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$\mathfrak{p}$-rigidity and Iwasawa μ-invariants

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Let F be a totally real field with ring of integers O and p be an odd prime unramified in F. Let $\mathfrak{p}$ be a prime above p. We prove that a mod p Hilbert modular form associated to F is determined by its restriction to the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ ($\mathfrak{p}$-rigidity). Let K/F be an imaginary quadratic CM extension such that each prime of F above p splits in K and λ a Hecke character of K. Partly based on $\mathfrak{p}$-rigidity, we prove that the μ-invariant of the anticyclotomic Katz $\mathfrak{p}$-adic L-function of λ equals the μ-invariant of the full anticyclotomic Katz p-adic L-function of λ. An analogue holds for a class of Rankin–Selberg p-adic L-functions. When λ is self-dual with the root number -1, we prove that the μ-invariant of the cyclotomic derivatives of the Katz $\mathfrak{p}$-adic L-function of λ equals the μ-invariant of the cyclotomic derivatives of the Katz p-adic L-function of λ. Based on previous works of the authors and Hsieh, we consequently obtain a formula for the μ-invariant of these $\mathfrak{p}$-adic L-functions and derivatives. We also prove a $\mathfrak{p}$-version of a conjecture of Gillard, namely the vanishing of the μ-invariant of the Katz $\mathfrak{p}$-adic L-function of λ.

1. Introduction

Zeta values seem to suggest deep phenomena in Mathematics. They seem to mysteriously encode deep arithmetic information. They also seem to suggest surprising modular and Iwasawa-theoretic phenomena.

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Sometimes, they are a sum of evaluations of modular forms at CM points. Such an expression for critical Hecke L-values and conjectural nontriviality of the corresponding anticyclotomic p-adic L-function, suggested to the second named author a linear independence of mod p Hilbert modular forms. Based on Chai's theory of Hecke-stable subvarieties of a Shimura variety (see [Chai 1995; 2003; 2006]), this guess was proven in [Hida 2010]. Let $\mathfrak{p}$ be a prime above p (see the abstract). Recently, the expression and conjectural nontriviality of the corresponding anticyclotomic $\mathfrak{p}$-adic L-function suggested to us a rather surprising rigidity property of mod p Hilbert modular forms. Partly based on the rigidity, we obtain intriguing equalities of Iwasawa μ-invariants of seemingly independent p-adic L-functions.

Let F be a totally real field of degree d and O the integers ring. Let p be an odd prime unramified in F. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the primes above p. Fix two embeddings $\iota_\infty\colon \overline{\boldsymbol{Q}} \to \boldsymbol{C}$ and $\iota_p\colon \overline{\boldsymbol{Q}} \to \boldsymbol{C}_p$. Let v_p be the p-adic valuation of $\boldsymbol{C}_p$ normalised such that $v_p(p) = 1$. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$.

Let $\mathrm{Sh}_{/\mathbb{F}}$ be the Kottwitz model of the prime-to-p Hilbert modular Shimura variety associated to F. We refer to Section 2B for the definition. Here we only mention that in the moduli interpretation for $\mathrm{Sh}_{/\boldsymbol{Z}_{(p)}}$, the full prime-to-$p$ level structure appears. Let $x \in \mathrm{Sh}$ be a closed ordinary point. From Serre–Tate deformation theory, a p^∞-level structure on x induces a canonical isomorphism

$$\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathrm{Sh},x}) \simeq \prod_i \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}_i}. \tag{1-1}$$

Let $\mathfrak{p} = \mathfrak{p}_i$, for some i. Let f be a mod p Hilbert modular form in the sense of Section 2D. In view of the irreducibility of the connected components of Sh, the form f is determined by its restriction to $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathrm{Sh},x})$. In fact, we have the following rigidity result.

Theorem 1.1 ($\mathfrak{p}$-rigidity). *Let $F/\boldsymbol{Q}$ be a totally real extension with integer ring O, p an odd prime unramified in F and $\mathfrak{p}|p$ a prime in F. Let f be a nonzero mod p Hilbert modular form over F as above.*

Then, f does not vanish identically on the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. In particular, a mod p Hilbert modular form is determined by its restriction to the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

We now describe the results regarding the Iwasawa μ-invariants.

Let K be a totally imaginary quadratic extension of F. Let $h_?$ denote the class number of ?, for ? $= K, F$. Let c denote the complex conjugation on $\boldsymbol{C}$ which induces the unique nontrivial element of $\mathrm{Gal}(K/F)$ via ι_∞. We assume the following hypothesis:

(*ord*) Every prime of F above p splits in K.

The condition (*ord*) guarantees the existence of a p-adic CM type Σ, i.e., Σ is a CM type of K such that the p-adic places induced by elements in Σ via ι_p are disjoint from the ones induced by Σc. We fix such a CM type. We also identify it with the set of infinite places of F. Let K_∞^- (resp. K_∞^+) be the anticyclotomic $\boldsymbol{Z}_p^d$-extension (resp. cyclotomic $\boldsymbol{Z}_p$-extension) of K and $K_{\mathfrak{p},\infty}^- \subset K_\infty^-$ be the $\mathfrak{p}$-anticyclotomic subextension, i.e., the maximal subextension unramified outside the primes above $\mathfrak{p}$ in K. Let $K_{\mathfrak{p},\infty} = K_{\mathfrak{p},\infty}^- K_\infty^+$. Let $\Gamma^\pm := \mathrm{Gal}(K_\infty^\pm/K)$, $\Gamma_\mathfrak{p}^- := \mathrm{Gal}(K_{\mathfrak{p},\infty}^-/K)$ and $\Gamma_\mathfrak{p} := \mathrm{Gal}(K_{\mathfrak{p},\infty}/K)$.

Let $\mathfrak{C}$ be a prime-to-p integral ideal of K. Let λ be an arithmetic Hecke character over K. Suppose that $\mathfrak{C}$ is the prime-to-p conductor of λ. Associated to this data, a natural $(d+1)$-variable Katz p-adic L-function

$$L_{\Sigma,\lambda} = L_{\Sigma,\lambda}(T_1, \ldots, T_d, S) \in \bar{\boldsymbol{Z}}_p[\![\Gamma]\!]$$

is constructed in [Katz 1978; Hida and Tilouine 1993]. Here, the T_i are the anticyclotomic variables and S is the cyclotomic variable. The Katz p-adic L-function interpolates critical Hecke L-values $L(0, \lambda\chi)$ as χ varies over certain Hecke characters over K factoring through the Ray class group with conductor $\mathfrak{C}p^\infty$ (see [Hida and Tilouine 1993, Theorem II]). Let $L_{\Sigma,\lambda}^- \in \bar{\boldsymbol{Z}}_p[\![\Gamma^-]\!]$ (resp. $L_{\Sigma,\lambda,\mathfrak{p}}^- \in \bar{\boldsymbol{Z}}_p[\![\Gamma_\mathfrak{p}^-]\!]$) be the anticyclotomic (resp. $\mathfrak{p}$-anticyclotomic) projection obtained from the projection $\pi^- : \bar{\boldsymbol{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \bar{\boldsymbol{Z}}_p[\![\Gamma^-]\!]$ (resp. $\pi_\mathfrak{p}^- : \bar{\boldsymbol{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \bar{\boldsymbol{Z}}_p[\![\Gamma_\mathfrak{p}^-]\!]$). Let $L_{\Sigma,\lambda,\mathfrak{p}} \in \bar{\boldsymbol{Z}}_p[\![\Gamma_\mathfrak{p}]\!]$ be obtained from the projection $\pi_\mathfrak{p} : \bar{\boldsymbol{Z}}_p[\![\Gamma]\!] \twoheadrightarrow \bar{\boldsymbol{Z}}_p[\![\Gamma_\mathfrak{p}]\!]$. We call $L_{\Sigma,\lambda,\mathfrak{p}}$ the Katz $\mathfrak{p}$-adic L-function to emphasise the consideration of the $\mathfrak{p}$-component. This is a slightly nontraditional terminology as the construction is still under the same embedding ι_p.

For the notion of Iwasawa μ-invariants, we refer to [Hida 2010, §1]. Here we only mention that the μ-invariant measures nontriviality modulo p. We now state our results regarding the Iwasawa μ-invariants of p-adic L-functions and derivatives arising in the context of the Iwasawa theory of an arithmetic Hecke character over a CM field as above.

The μ-invariant of $L_{\Sigma,\lambda,\mathfrak{p}}^-$ is given by the following theorem.

Theorem 1.2. *Let $F/\boldsymbol{Q}$ be a totally real extension, p an odd prime unramified in F and $\mathfrak{p}|p$ a prime in F. Let K/F be a p-ordinary CM quadratic extension. Let λ be an arithmetic Hecke character over K. Let $L_{\Sigma,\lambda}^-$ (resp. $L_{\Sigma,\lambda,\mathfrak{p}}^-$) be the corresponding anticyclotomic Katz p-adic (resp. anticyclotomic Katz $\mathfrak{p}$-adic) L-function as above. Then, we have*

$$\mu(L_{\Sigma,\lambda}^-) = \mu(L_{\Sigma,\lambda,\mathfrak{p}}^-).$$

In most of the cases, $\mu(L_{\Sigma,\lambda}^-)$ has been explicitly determined (see [Hida 2010; Hsieh 2014a]). Thus, we obtain a formula for $\mu(L_{\Sigma,\lambda,\mathfrak{p}}^-)$.

We show a result analogous to Theorem 1.2 for a class of Rankin–Selberg anticyclotomic $\mathfrak{p}$-adic L-functions.

When λ is self-dual with the root number -1, all the Hecke L-values appearing in the interpolation property of $L^-_{\Sigma,\lambda}$ vanish. Accordingly, $L^-_{\Sigma,\lambda}$ and $L^-_{\Sigma,\lambda,\mathfrak{p}}$ identically vanish. The anticyclotomic arithmetic information contained in $L^-_{\Sigma,\lambda}$ and $L^-_{\Sigma,\lambda,\mathfrak{p}}$ may seem to have disappeared. However, we can look at the cyclotomic derivatives

$$L'_{\Sigma,\lambda}=\left(\frac{\partial}{\partial S}L_{\Sigma,\lambda}(T_1,\dots,T_d,S)\right)\bigg|_{S=0} \tag{1-2}$$

and $L'_{\Sigma,\lambda,\mathfrak{p}}$ (defined analogously).

The μ-invariant of $L'_{\Sigma,\lambda,\mathfrak{p}}$ is given by the following theorem.

Theorem 1.3. *Let $F/\mathbf{Q}$ be a totally real extension, p an odd prime unramified in F and $\mathfrak{p}|p$ a prime in F. Let K/F be a p-ordinary CM quadratic extension. Let λ be an arithmetic self-dual Hecke character over K with root number -1. Let $L'_{\Sigma,\lambda}$ (resp. $L'_{\Sigma,\lambda,\mathfrak{p}}$) be the corresponding cyclotomic derivative of the Katz p-adic (resp. Katz $\mathfrak{p}$-adic) L-function as above.*

Suppose that $p\nmid h^-_K$, where h^-_K is the relative class number given by $h^-_K=h_K/h_F$. Then, we have

$$\mu(L'_{\Sigma,\lambda})=\mu(L'_{\Sigma,\lambda,\mathfrak{p}}).$$

In most of the cases, $\mu(L'_{\Sigma,\lambda})$ has been explicitly determined (see [Burungale 2015]). Thus, we obtain a formula for $\mu(L'_{\Sigma,\lambda,\mathfrak{p}})$.

Finally, we consider the $\mathfrak{p}$-adic L-function. We have the following $\mathfrak{p}$-version of a conjecture of Gillard [1991, Conjecture (i)] regarding the vanishing of the μ-invariant of Katz p-adic L-function.

Theorem 1.4. *Let $F/\mathbf{Q}$ be a totally real extension, p an odd prime unramified in F and $\mathfrak{p}|p$ a prime in F. Let K/F be a p-ordinary CM quadratic extension. Let λ be an arithmetic Hecke character over K. Let $L_{\Sigma,\lambda,\mathfrak{p}}$ be the corresponding cyclotomic Katz $\mathfrak{p}$-adic L-function as above. Then, we have*

$$\mu(L_{\Sigma,\lambda,\mathfrak{p}})=0.$$

We now describe the strategy of the proof of Theorem 1.1. Some of the notation used here is not followed in the rest of the article.

