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By a classical method due to Roitman, a complete intersection X of sufficiently small degree admits a rational decomposition of the diagonal. This means that some multiple of the diagonal by a positive integer N, when viewed as a cycle in the Chow group, has support in $X \times D \cup F \times X$, for some divisor D and a finite set of closed points F. The minimal such N is called the torsion order. We study lower bounds for the torsion order following the specialization method of Voisin, Colliot-Thélène, and Pirutka. We give a lower bound for the generic complete intersection with and without point. Moreover, we use methods of Kollár and Totaro to exhibit lower bounds for the very general complete intersection.

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Introduction

Decomposition of the diagonal has played a prominent role in recent progress on stable rationality questions. For a rationally connected variety over a field k, there is a minimal integer $\mathrm{Tor}_k(X) \geq 1$ such that the multiple of the diagonal $\mathrm{Tor}_k(X) \cdot \Delta_X$, when viewed in the Chow group of $X \times X$, is supported in $X \times D \cup F \times X$, for some divisor D and some finite set of closed points F. We will call $\mathrm{Tor}_k(X)$ the *torsion order* of X; it is a stable birational invariant which equals 1 if X is stably

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rational and in general gives an upper bound on the exponent of the unramified cohomology of X. This invariant is also studied by Kahn [2016]. In a proper flat family the torsion order of a fiber divides the torsion order of the generic fiber (see Lemma 1.5 for the precise statement). One can thus deduce a nontrivial torsion order from a nontrivial torsion order of a cleverly chosen degeneration. In all current implementations of this strategy divisors of the torsion order of the degeneration are computed by finding a good resolution of singularities. On the resolution, the action of algebraic correspondences on a suitable cohomology can be used to produce divisors of the torsion order.

This method was pioneered by Voisin [2015]. It was significantly simplified and applied by Colliot-Thélène and Pirutka [2016b] to show the nonrationality of a very general quartic threefold by using a degeneration to a classical example of Artin and Mumford (after a "universally CH_0-trivial" resolution of singularities [Colliot-Thélène and Pirutka 2016b, Définitions 1.1 and 1.2]), which is a unirational but nonrational variety. The nontrivial 2-torsion in its Brauer group forces nontriviality of the torsion order (in fact, it implies that the torsion order is even). The degeneration method is also used in the recent work of Hasset, Pirutka, and Tschinkel [Hassett et al. 2016] exhibiting a family of smooth projective fourfolds containing both rational and nonstably rational members. Totaro [2016] used [Colliot-Thélène and Pirutka 2016b] and Voisin's method combined with work of Kollár [1995] to improve Kollár's nonrationality results for hypersurfaces in [loc. cit.]. Roughly speaking, Totaro showed how, for large enough degree, a general hypersurface of even degree degenerates to an inseparable degree-2 cover in characteristic 2 with a universally CH_0-trivial resolution of singularities that supports nonvanishing differential forms. An action of correspondences on differentials shows that the torsion order is even.

In this paper we study the torsion order of complete intersections in projective space. The method used by Roitman to show that a degree-zero 0-cycle on a hypersurface of degree $d \le n$ in $\mathbb{P}^n$ over an algebraically closed field is d-torsion is applied in Proposition 5.2 to establish an upper bound for the torsion index, more precisely, that a complete intersection X of multidegree $d_1, \dots, d_r$ in $\mathbb{P}_k^{n+r}$ (over any field k) with $\sum_{i=1}^r d_i \le n+r$ satisfies $\mathrm{Tor}_k(X) \mid \prod_{i=1}^r (d_i!)$. Our first result is a lower bound for a generic complete intersection.

Theorem (Theorem 6.5 and Corollary 6.6). *Let* $\mathcal{Y} := \prod_{i=1}^r \mathbb{P}(H^0(\mathbb{P}_k^{n+r}, \mathcal{O}(d_i))^\vee)$, *and let* $\mathcal{X} \subset \mathcal{Y} \times_k \mathbb{P}_k^{n+r}$ *be the incidence variety*

$$\mathcal{X} = \{(f_1, \dots, f_r, x) \in \mathcal{Y} \times_k \mathbb{P}_k^{n+r} \mid f_1(x) = \cdots = f_r(x) = 0\}.$$

We denote by K the quotient field of $\mathcal{Y}$, and let X/K be the generic fiber of the family $\mathcal{X} \to \mathcal{Y}$. For an integer $d \ge 1$, let $d!^*$ be the l.c.m. of the integers $1, \dots, d$. *Then*:

(i) $\mathrm{Tor}_K(X)$ *is divisible by* $\prod_{i=1}^r d_i!^*$.

(ii) $\mathrm{Tor}_{K(X)}(X \otimes_K K(X))$ *is divisible by* $\prod_{i=1}^r d_i!^*/(d_1 \cdots d_r)$.

The invariant which detects divisors of the torsion order in the first part of the theorem is the index of a variety, that is, the image of the Chow group of zero cycles via the degree map. The index of X/K is given by $d_1 \cdots d_r$. Divisibility of the torsion order by other integers of the form $i_1 \cdots i_r$ with $1 \le i_j \le d_j$ is shown by degeneration to a union of complete intersections with lower degrees and using induction.

We also consider the generic cubic hypersurface with a line, and use Theorem 6.5 to show that this has torsion order exactly 2 (Example 6.8). We show the existence of a cubic threefold over $K = \mathbb{Q}_p((x))$ or $K = \mathbb{F}_p((t))((x))$, having a K-point and torsion order divisible by 2 (Example 6.9); more generally, we construct examples of cubic hypersurfaces of dimension n over a field $K = k((x))$, where k is a field of characteristic zero and u-invariant at least $n+1$, which have a K-point and for which 2 divides the torsion order. This last series of examples is taken over from [Colliot-Thélène 2016] (without the assumption that the u-invariant is a power of 2), with the kind permission of the author, and it gives an improvement over a construction in an earlier version of this paper, which relied on Rost's degree formula. We should mention that other examples of this kind already exist in the literature; see for example [Colliot-Thélène and Pirutka 2016b, Théorème 1.21], where cubic threefolds over a p-adic field with nonzero torsion order are constructed, as well as examples over $\mathbb{F}_p((x))$ [Colliot-Thélène and Pirutka 2016b, Remarque 1.23]; both examples have a rational point.

Our second result concerns the torsion order of very general complete intersections over algebraically closed fields of characteristic zero. The idea of the proof is as in the papers of Kollár and Totaro. We are able to generalize the results on the Hodge cohomology of the degeneration in characteristic p to Hodge–Witt cohomology. In this way we can establish results on divisibility by powers of p.

Theorem (Theorem 8.2). *Let k be an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}_k^{n+r}$ be a very general complete intersection of multidegree $d_1, d_2, \ldots, d_r$ such that $d' := \sum_{i=1}^r d_i \le n + r$ and $n \ge 3$. Let p be a prime, let $m \ge 1$, and suppose*

$$d_i \ge p^m \cdot \left\lceil \frac{n+r+1-d'+d_i}{p^m+1} \right\rceil$$

for some i, where $\lceil \cdot \rceil$ denotes the ceiling function. Assume that p is odd or n is even. Then $p^m \mid \mathrm{Tor}_k(X)$.

For example, it is easy to see that if $\sum_{i=1}^r d_i = n + r$ and $n \ge 3$, which is the extreme case, then $d_i \mid \mathrm{Tor}_k(X)$ if d_i is odd or n is even. For hypersurfaces and

$m = 1$, the theorem is due to Totaro, and we give a short proof of the straightforward generalization to complete intersections and the case $m = 1$ in Theorem 7.1. We should mention that our Theorems 7.1 and 8.2 are actually a bit stronger, in that we prove the same divisibility result for the torsion orders of level $n-2$ (see below), which automatically divide the torsion orders described above.

The paper is divided into seven sections. Section 1 contains the definition and basic properties of the torsion order. Following a suggestion of Claire Voisin, we consider decompositions of the diagonal of higher level and the associated torsion invariants; we also describe some elementary specialization results. In Section 3 we recall from Colliot-Thélène and Pirutka the notion of a universally CH_0-trivial morphism and a related notion, that of a totally CH_0-trivial morphism. Behavior under a combination of degeneration and modification by a birational totally CH_0-trivial morphism, which is the basic tool used for divisibility results, is the focus of Section 4; in this section we follow Colliot-Thélène and Pirutka [2016b] and extend their specialization results to cover decompositions of higher level. We recall Roitman's theorem in Section 5 and discuss the case of the generic complete intersection in Section 6. We recall Totaro's arguments leading to the divisibility results for the torsion order of a very general complete intersection in Section 7 and conclude by proving our refined version in Section 8.

1. Torsion orders

For a noetherian scheme Y, we let $\mathscr{Z}(Y)$ denote the group of algebraic cycles on Y, that is, the free abelian group on the integral closed subschemes of Y. If Y is a scheme of finite type over a field k, we grade $\mathscr{Z}(Y)$ by dimension over k. For such a scheme, we have the n-th Chow group $\mathrm{CH}_n(Y) := \mathscr{Z}_n(Y)/\sim$, where $\sim$ is the relation of rational equivalence (see [Fulton 1984, §1.3], where this group is denoted $A_n(Y)$). By an *integral component* of Y, we mean an irreducible component of Y, endowed with the reduced scheme structure.

Let k be a field and X a k-scheme of finite type. If A is a presheaf on X_{Zar}, we let

$$A(X(i)) := \mathrm{colim}_F\, A(X \setminus F)$$

where F runs over all closed subsets of X with $\dim_k F \le i$. We extend this notation to products, defining for a presheaf A on $(X \times_k Y)_{\mathrm{Zar}}$

$$A(X(i) \times Y(j)) = \mathrm{colim}_{F,G}\, A((X \setminus F) \times_k (Y \setminus G)).$$

For example, the contravariant functoriality of the classical Chow groups for open immersions [Fulton 1984, §1.7] allows us to apply this notation to $A(X) := \mathrm{CH}_n(X)$ for some n.

Let k be a field with algebraic closure $\bar{k}$. Let P be a property of k-schemes, such as "reduced" or "smooth over k". We say that a finite type k-scheme X is *generically* P if there exists an open Zariski dense subset $U \subset X$ having property P. The property of being generically smooth over k will be used frequently in this paper. Recall that X is generically smooth over k if and only if $X \times_k \bar{k}$ is generically smooth over $\bar{k}$. Moreover, this notion is stable under taking products; that is, if X and Y are generically smooth over k, then so is $X \times_k Y$.

A closed subset D of a finite type k-scheme X is called *nowhere dense* if the complement $X \setminus D$ is Zariski dense. We denote by $k(X)$ the product over the residue fields at the generic points of X, that is,

$$k(X) := \prod_{\eta \in X} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta,$$

where η runs over the generic points of the irreducible components of X (we note that $\mathcal{O}_{X,\eta}$ is a field if X is generically reduced). We have an evident morphism of schemes $\operatorname{Spec} k(X) \to X$. If X is equidimensional of dimension d, then we can see from the definition of Chow groups that

$$\begin{aligned} \varinjlim_{\substack{D \subset X \\ D \text{ nowhere dense}}} \mathrm{CH}_d(Y \times_k (X \setminus D)) &\xrightarrow{\cong} \mathrm{CH}_0(Y \times_k \operatorname{Spec} k(X)) \\ &= \bigoplus_{\eta \in X} \mathrm{CH}_0(Y \times_k \operatorname{Spec} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta), \end{aligned}$$

for any Y. For any class $\alpha \in \mathrm{CH}_d(Y \times_k X)$, we will call its image in $\bigoplus_{\eta \in X} \mathrm{CH}_0(Y \times_k \operatorname{Spec} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta)$ under this composition the pullback under the morphism $Y \times_k \operatorname{Spec} k(X) \to Y \times_k X$.

Definition 1.1. Let k be a field, and let X be a reduced proper k-scheme that is equidimensional of dimension d.

(1) For $i = 0, 1, 2, \ldots$, the i-th torsion order of X, $\mathrm{Tor}_k^{(i)}(X) \in \mathbb{N}_+ \cup \{\infty\}$, is the order of the image of the diagonal $\Delta_X \subset X \times_k X$ in $\mathrm{CH}_d(X(i) \times X(d-1))$. We write $\mathrm{Tor}_k(X)$ for $\mathrm{Tor}_k^{(0)}(X)$ and call this the torsion order of X.

(2) Suppose X is generically smooth over k. For $1 \le i < j \le 3$, let $p_{ij} : X \times_k X \times_k X \to X \times_k X$ denote the projection on the i-th and j-th factors, and let $\Delta_{ij} \subset X \times_k X \times_k X$ denote the pullback $p_{ij}^{-1}(\Delta_X)$. Consider the Cartesian diagram

$$\begin{tikzcd} X_{k(X \times_k X)} \arrow[r, "\tilde{j}"] \arrow[d] & X \times_k X \times_k X \arrow[d, "p_{23}"] \\ \operatorname{Spec} k(X \times_k X) \arrow[r, "i"'] & X \times_k X \end{tikzcd}$$

Let $\eta_1 - \eta_2 \in \mathrm{CH}_0(X_{k(X\times_k X)})$ denote the class of the pullback $\tilde{j}^*(\Delta_{12} - \Delta_{13})$, via the flat morphism $\tilde{j}$. The *generic torsion order* of X, $\mathrm{gTor}_k(X) \in \mathbb{N}_+ \cup \{\infty\}$, is the order of $\eta_1 - \eta_2$ in $\mathrm{CH}_0(X_{k(X\times_k X)})$.

(3) We say that X admits a decomposition of the diagonal of order N and level i if there is a nowhere dense closed subset D, a closed subset Z of X with $\dim_k Z \le i$, and cycles γ, γ' on $X \times_k X$, with γ supported in $X \times_k D$, with γ' supported in $Z \times_k X$, and with

$$N \cdot [\Delta_X] = \gamma' + \gamma$$

in $\mathrm{CH}_d(X \times_k X)$.

(4) Suppose X is geometrically integral. For an integer $N \geq 1$, we say that X admits a decomposition of the diagonal of order N if there is a 0-cycle x on X, a proper closed subset D of X, and a dimension-d cycle γ on $X \times_k X$, supported in $X \times_k D$, such that

$$N \cdot [\Delta_X] = x \times X + \gamma$$

in $\mathrm{CH}_d(X \times_k X)$. Then x has degree N over k, which can be seen by pushing forward along the second projection. We say that X admits a $\mathbb{Q}$-decomposition of the diagonal if X admits a decomposition of order N for some N, and that X admits a $\mathbb{Z}$-decomposition of the diagonal if X admits a decomposition of the diagonal of order 1.

(5) Let $\deg_k : \mathrm{CH}_0(X) \to \mathbb{Z}$ be the degree map. For X smooth and integral, the *index* of X is the positive generator I_X of the subgroup $\deg_k \mathrm{CH}_0(X) \subset \mathbb{Z}$. Equivalently, I_X is the g.c.d. of all degrees $[k(x) : k]$ as x runs over closed points of X. We extend the definition of the index to proper, integral, generically smooth k-schemes Y by defining I_Y to be the g.c.d. of all degrees $[k(y) : k]$ as y runs over closed points of the smooth locus Y_{sm} of Y (which is dense in Y).

Remarks 1.2. (1) Suppose X is equidimensional of dimension d and is geometrically integral. Since the only dimension-d cycles $\gamma_{(0)}$ on $X \times_k X$, supported on $Z_{(0)} \times_k X$ with $Z_{(0)} \subset X$ a dimension-zero closed subset, are of the form $\gamma_{(0)} = x \times X$ for some 0-cycle x on X, a decomposition of the diagonal of order N and level 0 is the same as decomposition of the diagonal of order N.

(2) We extend the definition of $\mathrm{Tor}_k^{(i)}(X)$ to all proper, equidimensional k-schemes by setting $\mathrm{Tor}_k^{(i)}(X) := \mathrm{Tor}_k^{(i)}(X_{\mathrm{red}})$.

(3) We will often use an equivalent formulation of Definition 1.1(3), namely, that X admits a decomposition of the diagonal of order N and level i if there is a closed subset D containing no generic point of X and a closed subset Z of X with $\dim_k Z \le i$ such that

$$N \cdot j^*[\Delta_X] = 0$$

in $\mathrm{CH}_d((X\setminus Z)\times_k(X\setminus D))$, where $j:(X\setminus Z)\times_k(X\setminus D)\to X\times_k X$ is the inclusion. This equivalence follows from the localization sequence

$$\mathrm{CH}_d(Z\times_k X\cup X\times_k D)\xrightarrow{i_*}\mathrm{CH}_d(X\times_k X)\xrightarrow{j^*}\mathrm{CH}_d((X\setminus Z)\times_k(X\setminus D))\to 0$$

and the surjection

$$\mathrm{CH}_d(Z\times_k X)\oplus\mathrm{CH}_d(X\times_k D)\to\mathrm{CH}_d(Z\times_k X\cup X\times_k D).$$

(4) Decompositions of the diagonal for *smooth* proper k-varieties have been considered in [Bloch and Srinivas 1983; Colliot-Thélène and Pirutka 2016b; Totaro 2016] and by many others. Here we have extended the definition to proper, equidimensional, but not necessarily smooth k-schemes.

(5) In the same way as in [Colliot-Thélène and Pirutka 2016b, Proposition 1.4], one can prove the following equivalence for a smooth, proper, and equidimensional k-scheme X, namely, X admits a decomposition of the diagonal of order N and level i if and only if there exists a closed subvariety $\iota: Z\subset X$ with $\dim Z\le i$ and

$$\mathrm{image}((\iota\times_k K)_*:\mathrm{CH}_0(Z\times_k K)\to\mathrm{CH}_0(X\times_k K))\supset N\cdot\mathrm{CH}_0(X\times_k K)$$

for all field extensions $k\subset K$.

Lemma 1.3. *Let X be a k-scheme that is proper and equidimensional of dimension d over k.*

(1) *If $\mathrm{Tor}_k^{(i)}(X)$ is finite, then so is $\mathrm{Tor}_k^{(i+1)}(X)$ and in this case, $\mathrm{Tor}_k^{(i+1)}(X)$ divides $\mathrm{Tor}_k^{(i)}(X)$.*

(2) *X admits a decomposition of the diagonal of order N and level i if and only if $\mathrm{Tor}_k^{(i)}(X)$ divides N; if X is geometrically integral, then X admits a decomposition of the diagonal of order N if and only if $\mathrm{Tor}_k(X)$ divides N and X does not admit a $\mathbb{Q}$-decomposition of the diagonal if and only if $\mathrm{Tor}_k(X)=\infty$.*

(3) *Suppose X is smooth over k and geometrically integral. If $\mathrm{Tor}_k(X)$ is finite, then so is $\mathrm{gTor}_k(X)$ and $\mathrm{gTor}_k(X)$ divides $\mathrm{Tor}_k(X)$.*

(4) *Suppose X is generically smooth over k, and let $L\supset k$ be a field extension. If $\mathrm{Tor}_k^{(i)}(X)$ is finite, then so is $\mathrm{Tor}_L^{(i)}(X_L)$ and in this case $\mathrm{Tor}_L^{(i)}(X_L)$ divides $\mathrm{Tor}_k^{(i)}(X)$. If L is finite over k, then $\mathrm{Tor}_k^{(i)}(X)$ is finite if and only if $\mathrm{Tor}_L^{(i)}(X_L)$ is finite and in this case $\mathrm{Tor}_k^{(i)}(X)$ divides $[L:k]\cdot\mathrm{Tor}_L^{(i)}(X_L)$. The corresponding statements hold replacing $\mathrm{Tor}^{(i)}$ with gTor.*

(5) *X admits a decomposition of the diagonal of level i and order N if and only if there is a closed subset $Z\subset X$ of dimension $\le i$ such that the pullback of Δ_X*

to $(X \setminus Z) \times_k \operatorname{Spec} k(X)$ *via the inclusion*

$$(X \setminus Z) \times_k \operatorname{Spec} k(X) \to X \times_k X$$

has order dividing N *in* $\mathrm{CH}_0((X \setminus Z) \times_k \operatorname{Spec} k(X))$.

Proof. Statement (1) follows from the existence of the restriction homomorphism

$$\mathrm{CH}_d((X \setminus F) \times_k (X \setminus D)) \to \mathrm{CH}_d((X \setminus F') \times_k (X \setminus D))$$

for $F \subset F'$. Statement (2) follows from the localization sequence for $\mathrm{CH}_*(\,\cdot\,)$, as in Remarks 1.2(3).

For (3), suppose

$$N \cdot [\Delta_X] = x \times X + \gamma$$

in $\mathrm{CH}_d(X \times_k X)$ for x and γ as in Definition 1.1. Since X is smooth and proper, we have for every field extension F of k the action of $\mathrm{CH}_d(X_F \times_F X_F)$ on $\mathrm{CH}_n(X_F)$ as correspondences [Fulton 1984, Chapter 16]; that is, for $\alpha \in \mathrm{CH}_d(X_F \times_F X_F)$ and $\rho \in \mathrm{CH}_n(X_F)$, one has the well defined element

$$\alpha^*(\rho) := p_{1*}(p_2^*\rho \cdot \alpha).$$

Acting by the correspondence $N \cdot \Delta^*_{X_{k(X \times_k X)}}$ on $\mathrm{CH}_0(X_{k(X \times_k X)})$ gives

$$N \cdot (\eta_1 - \eta_2) = x - x = 0$$

and thus $\mathrm{gTor}_k(X)$ divides N. Applying (2) gives (3).

For (4), the first assertion follows by applying the pullback in CH_d for $X_L \times_L X_L \to X \times_k X$ and using (2). The second part follows by applying the pushforward map $\mathrm{CH}_d(X_L \times_L X_L) \to \mathrm{CH}_d(X \times_k X)$ and using (2), and the assertion for $\mathrm{gTor}_k(X)$ follows similarly by applying the pushforward map $\mathrm{CH}_d(X_{L(X_L \times_L X_L)}) \to \mathrm{CH}_d(X_{k(X \times_k X)})$.

The last assertion (5) follows from the identity

$$\mathrm{CH}_0((X \setminus Z) \times_k \operatorname{Spec} k(X)) = \varinjlim_{D \subset X} \mathrm{CH}_d((X \setminus Z) \times_k (X \setminus D))$$

where the limit is over all closed nowhere dense $D \subset X$. $\square$

Remark 1.4. We have restricted our attention to proper k-schemes for the definitions of torsion orders and decompositions of the diagonal. Even though the definitions would make sense for nonproper equidimensional k-schemes, a naive extension is probably not useful. Possibly replacing Chow groups with Suslin homology would make more sense: following Lemma 1.3, one could define $\mathrm{Tor}^{(i)}(X)$ for an equidimensional finite type k-scheme as the order of the restriction of Δ_X

to $X \times_k \operatorname{Spec} k(X)$ in the quotient group

$$\varinjlim_{Z\subset X} H_0^{\mathrm{Sus}}(X \times_k \operatorname{Spec} k(X))/im(H_0^{\mathrm{Sus}}(Z \times_k \operatorname{Spec} k(X)))$$

where $Z \subset X$ runs over all closed subsets of dimension at most i. We will not investigate properties of these torsion orders for nonproper k-schemes here.

Here is the first in a series of elementary but useful specialization lemmas.

Lemma 1.5. *Let $\mathcal{O}$ be a noetherian regular local ring and $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}$ a proper flat morphism, with $\mathcal{X}$ equidimensional over $\operatorname{Spec} \mathcal{O}$ of relative dimension d, $X \to \operatorname{Spec} K$ the generic fiber and $Y \to \operatorname{Spec} k$ the special fiber. Fix an integer i. Suppose that, for each $z \in \operatorname{Spec} \mathcal{O}$, the geometric fiber $\mathcal{X}_{\bar{z}}$ is generically reduced over $\overline{k(z)}$.*

(1) *If $\operatorname{Tor}_K^{(i)}(X)$ is finite, then so is $\operatorname{Tor}_k^{(i)}(Y)$, and $\operatorname{Tor}_k^{(i)}(Y)$ divides $\operatorname{Tor}_K^{(i)}(X)$.*

(2) *Suppose that, for each $z \in \operatorname{Spec} \mathcal{O}$, the fiber $\mathcal{X}_z$ is generically smooth over $k(z)$. If $\operatorname{gTor}_K(X)$ is finite, then so is $\operatorname{gTor}_k(Y)$, and $\operatorname{gTor}_k(Y)$ divides $\operatorname{gTor}_K(X)$.*

(3) *Let $\bar{k}$ and $\bar{K}$ be the algebraic closures of k and K, respectively, and suppose either K has characteristic zero, or that $\mathcal{O}$ is excellent. If $\operatorname{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}})$ is finite, then so is $\operatorname{Tor}_{\bar{k}}^{(i)}(Y_{\bar{k}})$, and $\operatorname{Tor}_{\bar{k}}^{(i)}(Y_{\bar{k}})$ divides $\operatorname{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}})$.*

Proof. We use the definition of $\mathrm{CH}_d(X(i) \times X(d-1))$ as a limit to reduce to making computations in groups of the form $\mathrm{CH}_d((X \setminus Z) \times_K (X \setminus D))$ where Z, D are closed subsets of X with $\dim Z \le i$ and $\dim D \le d-1$. We can find a chain of regular closed subschemes $Z_0 \subset \cdots \subset Z_r = \operatorname{Spec} \mathcal{O}$, with Z_i of Krull dimension i. This gives us the DVRs $\mathcal{O}_i := \mathcal{O}_{Z_i, Z_{i-1}}$ and the restriction of $\mathcal{X}$ to $\mathcal{X}_i \to \operatorname{Spec} \mathcal{O}_i$. Regarding the proof of (3), if the original local ring $\mathcal{O}$ has characteristic-zero quotient field, we can find a chain as above such that each DVR $\mathcal{O}_i$ has characteristic-zero quotient field, and if $\mathcal{O}$ is excellent, so are each of the $\mathcal{O}_i$. Proving the result for each of the families $\mathcal{X}_i$ gives the result for $\mathcal{X}$, which reduces us to the case of a DVR $\mathcal{O}$.

