) *The set ${}^{\mathsf{t}}\operatorname{Spc} \mathsf{D}^-(R)$ is dense in $\operatorname{Spc} \mathsf{D}^-(R)$.*

(2) *The composition $\pi\theta$ is a continuous bijection.*

(3) *The maps $\mathcal{S}, \mathfrak{s}$ induce the bijections $\mathcal{S}', \widetilde{\mathcal{S}}, \mathfrak{s}', \tilde{\mathfrak{s}}$ which make the following diagram commute:*

$$\begin{tikzcd}
 & {}^{\mathsf{t}}\operatorname{Spc} \mathsf{D}^-(R) \arrow[d, "\theta"] \arrow[dr, "\mathfrak{s}'"] & \\
\operatorname{Spec} R \arrow[ur, "\mathcal{S}'"] \arrow[r, "\mathcal{S}"] \arrow[dr, "\widetilde{\mathcal{S}}"'] & \operatorname{Spc} \mathsf{D}^-(R) \arrow[r, "\mathfrak{s}"] \arrow[d, "\pi"] & \operatorname{Spec} R \\
 & \operatorname{Spc} \mathsf{D}^-(R)/\operatorname{Supp} \arrow[ur, "\tilde{\mathfrak{s}}"'] &
\end{tikzcd}$$

*In particular, one has* $\mathfrak{s}\mathcal{S} = \mathfrak{s}'\mathcal{S}' = \tilde{\mathfrak{s}}\widetilde{\mathcal{S}} = 1$.

(4) *The maps* $\mathfrak{s}, \mathfrak{s}', \tilde{\mathfrak{s}}$ *are continuous. The maps* $\mathcal{S}', \widetilde{\mathcal{S}}$ *are open and closed.*

*Proof.* First of all, recall from Corollary 3.14 that the image of $\mathcal{S}$ coincides with ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$.

(1) Let $X$ be a complex in $\mathsf{D}^-(R)$, and suppose that $U := \mathtt{U}(X)$ is nonempty. Then $U$ contains a prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$, and $X$ is in $\mathcal{P}$. It is seen from Theorem 3.9 that $\mathcal{P}$ is contained in $\mathcal{S}(\mathfrak{s}(\mathcal{P}))$, and hence, $X$ is in $\mathcal{S}(\mathfrak{s}(\mathcal{P}))$. Therefore, $\mathcal{S}(\mathfrak{s}(\mathcal{P}))$ belongs to the intersection $U \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, and we have $U \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R) \neq \varnothing$. This shows that any nonempty open subset of $\mathtt{Spc}\,\mathsf{D}^-(R)$ meets ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$.

(2) Since $\pi$ and $\theta$ are continuous, so is $\pi\theta$. Let $\mathcal{P}, \mathcal{Q}$ be tame prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$. Then $\mathcal{P} = \mathcal{S}\mathfrak{s}(\mathcal{P})$ and $\mathcal{Q} = \mathcal{S}\mathfrak{s}(\mathcal{Q})$ by Theorem 3.9. One has $\mathcal{P} \sim \mathcal{Q}$ if and only if $\mathfrak{s}(\mathcal{P}) = \mathfrak{s}(\mathcal{Q})$ by Corollary 3.11, if and only if $\mathcal{P} = \mathcal{Q}$ by Theorem 3.9 again. This shows that the map $\pi\theta$ is well defined and injective. To show the surjectivity, pick a prime thick $\otimes$-ideal $\mathcal{R}$ of $\mathsf{D}^-(R)$. It is seen from Proposition 1.10(1) that $\mathcal{R} \sim \operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{R}$, and the latter thick $\otimes$-ideal is tame. Consequently, $\pi\theta$ is a bijection.

(3) Using Theorem 3.9, we obtain the bijection $\mathcal{S}'$ satisfying $\theta\mathcal{S}' = \mathcal{S}$. Set $\widetilde{\mathcal{S}} = \pi\mathcal{S}$ and $\mathfrak{s}' = \mathfrak{s}\theta$. Define the map $\tilde{\mathfrak{s}} : \mathtt{Spc}\,\mathsf{D}^-(R)/\operatorname{Supp} \to \operatorname{Spec} R$ by $\tilde{\mathfrak{s}}([\mathcal{P}]) = \mathfrak{s}(\mathcal{P})$ for $\mathcal{P} \in \mathtt{Spc}\,\mathsf{D}^-(R)$. Corollary 3.11 guarantees that this is well defined, and by definition we have $\tilde{\mathfrak{s}}\pi = \mathfrak{s}$. Thus, the commutative diagram in the assertion is obtained, which and Theorem 3.9 yield $1 = \mathfrak{s}\mathcal{S} = \mathfrak{s}'\mathcal{S}' = \tilde{\mathfrak{s}}\widetilde{\mathcal{S}}$. It follows that the map $\mathcal{S}'$ is bijective, and so is $\mathfrak{s}'$. We have $\widetilde{\mathcal{S}} = (\pi\theta)\mathcal{S}'$, which is bijective by (2), and so is $\tilde{\mathfrak{s}}$.

(4) Let $\mathcal{P} \in \mathtt{Spc}\,\mathsf{D}^-(R)$. An ideal $I$ of $R$ is contained in $\mathfrak{s}(\mathcal{P})$ if and only if $\mathrm{V}(I)$ is not contained in $\operatorname{Supp}\mathcal{P}$, if and only if $R/I$ does not belong to $\mathcal{P}$ by Corollary 2.11. We obtain an equality

$$\mathfrak{s}^{-1}(\mathrm{V}(I)) = \mathtt{Spp}\, R/I,$$

which shows that $\mathfrak{s}$ is a continuous map. Since the map $\theta$ is continuous, so is the composition $\mathfrak{s}' = \mathfrak{s}\theta$. The equality $\mathfrak{s}' = \mathcal{S}'^{-1}$ from (3) and the continuity of $\mathfrak{s}'$ imply that the map $\mathcal{S}'$ is open and closed.

Fix an ideal $I$ of $R$. A prime ideal $\mathfrak{p}$ of $R$ is in $\mathrm{D}(I)$ if and only if $\mathcal{S}(\mathfrak{p})$ is in $\mathtt{U}(R/I)$. This shows $\mathcal{S}(\mathrm{D}(I)) = \mathtt{U}(R/I) \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, and we get $\pi^{-1}\widetilde{\mathcal{S}}(\mathrm{D}(I)) = \pi^{-1}\pi\mathcal{S}(\mathrm{D}(I)) = \pi^{-1}\pi(\mathtt{U}(R/I) \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R))$. Let $\mathcal{P} \in \mathtt{Spc}\,\mathsf{D}^-(R)$ and $\mathcal{Q} \in {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$. One has $\pi(\mathcal{P}) = \pi(\mathcal{Q})$ if and only if $\operatorname{Supp}\mathcal{P} = \operatorname{Supp}\mathcal{Q}$, if and only if $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{P} = \mathcal{Q}$ since $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{Q} = \mathcal{Q}$ by Proposition 1.10. Hence, $\mathcal{P}$ is in $\pi^{-1}\pi(\mathtt{U}(R/I) \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R))$ if and only if $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{P}$ contains $R/I$ (note here that $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{P}$ is in ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ by Theorem 3.9), if and only if $\operatorname{Supp}\mathcal{P}$ contains $\mathrm{V}(I)$, if and only if $R/I$ belongs to $\mathcal{P}$ by Corollary 2.11. Thus, we obtain $\pi^{-1}\widetilde{\mathcal{S}}(\mathrm{D}(I)) = \mathtt{U}(R/I)$, which shows that $\widetilde{\mathcal{S}}(\mathrm{D}(I))$ is an open subset of

$\operatorname{Spc} \mathsf{D}^{-}(R)/\operatorname{Supp}$. Therefore, $\widetilde{\mathcal{S}}$ is an open map. This map is also closed since it is bijective. Combining the equality $\tilde{\mathfrak{s}} = \widetilde{\mathcal{S}}^{-1}$ from (3) and the openness of $\widetilde{\mathcal{S}}$, we observe that $\tilde{\mathfrak{s}}$ is a continuous map. □

The assertions of the above theorem naturally lead us to ask when the maps in the diagram in the theorem are homeomorphisms. We start by establishing a lemma.

**Lemma 4.6.** *The following are equivalent*:

(1) *the set* Spec $R$ *is finite*,

(2) *there are only finitely many specialization-closed subsets of* Spec $R$,

(3) *there are only finitely many closed subsets of* Spec $R$, *and*

(4) *every specialization-closed subset of* Spec $R$ *is closed.*

*Proof.* (1) $\Longrightarrow$ (2). If Spec $R$ is finite, then there are only finitely many subsets of Spec $R$.

(2) $\Longrightarrow$ (3). This implication follows from the fact that any closed subset is specialization-closed.

(3) $\Longrightarrow$ (4). Every specialization-closed subset is a union of closed subsets. This is a finite union by assumption, and hence, it is closed.

(4) $\Longrightarrow$ (1). Since Max $R$ is specialization-closed, it is closed by our assumption. Hence, Max $R$ possesses only finitely many minimal elements with respect to the inclusion relation, which means that it is a finite set. Therefore, the ring $R$ is semilocal. In particular, it has finite Krull dimension, say $d$.

Suppose that $R$ possesses infinitely many prime ideals. Then there exists an integer $0 \leqslant n \leqslant d$ such that the set $S$ of prime ideals of $R$ with height $n$ is infinite. Then the specialization-closed subset $W = \bigcup_{\mathfrak{p} \in S} \mathrm{V}(\mathfrak{p})$ is not closed because $S$ consists of the minimal elements of $W$, which is an infinite set. This provides a contradiction, and consequently, $R$ has only finitely many prime ideals. □

Now we can prove the following theorem, which answers the question stated just before the lemma.

**Theorem 4.7.** *Consider the following seven conditions*: (1) $\mathcal{S}$ *is continuous*, (2) $\mathcal{S}'$ *is homeomorphic*, (3) $\mathfrak{s}'$ *is homeomorphic*, (4) $\widetilde{\mathcal{S}}$ *is homeomorphic*, (5) $\tilde{\mathfrak{s}}$ *is homeomorphic*, (6) $\pi\theta$ *is homeomorphic*, *and* (7) Spec $R$ *is finite. Then the following implications hold*:

$$\begin{tikzcd}[arrows=Leftrightarrow]
(1) \arrow[r] & (2) \arrow[r] & (3) \arrow[r] & (5+6) \arrow[r] \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & (7) \\
& (4) \arrow[r] & (5) & & (6)
\end{tikzcd}$$

*Proof.* In this proof we tacitly use Theorem 4.5.

(2) $\iff$ (3). Note that $\mathcal{S}'$ and $\mathfrak{s}'$ are mutually inverse bijections. The equivalence follows from this.

(4) $\iff$ (5). As $\widetilde{\mathcal{S}}$, $\tilde{\mathfrak{s}}$ are mutually inverse bijections, we have the equivalence.

(7) $\Longrightarrow$ (2). For each $X \in \mathsf{D}^-(R)$ we have $\mathcal{S}'^{-1}(\mathtt{Spp}\,X \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathcal{S}(\mathfrak{p}) \in \mathtt{Spp}\,X\} = \operatorname{Supp} X$. As $\operatorname{Supp} X$ is specialization-closed, it is closed by Lemma 4.6. Hence, the map $\mathcal{S}'$ is continuous.

(2) $\Longrightarrow$ (1). This follows from the fact that $\mathcal{S}$ is the composition of the continuous maps $\mathcal{S}'$ and $\theta$.

(1) $\Longrightarrow$ (7). It is easy to observe that for any complex $X \in \mathsf{D}^-(R)$ one has

$$\mathcal{S}^{-1}(\mathtt{Spp}\,X) = \operatorname{Supp} X. \tag{4.7.1}$$

Since $\mathcal{S}$ is continuous, $\operatorname{Supp} X$ is closed in $\operatorname{Spec} R$ for all $X \in \mathsf{D}^-(R)$ by (4.7.1). Suppose that $\operatorname{Spec} R$ is an infinite set. Then by Lemma 4.6 there is a nonclosed specialization-closed subset $W$ of $\operatorname{Spec} R$. There are infinitely many minimal elements of $W$ with respect to the inclusion relation, and we can choose countably many pairwise distinct minimal elements $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \dots$ of $W$. Consider the complex $X = \bigoplus_{i=1}^{\infty}(R/\mathfrak{p}_i)[i] \in \mathsf{D}^-(R)$. Then $\operatorname{Supp} X = \bigcup_{i=1}^{\infty} \mathrm{V}(\mathfrak{p}_i)$ is not closed since it has infinitely many minimal elements. This contradiction shows that $\operatorname{Spec} R$ is a finite set.

(2) $\Longrightarrow$ (4 + 6). Since $\pi, \theta, \mathcal{S}'$ are all continuous, so is $\widetilde{\mathcal{S}} = \pi\theta\mathcal{S}'$. Combining this with the fact that $\widetilde{\mathcal{S}}$ is bijective and open, we see that $\widetilde{\mathcal{S}}$ is a homeomorphism. As $\mathcal{S}'$ is homeomorphic, so is $\pi\theta = \widetilde{\mathcal{S}}\mathcal{S}'^{-1}$.

(4 + 6) $\Longrightarrow$ (2). We have $\mathcal{S}' = (\pi\theta)^{-1}\widetilde{\mathcal{S}}$. Since $\pi\theta$ and $\widetilde{\mathcal{S}}$ are homeomorphisms, so is $\mathcal{S}'$. □

Next we consider the maximal and minimal elements of $\mathtt{Spc}\,\mathsf{D}^-(R)$ with respect to the inclusion relation.

**Definition 4.8.** Let $\mathcal{T}$ be an essentially small tensor triangulated category.

(1) A thick $\otimes$-ideal $\mathcal{M}$ of $\mathcal{T}$ is said to be *maximal* if $\mathcal{M} \neq \mathcal{T}$ and there is no thick $\otimes$-ideal $\mathcal{X}$ of $\mathcal{T}$ with $\mathcal{M} \subsetneq \mathcal{X} \subsetneq \mathcal{T}$. We denote the set of maximal thick $\otimes$-ideals of $\mathcal{T}$ by $\mathtt{Mx}\,\mathcal{T}$. According to [Balmer 2005, Proposition 2.3(c)], any maximal thick $\otimes$-ideal is prime, or in other words, it holds that $\mathtt{Mx}\,\mathcal{T} \subseteq \mathtt{Spc}\,\mathcal{T}$.

(2) A prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathcal{T}$ is said to be *minimal* if it is minimal in $\mathtt{Spc}\,\mathcal{T}$ with respect to the inclusion relation. We denote the set of minimal prime thick $\otimes$-ideals of $\mathcal{T}$ by $\mathtt{Mn}\,\mathcal{T}$.

By Proposition 4.2 the Balmer spectrum of $\mathsf{D}^-(R)$ is decomposed into the fibers by $\mathfrak{s} : \mathtt{Spc}\,\mathsf{D}^-(R) \to \operatorname{Spec} R$ as a set. Concerning the fibers of maximal ideals and minimal primes of $R$, we have the following:

**Proposition 4.9.** *Let* $\mathfrak{m} \in \operatorname{Max} R$ *and* $\mathfrak{p} \in \operatorname{Min} R$. *Then*

$$\min \mathfrak{s}^{-1}(\mathfrak{m}) \subseteq \mathtt{Mn}\,\mathsf{D}^-(R), \qquad \max \mathfrak{s}^{-1}(\mathfrak{p}) \subseteq \mathtt{Mx}\,\mathsf{D}^-(R).$$

*Proof.* Take $\mathcal{P} \in \min \mathfrak{s}^{-1}(\mathfrak{m})$, and let $\mathcal{Q}$ be a prime thick $\otimes$-ideal contained in $\mathcal{P}$. Then $\mathfrak{m} = \mathfrak{s}(\mathcal{P}) \subseteq \mathfrak{s}(\mathcal{Q})$ by Theorem 3.9. Since $\mathfrak{m}$ is a maximal ideal, we get $\mathfrak{m} = \mathfrak{s}(\mathcal{P}) = \mathfrak{s}(\mathcal{Q})$, and $\mathcal{Q} \in \mathfrak{s}^{-1}(\mathfrak{m})$. The minimality of $\mathcal{P}$ implies $\mathcal{P} = \mathcal{Q}$. Thus, the first inclusion follows. The second inclusion is obtained similarly. $\square$

To prove our next theorem, we establish a lemma and a proposition. Recall that a topological space is called $\mathrm{T}_1$-*space* if every one-point subset is closed.

**Lemma 4.10.** (1) *The subspaces* $\operatorname{Max} R$, $\operatorname{Min} R$ *of* $\operatorname{Spec} R$ *are* $\mathrm{T}_1$-*spaces, so every finite subset is closed.*

(2) *Let* $\mathcal{T}$ *be an essentially small* $\otimes$-*triangulated category. The subspaces* $\mathtt{Mx}\,\mathcal{T}$, $\mathtt{Mn}\,\mathcal{T}$ *of* $\mathtt{Spc}\,\mathcal{T}$ *are* $\mathrm{T}_1$-*spaces, so every finite subset is closed.*

*Proof.* (1) Let $A$ be either $\operatorname{Max} R$ or $\operatorname{Min} R$. For each $\mathfrak{p} \in A$ the closure of $\{\mathfrak{p}\}$ in $A$ is $\mathrm{V}(\mathfrak{p}) \cap A$, which coincides with $\{\mathfrak{p}\}$. Hence, $A$ is a $\mathrm{T}_1$-space.

(2) Let $B$ be either $\mathtt{Mx}\,\mathcal{T}$ or $\mathtt{Mn}\,\mathcal{T}$. For each $\mathcal{P} \in B$ the closure of $\{\mathcal{P}\}$ in $B$ is $\{\mathcal{Q} \in B \mid \mathcal{Q} \subseteq \mathcal{P}\}$ by [Balmer 2005, Proposition 2.9], which coincides with $\{\mathcal{P}\}$. Hence, $B$ is a $\mathrm{T}_1$-space. $\square$

**Proposition 4.11.** *For each complex* $X \in \mathsf{D}^-(R)$ *it holds that*

$$\operatorname{Supp} X = \operatorname{Spec} R \iff \mathsf{thick}^{\otimes} X = \mathsf{D}^-(R) \iff \mathsf{Spp}\, X = \mathsf{Spc}\,\mathsf{D}^-(R).$$

*Proof.* The second equivalence follows from [Balmer 2005, Corollary 2.5]. Let us prove the first equivalence. Proposition 1.10(2) implies $\operatorname{Supp} X = \operatorname{Supp}(\mathsf{thick}^{\otimes} X)$, which shows ($\Longleftarrow$). As for ($\Longrightarrow$), for every $M \in \mathsf{D}^-(R)$ we have $\mathrm{V}(\operatorname{Ann} M) \subseteq \operatorname{Spec} R = \operatorname{Supp} X$, by which and Proposition 2.9 we get $M \in \mathsf{thick}^{\otimes} X$. $\square$

Now we can prove the following theorem. This especially says that $\mathsf{D}^-(R)$ is "semilocal" in the sense that $\mathsf{D}^-(R)$ admits only a finite number of maximal thick tensor ideals. If $R$ is an integral domain, then $\mathsf{D}^-(R)$ is "local" in the sense that $\mathsf{D}^-(R)$ has a unique maximal thick tensor ideal.

**Theorem 4.12.** *The restriction of* $\mathcal{S}$ *to* $\operatorname{Min} R$ *induces a homeomorphism*

$$\mathcal{S}|_{\operatorname{Min} R} : \operatorname{Min} R \xrightarrow{\cong} \mathtt{Mx}\,\mathsf{D}^-(R).$$

*Proof.* Let us show that there is an equality

$$\mathtt{Mx}\,\mathsf{D}^{-}(R) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min} R\}. \tag{4.12.1}$$

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals of $R$.

