Algebra & Number Theory

Volume 11

2017

No. 8

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ALGEBRA AND NUMBER THEORY 11:8 (2017)
dx.doi.org/10.2140/ant.2017.11.1837

Integral canonical models for automorphic vector bundles of abelian type

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We define and construct integral canonical models for automorphic vector bundles over Shimura varieties of abelian type.

More precisely, we first build on Kisin's work to construct integral canonical models over $\mathbb{O}_E[1/N]$ for Shimura varieties of abelian type with hyperspecial level at all primes not dividing N compatible with Kisin's construction. We then define a notion of an integral canonical model for the standard principal bundles lying over Shimura varieties and proceed to construct them in the abelian type case. With these in hand, one immediately also gets integral models for automorphic vector bundles.

MSC2010: primary 11G18; secondary 14G35.
Keywords: Shimura varieties, automorphic vector bundles, integral models, Shimura, abelian type.

1. Introduction

Since the introduction of the abstract theory of Shimura varieties and their canonical models by Deligne [1971; 1979] following Shimura, and given its promise as a rich generalisation of the classical theory of modular curves, substantial bodies of literature have arisen whose aim is to extend features of the classical theory to this wider context.

One such feature is the existence of smooth integral models at primes not dividing the level, and their subsequent utility for studying the action of Frobenius on the Galois representations arising from Shimura varieties crucial for the Langlands programme. Such results have been available in the PEL type case for a long time, thanks largely to the programme of Kottwitz, but have recently been extended to the much more general case of abelian type Shimura varieties in work culminating with the recent papers of Kisin [2010; 2017].

Another feature is the manifestation of certain automorphic forms as algebraic sections of a vector bundle over a Shimura variety, generalising the classical algebraic description of modular forms. The vector bundles playing the role analogous to the tensor powers of the Hodge bundle in the theory of modular forms are the automorphic vector bundles, and canonical models were defined and shown to exist in some cases by Harris [1985] and more generally by Milne [1988; 1990].

In the present paper we start to draw these two threads of the literature together, working in the case of general[1] abelian type Shimura varieties, first filling a gap in the existing literature and showing that Kisin's good integral models can be spread out to smooth models over $\mathcal{O}_E[1/N]$, then defining a notion of integral canonical models for automorphic vector bundles with a uniqueness property, and finally proving existence in the abelian type case.

More precisely, suppose $(G, \mathfrak{X})$ is a Shimura datum, with reflex field $E = E(G, \mathfrak{X})$, $N > 1$ and suppose G admits a reductive[2] model $\mathcal{G}/\mathbb{Z}[1/N]$. Take an open compact $K = K^N K_N \subset G(\mathbb{A}^\infty)$ where $K^N = \prod_{p\nmid N} \mathcal{G}(\mathbb{Z}_p)$ and $K_N \subset \prod_{p|N} G(\mathbb{Q}_p)$ is open compact. Consider the tower

$$\mathrm{Sh}_{K^N}(G, \mathfrak{X}) := \varprojlim_{K_N} \mathrm{Sh}_{K^N K_N}(G, \mathfrak{X})$$

of quasiprojective E-schemes, which comes equipped with an algebraic $\prod_{p|N} G(\mathbb{Q}_p)$ action (in fact it carries the action of a slightly larger group as in [Deligne 1979]). The main result of Kisin's first integral models paper [2010] tells us that when $(G, \mathfrak{X})$ is of abelian type this tower admits, for every $v \nmid N$, a smooth integral model

[1]We do require a small technical restriction: that $Z(G)^\circ$ is split by a CM field, but feel it should be possible to remove this restriction, and that it ought to be harmless for most applications.

[2]Recall that a (connected) reductive group scheme $G \to S$ is a smooth affine group scheme with connected reductive geometric fibres.

$\mathcal{S}^{\mathrm{Kis},v}_{K^N}/\mathbb{O}_{E,v}$ to which the $\prod_{p|N} G(\mathbb{Q}_p)$-action extends. Note that by a "smooth" model, we mean one which is smooth quasiprojective at any finite level and for which the maps between different finite levels are finite étale.

It is natural to ask whether these all come from a global model over $\mathbb{O}_E[1/N]$. Moreover, Kisin's models also enjoy an "extension property" which characterises them uniquely, so it is natural to ask if furthermore we can find a global model having this extension property. Our first theorem answers this in the affirmative.

Theorem. *With the above setup, suppose $(G, \mathfrak{X})$ is of abelian type. Then the tower $\mathrm{Sh}_{K^N}(G, \mathfrak{X})$ admits a smooth integral model $\mathcal{S}_{K^N}(G, \mathfrak{X})/\mathbb{O}_E[1/N]$ to which the $\prod_{p|N} G(\mathbb{Q}_p)$-action extends and having the extension property.*

Moreover, for any $v\nmid N$ a place of E, the model $\mathcal{S}_{K^N}(G, \mathfrak{X}) \otimes_{\mathbb{O}_E[1/N]} \mathbb{O}_{E,v}$ is canonically identified[3] with the model $\mathcal{S}^{\mathrm{Kis},v}_{K^N}$ obtained from Kisin's theory.

We call these models "integral canonical models" because the extension property guarantees their uniqueness.

We make some remarks about the proof. The obvious direct "patching" argument to obtain the result formally from Kisin's models fails because one cannot automatically get the extension property for the models obtained by spreading out, so instead we give a direct construction. The proof in the Hodge type case very closely follows that of [Kisin 2010]. In the abelian type case we need several new ideas.

Firstly, we have no guarantee that $\prod_{p|N} G(\mathbb{Q}_p)$ acts transitively on the components of $\mathrm{Sh}_{K^N}(G, \mathfrak{X})$ so we replace the group theoretic "Deligne-induction" argument by an argument that uses the extension property to reduce the problem to constructing models for each component individually over the ring of integers of the maximal abelian unramified extension of E. Secondly, to prove the analogue of [Kisin 2010, 3.4.6] without the lemma which follows it, which may not be true in our context, we show that Kisin's "twisting abelian varieties" construction can be carried out using torsors for the centre of G^{der} rather than G, and since such torsors are finite, they are simpler to work with. Finally, we believe in fact that unless $G^{\mathrm{der}} = G^{\mathrm{ad}}$, the group $\Delta(G, G^{\mathrm{ad}})$ [ibid.] is not finite, which is necessary to descend the extension property. Fortunately we are able to prove that the corresponding group at level K^N (our situation) is finite, which perhaps also helps to fill a small gap in the original argument.

We then turn to automorphic vector bundles, and following [Milne 1990, III], the heart of the matter is really to define and construct integral canonical models for the standard principal bundles $P_{K^N}(G, \mathfrak{X}) \to \mathrm{Sh}_{K^N}(G, \mathfrak{X})$. Such bundles are torsors for the quotient

$$G^c = G/Z_{\mathrm{nc}}$$

[3]This identification is as an $\mathbb{O}_{E,v}$-scheme with $\prod_{p|N} G(\mathbb{Q}_p)$-action and equivariant identification ι of its generic fibre with $\mathrm{Sh}_{K^N}(G, \mathfrak{X})$.

by the maximal subtorus $Z_{nc} \subset Z(G)^\circ$ which is $\mathbb{R}$-split but $\mathbb{Q}$-anisotropic. They come equipped with a flat connection ∇, an equivariant $\prod_{p|N} G(\mathbb{Q}_p)$-action and a "filtration" which can be written down by giving a map

$$\gamma : P_{K^N}(G, \mathfrak{X}) \to \mathrm{Gr}_\mu$$

where Gr_μ is the flag variety corresponding to a Hodge cocharacter $\mu : \mathbb{G}_m \to G$ coming from $\mathfrak{X}$. Informally, it is helpful to think of the fibre functors ω attached to these bundles as giving one the sheaf of de Rham cohomology of a family of motives over Sh_{K^N} with its Gauss–Manin connection and Hodge filtration.

Any smooth integral model for $P_{K^N}(G, \mathfrak{X})$ that is a $\mathcal{G}^c$-torsor would give, for each representation $\rho : \mathcal{G}^c \to \mathrm{GL}(V_{\mathbb{Z}[1/N]})$, a $\mathbb{Z}[1/N]$ lattice inside $\omega(V \otimes \mathbb{Q})$. We define an integral canonical model to be one where at "crystalline points" this lattice coincides with a different lattice constructed using Kisin's theory of $\mathfrak{S}$-modules [2006]. Roughly speaking, working on the generic fibre one has a Galois cover $\mathrm{Sh}_{K^{Np}}(G, \mathfrak{X}) \to \mathrm{Sh}_{K^N}(G, \mathfrak{X})$ with Galois group $\mathcal{G}^c(\mathbb{Z}_p)$ so we obtain attached to ρ a lisse $\mathbb{Z}_p$-sheaf

$$\mathcal{L} := \mathrm{Sh}_{K^{Np}}(G, \mathfrak{X}) \times V_{\mathbb{Z}_p}/\mathcal{G}^c(\mathbb{Z}_p)$$

on $\mathrm{Sh}_{K^N}(G, \mathfrak{X})$. Restricting this to a crystalline point s, the theory of $\mathfrak{S}$-modules gives a lattice $\mathbb{D}(s^*\mathcal{L}) \subset D_{\mathrm{dR}}(s^*\mathcal{L}[1/p])$. Thus we have defined lattices (at each prime) inside $\omega(V \otimes \mathbb{Q})$, and we say a model

$$(\mathcal{P}_{K^N}(G, \mathfrak{X}), \iota : \mathcal{P}_{K^N}(G, \mathfrak{X}) \otimes_{\mathcal{O}_E[1/N]} E \xrightarrow{\sim} P_{K^N}(G, \mathfrak{X}))$$

is canonical if the lattices it generates agree with these lattices coming from p-adic Hodge theory. We then check that if such a model exists and there are enough crystalline points (which we check in the abelian type case), it is unique up to canonical isomorphism. We also reserve the term "canonical" for models for which the connection, Hecke action and filtration extend, but these seem to be automatic properties of the models satisfying the lattice condition in the abelian type case (and we would assume in general).

Of course, with this definition, our main theorem is the following.

Theorem. *Let $(G, \mathfrak{X})$ be a Shimura datum of abelian type and $\mathcal{G}/\mathbb{Z}[1/N]$ a reductive model for G. Then $P_{K^N}(G, \mathfrak{X})$ has an integral canonical model $(\mathcal{P}_{K^N}, \iota)$.*

We give a quick summary of the proof. For the "special type" case where G is a torus, we use the theory of CM motives to find lattices in the de Rham cohomology of abelian varieties. For the Hodge type case, the universal abelian variety has an integral model so we take as our starting point the sheaf $\mathcal{V}$ of its relative de Rham cohomology, and note that the Hodge tensors of [Kisin 2010, 2.2] $s_{\alpha,\mathrm{dR}} \in \mathcal{V}_E^{\otimes}$ extend to $\mathcal{V}^{\otimes}$, at which point they may be used to define a functor

$$\mathcal{P}_{K^N} := \underline{\mathrm{Isom}}_{s_\alpha}(V, \mathcal{V})$$

which we show is a $\mathcal{G}$-torsor using several ingredients from [Kisin 2010], and is the required integral canonical model.

For passing from the Hodge type to abelian type case, we need a new idea which may be more widely applicable. Suppose $(G_2, \mathfrak{X}_2)$ is an abelian type Shimura datum of interest. If we let $(G, \mathfrak{X})$ be a Hodge type datum such that there is an isogeny $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ witnessing that $(G_2, \mathfrak{X}_2)$ is of abelian type, we do not have a map relating G_2 to G but only between the derived groups. However, the torsors $P_{K^N}(G, \mathfrak{X})$ cannot be reduced to G^{der} without passing to $\mathbb{C}$, which loses the information we are interested in.

Our solution, inspired by Deligne [1979, 2.5], is to define for each connected Shimura datum $(G^{\mathrm{der}}, \mathfrak{X}^+)$ and field $E \supset E(G^{\mathrm{der}}, \mathfrak{X}^+) := E(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}})$ a new Shimura datum $(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ with the property that any Shimura datum $(G, \mathfrak{X})$ whose reflex field is contained in E and whose connected Shimura datum is $(G^{\mathrm{der}}, \mathfrak{X}^+)$ admits a canonical map

$$(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) \to (G, \mathfrak{X}).$$

We then pass from Hodge type to abelian type by first giving a direct construction of a canonical model for $P_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ given one for $P_{K^N}(G, \mathfrak{X})$. With this, we are in business, because if we let $(\mathcal{B}_2, \mathfrak{X}_{\mathcal{B},2})$ be the corresponding pair for $(G_2, \mathfrak{X}_2)$ there is a map

$$(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) \to (\mathcal{B}_2, \mathfrak{X}_{\mathcal{B},2}) \to (G_2, \mathfrak{X}_2)$$

and we are able to descend the torsor through the map on connected components while pushing out the $\mathcal{B}^c$-action to a G_2^c-action. Finally we translate our results into results for automorphic vector bundles.

While we do not give applications in this paper, we anticipate this construction playing a useful role in several places. It is already used to study integrality of periods in a preprint of Ichino and Prasanna [2016], and we expect it should be useful in much more general contexts along these lines.

Working at a formal completion at a single place the same construction gives families of strongly divisible filtered F-crystals. This theory is developed and used in [Lovering 2017] to establish a new result that the Galois representations formed by taking the cohomology of the Shimura variety with coefficients in the usual lisse sheaves are crystalline, and in appropriate situations Fontaine Laffaille. In particular this gives a new proof of part of local global compatibility at $l = p$ in certain cases, as well as providing a new tool for studying the p-adic geometry of Shimura varieties and p-adic automorphic forms.

2. Integral canonical models for Shimura varieties of abelian type

2.1. *Extension property.*

2.1.1. We first recall the extension property used to characterise Shimura varieties at infinite level. Let R be a domain with field of fractions K. We call S/R a *test scheme* if it is regular and formally smooth over R.[4] Suppose we are given a scheme X/R. We say X *has the extension property* if for any test scheme S/R any map $S_K \to X_K$ extends over R.

The following uniqueness statement is well known.

Lemma 2.1.2. *Suppose Y/K is a scheme. Then if X/R is a model for it over R and is both a test scheme and has the extension property, X is the unique such model up to canonical isomorphism.*

Proof. Let X and X' be two such models. Then we are (as part of the data of a model) given maps

$$X_K \xrightarrow{\sim} Y \xrightarrow{\sim} X'_K.$$

Since X is a test scheme, and since X' has the extension property, this isomorphism extends to an isomorphism $X \xrightarrow{\sim} X'$ of R-schemes. □

We also record the following useful formal properties.

Lemma 2.1.3. *Let R'/R be an étale or indétale (or formally smooth) extension of domains with fraction field extension K'/K. Suppose X/R satisfies the extension property. Then so does $X \otimes_R R'/R'$.*

Conversely, if R'/R is also faithfully flat, then if $X \otimes_R R'/R'$ satisfies the extension property, so does X/R.

Proof. Let S'/R' be a test scheme. Then S' is regular, and it is formally smooth over R since indétale algebras are formally étale. Suppose we are given $S' \otimes_{R'} K' \to X_{R'}$, and first compose it with the map $X_{R'} \to X$. By the extension property for X/R, $S' \otimes_{R'} K' = S' \otimes_R K \to X$ extends to a map $S' \to X$ over R. But since we have a diagram

$$\begin{array}{ccc} S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \end{array}$$

this map must factor through $X \otimes_R R'$, as required, giving the extension property for $X \otimes_R R'/R'$.

For the converse, a test scheme S/R gives rise to a test scheme $S_{R'}/R'$ together with a descent datum θ_S for R'/R. Given a map $S_K \to X_K$ we obtain $S_{K'} \to X_{K'}$

[4] I believe there is still some controversy over the "correct" definition of a test scheme but this should do for our purposes.

compatible with the descent data on both sides. By the extension property this map extends to $S_{R'} \to X_{R'}$, and being a map of descent data this descends to the desired extension $S \to X$. □

Lemma 2.1.4. *Let X/R have the extension property and $Y \to X$ be finite or profinite étale. Then Y/R has the extension property.*

Proof. It clearly suffices to do the finite étale case (the profinite one then following formally). Let $K = \mathrm{Frac}(R)$ and suppose we have a test scheme S and a map $S_K \to Y_K$. Then this map composed with $Y_K \to X_K$ extends to a map $S \to X$. Pulling this back we get $S \times_X Y \to Y$. The map $S_K \to Y_K$ gives a section of $(S \times_X Y)_K \to S_K$. But $S \times_X Y \to S$ is finite étale so such sections extend uniquely and we get the required $S \to Y$. □

Lemma 2.1.5. *Let $Y \to X$ be finite étale, and suppose Y/R has the extension property. Then X/R has the extension property.*

Proof. By the theory of the étale fundamental group and Lemma 2.1.4 we may assume $Y \to X$ is Galois. The argument then follows from that of [Moonen 1998, 3.21.4]. □

We remark that the above result fails for pro(finite étale) extensions. Indeed, Shimura varieties at finite level certainly do not generally have the extension property. For example, for the modular curve at finite level this would imply all elliptic curves have good reduction.

2.2. *Main theorem.* Now, let $(G, \mathfrak{X})$ be a Shimura datum of abelian type with reflex field E.

Fix S a finite nonempty set of finite primes containing all those at which G is ramified, set $N = \prod_{p \in S} p$, a reductive integral model $G_{\mathbb{Z}[1/N]}$ of G (which exists by taking an arbitrary integral model, observing that it is reductive at all but finitely many primes and then gluing in models for the remaining primes). We abusively denote this model by G, and let $K^N = \prod_{p \notin S} G(\mathbb{Z}_p)$.

Consider the tower

$$\mathrm{Sh}_{K^N} = \varprojlim_{K_N \subset \prod_{p \in S} G(\mathbb{Q}_p)} \mathrm{Sh}_{K_N K^N}$$

of Shimura varieties over E with infinite level at the primes dividing N but hyperspecial level at all other primes.

Recall that in this context a *smooth integral model* $\mathscr{S}_{K^N}$ for Sh_{K^N} over $\mathbb{O}_E[1/N]$ is an integral model on which $\prod_{p|N} G(\mathbb{Q}_p)$ acts such that whenever $K_N \subset \prod_{p|N} G(\mathbb{Q}_p)$ compact open is sufficiently small that $\mathrm{Sh}_{K_N K^N}$ is a scheme, the model $\mathscr{S}_{K^N}/K_N$ is smooth quasiprojective, and the maps between such finite level schemes induced by Hecke operators are finite étale.

In this chapter we prove the following theorem.

Theorem 2.2.1. *The tower* Sh_{K^N} *has a smooth integral model* $\mathscr{S}_{K^N}/\mathbb{O}_E[1/N]$ *satisfying the extension property. For any* $v \nmid N$ *the localisation of this model at* v *agrees with Kisin's smooth integral model.*

The following together with Lemma 2.1.2 gives us that this model is canonical.

Lemma 2.2.2. *If* $\mathscr{S}_{K^N}$ *is a smooth integral model in the above sense, then it is regular and formally smooth* (*in the usual sense*).

Proof. That it is formally smooth follows formally because it is a limit of smooth schemes with finite étale transition maps. That it is regular follows from the same argument as [Milne 1992, 2.4]. □

2.2.3. We now set out to prove the theorem, following the outline of Kisin's strategy, first using the modular interpretation of Siegel varieties to get going, then making constructions in the Hodge and abelian cases close enough to his that the smoothness, extension and comparison properties follow by direct comparison or in a very similar way.

2.3. ***Siegel case.*** For this case we recall the following theorem.

Theorem 2.3.1 [Mumford 1965, 7.9]. *If* $n \geq 6^g d\sqrt{g!}$ *then the fine moduli scheme* $\mathscr{A}_{g,d,n}$ *of abelian schemes of dimension* g *together with a polarisation of degree* d *and a level* n *structure exists, and is quasiprojective over* $\mathbb{Z}$.

Moreover, for $N = \mathrm{lcm}(d, n)$ *these moduli schemes are smooth over* $\mathbb{Z}[1/N]$.

2.3.2. We may also (since we may take quotients of quasiprojective schemes by finite free group actions) form moduli $\mathscr{A}_{g,d,K_N}$ with $\mathrm{GSp}_{2g}(\hat{\mathbb{Z}}^N)K_N$-level structures for all $K_N \subset \prod_{p|N} \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ sufficiently small. These are also smooth over $\mathbb{Z}[1/N]$ by étale descent.

2.3.3. Moreover, recall that for $K = K_N\,\mathrm{GSp}_{2g}(\hat{\mathbb{Z}}^N) \subset \mathrm{GSp}_{2g}(\mathbb{A}^\infty)$, the Shimura variety $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, S^\pm)$ is defined over $\mathbb{Q}$ and has a moduli interpretation giving an embedding $\mathrm{Sh}_K(\mathrm{GSp}_{2g}, S^\pm) \hookrightarrow \mathscr{A}_{g,d,K_N}$. We define its integral model

$$\mathscr{S}_K := \overline{\mathrm{Sh}_K(\mathrm{GSp}_{2g}, S^\pm)} \subset \mathscr{A}_{g,d,K_N}.$$

It is well known that such models are smooth and admit an explicit description as moduli schemes. In particular, they carry a universal abelian scheme defined over $\mathbb{Z}[1/N]$. Moreover the transition maps as we vary K_N are finite étale, so we obtain the desired smooth integral models

$$\mathscr{S}_N = \varprojlim_{K_N} \mathscr{S}_K/\mathbb{Z}\big[\tfrac{1}{N}\big].$$

We are required to check the following. Note that in preparation for the Hodge type case we need to observe the following holds in the slightly more general context where we assume that $K = \prod K_p$ as above except for some finitely many $p \nmid N$, K_p is not necessarily hyperspecial but merely maximal compact. Luckily with this remark made the argument goes through unchanged.

Proposition 2.3.4. *The scheme $\mathscr{S}_N$ satisfies the extension property.*

Proof. This follows because the argument of Milne [1992, 2.10] adapts practically unchanged to our situation. For the reader's convenience we sketch the argument. Let S be a test scheme. A map

$$S_{\mathbb{Q}} \to \mathscr{S}_N$$

gives the data of a triple (A, λ, η) where $A/S_{\mathbb{Q}}$ is an abelian scheme, λ a polarisation (defined up to a constant) and η an infinite level structure at the primes $l|N$. Since $N > 1$, the set of such primes is nonempty, and we may let l be one of them.

Now, since S is assumed regular, in particular each of its components is integral. Let S^0 be such a component, and denote by η its generic point. The infinite level structure at l defined over $S_{\mathbb{Q}}$ in particular trivialises the l-adic Tate module of A_η. By the "generalised Neron criterion" [ibid., 2.13] we see that this implies A_η extends over S^0. Since it does so for all components S^0 of S and S is normal (so these components do not meet), we deduce that we have extended A to some $\mathscr{A}/S$.

The polarisation λ also extends by [ibid., 2.14], and the level structures (being at primes away from the characteristic of the base) also obviously extend. This suffices to show the extension property. □

2.4. ***Hodge type case.*** Suppose we have $(G, \mathfrak{X})$ a Shimura datum of Hodge type, and fix a symplectic embedding

$$i : (G, \mathfrak{X}) \hookrightarrow (\mathrm{GSp}(V_{\mathbb{Q}}, \psi), S^{\pm}).$$

Let N be the product of all primes where G is ramified. Recall that we are fixing an integral model $G/\mathbb{Z}[1/N]$ for G and taking the hyperspecial level $K^N = \prod_{p \nmid N} G(\mathbb{Z}_p)$ away from N. We begin with some group theoretic preliminaries.

Lemma 2.4.1. *Let $V/\mathbb{Q}$ be a finite dimensional vector space. Take $N \geq 1$ and $p \nmid N$, and suppose we have a $\mathbb{Z}[1/pN]$-lattice $\Lambda \subset V$ and a $\mathbb{Z}_p$-lattice $L \subset V \otimes \mathbb{Q}_p$. Then $L \cap \Lambda$ is a $\mathbb{Z}[1/N]$-lattice in V.*

Proof. We first observe that for any $\mathbb{Z}[1/pN]$-basis $e_1, \dots, e_n$ of Λ, for all $i = 1, \dots, n$ and some m sufficiently large $p^m e_i \subset L$. In particular $\Lambda \cap L$ contains $\mathbb{Z}[1/pN]$-bases for Λ. Let us take some such basis $e_1, \dots, e_n$ such that the p-adic volume is maximal (or equivalently such that the index of its $\mathbb{Z}_p$-span in L is minimal). We claim this basis generates $\Lambda' = \Lambda \cap L$ as a free $\mathbb{Z}[1/N]$-module.

