Algebra & Number Theory

Volume 11

2017

No. 9

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ALGEBRA AND NUMBER THEORY 11:9 (2017)
dx.doi.org/10.2140/ant.2017.11.2131

On the density of zeros of linear combinations of Euler products for $\sigma > 1$

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It has been conjectured by Bombieri and Ghosh that the real parts of the zeros of a linear combination of two or more L-functions should be dense in the interval $[1, \sigma^*]$, where σ^* is the least upper bound of the real parts of such zeros. In this paper we show that this is not true in general. Moreover, we describe the optimal configuration of the zeros of linear combinations of orthogonal Euler products by showing that the real parts of such zeros are dense in subintervals of $[1, \sigma^*]$ whenever $\sigma^* > 1$.

1. Introduction

Let $L(s)$ be a Dirichlet series and let $\sigma^* = \sigma^*(L)$ be the least upper bound of the real parts of the zeros of $L(s)$. It is well known that σ^* is finite (see, e.g., Titchmarsh [1975, §9.41]). For the Riemann zeta function we know that $\sigma^* \leq 1$, and it is expected that the Riemann hypothesis holds, i.e., $\sigma^* = \frac{1}{2}$. A similar situation is expected for many Euler products (see, e.g., Selberg [1992]).

On the other hand, we have recently proved [Righetti 2016a], for a large class of L-functions with a polynomial Euler product, that nontrivial linear combinations have zeros for $\sigma > 1$. This is not surprising since many examples of such linear combinations were already known to have zeros for $\sigma > 1$ from work of Davenport and Heilbronn [1936a; 1936b] on the Hurwitz and Epstein zeta functions. We also refer to later important works of Cassels [1961], Conrey and Ghosh [1994], Saias and Weingartner [2009], and Booker and Thorne [2014].

Since for this type of Dirichlet series we know that there are zeros in the region of absolute convergence, which we may always suppose to be $\sigma > 1$, it is of interest to know the distribution of such zeros in this half-plane. With respect to the distribution of the imaginary parts the problem was completely solved by Jessen and Tornehave [1945]. Indeed it is known that the number of zeros in any rectangle

MSC2010: primary 11M41; secondary 11M26.
Keywords: zeros of Dirichlet series, value distribution, asymptotic distribution functions, convexity.

$[\sigma_1, \sigma_2] \times [T_1, T_2]$, with $1 < \sigma_1 < \sigma_2$, satisfies (cf. Theorem 31 of [Jessen and Tornehave 1945])

$$N(\sigma_1, \sigma_2, T_1, T_2) = c(T_2 - T_1) + o(|T_2 - T_1|), \quad \text{when } |T_2 - T_1| \to \infty, \tag{1-1}$$

for some nonnegative constant $c = c(\sigma_1, \sigma_2)$. Note that by a classical application of the Bohr almost periodicity of Dirichlet series and Rouché's theorem we easily have that $c > 0$ whenever $N(\sigma_1, \sigma_2, T_1, T_2) > 0$.

On the other hand the situation regarding the distribution of the real parts of the zeros is much more complicated. In fact some Epstein zeta functions studied by Bombieri and Mueller [2008] are known to have the property that the real parts of their zeros are dense in the interval $[1, \sigma^*]$. Note that these functions may be written as a linear combination of two Hecke L-functions. Other examples of linear combinations with this property may be found in Bombieri and Ghosh [2011], although not explicitly stated. Moreover, we remarked in [Righetti 2016a] that, as a consequence of the technique used to prove the main result there, the real parts of the zeros of nontrivial combinations of orthogonal L-functions are dense in a small interval $[1, 1 + \eta]$, for some $\eta > 0$ (cf. Corollary 1 of [Righetti 2016a]). Hence one might expect, as conjectured by Bombieri and Ghosh [2011, p. 230], that the real parts of the zeros of linear combinations of two or more L-functions should be dense in the whole interval $[1, \sigma^*]$. However this is too much to hope for as one can see from the following general counterexample.

Theorem 1.1. *Let $N \geq 2$ be an integer and let $F_j(s) = \sum_{n=1}^{\infty} a_j(n)n^{-s}$ be distinct nonidentically zero Dirichlet series absolutely convergent for $\sigma > 1$, $j = 1, \ldots, N$, with $\sum_{j=1}^{N} |a_j(1)| \neq 0$. Then, for any $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{C}^N$ such that $\sum_{j=1}^{N} x_j a_j(1) = 0$ but the Dirichlet series $L_{\boldsymbol{x}}(s) = \sum_{j=1}^{N} x_j F_j(s)$ is not identically zero, there exist infinitely many projectively inequivalent vectors $\boldsymbol{c} \in \mathbb{C}^N$ such that $L_{\boldsymbol{c}}(s)$ has no zeros in some vertical strip $\sigma_1 < \sigma < \sigma_2$ with $1 < \sigma_1 < \sigma_2 < \sigma^*(L_{\boldsymbol{c}})$.*

Remark. The above statement is very general, but in particular may be applied to linear combinations of linearly independent L-functions. Moreover, it is easy to show that the same argument works also for a-values with $a \neq 0$.

This has to be compared with what happens for $\frac{1}{2} < \sigma < 1$. There it is known that *joint universality* of L-functions implies that the real parts of the zeros of any linear combination of these L-functions are dense in $\left[\frac{1}{2}, 1\right]$ (see, e.g., [Bombieri and Gosh 2011, p. 230]). Furthermore joint universality is known to hold for many families of L-functions and recently Lee, Nakamura and Pańkowski [Lee et al. 2017] have shown that this property holds in an axiomatic setting such as the Selberg class under a strong Selberg orthonormality conjecture.

We can actually prove more, i.e., it is in general possible to construct Dirichlet series, given by a linear combination of L-functions, which have many *distinct* vertical strips without zeros, i.e., such that between every two vertical strips without zeros there is at least one zero.

Theorem 1.2. *Let $k \geq 1$ be an integer and, for $j = 1, \dots, k+1$, let $F_j(s) = \sum_{n=1}^{\infty} a_j(n) n^{-s}$ be a Dirichlet series absolutely convergent for $\sigma > 1$ with $a_j(1) \neq 0$. Suppose that*

$$\det \begin{pmatrix} a_1(1) & a_1(2) & \cdots & a_1(k+1) \\ a_2(1) & a_2(2) & \cdots & a_2(k+1) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1}(1) & a_{k+1}(2) & \cdots & a_{k+1}(k+1) \end{pmatrix} \neq 0. \tag{1-2}$$

Then there exists at least one $\boldsymbol{c} \in \mathbb{C}^{k+1}$ such that the Dirichlet series $L_{\boldsymbol{c}}(s) = \sum_{j=1}^{k+1} c_j F_j(s)$ has at least k distinct vertical strips without zeros in the region $1 < \sigma < \sigma^*(L_{\boldsymbol{c}})$.

Remark. Note that trivially every nonzero scalar multiple of a vector $\boldsymbol{c}$ of Theorems 1.1 or 1.2 has the same property. On the other hand, in Theorem 1.1, for every $\boldsymbol{x}$ the vectors $\boldsymbol{c}$ are given by the intersection of a ball and a hyperplane in $\mathbb{C}^N$, hence there are clearly infinitely many projectively inequivalent such vectors; see Section 6 for details. Besides, the proof of Theorem 1.2 seems to suggest that there may actually be infinitely many projectively inequivalent vectors $\boldsymbol{c}$ with the same property in this case too.

The proof of Theorem 1.2 is actually constructive and may be used to explicitly obtain coefficients $\boldsymbol{c}$. As a concrete example we apply it to $\zeta(s)$, $L(s, \chi_1)$ and $L(s, \overline{\chi_1})$, where χ_1 is the unique Dirichlet character mod 5 such that $\chi_1(2) = i$, which satisfy the hypotheses of Theorem 1.2. We thus obtain the Dirichlet series

$$L(s) = c_1 L(s, \chi_1) + c_2 L(s, \overline{\chi_1}) + c_3 \zeta(s),$$

where

$$c_1 = -\frac{1}{L(8, \overline{\chi_1})} \frac{L(16, \overline{\chi_1})\zeta(8) - L(8, \overline{\chi_1})\zeta(16)}{L(16, \chi_1)\zeta(8) - L(8, \chi_1)\zeta(16)} = -0.08260584\ldots - i0.99658995\ldots,$$

$$c_2 = \frac{1}{L(8, \overline{\chi_1})} = 1.00000059\ldots + i0.00375400\ldots,$$

$$c_3 = \frac{1}{L(8, \overline{\chi_1})} \frac{L(8, \chi_1)L(16, \overline{\chi_1}) - L(8, \overline{\chi_1})L(16, \chi_1)}{\zeta(8)L(16, \chi_1) - L(8, \chi_1)\zeta(16)} = -0.91739597\ldots + i0.99283727\ldots.$$

In Figure 1 we see part of two distinct vertical strips without zeros of $L(s)$ within

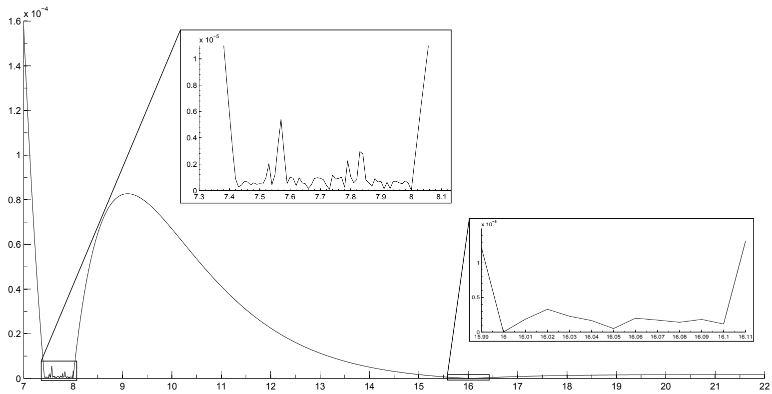

Figure 1. Approximate plot of

$$\min_t \left| c_1 L(\sigma + it, \chi_1) + c_2 L(\sigma + it, \overline{\chi_1}) + c_3 \zeta(\sigma + it) \right|$$

for $\sigma \in [7, 22]$ and $t \in [0, 2000]$ with step 0.01.

the vertical strip $1 < \sigma < \sigma^*$. We recall that, by [Saias and Weingartner 2009], there are zeros in the vertical strip $1 < \sigma < 1 + \eta$ for some $\eta > 0$.

Actually Figure 1 shows that another interesting phenomenon happens for linear combinations of *orthogonal* (see (1-3)) L-functions: it looks like that whenever there is one zero then there should be a small closed interval, either around or beside its real part, where the real parts of the zeros are dense. The bulk of this paper is devoted to showing that this is indeed true.

We first recall that, as a consequence of the work of Jessen and Tornehave [1945] on the asymptotic number of zeros mentioned above, we have the following general result. We denote by $\sigma_u(L)$ the abscissa of uniform convergence of $L(s)$.

Theorem 1.3. *Suppose* $L(s) = \sum_{n=n_0}^{\infty} a(n)/n^s$ *has* $a(n_0) \neq 0$ *and* $\sigma^*(L) > \sigma_u(L)$. *Then in any vertical strip* $\sigma_u(L) < \alpha \leq \sigma \leq \sigma^*(L)$, $L(s)$ *has only a finite number of zero-free vertical strips and a finite number of isolated vertical lines containing zeros. In particular, if* $\rho_0 = \beta_0 + i\gamma_0$ *is a zero of* $L(s)$ *with* $\beta_0 > \sigma_u(L)$, *then either* $\sigma = \beta_0$ *is an isolated vertical line as above or there exist* $\sigma_1 \leq \beta_0 \leq \sigma_2$, *with* $\sigma_1 < \sigma_2$, *such that the set*

$$\left\{ \beta \in [\sigma_1, \sigma_2] \mid \exists \gamma \in \mathbb{R} \text{ such that } L(\beta + i\gamma) = 0 \right\}$$

is dense in $[\sigma_1, \sigma_2]$.

The first part of Theorem 1.3 is a reinterpretation of Theorem 31 of [Jessen and Tornehave 1945] in view of Theorem 8 of the same paper. The second part follows from the first one by a simple set-theoretic argument.

Therefore we just need to prove that a linear combination of orthogonal Euler products has no isolated vertical lines containing zeros. As in [Righetti 2016a] we work in an axiomatic setting, and at the end of the introduction we briefly mention some important families of L-functions satisfying the required properties. Given a complex function $F(s)$ we consider the following properties:

(I) $F(s) = \sum_{n=1}^{\infty} a_F(n)/n^s$ is absolutely convergent for $\sigma > 1$;

(II) $\log F(s) = \sum_p \sum_{k=1}^{\infty} b_F(p^k)/p^{ks}$ is absolutely convergent for $\sigma > 1$, with $|b_F(p^k)| \ll p^{k\theta}$ for every prime p and every $k \geq 1$, for some $\theta < \frac{1}{2}$;

(III) for any $\varepsilon > 0$, $|a_F(n)| \ll n^{\varepsilon}$ for every $n \geq 1$.

Definition. For any integer $N \geq 1$, we say that $F_1(s), \dots, F_N(s)$ satisfying (I) and (II) are *orthogonal* if

$$\sum_{p \leq x} \frac{a_{F_i}(p)\overline{a_{F_j}(p)}}{p} = (m_{i,j} + o(1)) \log\log x, \quad x \to \infty, \tag{1-3}$$

with $m_{i,i} > 0$ and $m_{i,j} = 0$ if $i \neq j$.

Remark. There are some differences between the axioms that in [Righetti 2016a] define the class $\mathcal{E}$ and the above axioms (I)–(III), so that in principle we cannot say that the results that we obtained in [Righetti 2016a] may be applied here or *vice versa*. However most of the known families of L-functions satisfy, or are supposed to satisfy, both the axioms of $\mathcal{E}$ and (I)–(III).

We can now state the main theorems. We consider separately the cases $N = 2$ and $N \geq 3$ since they are handled in different ways and yield different results, although the underling idea is the same.