Let $\mathcal{M}$ be a maximal thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Suppose that $\mathcal{M}$ is not contained in $\mathcal{S}(\mathfrak{p}_i)$ for any $1 \leqslant i \leqslant n$. Then for each $i$ we find an object $M_i \in \mathcal{M}$ such that $(M_i)_{\mathfrak{p}_i}$ is nonzero. Set $M = M_1 \oplus \cdots \oplus M_n$. This object belongs to $\mathcal{M}$, and $M_{\mathfrak{p}_i}$ is nonzero for all $1 \leqslant i \leqslant n$. Hence, $\operatorname{Supp} M$ contains all the $\mathfrak{p}_i$, and we get $\operatorname{Supp} M = \operatorname{Spec} R$ because $\operatorname{Supp} M$ is specialization-closed. Proposition 4.11 yields $\operatorname{thick}^{\otimes} M = \mathsf{D}^{-}(R)$, and hence, we have $\mathcal{M} = \mathsf{D}^{-}(R)$, which contradicts the definition of a maximal thick $\otimes$-ideal. Thus, $\mathcal{M}$ is contained in $\mathcal{S}(\mathfrak{p}_l)$ for some $1 \leqslant l \leqslant n$. The maximality of $\mathcal{M}$ implies that $\mathcal{M} = \mathcal{S}(\mathfrak{p}_l)$.

Fix an integer $1 \leqslant i \leqslant n$. By [Balmer 2005, Proposition 2.3(b)] there exists a maximal thick $\otimes$-ideal $\mathcal{M}_i$ of $\mathsf{D}^{-}(R)$ that contains $\mathcal{S}(\mathfrak{p}_i)$. Applying the above argument to $\mathcal{M}_i$, we see that $\mathcal{M}_i$ coincides with $\mathcal{S}(\mathfrak{p}_j)$ for some $1 \leqslant j \leqslant n$. Hence, $\mathcal{S}(\mathfrak{p}_i)$ is contained in $\mathcal{S}(\mathfrak{p}_j)$, and Theorem 3.9 shows that $\mathfrak{p}_i$ contains $\mathfrak{p}_j$. The fact that $\mathfrak{p}_i, \mathfrak{p}_j$ are minimal prime ideals of $R$ forces us to have $i = j$. Therefore, we obtain $\mathcal{M}_i = \mathcal{S}(\mathfrak{p}_i)$, which especially says that $\mathcal{S}(\mathfrak{p}_i)$ is a maximal thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$.

Thus, we get the equality (4.12.1). This shows that the restriction of the map $\mathcal{S} : \operatorname{Spec} R \to \mathtt{Spc}\,\mathsf{D}^{-}(R)$ to $\operatorname{Min} R$ gives rise to a surjection $\operatorname{Min} R \to \mathtt{Mx}\,\mathsf{D}^{-}(R)$. As Theorem 3.9 says that $\mathcal{S}$ is an injection, the map $\mathcal{S}|_{\operatorname{Min} R}$ is a bijection. By Lemma 4.10 we see that $\mathcal{S}|_{\operatorname{Min} R}$ is a homeomorphism. □

Theorem 4.12 yields the following result concerning the structure of the Balmer spectrum of $\mathsf{D}^{-}(R)$.

**Corollary 4.13.** (1) *There are equalities*

$$\mathtt{Spc}\,\mathsf{D}^{-}(R) = \bigcup_{\mathfrak{p} \in \operatorname{Spec} R} \overline{\{\mathcal{S}(\mathfrak{p})\}} = \bigcup_{\mathfrak{p} \in \operatorname{Min} R} \overline{\{\mathcal{S}(\mathfrak{p})\}}.$$

(2) *The topological space* $\mathtt{Spc}\,\mathsf{D}^{-}(R)$ *is irreducible if and only if so is* $\operatorname{Spec} R$.

*Proof.* (1) The inclusions $\mathtt{Spc}\,\mathsf{D}^{-}(R) \supseteq \bigcup_{\mathfrak{p} \in \operatorname{Spec} R} \overline{\{\mathcal{S}(\mathfrak{p})\}} \supseteq \bigcup_{\mathfrak{p} \in \operatorname{Min} R} \overline{\{\mathcal{S}(\mathfrak{p})\}}$ clearly hold. Pick a prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^{-}(R)$. Then one finds a maximal thick $\otimes$-ideal $\mathcal{M}$ containing $\mathcal{P}$ by [Balmer 2005, Proposition 2.3(b)]. Theorem 4.12 implies that $\mathcal{M} = \mathcal{S}(\mathfrak{p})$ for some minimal prime ideal $\mathfrak{p}$ of $R$, and it follows from [Balmer 2005, Proposition 2.9] that $\mathcal{P}$ belongs to the closure $\overline{\{\mathcal{S}(\mathfrak{p})\}}$.

(2) First of all, $\mathtt{Spc}\,\mathsf{D}^{-}(R)$ is irreducible if and only if $\mathtt{Spc}\,\mathsf{D}^{-}(R) = \overline{\{\mathcal{P}\}}$ for some $\mathcal{P} \in \mathtt{Spc}\,\mathsf{D}^{-}(R)$. In fact, the "if" part is obvious, while the "only if" part follows from [Balmer 2005, Proposition 2.18]. By [Balmer 2005, Proposition 2.9], the set $\overline{\{\mathcal{P}\}}$ consists of the prime thick $\otimes$-ideals contained in $\mathcal{P}$. Hence, $\mathtt{Spc}\,\mathsf{D}^{-}(R) = \overline{\{\mathcal{P}\}}$ for some $\mathcal{P} \in \mathtt{Spc}\,\mathsf{D}^{-}(R)$ if and only if $\mathsf{D}^{-}(R)$ has a unique maximal element, which

is equivalent to Spec $R$ having a unique minimal element by Theorem 4.12. This is equivalent to saying that Spec $R$ is irreducible. $\square$

The following theorem is opposite to Theorem 4.12. The third assertion says that if $R$ is local, then $\mathsf{D}^-(R)$ is an "integral domain" in the sense that $\mathbf{0}$ is a (unique) minimal prime thick tensor ideal of $\mathsf{D}^-(R)$.

**Theorem 4.14.** (1) *For every maximal ideal* $\mathfrak{m}$ *of* $R$, *the subcategory* $\mathcal{S}(\mathfrak{m})$ *is a minimal prime thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$, *or in other words*, *the restriction of* $\mathcal{S}$ *to* Max $R$ *induces an injection*

$$\mathcal{S}|_{\operatorname{Max} R} : \operatorname{Max} R \hookrightarrow \mathtt{Mn}\,\mathsf{D}^-(R). \tag{4.14.1}$$

(2) *The ring* $R$ *is semilocal if and only if* $\mathsf{D}^-(R)$ *has only finitely many minimal prime thick* $\otimes$*-ideals. When this is the case*, *the map* (4.14.1) *is a homeomorphism.*

(3) *If* $(R, \mathfrak{m})$ *is a local ring*, *then* $\mathcal{S}(\mathfrak{m}) = \mathbf{0}$ *is a unique minimal prime thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$.

*Proof.* (1) Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ contained in $\mathcal{S}(\mathfrak{m})$. Take any object $X \in \mathcal{S}(\mathfrak{m})$. Then $\operatorname{Supp}(X \otimes_R^{\mathbf{L}} R/\mathfrak{m}) = \operatorname{Supp} X \cap \{\mathfrak{m}\} = \varnothing$ by Lemma 1.9(4). Remark 1.8 shows $X \otimes_R^{\mathbf{L}} R/\mathfrak{m} = 0$, which belongs to $\mathcal{P}$. As $\mathcal{P}$ is prime, either $X$ or $R/\mathfrak{m}$ is in $\mathcal{P}$. Since $\mathcal{S}(\mathfrak{m})$ does not contain $R/\mathfrak{m}$, neither does $\mathcal{P}$. Therefore, $X$ must be in $\mathcal{P}$, and we obtain $\mathcal{P} = \mathcal{S}(\mathfrak{m})$. This shows that the prime thick $\otimes$-ideal $\mathcal{S}(\mathfrak{m})$ is minimal. Thus, $\mathcal{S}$ induces a map $\operatorname{Max} R \to \mathtt{Mn}\,\mathsf{D}^-(R)$. The injectivity follows from Theorem 3.9.

(2) The first assertion of the theorem implies the "if" part, and it suffices to show that if $R$ is semilocal, then (4.14.1) is a homeomorphism. Let us first prove the surjectivity of the map (4.14.1). Take a minimal prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$. What we want is that there is a maximal ideal $\mathfrak{m}$ of $R$ such that $\mathcal{P} = \mathcal{S}(\mathfrak{m})$.

Suppose that $\mathcal{P}$ does not contain $\mathcal{S}(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max} R$. Write $\operatorname{Max} R = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$. For each $1 \leqslant i \leqslant t$ we find an object $X_i$ of $\mathsf{D}^-(R)$ with $X_i \in \mathcal{S}(\mathfrak{m}_i)$ and $X_i \notin \mathcal{P}$. Setting $X = X_1 \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} X_t$, for each $i$ we have $X_{\mathfrak{m}_i} = X_1 \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} (X_i)_{\mathfrak{m}_i} \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} X_t = 0$. Hence, $X_{\mathfrak{m}} = 0$ for all $\mathfrak{m} \in \operatorname{Max} R$, which implies $X_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. This means that $\operatorname{Supp} X$ is empty, and Remark 1.8 yields $X = 0$. In particular, $X = X_1 \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} X_t$ is in $\mathcal{P}$. As $\mathcal{P}$ is prime, it contains some $X_u$, which is a contradiction.

Consequently, $\mathcal{P}$ must contain $\mathcal{S}(\mathfrak{m})$ for some $\mathfrak{m} \in \operatorname{Max} R$. The minimality of $\mathcal{P}$ shows that $\mathcal{P} = \mathcal{S}(\mathfrak{m})$. We conclude that the map (4.14.1) is surjective, whence it is bijective. Since the set Max $R$ is finite, so is $\mathtt{Mn}\,\mathsf{D}^-(R)$. Applying Lemma 4.10, we observe that (4.14.1) is a homeomorphism.

(3) As $R$ is a local ring with maximal ideal $\mathfrak{m}$, the equality $\mathcal{S}(\mathfrak{m}) = \mathbf{0}$ holds, which especially says that $\mathbf{0}$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ by Proposition 3.4. If $\mathcal{P}$ is a minimal prime thick $\otimes$-ideal, then $\mathcal{P}$ contains $\mathbf{0}$, and the minimality of $\mathcal{P}$ implies $\mathcal{P} = \mathbf{0}$. Thus, $\mathbf{0}$ is a unique minimal prime thick $\otimes$-ideal. □

**Question 4.15.** Is the map (4.14.1) bijective even if $R$ is not semilocal?

Recall that a topological space $X$ is called *noetherian* if any descending chain of closed subsets of $X$ stabilizes. Applying the above two theorems to the artinian case gives rise to the following result.

**Corollary 4.16.** *Let $R$ be an artinian ring. Then the map* $\mathcal{S} : \operatorname{Spec} R \to \operatorname{\mathsf{Spc}} \mathsf{D}^-(R)$ *is a homeomorphism. Hence, the topological space* $\operatorname{\mathsf{Spc}} \mathsf{D}^-(R)$ *is noetherian, and one has* $\dim(\operatorname{\mathsf{Spc}} \mathsf{D}^-(R)) = \dim R = 0 < \infty$.

*Proof.* Since $\operatorname{Spec} R = \operatorname{Min} R = \operatorname{Max} R$, the assertion is deduced from Theorems 4.12 and 4.14(2). □

From here we consider when $\mathsf{D}^-(R)$ is a local tensor triangulated category. Let us recall the definition.

**Definition 4.17.** (1) A topological space $X$ is called *local* if for any open cover $X = \bigcup_{i \in I} U_i$ of $X$ there exists $t \in I$ such that $X = U_t$. In particular, any local topological space is quasicompact.

(2) An essentially small tensor triangulated category $\mathcal{T}$ is called *local* if $\operatorname{\mathsf{Spc}} \mathcal{T}$ is a local topological space.

**Remark 4.18.** It is clear that the topological space $\operatorname{Spec} R$ is local if and only if the ring $R$ is local.

For an essentially small $\otimes$-triangulated category $\mathcal{T}$ the following are equivalent [Balmer 2010a, Proposition 4.2]:

(i) $\mathcal{T}$ is local,

(ii) $\mathcal{T}$ has a unique minimal prime thick $\otimes$-ideal, and

(iii) the radical thick $\otimes$-ideal $\sqrt{\mathbf{0}}$ of $\mathcal{T}$ is prime.

If moreover $\mathcal{T}$ is *rigid*, then the above three conditions are equivalent to

(iv) the zero subcategory $\mathbf{0}$ of $\mathcal{T}$ is a prime thick $\otimes$-ideal.

Also, it follows from [Balmer 2010a, Example 4.4] that $\mathsf{K}^{\mathsf{b}}(\operatorname{\mathsf{proj}} R)$ is local if and only if so is $R$.

The following result says that the same statements hold for $\mathsf{D}^-(R)$. Also, we emphasize that it contains the equivalent condition (4), even though $\mathsf{D}^-(R)$ is not rigid; see Remark 1.3.

**Corollary 4.19.** *The following are equivalent*:

(1) *the* $\otimes$*-triangulated category* $\mathsf{D}^-(R)$ *is local,*

(2) *there is a unique minimal thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$*,*

(3) *the radical thick* $\otimes$*-ideal* $\sqrt{\mathbf{0}}$ *of* $\mathsf{D}^-(R)$ *is prime,*

(4) *the zero subcategory* $\mathbf{0}$ *of* $\mathsf{D}^-(R)$ *is a prime thick* $\otimes$*-ideal, and*

(5) *the ring* $R$ *is local.*

*Proof.* Combining Theorem 4.14 with the result given just before the corollary, we observe that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (5) $\Longrightarrow$ (4) holds. If $\mathbf{0}$ is prime, then it is easy to see that $\sqrt{\mathbf{0}} = \mathbf{0}$. Thus, (4) implies (3). $\square$

One can indeed obtain more precise information on the structure of $\mathtt{Spc}\,\mathsf{D}^-(R)$ than Corollary 4.19:

**Proposition 4.20.** *One has*

$$\mathtt{Spc}\,\mathsf{D}^-(R) = \begin{cases} \mathtt{U}(R/\mathfrak{m}) \sqcup \{\mathbf{0}\} & \text{if } (R,\mathfrak{m}) \text{ is local,} \\ \bigcup_{\mathfrak{m}\in\operatorname{Max} R} \mathtt{U}(R/\mathfrak{m}) & \text{if } R \text{ is nonlocal.} \end{cases}$$

*If* $\mathfrak{m}, \mathfrak{n}$ *are distinct maximal ideals of* $R$*, then* $\mathtt{Spc}\,\mathsf{D}^-(R) = \mathtt{U}(R/\mathfrak{m}) \cup \mathtt{U}(R/\mathfrak{n})$.

*Proof.* Suppose that $(R, \mathfrak{m}, k)$ is a local ring. Corollary 4.19 implies that $\mathbf{0}$ is prime, and $\mathtt{Spc}\,\mathsf{D}^-(R)$ contains $\mathtt{U}(k) \cup \{\mathbf{0}\}$. Let $\mathcal{P}$ be a nonzero prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Then there exists an object $X \neq 0$ in $\mathcal{P}$. By Remark 1.8 the support of $X$ is nonempty and specialization-closed, whence contains $\mathfrak{m}$. Using Lemma 1.9(4), we have $\operatorname{Supp}(X \otimes_R^{\boldsymbol{L}} k) = \operatorname{Supp} X \cap \operatorname{Supp} k = \{\mathfrak{m}\} \neq \varnothing$. Hence, $X \otimes_R^{\boldsymbol{L}} k$ is nonzero by Remark 1.8 again. Since $X \otimes_R^{\boldsymbol{L}} k$ is isomorphic to a direct sum of shifts of $k$-vector spaces, it contains $k[n]$ as a direct summand for some $n \in \mathbb{Z}$. As $X \otimes_R^{\boldsymbol{L}} k$ is in $\mathcal{P}$, so is $k$. Therefore, $\mathcal{P}$ is in $\mathtt{U}(k)$, and we obtain $\mathtt{Spc}\,\mathsf{D}^-(R) = \mathtt{U}(k) \cup \{\mathbf{0}\}$. It is obvious that $\mathtt{U}(k) \cap \{\mathbf{0}\} = \varnothing$. We conclude that $\mathtt{Spc}\,\mathsf{D}^-(R) = \mathtt{U}(k) \sqcup \{\mathbf{0}\}$.

Now, let $\mathfrak{m}$ and $\mathfrak{n}$ be distinct maximal ideals of $R$. Applying Lemma 1.9(4), we have $\operatorname{Supp}(R/\mathfrak{m} \otimes_R^{\boldsymbol{L}} R/\mathfrak{n}) = \{\mathfrak{m}\} \cap \{\mathfrak{n}\} = \varnothing$, and hence, $R/\mathfrak{m} \otimes_R^{\boldsymbol{L}} R/\mathfrak{n} = 0$ by Remark 1.8. Therefore, we obtain $\mathtt{U}(R/\mathfrak{m}) \cup \mathtt{U}(R/\mathfrak{n}) = \mathtt{U}(R/\mathfrak{m} \otimes_R^{\boldsymbol{L}} R/\mathfrak{n}) = \mathtt{U}(0) = \mathtt{Spc}\,\mathsf{D}^-(R)$, where the first equality follows from [Balmer 2005, Lemma 2.6(e)]. Thus, the last assertion of the proposition follows, which shows the first assertion in the nonlocal case. $\square$

So far we have investigated the irreducible and local properties of $\mathtt{Spc}\,\mathsf{D}^-(R)$. In general, there is no implication between the local and irreducible properties of $\mathtt{Spc}\,\mathsf{D}^-(R)$:

**Remark 4.21.** If $R$ is a local ring possessing at least two minimal prime ideals, then $\mathtt{Spc}\,\mathsf{D}^-(R)$ is local by Corollary 4.19, but not irreducible by Corollary 4.13(2). Similarly, if $R$ is a nonlocal ring with unique minimal prime ideal, then $\mathtt{Spc}\,\mathsf{D}^-(R)$ is irreducible but not local.

## 5. Relationships among thick tensor ideals and specialization-closed subsets

This section compares compact, tame, and radical thick tensor ideals of $\mathsf{D}^-(R)$, relating them to specialization-closed subsets of Spec $R$ and ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$ and Thomason subsets of $\mathsf{Spc}\,\mathsf{D}^-(R)$. We start with some notation.

**Definition 5.1.** (1) Let $\mathcal{T}$ be a tensor triangulated category. Let $\mathbb{P}$ be a property of thick $\otimes$-ideals of $\mathcal{T}$. For a subcategory $\mathcal{X}$ of $\mathcal{C}$ we denote by $\mathcal{X}^{\mathbb{P}}$ the $\mathbb{P}$-*closure* of $\mathcal{X}$, that is to say, the smallest thick $\otimes$-ideal of $\mathcal{T}$ which contains $\mathcal{X}$ and satisfies the property $\mathbb{P}$. Furthermore, we denote by $\mathcal{X}_{\mathbb{P}}$ the $\mathbb{P}$-*interior* of $\mathcal{X}$, namely the largest thick $\otimes$-ideal of $\mathcal{T}$ which is contained in $\mathcal{X}$ and satisfies the property $\mathbb{P}$. We define these only when they exist.

(2) Let $X$ be a topological space. Let $\mathbb{P}$ be a property of subsets of $X$. For a subset $A$ of $X$ we denote by $A^{\mathbb{P}}$ the $\mathbb{P}$-*closure* of $A$, namely, the smallest subset of $X$ that contains $A$ and satisfies $\mathbb{P}$. Furthermore, we denote by $A_{\mathbb{P}}$ the $\mathbb{P}$-*interior* of $A$ — that is, the largest subset of $X$ that is contained in $A$ and satisfies $\mathbb{P}$. We define these only when they exist.

Here is a list of properties $\mathbb{P}$ as in the above definition which we consider:

$$\mathsf{rad} = \text{radical}, \quad \mathsf{tame} = \text{tame}, \quad \mathsf{cpt} = \text{compact}, \quad \mathsf{spcl} = \text{specialization-closed}.$$

**Notation 5.2.** We denote by **Rad**, **Tame**, and **Cpt** the sets of radical, tame, and compact thick $\otimes$-ideals of $\mathsf{D}^-(R)$, respectively. Also, **Spcl**(Spec) and **Spcl**(${}^{\mathsf{t}}\mathsf{Spc}$) stand for the sets of specialization-closed subsets of the topological spaces Spec $R$ and ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$, respectively.