We begin with general remarks on rigidity. Here is a geometric reason for this rigidity, which we refer to as L-rigidity. Let $\widehat{S}$ be a formal torus over a field which is the residue field of a mixed characteristic ring. Suppose that we have a rational structure coming from an algebraic subscheme S over the mixed characteristic ring whose formal completion at a closed point $x\in S$ gives $\widehat{S}$. Here S is not necessarily an algebraic torus. We suppose that there exists a positive dimensional transcendental linear subvariety L of $\widehat{S}$ with strictly smaller dimension, i.e., a nontrivial formal subtorus which does not equal the formal completion along x of an algebraic subscheme of S. Transcendence of L implies the "algebraic" Zariski closure of L in S is the entire S. This may not happen for the formal Zariski closure. Let $A = \mathcal{O}_{S,x}$ and $L = \mathrm{Spf}(B)$. Thus, the transcendence of L is equivalent to the injectivity of the natural morphism $A \to B$ given by $\phi \mapsto \phi|_L$ for $\phi \in A$ and hence the L-rigidity, i.e., ϕ is determined by its restriction to the formal subtorus L. The rigidity remains true for $\phi \circ a$ for any automorphism a of $\widehat{S}$. It turns out that often the rigidity also remains true for $\sum_i \phi_i \circ a_i$ for a well chosen set of automorphisms a_i of $\widehat{S}$ and $\phi_i \in \Gamma(S, \mathcal{O}_S)$, i.e.,

$$\sum_i \phi_i \circ a_i = 0 \Rightarrow \phi_i = 0$$

for all i. As the notion of well chosen may vary from context to context, we only mention that $\{a_i\}$'s typically satisfy a transcendental property which we later specify in this context. We study the case for the Serre–Tate deformation space $\widehat{S}$ with rational structure induced from the Hilbert modular Shimura variety. The formal subtorus we study is the partial $\mathfrak{p}$-deformation subspace of $\widehat{S}$. We consider certain $\{a_i\}_i$ such that the differences $\{a_i a_j^{-1}\}_{i \neq j}$ are "transcendental" automorphisms of the Serre–Tate deformation space (see Section 4A).

Let $G = \mathrm{Res}_{O/\mathbf{Z}}(\mathrm{GL}_2)$. The group $G(\mathbf{Z}_{(p)})$ acts on the prime-to-p Hilbert modular Shimura variety $\mathrm{Sh}_{/\mathbf{Z}_{(p)}}$. We refer to Section 3B for the action. Here we only mention that in terms of the moduli interpretation, the action corresponds to the one on the level structure. Let V be an irreducible component of Sh containing x. Let $H_x(\mathbf{Z}_{(p)})$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. It acts on $\mathrm{Spec}(\mathcal{O}_{V,x})$ and thus on the Serre–Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$. In view of the description of the action on the Serre–Tate coordinates, we observe that the formal subtorus

$$\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$$

is stable under the action of $H_x(\mathbf{Z}_{(p)})$. Chai and the second named author have proven that a positive dimensional closed irreducible subvariety of V containing x and stable under $H_x(\mathbf{Z}_{(p)})$ equals V itself. In this sense, the formal subtorus $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ is transcendental in the Shimura variety. Recall that the Igusa tower is étale over V. As a mod p Hilbert modular form is an algebraic function on the Igusa tower, we prove $\mathfrak{p}$-rigidity (Theorem 1.1) based on the transcendence.

We now describe the strategy of the proof of Theorem 1.2. Some of the notation used here is not followed in the rest of the article.

Let us first recall the second named author's strategy [2010] to determine $\mu(L^-_{\Sigma,\lambda})$. Let $O_p = O \otimes \mathbb{Z}_p$. Let

$$G_{\Sigma,\lambda} \in \bar{\mathbf{Z}}_p[\![T_1, \dots, T_d]\!]$$

be the power series expansion of the measure $L^{-}_{\Sigma,\lambda}$ regarded as a p-adic measure on O_p with support in $1+pO_p$, given by

$$\begin{aligned} G_{\Sigma,\lambda} &= \int_{1+pO_p} t^y \, dL^{-}_{\Sigma,\lambda}(y) \\ &= \sum_{(k_1,\dots,k_d)\in(\boldsymbol{Z}_{\geq 0})^d} \left(\int_{1+pO_p} \binom{y}{k_1,\dots,k_d} dL^{-}_{\Sigma,\lambda}(y) \right) T_1^{k_1} \cdots T_d^{k_d}. \end{aligned} \tag{1-3}$$

The starting point is the observation that there are classical Hilbert modular Eisenstein series $(f_{\lambda,i})_i$ such that

$$G_{\Sigma,\lambda} = \sum_i a_i \circ (f_{\lambda,i}(t)), \tag{1-4}$$

where $f_{\lambda,i}(t)$ is the t-expansion of $f_{\lambda,i}$ around a well chosen CM point y with the CM type (K,Σ) on the Hilbert modular Shimura variety Sh and a_i is an automorphism of the Serre–Tate deformation space $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathrm{Sh},y})$, i.e., $a_i \in H_y(\boldsymbol{Z}_p)$ (see Section 3B). Based on Chai's study of Hecke-stable subvarieties of a Shimura variety, the second named author has proven the linear independence of $(a_i \circ f_{\lambda,i})_i$ modulo p. It follows that

$$\mu(L^{-}_{\Sigma,\lambda}) = \min_i \mu(f_{\lambda,i}(t)).$$

Let $G_{\Sigma,\lambda,\mathfrak{p}} \in \bar{\boldsymbol{Z}}_p[[\Gamma^{-}_{\mathfrak{p}}]]$ be the analogous power series expansion of the measure $L^{-}_{\Sigma,\lambda,\mathfrak{p}}$. Based on the action of p-adic differential operators on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre–Tate coordinates, we show that

$$G_{\Sigma,\lambda,\mathfrak{p}} = \sum_i a_{i,\mathfrak{p}} \circ (f_{\lambda,i}(t_{\mathfrak{p}})), \tag{1-5}$$

where $a_{i,\mathfrak{p}}$ is the projection of a_i to $H_y(\boldsymbol{Z}_p)_{\mathfrak{p}}$ (see Section 3B) and $f_{\lambda,i}(t_{\mathfrak{p}})$ is the $\mathfrak{p}$-adic Serre–Tate expansion of f around y (see Section 6A). Based on $\mathfrak{p}$-rigidity and Chai's theory of Hecke-stable subvarieties of a Shimura variety, we prove $\mathfrak{p}$-independence, i.e., the linear independence of $\big(a_{i,\mathfrak{p}} \circ (f_{\lambda,i}(t_{\mathfrak{p}}))\big)_i$ modulo p. It follows that

$$\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}}) = \min_i \mu(f_{\lambda,i}(t_{\mathfrak{p}})).$$

In view of the $\mathfrak{p}$-rigidity, we have

$$\mu(f_{\lambda,i}(t)) = \mu(f_{\lambda,i}(t_{\mathfrak{p}})).$$

This concludes the proof of Theorem 1.2.

The strategy of the proof of Theorem 1.3 is similar to the above strategy. It involves p-adic modular forms $f'_{\lambda,i}$ arising from the p-adic derivative of $f_{\lambda N^s,i}$

with N being the norm Hecke character. In this sense, the proof involves $\mathfrak{p}$-rigidity for nonclassical p-adic modular forms. Finally, Theorem 1.4 is proven based on Theorem 1.2 and the proof of results in [Hida 2011; Burungale and Hsieh 2013]. As in [Hida 2010], we would like to emphasise that Chai's theory plays an underlying role in all of the above results.

In this article, we give an elementary construction of p-adic differential operators on the space of p-adic Hilbert modular forms avoiding the use of Gauss–Manin connection in Katz's construction. This is based on the ideas of the second named author in the early nineties. Based on this construction, we determine the action of p-adic differential operators on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre–Tate coordinates.

Along with the p-adic Gross–Zagier formula, Theorem 1.2 and Theorem 1.3 have application towards generic nonvanishing of p-adic heights on CM abelian varieties (see [Burungale and Disegni $\geq$ 2017]). This provides evidence for Schneider's conjecture on the nonvanishing of p-adic heights in the CM case. In view of [Burungale 2017], the underlying ideas also have application towards generic nonvanishing of the p-adic Abel–Jacobi image of generalised Heegner cycles in the non-CM case. We refer to [Burungale 2016b] for a survey.

Let $\mathfrak{p}_i$ and $\mathfrak{p}_j$ be primes above p as above. Theorem 1.2 implies an intriguing equality

$$\mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}_i}) = \mu(L^{-}_{\Sigma,\lambda,\mathfrak{p}_j})$$

of Iwasawa μ-invariants. This is rather surprising as theses μ-invariants could be nonzero and one in general does not expect any relation between the p-adic L-functions $L_{\Sigma,\lambda,\mathfrak{p}_i}$. These p-adic L-functions correspond to independent variables whose number may vary with i. As far as we know, Theorem 1.2 is a first phenomena possibly suggesting a relation. It would be interesting to see whether an analogue holds for Iwasawa λ-invariants. One can perhaps first collect experimental data. In the case of self-duality and the root number being -1, the equality of the μ-invariants persists even after taking the cyclotomic derivative. It seems tempting to suppose that a deeper phenomena mediates the relation.

In view of the anticyclotomic main conjectures, Theorem 1.2 would imply an equality of the corresponding algebraic μ-invariants. Note that the underlying Selmer groups correspond to rather different local conditions. In many cases, the anticyclotomic main conjecture has been proven (see [Hida 2006; 2009b]). It would be interesting to prove the equality of the algebraic μ-invariants directly. This sort of equality does not seem to be conjectured in the literature.

In [Burungale $\geq$ 2017], the first named author proves the analogue of $\mathfrak{p}$-rigidity and $\mathfrak{p}$-independence for quaternionic modular forms over totally real fields. These results concern a quaternion algebra which is not totally definite. In the near future, he hopes to consider an analogue of Theorem 1.2 for a class of quaternionic Rankin–Selberg p-adic L-functions. In [Harris et al. 2006], a construction of p-adic L-functions for unitary Shimura varieties is announced (see [Eischen et al. 2016]). In such a case, the number of variables of the p-adic L-function is typically less than the dimension of the Shimura variety. The variables of the p-adic L-function may correspond to an analogue of the $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$-variables. In the future, we hope to consider this question starting with the case of $U(n, 1)$ Shimura varieties.

Formulating a $\mathfrak{p}$-rigidity type statement for a PEL Shimura variety may have an independent interest. In characteristic zero, we hope to explore rigidity based on the approach in [Burungale and Hida 2016].

The article is organised as follows. In Section 2, we recall basic facts about the Hilbert modular Shimura variety Sh. In Section 3, we prove Theorem 1.1. In Sections 3A–3B, we firstly recall some facts about Serre–Tate deformation theory of an ordinary closed point in Sh. In Section 3C, we prove the theorem. In Section 4, we prove $\mathfrak{p}$-rigidity, i.e., the linear independence of mod p Hilbert modular forms restricted to the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. In Section 5A, we give an elementary construction of p-adic differential operators on the space of p-adic Hilbert modular forms. In Section 5B, we use it to compute the action of the p-adic differential operators on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre–Tate coordinates. In Section 6, we consider Iwasawa μ-invariants as in Theorems 1.2–1.4. In Section 6A, we determine the μ-invariant of certain anticyclotomic $\mathfrak{p}$-adic L-functions (see Theorem 1.2). In Section 6B, we determine the μ-invariant of the cyclotomic derivative $L'_{\Sigma,\lambda,\mathfrak{p}}$ of the Katz $\mathfrak{p}$-adic L-function, when the branch character λ is self-dual with the root number -1 (see Theorem 1.3). In Section 6C, we prove a $\mathfrak{p}$-version of a conjecture Gillard regarding the vanishing of the μ-invariant of the Katz p-adic L-function (see Theorem 1.4).

Notation. We use the following notation unless otherwise stated.

For a number field L, let $\boldsymbol{A}_L$ be the adele ring and $\boldsymbol{A}_L^f$ the finite adeles of L. Let h_L denote the ideal class number. Let G_L be the absolute Galois group of L and G_L^{ab} the maximal abelian quotient. Let $\mathrm{rec}_L : \boldsymbol{A}_L^\times \to G_L^{\mathrm{ab}}$ be the geometrically normalised reciprocity law.