Theorem 1.4. *Let $F_1(s)$, $F_2(s)$ be orthogonal functions satisfying* (I) *and* (II), $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, *and $L(s) = c_1F_1(s) + c_2F_2(s)$. Then $L(s)$ has no isolated vertical lines containing zeros in the half-plane $\sigma > 1$.*

Theorem 1.5. *Suppose $N \geq 3$ is an integer, $c_1, \dots, c_N \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and $F_1(s), \dots, F_N(s)$ are orthogonal functions satisfying* (I)–(III). *If we write $L(s) = \sum_{j=1}^{N} c_jF_j(s) - c$, then $L(s)$ has no isolated vertical lines containing zeros in the half-plane $\sigma > 1$.*

Theorems 1.4 and 1.5 are obtained by suitably adapting the works of Bohr and Jessen [1930; 1932], Jessen and Wintner [1935], Jessen and Tornehave [1945], Borchsenius and Jessen [1948], and Lee [2014] on the value distribution of Dirichlet series. Note that, however, most of these papers refer to results on particular Dirichlet series in the strip $\frac{1}{2} < \sigma < 1$, while we work in the half-plane $\sigma > 1$ with far more general Dirichlet series. Hence, although the ideas are similar, the results are quite different in nature and technical difficulty. The proofs will be given in Sections 4 and 5 respectively.

Remark. Note that orthogonality is necessary in Theorems 1.4 and 1.5 as is shown by the following simple example

$$(1-2^{-s})\zeta(s)-\tfrac{3}{4}\zeta(s)=\Big(\frac{1}{4}-\frac{1}{2^s}\Big)\zeta(s),$$

which clearly vanishes, in the half-plane of absolute convergence $\sigma > 1$, only on the vertical line $\sigma = 2$. We mention here that in the proof of Theorems 1.4 and 1.5, roughly speaking, orthogonality is just used to bound particular oscillatory integrals (see the end of Section 2) and therefore to show that certain distribution functions behave "nicely" (see Section 3).

From Theorems 1.4 and 1.5 we obtain the following interesting consequence, which should be compared with Corollary 1 of [Righetti 2016a].

Corollary 1.6. *Let $L(s)$ be as in Theorems 1.4 or 1.5. If $\sigma^*(L) > 1$, then there exists $\eta > 0$ such that the set*

$$\big\{\beta\in[\sigma^*(L)-\eta,\sigma^*(L)]\mid \exists\gamma \text{ such that } L(\beta+i\gamma)=0\big\}$$

is dense in $[\sigma^*(L)-\eta,\sigma^*(L)]$.

Proof. If $\sigma^* = \sigma^*(L)$ is itself the real part of a zero, the result follows immediately from the second part of Theorem 1.3 and Theorems 1.4 and 1.5, choosing $\eta = \sigma^* - \sigma_1 > 0$ and $\sigma_2 = \sigma^*$. Suppose otherwise that σ^* is not the real part of a zero. Then by definition σ^* is the limit point of the real part of certain zeros of $L(s)$. Note that in general if $L(\sigma+it)\neq 0$, then either for any $\varepsilon > 0$ there exist β_ε with $|\sigma-\beta_\varepsilon| < \varepsilon$ and $\gamma_\varepsilon\in\mathbb{R}$ such that $L(\beta_\varepsilon+i\gamma_\varepsilon)=0$, i.e., σ is the limit point of the real part of certain zeros of $L(s)$, or there exists an open interval $(\sigma-\delta,\sigma+\delta)$, for some $\delta > 0$, which does not contain any real part of the zeros. Since by Theorem 1.3 the number of zero-free vertical strips in $\sigma^*-\varepsilon<\sigma<\sigma^*$ is finite for every small $\varepsilon > 0$, we can take $\eta = \varepsilon$ small enough so that there are none. □

By Theorems 1.1 and 1.2 we see that Theorems 1.4 and 1.5 are optimal, in the sense that without conditions on the coefficients $\boldsymbol{c}$ we cannot expect stronger results on the density of the real parts of the zeros. On the other hand it may be true that one could provide necessary and sufficient conditions on the coefficients of a linear combination of L-functions to guarantee Bombieri and Ghosh's conjecture to hold, but this seems out of reach at the moment. Here we just mention the following example with the Davenport–Heilbronn type L-functions studied by Bombieri and Ghosh [2011]. As we already remarked, Bombieri and Ghosh do not say whether these functions do have the property that the real parts of their zeros are dense in $[1,\sigma^*]$. However, in our Ph.D. thesis [Righetti 2016b] we gave necessary and

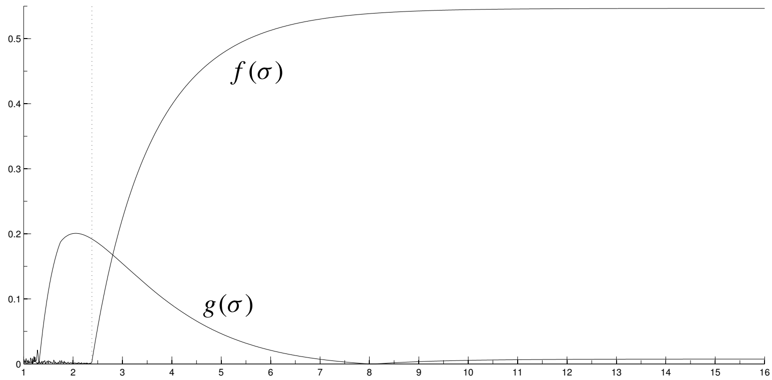

Figure 2. Approximate plot of

$$f(\sigma) = \min_t \left| \frac{L(\sigma+it, \chi_1)}{L(\sigma+it, \overline{\chi_1})} + \frac{1+i\tau}{1-i\tau} \right|,$$

$$g(\sigma) = \min_t \left| \frac{L(\sigma+it, \chi_1)}{L(\sigma+it, \overline{\chi_1})} - \frac{L(8, \chi_1)}{L(8, \overline{\chi_1})} \right|,$$

where $\sigma \in [1.01, 16.01]$ and $t \in [0, 2000]$ with step 0.01.

sufficient conditions on the coefficients of these Dirichlet series for this to happen, namely:

Theorem 1.7. *Let $\xi \in \mathbb{R}$, χ_1 be the unique Dirichlet character mod 5 such that $\chi_1(2) = i$, q be a positive integer and χ_0 be the principal character mod q. Then there exists $\xi_{\max}(q)$, such that the real parts of the zeros for $\sigma > 1$ of*

$$f(s, \xi, q) = \tfrac{1}{2}\big[(1-i\xi)L(s, \chi_1\chi_0) + (1+i\xi)L(s, \overline{\chi_1}\chi_0)\big]$$

are dense in the interval $[1, \sigma^*(\xi, q)]$ if and only if $|\xi| \le \xi_{\max}(q)$. In particular, if $6 \nmid q$ it is sufficient to take $|\xi| \le 6.5851599$.

Proof. The proof is a continuation of the proof of Theorem 7 of [Bombieri and Gosh 2011] using results of Kershner [1936, Theorems II–III] on the support function of the inner border of the sum of convex curves. We refer to Theorem 4.1.3 of [Righetti 2016b] for details. □

As an example we see in Figure 2 that the real parts of the zeros of Davenport–Heilbronn type L-function

$$f(s, \tau) = \tfrac{1}{2}\big[(1-i\tau)L(s, \chi_1) + (1+i\tau)L(s, \overline{\chi_1})\big], \quad \tau = -\frac{1+\sqrt{5}}{2} - \sqrt{1+\Big(\frac{1+\sqrt{5}}{2}\Big)^2},$$

are dense up to $\sigma^* = 2.3822861089\ldots$. On the other hand, we see that the real parts of the zeros of $L(s, \chi_1) - cL(s, \overline{\chi_1})$, where

$$c = \frac{L(8, \chi_1)}{L(8, \overline{\chi_1})} = 0.99997181\ldots + i0.00750790\ldots,$$

are dense close to $\sigma = 1$ (cf. Corollary 1 of [Righetti 2016a]), there are no zeros with real part in the interval $[2, 7]$, but $s = 8$ is clearly a zero.

Note that in the previous results we don't ask for a functional equation or meromorphic continuation to the whole complex plane. However, in many concrete cases these are known to hold, so one might ask what happens if one adds these conditions. On account of this we show that Theorem 1.1 may be modified so that the resulting Dirichlet series is an L-function with functional equation and, of course, without Euler product. We therefore consider functions $F(s)$ satisfying (I) and

(IV) $(s-1)^m F(s)$ is an analytic continuation as an entire function of finite order for some $m \geq 0$,

(V) $F(s)$ satisfies a functional equations of the form $\Phi(s) = \omega\overline{\Phi(1-\bar{s})}$, where $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \gamma(s) F(s),$$

say, with $r \geq 0$, $Q > 0$, $\lambda_j > 0$ and $\operatorname{Re} \mu_j \geq 0$,

although such requirements can actually be relaxed.

Theorem 1.8. *Let $N \geq 3$ be an integer, $(r, Q, \boldsymbol{\lambda}, \boldsymbol{\mu})$ fixed parameters, and let $F_1(s), \ldots, F_N(s)$ be functions satisfying* (I), (II), (IV) *and* (V) *for some $|\omega_j| = 1$, $j = 1, \ldots, N$. Suppose furthermore that $\omega_h \neq \omega_k$ for some $h, k \in \{1, \ldots, N\}$. Then there exist infinitely many $\boldsymbol{c} \in \mathbb{C}^N$ such that $L_{\boldsymbol{c}}(s) = \sum_{j=1}^{N} c_j F_j(s)$ satisfies* (IV), (V) *and has no zeros in some vertical strip $\sigma_1 < \sigma < \sigma_2$ with $1 < \sigma_1 < \sigma_2 < \sigma^*(L_{\boldsymbol{c}})$.*

To give a concrete example of the above result, we fix an integer $q \geq 7$, square-free, $(q, 6) = 1$ and $q \not\equiv 2 \pmod 4$, and consider the Dirichlet L-functions associated with primitive characters $\chi \pmod q$. Their number is $\varphi^*(q) = \prod_{p|q}(p-2)$ and at least half of them have the same *parity*. We denote by $\mathcal{W}(q)$ the set of such characters and we have that $|\mathcal{W}(q)| \geq 3$. As a consequence of Theorem 1 of Kaczorowski, Molteni and Perelli [Kaczorowski et al. 2010], we have that $\omega_{\chi_1} \neq \omega_{\chi_2}$ if $\chi_1 \neq \chi_2$ for $\chi_1, \chi_2 \in \mathcal{W}(q)$, so we may apply Theorem 1.8 to the Dirichlet L-functions associated with distinct characters of $\mathcal{W}(q)$.

On the other hand, we mention that Bombieri and Hejhal [1995] have shown that, under the generalized Riemann hypothesis and a weak pair correlation of the zeros, linear combinations with real coefficients of Euler products with the same functional equation have asymptotically almost all of their zeros on the line $\sigma = \frac{1}{2}$.

As concrete examples of families of L-functions satisfying the properties required by Theorems 1.4 and 1.5 we refer to [Righetti 2016a] for Artin L-functions, automorphic L-functions and the Selberg class. Here we only recall that the relevant analytic properties of the automorphic L-functions and their orthogonality can be found in the papers of Rudnick and Sarnak [1996], Iwaniec and Sarnak [2000], Bombieri and Hejhal [1995], Kaczorowski and Perelli [2000], Kaczorowski, Molteni and Perelli [Kaczorowski et al. 2007], Liu and Ye [2005], and Avdispahić and Smajlović [2010]. Moreover, we refer to Selberg [1992] and the surveys of Kaczorowski [2006], Kaczorowski and Perelli [1999], and Perelli [2005] for a thorough discussion on the Selberg class.

For the computations we have used the software packages PARI/GP [2016] and MATLAB®. These were made by truncating the Dirichlet series to the first 70 000 terms, which guarantees accuracy to eight decimal places for the values given above.

2. Radii of convexity of power series

Let $F(s)$ be a function satisfying (I) and (II). Then we can write $F(s)$ as an absolutely convergent Euler product $F(s) = \prod_p F_p(s)$ for $\sigma > 1$, where the local factor $F_p(s)$ is determined by $\log F_p(s) = \sum_{k=1}^{\infty} b_F(p^k) p^{-ks}$. Then, in most of the results on the value distribution of $F(s)$ for some fixed σ, a fundamental ingredient is the convexity of the curves $\log F_p(\sigma + it)$, $t \in \mathbb{R}$, at least for infinitely many primes p. In this section we collect and prove some results on this matter which will be needed later.

Let $\mathcal{A}$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} b(n) z^n$ which are regular on $D = \{|z| < 1\}$. Let $\mathcal{F}$ be any subclass of $\mathcal{A}$, then we write $r_c(\mathcal{F})$ for the largest r, with $0 < r \leq 1$, such that $f(\{|z| < r\})$ is convex.

Proposition 2.1 [Yamashita 1982, Theorem 2]. *Let $\mathcal{B} = \{f \in \mathcal{A} \mid |b(n)| \leq n,\ n \geq 2\}$. Then $r_c(\mathcal{B}) \geq R_1$, where R_1 is the smallest root in $(0, 1)$ of $2(1-X)^4 = 1 + 4X + X^2$. Let $K > 0$ and $\mathcal{G}(K) = \{f \in \mathcal{A} \mid |b(n)| \leq K,\ n \geq 2\}$. Then $r_c(\mathcal{G}(K)) \geq R_2(K)$, where $R_2(K)$ is the smallest root in $(0, 1)$ of $X^3 - 3X^2 + 4X = (1 - X)^3/K$.*

The proof of the above proposition is actually a simple consequence of the following result of Alexander and Remak (see Theorem 1 of [Goodman 1957]).

Theorem 2.2 (Alexander–Remak). *If $f(z) = z + \sum_{n=2}^{\infty} b(n) z^n \in \mathcal{A}$ and*

$$\sum_{n=2}^{\infty} n^2 |b(n)| \leq 1,$$

then $f(D)$ is convex.

Adapting Yamashita's proof [1982, §2] we obtain the following:

Proposition 2.3. *Let* $K > 0$ *and* $\mathcal{H}(K) = \{f \in \mathcal{A} \mid |b(n)| \le Kn^2,\ n \ge 2\}$. *Then* $r_c(\mathcal{H}(K)) \ge R_3(K)$, *where* $R_3(K)$ *is the smallest root in* $(0, 1)$ *of*

$$X^5 - 5X^4 + 11X^3 + X^2 + 16X = (1 - X)^5/K.$$

Remark 2.4. Note that $R_3(K)$ is a strictly decreasing function of K, with

$$\sup_{K>0} R_3(K) = \lim_{K\to 0^+} R_3(K) = 1 \quad \text{and} \quad \inf_{K>0} R_3(K) = \lim_{K\to+\infty} R_3(K) = 0.$$

Moreover, for any $K > 0$ we have $R_3(K) \le R_2(K)$.