Our first purpose in this section is to give a certain commutative diagram of bijections. To achieve this, we prepare several propositions. We state here two propositions. The first one is shown by using Proposition 1.10, while the second one is nothing but Theorem 2.19.

**Proposition 5.3.** *There is a one-to-one correspondence* $\operatorname{Supp} : \mathbf{Tame} \rightleftarrows \mathbf{Spcl}(\mathrm{Spec}) : \operatorname{Supp}^{-1}$.

**Proposition 5.4.** *There is a one-to-one correspondence* $\operatorname{Supp} : \mathbf{Cpt} \rightleftarrows \mathbf{Spcl}(\mathrm{Spec}) : \langle \cdot \rangle$.

**Notation 5.5.** For an object $M$ of $\mathsf{D}^-(R)$ we denote by $\mathsf{Sp}\,M$ the set of tame prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ not containing $M$; i.e., $\mathsf{Sp}\,M = \mathsf{Spp}\,M \cap {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$. For a subcategory $\mathcal{X}$ of $\mathsf{D}^-(R)$ we set $\mathsf{Sp}\,\mathcal{X} = \bigcup_{M\in\mathcal{X}} \mathsf{Sp}\,M$. For a subset $A$ of $\mathsf{Spc}\,\mathsf{D}^-(R)$ we denote by $\mathsf{Sp}^{-1} A$ the subcategory of $\mathsf{D}^-(R)$ consisting of objects $M$ such that $\mathsf{Sp}\,M$ is contained in $A$.

The second following assertion is a variant of [Balmer 2005, Lemma 4.8].

**Lemma 5.6.** (1) *For a subcategory* $\mathscr{X}$ *of* $\mathsf{D}^-(R)$, *the subset* $\mathtt{Sp}\,\mathscr{X}$ *of* ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ *is specialization-closed.*

(2) *For a subset* $A$ *of* ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ *one has* $\mathtt{Sp}^{-1} A = \bigcap_{\mathscr{P}\in A^{\complement}} \mathscr{P}$, *where* $A^{\complement} = {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R) \setminus A$.

(3) *Let* $\{\mathscr{X}_\lambda\}_{\lambda\in\Lambda}$ *be a collection of tame thick* $\otimes$*-ideals of* $\mathsf{D}^-(R)$. *Then the intersection* $\bigcap_{\lambda\in\Lambda}\mathscr{X}_\lambda$ *is also a tame thick* $\otimes$*-ideal of* $\mathsf{D}^-(R)$.

*Proof.* (1) We have $\mathtt{Sp}\,\mathscr{X} = \bigcup_{X\in\mathscr{X}} \mathtt{Sp}\,X$, and $\mathtt{Sp}\,X = \mathtt{Spp}\,X \cap {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ is closed in ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ since $\mathtt{Spp}\,X$ is closed in $\mathtt{Spc}\,\mathsf{D}^-(R)$. Therefore, $\mathtt{Sp}\,\mathscr{X}$ is specialization-closed in ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$.

(2) An object $X$ of $\mathsf{D}^-(R)$ belongs to $\mathtt{Sp}^{-1} A$ if and only if $\mathtt{Sp}\,X$ is contained in $A$, if and only if $A^{\complement}$ is contained in $(\mathtt{Sp}\,X)^{\complement} = \{\mathscr{P} \in {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R) \mid X \in \mathscr{P}\}$, if and only if $X$ belongs to $\bigcap_{\mathscr{P}\in A^{\complement}} \mathscr{P}$.

(3) For each $\lambda\in\Lambda$ there is a subset $S_\lambda$ of $\operatorname{Spec} R$ such that $\mathscr{X}_\lambda = \operatorname{Supp}^{-1} S_\lambda$. Then it is clear that the equality $\bigcap_{\lambda\in\Lambda}\mathscr{X}_\lambda = \operatorname{Supp}^{-1}\big(\bigcap_{\lambda\in\Lambda} S_\lambda\big)$ holds, which shows the assertion. □

Using the above lemma, we obtain a bijection induced by $\mathtt{Sp}$.

**Proposition 5.7.** *There is a one-to-one correspondence* $\mathtt{Sp} : \mathbf{Tame} \rightleftarrows \mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spc}) : \mathtt{Sp}^{-1}$.

*Proof.* Fix a tame thick $\otimes$-ideal $\mathscr{X}$ of $\mathsf{D}^-(R)$ and a specialization-closed subset $U$ of ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$. Lemma 5.6(1) implies that $\mathtt{Sp}\,\mathscr{X}$ is specialization-closed in ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, that is, $\mathtt{Sp}\,\mathscr{X} \in \mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spc})$. Lemma 5.6(2) implies that $\mathtt{Sp}^{-1} U = \bigcap_{\mathscr{P}\in U^{\complement}} \mathscr{P}$, and each $\mathscr{P} \in U^{\complement}$ is a tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Hence, $\mathtt{Sp}^{-1} U$ is also a tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$ by Lemma 5.6(3); namely, $\mathtt{Sp}^{-1} U \in \mathbf{Tame}$.

Let us show that $\mathtt{Sp}(\mathtt{Sp}^{-1} U) = U$. It is evident that $\mathtt{Sp}(\mathtt{Sp}^{-1} U)$ is contained in $U$. Pick any $\mathscr{P} \in U$. Corollary 3.14 says $\mathscr{P} = \mathscr{S}(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of $R$. Since $U$ is specialization-closed in ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, the closure $C$ of $\mathscr{S}(\mathfrak{p})$ in ${}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$ is contained in $U$. Using [Balmer 2005, Proposition 2.9], we see that $C$ consists of the prime thick $\otimes$-ideals of the form $\mathscr{S}(\mathfrak{q})$, where $\mathfrak{q}$ is a prime ideal of $R$ with $\mathscr{S}(\mathfrak{q}) \subseteq \mathscr{S}(\mathfrak{p})$. In view of Theorem 3.9, we have $C = \{\mathscr{S}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{p})\}$, and it is easy to observe that this coincides with $\mathtt{Sp}(R/\mathfrak{p})$. Hence, $R/\mathfrak{p}$ is in $\mathtt{Sp}^{-1} U$, and $\mathscr{P} = \mathscr{S}(\mathfrak{p})$ belongs to $\mathtt{Sp}(\mathtt{Sp}^{-1} U)$. Now we obtain $\mathtt{Sp}(\mathtt{Sp}^{-1} U) = U$.

It remains to prove that $\mathtt{Sp}^{-1}(\mathtt{Sp}\,\mathscr{X}) = \mathscr{X}$. We have $\mathtt{Sp}^{-1}(\mathtt{Sp}\,\mathscr{X}) = \bigcap_{\mathscr{P}\in(\mathtt{Sp}\,\mathscr{X})^{\complement}} \mathscr{P}$ by Lemma 5.6(2). Fix a prime thick $\otimes$-ideal $\mathscr{P}$ of $\mathsf{D}^-(R)$. Then $\mathscr{P}$ is in $(\mathtt{Sp}\,\mathscr{X})^{\complement}$ if and only if $\mathscr{P}$ is tame and $\mathscr{P}$ is not in $\mathtt{Sp}\,\mathscr{X}$. The former statement is equivalent to saying that $\mathscr{P} = \mathscr{S}(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$ by Corollary 3.14, while the latter is equivalent to saying that $\mathscr{X}$ is contained in $\mathscr{P}$. Hence, $\mathtt{Sp}^{-1}(\mathtt{Sp}\,\mathscr{X}) = \bigcap_{\mathfrak{p}\in\operatorname{Spec} R,\, \mathscr{X}\subseteq\mathscr{S}(\mathfrak{p})} \mathscr{S}(\mathfrak{p})$. Thus, an object $Y$ of $\mathsf{D}^-(R)$ belongs to $\mathtt{Sp}^{-1}(\mathtt{Sp}\,\mathscr{X})$ if and only if $Y$ belongs to $\mathscr{S}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathscr{X} \subseteq \mathscr{S}(\mathfrak{p})$, if and only if $Y_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with

$\mathscr{X}_\mathfrak{p} = 0$, if and only if $\operatorname{Supp} Y$ is contained in $\operatorname{Supp} \mathscr{X}$, if and only if $Y$ belongs to $\mathscr{X}$ by Proposition 5.3. $\square$

Here we consider describing rad-closures, tame-closures, and cpt-interiors, and their supports.

**Lemma 5.8.** *Let $\mathscr{X}$ be a subcategory of $\mathsf{D}^-(R)$, and let $\mathscr{Y}$ be a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. One has*

(1) $(\mathsf{thick}^{\otimes} \mathscr{X})_{\mathsf{cpt}} = \langle \operatorname{Supp} \mathscr{X} \rangle$, $\mathscr{X}^{\mathsf{rad}} = \sqrt{\mathsf{thick}^{\otimes} \mathscr{X}}$, $\mathscr{X}^{\mathsf{tame}} = \operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{X}$ *and*

(2) $\mathscr{Y}_{\mathsf{cpt}} \subseteq \mathscr{Y} \subseteq \mathscr{Y}^{\mathsf{rad}} \subseteq \mathscr{Y}^{\mathsf{tame}}$, $\operatorname{Supp}(\mathscr{Y}_{\mathsf{cpt}}) = \operatorname{Supp} \mathscr{Y} = \operatorname{Supp}(\mathscr{Y}^{\mathsf{rad}}) = \operatorname{Supp}(\mathscr{Y}^{\mathsf{tame}})$.

*Proof.* (1) It follows from [Balmer 2005, Lemma 4.2] and Remark 3.13 that $\sqrt{\mathsf{thick}^{\otimes} \mathscr{X}}$ and $\operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{X}$ are thick $\otimes$-ideals of $\mathsf{D}^-(R)$, respectively. It is clear that $\sqrt{\mathsf{thick}^{\otimes} \mathscr{X}}$ and $\operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{X}$ are radical and tame, respectively, and contain $\mathscr{X}$. If $\mathscr{C}$ is a radical or tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$ containing $\mathscr{X}$, then we have $\sqrt{\mathsf{thick}^{\otimes} \mathscr{X}} \subseteq \sqrt{\mathsf{thick}^{\otimes} \mathscr{C}} = \sqrt{\mathscr{C}} = \mathscr{C}$ or $\operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{X} \subseteq \operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{C} = \mathscr{C}$ by Proposition 5.3, respectively. Thus, we obtain the two equalities $\mathscr{X}^{\mathsf{rad}} = \sqrt{\mathsf{thick}^{\otimes} \mathscr{X}}$ and $\mathscr{X}^{\mathsf{tame}} = \operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{X}$. It remains to show the equality $(\mathsf{thick}^{\otimes} \mathscr{X})_{\mathsf{cpt}} = \langle \operatorname{Supp} \mathscr{X} \rangle$. Clearly, $\langle \operatorname{Supp} \mathscr{X} \rangle$ is a compact thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Applying Corollary 2.11, we observe that $\langle \operatorname{Supp} \mathscr{X} \rangle$ is contained in $\mathsf{thick}^{\otimes} \mathscr{X}$. Let $\mathscr{C}$ be a compact thick $\otimes$-ideal of $\mathsf{D}^-(R)$ contained in $\mathsf{thick}^{\otimes} \mathscr{X}$. Then it follows from Proposition 5.4 that $\mathscr{C} = \langle \operatorname{Supp} \mathscr{C} \rangle$, which is contained in $\langle \operatorname{Supp}(\mathsf{thick}^{\otimes} \mathscr{X}) \rangle = \langle \operatorname{Supp} \mathscr{X} \rangle$ by Proposition 1.10(2). We now conclude $(\mathsf{thick}^{\otimes} \mathscr{X})_{\mathsf{cpt}} = \langle \operatorname{Supp} \mathscr{X} \rangle$.

(2) Fix a prime ideal $\mathfrak{p}$ of $R$. Proposition 3.4 says that $\mathscr{S}(\mathfrak{p})$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$, whence it is radical. Therefore, $\mathscr{Y}_\mathfrak{p} = 0$ if and only if $(\sqrt{\mathscr{Y}})_\mathfrak{p} = 0$. This shows $\operatorname{Supp}(\sqrt{\mathscr{Y}}) = \operatorname{Supp} \mathscr{Y}$. Hence, $\sqrt{\mathscr{Y}}$ is contained in $\operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{Y}$, meaning that $\mathscr{Y}^{\mathsf{rad}}$ is contained in $\mathscr{Y}^{\mathsf{tame}}$ by (1). Thus, we get the inclusions $\mathscr{Y}_{\mathsf{cpt}} \subseteq \mathscr{Y} \subseteq \mathscr{Y}^{\mathsf{rad}} \subseteq \mathscr{Y}^{\mathsf{tame}}$, which implies $\operatorname{Supp}(\mathscr{Y}_{\mathsf{cpt}}) \subseteq \operatorname{Supp} \mathscr{Y} \subseteq \operatorname{Supp}(\mathscr{Y}^{\mathsf{rad}}) \subseteq \operatorname{Supp}(\mathscr{Y}^{\mathsf{tame}})$. By (1) and Proposition 1.10 we get $\operatorname{Supp}(\mathscr{Y}^{\mathsf{tame}}) = \operatorname{Supp} \mathscr{Y} = \operatorname{Supp}(\mathscr{Y}_{\mathsf{cpt}})$. The equalities in the assertion follow. $\square$

The inclusion $\mathscr{Y}^{\mathsf{rad}} \subseteq \mathscr{Y}^{\mathsf{tame}}$ in Lemma 5.8 in particular says:

**Corollary 5.9.** *Every tame thick $\otimes$-ideal of $\mathsf{D}^-(R)$ is radical.*

We now obtain a bijection, using the above lemma.

**Proposition 5.10.** *There is a one-to-one correspondence* $(\,\cdot\,)^{\mathsf{tame}} : \mathbf{Cpt} \rightleftarrows \mathbf{Tame} : (\,\cdot\,)_{\mathsf{cpt}}$.

*Proof.* Fix a compact thick $\otimes$-ideal $\mathscr{X}$, and a tame thick $\otimes$-ideal $\mathscr{Y}$ of $\mathsf{D}^-(R)$. We have $(\mathscr{X}^{\mathsf{tame}})_{\mathsf{cpt}} = \langle \operatorname{Supp}(\mathscr{X}^{\mathsf{tame}}) \rangle = \langle \operatorname{Supp} \mathscr{X} \rangle = \mathscr{X}$, where the first equality follows from Lemma 5.8(1), the second from Lemma 5.8(2), and the last from Proposition 5.4. Also, it holds that $(\mathscr{Y}_{\mathsf{cpt}})^{\mathsf{tame}} = \operatorname{Supp}^{-1} \operatorname{Supp}(\mathscr{Y}_{\mathsf{cpt}}) = \operatorname{Supp}^{-1} \operatorname{Supp} \mathscr{Y} = \mathscr{Y}$, where the first equality follows from Lemma 5.8(1), the second from Lemma 5.8(2), and the last from Proposition 5.3. Thus, we obtain the one-to-one correspondence in the proposition. $\square$

For each subset $A$ of $\operatorname{Spec} R$, we put $\mathcal{S}(A) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in A\}$. For each subset $B$ of $\operatorname{Spc} \mathsf{D}^-(R)$, we put $\mathfrak{s}(B) = \{\mathfrak{s}(\mathcal{P}) \mid \mathcal{P} \in B\}$. We get another bijection.

**Proposition 5.11.** *There exists a one-to-one correspondence* $\mathcal{S} : \mathbf{Spcl}(\mathrm{Spec}) \rightleftarrows \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spc}) : \mathfrak{s}$.

*Proof.* First of all, applying Theorem 3.9 and Corollary 3.14, we observe that

$$\begin{aligned} \mathfrak{s}(\mathcal{S}(\mathfrak{p})) &= \mathfrak{p} \quad \text{for all } \mathfrak{p} \in \operatorname{Spec} R, \\ \mathcal{S}(\mathfrak{s}(\mathcal{P})) &= \mathcal{P} \quad \text{for all } \mathcal{P} \in {}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R). \end{aligned} \tag{5.11.1}$$

Fix a specialization-closed subset $W$ of $\operatorname{Spec} R$ and a specialization-closed subset $U$ of ${}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$. It follows from (5.11.1) that $\mathfrak{s}(\mathcal{S}(W)) = W$ and $\mathcal{S}(\mathfrak{s}(U)) = U$.

Pick a prime ideal $\mathfrak{p}$ in $W$. Let $X$ be the closure of $\{\mathcal{S}(\mathfrak{p})\}$ in ${}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$. Then $X = Y \cap {}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$, where $Y$ is the closure of $\{\mathcal{S}(\mathfrak{p})\}$ in $\mathtt{Spc}\, \mathsf{D}^-(R)$, and hence,

$$\begin{aligned} X = \{\mathcal{P} \in {}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R) \mid \mathcal{P} \subseteq \mathcal{S}(\mathfrak{p})\} &= \{\mathcal{S}(\mathfrak{q}) \mid \mathfrak{q} \in \operatorname{Spec} R,\ \mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{p})\} \\ &= \{\mathcal{S}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{p})\} \subseteq \mathcal{S}(W), \end{aligned}$$

where the first equality follows from [Balmer 2005, Proposition 2.9], the second from Corollary 3.14, and the third from Theorem 3.9. The inclusion holds since $W$ is a specialization-closed subset of $\operatorname{Spec} R$. Therefore, $\mathcal{S}(W)$ is a specialization-closed subset of ${}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$; namely, $\mathcal{S}(W) \in \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spc})$.

Pick $\mathcal{P} \in U$. As $U$ is a subset of ${}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$, the prime thick $\otimes$-ideal $\mathcal{P}$ is tame. Let $\mathfrak{q}$ be a prime ideal of $R$ containing $\mathfrak{s}(\mathcal{P})$. We then get $\mathcal{S}(\mathfrak{q}) \subseteq \mathcal{S}(\mathfrak{s}(\mathcal{P})) = \mathcal{P}$ by Theorem 3.9 and (5.11.1), which says that $\mathcal{S}(\mathfrak{q})$ belongs to the closure of the set $\{\mathcal{P}\}$ in ${}^{\mathsf{t}}\mathtt{Spc}\, \mathsf{D}^-(R)$ by [Balmer 2005, Proposition 2.9]. The specialization-closed property of $U$ implies that $\mathcal{S}(\mathfrak{q})$ belongs to $U$. We have $\mathfrak{q} = \mathfrak{s}(\mathcal{S}(\mathfrak{q}))$ by (5.11.1), which belongs to $\mathfrak{s}(U)$. Consequently, the subset $\mathfrak{s}(U)$ of $\operatorname{Spec} R$ is specialization-closed; that is, $\mathfrak{s}(U) \in \mathbf{Spcl}(\mathrm{Spec})$. $\square$

Here we note an elementary fact on commutativity of a diagram of maps.

**Remark 5.12.** Consider the following diagram of bijections:

$$\begin{tikzcd} & A \arrow[dl, shift left, "a"] \arrow[dr, shift left, "c^{-1}"] & \\ B \arrow[ur, shift left, "a^{-1}"] \arrow[rr, shift left, "b"] & & C \arrow[ul, shift left, "c"] \arrow[ll, shift left, "b^{-1}"] \end{tikzcd}$$

One can choose infinitely many compositions of maps in the diagram, but once one of them is equal to another, this triangle with edges having any direction commutes.

To be more explicit, if $c = ba$ for instance, then the set $\{1, a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ is closed under possible compositions.

Now we can state and prove our first main result in this section.

**Theorem 5.13.** *There is a commutative diagram of mutually inverse bijections*:

$$\begin{tikzcd}
 & \mathbf{Spcl}(\mathrm{Spec}) \arrow[dl, shift left, "\langle\,\cdot\,\rangle"] \arrow[dd, shift left, "\mathrm{Supp}^{-1}"] \arrow[dr, shift left, "\mathscr{S}"] & \\
\mathbf{Cpt} \arrow[ur, shift left, "\mathrm{Supp}"] \arrow[ur, phantom, "\cong"] \arrow[dr, shift left, "(\,\cdot\,)^{\mathsf{tame}}"] \arrow[dr, phantom, "\cong"] & & \mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spc}) \arrow[ul, shift left, "\mathfrak{s}"] \arrow[ul, phantom, "\cong"] \arrow[dl, shift left, "\mathtt{Sp}^{-1}"] \\
 & \mathbf{Tame} \arrow[ul, shift left, "(\,\cdot\,)_{\mathsf{cpt}}"] \arrow[uu, shift left, "\mathrm{Supp}"] \arrow[uu, phantom, "\cong"] \arrow[ur, shift left, "\mathtt{Sp}"] \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

*Proof.* The five one-to-one correspondences in the diagram are shown in Propositions 5.3, 5.4, 5.7, 5.10, and 5.11. It remains to show the commutativity, and for this we take Remark 5.12 into account.