2. Hilbert modular Shimura variety

In this section, we recall basic facts about Hilbert modular Shimura varieties. We follow [Hida 2004].

2A. ***Setup.*** In this subsection, we recall a basic setup regarding Hilbert modular Shimura varieties.

Let $G = \mathrm{Res}_{F/\boldsymbol{Q}}\, GL_2$ and $h_0 : \mathrm{Res}_{\boldsymbol{C}/\boldsymbol{R}}\, \mathbb{G}_m \to G_{/\boldsymbol{R}}$ be the morphism of real group schemes arising from

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \boldsymbol{C}^{\times}$. Let X be the set of $G(\boldsymbol{R})$-conjugacy classes of h_0. We have a canonical isomorphism $X \simeq (\boldsymbol{C} - \boldsymbol{R})^I$, where I is the set of real places of F. The pair (G, X) satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(\mathrm{Sh}_K = \mathrm{Sh}_K(G, X))_K$ of quasiprojective smooth varieties over $\boldsymbol{Q}$ indexed by open compact subgroups K of $G(\boldsymbol{A}^f)$. The pro-algebraic variety $\mathrm{Sh}_{/\boldsymbol{Q}}$ is the projective limit of these varieties. The complex points of these varieties are given as follows

$$\mathrm{Sh}_K(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash X \times G(\boldsymbol{A}^f)/K, \quad \mathrm{Sh}(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash X \times G(\boldsymbol{A}^f)/\overline{Z(\boldsymbol{Q})}. \tag{2-1}$$

Here, $\overline{Z(\boldsymbol{Q})}$ is the closure of the center $Z(\boldsymbol{Q})$ in $G(\boldsymbol{A}^f)$ under the adélic topology. From (2-1) and the general theory of Shimura varieties, it follows that $\mathrm{Sh}_{/\boldsymbol{Q}}$ is endowed with an action of $G(\boldsymbol{A}^f)$ (see [Hida 2004, §4.2]). This gives rise to the Hecke action.

2B. *p-integral model.* In this subsection, we briefly recall a canonical p-integral smooth model $\mathrm{Sh}^{(p)}_{/\boldsymbol{Z}_{(p)}}$ of the Shimura variety $\mathrm{Sh}/G(\boldsymbol{Z}_p)_{/\boldsymbol{Q}}$.

Hilbert modular Shimura variety $\mathrm{Sh}_{/\boldsymbol{Q}}$ represents a functor $\mathcal{F}$ classifying abelian schemes having multiplication by O along with additional structure, where O is the ring of integers of F (see [Hida 2004, §4.2; Shimura 1963]). As in the introduction, let p be an odd prime unramified in F. Under this hypothesis, a p-integral interpretation $\mathcal{F}^{(p)}$ of $\mathcal{F}$ leads to a p-integral smooth model of $\mathrm{Sh}/G(\boldsymbol{Z}_p)_{/\boldsymbol{Q}}$.

The functor $\mathcal{F}^{(p)}$ is given by

$$\begin{aligned} \mathcal{F}^{(p)} :\ & \mathrm{SCH}_{/\boldsymbol{Z}_{(p)}} \to \mathrm{SETS}, \\ & S \mapsto \{(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S}\}/\sim . \end{aligned} \tag{2-2}$$

Here,

(PM1) A is abelian scheme over S of dimension of d.

(PM2) $\iota : O \hookrightarrow \mathrm{End}_S\, A$ is an algebra embedding.

(PM3) $\bar{\lambda}$ is the polarisation class of a homogeneous polarisation λ up to scalar multiplication by $\iota(O^{\times}_{(p),+})$, where $O_{(p),+} := \{a \in O_{(p)} \mid \sigma(a) > 0,\ \forall \sigma \in I\}$. Also, the Rosati involution of $\mathrm{End}_S\, A$ takes $\iota(l)$ to $\iota(l^*)$, for $l \in O$.

(PM4) Let $\mathcal{T}^{(p)}(A)$ be the prime-to-p Tate module $\varprojlim_{p \nmid N} A[N]$. Then $\eta^{(p)}$ is a prime-to-p level structure given by an O-linear isomorphism

$$\eta^{(p)} : O^2 \otimes_{\boldsymbol{Z}} \widehat{\boldsymbol{Z}}^{(p)} \simeq \mathcal{T}^{(p)}(A),$$

where $\widehat{\boldsymbol{Z}}^{(p)} = \prod_{l \neq p} \boldsymbol{Z}_l$.

(PM5) Let $\mathrm{Lie}_S(A)$ be the relative Lie algebra of A. There exists an $O\otimes_{\boldsymbol{Z}}\mathcal{O}_S$-module isomorphism

$$\mathrm{Lie}_S(A) \simeq O \otimes_{\boldsymbol{Z}} \mathcal{O}_S,$$

locally under the Zariski topology of S.

The notation $\sim$ denotes up to a prime-to-p isogeny.

Theorem 2.1 (Kottwitz). *Let the notation and assumptions be as above. Then, the functor $\mathcal{F}^{(p)}$ is represented by a pro-algebraic scheme $\mathrm{Sh}^{(p)}(G,X)_{/\boldsymbol{Z}_{(p)}}$. Moreover, there exists an isomorphism given by*

$$\mathrm{Sh}^{(p)} \times \boldsymbol{Q} \simeq \mathrm{Sh}\,/G(\boldsymbol{Z}_p)_{/\boldsymbol{Q}}.$$

(see [Hida 2004, §4.2.1]).

The pro-algebraic scheme $\mathrm{Sh}^{(p)}(G,X)_{/\boldsymbol{Z}_{(p)}}$ is usually referred as the Kottwitz model. In what follows, we let $\mathrm{Sh}^{(p)}_{/\boldsymbol{Z}_{(p)}}$ denote $\mathrm{Sh}^{(p)}(G,X)_{/\boldsymbol{Z}_{(p)}}$ for simplicity of notation.

2C. *Igusa tower.* In this subsection, we briefly recall the notion of p-ordinary Igusa tower over the p-integral model (see Section 2B).

Let $\overline{\boldsymbol{Q}}$ be an algebraic closure of $\boldsymbol{Q}$ and $\overline{\boldsymbol{Q}}_p$ be an algebraic closure of $\boldsymbol{Q}_p$. We fix a complex embedding $\iota_\infty : \overline{\boldsymbol{Q}} \hookrightarrow \boldsymbol{C}$ and a p-adic embedding $\iota_p : \overline{\boldsymbol{Q}} \hookrightarrow \overline{\boldsymbol{Q}}_p$.

Let $\mathcal{W}$ be the strict Henselisation inside $\overline{\boldsymbol{Q}}$ of the local ring of $\boldsymbol{Z}_{(p)}$ corresponding to ι_p. Let $\mathbb{F}$ be the residue field of $\mathcal{W}$. Note that $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$.

Let $\mathrm{Sh}^{(p)}_{/\mathcal{W}} = \mathrm{Sh}^{(p)} \times_{\boldsymbol{Z}_{(p)}} \mathcal{W}$ and $\mathrm{Sh}^{(p)}_{/\mathbb{F}} = \mathrm{Sh}^{(p)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.

From now, let Sh denote $\mathrm{Sh}^{(p)}_{/\mathbb{F}}$. Let $\mathcal{A}$ be the universal abelian scheme over Sh.

Let $\mathrm{Sh}^{\mathrm{ord}}$ be the subscheme of Sh on which the Hasse-invariant does not vanish. It is an open dense subscheme. Over $\mathrm{Sh}^{\mathrm{ord}}$, the connected part $\mathcal{A}[p^m]^\circ$ of $\mathcal{A}[p^m]$ is étale-locally isomorphic to $\mu_{p^m} \otimes_{\boldsymbol{Z}_p} O^*$ as an O_p-module, where $O^* = O^{-1}\mathfrak{d}_F^{-1}$, $\mathfrak{d}_F$ is the different of $F/\boldsymbol{Q}$ and $O_p = O \otimes \boldsymbol{Z}_p$.

We now define the Igusa tower. For $m \in \mathbb{N}$, the m-th layer of the Igusa tower over $\mathrm{Sh}^{\mathrm{ord}}$ is defined by

$$\mathrm{Ig}_m = \underline{\mathrm{Isom}}_{O_p}(\mu_{p^m} \otimes_{\boldsymbol{Z}_p} O^*, \mathcal{A}[p^m]^\circ). \tag{2-3}$$

Note that the projection $\pi_m : \mathrm{Ig}_m \to \mathrm{Sh}^{\mathrm{ord}}$ is finite and étale. The full Igusa tower over $\mathrm{Sh}^{\mathrm{ord}}$ is defined by

$$\mathrm{Ig} = \mathrm{Ig}_\infty = \varprojlim \mathrm{Ig}_m = \underline{\mathrm{Isom}}_{O_p}(\mu_{p^\infty} \otimes_{\boldsymbol{Z}_p} O^*, \mathcal{A}[p^\infty]^\circ). \tag{2-4}$$

(Ét) Note that the projection $\pi : \mathrm{Ig} \to \mathrm{Sh}^{\mathrm{ord}}$ is étale.

Let x be a closed ordinary point in Sh. We have the following description of the level p^∞-structure on the corresponding p-divisible group $A_x[p^\infty]$.

(PL) Let η_p° be a level p^∞-structure on $A_x[p^\infty]^\circ$. For the primes $\mathfrak{p}$ in O dividing p, it is a collection of level $\mathfrak{p}^\infty$ structures $\eta_{\mathfrak{p}}^\circ$, given by isomorphisms

$$\eta_{\mathfrak{p}}^\circ : O_{\mathfrak{p}}^* \simeq A_x[\mathfrak{p}^\infty]^\circ,$$

where $O_{\mathfrak{p}}^* = O^* \otimes O_{\mathfrak{p}}$. The Cartier duality and the polarisation $\bar{\lambda}_x$ induces an isomorphism

$$\eta_{\mathfrak{p}}^{\text{ét}} : O_{\mathfrak{p}} \simeq A_x[\mathfrak{p}^\infty]^{\text{ét}}.$$

Thus, we get a level p^∞-structure $\eta_p^{\text{ét}}$ on $A_x[p^\infty]^{\text{ét}}$ from η_p°.

Let V be an irreducible component of Sh and V^{ord} be $V \cap \text{Sh}^{\text{ord}}$. Let I be the inverse image of V^{ord} under π. In [Hida 2004, Chapter 8; 2009a], it has been shown that

(Ir) I is an irreducible component of Ig.

2D. *Mod p modular forms.* In this subsection, we briefly recall the notion of mod p modular forms on an irreducible component of the Hilbert modular Shimura variety (see Section 2B).

Let V and I be as in Section 2C. Let B be an $\mathbb{F}$-algebra. The space of mod p modular forms on V over B is defined by

$$M(V, B) = H^0(I_{/B}, \mathcal{O}_{I_{/B}}), \tag{2-5}$$

where $I_{/B} := I \times_{\mathbb{F}} B$. In view of Sections 2B–2C, we have the following geometric interpretation of mod p modular forms.

A mod p modular form is a function f of isomorphism classes of $\tilde{x} = (x, \eta_p^\circ)_{/B'}$, where B' is a B-algebra, $x = (A, \iota, \bar{\lambda}, \eta^{(p)})_{/B'} \in \mathcal{F}^{(p)}(B')$ and

$$\eta_p^\circ : \mu_{p^\infty} \otimes_{\mathbf{Z}_p} O^* \simeq A[p^\infty]^\circ$$

is an O_p-linear isomorphism, such that the following conditions are satisfied.

(G1) $f(\tilde{x}) \in B'$.

(G2) If $\tilde{x} \simeq \tilde{x}'$, then $f(\tilde{x}) = f(\tilde{x}')$, where $\tilde{x} \simeq \tilde{x}'$ means $x \simeq x'$ and the corresponding isomorphism between A and A' induces an isomorphism between η_p° and $\eta_p'^\circ$.

(G3) $f(\tilde{x} \times_{B'} B'') = h(f(\tilde{x}))$ for any B-algebra homomorphism $h : B' \to B''$.

We also have the key notion of q-expansion and q-expansion principle for mod p modular forms (see [Hida 2004, Theorem 4.21]).

2E. ***p-adic modular forms.*** In this subsection, we briefly recall the notion of p-adic modular forms on an irreducible component of the Hilbert modular Shimura variety (see Section 2B).