Proof of Proposition 2.3. For $f(z) = z + \sum_{n=2}^{\infty} b(n)z^n \in \mathcal{H}(K)$ and any $r \le R_3 = R_3(K)$ we have

$$\sum_{n=2}^{\infty} n^2|b(n)|r^{n-1} \le K \sum_{n=2}^{\infty} n^4 R_3^{n-1} = K\,\frac{R_3^5 - 5R_3^4 + 11R_3^3 + R_3^2 + 16R_3}{(1 - R_3)^5} = 1,$$

where the last equality follows from the fact that R_3 is chosen as the smallest real root in $(0, 1)$ of $X^5 - 5X^4 + 11X^3 + X^2 + 16X = (1 - X)^5/K$. Therefore we can apply Theorem 2.2 to $h(z) = r^{-1} f(rz)$, which is thus convex on $|z| < 1$. Hence $f(\{|z| < r\})$ is convex for any $r \le R_3$ and thus $R_3 \le r_c(\mathcal{H}(K))$. □

From this we obtain an explicit version of Theorem 13 of [Jessen and Wintner 1935] and Lemma 2.5 of [Lee 2014].

Proposition 2.5. *Let* N *be a fixed positive integer*,

$$G_j(z) = \sum_{n=1}^{\infty} a_j(n)z^n, \quad j = 1, \dots, N,$$

and suppose there exist positive real numbers ρ_j *and* K_j *such that* $|a(n)| \le K_j \rho_j^{1-n}$ *for every* $n \ge 2$. *For any* $\boldsymbol{y} = (y_1, \dots, y_J) \in \mathbb{C}^N$, *define*

$$g(r, \theta, \boldsymbol{y}) = \sum_{j=1}^{N} \operatorname{Re}\bigl(G_j(re^{2\pi i\theta})\overline{y_j}\bigr),$$

where $0 < r < \min_j \rho_j$ *and* $\theta \in [0, 1]$. *If* $\sum_{j=1}^{N} \overline{y_j} a_j(1) \ne 0$, *then there exists a positive constant* C *such that for any* $\delta > 0$ *we have*

$$\left|\int_0^1 e^{ig(r,\theta,\boldsymbol{y})}\, d\theta\right| \le \frac{24}{\sqrt{C\delta r\|\boldsymbol{y}\|}} \tag{2-1}$$

for every $0 < r \le R_3\bigl(\frac{1}{\delta}\sqrt{\sum_j |K_j|^2}\bigr) \min_j \rho_j$ *and every* $\boldsymbol{y}$ *such that* $\bigl|\sum_{j=1}^{N} \overline{y_j} a_j(1)\bigr| \ge \delta\|\boldsymbol{y}\| > 0$.

Proof. The proof is a combination of Theorems 12 and 13 of [Jessen and Wintner 1935] and Lemma 2.5 of [Lee 2014], and we use the aforementioned results to obtain explicit constants. Consider the power series

$$f(z) = \sum_{n=1}^{\infty} \biggl(\sum_{j=1}^{N} \overline{y}_j a_j(n) \biggr) z^n \quad \text{and} \quad h(z) = \sum_{n=1}^{\infty} n^2 \biggl(\sum_{j=1}^{N} \overline{y}_j a_j(n) \biggr) z^n.$$

Since, by hypothesis and the Cauchy–Schwarz inequality, we have

$$\biggl| \sum_{j=1}^{N} \overline{y}_j a_j(n) \biggr| \le \frac{\|\boldsymbol{y}\| \sqrt{\sum_j |K_j|^2}}{(\min_j \rho_j)^{n-1}} \quad \forall n \ge 2, \tag{2-2}$$

$f(z)$ and $h(z)$ are both holomorphic for $|z| < \min_j \rho_j$ and, by definition, we have

$$g(r, \theta, \boldsymbol{y}) = \operatorname{Re} f(re^{2\pi i\theta}) \quad \text{and} \quad g''(r, \theta, \boldsymbol{y}) = \frac{\partial^2}{\partial \theta^2} g(r, \theta, \boldsymbol{y}) = -4\pi^2 \operatorname{Re} h(re^{2\pi i\theta}).$$

By Proposition 2.1 we have that $f(re^{2\pi i\theta})$ is a parametric representation of a convex curve if

$$r \le R_2 \biggl(\frac{\|\boldsymbol{y}\| \sqrt{\sum_j |K_j|^2}}{\bigl| \sum_{j=1}^{N} \overline{y}_j a_j(1) \bigr|} \biggr) \min_j \rho_j.$$

Indeed, substituting $w = z / \min_j \rho_j$, we have

$$\tilde{f}(w) = \frac{f(z/\min_j \rho_j)}{(\min_j \rho_j) \bigl(\sum_{j=1}^{N} \overline{y}_j a_j(1) \bigr)} = w + \sum_{n=2}^{\infty} (\min_j \rho_j)^{n-1} \biggl(\frac{\sum_{j=1}^{N} \overline{y}_j a_j(n)}{\sum_{j=1}^{J} \overline{y}_j a_j(1)} \biggr) w^n$$

and, by (2-2),

$$\tilde{f}(w) \in \mathcal{G} \biggl(\frac{\|\boldsymbol{y}\| \sqrt{\sum_j |K_j|^2}}{\bigl| \sum_{j=1}^{J} \overline{y}_j a_j(1) \bigr|} \biggr).$$

Analogously, by Proposition 2.3 we have that $h(re^{2\pi i\theta})$ is a parametric representation of a convex curve if

$$r \le R_3 \biggl(\frac{\|\boldsymbol{y}\| \sqrt{\sum_j |K_j|^2}}{\bigl| \sum_{j=1}^{N} \overline{y}_j a_j(1) \bigr|} \biggr) \min_j \rho_j. \tag{2-3}$$

Therefore, by Remark 2.4, both $f(re^{2\pi i\theta})$ and $h(re^{2\pi i\theta})$ are parametric representations of convex curves for any fixed r satisfying (2-3). This implies that both $g(r, \theta, \boldsymbol{y})$ and $g''(r, \theta, \boldsymbol{y})$ have exactly two zeros mod 1. By the mean value theorem, we have that also $g'(r, \theta, \boldsymbol{y})$ has exactly two zeros mod 1, which separate those of $g''(r, \theta, \boldsymbol{y})$. Note that the zeros of $g'(r, \theta, \boldsymbol{y})$ and $g''(r, \theta, \boldsymbol{y})$ depend continuously on r and $\boldsymbol{y}$ since $g'(r, \theta, \boldsymbol{y})$ and $g''(r, \theta, \boldsymbol{y})$ are continuous functions in each variable.

We now consider the midpoints of the four arcs mod 1 determined by the zeros of $g'(r,\theta,\boldsymbol{y})$ and $g''(r,\theta,\boldsymbol{y})$. These midpoints clearly depend continuously on r and $\boldsymbol{y}$, and divide [0, 1] into four arcs, namely I_1, I_2, I_3 and I_4, such that I_1 and I_3 each contain one zero of $g'(r,\theta,\boldsymbol{y})$, while I_2 and I_4 each contain one zero of $g''(r,\theta,\boldsymbol{y})$. By van der Corput's lemma for oscillatory integrals (see [Titchmarsh 1986, Lemmas 4.2 and 4.4]) we have

$$\left|\int_{I_2\cup I_4} e^{ig(r,\theta,y)}d\theta\right| \le \frac{8}{\min\limits_{I_2\cup I_4}|g'(r,\theta,\boldsymbol{y})|}$$

and

$$\left|\int_{I_1\cup I_3} e^{ig(r,\theta,y)}d\theta\right| \le \frac{16}{\sqrt{\min\limits_{I_1\cup I_3}|g''(r,\theta,\boldsymbol{y})|}}.$$

Writing

$$g(r,\theta,\boldsymbol{y}) = r\left|\sum_{j=1}^{N}\overline{y}_j a_j(1)\right|\cos(2\pi(\theta-\xi)) + r^2 O(\|\boldsymbol{y}\|)$$

for some ξ, we see that by continuity there exists a positive constant C such that

$$\frac{g'(r,\theta,\boldsymbol{y})}{r\left|\sum_{j=1}^{N}\overline{y}_j a_j(1)\right|} \ge C \text{ on } I_2 \text{ and } I_4, \quad \text{and} \quad \frac{g''(r,\theta,\boldsymbol{y})}{r\left|\sum_{j=1}^{N}\overline{y}_j a_j(1)\right|} \ge C \text{ on } I_1 \text{ and } I_3$$

for every r satisfying (2-3) and $\boldsymbol{y}\in\mathbb{C}^N$.

We fix $\delta > 0$, $\boldsymbol{y} \ne \boldsymbol{0}$ such that

$$\left|\sum_{j=1}^{J}\overline{y}_j a_j(1)\right| \ge \delta\|\boldsymbol{y}\|, \quad r \le R_3\left(\frac{1}{\delta}\sqrt{\sum_j |K_j|^2}\right)\min_j \rho_j,$$

and we obtain

$$\left|\int_0^1 e^{ig(r,\theta,y)}d\theta\right| \le \frac{8}{C\delta r\|\boldsymbol{y}\|} + \frac{16}{\sqrt{C\delta r\|\boldsymbol{y}\|}}.$$

Since $1/(C\delta r\|\boldsymbol{y}\|) \le 1/\sqrt{C\delta r\|\boldsymbol{y}\|}$ when $C\delta r\|\boldsymbol{y}\| \ge 1$,

$$\left|\int_0^1 e^{ig(r,\theta,y)}d\theta\right| \le \frac{24}{\sqrt{C\delta r\|\boldsymbol{y}\|}} \quad \text{for } \|\boldsymbol{y}\| \ge \frac{1}{C\delta r}.$$

On the other hand, we clearly have that $\left|\int_0^1 e^{ig(r,\theta,y)}d\theta\right| \le 1$, hence (2-1) holds whenever the RHS is ≥ 1. Therefore the result follows from the simple fact that the RHS of (2-1) is > 24 when $0 < \|\boldsymbol{y}\| < 1/(C\delta r)$. □

Theorem 2.6. *Let $F_1(s), \dots, F_N(s)$ be orthogonal functions satisfying* (I) *and* (II). *Then there exists a positive constant A and infinitely many primes p such that*

$$\left|\int_0^1 \exp\biggl(i\sum_{j=1}^N \mathrm{Re}\Bigl(\overline{y_j}\log F_{j,p}\Bigl(\sigma+i\frac{2\pi\theta}{\log p}\Bigr)\Bigr)\biggr)\,d\theta\right| \le \frac{A}{\sqrt{\|\boldsymbol{y}\|}}p^{\sigma/2} \tag{2-4}$$

for every $\sigma \ge 1$ and every $\boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{C}^N \setminus \{\boldsymbol{0}\}$.

Proof. We want to apply Proposition 2.5 to

$$G_j(z) = \sum_{n=1}^{\infty} \frac{b_{F_j}(p^n)}{\sqrt{m_{j,j}}} z^n, \quad j = 1, \dots, N,$$

where the $m_{j,j}$ are as in (1-3). By (II) there exist K_{F_j} and $\theta_j < \frac{1}{2}$ such that for every prime p and every $n \ge 2$ we have $|b_{F_j}(p^n)| \le K_{F_j} p^{n\theta_j} \le K_{F_j} p^{2(n-1)\theta_j}$, $j = 1, \dots, N$. Thus, for $j = 1, \dots, N$ and every prime p we may take $K_j = K_{F_j}/\sqrt{m_{j,j}}$ and $\rho_j = p^{-2\theta_j}$.

On the other hand, by orthogonality we have that for any $\boldsymbol{y} \ne \boldsymbol{0}$

$$\sum_{p \le x} \left|\frac{\overline{y_1} b_{F_1}(p)}{\sqrt{m_{1,1}}} + \dots + \frac{\overline{y_N} b_{F_N}(p)}{\sqrt{m_{N,N}}}\right|^2 \Big/ p \sim \|\boldsymbol{y}\|^2 \log\log x, \quad \text{as } x \to \infty.$$

In particular this implies that there are infinitely many primes p such that

$$\left|\frac{\overline{y_1} b_{F_1}(p)}{\sqrt{m_{1,1}}} + \dots + \frac{\overline{y_N} b_{F_N}(p)}{\sqrt{m_{N,N}}}\right| \ge \frac{\|\boldsymbol{y}\|}{4}$$

for every $\boldsymbol{y} \ne \boldsymbol{0}$. For each such prime p we take $r = p^{-\sigma}$ and $\delta = \frac{1}{4}$. Then Proposition 2.5 yields

$$\left|\int_0^1 \exp\biggl(i\sum_{j=1}^N \mathrm{Re}\Bigl(\frac{\overline{y_j}}{\sqrt{m_{j,j}}}\log F_{j,p}\Bigl(\sigma+i\frac{2\pi\theta}{\log p}\Bigr)\Bigr)\biggr)\,d\theta\right| \le \frac{48}{\sqrt{C\|\boldsymbol{y}\|}}p^{\sigma/2} \tag{2-5}$$

when

$$p^{-\sigma} \le R_3\left(4\sqrt{\sum_j \frac{|K_{F_j}|^2}{m_{j,j}}}\right)p^{-2\max_j \theta_j} \tag{2-6}$$

and $\boldsymbol{y} \ne \boldsymbol{0}$. Note that (2-6) holds for every $\sigma \ge 1$ if p is sufficiently large since $\max_j \theta_j < \frac{1}{2}$. Now, substituting

$$\boldsymbol{y}' = (y_1', \dots, y_N') = \left(\frac{y_1}{\sqrt{m_{1,1}}}, \dots, \frac{y_N}{\sqrt{m_{N,N}}}\right)$$

in (2-5) we obtain that there are infinitely many primes p such that

$$\left|\int_0^1 \exp\left(i\sum_{j=1}^{N}\operatorname{Re}\left(\overline{y_j'}\log F_{j,p}\left(\sigma+i\frac{2\pi\theta}{\log p}\right)\right)\right)d\theta\right| \leq \frac{48}{\sqrt{C}\sqrt[4]{m_{1,1}|y_1'|^2+\cdots+m_{N,N}|y_N'|^2}}p^{\sigma/2}$$

for every $\sigma \geq 1$ and every $\boldsymbol{y'} \in \mathbb{C}^N \setminus \{\boldsymbol{0}\}$. Since clearly there exists a positive constant D such that $\sqrt{m_{1,1}|y_1'|^2+\cdots+m_{N,N}|y_N'|^2} \geq D\|\boldsymbol{y'}\|$, the result follows immediately with $A = 48/\sqrt{DC}$. □

Remark 2.7. From the proof we have that (2-4) holds for $\sigma \geq 1$ because $\max_j \theta_j < \frac{1}{2}$ by (II). Therefore if we had that $\max_j \theta_j < \frac{\kappa}{2}$ for some $0 < \kappa < 1$, we would immediately have that (2-4) holds for every $\sigma \geq \kappa$.