For a thick $\otimes$-ideal $\mathscr{X}$ of $\mathsf{D}^-(R)$, we have $\mathrm{Supp}(\mathscr{X}^{\mathsf{tame}}) = \mathrm{Supp}\,\mathscr{X}$ by Lemma 5.8(2), which shows that the left triangle in the diagram commutes. It is easy to observe from Corollary 3.14 that

$$\mathtt{Sp}\,\mathscr{X} = \mathscr{S}(\mathrm{Supp}\,\mathscr{X}) \text{ for any subcategory } \mathscr{X} \text{ of } \mathsf{D}^-(R). \tag{5.13.1}$$

The commutativity of the right triangle in the diagram follows from (5.13.1). □

**Remark 5.14.** The bijections in the diagram of Theorem 5.13 induce lattice structures in **Tame** and $\mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spc})$, so that the maps are lattice isomorphisms. However, we do not know if there is an explicit way to define lattice structures like the one of **Cpt** given in Proposition 2.18(2).

Let $f : A \to B$ and $g : B \to A$ be maps with $gf = 1$. Then we say that $(f, g)$ is a *section-retraction pair*, and write $f \dashv g$. Our next goal is to construct a certain commutative diagram of section-retraction pairs, and for this we again give several propositions. The first one is a consequence of [Balmer 2005, Theorem 4.10].

**Proposition 5.15.** *There is a one-to-one correspondence* $\mathtt{Spp} : \mathbf{Rad} \rightleftarrows \mathbf{Thom} : \mathtt{Spp}^{-1}$.

**Proposition 5.16.** *There is a section-retraction pair* $(\,\cdot\,)^{\mathsf{rad}} : \mathbf{Cpt} \rightleftarrows \mathbf{Rad} : (\,\cdot\,)_{\mathsf{cpt}}$.

*Proof.* For every $\mathscr{X} \in \mathbf{Cpt}$, we have $(\mathscr{X}^{\mathsf{rad}})_{\mathsf{cpt}} = \langle \mathrm{Supp}(\mathscr{X}^{\mathsf{rad}}) \rangle = \langle \mathrm{Supp}\,\mathscr{X} \rangle = \mathscr{X}_{\mathsf{cpt}} = \mathscr{X}$ by Lemma 5.8. □

Let $X$ be a topological space. A subset $T$ of $X$ is called *Thomason* if $T$ is a union of closed subsets of $X$ whose complements are quasicompact. Note that a Thomason subset is specialization-closed.

For each subset $A$ of Spec $R$, we set $\overline{\mathcal{S}}(A)=\bigcup_{\mathfrak{p}\in A}\overline{\{\mathcal{S}(\mathfrak{p})\}}$. For each subset $B$ of $\mathsf{Spc}\,\mathsf{D}^-(R)$, we set $\mathcal{S}^{-1}(B)=\{\mathfrak{p}\in\operatorname{Spec}R\mid\mathcal{S}(\mathfrak{p})\in B\}$. We obtain another section-retraction pair.

**Proposition 5.17.** *There is a section-retraction pair* $\overline{\mathcal{S}}:\mathbf{Spcl}(\mathrm{Spec})\rightleftarrows\mathbf{Thom}:\mathcal{S}^{-1}$.

*Proof.* Corollary 3.10 and [Balmer 2005, Proposition 2.9] yield

$$\mathsf{Spp}(R/\mathfrak{p})=\overline{\{\mathcal{S}(\mathfrak{p})\}}\ \text{ for any prime ideal } \mathfrak{p} \text{ of } R, \tag{5.17.1}$$

whence $(\overline{\{\mathcal{S}(\mathfrak{p})\}})^{\complement}=\mathtt{U}(R/\mathfrak{p})$, which is quasicompact by [Balmer 2005, Proposition 2.14(a)]. Hence, $\overline{\mathcal{S}}(A)$ is a Thomason subset of $\mathsf{Spc}\,\mathsf{D}^-(R)$ for any subset $A$ of Spec $R$. In particular, we get a map $\overline{\mathcal{S}}:\mathbf{Spcl}(\mathrm{Spec})\to\mathbf{Thom}$.

Let $T$ be a Thomason subset of $\mathsf{Spc}\,\mathsf{D}^-(R)$. Let $\mathfrak{p}$, $\mathfrak{q}$ be prime ideals of $R$ with $\mathfrak{p}\subseteq\mathfrak{q}$ and $\mathcal{S}(\mathfrak{p})\in T$. Then $\mathcal{S}(\mathfrak{q})$ belongs to $\overline{\{\mathcal{S}(\mathfrak{p})\}}$ by Theorem 3.9 and [Balmer 2005, Proposition 2.9]. Since $T$ is Thomason, it contains $\overline{\{\mathcal{S}(\mathfrak{p})\}}$. Hence, $\mathcal{S}(\mathfrak{q})$ belongs to $T$. Thus, the assignment $T\mapsto\mathcal{S}^{-1}(T)$ defines a map $\mathcal{S}^{-1}:\mathbf{Thom}\to\mathbf{Spcl}(\mathrm{Spec})$.

For a specialization-closed subset $W$ of Spec $R$ and a prime ideal $\mathfrak{p}$ of $R$,

$$\begin{aligned}\mathcal{S}(\mathfrak{p})\in\overline{\{\mathcal{S}(\mathfrak{q})\}} \text{ for some } \mathfrak{q}\in W \iff \mathcal{S}(\mathfrak{p})\subseteq\mathcal{S}(\mathfrak{q}) \text{ for some } \mathfrak{q}\in W\\ \iff \mathfrak{p}\supseteq\mathfrak{q} \text{ for some } \mathfrak{q}\in W \iff \mathfrak{p}\in W,\end{aligned}$$

where the first and second equivalences follow from [Balmer 2005, Proposition 2.9] and Theorem 3.9, and the last equivalence holds by the fact that $W$ is specialization-closed. This yields $\mathcal{S}^{-1}(\overline{\mathcal{S}}(W))=W$. □

Now we consider describing spcl-closures and spcl-interiors.

**Proposition 5.18.** *Let $A$ be a specialization-closed subset of* $\mathsf{Spc}\,\mathsf{D}^-(R)$, *and let $B$ be a specialization-closed subset of* ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$.

(1) *Let $A_{\mathsf{spcl}}$ stand for the* spcl-*interior of $A$ in* ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$. *Then*

$$A_{\mathsf{spcl}}=A\cap{}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R).$$

(2) *Let $B^{\mathsf{spcl}}$ stand for the* spcl-*closure of $B$ in* $\mathsf{Spc}\,\mathsf{D}^-(R)$. *Then*

$$B^{\mathsf{spcl}}=\{\mathcal{P}\in\mathsf{Spc}\,\mathsf{D}^-(R)\mid\mathcal{P}^{\mathsf{tame}}\in B\}=\bigcup_{\mathcal{P}\in B^{\mathsf{spcl}}}\mathsf{Spp}(R/\mathfrak{s}(\mathcal{P})).$$

*In particular, $B^{\mathsf{spcl}}$ is a Thomason subset of* $\mathsf{Spc}\,\mathsf{D}^-(R)$.

*Proof.* (1) We easily observe that $A\cap{}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$ is a specialization-closed subset of the topological space ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$ contained in $A$. Also, it is obvious that if $X$ is a specialization-closed subset of ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$ contained in $A$, then $X$ is contained in $A\cap{}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$. Hence, $A\cap{}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$ coincides with $A_{\mathsf{spcl}}$.

(2) Let $C$ be the set of prime thick $\otimes$-ideals $\mathcal{P}$ of $\mathsf{D}^-(R)$ with $\mathcal{P}^{\mathsf{tame}}\in B$. We proceed step by step.

(a) Each $\mathcal{Q} \in B$ is tame. Hence, $\mathcal{Q}^{\mathsf{tame}} = \mathcal{Q} \in B$. This shows that $C$ contains $B$.

(b) Let $Y$ be a specialization-closed subset of $\mathsf{Spc}\,\mathsf{D}^-(R)$ containing $B$. Take any element $\mathcal{P}$ of $C$. Then $\mathcal{P}^{\mathsf{tame}}$ belongs to $B$, and hence to $Y$. Since $Y$ is specialization-closed, $\overline{\{\mathcal{P}^{\mathsf{tame}}\}}$ is contained in $Y$. Hence, $\mathcal{P}$ belongs to $Y$ by [Balmer 2005, Proposition 2.9]. It follows that $C$ is contained in $Y$.

(c) We prove $C = \bigcup_{\mathcal{P}\in C} \mathsf{Spp}(R/\mathfrak{s}(\mathcal{P}))$. Combining Theorem 3.9, Lemma 5.8(1), and (5.17.1) gives rise to $\mathsf{Spp}(R/\mathfrak{s}(\mathcal{P})) = \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$, and thus, it is enough to verify $C = \bigcup_{\mathcal{P}\in C} \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$. By [Balmer 2005, Proposition 2.9] we see that $C$ is contained in $\bigcup_{\mathcal{P}\in C} \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$. Conversely, let $\mathcal{P} \in \mathcal{C}$ and $\mathcal{Q} \in \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$. Then $\mathcal{P}^{\mathsf{tame}}$ belongs to $B$, and $\mathcal{Q}$ is contained in $\mathcal{P}^{\mathsf{tame}}$ by [Balmer 2005, Proposition 2.9], which shows that $\mathcal{Q}^{\mathsf{tame}}$ is contained in $\mathcal{P}^{\mathsf{tame}}$. Hence, $\mathcal{Q}^{\mathsf{tame}}$ is in $\overline{\{\mathcal{P}^{\mathsf{tame}}\}} \cap {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$. As $B$ is specialization-closed in ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$, it contains $\overline{\{\mathcal{P}^{\mathsf{tame}}\}} \cap {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$, and therefore, $\mathcal{Q}^{\mathsf{tame}}$ is in $B$. Thus, $\mathcal{Q}$ belongs to $C$. We obtain $C = \bigcup_{\mathcal{P}\in C} \overline{\{\mathcal{P}^{\mathsf{tame}}\}}$.

The equality $C = \bigcup_{\mathcal{P}\in C} \mathsf{Spp}(R/\mathfrak{s}(\mathcal{P}))$ shown in (c) especially says that $C$ is specialization-closed. By this together with (a) and (b) we obtain $C = B^{\mathsf{spcl}}$, and it follows that $C = \bigcup_{\mathcal{P}\in B^{\mathsf{spcl}}} \mathsf{Spp}(R/\mathfrak{s}(\mathcal{P}))$. □

We now obtain another section-retraction pair:

**Proposition 5.19.** *The operations* $(\cdot)^{\mathsf{spcl}}$ *and* $(\cdot)_{\mathsf{spcl}}$ *defined in Proposition 5.18 make a section-retraction pair* $(\cdot)^{\mathsf{spcl}} : \mathbf{Spcl}({}^{\mathsf{t}}\mathsf{Spc}) \rightleftarrows \mathbf{Thom} : (\cdot)_{\mathsf{spcl}}$.

*Proof.* Let $U$ be a specialization-closed subset of ${}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R)$. By Proposition 5.18, $U^{\mathsf{spcl}}$ is a Thomason subset of $\mathsf{Spc}\,\mathsf{D}^-(R)$, and $(U^{\mathsf{spcl}})_{\mathsf{spcl}} = U^{\mathsf{spcl}} \cap {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R) = \{\mathcal{P} \in {}^{\mathsf{t}}\mathsf{Spc}\,\mathsf{D}^-(R) \mid \mathcal{P}^{\mathsf{tame}} \in U\} = U$. □

We can prove our second main result in this section.

**Theorem 5.20.** *There is a diagram*

$$\begin{tikzcd}
\mathbf{Rad} \arrow[r, equal, "\sim"] \arrow[d, shift left, "(\cdot)_{\mathsf{cpt}}"] \arrow[d, phantom, "\dashv"] & \mathbf{Thom} \arrow[r, equal] \arrow[d, shift left, "\mathcal{S}^{-1}"] \arrow[d, phantom, "\dashv"] & \mathbf{Thom} \arrow[d, shift left, "(\cdot)_{\mathsf{spcl}}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Cpt} \arrow[r, equal, "\sim"] \arrow[u, shift left, "(\cdot)^{\mathsf{rad}}"] & \mathbf{Spcl}(\mathrm{Spec}) \arrow[r, equal, "\sim"] \arrow[u, shift left, "\bar{\mathcal{S}}"] & \mathbf{Spcl}({}^{\mathsf{t}}\mathsf{Spc}) \arrow[u, shift left, "(\cdot)^{\mathsf{spcl}}"]
\end{tikzcd}$$

*where the upper horizontal bijections are the one given in Proposition 5.15 and an equality, and the lower horizontal bijections are the ones appearing in Theorem 5.13. The diagrams with vertical arrows from the bottom to the top and the top to the bottom are commutative.*

*Proof.* The three section-retraction pairs are obtained in Propositions 5.16, 5.17, and 5.19.

We claim that for any thick $\otimes$-ideal $\mathcal{X}$ of $\mathsf{D}^-(R)$ one has

$$\mathsf{Spp}(\mathcal{X}^{\mathsf{rad}}) = \mathsf{Spp}\,\mathcal{X}. \tag{5.20.1}$$

Indeed, Lemma 5.8(1) shows $\mathscr{X}^{\mathsf{rad}} = \sqrt{\mathscr{X}}$. The inclusion $\mathscr{X} \subseteq \sqrt{\mathscr{X}}$ implies $\operatorname{Spp}\mathscr{X} \subseteq \operatorname{Spp}\sqrt{\mathscr{X}}$. Let $\mathscr{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. If $\mathscr{X}$ is contained in $\mathscr{P}$, then so is $\sqrt{\mathscr{X}}$ as $\mathscr{P}$ is prime. Therefore, we obtain $\operatorname{Spp}\sqrt{\mathscr{X}} = \operatorname{Spp}\mathscr{X}$, and the claim follows.

Fix a thick $\otimes$-ideal $\mathscr{C}$ of $\mathsf{D}^-(R)$. For a prime ideal $\mathfrak{p}$ of $R$ one has $\mathscr{S}(\mathfrak{p}) \in \operatorname{Spp}\mathscr{C}$ if and only if $\mathscr{C} \nsubseteq \mathscr{S}(\mathfrak{p})$, if and only if $\mathscr{C}_{\mathfrak{p}} \neq 0$, if and only if $\mathfrak{p} \in \operatorname{Supp}\mathscr{C}$. This shows $\mathscr{S}^{-1}(\operatorname{Spp}\mathscr{C}) = \operatorname{Supp}\mathscr{C}$. Lemma 5.8(2) gives $\operatorname{Supp}(\mathscr{C}_{\mathsf{cpt}}) = \mathscr{S}^{-1}(\operatorname{Spp}\mathscr{C})$. Next, suppose that $\mathscr{C}$ is compact. Lemma 5.8(1), (5.17.1), and (5.20.1) yield

$$\begin{aligned}\operatorname{Spp}(\mathscr{C}^{\mathsf{rad}}) = \operatorname{Spp}\mathscr{C} = \operatorname{Spp}(\mathscr{C}_{\mathsf{cpt}}) = \operatorname{Spp}(\langle \operatorname{Supp}\mathscr{C}\rangle) \\ = \operatorname{Spp}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}\mathscr{C}\} = \bar{\mathscr{S}}(\operatorname{Supp}\mathscr{C}).\end{aligned}$$

Thus, we obtain the commutativity of the left square of the diagram.

Let $A$ be any subset of $\operatorname{Spec} R$. It is clear that $\mathscr{S}(A) = \{\mathscr{S}(\mathfrak{p}) \mid \mathfrak{p} \in A\}$ is contained in $\bar{\mathscr{S}}(A)$. As $\bar{\mathscr{S}}(A)$ is a union of closed subsets of the topological space $\operatorname{Spc}\mathsf{D}^-(R)$, it is a specialization-closed subset of $\operatorname{Spc}\mathsf{D}^-(R)$. Note that any specialization-closed subset of $\operatorname{Spc}\mathsf{D}^-(R)$ containing $\mathscr{S}(A)$ contains $\bar{\mathscr{S}}(A)$. Hence, we have $\bar{\mathscr{S}}(A) = (\mathscr{S}(A))^{\mathsf{spcl}}$. Let $B$ be a specialization-closed subset of $\operatorname{Spc}\mathsf{D}^-(R)$. Then $\mathscr{S}(\mathscr{S}^{-1}(B)) = \{\mathscr{S}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R,\ \mathscr{S}(\mathfrak{p}) \in B\} = B \cap {}^{\mathsf{t}}\operatorname{Spc}\mathsf{D}^-(R) = B_{\mathsf{spcl}}$ by Corollary 3.14 and Proposition 5.18(1). Now it follows that the right square of the diagram commutes. $\square$

We close this section by producing another commutative diagram, coming from the above theorem.

**Corollary 5.21.** *There is a commutative diagram*

$$\begin{tikzcd}
 & & & \mathbf{Rad} \arrow[dlll, "(\,\cdot\,)_{\mathsf{cpt}}"'] \arrow[dl, "\operatorname{Supp}"'] \arrow[dr, "(\,\cdot\,)^{\mathsf{tame}}"] \arrow[drrr, "\operatorname{Sp}"] & & & \\
\mathbf{Cpt} \arrow[rr, equal, "\sim"] & & \mathbf{Spcl}(\operatorname{Spec}) \arrow[rr, equal, "\sim"] & & \mathbf{Tame} \arrow[rr, equal, "\sim"] & & \mathbf{Spcl}({}^{\mathsf{t}}\operatorname{Spc})
\end{tikzcd}$$

*Here, the three bijections are the ones appearing in Theorem 5.13, and the other maps are retractions.*

*Proof.* We have the following diagram:

$$\begin{tikzcd}
\mathbf{Rad} \arrow[d, shift left, "(\,\cdot\,)_{\mathsf{cpt}}"] \arrow[d, phantom, "\dashv"] & & & \\
\mathbf{Cpt} \arrow[u, shift left, "(\,\cdot\,)^{\mathsf{rad}}"] \arrow[r, shift left, "\operatorname{Supp}"] \arrow[r, phantom, "\cong"] & \mathbf{Spcl}(\operatorname{Spec}) \arrow[l, shift left, "\langle\,\cdot\,\rangle"] \arrow[r, shift left, "\operatorname{Supp}^{-1}"] \arrow[r, phantom, "\cong"] & \mathbf{Tame} \arrow[l, shift left, "\operatorname{Supp}"] \arrow[r, shift left, "\operatorname{Sp}"] \arrow[r, phantom, "\cong"] & \mathbf{Spcl}({}^{\mathsf{t}}\operatorname{Spc}) \arrow[l, shift left, "\operatorname{Sp}^{-1}"]
\end{tikzcd}$$

Thus, it suffices to verify the equalities of compositions of maps $\mathrm{Supp}\circ(\,\cdot\,)_{\mathsf{cpt}} = \mathrm{Supp}$, $\mathrm{Supp}^{-1}\circ\mathrm{Supp} = (\,\cdot\,)^{\mathsf{tame}}$, and $\mathtt{Sp}\circ(\,\cdot\,)^{\mathsf{tame}} = \mathtt{Sp}$. This is equivalent to showing that the equalities

(i) $\mathrm{Supp}(\mathscr{X}_{\mathsf{cpt}}) = \mathrm{Supp}\,\mathscr{X}$,

(ii) $\mathrm{Supp}^{-1}\,\mathrm{Supp}\,\mathscr{X} = \mathscr{X}^{\mathsf{tame}}$, and

(iii) $\mathtt{Sp}(\mathscr{X}^{\mathsf{tame}}) = \mathtt{Sp}\,\mathscr{X}$

hold for each (radical) thick $\otimes$-ideal $\mathscr{X}$ of $\mathsf{D}^-(R)$. The equalities (i) and (ii) immediately follow from Lemma 5.8. We have $\mathtt{Sp}(\mathscr{X}^{\mathsf{tame}}) = \mathtt{Sp}(\mathrm{Supp}^{-1}\,\mathrm{Supp}\,\mathscr{X}) = (\mathtt{Sp}\circ\mathrm{Supp}^{-1})(\mathrm{Supp}\,\mathscr{X}) = \mathscr{S}(\mathrm{Supp}\,\mathscr{X}) = \mathtt{Sp}\,\mathscr{X}$, where the first and last equalities follow from Lemma 5.8(1) and (5.13.1). Proposition 1.10(2) says that $\mathrm{Supp}\,\mathscr{X}$ belongs to **Spcl**(Spec), and the third equality above is obtained by Theorem 5.13. Now the assertion (iii) follows, and the proof of the corollary is completed. □

## 6. Distinction between thick tensor ideals, and Balmer's conjecture

In this section, we consider when the section-retraction pairs in Theorem 5.20 and Corollary 5.21 are one-to-one correspondences, and construct a counterexample to the conjecture of Balmer. We begin with a lemma on the annihilator of an object in the thick $\otimes$-ideal closure.