In this case, suppose we have a relation

$$N \cdot \Delta_X = 0 \tag{1-1}$$

in $\mathrm{CH}_d((X \setminus Z) \times_K (X \setminus D))$, with $\dim_K Z \le i$ and D nowhere dense. Taking the closures $\bar{Z}$ and $\bar{D}$ in $\mathcal{X}$, and letting $Z_0 = Y \cap \bar{Z}$ and $D_0 = Y \cap \bar{D}$ (as intersections of closed subsets of $\mathcal{X}$), we have the specialization homomorphism (see for example [Fulton 1984, §20.3])

$$\mathrm{sp} : \mathrm{CH}_d((X \setminus Z) \times_K (X \setminus D)) \to \mathrm{CH}_d((Y \setminus Z_0) \times_k (Y \setminus D_0))$$

associated to the family

$$\mathcal{X} \times_{\mathcal{O}} \mathcal{X} \setminus \bar{Z} \times_{\mathcal{O}} \mathcal{X} \cup \mathcal{X} \times_{\mathcal{O}} \bar{D} \to \operatorname{Spec} \mathcal{O}.$$

Since $\mathcal{O}$ is a DVR, the closure $\bar{Z}$ is automatically flat over $\operatorname{Spec}\mathcal{O}$, and thus $\dim_k Z_0 \le i$; similarly, D_0 is nowhere dense in Y. We have two cartesian diagrams

$$\begin{tikzcd} Y \times_k Y \arrow[r] & \mathscr{X} \times_{\mathcal{O}} \mathscr{X} & X \times_K X \arrow[l] \\ Y \arrow[u, "\Delta_Y"] \arrow[r] & \mathscr{X} \arrow[u, "\Delta_{\mathscr{X}}"] & X \arrow[l] \arrow[u, "\Delta_X"'] \end{tikzcd}$$

From [Fulton 1984, Proposition 20.3(a)] applied to $f = \Delta_{\mathscr{X}}$, [Fulton 1984, Example 6.2.1], and our assumption that X and Y are generically reduced, we conclude

$$\begin{aligned} \operatorname{sp}([\Delta_{X_{\mathrm{red}}}]) = \operatorname{sp}([\Delta_X]) = \operatorname{sp}(\Delta_{X*}([X])) = \Delta_{Y*}(\operatorname{sp}([X])) \\ = \Delta_{Y*}([Y]) = [\Delta_Y] = [\Delta_{Y_{\mathrm{red}}}] \end{aligned}$$

in $\mathrm{CH}_d(Y \times_k Y)$. We used [Fulton 1984, Example 6.2.1] in order to obtain $\iota^*([\mathscr{X}]) = [Y]$, where ι is the evident (regular) closed immersion, which implies $\operatorname{sp}([X]) = [Y]$ by definition of the specialization map. By using compatibility of sp with pullback along open immersions and applying sp to (1-1), we have proved (1).

The proof of (2) is a similar specialization argument. Indeed, we reduce as before to the case of a DVR $\mathcal{O}$. Due to the generic smoothness assumption, there is a dense open subscheme $\mathscr{U}$ of $\mathscr{X} \times_{\mathcal{O}} \mathscr{X}$ that is smooth over $\operatorname{Spec}\mathcal{O}$, with special fiber dense in $Y \times_k Y$. If now τ is a generic point of $Y \times_k Y$, let $\mathscr{R}$ be the local ring $\mathcal{O}_{\mathscr{U},\tau}$. Then $\mathscr{R}$ is a DVR and we may consider the $\mathscr{R}$-scheme $\mathscr{X} \otimes_{\mathcal{O}} \mathscr{R} \to \operatorname{Spec}\mathscr{R}$. The quotient field F of $\mathscr{R}$ is one of the field factors of $k(X \times_K X)$, and the residue field $\mathfrak{f}$ of $\mathscr{R}$ is the factor of $k(Y \times_k Y)$ corresponding to τ. Let η_i^X, η_i^Y, $i = 1, 2$, denote the images of the "generic" points used to define $\operatorname{gTor}_K(X)$ and $\operatorname{gTor}_k(Y)$ in $\mathrm{CH}_0(\mathscr{X}_F)$ and $\mathrm{CH}_0(Y_{\mathfrak{f}})$, respectively. Applying the specialization homomorphism

$$\operatorname{sp} : \mathrm{CH}_0(\mathscr{X}_F) \to \mathrm{CH}_0(Y_{\mathfrak{f}})$$

to a relation $N \cdot (\eta_1^X - \eta_2^X)$ in $\mathrm{CH}_0(\mathscr{X}_F)$ shows that $N \cdot (\eta_1^Y - \eta_2^Y) = 0$ in $\mathrm{CH}_0(Y_{\mathfrak{f}})$ for each generic point τ, and thus $\operatorname{gTor}_k(Y)$ divides N.

For (3), we note that there is a finite extension L of K so that

$$\operatorname{Tor}^{(i)}_{\bar{K}}(X_{\bar{K}}) = \operatorname{Tor}^{(i)}_L(X_L) = \operatorname{Tor}^{(i)}_F(X_F)$$

for all finite extensions F of L. Since either K has characteristic zero or $\mathcal{O}$ is excellent, the normalization $\mathcal{O}^N$ of $\mathcal{O}$ in L is a semilocal principal ideal ring, finite over $\mathcal{O}$ (the characteristic-zero case follows from [Zariski and Samuel 1975, Chapter V, Theorem 7], and the excellent case follows from [Matsumura 1980, Theorem 78]). Thus, after replacing $\mathcal{O}$ with the localization $\mathcal{O}'$ of $\mathcal{O}^N$ at a maximal ideal, and replacing $\mathscr{X}$ with $\mathscr{X}' := \mathscr{X} \otimes_{\mathcal{O}} \mathcal{O}'$, we may assume that $\operatorname{Tor}^{(i)}_K(X) = \operatorname{Tor}^{(i)}_{\bar{K}}(X_{\bar{K}})$. Since $\operatorname{Tor}^{(i)}_{\bar{k}}(Y_{\bar{k}})$ divides $\operatorname{Tor}^{(i)}_k(Y)$ by Lemma 1.3(4), (3) follows from (1). $\square$

Remark 1.6. We did not use the properness of $\mathscr{X} \to \operatorname{Spec} \mathcal{O}$ in the proof of Lemma 1.5, but we have defined $\operatorname{Tor}^{(i)}$ for proper k-schemes only.

Next, we prove a modification of the specialization Lemma 1.5. A related result may be found in [Totaro 2016, Lemma 2.4].

Lemma 1.7. *Let $\mathcal{O}$ be a discrete valuation ring with quotient field K and residue field k. Let $f : \mathscr{X} \to \operatorname{Spec} \mathcal{O}$ be a flat morphism of dimension d over $\operatorname{Spec} \mathcal{O}$ with generic fiber X and special fiber Y. We suppose Y is a union of closed subschemes, $Y = Y_1 \cup Y_2$, with Y_1 and Y_2 having no common components. Suppose in addition that X admits a decomposition of the diagonal of order N and level i. Then there is an identity in $\mathrm{CH}_d(Y_1 \times_k Y_1)$*

$$N \Delta_{Y_1} = \gamma + \gamma_1 + \gamma_2$$

with γ supported in $Z_1 \times_k Y_1$ for some closed subset $Z_1 \subset Y_1$ of dimension $\leq i$, γ_1 supported on $Y_1 \times_k D_1$ for some nowhere dense closed subset $D_1 \subset Y_1$, and γ_2 supported in $(Y_1 \cap Y_2) \times_k Y_1$.

Proof. We consider the (nonproper) $\mathcal{O}$-scheme $(\mathscr{X} \setminus Y_2) \times_{\mathcal{O}} (\mathscr{X} \setminus Y_2) \to \operatorname{Spec} \mathcal{O}$, closed subsets Z, D of X with $\dim_K Z \leq i$, D nowhere dense, and a relation

$$N \cdot [\Delta_X] = 0$$

in $\mathrm{CH}_d((X \setminus Z) \times_K (X \setminus D))$, where $[\Delta_X]$ denotes the cycle class represented by the restriction of the diagonal.

As in the proof of Lemma 1.5(1), we have closed subsets Z_0, D_0 of $Y_1^0 := Y_1 \setminus Y_2$ with $\dim_k Z_0 \leq i$, D_0 nowhere dense, and a specialization homomorphism

$$\mathrm{sp} : \mathrm{CH}_d((X \setminus Z) \times_K (X \setminus D)) \to \mathrm{CH}_d((Y_1^0 \setminus Z_0) \times_k (Y_1^0 \setminus D_0)), \tag{1-2}$$

which is induced by

$$\mathrm{sp} : \mathrm{CH}_d(X \times_K X) \to \mathrm{CH}_d(Y \times_k Y).$$

As in the proof of Lemma 1.5(1), we have $\mathrm{sp}([\Delta_X]) = [\Delta_Y]$ in $\mathrm{CH}_d(Y \times_k Y)$. It follows immediately that $\mathrm{sp}([\Delta_X]) = [\Delta_{Y_1^0}]$ in $\mathrm{CH}_d(Y_1^0 \times_k Y_1^0)$, where $[\Delta_{Y_1^0}]$ is the cycle class of the restriction of the diagonal on Y_1^0. Applying (1-2) thus gives the relation

$$N \cdot [\Delta_{Y_1^0}] = 0$$

in $\mathrm{CH}_d((Y_1^0 \setminus Z_0) \times_k (Y_1^0 \setminus D_0))$.

Let $Z_1 := \bar{Z}_0$ be the closures of Z_0 in Y_1, let $\bar{D}_0$ be the closure of D_0 in Y_1, and let $D_1 = \bar{D}_0 \cup (Y_1 \cap Y_2)$. Using the localization sequence

$$\begin{aligned}\mathrm{CH}_d(Z_1 \times_k Y_1 \cup Y_1 \times_k D_1 \cup (Y_1 \cap Y_2) \times_k Y_1)& \\ \to \mathrm{CH}_d(Y_1 \times_k Y_1) \to \mathrm{CH}_d&((Y_1^0 \setminus Z_0) \times_k (Y_1^0 \setminus D_0)) \to 0\end{aligned}$$

and the surjection

$$\mathrm{CH}_d(Z_1\times_k Y_1)\oplus\mathrm{CH}_d(Y_1\times_k D_1)\oplus\mathrm{CH}_d((Y_1\cap Y_2)\times_k Y_1)\\ \to\mathrm{CH}_d(Z_1\times_k Y_1\cup Y_1\times_k D_1\cup(Y_1\cap Y_2)\times_k Y_1),$$

the relation $N\cdot[\Delta_{Y_1^0}]=0$ in $\mathrm{CH}_d((Y_1^0\setminus Z_0)\times_k(Y_1^0\setminus D_0))$ lifts to a relation of the desired form in $\mathrm{CH}_d(Y_1\times_k Y_1)$. □

We conclude this series of specialization results with the following variation on Lemma 1.7; a similar result may be found in [Colliot-Thélène 2016, Lemme 2.2].

Lemma 1.8. *Let $\mathbb{O}$ be a discrete valuation ring with quotient field K and residue field k. Let $f:\mathscr{X}\to\operatorname{Spec}\mathbb{O}$ be a flat and proper morphism of dimension d over $\operatorname{Spec}\mathbb{O}$ with generic fiber X and special fiber Y. We suppose Y is a union of closed subschemes, $Y=Y_1\cup Y_2$, with X and Y_1 geometrically irreducible, X generically smooth over K, and Y_1 generically smooth over k. Suppose that X admits a decomposition of the diagonal of order N. Let $Z=(Y_1\cap Y_2)_{\mathrm{red}}$ with inclusion $i_Z:Z\to Y_1$. Suppose further that $Y_{2k(Y_1)}$ admits a zero-cycle y_2 of degree r supported in the smooth locus of $Y_{2k(Y_1)}$.*

Then there is an identity in $\mathrm{CH}_d(Y_1\times_k Y_1)$

$$Nr\Delta_{Y_1}=\gamma_1+\gamma_2$$

with γ_1 supported on $Y_1\times_k D_1$, for some divisor $D_1\subset Y_1$, and γ_2 supported in $Z\times_k Y_1$.

Proof. Let η_1 be the generic point of Y_1, let $\mathbb{O}_1=\mathbb{O}_{\mathscr{X},\eta_1}$, and let $\mathfrak{D}$ be the henselization of $\mathbb{O}_1$. Let L be the quotient field of $\mathfrak{D}$; clearly $\mathfrak{D}$ has residue field $k(Y_1)$. Then as $\operatorname{Spec}\mathbb{O}_1\to\operatorname{Spec}\mathbb{O}$ is essentially smooth, the base-change $\mathscr{X}_{\mathfrak{D}}:=\mathscr{X}\otimes_{\mathbb{O}}\mathfrak{D}\to\operatorname{Spec}\mathfrak{D}$ has generic fiber $\mathscr{X}_L$ and special fiber $Y_{k(Y_1)}=Y_{1k(Y_1)}\cup Y_{2k(Y_1)}$. Let $\mathscr{X}^{\mathrm{sm}}_{\mathfrak{D}}\subset\mathscr{X}_{\mathfrak{D}}$ be the maximal open subscheme of $\mathscr{X}_{\mathfrak{D}}$ that is smooth over $\mathfrak{D}$.

Fix a rational equivalence

$$N\cdot\Delta_X\sim x\times X+\gamma$$

with x a 0-cycle on X and γ supported on $X\times_K E$ for some divisor E. Pulling this back to X_L gives the rational equivalence

$$N\cdot\Delta_{X_L}\sim x_L\times X_L+\gamma_L$$

with γ_L supported on $X_L\times_L E_L$. Let $\mathscr{E}$ be the closure of E_L in $\mathscr{X}_{\mathfrak{D}}$, and let $E_0=\mathscr{E}\cap Y_{k(Y_1)}$; E_0 contains no generic point of $Y_{k(Y_1)}$. Furthermore, since the 0-cycle y_2 on $Y_{2k(Y_1)}$ is contained in the smooth locus of $Y_{2k(Y_1)}$, we may find a 0-cycle y_2' on $Y_{2k(Y_1)}$, rationally equivalent to y_2, and with support in the smooth locus of $Y_{2k(Y_1)}\setminus(E_0\cup Z_{k(Y_1)})$. Changing notation, we may assume that y_2 is supported in the smooth locus of $Y_{2k(Y_1)}\setminus(E_0\cup Z_{k(Y_1)})$.

Since $\mathfrak{D}$ is hensel, we may lift $\eta_1 \in Y_1(k(Y_1))$ to a section $s_1 : \operatorname{Spec} \mathfrak{D} \to \mathscr{X}_{\mathfrak{D}}$. Since y_2 is supported in the smooth locus of $Y_{k(Y_1)}$, we may similarly lift the 0-cycle y_2 on $Y_{2k(Y_1)}$ to a cycle $\mathfrak{y}_2$ on $\mathscr{X}_{\mathfrak{D}}$ of relative dimension zero and relative degree r over $\mathfrak{D}$. This gives us the 0-cycle of degree zero $\rho_L := r \cdot s_1(\operatorname{Spec} L) - \mathfrak{y}_{2L}$ on X_L. Since $\mathfrak{D}$ is local, $\mathscr{X}_{\mathfrak{D}}$ is flat over $\mathfrak{D}$ and both y_2 and η_1 are supported in the smooth locus of $Y \setminus E_0$, it follows that both $s_1(\operatorname{Spec} \mathfrak{D})$ and $\mathfrak{y}_2$ are supported in $\mathscr{X}^{\mathrm{sm}}_{\mathfrak{D}} \setminus \mathscr{E}$, and thus ρ_L is supported in the smooth locus of $X_L \setminus E$.

Let p be a closed point in the smooth locus of X_L, inducing the inclusion $i_p : X_L \times_L p \to X_L \times_L X_L$. Since i_p is a regular codimension-$d = \dim X$ embedding, we have the pullback map (see [Fulton 1984, §6.2, pp. 97–98], where this map is called the *Gysin homomorphism*)

$$i_p^* : \mathrm{CH}_d(X_L \times_L X_L) \to \mathrm{CH}_0(X_L \times_L p).$$

If $\mathfrak{z}$ is a 0-cycle supported in the smooth locus of X_L, $\mathfrak{z} = \sum_j n_j p_j$, we have the map

$$\mathfrak{z}^* : \mathrm{CH}_d(X_L \times_L X_L) \to \mathrm{CH}_0(X_L)$$

defined as the sum $\sum_j n_j p_{1*} \circ i^*_{p_j}$. If γ is a d-cycle on $X_L \times_L X_L$ such that each component of γ intersects each subvariety $X_L \times p_j$ properly, then $\gamma^*(\mathfrak{z})$ is well defined and

$$\mathfrak{z}^*(\gamma) = \gamma^*(\mathfrak{z}).$$

We apply these comments to the 0-cycle ρ_L and the cycles $N \cdot \Delta_{X_L}$, $x_L \times_L X_L$, and γ_L. We get the identities in $\mathrm{CH}_0(X_L)$

$$\begin{aligned} N \cdot \rho_L &= \rho_L^*(N \cdot \Delta_{X_L}) \\ &= \rho_L^*(x_L \times_L X_L) + \rho_L^*(\gamma_L). \end{aligned}$$

Both terms in this last line are zero: the first since, as X_L is irreducible, we have $\rho_L^*(x_L \times_L X_L) = \deg_L(\rho_L) \cdot x_L = 0$, and the second since $X_L \times_L \operatorname{supp}(\rho_L) \cap \operatorname{supp}(\gamma_L) = \varnothing$. In other words, $N \cdot \rho_L = 0$ in $\mathrm{CH}_0(X_L)$.

We apply the specialization map

$$\mathrm{sp} : \mathrm{CH}_0(X_L) \to \mathrm{CH}_0(Y_{k(Y_1)})$$

and find $N(r \cdot \eta_1 - y_2) = 0$ in $\mathrm{CH}_0(Y_{k(Y_1)})$. Thus, $Nr \cdot \eta_1 = 0$ in $\mathrm{CH}_0(Y_{1k(Y_1)} \setminus Z_{k(Y_1)})$, and by using the localization sequence for the inclusion $Z_{k(Y_1)} \to Y_{k(Y_1)}$, there is a 0-cycle $\gamma_{2k(Y_1)}$ on $Z_{k(Y_1)}$ with

$$Nr \cdot \eta_1 = i_{Z*}(\gamma_{2k(Y_1)})$$

in $\mathrm{CH}_0(Y_{1k(Y_1)})$. Spreading this relation out over Y_1 as in previous proofs gives the desired decomposition of $Nr \cdot \Delta_{Y_1}$. $\square$

Remark 1.9. Suppose we have $\mathscr{X}$, $Y = Y_1 \cup Y_2$, and $Z = Y_1 \cap Y_2$ satisfying the hypotheses of Lemma 1.8; suppose in addition that Y_1 is smooth over k. Then for all fields $F \supset k$, the quotient group $\mathrm{CH}_0(Y_{1F})/i_{Z*}(\mathrm{CH}_0(Z_F))$ is Nr-torsion. Indeed, since Y_1 is smooth, we have an operation of correspondences on $\mathrm{CH}_0(Y_{1F})$, the correspondence γ_1^* of Lemma 1.8 acts trivially on $\mathrm{CH}_0(Y_{1F})$, γ_2^* maps $\mathrm{CH}_0(Y_{1F})$ to $i_{Z*}(\mathrm{CH}_0(Z_F))$, and the sum acts by multiplication by Nr.

The torsion orders behave well with respect to base-change.

Lemma 1.10. *Let X and Y be proper generically smooth k-schemes, with Y integral and with X equidimensional over k. Let K be the function field $k(Y)$ and I_Y the index of Y.*

(1) *For all i, $\mathrm{Tor}_k^{(i)}(X)$ is finite if and only if $\mathrm{Tor}_K^{(i)}(X_K)$ is finite and in this case, $\mathrm{Tor}_k^{(i)}(X)$ divides $I_Y\,\mathrm{Tor}_K^{(i)}(X_K)$.*

(2) *Suppose X is geometrically integral. If $\mathrm{gTor}_k(X)$ is finite, then so is $\mathrm{Tor}_k(X)$ and $\mathrm{Tor}_k(X)$ divides $I_X \cdot \mathrm{gTor}_k(X)$.*

Proof. For (1), if $\mathrm{Tor}_k^{(i)}(X)$ is finite, then so is $\mathrm{Tor}_K^{(i)}(X_K)$ by Lemma 1.3(4). Suppose $\mathrm{Tor}_K^{(i)}(X_K)$ is finite. Let y be a closed point of Y, contained in the smooth locus of Y over k, and let $\mathcal{O} := \mathcal{O}_{Y,y}$. Applying Lemma 1.5 to the constant family $\mathscr{X} := X \times_k \mathcal{O}$, we see that $\mathrm{Tor}_{k(y)}^{(i)}(X_{k(y)})$ is finite and $\mathrm{Tor}_{k(y)}^{(i)}(X_{k(y)})$ divides $\mathrm{Tor}_K^{(i)}(X_K)$. Applying Lemma 1.3(4) again, $\mathrm{Tor}_k^{(i)}(X)$ is finite and divides $[k(y):k]\cdot \mathrm{Tor}_{k(y)}^{(i)}(X_{k(y)})$. This proves the first assertion.

For (2), let y be a closed point of X, contained in the smooth locus of X over k, let $\mathcal{O} := \mathcal{O}_{X,y}$, and let $\eta \in X(k(X))$ be the canonical point, that is, the restriction of the diagonal section $X \to X \times_k X$ to $\operatorname{Spec} k(X)$. As in the proof of Lemma 1.5, we may find a sequence of regular closed subschemes $y = Z_0 \subset \cdots \subset Z_d = \operatorname{Spec} \mathcal{O}$, $d = \dim_k X$, and thereby define specialization homomorphisms

$$\mathrm{sp}_i : \mathrm{CH}_0(X_{k(Z_i)(X)}) \to \mathrm{CH}_0(X_{k(Z_{i-1})(X)}), \quad i = 1, \ldots, d.$$

Letting $\mathrm{sp}_y : \mathrm{CH}_0(X_{k(X\times_k X)}) \to \mathrm{CH}_0(X_{k(y)(X)})$ be the composition of the sp_i, we have $\mathrm{sp}_y(\eta_1 - \eta_2) = \eta_y - y_{\mathrm{gen}}$, where $\eta_y \in X(k(y)(X))$ is the base-change of $y \in X(k(y))$ and $y_{\mathrm{gen}} \in X(k(y)(X))$ is the base-change of $\eta \in X(k(X))$. Thus, $\mathrm{gTor}_k(X)\cdot(\eta_y - y_{\mathrm{gen}}) = 0$ in $\mathrm{CH}_0(X_{k(y)(X)})$; pushing forward to $\mathrm{CH}_0(X_{k(X)})$ gives $[k(y):k]\cdot \mathrm{gTor}_k(X)\cdot \eta - \mathrm{gTor}_k(X)\cdot y \times_k k(X) = 0$ in $\mathrm{CH}_0(X_{k(X)})$. Applying localization gives us the decomposition of the diagonal Δ_X of order $[k(y):k]\cdot \mathrm{gTor}_k(X)$; doing this for each closed point y gives us the decomposition of the diagonal of order $I_X \cdot \mathrm{gTor}_k(X)$. Hence, $\mathrm{Tor}_k(X)$ is finite and divides $I_X \cdot \mathrm{gTor}_k(X)$. □

For example, $\mathrm{Tor}_k^{(i)}(X) = \mathrm{Tor}_L^{(i)}(X_L)$ if L is a pure transcendental extension of a field k.

Lemma 1.11. *Let X be a proper k-scheme. Let $k \subset L$ be an extension of fields with k algebraically closed. The following hold*:

(1) *For all i*, $\mathrm{Tor}_k^{(i)}(X) = \mathrm{Tor}_L^{(i)}(X_L)$.

(2) *Suppose in addition X is smooth and integral. Then* $\mathrm{gTor}_k(X) = \mathrm{gTor}_L(X_L)$ *and* $\mathrm{Tor}_k(X) = \mathrm{gTor}_k(X)$.

Proof. We may assume that L is finitely generated over k. Using openness of the regular locus for finite type k-schemes and k algebraically closed, we can find a noetherian local regular k-algebra $\mathcal{O}$ with quotient field L and residue field k. Applying Lemma 1.5(1) to $\mathscr{X} := X \times_k \mathcal{O} \to \operatorname{Spec} \mathcal{O}$ implies (1).

The assertions about gTor follow from (1), Lemma 1.10(2), and Lemma 1.3. □

Definition 1.12. Let X be a proper, generically smooth k-scheme. Let $\bar{k}$ be the algebraic closure of k, and define $\mathrm{Tor}^{(i)}(X) := \mathrm{Tor}_{\bar{k}}^{(i)}(X_{\bar{k}})$. We call $\mathrm{Tor}^{(i)}(X)$ the i-th *geometric torsion order* of X. We write $\mathrm{Tor}(X)$ for $\mathrm{Tor}^{(0)}(X)$.

Note that $\mathrm{Tor}^{(i)}(X)$ is invariant under base-extension $X \rightsquigarrow X_L$ for a field extension $L \supset k$. Also, assuming X to be smooth and geometrically integral, $\mathrm{Tor}(X)$ is equal to $\mathrm{gTor}_{\bar{k}}(X_{\bar{k}})$.

In much the same vein as Lemma 1.3, we show that the generic torsion order measures the torsion order after adjoining a "generic" rational point, that is:

Lemma 1.13. *Let X be a smooth proper geometrically integral k-scheme*, *and let $K = k(X)$. Then* $\mathrm{gTor}_k(X) = \mathrm{Tor}_K(X_K)$.

Proof. If $N \cdot (\eta_1 - \eta_2) = 0$ in $\mathrm{CH}_0(X_{k(X \times_k X)})$, then we have a decomposition of the diagonal of order N for $X_{k(X)}$:

$$N \cdot \Delta_{X_K} = N \cdot [\eta] \times_K X_K + \gamma$$

with γ supported in $X_K \times_K D$, with $D \subsetneq X_K$, and with η the restriction of Δ_X to $X \times_k k(X) \subset X \times_k X$. In other words, η is the K-rational point of X_K induced by the generic point of X. Thus, $\mathrm{Tor}_K(X_K)$ divides $\mathrm{gTor}_k X$. Conversely, if X_K admits a decomposition of the diagonal of order n,

$$n \cdot \Delta_{X_K} = x \times X_K + \gamma \tag{1-3}$$

with x a 0-cycle on X_K and γ supported on $X_K \times_K D$ for some divisor $D \subset X_K$, then pulling (1-3) back along $(\mathrm{id}_{X_K}, \eta) : X_K \to X_K \times_K X_K$ gives us $x = n \cdot [\eta]$ in $\mathrm{CH}_0(X_K)$, so $n \cdot \Delta_{X_K} = n \cdot [\eta] \times X_K + \gamma$ in $\mathrm{CH}_d(X_K \times_K X_K)$. Restriction to $X_K \times_K K(X_K)$ gives $n \cdot \eta_1 = n \cdot \eta_2$ in $\mathrm{CH}_0(X_{k(X \times_k X)})$, so $\mathrm{gTor}_k(X)$ divides $\mathrm{Tor}_K(X_K)$. □

One last elementary property of the torsion indices concerns the behavior with respect to morphisms.