Let W denote the Witt ring $W(\mathbb{F})$. The construction of the Igusa tower in Section 2C is well defined for the base W. Let $I_{/W}$ be the irreducible component of the Igusa tower $\mathrm{Ig}_{/W}$ over an irreducible component $V_{/W}$ of the Shimura variety $\mathrm{Sh}_{/W}$. Let C be a p-adically complete local W-algebra with maximal ideal $\mathfrak{m}_C$. The space of p-adic modular forms on V over C is defined by

$$M(V, C) = H^0(I_{/C}, \mathcal{O}_{I_{/C}}), \tag{2-6}$$

where $I_{/C} := I_{/W} \times_{/W} C$.

By definition, a p-adic modular form over C modulo $\mathfrak{m}_C$ is a mod p modular form.

We have an analogous moduli interpretation as in Section 2D and also the q-expansion principle, for p-adic modular forms (see [Hida 2010, §4.1]).

3. $\mathfrak{p}$-rigidity

In this section, we prove the rigidity of mod p modular forms (see Theorem 1.1). In Sections 3A–3B, we firstly recall basic facts about Serre–Tate deformation theory of an ordinary closed point on the Hilbert modular variety (see Section 2B). In Section 3C, we prove the rigidity.

3A. ***Serre–Tate deformation theory.*** In this subsection, we briefly recall Serre–Tate deformation theory of an ordinary closed point on the Hilbert modular variety (see Section 2B). We follow [Hida 2004, §8.2; 2010, §2; 2013b, §1].

Let the notation and assumptions be as in Section 2. Let x be a closed point in $\mathrm{Sh}^{\mathrm{ord}}$ carrying $(A_x, \iota_x, \bar{\lambda}_x, \eta_x^{(p)})_{/\mathbb{F}}$. Let V be the irreducible component of Sh containing x.

Let CL_W be the category of complete local W-algebras with residue field $\mathbb{F}$. Let $\mathcal{D}_{/W}$ be the fiber category over CL_W of deformations of $x_{/\mathbb{F}}$ defined as follows. Let $R \in \mathrm{CL}_W$. The objects of $\mathcal{D}_{/W}$ over R consist of $x'^{*} = (x', \iota_{x'})$, where $x' \in \mathcal{F}^{(p)}(R)$ and

$$\iota_{x'} : x' \times_R \mathbb{F} \simeq x.$$

Let x'^{*} and x''^{*} be in $\mathcal{D}_{/W}$ over R. By definition, a morphism ϕ between x'^{*} and x''^{*} is a morphism (still denoted by) ϕ between x' and x'' satisfying [Hida 2004, (7.3)] and the following condition.

(M) Let ϕ_0 be the special fiber of ϕ. The automorphism $\iota_{x''} \circ \phi_0 \circ \iota_{x'}^{-1}$ of x equals the identity.

Let $\widehat{\mathcal{F}}_x$ be the deformation functor given by

$$\begin{aligned} \widehat{\mathcal{F}}_x : \mathrm{CL}_{/W} &\to \mathrm{SETS}, \\ R &\mapsto \{x'^{*}_{/R} \in \mathcal{D}\}/\simeq . \end{aligned} \tag{3-1}$$

The notation $\simeq$ denotes up to an isomorphism.

Recall, $R \in \mathrm{CL}_W$. As R is a projective limit of local W-algebras with nilpotent maximal ideal, we can (and do) suppose that R is a local Artinian W-algebra with nilpotent maximal ideal m_R. Let $x'^{*}_{/R} \in \mathcal{D}$ and A denote $A_{x'}$. By Drinfeld's theorem (see [Hida 2004, §8.2.1]), $A[p^\infty]^\circ(R)$ is killed by p^{n_0} for sufficiently large n_0. Let $y \in A(\mathbb{F})$ and $\tilde{y} \in A(R)$ such that $\tilde{y}_0 = y$, where $\tilde{y}_0$ denotes the special fiber of $\tilde{y}$ (as $A_{/R}$ is smooth, such a lift always exists). By definition, $\tilde{y}$ is determined modulo $\ker(A(R) \to A(\mathbb{F})) = A[p^\infty]^\circ(R)$. Thus, for $n \geq n_0$, "p^n"$y_0 := p^n\tilde{y}$ is well defined. From now, we suppose that $n \geq n_0$. If $y \in A[\mathfrak{p}^n](\mathbb{F})$, then "$p^n$"$y \in A[\mathfrak{p}^\infty]^\circ(R)$. Strictly speaking, we apply the idempotent $e_{\mathfrak{p}}$ corresponding to $\mathfrak{p}$ so that $e_{\mathfrak{p}}$"p^n"$y \in A[\mathfrak{p}^\infty]^\circ(R)$. We let "$p^n$"$y$ denote $e_{\mathfrak{p}}$"p^n"y for simplicity of notation.

Thus, we have a homomorphism

$$\text{“}p^n\text{”} : A[\mathfrak{p}^n](\mathbb{F}) \to A[\mathfrak{p}^\infty]^\circ(R). \tag{3-2}$$

We also have the commutative diagram

$$\begin{array}{ccccc} A[\mathfrak{p}^{n+1}]^{\text{ét}}(R) & \xrightarrow{\simeq} & A[\mathfrak{p}^{n+1}](\mathbb{F}) & \xrightarrow{\text{“}p^{n+1}\text{”}} & A[\mathfrak{p}^\infty]^\circ(R) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle =} \\ A[\mathfrak{p}^{n}]^{\text{ét}}(R) & \xrightarrow{\simeq} & A[\mathfrak{p}^{n}](\mathbb{F}) & \xrightarrow{\text{“}p^{n}\text{”}} & A[\mathfrak{p}^\infty]^\circ(R). \end{array}$$

Passing to the projective limit, this gives rise to a homomorphism

$$\text{“}p^\infty\text{”} : A[\mathfrak{p}^\infty](\mathbb{F}) \to A[\mathfrak{p}^\infty]^\circ(R). \tag{3-3}$$

(CC) For $\varprojlim y_n \in \varprojlim A[\mathfrak{p}^n](\mathbb{F})$, let

$$y = \varprojlim y_n \in A[\mathfrak{p}^\infty](\mathbb{F}) \simeq A_x[\mathfrak{p}^\infty]^{\text{ét}}.$$

The isomorphism is induced by $\iota_{x'}$.

Let $q_{n,\mathfrak{p}}(y_n) =$ "p^n"y_n and $q_{\mathfrak{p}}(y) = \lim q_{n,\mathfrak{p}}(y_n)$. By definition,

$$q_{\mathfrak{p}}(y) \in A[\mathfrak{p}^\infty]^\circ(R) \simeq \mathrm{Hom}(A_x^\vee[\mathfrak{p}^\infty]^{\text{ét}}, \widehat{\mathbb{G}}_m(R)),$$

where $A_x^\vee$ is the dual of the abelian variety A_x. Let $q_{A,\mathfrak{p}}$ be the pairing given by

$$\begin{aligned} q_{A,\mathfrak{p}} : A_x[\mathfrak{p}^\infty]^{\text{ét}} \times A_x^\vee[\mathfrak{p}^\infty]^{\text{ét}} &\to \widehat{\mathbb{G}}_m(R), \\ q_{A,\mathfrak{p}}(y,z) &= q_{\mathfrak{p}}(y)(z). \end{aligned} \tag{3-4}$$

We have the following fundamental result.

Theorem 3.1 (Serre–Tate). (1) *There exists a canonical isomorphism*

$$\widehat{\mathcal{F}}_x(R) \simeq \prod_{\mathfrak{p}} \mathrm{Hom}_{\mathbf{Z}_p}\big(A_x[\mathfrak{p}^\infty]^{\text{ét}} \times A_x^\vee[\mathfrak{p}^\infty]^{\text{ét}}, \widehat{\mathbb{G}}_m(R)\big) \tag{3-5}$$

given by $x'^* \mapsto q_A := \prod_{\mathfrak{p}} q_{A_{x'},\mathfrak{p}}$.

(2) *The deformation functor* $\widehat{\mathcal{F}}_x$ *is represented by the formal scheme* $\widehat{S}_{/W} :=$ $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$. *A level* p^∞*-structure as in* (PL), *gives rise to a canonical isomorphism of the deformation space* $\widehat{S}_{/W}$ *with the formal torus* $\prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}$ (see [Hida 2013b, Proposition 1.2]).

Let x_{ST} be the universal deformation of the closed ordinary point x.

We now recall key underlying notions.

Definition 3.2. Let x be a closed ordinary point on the Hilbert modular variety as above. Recall that a level p^∞-structure on x (see (PL)) gives rise to a canonical isomorphism of the deformation space $\widehat{S}_{/W}$ with the formal torus $\prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}$ (see part (2) of Theorem 3.1). Under this identification, let

$$t = (t_{\mathfrak{p}})_{\mathfrak{p}}$$

be the coordinates of the deformation space $\widehat{S}_{/W}$, where $t_{\mathfrak{p}}$ is the coordinate of $\widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}}$. We call $t = (t_{\mathfrak{p}})_{\mathfrak{p}}$ the Serre–Tate coordinates of the deformation space $\widehat{S}_{/W}$.

We have $\widehat{S} = \mathrm{Spf}(\widehat{W[O]})$, where $S = \mathbb{G}_m \otimes O^*$, $W[O] = W[X(S)]$ and $\widehat{W[O]}$ is the completion at the augmentation ideal. Here, $X(S)$ is the character group of S. Note that $W[O]$ is the ring consisting of formal finite sums $\sum_{\xi \in O} a(\xi) t^\xi$, where $a(\xi) \in W$ and t is the coordinate of $\mathbb{G}_m$. Here, t^ξ is the character given by

$$t^\xi(t \otimes u) = t^{\mathrm{Tr}_{F/\mathbf{Q}}(\xi u)}$$

for $u \in O^*$.

Definition 3.3. Let f be a mod p modular form over $\mathbb{F}$ (see Section 2D). A level p^∞-structure η_p° of x gives rise to a canonical level p^∞-structure $\eta_{p,\mathrm{ST}}^\circ$ of the universal deformation x_{ST}. We denote by

$$f((x_{\mathrm{ST}}, \eta_{p,\mathrm{ST}}^\circ)) \in \widehat{\mathbb{F}[O]}$$

the t-expansion of f around x.

We have the following t-expansion principle.

(t-expansion principle) The above t-expansion of a mod p modular form f around a closed ordinary point determines f uniquely (see (Ir)).

We have an analogous t-expansion principle for p-adic modular forms (see [Hida 2010, §4.1]).

3B. ***Reciprocity law for the deformation space.*** In this subsection, we recall the action of the local algebraic stabiliser of a closed ordinary point x on the Serre–Tate coordinates of the deformation space of x. This can be considered as an infinitesimal analogue of Shimura's reciprocity law.

Let $g \in G(\mathbf{Z}_{(p)})$ act on Sh through the right multiplication on the prime-to-p level structure, i.e.,

$$(A, \iota, \bar{\lambda}, \eta^{(p)})_{/S} \mapsto (A, \iota, \bar{\lambda}, \eta^{(p)} \circ g)_{/S}$$

(see Section 2B).

Recall that x is a closed ordinary point in Sh with a p^∞-level structure η_p^{ord}. Let (K_x, Σ_x) be the CM-type of x. We suppose that $\iota : O \hookrightarrow \mathrm{End}(A)$ extends to $\iota_x : \mathcal{O} \hookrightarrow \mathrm{End}(A)$, where $\mathcal{O}$ is the ring of integers of K_x. Let $H_x(\mathbf{Z}_{(p)})$ be the stabiliser of x in $G(\mathbf{Z}_{(p)})$. Note that

$$H_x(\mathbf{Z}_{(p)}) = (\mathrm{Res}_{\mathcal{O}_{(p)}/\mathbf{Z}_{(p)}} \mathbb{G}_m)(\mathbf{Z}_{(p)}) = \mathcal{O}_{(p)}^\times, \tag{3-6}$$

where $\mathcal{O}_{(p)} = \mathcal{O} \otimes \mathbf{Z}_{(p)}$ (see [Hida 2010, §3.2; Shimura 1998]). We call $H_x(\mathbf{Z}_{(p)})$ the local algebraic stabiliser of x.

Let c_x be the complex conjugation of K_x.