Since it's a $\mathbb{Z}[1/pN]$-basis for Λ, its span certainly is a free $\mathbb{Z}[1/N]$-module. Suppose it doesn't generate. Then there is some $y \in \Lambda'$ not in the span of the e_i.

Since the e_i are a $\mathbb{Z}[1/pN]$-basis we may write y uniquely as

$$y = \sum_i \lambda_i e_i,$$

with $\lambda_i \in \mathbb{Z}[1/pN]$. After reordering, let us assume $\lambda_1 = p^{-k}\lambda$ with $\lambda \in \mathbb{Z}[1/N]$ and $k \geq 1$ is the coefficient with the largest p-adic norm amongst the λ_i. Then we can take the new basis $y, e_2, \ldots, e_n$ and it visibly has strictly larger p-adic volume, contradicting our original choice of basis and hence the existence of y. □

Proposition 2.4.2. *Let $G/\mathbb{Q}$ be a reductive group unramified away from N, let $N \mid M$ and T be the set of primes dividing M/N. If $2 \in T$, assume further that any factors of G of type B have simply connected derived group.*

Then given choices of reductive models $G^M/\mathbb{Z}[1/M]$ and $G_p/\mathbb{Z}_p$ for $p \in T$ for G, we can find a model $\mathcal{G}/\mathbb{Z}[1/N]$ isomorphic to each of these.

Moreover, if we have a faithful representation $i : G^M \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}[1/M]})$ there is a $\mathbb{Z}[1/N]$-lattice $\Lambda' \subset V_{\mathbb{Z}[1/M]}$ such that i extends to $\tilde{i} : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda')$.

Proof. It obviously suffices to consider the case where $M = pN$. Let $i : G^M \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}[1/pN]})$ be a faithful representation. By [Kisin 2010, 2.3.1], and the remark of Madapusi Pera [2012, 4.3, footnote] in the $2 \in T$ case, we can find a $\mathbb{Z}_p$-lattice $\Lambda \subset V \otimes \mathbb{Q}_p$ such that $G_p(\mathbb{Q}_p) = G(\mathbb{Q}_p) = G^M_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(V_{\mathbb{Q}_p})$ is induced from a map $G_p \to \mathrm{GL}(\Lambda)$ over $\mathbb{Z}_p$.

By Lemma 2.4.1 we obtain a $\mathbb{Z}[1/N]$-lattice $\Lambda' = \Lambda \cap V_{\mathbb{Z}[1/pN]}$. Of course we can canonically identify $\Lambda' \otimes \mathbb{Z}[1/pN] \cong V_{\mathbb{Z}[1/pN]}$, in the context of which we take $\mathcal{G}/\mathbb{Z}[1/N]$ to be the closure of

$$\mathrm{Im}(G^M \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}[1/pN]}) \hookrightarrow \mathrm{GL}(\Lambda')).$$

We claim this $\mathcal{G}$ does the job. It's evident that $\mathcal{G}|_{\mathbb{Z}[1/pN]} \cong G^M$. Since we have a canonical isomorphism $\Lambda' \otimes \mathbb{Z}_p \cong \Lambda$ and by the other identifications in the construction we also have $\mathcal{G}|_{\mathbb{Z}_p} \cong G_p$. We also need it to be reductive (i.e., smooth affine with connected reductive geometric fibres). Being a closed subgroup of $\mathrm{GL}(\Lambda')$ it's visibly affine, its geometric fibres are reductive by what we already know, and smoothness can be checked fpqc locally, whence it also follows by the identifications we have made.

The second part of the proposition is an immediate consequence of our argument. □

We also echo the remarks in [Madapusi Pera 2012, 4.3], that in the case of G coming from a Hodge type Shimura datum, the condition on factors of type B is always satisfied, by Deligne's classification of symplectic representations. We shall also need the following modification of [Kisin 2010, 2.1.2].

Lemma 2.4.3. *Let $i : (G_1, \mathfrak{X}_1) \hookrightarrow (G_2, \mathfrak{X}_2)$ be an embedding of Shimura data with $K_2^N \subset \prod'_{p\nmid N} G_2(\mathbb{Q}_p) =: G_2(\mathbb{A}^{\infty,N})$ compact open, and $K_1 = K_1^N K_{1,N}$ a compact open of $G_1(\mathbb{A}^\infty)$ such that $K_1^N = K_2^N \cap G_1(\mathbb{A}^{\infty,N})$. Suppose G_1 and G_2 both have centre which is compact modulo its split part.*[5] *Then there exists an open compact subgroup $K_{2,N} \subset \prod_{p\mid N} G_2(\mathbb{Q}_p)$ such that $K_2 := K_{2,N} K_2^N \supset K_1$ and the induced map of $E(G_1, \mathfrak{X}_1)$-schemes*

$$\mathrm{Sh}_{K_1}(G_1, \mathfrak{X}_1) \to \mathrm{Sh}_{K_2}(G_2, \mathfrak{X}_2)_{E(G_1,\mathfrak{X}_1)}$$

is a closed embedding.

Proof. By the same argument as [Deligne 1971, 1.15] it suffices to check

$$\alpha : \mathrm{Sh}_{K_1^N}(\mathbb{C}) \to \mathrm{Sh}_{K_2^N}(\mathbb{C})$$

is injective.

On the level of complex points (by the assumption on centres, which removes the technicalities involving units) this map is

$$\alpha : G_1(\mathbb{Q})\backslash \mathfrak{X}_1 \times G_1(\mathbb{A}^\infty)/K_1^N \to G_2(\mathbb{Q})\backslash \mathfrak{X}_2 \times G_2(\mathbb{A}^\infty)/K_2^N.$$

We prove this is injective by first noting that

$$G_1(\mathbb{Q})\backslash \prod_{p\mid N} G_1(\mathbb{Q}_p) \to G_2(\mathbb{Q})\backslash \prod_{p\mid N} G_2(\mathbb{Q}_p)$$

is injective as in [Deligne 1971, 1.15.3]. Now fix a set of coset representatives of $G_1(\mathbb{Q})$ in $\prod_{p\mid N} G_1(\mathbb{Q}_p)$ and note that translating by these, the fibres of $G_1(\mathbb{Q})\backslash \mathfrak{X}_1 \times G_1(\mathbb{A}^\infty)/K_1^N \to G_1(\mathbb{Q})\backslash \prod_{p\mid N} G_1(\mathbb{Q}_p)$ may be identified with $\mathfrak{X}_1 \times G_1(\mathbb{A}^{\infty,N})/K_1^N$. But now since $K_1^N = K_2^N \cap G_1(\mathbb{A}^{\infty,N})$, we have that

$$\mathfrak{X}_1 \times G_1(\mathbb{A}^{\infty,N})/K_1^N \to \mathfrak{X}_2 \times G_2(\mathbb{A}^{\infty,N})/K_2^N,$$

is injective, and the lemma follows. □

2.4.4. We now proceed with the construction in the Hodge type case.

Firstly, by finite-presentedness we note that i is in fact defined over $\mathbb{Z}[1/M]$ for some M divisible by N. By Proposition 2.4.2 we can find a lattice $V_{\mathbb{Z}[1/N]} \subset V$ and $\mathcal{G}/\mathbb{Z}[1/N]$ a reductive model such that (forgetting the symplectic pairing) i is obtained from a map $\mathcal{G} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}[1/N]})$ and $\mathcal{G}(\mathbb{Z}_p) = G(\mathbb{Z}_p)$ for all $p\nmid N$, in particular giving $K^N = \prod_{p\nmid N} \mathcal{G}(\mathbb{Z}_p)$.

Let K'^N be the stabiliser of $V_{\mathbb{Z}[1/N]}$ in $\prod_{p\nmid N} \mathrm{GSp}(V \otimes \mathbb{Q}_p, \psi)$, noting that K'^N will be maximal compact but need not be hyperspecial at the primes dividing M/N. We also fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Z}[1/N]}$ and note that for all $K_N \subset \prod_{p\mid N} G(\mathbb{Q}_p)$ sufficiently small $K = K_N K^N$ fixes $V_{\hat{\mathbb{Z}}}$.

[5] We expect the argument can be modified slightly as in Deligne to remove this hypothesis.

2.4.5. Applying the lemma Lemma 2.4.3 in our setting (since G is Hodge type the hypothesis on the centre holds), and letting $E = E(G, \mathfrak{X})$, we obtain a closed embedding of Shimura varieties

$$\mathrm{Sh}_{K^N}(G, \mathfrak{X}) \hookrightarrow \mathrm{Sh}_{K'^N}(\mathrm{GSp}, S^{\pm})_E \subset \mathscr{S}_{N,\mathbb{O}_E[1/N]}.$$

Letting $\overline{\mathrm{Sh}_{K^N}} \subset \mathscr{S}_{N,\mathbb{O}_E[1/N]}$ be the scheme theoretic closure of this map, the extension property for $\mathscr{S}_N$ implies the extension property for $\overline{\mathrm{Sh}_{K^N}}$. Now let $\mathscr{S}_{K^N}$ be the normalisation of $\overline{\mathrm{Sh}_{K^N}}$. Since test schemes are regular and a fortiori normal, the universal property of normalisation implies that $\mathscr{S}_{K^N}$ also has the extension property.

2.4.6. We need to check smoothness at finite level. It suffices to check smoothness in a formal neighbourhood of any closed point. When the point has characteristic zero it is in the generic fibre and smoothness is guaranteed. When it has characteristic p, we observe that at finite level our construction exactly follows that of Kisin [2010, 2.3] and Kim and Madapusi Pera [2016, 3.5] for the case $p = 2$, and so in particular the necessary local rings are smooth. Hence the $\mathscr{S}_{K^N}$ give the required integral canonical models in the Hodge type case, compatible with Kisin's by construction.

2.5. ***Abelian type case.*** We begin by making some more observations about the Hodge type setting. As before we fix connected reductive $G/\mathbb{Z}[1/N]$ belonging to a Shimura datum $(G, \mathfrak{X})$ of Hodge type with reflex field E, and consider the tower

$$\mathrm{Sh}_{K^N} = \varprojlim_{K_N} \mathrm{Sh}_{K_N K^N}$$

obtained by fixing $K^N = G(\hat{\mathbb{Z}}^N)$ and letting the level at $p|N$ go to infinity.

Lemma 2.5.1. *The connected component* $\mathrm{Sh}^+_{K^N} \subset \mathrm{Sh}_{K^N,\bar{E}}$ *is defined over the maximal abelian extension* E_N/E *unramified away from* N.

Proof. By [Kisin 2010, 2.2.4], and taking a suitable quotient, we see that $\mathrm{Sh}^+_{K^N}$ is defined over the maximal abelian extension E^p/E unramified away from p for all $p \nmid N$. By the identification $\bigcap_{p \nmid N} E^p = E_N$ inside E^{ab}, we deduce that $\mathrm{Sh}^+_{K^N}$ is defined over E_N. □

2.5.2. Let $q \nmid N$ be a prime, and recall the following groups which are used to construct Kisin's integral models, and the following results from [Kisin 2010, 3.3]. We adopt the usual notations where $G^{\mathrm{ad}}(\mathbb{Q})^+$ is the intersection of $G^{\mathrm{ad}}(\mathbb{Q})$ with the connected component $G^{\mathrm{ad}}(\mathbb{R})$, $G(\mathbb{Q})_+$ the inverse image of $G^{\mathrm{ad}}(\mathbb{Q})^+$ in $G(\mathbb{Q})$, $G^{\mathrm{ad}}(\mathbb{Z}_{(q)})^+ = G^{\mathrm{ad}}(\mathbb{Q})^+ \cap G^{\mathrm{ad}}(\mathbb{Z}_{(q)})$ and $G(\mathbb{Z}_{(q)})_+ = G(\mathbb{Q})_+ \cap G(\mathbb{Z}_{(q)})_+$.

We have

$$\mathscr{A}_q(G) := G(\mathbb{A}^{\infty,q})/\overline{Z(\mathbb{Z}_{(q)})} *_{G(\mathbb{Z}_{(q)})_+/Z(\mathbb{Z}_{(q)})} G^{\mathrm{ad}}(\mathbb{Z}_{(q)})^+$$

which acts on $\mathrm{Sh}_{K_q} = \varprojlim_{K^q} \mathrm{Sh}_{G(\mathbb{Z}_q)K^q}$, and the subgroup

$$\mathscr{A}_q^\circ(G) := \overline{G(\mathbb{Z}_{(q)})_+}/\overline{Z(\mathbb{Z}_{(q)})} *_{G(\mathbb{Z}_{(q)})_+/Z(\mathbb{Z}_{(q)})} G^{\mathrm{ad}}(\mathbb{Z}_{(q)})^+$$

which acts on a connected component $\mathrm{Sh}^+_{K_q}$. It follows from the argument in [Kisin 2010, 3.3.7] that this is precisely the subgroup sending $\mathrm{Sh}^+_{K_q}$ into itself.

2.5.3. We make the following new remarks. The construction of twisting abelian varieties by a Z-torsor from [Kisin 2010, §3] can be carried out by twisting by a $Z^{\mathrm{der}} = Z(G^{\mathrm{der}})$-torsor instead. Indeed, given $\gamma \in G^{\mathrm{ad}}(\mathbb{Q})^+$ we may take $\mathscr{P}$ to be the fibre of γ along $G^{\mathrm{der}} \to G^{\mathrm{ad}}$. It will suffice to check the following.

Proposition 2.5.4. (1) *With notation as above, if $\mathscr{P}'$ is the fibre of γ along $G \to G^{\mathrm{ad}}$ then $\mathscr{P}' = \mathscr{P} \times^{Z^{\mathrm{der}}} Z$.*

(2) *If V is a $\mathbb{Q}$-vector space with an $\mathbb{O}_Z$-comodule action, $\mathscr{P}$ and $\mathscr{P}'$ as above, there is a natural isomorphism*

$$(V \otimes_{\mathbb{Q}} \mathbb{O}_{\mathscr{P}})^{Z^{\mathrm{der}}} \xrightarrow{\sim} (V \otimes_{\mathbb{Q}} \mathbb{O}_{\mathscr{P}'})^{Z}.$$

Proof. For (1) we can define (using $\mathscr{P}' \subset G$) a map

$$\mathscr{P} \times^{Z^{\mathrm{der}}} Z \ni (p, z) \mapsto p.z \in \mathscr{P}'$$

which is obviously an isomorphism of Z-torsors and proves the claim.

Part (2) can be seen easily by appealing to the general Tannakian framework that Z-torsors over $\operatorname{Spec}\mathbb{Q}$ correspond to fibre functors $\omega : \mathrm{Rep}_{\mathbb{Q}} Z \to \mathrm{Vec}_{\mathbb{Q}}$, and that $\mathscr{P} \times^{Z^{\mathrm{der}}} Z = \mathscr{P}'$ implies that we can factor $\omega_{\mathscr{P}'}$ as

$$\omega_{\mathscr{P}'} : \mathrm{Rep}_{\mathbb{Q}} Z \xrightarrow{\mathrm{Res}} \mathrm{Rep}_{\mathbb{Q}} Z^{\mathrm{der}} \xrightarrow{\omega_{\mathscr{P}}} \mathrm{Vec}_{\mathbb{Q}}.$$

Writing out what this statement actually means algebraically (and writing an arbitrary $\mathbb{O}_Z$-comodule as a filtered colimit of finite dimensional ones), we recover (2). □

This has the technical advantage given by the following lemma. We thank Kestutis Cesnavicius for pointing out to us that the analogous result from [Kisin 2010, 3.4.8] where $G = Z$ is a general group of multiplicative type is false over the base $\mathbb{Z}[1/N]$ in general.

Lemma 2.5.5. *Let R be an integrally closed domain with fraction field K, and G/R a finite group scheme. Then a torsor T/R for G is trivial if and only if T_K is a trivial G_K-torsor.*

Proof. Since G/R is finite, it is proper, which as a property stable under fpqc descent is inherited by T, and so $T(R) = \bigcap_{R_v \subset K} T(R_v) = T(K)$. In particular the latter is nonempty if and only if the former is. □

2.5.6. With these remarks in mind, we can rewrite the action of $\mathscr{A}_q(G)$ on the integral model $\mathscr{S}_{K_q}/\mathbb{O}_{E,v}$ explicitly following [Kisin 2010, 3.4.5]. A point $x \in \mathscr{S}_{K_q}(T)$ gives rise to a triple (A, λ, ϵ^q) where A/T is an abelian scheme up to prime to q-isogeny, λ a weak polarisation of A and ϵ^q a section of $\Gamma(T, \underline{\mathrm{Isom}}(V_{\mathbb{A}^{\infty,q}}, \hat{V}^q(A)))$.

By [Kisin 2010, 3.4.5] and our previous remarks, if we take $(h, \gamma^{-1}) \in \mathscr{A}_q(G)$ and x associated to the triple (A, λ, ϵ^q) then the triple associated to $x.(h, \gamma^{-1})$ is isogenous to

$$(A^{\mathscr{P}}, \lambda^{\mathscr{P}}, \epsilon^{q,\mathscr{P}} \circ \tilde{\gamma} h \tilde{\gamma}^{-1})$$

where $\mathscr{P}$ is the torsor for $Z^{\mathrm{der}} \subset G^{\mathrm{der}}$ given by the fibre of γ, $\tilde{\gamma}$ is an element of $G^{\mathrm{der}}(F)$ for some finite Galois extension $F/\mathbb{Q}$ mapping to γ under $G^{\mathrm{der}}(F) \to G^{\mathrm{ad}}(F)$, and the notations $A^{\mathscr{P}}, \lambda^{\mathscr{P}}, \epsilon^{q,\mathscr{P}}$ are the "twists by $\mathscr{P}$" as defined in [Kisin 2010, 3.1.3].

2.5.7. We introduce the notation $\mathbb{Z}^N := \mathbb{Z}[1/N]$, to provide a slight simplification in situations where the notation quickly becomes messy. Let (where plus notation denotes intersection with $G(\mathbb{R})_+$,$G^{\mathrm{ad}}(\mathbb{R})^+$ as usually defined, and overline notation denotes closures in $\prod_{p|N} G(\mathbb{Q}_p)$)

$$\mathscr{A}^N(G) := \frac{\prod_{p|N} G(\mathbb{Q}_p)}{\overline{Z(\mathbb{Z}^N)}} *_{G(\mathbb{Z}^N)_+/Z(\mathbb{Z}^N)} G^{\mathrm{ad}}(\mathbb{Z}^N)^+,$$

$$\mathscr{A}^{N,\circ}(G) := \frac{\overline{G(\mathbb{Z}^N)_+}}{\overline{Z(\mathbb{Z}^N)}} *_{G(\mathbb{Z}^N)_+/Z(\mathbb{Z}^N)} G^{\mathrm{ad}}(\mathbb{Z}^N)^+,$$

and (where here overline notation means closures in $G(\mathbb{A}^{\infty,q})$)

$$\tilde{\mathscr{A}}_q^N(G) := \frac{\prod_{p|N} G(\mathbb{Q}_p) \times \prod_{p\nmid qN} G(\mathbb{Z}_p)}{\overline{Z(\mathbb{Z}^N)}} *_{G(\mathbb{Z}^N)_+/Z(\mathbb{Z}^N)} G^{\mathrm{ad}}(\mathbb{Z}^N)^+ \subset \mathscr{A}_q(G).$$

Let $K^N = \prod_{p\nmid N} G(\mathbb{Z}_p)$, $K^{Nq} = \prod_{p\nmid Nq} G(\mathbb{Z}_p)$ and

$$\Delta^N = \mathrm{Ker}(\mathscr{A}^{N,\circ}(G) \to \mathscr{A}^{N,\circ}(G^{\mathrm{ad}})).$$

First, a group theoretic lemma following Deligne's Corvallis paper.

Lemma 2.5.8. (1) *The group $\mathscr{A}^{N,\circ}(G)$ is canonically the completion of $G^{\mathrm{ad}}(\mathbb{Z}^N)^+$ with respect to the topology generated by the images of congruence subgroups of G^{der} the form $K_N \times \prod_{p\nmid N} G^{\mathrm{der}}(\mathbb{Z}_p)$ as K_N varies. In particular $\mathscr{A}^{N,\circ}(G^{\mathrm{der}}) := \mathscr{A}^{N,\circ}(G)$ canonically depends only on G^{der}.*

(2) *We can naturally identify*

$$\mathscr{A}^{N,\circ}(G^{\mathrm{der}}) = \overline{G^{\mathrm{der}}(\mathbb{Z}^N)_+} *_{G^{\mathrm{der}}(\mathbb{Z}^N)_+} G^{\mathrm{ad}}(\mathbb{Z}^N)^+$$

(*where the closure is taken in* $\prod_{p|N} G^{\mathrm{der}}(\mathbb{Q}_p)$).

Proof. For (1) we have a natural inclusion $G^{\mathrm{ad}}(\mathbb{Z}^N)^+ \subset \mathscr{A}^{N,\circ}(G)$, and it is easily checked that the image is dense. Moreover, a neighbourhood of the identity is $\overline{G(\mathbb{Z}^N)_+}/\overline{Z(\mathbb{Z}^N)}$ whose topology is generated by that of the congruence subgroups of G with fixed hyperspecial level away from N. But by [Deligne 1979, 2.0.13] this topology is the same as that generated by the congruence subgroups of G^{der} with fixed hyperspecial level away from N. Finally this neighbourhood of the identity is obviously complete, so we are done. From this description (2) follows immediately. □

Our key result is the following, whose proof follows that of [Kisin 2010, 3.4.6].

Proposition 2.5.9. *The group $\mathscr{A}^N(G)$ acts naturally on the integral canonical model $\mathscr{S}_{K^N}$, and Δ^N acts freely.*

Proof. By the extension property, the first part can be checked on the generic fibre, where it follows from the self evident isomorphism

$$\tilde{\mathscr{A}}_q^N(G)/K^{Nq} \xrightarrow{\sim} \mathscr{A}^N(G).$$

This (together with the compatibility with Kisin's construction) gives us the additional information that for $v \mid q$ and $q \nmid N$, the action on $\mathscr{S}_{K^N,v}$ can be described on the level of triples (A, λ, ϵ^q) where ϵ^q is now given modulo K^{Nq}.

Take $(h, \gamma^{-1}) \in \Delta^N \subset \overline{G^{\mathrm{der}}(\mathbb{Z}^N)_+} *_{G^{\mathrm{der}}(\mathbb{Z}^N)_+} G^{\mathrm{ad}}(\mathbb{Z}^N)^+$, and $\tilde{\gamma} \in G^{\mathrm{der}}(F)$ mapping to $\gamma \in G^{\mathrm{ad}}(F)$ with $F/\mathbb{Q}$ finite Galois. We also let $\mathscr{P}$ denote the Z^{der}-torsor of elements of G^{der} mapping to γ. Since $(h, \gamma^{-1}) \in \Delta^N$ we see that $h\tilde{\gamma}^{-1} \in \prod_{p|N} Z^{\mathrm{der}}(\mathbb{Q}_p \otimes F)$.

Suppose $x \in \mathscr{S}_{K^N}(\kappa)$ for some algebraically closed field κ of characteristic q or 0, that $x.(h, \gamma^{-1}) = x$ and associated to x is the triple $(A, \lambda, \epsilon^q/K^{qN})$. We need to show that $(h, \gamma^{-1}) = 1$.