Lemma 1.14. *Let $f : Y \to X$ be a surjective morphism of integral reduced proper k-schemes of the same dimension d. Then $\mathrm{Tor}_k^{(i)} X$ divides $\deg f \cdot \mathrm{Tor}_k^{(i)} Y$ for all i. If X and Y are generically smooth over k, then $\mathrm{gTor}_k X$ divides $(\deg f)^2 \cdot \mathrm{gTor}_k Y$.*

Proof. Suppose the diagonal for Y admits a decomposition of order N and level i:

$$N \cdot \Delta_Y = \gamma_i + \gamma'$$

with γ' supported on $Y \times_k D$ for some divisor D and γ_i supported on $Z \times_k Y$ for some closed subset Z of Y with $\dim_k Z \leq i$. Pushing forward by $f \times f$ gives

$$\deg f \cdot N \cdot \Delta_X = (f \times f)_* \gamma_i + (f \times f)_* \gamma',$$

and thus $\mathrm{Tor}_k^{(i)} X$ divides $\deg f \cdot \mathrm{Tor}_k^{(i)} Y$. Similarly, we have $(f \times f \times f)_*(\Delta_{Y,ij}) = (\deg f)^2 \cdot \Delta_{X,ij}$ for $ij = 12, 13$, which shows $\mathrm{gTor}_k X$ divides $(\deg f)^2 \cdot \mathrm{gTor}_k Y$. $\square$

The behavior of the torsion indices with respect to rational and birational maps will be discussed in Section 3.

2. Torsion orders for very general fibers

The following global version of Lemma 1.5(3) follows by an argument using Hilbert schemes. See [Voisin 2015, Theorem 1.1 and Proposition 1.4] or [Colliot-Thélène and Pirutka 2016b, Appendice B] for similar statements. This result will only be used in Sections 7 and 8.

Proposition 2.1. *Let $p : \mathscr{X} \to B$ be a flat, equidimensional, and projective family over a scheme B of finite type over a field k, and let b_0 be a point of B. We suppose that each geometric fiber of p is generically reduced. Fix an integer $i \geq 0$. Then there is a countable union of closed subsets $F = \bigcup_{j=1}^{\infty} F_j$ with $b_0 \notin F$ such that for all $b \in B \setminus F$, the geometric fiber $\mathscr{X}_{\overline{k(b)}}$ satisfies $\mathrm{Tor}^{(i)}(\mathscr{X}_{\overline{k(b_0)}}) \mid \mathrm{Tor}^{(i)}(\mathscr{X}_{\overline{k(b)}})$. Here we use the convention that $N \mid \infty$ for all $N \in \mathbb{N}_+ \cup \{\infty\}$ and $\infty \mid N \implies N = \infty$.*

The proof uses the following elementary lemma, which we were not able to find in the literature.

Let X be a noetherian equidimensional scheme with integral components $X_1, \ldots, X_t$. Let $x_i \in X_i$ be the generic point. The *associated cycle* [Fulton 1984, §1.5] of X is the cycle $\mathrm{cyc}(X) := \sum_{i=1}^{s} e_i X_i \in \mathscr{Z}(X)$ with e_i defined as

$$e_i := \mathrm{lng}_{\mathscr{O}_{X_i,x_i}} \mathscr{O}_{X_i,x_i}.$$

Lemma 2.2. *Let B be a noetherian scheme, let $p : \mathscr{Y} \to B$ be a flat morphism, and let $\mathscr{W}_0, \mathscr{W}_1, \ldots, \mathscr{W}_s$ be closed subschemes of $\mathscr{Y}$, flat and equidimensional of dimension r over B. For each $b \in B$, let $\mathscr{W}_{ib} \subset \mathscr{Y}_b$ be the respective fibers over b. Fix integers $m_0, \ldots, m_s$.*

(1) *The subset*

$$\mathcal{T}_{\mathcal{Y}}(B) := \left\{ b \in B \;\middle|\; \sum_{i=0}^{s} m_i \cdot \operatorname{cyc}(\mathcal{W}_{ib}) = 0 \textit{ in } \mathcal{Z}(\mathcal{Y}_b) \right\}$$

is a constructible subset of B.

(2) *For each $b \in B$, let $\bar{b}$ be a geometric point mapping to b, and $\mathcal{W}_{i\bar{b}} \subset \mathcal{Y}_{\bar{b}}$ be the respective fibers over $\bar{b}$. Fix integers $m_0, \dots, m_s$. Then, the equality*

$$\mathcal{T}_{\mathcal{Y}}(B) = \left\{ b \in B \;\middle|\; \sum_{i=0}^{s} m_i \cdot \operatorname{cyc}(\mathcal{W}_{i\bar{b}}) = 0 \textit{ in } \mathcal{Z}(\mathcal{Y}_{\bar{b}}) \right\} \tag{2-1}$$

holds.

Proof. We first prove (1); we proceed by a series of reductions. Firstly, we may assume that B is integral and separated. As the assertion is obvious if B is a point, we may use noetherian induction and replace B with any dense open subscheme. Moreover, if $\mathcal{Y} = \bigcup_i \mathcal{U}_i$ is a finite open covering, then $\mathcal{T}_{\mathcal{Y}}(B) = \bigcap_i \mathcal{T}_{\mathcal{U}_i}(B)$; hence, it suffices to prove the assertion for each i.

Let S be the set of all integral components of the subschemes $\mathcal{W}_0, \dots, \mathcal{W}_s$. Let us consider the elements of S as integral schemes. We claim that there is an open dense subset U of B such that for all $b \in U$ and $\mathcal{W}, \mathcal{V} \in S$ with $\mathcal{W} \neq \mathcal{V}$, $\mathcal{W}_b$ and $\mathcal{V}_b$ have no common integral component. Indeed, for $\mathcal{W}, \mathcal{V} \in S$ with $\mathcal{W} \neq \mathcal{V}$, there is an open dense $U_{\mathcal{V},\mathcal{W}} \subset B$ such that $\mathcal{W} \times_{\mathcal{Y}} \mathcal{V} \times_{\mathcal{Y}} p^{-1}(U_{\mathcal{V},\mathcal{W}}) \to U_{\mathcal{V},\mathcal{W}}$ is flat of relative dimension $\leq r-1$ (it need not be equidimensional). All integral components of $\mathcal{W}_b$ and $\mathcal{V}_b$ have dimension r, and therefore $\mathcal{W}_b$ and $\mathcal{V}_b$ do not have a common integral component if $b \in U_{\mathcal{V},\mathcal{W}}$. Taking the intersection of the $U_{\mathcal{V},\mathcal{W}}$ for all $\mathcal{V} \neq \mathcal{W}$ in S gives the desired open dense subset U.

After passing from B to U, we get $\mathcal{T}_{\mathcal{Y}}(B) = \mathcal{T}_{\mathcal{Y}'}(B)$ with

$$\mathcal{Y}' = \mathcal{Y} \setminus \bigcup_{\substack{\mathcal{W},\mathcal{V} \in S \\ \mathcal{W} \neq \mathcal{V}}} \mathcal{W} \cap \mathcal{V} = \bigcup_{\mathcal{W} \in S} \mathcal{Y} \setminus \Biggl(\bigcup_{\mathcal{V} \in S \setminus \{\mathcal{W}\}} \mathcal{V} \Biggr).$$

Therefore, we may suppose $S = \{\mathcal{W}\}$, in other words, there is only one integral component.

For each i, define n_i by $\operatorname{cyc}(\mathcal{W}_i) = n_i \cdot \mathcal{W}$. We claim that there is an open dense $U \subset B$ such that $\operatorname{cyc}(\mathcal{W}_{ib}) = n_i \cdot \operatorname{cyc}(\mathcal{W}_b)$ for all $b \in U$. This will imply the assertion, because after shrinking U further so that $\mathcal{W}_b \neq \varnothing$ holds, either $\mathcal{T}_{\mathcal{Y} \cap p^{-1}(U)}(U) = U$ or $\mathcal{T}_{\mathcal{Y} \cap p^{-1}(U)}(U) = \varnothing$, depending on whether $0 = \sum_i m_i \cdot n_i$ holds.

In order to prove our claim, let η be the generic point of $\mathcal{W}$ and $\mathcal{W}_i$. Since $\mathcal{O}_{\mathcal{W},\eta}$ is a field, there is a filtration

$$\mathcal{O}_{\mathcal{W}_i,\eta} = F^0 \subset F^1 \supset \cdots \supset F^r = 0$$

by $\mathbb{O}_{\mathcal{W}_i,\eta}$ submodules such that the quotients F^{j+1}/F^j are free $\mathbb{O}_{\mathcal{W},\eta}$ modules of rank r_j. By definition, the equality $n_i = \sum_j r_j$ holds. We can extend this filtration to a nonempty open subset $\mathcal{W}_i'$ of $\mathcal{W}_i$ such that the quotients are free $\mathbb{O}_{\mathcal{W}'}$ modules of rank r_j, where $\mathcal{W}' = \mathcal{W} \cap \mathcal{W}_i'$. We denote it by

$$\mathbb{O}_{\mathcal{W}_i'} = \tilde{F}^0 \subset \tilde{F}^1 \supset \cdots \supset \tilde{F}^r = 0.$$

Define $U \subset B$ to be a nonempty open subset such that $(\mathcal{W} \setminus \mathcal{W}') \cap p^{-1}(U) \to U$ is flat of relative dimension $\leq r-1$ and $\mathcal{W}' \cap p^{-1}(U) \to U$ is flat. For $b \in U$, the generic points of the integral components of $\mathcal{W}_{ib}$ (= integral components of $\mathcal{W}_b$) are contained in $\mathcal{W}_i'$. Flatness of $\mathcal{W}' \cap p^{-1}(U) \to U$ implies that we get an induced filtration

$$\mathbb{O}_{\mathcal{W}_{ib}'} = \tilde{F}_b^0 \subset \tilde{F}_b^1 \supset \cdots \supset \tilde{F}_b^r = 0$$

with quotients $\tilde{F}_b^{j+1}/\tilde{F}_b^j$ free of rank r_j as $\mathbb{O}_{\mathcal{W}_b'}$-modules. For every generic point ϵ of $\mathcal{W}_{ib}$ (hence $\mathcal{W}_{ib}'$) we get

$$\operatorname{lng}_{\mathbb{O}_{\mathcal{W}_{ib},\epsilon}} \mathbb{O}_{\mathcal{W}_{ib},\epsilon} = \Bigl(\sum_j r_j\Bigr) \cdot \operatorname{lng}_{\mathbb{O}_{\mathcal{W}_b,\epsilon}} \mathbb{O}_{\mathcal{W}_b,\epsilon} = n_i \cdot \operatorname{lng}_{\mathbb{O}_{\mathcal{W}_b,\epsilon}} \mathbb{O}_{\mathcal{W}_b,\epsilon},$$

which proves the claim.

For (2), we note that for each $\bar{b} \to b \in B$, the map $\bar{b} \to b$ is flat and the pullback map

$$\mathfrak{X}(Y_b) \to \mathfrak{X}(Y_{\bar{b}})$$

is injective; (2) follows directly from this and (1). □

Proof of Proposition 2.1. Let d be the relative dimension of $\mathcal{X}$ over B. For a positive integer M, let $\mathcal{S}(M)$ be the set of $b \in B$ such that M does not divide $\operatorname{Tor}^{(i)}(\mathcal{X}_{\overline{k(b)}})$. Taking $M = \operatorname{Tor}^{(i)}(\mathcal{X}_{\overline{k(b_0)}})$ and $F = \mathcal{S}(M)$, it suffices to show that $\mathcal{S}(M)$ is a countable union of closed subsets of B.

We first show that $\mathcal{S}(M)$ is closed under specialization. Indeed, if we have a specialization $b \rightsquigarrow \tilde{b}$ with $b \in \mathcal{S}(M)$, then there is an excellent DVR $\mathbb{O}$ and a morphism $\operatorname{Spec} \mathbb{O} \to B$ with b the image of the generic point of $\operatorname{Spec} \mathbb{O}$ and $\tilde{b}$ the image of the closed point. Indeed, let C be the closure of b in B, blow-up $\operatorname{Spec} \mathbb{O}_{C,\tilde{b}}$ along $\tilde{b}$, normalize to obtain a normal scheme $\pi : T \to \operatorname{Spec} \mathbb{O}_{C,\tilde{b}}$ of finite type over $\mathbb{O}_{C,\tilde{b}}$, choose a generic point t of the Cartier divisor $\pi^{-1}(\tilde{b})$ on T, and take $\mathbb{O} := \mathbb{O}_{T,t}$. The local ring $\mathbb{O}_{C,\tilde{b}}$ is excellent since C is of finite type over a field, and the operations used in constructing $\mathbb{O}$ from $\mathbb{O}_{C,\tilde{b}}$ all preserve excellence [Matsumura 1980, Chapters 12 and 13]. Pulling back $\mathcal{X}$ to $\operatorname{Spec} \mathbb{O}$, it follows from Lemmas 1.5(3) and 1.11(1) that $\tilde{b}$ is also in $\mathcal{S}(M)$.

Since $\mathcal{S}(M)$ is closed under specialization, it suffices to show that, for each affine open subscheme U of B, $\mathcal{S}(M) \cap U$ is a countable union of constructible subsets of U. Thus, we may assume that B is affine, and that $\mathscr{X}$ is a closed subscheme of $B \times_k \mathbb{P}^n_k$ for some n, with $p : \mathscr{X} \to B$ the restriction of the projection.

By standard Hilbert scheme arguments, there is a projective B-scheme $q : \mathscr{Y}_{\alpha,\beta} \to B$ such that the geometric points of $\mathscr{Y}_{\alpha,\beta}$ consist of triples (b, Z_b, D_b), with b a geometric point of B, $Z_b \subset \mathscr{X}_b$ a closed subscheme of dimension $j \leq i$, and $D_b \subset \mathscr{X}_b$ a closed subscheme of dimension $< d$, and with Z_b and D_b having fixed Hilbert polynomials α, β. Let $\mathscr{Z} \subset \mathscr{X} \times_B \mathscr{Y}_{\alpha,\beta}$ and $\mathscr{D} \subset \mathscr{X} \times_B \mathscr{Y}_{\alpha,\beta}$ be the universal subschemes. We set

$$\mathscr{U} := \mathscr{X} \times_B \mathscr{X} \times_B \mathscr{Y}_{\alpha,\beta} \setminus (\mathscr{X} \times_B \mathscr{D} \cup \mathscr{Z} \times_B \mathscr{X}).$$

Similarly, there is a finite type B-scheme $g : \mathscr{W}_\phi \to B$ whose geometric points consist of pairs (b, W) with $W \subset \mathscr{X}_b \times_b \mathscr{X}_b \times_b \mathbb{P}^1_b$ a closed subscheme of dimension $d+1$, having Hilbert polynomial ϕ and being flat over $\mathbb{P}^1_b$. Indeed, denoting by

$$H := \mathrm{Hilb}_\phi(\mathscr{X} \times_B \mathscr{X} \times_B \mathbb{P}^1_B)$$

the Hilbert scheme, we can consider the subfunctor F of H defined by

$$F(T) = \{W \in H(T) \mid W \to T \times_B \mathbb{P}^1_B \text{ is flat}\}.$$

If $W_{\mathrm{uni}} \subset H \times_B \mathscr{X} \times_B \mathscr{X} \times_B \mathbb{P}^1_B$ denotes the universal subscheme, then we let $W'_{\mathrm{uni}} \subset W_{\mathrm{uni}}$ be the closed subset where $W_{\mathrm{uni}} \to H \times_B \mathbb{P}^1_B$ is not flat. We define $\mathscr{W}_\phi$ as the complement of the image of W'_{uni} in H. By using critère de platitude par fibres [EGA IV$_3$ 1966, Théorème 11.3.10], we conclude that $\mathscr{W}_\phi$ represents F.

For each integer $r \geq 1$, and each choice of Hilbert polynomials α, β and $\phi_1, \dots, \phi_r$, we obtain subschemes $\mathscr{W}^0_1, \mathscr{W}^\infty_1, \dots, \mathscr{W}^0_r, \mathscr{W}^\infty_r$ of $\mathscr{U} \times_B \mathscr{W}_{\phi_1} \times_B \cdots \times_B \mathscr{W}_{\phi_r}$ that are flat of relative dimension d over $\mathscr{Y}_{\alpha,\beta} \times_B \mathscr{W}_{\phi_1} \times_B \cdots \times_B \mathscr{W}_{\phi_r}$ as follows. Let $\mathscr{V}_i \subset \mathscr{X} \times_B \mathscr{X} \times_B \mathbb{P}^1_B \times_B \mathscr{W}_{\phi_i}$ be the universal subscheme. Since $\mathscr{V}_i \to \mathbb{P}^1_B \times_B \mathscr{W}_{\phi_i}$ is flat, the base-change $\mathscr{V}^\epsilon_i$ to $\mathscr{W}_{\phi_i}$ via $B \xrightarrow{\epsilon} \mathbb{P}^1_B$, for $\epsilon = 0$ and $\epsilon = \infty$, is flat. We define $\mathscr{W}^\epsilon_1$ to be the restriction of $\mathscr{V}^\epsilon_1 \times_B \mathscr{Y}_{\alpha,\beta} \times_B \mathscr{W}_{\phi_2} \times_B \cdots \times_B \mathscr{W}_{\phi_2}$ to $\mathscr{U} \times_B \mathscr{W}_{\phi_1} \times_B \cdots \times_B \mathscr{W}_{\phi_r}$, and similarly for $\mathscr{W}^\epsilon_i$.

Fix a sequence of integers $m_1, \dots, m_r$ and an integer $N > 0$. By Lemma 2.2, the image of all geometric points $(b, Z_b, D_b, W_1, \dots, W_r)$ satisfying

$$N \cdot \Delta_{\mathscr{X}_b \times_b \mathscr{X}_b}|_{\mathscr{X}_b \times_b \mathscr{X}_b \setminus (\mathscr{X}_b \times_b D_b \cup \mathscr{Z}_b \times_b \mathscr{X}_b)} = \sum_i m_i \cdot (\mathrm{cyc}(W^0_i) - \mathrm{cyc}(W^\infty_i)),$$

where W^ϵ_i is the scheme theoretic intersection of W_i with $\mathscr{X}_b \times_b \mathscr{X}_b \times \epsilon$ in $\mathscr{X}_b \times_b \mathscr{X}_b \times_b \mathbb{P}^1_b$, forms a constructible subset $\mathscr{T}_{r,\phi_*,\alpha,\beta,m_*,N}$ of $\mathscr{Y}_{\alpha,\beta} \times_B \mathscr{W}_{\phi_1} \times_B \cdots \times_B \mathscr{W}_{\phi_r}$. Let $\mathscr{R}_{r,\phi_*,\alpha,\beta,m_*,N}$ be the image of $\mathscr{T}_{r,\phi_*,\alpha,\beta,m_*,N}$ in B.

If b is a geometric point of B with image in $\mathscr{R}_{r,\phi_*,\alpha,\beta,m_*,N}$, and if we choose a geometric point $(b, Z_b, D_b, W_1, \dots, W_r)$ of $\mathscr{Y}_{\alpha,\beta} \times_B \mathscr{W}_{\phi_1} \times_B \cdots \times_B \mathscr{W}_{\phi_r}$ lying over b, then the cycle $\sum_i m_i \cdot \mathrm{cyc}(W_i)$ on $\mathscr{X}_b \times_b \mathscr{X}_b \times \mathbb{P}^1$ gives a rational equivalence showing that $\mathrm{Tor}^{(i)}(\mathscr{X}_b) \mid N$. Conversely, as each integral closed subscheme $W \subset$

$\mathscr{X}_b \times_b \mathscr{X}_b \times \mathbb{P}^1$ that dominates $\mathbb{P}^1$ is flat over $\mathbb{P}^1$, each geometric point $b \in B$ such that $\mathrm{Tor}^{(i)}(\mathscr{X}_b) \mid N$ is in $\mathscr{R}_{r,\phi_*,\alpha,\beta,m_*,N}$ for some choice of Hilbert polynomials α, β, ϕ_* and integers r and $m_1, \dots, m_r$. Thus, $\mathscr{S}(M)$ is the union of the subsets $\mathscr{R}_{r,\phi_*,\alpha,\beta,m_*,N}$ over all α, β, r, ϕ_*, m_*, and all $N > 0$ not divisible by M. As this set of choices is countable, it follows that $\mathscr{S}(M)$ is a countable union of constructible subsets of B. Since $\mathscr{S}(M)$ is closed under specialization, the proof is complete. □

3. Universally and totally CH_0-trivial morphisms

We recall the notion of a universally CH_0-trivial morphism and a related notion, that of a totally CH_0-trivial morphism.

Definition 3.1 [Colliot-Thélène and Pirutka 2016b, Définitions 1.1 and 1.2]. Let $p : Z \to Y$ be a proper morphism of finite type k-schemes for some field k. The morphism p is *universally* CH_0-*trivial* if for all field extensions $F \supset k$, the map $p_* : \mathrm{CH}_0(Z_F) \to \mathrm{CH}_0(Y_F)$ is an isomorphism. A proper k-scheme $\pi_Y : Y \to \operatorname{Spec} k$ is called a universally CH_0-trivial k-scheme if π_Y is a universally CH_0-trivial morphism.

Definition 3.2. A proper morphism $p : Z \to Y$ of k-schemes is *totally* CH_0-*trivial* if for each point $y \in Y$, the fiber $p^{-1}(y)$ is a universally CH_0-trivial $k(y)$-scheme.

It follows directly from the definition that the property of a proper morphism being totally CH_0-trivial is stable under arbitrary base-change.

We rephrase a result of Colliot-Thélène and Pirutka.

Proposition 3.3 [Colliot-Thélène and Pirutka 2016b, Proposition 1.8]. *Let $p : Z \to Y$ be a totally* CH_0-*trivial morphism. Then p is universally* CH_0-*trivial.*

Remarks 3.4. (1) By the base-change property of totally CH_0-trivial morphisms, we see that for $p : Z \to Y$ a totally CH_0-trivial morphism and $W \to Y$ a morphism of k-schemes, the projection $Z \times_Y W \to W$ is universally CH_0-trivial.

(2) There are examples of universally CH_0-trivial morphisms that are not totally CH_0-trivial;[1] in particular, the property of a morphism being universally CH_0-trivial is not stable under base-change.

Corollary 3.5. (1) *Universally* CH_0-*trivial morphisms and totally* CH_0-*trivial morphisms are closed under composition.*

[1]For example, let k be an algebraically closed field of characteristic $\neq 2$, let S be the cone in $\mathbb{P}^3_k$ over a smooth plane curve C of degree ≥ 3, let $Y \to S$ be the double cover branched over the transverse intersection of S with a quadric, and let $y_1, y_2 \in Y$ be the points lying over the vertex of S. Let $p : Z \to Y$ be the blow-up of Y at y_1, and let $z = p^{-1}(y_2)$. Then for all fields $L \supset k$, $\mathrm{CH}_0(z_L) \xrightarrow{i_{z*}} \mathrm{CH}_0(Z_L)$ and $\mathrm{CH}_0(y_{2L}) \xrightarrow{i_{y_2*}} \mathrm{CH}_0(Y_L)$ are isomorphisms, and thus p is universally CH_0-trivial. However, $p^{-1}(y_1) \cong C$, so p is not totally CH_0-trivial.

(2) *Let $p : Z \to Y$ be a morphism of smooth k-schemes that is a sequence of blow-ups with smooth centers. Then p is a totally* CH_0*-trivial morphism.*

(3) *Suppose that the field k admits resolution of singularities of birational morphisms for smooth k-schemes of dimension $\leq d$; that is, if $p : Z \to Y$ is a proper birational morphism of smooth k-schemes of dimension $\leq d$, there is a sequence of blow-ups of Y with smooth centers, $q : W \to Y$, such that the resulting birational map $r : W \to Z$ is a morphism. Then each proper birational morphism $p : Z \to Y$ of smooth k-schemes of dimension $\leq d$ is totally* CH_0*-trivial. In particular, this holds for k of characteristic zero, or for $d \leq 3$ and k algebraically closed* [Abhyankar 1966].

Proof. Statement (1) for universally CH_0-trivial morphisms is obvious from the definition, and for totally CH_0-trivial morphisms this follows with the help of Proposition 3.3.

For (2), we use (1) to reduce to checking for the blow-up of Y along a smooth closed subscheme F, for which the assertion is clear.

For (3), let y be a point of Y and $L \supset k(y)$ a field extension. Dominating Z by a $q : W \to Y$ as above, we have the maps

$$\mathrm{CH}_0(q^{-1}(y)_L) \xrightarrow{r_*} \mathrm{CH}_0(p^{-1}(y)_L) \xrightarrow{p_*} \mathrm{CH}_0(\operatorname{Spec} L) = \mathbb{Z}$$

which, as $\mathrm{CH}_0(q^{-1}(y)_L) \to \mathrm{CH}_0(\operatorname{Spec} L)$ is an isomorphism, gives us a splitting to p_*. Applying resolution of singularities to $r : W \to Z$ gives a sequence of blow-ups with smooth centers $s : X \to Z$ such that $t := r^{-1}s : X \to W$ is a morphism. Since $X \to Z$ is totally CH_0-trivial, the sequence

$$\mathrm{CH}_0(t^{-1}(q^{-1}(y))_L) \xrightarrow{t_*} \mathrm{CH}_0(q^{-1}(y)_L) \xrightarrow{r_*} \mathrm{CH}_0(p^{-1}(y)_L)$$

gives a splitting to r_*, so p_* is an isomorphism. □

Lemma 3.6. (1) *Let $q : Z \to Y$ be a birational totally* CH_0*-trivial morphism of integral, generically smooth k-schemes. Let $N > 0$ be an integer, let $Y_i, W, D \subset Y$ be proper closed subsets with* $\dim Y_i \leq i$, *and suppose we have a decomposition of Δ_Y as*

$$N \cdot \Delta_Y = \gamma + \gamma_1 + \gamma_2,$$

with γ supported on $Y_i \times_k Y$, γ_1 supported on $Y \times_k D$, and γ_2 supported on $W \times_k Y$. Then there are proper closed subsets $Z_i, D' \subset Z$ with $\dim Z_i \leq i$ *and a decomposition of Δ_Z as*

$$N \cdot \Delta_Z = \gamma' + \gamma'_1 + \gamma'_2,$$

with γ' supported on $Z_i \times_k Z$, γ'_1 supported on $Z \times_k D'$, and γ'_2 supported on $q^{-1}(W) \times_k Z$.