**Lemma 6.1.** *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects of $\mathsf{D}^-(R)$. For $M \in \mathsf{thick}^{\otimes}\{X_\lambda\}_{\lambda\in\Lambda}$ there are (pairwise distinct) indices $\lambda_1, \dots, \lambda_n \in \Lambda$ and integers $e_1, \dots, e_n > 0$ such that* $\mathrm{Ann}\,M$ *contains* $\prod_{i=1}^n(\mathrm{Ann}\,X_{\lambda_i})^{e_i}$.

*Proof.* Let $\mathscr{C}$ be the subcategory of $\mathsf{D}^-(R)$ consisting of objects $C$ such that there are $\lambda_1, \dots, \lambda_n \in \Lambda$ and $e_1, \dots, e_n > 0$ such that $\mathrm{Ann}\,C$ contains $\prod_{i=1}^n(\mathrm{Ann}\,X_{\lambda_i})^{e_i}$. The following statements hold in general.

- If $A$ is an object of $\mathsf{D}^-(R)$ and $B$ is a direct summand of $A$, then $\mathrm{Ann}\,A \subseteq \mathrm{Ann}\,B$.
- For each object $A \in \mathsf{D}^-(R)$ one has $\mathrm{Ann}(A[\pm 1]) = \mathrm{Ann}\,A$.
- If $A \to B \to C \to A[1]$ is an exact triangle in $\mathsf{D}^-(R)$, then $\mathrm{Ann}\,B$ contains $\mathrm{Ann}\,A \cdot \mathrm{Ann}\,C$.
- For any objects $A, B$ of $\mathsf{D}^-(R)$ one has $\mathrm{Ann}(A \otimes_R^{\boldsymbol{L}} B) \supseteq \mathrm{Ann}\,A$.

It follows from these that $\mathscr{C}$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Since $X_\lambda$ is in $\mathscr{C}$ for all $\lambda \in \Lambda$, it holds that $\mathscr{C}$ contains $\mathsf{thick}^{\otimes}\{X_\lambda\}_{\lambda\in\Lambda}$. The assertion of the lemma now follows. □

The proposition below says in particular that in the case where $R$ is a local ring $\mathsf{D}^-(R)$ has a compact prime thick tensor ideal. On the other hand, in the nonlocal case it is often that $\mathsf{D}^-(R)$ has no such one.

**Proposition 6.2.** (1) *If $R$ is a local ring with maximal ideal $\mathfrak{m}$, then* $\mathbf{Cpt} \cap \mathfrak{s}^{-1}(\mathfrak{m}) = \{\mathbf{0}\} \neq \varnothing$.

(2) *Let $R$ be a nonlocal semilocal domain. Then there exists no compact prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. In particular, one has $\mathcal{P}_{\mathsf{cpt}} \subsetneq \mathcal{P} = \mathcal{P}^{\mathsf{rad}}$ for all $\mathcal{P} \in \operatorname{Spc} \mathsf{D}^-(R)$.*

*Proof.* (1) Let $\mathcal{P}$ be in $\operatorname{Spc} \mathsf{D}^-(R)$. Then $\mathcal{P}$ is in $\mathfrak{s}^{-1}(\mathfrak{m})$ if and only if $\operatorname{Supp} \mathcal{P} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \nsubseteq \mathfrak{m}\} = \varnothing$ by Proposition 4.2, if and only if $\mathcal{P} = \mathbf{0}$ by Remark 1.8. Since $\mathbf{0}$ is compact, we are done.

(2) Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the (pairwise distinct) maximal ideals of $R$ with $n \geqslant 2$. For each $1 \leqslant i \leqslant n$ one finds an element $x_i \in \mathfrak{m}_i$ that does not belong to any other maximal ideals. As $R$ is a domain of positive dimension, $x_i$ is a nonzerodivisor of $R$. Set $C_i = \bigoplus_{t \geqslant 0} R/x_i^{t+1}[t]$; note that this is an object of $\mathsf{D}^-(R)$. We have $\operatorname{Supp}(C_1 \otimes_R^{\boldsymbol{L}} \cdots \otimes_R^{\boldsymbol{L}} C_n) = \bigcap_{i=1}^n \operatorname{Supp} C_i = \bigcap_{i=1}^n \mathrm{V}(x_i) = \mathrm{V}(x_1, \dots, x_n) = \varnothing$ by Lemma 1.9(4) and the fact that $(x_1, x_2)$ is a unit ideal of $R$. Remark 1.8 gives $C_1 \otimes_R^{\boldsymbol{L}} \cdots \otimes_R^{\boldsymbol{L}} C_n = 0$.

Suppose that there exists a compact prime thick $\otimes$-ideal $\mathcal{P}$ of $\mathsf{D}^-(R)$. Then $C_1 \otimes_R^{\boldsymbol{L}} \cdots \otimes_R^{\boldsymbol{L}} C_n = 0$ is contained in $\mathcal{P}$, and so is $C_\ell$ for some $1 \leqslant \ell \leqslant n$. We have $\mathcal{P} = \langle \operatorname{Supp} \mathcal{P} \rangle$ by Proposition 5.4, and by Lemma 6.1 there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r \in \operatorname{Supp} \mathcal{P}$ and integers $e_1, \dots, e_r > 0$ such that $\operatorname{Ann} C_\ell$ contains $(\operatorname{Ann} R/\mathfrak{p}_1)^{e_1} \cdots (\operatorname{Ann} R/\mathfrak{p}_r)^{e_r} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$. Since $R$ is a domain and $x_\ell$ is a nonunit of $R$, we have $\operatorname{Ann} C_\ell = \bigcap_{t \geqslant 0} x_\ell^{t+1} R = 0$ by Krull's intersection theorem. Therefore, $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} = 0$, and $\mathfrak{p}_s = 0$ for some $1 \leqslant s \leqslant r$ as $R$ is a domain. Thus, the zero ideal 0 of $R$ belongs to $\operatorname{Supp} \mathcal{P}$, which implies $\operatorname{Supp} \mathcal{P} = \operatorname{Spec} R$. We obtain $\mathcal{P} = \mathsf{D}^-(R)$ by Proposition 4.11, which is a contradiction. □

To show a main result of this section, we make two lemmas. The first one concerns the structure of the radical and tame closures, while the second one gives an elementary characterization of artinian rings.

**Lemma 6.3.** *Let $\mathcal{X}$ be a subcategory of $\mathsf{D}^-(R)$. One has*

$$\mathcal{X}^{\mathsf{rad}} = \bigcap_{\mathcal{X} \subseteq \mathcal{P} \in \operatorname{Spc} \mathsf{D}^-(R)} \mathcal{P}, \qquad \mathcal{X}^{\mathsf{tame}} = \bigcap_{\mathcal{X} \subseteq \mathcal{P} \in {}^{\mathsf{t}}\operatorname{Spc} \mathsf{D}^-(R)} \mathcal{P}.$$

*Proof.* Lemma 5.8(1) implies $\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathsf{thick}^{\otimes} \mathcal{X}}$, which coincides with the intersection of the prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$ containing $\mathsf{thick}^{\otimes} \mathcal{X}$ by [Balmer 2005, Lemma 4.2]. This is equal to the intersection of the prime thick $\otimes$-ideals containing $\mathcal{X}$, and thus, the first equality holds. As for the second equality, if $\mathcal{P}$ is a tame thick $\otimes$-ideal containing $\mathcal{X}$, then we have $\mathcal{X}^{\mathsf{tame}} \subseteq \mathcal{P}^{\mathsf{tame}} = \mathcal{P}$, which shows the inclusion $(\subseteq)$. Let $M$ be an object of $\mathsf{D}^-(R)$ belonging to all $\mathcal{P} \in {}^{\mathsf{t}}\operatorname{Spc} \mathsf{D}^-(R)$ with $\mathcal{X} \subseteq \mathcal{P}$. Corollary 3.14 says that $M$ is in $\mathcal{S}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of $R$

with $\mathscr{X} \subseteq \mathcal{S}(\mathfrak{p})$. This means that Supp $M$ is contained in Supp $\mathscr{X}$. Hence, $M$ is in $\mathrm{Supp}^{-1}$ Supp $\mathscr{X}$, which coincides with $\mathscr{X}^{\mathsf{tame}}$ by Lemma 5.8(1). Thus, the second equality follows. □

**Lemma 6.4.** *The ring $R$ is artinian if and only if for any sequence $I_1, I_2, \dots$ of ideals of $R$ it holds that $\mathrm{V}\big(\bigcap_{n\geqslant 1} I_n\big) = \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$.*

*Proof.* First of all, note that the inclusion $\mathrm{V}\big(\bigcap_{n\geqslant 1} I_n\big) \supseteq \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$ always holds.

If $R$ is artinian, then there exists an integer $m \geqslant 1$ such that $\bigcap_{n\geqslant 1} I_n = \bigcap_{j=1}^{m} I_j$. From this we obtain $\mathrm{V}\big(\bigcap_{n\geqslant 1} I_n\big) = \mathrm{V}\big(\bigcap_{j=1}^{m} I_j\big) = \bigcup_{j=1}^{m} \mathrm{V}(I_j) \subseteq \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$. This shows the "only if" part.

Let us prove the "if" part. Assume first that $R$ has infinitely many maximal ideals, and take a sequence $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ of pairwise distinct maximal ideals of $R$. By assumption, we get $\mathrm{V}\big(\bigcap_{n\geqslant 1} \mathfrak{m}_n\big) = \bigcup_{n\geqslant 1} \mathrm{V}(\mathfrak{m}_n)$. Since $\mathrm{V}\big(\bigcap_{n\geqslant 1} \mathfrak{m}_n\big)$ is a closed subset of Spec $R$, it has only finitely many minimal elements with respect to the inclusion relation. However, $\bigcup_{n\geqslant 1} \mathrm{V}(\mathfrak{m}_n) = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots\}$ has infinitely many minimal elements, which is a contradiction. Thus, $R$ is a semilocal ring. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ be the maximal ideals of $R$, and $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$ the Jacobson radical of $R$. Applying the assumption to the sequence $\{J^n\}_{n\geqslant 1}$ of ideals gives $\mathrm{V}\big(\bigcap_{n\geqslant 1} J^n\big) = \bigcup_{n\geqslant 1} \mathrm{V}(J^n) = \mathrm{V}(J)$. By Krull's intersection theorem, we obtain $\bigcap_{n\geqslant 1} J^n = 0$, whence $\mathrm{V}(J) = \mathrm{Spec}\, R$. Hence, $\mathrm{Spec}\, R = \{\mathfrak{m}_1, \dots, \mathfrak{m}_t\} = \mathrm{Max}\, R$, and we conclude that $R$ is artinian. □

Now we can prove our first main result in this section. Roughly speaking, if our ring $R$ is artinian, then everything is explicit and behaves well, and vice versa. Note that this result includes Corollary 4.16.

**Theorem 6.5.** *The following are equivalent.*

(1) *The ring $R$ is artinian.*

(2) *Every thick $\otimes$-ideal of $\mathsf{D}^-(R)$ is compact, tame, and radical.*

(3) *The maps $\mathcal{S} : \mathrm{Spec}\, R \rightleftarrows \mathtt{Spc}\, \mathsf{D}^-(R) : \mathfrak{s}$ are mutually inverse homeomorphisms.*

(4) *The section-retraction pair $\mathcal{S} : \mathrm{Spec}\, R \rightleftarrows \mathtt{Spc}\, \mathsf{D}^-(R) : \mathfrak{s}$ is a one-to-one correspondence.*

(5) *The section-retraction pair $(\,\cdot\,)_{\mathsf{cpt}} : \mathbf{Rad} \rightleftarrows \mathbf{Cpt} : (\,\cdot\,)^{\mathsf{rad}}$ is a one-to-one correspondence.*

(6) *The section-retraction pair $\mathcal{S}^{-1} : \mathbf{Thom} \rightleftarrows \mathbf{Spcl}(\mathrm{Spec}) : \overline{\mathcal{S}}$ is a one-to-one correspondence.*

(7) *The section-retraction pair $(\,\cdot\,)_{\mathsf{spcl}} : \mathbf{Thom} \rightleftarrows \mathbf{Spcl}({}^{\mathsf{t}}\mathtt{Spc}) : (\,\cdot\,)^{\mathsf{spcl}}$ is a one-to-one correspondence.*

(8) *The retraction* Supp : $\mathbf{Rad} \to \mathbf{Spcl}(\mathrm{Spec})$ *is a bijection.*

(9) *The retraction* $(\,\cdot\,)^{\mathsf{tame}} : \mathbf{Rad} \to \mathbf{Tame}$ *is a bijection.*

(10) *The retraction* $\mathtt{Sp} : \mathbf{Rad} \to \mathbf{Spcl}({}^{\mathtt{t}}\mathtt{Spc})$ *is a bijection.*

(11) *The inclusion* $\mathbf{Rad} \supseteq \mathbf{Tame}$ *is an equality.*

*Proof.* Theorems 3.9, 5.13, and 5.20 and Corollary 5.9 imply that the pairs in (4), (5), (6), and (7) are section-retraction pairs, the maps in (8), (9), and (10) are retractions, and one has the inclusion in (11).

The equivalences (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) and (5) $\Longleftrightarrow$ (8) $\Longleftrightarrow$ (9) $\Longleftrightarrow$ (10) follow from Theorem 5.20 and Corollary 5.21, respectively. It is trivial that (3) implies (4), while (1) implies (2) by Corollaries 2.20 and 5.9 and Proposition 1.10(1). If $\mathtt{Spc}\,\mathsf{D}^-(R) = {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, then $\mathcal{S} = \mathcal{S}'$ and $\mathfrak{s} = \mathfrak{s}'$. From Theorems 4.5(3) and 4.7 we see that (2) implies (3). Corollary 5.9 says $\mathcal{X}^{\mathsf{tame}} \in \mathbf{Rad}$ for each $\mathcal{X} \in \mathbf{Rad}$. Hence, if $(\,\cdot\,)^{\mathsf{tame}} : \mathbf{Rad} \to \mathbf{Tame}$ is injective, then $\mathcal{X} = \mathcal{X}^{\mathsf{tame}}$ holds. This shows that (9) implies (11). It is easily seen that the converse is also true, and we get the equivalence (9) $\Longleftrightarrow$ (11). When $\mathcal{S} : \operatorname{Spec} R \to \mathtt{Spc}\,\mathsf{D}^-(R)$ is surjective, we have $\mathtt{Spc}\,\mathsf{D}^-(R) = {}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)$, and for a radical thick $\otimes$-ideal $\mathcal{X}$ it holds that $\mathcal{X} = \mathcal{X}^{\mathsf{rad}} = \bigcap_{\mathcal{X}\subseteq\mathcal{P}\in\mathtt{Spc}\,\mathsf{D}^-(R)} \mathcal{P} = \bigcap_{\mathcal{X}\subseteq\mathcal{P}\in{}^{\mathtt{t}}\mathtt{Spc}\,\mathsf{D}^-(R)} \mathcal{P} = \mathcal{X}^{\mathsf{tame}}$ by Lemma 6.3, whence $\mathcal{X}$ is tame. Therefore, (4) implies (11).

Now it remains to prove that (11) implies (1). By Lemma 6.4, it suffices to prove that $\mathrm{V}\big(\bigcap_{n\geqslant 1} I_n\big)$ is contained in $\bigcup_{n\geqslant 1} \mathrm{V}(I_n)$ for any sequence $I_1, I_2, \ldots$ of ideals of $R$. For each $n \geqslant 1$, fix a system of generators $\boldsymbol{x}(n)$ of $I_n$. Set $C = \bigoplus_{n\geqslant 1} \mathrm{K}(\boldsymbol{x}(n), R)[n]$; note that this is defined in $\mathsf{D}^-(R)$. Then $\operatorname{Supp} C = \bigcup_{n\geqslant 1} \operatorname{Supp} \mathrm{K}(\boldsymbol{x}(n), R) = \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$ by Proposition 2.3(3). The radical closure $\mathcal{E}$ of $\big\langle\bigcup_{n\geqslant 1} \mathrm{V}(I_n)\big\rangle$ is tame by assumption. Lemma 5.8 implies $\operatorname{Supp}\mathcal{E} = \bigcup_{n\geqslant 1} \mathrm{V}(I_n) = \operatorname{Supp} C$. Thus, $C$ is in $\operatorname{Supp}^{-1}\operatorname{Supp}\mathcal{E} = \mathcal{E}$ by Proposition 5.3, and $C^{\otimes r} \in \big\langle\bigcup_{n\geqslant 1} \mathrm{V}(I_n)\big\rangle$ for some $r > 0$. Using [Bruns and Herzog 1998, Proposition 1.6.21], we have

$$
\begin{aligned}
C^{\otimes r} &= \bigoplus_{n\geqslant 1}\Bigg(\bigoplus_{i_1+\cdots+i_r=n} \mathrm{K}(\boldsymbol{x}(i_1), R) \otimes^{\boldsymbol{L}}_R \cdots \otimes^{\boldsymbol{L}}_R \mathrm{K}(\boldsymbol{x}(i_r), R)\Bigg)[n] \\
&\gtrdot \bigoplus_{n\geqslant 1} \mathrm{K}(\boldsymbol{x}(n), R)^{\otimes r}[nr] = \bigoplus_{n\geqslant 1} \mathrm{K}(\underbrace{\boldsymbol{x}(n), \ldots, \boldsymbol{x}(n)}_{r}, R)[nr] \\
&\gtrdot \bigoplus_{n\geqslant 1} \mathrm{K}(\boldsymbol{x}(n), R)[nr] =: B. \qquad (6.5.1)
\end{aligned}
$$

Thus, $B$ is in $\big\langle\bigcup_{n\geqslant 1} \mathrm{V}(I_n)\big\rangle$, and Corollary 2.13(3) implies $\mathrm{V}(\operatorname{Ann} B) \subseteq \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$. We have $\operatorname{Ann} B = \bigcap_{n\geqslant 1} \operatorname{Ann} \mathrm{K}(\boldsymbol{x}(n), R) = \bigcap_{n\geqslant 1} I_n$ by Proposition 2.3(3). It follows that $\mathrm{V}\big(\bigcap_{n\geqslant 1} I_n\big) \subseteq \bigcup_{n\geqslant 1} \mathrm{V}(I_n)$. $\square$

Our second main result in this section deals with the difference between the radical and tame closures.

**Theorem 6.6.** *Let $W$ be a specialization-closed subset of* $\operatorname{Spec} R$. *Set* $\mathcal{X} = \langle W \rangle$ *and* $\mathcal{Y} = \operatorname{Supp}^{-1} W$.

(1) *The subcategory $\mathcal{X}$ is compact, and satisfies $\mathcal{X}^{\mathsf{rad}} = \sqrt{\mathcal{X}}$ and $\mathcal{X}^{\mathsf{tame}} = \mathcal{Y}$.*

(2) *The subcategories $\mathcal{X}$ and $\mathcal{Y}$ are the smallest and largest thick $\otimes$-ideals of* $\mathsf{D}^-(R)$ *whose supports are $W$, respectively. In particular, $\mathcal{X} \subseteq \sqrt{\mathcal{X}} \subseteq \mathcal{Y}$.*

(3) *Assume that $R$ is either a domain or a local ring, and that $W$ is nonempty and proper. Then one has $\sqrt{\mathcal{X}} \subsetneq \mathcal{Y}$. Hence, $\mathcal{Y}$ is not compact, and $\mathcal{X}^{\mathsf{rad}} \subsetneq \mathcal{X}^{\mathsf{tame}}$.*

*Proof.* (1) The first statement is evident. The equalities follow from Lemma 5.8 and Proposition 1.10.