As in the proof of [Kisin 2010, 3.4.6] we can find a unique quasiisogeny $\alpha : A \to A^{\mathscr{P}}$ such that

$$\begin{array}{ccccc}
V \otimes \mathbb{A}^{\infty,q} \otimes F & \xrightarrow{\tilde{\gamma} h \tilde{\gamma}^{-1}} & V \otimes \mathbb{A}^{\infty,q} \otimes F & \xrightarrow{\tilde{\gamma}^{-1}} & V \otimes \mathbb{A}^{\infty,q} \otimes F \\
\Big\downarrow{\scriptstyle \epsilon^q} & & \Big\downarrow{\scriptstyle \epsilon^{q,\mathscr{P}}} & & \Big\downarrow{\scriptstyle \epsilon^q} \\
\hat{V}^q(A) \otimes F & \xrightarrow[\alpha_* \otimes 1]{} & \hat{V}^q(A^{\mathscr{P}}) \otimes F & \xrightarrow[\iota_{\tilde{\gamma}}]{} & \hat{V}^q(A) \otimes F.
\end{array}$$

commutes. This demonstrates that also $h\tilde{\gamma}^{-1} \in \mathrm{Aut}_{\mathbb{Q}}(A) \otimes F$. But the intersection of $\prod_{p|N} Z^{\mathrm{der}}(\mathbb{Q}_p \otimes F)$ and $\mathrm{Aut}_{\mathbb{Q}}(A) \otimes F$ inside $\prod_{p|N} \mathrm{Aut}_{\mathbb{Q}}(A)(\mathbb{Q}_p \otimes F)$ is $Z^{\mathrm{der}}(F)$, so we conclude that

$$h\tilde{\gamma}^{-1} \in Z^{\mathrm{der}}(F).$$

As in [Kisin 2010, 3.4.6] this demonstrates that $\mathscr{P}$ is trivial as a $Z^{\mathrm{der}}_{\mathbb{Q}}$-torsor over $\mathbb{Q}$. But Z^{der} is a finite group, Lemma 2.5.5 implies that $\mathscr{P}_{\mathbb{Z}[1/N]}$ (the torsor given by the inverse image of γ in $G^{\mathrm{der}}_{\mathbb{Z}[1/N]}$) is a trivial $\mathbb{Z}[1/N]$-torsor, so we can assume $\tilde{\gamma} \in G^{\mathrm{der}}(\mathbb{Z}[1/N])_+$ and replacing h by $h\tilde{\gamma}^{-1}$, assume $\gamma = 1$. Moreover, we may take $F = \mathbb{Q}$, and so we deduce $h \in Z^{\mathrm{der}}(\mathbb{Q}) = Z^{\mathrm{der}}(\mathbb{Z}[1/N])$: that is to say, it is trivial as an element of Δ^N, as required. □

We also need the following ingredient.[6]

Lemma 2.5.10. *The group Δ^N is finite.*

Proof. Let $\rho : G^{\mathrm{der}} \to G^{\mathrm{ad}}$ be the usual finite isogeny. By the discussion in [Deligne 1979, 2.0] there is a diagram with exact rows.

$$\begin{array}{ccccc}
\overline{G^{\mathrm{der}}(\mathbb{Z}^N)_+} & \longrightarrow & \mathscr{A}^{N,\circ}(G^{\mathrm{der}}) & \longrightarrow & G^{\mathrm{ad}}(\mathbb{Z}^N)^+/\rho G^{\mathrm{der}}(\mathbb{Z}^N)_+ \\
\downarrow & & \downarrow & & \downarrow \\
\overline{G^{\mathrm{ad}}(\mathbb{Z}^N)^+} & \longrightarrow & \mathscr{A}^{N,\circ}(G^{\mathrm{ad}}) & \longrightarrow & 1
\end{array}$$

Considering the kernels of the vertical maps, this puts Δ^N in an exact sequence between a subgroup of $Z^{\mathrm{der}}(\mathbb{A}_N)$ and the group $G^{\mathrm{ad}}(\mathbb{Z}^N)^+/\rho G^{\mathrm{der}}(\mathbb{Z}^N)_+$. The former as a product of finite groups is visibly finite. The latter group is a subgroup of $G^{\mathrm{ad}}(\mathbb{Z}^N)/\rho G^{\mathrm{der}}(\mathbb{Z}^N)$, which is in turn a subgroup of $H^1_{\mathrm{fppf}}(\mathbb{Z}^N, Z^{\mathrm{der}})$, so it suffices to check that this is finite.

Since Z^{der} is a finite group of multiplicative type, we may find a finite Zariski cover U_i of $\operatorname{Spec} \mathbb{Z}[1/N]$ and finite étale covers $V_i \to U_i$ such that $Z^{\mathrm{der}}|_{V_i}$ is isomorphic to a product of split finite multiplicative groups μ_k. By the Čech to derived functor spectral sequence for this cover, we may reduce our claim to checking that for L a number field and k, M positive integers, the groups $H^1_{\mathrm{fppf}}(\mathbb{O}_L[1/M], \mu_k)$ are finite. But the Kummer exact sequence gives an exact sequence

$$1 \to \mathbb{O}_L\big[\tfrac{1}{M}\big]^*/\mathbb{O}_L\big[\tfrac{1}{M}\big]^{*k} \to H^1_{\mathrm{fppf}}\big(\mathbb{O}_L\big[\tfrac{1}{M}\big], \mu_k\big) \to \operatorname{Pic}\big(\mathbb{O}_L\big[\tfrac{1}{M}\big]\big),$$

and both the outer terms are finite by classical algebraic number theory. □

2.5.11. We now set out to prove the main theorem Theorem 2.2.1 in the abelian type case. Let $(G_2, \mathfrak{X}_2)$ be a Shimura datum of abelian type, with $G_2/\mathbb{Z}[1/N]$ reductive. We first need to relate it to a datum of Hodge type, modifying [Kisin 2010, 3.4.13] slightly.

Lemma 2.5.12. *Let $(H, \mathfrak{Y})$ be a Shimura datum of abelian type with H adjoint. Then there exists a central isogeny $H' \to H$ such that whenever $(G, \mathfrak{X})$ is of Hodge type with $(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}}) \cong (H, \mathfrak{Y})$ then G^{der} is a quotient of H'.*

[6] Note that we believe the corresponding result of [Kisin 2010] is not true, so a result like this is perhaps also needed to deduce the extension property in that case.

Assume that H is quasisplit and unramified at all $p\nmid N$. Then there exists a Shimura datum $(G, \mathfrak{X})$ of Hodge type such that $(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}}) \cong (H, \mathfrak{Y})$, $G^{\mathrm{der}} = H'$ and G is quasisplit and unramified at all $p\nmid N$.

Proof. Most of this is proved in [Kisin 2010, 3.4.13]; the only thing to check is that we can arrange for G to be quasisplit and unramified at all $p\nmid N$. But since H has this property, clearly H' does, and so it suffices to control the centre, whose ramification is given by the totally imaginary quadratic extension L/F of [Deligne 1979, 2.3.10] which can be chosen arbitrarily. In particular, we may take any prime $q \mid N$, construct L by adjoining a q-th (or 4th if $q = 2$) root of unity and passing to a quadratic subfield. Then note that L/F is unramified at all primes $v\nmid q$, in particular at places over $p\nmid N$. □

Lemma 2.5.13. *Suppose we are given G and G_2, reductive over $\mathbb{Q}$ and unramified away from N, and a central isogeny $f : G^{\mathrm{der}} \to G_2^{\mathrm{der}}$. Suppose we are also given a reductive model $\mathcal{G}_2/\mathbb{Z}[1/N]$ of G_2.*

Then there exists a reductive model $\mathcal{G}/\mathbb{Z}[1/N]$ of G such that f extends to

$$f : \mathcal{G}^{\mathrm{der}} \to \mathcal{G}_2^{\mathrm{der}}.$$

Proof. We do the usual patching argument. Take any integral model $\mathcal{G}/\mathbb{Z}[1/N]$ and note that there will be some M such that $\mathcal{G}[1/M]$ is reductive and $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ extends to $\mathcal{G}^{\mathrm{der}}[1/M] \to \mathcal{G}_2^{\mathrm{der}}[1/M]$.

By Proposition 2.4.2 we will be done if for every $p \mid M$ and $p\nmid N$ we can find $G_p/\mathbb{Z}_p$ a reductive model such that f extends to $G_p^{\mathrm{der}} \to \mathcal{G}_{2,\mathbb{Z}_p}^{\mathrm{der}}$. But this is the case by the argument of [Kisin 2010, 3.4.14]. □

2.5.14. Now given our $(G_2, \mathfrak{X}_2)$ of abelian type, it gives rise to an adjoint Shimura datum, which by Lemma 2.5.12 is covered by $(G, \mathfrak{X})$ of Hodge type and by Lemma 2.5.13 we may take $G/\mathbb{Z}[1/N]$ reductive and $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ a central isogeny inducing a morphism $(G^{\mathrm{der}}, \mathfrak{X}^+) \to (G_2^{\mathrm{der}}, \mathfrak{X}_2^+)$ of connected Shimura data. Let $E = E(G, \mathfrak{X}) \subset \overline{\mathbb{Q}}$. Since for every $p\nmid N$, G and G_2 split over an unramified extension of p, we deduce that $E/\mathbb{Q}$ is unramified at all $p\nmid N$, and by Lemma 2.5.1 we see their connected Shimura varieties at levels $K^N := G(\hat{\mathbb{Z}}^N)$ and $K_2^N := G_2(\hat{\mathbb{Z}}^N)$ are defined over E_N/E, the maximal abelian extension of E unramified away from N. Let $\mathcal{O}_N$ be its ring of integers, and note that $\mathcal{O}_N[1/N]/\mathbb{Z}[1/N]$ is indétale.

Now, comparing our description (2.5.8(1)) of $\mathscr{A}^{N,\circ}(G^{\mathrm{der}})$ with Deligne's description [1979, 2.1.6] of the group acting on a connected Shimura variety, it is clear that $\mathscr{A}^{N,\circ}(G)$ acts on $\mathrm{Sh}^+_{K^N}(G, \mathfrak{X})_{E_N}$, and considering the subgroup

$$\Delta^N(G, G_2) := \mathrm{Ker}(\mathscr{A}^{N,\circ}(G^{\mathrm{der}}) \to \mathscr{A}^{N,\circ}(G_2^{\mathrm{der}})) \subset \Delta^N,$$

that the morphism $\mathrm{Sh}^+_{K^N}(G, \mathfrak{X})_{E_N} \to \mathrm{Sh}^+_{K_2^N}(G_2, \mathfrak{X}_2)_{E_N}$ is given by taking the quotient by $\Delta^N(G, G_2)$. Moreover, by the previous section we have an integral model

$\mathscr{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}_N[1/N]}$ for $\mathrm{Sh}^+_{K^N}(G, \mathfrak{X})_{E_N}$ satisfying the extension property, with a free Proposition 2.5.9 action of the finite Lemma 2.5.10 group Δ^N. We may therefore form the quotient by the finite subgroup $\Delta^N(G, G_2)$ and obtain a model $\mathscr{S}^+_{K_2^N}(G_2, \mathfrak{X}_2)_{\mathbb{O}_N[1/N]}$ for $\mathrm{Sh}^+_{K_2^N}(G_2, \mathfrak{X}_2)_{E_N}$ over $\mathbb{O}_N[1/N]$. By Lemma 2.1.5 this model enjoys the extension property, and passing to finite levels we see it is smooth.

2.5.15. Unlike at infinite or K_p-level, we do not know whether $\prod_{p|N} G_2(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sh}_{K^N}(G_2, \mathfrak{X}_2)_{\overline{\mathbb{Q}}})$, so to conclude our proof we need an alternative to the usual "Deligne induction." Noting that our argument up to this point holds for any model $\mathscr{G}_2/\mathbb{Z}[1/N]$ for G_2, the following lemma is enough to conclude our argument.

We will need the Shimura variety

$$\mathrm{Sh}(G, \mathfrak{X}) = \varprojlim_K \mathrm{Sh}_K(G, \mathfrak{X})$$

at full infinite level, together with the usual fixed connected component $\mathrm{Sh}^+_{\overline{\mathbb{Q}}} \subset \mathrm{Sh}(G, \mathfrak{X})_{\overline{\mathbb{Q}}}$ containing the complex point $[x, 1]$ for $x \in \mathfrak{X}^+$.

Lemma 2.5.16. *Let $(G, \mathfrak{X})$ be any Shimura datum unramified away from N, $E \supset E(G, \mathfrak{X})$, $K^N = \prod_{p\nmid N} G(\mathbb{Z}_p)$ for some choice of integral model for G, and E_N/E the maximal abelian extension unramified away from N.*

1. *Let $X^+_{K^N}$ be any component of $\mathrm{Sh}_{K^N}(G, \mathfrak{X})_{\overline{\mathbb{Q}}}$. Then $X^+_{K^N}$ is defined over E_N, and given for every choice of hyperspecial levels of the form $U^N = \prod_{p\nmid N} \mathscr{G}(\mathbb{Z}_p)$ for $\mathscr{G}/\mathbb{Z}[1/N]$ a reductive model for G a smooth integral model $\mathscr{S}^+_{U^N}/\mathbb{O}_N[1/N]$ for $\mathrm{Sh}^+_{U^N, E_N}$ with the extension property, we can construct a smooth integral model for $X^+_{K^N, E_N}$ with the extension property.*
2. *Given any smooth integral model for every $X^+_{K^N, E_N}$ with the extension property, their disjoint union gives a smooth integral canonical model for Sh_{K^N, E_N} which descends to $\mathbb{O}_E[1/N]$ and to which the $\mathscr{A}^N(G)$ action extends.*

Proof. For (1), let $\pi : \mathrm{Sh}(G, \mathfrak{X})_{\overline{\mathbb{Q}}} \to \mathrm{Sh}_{K^N}(G, \mathfrak{X})_{\overline{\mathbb{Q}}}$ be the canonical projection, and take $a \in G(\mathbb{A}^\infty)$ such that $\pi(\mathrm{Sh}^+ .a) = X^+_{K^N}$. Note that this a descends to an identification

$$\mathrm{Sh}_{K^N}(G, \mathfrak{X})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} \mathrm{Sh}_{a^{-1}K^N a}(G, \mathfrak{X})_{\overline{\mathbb{Q}}}$$

defined over E and under which $X^+_{K^N}$ is identified with $\mathrm{Sh}^+_{a^{-1}K^N a, \overline{\mathbb{Q}}}$. Since the latter is defined over E_N by Lemma 2.5.1, the former must be also. Moreover, since $a_p \in G(\mathbb{Z}_p)$ for all but finitely many p, and in all other cases we are taking the conjugate of a hyperspecial subgroup, which are always hyperspecial and so have local reductive models, by Proposition 2.4.2 there exists a reductive model $\mathscr{G}/\mathbb{Z}[1/N]$ for G giving rise to the level $a^{-1}K^N a$. Hence by hypothesis we have a smooth integral model for $\mathrm{Sh}^+_{a^{-1}K^N a, E_N}$ with the extension property. Composing with the isomorphism induced by a gives the required model for $X^+_{K^N, E_N}$.

For (2), we assume we are given for each $X^+_{K^N,E_N}$ some smooth integral model $\mathscr{X}^+_{K^N}$ with the extension property. Letting $\mathscr{S}_{K^N,\mathbb{O}_N[1/N]}$ together with

$$\iota : \mathscr{S}_{K^N,\mathbb{O}_N[1/N]} \otimes_{\mathbb{O}_N[1/N]} E_N \xrightarrow{\sim} \mathrm{Sh}_{K^N,E_N}$$

be their disjoint union, note that it still has the extension property. In particular the $\mathrm{Gal}(E_N/E)$-action on Sh_{K^N,E_N} extends to $\mathscr{S}_{K^N,\mathbb{O}_N[1/N]}$, and since E_N/E is unramified away from N this gives an étale descent datum from $\mathbb{O}_N[1/N]$ to $\mathbb{O}_E[1/N]$. Thus we may descend our model to $\mathscr{S}_{K^N}/\mathbb{O}_E[1/N]$ and by Lemma 2.1.3 this model still has the extension property. The $\mathscr{A}^N(G)$-action also extends to $\mathscr{S}_{K^N,\mathbb{O}_N[1/N]}$ by the extension property, and commutes with the $\mathrm{Gal}(E_N/E)$-action, so it descends, and for any $K_N \subset \prod_{p|N} G(\mathbb{Q}_p)$ we have that $(\mathscr{S}_{K^N}/K_N)\otimes\mathbb{O}_N[1/N]$ is smooth, which is a property stable by fpqc descent and allows us to see that $\mathscr{S}_{K^N}$ is a smooth canonical model. □

3. Automorphic vector bundles and filtered G-bundles

3.1. *Review of characteristic zero.* We sketch the main results of [Milne 1990, III], on which our results will build.

3.1.1. Let $(G, \mathfrak{X})$ be a Shimura datum with reflex field E, and $\mu : \mathbb{G}_{m,\bar{E}} \to G_{\bar{E}}$ a Hodge cocharacter of $\mathfrak{X}$. Then we can form the *compact dual* Gr_μ which represents the following functor. Fix a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ and tensors $s_\alpha \in V^\otimes$ such that G is exactly the subgroup fixing these tensors, and note that μ induces a filtration $\mathrm{Fil}^\bullet \subset V \otimes \bar{E}$. For $\pi : S \to \operatorname{Spec} \mathbb{Q}$ we let $\mathscr{V}_S := \pi^* V$ be the constant vector bundle, which also carries tensors $s_{\alpha,S} = \pi^* s_\alpha$. Then as a functor on E-schemes

$$\mathrm{Gr}_\mu(S) = \big\{\text{filtrations } \mathscr{F}^\bullet \text{ of } \mathscr{V}_S \text{ such that } (V_{\bar{s}}, \mathrm{Fil}^\bullet_{\bar{s}}, s_\alpha) \cong (\mathscr{V}_{S,\bar{s}}, \mathscr{F}^\bullet_{\bar{s}}, s_{\alpha,S}) \\ \text{for each geometric point } \bar{s} \in S\big\}.$$

This depends only on the conjugacy class of μ and is representable as an E-scheme. Moreover by construction it carries an algebraic action of G_E, which we write as a right action

$$\mathrm{Gr}_\mu(S) \times G(S) \ni (\mathscr{F}^\bullet, g) \mapsto g^{-1}(\mathscr{F}^\bullet) \in \mathrm{Gr}_\mu(S).$$

3.1.2. Let $Z_{nc} \subset Z(G)$ be the largest subtorus of $Z(G)$ split over $\mathbb{R}$ but with no subtorus split over $\mathbb{Q}$, and $G^c = G/Z_{nc}$. Then there is a G^c-torsor $P = P(G, \mathfrak{X})$ over $\mathrm{Sh}(G, \mathfrak{X})$ with an equivariant $G(\mathbb{A}^\infty)$-action,[7] which can be easily defined analytically over $\mathbb{C}$ and by [Milne 1990, III,4.3] admits a canonical model (in Milne's sense) over E, together with an integrable connection with regular singularities at infinity.

[7]It is natural to ask whether this extends to an equivariant action of Deligne's extension $G(\mathbb{A}^\infty)/\overline{Z(\mathbb{Q})} *_{G(\mathbb{Q})_+/Z(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+$: in fact this group does act but not quite in a way that commutes with the algebraic G^c-action, as we shall later see.

Moreover $P(G, \mathfrak{X})$ has a G-action, via $G \to G^c$, and there is a G-equivariant map [Milne 1990, III,4.6] $\gamma : P(G, \mathfrak{X}) \to \mathrm{Gr}_\mu$ which complex analytically is given by the Hodge filtration coming from $\mathfrak{X}$, but is algebraic and descends to E.

To begin discussing any functoriality of this construction we need a lemma.

Lemma 3.1.3. *Let $f : (G_1, \mathfrak{X}_1) \to (G_2, \mathfrak{X}_2)$ be a morphism of Shimura data. Then there is an induced map*

$$G_1^c \to G_2^c.$$

Proof. This comes down to showing that $f(Z(G_1)_{nc}) \subset Z(G_2)$. Suppose for contradiction we have $\mathbb{R}^* \cong \Psi \subset Z(G_1)(\mathbb{R})$ with $f(\Psi)$ intersecting trivially with $Z(G_2)(\mathbb{R})$. Let $h \in \mathfrak{X}_1$, and recall that $\operatorname{ad} f(h(i))$ is a Cartan involution acting on G_2^{ad}, so the real group

$$H = \{g \in G_2^{\mathrm{ad}}(\mathbb{C}) : f(h(i))gf(h(i))^{-1} = \bar{g}\}$$

is compact. On the other hand for any $\psi \in \Psi$ we have

$$h(i)\psi h(i)^{-1} = \psi = \overline{\psi}$$

since ψ is central in G_1, $h(i) \in G_1(\mathbb{R})$, and ψ is real. Hence, we have an embedding of $\Psi \cong \mathbb{R}^*$ into the compact group $H(\mathbb{R})$, which is absurd. □

Note that there is no such functoriality for general group morphisms. For example letting F be a totally real field of degree d acting on itself by multiplication we get a morphism

$$F^\times \hookrightarrow \mathrm{GL}_d$$

failing to have the required property for all $d > 1$.

3.1.4. Automorphic vector bundles are typically parametrised by complex representations of the parabolic subgroup P_μ associated with the Hodge cocharacter μ in the usual fashion (for example one may define $P_\mu \subset G_\mathbb{C}$ as the subgroup preserving the filtrations μ induces on $\operatorname{Rep} G_\mathbb{C}$). By a complex analytic interpretation of the Grassmannian it is easy to show these are in correspondence with $G_\mathbb{C}$-equivariant vector bundles on $\mathrm{Gr}_{\mu,\mathbb{C}}$. We therefore take as our input data a G-equivariant vector bundle $\mathcal{J}$ on Gr_μ defined over L/E some number field, and we assume the G-action factors through G^c.[8]

Given this data, we can pull it back along γ to get a $G(\mathbb{A}^\infty) \times G^c$-equivariant vector bundle on $P(G, \mathfrak{X})_L$ and therefore a $G(\mathbb{A}^\infty)$-equivariant vector bundle $\mathcal{V}(\mathcal{J})$ on $\mathrm{Sh}(G, \mathfrak{X})_L$. This is the construction of canonical models for automorphic vector bundles we seek to perform integrally.

[8]This is a reasonable condition to impose since $Z \subset P_\mu$ acts trivially on Gr_μ.

3.1.5. We need a basic functoriality property for which we could not find a direct reference but which follows easily from Milne's definition together with the map on complex points induced by $\mathfrak{X} \times G(\mathbb{A}^\infty) \times G^c(\mathbb{C}) \to \mathfrak{X}' \times G'(\mathbb{A}^\infty) \times G'^c(\mathbb{C})$ which we note relies on Lemma 3.1.3.

Lemma 3.1.6. *Let $f : (G, \mathfrak{X}) \to (G', \mathfrak{X}')$ be a morphism of Shimura data, μ and μ' Hodge cocharacters of $\mathfrak{X}$ and $\mathfrak{X}'$, respectively. Then there is a diagram*

$$\begin{array}{ccccc} \mathrm{Sh}(G, \mathfrak{X}) & \longleftarrow & P(G, \mathfrak{X}) & \longrightarrow & \mathrm{Gr}_\mu \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sh}(G', \mathfrak{X}') & \longleftarrow & P(G', \mathfrak{X}') & \longrightarrow & \mathrm{Gr}_{\mu'} \end{array}$$

defined over $E(G, \mathfrak{X})$ which is G-equivariant and $G(\mathbb{A}^\infty)$-equivariant in the obvious senses.

3.1.7. In the case where $G = T$ a torus we also need the following, the main content of which is due to Blasius. Let p be a prime containing a place v of $E \supset E(T, h)$, ω_{et} the usual fibre functor giving étale local systems on the Shimura variety, and fix $\overline{E}_v$ an algebraic closure of E_v letting $\Gamma_{E_v} := \mathrm{Gal}(\overline{E}_v/E_v)$. Consider the fibre functor coming from p-adic Hodge theory

$$\omega_{v,\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(T^c) \ni V \mapsto (\omega_{\mathrm{et}}(V) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_{E_v}} \in \mathrm{Vec}_{E_v} .$$

Proposition 3.1.8. *Suppose T is split by a CM field, and let $X = \mathrm{Spec}\, E \subset \mathrm{Sh}_U(T, h)$ be a component of the Shimura variety for $U \subset T(\mathbb{A}^\infty)$ open compact. There is a natural isomorphism between $\omega_{P,X,v} : V \mapsto (V \times^{T^c} P_U(T, h)|_X) \otimes_E E_v$ and $\omega_{v,\mathrm{dR}}$.*

Proof. By Lemma 3.1.6 it suffices to do the case $T = T^c$. We observe that any cocharacter $\mu : \mathbb{G}_{m,E} \to T^c_E$ has weight defined over $\mathbb{Q}$. Indeed, by definition $X_*(T^c)_{\mathbb{Q}}$ is the summand of $X_*(T)_{\mathbb{Q}}$ on which either Galois acts trivially or complex conjugation acts via -1, so for any $\mu \in X_*(T^c)$, $\mu + \mu^c$ lands in the summand on which Galois acts trivially.