(2) *Let $q : Z \to Y$ be a birational totally* CH_0*-trivial morphism of integral, generically smooth, proper k-schemes. Then* $\mathrm{Tor}_k^{(i)}(Z) = \mathrm{Tor}_k^{(i)}(Y)$ *for all i.*

(3) *Let $q : Z \to Y$ be a birational universally* CH_0*-trivial morphism of integral proper k-schemes. Then* $\mathrm{Tor}_k(Z) = \mathrm{Tor}_k(Y)$. *If moreover Z and Y are geometrically integral, then* $\mathrm{gTor}_k(Z) = \mathrm{gTor}_k(Y)$.

Proof. We note that (2) follows easily from (1). Indeed, (1) with $W = \varnothing$ shows that $\mathrm{Tor}_k^{(i)}(Z)$ divides $\mathrm{Tor}_k^{(i)}(Y)$ for all i; as $(q \times q)_*(\Delta_Z) = \Delta_Y$, it follows that a decomposition of Δ_Z of order N and level i gives a similar decomposition of Δ_Y by applying $(q \times q)_*$.

We now prove (1). We may assume that $W = \varnothing$. Indeed, if we replace Y with $Y' := Y \setminus W$ and Z with $Z' := Z \setminus q^{-1}(W)$, the result for $q|_{Z'} : Z' \to Y'$ and the decomposition

$$N \cdot \Delta_{Y'} = \gamma|_{Y' \times_k Y'} + \gamma_{1|Y' \times_k Y'},$$

together with localization gives (1) for the original data.

Suppose then we have

$$N \cdot \Delta_Y = \gamma + \gamma_1$$

with γ supported on $Y_i \times_k Y$ and γ_1 supported on $Y \times_k D$. Let $K = k(Y)$, and let $\eta_Y \in Y$ be the generic point. We have a rational equivalence of 0-cycles on $Y \times_k \eta_Y$

$$N \cdot \eta_Y \times \eta_Y \sim \gamma_{\eta_Y}$$

with γ_{η_Y} a 0-cycle supported on $Y_i \times_k \eta_Y$. Thus, $N \cdot \eta_Y \times \eta_Y \sim 0$ on $(Y \setminus Y_i) \times_k \eta_Y$.

Since $Z \setminus q^{-1}(Y_i) \to Y \setminus Y_i$ is birational and universally CH_0-trivial (Remarks 3.4), there is a rational equivalence of 0-cycles

$$N \cdot \eta_Z \times \eta_Z \sim 0$$

on $(Z \setminus q^{-1}(Y_i)) \times_k \eta_Z$, where $\eta_Z \in Z$ is the generic point. We claim that there is a dimension $\leq i$ closed subset Z' of Z and a rational equivalence of 0-cycles on $Z \times_k \eta_Z$

$$N \cdot \eta_Z \times \eta_Z \sim \rho_Z$$

with ρ_Z a 0-cycle supported on $Z' \times_k \eta_Z$. We proceed by a noetherian induction. We assume there is a closed subset $Y^j \subset Y_i$, a dimension $\leq i$ closed subset Z_j of $q^{-1}(Y_i)$, and a rational equivalence of 0-cycles on $(Z \setminus q^{-1}(Y^j)) \times_k \eta_Z$

$$N \cdot \eta_Z \times \eta_Z \sim \rho_j$$

with ρ_j a 0-cycle supported on $Z_j \times_k \eta_Z$, and we show the parallel statement for a proper closed subset Y^{j+1} of Y^j. The induction starts with $Y^0 = Y_i$.

Choose an integral component Y_0^j of Y^j, and let ν be its generic point. Let Y' be the union of the components of Y^j different from Y_0^j. We have the exact localization sequence

$$\mathrm{CH}_0((q^{-1}(Y_0^j \setminus Y')) \times_k \eta_Z) \xrightarrow{i_*} \mathrm{CH}_0((Z \setminus q^{-1}(Y')) \times_k \eta_Z) \\ \to \mathrm{CH}_0((Z \setminus q^{-1}(Y^j)) \times_k \eta_Z) \to 0,$$

and thus there is a 0-cycle ρ' on $q^{-1}(Y_0^j \setminus Y') \times_k \eta_Z$ and a rational equivalence

$$N \cdot \eta_Z \times \eta_Z \sim \rho_j + i_*(\rho')$$

on $(Z \setminus q^{-1}(Y')) \times_k \eta_Z$.

Write

$$\rho' = \sum_i m_i x_i + \sum_j n_j x'_j,$$

where the x_i, x'_j are closed points of $q^{-1}(Y_0^j \setminus Y') \times_k \eta_Z$, such that $q \circ p_1(x_i) = \nu$ for all i and $q \circ p_1(x'_j)$ is contained in some proper closed subset (say Y'') of Y_0^j for all j. Replacing Y' with $Y' \cup Y''$ and changing notation, we may assume that $\rho' = \sum_i m_i x_i$.

By assumption, the map $q^{-1}(\nu) \to \nu$ is universally CH_0-trivial, so there is a degree-one 0-cycle ϵ on $q^{-1}(\nu)$ so that ϵ_L generates $\mathrm{CH}_0(q^{-1}(\nu)_L)$ for all field extensions $L \supset k(\nu)$; in particular, $\epsilon \times \eta_Z$ generates $\mathrm{CH}_0(q^{-1}(\nu) \times_k \eta_Z)$. Enlarging Y' again by a proper closed subset of Y_0^j, we may assume that

$$\rho' = m \cdot \epsilon \times \eta_Z$$

in $\mathrm{CH}_0(q^{-1}(Y_0^j \setminus Y') \times_k \eta_Z)$, for some $m \in \mathbb{Z}$. Since ϵ is a 0-cycle on $q^{-1}(\nu)$, the dimension of the closure Z' of the support of ϵ in $q^{-1}(Y_0^j)$ is bounded by the transcendence dimension of $k(\nu)$ over k, that is, by $\dim_k Y_0^j$; since $Y_0^j \subset Y_i$,

$$\dim_k Z' \le i.$$

Taking $Y^{j+1} = Y'$, $Z_{j+1} = Z_j \cup Z'$, and $\rho_{j+1} = \rho_j + m \cdot \epsilon \times \eta_Z$, the 0-cycle ρ_{j+1} is supported on $Z_{j+1} \times_k \eta_Z$, $\dim_k Z_{j+1} \le i$, and we have

$$N \cdot \eta_Z \times \eta_Z = \rho_{j+1}$$

in $\mathrm{CH}_0((Z \setminus q^{-1}(Y^{j+1})) \times_k \eta_Z)$. The induction thus goes through, proving the result.

The proof of (3) is similar but easier. We have already seen that if Z has a decomposition of the diagonal of order N, then so does Y. If conversely Y has a decomposition of the diagonal of order N, then there is a 0-cycle y on Y with

$$N \cdot \eta_Y \times \eta_Y = y \times \eta_Y$$

in $\mathrm{CH}_0(Y \times \eta_Y)$. As $q : Z \to Y$ is universally CH_0 trivial, there is a 0-cycle z on Z with $q_* z = y$ in $\mathrm{CH}_0(Y)$ and since $(q \times q)_* : \mathrm{CH}_0(Z \times_k \eta_Z) \to \mathrm{CH}_0(Y \times_k \eta_Y)$ is an isomorphism, we have

$$N \cdot \eta_Z \times \eta_Z = z \times \eta_Z$$

in $\mathrm{CH}_0(Z \times_k \eta_Z)$. The proof for gTor is the same. □

We note some consequences of Lemma 3.6.

Proposition 3.7. *Let $f : Y \to X$ be a dominant rational map of smooth integral proper k-schemes of the same dimension d.*

(1) *Suppose k admits resolution of singularities for rational maps of varieties of dimension $\leq d$; that is, if $p : Y \to X$ is a rational morphism of smooth k-schemes of dimension $\leq d$, there is a sequence of blow-ups of Y with smooth center, $q : W \to Y$, such that the resulting rational map $r : W \to X$ is a morphism. Then $\mathrm{Tor}_k^{(i)} X$ divides $\deg f \cdot \mathrm{Tor}_k^{(i)} Y$ for all i.*

(2) *Without assumption on k, $\mathrm{Tor}_k X$ divides $\deg f \cdot \mathrm{Tor}_k Y$ and $\mathrm{gTor}_k X$ divides $(\deg f)^2 \cdot \mathrm{gTor}_k Y$.*

Proof. For (1) we may find a sequence of blow-ups with smooth centers, $g : Z \to Y$, so that the induced rational map $h : Z \to X$ is a morphism. Since g is a totally CH_0-trivial morphism, $\mathrm{Tor}_k^{(i)} Z = \mathrm{Tor}_k^{(i)} Y$ by Lemma 3.6(2), so we may assume that g is a morphism; the result then follows from Lemma 1.14.

For (2), let $Z \subset Y \times_k X$ be the graph of f, that is, the closure of the graph of $f : V \to X$ for a nonempty open subset $V \subset Y$ on which f is defined. The map $p_1 : Z \to Y$ is birational and there is a nonempty open $X_0 \subset X$ such that $p_1 : p_2^{-1}(X_0) \cap Z \to Y$ is an open immersion; set $Y_0 := p_1(p_2^{-1}(X_0) \cap Z)$. The correspondence $Z \times_k Z$ yields a homomorphism

$$g : \mathrm{CH}_d(Y \times_k Y) \to \mathrm{CH}_d(X \times_k X).$$

We claim that $g(\Delta_Y) = \deg(f) \cdot \Delta_X + \gamma$ where γ is a cycle supported on $X \times_k (X \setminus X_0)$, which implies the assertion for $\mathrm{Tor}_k X$. Keeping track of supports and using localization, we have an identity in $\mathrm{CH}_d(Z \times_k Z)$ of the form

$$[Z \times_k Z] \cdot (p_1 \times p_1)^*(\Delta_Y) = \Delta_Z + \gamma', \tag{3-1}$$

where γ' has support in $(p_1^{-1}(Y \setminus Y_0) \cap Z) \times_{Y \setminus Y_0} (p_1^{-1}(Y \setminus Y_0) \cap Z)$. Therefore, $(p_2 \times p_2)_*(\gamma')$ has support in $X \times_k (X \setminus X_0)$. Applying $(p_2 \times p_2)_*$ to (3-1) we prove our claim.

The proof for gTor_k is similar. □

In particular, if we have resolution of singularities of birational maps, $\mathrm{Tor}_k^{(i)}$ is a birational invariant and in general Tor_k is a birational invariant; from this it follows easily that $\mathrm{Tor}_k^{(i)}$ is a stable birational invariant if we have resolution of singularities of birational maps and in general Tor_k is a stable birational invariant.

4. Specialization and degeneration

The next result, in a somewhat different form, is proven in [Colliot-Thélène and Pirutka 2016b, Théorème 1.12]. In a less general setting, a similar result may be found in [Voisin 2015, Theorem 1.1].

Proposition 4.1. *Let $\mathcal{O}$ be a regular local ring with quotient field K and residue field k. Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}$ be a flat and proper morphism with geometrically integral fibers, and let X be the generic fiber $\mathcal{X}_K$ and Y the special fiber $\mathcal{X}_k$. We suppose that Y admit a resolution of singularities $q : Z \to Y$ such that q is a universally CH_0-trivial morphism. Suppose in addition that X admits a decomposition of the diagonal of order N. Then Z also admits a decomposition of the diagonal of order N. In particular, if $\mathrm{Tor}_K(X)$ is finite, then so is $\mathrm{Tor}_k(Z)$, and in this case $\mathrm{Tor}_k(Z) \mid \mathrm{Tor}_K(X)$.*

In [Colliot-Thélène and Pirutka 2016b] it is assumed that X has a resolution of singularities $\tilde{X} \to X$ such that $\tilde{X}_K$ admits a decomposition of the diagonal of order N, which implies the same condition on X by pushing forward; there is also an assumption that Z has a 0-cycle of degree 1. This resolution of singularities in [Colliot-Thélène and Pirutka 2016b] arises because they consider decompositions of the diagonal only on smooth proper varieties; the existence of a degree-1 0-cycle comes from considering only the case $N = 1$. The modified version stated above is proved exactly as in [loc. cit.].

We prove an extension of this specialization result which takes the decompositions of higher level into account.

Proposition 4.2. *Let $\mathcal{O}$ be a regular local ring with quotient field K and residue field k. Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}$ be a flat and proper morphism with geometrically integral fibers, and let X be the generic fiber $\mathcal{X}_K$ and Y the special fiber $\mathcal{X}_k$. Suppose that there is a birational totally CH_0-trivial morphism $q : Z \to Y$ of geometrically integral proper k-schemes.*

(1) *Suppose X admits a decomposition of the diagonal of order N and level i. Then Z also admits a decomposition of the diagonal of order N and level i. If $\mathrm{Tor}_K^{(i)}(X)$ is finite, then so is $\mathrm{Tor}_k^{(i)}(Z)$ and in this case $\mathrm{Tor}_k^{(i)}(Z) \mid \mathrm{Tor}_K^{(i)}(X)$.*

(2) *Let $\bar{K}$ and $\bar{k}$ be the respective algebraic closures of K and k, and suppose that $X_{\bar{K}}$ admits a decomposition of the diagonal of order N and level i. Suppose that K has characteristic zero, or that $\mathcal{O}$ is excellent. Then $Z_{\bar{k}}$ also admits a decomposition of the diagonal of order N and level i. If $\mathrm{Tor}^{(i)}(X)$ is finite, then so is $\mathrm{Tor}^{(i)}(Z)$ and in this case $\mathrm{Tor}^{(i)}(Z) \mid \mathrm{Tor}^{(i)}(X)$.*

Proof. The assertion (2) follows from (1) by first stratifying $\operatorname{Spec}\mathcal{O}$ as in the proof of Lemma 1.5 to reduce to the case of a DVR. We then take a finite extension L of K so that $\operatorname{Tor}^{(i)}(X) = \operatorname{Tor}_L^{(i)}(X_L)$, take the normalization $\mathcal{O} \to \mathcal{O}^N$ of $\mathcal{O}$ in L, and replace $\mathcal{O}$ with the localization $\mathcal{O}'$ of $\mathcal{O}^N$ at some maximal ideal. Letting k' be the residue field of $\mathcal{O}'$, $\operatorname{Tor}^{(i)}(Z)$ divides $\operatorname{Tor}_{k'}^{(i)}(Z_{k'})$, so (1) implies (2). We now prove (1).

By Lemma 1.5, Y admits a decomposition of the diagonal of order N and level i. By Lemma 3.6, Z also admits a decomposition of the diagonal of order N and level i, proving (1). □

We also have a version that incorporates Totaro's extended specialization Lemma 1.7.

Proposition 4.3. *Let $\mathcal{O}$ be a discrete valuation ring with quotient field K and residue field k. Let $f : \mathcal{X} \to \operatorname{Spec}\mathcal{O}$ be a flat and proper morphism of dimension d over $\operatorname{Spec}\mathcal{O}$ with generic fiber X and special fiber Y. We suppose Y is a union of closed subschemes, $Y = Y_1 \cup Y_2$, and that X and Y_1 are geometrically integral. Suppose there is a birational totally CH_0-trivial morphism $q : Z \to Y_1$ of geometrically integral proper k-schemes and that X admits a decomposition of the diagonal of order N and level i. Then there are proper closed subsets $Z_i, D \subset Z$ with $\dim Z_i \leq i$ and a decomposition*

$$N \cdot \Delta_Z = \gamma + \gamma_1 + \gamma_2$$

with γ supported in $Z_i \times_k Z$, γ_1 supported in $Z \times_k D$, and γ_2 supported in $q^{-1}(Y_1 \cap Y_2) \times_k Z$.

Proof. This follows directly from Lemmas 1.7 and 3.6. □

Remark 4.4. As in the second part of Proposition 4.2, we may take the N in Proposition 4.3 to be $\operatorname{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}})$ if $\mathcal{O}$ is excellent or if K has characteristic zero, by replacing $\mathcal{O}$ with its normalization $\mathcal{O}'$ in a finite extension L of K so that $\operatorname{Tor}_{\bar{K}}^{(i)}(X_{\bar{K}}) = \operatorname{Tor}_L^{(i)}(X_L)$, replacing $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{O}} \mathcal{O}'$, replacing k with a residue field k' of $\mathcal{O}'$, and replacing Z with $Z \otimes_k k'$.

5. Torsion order for complete intersections in a projective space: an upper bound

We concentrate on the 0-th torsion order of a (reduced, generically smooth) complete intersection $X = X^n_{d_1,\dots,d_r}$ in $\mathbb{P}^{n+r}$ of dimension n and multidegree $d_1, d_2, \dots, d_r$. In this section, we recall the construction of Roitman [1980], which when suitably refined gives an upper bound for $\operatorname{Tor}_k(X)$; by Lemma 1.3(1), this gives an upper bound for $\operatorname{Tor}_k^{(i)}(X)$ for all i.

Remark 5.1. Roitman [1980] considered 0-cycles modulo rational equivalence on a smooth hypersurface X of degree $d \leq n$ in $\mathbb{P}^n_k$ for k an algebraically closed field.

His argument (in part) consisted in showing that through each point x of X, there is a line ℓ in $\mathbb{P}^n$ containing x with either $\ell \subset X$ or $\ell \cap X = \{x\}$ (set-theoretically). To do this, he showed how the defining equation for X gives equations for the set of all ℓ with the above properties, as a closed subset of the projective space $\mathbb{P}^{n-1}(x)$ of lines through x. For his purpose, it is enough to show that the closed subset of all ℓ containing x, with $\ell \subset X$ or with $\ell \cap X = \{x\}$, is nonempty; for our purposes, we need the degree of this closed subscheme. More concretely, if x, x' are points of $X(k)$ with k algebraically closed, Roitman's argument shows that $d \cdot x \sim d \cdot x'$ by finding lines ℓ, ℓ' as described above, whereas we need to consider points x, x' in $X(k(X))$, so the factor d becomes multiplied by the degree of the closed subscheme of lines through x or x'. Finally, Roitman eventually shows that $x \sim x'$ for all $x \in X(k)$, k algebraically closed, by applying his famous theorem on the torsion in the group of 0-cycles modulo rational equivalence.

We often shorten the notation by writing d_* for a sequence $d_1, d_2, \ldots, d_r$.

Proposition 5.2. *Let k be a field, and let $X = X^n_{d_1,\ldots,d_r}$ in $\mathbb{P}^{n+r}_k$ with $\sum_i d_i \le n + r$ be a reduced, generically smooth complete intersection of multidegree $d_1, \ldots, d_r$, with $n \ge 1$. Then $\mathrm{Tor}_k(X)$ is finite and divides $\prod_{i=1}^r d_i!$.*

Proof. The reduced, generically smooth complete intersections in $\mathbb{P}^{n+r}$ and of multidegree $d_1, \ldots, d_r$ are parametrized by an open subscheme $\mathcal{U}_{d_*;n}$ of a product of projective spaces; by Lemma 1.5 it suffices to prove the result for the subscheme $X := X_{d_*,\mathrm{gen}}$ of $\mathbb{P}^{n+r}_K$ defined over the field $K := k(\mathcal{U}_{d_*;n})$ corresponding to the generic point of $\mathcal{U}_{d_*;n}$. For such an X, there is an open subset $V \subset X$, such that, for $x \in V$, the set of lines $\ell \subset \mathbb{P}^{n+r}$ such that $x \in \ell$ and $(\ell \cap X)_{\mathrm{red}}$ is either $\{x\}$ or is ℓ is defined by a complete intersection W_x of multidegree

$$d_1 - 1, d_1 - 2, \ldots, 2, 1, d_2 - 1, d_2 - 2, \ldots 2, 1, \ldots, d_r - 1, \ldots, 2, 1$$

in the projective space $\mathbb{P}^{n+r-1}_{K(x)}$ of lines through x. Indeed, we may choose a standard affine open U in $\mathbb{P}^{n+r}_{K(x)}$ containing x and choose affine coordinates $t_0, \ldots, t_{n+r-1}$ for U so that x is the origin, and $X \cap U$ is defined by inhomogeneous equations $F_1 = \cdots = F_r = 0$. Writing each F_i as a sum of homogeneous terms $F_i^{(j)}$ of degree j,

$$F_i = \sum_{j=1}^{d_i} F_i^{(j)},$$

W_x is defined by ideal $(\cdots F_i^{(j)} \cdots)$, $i = 1, \ldots, r$ and $j = 1, \ldots, d_i - 1$. Since we are choosing X to be the generic hypersurface, and as we may also choose x to lie outside any proper closed subset of X, the homogeneous terms $F_i^{(j)} \in K(x)[t_0, \ldots, t_{n+r-1}]_j$ will define a complete intersection in $\mathbb{P}^{n+r-1}_{K(x)}$. In particular W_x has codimension $\sum_{i=1}^r (d_i - 1) \le n + r - 1$ in $\mathbb{P}^{n+r-1}_{K(x)}$, is nonempty, and has degree $\prod_{i=1}^r (d_i - 1)!$.

Let $W_x^0 \subset W_x$ be the closed subset of lines ℓ containing x with $\ell \subset X$; this is defined by the r additional equations $F_i^{(d_i)} = 0$. Thus, for general (X, x), W_x^0 has codimension r on W_x (or is empty).

Since $n + r - 1 - \sum_{i=1}^r (d_i - 1) \geq r - 1$, we may intersect W_x with a suitably general linear space $L \subset \mathbb{P}^{n+r-1}_{K(x)}$ to form a closed subscheme $\overline{W}_x \subset W_x$ of dimension $r - 1$ and degree $\prod_{i=1}^r (d_i - 1)!$ and we may choose L with $L \cap W_x^0 = \varnothing$. The cone over $\overline{W}_x$ with vertex x, $\mathscr{C}_x \subset \mathbb{P}^{n+r}_{K(x)}$, is thus a dimension-$r$ closed subscheme of degree $\prod_{i=1}^r (d_i - 1)!$ with intersection (set) $\mathscr{C}_x \cap X = \{x\}$. Thus, as cycles

$$\mathscr{C}_x \cdot X = \left(\prod_{i=1}^r d_i!\right) \cdot x.$$

Let η be the generic point of X. Taking $x = \eta$ in the above discussion gives

$$\prod_{i=1}^r d_i! \cdot \eta = \mathscr{C}_\eta \cdot X.$$

But $\mathscr{C}_\eta$ is an r-cycle on $\mathbb{P}^{n+r}_{K(\eta)}$ of degree $\prod_{i=1}^r (d_i - 1)!$, so $\mathscr{C}_\eta = \prod_{i=1}^r (d_i - 1)! \cdot L_r$ in $\mathrm{CH}_r(\mathbb{P}^{n+r}_{K(\eta)})$, where $L_r \subset \mathbb{P}^{n+r}_K$ is any dimension-r linear subspace. Since K is infinite, we may choose L_r so that the intersection $L_r \cap X$ has dimension zero. Thus, letting $z = \prod_{i=1}^r (d_i - 1)! \cdot (L_r \cdot X)$, we have

$$\prod_{i=1}^r d_i! \cdot \eta - z_{K(\eta)} = 0$$

in $\mathrm{CH}_0(X_{K(\eta)})$, which gives a decomposition of the diagonal in X of order $\prod_{i=1}^r d_i!$. Thus, $\mathrm{Tor}_K(X)$ is finite and divides $\prod_{i=1}^r d_i!$, as desired. □

Corollary 5.3. *Let $X = X^n_{d_1,\ldots,d_r}$ in $\mathbb{P}^{n+r}_k$ be a smooth complete intersection of multidegree $d_1, \ldots, d_r$ and of dimension $n \geq 1$ with $\sum_i d_i \leq n + r$. Then* $\mathrm{gTor}_k(X)$ *and* $\mathrm{Tor}(X)$ *are both finite and both divide $\prod_{i=1}^r d_i!$.*

Proof. Both $\mathrm{gTor}_k(X)$ and $\mathrm{Tor}(X) := \mathrm{Tor}_{\bar{k}}(X_{\bar{k}})$ divide $\mathrm{Tor}_k(X)$ (Lemma 1.3), so the result follows from Proposition 5.2. □

6. A lower bound in the generic case

In this section we discuss the case of the generic complete intersection. Let k denote a fixed base-field, for instance the prime field. The bounds we find for the generic case are independent of k, so one could equally well take k to be the reader's favorite field, even an algebraically closed one.

Before going into details, we outline the case of hypersurfaces, which uses all the main ideas.

Let $d!^*$ denote the l.c.m. of the integers $2, \dots, d$. Note that $d!^*$ is inductively the l.c.m. of d and $(d-1)!^*$ (Lemma 6.4). Our main result in the case of hypersurfaces is that the torsion order of level 0 of the generic hypersurface of degree $d \le n+1$ in $\mathbb{P}^{n+1}$ is divisible by $d!^*$, in other words, if the generic hypersurface admits a decomposition of the diagonal of degree N, then $d!^*$ divides N.

The hypersurfaces of degree $d \le n+1$ in $\mathbb{P}_k^{n+1}$ are parametrized by a projective space $\mathbb{P}^{N_{n,d}}$, and it is not hard to show that the index over $k(\mathbb{P}^{N_{n,d}})$ of the generic degree-d hypersurface X is d. In fact, we have a much stronger statement, namely $\mathrm{CH}_0(X) = \mathbb{Z}$, generated by $X \cdot \ell$ for $\ell \subset \mathbb{P}^{n+1}$ a line (Lemma 6.1(1)). In particular, for any zero cycle x on X, we have $d \mid \deg_{k(\mathbb{P}^{N_{n,d}})} x$.

If we have a decomposition of order N of the diagonal on X,

$$N \cdot \Delta_X \sim x \times X + \gamma,$$

then, as projecting this identity on the second factor shows that $N = \deg_{k(\mathbb{P}^{N_{n,d}})} x$, it follows that $d \mid N$. Now degenerate X to the generic degree-$(d-1)$ hypersurface Y in $\mathbb{P}^{n+1}$ plus the hyperplane H given by $x_{n+1} = 0$, and let $Z = Y \cap H$. Here Y and Z are defined over $L := k(\mathbb{P}^{N_{n,d-1}})$. Specializing the above rational equivalence using Lemma 1.7 gives a rational equivalence on $Y \times_L Y$ of the form

$$N \cdot \Delta_Y \sim \bar{x} \times Y + \gamma_1 + \gamma_2$$

with $\bar{x}$ a zero-cycle on Y, γ_1 a dimension-n cycle on $Z \times_L Y$, and γ_2 supported in $Y \times_L D$ for some divisor D on Y. Passing to the generic point of Y, γ_1 gives a 0-cycle on $Z \times_L L(Y)$. The main point is to show that $\mathrm{CH}_0(Z \times_L L(Y))$ is also $\mathbb{Z}$, generated by intersections from $\mathbb{P}^{n-1}$ (Lemma 6.1(3)), so we can replace γ_1 with $y \times Y + \gamma_3$, where y is a 0-cycle on Z and γ_3 is supported on $Z \times_L D'$ for some divisor D' on Y (Lemma 6.2). In other words,

$$N \cdot \Delta_Y \sim (\bar{x} + y) \times Y + \gamma_2 + \gamma_3,$$

so Y admits a decomposition of the diagonal of degree N. Now use induction on d to conclude that $(d-1)!^* \mid N$. As we already know that $d \mid N$, we find $d!^* \mid N$.