As x is ordinary, Σ_x is a p-ordinary CM type. When considered as a p-adic CM type, we denote it by $\Sigma_{x,p}$. Let $\boldsymbol{p} = \prod_{v \in \Sigma_{x,p}} \mathfrak{p}_v$, for the primes $\mathfrak{p}_v$ associated to the valuation $v \in \Sigma_{x,p}$. Note that $\mathcal{O}_{\boldsymbol{p}} = O_p = \prod_{\mathfrak{p}} O_{\mathfrak{p}}$ and $\mathcal{O}_p = \mathcal{O}_{\boldsymbol{p}} \times \mathcal{O}_{\boldsymbol{p}^{c_x}}$. Let $H_x(\mathbf{Z}_p)_{\mathfrak{p}}$ be the $\mathfrak{p}$-component $O_{\mathfrak{p}}^\times$ of $H_x(\mathbf{Z}_p) = \mathcal{O}_p^\times$. We have a natural inclusion $\mathcal{O}_{(p)} \subset \mathcal{O}_p$. Thus, we regard $H_x(\mathbf{Z}_{(p)}) \subset H_x(\mathbf{Z}_p)$. Let $H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ be the projection of $H_x(\mathbf{Z}_{(p)})$ to $H_x(\mathbf{Z}_p)_{\mathfrak{p}}$. Note that we have an isomorphism $H_x(\mathbf{Z}_{(p)}) \simeq H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$. Let $\alpha \in H_x(\mathbf{Z}_{(p)})$. Let $\alpha_{\mathfrak{p}}$ be the projection of α to the $\mathfrak{p}$-component $O_{\mathfrak{p}}^\times$ of $O_{\boldsymbol{p}}^\times$. As $H_x(\mathbf{Z}_{(p)})$ stabilises x, it follows that α acts on $\mathrm{Spec}(\mathcal{O}_{V,x})$ and thus on $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$. In particular, it acts on the Serre–Tate coordinates (see Definition 3.2). The action is given by the following lemma.

Lemma 3.4 [Hida 2010, Lemma 3.3]. *The endomorphism α acts on the Serre–Tate coordinates $t = (t_{\mathfrak{p}})_{\mathfrak{p}}$ by*

$$t \mapsto t^{\alpha^{1-c_x}}$$

for $t^{\alpha^{1-c_x}} = (t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}^{1-c_x}})_{\mathfrak{p}}$.

We have the following immediate corollary.

Corollary 3.5. *The partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$ is stable under the action of the local algebraic stabiliser $H_x(\mathbf{Z}_{(p)})$.*

This simple corollary is crucial for $\mathfrak{p}$-rigidity. It may lead to rigidity-style phenomena with different flavour.

3C. $\mathfrak{p}$-*rigidity.* In this subsection, we prove the rigidity of mod p modular forms (see Theorem 1.1).

Recall that x is a closed ordinary point in V with p^∞-level structure η_p° and $V = \varprojlim V_K$, where V_K is the projection of V to $\mathrm{Sh}_K = \mathrm{Sh}/K$ for K small and maximal at p. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in the Igusa tower I over x. In view of (Ét), there exists a canonical isomorphism

$$\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}.$$

Thus, a mod p modular form can be considered as a function on $\mathrm{Spec}(\mathcal{O}_{V,x})$.

Let f be a nonzero mod p modular form. We suppose that f is a nonunit in $\mathcal{O}_{V,x}$. Let $\mathfrak{b} \subset \mathcal{O}_{V,x}$ be the zero ideal of f, i.e., $\mathfrak{b} = (f) \cap \mathcal{O}_{V,x}$. As f is an algebraic function on I, it follows that $V(\mathfrak{b})$ is nonempty not only in $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$ but also in $\mathrm{Spec}(\mathcal{O}_{V,x})$. In particular, we have $\mathfrak{b} \neq 0$. Let X be the Zariski closure of $\mathrm{Spec}(\mathcal{O}_{V,x}/\mathfrak{b})$ in V. Note that $X \subset V$ is a closed irreducible pro-subscheme containing x and $X = \varprojlim X_K$, where X_K is the projection of X to V_K.

We start with a preparatory lemma.

Lemma 3.6. *Let Y_i be a family of closed subschemes of an irreducible noetherian scheme Y such that there exists a closed point $y \in Y_i$, for all i, i.e., $y \in \bigcap_i Y_i$. Suppose that the intersection of the* $\mathrm{Spf}(\widehat{\mathcal{O}}_{Y_i,y})$ (*viewed inside* $\mathrm{Spf}(\widehat{\mathcal{O}}_{Y,y})$) *is positive dimensional. Then, the intersection of the Y_i is positive dimensional.*

Proof. It suffices to consider the affine case.

Let A be a noetherian integral domain and $R_i = A/a_i$, for ideals a_i. Take nonunits $t_1, \dots, t_r$ in A and let $(t) = (t_1, \dots, t_r)$. Suppose that

$$V(t) = \mathrm{Spec}(R/(t)) \subset V_i = \mathrm{Spec}(R/a_i), \quad \text{for all } i.$$

This implies $a_i \subset (t)$. In particular, $\sum_i a_i \subset (t)$.

If A is local noetherian with maximal ideal m, by faithful flatness of $\widehat{A} = \varprojlim A/m^n$ over A,

$$\widehat{\sum_i a_i} = \sum_i \widehat{a_i}.$$

So, the above argument can be applied to $(t) \in \widehat{m}$, $\widehat{A}$ and $\widehat{(A/a_i)}$.

This finishes the proof as

$$\dim\Big(\bigcap_i \mathrm{Spec}(A/a_i)\Big) = \dim\Big(\mathrm{Spec}\Big(A/\sum_i a_i\Big)\Big) = \dim\Big(\mathrm{Spec}\Big(\widehat{A}/\sum_i \widehat{a_i}\Big)\Big)$$
$$= \dim\Big(\bigcap_i \mathrm{Spec}(\widehat{A}/\widehat{a_i})\Big) \geq \dim(\widehat{A}/(t)). \qquad \square$$

We are now ready to prove the rigidity.

Theorem 3.7 ($\mathfrak{p}$-rigidity). *Let the notation and assumptions be as above. The partial Serre–Tate deformation space* $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}$ *is not a formal subscheme of* $\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$ (*viewed inside* $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x})$).

Proof. Suppose that it is a formal subscheme.

We now consider $Z = \bigcap_{\alpha \in H_x(\mathbb{Z}_{(p)})} \alpha(X)$ (viewed inside V). As above, recall that $Z = \varprojlim Z_K$.

Note that $\mathrm{Spf}(\widehat{\mathcal{O}}_{\alpha(X),x}) = \alpha(\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x}))$ (viewed inside $\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$). It follows that if $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$, then $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{\alpha(X),x})$ (see Corollary 3.5). In particular, we have $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p} \subset \mathrm{Spf}(\widehat{\mathcal{O}}_{\alpha(X)_K,x})$. In the last equality, $\alpha(X)_K$ is the projection of $\alpha(X)$ to V_K. By Lemma 3.6, Z_K is positive dimensional. As the projection $\pi_K : V \to V_K$ is étale, we conclude that Z itself is positive dimensional.

Thus, Z is a closed irreducible pro-subscheme of V containing x and stable under $H_x(\mathbb{Z}_{(p)})$. We conclude that $Z = V$ (see [Hida 2010, Proposition 3.8]). It follows that $\mathfrak{b} = 0$. This is a contradiction for f being nonzero as noted in the paragraph before Lemma 3.6. □

In particular, Theorem 1.1 holds. The following consequence is immediate.

Corollary 3.8. *Let g be a p-adic modular form. Then,*

$$\mu(g) = \mu(g|_{\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}}).$$

Proof. Let g be nonzero and defined over a p-adically complete local W-algebra C.

If g is a unit in $\mathcal{O}_{V,x}$, the corollary follows instantly. We thus suppose that g is a nonunit in $\mathcal{O}_{V,x}$.

In view of the t-expansion principle, we have

$$\mu(g) = \mu(g|_{\prod_\mathfrak{p} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_\mathfrak{p}}).$$

Let $c \in C$ such that $\mu(g/c) = 0$. By definition, g/c modulo the maximal ideal $\mathfrak{m}_C$ is a nonzero mod p modular form. In view of Theorem 3.7, it thus follows that

$$\mu(g/c|_{\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}}) = 0. \qquad \square$$

4. $\mathfrak{p}$-independence

In this section, we consider $\mathfrak{p}$-independence, i.e., a linear independence of mod p modular forms restricted to the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}$ (see Section 3A). In Section 4A, we first state the formulation and in Section 4B, we prove the independence.

4A. ***Formulation.*** In this subsection, we give a formulation of the linear independence of mod p modular forms restricted to the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}$ (Section 3A).

Let the notation and assumptions be as in Section 3. Recall that x is a closed ordinary point in V with p^∞-level structure η_p°. This gives rise to a closed point $\tilde{x} = (x, \eta_p^\circ)$ in the Igusa tower I over x and a canonical isomorphism

$$\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x}) \simeq \prod_{\mathfrak{p}} \widehat{\mathbb{G}}_m \otimes_{\mathbf{Z}_p} O_{\mathfrak{p}} \tag{4-1}$$

(see Theorem 3.1).

Recall that we have a canonical isomorphism $\widehat{\mathcal{O}}_{V,x} \simeq \widehat{\mathcal{O}}_{I,\tilde{x}}$ (see (Ét)). Thus, the p-completed stabiliser $H_x(\mathbf{Z}_p)$ acts on $\widehat{\mathcal{O}}_{I,\tilde{x}}$.

Let f be a mod p modular form and $a \in H_x(\mathbf{Z}_p)$. Note that

$$(a(f))|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}} = a_{\mathfrak{p}}(f|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}})$$

(see Corollary 3.5). Here $a_{\mathfrak{p}}$ is as in Section 3B.

To provide context for the following independence, note that there exists $a \in H_x(\mathbf{Z}_p)$ such that $a_{\mathfrak{p}_i} \in H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}_i}$ and $a_{\mathfrak{p}_j} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}_j}$, for $j \neq i$. Indeed,

$$H_x(\mathbf{Z}_p) = O_p^\times = \prod_{\mathfrak{p}} O_{\mathfrak{p}}^\times$$

and we may choose a to be nonidentity precisely at the $\mathfrak{p}_i$-th component.

Let n be a positive integer. For $1 \leq i \leq n$, let $a_i \in H_x(\mathbf{Z}_p)$ such that $(a_i a_j^{-1})_{\mathfrak{p}} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ for all $i \neq j$ (see Section 3B). Let $f_1, \ldots, f_n$ be n nonconstant mod p modular forms on V (see Section 2D).

Our formulation of the linear independence is the following.

Theorem 4.1 ($\mathfrak{p}$-independence). *Let the notation and assumptions be as above, and suppose that $(a_i a_j^{-1})_{\mathfrak{p}} \notin H_x(\mathbf{Z}_{(p)})_{\mathfrak{p}}$ for all $i \neq j$. Then, the $(a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}))_i$ are linearly independent in the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.*

Note that $a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}})$ is not necessarily the restriction of a mod p modular form to $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$.

The above independence is $\mathfrak{p}$-analogue of the linear independence in [Hida 2010, §3.5]. As expounded in Section 4B, the approach in [Hida 2010, §3] is fundamental to the $\mathfrak{p}$-independence.

4B. *$\mathfrak{p}$-independence.* In this subsection, we prove the linear independence of mod p modular forms restricted to the partial Serre–Tate deformation space $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ (see Section 4A).

The approach is based on $\mathfrak{p}$-rigidity and Chai's theory of Hecke-stable subvarieties of a mod p Shimura variety adopted for local algebraic stabilisers in [Hida 2010, §3]. For a detailed treatment of the latter, we refer to [Chai 1995; 2003; 2006; Hida 2010].

Let n be a positive integer. In this subsection, any tensor product is taken n-times.

We consider an $\mathbb{F}$-algebra homomorphism

$$\phi_I : \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}} \to \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \tag{4-2}$$

given by

$$f_1 \otimes \cdots \otimes f_n \mapsto \prod_{i=1}^{i=n} a_{i,\mathfrak{p}} \circ (f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}). \tag{4-3}$$

As we are interested in the linear independence of $(a_{i,\mathfrak{p}}(f_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}))_i$, we consider $\mathfrak{b}_I := \ker(\phi_I)$.