3. On some distribution functions

This section is an adaptation of Chapter II of [Borchsenius and Jessen 1948]. We will also use Theorem 2.6 similarly to how Borchsenius and Jessen use Theorem 13 of [Jessen and Wintner 1935]. The particular distribution functions under investigation in this section may be found in [Lee 2014] and they will be used in Sections 4 and 5 for the proofs of Theorems 1.4 and 1.5. We refer to [Lee 2014] for a brief introduction to the theory developed by Jessen and Tornehave [1945] and Borchsenius and Jessen [1948] and how it may be applied to linear combinations of Euler products.

Given a function $F(s)$ satisfying (I) and (II), and a positive integer n we write

$$F_n(s) = \prod_{m=1}^{n} F_{p_m}(s) \quad \text{and} \quad F_n(\sigma,\boldsymbol{\theta}) = F_n(\sigma,\theta_1,\ldots,\theta_n) = \prod_{m=1}^{n} F_{p_m}\left(\sigma+i\frac{2\pi\theta_m}{\log p_m}\right),$$

where p_m is the m-th prime and $F_p(s)$ is determined by

$$\log F_p(s) = \sum_{k=1}^{\infty} b_F(p^k)p^{-ks}.$$

Remark 3.1. For any $n \geq 1$, $F_n(s)$ is well defined as a Dirichlet series (and Euler product) absolutely convergent for $\sigma > \theta = \theta_F$ by (II). Moreover, $F_n(s)$ and $\log F_n(s)$ converge uniformly for $\sigma \geq \sigma_0 > 1$ to $F(s)$ and $\log F(s)$, respectively.

Let $F_1(s), \ldots, F_N(s)$ be orthogonal functions satisfying (I) and (II). For $\boldsymbol{\theta} \in [0,1]^n$, we define

$$\boldsymbol{F}_n(\sigma,\boldsymbol{\theta}) = \big(F_{1,n}(\sigma,\boldsymbol{\theta}), \ldots, F_{N,n}(\sigma,\boldsymbol{\theta})\big)$$

and

$$\mathbf{log}\,\boldsymbol{F}_n(\sigma,\boldsymbol{\theta}) = \big(\log F_{1,n}(\sigma,\boldsymbol{\theta}),\ldots,\log F_{N,n}(\sigma,\boldsymbol{\theta})\big).$$

To these functions we attach some distribution functions, namely for any Borel set $\boldsymbol{E} \subseteq \mathbb{C}^N$, $j, l \in \{1,\ldots,N\}$, $j \neq l$ and $\sigma > 1$, we set

$$\boldsymbol{\lambda}_{\sigma,n;j}(\boldsymbol{E}) = \int_{\boldsymbol{W}_{\mathbf{log}\,\boldsymbol{F}_n}(\sigma,\boldsymbol{E})} \left|\frac{F'_{j,n}}{F_{j,n}}(\sigma,\boldsymbol{\theta})\right|^2 d\boldsymbol{\theta} \tag{3-1}$$

and

$$\boldsymbol{\lambda}_{\sigma,n;j,l;\tau}(\boldsymbol{E}) = \int_{\boldsymbol{W}_{\mathbf{log}\,\boldsymbol{F}_n}(\sigma,\boldsymbol{E})} \left|\frac{F'_{j,n}}{F_{j,n}}(\sigma,\boldsymbol{\theta}) + \tau\frac{F'_{l,n}}{F_{l,n}}(\sigma,\boldsymbol{\theta})\right|^2 d\boldsymbol{\theta}, \tag{3-2}$$

where $\boldsymbol{W}_{\mathbf{log}\,\boldsymbol{F}_n}(\sigma,\boldsymbol{E}) = \{\boldsymbol{\theta} \in [0,1)^n \mid \mathbf{log}\,\boldsymbol{F}_n(\sigma,\boldsymbol{\theta}) \in \boldsymbol{E}\}$, and $\tau = \pm 1, \pm i$.

A distribution function μ on $\mathbb{C}^n$ is *absolutely continuous* (with respect to the Lebesgue measure, meas) if for every Borel set $\boldsymbol{E} \subseteq \mathbb{C}^n$, $\mathrm{meas}(\boldsymbol{E}) = 0$ implies $\mu(\boldsymbol{E}) = 0$ (cf. [Bogachev 2007, Definition 3.2.1]). By the Radon–Nikodym theorem (see, e.g., Theorem 3.2.2 in [Bogachev 2007]) this holds if and only if there exists a Lebesgue integrable function $G_\mu : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ such that

$$\mu(\boldsymbol{E}) = \int_{\boldsymbol{E}} G_\mu(\boldsymbol{x})\,d\boldsymbol{x}$$

for any Borel set $\boldsymbol{E} \subseteq \mathbb{C}^n$; $G_\mu(\boldsymbol{x})$ is the *density* of μ.

As a sufficient condition for absolute continuity we recall here the following result (cf. [Borchsenius and Jessen 1948, §6; Bogachev 2007, §3.8]).

Lemma 3.2. *Let μ be a distribution function on $\mathbb{C}^n$ and let $\hat{\mu}$ be its Fourier transform. If $\int_{\mathbb{C}^n} \|\boldsymbol{y}\|^q |\hat{\mu}(\boldsymbol{y})|\,d\boldsymbol{y} < \infty$ for some integer $q \geq 0$, then μ is absolutely continuous with density $G_\mu(\boldsymbol{x}) \in \mathcal{C}^q(\mathbb{C}^n)$ determined by the Fourier inversion formula*

$$G_\mu(\boldsymbol{x}) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{C}^n} e^{-i\langle \boldsymbol{x},\boldsymbol{y}\rangle} \hat{\mu}(\boldsymbol{y})\,d\boldsymbol{y}.$$

We have the following result on the distribution functions defined above.

Theorem 3.3. *Let $F_1(s),\ldots,F_N(s)$ be orthogonal functions satisfying* (I) *and* (II). *Then there exists $n_0 \geq 1$ such that the distribution functions $\boldsymbol{\lambda}_{\sigma,n;j}$ and $\boldsymbol{\lambda}_{\sigma,n;j,l;\tau}$ are absolutely continuous with continuous densities $G_{\sigma,n;j}(\boldsymbol{x})$ and $G_{\sigma,n;j,l;\tau}(\boldsymbol{x})$ for every $n \geq n_0$, $\sigma \geq 1$, $j, l \in \{1,\ldots,N\}$, $j \neq l$ and $\tau = \pm 1, \pm i$. More generally for any $k \geq 0$ there exists $n_k \geq 1$ such that $G_{\sigma,n;j}(\boldsymbol{x}),\ G_{\sigma,n;j,l;\tau}(\boldsymbol{x}) \in \mathcal{C}^k(\mathbb{C}^N)$ for every $n \geq n_k$, $\sigma \geq 1$.*
Moreover, $\boldsymbol{\lambda}_{\sigma,n;j}$ and $\boldsymbol{\lambda}_{\sigma,n;j,l;\tau}$ converge weakly to some distribution functions $\boldsymbol{\lambda}_{\sigma;j}$ and $\boldsymbol{\lambda}_{\sigma;j,l;\tau}$ as $n \to \infty$, which are absolutely continuous with densities $G_{\sigma;j}(\boldsymbol{x}),\ G_{\sigma;j,l;\tau}(\boldsymbol{x}) \in \mathcal{C}^\infty(\mathbb{C}^N)$ for every $\sigma \geq 1$, $j, l \in \{1,\ldots,N\}$, $j \neq l$ and $\tau = \pm 1, \pm i$. The functions $G_{\sigma,n;j}(\boldsymbol{x})$ and $G_{\sigma,n;j,l;\tau}(\boldsymbol{x})$ and their partial derivatives

converge uniformly for $\boldsymbol{x} \in \mathbb{C}^n$ and $1 \le \sigma \le M$ to $G_{\sigma;j}(\boldsymbol{x})$ and $G_{\sigma;j,l;\tau}(\boldsymbol{x})$ and their partial derivatives as $n \to \infty$ for every $M > 1$.

Proof. The proof is an adaptation of Theorem 5 of Borchsenius and Jessen [1948] (see also [Lee 2014, pp. 1827–1830]). We prove it just for $\boldsymbol{\lambda}_{\sigma,n;j}$ since the proof for the other distributions is completely similar.

We compute the Fourier transform of the functions $\boldsymbol{\lambda}_{\sigma,n;j}$ and get

$$\widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y}) = \int_{[0,1]^n} \exp\biggl(i\sum_{h=1}^{N} \operatorname{Re}(\log F_{h,n}(\sigma,\boldsymbol{\theta})\overline{y_h})\biggr)\biggl|\frac{F'_{j,n}}{F_{j,n}}(\sigma,\boldsymbol{\theta})\biggr|^2 d\boldsymbol{\theta}, \tag{3-3}$$

for any $\boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{C}^N$. By Lemma 3.2, to prove the first part it is sufficient to show that for every $k \ge 0$ there exists n_k such that, for any $M > 1$, $\|\boldsymbol{y}\|^k \widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y})$ is Lebesgue integrable for every $n \ge n_k$ and $1 \le \sigma \le M$. We recall that by (II) there exist K_{F_j} and $\theta_{F_j} < \frac{1}{2}$ such that

$$|b_{F_j}(p^n)| \le K_{F_j} p^{n\theta_{F_j}}$$

for every prime p and $k \ge 1$, $j = 1, \dots, N$. Then we have

$$\begin{aligned} |\widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y})| \le \sup_{\sigma>1}\biggl|\frac{F'_{j,n}}{F_{j,n}}(\sigma,\boldsymbol{\theta})\biggr|^2 &\le \sum_{m=1}^{n} \log^2 p_m \sum_{k=1}^{\infty} \frac{|b_{F_j}(p_m^k)|^2}{p_m^{2k\sigma}} \\ &\le K_{F_j}^2 \sum_p \frac{\log^2 p}{p^{2(\sigma-\theta_{F_j})}} < \infty \end{aligned} \tag{3-4}$$

for every $n \ge 1$ and $\sigma \ge 1$. Hence it is sufficient to show that there exist constants $C_k > 0$ and $n_k \ge 1$ such that for any $M > 1$ we have

$$|\widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y})| \le C_k \|\boldsymbol{y}\|^{-\frac{5}{2}-k} \quad \text{as } \|\boldsymbol{y}\| \to \infty$$

for every $n \ge n_k$ and $1 \le \sigma \le M$. To prove this, note that we can write (cf. [Borchsenius and Jessen 1948, (47); Lee 2014, (3.24)])

$$\begin{aligned} \widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y}) = \sum_{m=1}^{n} K_{2,j}(p_m,\boldsymbol{y}) \prod_{\substack{\ell=1\\ \ell\ne m}}^{n} K_{0,j}(p_\ell,\boldsymbol{y}) \\ + \sum_{\substack{m,k=1\\ m\ne k}}^{n} K_{1,j}(p_m,\boldsymbol{y})\overline{K_{1,j}(p_k,-\boldsymbol{y})} \prod_{\substack{\ell=1\\ \ell\ne m,k}}^{n} K_{0,j}(p_\ell,\boldsymbol{y}), \end{aligned} \tag{3-5}$$

where, for any prime p and $j \in \{1, \dots, N\}$, we take

$$K_{0,j}(p, \boldsymbol{y}) = \int_0^1 \exp\biggl(i \sum_{h=1}^{N} \mathrm{Re}\biggl(\log F_{h,p}\biggl(\sigma + i\frac{2\pi\theta}{\log p}\biggr)\overline{y_h}\biggr)\biggr)\, d\theta,$$

$$\begin{aligned} &K_{1,j}(p, \boldsymbol{y}) \\ &\quad = \int_0^1 \exp\biggl(i \sum_{h=1}^{N} \mathrm{Re}\biggl(\log F_{h,p}\biggl(\sigma + i\frac{2\pi\theta}{\log p}\biggr)\overline{y_h}\biggr)\biggr) \frac{F'_{j,p}}{F_{j,p}}\biggl(\sigma + i\frac{2\pi\theta}{\log p}\biggr)\, d\theta, \end{aligned} \tag{3-6}$$

$$\begin{aligned} &K_{2,j}(p, \boldsymbol{y}) \\ &\quad = \int_0^1 \exp\biggl(i \sum_{h=1}^{N} \mathrm{Re}\biggl(\log F_{h,p}\biggl(\sigma + i\frac{2\pi\theta}{\log p}\biggr)\overline{y_h}\biggr)\biggr) \biggl|\frac{F'_{j,p}}{F_{j,p}}\biggl(\sigma + i\frac{2\pi\theta}{\log p}\biggr)\biggr|^2 d\theta. \end{aligned}$$

Hence, we just need to estimate the functions defined in (3-6).