Combining this with fact that T hence T^c is split by a CM field, the induced Shimura datum (T^c, h^c) is of CM type, and thus admits a characterisation as a moduli space of CM motives M, which we take as those for which the Betti fibre functor $\omega_B = H_B(M(\rho)) : \mathrm{Rep}_{\mathbb{Q}}\, T \to \mathrm{Vec}_{\mathbb{Q}}$ is trivial.

Therefore a point $x \in X(\overline{\mathbb{Q}})$ has attached to it a T-valued CM motive $M : \mathrm{Rep}_{\mathbb{Q}}(T) \to (CM/\overline{\mathbb{Q}})$, and the fibre functor attached to $x^* P_U(T, h)$ is given by $\omega_P(\rho) := H_{\mathrm{dR}}(M(\rho)/\overline{\mathbb{Q}})$ Noting that for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ we have a canonical isomorphism

$$H_{\mathrm{dR}}(M^\sigma(\rho)/\overline{\mathbb{Q}}) \cong H_{\mathrm{dR}}(M(\rho)/\overline{\mathbb{Q}}) \otimes_\sigma \overline{\mathbb{Q}},$$

we see that ω_P carries a canonical descent datum to $X = \operatorname{Spec} E$ which defines $P_U(T, h)|_X$. We may also use the reciprocity law to canonically identify the p-adic étale cohomology fibre functors

$$H_{\mathrm{et}} \circ M = \omega_{\mathrm{et}} : \mathrm{Rep}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p}) \to \mathrm{Vec}_{\mathbb{Q}_p},$$

which in particular have an equivariant Γ_E-action coming from ω_{et}.

Now fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{E_v}$, and a faithful representation $T \hookrightarrow \mathrm{GL}(V)$, and s_α cycles on $V^\otimes$ fixed precisely by T. For any $M \in X(\overline{\mathbb{Q}})$, these give rise to absolute Hodge cycles on $H_B(M(V))^\otimes$, which in turn give rise to de Rham cycles $s_{\alpha,\mathrm{dR}} \in H_{\mathrm{dR}}(M(V))^\otimes$ via the Betti–de Rham comparison, and étale cycles $s_{\alpha,et} \in \omega_{\mathrm{et}}(V)^\otimes$. These $s_{\alpha,et}$ are Γ_E-invariant because the action factors

$$\Gamma_E \to U_p \subset T(\mathbb{Q}_p) \to \mathrm{GL}(\omega_{\mathrm{et}}(V))$$

by construction, and as in [Kisin 2010, 2.2.1] this implies the $s_{\alpha,\mathrm{dR}} \in \omega_{P,X,v}(V)^\otimes$ because an absolute Hodge cycle is determined by either component.

Let us fix Y/L with $\overline{\mathbb{Q}} \supset L$ a finite Galois extension of E such that $M(V)$ is realised in the cohomology of Y, and a place $w|v$ of L determining $L_w \subset \overline{E}_v$. By Blasius' theorem on de Rham cycles [1994], the p-adic Hodge theoretic comparison map

$$H_{\mathrm{et}}(M(V), \mathbb{Q}_p)^\otimes \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow{\sim} H_{\mathrm{dR}}(M(V)_{L_w}/L_w)^\otimes \otimes_{L_w} B_{\mathrm{dR}}$$

identifies $s_{\alpha,et}$ with $s_{\alpha,\mathrm{dR}}$. Unravelling the definitions, we see that

$$P_U(T, h)_L = \underline{\mathrm{Isom}}_{s_\alpha}(V_L, H_{\mathrm{dR}}(M(V))_L)$$

and

$$P_{\omega_{v,\mathrm{dR}}} \otimes L_w = \underline{\mathrm{Isom}}_{s_\alpha}(V_{L_w}, (\omega_{\mathrm{et}}(V) \otimes_{L_w} B_{\mathrm{dR}})^{\Gamma_{L_w}}).$$

Thus putting it all together we get a canonical[9] identification

$$\omega_{P,X,v} \otimes L_w \cong \omega_{v,\mathrm{dR}} \otimes L_w.$$

We conclude by checking that this descends to E_v. Indeed, Γ_{E_v} acts canonically on both sides of the p-adic comparison map compatibly, so we have an isomorphism of $\mathrm{Gal}(L_w/E_v)$-modules

$$(\omega_{\mathrm{et}}(V) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_{L_w}} \xrightarrow{\sim} H_{\mathrm{dR}}(M(V)_{L_w}/L_w) = \omega_{P,X,v}(V) \otimes_{E_v} L_w$$

and taking $\mathrm{Gal}(L_w/E_v)$-invariants therefore $\omega_{v,\mathrm{dR}}(V) = \omega_{P,X,v}(V)$. Finally, we have already observed that the $s_{\alpha,et}$ and $s_{\alpha,\mathrm{dR}}$ are Galois invariant, so these too descend to E_v. □

[9]The choice of s_α does not affect the identification. To see this note that it does not change under adding more tensors so given two sets of choices one can just compare both with the union.

3.2. ***Moduli of μ-filtrations of G.*** In §3.2 and §3.3 we make a brief digression from the theory of Shimura varieties to discuss Grassmannians and filtrations more generally. Let R be a domain with fraction field K of characteristic zero, R'/R an étale cover, and G/R a connected reductive group.

3.2.1. Suppose we are given a cocharacter $\mu : \mathbb{G}_{m,R'} \to G_{R'}$. This cocharacter gives us a parabolic subgroup $P_\mu \subset G_{R'}$, which we can view as a point $x_\mu \in \mathrm{Par}_{G/R}(R')$, where we recall [Conrad 2014, 5.2.9] that $\mathrm{Par}_{G/R}$, the functor assigning to each R'/R the set of parabolic subgroups of $G_{R'}$ defined over R', is a proper smooth scheme over R.

After making a base change $R \subset K \to \overline{K}$ we get a well-known finite decomposition

$$\mathrm{Par}_{G/R} \otimes_R \overline{K} = \mathrm{Par}_{G_{\overline{K}}/\overline{K}} = \coprod_i G_{\overline{K}}/P_i$$

where the P_i are representatives of the finitely many $\overline{K}$ conjugacy classes of parabolic subgroups of $G_{\overline{K}}$.

For $P_\mu \subset G_{R'}$ a parabolic subgroup, we denote its conjugacy class by $[P_\mu]$, in a precise sense we will soon make clear. Let us say that $[P_\mu]$ *is defined over R* if there is a component[10] $Z_\mu \subset \mathrm{Par}_{G/R}$ defined over R such that $x_\mu \in Z_\mu(R')$ and $Z_{\mu,\overline{K}}$ is connected.

This definition in a sense is saying that "being étale locally conjugate to P_μ" is a notion that is defined over R, even if P_μ itself is not. More precisely, we have the following.

Lemma 3.2.2. *Let μ be a cocharacter as above defined over R' such that $[P_\mu]$ is defined over R. For S an R-algebra, and $P \subset G_S$ a parabolic subgroup, P is étale locally conjugate to P_μ if and only if $x_P \in \mathrm{Par}_{G/R}(S)$ factors through Z_μ.*

Proof. The statement may be checked étale locally, so we may work over R', at which point by the construction of $\mathrm{Par}_{G/R}$, $G_{R'}/P_\mu \subset Z_{\mu,R'}$ is a union of components. Working over $R' \otimes_R \overline{K}$ using the fact that the fibres of Z_μ over each generic point are connected, we see that in fact $G_{R'}/P_\mu = Z_{\mu,R'}$, so the desired statement follows immediately from [Conrad 2014, 5.2.8]. □

For us an important context where the above holds will be the following.

Proposition 3.2.3. *Assume R is a Dedekind domain with $K = \mathrm{Frac}\, R$ of characteristic zero, G/R connected reductive and L/K a finite extension. Suppose $\mu : \mathbb{G}_{m,L} \to G_L$ is a cocharacter whose conjugacy class is defined over K.*

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[10] Here and in much of what follows we use "component" in the relative sense of a "component" of a morphism $X \to Y$ being a Y-subscheme $X' \subset X$ such that the preimage of any connected open subscheme $U \subset Y$ is connected.

Then there exists an étale cover R'/R and a cocharacter $\mu' : \mathbb{G}_{m,R'} \to G_{R'}$ such that if we write $R' \otimes_R K \cong \prod_i L_i$, there are embeddings $L \hookrightarrow L_i$ such that $\mu' \otimes_R K$ is $G(R' \otimes_R K)$-conjugate to $\mu \otimes_L \prod_i L_i$.

Furthermore, this μ' has the property that $[P_{\mu'}]$ is defined over R and independent of the choice of μ'.

Proof. For existence, take R'/R an étale cover which splits G and whose generic points contain L, and fix embeddings $L \hookrightarrow L_i$. Let $T \subset G_{R'}$ be a split maximal torus and let $T_\mu \subset G_{R' \otimes_R K}$ be a maximal torus containing the image of μ which perhaps after enlarging R' we may assume is also split. Then there exists $g \in G\big(\prod_i L_i\big) = \prod_i G(L_i)$ such that $gT_K g^{-1} = T_\mu$ and so

$$\mu' = g^{-1}\mu g \in \mathrm{Hom}_{R' \otimes_R K}(\mathbb{G}_m, T_K) = \mathrm{Hom}_{R'}(\mathbb{G}_m, T)$$

is a cocharacter with the desired property.

Let us next show that $[P_{\mu'}]$ is defined over R. By definition we see that x_μ and $x_{\mu'}$ lie on the same component of $\mathrm{Par}_{G/R} \otimes (R' \otimes_R K)$, whence certainly on that of $\mathrm{Par}_{G/R} \otimes R'$. But since the conjugacy class of μ is defined over K, x_μ lies on a component $Z_{\mu,K} \subset \mathrm{Par}_{G_K/K}$ which is geometrically connected. Letting Z_μ be its closure in (i.e., the corresponding component of) $\mathrm{Par}_{G/R}$, we obtain our witness to the fact that $x_{\mu'}$ is a defined over R. Also since Z_μ is determined by μ, in particular it does not depend on μ'. □

3.2.4. Let S/R be a scheme. We let Vec_S denote the category of vector bundles (projective finitely generated modules) on S, and $\mathrm{Rep}_R(G)$ the category of algebraic representations $G \to \mathrm{Aut}(V)$ for $V \in \mathrm{Vec}_R = \mathrm{Vec}_{\mathrm{Spec}\, R}$.

A *filtered bundle* over S is a vector bundle M/S together with a decreasing complete exhaustive filtration $F^\bullet \subset M$ by flat submodules such that $\mathrm{gr}^\bullet_F M := \bigoplus_p F^p/F^{p+1}$ is flat over S. These form an exact category Fil_S, allowing us to define a *filtered G-bundle* over S to be a faithful exact tensor functor

$$\mathcal{F} : \mathrm{Rep}_R(G) \to \mathrm{Fil}_S\,.$$

We say that $\mathcal{F}$ is a *filtration of G over S* (or just "filtration of G" if no confusion will arise) if its composite with the forgetful functor $\mathrm{Fil}_S \to \mathrm{Vec}_S$ is naturally isomorphic to the usual forgetful functor $\mathrm{Rep}_R(G) \to \mathrm{Vec}_R \xrightarrow{\otimes \mathcal{O}_S} \mathrm{Vec}_S$.

Let $\mu : \mathbb{G}_{m,R'} \to G_{R'}$ be a cocharacter defined over some étale cover R'/R such that $[P_\mu]$ is defined over R. It induces a grading on each $V \in \mathrm{Rep}_R(G) \otimes R'$, each of which in turn gives such V the structure of a filtered bundle, so we can define a canonical filtration of G associated with μ

$$\mathcal{F}_\mu : \mathrm{Rep}_R(G) \xrightarrow{\mathrm{Fil} \circ \mu_*} \mathrm{Fil}_R(G) \otimes R'.$$

3.2.5. Let us say that a filtration $\mathcal{F}$ of G over S is a *μ-filtration* if étale locally on S we have $\mathcal{F} \cong \mathcal{F}_\mu$. We make the necessary remark that of course given $f : S' \to S$, whenever $\mathcal{F}$ is a μ-filtration of G over S, we may take an étale cover $\{U_\alpha \to S\}$ witnessing that $\mathcal{F}$ is a μ filtration and pulling it back along f it will give an étale cover of S' witnessing that $f^* \circ \mathcal{F}$ is a μ-filtration of G over S'.

Thus we have a natural functor on R'-schemes

$$\mathrm{Gr}_{\mu,R'}(S) = \{\mu\text{-filtrations of } G \text{ over } S\}.$$

Proposition 3.2.6. *This has the following properties*:

(1) *The functor* $\mathrm{Gr}_{\mu,R'}$ *is representable by a smooth proper R'-scheme which in fact canonically descends to* Gr_μ /R.

(2) *The scheme* Gr_μ /R *comes equipped with a natural G-action.*

(3) *If R is a field*, *this agrees with the construction of Section 3.1.1.*

Proof. In the usual fashion μ determines a parabolic $P_\mu \subset G_{R'}$, and we claim that $\mathrm{Gr}_{\mu,R'} \cong G/P_\mu$ which in turn is smooth and projective by [Conrad 2014, 5.2.8], and canonically descends by the definition of $[P_\mu]$ being defined over R.

In [ibid.] it is also shown that G/P_μ represents the functor of subgroups of G which are étale locally conjugate to P_μ, so it suffices to check this coincides with our functor. Given a μ-filtration $\mathcal{F}$ over S we can define the subgroup $P_{\mathcal{F}} \subset G$ of elements which preserve $\mathcal{F}$ (if you like, acting on all representations). Since $\mathcal{F}$ is a μ-filtration, étale locally there is an identification of $\mathcal{F}$ with $\mathcal{F}_\mu$, which conjugates $P_{\mathcal{F}}$ onto P_μ. Thus $\mathcal{F} \mapsto P_{\mathcal{F}}$ gives a map $\mathrm{Gr}_\mu(S) \to G/P_\mu(S)$.

Let us construct an inverse. Given P/S étale locally conjugate to $P_{\mu,S}$, after passing to an étale cover $S' \to S$ there exists $g \in G(S')$ such that $P_{S'} = g P_{\mu,S'} g^{-1}$. In particular $P_{S'}$ is a parabolic subgroup of $G_{S'}$, and letting $\mu_P := g \circ \mu$ we see that $P_{S'}$ preserves the filtration $\mathcal{F}_P$ defined by μ_P of G over S'. We must now check this filtration is independent of the choice of g and so in particular is canonical and descends to S.

Suppose we take $h \in G(S')$ such that $h P_{\mu,S'} h^{-1} = g P_{\mu,S'} g^{-1} = P_{S'}$. Then

$$hg^{-1} P_{S'} g h^{-1} = h P_{\mu,S'} h^{-1} = P_{S'}$$

so $hg^{-1} \in N_G(P)(S') = P(S')$, where the final equality is again by [Conrad 2014, 5.2.8]. It follows that $g\mu$ and $h\mu$ induce the same filtration.

Part (2) is now obvious, since G acts by conjugation on G/P_μ, and part (3) is immediate from [Conrad 2014, 5.2.7 (1)] and an easy verification shows that the G-actions agree. □

3.2.7. In the context of Proposition 3.2.3 where we are given a conjugacy class of cocharacters $\mu : \mathbb{G}_{m,L} \to G_L$ defined over $K = \mathrm{Frac}(R)$ and deduce the existence of a μ' defined over an étale cover of R inducing a conjugacy class of parabolics defined over R and hence a $\mathrm{Gr}_{\mu'}/R$ we use the notation $\mathcal{GR}_\mu := \mathrm{Gr}_{\mu'}$ noting the canonical identification

$$\mathcal{GR}_\mu \otimes_R K \cong \mathrm{Gr}_\mu .$$

Lemma 3.2.8. *Suppose $G_1 \to G_2$ is a map of connected reductive groups over R and $\mu_1 : \mathbb{G}_m \to G_{1,R'}$ giving a conjugacy class defined over R and inducing $\mu_2 : \mathbb{G}_m \to G_{1,R'} \to G_{2,R'}$. There is a natural map*

$$\mathrm{Gr}_{\mu_1} \to \mathrm{Gr}_{\mu_2} .$$

Proof. Given S/R and $\mathcal{F} \in \mathrm{Gr}_{\mu_1}(S)$, recall that $\mathcal{F}$ is specified by a fibre functor

$$\mathrm{Rep}_R(G_1) \to \mathrm{Fil}_S .$$

Composing with the restriction map $\mathrm{Rep}_R(G_2) \to \mathrm{Rep}_R(G_1)$, we get a new fibre functor from $\mathrm{Rep}_R(G_2)$ which it is easy to check gives an element of $\mathrm{Gr}_{\mu_2}(S)$, defining the map required. □

3.3. ***Filtered G-bundles.*** Fix G, μ as above and suppose we have X/R a scheme and $P \to X$ a G-bundle on X. A *μ-filtration* of P is a G-equivariant map of R-schemes

$$\gamma : P \to \mathrm{Gr}_\mu .$$

Lemma 3.3.1. *To give a μ-filtration γ on P is to give a fibre functor*

$$\omega_P^\gamma : \mathrm{Rep}_R(G) \to \mathrm{Fil}_X$$

which étale locally is isomorphic to $\mathcal{F}_\mu$ and such that the composite with the forgetful functor

$$\mathrm{Rep}_R(G) \xrightarrow{\omega_P^\gamma} \mathrm{Fil}_X \to \mathrm{Vec}_X$$

is equal to the fibre functor ω_P defined by P.

Proof. Suppose we are given a μ-filtration $\gamma : P \to \mathrm{Gr}_\mu$ of P. Pulling back the universal μ-filtration of G, we obtain a G-equivariant μ-filtration of G over P, which descends to a μ-filtration of G over X, i.e., we obtain a fibre functor

$$\omega_P^\gamma : \mathrm{Rep}_R(G) \to \mathrm{Fil}_X .$$

Since it comes from a G-equivariant μ-filtration of G over P, whose forgetful functor to Vec_P by definition is the canonical one from $\mathrm{Rep}_R(G)$ which descends to ω_P, we see that it satisfies the condition in the lemma.

Conversely, if we are given ω_P^γ satisfying the condition, we can pull it back along $P \to X$ to obtain a G-equivariant μ-filtration on P, which is the same as a G-equivariant map $\gamma : P \to \mathrm{Gr}_\mu$. □

We also need the following criterion for extending μ-filtrations over K to μ-filtrations over R.

Lemma 3.3.2. *Suppose we are given $\mu : \mathbb{G}_{m,L} \to G_L$ for $L/K = \mathrm{Frac}(R)$ a finite extension whose conjugacy class is defined over K, X/R a scheme and $P \to X$ a G-bundle. A μ-filtration*

$$\gamma : P_K \to \mathrm{Gr}_\mu$$

extends to a G-equivariant

$$P \to \mathcal{GR}_\mu$$

if for $\rho : G \hookrightarrow \mathrm{GL}(V)$ a faithful representation on V/R finite projective, the bundle $\mathcal{V} := V \times^G P/X$ carries a filtration $\mathcal{V}^\bullet$ (making $\mathcal{V}$ into a filtered bundle in the above sense) extending that of $\mathcal{V}_K = \omega^\gamma_{P_K}(V)$.

Proof. Let R'/R be an étale cover and $\mu : \mathbb{G}_{m,R'} \to G_{R'}$ a cocharacter conjugate to μ as in Proposition 3.2.3. We let $\mathrm{Gr}^G_{\mu'}$ and $\mathrm{Gr}^{\mathrm{GL}(V)}_{\mu'}$ be the two Grassmannians formed from considering $\mu' : \mathbb{G}_m \to G$ and $\mu' : \mathbb{G}_m \to G \xrightarrow{\rho} \mathrm{GL}(V)$ respectively, both defined over R.

By Lemma 3.2.8 we obtain a map

$$\rho_* : \mathrm{Gr}^G_{\mu'} \to \mathrm{Gr}^{\mathrm{GL}(V)}_{\mu'} .$$

We claim this map is a closed immersion, and this suffices to prove the lemma because we are assuming that $\rho_*(\gamma) \in \mathrm{Gr}^{\mathrm{GL}(V)}_{\mu'} \otimes_R K$ extends to a map $\tilde{\gamma} : P \to \mathrm{Gr}^{\mathrm{GL}(V)}_{\mu'}$, and since $\mathrm{Gr}^{\mathrm{GL}(V)}_{\mu'}$ is flat that the ideal sheaf of $\mathrm{Gr}^G_{\mu'}$ is killed by $\tilde{\gamma}^*$ may be checked on the generic fibre.

Since it is clearly proper (as both the source and target are proper), it suffices to show it is a monomorphism, for which it suffices to check the functor of points is injective. But this is obvious: given two μ'-filtrations of G_T for a test scheme T, if they induce the same filtration on V_T then they are equal, since any other representation of G_T can be embedded in a tensor construction on V_T and will have to receive the induced filtration. □

4. Integral Models for the standard principal bundle

4.1. ***Breuil–Kisin modules and lattices in de Rham cohomology.*** We consider the following adaptation of the results of [Kisin 2006], as packaged in [Kisin 2010, 1.2]. Fix $F = \prod_{i=1}^s F_i$ some finite étale algebra over $\mathbb{Q}_p$, and let κ_i be the residue field of F_i, and $E_i(u)$ the monic minimal polynomial of a uniformiser ϖ_i for F_i over $W(\kappa_i)$. Consider the ring $\mathfrak{S} = \mathfrak{S}_F := \prod_i W(\kappa_i)[\![u]\!]$ and $E(u) = (E_1(u), \ldots, E_s(u))$. We equip $\mathfrak{S}$ with the Frobenius φ raising $u \mapsto u^p$ and the canonical Frobenius on each $W(\kappa_i)$.

Define the category $\mathrm{Lisse}^{\mathrm{crys}}_{\mathbb{Z}_p}(F)$ to be that of crystalline constant rank lisse $\mathbb{Z}_p$-sheaves on $\operatorname{Spec} F$, i.e., Galois-stable $\mathbb{Z}_p$-lattices in tuples of crystalline p-adic representations $(\sigma_i : \mathrm{Gal}(\overline{F_i}/F_i) \to \mathrm{GL}_n(\mathbb{Q}_p))$. Define the category $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ of $\mathfrak{S}$-*modules* to consist of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ together with a φ-semilinear isomorphism

$$1 \otimes \varphi : \varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)].$$

Theorem 4.1.1. *There is a fully faithful tensor functor*

$$\mathfrak{M} : \mathrm{Lisse}^{\mathrm{crys}}_{\mathbb{Z}_p}(F) \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}_F}$$

compatible with the formation of symmetric and exterior powers, unramified base change $F \to F'$ of finite étale algebras over $\mathbb{Q}_p$, and with the property that

$$\mathbb{D}(L) := \varphi^*\mathfrak{M}(L) \otimes_{\mathfrak{S}_F} \mathbb{O}_F \subset D_{\mathrm{dR}}(L \otimes \mathbb{Q}_p)$$

obtained by tensoring along the map $u \mapsto (\varpi_i)$ is a natural $\mathbb{O}_F$-lattice in $D_{\mathrm{dR}}(L \otimes \mathbb{Q}_p)$. Moreover, if $\mathcal{A}$ is an abelian variety over $\mathbb{O}_F$, and $L = T_p(\mathcal{A}_F)^*$, then this lattice is identified with integral de Rham cohomology

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(\mathcal{A}/\mathbb{O}_F) & \xrightarrow{\subset} & H^1_{\mathrm{dR}}(\mathcal{A}_F/F) \\ \| & & \downarrow \wr \\ \mathbb{D}(L) & \xrightarrow{\subset} & D_{\mathrm{dR}}(L \otimes \mathbb{Q}_p). \end{array}$$

Proof. The first part is just [Kisin 2010, 1.2.1] and the second follows perhaps most quickly from [Bhatt et al. 2016, 1.8 (ii)] (although since we are in the case of abelian varieties the theorem probably also can be deduced directly from the theory of Breuil and Kisin). □

4.1.2. For our application to integral models of Shimura varieties over $\mathbb{O}_E[1/N]$, we also need the following abstract lemmas. We thank one of the anonymous referees for pointing out the ideas for the argument for Lemma 4.1.3 in [Maulik 2014, 6.15].