(2) Let $\mathcal{Z}$ be a thick $\otimes$-ideal of $\mathsf{D}^-(R)$ whose support is $W$. Then it is clear that $\mathcal{Z}$ is contained in $\mathcal{Y}$. Proposition 2.9 implies that $R/\mathfrak{p}$ belongs to $\mathcal{Z}$ for each $\mathfrak{p} \in W$, which shows that $\mathcal{Z}$ contains $\mathcal{X}$.

(3) Since $W$ is nonempty, there is a prime ideal $\mathfrak{p} \in W$. Let $\boldsymbol{x} = x_1, \dots, x_r$ be a system of generators of $\mathfrak{p}$, and put $C = \bigoplus_{i \geqslant 0} \mathrm{K}(\boldsymbol{x}^{i+1}, R)[i]$, which is an object of $\mathsf{D}^-(R)$. The support of $C$ is equal to $\mathrm{V}(\mathfrak{p})$ by Proposition 2.3(3), which is contained in $W$ as it is specialization-closed. Hence, $C$ is in $\operatorname{Supp}^{-1} W = \mathcal{Y}$.

Suppose that $\sqrt{\mathcal{X}}$ coincides with $\mathcal{Y}$, and let us derive a contradiction. There exists an integer $n > 0$ such that the $n$-fold tensor product $D := C \otimes_R^{\boldsymbol{L}} \cdots \otimes_R^{\boldsymbol{L}} C$ belongs to $\mathcal{X}$. An analogous argument to (6.5.1) yields that $D$ contains $E := \bigoplus_{k \geqslant 0} \mathrm{K}(\boldsymbol{x}^{k+1}, R)[nk]$ as a direct summand, whence $E$ belongs to $\mathcal{X}$. We use a similar technique to the one in the latter half of the proof of Proposition 6.2. By Lemma 6.1, there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m \in W$ and integers $e_1, \dots, e_m > 0$ such that $\operatorname{Ann} E$ contains $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$. We have

$$\operatorname{Ann} E = \bigcap_{k \geqslant 0} \operatorname{Ann} \mathrm{K}(\boldsymbol{x}^{k+1}, R) = \bigcap_{k \geqslant 0} \boldsymbol{x}^{k+1} R = 0 \tag{6.6.1}$$

by Proposition 2.3(3) and Krull's intersection theorem. This yields $\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m} = 0$, which says that each prime ideal of $R$ contains $\mathfrak{p}_i$ for some $1 \leqslant i \leqslant m$. As $W$ is specialization-closed, we observe that $W = \operatorname{Spec} R$, which is contrary to the assumption. Consequently, $\sqrt{\mathcal{X}}$ is strictly contained in $\mathcal{Y}$.

If $\mathcal{Y}$ is compact, then we have $\mathcal{Y} = \langle \operatorname{Supp} \mathcal{Y} \rangle = \langle W \rangle = \mathcal{X} \subseteq \sqrt{\mathcal{X}}$ by Propositions 5.4 and 1.10(1), which is a contradiction. Hence, $\mathcal{Y}$ is not compact. □

**Remark 6.7.** (1) Let $\mathfrak{p}, C$ be as in the proof of Theorem 6.6(3). Then

(a) $\operatorname{Supp} C$ is contained in $\operatorname{Supp} R/\mathfrak{p}$, but $C$ does not belong to $\mathsf{thick}^{\otimes} R/\mathfrak{p}$.
(b) $\mathrm{V}(\operatorname{Ann} R)$ is contained in $\mathrm{V}(\operatorname{Ann} C)$, but $R$ does not belong to $\mathsf{thick}^{\otimes} C$.

This guarantees in Proposition 2.9 one cannot replace $\mathrm{V}(\operatorname{Ann} X)$ by $\operatorname{Supp} X$, or $\operatorname{Supp} \mathcal{Y}$ by $\mathrm{V}(\operatorname{Ann} \mathcal{Y})$.

Indeed, we have $\operatorname{Supp} C = \operatorname{Supp} R/\mathfrak{p} = \mathrm{V}(\mathfrak{p}) \subseteq W \neq \operatorname{Spec} R$ and $\operatorname{Ann} C = \bigcap_{i \geqslant 0} \boldsymbol{x}^{i+1} R = 0$. The former together with Proposition 4.11 shows $R \notin \mathsf{thick}^{\otimes} C$, while the latter implies $\mathrm{V}(\operatorname{Ann} R) = \mathrm{V}(0) = \mathrm{V}(\operatorname{Ann} C)$. Assume $C$ is in $\mathsf{thick}^{\otimes} R/\mathfrak{p}$. Then $\operatorname{Ann} C = 0$ contains some power of $\operatorname{Ann} R/\mathfrak{p} = \mathfrak{p}$ by Lemma 6.1. Hence, $\mathrm{V}(\mathfrak{p}) = \operatorname{Spec} R$, which is a contradiction. Therefore, $C$ is not in $\mathsf{thick}^{\otimes} R/\mathfrak{p}$.

(2) The assumption in Theorem 6.6(3) that $R$ is either domain or local is indispensable. In fact, let $R = A \times B$ be a direct product of two commutative noetherian rings. Then $\operatorname{Spec} R = \operatorname{Spec} A \sqcup \operatorname{Spec} B$ and $\mathsf{D}^-(R) \cong \mathsf{D}^-(A) \times \mathsf{D}^-(B)$, which imply that $\operatorname{Supp}^{-1}_{\mathsf{D}^-(R)}(\operatorname{Spec} A) = \mathsf{D}^-(A) = \langle \operatorname{Spec} A \rangle_{\mathsf{D}^-(R)}$.

(3) Recall that we have the following first section-retraction pair (Proposition 5.16), while Corollary 5.9 gives rise to the following second section-retraction pair:

$$(\,\cdot\,)^{\mathsf{rad}} : \mathbf{Cpt} \rightleftarrows \mathbf{Rad} : (\,\cdot\,)_{\mathsf{cpt}}, \qquad \mathsf{inc} : \mathbf{Tame} \rightleftarrows \mathbf{Rad} : (\,\cdot\,)^{\mathsf{tame}}.$$

Corollary 5.21 implies that the left diagram below commutes. Therefore, it is natural to ask whether the right diagram below also commutes:

$$\begin{tikzcd} & \mathbf{Rad} \arrow[dl, "(\,\cdot\,)^{\mathsf{cpt}}"'] \arrow[dr, "(\,\cdot\,)^{\mathsf{tame}}"] & \\ \mathbf{Cpt} \arrow[rr, equal, "\sim"] & & \mathbf{Tame} \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{Rad} & \\ \mathbf{Cpt} \arrow[ur, "(\,\cdot\,)^{\mathsf{rad}}"] \arrow[rr, equal, "\sim"] & & \mathbf{Tame} \arrow[ul, "\mathsf{inc}"'] \end{tikzcd}$$

This is equivalent to asking if $(\mathscr{X}_{\mathsf{cpt}})^{\mathsf{rad}} = \mathscr{X}$ for all $\mathscr{X} \in \mathbf{Tame}$, and to asking if $\mathscr{Y}^{\mathsf{tame}} = \mathscr{Y}^{\mathsf{rad}}$ for all $\mathscr{Y} \in \mathbf{Cpt}$. Theorem 6.6 gives rise to a negative answer to this question.

Finally, we consider a conjecture of Balmer. Let $\mathcal{T}$ be an arbitrary essentially small tensor triangulated category. Balmer [2010a] constructs a continuous map

$$\rho^{\bullet}_{\mathcal{T}} : \mathsf{Spc}\, \mathcal{T} \to \operatorname{Spec}^{\mathrm{h}} \mathrm{R}^{\bullet}_{\mathcal{T}}$$

given by $\rho^{\bullet}_{\mathcal{T}}(\mathcal{P}) = (f \in \mathrm{R}^{\bullet}_{\mathcal{T}} \mid \mathsf{cone}\, f \notin \mathcal{P})$, where $\mathrm{R}^{\bullet}_{\mathcal{T}} = \operatorname{Hom}_{\mathcal{T}}(\mathbf{1}, \Sigma^{\bullet}\mathbf{1})$ is a graded-commutative ring. (The ideal generated by a subset $S$ of a ring $A$ is denoted by $(S)$.) Recall that a triangulated category is called *algebraic* if it arises as the stable category of some Frobenius exact category. Balmer [2010b, Conjecture 72] conjectures the following:

**Conjecture 6.8** (Balmer). The map $\rho^{\bullet}_{\mathcal{T}}$ is (locally) injective when $\mathcal{T}$ is algebraic.

Here, recall that a continuous map $f : X \to Y$ of topological spaces is called *locally injective at* $x \in X$ if there exists a neighborhood $N$ of $x$ such that the restriction $f|_N : N \to Y$ is an injective map. We say that $f$ is *locally injective* if it is locally injective at every point in $X$. If for any $x \in X$ there exists a neighborhood $E$

of $f(x)$ such that the induced map $f^{-1}(E) \to E$ is injective, then $f$ is locally injective.

Let us consider the above conjecture for our tensor triangulated category $\mathsf{D}^-(R)$. It turns out that for $\mathcal{T} = \mathsf{D}^-(R)$, Balmer's constructed map $\rho^\bullet_{\mathcal{T}}$ coincides with our constructed map $\mathfrak{s} : \mathsf{Spc}\,\mathsf{D}^-(R) \to \operatorname{Spec} R$.

**Proposition 6.9.** *Let $\mathcal{P}$ be a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. One then has*

(1) $\mathfrak{s}(\mathcal{P}) = (a \in R \mid R/a \notin \mathcal{P}) = \{a \in R \mid R/a \notin \mathcal{P}\}$ *and*

(2) $\mathfrak{s}(\mathcal{P}) = \rho^\bullet_{\mathsf{D}^-(R)}(\mathcal{P})$.

*Proof.* Corollary 2.11 and (1) imply (2). Let us show (1). Set $J = (a \in R \mid R/a \notin \mathcal{P})$. As $R$ is noetherian, we find a finite number of elements $x_1, \dots, x_n$ with $R/x_1, \dots, R/x_n \notin \mathcal{P}$ and $J = (x_1, \dots, x_n)$. Therefore, $\mathrm{K}(x_1, \dots, x_n, R) = \mathrm{K}(x_1, R) \otimes^{\boldsymbol{L}}_R \cdots \otimes^{\boldsymbol{L}}_R \mathrm{K}(x_n, R)$ is not in $\mathcal{P}$ by Corollary 2.11 and the fact that $\mathcal{P}$ is prime. Using Corollary 2.11 again shows $J \in \mathbb{I}(\mathcal{P})$, whence $J$ is contained in $\mathfrak{s}(\mathcal{P})$. Next, take any $a \in \mathfrak{s}(\mathcal{P})$. Since $\mathrm{V}(\mathfrak{s}(\mathcal{P}))$ is not contained in $\operatorname{Supp}\mathcal{P}$, neither is $\mathrm{V}(a)$. This implies $R/a \notin \mathcal{P}$ by Corollary 2.11. □

As an application of our Theorem 6.6, we confirm that Conjecture 6.8 is not true in general; our $\mathsf{D}^-(R)$ is an algebraic triangulated category, but does not satisfy Conjecture 6.8 under quite mild assumptions.

**Corollary 6.10.** *Assume that $R$ has positive dimension, and that $R$ is either a domain or a local ring. Then the map $\mathfrak{s} : \mathsf{Spc}\,\mathsf{D}^-(R) \to \operatorname{Spec} R$ is not locally injective. Hence, Conjecture 6.8 does not hold.*

*Proof.* We can choose a nonunit $x \in R$ such that the ideal $xR$ of $R$ has positive height (hence, it has height 1). Put $\mathcal{X} = \langle \mathrm{V}(x) \rangle$. Using Theorem 6.6(3) and Lemma 6.3, we find a prime thick $\otimes$-ideal $\mathcal{P}$ such that $\mathcal{X} \subseteq \mathcal{P} \subsetneq \mathcal{P}^{\mathsf{tame}}$. Suppose that $\mathfrak{s}$ is locally injective at $\mathcal{P}$. Then there exists a complex $M \in \mathsf{D}^-(R)$ with $\mathcal{P} \in \mathtt{U}(M)$ such that the restriction $\mathfrak{s}|_{\mathtt{U}(M)} : \mathtt{U}(M) \to \operatorname{Spec} R$ is injective. Since $M$ is in $\mathcal{P}$, it is also in $\mathcal{P}^{\mathsf{tame}}$. Hence, both $\mathcal{P}$ and $\mathcal{P}^{\mathsf{tame}}$ belong to $\mathtt{U}(M)$. However, these two prime thick $\otimes$-ideals are sent by $\mathfrak{s}$ to the same point; see Theorem 3.9. This contradicts the injectivity of $\mathfrak{s}|_{\mathtt{U}(M)}$, and we conclude that $\mathfrak{s}$ is not locally injective at $\mathcal{P}$. The last assertion of the corollary follows from Proposition 6.9(2). □

**Remark 6.11.** The reader may think that Corollary 6.10 can also be obtained by showing that the map

$$f : \mathsf{Spc}\,\mathsf{D}^-(R) \to \mathsf{Spc}\,\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R), \quad \mathcal{P} \mapsto \mathcal{P} \cap \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$$

is not injective. We are not sure whether the noninjectivity of the map $f$ implies Corollary 6.10, but at least showing the noninjectivity of $f$ is equivalent to our approach. Using Proposition 2.9, we see that $\mathcal{P} \cap \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)$ contains the

Koszul complex of a system of generators of each prime ideal belonging to $\operatorname{Supp}\mathcal{P}$. Hence, $\operatorname{Supp}(\mathcal{P}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R))=\operatorname{Supp}\mathcal{P}$, and the Hopkins–Neeman theorem implies $\mathcal{P}\cap\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)=\operatorname{Supp}^{-1}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,R)}\operatorname{Supp}\mathcal{P}$. Therefore, for $\mathcal{P},\mathcal{Q}\in\operatorname{Spc}\mathsf{D}^-(R)$,

$$f(\mathcal{P})=f(\mathcal{Q})\iff\operatorname{Supp}\mathcal{P}=\operatorname{Supp}\mathcal{Q},$$

which says that the map $f$ is injective if and only if all the prime thick tensor ideals of $\mathsf{D}^-(R)$ are tame. In the end, even if we intend to prove Corollary 6.10 by showing the noninjectivity of the map $f$, we must find a nontame prime thick tensor ideal of $\mathsf{D}^-(R)$, which is what we have done in this section.

## 7. Thick tensor ideals over discrete valuation rings

In this section, we concentrate on handling the case where $R$ is a discrete valuation ring. Several properties that are specific to this case are found out in this section. Just for convenience, we write complexes as chain complexes, rather than as cochain complexes. We start by studying complexes with zero differentials.

**Proposition 7.1.** *Let* $X=\bigoplus_{i\geqslant 0}X_i[i]=(\cdots\xrightarrow{0}X_3\xrightarrow{0}X_2\xrightarrow{0}X_1\xrightarrow{0}X_0\to 0)$ *be a complex in* $\mathsf{D}^-(R)$. *Then it holds that* $\mathsf{thick}^{\otimes}X=\mathsf{thick}^{\otimes}Y$ *in* $\mathsf{D}^-(R)$, *where*

$$\begin{aligned}Y&=\bigoplus_{i\geqslant 0}\Bigl(\bigoplus_{j=0}^{i}X_j\Bigr)[i]\\&=(\cdots\xrightarrow{0}X_3\oplus X_2\oplus X_1\oplus X_0\xrightarrow{0}X_2\oplus X_1\oplus X_0\xrightarrow{0}X_1\oplus X_0\xrightarrow{0}X_0\to 0).\end{aligned}$$

*Proof.* Putting $F=\bigoplus_{j\geqslant 0}R[j]$, we have $X\otimes^{\boldsymbol{L}}_R F=\bigl(\bigoplus_{i\geqslant 0}X_i[i]\bigr)\otimes^{\boldsymbol{L}}_R\bigl(\bigoplus_{j\geqslant 0}R[j]\bigr)=\bigoplus_{i,j\geqslant 0}X_i[i+j]=Y$. Hence, $\mathsf{thick}^{\otimes}X$ contains $\mathsf{thick}^{\otimes}Y$. The opposite inclusion also holds as $X$ is a direct summand of $Y$. ☐

**Proposition 7.2.** *Let* $X=\bigoplus_{i\geqslant 0}X_i[i]=(\cdots\xrightarrow{0}X_3\xrightarrow{0}X_2\xrightarrow{0}X_1\xrightarrow{0}X_0\to 0)$ *be a complex in* $\mathsf{D}^-(R)$. *Then for all integers* $a_i\geqslant 0$, *the thick* $\otimes$*-ideal closure* $\mathsf{thick}^{\otimes}X$ *in* $\mathsf{D}^-(R)$ *contains*

$$\bigoplus_{i\geqslant 0}X_i^{\oplus a_i}[2i]=(\cdots\to X_3^{\oplus a_3}\to 0\to X_2^{\oplus a_2}\to 0\to X_1^{\oplus a_1}\to 0\to X_0^{\oplus a_0}\to 0).$$

*Proof.* The complex $\bigoplus_{i\geqslant 0}X_i^{\oplus a_i}[2i]=\bigoplus_{i\geqslant 0}(X_i\otimes^{\boldsymbol{L}}_R R^{\oplus a_i})[2i]$ is a direct summand of $\bigoplus_{i,j\geqslant 0}(X_i\otimes^{\boldsymbol{L}}_R R^{\oplus a_j})[i+j]=\bigl(\bigoplus_{i\geqslant 0}X_i[i]\bigr)\otimes^{\boldsymbol{L}}_R\bigl(\bigoplus_{j\geqslant 0}R^{\oplus a_j}[j]\bigr)=X\otimes^{\boldsymbol{L}}_R Y$ in the category $\mathsf{D}^-(R)$, where $Y=\bigoplus_{j\geqslant 0}R^{\oplus a_j}[j]=(\cdots\xrightarrow{0}R^{\oplus a_2}\xrightarrow{0}R^{\oplus a_1}\xrightarrow{0}R^{\oplus a_0}\to 0)$ is a complex in $\mathsf{D}^-(R)$. Thus, the assertion follows. ☐

**Corollary 7.3.** *Let* $X=\bigoplus_{i\geqslant 0}X_i[i]=(\cdots\xrightarrow{0}X_3\xrightarrow{0}X_2\xrightarrow{0}X_1\xrightarrow{0}X_0\to 0)$ *be a complex in* $\mathsf{D}^-(R)$. *Then for any integers* $a_i\geqslant 0$ *the complex*

$$Y=\bigoplus_{i\geqslant 0}X_i^{\oplus a_i}[i]=(\cdots\xrightarrow{0}X_3^{\oplus a_3}\xrightarrow{0}X_2^{\oplus a_2}\xrightarrow{0}X_1^{\oplus a_1}\xrightarrow{0}X_0^{\oplus a_0}\to 0)$$

*is in* $\mathsf{thick}^{\otimes}\{X_{\mathrm{even}}, X_{\mathrm{odd}}\}$, *where* $X_{\mathrm{even}} = \bigoplus_{i\geqslant 0} X_{2i}[i] = (\cdots \xrightarrow{0} X_6 \xrightarrow{0} X_4 \xrightarrow{0} X_2 \xrightarrow{0} X_0 \to 0)$ *and* $X_{\mathrm{odd}} = \bigoplus_{i\geqslant 0} X_{2i+1}[i] = (\cdots \xrightarrow{0} X_7 \xrightarrow{0} X_5 \xrightarrow{0} X_3 \xrightarrow{0} X_1 \to 0)$.