Now we address the details and the case of a general complete intersection. Fix integers $n, r \ge 1$. For an integer d, let $\mathcal{I}_{d,n+r}$ be the set of indices $I = (i_0, \dots, i_{n+r})$ with $0 \le i_j$ and $\sum_j i_j = d$. We let $\mathcal{I}_i = \mathcal{I}_{d_i,n+r}$ and let $N_i := \#\mathcal{I}_i - 1$. Let $\{u_i^{(I)} \mid I \in \mathcal{I}_i\}$ be homogeneous coordinates for $\mathbb{P}^{N_i}$, and let $x_0, \dots, x_{n+r}$ be homogeneous coordinates for $\mathbb{P}^{n+r}$. The universal family of intersections of multidegree $d_1, \dots, d_r$ in $\mathbb{P}^{n+r}$, $\mathcal{X}^{d_*,n}$, is the subscheme of $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r} \times \mathbb{P}^{n+r}$ defined by the multihomogeneous ideal in the polynomial ring $k[\{u_i^{(I)}\}_{I \in \mathcal{I}_i,\, i=1,\dots,r}, x_0, \dots, x_{n+r}]$ generated by the elements

$$\sum_{I\in\mathcal{S}_i} u_i^{(I)}x^I, \quad i=1,\dots,r,$$

where as usual $x^I = x_0^{i_0}\cdots x_{n+r}^{i_{n+r}}$ for $I=(i_0,\dots,i_{n+r})$. We let $\eta := \eta_{d_*;n}$ denote the generic point of $\mathbb{P}^{N_1}\times\dots\times\mathbb{P}^{N_r}$ and let $\mathscr{X}_\eta^{d_*,n}$ denote the fiber product

$$\mathscr{X}_\eta^{d_*,n} := \mathscr{X}^{d_*,n}\times_{\mathbb{P}^{N_1}\times\cdots\times\mathbb{P}^{N_r}}\eta\subset\mathbb{P}_\eta^{n+r}.$$

By Proposition 5.2, we know that if $\sum_{i=1}^r d_i\le n+r$, then $\mathrm{Tor}_{k(\eta)}(\mathscr{X}_\eta^{d_*,n})$ is finite and divides $\prod_i d_i!$. We turn to a computation of a lower bound.

Let $H\subset\mathbb{P}^{N_1}\times\cdots\times\mathbb{P}^{N_r}\times\mathbb{P}^{n+r}$ be the subscheme defined by $(x_{n+r}=0)$, let $\mathscr{X}_H^{d_*,n}:=\mathscr{X}^{d_*,n}\cap H$, and let $\mathscr{X}_{H,\eta}^{d_*,n}:=\mathscr{X}_\eta^{d_*,n}\cap H$. Let $\eta':=\eta_{d_*,n-1}$.

We separate the indices $\mathcal{S}_i$ into two disjoint subsets $\mathcal{S}_i^0$ and $\mathcal{S}_i^1$, with $\mathcal{S}_i^0$ the set of $(i_0,\dots,i_{n+r})$ with $i_{n+r}=0$ and $\mathcal{S}_i^1$ those with $i_{r+n}>0$. We set $v_i^{(I)}=u_i^{(I)}$ for $I\in\mathcal{S}_i^0$ and $w_i^{(I)}=u_i^{(I)}$ for $I\in\mathcal{S}_i^1$. We write $k(\{u_i^{(I)}\}_0)$ for the field extension of k generated by the ratios $u_i^{(I)}/u_i^{(I')}$, $I\neq I'$, and similarly for $k(\{v_i^{(I)}\}_0)$, giving us the field extension $k(\{v_i^{(I)}\}_0)\subset k(\{u_i^{(I)}\}_0)$. We note that $k(\{u_i^{(I)}\}_0)=k(\eta)$, $k(\{v_i^{(I)}\}_0)=k(\eta')$, and the $k(\eta)$-scheme $\mathscr{X}_{H,\eta}^{d_*,n}$ is canonically isomorphic to the base-change of the $k(\eta')$-scheme $\mathscr{X}_{\eta'}^{d_*,n-1}$ via the base-extension $k(\eta')\subset k(\eta)$:

$$\mathscr{X}_{H,\eta}^{d_*,n}\cong\mathscr{X}_{\eta'}^{d_*,n-1}\otimes_{k(\eta')}k(\eta).$$

This defines for us the projection $q_1:\mathscr{X}_{H,\eta}^{d_*,n}\to\mathscr{X}_{\eta'}^{d_*,n-1}$.

Let $K=k(\eta)(\mathscr{X}_\eta^{d_*;n})=k(\mathscr{X}^{d_*,n})$. We have the morphism of $k(\eta')$-schemes

$$\pi:\mathscr{X}_{H,\eta}^{d_*,n}\otimes_{k(\eta_{d_*;n})}K\to\mathscr{X}_{\eta'}^{d_*,n-1}$$

formed by the composition

$$\mathscr{X}_{H,\eta}^{d_*,n}\otimes_{k(\eta)}K\xrightarrow{p_1}\mathscr{X}_{H,\eta}^{d_*,n}\xrightarrow{q_1}\mathscr{X}_{\eta'}^{d_*,n-1}.$$

Lemma 6.1. (1) *For $i=0,\dots,n$, the intersection map*

$$\mathrm{CH}_{r+i}(\mathbb{P}_{k(\eta)}^{n+r})\to\mathrm{CH}_i(\mathscr{X}_\eta^{d_*;n})$$

is an isomorphism.

(2) *For $i=0,\dots,n-1$, the pullback*

$$\pi^*:\mathrm{CH}_i(\mathscr{X}_{\eta'}^{d_*,n-1})\to\mathrm{CH}_i(\mathscr{X}_{H,\eta}^{d_*,n}\otimes_{k(\eta)}K)$$

is an isomorphism.

(3) *For $i=0,\dots,n-1$, the intersection map*

$$\mathrm{CH}_{r+i+1}(\mathbb{P}_K^{n+r})\to\mathrm{CH}_i(\mathscr{X}_{H,\eta}^{d_*,n}\otimes_{k(\eta)}K)$$

is an isomorphism.

Proof. Noting that the base-extension $\mathrm{CH}_*(\mathbb{P}^{n+r}_{k(\eta)}) \to \mathrm{CH}_*(\mathbb{P}^{n+r}_K)$ is an isomorphism, the assertion (3) follows from (1) (for $n-1$) and (2). For (1), the projection

$$p_2 : \mathscr{X}^{d_*,n} \to \mathbb{P}^{n+r}$$

expresses $\mathscr{X}^{d_*,n}$ as a $\mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_r-1}$-bundle over $\mathbb{P}^{n+r}$, with fibers embedded in $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r}$ linearly in each factor. Thus, $\mathrm{CH}_*(\mathscr{X}^{d_*;n})$ is generated by $\mathrm{CH}_*(\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r} \times \mathbb{P}^{n+r})$ via restriction. After localization at η, this shows that $\mathrm{CH}_*(\mathscr{X}^{d_*;n}_\eta)$ is generated by $\mathrm{CH}_*(\mathbb{P}^{n+r}_{k(\eta)})$ via restriction. The fact that the surjective map $\mathrm{CH}_{r+i}(\mathbb{P}^{n+r}_{k(\eta)}) \to \mathrm{CH}_i(\mathscr{X}^{d_*;n}_\eta)$ is also injective in the stated range follows by noting that the intersection pairing on $\mathscr{X}^{d_*;n}_\eta$ is nondegenerate when restricted to these cycles. This proves (1).

For (2), fix for each i the index $I^0_i := (d_i, 0, \ldots, 0)$, and the index $I^1_i := (0, \ldots, 0, d_i)$, and for each homogeneous variable $w^{(I)}_i$, let $w^{(I)0}_i$ be the corresponding affine coordinate $w^{(I)}_i / v^{(I^0_i)}_i$. Similarly, we let $v^{(I)0}_i = v^{(I)}_i / v^{(I^0_i)}_i$. Let $y_i = x_i / x_{n+r}$, $i = 0, \ldots, n+r-1$ and $y_{n+r} = 1$. The field extension $k(\eta') \to K$ is isomorphic to the field extension given by including the constants $k(\{v^{(I)}_i\}_0)$ of the $k(\{v^{(I)}_i\}_0)$-algebra A,

$$A := k(\{v^{(I)}_i\}_0, y_0, \ldots, y_{n+r})[\{w^{(I)0}_i\}] \Big/ \Bigg(\ldots, \sum_{I \in \mathscr{S}^0_i} v^{(I)0}_i \cdot y^I + \sum_{I' \in \mathscr{S}^1_i} w^{(I')0}_i \cdot y^{I'}, \ldots \Bigg)$$

into the quotient field L of A. In each defining relation for A, we can solve for $w^{(I^1_i)0}_i$ in terms of the y_i and the other $w^{(I')0}_i$. After eliminating each $w^{(I^1_i)0}_i$ in this way, we see that A is a polynomial algebra over $k(\{v^{(I)}{}_i\}_0, y_0, \ldots, y_{n+r-1})$. The y_i and the $w^{(I')0}_i$, after removing $w^{(I^1_i)0}_i$ for each i, therefore form an algebraically independent set of generators for L over $k(\{v^{(I)}_i\}_0)$, and thus K is a pure transcendental extension of $k(\eta')$. As Chow groups are invariant under base-change by purely transcendental field extensions, this proves (2). $\square$

Lemma 6.2. *Take γ in $\mathrm{CH}_n(\mathscr{X}^{d_*,n}_{H,\eta} \times_{k(\eta)} \mathscr{X}^{d_*,n}_\eta)$. Then there is a zero cycle y on $\mathscr{X}^{d_*,n}_{H,\eta}$, a proper closed subset D' of $\mathscr{X}^{d_*,n}_\eta$, and a cycle γ' supported on $\mathscr{X}^{d_*,n}_{H,\eta} \times_{k(\eta)} D'$ such that*

$$\gamma = y \times \mathscr{X}^{d_*,n}_\eta + \gamma'$$

in $\mathrm{CH}_n(\mathscr{X}^{d_*,n}_{H,\eta} \times_{k(\eta)} \mathscr{X}^{d_*,n}_\eta)$. Furthermore, the degree of y is divisible by $\prod_{i=1}^r d_i$.

Proof. Let ξ denote the generic point of $\mathscr{X}^{d_*,n}_\eta$. By Lemma 6.1(3), the class of the restriction $j^*\gamma$ of γ to $\mathscr{X}^{d_*,n}_{H,\eta} \times_{k(\eta)} \xi$ is of the form

$$j^*\gamma = M \cdot L \cdot \mathscr{X}^{d_*,n}_{H,\eta} \times_{k(\eta)} \xi,$$

where L is a linear subspace of $H \subset \mathbb{P}^{n+r}$, M an integer. Letting $y \in \mathrm{CH}_0(\mathscr{X}^{d_*,n}_{H,\eta})$ be the 0-cycle $M \cdot L \cdot \mathscr{X}^{d_*,n}_{H,\eta}$, the result follows from the localization theorem for the Chow groups; the assertion on the degree follows from the fact that $\mathscr{X}^{d_*,n}_{H,\eta}$ has degree $\prod_{i=1}^r d_i$ and hence y has degree $M\cdot\prod_{i=1}^r d_i$. $\square$

Definition 6.3. For a natural number $n\geq 1$, we let $n!^*$ denote the l.c.m. of the numbers $1,2,\dots,n$.

Lemma 6.4. *Let $d_1,\dots,d_r$ be a sequence of positive natural numbers. Then the product $\prod_{i=1}^r(d_i!^*)$ is equal to the l.c.m. M of all products $i_1\cdots i_r$ with $1\leq i_j\leq d_j$, $j=1,\dots,r$.*

Proof. Fix a prime number p. For each $j=1,\dots,r$, let i_j^* be an integer with $1\leq i_j^*\leq d_j$ and with p-adic valuation $\nu_p(i_j^*)$ equal to $\nu_p(d_j!^*)$. Then

$$\nu_p\Bigl(\prod_{j=1}^r i_j^*\Bigr)=\nu_p\Bigl(\prod_{i=1}^r(d_i!^*)\Bigr)$$

and $\nu_p\bigl(\prod_{j=1}^r i_j\bigr)\leq\nu_p\bigl(\prod_{j=1}^r i_j^*\bigr)$ for all sequences $i_1,\dots,i_r$ with $1\leq i_j\leq d_j$. Thus, $\nu_p(M)=\nu_p\bigl(\prod_{i=1}^r i_j^*\bigr)=\nu_p\bigl(\prod_{i=1}^r(d_i!^*)\bigr)$. Since p was arbitrary, this gives $M=\prod_{i=1}^r(d_i!^*)$. $\square$

Theorem 6.5. *For integers $d_1,\dots,d_r$ with $\sum_i d_i\leq n+r$, $\prod_{i=1}^r d_i!^*\mid\mathrm{Tor}_{k(\eta)}(\mathscr{X}^{d_*,n}_\eta)$.*

Proof. We may suppose that $d_1>1$. Let $d'_*=(d_1-1,d_2,\dots,d_r)$. Let $\mathbb{O}$ be the local ring of the origin in $\mathbb{A}^1_{k(\eta)}=\operatorname{Spec}k(\eta)[t]$, and let $\widetilde{\mathscr{X}}$ be the subscheme of $\mathbb{P}^{N_1}\times\cdots\times\mathbb{P}^{N_r}_{\mathbb{O}}$ defined by the homogeneous ideal $(f_1,\dots,f_r)$, with

$$f_j=\begin{cases}\sum_{I\in\mathscr{S}_{d_j,n+r}}u_j^{(I)}x^I & \text{for } j\neq 1,\\ t\cdot\sum_{I\in\mathscr{S}_{d_1,n+r}}u_1^{(I)}x^I+(1-t)\cdot x_{n+r}\cdot\sum_{J\in\mathscr{S}_{d_1-1,n+r}}u_1^{(J)}x^J & \text{for } j=1.\end{cases}$$

The generic fiber of $\mathscr{X}$ is thus isomorphic to $\mathscr{X}^{d_*,n}_\eta\times_{k(\eta)}k(\eta,t)$, and the special fiber is $\mathscr{X}^{d'_*,n}_\eta\cup H$.

Suppose that $\mathscr{X}^{d_*,n}_\eta$ admits a decomposition of the diagonal of order N:

$$N\cdot\Delta_{\mathscr{X}^{d_*,n}_\eta}=x\times\mathscr{X}^{d_*,n}_\eta+\gamma$$

with γ supported on $\mathscr{X}^{d_*,n}_\eta\times D$ for some divisor D. By Lemma 6.1, $\deg x$ is divisible by $\prod_{i=1}^r d_i$, and thus $\prod_{i=1}^r d_i$ divides N.

By applying Totaro's specialization lemma (Lemma 1.7) to the family $\mathscr{X}\to\operatorname{Spec}\mathbb{O}$, the diagonal for $\mathscr{X}^{d'_*,n}_\eta$ admits a decomposition of the form

$$N\cdot\Delta_{\mathscr{X}^{d'_*,n}_\eta}=\bar{x}\times\mathscr{X}^{d'_*,n}_\eta+\gamma_1+\gamma_2$$

with γ_1 supported in $\mathscr{X}^{d'_*,n}_{H,\eta}\times\mathscr{X}^{d'_*,n}_\eta$ and γ_2 supported in $\mathscr{X}^{d'_*,n}_\eta\times D_2$ for some divisor D_2 on $\mathscr{X}^{d'_*,n}_\eta$. By Lemma 6.2, we have the identity

$$\gamma_1=y\times\mathscr{X}^{d'_*,n}_\eta+\gamma_3$$

with y a zero-cycle on $\mathscr{X}_\eta^{d'_*,n}$ and γ_3 supported on $\mathscr{X}_\eta^{d'_*,n} \times D_3$ for some divisor D_3. Thus, the diagonal on $\mathscr{X}_\eta^{d'_*,n}$ admits a decomposition of order N as well. By induction $(d_1-1)!^* \cdot \prod_{i=2}^r (d_i!^*)$ divides N; by symmetry $(d_j-1)!^* \cdot \prod_{i=1, i\neq j}^r (d_i!^*)$ divides N for all j with $d_j > 1$. As we have already seen that $\prod_i d_i$ divides N, Lemma 6.4 completes the proof. □

We also have a lower bound for the generic complete intersection with a rational point.

Corollary 6.6. *For integers $d_1, \dots, d_r$ with $\sum_i d_i \le n+r$, let K be the function field of the generic complete intersection of multidegree $d_1, \dots, d_r$, $K := k(\eta)(\mathscr{X}_\eta^{d_*,n})$. Then $\bigl(1/\prod_{i=1}^r d_i\bigr)\prod_{i=1}^r (d_i!^*)$ divides $\mathrm{Tor}_K(\mathscr{X}_\eta^{d_*,n} \times_{k(\eta)} K)$.*

Proof. Let $X = \mathscr{X}_\eta^{d_*,n}$. By Lemma 6.1, $I_X = \prod_{i=1}^r d_i$ and thus by Lemma 1.10, $\mathrm{Tor}_{k(\eta)}(\mathscr{X}_\eta^{d_*,n})$ divides $I_X \cdot \mathrm{Tor}_K(\mathscr{X}_\eta^{d_*,n} \times_{k(\eta)} K)$. Clearly $\prod_{i=1}^r d_i$ divides $\prod_{i=1}^r (d_i!^*)$, whence the result. □

Example 6.7 (generic cubic hypersurfaces). For the generic cubic hypersurface $X := \mathscr{X}_\eta^{3,n}$, $n \ge 2$, we thus have $\mathrm{Tor}_{k(\eta)} X = 6$ and the generic cubic hypersurface with a rational point X_K, $K = k(\eta)(X)$, has $2 \mid \mathrm{Tor}_K X_K \mid 6$.

It follows from [Colliot-Thélène 2016, Théorème 4.1] that the generic cubic hypersurface with a rational point does have $\mathrm{Tor}_K X_K = 6$, at least if k has characteristic not equal to 3. Indeed, in view of Lemma 1.3(4), we may enlarge k. First we may suppose that k contains a primitive third root of unity. Then we pass from k to $k(\lambda_0, \dots, \lambda_{n-2})$. The smooth cubic $Y \subset \mathbb{P}^{n+1}_{k(\lambda_0,\dots,\lambda_{n-2})}$ given by $x_0^3 + x_1^3 + x_2^3 + \sum_{i=0}^{n-2} \lambda_i x_{i+3}^3 = 0$ has a rational point but also has nontrivial higher unramified cohomology with $\mathbb{Z}/3$-coefficients. We apply Lemma 1.5(1) to conclude that $3 \mid \mathrm{Tor}_K X_K$.

In particular, the generic dimension-n cubic hypersurface with a rational point does not admit a rational map $\mathbb{P}^n \dashrightarrow X_K$ of degree not divisible by 6 by Proposition 3.7(2).

Example 6.8 (generic cubic hypersurfaces with a line). Take $n \ge 2$. For X a cubic hypersurface in $\mathbb{P}^{n+1}_L$ (defined over some field $L \supset k$), we have the Fano variety of lines on X, F_X, a closed subscheme of the Grassmann variety $\mathrm{Gr}(2, n+2)_L$. In fact, if $U \to \mathrm{Gr}(2, n+2)$ is the universal rank-two bundle, and f is the defining equation for X, then F_X is the closed subscheme defined by the vanishing of the section of the rank-four bundle $\mathrm{Sym}^3 U$ determined by f. In particular, the class of F_X in $\mathrm{CH}^4(\mathrm{Gr}(2, n+2)_L)$ is given by the Chern class $c_4(\mathrm{Sym}^3 U)$. One computes this easily as $c_4 = 9c_2^2(U) + 18c_1(U)^2 c_2(U)$. As $c_2(U)^n$ and $c_2(U)^{n-2}c_1(U)^2$ both have degree one, we see that $F_X \cdot c_2(U)^{n-2}$ has degree 27, and thus I_{F_X} divides 27. This 27 is of course the famous 27 lines on a cubic surface, as intersecting F_X with $c_2(U)^{n-2}$ in $\mathrm{Gr}(2, n+2)$ is the same as taking the Fano variety of the intersection of X with a general $\mathbb{P}^3$ in $\mathbb{P}^{n+1}$. See for example [Fulton 1984, Example 14.7.13] for details of the Chern class computation.

Taking $X = \mathscr{X}^{3,n}_\eta$, and letting $K = k(\eta)(F_X)$, it follows from Lemma 1.10(1) that $6 = \mathrm{Tor}_{k(\eta)}(\mathscr{X}^{3,n}_\eta)$ divides $27 \cdot \mathrm{Tor}_K(\mathscr{X}^{3,n}_\eta \times_{k(\eta)} K)$; since we have the degree-two rational map $\mathbb{P}^n_K \dashrightarrow \mathscr{X}^{3,n}_\eta \times_{k(\eta)} K$, we have $\mathrm{Tor}_K(\mathscr{X}^{3,n}_\eta \times_{k(\eta)} K) = 2$. In particular, the generic cubic with a line is not stably rational over its natural field of definition $k(\eta)(F_X)$.

We are indebted to J.-L. Colliot-Thélène [2016, Théorème 3.2] for the next example, which improves the bounds and simplifies the argument of an example in an earlier version of this paper.

Example 6.9 (cubics over a "small" field). Take $n \geq 2$. We consider a DVR $\mathscr{O}$ with quotient field K and residue field k (of characteristic $\neq 2$), and a degree-3 hypersurface $\mathscr{X} \subset \mathbb{P}^{n+1}_{\mathscr{O}}$. Let $X = \mathscr{X}_K$ and $Y = \mathscr{X}_k$. We suppose that X is smooth and $Y = Q \cup H$, with Q a smooth quadric and H a hyperplane. Furthermore, we assume

(1) $I_Q = 1$,

(2) Q and H intersect transversely, and

(3) $I_{Q\cap H} = 2$.

From Proposition 5.2, we know that $\mathrm{Tor}_K(X)$ is finite and divides 6. We will show that 2 divides $\mathrm{Tor}_K(X)$.

For this, suppose we have a decomposition of the diagonal of X of order N. We note that our family $\mathscr{X}$ satisfies the hypotheses of Lemma 1.8, with $Y_1 = Q$, $Y_2 = H$, and $r = 1$. By Remark 1.9, $N \cdot (\mathrm{CH}_0(Q)/i_{Q\cap H*}(\mathrm{CH}_0(Q \cap H))) = 0$; considering degrees, we see that $2 \mid N$.

To construct an explicit example, recall [Lam 1980, Chapter 11, Definition 4.1] that the *u-invariant* $u(k)$ of a field k is the maximum r such that there exists an anisotropic quadratic form over k of dimension r, or is ∞ if no maximum exists. For example, for p odd, $\mathbb{F}_p$ has u-invariant 2, and $\mathbb{Q}_p$ has u-invariant 4; more generally, for a field k of characteristic different from 2, $k((t))$ has u-invariant $2 \cdot u(k)$ [Lam 1980, Chapter 4, Examples 4.2].

The above construction gives us a cubic hypersurface X of dimension $n \geq 2$ over $K := k((x))$ with $2 \mid \mathrm{Tor}_K(X)$ and $X(K) \neq \varnothing$ if k is an infinite field of characteristic $\neq 2$ with u-invariant $\geq n+1$. Indeed, take an anisotropic quadratic form q_0 in $(n+1)$-variables $X_0, \dots, X_n$, choose $\alpha \in k^\times$ represented by q_0, and let $q = q_0 - \alpha \cdot X^2_{n+1}$, so q is nondegenerate. Let $Q \subset \mathbb{P}^{n+1}_k$ be the quadric defined by q, and let H be the hyperplane $X_{n+1} = 0$. Take a cubic form $c_0 \in k[X_0, \dots, X_{n+1}]$, and let $c = xc_0 + q \cdot X_{n+1} \in k[[x]][X_0, \dots, X_{n+1}]$. Since k is infinite, we can choose c_0 so that the subscheme X of $\mathbb{P}^{n+1}_{k((x))}$ defined by c is smooth (and hence geometrically integral); it suffices to choose c_0 so that $c_0 = 0$ is smooth and intersects Q and H transversely. Clearly $I_Q = 1$, Q and H intersect transversely, and $I_{Q\cap H} = 2$, giving us the desired example.

Thus, there are cubic threefolds X over $K := \mathbb{Q}_p((x))$ with $2 \mid \mathrm{Tor}_K(X)$ and with $X(K) \neq \varnothing$. Similarly, there are examples of such cubic threefolds over $K = \mathbb{F}_p((t))((x))$ for $p \neq 2$. Over $K = \mathbb{Q}((x))$ or even over $K = \mathbb{R}((x))$ there are cubic hypersurfaces X of dimension n over K for arbitrary $n \geq 2$, with $2 \mid \mathrm{Tor}_K(X)$ and $X(K) \neq \varnothing$. As in the previous example, we may pass to an odd-degree field extension L of K to find a cubic hypersurface X_L with a line, and with $\mathrm{Tor}_L(X_L) = 2$; all these cubics are thus not stably rational over their corresponding field of definition.

Remark 6.10. As mentioned in the introduction, Colliot-Thélène and Pirutka have constructed cubic threefolds over a p-adic field [2016b, Théorème 1.21] and over $\mathbb{F}_p((x))$ [2016b, Remarque 1.23] with nonzero torsion order and having a rational point.

7. Torsion order for very general complete intersections in a projective space: a lower bound

As in the previous sections, we consider smooth complete intersection subschemes X of $\mathbb{P}^{n+r}$ of multidegree $d_1, \ldots, d_r$.

By saying a property holds for a very general complete intersection in $\mathbb{P}^{n+r}_k$ of multidegree $d_1, \ldots, d_r$, we mean that there is a countable union F of proper closed subsets of the parameter scheme of such complete intersections (an open subset in a product of projective spaces over k) such that the property holds for X_b if $b \notin F$.