For all primes p and $j \in \{1, \dots, N\}$ we clearly have

$$|K_{0,j}(p, \boldsymbol{y})| \le 1. \tag{3-7}$$

On the other hand, by the hypotheses on $F_1(s), \dots, F_N(s)$ we can apply Theorem 2.6 and obtain a positive constant A and infinitely many primes p such that

$$|K_{0,j}(p, \boldsymbol{y})| \le \frac{A}{\sqrt{\|\boldsymbol{y}\|}} p^{\sigma/2} \tag{3-8}$$

for every $\sigma \ge 1$, $\boldsymbol{y} \ne \boldsymbol{0}$ and $j \in \{1, \dots, N\}$. Thus, putting together (3-7) and (3-8) we obtain that for any fixed integer $q \ge 1$ there exists m_q such that

$$\prod_{\substack{\ell=1 \\ \ell \ne m,k}}^{n} |K_{0,j}(p_\ell, \boldsymbol{y})| \le \biggl[\frac{A}{\sqrt{\|\boldsymbol{y}\|}} p_{m_q}^{\sigma/2}\biggr]^q \tag{3-9}$$

for every $m, k \le n$, $n \ge m_q$, $\sigma \ge 1$, $\boldsymbol{y} \ne \boldsymbol{0}$ and $j \in \{1, \dots, N\}$. Since we shall need it later, we also note that from the fact that $|e^{it} - 1 - it| \le t^2/2$ and by (II), for every prime p we get (cf. [Borchsenius and Jessen 1948, (50); Lee 2014, p. 1830])

$$|K_{0,j}(p, \boldsymbol{y}) - 1| \le \frac{\|\boldsymbol{y}\|^2}{2} \biggl(\sum_{h=1}^{N} K_{F_j}^2\biggr) \frac{1}{p^{2(\sigma - \max_h \theta_{F_h})}}. \tag{3-10}$$

For $K_{1,j}(p, \boldsymbol{y})$, using the fact that $|e^{it} - 1| \le |t|$ and (II), we obtain for any $\sigma \ge 1$ and any prime p (cf. [Borchsenius and Jessen 1948, (52); Lee 2014, (3.27)])

$$|K_{1,j}(p, \boldsymbol{y})| \le \|\boldsymbol{y}\| K_{F_j} \sqrt{\sum_{h=1}^{N} K_{F_h}^2}\, \frac{\log p}{p^{2(\sigma - \max_h \theta_{F_h})}}. \tag{3-11}$$

Finally, for any prime p, $\sigma \geq 1$ and $j \in \{1, \dots, N\}$, we simply have (cf. [Borchsenius and Jessen 1948, (53); Lee 2014, (3.26)])

$$|K_{2,j}(p, \boldsymbol{y})| \leq \int_0^1 \left| \frac{F'_{j,p}}{F_{j,p}}\left(\sigma + i\frac{2\pi\theta}{\log p}\right)\right|^2 d\theta \overset{\text{(II)}}{\leq} K_{F_j}^2 \frac{\log^2 p}{p^{2(\sigma-\theta_{F_j})}}. \tag{3-12}$$

Putting (3-7), (3-9), (3-11) and (3-12) into (3-5), for any fixed $M > 1$, $j \in \{1, \dots, N\}$ and $q \geq 0$, we get

$$\begin{aligned} |\widehat{\lambda_{\sigma,n;j}}(\boldsymbol{y})| \leq K_{F_j}^2 A^q \|\boldsymbol{y}\|^{-q/2} p_{m_q}^{q\sigma/2} \sum_{m=1}^{n} \frac{\log^2 p_m}{p_m^{2(\sigma-\theta_{F_j})}} \\ + K_{F_j}^2 \left(\sum_{h=1}^{N} K_{F_h}^2\right) A^q \|\boldsymbol{y}\|^{2-q/2} p_{m_q}^{q\sigma/2} \left(\sum_{m=1}^{n} \frac{\log p_m}{p_m^{2(\sigma-\max_h \theta_{F_h})}}\right)^2 \end{aligned}$$

for any $n \geq m_q$, $\sigma \geq 1$ and $\boldsymbol{y} \neq \boldsymbol{0}$. Choosing $q = 9 + 2k$, $n_k = m_{9+2k}$ and

$$\begin{aligned} C_k = \left(\sum_{h=1}^{N} K_{F_h}^2\right) A^{9+2k} p_{n_k}^{(9+2k)M/2} \left(1 + \left(\sum_{h=1}^{N} K_{F_h}^2\right)^2 \sum_p \frac{\log p}{p^{2(\sigma-\max_h \theta_{F_h})}}\right) \\ \times \sum_p \frac{\log p}{p^{2(\sigma-\max_h \theta_{F_h})}} \end{aligned}$$

we have

$$|\widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y})| \leq C_k \|\boldsymbol{y}\|^{-\frac{5}{2}-k} \quad \text{when } \|\boldsymbol{y}\| \geq 1, \tag{3-13}$$

for every $n \geq n_k = m_{9+2k}$, $1 \leq \sigma \leq M$ and $j \in \{1, \dots, N\}$. Therefore, by Lemma 3.2, since n_k doesn't depend on M and since M is arbitrary, it follows that $\boldsymbol{\lambda}_{\sigma,n;j}$, $j = 1, \dots, N$, are absolutely continuous with continuous density for every $n \geq n_0$ and every $\sigma \geq 1$, while $G_{\sigma,n;j}(\boldsymbol{x}) \in \mathcal{C}^k(\mathbb{C}^N)$ for every $j \in \{1, \dots, N\}$, $n \geq n_k$ and $\sigma \geq 1$.

On the other hand, by (3-4), (3-5), (3-7), (3-10), (3-11), and (3-12), we have (cf. [Borchsenius and Jessen 1948, (60); Lee 2014, p. 1830])

$$|\widehat{\boldsymbol{\lambda}_{\sigma,n+1;j}}(\boldsymbol{y}) - \widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y})| \ll \|\boldsymbol{y}\|^2 \frac{\log p_{n+1}}{p_{n+1}^{2(\sigma-\max_h \theta_{F_h})}}$$

for every $n \geq 1$, $\sigma \geq 1$ and $j \in \{1, \dots, N\}$. By the triangle inequality we thus get

$$\begin{aligned} |\widehat{\boldsymbol{\lambda}_{\sigma,n+k;j}}(\boldsymbol{y}) - \widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y})| &\ll \|\boldsymbol{y}\|^2 \sum_{m=n+1}^{n+k} \frac{\log p_m}{p_m^{2(\sigma-\max_h \theta_{F_h})}} \\ &\leq \|\boldsymbol{y}\|^2 \sum_{m=n+1}^{\infty} \frac{\log p_m}{p_m^{2(\sigma-\max_h \theta_{F_h})}} \end{aligned} \tag{3-14}$$

for every $n,\ k \geq 1$ and $\sigma \geq 1$. Hence, by Cauchy's criterion, there exist the limit functions

$$\widehat{\boldsymbol{\lambda}_{\sigma;j}}(\boldsymbol{y}) = \lim_{n\to\infty} \widehat{\boldsymbol{\lambda}_{\sigma,n;j}}(\boldsymbol{y}), \quad j = 1, \dots, N,$$

and by (3-14) it is clear that the convergence is uniform in $\|\boldsymbol{y}\| \leq a$, for every $a > 0$. Therefore, by Lévy's convergence theorem (see, e.g., Theorem 8.8.1 in [Bogachev 2007]), we have that $\widehat{\boldsymbol{\lambda}_{\sigma;j}}(\boldsymbol{y})$ is the Fourier transform of some distribution function $\boldsymbol{\lambda}_{\sigma;j}$ and $\boldsymbol{\lambda}_{\sigma,n;j} \to \boldsymbol{\lambda}_{\sigma;j}$ weakly as $n \to \infty$, for $j = 1, \dots, N$. Moreover by (3-13) we have that we may apply the dominated convergence theorem and thus $\boldsymbol{\lambda}_{\sigma;j}$ are absolutely continuous for every $\sigma \geq 1$ and $j \in \{1, \dots, N\}$, with density $G_{\sigma;j}(\boldsymbol{x}) \in \mathcal{C}^\infty(\mathbb{C})$ (for the arbitrariness of M and k). Moreover, since $G_{\sigma,n;j}(\boldsymbol{x})$ and $G_{\sigma;j}(\boldsymbol{x})$ are determined by the inverse Fourier transform (see Lemma 3.2), the dominated convergence theorem yields that $G_{\sigma,n;j}(\boldsymbol{x})$ and their partial derivatives converge uniformly for $\boldsymbol{x} \in \mathbb{C}^n$ and $1 \leq \sigma \leq M$ toward $G_{\sigma;j}(\boldsymbol{x})$ and their partial derivatives for every $j \in \{1, \dots, N\}$. □

Theorem 3.4. *For any $\alpha > 0$ and $q \geq 0$ the densities $G_{\sigma;j}(\boldsymbol{x})$ and $G_{\sigma,n;j}(\boldsymbol{x})$, $n \geq n_q$, together with their partial derivatives of order $\leq q$, have a majorant of the form $K_q e^{-\alpha\|\boldsymbol{x}\|^2}$ for every $\sigma \geq 1$, $j, l \in \{1, \dots, N\}$, $j \neq l$ and $\tau = \pm 1, \pm i$.*

Proof. This is a straightforward adaptation of Theorems 6 and 9 of [Borchsenius and Jessen 1948]. □

Theorem 3.5. *The distribution functions $\boldsymbol{\lambda}_{\sigma;j}$, $\boldsymbol{\lambda}_{\sigma;j,l;\tau}$, $\boldsymbol{\lambda}_{\sigma,n;j}$ and $\boldsymbol{\lambda}_{\sigma,n;j,l;\tau}$, for $n \geq n_0$, depend continuously on σ, and their densities $G_{\sigma;j}(\boldsymbol{x})$, $G_{\sigma;j,l;\tau}(\boldsymbol{x})$, $G_{\sigma,n;j}(\boldsymbol{x})$ and $G_{\sigma,n;j,l;\tau}(\boldsymbol{x})$, together with their partial derivatives of order $\leq q$ if $n \geq n_q$, are continuous in σ for every $\sigma \geq 1$, $j, l \in \{1, \dots, N\}$, $j \neq l$ and $\tau = \pm 1, \pm i$.*

Proof. As in Theorem 9 of [Borchsenius and Jessen 1948] the result follows from (3-13), (3-14) and the Fourier inversion formula. □

Remark 3.6. As for Remark 2.7, note that Theorems 3.3, 3.4 and 3.5 hold for $\sigma > 1$ because $\max_j \theta_{F_j} < \frac{1}{2}$ by (II). Therefore if we had that $\max_j \theta_{F_j} < \kappa/2$ for some $0 < \kappa < 1$ we would immediately have that (2-4) holds for every $\sigma > \kappa$.

4. Zeros of sums of two Euler products

Let $F_1(s)$ and $F_2(s)$ be functions satisfying (I) and (II), and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. We then set

$$L(s) = c_1 F_1(s) + c_2 F_2(s).$$

To study the distribution of the zeros of $L(s)$ for $\sigma > 1$, we note that, since $F_1(s)F_2(s) \neq 0$ for $\sigma > 1$,

$$L(s) = 0 \quad \Leftrightarrow \quad \log\left(\frac{F_1(s)}{F_2(s)}\right) = \log\left(-\frac{c_2}{c_1}\right).$$

This idea was used by Gonek [1981], and later by Bombieri and Mueller [2008] and Bombieri and Ghosh [2011]. Moreover, if $F_1(s)$ and $F_2(s)$ are orthogonal, then it is easy to show that $\frac{F_1}{F_2}(s)$ satisfies (I), (II) and, if we write $\frac{F_1}{F_2}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$,

$$\sum_{p \le x} \frac{|a(p)|^2}{p} = (\kappa + o(1)) \log\log x, \quad x \to \infty, \tag{4-1}$$

for some constant $\kappa > 0$. Therefore Theorem 1.4 follows immediately from the following more general result on the value distribution of the logarithm of an Euler product.

Theorem 4.1. *Let $F(s)$ be a function satisfying* (I), (II) *and* (4-1), *and $c \in \mathbb{C}$. Then the Dirichlet series* $\log F(s) - c$ *has no isolated vertical lines containing zeros in the half-plane $\sigma > 1$.*

Proof. The first part of the proof is similar to Borchsenius and Jessen's application [1948, Theorems 11 and 13] of their Theorems 5–9 to the Riemann zeta function.

For every $n \ge 1$ consider the Dirichlet series $\log F_n(s)$, which are absolutely convergent for $\sigma > \theta_F$ by Remark 3.1. Let $\nu_{\sigma,n}$ be, for every $\sigma > \theta_F$, the *asymptotic distribution function* of $\log F_n(s)$ with respect to $|(F_n'/F_n)(s)|^2$, defined for any Borel set $E \subseteq \mathbb{C}$ by (cf. [Borchsenius and Jessen 1948, §7])

$$\nu_{\sigma,n}(E) = \lim_{T_2 - T_1 \to \infty} \frac{1}{T_2 - T_1} \int_{V_{\log F_n}(\sigma, T_1, T_2, E)} \left| \frac{F_n'}{F_n}(s) \right|^2 dt,$$

where $V_{\log F_n}(\sigma, T_1, T_2, E) = \{t \in (T_1, T_2) \mid \log F_n(\sigma + it) \in E\}$. For $\sigma \ge 1$, we compute its Fourier transform and, by the Kronecker–Weyl theorem (see, e.g., [Karatsuba and Voronin 1992, §A.8]) we get (cf. [Borchsenius and Jessen 1948, p. 160] or [Lee 2014, p. 1819])

$$\widehat{\nu_{\sigma,n}}(y) = \int_{[0,1]^n} \exp\bigl(i \operatorname{Re}(\log F_n(\sigma, \boldsymbol{\theta}) \bar{y})\bigr) \left| \frac{F_n'}{F_n}(\sigma, \boldsymbol{\theta}) \right|^2 d\boldsymbol{\theta} \overset{(3\text{-}3)}{=} \widehat{\boldsymbol{\lambda}_{\sigma,n;1}}(y),$$

with $N = 1$. For simplicity we write $\lambda_{\sigma,n} = \boldsymbol{\lambda}_{\sigma,n;1}$. By the uniqueness of the Fourier transform (see, e.g., Proposition 3.8.6 in [Bogachev 2007]) we have that $\nu_{\sigma,n} = \lambda_{\sigma,n}$ as distribution functions for every $\sigma \ge 1$ and $n \ge 1$.

By Theorem 3.3 we know that $\nu_{\sigma,n} = \lambda_{\sigma,n}$ is absolutely continuous for $n \ge n_0$ with density $G_{\sigma,n}(x)$ which is a continuous function of both σ and x (see Theorem 3.5). Hence for any $n \ge n_0$, $x \in \mathbb{C}$ and $\sigma > \theta_F$ we have that the *Jensen function* $\varphi_{\log F_n - x}(\sigma)$ (see, e.g., Theorem 5 of [Jessen and Tornehave 1945]) is twice differentiable with continuous second derivative (cf. [Borchsenius and Jessen 1948, §9])

$$\varphi''_{\log F_n - x}(\sigma) = 2\pi G_{\sigma,n}(x). \tag{4-2}$$

Note that in order to apply Theorems 3.3 and 3.5 we have implicitly made use of the orthogonality hypothesis.