Lemma 4.1.3. (1) *Let A be a Dedekind domain with fraction field K, X/A a smooth scheme and suppose we are given two vector bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over X together with an identification*

$$\theta : \mathcal{L}_1 \otimes_A K \xrightarrow{\sim} \mathcal{L}_2 \otimes_A K.$$

Suppose further that θ extends as an isomorphism to the formal completion of X at all maximal ideals of A. Then θ extends over X.

(2) *Let $W = W(k)$ for k a perfect field with fraction field K. Suppose X/W is a smooth scheme, with p-adic completion $\hat{X}$, and special fibre X_0, and suppose we are given vector bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ over $\hat{X}$ together with an identification*

$$\theta : \mathscr{L}_1\left[\tfrac{1}{p}\right] \xrightarrow{\sim} \mathscr{L}_2\left[\tfrac{1}{p}\right].$$

Suppose further that there is a Zariski dense subset $U_0 \subset X_0$ of the special fibre such that each $x_0 \in U_0$ admits a lift $\tilde{x}_0 \in \hat{X}(W(\kappa(x_0)))$ with the property that $\tilde{x}_0^*(\theta)$ extends to an isomorphism $\tilde{x}_0^*(\mathscr{L}_1) \xrightarrow{\sim} \tilde{x}_0^*(\mathscr{L}_2)$. Then θ extends over $\hat{X}$.

Proof. Note that in both cases the claim may be checked locally on X, so we may assume $\mathscr{L}_1, \mathscr{L}_2$ are both free and that $X = \operatorname{Spec} R$ is affine, and by picking bases and considering the matrices of both θ and θ^{-1} the question can be reduced to a question about the matrix coefficients.

For (1), we wish to show matrix values lying in $R \otimes_A K$ in fact lie in R. The matrix values all lie in $R[1/D]$ for some $D \in A$, since R is of finite type over A, and the $\mathscr{L}_i$ of finite rank. But by the hypothesis for each maximal ideal $\mathfrak{p}$ containing D, we know θ and θ^{-1} extend over $\hat{X}_{/\mathfrak{p}}$, implying that the matrix values also lie in $R \otimes_A A_{\mathfrak{p}}$, hence they lie in R as required.

For (2), letting $\hat{R}$ be the p-adic completion of R, we are required to show that a matrix coefficient $f \in \hat{R}[1/p]$ in fact lies in $\hat{R}$. Suppose for contradiction it does not, and let $r > 0$ be minimal such that $F = p^r f \in \hat{R}$. The hypothesis tells us that for any $x_0 \in U_0$ and some lift $\tilde{x}_0$ we have that $\tilde{x}_0^*(f) \in W(\kappa(x_0))$. Therefore $\tilde{x}_0^*(F) \in p^r W(\kappa(x_0))$ and in particular $x_0^*(F) = 0$, or $F \in \mathfrak{m}_{x_0} \subset R \otimes_W k$. Since the set of such x_0 is dense, and $R \otimes_W k$ a reduced algebra of finite type over a field, we deduce that the $F = 0$ restricted to the special fibre. I.e. $F = pF'$ for some $F' \in \hat{R}$. But then $p^{r-1} f = F' \in R$ giving the required contradiction. □

Lemma 4.1.4. *Let R be a PID with fraction field K, X/R a flat scheme, G/R a flat affine group scheme of finite type, and P/X_K a G_K-torsor. Suppose we have two pairs $(\mathscr{P}_i, \iota_i)$ $i = 1, 2$ where $\mathscr{P}_i$ is a G-torsor over X and $\iota_i : \mathscr{P}_i \otimes_R K \xrightarrow{\sim} P$ an isomorphism of G_K-torsors over X_K.*

Then the map $\iota_2^{-1}\iota_1 : \mathscr{P}_1 \otimes K \xrightarrow{\sim} \mathscr{P}_2 \otimes K$ extends over R if and only if for every representation $G \to \mathrm{GL}(V)$ with V/R finite free the composite isomorphism

$$\omega_{\mathscr{P}_1}(V) \otimes K \xrightarrow{\iota_{1*}} \omega_P(V_K) \xrightarrow{\iota_{2*}^{-1}} \omega_{\mathscr{P}_2}(V) \otimes K$$

identifies the lattices $\omega_{\mathscr{P}_1}(V)$ and $\omega_{\mathscr{P}_2}(V)$.

Proof. Recall the natural equivalence [Broshi 2013, 1.2] under our hypotheses between the groupoid of G_X-torsors and the groupoid of fibre functors $\operatorname{Rep}(G) \to \operatorname{Vec}(X)$, and that it is functorial in X/R. In particular, $\iota_2^{-1} \circ \iota_1$ extends over R if and only if the composite induced map $\omega_{\mathscr{P}_1} \otimes_R K \xrightarrow{\sim} \omega_{\mathscr{P}_2} \otimes_R K$ on fibre functors extends over R, and this is the case if and only if the condition on lattices holds. □

4.2. *Special type case.*

4.2.1. Let (T, h) be the Shimura datum defined by a torus split by a CM field, $E_T := E(T, h)$. Then $\mathrm{Sh}(T, h)/E_T$ is a product of algebraic field extensions, and we recall that its integral canonical model $\mathcal{S}(T, h)$ is that obtained by taking integral closures. In particular, recall that if T is unramified at all $p \nmid N$, we get a unique integral model (also abusively written) $T/\mathbb{Z}[1/N]$ and letting $K^N = \prod_{p\nmid N} T(\mathbb{Z}_p)$ we get $\mathcal{S}_{K^N}(T, h)$ indétale over $\mathbb{O}_{E_T}[1/N]$ with a natural $\prod_{p|N} T(\mathbb{Q}_p)$ action extending that on the generic fibre.

Recall that we also have $P_{K^N}(T, h) \to \mathrm{Sh}_{K^N}(T, h)$ the standard principal T^c-bundle, defined over E_T. Our aim here is to construct for it a canonical integral model. We assume henceforth that T is unramified at $p \nmid N$.

4.2.2. Suppose $\mathcal{P}/\mathcal{S}_{K^N}(T, h)$ together with $\iota : \mathcal{P}_{E_T} \xrightarrow{\sim} P_{K^N}(T, h)$ is such a model. We can study the associated fibre functor

$$\omega_{\mathcal{P}} : \mathrm{Rep}_{\mathbb{Z}[1/N]}(T^c) \ni W \mapsto W \times^{T^c} \mathcal{P} \in \mathrm{Vec}(\mathcal{S}_{K^N}(T, h)).$$

The identification ι realises the image of such as lattices

$$\omega_{\mathcal{P}}(W) \subset \omega_{P_{K^N}(T,h)}(W \otimes \mathbb{Q})$$

in the "de Rham sheaves" defined by $P_{K^N}(T, h)$.

On the other hand, for each prime $q \nmid N$, and $v|q$ a place of E_T we can let $K^{Nq} = \prod_{p\nmid Nq} T(\mathbb{Z}_p)$ and it is immediate from the setup and class field theory that the pro-étale $T^c(\mathbb{Z}_q)$-cover $\mathrm{Sh}_{K^{Nq}}(T, h) \to \mathrm{Sh}_{K^N}$ gives rise to, for each representation W_q of $T^c(\mathbb{Z}_q)$, a crystalline lisse $\mathbb{Z}_q$-sheaf $\omega_{\mathrm{et}}(W_q)$ on Sh_{K^N}. By Theorem 4.1.1 we have associated to such a datum a canonical lattice $\mathbb{D}(\omega_{\mathrm{et}}(W_q)) \subset D_{\mathrm{dR}}(\omega_{\mathrm{et}}(W_q) \otimes_{\mathbb{Z}_q} \mathbb{Q}_q)$.

Finally recall Proposition 3.1.8 which gives a natural identification

$$\theta : \omega_{P_{K^N}(T,h)}(W \otimes \mathbb{Q}) \otimes_{E_T} E_{T,v} \xrightarrow{\sim} D_{\mathrm{dR}}(\omega_{\mathrm{et}}(W_q) \otimes_{\mathbb{Z}_q} \mathbb{Q}_q).$$

We say that the model $\mathcal{P}$ is *canonical* if for every $q \nmid N$ and $W \in \mathrm{Rep}_{\mathbb{Z}[1/N]}(T)$ the two lattices $\omega_{\mathcal{P}}(W) \otimes_{\mathbb{O}_{E_T}[1/N]} \mathbb{O}_{E_T,v}$ and $\mathbb{D}(\omega_{\mathrm{et}}(W_q))$ constructed above are, under the map θ, identified.

Proposition 4.2.3. *With* (T, h) *as above*, *there exists a unique integral canonical model for* $P_{K^N}(T, h)$.

Proof. We first remark that uniqueness follows directly from (4.1.3 (1)) and Lemma 4.1.4.

For existence, let us first remark that we have the map of Shimura data $(T, h) \to (T^c, h^c)$ giving rise to a map of integral canonical models for the Shimura varieties $i : \mathcal{S}_{K^N}(T, h) \to \mathcal{S}_{K^{cN}}(T^c, h^c)$. Suppose $\mathcal{P}^c$ is an integral canonical model for $P_{K^{cN}}(T^c, h^c)$. Then $i^*\mathcal{P}^c$ is an integral canonical model for $P_{K^N}(T, h)$ because

$i^*\omega_{\mathrm{et},(T^c,h^c)} = \omega_{\mathrm{et},(T,h)}$. Since any $h : \mathbb{S} \to T^c_{\mathbb{R}}$ has weight defined over $\mathbb{Q}$, we are therefore reduced to the case where (T, h) is a CM pair (i.e., where T is split by a CM field and the weight of h is defined over $\mathbb{Q}$).

Let $T \to \mathrm{GL}(W)$ be a representation of $T/\mathbb{Z}[1/N]$. Since (T, h) is a CM pair, $\mathrm{Sh}_{K^N}(T, h)$, a union of copies of $\operatorname{Spec} E$ for some field E, can be interpreted as a moduli space of CM motives with level structures. Let $P_{K^N}(T, h)^0 \to \mathrm{Sh}_{K^N}(T, h)^0 \xrightarrow{\sim} \operatorname{Spec} E$ be any single component together with the restriction of $P_{K^N}(T, h)$ over it. There exists a finite extension F/E and a CM abelian variety A/F for such that $\omega_{P_{K^N}(T,h)^0}(W \otimes \mathbb{Q}) \otimes_E F \subset H^1_{\mathrm{dR}}(A/F)^{\otimes}$ and $\omega_{\mathrm{et}}(W) = H^1_{\mathrm{et}}(A, \mathbb{Z}_p)^{\otimes} \cap \omega_{\mathrm{et}}(W) \otimes \mathbb{Q}_p$ for all $p \nmid N'$ some N' divisible by N. Let $\mathscr{A}/\mathbb{O}_F$ be the Neron model of A, and notice that it is an abelian variety over $\mathbb{O}_F[1/M]$ for some M which we may assume is divisible by N'. Thus we construct an $\mathbb{O}_E[1/M]$-lattice

$$\Lambda' := \omega_{P_{K^N}(T,h)^0}(W \otimes \mathbb{Q}) \cap H^1_{\mathrm{dR}}\big(\mathscr{A}/\mathbb{O}_F\big[\tfrac{1}{M}\big]\big)^{\otimes} \subset \omega_{P_{K^N}(T,h)^0}(W \otimes \mathbb{Q}).$$

The second part of Theorem 4.1.1 assures us that at all $p \nmid M$ this lattice agrees with that obtained via first taking the dual Tate module and applying Breuil–Kisin theory.

Moreover, using the construction of Theorem 4.1.1 together with Lemma 2.4.1 for the finitely many p which divide M but do not divide N, we are able to extend Λ' to a lattice Λ over $\mathbb{O}_F[1/N]$. This Λ (applying the construction for all components) gives us the lattice in $\omega_{P_{K^N}(T,h)}(W \otimes \mathbb{Q})$ we require to prove existence. Note that since we already have uniqueness, we may observe that the construction does not depend on the choices of F and A/F. □

4.3. ***Connections on G-bundles.*** It will help to collect some basic facts about connections on G-bundles. Let S be a scheme and G/S a flat affine group scheme of finite type.

Suppose X/S a scheme, and let $\Delta^2(1) = \Delta^2_{X/S}(1)$ be the first order neighbourhood of the diagonal in $X \times_S X$, $\delta : X \hookrightarrow \Delta(1)$ and $p_1, p_2 : \Delta(1) \to X$ the two projection maps. We also will need $\Delta^3(1)$, the first order neighbourhood of the diagonal in $X \times_S X \times_S X$ and its three projections $p_{12}, p_{23}, p_{13} : \Delta^3(1) \to \Delta^2(1)$.

4.3.1. Recall that if we have a vector bundle $\mathcal{V}$ on X a *connection* on $\mathcal{V}/X$ (relative to S) is given by an isomorphism

$$\nabla : p_1^*\mathcal{V} \xrightarrow{\sim} p_2^*\mathcal{V}$$

such that $\delta^*\nabla = \mathrm{id}$ (under the canonical identification $(\delta \circ p_i)^*\mathcal{V} = \mathrm{id}^*\mathcal{V} \cong \mathcal{V}$). Such a connection is said to be *flat* if

$$p_{13}^*(\nabla) = p_{23}^*(\nabla) \circ p_{12}^*(\nabla).$$

It is well known that these definitions are equivalent to the more usual definitions from differential geometry.

4.3.2. Let $P \to X$ be a G_X-torsor. A *connection* on P is an isomorphism

$$\nabla : p_1^* P \xrightarrow{\sim} p_2^* P$$

such that $\delta^* \nabla = \mathrm{id}_P$, and ∇ is said to be flat, again if

$$p_{13}^*(\nabla) = p_{23}^*(\nabla) \circ p_{12}^*(\nabla).$$

Lemma 4.3.3. *Let X/S be a scheme, and assume S is Dedekind. Let $P \to X$ be a G-bundle, with associated fibre functor ω_P. To give the following pieces of data are equivalent*:

(1) *A (flat) connection on $P \to X$.*

(2) *For each representation $V \in \mathrm{Rep}_S(G)$ a (flat) connection on $\omega_P(V)$ in such a way that the isomorphisms*

$$\omega_P(V \otimes W) \cong \omega_P(V) \otimes \omega_P(W)$$

and

$$\omega_P(V^\vee) \cong \omega_P(V)^\vee$$

are isomorphisms of bundles with connection.

(3) *Given a faithful representation $G \hookrightarrow \mathrm{GL}_S(V)$ and tensors $s_\alpha \in V^\otimes$ such that $G = \mathrm{GL}_{S,s_\alpha}(V)$, a (flat) connection on $\omega_P(V)$ with the property that each $\omega_P(s_\alpha \mathcal{O}_S) \subset \omega_P(V^\otimes) = \omega_P(V)^\otimes$ receives the trivial connection.*

Proof. The equivalence of (1) and (2) is formal given Broshi's Tannakian formalism over a Dedekind scheme [Broshi 2013, 1.2]. Indeed the conditions in (2) are exactly those needed to say that the connections $\nabla_{\omega_P(V)}$ define an isomorphism of tensor functors $p_1^*\omega_P \xrightarrow{\sim} p_2^*\omega_P$, which is the same as an isomorphism of torsors $p_1^* P \xrightarrow{\sim} p_2^* P$, and one easily verifies that the conditions translate across.

It is obvious how to pass from (1) to (3). For going from (3) to (1), we note that P can be canonically identified with the frame bundle

$$P \cong \underline{\mathrm{Isom}}_{s_\alpha}(V \otimes \mathcal{O}_X, P \times^G V),$$

(via $p \mapsto (v \mapsto (p, v))$). Let $\mathcal{V} = \omega_P(V) = P \times^G V$ and $s_{\alpha,1} = (1, s_\alpha) \in \mathcal{V}^\otimes$. If we are given a connection $\nabla : p_1^*\mathcal{V} \xrightarrow{\sim} p_2^*\mathcal{V}$, which is trivial on each line containing $s_{\alpha,1}$, that is to say $\nabla(p_1^* s_{\alpha,1}) = p_2^* s_{\alpha,1}$, we obtain a well-defined isomorphism

$$\nabla_P : p_1^* P \ni \phi \mapsto \nabla \circ \phi \in p_2^* P,$$

giving the desired connection on P. Again it is easy to check that if ∇ is flat then so is ∇_P. ☐

4.4. ***Definition of canonical models.*** Let $(G, \mathfrak{X})$ be a Shimura datum with reflex $E = E(G, \mathfrak{X})$, and for what follows assume it admits an integral canonical model over $\mathbb{O}_E[1/N]$ at any level hyperspecial away from N.

4.4.1. We shall need the mild assumption that $Z(G)^\circ$ is split by a CM field, which we impose for the rest of the paper. This can be removed if the arguments relying on CM motives in proving Proposition 3.1.8 and Proposition 4.2.3 can be replaced by arguments that work with greater generality.

4.4.2. Recall Section 3.1.2 that the Shimura variety $\mathrm{Sh}(G, \mathfrak{X})$ is equipped with a *standard principal bundle* $P(G, \mathfrak{X})$, which is a $G(\mathbb{A}^\infty)$-equivariant G^c-torsor defined over E, with a flat connection ∇ and a G-equivariant map $\gamma : P(G, \mathfrak{X}) \to \mathrm{Gr}_\mu$. Suppose also G is quasisplit and unramified away from N, fix $G = G_{\mathbb{Z}[1/N]}$ a reductive model, $K^N = \prod_{p \nmid N} G(\mathbb{Z}_p)$ and for μ the Hodge cocharacter note that over an integral model one has the canonical inclusion $\mathrm{Gr}_\mu \subset \mathcal{GR}_\mu$.

We also let

$$\omega_{\mathrm{et}} : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{Lisse}_{\mathbb{Z}_p}(\mathrm{Sh}_K(G, \mathfrak{X}))$$

denote the standard étale $\mathbb{Z}_p$-sheaves coming from the tower at p, and $\omega_{\mathrm{et},\eta} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c) \to \mathrm{Lisse}_{\mathbb{Q}_p}(\mathrm{Sh}_K(G, \mathfrak{X}))$ the corresponding lisse $\mathbb{Q}_p$-sheaves.

4.4.3. We will say that a $(G, \mathfrak{X})$ *has enough crystalline points* if for every prime $\mathfrak{p}$ of E with $\mathfrak{p} | p \nmid N$, and every $K_N \subset \prod_{q|N} G(\mathbb{Q}_q)$ compact open, $K = K_N K^N$, there is a dense subset U_0 of the special fibre of $\mathcal{S}_{K_N K^N}/\mathfrak{p}$ with the property that for each $x_0 \in U_0$ there is a lift $\tilde{x}_0 \in \mathcal{S}_{K_N K^N}(W(\kappa(x_0))$ such that $\tilde{x}_0[1/p]^*\omega_{\mathrm{et},\eta}$ takes values in crystalline representations of $\Gamma_{W(\kappa(x_0))[1/p]}$.

We say $(G, \mathfrak{X})$ has *all the crystalline points* if for every $W(k)$-valued point $x \in \mathcal{S}_{K_N K^N}$ for k a finite field of characteristic $p \nmid N$, $x[1/p]^*\omega_{\mathrm{et},\eta}$ takes values in crystalline representations of $\Gamma_{W(k)[1/p]}$. Clearly in the present context where the base is smooth and so every k-point admits a $W(k)$-lift, if you have all the crystalline points you have enough crystalline points.

Moreover, we note that when $(G, \mathfrak{X})$ is of Hodge type or special type it has all the crystalline points. In the Hodge type case, this is because the relevant Galois representations live inside $H^1_{\mathrm{et}}(\mathcal{A}_x[1/p], \mathbb{Q}_p)^\otimes$, where $\mathcal{A}_x$ is the fibre of the universal abelian scheme at x, and since $\mathcal{A}_x/W(\kappa(x))$ is an abelian scheme, this representation is crystalline. In the special type case it follows from the explicit reciprocity law computation as in Section 4.2.2. We shall later use the methods of §4.6 to establish that $(G, \mathfrak{X})$ of abelian type also has all the crystalline points.

In what follows, a *crystalline point* is understood to be, for some K_N and some finite field k of characteristic $p \nmid N$, a point of $\mathcal{S}_{K_N K^N}(W(k))$ such that $x[1/p]^*\omega_{\mathrm{et},\eta}$ takes values in crystalline representations, and any $p, K_N, k, K_0 = W(k)[1/p]$ appearing will be understood to be part of this data.

4.4.4. We will also say that to give $(G, \mathfrak{X})$ a *de Rham structure* is to give for every crystalline point $x \in \mathcal{S}_{K_N K^N}(W(k))$, an identification

$$D_{\mathrm{dR}} \circ x\left[\tfrac{1}{p}\right]^* \omega_{\mathrm{et},\eta} \xrightarrow{\sim} x\left[\tfrac{1}{p}\right]^* \omega_{P_{K_N K^N}(G,\mathfrak{X})} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c) \to \mathrm{Vec}_{K_0}.$$

In the case where $(G, \mathfrak{X})$ is of abelian type (and we expect in general), there is a canonical de Rham structure. This follows using functoriality and the morphism $(G, \mathfrak{X}) \to (G^c, \mathfrak{X}^c)$ to reduce to the case where the weight is defined over $\mathbb{Q}$. Here we may use Milne's moduli interpretation [1994] in such a situation to interpret $x[1/p]$ as an abelian motive, and get the required identification by the same argument as Proposition 3.1.8. It is also straightforward to check that these canonical de Rham structures are compatible under morphisms of Shimura varieties induced by maps of Shimura data. In what follows, we will usually assume $(G, \mathfrak{X})$ is of abelian type in which case we are always working with reference to the canonical de Rham structure.

4.4.5. Let $(G, \mathfrak{X})$ be a Shimura datum as above equipped with a de Rham structure. An *integral canonical model* $\mathcal{P} = \mathcal{P}_{K^N}(G, \mathfrak{X})$ for $P_{K^N}(G, \mathfrak{X})$ is a G^c-torsor over the integral canonical model $\mathcal{S}_{K^N}(G, \mathfrak{X})$ for $\mathrm{Sh}_{K^N}(G, \mathfrak{X})$ with an identification

$$\iota : \mathcal{P}_{K^N}(G, \mathfrak{X}) \otimes_{\mathbb{O}_E[1/N]} E \xrightarrow{\sim} P_{K^N}(G, \mathfrak{X})$$

and equivariant $\prod_{p|N} G(\mathbb{Q}_p)$-action such that we have the *lattice property* given below for any crystalline point $x \in \mathcal{S}_{K_N K^N}(W(k))$.

Consider

$$\omega_{\mathrm{dR},x} : D_{\mathrm{dR}} \circ x\left[\tfrac{1}{p}\right]^* \omega_{\mathrm{et},\eta} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c) \to \mathrm{Vec}_{K_0}.$$

This functor comes with two canonical lattices. First, there is the lattice given by $\omega_{\mathcal{P}_{K_N K^N}}$ coming via ι and the de Rham structure. Second, there is the lattice $\mathbb{D} \circ x[1/p]^* \omega_{\mathrm{et}}$ coming from the theory Theorem 4.1.1 of $\mathfrak{S}$-modules. The lattice property requires that these lattices are equal. Note that since everything is Hecke equivariant, it suffices to check the lattice property at infinite level, an observation of which we shall make liberal use.

We will also insist that for $\mathcal{P}_{K^N}$ to qualify as a canonical model the connection ∇ and the μ-filtration γ extend to $\mathcal{P}$, although we shall see that in the abelian type situation these additional properties are automatic given the condition on crystalline points.