Recall that for X a proper, generically smooth L-scheme for some field L, and $\bar{L}$ the algebraic closure of L, we have defined $\mathrm{Tor}^{(i)}(X) := \mathrm{Tor}^{(i)}_{\bar{L}}(X_{\bar{L}})$ (Definition 1.12).

Theorem 7.1. *Let k be a field of characteristic zero. Let $d_1, \ldots, d_r$ and $n \geq 3$ be integers with $d' := \sum_{j=1}^{r} d_j \leq n + r$. Let p be a prime number. Suppose that*

$$d_i \geq p \cdot \left\lceil \frac{n + r + 1 - d' + d_i}{p + 1} \right\rceil \tag{7-1}$$

for some i, $1 \leq i \leq r$. Then $p \mid \mathrm{Tor}^{(n-2)}(X)$ for all very general $X = X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n+r}_k$.

Corollary 7.2. *Let k, $d_1, \ldots, d_r$, n, and p be as in Theorem 7.1, and suppose that d_i satisfies (7-1). Then $p \mid \mathrm{Tor}(X)$ for all very general $X = X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n+r}_k$.*

Proof. $\mathrm{Tor}^{(n-2)}(X)$ divides $\mathrm{Tor}(X) := \mathrm{Tor}^{(0)}(X)$ by Lemma 1.3(1). □

Remarks 7.3. (1) We know that $\mathrm{Tor}(X)$ is finite for all $X = X_{d_1,\ldots,d_r} \subset \mathbb{P}^{n+r}$ with $\sum_j d_j \leq n + r$ by Proposition 5.2 and hence $\mathrm{Tor}^{(n-2)}(X)$ is also finite.

(2) For $p = 2$ and for hypersurfaces, the corollary follows directly from the results in [Totaro 2016].

(3) We only use the hypothesis of characteristic zero to allow for a specialization to characteristic p, where p is the prime number in the statement. For k a field of positive characteristic, the analogous result holds, but only for $p = \operatorname{char} k$.

(4) There are two interesting cases of complete intersection threefolds we would like to mention: that of a multidegree-(3, 2) complete intersection in $\mathbb{P}^5$ and a multidegree-(2, 2, 2) complete intersection in $\mathbb{P}^6$ (see the recent results of Hassett and Tschinkel [2016]). In both cases we take $d_i = 2$ and get a divisibility by 2. Notice that in the (2, 3) case taking $d_i = 3$ and $p = 3$ works.

Proof of Theorem 7.1. This is another application of the argument of Kollár [1995], as used for example by Totaro [2016], Colliot-Thélène and Pirutka [2016a], or Okada [2016]. We may reorder the d_j so that $d_i = d_1$. We first assume that p divides d_1, $d_1 = q \cdot p$. Take f and g suitably general homogeneous polynomials of degree d_1 and q, respectively, and let $f_2, \dots, f_r$ be suitably general homogeneous polynomials, with f_j of degree d_j, $j = 2, \dots, r$. We take these to be in the polynomial ring $\mathbb{O}[X_0, \dots, X_{n+r}]$, where $\mathbb{O}$ is a complete (hence excellent) discrete valuation ring with maximal ideal (t), with residue field $k = \overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$, and with quotient field K a field of characteristic zero. We let $\mathscr{X} \to \operatorname{Spec} \mathbb{O}$ be the closed subscheme of a weighted projective space $\mathbb{P} = \operatorname{Proj} \mathbb{O}[X_0, \dots, X_{n+r}, Y]$, with the X_i having weight 1 and Y having weight q, defined by the homogeneous ideal

$$(f_2, \dots, f_r, Y^p - f, g - tY).$$

The generic fiber $X := \mathscr{X}_K$ is isomorphic to the complete intersection subscheme of $\mathbb{P}^{n+r}_K$ defined by $g^p - t^p f = f_2 = \cdots = f_r = 0$, and the special fiber $Y := \mathscr{X}_k$ is the cyclic p to 1 cover $Y \to W$, with $W \subset \mathbb{P}^{n+r}_k$ the complete intersection defined by $\bar{g} = \bar{f}_2 = \cdots = \bar{f}_r = 0$, and $y^p = f|_W$.

For general $f, g, f_2, \dots, f_r$, X and W are smooth, and Y has only finitely many singularities, which may be resolved by an explicit iterated blow-up $q : Z \to Y$ which is totally CH_0-trivial: for details, see Proposition 8.5 if $p \geq 3$. If $p = d_1 = 2$, then we use Lemma 8.7 and Proposition 8.8 for the construction of the resolution of singularities and the proof that the resolution morphism q is totally CH_0-trivial. Kollár shows in addition that, under the assumption (7-1), one has $H^0(Z, \Omega^{n-1}_{Z/k}) \neq \{0\}$. In somewhat more detail, Kollár [1995, §15, Lemma 16] defines an invertible sheaf Q (denoted $\pi^* Q(L, s)$ in [loc. cit.]) with an injection $Q \to (\Omega^{n-1}_{Y/k})^{**}$, where ** denotes the double dual. A local computation (see [Colliot-Thélène and Pirutka 2016a], [Okada 2016], or Remark 8.18 for details) in a neighborhood of the finitely many singularities of Y shows that this injection extends to an injection $q^*Q \to \Omega^{n-1}_{Z/k}$; here is where the condition $n \geq 3$ is used. In addition, q^*Q is isomorphic to the pullback to Z of $\omega_W \otimes \mathcal{O}_W(d_1)$, where ω_W is the canonical sheaf on W. As $\omega_W = \mathcal{O}_W\big(d_1/p + \sum_{j\geq 2} d_j - n - r - 1\big)$, we have a nonzero section of $\Omega^{n-1}_{Z/k}$ if $d_1(p+1)/p \geq n + r + 1 - \sum_{i=2}^r d_i$, which is exactly the condition in the statement of the theorem.

By Proposition 5.2, we know that $\mathrm{Tor}(X_{\bar{K}})$ is finite and thus $\mathrm{Tor}^{(n-2)}(X_{\bar{K}})$ is finite as well. The specialization result Proposition 4.2 thus implies that $\mathrm{Tor}^{(n-2)}(Z_{\bar{k}})$ is finite and divides $\mathrm{Tor}^{(n-2)}(X_{\bar{K}})$. By [Gros 1985, Chapitre II, Proposition 4.2.33; Chatzistamatiou and Rülling 2011, Theorem 3.1.8; El Zein 1978, §3.3, Proposition 4], correspondences on $Z \times_k Z$ act on $H^0(Z, \Omega^{n-1}_{Z/k})$, and if γ is a correspondence on $Z \times_k Z$ supported in some $Z' \times_k Z$ with $\dim_k Z' \leq n-2$, then by [Chatzistamatiou and Rülling 2011, Proposition 3.2.2(2)], γ_* acts by zero on $H^0(Z, \Omega^{n-1}_{Z/k})$. Similarly, if γ is a correspondence on $Z \times_k Z$, supported in $Z \times_k D$ for some divisor $D \subset Z$, then $\gamma_*(\omega)|_{Z\setminus D} = 0$ for each $\omega \in H^0(Z, \Omega^{n-1}_{Z/k})$; as $\Omega^{n-1}_{Z/k}$ is locally free, it follows that $\gamma_*(\omega) = 0$. Thus, if Δ_Z admits a decomposition of order N and level $n-2$, this implies that $N \cdot \omega = 0$ for all $\omega \in H^0(Z, \Omega^{n-1}_{Z/k})$, and since $H^0(Z, \Omega^{n-1}_{Z/k})$ is a nonzero k-vector space, this implies that $p \mid N$. Since $\mathrm{Tor}^{(n-2)}(Z_{\bar{k}})$ divides $\mathrm{Tor}^{(n-2)}(X_{\bar{K}})$, it follows that $p \mid \mathrm{Tor}^{(n-2)}(X_{\bar{K}})$ and Proposition 2.1 finishes the proof in this case.

In the case of a general d_1, write $d_1 = q \cdot p + c$, $0 < c < p$, and consider a family $\mathscr{X} \to \operatorname{Spec} \mathcal{O}$ defined by a homogeneous ideal of the form

$$(f_2, \ldots, f_r, (Y^p - h)s + tu, g - tY),$$

with $u, h, g, s \in \mathcal{O}[X_0, \ldots, X_{n+r}]$, u of degree d_1, h of degree pq, g of degree q, and s of degree c, suitably general, and with Y as above of weight q. The generic fiber X is the complete intersection $f_1 = f_2 = \cdots = f_r = 0$, with $f_1 = (g^p - t^p h)s + t^{p+1}u$; the special fiber Y has two components Y_1, Y_2, with Y_1 the p to 1 cyclic cover of $W := (\bar{f}_2 = \cdots = \bar{f}_r = \bar{g} = 0)$, branched along $W \cap (h = 0)$. We take $q : Z \to Y_1$ to be the resolution as in the previous case. Having chosen h, g, s, we may take u sufficiently general so that X is a smooth complete intersection.

Since $\mathcal{O}$ is excellent, we are free to make a finite extension L of K, take the integral closure $\mathcal{O}_L$ of $\mathcal{O}$ in L, replace $\mathcal{O}$ with the localization $\mathcal{O}'$ at a maximal ideal of $\mathcal{O}_L$, and replace $\mathscr{X}$ with $\mathscr{X} \otimes_{\mathcal{O}} \mathcal{O}'$; changing notation, we may assume that $\mathrm{Tor}^{(n-2)}_K(X)$ is the geometric torsion order $\mathrm{Tor}^{(n-2)}(X)$. By Proposition 4.3, the smooth proper k-scheme Z admits a decomposition of the diagonal as

$$N \cdot \Delta_Z = \gamma + \gamma_1 + \gamma_2,$$

with $N = \mathrm{Tor}^{(n-2)}(X)$, γ supported in $Z_{n-2} \times_k Z$ with $\dim Z_{n-2} \leq n-2$, γ_1 supported in $q^{-1}(Y_1 \cap Y_2) \times_k Z$, and γ_2 supported in $Z \times_k D$ for some divisor D on Z.

We may take the degree-c part s as general as we like. In particular, we may assume that $Y_1 \cap Y_2$ is contained in the smooth locus of Y_1 and is thus isomorphic to a closed subscheme Z' of Z.

Our decomposition of the diagonal on Z gives the relation

$$N \cdot \omega = \gamma_{1*}\omega$$

for each $\omega \in H^0(Z, \Omega_Z^{n-1})$. Indeed,

$$N \cdot \omega = N \cdot \Delta_{Z*}\omega = \gamma_{1*}\omega + \gamma_{2*}\omega + \gamma_*\omega.$$

But γ_* factors through the restriction to Z_{n-2}, so $\gamma_*\omega = 0$. Similarly, $\gamma_{2*}\omega$ is a global section of Ω_Z^{n-1} supported in D, which is zero, since Ω_Z^{n-1} is a locally free sheaf.

One computes that the canonical class of $Y_1 \cap Y_2$ is antiample, and thus the canonical line bundle on the dimension-$(n-1)$ subscheme Z' has no sections. Note that Z' is a cyclic p to 1 cover of the complete intersection $W \cap V(\bar{s})$. If s is general, then there is a rational resolution of singularities $\widetilde{Z}'$ (Proposition 8.8 and Lemma 8.9), hence the canonical line bundle of $\widetilde{Z}'$ has no nonvanishing sections. But $\gamma_{1*}\omega$ factors through the restriction of ω to $\widetilde{Z}'$; hence, $\gamma_{1*}\omega = 0$. Since h has degree $q \cdot p$ in the range needed to give the existence of a nonzero ω in $H^0(Z, \Omega_Z^{n-1})$, we conclude as before that $p \mid N$. $\square$

Example 7.4. We consider the case of hypersurfaces of degree d in $\mathbb{P}^{n+1}$, $n \geq 3$. The theorem says that p divides $\mathrm{Tor}^{(n-2)}(X)$ for very general degree $d \leq n+1$ hypersurfaces X in $\mathbb{P}^{n+1}$ if

$$d \geq p \cdot \left\lceil \frac{n+2}{p+1} \right\rceil.$$

For $p = 2$, this is the range considered by Totaro; for $p = 3$, the first case is degree 6 in $\mathbb{P}^6$. For the extreme case of degree $d = n+1$ in $\mathbb{P}^{n+1}$, we have $p \mid \mathrm{Tor}^{(n-2)}(X)$ for all p dividing $n+1$.

8. An improved lower bound for the very general complete intersection

In this section we extend Theorem 7.1 to cover prime powers. The basic idea is to replace the differential forms with Hodge–Witt cohomology. We are grateful to Kay Rülling for providing the argument for the next lemma which shows that a cycle on $Z \times_k Z$, supported on $Z' \times_k Z$ with $\dim Z' \leq n-2$, acts trivially on $H^0(Z, W_m\Omega_Z^{n-1})$.

Lemma 8.1. *Let k be a perfect field of positive characteristic p, and let X, Y be smooth, equidimensional, and quasiprojective k-schemes. Set $n = \dim X$ and $\mathrm{CH}^n_{\mathrm{prop}/Y}(X \times_k Y) = \varinjlim_Z \mathrm{CH}_{\dim Y}(Z)$, where the limit is over all closed subsets*

$Z \subset X \times_k Y$ that are proper over Y. For $\alpha \in \mathrm{CH}^n_{\mathrm{prop}/Y}(X \times_k Y)$ denote by

$$\alpha_* : \bigoplus_{i,j} H^i(X, W_m\Omega^j) \to \bigoplus_{i,j} H^i(Y, W_m\Omega^j)$$

the map induced by α via the cycle action from [Chatzistamatiou and Rülling 2012, §3.5]. Assume α is supported on $A \times_k Y$, where $A \subset X$ is a closed subset of codimension $\geq r$. Then α_* vanishes on $\bigoplus_{i,j+r>n} H^i(X, W_m\Omega^j)$.

Proof. We may assume $\alpha = [Z]$, with $Z \subset X \times_k Y$ an integral closed subscheme of codimension n supported on $A \times_k Y$. Denote by p_X, p_Y the respective projections from $X \times_k Y$. It suffices to show for $i \geq 0$, $j + r > n$, and $b \in H^i(X, W_m\Omega^j)$ that

$$p_X^*(b) \cup \mathrm{cl}[Z] = 0 \quad \text{in } H_Z^{i+n}(X \times_k Y, W_m\Omega^{j+n}_{X\times_k Y}). \tag{8-1}$$

Then $\alpha_*(b) = p_{Y*}(p_X^*(b) \cup \mathrm{cl}[Z])$ will also vanish.

We first prove (8-1) for $i = 0$. Denote by $\eta \in X \times_k Y$ the generic point of Z. Since $W_m\Omega^{j+n}_{X\times_k Y}$ is Cohen–Macaulay, the natural map $H^n_Z(X \times_k Y, W_m\Omega^{j+n}_{X\times_k Y}) \to H^n_\eta(X \times_k Y, W_m\Omega^{j+n}_{X\times_k Y})$ is injective. Set $B = \mathcal{O}_{X\times_k Y,\eta}$ and $C = \mathcal{O}_{X,p_X(\eta)}$; by assumption we have $\dim C \geq r$. Since B is formally smooth over C we find $t_1, \dots, t_r \in C$ and $s_{r+1}, \dots, s_n \in B$ such that $p_X^*(t_1), \dots, p_X^*(t_r), s_{r+1}, \dots, s_n$ form a regular sequence of parameters of B. Hence, by [Gros 1985, Chapitre II, §3.5] (see also [Chatzistamatiou and Rülling 2012, Proposition 2.4.1]) and [Chatzistamatiou and Rülling 2012, Lemma 3.1.5] and in the notation of [Chatzistamatiou and Rülling 2012, §1.11.1] the image of $p_X^*(b) \cup \mathrm{cl}[Z] = \Delta^*(p_X^*(b) \times \mathrm{cl}[Z])$ in $H^n_\eta(X \times_k Y, W_m\Omega^{j+n}_{X\times_k Y})$ is up to a sign given by

$$\begin{bmatrix} p_X^*(b \cdot d[t_1] \cdots d[t_r]) \cdot d[s_{r+1}] \cdots d[s_n] \\ p_X^*([t_1]), \dots, p_X^*([t_r]), [s_{r+1}], \dots, [s_n] \end{bmatrix}.$$

Hence, the vanishing follows from $b \cdot d[t_1] \cdots d[t_r] \in W_m\Omega_X^{j+r} = 0$.

For the general case $i \geq 0$, we first observe that the CM property of $W_m\Omega^{j+n}_{X\times_k Y}$ implies $R\underline{\Gamma}_Z(W_m\Omega^{j+n}_{X\times_k Y}) \cong \mathcal{H}^n_Z(W_m\Omega^{j+n}_{X\times_k Y})[-n]$. Therefore,

$$H_Z^{i+n}(X \times_k Y, W_m\Omega^{j+n}_{X\times_k Y}) = H^i(X \times Y, \mathcal{H}^n_Z(W_m\Omega^{j+n}_{X\times_k Y})).$$

Let $\mathcal{U}$ be an open affine cover of X, and denote by $\mathcal{U} \times_k Y$ the open (not necessarily affine) cover of $X \times_k Y$. We can consider the Cech cohomology with respect to $\mathcal{U} \times_k Y$ and obtain a natural map

$$\check{H}^i(\mathcal{U} \times_k Y, \mathcal{H}^n_Z(W_m\Omega^{j+n}_{X\times_k Y})) \to H^i(X \times_k Y, \mathcal{H}^n_Z(W_m\Omega^{j+n}_{X\times_k Y})). \tag{8-2}$$

Since $\check{H}^i(\mathcal{U}, W_m\Omega^j_X) = H^i(X, W_m\Omega^j_X)$ and pullback and cup product are compatible with restriction to open subsets, we see that $p_X^*(\,\cdot\,) \cup \mathrm{cl}[Z] : H^i(X, W_m\Omega^j_X) \to$

$H_Z^{i+n}(X \times_k Y, W_m\Omega^{j+n}_{X\times_k Y})$ naturally factors via (8-2). Therefore, the case $i \geq 0$ follows from the case $i = 0$. □

Theorem 8.2. *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}^{n+r}_k$ be a very general complete intersection of multidegree $d_1, d_2, \ldots, d_r$ such that $d' := \sum_{i=1}^r d_i \leq n+r$ and $n \geq 3$. Let p be a prime and $m \geq 1$, and suppose*

$$d_i \geq p^m \cdot \left\lceil \frac{n+r+1-d'+d_i}{p^m+1} \right\rceil \tag{8-3}$$

for some i. Furthermore, suppose that p is odd or n is even. Then $p^m \mid \mathrm{Tor}^{(n-2)}(X)$.

Remark 8.3. Just as for Theorem 7.1, the same result holds for k a field of positive characteristic, but only for $p = \mathrm{char}\, k$.

Proof. The proof relies on Theorem 8.17, which we prove later in this section.

By Proposition 2.1, we need to find only one smooth complete intersection $X \subset \mathbb{P}^{n+r}_k$ such that $p^m \mid \mathrm{Tor}^{(n-2)}(X)$.

For a scheme X with locally free sheaf $\mathscr{E}$ and a section $s : \mathbb{O}_X \to \mathscr{E}$, we let $V(s)$ denote the closed subscheme of X defined by s.

We set $d = d_i$, $a = \lceil (n+r+1-d'+d)/(p^m+1) \rceil$, and $c = d - p^m \cdot a$. Let $\mathbb{O} = W(\bar{\mathbb{F}}_p)$ and $K = \mathrm{Frac}(\mathbb{O})$; we take r, f, g, l, and $f_2, \ldots, f_r$ suitably general (we will make this precise) homogeneous polynomials in $\mathbb{O}[X_0, \ldots, X_{n+r}]$ of degrees d, $d-c$, a, 1, and $d_2, \ldots, d_r$, respectively. We let $\mathscr{X} \to \mathrm{Spec}\,\mathbb{O}$ be the closed subscheme of the weighted projective space $\mathbb{P} = \mathrm{Proj}\,\mathbb{O}[X_0, \ldots, X_{n+r}, Y]$, with the X_i having weight 1, and Y having weight a, defined by the homogeneous ideal

$$l^c \cdot (Y^{p^m} - f) + p \cdot r,\ g - p \cdot Y,\ f_2, \ldots, f_r. \tag{8-4}$$

The generic fiber $X := \mathscr{X}_K$ is isomorphic to the complete intersection of $\mathbb{P}^{n+r}_K$ defined by $l^c \cdot (g^{p^m} - p^{p^m} \cdot f) + p^{p^m+1} \cdot r$, $f_2, \ldots, f_r$. For $r, f_2, \ldots, f_r$ general, it is smooth. By replacing $\mathbb{O}$ with its normalization in a suitable finite extension of K and changing notation, we may assume that $\mathrm{Tor}^{(n-2)}_K(X)$ is equal to the geometric torsion order $\mathrm{Tor}^{(n-2)}(X)$.

The special fiber $Y := \mathscr{X}_{\bar{\mathbb{F}}_p}$ is $Y = Y_1 + c \cdot Y_2$. Here, Y_1 is the cyclic p^m cover $Y_1 \to W$ defined by $f \in H^0(W, \mathbb{O}(a)^{\otimes p^m})$, with $W \subset \mathbb{P}^{n+r}_{\bar{\mathbb{F}}_p}$ the complete intersection defined by $g, f_2, \ldots, f_r$. We will take $f, g, f_2, \ldots, f_r$ general enough so that

(1) W is smooth,

(2) Y_1 has nondegenerate singularities (see Section 8A), and

(3) the assumption (3) of Theorem 8.17 is satisfied for Y_1.

For (2) we use Proposition 8.5 if $d - c \geq 3$. If $d - c = 2$ and hence $p = 2$, then we use Lemma 8.7. For (3) we use the theorem of Illusie [1990, Théorème 2.2] about ordinarity of a general complete intersection. Let us check that all other assumptions of Theorem 8.17 are satisfied. Assumption (1) is evident, and (2) is equivalent to $(p^m+1)\cdot a-n-r-1+d'-d\geq 0$, which follows immediately from the definition of a. Assumption (4) is equivalent to $i\cdot a+a-n-r-1+d'-d<0$, for all $i=0,\ldots,p^m-1$, which follows from $d'<n+r+1$; (5) is obvious.

The variety Y_2 is defined by $l, g, f_2, \ldots, f_r$, and only exists if $c\neq 0$. We take l general so that Y_2 does not contain the singular points of Y_1, $W\cap V(l)$ is smooth, and the p^m cyclic covering of $W\cap V(l)$ corresponding to $f|_{W\cap V(l)}$ has nondegenerate singularities.

Let $r:\tilde{Y}_1\to Y_1$ be the resolution of singularities constructed in Proposition 8.8; the map $r:\tilde{Y}_1\to Y_1$ is totally CH_0-trivial. By Proposition 4.3,

$$\mathrm{Tor}^{(n-2)}(X)\cdot\Delta_{\tilde{Y}_1}=\gamma+Z+Z_2,$$

where γ is a cycle with support in $A\times_{\bar{\mathbb{F}}_p}\tilde{Y}_1$ with $\dim A\leq n-2$, Z has support in $\tilde{Y}_1\times_{\bar{\mathbb{F}}_p}D$ with D a divisor, and Z_2 has support in $(Y_1\cap Y_2)\times_{\bar{\mathbb{F}}_p}\tilde{Y}_1$.

In view of Theorem 8.17, we have $\mathbb{Z}/p^m\subset H^0(\tilde{Y}_1, W_m\Omega^{n-1})$. By the work [Chatzistamatiou and Rülling 2012] on Hodge–Witt cohomology, we have an action of algebraic correspondences on $H^0(\tilde{Y}_1, W_m\Omega^{n-1})$ (relying on the cycle class of Gros [1985, Chapitre II, §3.4]; see [Chatzistamatiou and Rülling 2012, Proposition 2.4.1]). Let us show that Z_2 acts trivially. Note that $T:=Y_1\cap Y_2$ is the p^m cyclic covering of $W\cap V(l)$ corresponding to $f|_{W\cap V(l)}$. An easy computation shows $H^{>0}(Y_1\cap Y_2,\mathbb{O})=0$; hence, $H^{>0}(\tilde{T},\mathbb{O})=0$ by Lemma 8.9, where $\tilde{T}$ is the resolution constructed in Proposition 8.8, and $H^{>0}(\tilde{T}, W_m(\mathbb{O}))=0$. By Ekedahl duality [Ekedahl 1984, Chapter I, Theorem 4.1, Chapter II, Theorem 2.2, and Chapter III, Proposition 2.4] (see [Chatzistamatiou and Rülling 2012, Theorems 1.10.1 and 1.10.3]), we get $H^{<n-1}(\tilde{T}, W_m\Omega^{n-1})=0$. Let $\tilde{Z}_2$ be a lift of Z_2 to $\tilde{T}\times_{\bar{\mathbb{F}}_p}\tilde{Y}_1$. The action of Z_2 factors as

$$H^0(\tilde{Y}_1, W_m\Omega^{n-1})\to H^0(\tilde{T}, W_m\Omega^{n-1})\xrightarrow{\tilde{Z}_2}H^0(\tilde{Y}_1, W_m\Omega^{n-1}),$$

the first map being the pullback for the map $\tilde{T}\to\tilde{Y}_1$; thus, it is zero.

Lemma 8.1 implies that the action of γ on $H^0(\tilde{Y}_1, W_m\Omega^{n-1})$ vanishes. Therefore,

$$H^0(\tilde{Y}_1, W_m\Omega^{n-1})\xrightarrow{\mathrm{Tor}^{(n-2)}(X)\cdot}H^0(\tilde{Y}_1, W_m\Omega^{n-1})\xrightarrow{\text{restriction}}H^0(\tilde{Y}_1\setminus D, W_m\Omega^{n-1})$$

is zero. Since the restriction map is injective, we get $p^m\mid\mathrm{Tor}^{(n-2)}(X)$. □

Corollary 8.4. *Let k be a field of characteristic zero. Let $X\subset\mathbb{P}^{n+r}_k$ be a very general complete intersection of multidegree $d_1,\ldots,d_r$ with $\sum_i d_i=n+r$ and $n\geq 3$. For each i, $d_i\mid\mathrm{Tor}^{(n-2)}(X)$ if d_i is odd or if n is even.*

8A. Let X be a smooth variety over an algebraically closed field k of characteristic p. Suppose that $n:=\dim X\geq 2$. Let L be a line bundle on X, and let $s\in H^0(X, L^{\otimes p^m})$.

We denote by $\pi : Y \to X$ the p^m cyclic covering corresponding to s. It is an inseparable morphism and induces a homeomorphism on the underlying topological spaces.