Similarly, we consider an $\mathbb{F}$-algebra homomorphism

$$\phi = \phi_V : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}} \tag{4-4}$$

given by

$$h_1 \otimes \cdots \otimes h_n \mapsto \prod_{i=1}^{i=n} a_{i,\mathfrak{p}} \circ (h_i|_{\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}}). \tag{4-5}$$

In view of Theorem 3.7, it follows that ϕ_I and ϕ_V are both nontrivial.

(EQ) We note that ϕ is equivariant with the $H_x(\mathbf{Z}_{(p)})$-action.

Let $\mathfrak{b}_I = \ker(\phi_V)$ and $\mathfrak{b} = \ker(\phi_V)$.

Lemma 4.2. $\mathfrak{b}_I = 0$ *if and only if* $\mathfrak{b} = 0$.

Proof. In view of (Ét), we have an étale morphism

$$\pi^m : \mathcal{O}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{V,x} \to \mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}.$$

Note that $\mathfrak{b}_I$ is the unique prime ideal of $\mathcal{O}_{I,\tilde{x}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{I,\tilde{x}}$ over $\mathfrak{b}$. □

As ϕ is equivariant with the $H_x(\mathbf{Z}_{(p)})$-action (see (EQ)), it follows that $\mathfrak{b}$ is a prime ideal of $\mathcal{O}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{V,x}$ stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. Let Y be the Zariski closure of $\mathrm{Spec}(\mathcal{O}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{V,x}/\mathfrak{b})$ in V^n. Thus, Y is a closed irreducible subscheme of V^n containing x^n stable under the diagonal action of $H_x(\mathbf{Z}_{(p)})$. We also have an analogue of the commutative diagram [Hida 2010, (3.22)] with $\widehat{\mathcal{O}}_S$ replaced by $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$. For $n \geq 2$, the subscheme Y thus satisfies the hypothesis in [Hida 2010, Corollary 3.19].

Theorem 4.3. *The subscheme Y equals V^n.*

Proof. When $n = 1$, this is nothing but [Hida 2010, Proposition 3.8].

We thus suppose that $n \geq 2$. From [Hida 2010, Corollary 3.19], we have two possibilities, namely $Y = V^{n-2} \times \Delta_{\alpha,\beta}$ (up to a permutation of the factors) for some $\alpha, \beta \in H_x(\mathbf{Z}_{(p)})$, or $Y = V^n$. The skewed diagonal $\Delta_{\alpha,\beta}$ is given by

$$\Delta_{\alpha,\beta} = \{(\alpha(v), \beta(v)) \mid v \in V\}.$$

Suppose that $Y = V^{n-2} \times \Delta_{\alpha,\beta}$ (up to a permutation of the factors). Let (t, t') be the Serre–Tate coordinates of the last two factors $\Delta_{\alpha,\beta}$ at (x, x), respectively. It follows that

$$t^{\beta^{1-c_x}} = t'^{\alpha^{1-c_x}}.$$

On the other hand, from the definition of Y it follows that

$$t_{\mathfrak{p}}^{a_{n,\mathfrak{p}}} = t_{\mathfrak{p}}'^{a_{n-1,\mathfrak{p}}}.$$

Thus, $(a_n a_{n-1}^{-1})_{\mathfrak{p}} = (\beta\alpha^{-1})_{\mathfrak{p}}^{1-c_x} \in H_x(\boldsymbol{Z}_{(p)})_{\mathfrak{p}}$. This contradicts the hypothesis on the a_i. We conclude that $Y = V^n$. □

We have the following immediate consequence.

Corollary 4.4. *Theorem 4.1 holds.*

Proof. In view of Theorem 4.3, it follows that $\mathfrak{b} = 0$. Thus, $\mathfrak{b}_I = 0$ (see Lemma 4.2). □

5. *p*-adic differential operators

In this section, we consider p-adic operators on the space of p-adic Hilbert modular forms. This is a p-adic analogue of the Maass–Shimura differential operators on complex Hilbert modular forms. In Section 5A, we give an elementary construction of these operators. In Section 5B, we compute their action on the t-expansion of a p-adic Hilbert modular form around an ordinary point in terms of the partial Serre–Tate coordinates. For a more detailed account in the elliptic modular case, we refer to [Hida 2013a, §1.3.6]. In Section 6, we will use these results to compute the power series expansion of anticyclotomic Katz p-adic L-functions and variants (see Section 1).

5A. ***Elementary construction.*** In this subsection, we give an elementary construction of p-adic differential operators on the space of p-adic Hilbert modular forms.

Let the notation and assumptions be as in Section 2. For the geometric definition of classical and p-adic modular forms on the Hilbert modular Shimura variety, we refer to [Hida 2004, §4.2; 2010, §4.1].

Recall that $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$, W the Witt ring $W(\mathbb{F})$, $\iota_p : \overline{\boldsymbol{Q}} \hookrightarrow \boldsymbol{C}_p$ a p-adic embedding and $\iota_\infty : \overline{\boldsymbol{Q}} \hookrightarrow \boldsymbol{C}$ a complex embedding. Let $\mathcal{W}$ denote $\iota_p^{-1}(W)$. Possibly enlarging $\mathcal{W}$, we suppose that $\tau(O) \subset \mathcal{W}$, for all $\tau \in \Sigma$.

In this subsection, we suppose the prime-to-p level of classical or p-adic Hilbert modular forms is one. This is only to simplify the notation.

Let $G_\kappa(\Gamma_1(p^m), \mathcal{W})$ be the space of classical Hilbert modular forms of weight κ and level $\Gamma_1(p^m)$ over $\mathcal{W}$, where $\kappa \in \boldsymbol{Z}_{\geq 0}[\Sigma]$ and m is a nonnegative integer. Let $f \in G_\kappa(\Gamma_1(p^m), \mathcal{W})$. Via ι_∞, we regard f as a Hilbert modular form over $\boldsymbol{C}$. Let

$z = (z_\tau)_{\tau\in\Sigma}$ be the complex variables of the Hilbert modular Shimura variety or those of $\mathfrak{H}^\Sigma$, where $\mathfrak{H}$ is the upper half plane (see [Hida 2010, §4.1]). Let $(\mathfrak{a}, \mathfrak{b})$ be a pair giving rise to a cusp of the Hilbert modular Shimura variety (see [Hida 2010, §4.1]). The Fourier expansion of f at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ is given by

$$f(z) = \sum_{\xi\in O} a(\xi) e_F(\xi z)$$

for $e_F(\xi z) = \exp\bigl(2\pi i \sum_{\tau\in\Sigma} \tau(\xi) z_\tau\bigr)$.

Let $\phi : O/p^r O \to \mathcal{W}$ be an arbitrary function with the normalised Fourier transform ϕ^* given by

$$\phi^*(y) = \frac{1}{p^{rd/2}} \sum_{u\in O/p^r O} \phi(u) e_F(yu/p^r), \tag{5-1}$$

where $y \in O/p^r O$ and $e_F(w) = \exp(2\pi i \operatorname{Tr}_{F/\mathbb{Q}}(w))$ for $w \in F$.

Let $f|\phi$ be the classical Hilbert modular form given by

$$f|\phi(z) = \sum_{u'=(\sigma_1(u),\dots,\sigma_d(u)),\, u\in O/p^r O} \phi^*(-u) f(z + u'/p^r), \tag{5-2}$$

where $z + u'/p^r = (z_\tau + \tau(u)/p^r)_\tau$.

Note that $f|\phi \in G_\kappa(\Gamma_1(p^s), \mathcal{W})$, where $s = \max(m, 2r)$. In view of the Fourier inversion formula, it follow that the Fourier expansion of $f|\phi$ at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ is given by

$$f|\phi(z) = \sum_{\xi\in O} \phi(\xi) a(\xi) e_F(\xi z). \tag{5-3}$$

Let $(\phi_n^\sigma : O/p^n O \to \mathcal{W})_n$ be a sequence of functions such that

$$\phi_n^\sigma(\xi) \equiv \sigma(\xi) \pmod{p^n \mathcal{W}}.$$

In view of the q-expansion principle, it follow that the sequence $(f|\phi_n^\sigma)_n$ of classical Hilbert modular forms converges p-adically to a p-adic Hilbert modular form $d^\sigma f$ whose formal Fourier expansion at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ is given by

$$d^\sigma f(z) = \sum_{\xi\in O} \sigma(\xi) a(\xi) e_F(\xi z). \tag{5-4}$$

In other words, the operator d^σ equals the Maass–Shimura differential operator

$$\delta_0^\sigma = \frac{1}{2\pi i} \frac{\partial}{\partial z_\sigma}$$

on $G_\kappa(\Gamma_1(p^m), \mathcal{W})$.

This construction extends to the space of p-adic Hilbert modular forms over W as follows. Let $V(W)$ be the space of p-adic Hilbert modular forms of prime-to-p

level one over W. Via ι_p, we can regard $G_\kappa(\Gamma_1(p^m), \mathcal{W})$ as a subspace of $V(W)$ (see [Hida 2004, §8.1]). The space $G_\kappa(\Gamma_1(p^\infty), \mathcal{W}) = \bigcup_m G_\kappa(\Gamma_1(p^m), \mathcal{W})$ is p-adically dense in $V(W)$ (see [Hida 2004, §8.1]). Thus, the differential operator d^σ extends to $V(W)$.

Remark. In [Katz 1978, Chapter II], the operator d^σ is constructed based on the Gauss–Manin connection of the universal abelian scheme over the Shimura variety. The above approach can be generalised for a class of PEL Shimura varieties.

5B. ***Action on the t-expansion.*** In this subsection, we compute the action of the differential operator on the t-expansion of a p-adic Hilbert modular form around a p-ordinary point in terms of the partial Serre–Tate coordinates.

Let $(\zeta_{p^n} = \exp(2\pi i/p^n))_n \in \overline{\boldsymbol{Q}}$ be a compatible system of p-power roots of unity. Via ι_p, we regard it as a compatible system in $\boldsymbol{C}_p$.

Let $\mathfrak{p}$ be the prime corresponding to σ via ι_p and Σ_p be the set of places above p in F. For $\mathfrak{q} \in \Sigma_p$, let $\Sigma_\mathfrak{q} \subset \Sigma$ be the subset giving rise to $\mathfrak{q}$ under ι_p.

Let $W_n = W[\mu_{p^n}]$ and m_n be the maximal ideal. We have

$$(\widehat{\mathbb{G}}_m \otimes O_\mathfrak{p}^*)(W_n) = (1+m_n) \otimes O_\mathfrak{p}^* \tag{5-5}$$

and

$$\widehat{\mathbb{G}}_m \otimes O_\mathfrak{q}^* = \prod_{\tau \in \Sigma_\mathfrak{q}} \widehat{\mathbb{G}}_m. \tag{5-6}$$

Let $u \in O$ and $\alpha(u/p^m) \in G(\boldsymbol{A}^f)$ such that

$$\alpha(u/p^m)_\mathfrak{p} = \begin{bmatrix} 1 & u/p^m \\ 0 & 1 \end{bmatrix}$$

and $\alpha(u/p^m)_\mathfrak{l} = 1$, for $\mathfrak{l} \neq \mathfrak{p}$.

Let us recall some notation. Let $\pi : \mathrm{Ig} \to \mathrm{Sh}$ be the Igusa tower over W. Let $x \in \mathrm{Sh}^{\mathrm{ord}}_{/\mathbb{F}}$ be a closed point and $\tilde{x}$ be a point above it in Sh. Let $\widehat{S}_{/W}$ be the deformation space of $\tilde{x}$. For $q \in \Sigma_p$, recall $t_\mathfrak{q} = t \otimes 1_\mathfrak{q} \in \widehat{\mathbb{G}}_m \otimes O_\mathfrak{q}$ is the Serre–Tate coordinate of the partial deformation space $\widehat{\mathbb{G}}_m \otimes O_\mathfrak{q}$ (see Section 3A). We regard $1 \otimes u \in \widehat{\mathbb{G}}_m \otimes O_\mathfrak{q}$ via the image $u \in O_\mathfrak{q}$.

We start with a preparatory lemma.