*Proof.* The complex $Y$ is the direct sum of $A = (\cdots \to 0 \to X_4^{\oplus a_4} \to 0 \to X_2^{\oplus a_2} \to 0 \to X_0^{\oplus a_0} \to 0)$ and $B = (\cdots \to X_5^{\oplus a_5} \to 0 \to X_3^{\oplus a_3} \to 0 \to X_1^{\oplus a_1} \to 0 \to 0)$. Proposition 7.2 shows that $A$ is in $\mathsf{thick}^{\otimes} X_{\mathrm{even}}$ and $B$ is in $\mathsf{thick}^{\otimes} X_{\mathrm{odd}}$. Therefore, $Y$ belongs to $\mathsf{thick}^{\otimes}\{X_{\mathrm{even}}, X_{\mathrm{odd}}\}$. □

A natural question arises from Proposition 7.2 and Corollary 7.3:

**Question 7.4.** Does $\mathsf{thick}^{\otimes}(\cdots \to 0 \to X_2 \to 0 \to X_1 \to 0 \to X_0 \to 0)$ contain $(\cdots \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$? Does $\mathsf{thick}^{\otimes}(\cdots \xrightarrow{0} X_1 \xrightarrow{0} X_0 \to 0)$ contain $(\cdots \xrightarrow{0} X_1^{\oplus a_1} \xrightarrow{0} X_0^{\oplus a_0} \to 0)$ for all integers $a_i \geqslant 0$?

We do not know the general answer to this question. The following example gives an affirmative answer.

**Example 7.5.** Let $(R, xR)$ be a discrete valuation ring. Then

$$\begin{aligned}\mathsf{thick}^{\otimes}&(\cdots \xrightarrow{0} R/x^3 \xrightarrow{0} R/x^2 \xrightarrow{0} R/x \to 0)\\ &= \mathsf{thick}^{\otimes}(\cdots \to 0 \to R/x^3 \to 0 \to R/x^2 \to 0 \to R/x \to 0).\end{aligned}$$

*Proof.* In fact, the inclusion $(\supseteq)$ follows from Proposition 7.2. To check the inclusion $(\subseteq)$, set $A = (\cdots \xrightarrow{0} R/x^3 \xrightarrow{0} R/x^2 \xrightarrow{0} R/x \to 0)$ and $B = (\cdots \to 0 \to R/x^3 \to 0 \to R/x^2 \to 0 \to R/x \to 0)$. Note that for each integer $n \geqslant 0$ there is an exact sequence $0 \to R/x^n \xrightarrow{x^{n+1}} R/x^{2n+1} \to R/x^{n+1} \to 0$ of $R$-modules. This induces an exact sequence $0 \to C \to A \to B \to 0$ of complexes of $R$-modules, where

$$\begin{aligned}C = \Big(\cdots \xrightarrow{0} R/x^n \xrightarrow{0} R/x^{2n} \xrightarrow{0} R/x^{n-1} \xrightarrow{0} R/x^{2(n-1)}&\\ \xrightarrow{0} \cdots \xrightarrow{0} R/x^2 \xrightarrow{0} R/x^4 \xrightarrow{0} R/x \xrightarrow{0} R/x^2 \to 0\Big).&\end{aligned}$$

We see that $C = B[2] \oplus D$, where $D = (\cdots \to 0 \to R/x^{2n} \to 0 \to \cdots \to 0 \to R/x^4 \to 0 \to R/x^2 \to 0)$, and have an exact sequence $0 \to B[1] \to D \to B[1] \to 0$ of complexes. The assertion now follows. □

The *Loewy length* of a finitely generated $R$-module $M$, denoted by $\ell\ell_R(M)$, is by definition the infimum of integers $i$ such that the ideal $(\operatorname{rad} R)^i$ kills $M$. Let us consider thick $\otimes$-ideals defined by Loewy lengths.

**Notation 7.6.** Let $R$ be a local ring with maximal ideal $\mathfrak{m}$. Let $c \geqslant 0$ be an integer.

(1) Let $\mathscr{L}_c$ be the subcategory of $\mathsf{D}^-_{\mathsf{fl}}(R)$ consisting of complexes $X$ such that there exists an integer $t \geqslant 0$ with $\ell\ell(\mathsf{H}_i X) \leqslant t i^{c-1}$ for all $i \gg 0$.

(2) When $c \geqslant 1$, let $G_c$ be the complex $\bigoplus_{i>0}(R/\mathfrak{m}^{i^{c-1}})[i] = (\cdots \xrightarrow{0} R/\mathfrak{m}^{3^{c-1}} \xrightarrow{0} R/\mathfrak{m}^{2^{c-1}} \xrightarrow{0} R/\mathfrak{m} \to 0)$.

**Proposition 7.7.** *Let $(R, \mathfrak{m}, k)$ be local. One has $\mathscr{L}_0 \subsetneq \mathscr{L}_1 \subsetneq \mathscr{L}_2 \subsetneq \cdots$ and $\mathscr{L}_0 = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) = \mathsf{thick}_{\mathsf{D}^-(R)}\, k$.*

*Proof.* Fix an integer $n \geqslant 0$. It is clear that $\mathscr{L}_n$ is contained in $\mathscr{L}_{n+1}$. We have $\ell\ell(\mathsf{H}_i G_{n+1}) = i^n$ for each $i \geqslant 0$, which shows $\mathscr{L}_n \neq \mathscr{L}_{n+1}$. Hence, the chain $\mathscr{L}_0 \subsetneq \mathscr{L}_1 \subsetneq \mathscr{L}_2 \subsetneq \cdots$ is obtained. Let $X$ be a complex in $\mathsf{D}^-(R)$. Suppose that there exists an integer $t \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant t i^{-1}$ for $i \gg 0$. Then we have to have $\ell\ell(\mathsf{H}_i X) = 0$ for $i \gg 0$, which says that $\mathsf{H}_j X = 0$ for $j \gg 0$. Thus, we obtain $\mathscr{L}_0 = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) = \mathsf{thick}_{\mathsf{D}^-(R)}\, k$, where the second equality is shown in Proposition 1.4. □

Recall that an abelian category $\mathscr{A}$ is called *hereditary* if it has global dimension at most one, that is, if $\mathrm{Ext}^2_{\mathscr{A}}(\mathscr{A}, \mathscr{A}) = 0$. Recall also that a ring $R$ is called *hereditary* if $R$ has global dimension at most one.

From now on, we study thick $\otimes$-ideals of $\mathsf{D}^-(R)$ when $R$ is local and hereditary. In this case, $R$ is either a field or a discrete valuation ring. If $R$ is a field, then by Corollary 2.20 there are only trivial thick $\otimes$-ideals. So, we mainly consider the case of a discrete valuation ring. First, we mention a well known fact, saying that each complex in the derived category of a hereditary abelian category has zero differentials.

**Lemma 7.8** [Krause 2007, §1.6]. *Let $\mathscr{A}$ be a hereditary abelian category. Then for each object $M \in \mathsf{D}(\mathscr{A})$ there exists an isomorphism $M \cong \mathsf{H}(M) = \bigoplus_{i \in \mathbb{Z}} \mathsf{H}_i(M)[i]$ in $\mathsf{D}(\mathscr{A})$.*

The lemma below is part of our first main result in this section.

**Lemma 7.9.** *Let $R$ be a discrete valuation ring. Then $\mathscr{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$ for every $c \geqslant 1$.*

*Proof.* By Proposition 1.10(3), it suffices to show $\mathscr{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-_{\mathsf{fl}}(R)$. We do this step by step.

(1) Take any complex $X$ in $\mathscr{L}_c$. There exist integers $t, u \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant t i^{c-1}$ for all $i \geqslant u$. Let $Y$ be a direct summand of $X$ in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Then $\mathsf{H}_i Y$ is a direct summand of $\mathsf{H}_i X$, and we have $\ell\ell(\mathsf{H}_i Y) \leqslant \ell\ell(\mathsf{H}_i X) \leqslant t i^{c-1}$ for all $i \geqslant u$. Hence, $Y$ belongs to $\mathscr{L}_c$.

(2) Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Suppose that both $X$ and $Z$ belong to $\mathscr{L}_c$. Then there exist integers $t, u, a, b \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant t i^{c-1}$ and $\ell\ell(\mathsf{H}_j Z) \leqslant u j^{c-1}$ for all $i \geqslant a$ and $j \geqslant b$. An exact sequence $\cdots \to \mathsf{H}_k X \to \mathsf{H}_k Y \to \mathsf{H}_k Z \to \cdots$ is induced, and from this we see that $\ell\ell(\mathsf{H}_k Y) \leqslant \ell\ell(\mathsf{H}_k X) + \ell\ell(\mathsf{H}_k Z) \leqslant (t+u)k^{c-1}$ for all $k \geqslant \max\{a, b\}$. Therefore, $Y$ belongs to $\mathscr{L}_c$.

(3) Let $X$ be a complex in $\mathscr{L}_c$. Then there exist integers $t, u \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant t i^{c-1}$ for all $i \geqslant u$. It holds that $\ell\ell(\mathsf{H}_i(X[1])) = \ell\ell(\mathsf{H}_{i-1} X) \leqslant t(i-1)^{c-1} \leqslant t i^{c-1}$ for all $i \geqslant u+1$ for all $i \geqslant u+1$, where the second inequality holds as $c \geqslant 1$. Also,

$\ell\ell(\mathsf{H}_i(X[-1])) = \ell\ell(\mathsf{H}_{i+1}X) \leqslant t(i+1)^{c-1} \leqslant t(i+i)^{c-1} = (2^{c-1}t)\cdot i^{c-1}$ for all $i \geqslant \max\{1, u-1\}$, where the first inequality holds as $i \geqslant u-1$, and the second one holds since $i \geqslant 1$ and $c \geqslant 1$. Thus, the complexes $X[1]$ and $X[-1]$ belong to $\mathscr{L}_c$.

(4) Let $X, Y$ be complexes in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Suppose that $X$ belongs to $\mathscr{L}_c$. We want to show that $X \otimes^{\boldsymbol{L}}_R Y$ also belongs to $\mathscr{L}_c$. Taking into account (3) and Lemma 7.8, we may assume that $X = \bigoplus_{i\geqslant 1} X_i[i]$ and $Y = \bigoplus_{j\geqslant 0} Y_j[j]$ with $X_i, Y_j$ being $R$-modules, and that there exist $s \geqslant 1$ and $t \geqslant 0$ such that $\ell\ell(X_i) \leqslant ti^{c-1}$ for all $i \geqslant s$. Set $u = \max\{\ell\ell(X_i) \mid 1 \leqslant i \leqslant s-1\}$; note that each $X_i$ has finite length, whence has finite Loewy length. We have $X \otimes^{\boldsymbol{L}}_R Y = \bigoplus_{i\geqslant 1,\, j\geqslant 0}(X_i \otimes^{\boldsymbol{L}}_R Y_j)[i+j]$, and from this we get $\mathsf{H}_k(X \otimes^{\boldsymbol{L}}_R Y) = \bigoplus_{i\geqslant 1,\, j\geqslant 0,\, i+j\leqslant k} \operatorname{Tor}^R_{k-i-j}(X_i, Y_j)$ for all integers $k$. Note here that $\operatorname{Tor}^R_{k-i-j}(X_i, Y_j) = 0$ for $i + j > k$.

We claim that $\ell\ell(X_i) \leqslant (t+u)i^{c-1}$ for all $i \geqslant 1$. In fact, recall $c \geqslant 1$ and $t, u \geqslant 0$. If $i \geqslant s$, then $\ell\ell(X_i) \leqslant ti^{c-1} \leqslant (t+u)i^{c-1}$. If $1 \leqslant i \leqslant s-1$, then $\ell\ell(X_i) \leqslant u \leqslant t+u \leqslant (t+u)i^{c-1}$. The claim follows.

Fix three integers $i, j, k$ with $i \geqslant 1$, $j \geqslant 0$, and $i + j \leqslant k$. Then $(t+u)k^{c-1} \geqslant (t+u)i^{c-1}$ since $k \geqslant i$ and $c \geqslant 1$. The claim shows that $X_i$ is killed by $\mathfrak{m}^{(t+u)k^{c-1}}$, and so is $\operatorname{Tor}^R_{k-i-j}(X_i, Y_j)$, where $\mathfrak{m}$ stands for the maximal ideal of $R$. Hence, $\ell\ell(\mathsf{H}_k(X \otimes^{\boldsymbol{L}}_R Y)) \leqslant (t+u)k^{c-1}$ for all $k \in \mathbb{Z}$, which implies $X \otimes^{\boldsymbol{L}}_R Y \in \mathscr{L}_c$.

It follows from the above arguments (1)–(4) that $\mathscr{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-_{\mathsf{fl}}(R)$. $\square$

**Remark 7.10.** Let $(R, \mathfrak{m}, k)$ be a local ring. When $c = 0$, the subcategory $\mathscr{L}_c$ is never a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Indeed, by Proposition 7.7 we have $\mathscr{L}_0 = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$. The module $k$ is in $\mathscr{L}_0$, but the complex $(\cdots \xrightarrow{0} k \xrightarrow{0} k \to 0) = k \otimes^{\boldsymbol{L}}_R (\cdots \xrightarrow{0} R \xrightarrow{0} R \to 0)$ is not in $\mathscr{L}_0$.

Now we have our first theorem concerning the subcategories $\mathscr{L}_c$ of $\mathsf{D}^-(R)$ for a discrete valuation ring $R$. This especially says that the equality of Proposition 4.3(2) does not necessarily hold.

**Theorem 7.11.** *Let $R$ be a discrete valuation ring. Then $\mathscr{L}_c$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$ for all integers $c \geqslant 1$. In particular, one has*

$$\dim(\mathsf{Spc}\,\mathsf{D}^-(R)) = \infty > 1 = \dim R.$$

*Proof.* Lemma 7.9 says that $\mathscr{L}_c$ is a thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Proposition 7.7 especially says $\mathscr{L}_c \neq \mathsf{D}^-(R)$. Let $X, Y$ be complexes in $\mathsf{D}^-(R)$ with $X \otimes^{\boldsymbol{L}}_R Y \in \mathscr{L}_c$, and we shall prove that either $X$ or $Y$ is in $\mathscr{L}_c$. Applying Lemma 7.8 and taking shifts if necessary, we may assume $X = \bigoplus_{i\geqslant 0} X_i[i]$ and $Y = \bigoplus_{j\geqslant 0} Y_j[j]$, where $X_i, Y_j$ are finitely generated $R$-modules. Assume that $X$ is not in $\mathsf{D}^-_{\mathsf{fl}}(R)$. Then $X_a$ has infinite length for some $a \geqslant 0$. As $R$ is a discrete valuation ring, $X_a$ has a nonzero free direct summand. Hence, $R[a]$ is a direct summand of $X$, and $Y[a] = R[a] \otimes^{\boldsymbol{L}}_R Y$ is a direct summand of $X \otimes^{\boldsymbol{L}}_R Y$. As $X \otimes^{\boldsymbol{L}}_R Y$ is in $\mathscr{L}_c$, so is $Y$. Similarly, if $Y \notin \mathsf{D}^-_{\mathsf{fl}}(R)$, then $X \in \mathscr{L}_c$. This argument shows that we may assume that both $X$ and $Y$ belong to $\mathsf{D}^-_{\mathsf{fl}}(R)$, or equivalently, that all $X_i$ and $Y_j$ have finite length as $R$-modules. Since $X \otimes^{\boldsymbol{L}}_R Y$ belongs to $\mathscr{L}_c$, there exist integers $t, u \geqslant 0$ such that $\mathsf{H}_n(X \otimes^{\boldsymbol{L}}_R Y)$ has Loewy length at most $tn^{c-1}$ for all $n \geqslant u$. Assume that $X$ is not in $\mathscr{L}_c$. Then we can find an integer $e \geqslant u$ such that $\ell\ell(X_e) > te^{c-1}$. We have $X \otimes^{\boldsymbol{L}}_R Y = \bigoplus_{i,j \geqslant 0}(X_i \otimes^{\boldsymbol{L}}_R Y_j)[i+j]$, which gives rise to $\mathsf{H}_n(X \otimes^{\boldsymbol{L}}_R Y) = \bigoplus_{i,j \geqslant 0} \operatorname{Tor}_{n-i-j}(X_i, Y_j)$ for all integers $n$. Setting $a_i = \ell\ell(X_i)$ and $b_j = \ell\ell(Y_j)$ for $i, j \geqslant 0$, we obtain for every integer $n \geqslant e$

$$\begin{aligned}\mathsf{H}_n(X \otimes^{\boldsymbol{L}}_R Y) &\gtrdot \operatorname{Tor}_{n-e-(n-e)}(X_e, Y_{n-e}) \\ &= X_e \otimes_R Y_{n-e} \gtrdot R/x^{a_e} \otimes_R R/x^{b_{n-e}} = R/x^{\min\{a_e, b_{n-e}\}}.\end{aligned}$$

It is seen that $\min\{a_e, b_{n-e}\} \leqslant tn^{c-1}$ for all $n \geqslant e$. As $a_e > te^{c-1}$, we must have $a_e > b_{n-e}$, and $b_{n-e} \leqslant tn^{c-1}$ for all $n \geqslant e$. Hence, $\ell\ell(\mathsf{H}_n(Y[e])) = \ell\ell(Y_{n-e}) = b_{n-e} \leqslant tn^{c-1}$ for $n \geqslant e$, which implies that $Y[e]$ is in $\mathscr{L}_c$, and so is $Y$. Similarly, if $Y$ is not in $\mathscr{L}_c$, then $X$ is in $\mathscr{L}_c$. Thus, $\mathscr{L}_c$ is a prime thick $\otimes$-ideal of $\mathsf{D}^-(R)$. Now $\mathscr{L}_1 \subsetneq \mathscr{L}_2 \subsetneq \mathscr{L}_3 \subsetneq \cdots$ from Lemma 7.9 is an ascending chain of prime thick $\otimes$-ideals with infinite length, which shows the inequality in the proposition; see Proposition 4.3(1). ☐

To give an application of the above theorem, we state and prove a lemma.

**Lemma 7.12.** *For each prime ideal* $\mathfrak{p}$ *of* $R$, $\dim \operatorname{Spc} \mathsf{D}^-(R_\mathfrak{p}) \leqslant \dim \operatorname{Spc} \mathsf{D}^-(R)$.

*Proof.* We first show that the localization functor $L : \mathsf{D}^-(R) \to \mathsf{D}^-(R_\mathfrak{p})$ is an essentially surjective. Let $X = (\cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \to 0)$ be a complex in $\mathsf{D}^-(R_\mathfrak{p})$. What we want is a complex $Y \in \mathsf{D}^-(R)$ such that $X \cong L(Y)$. For each integer $i \geqslant 0$, choose a finitely generated $R$-module $Y_i$ with $(Y_i)_\mathfrak{p} = X_i$, and $R$-linear maps $d_i^Y : Y_i \to Y_{i-1}$ and $s_i \in R \setminus \mathfrak{p}$ such that $d_i^X = d_i^Y/s_i$ in $\operatorname{Hom}_{R_\mathfrak{p}}(X_i, X_{i-1}) = \operatorname{Hom}(Y_i, Y_{i-1})_\mathfrak{p}$. Then $(d_{i-1}^Y d_i^Y)/(s_{i-1}s_i) = d_{i-1}^X d_i^X = 0$, and there is an element $t_i \in R \setminus \mathfrak{p}$ such that $t_i d_{i-1}^Y d_i^Y = 0$. Define a complex $Y = (\cdots \xrightarrow{t_{i+1}d_{i-1}^Y} Y_i \xrightarrow{t_i d_i^Y} \cdots \xrightarrow{t_2 d_2^Y} Y_1 \xrightarrow{t_1 d_1^Y} Y_0 \to 0)$ in $\mathsf{D}^-(R)$. Then there is an isomorphism

$$\begin{array}{ccccccccccccc}
Y_\mathfrak{p} = (\cdots \longrightarrow & (Y_i)_\mathfrak{p} & \xrightarrow{\frac{t_i d_i^Y}{1}} & (Y_{i-1})_\mathfrak{p} & \longrightarrow \cdots \longrightarrow & (Y_2)_\mathfrak{p} & \xrightarrow{\frac{t_2 d_2^Y}{1}} & (Y_1)_\mathfrak{p} & \xrightarrow{\frac{t_1 d_1^Y}{1}} & (Y_0)_\mathfrak{p} & \longrightarrow 0) \\
\Big\downarrow & \cong\Big\downarrow u_i & & \cong\Big\downarrow u_{i-1} & & \cong\Big\downarrow u_2 & & \cong\Big\downarrow u_1 & & \Big\| & \\
X = (\cdots \longrightarrow & X_i & \xrightarrow{d_i^X} & X_{i-1} & \longrightarrow \cdots \longrightarrow & X_2 & \xrightarrow{d_2^X} & X_1 & \xrightarrow{d_1^X} & X_0 & \longrightarrow 0)
\end{array}$$

of complexes, where $u_i := t_1 \cdots t_i s_1 \cdots s_i$. Thus, we obtain $L(Y) = Y_\mathfrak{p} \cong X$.