On the other hand, for any $1 < \sigma_1 < \sigma_2$, by the uniform convergence of $\log F_n(s)$ of Remark 3.1 and by Theorem 6 of [Jessen and Tornehave 1945], we have that

$$\varphi_{\log F_n - x}(\sigma) \to \varphi_{\log F - x}(\sigma) \quad \text{as } n \to \infty \tag{4-3}$$

uniformly for $\sigma_1 \le \sigma \le \sigma_2$. Moreover, by Theorem 3.3, $G_{\sigma,n}(x)$ converges uniformly for $\sigma_1 \le \sigma \le \sigma_2$ toward $G_\sigma(x)$, which is continuous in both σ and x. Then, by (4-2), (4-3), the convexity of $\varphi_{\log F_n - x}$ and Theorem 7.17 in [Rudin 1976] we obtain that for any $x \in \mathbb{C}$ the Jensen function $\varphi_{\log F-x}(\sigma)$ is twice differentiable with continuous second derivative

$$\varphi''_{\log F-x}(\sigma) = 2\pi G_\sigma(x).$$

We fix an arbitrary $c \in \mathbb{C}$ and we note the following: Suppose that $\varphi''_{\log F-c}(\sigma_0) > 0$ for some $\sigma_0 > 1$. Then, by continuity, there exists $\varepsilon_0 > 0$ such that $\varphi''_{\log F-c}(\sigma) > 0$ for every $\sigma \in (\sigma_0 - \varepsilon_0, \sigma_0 + \varepsilon_0)$. Then, for any $0 < \varepsilon < \varepsilon_0$, by Theorem 31 of [Jessen and Tornehave 1945] and the mean value theorem, we have

$$\begin{aligned}\lim_{T_2 - T_1 \to \infty} & \frac{N_{\log F-c}(\sigma_0 - \varepsilon, \sigma_0 + \varepsilon, T_1, T_2)}{T_2 - T_1} \\ &= \frac{1}{2\pi}\big(\varphi'_{\log F-c}(\sigma_0 + \varepsilon) - \varphi'_{\log F-c}(\sigma_0 - \varepsilon)\big) = \frac{\varepsilon}{2\pi}\varphi''_{\log F-c}(\sigma_\varepsilon) > 0,\end{aligned}$$

for some $\sigma_\varepsilon \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$, i.e., there are infinitely many zeros with real part $\sigma \in (\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$. This means, by letting $\varepsilon \to 0^+$, that σ_0 is the limit point of the real parts of some zeros of $\log F(s) - c$ (or σ_0 is itself a zero).

Now, suppose there exists $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > 1$ such that $\log F(\rho_0) - c = 0$. If we suppose that $\varphi''_{\log F-c}(\beta_0) > 0$, then $\sigma = \beta_0$ cannot be an isolated vertical line containing zeros since β_0 is the limit point of the real parts of some zeros. Suppose otherwise that $\varphi''_{\log F-c}(\tilde{\sigma}) = 0$, and for any $\delta > 0$ consider the intervals $I_\delta^+ = (\tilde{\sigma}, \tilde{\sigma} + \delta)$ and $I_\delta^- = (\tilde{\sigma} - \delta, \tilde{\sigma})$. Note that in general, if $\varphi''_{\log F-c}(\sigma) = 0$ for every $\sigma \in (\sigma_1, \sigma_2)$, for some $1 < \sigma_1 < \sigma_2$, then Theorem 31 of [Jessen and Tornehave 1945] and the mean value theorem imply that $\log F(s) - c$ has no zeros for $\sigma_1 < \sigma < \sigma_2$. Therefore, in at least one of I_δ^+ or I_δ^- there are infinitely many σ such that $\varphi''_{\log F-c}(\sigma) > 0$, for any $\delta > 0$, by almost periodicity. Hence, letting $\delta \to 0$, we see that there exists a sequence $\{\sigma_\delta\}_\delta$ such that $\varphi''_{\log F-c}(\sigma_\delta) > 0$ and $\sigma_\delta \to \beta_0$. Since every σ_δ is the limit point of the real parts of some zeros, we conclude that also β_0 is the limit point of the real parts of some zeros. □

5. *c*-values of sums of at least three Euler products

We first state the following simple result which is a generalization of Lemma 2.4 of [Lee 2014].

Lemma 5.1. *Let $F(s)$ be a function satisfying* (I), (II) *and* (III), $\sigma_0 > \frac{1}{2}$ *and k be a fixed positive integer. Then there exists a positive constant $A_k(\sigma_0)$ such that*

$$\int_{[0,1]^n} |F_n(\sigma,\boldsymbol{\theta})|^{2k}\, d\boldsymbol{\theta} \le A_k(\sigma_0) \quad \text{and} \quad \int_{[0,1]^n} |F_n'(\sigma,\boldsymbol{\theta})|^{2k}\, d\boldsymbol{\theta} \le A_k(\sigma_0)$$

for every $n \ge 1$ and $\sigma \ge \sigma_0$.

Proof. As in Lemma 2.4 of [Lee 2014] the proof follows from a bound of

$$\mathcal{J}_k(z_1,\dots,z_n,w_1,\dots,w_n) = \int_{[0,1]^n} \prod_{j=1}^{k} F_n(\sigma+z_j,\boldsymbol{\theta})\overline{F_n(\sigma+\overline{w_j},\boldsymbol{\theta})}\, d\boldsymbol{\theta}$$

and Cauchy's integral formula on polydiscs. This bound may be obtained with the same computations as in Lemma 2.5 of [Lee 2014] by replacing the Ramanujan bound $|a(n)| \le 1$ with the weaker Ramanujan conjecture $|a(n)| \ll_\varepsilon n^\varepsilon$, where we take $0 < \varepsilon < (2\sigma_0 - 1)/4$. □

Proof of Theorem 1.5. To handle this case we follow an idea of Lee [2014, §3.2] and we use the distribution functions studied in Section 3, similarly to what we have done in the previous section for $N = 2$. We give only a sketch of the proof.

For every $n \ge 1$ we write

$$L_n(s) = \sum_{j=1}^{N} c_j F_{j,n}(s),$$
$$L_n(\sigma,\boldsymbol{\theta}) = L_n(\sigma,\theta_1,\dots,\theta_n) = \sum_{j=1}^{N} c_j F_{j,n}(s,\theta_1,\dots,\theta_n).$$

Let $\nu_{\sigma,n}$ be the asymptotic distribution function of $L_n(s)$ with respect to $|L_n'(s)|^2$ defined for any Borel set $E \subseteq \mathbb{C}$ by (cf. [Borchsenius and Jessen 1948, §7])

$$\nu_{\sigma,n}(E) = \lim_{T_2-T_1\to\infty} \frac{1}{T_2-T_1} \int_{V_{L_n}(\sigma,T_1,T_2,E)} |L_n'(s)|^2\, dt,$$

where $V_{L_n}(\sigma,T_1,T_2,E) = \{t \in (T_1,T_2) \mid L_n(\sigma+it) \in E\}$. As in Theorem 4.1, by the Kronecker–Weyl theorem and the uniqueness of the Fourier transform, we have that $\nu_{\sigma,n} = \lambda_{\sigma,n}$, for any $n \ge 1$ and $\sigma \ge 1$, where $\lambda_{\sigma,n}$ is the distribution function of $L_n(s,\boldsymbol{\theta})$ with respect to $|L_n'(s,\boldsymbol{\theta})|^2$, defined for every Borel set $E \subseteq \mathbb{C}$ by

$$\lambda_{\sigma,n}(E) = \int_{W_{L_n}(\sigma,E)} |L_n'(\sigma,\boldsymbol{\theta})|^2\, d\boldsymbol{\theta},$$

with $W_{L_n}(\sigma,E) = \{\boldsymbol{\theta} = (\theta_1,\dots,\theta_n) \in [0,1)^n \mid L_n(\sigma,\boldsymbol{\theta}) \in E\}$. We want to show that there exists $\tilde{n} \ge 1$ such that $\lambda_{\sigma,n}$, and hence $\nu_{\sigma,n}$, is absolutely continuous with continuous density, which we call $H_{\sigma,n}(x)$, for every $n \ge \tilde{n}$ and $\sigma \ge 1$.

As in [Lee 2014, pp. 1830–1831], we compute the Fourier transform of $\lambda_{\sigma,n}$ and, for $\sigma \geq 1$ and $n \geq n_0$, we get

$$\widehat{\lambda_{\sigma,n}}(y) = \sum_{j,l=1}^{N} \overline{c_j} c_l (2\pi)^N \int_{\mathbb{R}_+^N} \int_{\mathbb{R}^N} \exp\biggl(i \sum_{h=1}^{N} |c_h \bar{y}| r_h \sin(2\pi(\theta_h - \alpha_h)) - 2\pi i \theta_j + 2\pi i \theta_l \biggr) \times r_j r_l G_{\sigma,n;j,l}(\boldsymbol{r}) \frac{dr_1}{r_1} \cdots \frac{dr_N}{r_N} \, d\theta_1 \cdots d\theta_N,$$

where $\boldsymbol{r} = (\log r_1 + 2\pi i \theta_1, \ldots, \log r_N + 2\pi i \theta_N)$, α_h is determined by the argument of $c_h \bar{y}$, for $h = 1, \ldots, N$, and

$$G_{\sigma,n;j,l}(\boldsymbol{x}) = \begin{cases} G_{\sigma,n;j}(\boldsymbol{x}), & j = l, \\ \sum_{\tau=\pm 1, \pm i} \bar{\tau} G_{\sigma,n;j,l;\tau}(\boldsymbol{x}), & j \neq l \end{cases}$$

is defined from the densities of the distribution functions $\boldsymbol{\lambda}_{\sigma,n;j}$ and $\boldsymbol{\lambda}_{\sigma,n;j,l;\tau}$ of Section 3.

For any $h \in \{1, \ldots, N\}$ and any $\varepsilon > 0$ let

$$A_{h,\varepsilon} = \bigl\{ \theta \in \mathbb{R} \mid |\theta - \alpha_h - m\pi| < \varepsilon \text{ for some } m \in \mathbb{Z} \bigr\}.$$

Then we note that integrating by parts with respect to r_h, $h = 1, \ldots, N$, and using the majorant $K_N \exp\bigl(-\bigl[\sum_{h=1}^{N} \log^2 r_h + \theta_h^2\bigr]\bigr)$ of Theorem 3.4 for the partial derivatives up to order N of the density $G_{\sigma,n;j,l}(\boldsymbol{r})$, for $n \geq n_N$ and $\sigma \geq 1$, we obtain (cf. [Lee 2014, p. 1832])

$$\begin{aligned} &\int_{\mathbb{R}\setminus A_{1,\varepsilon}} \cdots \int_{\mathbb{R}\setminus A_{N,\varepsilon}} \int_{\mathbb{R}_+^N} \exp\biggl(i \operatorname{Re}\biggl(\sum_{h=1}^{N} r_h c_h \bar{y} e^{2\pi i \theta_h} \biggr) - 2\pi i \theta_j + 2\pi i \theta_l \biggr) \\ &\qquad\qquad \times r_j r_l G_{\sigma,n;j,l}(\boldsymbol{r}) \frac{dr_1}{r_1} \cdots \frac{dr_N}{r_N} \, d\theta_1 \cdots d\theta_N \\ &\quad \ll \prod_{h=1}^{N} \int_{\mathbb{R}\setminus A_{h,\varepsilon}} \frac{1}{|c_h \bar{y}| \sin(2\pi(\theta_h - \alpha_h))} e^{-\theta_h^2} d\theta_h \\ &\quad \ll \frac{1}{(\varepsilon |y|)^N} \end{aligned} \tag{5-1}$$

for every $n \geq n_N$, $\sigma \geq 1$ and $y \neq 0$. Analogously, integrating by parts with respect to θ_h, $h = 1, \ldots, N$, using van der Corput's lemma for oscillatory integrals (see, e.g., Lemma 4.2 in [Titchmarsh 1986]) on each interval $[\alpha_h + m_h/2 - \varepsilon, \alpha_h + m_h/2 + \varepsilon]$ with $\varepsilon < \frac{1}{2}$, and the majorant $K_N \exp\bigl(-\bigl[\sum_{h=1}^{N} \log^2 r_h + \theta_h^2\bigr]\bigr)$ of Theorem 3.4 for the partial derivatives up to order N of the density $G_{\sigma,n;j,l}(\boldsymbol{r})$, $n \geq n_N$ and $\sigma \geq 1$, we obtain (cf. [Lee 2014, p. 1832])

$$\begin{aligned}\int_{\mathbb{R}_+^N}\int_{A_{1,\varepsilon}}\cdots\int_{A_{N,\varepsilon}} \exp\biggl(i\,\mathrm{Re}\biggl(\sum_{h=1}^{N} r_h c_h \bar{y} e^{2\pi i\theta_h}\biggr) - 2\pi i\theta_j + 2\pi i\theta_l\biggr)&\\ \times r_j r_l G_{\sigma,n;\,j,l}(\boldsymbol{r})\frac{dr_1}{r_1}\cdots\frac{dr_N}{r_N}\,d\theta_1\cdots d\theta_N&\\ \ll \prod_{h=1}^{N}\int_{\mathbb{R}_+}\frac{1}{|c_h\bar{y}|}e^{-\log^2 r_h}dr_h&\\ \ll \frac{1}{|y|^N},&\end{aligned} \tag{5-2}$$

for every $n \ge n_N$, $\sigma \ge 1$, $|y| \ge \max_h 1/|c_h|$ and $\varepsilon > 0$ sufficiently small. Note that to apply Theorem 3.4 we have implicitly made use of the orthogonality hypothesis. Fixing $\varepsilon > 0$ sufficiently small so that (5-2) holds and putting together (5-1) and (5-2), we obtain

$$|\widehat{\nu_{\sigma,n}}(y)| = |\widehat{\lambda_{\sigma,n}}(y)| \ll |y|^{-N} \ll |y|^{-3} \tag{5-3}$$

since $N \ge 3$, for every $n \ge n_N$, $\sigma \ge 1$ and $|y| \ge \max(1, \max_h |c_h|^{-1})$. By Lemma 3.2 we have thus proved that $\nu_{\sigma,n}$ is absolutely continuous for every $n \ge \tilde{n} = n_N$ and $\sigma \ge 1$. Moreover, since $\nu_{\sigma,n}$ depends continuously on σ (cf. [Borchsenius and Jessen 1948, §7]), we have that $\widehat{\nu_{\sigma,n}}$ is continuous in σ. Therefore (5-3) and the Fourier inversion formula imply that $H_{\sigma,n}(x)$ is continuous in both σ and x. Note that all implied constants in (5-3) are independent of n.

Now we prove that the absolutely continuous distribution functions $\lambda_{\sigma,n}$ converge weakly as $n \to \infty$ toward the absolutely continuous distribution function λ_σ with density $H_\sigma(x)$ which is continuous in both σ and x. Moreover, we want to show that, for any $1 < \sigma_1 < \sigma_2$, $H_{\sigma,n}(x)$ converges uniformly for $\sigma_1 \le \sigma \le \sigma_2$ toward $H_\sigma(x)$ as $n \to \infty$.