Lemma 5.1. *The isogeny action of $\alpha(u/p^m)$ on the Igusa tower $\pi : \mathrm{Ig} \to \mathrm{Sh}_{/W_m}$ preserves the deformation space $\widehat{S}$ and induces $t_\mathfrak{p} \mapsto \zeta_{p^m} t \otimes u$ (see* (5-5)*) and $t_{\mathfrak{p}'} \mapsto t_{\mathfrak{p}'}$, for $\mathfrak{p}' \neq \mathfrak{p}$.*

Proof. Let $x_{\mathrm{ST}} = (\mathcal{A}_x, \cdot)$ be the universal deformation and $x_0 = (A_0, \cdot)$ be the origin of the deformation space $\widehat{S}$. In particular, we have

$$A_0[\mathfrak{p}^m] = \mu_{p^m} \otimes O_\mathfrak{p}^* \oplus O_\mathfrak{p}/\mathfrak{p}^m O_\mathfrak{p}. \tag{5-7}$$

By the universal level structure, we have an exact sequence

$$0 \longrightarrow \mu_{p^m} \otimes O_{\mathfrak{p}}^* \longrightarrow \mathcal{A}_x[\mathfrak{p}^m] \xrightarrow{h} O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}} \longrightarrow 0.$$

A section of the morphism h determines an $O_{\mathfrak{p}}$-cyclic subgroup C_u isomorphic to $O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}}$ defined over W_m, which specialises at the origin A_0 to a cyclic subgroup generated by $(\zeta_{p^m} \otimes u, \gamma_0)$ in $A_0[\mathfrak{p}^m]$ (see (5-7)). Here γ_0 denotes the image of 1 under the identification $A_0[\mathfrak{p}^m]^{\text{ét}} = O_{\mathfrak{p}}/\mathfrak{p}^m O_{\mathfrak{p}}$.

The isogeny action $\alpha(u/p^m)$ corresponds to the isogeny $\mathcal{A}_x \to \mathcal{A}_{x,u} := \mathcal{A}_x/C_u$. By an argument similar to the proof of [Brakočević 2011, Lemmas 7.1, 7.2], it follows that the $\mathfrak{p}$-Serre–Tate coordinates of $\mathcal{A}_{x,u}$ is given by $\zeta_{p^m} t \otimes u$. For $\mathfrak{p}' \neq \mathfrak{p}$, in view of the construction of the Serre–Tate coordinates (see Section 3A), it follows that the $\mathfrak{p}'$-Serre–Tate coordinate is given by $t_{\mathfrak{p}'}$. □

For $\tau \in \Sigma_{\mathfrak{q}}$, let $t_{\mathfrak{q}}^{\tau}$ be the τ-component of $t_{\mathfrak{q}}$ (see (5-6)).

Proposition 5.2. *The action of the p-adic differential operator d^{σ} on the deformation space $\widehat{S}$ is given by*

$$t_{\mathfrak{p}}^{\sigma} \frac{\partial}{\partial t_{\mathfrak{p}}^{\sigma}}.$$

Proof. As a p-adic Hilbert modular form is a p-adic limit of classical Hilbert modular forms, it suffices to verify the proposition for classical Hilbert modular forms.

Let $f \in G_{\kappa}(\Gamma_1(p^m), \mathcal{W})$ and the t-expansion of f around x be given by

$$f(t) = \sum_{\omega} a(\omega) \prod_{\mathfrak{q} \in \Sigma_p} t_{\mathfrak{q}}^{\omega_{\mathfrak{q}}}. \tag{5-8}$$

Here by $t_{\mathfrak{q}}^{\omega_{\mathfrak{q}}}$, we mean the character $t^{\omega_{\mathfrak{q}}}$ (see Section 3A) and the summation is over $\omega \in O$ (see [Hida 2010, pp. 106–107]). To emphasise the notion of t-expansion around a point, we use the indexing notation ω instead of the traditional notation ξ for q-expansion around a cusp.

We have

$$f|\phi = \sum_{u \in O/p^r O} \phi^*(-u) f|\alpha(u/p^r),$$

as $\alpha(u/p^m)$ acts on $\mathfrak{H}^{\Sigma}$ by $z \mapsto z + u'/p^r$ (see (5-2)).

In view of Lemma 5.1, it follows that

$$\begin{aligned} f|\phi_n^{\sigma}(t) &= \sum_{\omega} a(\omega) \Bigg(\sum_{u \in O/p^n O} (\phi_n^{\sigma})^*(-u) \zeta_{p^n}^{\mathrm{Tr}_{F/\mathbb{Q}}(u\omega_{\mathfrak{p}})} \Bigg) t_{\mathfrak{p}}^{\omega_{\mathfrak{p}}} \prod_{\mathfrak{p}' \neq \mathfrak{p}} t_{\mathfrak{p}'}^{\omega_{\mathfrak{p}'}} \\ &= \sum_{\omega} \phi_n^{\sigma}(\omega_{\mathfrak{p}}) a(\omega) \prod_{\mathfrak{q} \in \Sigma_p} t_{\mathfrak{q}}^{\omega_{\mathfrak{q}}}. \end{aligned} \tag{5-9}$$

The last equality follows from the Fourier inversion formula.

Thus, we have

$$d^\sigma f(t) = \lim_n f|\phi_n^\sigma(t) = \sum_\omega \sigma(\omega_\mathfrak{p})a(\omega) \prod_{\mathfrak{q}\in\Sigma_p} t_\mathfrak{q}^{\omega_\mathfrak{q}} = t_{\mathfrak{p},\sigma}\frac{\partial f}{\partial t_{\mathfrak{p},\sigma}}. \qquad \square$$

We have the following immediate consequence.

Corollary 5.3. *The p-adic differential operator d^σ is $\widehat{S}$-invariant.*

Remark. The above corollary is proven in [Katz 1981, §4.3] based on the computation of the Gauss–Manin connection in terms of the Serre–Tate coordinates. The above approach can be generalised for a class of PEL Shimura varieties.

6. Iwasawa μ-invariants

In this section, we consider Iwasawa μ-invariants as in Theorems 1.2–1.4. In Section 6A, we determine the μ-invariant of certain anticyclotomic $\mathfrak{p}$-adic L-functions (see Theorem 1.2). In Section 6B, we determine the μ-invariant of the cyclotomic derivative $L'_{\Sigma,\lambda,\mathfrak{p}}$ of the Katz $\mathfrak{p}$-adic L-function, when the branch character λ is self-dual with the root number -1 (see Theorem 1.3). In Section 6C, we prove a $\mathfrak{p}$-version of a conjecture Gillard regarding the vanishing of the μ-invariant of Katz p-adic L-function (see Theorem 1.4).

6A. ***μ-invariant of anticyclotomic $\mathfrak{p}$-adic L-functions.*** In this subsection, we first obtain a formula for the μ-invariant anticyclotomic Katz $\mathfrak{p}$-adic L-function (see Theorem 1.2). Towards the end, we comment on a similar formula for a class of Rankin–Selberg anticyclotomic $\mathfrak{p}$-adic L-functions.

Let the notation and assumptions be as in the introduction. Let $\mathfrak{C}$ be a prime-to-p ideal of K. Let $Z(\mathfrak{C})$ be the Ray class group of K modulo $\mathfrak{C}p^\infty$. Let $Z(\mathfrak{C})^-$ be the anticyclotomic quotient. The reciprocity law $\mathrm{rec}_K : (\boldsymbol{A}_K^f)^\times \to Z(\mathfrak{C})^-$ induces the isomorphism

$$\mathrm{rec}_K : \varprojlim_n K^\times(\boldsymbol{A}_F^f)^\times\backslash(\boldsymbol{A}_K^f)^\times/U_K(\mathfrak{C}p^n) \xrightarrow{\sim} Z(\mathfrak{C})^-$$

for $U_K = (\mathcal{O}_K \otimes \widehat{\boldsymbol{Z}})^\times$ and

$$U_K(\mathfrak{C}p^n) = \{u \in U_K \mid u \equiv 1 \pmod{\mathfrak{C}p^n}\}.$$

Let Γ^- be the maximal $\boldsymbol{Z}_p$-free quotient of $Z(\mathfrak{C})^-$ and $\Gamma_\mathfrak{p}^-$ be the $\mathfrak{p}$-part of Γ^- (see Section 1).

Let Γ' and $\Gamma'_\mathfrak{p}$ be the open subgroups of Γ^- generated by the images via rec_K of

$$O_p^\times \times \prod_{v|D_{K/F}} K_v^\times \quad \text{and} \quad O_\mathfrak{p}^\times \times \prod_{v|D_{K/F}} K_v^\times,$$

respectively. Let Σ_p be the places above p in K induced by the p-ordinary CM type Σ. The reciprocity law rec_K at Σ_p induces an injective map

$$\mathrm{rec}_{\Sigma_p} : 1 + pO_p \hookrightarrow O_p^\times = \bigoplus_{w \in \Sigma_p} \mathcal{O}_{K_w}^\times \xrightarrow{\mathrm{rec}_K} Z(\mathfrak{C})^-$$

with finite cokernel as $p \nmid D_F$. It induces isomorphisms $\mathrm{rec}_{\Sigma_p} : 1 + pO_p \xrightarrow{\sim} \Gamma'$ and $\mathrm{rec}_{\Sigma_p} : 1 + \mathfrak{p}O_\mathfrak{p} \xrightarrow{\sim} \Gamma'_\mathfrak{p}$. Via these isomorphisms, we identify Γ' (resp. $\Gamma'_\mathfrak{p}$) with the subgroup $\mathrm{rec}_{\Sigma_p}(1 + pO_p)$ (resp. $\mathrm{rec}_{\Sigma_p}(1 + \mathfrak{p}O_\mathfrak{p})$) of the anticyclotomic quotient $Z(\mathfrak{C})^-$.

In [Katz 1978; Hida and Tilouine 1993], a $\bar{\boldsymbol{Z}}_p$-valued p-adic measure $\mathcal{L}_{\mathfrak{C},\Sigma}$ on $Z(\mathfrak{C})$ is constructed. It interpolates a class of critical Hecke L-values corresponding to Hecke characters over K with prime-to-p conductor $\mathfrak{C}$ (see [Katz 1978; Hida and Tilouine 1993; Burungale 2016a]). Let λ be an arithmetic Hecke character over K with prime-to-p conductor $\mathfrak{C}$. Let $\mathcal{L}^-_{\Sigma,\lambda}$ (resp. $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}$) be the p-adic measure on Γ^- (resp. $\Gamma^-_\mathfrak{p}$) obtained by the push-forward of $\mathcal{L}_{\mathfrak{C},\Sigma}$ along λ.

Recall that the μ-invariant $\mu(\varphi)$ of a $\bar{\boldsymbol{Z}}_p$-valued p-adic measure φ on a p-adic group H is defined by

$$\mu(\varphi) = \inf_{U \subset H \text{ open}} v_p(\varphi(U)).$$

The μ-invariants of the above measures are related by the following theorem.

Theorem 6.1. *Let the notation and assumptions be as above. Then, we have*

$$\mu(\mathcal{L}^-_{\Sigma,\lambda}) = \mu(\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}).$$

Proof. Let $L^-_{\Sigma,\lambda}$ (resp. $L^-_{\Sigma,\lambda,\mathfrak{p}}$) be the power series of the measure $\mathcal{L}^-_{\Sigma,\lambda}$ (resp. $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}$) in the sense of (1-3).

We first suppose that $p \nmid h_K^-$, where h_K^- is the relative class number given by $h_K^- = h_K / h_F$ and $h_?$ is the class number of ?, for ? $= F, K$. We thus have

$$Z(\mathfrak{C})^- \simeq \Gamma^-.$$

We now describe the approach of the second named author to determine $\mu(L^-_{\Sigma,\lambda})$ (see [Hida 2010]).

Under the hypothesis, there exist a finite number of classical Hilbert modular Eisenstein series $(f_{\lambda,i})_i$ such that

$$L^-_{\Sigma,\lambda} = \sum_i a_i \circ (f_{\lambda,i}(t)) \tag{6-1}$$

up to an automorphism of $\bar{\boldsymbol{Z}}_p[\![\Gamma^-]\!]$. Here $f_{\lambda,i}(t)$ is the t-expansion of $f_{\lambda,i}$ around a well chosen CM point x with the CM type (K, Σ) on the Hilbert modular Shimura variety Sh and a_i is an automorphism of the deformation space of x in Sh for each i

(see [Hida 2010, Theorem 5.1; Hsieh 2014a, §5.2]). Moreover, the a_i satisfy the hypothesis in Theorem 4.1.