The essentially surjective tensor triangulated functor $L$ induces an injective continuous map $\operatorname{Spc} L : \operatorname{Spc} \mathsf{D}^-(R_\mathfrak{p}) \to \operatorname{Spc} \mathsf{D}^-(R)$ given by $\mathscr{P} \mapsto L^{-1}(\mathscr{P})$ [Balmer

2005, Corollary 3.8]. This map sends a chain $\mathscr{P}_0 \subsetneq \cdots \subsetneq \mathscr{P}_n$ of prime thick $\otimes$-ideals of $\mathsf{D}^-(R_\mathfrak{p})$ to the chain $L^{-1}(\mathscr{P}_0) \subsetneq \cdots \subsetneq L^{-1}(\mathscr{P}_n)$ of prime thick $\otimes$-ideals of $\mathsf{D}^-(R)$. The lemma now follows. □

The following corollary of Theorem 7.11 provides a class of rings $R$ such that the Balmer spectrum of $\mathsf{D}^-(R)$ has infinite Krull dimension. This class includes normal local domains for instance.

**Corollary 7.13.** *If $R_\mathfrak{p}$ is regular for some $\mathfrak{p}$ with* $\operatorname{ht}\mathfrak{p} > 0$, *then* $\dim \mathsf{Spc}\, \mathsf{D}^-(R) = \infty$.

*Proof.* We may assume $\operatorname{ht}\mathfrak{p} = 1$. We have $\dim \mathsf{Spc}\, \mathsf{D}^-(R) \geqslant \dim \mathsf{Spc}\, \mathsf{D}^-(R_\mathfrak{p}) = \infty$, where the inequality follows from Lemma 7.12, and the equality is shown in Theorem 7.11. □

Next we study generation of the thick tensor ideals $\mathscr{L}_c$. In fact each of them possesses a single generator.

**Theorem 7.14.** *Let $(R, xR, k)$ be a discrete valuation ring, and let $c \geqslant 1$ be an integer. It then holds that $\mathscr{L}_c = \mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} G_c$. In particular, one has $\mathscr{L}_1 = \mathsf{thick}^{\otimes}_{\mathsf{D}^-(R)} k$.*

*Proof.* Clearly, $G_c$ is in $\mathscr{L}_c$. Lemma 7.9 implies that $\mathsf{thick}^{\otimes} G_c$ is contained in $\mathscr{L}_c$. We establish a claim.

**Claim.** *Let $0 \leqslant n \leqslant c-1$ be an integer. Let $X \in \mathsf{D}^-_{\mathsf{fl}}(R)$ be a complex. Suppose that there exists an integer $t \geqslant 0$ such that $\ell\ell(\mathsf{H}_i X) \leqslant t i^n$ for all $i \gg 0$. Then $X$ belongs to $\mathsf{thick}^{\otimes} G_c$.*

Once we show this claim, it will follow that $\mathscr{L}_c$ is contained in $\mathsf{thick}^{\otimes} G_c$, and we will be done.

First of all, note that $k$ is a direct summand of $G_c$. Combining this with Proposition 1.4, we have

$$\mathsf{thick}^{\otimes} G_c \supseteq \mathsf{thick}^{\otimes} k \supseteq \mathsf{thick}\, k = \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R). \tag{7.14.1}$$

Let $X$ be a complex as in the claim. Using Lemma 7.8, we may assume $X = \bigoplus_{i \geqslant s} X_i[i]$ for some integer $s$ and $R$-modules $X_i$ of finite length. There is an integer $u \geqslant s$ with $\ell\ell(X_i) \leqslant t i^n$ for all $i \geqslant u$. We have $X = \big(\bigoplus_{i \geqslant u} X_i[i]\big) \oplus \big(\bigoplus_{i=s}^{u-1} X_i[i]\big)$, whose latter summand is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$. In view of (7.14.1), replacing $X$ with the former summand, we may assume $u = s$. When $s \geqslant 0$, we set $X_i = 0$ for $0 \leqslant i \leqslant s-1$. When $s < 0$, we have $X = \big(\bigoplus_{i \geqslant 0} X_i[i]\big) \oplus \big(\bigoplus_{i=s}^{-1} X_i[i]\big)$, whose latter summand is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$. By similar replacement as above, we may assume $s = 0$. Thus, $X = \bigoplus_{i \geqslant 0} X_i[i]$ and $\ell\ell(X_i) \leqslant t i^n$ for all $i \geqslant 0$.

Since $R$ is a discrete valuation ring with maximal ideal $xR$, for every $i \geqslant 1$ there is an integer $a_{ij} \geqslant 0$ such that $X_i$ is isomorphic to $\bigoplus_{j=1}^{ti^n} (R/x^j)^{\oplus a_{ij}}$. Therefore, it holds that

$$X \cong \bigoplus_{i\geqslant 0}\Big(\bigoplus_{j=1}^{ti^n}(R/x^j)^{\oplus a_{ij}}\Big)[i] \lessdot \bigoplus_{i\geqslant 0}\Big(\bigoplus_{j=1}^{ti^n} R/x^j\Big)^{\oplus a_i}[i]$$

$$\in \mathsf{thick}^{\otimes}\Big\{\bigoplus_{i\geqslant 0}\Big(\bigoplus_{j=1}^{t(2i)^n} R/x^j\Big)[i], \bigoplus_{i\geqslant 0}\Big(\bigoplus_{j=1}^{t(2i+1)^n} R/x^j\Big)[i]\Big\}$$

$$= \mathsf{thick}^{\otimes}\Big\{A_1,\, A_2 \oplus \Big(\bigoplus_{j=1}^{t} R/x^j\Big)\Big\},$$

where $a_i := \max\{a_{ij} \mid 1 \leqslant j \leqslant ti^n\}$ and $A_l := \bigoplus_{i\geqslant 1}\big(\bigoplus_{j=t(2i-l)^n+1}^{t(2i-l+2)^n} R/x^j\big)[i]$ for $l = 1, 2$. The relations "$\in$" and "$=$" follow from Corollary 7.3 and Proposition 7.1, respectively. Since $\bigoplus_{j=1}^{t} R/x^j$ is in $\mathsf{thick}^{\otimes} G_c$ by (7.14.1), it suffices to show that $A_l$ belongs to $\mathsf{thick}^{\otimes} G_c$ for $l = 1, 2$.

We prove this by induction on $n$. When $n = 0$, we have $A_1 = A_2 = 0 \in \mathsf{thick}^{\otimes} G_c$, and are done. Let $n \geqslant 1$. Fix $l = 1, 2$. The exact sequences

$$0 \to R/x^{t(2i-l)^n} \xrightarrow{x^j} R/x^{j+t(2i-l)^n} \to R/x^j \to 0 \ \ (i \geqslant 1,\ 1 \leqslant j \leqslant tb_{il})$$

with $b_{il} = (2i - l + 2)^n - (2i - l)^n$ induce exact sequences

$$0 \to (R/x^{t(2i-l)^n})^{\oplus tb_{il}} \to \bigoplus_{j=t(2i-l)^n+1}^{t(2i-l+2)^n} R/x^j \to \bigoplus_{j=1}^{tb_{il}} R/x^j \to 0 \ \ (i \geqslant 1),$$

which induce an exact triangle $B_l \to A_l \to C_l \rightsquigarrow$ in $\mathsf{D}^-_{\mathsf{fl}}(R)$, where we set $B_l = \bigoplus_{i\geqslant 1}(R/x^{t(2i-l)^n})^{\oplus tb_{il}}[i]$ and $C_l = \bigoplus_{i\geqslant 1}\big(\bigoplus_{j=1}^{tb_{il}} R/x^j\big)[i]$. Since $\ell\ell(\mathsf{H}_i C_l) = tb_{il}$ has degree at most $n-1$ as a polynomial in $i$, the induction hypothesis implies that $C_l$ is in $\mathsf{thick}^{\otimes} G_c$. By Corollary 7.3, $B_l$ belongs to

$$\mathsf{thick}^{\otimes}\Big\{\bigoplus_{i\geqslant 0}(R/x^{t(4i+r)^n})[i] \;\Big|\; 0 \leqslant r \leqslant 3\Big\}.$$

Let $f(i)$ be a polynomial in $i$ over $\mathbb{N}$ with leading term $ei^n$. The exact sequences

$$0 \to R/x^{(t-1)f(i)} \xrightarrow{x^{f(i)}} R/x^{tf(i)} \to R/x^{f(i)} \to 0 \ \ (i \geqslant 0)$$

induce an exact triangle $D_{t-1} \to D_t \to D_1 \rightsquigarrow$ in $\mathsf{D}^-_{\mathsf{fl}}(R)$, where we put $D_t = \bigoplus_{i\geqslant 0} R/x^{tf(i)}[i]$. An inductive argument on $t$ shows that $D_t$ belongs to the thick closure of $D_1$. The exact sequences

$$0 \to R/x^{f(i)-(m+1)i^n} \xrightarrow{x^{i^n}} R/x^{f(i)-mi^n} \to R/x^{i^n} \to 0 \ \ (i \geqslant 0)$$

induce an exact triangle $E_{m+1} \to E_m \to G_c \rightsquigarrow$, where $E_m := \bigoplus_{i\geqslant 0}(R/x^{f(i)-mi^n})[i]$ for $0 \leqslant m \leqslant e$. Hence, $E_0$ is in the thick closure of $G_c$ and $E_e$. Since $\ell\ell(\mathsf{H}_i E_e) = f(i) - ei^n$ has degree at most $n-1$ as a polynomial in $i$, the induction hypothesis shows that $E_e$ is in $\mathsf{thick}^{\otimes} G_c$. Hence, $D_1 = E_0$ is also in $\mathsf{thick}^{\otimes} G_c$, and so is $D_t$. Therefore, $B_l$ is in $\mathsf{thick}^{\otimes} G_c$. Thus, $A_l$ belongs to $\mathsf{thick}^{\otimes} G_c$ for $l = 1, 2$. □

**Remark 7.15.** Let $(R, xR, k)$ be a discrete valuation ring, and let $c \geqslant 2$ be an integer. Then $\operatorname{Supp} G_c = \{xR\} = \operatorname{Supp} k$. In particular, we have $\operatorname{Supp} G_c \neq \operatorname{Spec} R$, so that $R$ is not in $\mathsf{thick}^{\otimes} G_c$ by Proposition 4.11. Krull's intersection theorem implies $\operatorname{Ann} G_c = 0 = \operatorname{Ann} R$. Proposition 7.7 and Theorem 7.14 imply that $G_c$ is not in $\mathscr{L}_1 = \mathsf{thick}^{\otimes} k$. In summary,

(1) $\operatorname{Supp} G_c$ is contained in $\operatorname{Supp} k$, but $G_c$ does not belong to $\mathsf{thick}^{\otimes} k$, and

(2) $\mathrm{V}(\operatorname{Ann} R)$ is contained in $\mathrm{V}(\operatorname{Ann} G_c)$, but $R$ does not belong to $\mathsf{thick}^{\otimes} G_c$.

This guarantees that in Proposition 2.9 one cannot replace $\mathrm{V}(\operatorname{Ann} X)$ by $\operatorname{Supp} X$, or $\operatorname{Supp} \mathcal{Y}$ by $\mathrm{V}(\operatorname{Ann} \mathcal{Y})$.

**Example 7.16.** Let us deduce the conclusion of Proposition 6.2(1) directly in the case where $(R, \mathfrak{m}, k)$ is a discrete valuation ring. In this case, we have $\mathbf{Spcl}(\mathrm{Spec}) = \{\varnothing, \{\mathfrak{m}\}, \operatorname{Spec} R\}$. Using Proposition 5.4, we obtain $\mathbf{Cpt} = \{\mathbf{0}, \mathsf{thick}^{\otimes} k, \mathsf{D}^{-}(R)\}$. Example 3.2 and Theorems 7.14 and 7.11 say that $\mathbf{0}$ and $\mathsf{thick}^{\otimes} k$ are prime. Thus, the compact prime thick $\otimes$-ideals of $\mathsf{D}^{-}(R)$ are $\mathbf{0}$ and $\mathsf{thick}^{\otimes} k$. It follows from Corollary 3.10 that $\mathfrak{s}(\mathsf{thick}^{\otimes} k)$ does not contain $\mathfrak{m}$, which implies $\mathfrak{s}(\mathsf{thick}^{\otimes} k) = 0$. Hence, $\mathbf{Cpt} \cap \mathfrak{s}^{-1}(\mathfrak{m}) = \{\mathbf{0}\}$.

Let us consider for a discrete valuation ring $R$ the tameness and compactness of the thick $\otimes$-ideals $\mathscr{L}_c$.

**Proposition 7.17.** *Let $R$ be a discrete valuation ring, and let $c \geqslant 1$ be an integer. Then $\mathscr{L}_c$ is a nontame prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. If $c \geqslant 2$, then $\mathscr{L}_c$ is noncompact.*

*Proof.* It is shown in Theorem 7.11 that $\mathscr{L}_c$ is a prime thick $\otimes$-ideal of $\mathsf{D}^{-}(R)$. Denote by $xR$ the maximal ideal of $R$. Using Proposition 7.7 and Theorem 7.14, we easily see that $\operatorname{Supp} \mathscr{L}_c = \mathrm{V}(x) = \{xR\}$.

Suppose that $\mathscr{L}_c$ is tame. Then $\mathscr{L}_c = \operatorname{Supp}^{-1}\{xR\}$ by Proposition 5.3. For example, consider the complex $E = \bigoplus_{i\geqslant 0}(R/x^{i!})[i]$. We have $\operatorname{Supp} E = \{xR\}$, which shows $E \in \mathscr{L}_c$. Hence, there exists an integer $t \geqslant 0$ such that $i! = \ell\ell(\mathsf{H}_i E) \leqslant ti^{c-1}$ for all $i \gg 0$. This contradiction shows that $\mathscr{L}_c$ is not tame.

Suppose that $\mathscr{L}_c$ is compact. Then $\mathscr{L}_c = \langle \operatorname{Supp} \mathscr{L}_c \rangle = \mathsf{thick}^{\otimes} k = \mathscr{L}_1$ by Proposition 5.4 and Theorem 7.14. This gives a contradiction when $c \geqslant 2$; see Proposition 7.7. Thus, $\mathscr{L}_c$ is not compact for all $c \geqslant 2$. □

**Remark 7.18.** Theorem 7.14 implies that $\mathscr{L}_c$ is generated by the complex $G_c$, whose support is the closed subset $\{\mathfrak{m}\}$ of $\operatorname{Spec} R$. Proposition 7.17 says that $\mathscr{L}_c$

is not compact for $c \geqslant 2$. This gives an example of a noncompact thick $\otimes$-ideal which is generated by objects with closed supports.

In the proof of Proposition 7.17, a complex defined by using factorials of integers played an essential role. In relation to this, a natural question arises.

**Question 7.19.** Let $(R, xR)$ be a discrete valuation ring. Consider the complex

$$\begin{aligned} E &= \bigoplus_{i\geqslant 0}(R/x^{i!})[i] \\ &= (\cdots \xrightarrow{0} R/x^{120} \xrightarrow{0} R/x^{24} \xrightarrow{0} R/x^{6} \xrightarrow{0} R/x^{2} \xrightarrow{0} R/x \xrightarrow{0} R/x \to 0) \end{aligned}$$

in $\mathsf{D}^{-}(R)$. Is it possible to establish a similar result to Theorem 7.14 for $\mathsf{thick}^{\otimes} E$? For example, can one characterize the objects of $\mathsf{thick}^{\otimes} E$ in terms of the Loewy lengths of their homologies?

We have no idea how to answer this question. In relation to it, in the next example we will consider complexes defined by using not factorials but polynomials. To do this, we provide a lemma.

**Lemma 7.20.** *Let $x$ be a nonzerodivisor of $R$. Then the complex $\bigoplus_{i\geqslant 0}(R/x^{a_i+b_i})[i]$ belongs to the thick closure of $\bigoplus_{i\geqslant 0}(R/x^{a_i})[i]$ and $\bigoplus_{i\geqslant 0}(R/x^{b_i})[i]$ for all integers $a_i, b_i \geqslant 0$. In particular, the complex $\bigoplus_{i\geqslant 0}(R/x^{ca_i})[i]$ is in the thick closure of $\bigoplus_{i\geqslant 0}(R/x^{a_i})[i]$ for all integers $c, a_i \geqslant 0$.*

*Proof.* For each $i \geqslant 0$ there is an exact sequence $0 \to R/x^{a_i} \xrightarrow{x^{b_i}} R/x^{a_i+b_i} \to R/x^{b_i} \to 0$. From this we induce an exact sequence $0 \to \bigoplus_{i\geqslant 0}(R/x^{a_i})[i] \to \bigoplus_{i\geqslant 0}(R/x^{a_i+b_i})[i] \to \bigoplus_{i\geqslant 0}(R/x^{b_i})[i] \to 0$. The first assertion follows from this. The second assertion is shown by induction and the first assertion. □

**Example 7.21.** Let $x \in R$ be a nonzerodivisor. For integers $a, b, c \geqslant 0$, define a complex

$$X(a, b, c) = \bigoplus_{i\geqslant 0}(R/f_i)[i] = \big(\cdots \xrightarrow{0} R/f_2 \xrightarrow{0} R/f_1 \xrightarrow{0} R/f_0 \to 0\big),$$

where $f_i = x^{ai^2+bi+c} \in R$. Then it holds that $\mathsf{thick}^{\otimes}\{X(a, b, c) \mid a, b, c \geqslant 0\} = \mathsf{thick}^{\otimes}\{X(1, 0, 0)\}$.

*Proof.* It is obvious that the left-hand side contains the right-hand side. In view of Lemma 7.20, the opposite inclusion will follow if we show that $X(1,0,0)$, $X(0,1,0)$, $X(0, 0, 1)$ are in $\mathsf{thick}^{\otimes}\{X(1, 0, 0)\}$, whose first containment is evident. The complex $X(1, 0, 0)$ has the direct summand $(R/x)[1]$, so the module $R/x$ belongs to $\mathsf{thick}^{\otimes}\{X(1, 0, 0)\}$. We have $X(0, 0, 1) = R/x \otimes_R^{\boldsymbol{L}} (\cdots \xrightarrow{0} R \xrightarrow{0} R \to 0)$, which is in $\mathsf{thick}^{\otimes}\{X(1, 0, 0)\}$. The exact sequences $0 \to R/x^{i^2} \xrightarrow{x^{2i+1}} R/x^{(i+1)^2} \to R/x^{2i+1} \to 0$ and $0 \to R/x^{2i+1} \xrightarrow{x} R/x^{2i+2} \to R/x \to 0$ with $i > 0$ induce exact sequences $0 \to X(1,0,0) \to X(1,0,0)[-1] \to X(0,2,1) \to 0$ and $0 \to X(0,2,1) \to X(0,2,2) \to X(0,0,1) \to 0$, which shows that $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$ contains $X(0,2,1) = (\cdots \xrightarrow{0} R/x^5 \xrightarrow{0} R/x^3 \xrightarrow{0} R/x \to 0)$ and $X(0,2,2) = (\cdots \xrightarrow{0} R/x^6 \xrightarrow{0} R/x^4 \xrightarrow{0} R/x^2 \to 0)$. Applying Corollary 7.3, we see that $X(0,1,0)$ belongs to $\mathsf{thick}^{\otimes}\{X(1,0,0)\}$. $\square$

**Remark 7.22.** One can consider a general statement of Example 7.21 by defining $f_i = x^{a_0 i^d + a_1 i^{d-1} + \cdots + a_d}$, so that it is nothing but the example for $d = 2$. We do not know if it holds for $d \geqslant 3$.

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