There is a tautological connection $d : L^{\otimes p^m} \to L^{\otimes p^m} \otimes \Omega^1_X$ which satisfies $d(t^{p^m}) = 0$ for all sections $t \in L$. In particular, we have $d(s) \in H^0(X, L^{\otimes p^m} \otimes \Omega^1_X)$. Note that $Y_{\mathrm{sing}} = \pi^{-1}(V(d(s)))$.

We say that Y has *nondegenerate* singularities if the following conditions hold:

(1) Y has at most isolated singularities, or equivalently, $\dim(V(d(s))) = 0$ or $V(d(s)) = \varnothing$.

(2) For all $x \in V(d(s))$, $\mathrm{length}(\mathbb{O}_{V(d(s)),x}) \le 1$, if p is odd or $p = 2$ and n is even. If $p = 2$ and n is odd, then we require $\mathrm{length}(\mathbb{O}_{V(d(s)),x}) \le 2$ and the blow-up $\mathrm{Bl}_x Y$ of x has an exceptional divisor that is a cone over a smooth quadric.

Around a nondegenerate singularity of Y, we can find local coordinates $x_1, \dots, x_n$ of X such that Y is defined by

$$y^{p^m} + x_1^2 + \cdots + x_n^2 + f_3 \qquad \text{if } p \text{ is odd,} \tag{8-5}$$

$$y^{p^m} + x_1x_2 + \cdots + x_{n-1}x_n + f_3 \qquad \text{if } p = 2 \text{ and } n \text{ is even,} \tag{8-6}$$

$$y^{p^m} + x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n + b \cdot x_1^3 + f_3 \qquad \text{if } p = 2 \text{ and } n \text{ is odd,} \tag{8-7}$$

where $f_3 \in (x_1, \dots, x_n)^3$, $b \in k^\times$, and f_3 has no x_1^3 term in the last case.

An easy dimension counting argument yields the following proposition (cf. [Kollár 1995, §18]).

Proposition 8.5. *Let $W \subset H^0(X, L^{\otimes p^m})$ be such that for every closed point $x \in X$ the restriction map*

$$W \to \mathbb{O}_{X,x}/m_x^4 \otimes L^{\otimes p^m}$$

is surjective. For a general section $s \in W$ the corresponding p^m cyclic covering has nondegenerate singularities.

Remark 8.6. If $p \neq 2$ or $\dim X$ even, then the following surjectivity is sufficient to conclude the assertion of the proposition:

$$W \to \mathbb{O}_{X,x}/m_x^3 \otimes L^{\otimes p^m} \quad \text{for every closed point } x \in X.$$

In order to handle the case $d_i = 2 = p$, $m = 1$, and $n + r + 1 - d' + 2 \le 3$ in Theorem 8.2 we need the following lemma.

Lemma 8.7. *For a general complete intersection X in $\mathbb{P}^{n+r}$ with $n \ge 2$ and multidegree $d_1, d_2, \dots, d_r$ such that $d_1 \ge 2$, and a general $s \in H^0(\mathbb{P}^{n+r}, \mathbb{O}(2))$, the double covering corresponding to $s|_X$ has nondegenerate singularities.*

Proof. Only the case $p = 2$ and n odd has to be proved. Consider the variety A consisting of points $(x, f_1, \dots, f_r, s)$ where $x \in \mathbb{P}^{n+r}$, $(f_1, \dots, f_r, s)$ are homogeneous of degree $d_1, \dots, d_r, 2$, $X = V(f_1) \cap \dots \cap V(f_r)$ is smooth at x, and $d(s)|_X$ is vanishing at x. Those points for which the double covering corresponding to $s|_X$ has nondegenerate singularities at x form an open set B. It is not difficult to show that it is nonempty. Indeed, take $x = [1 : 0 : 0 : \dots : 0]$, and (in coordinates $x_1, \dots, x_{n+r}$ around x) $s = 1 + x_1^2 + x_2x_3 + \dots + x_{n-1}x_n + x_1x_{n+1}$, $f_1 = x_{n+1} + x_1^2$, and $f_i = x_{n+i} +$ terms of degree ≥ 2.

Let $V \subset A$ be the open set consisting of points such that $V(f_1) \cap \dots \cap V(f_r)$ is smooth. Since $B \cap V \neq \varnothing$, we conclude that for a general complete intersection X there is an open nonempty set $U \subset X$ such that for any $x \in U$ the set

$$\bigl\{s \in H^0(\mathbb{P}^{n+r}, \mathcal{O}(2)) \bigm| d(s)|_X(x) = 0 \text{ and } s \text{ does not yield a nondegenerate double covering at } x\bigr\}$$

has codimension $\geq n + 1$. Counting dimensions yields the claim. □

The following proposition has been proved for the case $m = 1$ in [Colliot-Thélène and Pirutka 2016a], and for the general case in [Okada 2016].

Proposition 8.8. *Suppose Y has nondegenerate singularities. Then by successively blowing up singular points, we can construct a resolution of singularities $r : \tilde{Y} \to Y$ such that the exceptional divisor is a normal crossings divisor* (cf. [Kollár 1995]). *Over every singular point $y \in Y$ the fiber $r^{-1}(y)$ is a chain of smooth irreducible divisors, each component of which is either a projective space, a smooth quadric, or a projective bundle over a smooth quadric. The intersection of two irreducible components is a smooth quadric or is empty. In particular, since k is algebraically closed, the morphism r is totally* CH_0 *trivial.*

Proof. We distinguish three cases:

(1) p is odd,

(2) $p = 2$, and n is even, and

(3) $p = 2$, and n is odd.

In any case we will only blow up singular points, and over any singular s there will be at most one singular point appearing in the exceptional divisor of the blow-up of s.

We may assume that Y has only one singular point. In case (1), note that we have a singularity of the form (8-5). We need $(p^m - 1)/2 + 1$ blow-ups:

$$\tilde{Y} := Y_{(p^m-1)/2+1} \to Y_{(p^m-1)/2} \to \dots \to Y_1 \to Y_0 := Y.$$

Around the singularity of Y_i, for $0 \leq i < (p^m - 1)/2$, Y_i is defined by

$$y^{p^m - 2\cdot i} + x_1'^2 + \dots + x_n'^2 + f_3', \tag{8-8}$$

where $x'_i = x_i/y^i$ and $f'_3 \in y^i \cdot (x'_1, \dots, x'_n)^3$. Therefore, the exceptional divisor of $Y_{i+1} \to Y_i$ is the cone C defined by $x'^2_1 + \dots + x'^2_n$ in the projective space with homogeneous variables $y, x'_1, \dots, x'_n$. For $i = (p^m - 1)/2$, Y_i is also given by (8-8) around the vertex of the exceptional divisor; hence, $p^m - 2i = 1$ implies that it is smooth and the exceptional divisor of $Y_{(p^m-1)/2+1} \to Y_{(p^m-1)/2}$ is $\mathbb{P}^{n-1}$. Denoting by $\tilde{E}_i$ the strict transform in $\tilde{Y}$ of the exceptional divisor of $Y_i \to Y_{i-1}$, we conclude that $\tilde{E}_i$ is the blow-up of C in its vertex if $i \le (p^m - 1)/2$, and $\tilde{E}_{(p^m-1)/2+1} = \mathbb{P}^{n-1}$. Every $\tilde{E}_i$ has only nonempty intersection with $\tilde{E}_{i+1}$ (if $i \le (p^m - 1)/2$) and $\tilde{E}_{i-1}$ (if $i > 1$); the intersection is the smooth quadric given by $x'^2_1 + \dots + x'^2_n$ in the projective space with homogeneous variables $x'_1, \dots, x'_n$.

For case (2), this case is similar to (1). We need 2^{m-1} blow-ups to arrive at $\tilde{Y}$. Around the singularity of Y_i, for $0 \le i < 2^{m-1}$, Y_i is defined by

$$y^{2^m - 2\cdot i} + x'_1 x'_2 + \dots + x'_{n-1} x'_n + f'_3, \tag{8-9}$$

and the exceptional divisor of $Y_i \to Y_{i-1}$ is the cone C defined by $x'_1 x'_2 + \dots + x'_{n-1} x'_n$ in the projective space P with homogeneous variables $y, x'_1, \dots, x'_n$. The exceptional divisor of $\tilde{Y} := Y_{2^{m-1}} \to Y_{2^{m-1}-1}$ is the smooth quadric defined by $y^2 + x'_1 x'_2 + \dots + x'_{n-1} x'_n$ in P. Again, the intersection of $\tilde{E}_i$ with $\tilde{E}_{i-1}$ is the smooth quadric given by $x'_1 x'_2 + \dots + x'_{n-1} x'_n$ in the projective space with homogeneous variables $x'_1, \dots, x'_n$.

For case (3), we need 2^m blow-ups to arrive at $\tilde{Y}$. The case $m = 1$ is easy to check; we will assume $m > 1$. We start with Y and the singularity (8-7). After $2^{m-1} - 1$ blow-ups the singularity is of the form

$$b \cdot y^{2^{m-1}+2} + x_1^{[1]^2} + x'_2 x'_3 + \dots + x'_{n-1} x'_n + b \cdot x_1^{[1]} \cdot y^{2^{m-1}+1} + \text{h.o.t.},$$

where $x'_i = x_i/y^{2^{m-1}-1}$, $x_1^{[1]} = x'_1 + y$, and the higher order terms h.o.t. can be ignored. After 2^{m-2} more blow-ups we introduce $x_1^{[2]} = x_1^{[1]}/y^{2^{m-2}} + \sqrt{b} \cdot y$, after 2^{m-3} more blow-ups we introduce $x_1^{[3]} = x_1^{[2]}/y^{2^{m-3}} + \sqrt{\sqrt{b} \cdot b} \cdot y$, etc. The singularity is after $2^{m-1} - 1 + 2^{m-2} + 2^{m-3} + \dots + 2^{m-i}$ blow-ups of the form

$$b_i \cdot y^{2^{m-i}+2} + x_1^{[i]^2} + x'_2 x'_3 + \dots + x'_{n-1} x'_n + b \cdot x_1^{[i]} \cdot y^{2^{m-i}+1} + \text{h.o.t.}, \tag{8-10}$$

where $x'_i = x_i/y^{-1+\sum_{j=1}^{i} 2^{m-j}}$ and $b_i = b \cdot \sqrt{b_{i-1}}$ with $b_1 = b$. After $2^m - 2$ blow-ups we get a singularity (8-10) with $i = m$. After one more blow-up the variety becomes smooth, and we need one more blow-up to obtain an exceptional divisor with strict normal crossings.

The exceptional divisor E_i of $Y_i \to Y_{i-1}$ is a cone defined by $x_1^{[j]^2} + x'_2 x'_3 + \dots + x'_{n-1} x'_n$ in the projective space with homogeneous variables $y, x_1^{[j]}, x'_2, \dots, x'_n$, except for the last blow-up where it is a projective space. The strict transform $\tilde{E}_i$ is the blow-up of the vertex. $\square$

For p odd or n odd, we get a projective space as exceptional divisor in the last step. Denoting by E the sum over all components of the exceptional divisor of r, we set

$$E' := \begin{cases} E + (\text{exc. div. from last step}) & \text{if } p \text{ is odd or } n \text{ is odd,} \\ E & \text{if } p = 2 \text{ and } n \text{ is even.} \end{cases} \tag{8-11}$$

Thus, the exceptional divisor of the last blow-up (a projective space) has multiplicity 2 in E' in the first case. If the singularity is of the form (8-5), (8-6), or (8-7), then E' is the restriction of $\operatorname{div}(y)$ to the exceptional divisor of the resolution r.

Lemma 8.9. *The resolution $r : \tilde{Y} \to Y$ is rational*; *that is*, $Rr_*\mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$.

Proof. We may suppose that Y has only one singularity. We will show that for each $r_i : Y_i \to Y_{i-1}$, we have $Rr_{i*}\mathcal{O}_{Y_i} = \mathcal{O}_{Y_{i-1}}$. Since Y_{i-1} is normal, it suffices to prove $R^j r_{i*}\mathcal{O}_{Y_i} = 0$ for $j > 0$. We know that r_i is the blow-up of a point and the exceptional divisor D is a cone over a smooth quadric, a smooth quadric, or a projective space, and comes with a given embedding into projective space; we call the corresponding ample line bundle $\mathcal{O}_D(1)$. In any case, $H^{>0}(D, \mathcal{O}(-s \cdot D)) \cong H^{>0}(D, \mathcal{O}(s)) = 0$ for all $s \geq 0$, where $\mathcal{O}_D(s) = \mathcal{O}_D(1)^{\otimes s}$. This implies the claim. $\square$

Lemma 8.10. *Let E' be as defined in (8-11). For all $i \geq 2$ we have*

$$H^i(E', \mathcal{O}(E')) = 0.$$

Proof. We may suppose that Y has only one singular point. The exceptional divisor is $\sum_{i=1}^{s} \tilde{E}_i$, and $\tilde{E}_i$ has nonempty intersection only with $\tilde{E}_{i+1}$ and $\tilde{E}_{i-1}$. Recall that all intersections are smooth quadrics. If $i \neq s$, then $\tilde{E}_i$ is the blow-up at the vertex of a cone $C_i \subset \mathbb{P}^n$ over a smooth quadric $Q_i \subset \mathbb{P}^{n-1}$; let $r_i : \tilde{E}_i \to C_i$ denote the blow-up.

For $i = 1, \dots, s-2$, we have $\mathcal{O}_{\tilde{E}_i}(\tilde{E}_i + \tilde{E}_{i+1}) \cong r_i^*\mathcal{O}_{C_i}(-1)$, hence

$$\mathcal{O}_{\tilde{E}_i \cap \tilde{E}_{i+1}}(E') \cong \mathcal{O}_{\tilde{E}_i \cap \tilde{E}_{i+1}}. \tag{8-12}$$

For $i = 2, \dots, s-2$, we obtain $\mathcal{O}_{\tilde{E}_i}(E') \cong \mathcal{O}_{\tilde{E}_i}$.

If p or n is odd, then

$$\mathcal{O}_{\tilde{E}_{s-1}}(\tilde{E}_{s-1} + 2 \cdot \tilde{E}_s) \cong r_{s-1}^*\mathcal{O}_{C_{s-1}}(-1);$$

hence $\mathcal{O}_{\tilde{E}_{s-1}}(E') \cong \mathcal{O}_{\tilde{E}_{s-1}}$, and $\mathcal{O}_{\tilde{E}_s}(\tilde{E}_{s-1} + 2 \cdot \tilde{E}_s) \cong \mathcal{O}_{\mathbb{P}^{n-1}}$; thus (8-12) holds for $i = s-1$. If p and n are even, then $\mathcal{O}_{\tilde{E}_{s-1}}(\tilde{E}_{s-1} + \tilde{E}_s) \cong r_{s-1}^*\mathcal{O}_{C_{s-1}}(-1)$, hence $\mathcal{O}_{\tilde{E}_{s-1}}(E') \cong \mathcal{O}_{\tilde{E}_{s-1}}$. Moreover, $\mathcal{O}_{\tilde{E}_s}(E') \cong \mathcal{O}_{\tilde{E}_s}$. This implies the assertion easily. $\square$

8B. Again, we assume that Y has nondegenerate singularities. We denote by $U \subset X$

the complement of the critical points, $Y_{\rm sm} = \pi^{-1}(U)$; we have

$$W_l(\pi)^* W_l \Omega^1_{U/k} \to W_l \Omega^1_{Y_{\rm sm}/k},$$

but there is no Verschiebung on $W_l(\pi)^* W_l \Omega^1_{U/k}$. Therefore, we define

$$\mathrm{Im}_V(W_l \Omega^1_{U/k}) \subset W_l \Omega^1_{Y_{\rm sm}/k}$$

inductively on l by

$$\mathrm{Im}_V(W_l \Omega^1_{U/k}) = \mathrm{image}(W_l(\pi)^* W_l \Omega^1_{U/k} \to W_l \Omega^1_{Y_{\rm sm}/k}) + V(\mathrm{Im}_V(W_{l-1}\Omega^1_{U/k})).$$

We have an R, V, F calculus for $\mathrm{Im}_V(W_l \Omega^1_{U/k})$, that is, morphisms

$$\begin{aligned} R &: \mathrm{Im}_V(W_l \Omega^1_{U/k}) \to \mathrm{Im}_V(W_{l-1} \Omega^1_{U/k}), \\ V &: \mathrm{Im}_V(W_{l-1} \Omega^1_{U/k}) \to \mathrm{Im}_V(W_l \Omega^1_{U/k}), \\ F &: \mathrm{Im}_V(W_l \Omega^1_{U/k}) \to \mathrm{Im}_V(W_{l-1} \Omega^1_{U/k}) \end{aligned}$$

satisfying the relations induced by $W_*\Omega^1_{Y_{\rm sm}/k}$ [Illusie 1979, p. 541]. By abuse of notation, any composition of maps R will be also denoted by R.

We are going to need several statements on $\mathrm{Im}_V(W_l \Omega^1_{U/k})$ in Theorem 8.17 which we provide in the following:

Lemma 8.11. *The evident map*

$$\begin{aligned} &\ker(R : W_l(\pi)^* W_l \Omega^1_{U/k} \to \pi^*\Omega^1_U) \\ &\qquad \to \ker(R : \mathrm{Im}_V(W_l \Omega^1_{U/k}) \to \mathrm{Im}_V(W_1 \Omega^1_{U/k}))/V(\mathrm{Im}_V(W_{l-1}\Omega^1_{U/k})) \qquad (8\text{-}13) \end{aligned}$$

is surjective if $l \le m$.

Proof. The target is the image of $R^{-1}(\ker(\pi^*\Omega^1_U \to \Omega^1_{Y_{\rm sm}})) \subset W_l(\pi)^* W_l \Omega^1_{U/k}$ via the evident map $W_l(\pi)^* W_l \Omega^1_{U/k} \to \mathrm{Im}_V(W_l \Omega^1_{U/k})/V(\mathrm{Im}_V(W_{l-1}\Omega^1_{U/k}))$. Locally, $Y_{\rm sm}$ is defined by $y^{p^m} - f$, for $f \in \mathbb{O}_U$, and $\ker(\pi^*\Omega^1_U \to \Omega^1_{Y_{\rm sm}})$ is generated by $d(f)$. Since $d([f]) \in W_l(\pi)^* W_l \Omega^1_{U/k}$ is a lifting of $d(f)$ whose image vanishes in $\mathrm{Im}_V(W_l \Omega^1_{U/k})$ (here we use $l \le m$), the claim follows. □

Recall the subsheaves $B_n \Omega^1_{U/k}$ of $\Omega^1_{U/k}$, $n = 1, 2, \ldots$ (see for example [Illusie 1979, Chapitre I, §2.2]). We have a short exact sequence

$$W_{l-1}\Omega^1_{U/k} \xrightarrow{V} \ker(R : W_l \Omega^1_{U/k} \to \Omega^1_U) \xrightarrow{F^{l-1}} B_{l-1}\Omega^1_U \to 0.$$

With the appropriate $W_l(\mathbb{O}_U)$-module structures this becomes a short exact sequence of $W_l(\mathbb{O}_U)$-modules. We obtain the diagram

$$\begin{array}{c} W_l(\pi)^*\ker(R: W_l\Omega^1_{U/k} \to \Omega^1_U)/W_l(\pi)^*(V)(W_l(\pi)^* W_{l-1}\Omega^1_{U/k}) \xrightarrow{\cong} \pi^* B_{l-1}\Omega^1_U \\ \Big\downarrow \text{surjective by Lemma 8.11} \\ \ker(R: \mathrm{Im}_V(W_l\Omega^1_{U/k}) \to \mathrm{Im}_V(W_1\Omega^1_{U/k}))/V(\mathrm{Im}_V(W_{l-1}\Omega^1_{U/k})) \\ \Big\downarrow (*) \\ \ker(R: W_l\Omega^1_{Y_{\mathrm{sm}}/k} \to \Omega^1_{Y_{\mathrm{sm}}})/V(W_{l-1}\Omega^1_{Y_{\mathrm{sm}}/k}) \xrightarrow{\quad\cong\quad} B_{l-1}\Omega^1_{Y_{\mathrm{sm}}} \end{array} \tag{8-14}$$

The induced map

$$\pi^* B_{l-1}\Omega^1_U \to B_{l-1}\Omega^1_{Y_{\mathrm{sm}}} \tag{8-15}$$

is the natural one, that is, given by $a \otimes \pi^{-1}(\omega) \mapsto \mathrm{Frob}^{l-1}(a)\cdot \pi^{-1}(\omega)$. We would like to show that $(*)$ is injective, which we prove by computing the kernel of (8-15) and showing that it is killed in $\mathrm{Im}_V(W_l\Omega^1_{U/k})$.

It is convenient to use the isomorphism [Illusie 1979, Chapitre I, (3.11.4)]

$$F^{l-2}d : W_{l-1}(\mathbb{O}_U)/F(W_{l-1}(\mathbb{O}_U)) \xrightarrow{\cong} B_{l-1}\Omega^1_U. \tag{8-16}$$

The $W_l(\mathbb{O}_U)$-module structure on the left is via the Frobenius $F: W_l(\mathbb{O}_U) \to W_{l-1}(\mathbb{O}_U)$. We give $W_{l-1}(\mathbb{O}_{Y_{\mathrm{sm}}})/F(W_{l-1}(\mathbb{O}_{Y_{\mathrm{sm}}}))$ the analogous $W_l(\mathbb{O}_{Y_{\mathrm{sm}}})$-module structure.

Lemma 8.12. *Suppose Y_{sm} is defined by $y^{p^m} - f$ for $f \in \mathbb{O}_U$ (this is the local picture). The kernel of*

$$W_l(\pi)^*(W_{l-1}(\mathbb{O}_U)/F(W_{l-1}(\mathbb{O}_U))) \to W_{l-1}(\mathbb{O}_{Y_{\mathrm{sm}}})/F(W_{l-1}(\mathbb{O}_{Y_{\mathrm{sm}}}))$$

is generated by $V(W_{l-1}(\mathbb{O}_{Y_{\mathrm{sm}}})) \otimes W_l(\pi)^{-1}(W_{l-1}(\mathbb{O}_U))$, and elements of the form

$$[y^i]\otimes \pi^{-1}(V^j(b)) - [y^{i\% p^{m-1-j}}]\otimes \pi^{-1}(V^j([f^{(i:p^{m-1-j})}]\cdot b)), \tag{8-17}$$

for all $0 \le j \le l-2$, $i \ge p^{m-1-j}$, and $b \in W_{l-1-j}(\mathbb{O}_U)$. Here, $i\% p^{m-1-j}$ means the remainder of i in the division by p^{m-1-j}, and $i = (i : p^{m-1-j})\cdot p^{m-1-j} + i\% p^{m-1-j}$.

Proof. The kernel contains $V(W_{l-1}(\mathbb{O}_{Y_{\mathrm{sm}}}))\otimes W_l(\pi)^{-1}(W_{l-1}(\mathbb{O}_U))$, because $V(a)\otimes \pi^{-1}(b)$ maps to $F(V(a))\cdot\pi^{-1}(b) = pa\cdot\pi^{-1}(b) = F(V(a\cdot\pi^{-1}(b)))$. Moreover,

$$\begin{aligned}[] [y^i]\otimes\pi^{-1}(V^j(b)) &- [y^{i\% p^{m-1-j}}]\otimes\pi^{-1}(V^j([f^{(i:p^{m-1-j})}]\cdot b)) \\ &\mapsto [y^{pi}]\cdot V^j(b) - [y^{(i\% p^{m-1-j})\cdot p}]\cdot V^j([f^{(i:p^{m-1-j})}]\cdot b) \\ &= V^j(([y^{p^{1+j}\cdot i}] - [y^{(i\% p^{m-1-j})\cdot p^{1+j}}\cdot f^{(i:p^{m-1-j})}])\cdot b) = 0. \end{aligned}$$

To show that these are all elements in the kernel, we proceed by induction on l. First, we assume $l = 2$. Without loss of generality, we need only consider elements in the kernel that are of the form $\sum_i [y^i]\otimes\pi^{-1}(b_i)$. By étale base-change, we may assume that $U = \operatorname{Spec}(k[x_1, \dots, x_n])$ and $x_1 = f$; hence, $Y_{\mathrm{sm}} = \operatorname{Spec}(k[y, x_2, \dots, x_n])$. By using elements of the form (8-17), we may suppose that $b_i = b_i(x_2, \dots, x_n)$. Since $\sum_i y^{ip} b_i \in k[y^p, x_2^p, \dots, x_n^p]$ implies $b_i \in k[x_2^p, \dots, x_n^p]$, we are done.

Suppose now that $l > 2$. By induction, we need only consider elements in the kernel that are of the form

$$\sum_i [y^i] \otimes \pi^{-1}(V^{l-2}(b_i)),$$

and we may use the same argument as for the $l = 2$ case. $\square$

Proposition 8.13. *Suppose $l \le m$. The map*

$$\ker(R : \operatorname{Im}_V(W_l\Omega^1_{U/k}) \to \operatorname{Im}_V(W_1\Omega^1_{U/k}))/V(\operatorname{Im}_V(W_{l-1}\Omega^1_{U/k})) \\ \to \ker(R : W_l\Omega^1_{Y_{\mathrm{sm}}/k} \to \Omega^1_{Y_{\mathrm{sm}}})/V(W_{l-1}\Omega^1_{Y_{\mathrm{sm}}/k})$$

is injective.

Proof. In view of diagram (8-14) and Lemma 8.12, we need to prove that the following elements vanish in $\operatorname{Im}_V(W_l\Omega^1_{U/k})/V(\operatorname{Im}_V(W_{l-1}\Omega^1_{U/k}))$:

(1) $V(a) \cdot dV(b)$ for $a \in W_l(\mathbb{O}_{Y_{\mathrm{sm}}})$ and $b \in W_{l-1}(\mathbb{O}_U)$ and

(2) $[y^i] \cdot dV^{j+1}(b) - [y^{i\% p^{m-1-j}}] \cdot dV^{j+1}([f^{(i:p^{m-1-j})}] \cdot b)$ for $b \in W_{l-1-j}(\mathbb{O}_U)$.