For $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma]$, let ν_κ be the p-adic character of Γ' such that

$$\nu_\kappa(\mathrm{rec}_{\Sigma_p}(y)) = y^\kappa$$

for $y \in 1 + pO_p$. Let d^κ be the p-adic differential operator corresponding to κ (see Section 5A). From (6-1), we thus conclude

$$\left(d^\kappa \sum_i a_i \circ (f_{\lambda,i}(t)) \right)\Big|_{t=1} = \int_{\Gamma^-} \nu_\kappa \, d\mathcal{L}^-_{\Sigma,\lambda}. \tag{6-2}$$

In view of the linear independence of $(a_i \circ (f_{\lambda,i}))_i$ (see [Hida 2010, Theorem 3.20]), it follows that

$$\mu(\mathcal{L}^-_{\Sigma,\lambda}) = \min_i \mu(f_{\lambda,i}(t)) = \min_i \mu(f_{\lambda,i}). \tag{6-3}$$

We now turn towards the anticyclotomic Katz $\mathfrak{p}$-adic L-function.

For a p-adic Hilbert modular form f, let $f(t_\mathfrak{p})$ be obtained from $f(t)$ by substituting $t_{\mathfrak{p}'} = 1$, for all $\mathfrak{p}' \neq \mathfrak{p}$.

Let

$$f^-_{\Sigma,\lambda,\mathfrak{p}} = \sum_i a_{i,\mathfrak{p}} \circ (f_{\lambda,i}(t_\mathfrak{p})). \tag{6-4}$$

Recall that $\Sigma_\mathfrak{p} \subset \Sigma$ denotes the subset of infinite places of F corresponding to $\mathfrak{p}$, via ι_p. For $\sigma \in \Sigma_\mathfrak{p}$, let $d^{\sigma'}$ be the formal differential operator given by

$$f(t_\mathfrak{p}) \mapsto t^\sigma_\mathfrak{p} \frac{\partial f}{\partial t^\sigma_\mathfrak{p}}.$$

In view of Proposition 5.2, it follows that

$$d^{\sigma'}(f(t_\mathfrak{p})) = (d_\sigma f)(t_\mathfrak{p}). \tag{6-5}$$

From now on, let $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma_\mathfrak{p}]$. Let $d^{\kappa'}$ be the corresponding formal differential operator.

We now have

$$d^{\kappa'}(f^-_{\Sigma,\lambda,\mathfrak{p}})|_{t_\mathfrak{p}=1} = (d^\kappa L^-_{\Sigma,\lambda})|_{t=1} = \int_{\Gamma^-} \nu_\kappa \, d\mathcal{L}^-_{\Sigma,\lambda} = \int_{\Gamma^-_\mathfrak{p}} \nu_\kappa \, d\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}. \tag{6-6}$$

The first two equalities follow from (6-5) and (6-2), respectively. The last equality follows from the fact that for $\kappa \in \mathbf{Z}_{\geq 0}[\Sigma_\mathfrak{p}]$, the character ν_κ factors through $\Gamma^-_\mathfrak{p}$.

In other words, $d^{\kappa'}(f^-_{\Sigma,\lambda,\mathfrak{p}})|_{t_\mathfrak{p}=1}$ interpolates the κ-th moment of the measure $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}}$. Thus,

$$f^-_{\Sigma,\lambda,\mathfrak{p}} = L^-_{\Sigma,\lambda,\mathfrak{p}}$$

up to an automorphism of $\bar{\mathbf{Z}}_p[[\Gamma^-_\mathfrak{p}]]$.

In view of the linear independence of $(a_{i,\mathfrak{p}} \circ (f_{\lambda,i}|_{\widehat{\mathbb{G}}_m \otimes \mathcal{O}_\mathfrak{p}}))_i$ (see Theorem 4.1), it follows that

$$\mu(L^-_{\Sigma,\lambda,\mathfrak{p}}) = \min_i \mu(f_{\lambda,i}(t_\mathfrak{p})) = \min_i \mu(f_{\lambda,i}|_{\widehat{\mathbb{G}}_m \otimes \mathcal{O}_\mathfrak{p}})). \tag{6-7}$$

In view of Corollary 3.8, this finishes the proof of the theorem for the case $p \nmid h^-_K$.

When $p \mid h^-_K$, the power series $L^-_{\Sigma,\lambda}$ restricted to an explicit finite open cover of Γ^- is still of the form (5-1) (see [Hsieh 2014a, §5.2]). Thus, a similar argument proves the Theorem. □

In most of the cases, $\mu(L^-_{\Sigma,\lambda})$ has been explicitly determined [Hida 2010; Hsieh 2014a]. Thus, we obtain a formula for $\mu(L^-_{\Sigma,\lambda,\mathfrak{p}})$.

Remark. A class of Rankin–Selberg anticyclotomic p-adic L-functions for Hilbert modular forms is constructed in [Hsieh 2014b]. It also satisfies a property analogous to (6-1) [Hsieh 2014b, §6.2]. Thus, by an argument similar to the proof of Theorem 6.1, we get an analogue of Theorem 6.1. In the near future, the first named author hopes to consider Rankin–Selberg anticyclotomic p-adic L-functions for quaternionic modular forms. In these situations, the underlying Shimura variety turns out to be a quaternionic Shimura variety arising from a quaternion algebra over a totally real field which is not totally definite.

6B. ***μ-invariant of the cyclotomic derivative of the Katz 𝔭-adic L-function.*** In this subsection, we determine the μ-invariant of the cyclotomic derivative of the Katz 𝔭-adic L-function, when the branch character λ is self-dual with the root number -1 (see Theorem 1.3).

Let K^+_∞ be the cyclotomic $\mathbf{Z}_p$-extension of K and $K_{\mathfrak{p},\infty} = K^-_{\mathfrak{p},\infty}K^+_\infty$. Let $\Gamma = \mathrm{Gal}(K^-_\infty K^+_\infty/K)$ and $\Gamma_\mathfrak{p} = \mathrm{Gal}(K_{\mathfrak{p},\infty}/K)$. Let λ be an arithmetic Hecke character over K. Let $\mathcal{L}_{\Sigma,\lambda}$ (resp. $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}$) be the p-adic measure on Γ (resp. $\Gamma_\mathfrak{p}$) obtained by the pull-back of $\mathcal{L}_{\mathfrak{C},\Sigma}$ (see Section 6A) along λ. We call $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}$ the Katz 𝔭-adic L-function with branch character λ. Let $L_{\Sigma,\lambda}(T_1, T_2, \dots, T_d, S) \in \bar{\mathbf{Z}}_p[\![\Gamma]\!]$ (resp. $L_{\Sigma,\lambda,\mathfrak{p}}(\,\cdot\,, S) \in \bar{\mathbf{Z}}_p[\![\Gamma_\mathfrak{p}]\!]$) be the power series of $\mathcal{L}_{\Sigma,\lambda}$ (resp. $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}$). Here, $T_1, \dots, T_d$ are the anticyclotomic variables and S is the cyclotomic variable.

In this subsection, we now suppose that λ is self-dual, i.e.,

$$\lambda|_{\mathbf{A}_F^\times} = \tau_{K/F} |\cdot|_{\mathbf{A}_F},$$

where $\tau_{K/F}$ is the quadratic character associated to K/F and $|\cdot|_{\mathbf{A}_F}$ is the adelic norm. In particular, the global root number of λ is ± 1. Now, suppose that the global root number is -1. In view of the functional equation of the Hecke L-function, this root number condition forces all the Hecke L-values appearing in the interpolation property of $\mathcal{L}^-_{\Sigma,\lambda}$ to vanish. Accordingly, we have $\mathcal{L}^-_{\Sigma,\lambda} = 0$. This also follows from the functional equation of $\mathcal{L}_{\Sigma,\lambda}$ [Hida and Tilouine 1993, §5]. In particular, we have $\mathcal{L}^-_{\Sigma,\lambda,\mathfrak{p}} = 0$.

We can consider the cyclotomic derivatives

$$L'_{\Sigma,\lambda} = \left(\frac{\partial}{\partial S} L_{\Sigma,\lambda}(T_1, \dots, T_d, S)\right)\Big|_{S=0} \tag{6-8}$$

and $L'_{\Sigma,\lambda,\mathfrak{p}}$ (defined analogously).

The μ-invariants of these derivatives are related by the following theorem.

Theorem 6.2. *Let the notation and assumptions be as above. In addition, suppose that $p \nmid h^-_K$, where h^-_K is the relative class number given by $h^-_K = h_K / h_F$. Then,*

$$\mu(L'_{\Sigma,\lambda}) = \mu(L'_{\Sigma,\lambda,\mathfrak{p}}).$$

Proof. We follow the notation in the proof of Theorem 6.1.

In the proof of [Burungale 2015, Theorem 3.2], it is shown there are p-adic Hilbert modular forms $(\boldsymbol{f}'_{\lambda,i})_i$ such that

$$L'_{\Sigma,\lambda} = \frac{1}{\log_p(1+p)} \sum_i a_i \circ (f'_{\lambda,i}(t)), \tag{6-9}$$

up to an automorphism of $\bar{\boldsymbol{Z}}_p[[\Gamma^-]]$. More precisely, $f'_{\lambda,i}$ is the p-adic derivative of $f_{\lambda N^s,i}$ at $s = 0$ for N the norm Hecke character over K. Being a p-adic limit of classical Hilbert modular forms, it is a p-adic Hilbert modular form. We refer to the proof of [Burungale 2015, Theorem 3.2] for details.

In view of (6-4), (6-6) and by a similar argument as in the proof of [Burungale 2015, Theorem 3.2], it follows that

$$L'_{\Sigma,\lambda,\mathfrak{p}} = \frac{1}{\log_p(1+p)} \sum_i a_{i,\mathfrak{p}} \circ (f'_{i,\lambda}(t_\mathfrak{p})), \tag{6-10}$$

up to an automorphism of $\bar{\boldsymbol{Z}}_p[[\Gamma^-_\mathfrak{p}]]$.

We finish the proof in the same way as in Theorem 6.1. □

In most of the cases, $\mu(L'_{\Sigma,\lambda})$ has been explicitly determined [Burungale 2015, Theorem A]. Thus, we obtain a formula for $\mu(L'_{\Sigma,\lambda,\mathfrak{p}})$.

Remark. When $p \mid h^-_K$, we do not know an expression for $L_{\Sigma,\lambda}$ in terms of the t-expansion of certain Hilbert modular forms. Such an expression seems to be essential in the above approach.

6C. ***$\mathfrak{p}$-version of a conjecture of Gillard.*** In this subsection, we prove a $\mathfrak{p}$-version of a conjecture Gillard regarding the vanishing of the μ-invariant of Katz p-adic L-function (see Theorem 1.4).

Let λ be an arithmetic Hecke character over K. Recall that we have the Katz $\mathfrak{p}$-adic L-function $\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}} \in \bar{\boldsymbol{Z}}_p[[\Gamma_\mathfrak{p}]]$ (see Section 6B). As a consequence of Theorem 6.1

and the results of [Hida 2011; Burungale and Hsieh 2013], we prove a $\mathfrak{p}$-version of a conjecture of Gillard [1991, Conjecture (i)] regarding the vanishing of the μ-invariant of the Katz p-adic L-function. The conjecture was originally formulated for the $(d+1)$-variable Katz p-adic L-function.

Theorem 6.3. *Let the notation and assumptions be as above. Then, we have*

$$\mu(\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}) = 0.$$

Proof. Let $\mathfrak{X}^+$ be the set consisting of finite order characters $\epsilon : \Gamma^+ \to \mu_{p^\infty}$. For every $\epsilon \in \mathfrak{X}^+$, we regard ϵ as a Hecke character.

In [Hida 2011; Burungale and Hsieh 2013], it has been shown that

$$\liminf_{\epsilon \in \mathfrak{X}^+} \mu(\mathcal{L}^-_{\Sigma,\lambda\epsilon}) = 0. \tag{6-11}$$

In view of Theorem 6.1, this finishes the proof:

$$0 \leq \mu(\mathcal{L}_{\Sigma,\lambda,\mathfrak{p}}) \leq \liminf_{\epsilon} \mu(\mathcal{L}^-_{\Sigma,\lambda\epsilon}). \qquad \square$$

The theorem evidently implies the main results of [Hida 2011; Burungale and Hsieh 2013].

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