4.4.6. We remark that this definition is compatible with Section 4.2.2. We also remark that one can make definitions of integral canonical models defined over any unramified-away-from-N extension F/E, and for any such F containing the abelian extension over which $\mathrm{Sh}_{K^N}(G, \mathfrak{X})$ splits into its geometrically connected components we may also define integral canonical models for $P_{K^N}(G, \mathfrak{X})^+_F$. These definitions are made in exactly the same way and such models enjoy the properties we are about to note for the same reasons.

Lemma 4.4.7. *Suppose* $(G, \mathfrak{X})$ *has enough crystalline points. An integral canonical model* $(\mathcal{P}, \iota)$, *if it exists, is unique up to canonical isomorphism.*

This is true without assuming ∇ *or* γ *extend.*

Proof. Suppose we have $(\mathcal{P}, \iota)$ and $(\mathcal{P}', \iota')$, two integral canonical models for $P_{K^N}(G, \mathfrak{X})$. We will aim to show that $\iota'^{-1} \circ \iota : \mathcal{P} \otimes E \xrightarrow{\sim} \mathcal{P}' \otimes E$ extends over $\mathcal{O}_E[1/N]$. By Lemma 4.1.4 it suffices to check that for any representation of $G^c_{\mathbb{Z}[1/N]}$ on a finite free module V that the composite $(\iota'^{-1} \circ \iota)_* : \omega_{\mathcal{P}}(V) \otimes E \xrightarrow{\sim} \omega_{\mathcal{P}'}(V) \otimes E$ identifies the $\mathcal{O}_E[1/N]$-lattice $\omega_{\mathcal{P}}(V)$ with $\omega_{\mathcal{P}'}(V)$.

By Hecke equivariance we may reduce to checking this at every finite level $K_N K^N$. Noting that $\mathcal{S}_{K_N K^N}$ is smooth, we may invoke our lattice Lemma 4.1.3 to reduce the statement first (4.1.3(1)) to checking equality of lattices over the formal completion of $\mathcal{S}_{K_N K^N}$ at each (maximal) prime $\mathfrak{p}$ of $\mathcal{O}_E[1/N]$, and then (4.1.3 (2)) to checking equality of lattices on lifts of a dense subset of the special fibre at $\mathfrak{p}$. But since $(G, \mathfrak{X})$ has enough crystalline points, and by the lattice condition we have the necessary equality at these points, we get the desired identification.

Note that the connection and μ-filtration are extended from the generic fibre and that they extend is a property, so they have no impact on the uniqueness statement. □

The following functoriality properties can now be read off.

Proposition 4.4.8. *Let* $f : (G_1, \mathfrak{X}_1) \to (G_2, \mathfrak{X}_2)$ *be a morphism of Shimura data with enough crystalline points and compatible de Rham structures induced by a map* $G_{1,\mathbb{Z}[1/N]} \to G_{2,\mathbb{Z}[1/N]}$ *of reductive groups over* $\mathbb{Z}[1/N]$, *and* $K_i^N = \prod_{p \nmid N} G_i(\mathbb{Z}_p)$ *and* $(\mathcal{P}_i, \iota_i)$ *an integral canonical model for* $P_{K_i^N}(G_i, \mathfrak{X}_i)$, $i = 1, 2$.

(1) *We can canonically identify*

$$\mathcal{P}_1 \times^{G_1^c} G_2^c \xrightarrow{\sim} f^* \mathcal{P}_2.$$

(2) *The induced diagram*

$$\begin{array}{ccc} \mathcal{P}_1 & \xrightarrow{\gamma_1} & \mathcal{GR}_{\mu_1} \\ \downarrow & & \downarrow \\ \mathcal{P}_2 & \xrightarrow{\gamma_2} & \mathcal{GR}_{\mu_2} \end{array}$$

commutes.

Proof. Let $E = E(G_1, \mathfrak{X}_1)$. Note that by Lemma 3.1.6 we have a natural identification

$$\theta : P_{K_1^N}(G_1, \mathfrak{X}_1) \times^{G_1^c} G_2^c \xrightarrow{\sim} f^* P_{K_2^N}(G_2, \mathfrak{X}_2)$$

and so as in the previous lemma it will suffice to check that

$$f^*(\iota_2^{-1})\circ\theta\circ\iota_1 : (\mathcal{P}_1\times^{G_1^c} G_2^c)\otimes_{\mathcal{O}_E[1/N]} E \xrightarrow{\sim} (f^*\mathcal{P}_2)\otimes_{\mathcal{O}_E[1/N]} E$$

exchanges the natural lattices when taken on the level of fibre functors

$$\omega : \mathrm{Rep}_{\mathbb{Z}[1/N]} G_2^c \to \mathrm{Vec}(\mathcal{S}_{K_1^N}(G_1,\mathfrak{X}_1)).$$

As in the previous lemma, we may reduce to working at finite level and checking equality at crystalline points. But this is then immediate from the lattice condition, the compatibility of the étale fibre functors on both Shimura varieties, together with the (consequent) observation that the image of a crystalline point is always a crystalline point. This completes the proof of (1). Now (2) follows directly from the fact that after taking $\otimes_{\mathcal{O}_E[1/N]}E$ the diagram commutes by Lemma 3.1.6. Note that the right hand vertical map is that given by Lemma 3.2.8. □

Lemma 4.4.9. *Suppose $F/E = E(G,\mathfrak{X})$ a Galois extension unramified away from N, and $(\mathcal{P}'/\mathcal{O}_F[1/N],\iota')$ an integral canonical model for $P_{K^N}(G,\mathfrak{X})_F$.*

Then the descent data

$$\mathrm{Gal}(F/E)\ni\sigma\mapsto\theta_\sigma : \sigma^*P_{K^N}(G,\mathfrak{X})_F \xrightarrow{\sim} P_{K^N}(G,\mathfrak{X})_F$$

extend to $\mathcal{P}'$, and the pair $(\mathcal{P},\iota)$ obtained by étale descent is an integral canonical model for $P_{K^N}(G,\mathfrak{X})$ over $\mathcal{O}_E[1/N]$.

Proof. Take $\sigma\in\mathrm{Gal}(F/E)$. We claim that $\sigma^*\mathcal{P}'_{K^N}$ together with the composite

$$\sigma^*\mathcal{P}'_{K^N}\otimes F \xrightarrow{\sigma^*\iota} \sigma^*P_{K^N}(G,\mathfrak{X})_F \xrightarrow{\sigma} P_{K^N}(G,\mathfrak{X})_F$$

is an integral canonical model. From this claim it is immediate that the descent data extend and in its turn $\mathcal{P}'_{K^N}$ descends to an integral canonical model over $\mathcal{O}_E[1/N]$, because being a crystalline point and the lattices coming from $\mathfrak{S}$-modules are stable under finite unramified base change.

We may pull back the Hecke action, so it remains to check the lattice condition on crystalline points at finite level. Let $x\in\mathcal{S}_{K_NK^N}(G,\mathfrak{X})(W(\kappa))$ be a crystalline point and v the corresponding place of F, and $\sigma^*(x)$ its pullback along $\sigma:\mathcal{S}_{K_NK^N}\xrightarrow{\sim}\mathcal{S}_{K_NK^N}$. Let $\omega_{\mathrm{et},x}$ and $\omega_{\mathrm{et},\sigma^*(x)}$ be the $\mathbb{Z}_p$-linear étale fibre functors coming from the Shimura variety at infinite level at p and restricting to the fibres at $x[1/p]$ and $\sigma^*(x)[1/p]$ respectively, viewing both as taking values in Γ_{K_0}-representations. Note that by construction (since $\mathrm{Sh}(G,\mathfrak{X})$ is defined over E) there is an isomorphism $\omega_{\mathrm{et},\sigma^*(x)}\xrightarrow{\sim}\omega_{\mathrm{et},x}$ covering $\sigma:\sigma^*x\xrightarrow{\sim}x$. In particular, $\sigma^*(x)$ is also crystalline.

For $V\in\mathrm{Rep}_{\mathbb{Z}_p}(G^c)$ it is necessary to check an equality of two lattices in $D_{\mathrm{dR}}(\omega_{\mathrm{et},x}(V)[1/p])$. The first is the usual $\mathbb{D}(\omega_{\mathrm{et},x}(V))$ coming straight from $\mathfrak{S}$-modules, the second obtained as the composite

$$\mathbb{D}(\omega_{\mathrm{et},\sigma^*(x)}(V))\subset D_{\mathrm{dR}}\big(\omega_{\mathrm{et},\sigma^*(x)}(V)\big[\tfrac{1}{p}\big]\big)\xrightarrow{\sim} D_{\mathrm{dR}}\big(\omega_{\mathrm{et},x}(V)\big[\tfrac{1}{p}\big]\big).$$

Having thus spelt it out, we see that it follows immediately from functoriality of the $\mathbb{D}$ construction. Moreover, the G^c-action, connection and filtration are all defined over E on the generic fibre and it can be checked they extend to $\mathcal{O}_E[1/N]$ étale locally, so the fact they do over F immediately gives them all. □

4.5. ***Hodge type case.*** We turn our attention to existence.

Theorem 4.5.1. *Let $(G, \mathfrak{X})$ be a Shimura variety of Hodge type with G unramified away from N and $K^N = \prod_{p \nmid N} G(\mathbb{Z}_p)$. Then $P_{K^N}(G, \mathfrak{X})$ admits an integral canonical model.*

We let $E = E(G, \mathfrak{X})$ throughout this paragraph, and unless otherwise specified all Shimura varieties Sh_K, $\mathcal{S}_K$ or bundles P_K or $\mathcal{P}_K$ will be understood to be those attached to the Shimura datum $(G, \mathfrak{X})$. We also note that $(G, \mathfrak{X})$ being of Hodge type implies $G = G^c$.

4.5.2. First some more algebraic preliminaries. Let R be a ring, M a finite free R-module, and $M^{\otimes}$ the direct sum of all R-modules formed from M by the operations of taking duals, tensor products, symmetric powers and exterior powers. If $s_\alpha \in M^{\otimes}$ is a collection of tensors, we say that they *define* a subgroup $G \subset \mathrm{GL}(M)$ if it is precisely the group which acts trivially on all the s_α.

We need the following version of [Kisin 2010, 1.3.2], whose proof is basically identical.

Proposition 4.5.3. *Let R be a PID with field of fractions K, M a finite free R-module, and $G \subset \mathrm{GL}(M)$ a closed subgroup, flat over R with reductive generic fibre. Then it is defined by a finite collection of tensors s_α.*

Proof. Exactly as in [Kisin 2010, 1.3.2] we can reduce to showing that it suffices to find tensors defining G in some representation of $\mathrm{GL}(M)$ on a finite projective R-module, and we take $I \subset \mathcal{O}_{\mathrm{GL}(M)}$ the ideal of G in the Hopf algebra of $\mathrm{GL}(M)$, which has scheme-theoretic stabiliser precisely G.

By [Waterhouse 1979, 3.3] there is a finite dimensional $\mathrm{GL}(M)_K$-stable subspace $W_\eta \subset \mathcal{O}_{\mathrm{GL}(M)} \otimes K$ containing a finite set of generators for I as an ideal. Let $W = W_\eta \cap \mathcal{O}_{\mathrm{GL}(M)}$, which is visibly $\mathrm{GL}(M)$-stable, and it is finitely generated because $\mathcal{O}_{\mathrm{GL}(M)}$ is a free R-module and letting $e_1, \dots, e_n$ be a basis for W_η, writing them as elements of $\mathcal{O}_{\mathrm{GL}(M)} \otimes K$ we see they lie in $\tilde{W} \otimes K$ for some $\tilde{W} \subset \mathcal{O}_{\mathrm{GL}(M)}$ a finite free R-submodule, and such that $W \subset \tilde{W}$. But R is a PID, so we deduce W is finite free.

Now we see that G is the stabiliser of $W \cap I \subset W$, and the argument can be concluded exactly as in [Kisin 2010, 1.3.2]. □

4.5.4. Now recall Section 2.4.4 that for $(G, \mathfrak{X})$ our Shimura datum of Hodge type with G unramified away from $\mathbb{Z}[1/N]$ (taking $G_{\mathbb{Z}[1/N]} = \mathcal{G}$ from the discussion in

Section 2.4.4) we may choose a symplectic embedding $i : (G, \mathfrak{X}) \hookrightarrow (\mathrm{GSp}_{2g}, S^{\pm})$ extending to a closed immersion

$$i : G_{\mathbb{Z}[1/N]} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}[1/N]}).$$

Henceforth view V as a $\mathbb{Z}[1/N]$-module, and G as a reductive group over $\mathbb{Z}[1/N]$. By the above proposition we can find $s_\alpha \in V^{\otimes}$ such that $G = \mathrm{Aut}_{s_\alpha}(V)$.

4.5.5. The map i defines (pulling back the universal abelian variety) an abelian variety $\pi : A_{K^N} \to \mathcal{S}_{K^N}$, and we can study its sheaf of relative de Rham cohomology $\mathcal{V} = \mathcal{H}^1_{\mathrm{dR}}(A_{K^N}/\mathcal{S}_{K^N})$ with Gauss–Manin connection ∇.

By the absolute Hodge cycles argument of [Kisin 2010, 2.2.2] we obtain ∇-horizontal $\prod_{p|N} G(\mathbb{Q}_p)$-invariant tensors $s_{\alpha,\mathrm{dR}} \in \mathcal{V}^{\otimes} \otimes_{\mathbb{Z}[1/N]} \mathbb{Q}$, and by [ibid., 2.3.9] for any $v \nmid N$ a finite place of E these sections extend over $\mathcal{O}_{E,(v)}$, which is enough to conclude they in fact lie in $\mathcal{V}^{\otimes}$.

Let us therefore define the sheaf on $(\mathcal{S}_{K^N})_{\mathrm{fppf}}$

$$\mathcal{P}_{K^N} := \underline{\mathrm{Isom}}_{s_\alpha}(V, \mathcal{V})$$

whose sections over $u : U \to \mathcal{S}_{K^N}$ are those trivialisations $\theta : V \otimes_{\mathbb{Z}[1/N]} \mathcal{O}_U \xrightarrow{\sim} u^*\mathcal{V}$ such that (after applying $(-)^{\otimes}$ to both sides) each s_α is identified with $s_{\alpha,\mathrm{dR}}$. Note that since $\prod_{p|N} G(\mathbb{Q}_p)$ acts equivariantly on $\mathcal{V}$ and fixes $s_{\alpha,\mathrm{dR}}$ we may freely pass between infinite and finite level.

Proposition 4.5.6. *The sheaf $\mathcal{P}_{K^N}$ is a G-torsor over $\mathcal{S}_{K^N}$.*

Proof. It obviously suffices to check at finite level $K = K_N K^N$ for some sufficiently small K_N. We claim it also suffices to check in the formal neighbourhood of any closed point $x \in \mathcal{S}_K$.

First we may directly verify that $\mathcal{P}_K$ is representable because passing to a Zariski cover making $\mathcal{V}$ free, and then choosing arbitrary identifications $V \cong \mathcal{V}$, $\mathcal{P}_K$ can be constructed as a closed (hence affine) subvariety of $\mathrm{GL}(V)$. Now suppose we know it is a G-torsor in a formal neighbourhood of every closed point. Then since G is faithfully flat, $\mathcal{P}_K$ is faithfully flat over the formal neighbourhood of every closed point, which is enough to prove that $\mathcal{P}_K/\mathcal{S}_K$ is faithfully flat. It remains to show that the natural map

$$\mathcal{P}_K \times_{\mathcal{S}_K} G_{\mathcal{S}_K} \ni (p, g) \mapsto (p, pg) \in \mathcal{P}_K \times_{\mathcal{S}_K} \mathcal{P}_K$$

is an isomorphism, but this also is the case if and only if it is the case over a formal neighbourhood of each closed point, where again it follows from our assumption.

Thus we are reduced to the study of $\hat{P}_x := \mathcal{P}_K|_{\hat{\mathcal{S}}_{K,x}}$ for closed points $x \in \mathcal{S}_K$. If $\mathrm{char}(x) = 0$, the Betti–de Rham comparison theorem gives an identification $V_{\hat{\mathcal{S}}_{K,x}} \otimes \mathbb{C} \xrightarrow{\sim} \hat{\mathcal{V}}_x \otimes \mathbb{C}$, giving a fpqc-local section. This suffices because it shows

$\hat{P}_x$ is smooth by fpqc descent, and so $\hat{P}_x$ admits a section étale locally, which is enough to see it is a G-torsor.

If $\mathrm{char}(x) = p$, we may follow the proof of [Kisin 2010, 2.3.5] to study $N_x := \mathcal{O}_{\mathcal{S}^{\wedge}_{K,x}}$ together with the p-divisible group $\mathcal{G} = A_K|_{\mathcal{S}^{\wedge}_{K,x}}[p^\infty]$ over the formally smooth ring N_x.

First note that x determines a place v of E above p and since E is unramified at p we may identify $\mathcal{O}_{E,v} = W(\kappa(x)) =: W$, over which N_x is canonically an algebra. The crystalline-de Rham comparison gives an identification $\mathcal{V}_{N_x} = \mathbb{D}(\mathcal{G})(N_x)$. We let $\mathcal{G}_0$ denote the restriction of $\mathcal{G}$ to x.

Taking $\tilde{x} : N_x \to W$ a lift, we may follow the argument of [ibid., 2.3.5] to obtain from the Tate module of $\tilde{x}^*\mathcal{G}$ tensors $s_{\alpha,0} \in \mathbb{D}(\mathcal{G}_0)(W)^{\otimes}$ defining a reductive subgroup $G(\tilde{x}) \subset \mathrm{GL}(\mathbb{D}(\mathcal{G}_0)(W))$, which gives rise to an explicit versal deformation ring $R_{G(\tilde{x})}$ of $G(\tilde{x})$-adapted deformations of $\mathcal{G}_0$. This then has the property that we may identify $R_{G(\tilde{x})} \xrightarrow{\sim} N_x$ in such a way that induces an identification of $\mathcal{G}$ with the explicit versal deformation (let's also call it $\mathcal{G}$) one constructs over $R_{G(\tilde{x})}$, so in particular we have

$$\mathcal{V}_{N_x} = \mathbb{D}(\mathcal{G})(R_{G(\tilde{x})}).$$

As in [ibid., 2.3.9] one has by an explicit construction lifts $\tilde{s}_{\alpha,0}$ to $\mathbb{D}(\mathcal{G})(R_{G(\tilde{x})})^{\otimes}$ of $s_{\alpha,0}$ which are identified with $s_{\alpha,\mathrm{dR}} \in \mathcal{V}^{\otimes}_{N_x}$. This explicit construction comes about in [ibid., §1.5] by defining $\mathbb{D}(\mathcal{G})(R_{G(\tilde{x})}) = \mathbb{D}(\mathcal{G}_0)(W) \otimes_W R_{G(\tilde{x})}$ as a module and taking the lift $\tilde{s}_{\alpha,0} = s_{\alpha,0} \otimes 1$. Finally let us note that by [ibid., 1.4.3] (since G is connected) there exists a W-linear isomorphism

$$V_W \xrightarrow{\sim} \mathbb{D}(\mathcal{G}_0)(W)$$

identifying s_α with $s_{\alpha,0}$. In particular this identifies $G_W \xrightarrow{\sim} G(\tilde{x})$ uniquely up to inner automorphisms and the composite

$$V_W \otimes_W R_{G(\tilde{x})} \xrightarrow{\sim} \mathbb{D}(\mathcal{G}_0)(W) \otimes_W R_{G(\tilde{x})} \xrightarrow{\sim} \mathbb{D}(\mathcal{G})(R_{G(\tilde{x})}) = \mathcal{V}_{N_x}$$

gives a section of $\hat{P}_x$ as required. □

4.5.7. We would now like to show it is the desired canonical model for $P_{K^N}(G, \mathfrak{X})$. Firstly we must check that the additional structures extend.

We have already noted that the $\prod_{p|N} G(\mathbb{Q}_p)$-action extends. By Lemma 4.3.3, the connection ∇ on P_{K^N} extends to $\mathcal{P}_{K^N}$ because the Gauss–Manin connection extends to (i.e., can be constructed directly on) $\mathcal{V} = \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}_{K^N}/\mathcal{S}_{K^N})$ and by the construction of $s_{\alpha,\mathrm{dR}}$ in [Kisin 2010, 2.2] it is clear they are parallel sections. It is also immediate the μ-filtration extends, by Lemma 3.3.2 and that the Hodge filtration is naturally defined on the integral de Rham cohomology $\mathcal{V}$.

Finally, we need to verify that it is indeed a canonical model, and as usual we check the lattice condition at a crystalline point x. But in our situation this is very straightforward: we just use the facts that, taking $\mathscr{A}_x$ to be the pullback of the universal abelian scheme to x, that Theorem 4.1.1 $\mathbb{D}(H^1_{\text{et}}(\mathscr{A}_x, \mathbb{Z}_p)) = H^1_{\text{dR}}(\mathscr{A}_x/W(k))$ and that by compatibility of $\mathbb{D}$ with the p-adic comparison theorems, $\mathbb{D}(x^* s_{\alpha,et}) = x^* s_{\alpha,\text{dR}}$.

4.5.8. For making the transition from Hodge to abelian type, we will need to extend the equivariant $G^c \times \prod_{p|N} G(\mathbb{Q}_p)$-action to an equivariant action of the $\mathbb{Z}^N = \mathbb{Z}[1/N]$-group scheme

$$\mathscr{A}^N_P(G) = \left(G^c \times \frac{\prod_{p|N} G(\mathbb{Q}_p)}{\overline{Z(\mathbb{Z}^N)}} \right) *_{G(\mathbb{Z}^N)_+/Z(\mathbb{Z}^N)} G^{\text{ad}}(\mathbb{Z}^N)^+$$

which sits in an exact sequence

$$1 \to G^c \to \mathscr{A}^N_P(G) \to \mathscr{A}^N(G) \to 1.$$

To do this it suffices to give the $G^{\text{ad}}(\mathbb{Z}^N)^+$-action and then check it is compatible, for which we imitate the approach taken in [Kisin 2010, 3.2] and [Kissin and Pappas 2015, 4.5], with our modifications from Section 2.5.6 and further slight modifications.

To be precise, recall that a point $x \in \mathscr{P}_{K^N}(T)$ gives the data of an quadruple $(A, \lambda, \epsilon_{\text{et}}, \epsilon_{\text{dR}})$ where A/T is an abelian scheme up to an isogeny whose degree is supported on N, λ a weak polarisation of A, ϵ_{dR} a section of $\underline{\text{Isom}}_{s_\alpha}(V_T, \mathscr{V}_T)$, and $\epsilon_{\text{et}} \in \Gamma(T, \underline{\text{Isom}}_{s_\alpha}(V_{\prod_{p|N} \mathbb{Q}_p}, \prod_{p|N} \hat{V}_p(A)))$.

Given $\gamma \in G^{\text{ad}}(\mathbb{Z}^N)^+$, we may form the Z^{der}-torsor $\mathscr{P} = \{g \in G^{\text{der}} | \pi(g) = \gamma\}$. We may take $F/\mathbb{Q}$ Galois such that $\mathscr{P}(\mathbb{O}_F[1/N])$ is nonempty, and take $\tilde{\gamma} \in \mathscr{P}(\mathbb{O}_F[1/N])$. This can be used to define an action just as in [Kisin 2010, 3.2]. Indeed we can define

$$A^{\mathscr{P}}(T) = (A(T) \otimes_{\mathbb{Z}[1/N]} \mathbb{O}_{\mathscr{P}})^{Z^{\text{der}}}$$

and specialise the map

$$A^{\mathscr{P}} \otimes_{\mathbb{Z}[1/N]} \mathbb{O}_{\mathscr{P}} \xrightarrow{\sim} A \otimes_{\mathbb{Z}[1/N]} \mathbb{O}_{\mathscr{P}}$$

at $\tilde{\gamma}$ to obtain

$$\iota_{\tilde{\gamma}} : A^{\mathscr{P}} \otimes_{\mathbb{Z}[1/N]} \mathbb{O}_F\left[\tfrac{1}{N}\right] \xrightarrow{\sim} A \otimes_{\mathbb{Z}[1/N]} \mathbb{O}_F\left[\tfrac{1}{N}\right].$$

These allow one to give the action of γ as taking

$$(A, \lambda, \epsilon_{\text{et}}, \epsilon_{\text{dR}}) \mapsto (A^{\mathscr{P}}, \lambda^{\mathscr{P}}, \epsilon^{\mathscr{P}}_{\text{et}}, \epsilon^{\mathscr{P}}_{\text{dR}}),$$

where all but the last of these are as defined by Kisin. For the action on the de Rham component, we need the following modification of [Kisin 2010, 3.2.5].