For (1), we have

$$V(a) \cdot dV(b) = V(a \cdot d(b)) \in V(\operatorname{Im}_V(W_{l-1}\Omega^1_{U/k})).$$

For (2), we compute

$$\begin{aligned}
[y^i] \cdot dV^{j+1}(b) &= d([y^i] \cdot V^{j+1}(b)) - V^{j+1}(b) \cdot d([y^i]) \\
&= dV^{j+1}([y^{i \cdot p^{1+j}}] \cdot b) - V^{j+1}(b) \cdot d([y^i]) \\
&= dV^{j+1}([y^{(i\% p^{m-1-j}) \cdot p^{1+j}}] \cdot [f^{(i:p^{m-1-j})}] \cdot b) - V^{j+1}(b) \cdot d([y^i]) \\
&= d([y^{i\% p^{m-1-j}}] \cdot V^{j+1}([f^{(i:p^{m-1-j})}] \cdot b)) - V^{j+1}(b) \cdot d([y^i]) \\
&= V^{j+1}([f^{(i:p^{m-1-j})}] \cdot b) \cdot d([y^{i\% p^{m-1-j}}]) \\
&\qquad + [y^{i\% p^{m-1-j}}] \cdot dV^{j+1}([f^{(i:p^{m-1-j})}] \cdot b) - V^{j+1}(b) \cdot d([y^i]),
\end{aligned}$$

which together with

$$\begin{aligned}
&V^{j+1}([f^{(i:p^{m-1-j})}] \cdot b) \cdot d([y^{i\% p^{m-1-j}}]) - V^{j+1}(b) \cdot d([y^i]) \\
&\quad = V^{j+1}(b \cdot ([f^{(i:p^{m-1-j})}] \cdot F^{j+1}(d([y^{i\% p^{m-1-j}}])) - F^{j+1}(d([y^i])))) \\
&\qquad = V^{j+1}(b \cdot F^{j+1}(d([y^{(i:p^{m-1-j}) \cdot p^{m-1-j}}][y^{i\% p^{m-1-j}}] - [y^i]))) = 0
\end{aligned}$$

(note that $F^{j+1}(d([y^{(i:p^{m-1-j}) \cdot p^{m-1-j}}])) = 0$) implies the claim. $\square$

8C. We denote by $\jmath : r^{-1}(Y_{\mathrm{sm}}) \to \tilde{Y}$ the open immersion. We will work with the logarithmic de Rham–Witt complex

$$W_l\Omega^1_{\tilde{Y}/k}(\log E) \subset \jmath_* W_l\Omega^1_{Y_{\mathrm{sm}}/k}.$$

Locally, when $E = \bigcup_{i=1}^r V(f_i)$ with $V(f_i)$ smooth, $W_l\Omega^1_{\tilde{Y}/k}(\log E)$ is generated as a $W_l(\mathbb{O}_{\tilde{Y}})$ submodule of $\jmath_* W_l\Omega^1_{Y_{\mathrm{sm}}/k}$ by $W_l\Omega^1_{\tilde{Y}/k}$ and $\langle d[f_i]/[f_i] \mid i = 1, \dots, r\rangle$. As for the de Rham complex there is an exact sequence

$$0 \to W_l\Omega^1_{\tilde{Y}/k} \to W_l\Omega^1_{\tilde{Y}/k}(\log E) \to \bigoplus_{i=0}^{r} W_l(\mathbb{O}_{V(f_i)}) \to 0. \tag{8-18}$$

We have the usual F, V, R calculus for $W_*\Omega^1_{\tilde{Y}/k}(\log E)$.

We define

$$K_l := \jmath_* \operatorname{Im}_V(W_l\Omega^1_{U/k}) \cap W_l\Omega^1_{\tilde{Y}/k}(\log E) \subset \jmath_* W_l\Omega^1_{Y_{\mathrm{sm}}/k}.$$

We have an F, V, R calculus for K_* induced by the one for $\operatorname{Im}_V(W_*\Omega^1_{U/k})$ and $W_*\Omega^1_{\tilde{Y}/k}(\log E)$. We set $Q_* := W_*\Omega^1_{\tilde{Y}/k}(\log E)/K_*$.

Lemma 8.14. *Suppose that $p \neq 2$ or n is even. Then, for all $l \geq 1$, the following map is surjective*:

$$R : K_l \to K_1.$$

Proof. The first case is $p \neq 2$. We need to compute K_1. We may assume that Y has only one singularity as in the proof of Proposition 8.8. Recall that $\tilde{Y}$ is constructed as a sequence of blow-ups $\cdots \to Y_i \to Y_{i-1} \to \cdots \to Y$. We denote by $r_i : Y_i \to Y$ the evident composition; we let D_i be the exceptional divisor of r_i, and E_i denotes the exceptional divisor of $Y_i \to Y_{i-1}$. We would like to understand

$$\jmath_{Y_i \setminus D_i,*}(\mathrm{image}(r_i^*\pi^*\Omega^1_X|_{Y_i\setminus D_i} \to \Omega^1_{Y_i\setminus D_i})) \cap \Omega^1_{Y_{i,\mathrm{sm}}}(\log D_i|_{Y_{i,\mathrm{sm}}}), \tag{8-19}$$

in a neighborhood of $E_i \cap Y_{i,\mathrm{sm}}$, where $Y_{i,\mathrm{sm}}$ is the smooth locus of Y_i, and $\jmath_{Y_i\setminus D_i} : Y_i \setminus D_i \to Y_{i,\mathrm{sm}}$ is the open immersion.

As in the proof of Proposition 8.8, we have coordinates $y, x'_1, \dots, x'_n$ around the singular point of Y_{i-1}, where $x'_j = x_j/y^{i-1}$. We can cover E_i by $n+1$ open sets $V_0, V_1, \dots, V_n$, where V_0 is a hypersurface in the affine space with coordinates $y, x'_1/y, \dots, x'_n/y$, and V_j is a hypersurface in the affine space with coordinates $y/x'_j, x'_1/x'_j, \dots, x'_j, \dots, x'_n/x'_j$, for $j = 1, \dots, n$. On V_0 we have $E_i \cap V_0 = D_i \cap V_0 = V(y)$. Note that if $i = (p^m-1)/2+1$, which is the last blow-up, then $E_i \cap V_0$ is empty.

On V_j we have $E_i \cap V_j = V(x'_j)$ and $D_i \cap V_j = V(y)$ if $j = 1, \dots, n$ and $i \notin \{1, (p^m-1)/2+1\}$, that is, except for the first and the last blow-ups. For the first blow-up ($i = 1$), we have $E_i \cap V_j = D_i \cap V_j = V(x'_j)$. For the last blow-up ($i = (p^m-1)/2+1$), we have $E_i \cap V_j = V(x'_j)$ and $D_i \cap V_j = V(y/x'_j)$.

We claim that the restriction of (8-19) to V_0 is generated by $dx_1/y^i, \dots, dx_n/y^i$, and the restriction of (8-19) to V_j is generated by $dx_1/x_j, \dots, dx_j/x_j, \dots, dx_n/x_j$. It is obvious that all differential forms are contained in the left-hand side of (8-19), and we need to show that they are contained in $\Omega^1_{Y_{i,\mathrm{sm}}}(\log D_i|_{Y_{i,\mathrm{sm}}})$. Indeed, $dx_j/y^i = d(x'_j/y\cdot y^i)/y^i = d(x'_j/y) + i\cdot(x'_j/y)\cdot(dy/y)$, and

$$\frac{dx_k}{x_j} = d\left(\frac{x_k}{x_j}\right) + \frac{x_k}{x_j}\cdot\frac{dx_j}{x_j} = d\left(\frac{x'_k}{x'_j}\right) + \frac{x'_k}{x'_j}\cdot\frac{dx_j}{x_j} = d\left(\frac{x'_k}{x'_j}\right) + \frac{x'_k}{x'_j}\cdot\left(\frac{dx'_j}{x'_j} + (i-1)\cdot\frac{dy}{y}\right).$$

In order to show that the given differential forms are generators, we note that the quotient of $\Omega^1_{Y_{i,\mathrm{sm}}}(\log D_i|_{Y_{i,\mathrm{sm}}}) \cap V_j$ by the module generated by these forms is a quotient of a free rank $= 1$ module. Since the quotient of $\Omega^1_{Y_{\mathrm{sm}}}$ by the image of $\pi^*(\Omega^1_U)$ is free of rank 1, the claim follows.

The case $p = 2$ and n even can be proved in the same way.

In order to prove that $K_l \to K_1$ is surjective, we may argue by induction on i and only consider a neighborhood of $E_i \cap Y_{i,\mathrm{sm}}$ in $Y_{i,\mathrm{sm}}$. We note that dx_j/y^i can be lifted by $d[x_j]/[y^i] \in K_l(V_0)$, and dx_k/x_j can be lifted by $d[x_k]/[x_j] \in K_l(V_j)$. □

Remark 8.15. We do not know whether Lemma 8.14 holds if $p = 2$ and n is odd. We can still describe K_1, but the coordinate changes $x_1^{[1]}, x_1^{[2]}, \dots$ used in the resolution process are incompatible with the multiplicative Teichmüller map and evident liftings do not exist.

8D. Let us assume that $p \neq 2$ or n is even. In view of the lemma, the map

$$\ker\big(W_*\Omega^1_{\tilde{Y}/k}(\log E) \xrightarrow{R} W_1\Omega^1_{\tilde{Y}/k}(\log E)\big) \to \ker(Q_* \xrightarrow{R} Q_1) \tag{8-20}$$

is surjective.

As a consequence of Proposition 8.13 we obtain the following corollary.

Corollary 8.16. *For all $l \le m$, the composition*

$$\mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{Y}}^{p^{l-1}} \xrightarrow{dV^{l-2},\cong} \ker(V : W_{l-1}\Omega^1_{\tilde{Y}/k} \to W_l\Omega^1_{\tilde{Y}/k})$$
$$\to \ker\big(V : W_{l-1}\Omega^1_{\tilde{Y}/k}(\log E) \to W_l\Omega^1_{\tilde{Y}/k}(\log E)\big) \to \ker(V : Q_{l-1} \to Q_l)$$

is surjective on the open set Y_{sm}.

Proof. The first isomorphism follows from [Illusie 1979, Chapitre I, Proposition 3.11]. The second arrow is an isomorphism on Y_{sm}. Set

$$A_l := \ker(R : \mathrm{Im}_V(W_l\Omega^1_{U/k}) \to \mathrm{Im}_V(W_1\Omega^1_{U/k})),$$
$$B_l := \ker(R : W_l\Omega^1_{Y_{\mathrm{sm}}/k} \to \Omega^1_{Y_{\mathrm{sm}}}),$$
$$C_l := \ker(R : Q_{l|Y_{\mathrm{sm}}} \to Q_{1|Y_{\mathrm{sm}}}).$$

In view of (8-20) we have a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Im}_V(W_{l-1}\Omega^1_{U/k}) \arrow[r] \arrow[d, "V"] & W_{l-1}\Omega^1_{Y_{\mathrm{sm}}/k} \arrow[r] \arrow[d, "V"] & Q_{l-1|Y_{\mathrm{sm}}} \arrow[r] \arrow[d, "V"] & 0 \\ 0 \arrow[r] & A_l \arrow[r] & B_l \arrow[r] & C_l \arrow[r] & 0 \end{tikzcd}$$

and the snake lemma and Proposition 8.13 imply the assertion. □

Theorem 8.17. *Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic p. Suppose that p is odd or n is even. Let L be a line bundle on X, and let $s \in H^0(X, L^{\otimes p^m})$ for $m \geq 1$. Suppose that the p^m cyclic covering $\pi : Y \to X$ corresponding to s has only nondegenerate singularities; let $r : \tilde{Y} \to Y$ be the resolution from Proposition 8.8. Suppose that*

(1) $n \geq 3$,

(2) $H^0(X, L^{\otimes p^m} \otimes K_X) \neq 0$,

(3) *the Frobenius acts bijectively on* $H^{n-1}(V(s), \mathcal{O})$,

(4) $H^n(X, L^{\otimes -j}) = 0$ *for all* $j = 0, \dots, p^m - 1$, *and*

(5) $H^{n-1}(X, L^{\otimes -j}) = 0$ *for all* $j = 0, \dots, p^m$.

Then $W_m(k) \subset H^0(\tilde{Y}, W_m\Omega^{n-1})$.

Proof. We have

$$\mathrm{coker}(\pi^*(\Omega^1_U) \to \Omega^1_{Y_{\mathrm{sm}}}) = \pi^*(L^{-1}),$$

and this identity extends to

$$Q_1 = r^*\pi^*(L^{-1})(E')$$

on $\tilde{Y}$, with E' as defined in (8-11). If the singularity of Y is of the form (8-5), (8-6), or (8-7), then Q_1 is generated by dy/y.

In view of Lemmas 8.9 and 8.10, and conditions (2), (4), and (5), we obtain

$$H^{n-1}(\tilde{Y}, Q_1) = 0, \qquad H^n(\tilde{Y}, Q_1) \cong H^n(X, L^{\otimes -p^m}) \neq 0. \tag{8-21}$$

We will work with the short exact sequences

$$0 \to \ker(R : Q_l \to Q_1) \to Q_l \to Q_1 \to 0, \tag{8-22}$$

$$Q_{l-1} \xrightarrow{V} \ker(R : Q_l \to Q_1) \to T_l \to 0, \tag{8-23}$$

where T_l is simply defined to be the cokernel. We claim

$$H^{n-1}(\tilde{Y}, T_l) = 0 = H^n(\tilde{Y}, T_l) \tag{8-24}$$

for all $l \leq m$. The surjectivity of (8-20) yields the surjectivity of the composition

$$\ker\bigl(W_l\Omega^1_{\tilde{Y}/k}(\log E) \xrightarrow{R} \Omega^1_{\tilde{Y}}(\log E)\bigr)/V W_{l-1}\Omega^1_{\tilde{Y}/k}(\log E) \xrightarrow[\cong]{F^{l-1}} B_{l-1}\Omega^1_{\tilde{Y}} \to T_l \qquad (8\text{-}25)$$

[Illusie 1979, p. 575]. Note that

$$\ker(W_l\Omega^1_{\tilde{Y}/k} \xrightarrow{R} \Omega^1_{\tilde{Y}})/V W_{l-1}\Omega^1_{\tilde{Y}/k} \xrightarrow{\cong} \ker\bigl(W_l\Omega^1_{\tilde{Y}/k}(\log E) \xrightarrow{R} \Omega^1_{\tilde{Y}}(\log E)\bigr)/V W_{l-1}\Omega^1_{\tilde{Y}/k}(\log E)$$

is an isomorphism.

Now we need to find a complex of $W_l(\mathbb{O}_{\tilde{Y}})$-modules

$$R_1 \to R_0 \to \ker(B_{l-1}\Omega^1_{\tilde{Y}} \to T_l),$$

such that the following conditions hold:

- $R_{0|Y_{\mathrm{sm}}} \to \ker(B_{l-1}\Omega^1_{\tilde{Y}} \to T_l)|_{Y_{\mathrm{sm}}}$ is surjective and
- $H^n(\tilde{Y}, R_1) \to H^n(\tilde{Y}, R_0)$ is surjective.

It will follow that $H^n(\tilde{Y}, T_l) = 0 = H^{n-1}(\tilde{Y}, T_l)$. Indeed, we have

$$H^n(\tilde{Y}, B_{l-1}\Omega^1_{\tilde{Y}}) = 0 = H^{n-1}(\tilde{Y}, B_{l-1}\Omega^1_{\tilde{Y}})$$

by induction on l, and using the exact sequence (8-26). The case $l = 2$ follows from assumptions (4) and (5), Lemma 8.9, and the short exact sequence (8-27).

We take

$$R_{0,l} := r^*\pi^* B_{l-1}\Omega^1_X, \qquad R_{1,l} = \ker(R_{0,l} \to B_{l-1}\Omega^1_{\tilde{Y}}).$$

Clearly, the image of $r^*\pi^* B_{l-1}\Omega^1_X$ is contained in $\ker(B_{l-1}\Omega^1_{\tilde{Y}} \to T_l)$. The surjectivity of $R_{0|Y_{\mathrm{sm}}} \to \ker(B_{l-1}\Omega^1_{\tilde{Y}} \to T_l)|_{Y_{\mathrm{sm}}}$ follows from Lemma 8.11 and diagram (8-14).

We claim that $H^n(\tilde{Y}, R_{1,l}) \to H^n(\tilde{Y}, R_{0,l})$ is surjective. We will proceed by induction on l. We have an exact sequence of locally free $\mathbb{O}_X$-modules

$$0 \to \mathrm{Frob}^{l-2}_* B_1\Omega^1_X \to B_{l-1}\Omega^1_X \xrightarrow{C} B_{l-2}\Omega^1_X \to 0, \qquad (8\text{-}26)$$

where C is the Cartier operator. Therefore,

$$0 \to r^*\pi^*\mathrm{Frob}^{l-2}_* B_1\Omega^1_X \to R_{0,l} \xrightarrow{C} R_{0,l-1} \to 0$$

is exact. Lemma 8.12 shows that $R_{1,l|Y_{\mathrm{sm}}} \xrightarrow{C} R_{1,l-1|Y_{\mathrm{sm}}}$ is surjective; note that under the isomorphism $F^{l-2}d$ from (8-16) the Cartier operator corresponds to the restriction. By induction we need to prove that the image of

$$H^n(\tilde{Y}, r^*\pi^*\mathrm{Frob}_*^{l-2} B_1\Omega_X^1) \to H^n(\tilde{Y}, R_{0,l})$$

is contained in the image of $H^n(\tilde{Y}, R_{1,l})$. Rationality of the resolution r implies

$$\begin{aligned} H^n(\tilde{Y}, r^*\pi^*\mathrm{Frob}_*^{l-2} B_1\Omega_X^1) &= H^n(Y, \pi^*\mathrm{Frob}_*^{l-2} B_1\Omega_X^1) \\ &= H^n(X, \mathrm{Frob}_*^{l-2}(B_1\Omega_X^1) \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_Y). \end{aligned}$$

In view of the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\mathrm{Frob}} \mathrm{Frob}_*\mathcal{O}_X \to B_1\Omega_X^1 \to 0, \tag{8-27}$$

we obtain a surjective map

$$H^n\Bigl(X, \bigoplus_{i=p^{m-l+1}}^{p^m-1} L^{-i\cdot p^{l-1}}\Bigr) \to H^n(\tilde{Y}, r^*\pi^*\mathrm{Frob}_*^{l-2} B_1\Omega_X^1),$$

because

$$\mathrm{Frob}_*^{l-1}\mathcal{O}_X \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{p^m-1} \mathrm{Frob}_*^{l-1}(\mathrm{Frob}^{l-1,*} L^{-i}).$$

For every $p^m > i \geq p^{m-l+1}$, we have two morphisms $\mathrm{Frob}_*^{l-1}(\mathrm{Frob}^{l-1,*}(L^{-i})) \to \mathrm{Frob}_*^{l-1}(\mathrm{Frob}^{l-1,*}(\pi_*\mathcal{O}_Y))$; the first one is induced by $\mathrm{Frob}_*^{l-1}\mathrm{Frob}^{l-1,*}$ applied to $L^{-i} \subset \pi_*\mathcal{O}_Y$. The second one is induced by Frob_*^{l-1} applied to

$$\begin{aligned} \mathrm{Frob}^{l-1,*}(L^{-i}) = L^{-i\cdot p^{l-1}} &\xrightarrow{s^{(i:p^{m+1-l})}} L^{-(i\% p^{m+1-l})\cdot p^{l-1}} \\ &= \mathrm{Frob}^{l-1,*}(L^{-(i\% p^{m+1-l})}) \\ &\to \mathrm{Frob}^{l-1,*}(\pi_*\mathcal{O}_Y), \end{aligned}$$

where the last arrow comes from $L^{-(i\% p^{m+1-l})} \subset \pi_*\mathcal{O}_Y$. Note that after application of $H^n(X, \cdot)$ this map vanishes, because it factors over

$$H^n(X, L^{-(i\% p^{m+1-l})\cdot p^{l-1}}) = 0.$$

Subtracting the two maps yields a morphism

$$r^*\pi^*\mathrm{Frob}_*^{l-1}(\mathrm{Frob}^{l-1,*}(L^{-i})) \to (R_{1,l} \cap r^*\pi^*\mathrm{Frob}_*^{l-2} B_1\Omega_X^1)$$

which shows that the $H^n(X, L^{-i\cdot p^{l-1}})$ piece of $H^n(\tilde{Y}, r^*\pi^*\mathrm{Frob}_*^{l-2} B_1\Omega_X^1)$ is contained in the image of $H^n(\tilde{Y}, R_{1,l})$. This proves claim (8-24).

In view of the short exact sequences (8-22) and (8-23), Corollary 8.16, vanishing of $H^n(\tilde{Y}, \mathcal{O}_{\tilde{Y}}/\mathcal{O}_{\tilde{Y}}^{p^{l-1}})$, and (8-24), we obtain, for all $l \leq m$, a short exact sequence

$$0 \to H^n(\tilde{Y}, Q_{l-1}) \xrightarrow{V} H^n(\tilde{Y}, Q_l) \xrightarrow{R} H^n(\tilde{Y}, Q_1) \to 0. \tag{8-28}$$

This enables us to define

$$\psi_{l-1} : H^n(\tilde{Y}, Q_1) \to H^n(\tilde{Y}, Q_1), \quad a \mapsto F^{l-1}(R^{-1}(a)).$$

It is evident that $\psi_{l-1} = \psi_1^{l-1}$. In view of (8-21) we have

$$H^n(\tilde{Y}, Q_1) \cong H^n(X, L^{-p^m}).$$

Via this identification, the map ψ_1 is given by

$$H^n(X, L^{-p^m}) \to H^n(X, L^{-p^{m+1}}) \xrightarrow{\cdot s^{p-1}} H^n(X, L^{-p^m}),$$

where the first arrow is induced by the p-th power map $L^{-p^m} \to L^{-p^{m+1}}$, $a \mapsto a^p$. Indeed, denoting by $\iota : L^{-p^m} \to \pi_* r_* Q_1$ the evident map, we have a commutative diagram

$$\begin{tikzcd}
L^{-p^m} \arrow[r, "(\cdot)^p"] \arrow[d, "\iota"] & L^{-p^{m+1}} \arrow[r, "s^{p-1}"] & L^{-p^m} \arrow[d, "\iota"] \\
\pi_* r_* Q_1 \arrow[r, "\pi_* r_*(F \circ R^{-1})"] & \pi_* r_* \left(\dfrac{Q_1}{\mathrm{image}(B_1 \Omega^1_{\tilde{Y}})} \right) & \dfrac{\pi_* r_* Q_1}{\mathrm{image}(\pi_* r_* B_1 \Omega^1_{\tilde{Y}})} \arrow[l]
\end{tikzcd}$$

Moreover, ψ_1 equals the composition

$$H^n(\tilde{Y}, Q_1) \xrightarrow{=} H^n(X, \pi_* r_* Q_1) \xrightarrow{\pi_* r_*(F \circ R^{-1})} H^n(X, \pi_* r_* (Q_1 / \mathrm{image}(B_1 \Omega^1_{\tilde{Y}})))$$
$$\to H^n(\tilde{Y}, Q_1 / \mathrm{image}(B_1 \Omega^1_{\tilde{Y}})) \xrightarrow{\cong} H^n(\tilde{Y}, Q_1),$$

where the last morphism is the inverse of $H^n(\tilde{Y}, Q_1) \xrightarrow{\eta} H^n(\tilde{Y}, Q_1 / \mathrm{image}(B_1 \Omega^1_{\tilde{Y}}))$, which is injective because $H^n(\tilde{Y}, Q_1) \xrightarrow{V} H^n(\tilde{Y}, Q_2)$ factors through η.

In the notation of [Chatzistamatiou 2012, Definition 1.3.1], we therefore get

$$H^n_c(X \setminus V(s), \mathbb{O})_s \cong \bigcap_{i \geq 1} \mathrm{image}(\psi_1^i).$$

Since $H^{n-1}(X, \mathbb{O}_X) = 0 = H^n(X, \mathbb{O}_X)$, [Chatzistamatiou 2012, §1.4] implies

$$H^n_c(X \setminus V(s), \mathbb{O})_s \cong H^{n-1}(V(s), \mathbb{O})_s = \bigcap_{i \geq 1} \mathrm{image}(\mathrm{Frob}^i).$$

By using assumption (3), we obtain

$$H^n(\tilde{Y}, Q_l) \cong \bigoplus_{i=1}^{h} W(k)/p^l, \tag{8-29}$$

where $h = \dim_k H^n(X, L^{-p^m})$. Indeed, since the Frobenius acts bijectively on $H^{n-1}(V(s), \mathbb{O}) \cong H^n(X, L^{-p^m})$, ψ_1 is bijective on $H^n(\tilde{Y}, Q_1)$. In view of (8-28), any lifting of a basis of $H^n(\tilde{Y}, Q_1)$ via the map $R : H^n(\tilde{Y}, Q_l) \to H^n(\tilde{Y}, Q_1)$ will be a $W(k)/p^l$-basis of $H^n(\tilde{Y}, Q_l)$.

Finally, let us show that $W_l(k) \subset H^0(\tilde{Y}, W_l\Omega^{n-1}_{\tilde{Y}})$. In view of (8-29), there is a surjective morphism of $W(k)$-modules

$$H^n(\tilde{Y}, W_l\Omega^1_{\tilde{Y}/k}(\log E)) \to W(k)/p^l = W_l(k).$$

From the residue short exact sequence (8-18) we obtain a surjective map

$$H^n(\tilde{Y}, W_l\Omega^1_{\tilde{Y}/k}) \to W_l(k).$$

Ekedahl duality [1984] implies

$$R\Gamma(W_l\Omega^{n-1}_{\tilde{Y}}) \xrightarrow{\cong} R\,\mathrm{Hom}_{W_l(k)}(R\Gamma(W_l\Omega^1_{\tilde{Y}}), W_l(k)[-n]),$$

hence the claim. □

Remark 8.18. Even for the case $m = 1$ the approach is dual to the one in [Kollár 1995]. With the notation in the proof of Theorem 8.17, we show that the composition

$$H^n(\tilde{Y}, \Omega^1_{\tilde{Y}}) \to H^n(\tilde{Y}, Q_1) \xrightarrow{\cong} H^n(X, L^{\otimes -p^m})$$

is surjective. For the last isomorphism we use $n \geq 3$, because we need to use Lemma 8.10, where vanishing holds for $i > 1$ only. Since we don't use Lemma 8.14 for this part, the argument also works for $p = 2$ and n odd. Taking duals we obtain an inclusion

$$H^0(X, \omega_X \otimes L^{\otimes p^m}) \subset H^0(\tilde{Y}, \Omega^{n-1}_{\tilde{Y}}).$$

This corresponds to a result about extending $(n-1)$-forms from Y_{sm} to $\tilde{Y}$ in [Kollár 1995] (and [Colliot-Thélène and Pirutka 2016a; Okada 2016]).

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