For this, note that

$$\begin{aligned} L_{n+1}(\sigma,\boldsymbol{\theta},\theta_{n+1}) &= \sum_{j=1}^{N} c_j F_{j,n}(\sigma,\boldsymbol{\theta}) F_{j,p_{n+1}}\biggl(\sigma + i\frac{2\pi\theta_{n+1}}{\log p_{n+1}}\biggr)\\ &\overset{\text{(III)}}{=} \sum_{j=1}^{N} c_j F_{j,n}(\sigma,\boldsymbol{\theta})\biggl(1 + \frac{a_{F_j}(p_{n+1})}{p_{n+1}^{\sigma}}e^{2\pi i\theta_{n+1}} + O_\varepsilon\biggl(\frac{1}{p_{n+1}^{2(\sigma-\varepsilon)}}\biggr)\biggr)\\ &= L_n(\sigma,\boldsymbol{\theta}) + \frac{e^{2\pi i\theta_{n+1}}}{p_{n+1}^{\sigma}}\sum_{j=1}^{N} c_j a_{F_j}(p_{n+1}) F_{j,n}(\sigma,\boldsymbol{\theta}) + O_\varepsilon\biggl(\frac{\sum_j |F_{j,n}|}{p_{n+1}^{2(\sigma-\varepsilon)}}\biggr) \end{aligned} \tag{5-4}$$

for every $\sigma \geq 1$ and $0 < \varepsilon < \frac{1}{2}$. Similarly

$$\begin{aligned} L'_{n+1}&(\sigma, \boldsymbol{\theta}, \theta_{n+1}) \\ &= L'_n(\sigma, \boldsymbol{\theta}) + \frac{e^{2\pi i \theta_{n+1}}}{p_{n+1}^{\sigma}} \sum_{j=1}^{N} c_j a_{F_j}(p_{n+1})\big[F'_{j,n}(\sigma, \boldsymbol{\theta}) - \log p_{n+1} F_{j,n}(\sigma, \boldsymbol{\theta})\big] \\ &\qquad + O_{\varepsilon}\bigg(\frac{\log p_{n+1} \sum_j |F_{j,n}| + |F'_{j,n}|}{p_{n+1}^{2(\sigma-\varepsilon)}}\bigg) \end{aligned}$$

for every $\sigma \geq 1$ and $0 < \varepsilon < \frac{1}{2}$. Hence we have (cf. [Lee 2014, (3.20)])

$$\begin{aligned} \widehat{\lambda_{\sigma,n+1}}&(y) - \widehat{\lambda_{\sigma,n}}(y) \\ &= \int_{[0,1]^{n+1}} \big[e^{i \operatorname{Re}(L_{n+1}(\sigma,\boldsymbol{\theta},\theta_{n+1})\bar{y})} - e^{i \operatorname{Re}(L_n(\sigma,\boldsymbol{\theta})\bar{y})}\big] |L'_n(\sigma, \boldsymbol{\theta})|^2 \, d\boldsymbol{\theta} \, d\theta_{n+1} \\ &\quad + \frac{2}{p_{n+1}^{\sigma}} \int_{[0,1]^{n+1}} e^{i \operatorname{Re}(L_{n+1}(\sigma,\boldsymbol{\theta},\theta_{n+1})\bar{y})} \operatorname{Re}\bigg(\overline{L'_n(\sigma, \boldsymbol{\theta})} e^{2\pi i \theta_{n+1}} \\ &\qquad \times \sum_{j=1}^{N} c_j a_{F_j}(p_{n+1})\big(F'_{j,n}(\sigma, \boldsymbol{\theta}) - \log p_{n+1} F_{j,n}(\sigma, \boldsymbol{\theta})\big)\bigg) \\ &\qquad \times d\boldsymbol{\theta} \, d\theta_{n+1} \\ &\quad + O_{\varepsilon}\bigg(\frac{\log p_{n+1}}{p_{n+1}^{2(\sigma-\varepsilon)}} \int_{[0,1]^{n+1}} \Big(1 + \sum_j |F'_{j,n}|\Big)\Big(\sum_j |F_{j,n}| + |F'_{j,n}|\Big) \, d\boldsymbol{\theta} \, d\theta_{n+1}\bigg) \\ &\quad + O_{\varepsilon}\bigg(\frac{\log^2 p_{n+1}}{p_{n+1}^{4(\sigma-\varepsilon)}} \int_{[0,1]^{n+1}} \Big(\sum_j |F_{j,n}| + |F'_{j,n}|\Big)^2 \, d\boldsymbol{\theta} \, d\theta_{n+1}\bigg). \end{aligned} \tag{5-5}$$

for every $\sigma \geq 1$ and $0 < \varepsilon < \frac{1}{2}$.

For the first term, using again $|e^{it} - 1 - it| \leq t^2/2$, we obtain (cf. [Lee 2014, (3.22)])

$$\bigg|\int_0^1 \big[e^{i \operatorname{Re}(L_{n+1}(\sigma,\boldsymbol{\theta},\theta_{n+1})\bar{y})} - e^{i \operatorname{Re}(L_n(\sigma,\boldsymbol{\theta})\bar{y})}\big] \, d\theta_{n+1}\bigg| \ll_{\varepsilon,a} \frac{\sum_j |F_{j,n}| + |F_{j,n}|^2}{p_{n+1}^{2(\sigma-\varepsilon)}}$$

for $|y| \leq a$, $a > 0$, $\sigma \geq 1$ and $0 < \varepsilon < \frac{1}{2}$. For the second term we get directly from (5-4) and $|e^{it} - 1| \leq |t|$ that

$$\bigg|\int_0^1 e^{i \operatorname{Re}(L_{n+1}(\sigma,\boldsymbol{\theta},\theta_{n+1})\bar{y})} e^{\pm 2\pi i \theta_{n+1}} \, d\theta_{n+1}\bigg| \ll_{\varepsilon,a} \frac{\sum_j |F_{j,n}|}{p_{n+1}^{(\sigma-\varepsilon)}}$$

for $|y| \leq a$, $a > 0$, $\sigma \geq 1$ and $0 < \varepsilon < \frac{1}{2}$. We fix $0 < \varepsilon < \frac{1}{2}$, then putting these together, by triangle inequality and Lemma 5.1 with $\sigma_0 = 1$, we get (cf. [Lee 2014, p. 1826])

$$|\widehat{\lambda_{\sigma,n+1}}(y) - \widehat{\lambda_{\sigma,n}}(y)| \ll_{a,\varepsilon} \frac{\log p_{n+1}}{p_{n+1}^{2(\sigma-\varepsilon)}}$$

uniformly for $|y| \le a$, $a > 0$, and for every $\sigma \ge 1$. It follows that for any $k > 0$

$$|\widehat{\lambda_{\sigma,n+k}}(y) - \widehat{\lambda_{\sigma,n}}(y)| \ll_{a,\varepsilon} \sum_{m=n+1}^{n+k} \frac{\log p_m}{p_m^{2(\sigma-\varepsilon)}} \le \sum_{m=n+1}^{\infty} \frac{\log p_m}{p_m^{2(\sigma-\max_h \theta_{F_h})}} \tag{5-6}$$

for every $n,\ k \ge 1$ and $\sigma \ge 1$, uniformly for $|y| \le a,\ a > 0$. Hence, by Cauchy's criterion, there exists the limit function

$$\widehat{\lambda_\sigma}(y) = \lim_{n\to\infty} \widehat{\lambda_{\sigma,n}}(y)$$

and by (3-14) the convergence is uniform in $|y| \le a$ for every $a > 0$. Therefore, by Lévy's convergence theorem, we have that $\widehat{\lambda_\sigma}(y)$ is the Fourier transform of some distribution function λ_σ, and $\lambda_{\sigma,n} \to \lambda_\sigma$ weakly as $n \to \infty$. Moreover, since the constants in (5-3) are independent of n, we may apply the dominated convergence theorem and thus λ_σ is absolutely continuous for every $\sigma \ge 1$, with continuous (both in σ and x) density $H_\sigma(x)$. Furthermore, since $H_{\sigma,n}(x)$ and $H_\sigma(x)$ are determined by the Fourier inversion formula (see Lemma 3.2), the uniform convergence of $\widehat{\lambda_{\sigma,n}}(y) \to \widehat{\lambda_\sigma}(y)$ and (5-3) imply that $H_{\sigma,n}(x)$ converges, uniformly with respect to both $1 \le \sigma \le M$, $M > 1$, and $x \in \mathbb{C}$, toward $H_\sigma(x)$.

Now, similarly to Theorem 4.1, for $n \ge \tilde{n}$ and $c \in \mathbb{C}$ we have that the Jensen function $\varphi_{L_n-c}(\sigma)$ is twice differentiable with continuous second derivative (cf. [Borchsenius and Jessen 1948, §9])

$$\varphi''_{L_n-c}(\sigma) = 2\pi H_{\sigma,n}(c). \tag{5-7}$$

On the other hand, for any $1 < \sigma_1 < \sigma_2$, by the uniform convergence of $F_{j,n}(s)$, $j = 1, \ldots, N$, of Remark 3.1 and by Theorem 6 of [Jessen and Tornehave 1945], we have that

$$\varphi_{L_n-c}(\sigma) \to \varphi_{L-c}(\sigma) \quad \text{as } n \to \infty \tag{5-8}$$

uniformly for $\sigma_1 \le \sigma \le \sigma_2$. By (5-7), (5-8), the convexity of $\varphi_{L_n-c}(\sigma)$ and Theorem 7.17 in [Rudin 1976] we obtain that the Jensen function $\varphi_L(\sigma)$ is twice differentiable with continuous second derivative

$$\varphi''_{L-c}(\sigma) = 2\pi H_\sigma(c).$$

At this point, the same final argument of Theorem 4.1 yields the result. □

6. Dirichlet series with vertical strips without zeros

In this section we collect the proofs of Theorems 1.1, 1.2 and 1.8.

Proof of Theorem 1.1. Since $L_{\boldsymbol{x}}(s)$ is not identically zero, then $\sigma^*(L_{\boldsymbol{x}}) < +\infty$ and hence we fix

$$\sigma_2 > \sigma_1 > \max\Bigl(\sigma^*(L_{\boldsymbol{x}}), \max_{1\le j\le N} \sigma^*(F_j)\Bigr).$$

Then, by definition of $\sigma^*(L_{\boldsymbol{x}})$ and Theorem 8 of [Jessen and Tornehave 1945], there exists $\varepsilon > 0$ such that $|L_{\boldsymbol{x}}(s)| > \varepsilon$ for $\sigma_1 \le \sigma \le \sigma_2$. Moreover, there exists $M > 0$ such that $|F_j(s)| \le M$ for $\sigma_1 \le \sigma \le \sigma_2$. On the other hand, if we consider the hyperplanes $H(\sigma) = \{\boldsymbol{z} \in \mathbb{C}^N \mid L_{\boldsymbol{z}}(\sigma) = 0\}$ we have

$$\lim_{\sigma\to+\infty} \operatorname{dist}(\boldsymbol{x}, H(\sigma)) = \lim_{\sigma\to+\infty} \frac{|L_{\boldsymbol{x}}(\sigma)|}{\sqrt{\sum_j |F_j(\sigma)|^2}} = 0.$$

Therefore there exists $\beta > \sigma_2$ such that $\operatorname{dist}(\boldsymbol{x}, H(\beta)) < \varepsilon/(4\sqrt{N}M)$. Then for any $\boldsymbol{0} \ne \boldsymbol{c} \in B_{\varepsilon/(2\sqrt{N}M)}(\boldsymbol{x}) \cap H(\beta)$ we have $L_{\boldsymbol{c}}(\beta) = 0$ and, by the triangle and Cauchy–Schwartz inequalities,

$$|L_{\boldsymbol{c}}(s)| \ge |L_{\boldsymbol{x}}(s)| - |L_{\boldsymbol{c}-\boldsymbol{x}}(s)| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

for $1 \le \sigma^*(L_{\boldsymbol{x}}) < \sigma_1 \le \sigma \le \sigma_2 < \beta \le \sigma^*(L_{\boldsymbol{c}})$. This concludes the proof since $B_{\varepsilon/(2\sqrt{N}M)}(\boldsymbol{x}) \cap H(\beta)$ clearly contains infinitely many projectively inequivalent vectors $\boldsymbol{c}$.

Proof of Theorem 1.2. We write $N = k + 1 \ge 2$. If $N = 2$ then the result follows from Theorem 1.1; so we suppose that $N \ge 3$.

Note that $\boldsymbol{x} \in \mathbb{C}^N$ is such that $L_{\boldsymbol{x}}(\sigma) = 0$ for some $\sigma > 1$ if and only if $\boldsymbol{x} = (x_1, \dots, x_N)$ belongs to the hyperplane

$$F_1(\sigma)x_1 + \cdots + F_N(\sigma)x_N = 0. \tag{6-1}$$

If $\sigma > \max_{1\le j\le N} \sigma^*(F_j) = \tilde{\sigma}_0$, then the space of solutions of (6-1) has dimension $N - 1 \ge 2$ and is generated by

$$v_j^{(1)}(\sigma) = \Bigl(-\frac{1}{F_1(\sigma)}, 0, \dots, \frac{1}{F_j(\sigma)}, \dots, 0\Bigr), \quad j = 2, \dots, N.$$

Moreover we define inductively for $h = 2, \dots, N-1$ the vectors

$$v_j^{(h)}(\sigma_1, \dots, \sigma_h) = v_j^{(h-1)}(\sigma_1, \dots, \sigma_{h-1}) - \frac{L_{v_j^{(h-1)}(\sigma_1,\dots,\sigma_{h-1})}(\sigma_h)}{L_{v_h^{(h-1)}(\sigma_1,\dots,\sigma_{h-1})}(\sigma_h)} v_h^{(h-1)}(\sigma_1, \dots, \sigma_{h-1}),$$

$j = h+1, \dots, N$. Note that these are well defined linear combinations of $v_j^{(1)}(\sigma_1)$, $j = 2, \dots, N$, hence solutions of (6-1), if $\sigma_1 > \tilde{\sigma}_0$ and $\sigma_h > \sigma^*(L_{v_h^{(h-1)}(\sigma_1,\dots,\sigma_{h-1})})$,

$h = 2, \dots, N-1$. Actually, by definition it is clear that, under these conditions, $v_j^{(h)}(\sigma_1, \dots, \sigma_h)$ is a solution of

$$F_1(\sigma_1)x_1 + \dots + F_N(\sigma_1)x_N = 0,$$
$$\vdots$$
$$F_1(\sigma_h)x_1 + \dots + F_N(\sigma_h)x_N = 0.$$

Moreover, for any $1 \le m \le N-1$ we consider the vector

$$v_m(\sigma_1, \dots, \sigma_{m-1}, \infty, \dots, \infty) = \lim_{\sigma_m \to \infty} \cdots \lim_{\sigma_{N-1} \to \infty} v_N^{(N-1)}(\sigma_1, \dots, \sigma_{N-1}) \quad (6\text{-}2)$$

and for simplicity we write $v_N(\sigma_1, \dots, \sigma_{N-1}) = v_N^{(N-1)}(\sigma_1, \dots, \sigma_{N-1})$. Note that there exists a finite set of explicit conditions on $\sigma_1, \dots, \sigma_{N-1}$ for which these limits exist, i.e., there exist $\tilde{\sigma}_j$, $j = 1, \dots, N-1$, which depend only on the Dirichlet series $F_1, \dots, F_N$, such that $v_m(\sigma_1, \dots, \sigma_{m-1}, \infty, \dots, \infty)$ exists for every $1 \le m \le N-1$ if $\sigma_l > \tilde{\sigma}_l$ for every $l = 1, \dots, N-1$. These conditions actually correspond to the fact that the vector $v_m(\sigma_1, \dots, \sigma_{m-1}, \infty, \dots, \infty)$ is a generator of the one-dimensional vector space (by (1-2), reordering the functions if needed) defined by the system

$$F_1(\sigma_1)x_1 + \dots + F_N(\sigma_1)x_N = 0,$$
$$\vdots$$
$$F_1(\sigma_{m-1})x_1 + \dots + F_N(\sigma_{m-1})x_N = 0,$$
$$a_1(1)x_1 + \dots + a_N(1)x_N = 0,$$
$$\vdots$$
$$a_1(N-m)x_1 + \dots + a_N(N-m)x_N = 0.$$

Hence, in particular, this implies that the definition of $v_m(\sigma_1, \dots, \sigma_{m-1}, \infty, \dots, \infty)$ is independent from the order of the limits and that $L_{v_m(\sigma_1, \dots, \sigma_{m-1}, \infty, \dots, \infty)}(\sigma_l) = 0$, $l = 1, \dots, m-1$.