Lemma 4.5.9. *With notation as above, the composite*

$$V_T \otimes_{\mathbb{Z}^N} \mathbb{O}_F^N \xrightarrow{\tilde{\gamma}^{-1}} V_T \otimes_{\mathbb{Z}^N} \mathbb{O}_F^N \xrightarrow{\epsilon_{\mathrm{dR}}} H^1_{\mathrm{dR}}(A/T) \otimes_{\mathbb{Z}^N} \mathbb{O}_F^N \xrightarrow{\iota_{\tilde{\gamma}}^{-1}} H^1_{\mathrm{dR}}(A^{\mathcal{P}}/T) \otimes_{\mathbb{Z}^N} \mathbb{O}_F^N$$

is $\mathrm{Gal}(F/\mathbb{Q})$*-invariant, hence induces a section*

$$\epsilon^{\mathcal{P}}_{\mathrm{dR}} : V_T \xrightarrow{\sim} H^1_{\mathrm{dR}}(A^{\mathcal{P}}/T)$$

which takes s_α *to* $s_{\alpha,\mathrm{dR}}$.

Proof. The cocycle computation argument of [ibid., 3.2.5] goes through unchanged. For the claim about tensors, the only nonobvious point is that $\iota_{\tilde{\gamma}}^{-1}$ preserves the tensors, which one checks directly by working over $\mathbb{C}$, using the comparison with Betti cohomology as in [ibid., 3.2.6]. □

It is also clear that this action is compatible with the $G^c \times \prod_{p|N} G(\mathbb{Q}_p)/\overline{Z(\mathbb{Z}^N)}$-action in the sense that if we are given $g \in G^c(\mathbb{Z}^N)_+$ we may take $F = \mathbb{Q}$ and $\tilde{\gamma} = \gamma = g^{-1}$ and postcompose the whole quadruple by the quasiisogeny ι_γ to identify

$$(A, \lambda, \epsilon_{\mathrm{et}} \circ g, \epsilon_{\mathrm{dR}} \circ g) = (A^{\mathcal{P}}, \lambda^{\mathcal{P}}, \epsilon^{\mathcal{P}}_{\mathrm{et}}, \epsilon^{\mathcal{P}}_{\mathrm{dR}}).$$

4.6. ***Some distinguished Shimura data.*** In this section we digress a little. To get from the Hodge type to the abelian type case we would like to follow Deligne [1979] and pass via connected components, but unfortunately a straightforward reduction of $\mathcal{P}$ to the derived group is not possible to carry out over $\mathbb{O}_E[1/N]$ (roughly speaking because carrying out such a construction requires "trivialising the G^{ab} part of the motivic structure", which can only be done canonically over $\mathbb{C}$ using Betti–de Rham comparison). We address this by constructing some distinguished Shimura data $\mathcal{B}(G^{\mathrm{der}}, \mathfrak{X}^+, E)$ which are in some sense initial among Shimura data whose connected Shimura datum is $(G^{\mathrm{der}}, \mathfrak{X}^+)$ and whose reflex field is contained in E. Pulling back our torsors from the Hodge type case we may then reduce along $\mathcal{B} \to G$ while leaving all motivic structures intact.

4.6.1. Let $(G, \mathfrak{X})$ be a Shimura datum, and E its reflex field. Our goal is to construct a new Shimura datum $(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) \to (G, \mathfrak{X})$ with $\mathcal{B}^{\mathrm{der}} = G^{\mathrm{der}}$ and reflex field E and show it depends only on these data and $\mathfrak{X}^+$ and not on G.

Since E is the reflex field, we obtain a well-defined

$$\mu_E : \mathbb{G}_{m,E} \to G^{\mathrm{ab}}_E$$

taking the "determinant" (composite with $G \to G^{\mathrm{ab}}$) of any Hodge cocharacter. Taking Weil restriction along $E/\mathbb{Q}$ and the norm map, we get a composite

$$r : E^* := \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_{m,E} \to \mathrm{Res}_{E/\mathbb{Q}}\, G^{\mathrm{ab}}_E \to G^{\mathrm{ab}}.$$

Define $\mathcal{B} = G \times_{G^{\mathrm{ab}}} E^*$, where the first map is the natural $\delta : G \to G^{\mathrm{ab}}$. It is straightforward to check the following.

Lemma 4.6.2. *The scheme $\mathcal{B}$ is a reductive group scheme over $\mathbb{Q}$, sitting in a short exact sequence of $\mathbb{Q}$-group schemes*

$$0 \to G^{\mathrm{der}} \to \mathcal{B} \to E^* \to 0.$$

Suppose now that $(G, \mathfrak{X})$ induces a connected Shimura datum $(G^{\mathrm{der}}, \mathfrak{X}^+)$ with reflex field $E(G^{\mathrm{der}}, \mathfrak{X}^+) := E(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}}) \subset E$. We could make the above construction varying over all Shimura data with connected datum $(G^{\mathrm{der}}, \mathfrak{X}^+)$ and reflex field contained in E. The following is an adaptation of the argument in [Deligne 1979, 2.5].

Proposition 4.6.3. *The extension*

$$0 \to G^{\mathrm{der}} \to \mathcal{B} \to E^* \to 0$$

depends only on the connected Shimura datum $(G^{\mathrm{der}}, \mathfrak{X}^+)$ and the field E.

Proof. Let $(G, \mathfrak{X})$ and $(G', \mathfrak{X}')$ be two Shimura data whose reflex fields are both contained in E and which give rise to the same connected Shimura datum. By [Deligne 1979, 2.5.6] we can find a third such Shimura datum $(G'', \mathfrak{X}'')$ which admits maps

$$(G, \mathfrak{X}) \leftarrow (G'', \mathfrak{X}'') \to (G', \mathfrak{X}').$$

It will therefore suffice to show for any morphism $\alpha : (G, \mathfrak{X}, E) \to (G_2, \mathfrak{X}_2, E_2)$ of Shimura data with a field of definition (where $E \to E_2$ represents an inclusion $E_2 \subset E$) that we have an induced natural morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{\mathrm{der}} & \longrightarrow & \mathcal{B} & \longrightarrow & E^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_2^{\mathrm{der}} & \longrightarrow & \mathcal{B}_2 & \longrightarrow & E_2^* & \longrightarrow & 0 \end{array}$$

where the outer maps are the natural ones induced by $\alpha : G \to G_2$ and $N_{E/E_2} : E^* \to E_2^*$. Since in the above special case both these maps are isomorphisms, we will get a canonical identification

$$\mathcal{B} \xleftarrow{\simeq} \mathcal{B}'' \xrightarrow{\simeq} \mathcal{B}'$$

of extensions, proving the proposition.

Let us verify the claim. Recall that to give a map $\mathcal{B} \to \mathcal{B}_2$ is to give maps $f : \mathcal{B} \to G_2$ and $k : \mathcal{B} \to E_2^*$ that agree when extended to G_2^{ab}. Write $N\mu : E^* \to G^{\mathrm{ab}}$ and $\delta : G \to G^{\mathrm{ab}}$ and $N\mu_2$ and δ_2 for the analogues for G_2. Let us take f to be the natural composite

$$f : \mathcal{B} \xrightarrow{\mathrm{pr}_G} G \to G_2$$

and k to be

$$k : \mathcal{B} \xrightarrow{\mathrm{pr}_{E^*}} E^* \xrightarrow{N_{E/E_2}} E_2^*.$$

We must check that whenever $\delta(g) = N\mu(e)$, we have

$$\delta_2(\alpha(g)) = N\mu_2(N_{E/E_2}(e)).$$

But by functoriality of $(-)^{\mathrm{ab}}$

$$\delta_2(\alpha(g)) = \alpha^{\mathrm{ab}}(\delta(g)) = \alpha^{\mathrm{ab}}(N\mu(e)) = N\mu_2(N_{E/E_2}(e)),$$

where the final equality holds because since α is a morphism of Shimura data we have in particular that $\alpha^{\mathrm{ab}}\mu = \mu_2 : \mathbb{G}_{m,\mathbb{C}} \to G^{\mathrm{ab}}_{2,\mathbb{C}}$, from which it follows that $\alpha^{\mathrm{ab}} \circ N\mu = N\mu_2 \circ N_{E/E_2}$ by functoriality of the norm map between tori. Thus we have a map $\mathcal{B} \to \mathcal{B}_2$, and it is clear from its definition that it fits into the diagram described. □

4.6.4. Our next task is to define a Shimura datum. Take any $h_G \in \mathfrak{X}^+$, and we define a canonical map

$$h_E : \mathbb{C}^* \to E^*(\mathbb{R})$$

as follows. Let $\tau : E \hookrightarrow \mathbb{C}$ be the canonical inclusion of the reflex field E into $\mathbb{C}$, and write $E^*(\mathbb{R}) = E^*_\tau \times \prod_{\tau' \neq \tau} E^*_{\tau'}$. If τ is real, set

$$h_E(z) = (z\bar{z}; 1, 1, \ldots, 1).$$

If τ is complex,

$$h_E(z) = (z; 1, \ldots, 1)$$

(i.e., the entries away from the place τ are all trivial in both cases).

Proposition 4.6.5. *These give a map $h_G \times h_E : \mathbb{S} \to \mathcal{B}_{\mathbb{R}}$ which defines a Shimura datum with a natural map $(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) \to (G, \mathfrak{X})$. Moreover, it is independent of the choice of h_G, has reflex field E, and only depends on E and the connected Shimura datum $(G^{\mathrm{der}}, \mathfrak{X}^+)$ up to canonical isomorphism.*

Proof. We must check it is defined, which amounts to proving that $r_{\mathbb{R}}(h_E(z)) = \delta(h_G(z))$. First we note that

$$\delta(h_G(z)) = \mu_E(z)\mu_E(\bar{z}) = \mu_E(z)\overline{\mu_E(z)}$$

where we view μ_E as a character $\mathbb{G}_{m,\mathbb{C}} \to G^{\mathrm{ab}}_{\mathbb{C}}$ using the canonical embedding τ of E.

On the other hand recall that

$$r_{\mathbb{R}} = (N_{E/\mathbb{Q}} \circ \mu_E)_{\mathbb{R}} : E^*_{\mathbb{R}} \to G^{\mathrm{ab}}_{E\otimes_{\mathbb{Q}}\mathbb{R}} \to G^{\mathrm{ab}}_{\mathbb{R}}.$$

If τ is real then we have

$$(z\bar{z}, 1, 1, \ldots, 1) \mapsto (\mu_E(z\bar{z}), 1, \ldots, 1) \mapsto \mu_E(z\bar{z}) = \mu_E(z)\overline{\mu_E(z)}$$

and if τ is complex then

$$(z, 1, 1, 1, 1) \mapsto (\mu_E(z), 1, \dots, 1) \mapsto \mu_E(z)\overline{\mu_E(z)},$$

where again all occurrences of μ_E are viewed over $\mathbb{C}$ via τ.

In particular we have checked the necessary equality.

It defines a Shimura datum because after projecting along $\mathcal{B} \to G^{\text{ad}}$, $h_G \times h_E$ agrees with h_G. This is independent of the choice of h_G because in general $\mathfrak{X}$ is a union of copies of $\mathfrak{X}^+$ so every other choice for $h_G \in \mathfrak{X}^+$ must be obtained. For the statement about the reflex field, we note that it is immediate from the general fact that $E(\mathcal{B}, \mathfrak{X}) = E(\mathcal{B}^{\text{ad}}, \mathfrak{X}^{\text{ad}}_{\mathcal{B}})E(\mathcal{B}^{\text{ab}}, h) = E$.

To show it only depends on $(G^{\text{der}}, \mathfrak{X}^+)$ and E we just return to the argument above for $\mathcal{B}$ being independent. Given $(G, \mathfrak{X})$ and $(G', \mathfrak{X}')$ Shimura data with reflex fields contained in E and the same connected Shimura datum, we can again apply [Deligne 1979, 2.5.6] and find $(G'', \mathfrak{X}'')$ mapping to both. Since the construction of $(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ is functorial in $(G, \mathfrak{X})$, choosing $h \in \mathfrak{X}^+$ immediately gives a commuting diagram

$$\begin{array}{ccccc} \mathbb{S} & = & \mathbb{S} & = & \mathbb{S} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}_{\mathbb{R}} & \xleftarrow{\sim} & \mathcal{B}''_{\mathbb{R}} & \xrightarrow{\sim} & \mathcal{B}'_{\mathbb{R}}. \end{array}$$

From this it is clear that $(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ depends only on $(G^{\text{der}}, \mathfrak{X})$ and E. □

We would like to check the obvious functorialities for the pair $(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$.

Lemma 4.6.6. *The above construction is functorial in the following senses*:

(1) *Let* $u : (G_1, \mathfrak{X}_1^+) \to (G_2, \mathfrak{X}_2^+)$ *be a map of connected Shimura data*: *that is*, *a central isogeny* $G_1 \to G_2$ *of semisimple groups such that* $\mathfrak{X}_1^{+,ad} = \mathfrak{X}_2^{+,ad}$, *and* $E \supset E(G_1, \mathfrak{X}_1^+)$. *Let* $(\mathcal{B}_1, \mathfrak{X}_1)$ *and* $(\mathcal{B}_2, \mathfrak{X}_2)$ *be the Shimura data associated to the triples* $(G_1, \mathfrak{X}_1^+, E)$ *and* $(G_2, \mathfrak{X}_2^+, E)$ *by the above procedure. Then there is a canonical morphism*

$$u_* : (\mathcal{B}_1, \mathfrak{X}_1) \to (\mathcal{B}_2, \mathfrak{X}_2)$$

of Shimura data also induced by a natural central isogeny $\mathcal{B}_1 \to \mathcal{B}_2$.

(2) *Let* $(G^{\text{der}}, \mathfrak{X}^+)$ *be a connected Shimura datum*, $E(G^{\text{der}}, \mathfrak{X}^+) \subset F \subset E$ *two fields*, *and* $(\mathcal{B}_F, \mathfrak{X}_F)$ *and* $(\mathcal{B}_E, \mathfrak{X}_E)$ *the associated data. Then the norm map induces a canonical morphism of Shimura data*

$$N_{E/F} : (\mathcal{B}_E, \mathfrak{X}_E) \to (\mathcal{B}_F, \mathfrak{X}_F).$$

Proof. For (1), let $\Delta_1 \subset G_1 \subset \mathcal{B}_1$ be the kernel of $G_1 \to G_2$ viewed as a subgroup of the centre of $\mathcal{B}_1$. Then $(\mathcal{B}_1/\Delta_1, \mathfrak{X}_1/\Delta_1)$ is a Shimura datum with connected Shimura datum $(G_2, \mathfrak{X}_2^+)$ and reflex field contained in E. It therefore receives a natural map $(\mathcal{B}_2, \mathfrak{X}_2) \to (\mathcal{B}_1/\Delta_1, \mathfrak{X}_1/\Delta_1)$ which we claim is an isomorphism. It will suffice to check $\mathcal{B}_2 \to \mathcal{B}_1/\Delta_1$ is an isomorphism, which is immediate because it is a pullback of the identity map on E^*. Thus we obtain the canonical map

$$(\mathcal{B}_1, \mathfrak{X}_1) \to (\mathcal{B}_1/\Delta_1, \mathfrak{X}_1/\Delta_1) \xleftarrow{\sim} (\mathcal{B}_2, \mathfrak{X}_2)$$

required.

For (2), for any Shimura datum $(G, \mathfrak{X})$ with connected Shimura datum $(G^{\mathrm{der}}, \mathfrak{X}^+)$ and reflex field contained in F, one obviously has a canonical map $\mathcal{B}_E = G \times_{G^{\mathrm{ab}}} E^* \to G \times_{G^{\mathrm{ab}}} F^* = \mathcal{B}_F$ induced by the norm morphism[11] $N_{E/F} : E^* \to F^*$. Visibly $h_F = N_{E/F} \circ h_E$, so this induces the map of Shimura data claimed. □

4.7. *Abelian type case.* We now return to integral canonical models of standard principal bundles. Fix $(G_2, \mathfrak{X}_2)$ a Shimura datum of abelian type unramified away from N, $G_2/\mathbb{Z}[1/N]$ a reductive integral model and $K_2^N = \prod_{p \nmid N} G_2(\mathbb{Z}_p)$. The goal of this section is to prove the following.

Theorem 4.7.1. *There exists an integral canonical model for $P_{K_2^N}(G_2, \mathfrak{X}_2)$.*

4.7.2. As in Section 2.5.14 we may find $(G, \mathfrak{X})$ of Hodge type with G unramified away from N with $j : G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ a central isogeny inducing a morphism of connected Shimura data $(G^{\mathrm{der}}, \mathfrak{X}^+) \to (G_2^{\mathrm{der}}, \mathfrak{X}_2^+)$. By Lemma 2.5.13 we may take $G/\mathbb{Z}[1/N]$ reductive such that j is defined over $\mathbb{Z}[1/N]$.

By the construction of the previous section, we may obtain a diagram of Shimura data

$$\begin{array}{ccc} (\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) & \longrightarrow & (G, \mathfrak{X}) \\ \downarrow & & \\ (\mathcal{B}_2, \mathfrak{X}_{\mathcal{B},2}) & \longrightarrow & (G_2, \mathfrak{X}_2). \end{array}$$

The plan has three stages. We first combine our constructions in the special and Hodge type cases to construct a canonical integral model of $P(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$. We next pass to connected components and take a quotient to transfer this bundle from a $\mathcal{B}^c$-bundle on $\mathrm{Sh}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})^+$ to a G_2^c-bundle on $\mathrm{Sh}(G_2, \mathfrak{X}_2)^+$ defined over the field E_N over which the connected components split off. We finally then assemble over the whole Shimura variety and descend these models making liberal use of the uniqueness statement for canonical models in place of the extension property from §2.5. We believe this framework could be useful in other contexts where one wants to extend a "G-valued" construction over Hodge type Shimura varieties to abelian type Shimura varieties.

[11] Recall that the norm map $N_{E/F}$ may be defined for example as the composite of $\mathbb{Q}$-group maps $E^* \hookrightarrow \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_F(E) \xrightarrow{\det} F^*$.

4.7.3. We start out by checking this picture respects the integral structures on G and G_2. Since $E := E(G, \mathfrak{X})$ is absolutely unramified at $p \nmid N$ and $E_2 \subset E$ (hence E_2^* and E^* are smooth tori over $\mathbb{Z}[1/N]$) we have a diagram defined over $\mathbb{Z}[1/N]$

$$G \leftarrow \mathcal{B} = G \times_{G^{\mathrm{ab}}} E^* \rightarrow \mathcal{B}_2 = G_2 \times_{G_2^{\mathrm{ab}}} E_2^* \rightarrow G_2.$$

We remark that the middle map is a composite of Lemma 4.6.6(1) and (2), the first of which involves a change of derived group (and by construction $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ is defined over $\mathbb{Z}[1/N]$), and the second of which is a norm map $N_{E/E_2} : E^* \to E_2^*$ which clearly respects integral structures. We let K^N, K_2^N and $K_{\mathcal{B}}^N$ denote the obvious corresponding hyperspecial prime to N level structures and note there is a corresponding diagram of integral models of Shimura varieties

$$\mathcal{S}_{K^N}(G, \mathfrak{X}) \leftarrow \mathcal{S}_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) \rightarrow \mathcal{S}_{K_2^N}(G_2, \mathfrak{X}_2).$$

Let us, now this diagram is in place, check the following condition Section 4.4.3, which we recall is important in guaranteeing the uniqueness of canonical models.

Proposition 4.7.4. *With notation as above,* $\mathcal{S}_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ *and* $\mathcal{S}_{K_2^N}(G_2, \mathfrak{X}_2)$ *both have all the crystalline points.*

Proof. Let $x \in \mathcal{S}_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})(W(k))$ and $K_0 = W(k)[1/p]$. Also let $W(k)^{\mathrm{ur}}$ be the ring of integers of the maximal unramified (algebraic) extension K_0^{ur}/K_0. Note that its images in $\mathcal{S}_{K^N}(G, \mathfrak{X})$ and $\mathcal{S}_{\hat{E^*,N}}(E^*, h_E)$ are both crystalline (since these Shimura data have all the crystalline points). To check x is crystalline, it will suffice to check $\omega_{\mathrm{et},x}(V)$ is crystalline for V a faithful representation of $\mathcal{B}$. Let us take $G \hookrightarrow \mathrm{GL}(V_1)$ and $E^* \hookrightarrow \mathrm{GL}(V_2)$ and consider the representation

$$\mathcal{B} \subset G \times E^* \hookrightarrow \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \subset \mathrm{GL}(V_1 \oplus V_2).$$

Considering the diagram of Shimura varieties and towers at infinite level at p, it is clear that $\Gamma_{K_0} \to \mathrm{GL}(\omega_{\mathrm{et}}(V_1 \oplus V_2))$ acts via $\Gamma_{K_0} \to \mathrm{GL}(\omega_{\mathrm{et}}(V_1)) \times \mathrm{GL}(\omega_{\mathrm{et}}(V_2))$, each projection of which is crystalline, and this is enough to give the first part of our proposition.

We turn our attention to $\mathcal{S}_{K_2^N}(G_2, \mathfrak{X}_2)$. First, since it receives a map from the Shimura variety $\mathcal{S}_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ and we observe that the image of a crystalline point is crystalline, and moreover at finite level these maps are finite étale, we deduce that in particular all $W(k)^{\mathrm{ur}}$ points on the geometric connected component $\mathcal{S}_{K_2^N}{}^+(G_2, \mathfrak{X}_2)_{W(k)^{\mathrm{ur}}}$ are crystalline.

To show that any point $x \in \mathcal{S}_{K_2^N K_{2,N}}(W(k))$ is crystalline, recall that Kisin's integral model at level $K_{2,p} := G_2(\mathbb{Z}_p)$ over $W(k)$, $\mathcal{S}_{K_{2,p}}(G_2, \mathfrak{X}_2)_{W(k)}$ has a $G(\mathbb{A}^{\infty,p})$-action that acts transitively on geometric connected components and acts equivariantly with regard to the tower at p on the generic fibre. Therefore, taking a lift $\tilde{x}$ of x to level $K_{2,p}$, which we may assume is a $W(k)^{\mathrm{ur}}$ point since $\mathcal{S}_{K_{2,p},W(k)} \to$

$\mathscr{S}_{K_{2,p}K^p,W(k)}$ is formally étale, and taking a translate $\tilde{x}.a$ by a Hecke operator $a \in G(\mathbb{A}^{\infty,p})$ such that $\tilde{x}.a$ lies in the connected component, we may deduce that x is crystalline from the fact that all points on the connected component are, and that the crystalline property can be checked after restricting a Γ_{K_0} representation to $\Gamma_{K_0^{\mathrm{ur}}}$. $\square$

We now proceed with constructing the canonical models. The content of the next step can be extracted as a lemma.

Lemma 4.7.5. *Let S be a scheme, and $G \twoheadrightarrow \Delta \leftarrow H$ a diagram of S-groups, $\mathscr{B} = G \times_\Delta H$ the fibre product group. Let X be an S-scheme and suppose we are given P_G a G-torsor and P_H an H-torsor on X with the property that there is an isomorphism of Δ-torsors $\theta : P_H \times^H \Delta \xrightarrow{\sim} P_G \times^G \Delta$.*

Then there is a unique (up to canonical isomorphism) $\mathscr{B}$-torsor $P_{\mathscr{B}}$ together with torsor isomorphisms $\theta_G : P_{\mathscr{B}} \times^{\mathscr{B}} G \xrightarrow{\sim} P_G$ and $\theta_H : P_{\mathscr{B}} \times^{\mathscr{B}} H \xrightarrow{\sim} P_H$ such that the induced isomorphism

$$\theta_H \circ \theta_G^{-1} : P_G \times^G \Delta \xrightarrow{\sim} P_{\mathscr{B}} \times^{\mathscr{B}} \Delta \xrightarrow{\sim} P_H \times^H \Delta$$

is equal to θ, and it is given by

$$P_{\mathscr{B}} = P_G \times_\theta P_H,$$

with θ_G and θ_H the natural projections.

We use the notation $X \times_\theta Y$ in a situation where we are given maps $X \to Z_X$ and $Y \to Z_Y$ and an isomorphism $\theta : Z_X \xrightarrow{\sim} Z_Y$ between two schemes to mean the limit of the diagram

$$\begin{array}{ccccc} & & & & Y \\ & & & & \downarrow \\ X & \longrightarrow & Z_X & \underset{\sim}{\xrightarrow{\theta}} & Z_Y. \end{array}$$

Proof. Let us first show that $P_{\mathscr{B}} = P_G \times_\theta P_H$ is indeed a $\mathscr{B}$-torsor. We define the $\mathscr{B} = G \times_\Delta H$-action

$$(p_G, p_H).(g, h) =: (p_G.g, p_H.h)$$

which is visibly an action. Passing to an étale cover $X' \to X$ over which P_G and P_H admit sections p_G and p_H. Since G surjects onto Δ these may be chosen (perhaps at the cost of passing to a finer étale cover) such that if $\pi_G : P_G \to P_G \times^G \Delta$ and $\pi_H : P_H \to P_H \times^H \Delta$ are the projection maps, we have $\theta(\pi_G(p_G)) = \pi_H(p_H)$, giving a section $(p_G, p_H) \in P_{\mathscr{B}}(X')$, from which we may see immediately that it is a $\mathscr{B}$-torsor. It is also obvious that it has the required property.

Uniqueness is essentially formal, but we give the argument. Suppose we are given $P'_{\mathscr{B}}$ together with $\theta'_G : P'_{\mathscr{B}} \times^{\mathscr{B}} G \xrightarrow{\sim} P_G$ and $\theta'_H : P'_{\mathscr{B}} \times^{\mathscr{B}} H \xrightarrow{\sim} P_H$ satisfying the compatibility given. Then by the universal property of $P_{\mathcal{B}} = P_G \times_\theta P_H$ there exists a unique map $\alpha : P'_{\mathcal{B}} \to P_{\mathcal{B}}$ making everything commute. It is easy to check α is a map of $\mathcal{B}$-torsors, whence it is automatically an isomorphism. □

4.7.6. Let us apply this in the context of our standard principal bundles. If T is a torus unramified away from N we introduce the notation $\hat{T}^N = \prod_{p\nmid N} T(\mathbb{Z}_p)$.

Consider the diagram (whose arrows we have named for convenience)

$$\begin{array}{ccc} \mathcal{S}_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) & \xrightarrow[\delta']{} & \mathcal{S}_{\hat{E}^{*,N}}(E^*, h_E) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle N\mu} \\ \mathcal{S}_{K^N}(G, \mathfrak{X}) & \xrightarrow[\delta]{} & \mathcal{S}_{\hat{G}^{ab,N}}(G^{\mathrm{ab}}, h^{\mathrm{ab}}). \end{array}$$

By Theorem 4.5.1 we have $\mathcal{P}_{K^N}$ over $\mathcal{S}_{K^N}(G, \mathfrak{X})$ an integral canonical model for $P_{K^N}(G, \mathfrak{X})$, and by Proposition 4.2.3 we have $\mathcal{P}_{\hat{E}^{*,N}}$ and $\mathcal{P}_{\hat{G}^{ab,N}}$ canonical models over the right hand side of the diagram also. By Proposition 4.4.8 there are natural isomorphisms of G^{ab} torsors

$$N\mu^*\mathcal{P}_{\hat{G}^{ab,N}} \cong \mathcal{P}_{\hat{E}^{*,N}} \times^{E^{*c}} G^{\mathrm{ab}} \quad \text{and} \quad \delta^*\mathcal{P}_{\hat{G}^{ab,N}} \cong \mathcal{P}_{K^N} \times^G G^{\mathrm{ab}}.$$

Pulling these back further and using that the diagram commutes and so we have canonical identifications $\pi^*\delta^* \cong (\delta\pi)^* = (N\mu\delta')^* \cong \delta'^*(N\mu)^*$, we obtain a natural isomorphism

$$\theta : \pi^*(\mathcal{P}_{K^N} \times^G G^{\mathrm{ab}}) \xrightarrow{\sim} \delta'^*(\mathcal{P}_{\hat{E}^{*,N}} \times^{E^{*c}} G^{\mathrm{ab}}).$$

Note further that any integral canonical model $\mathcal{P}_{K^N_{\mathcal{B}}}$ for $P_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ will by Proposition 4.4.8 be required to satisfy the conditions of Lemma 4.7.5 with respect to this isomorphism θ. Therefore by Lemma 4.7.5 if it exists it is given by

$$\mathcal{P}_{K^N_{\mathcal{B}}} := \pi^*\mathcal{P}_{K^N} \times_\theta \delta'^*\mathcal{P}_{\hat{E}^{*,N}},$$

together with the identification

$$\begin{aligned} \iota_{\mathcal{B}} : \mathcal{P}_{K^N_{\mathcal{B}}} \otimes_{\mathcal{O}_E[1/N]} E &= (\pi^*\mathcal{P}_{K^N} \times_\theta \delta'^*\mathcal{P}_{\hat{E}^{*,N}}) \otimes_{\mathcal{O}_E[1/N]} E \\ &\xrightarrow{(\iota_G, \iota_{E^*})} \pi^* P_{K^N}(G, \mathfrak{X}) \times_{\theta\otimes E} \delta'^* P_{\hat{E}^{*,N}}(E^*, h_E) = P_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}). \end{aligned}$$

Proposition 4.7.7. *As defined above, $(\mathcal{P}_{K^N_{\mathcal{B}}}, \iota_{\mathcal{B}})$ is an integral canonical model for $P_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})$ with equivariant $\mathcal{A}_P^N(\mathcal{B})$-action.*[12]

Proof. Our task is to show the connection, filtration and $\mathcal{A}_P^N(\mathcal{B})$-actions extend, and check the lattice property.

[12] Recall we defined this group in Section 4.5.8.

We first study the connection. Let $p_i, i = 1, 2$ be the projections from a first order neighbourhood of the diagonal, abusively retaining the same notation for morphisms of schemes and their first order thickenings. Then[13]

$$p_i^* \mathcal{P}_{K^N_{\mathcal{B}}} = p_i^*(\pi^* \mathcal{P}_{K^N} \times_\theta \delta'^* \mathcal{P}_{\hat{E}^*,N}) \cong \pi^* p_i^* \mathcal{P}_{K^N} \times_{p_i^*(\theta)} \delta'^* p_i^* \mathcal{P}_{\hat{E}^*,N},$$

and in these coordinates, $\nabla_{\mathcal{B}} = (\pi^* \nabla_G, \delta'^* \operatorname{id})$ is the connection required so we see directly that it extends.

For the filtrations, it is easy to check using [Conrad 2014, 5.3.4] that in fact we may naturally identify

$$\mathcal{GR}_\mu = G/P_\mu = G^{\mathrm{der}}/(P_\mu \cap G^{\mathrm{der}}) = \mathcal{B}/\mathcal{P}_{\mu_{\mathcal{B}}} = \mathcal{GR}_{\mu_{\mathcal{B}}}$$

with G and $\mathcal{B}$-actions factoring through G^{ad}, so actually the composite

$$\gamma_{\mathcal{B}} : \mathcal{P}_{K^N_{\mathcal{B}}} \to \mathcal{P}_{K^N} \xrightarrow{\gamma} \mathcal{GR}_\mu = \mathcal{GR}_{\mu_{\mathcal{B}}}$$

does the job.

We also note that we already have a natural extension of the Hecke action via

$$\prod_{p|N} \mathcal{B}(\mathbb{Q}_p) \subset \Big(\prod_{p|N} G(\mathbb{Q}_p) \Big) \times \Big(\prod_{p|N} E^*(\mathbb{Q}_p) \Big)$$

which acts via its two projections on $\pi^* \mathcal{P}_{K^N} \times_\theta \delta'^* \mathcal{P}_{\hat{E}^*,N}$.

For the lattice property, we borrow the trick from the first part of the proof of Proposition 4.7.4, taking a faithful representation $V = V_1 \oplus V_2$ of $\mathcal{B}$, formed as a sum of faithful representations of G and E^*. It is easy to check from the construction that

$$\omega_{\mathcal{P}_{K^N_{\mathcal{B}}}}(V) = \omega_{\pi^* \mathcal{P}_{K^N}}(V_1) \oplus \omega_{\delta'^* \mathcal{P}_{\hat{E}^*,N}}(V_2)$$

whence the lattice property follows immediately from that already known for the two terms on the right hand side together with the fact that the image of a crystalline point is always crystalline. Now we know the lattice property, we get a uniqueness statement, and the $\mathcal{A}_P^N(\mathcal{B})$-action extends formally. □

4.7.8. We now pass to a connected component

$$\mathrm{Sh}^+_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) = \mathrm{Sh}^+_{K^N}(G, \mathfrak{X}),$$

recalling Lemma 2.5.1 that it is defined over an extension E_N/E unramified away from N and letting $\mathbb{O} := \mathbb{O}_{E_N}[1/N]$. We may therefore extend the above identification to

$$\mathcal{S}^+_{K^N_{\mathcal{B}}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})_{\mathbb{O}} = \mathcal{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}},$$

[13] We also abuse notation here and confuse π and δ' with their induced maps on first order neighbourhoods of the diagonal.

and restrict $\mathcal{P}_{K^N_{\mathcal{B}}}$ to get the $\mathcal{B}^c$-torsor

$$\mathcal{P}^+_{\mathcal{B}} := \mathcal{P}_{K^N_{\mathcal{B}}} \times_{\mathcal{S}_{K^N_{\mathcal{B}}}(\mathcal{B},\mathfrak{X}_{\mathcal{B}})_{\mathbb{O}}} \mathcal{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}} \to \mathcal{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}}.$$

Recall the group

$$\Delta^N(G, G_2) = \operatorname{Ker}(\mathcal{A}^{N,\circ}(G^{\mathrm{der}}) \to \mathcal{A}^{N,\circ}(G_2^{\mathrm{der}}))$$

with the property that

$$\mathcal{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}}/\Delta^N(G, G_2) = \mathcal{S}^+_{K_2^N}(G_2, \mathfrak{X}_2)_{\mathbb{O}}.$$

Let us also recall the extension of group schemes over $\mathbb{Z}[1/N]$

$$1 \to \mathcal{B}^c \to \mathcal{A}^N_P(\mathcal{B}) \to \mathcal{A}^N(\mathcal{B}) \to 1.$$

Forming the pullback along $\mathcal{A}^{N,\circ}(G^{\mathrm{der}}) = \mathcal{A}^{N,\circ}(\mathcal{B}) \subset \mathcal{A}^N(\mathcal{B})$ we get an extension

$$1 \to \mathcal{B}^c \to \mathcal{A}^{N,\circ}_P(\mathcal{B}) \to \mathcal{A}^{N,\circ}(G^{\mathrm{der}}) \to 1$$

which acts on $\mathcal{P}^+_{\mathcal{B}}$ equivariantly in the obvious fashion.

Lemma 4.7.9. *These group schemes have the following additional properties*:

(1) *The group* $\Xi := \operatorname{Ker}(\mathcal{B}^c \to G_2^c) \subset \mathcal{B}^c$ *is a normal subgroup of* $\mathcal{A}^{N,\circ}_P(\mathcal{B})$.

(2) *The kernel* $\Delta^N_P(\mathcal{B}, G_2) := \operatorname{Ker}(\mathcal{A}^{N,\circ}_P(\mathcal{B}) \to \mathcal{A}^{N,\circ}_P(G_2))$ *is canonically an extension*

$$1 \to \Xi \to \Delta^N_P(\mathcal{B}, G_2) \to \Delta^N(G, G_2) \to 1$$

which acts freely on $\mathcal{P}^+_{\mathcal{B}}$.

Proof. We first remark that (1) is obvious because $\Xi \subset Z(\mathcal{B}^c)$ which commutes with the whole of $\mathcal{A}^{N,\circ}_P(\mathcal{B})$.

For (2), that Ξ is a subgroup is clear and normality follows by (1). Existence of the maps and exactness in the middle follow in the usual way. That the final map is a surjection follows because for any $(g, \gamma^{-1}) \in \Delta^N(G, G_2)$, it is hit by $(1, g, \gamma^{-1}) \in \Delta^N_P(\mathcal{B}, G)$. Finally the action on $\mathcal{P}^+_{\mathcal{B}} \to \mathcal{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}}$ is free because Ξ acts freely on each fibre, and $\Delta^N(G, G_2)$ acts freely on $\mathcal{S}^+_{K^N}(G, \mathfrak{X})_{\mathbb{O}}$ by Proposition 2.5.9. □

4.7.10. Equipped with this lemma we may construct a G_2^c-bundle

$$\mathcal{P}^+_2 := (\mathcal{P}^+_{\mathcal{B}}/\Delta^N_P(\mathcal{B}, G_2)) \times^{\mathcal{B}^c/\Xi} G_2^c \to \mathcal{S}^+_{K_2^N}(G_2, \mathfrak{X}_2)_{\mathbb{O}}$$

with equivariant $\mathcal{A}^{N,\circ}_P(G_2)$-action. Moreover we have

$$\begin{aligned}\iota_2 : \mathcal{P}^+_2 \otimes_{\mathbb{O}} E_N &= (\mathcal{P}^+_{\mathcal{B}} \otimes_{\mathbb{O}} E_N/\Delta^N_P(\mathcal{B}, G_2)) \times^{\mathcal{B}^c/\Xi} G_2^c \\ &\xrightarrow{\iota_{\mathcal{B}}} (P^+_{K^N_{\mathcal{B}}, E_N}/\Delta^N_P(\mathcal{B}, G_2)) \times^{\mathcal{B}^c/\Xi} G_2^c = P^+_{K_2^N, E_N},\end{aligned}$$

where the final equality follows by working over $\mathbb{C}$ and the argument of [Milne 1988, 7.2].

Proposition 4.7.11. *The pair* $(\mathcal{P}_2^+, \iota_2)$ *is an integral canonical model for the bundle* $P^+_{K_2^N, E_N}(G_2, \mathfrak{X}_2)$.

Proof. The Hecke action extends by construction. Let us check that the connection extends. We know (from direct observation over $\mathbb{C}$) that the $\Delta_P^N(\mathcal{B}, G_2)$ action is horizontal, so the flat connection $\nabla : p_1^*\mathcal{P}^+_{\mathcal{B}} \xrightarrow{\sim} p_2^*\mathcal{P}^+_{\mathcal{B}}$ descends (and then can obviously be pushed out along $\mathcal{B}^c/\Xi \to G_2^c$) to a flat connection

$$\nabla : p_1^*\mathcal{P}_2^+ \xrightarrow{\sim} p_2^*\mathcal{P}_2^+.$$

Since a similar relationship also relates the connections on the generic fibre we have shown that this ∇ extends over $\mathbb{O}$ the connection on $P_{K_2^N, E_N}$.

To check the lattice condition at a crystalline point x, note the commutative diagram of E_N-schemes, letting $K_{\mathcal{B}}^{Np}$ and K_2^{Np} be the obvious full level structures at all primes except N and p

$$\begin{array}{ccc}
\mathrm{Sh}_{K_{\mathcal{B}}^{Np}}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})|_{\mathrm{Sh}^+_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})} & \longrightarrow & \mathrm{Sh}_{K_2^{Np}}(G_2, \mathfrak{X}_2)|_{\mathrm{Sh}^+_{K_2^N}(G_2, \mathfrak{X}_2)} \\
\downarrow & & \downarrow \\
\mathrm{Sh}^+_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}}) & \longrightarrow & \mathrm{Sh}^+_{K_2^N}(G_2, \mathfrak{X}_2).
\end{array}$$

Taking a lift $\tilde{x}$ of x to $\mathcal{S}^+_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})_{\mathbb{O}}$, which is crystalline since the map

$$\mathcal{S}^+_{K_{\mathcal{B}}^N}(\mathcal{B}, \mathfrak{X}_{\mathcal{B}})_{\mathbb{O}} \to \mathcal{S}^+_{K_2^N}(G_2, \mathfrak{X}_2)_{\mathbb{O}}$$

is finite étale and its source has all the crystalline points, the diagram gives an identification

$$\omega_{\mathrm{et},\tilde{x}} \circ \mathrm{Res}^{G_2^c(\mathbb{Z}_p)}_{\mathcal{B}^c(\mathbb{Z}_p)} = \omega_{\mathrm{et},x} : \mathrm{Rep}_{\mathbb{Z}_p}(G_2^c) \to \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_{\mathbb{Q}_p^{\mathrm{ur}}}).$$

But now the lattice condition is immediate, since equality of lattices can be checked after passing to a finite étale cover, and $\mathcal{P}^+_{2,x}$ pulls back to $\mathcal{P}^+_{\mathcal{B},\tilde{x}} \times^{\mathcal{B}^c} G_2^c$ which as a canonical model already has the required property that $\omega_{\mathcal{P}^+_{\mathcal{B},\tilde{x}}} = \mathbb{D} \circ \omega_{\mathrm{et},\tilde{x}}$ by its own lattice condition.

It remains to check the filtration $\gamma_B : \mathcal{P}^+_{\mathcal{B}} \to \mathcal{GR}_{\mu_{\mathcal{B}}}$ descends to $\gamma_2 : \mathcal{P}^+_2 \to \mathcal{GR}_{\mu_2}$, which amounts to showing that the composite

$$\mathcal{P}^+_{\mathcal{B}} \to \mathcal{GR}_{\mu_{\mathcal{B}}} \to \mathcal{GR}_{\mu_2}$$

is $\Delta_P^N(\mathcal{B}, G_2)$-invariant. But this is clear; since $\Xi = \mathrm{Ker}(\mathcal{B}^c \to G_2^c) \subset Z(\mathcal{B}^c)$ it acts trivially on $\mathcal{GR}_{\mu_2}$ and $\Delta^N(G, G_2)$ acts trivially because Hecke operators always act trivially on any Gr_μ. $\square$

4.7.12. With this in hand, we are finally able to construct an integral canonical model $\mathcal{P}_2$ for $P_{K_2^N}(G_2, \mathfrak{X}_2)$ in the spirit of Lemma 2.5.16, but with the uniqueness of canonical models used in place of the extension property, because it may be used to canonically extend any isomorphism.

Indeed, let us decompose $\mathcal{S}_{K_2^N,\mathbb{O}} = \coprod_{c\in\pi_0} \mathcal{S}_c$ into components each of which is geometrically integral. By the argument of Lemma 2.5.16(1) we can find for each $c \in \pi_0$ some $\mathcal{G}_c/\mathbb{Z}[1/N]$ a model for G_2 such that we may identify $\mathcal{S}_c \cong \mathcal{S}_{\prod_{p\nmid N}\mathcal{G}_c(\mathbb{Z}_p),\mathbb{O}}{}^+$ via a Hecke operator. For each c we may make such a choice and invoking Proposition 4.7.11 for $\mathcal{G} = \mathcal{G}_c$ and using the Hecke equivariance of P_{K^N} we obtain a canonical model $\mathcal{P}_c^+$ for $P_{K_2^N,E_N}(G_2, \mathfrak{X}_2)|_{\mathcal{S}_c\otimes_{\mathbb{O}} E_N}$. Taking the disjoint union of these we obtain

$$\mathcal{P}_{2,\mathbb{O}} := \coprod_{c\in\pi_0} \mathcal{P}_c^+,$$

an integral canonical model minus a full Hecke action. But by the uniqueness of canonical models (Lemma 4.4.7) it is formal to extend the Hecke operators acting between components. Finally by Lemma 4.4.9 it descends to $\mathcal{P}_2/\mathbb{O}_{E_2}[1/N]$, an integral canonical model for $P_{K_2^N}(G_2, \mathfrak{X}_2)/E_2$. Thus our main theorem has been proved.

4.8. ***Automorphic vector bundles.*** With our integral canonical models $\mathcal{P}_{K^N}(G, \mathfrak{X})$ constructed in the abelian type case, we should discuss the construction of automorphic vector bundles (recall the discussion Section 3.1.4) in this setting, although it requires no new ideas.

Theorem 4.8.1. *Let $(G, \mathfrak{X})$ be a Shimura datum of abelian type, $\mathcal{G}/\mathbb{Z}[1/N]$ a reductive model for G, $K^N = \prod_{p\nmid N} \mathcal{G}(\mathbb{Z}_p)$ and for μ a Hodge cocharacter of $(G, \mathfrak{X})$, and $L/E = E(G, \mathfrak{X})$ a finite extension.*

Then we have a canonical functor $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})$ from $\mathcal{G}_{\mathbb{O}_L[1/N]}$-equivariant vector bundles on $\mathcal{GR}_{\mu,\mathbb{O}_L[1/N]}$ to vector bundles on $\mathcal{S}_{K^N}$ which on the generic fibre is identified naturally with that of [Milne 1990, III 5.1].

Proof. Recall that we have the picture

$$\mathcal{S}_{K^N} \leftarrow \mathcal{P}_{K^N} \xrightarrow{\gamma} \mathcal{GR}_\mu.$$

The functor $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})$ is given by pulling back $\mathcal{J} \mapsto \gamma^*\mathcal{J}$ and then using the usual equivalence between $\mathcal{G}$-equivariant vector bundles on $\mathcal{P}_{K^N}$ and vector bundles on $\mathcal{S}_{K^N}$. This is obviously a functor, and the compatibility with the usual construction [Milne 1990, III 5.1] follows because $\mathcal{P}_{K^N}$ comes with a canonical identification

$$\iota : \mathcal{P}_{K^N} \otimes E \xrightarrow{\sim} P_{K^N}(G, \mathfrak{X})$$

under which γ is an extension of the G-filtration on the RHS. □

It is also easy to deduce from our construction the following additional functoriality property.

Proposition 4.8.2. *We are given a morphism $f : (G_1, \mathfrak{X}_1, \mathcal{G}_1) \to (G_2, \mathfrak{X}_2, \mathcal{G}_2)$ of Shimura data of abelian type together with reductive models for G_i over $\mathbb{Z}[1/N]$, $L/E(G_1, \mathfrak{X}_1)$ finite, and μ_1 and μ_2 Hodge cocharacters for $\mathfrak{X}_1$ and $\mathfrak{X}_2$ respectively. Suppose we are also given $\mathcal{J}_2$ a $\mathcal{G}_2$-equivariant vector bundle on $\mathcal{G}\mathcal{R}_{\mu_2,\mathcal{O}_L[1/N]}$. Pulling back $\mathcal{J}_2$ along*

$$\mathcal{G}\mathcal{R}_{\mu_1,\mathcal{O}_L[1/N]} \to \mathcal{G}\mathcal{R}_{\mu_2,\mathcal{O}_L[1/N]}$$

and restricting the $\mathcal{G}_2$ action to $\mathcal{G}_1$ we obtain a $\mathcal{G}_1$-equivariant vector bundle $\mathcal{J}_1$.

There is a canonical identification of vector bundles over $\mathcal{S}_{\prod_{p\nmid N} \mathcal{G}_1(\mathbb{Z}_p)}(G_1, \mathfrak{X}_1)$

$$\mathcal{V}(\mathcal{J}_1) \cong f^*\mathcal{V}(\mathcal{J}_2).$$

Proof. This is immediate from the construction of $\mathcal{V}(-)$ and Proposition 4.4.8. □

Acknowledgements

The author would like to extend the greatest of thanks to his PhD supervisor Mark Kisin, under whose guidance this project was completed. This project owes much to his sharp insights, but also his optimism, energy and patience.

We would also like to thank Jack Thorne, after a conversation with him in 2013 provided the initial motivation for this project. For other useful conversations we thank Chris Blake, George Boxer, Lukas Brantner, Justin Campbell, Kestutis Cesnavicius, Erick Knight, Ananth Shankar, Jack Shotton, Rong Zhou, and Yihang Zhu.

This paper is part of the author's PhD thesis completed at and funded by Harvard University, and begun while the author was on a Kennedy Scholarship.

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Communicated by Brian Conrad
Received 2016-08-29 Revised 2017-06-08 Accepted 2017-07-21

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

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Algebra & Number Theory

Volume 11 No. 8 2017

1937-0652(2017)11:8;1-D