We work by induction on $h \in [1, N-2]$. For $h = 1$ we fix

$$\sigma_{1,2} > \sigma_{1,1} > \max\bigl(\sigma^*(L_{v_1(\infty, \dots, \infty)}), \tilde{\sigma}_0\bigr),$$

and take

$$\varepsilon_1 = \min_{\sigma_{1,1} \le \sigma \le \sigma_{1,2}, t \in \mathbb{R}} |L_{v_1(\infty, \dots, \infty)}(\sigma + it)| > 0$$

and

$$M_1 = \max_{1 \le j \le N} \max_{\sigma_{1,1} \le \sigma \le \sigma_{1,2}, t \in \mathbb{R}} |F_j(\sigma + it)| < \infty.$$

Note that $M_1 > 0$ by the choice of $\sigma_{1,1}$ and $\sigma_{1,2}$. By (6-2), we can choose $\beta_1 > \sigma_{1,2}$ such that

$$\left\| v_1(\infty, \dots, \infty) - v_2(\beta_1, \infty, \dots, \infty) \right\| < \frac{\varepsilon_1}{2\sqrt{N} M_1}.$$

Then, since $v_2(\beta_1, \infty, \dots, \infty)$ is a solution of (6-1) with $\sigma = \beta_1$, we have that $L_{v_2(\beta_1,\infty,\dots,\infty)}(\beta_1) = 0$. Moreover for $\sigma_{1,1} \le \sigma \le \sigma_{1,2}$ we have, by the triangle and Cauchy–Schwartz inequalities,

$$\begin{aligned} |L_{v_2(\beta_1,\infty,\dots,\infty)}(s)| &\ge |L_{v_1(\infty,\dots,\infty)}(s)| - |L_{v_1(\infty,\dots,\infty)-v_2(\beta_1,\infty,\dots,\infty)}(s)| \\ &\ge \varepsilon_1 - \frac{\varepsilon_1}{2} = \frac{\varepsilon_1}{2} = \delta_1 > 0. \end{aligned}$$

By induction we suppose that for any fixed $1 < h \le N-2$ there exist

$$\sigma_{1,1} < \sigma_{1,2} < \beta_1 < \cdots < \sigma_{h,1} < \sigma_{h,2} < \beta_h$$

and $\delta_h > 0$ such that

$$\min_{1 \le l \le h} \min_{\sigma_{l,1} < \sigma < \sigma_{l,2}, t \in \mathbb{R}} |L_{v_{h+1}(\beta_1,\dots,\beta_h,\infty,\dots,\infty)}(\sigma + it)| > \delta_h.$$

These hypotheses mean that the Dirichlet series $L_{v_{h+1}(\beta_1,\dots,\beta_h,\infty,\dots,\infty)}(s)$, which vanishes for $s = \beta_1, \dots, \beta_h$, has at least h distinct vertical strips without zeros in the region $1 < \sigma < \sigma^*(L_{v_{h+1}(\beta_1,\dots,\beta_h,\infty,\dots,\infty)})$.

For the inductive step $h \mapsto h+1$, we take

$$\sigma_{h+1,2} > \sigma_{h+1,1} > \max\left(\sigma^*(L_{v_{h+1}(\beta_1,\dots,\beta_h,\infty,\dots,\infty)}), \max_{h+1 \le j \le N} \sigma^*(L_{v_j^{(h)}(\beta_1,\dots,\beta_h)}), \tilde{\sigma}_h\right),$$

$$\varepsilon_{h+1} = \min\left(\delta_h, \min_{\sigma_{h+1,1} \le \sigma \le \sigma_{h+1,2}, t \in \mathbb{R}} |L_{v_{h+1}(\beta_1,\dots,\beta_h,\infty,\dots,\infty)}(\sigma + it)|\right) > 0$$

and

$$M_{h+1} = \max_{1 \le j \le N} \max_{\sigma_{1,1} \le \sigma \le \sigma_{h+1,2}, t \in \mathbb{R}} |F_j(\sigma + it)| < \infty.$$

Note that since $\sigma_{h+1,1} > \sigma_{1,2}$ we have $M_{h+1} > 0$. Then we choose $\beta_{h+1} > \sigma_{h+1,2}$ such that

$$\begin{aligned} &\left\| v_{h+1}(\beta_1, \dots, \beta_h, \infty, \dots, \infty) - v_{h+2}(\beta_1, \dots, \beta_h, \beta_{h+1}, \infty, \dots, \infty) \right\| \\ &\qquad < \frac{\varepsilon_{h+1}}{2\sqrt{N} M_{h+1}}, \end{aligned}$$

which exists by definition. Moreover, by the triangle and Cauchy–Schwartz inequalities, we have that

$$
\begin{aligned}
&\left|L_{v_{h+2}(\beta_1,\dots,\beta_{h+1},\infty,\dots,\infty)}(s)\right| \\
&\quad\geq \left|L_{v_{h+1}(\beta_1,\dots,\beta_h,\infty,\dots,\infty)}(s)\right| - \left|L_{v_{h+2}(\beta_1,\dots,\beta_{h+1},\infty,\dots,\infty)-v_{h+2}(\beta_1,\dots,\beta_{h+1},\infty,\dots,\infty)}(s)\right| \\
&\quad\geq \delta_h - \frac{\varepsilon_{h+1}}{2} \geq \frac{\varepsilon_{h+1}}{2} = \delta_{h+1}
\end{aligned}
$$

for any $\sigma_{l,1} \leq \sigma \leq \sigma_{l,2},\ l = 1, \dots, h+1$.

When $h+1 = N-2+1 = N-1$ we have just one vector

$$\boldsymbol{c} = v_N(\beta_1, \dots, \beta_{N-1}) \in \mathbb{C}^N \setminus \{\boldsymbol{0}\}$$

and the corresponding Dirichlet series $L_{\boldsymbol{c}}(s)$ has, as noted above, at least $N-1$ distinct vertical strips without zeros in the region $1 < \sigma < \sigma^*(L_{\boldsymbol{c}})$.

Proof of Theorem 1.8. For any $j = 1, \dots, N$, let α_j be a square root of ω_j. Without loss of generality we may suppose that $h = 1$ and $k = 2$. Note that, since $|\omega_j| = 1$ and $\omega_1 \neq \omega_2$ then $\alpha_1 \neq \pm\alpha_2$ and we may suppose $\alpha_1 \notin \mathbb{R}$. It follows that the system of equations

$$
\begin{aligned}
\operatorname{Re}(\alpha_1)x_1 + \cdots + \operatorname{Re}(\alpha_N)x_N = 0,\\
\operatorname{Im}(\alpha_1)x_1 + \cdots + \operatorname{Im}(\alpha_N)x_N = 0
\end{aligned}
\tag{6-3}
$$

defines a real vector space V_∞ of dimension $N-2 \geq 1$ which may be written as

$$V_\infty = \left\{ \left(\sum_{j=3}^{\infty} \left(\frac{\operatorname{Im}(\alpha_2)\operatorname{Im}(\alpha_1\overline{\alpha_j})}{\operatorname{Im}(\alpha_1)\operatorname{Im}(\alpha_1\overline{\alpha_2})} - \frac{\operatorname{Im}(\alpha_j)}{\operatorname{Im}(\alpha_1)} \right) t_j, -\sum_{j=3}^{\infty} \frac{\operatorname{Im}(\alpha_1\overline{\alpha_j})}{\operatorname{Im}(\alpha_1\overline{\alpha_2})} t_j, t_3, \dots, t_N \right) \;\middle|\; t_3, \dots, t_N \in \mathbb{R} \right\}.$$

Let $v_\infty \in V_\infty$ be the vector corresponding to a fixed choice $(\tau_1, \dots, \tau_N) \in \mathbb{R}^{N-2} \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{c}_0 = (\overline{\alpha_1} v_{\infty,1}, \dots, \overline{\alpha_N} v_{\infty,N})$. We take $\sigma_2 > \sigma_1 > \max(\sigma^*(L_{\boldsymbol{c}_0}))$, then, by Theorem 8 of [Jessen and Tornehave 1945], there exists $\varepsilon > 0$ such that $|L_{\boldsymbol{c}_0}(s)| > \varepsilon$ for $\sigma_1 \leq \sigma \leq \sigma_2$. Moreover, there exists $M > 0$ such that $|F_j(s)| \leq M$ for $\sigma_1 \leq \sigma \leq \sigma_2$. On the other hand, for any fixed $\sigma > \sigma_2$, the system of equations

$$
\begin{aligned}
\operatorname{Re}(\alpha_1 F_1(\sigma))x_1 + \cdots + \operatorname{Re}(\alpha_N F_N(\sigma))x_N = 0,\\
\operatorname{Im}(\alpha_1 F_1(\sigma))x_1 + \cdots + \operatorname{Im}(\alpha_N F_N(\sigma))x_N = 0
\end{aligned}
\tag{6-4}
$$

defines a real vector space V_σ of dimension at least $N-2$. However, since $F_j(\sigma) \to a_j(1) = 1$ as $\sigma \to \infty$, $j = 1, 2$, there exists $\sigma_0 > \sigma_2$ such that V_σ has dimension

$N-2$ for every $\sigma > \sigma_0$ and

$$V_\sigma = \left\{ \left(\sum_{j=3}^{\infty} \left(\frac{\operatorname{Im}(\alpha_2 F_2(\sigma)) \operatorname{Im}(\alpha_1 \overline{\alpha_j} F_1(\sigma) \overline{F_j(\sigma)})}{\operatorname{Im}(\alpha_1 F_1(\sigma)) \operatorname{Im}(\alpha_1 \overline{\alpha_2} F_1(\sigma) \overline{F_2(\sigma)})} - \frac{\operatorname{Im}(\alpha_j) F_j(\sigma)}{\operatorname{Im}(\alpha_1) F_1(\sigma)} \right) t_j, \right.\right.$$
$$\left.\left. - \sum_{j=3}^{\infty} \frac{\operatorname{Im}(\alpha_1 \overline{\alpha_j} F_1(\sigma) \overline{F_j(\sigma)})}{\operatorname{Im}(\alpha_1 \overline{\alpha_2} F_1(\sigma) \overline{F_2(\sigma)})} t_j, t_3, \dots, t_N \right) \,\middle|\, t_3, \dots, t_N \in \mathbb{R} \right\}.$$

Let $v_\sigma \in V_\sigma$ be the vector corresponding to $(\tau_1, \dots, \tau_N)$, then $\|v_\infty - v_\sigma\| \to 0$ as $\sigma \to \infty$. Therefore there exists $\beta > \sigma_0$ such that, taking $\boldsymbol{c} = (\overline{\alpha_1} v_{\beta,1}, \dots, \overline{\alpha_N} v_{\beta,N})$, we have $\|\boldsymbol{c}_0 - \boldsymbol{c}\| < \varepsilon/(2\sqrt{N}M)$. Then by (6-4) we have that $L_{\boldsymbol{c}}(\beta) = 0$ and, by the triangle and Cauchy–Schwartz inequalities, that

$$|L_{\boldsymbol{c}}(s)| \geq |L_{\boldsymbol{c}_0}(s)| - |L_{\boldsymbol{c}-\boldsymbol{c}_0}(s)| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

for $1 \leq \sigma^*(L_{\boldsymbol{c}_0}) < \sigma_1 \leq \sigma \leq \sigma_2 < \sigma_0 < \beta \leq \sigma^*(L_{\boldsymbol{c}})$. Moreover

$$\begin{aligned} \Phi(s) &= \sum_{j=1}^{N} \overline{\alpha_j} v_{\beta,j} \Phi_j(s) = \sum_{j=1}^{N} \overline{\alpha_j} v_{\beta,j} \omega_j \overline{\Phi_j(1-\bar{s})} \\ &= \sum_{j=1}^{N} \alpha_j v_{\beta,j} \overline{\Phi_j(1-\bar{s})} = \overline{\Phi(1-\bar{s})}. \end{aligned}$$

Acknowledgments

This paper is part of my Ph.D. thesis at the Department of Mathematics of the University of Genova. I express my sincere gratitude to my supervisor Professor Alberto Perelli for his support and advice. I thank Professor Giuseppe Molteni for carefully reading the manuscript and suggesting several improvements. I also wish to thank Professor Enrico Bombieri for an enlightening discussion on this topic and for helpful comments concerning the paper. Finally, I would like to thank the referees for pointing out some inaccuracies and for the many suggestions which greatly improved the exposition.

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Communicated by Peter Sarnak
Received 2016-11-21 Revised 2017-07-18 Accepted 2017-08-14

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

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nonprofit scientific publishing

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Algebra & Number Theory

Volume 11 No. 9 2017

1937-0652(2017)11:9;1-C