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Local positivity of linear series on surfaces

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We study asymptotic invariants of linear series on surfaces with the help of Newton–Okounkov polygons. Our primary aim is to understand local positivity of line bundles in terms of convex geometry. We work out characterizations of ample and nef line bundles in terms of their Newton–Okounkov bodies, treating the infinitesimal case as well. One of the main results is a description of moving Seshadri constants via infinitesimal Newton–Okounkov polygons. As an illustration of our ideas we reprove results of Ein–Lazarsfeld on Seshadri constants on surfaces.

Introduction

The main purpose of this paper is to understand local positivity of line bundles on surfaces, by making use of the theory of Newton–Okounkov bodies. More precisely, we find ampleness and nefness criteria in terms of convex geometry, and relate the information obtained this way to Seshadri-type invariants.

For the past thirty-odd years there has been an increasing interest in describing positivity of line bundles around single points of varieties. Starting with the work of Demailly on Fujita's conjecture, where he introduced Seshadri constants, the topic developed quickly due to the work of Demailly [1992], Ein and Lazarsfeld [Ein et al. 1995; Ein et al. 2009], Nakamaye [2003; 2005], and others. In spite of all the effort, our understanding is still very limited, simple questions that remain unanswered to this day abound.

We aim at translating existing invariants of local positivity to the language of Newton–Okounkov bodies, thus enriching them with extra structure. Our model is the relationship between Newton–Okounkov bodies and the volume of a divisor: we intend to replace numbers functioning as invariants by collections of convex bodies. We would like to emphasize the special nature of linear series on surfaces, most of the time we will employ elementary surface-specific tools, Zariski decomposition will play a crucial role for instance.

Originating in the work of Khovanskii from the late seventies, and Okounkov's construction [1996] in representation theory, Newton–Okounkov bodies are a not-quite-straightforward generalization of Newton polytopes to the setting of arbitrary projective varieties. For an n-dimensional projective variety X, a full flag of subvarieties $Y_\bullet$, and a big divisor D on X, the Newton–Okounkov body $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$ is a convex set, encoding the set of all normalized valuation vectors coming from sections of multiples of D, where the rank n valuation of the function field of X is determined by $Y_\bullet$.

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These convex bodies display surprisingly good properties, and gave rise to a flurry of activities in projective geometry, combinatorics, and representation theory. For detailed descriptions and proofs the reader is referred to the original sources [Kaveh and Khovanskii 2012; Lazarsfeld and Mustaţă 2009] and the recent review [Boucksom 2014]. The main idea is that Newton–Okounkov bodies capture the vanishing behavior of all sections of all multiples of D at the same time.

Our philosophical starting point is the work of Jow [2010] observing that the collection of all Newton–Okounkov bodies attached to a given line bundle serves as a universal numerical invariant: whenever all the Newton–Okounkov bodies agree for two divisors D and D', then in fact they must be numerically equivalent. Jow's result leads to the expectation that one should be able to read off numerical properties of line bundles from the collection of attached Newton–Okounkov bodies.

Once we focus on local positivity, the above principle modifies it in the following manner: we expect that the local positivity of a Cartier divisor D at a point $x \in X$ to be governed by the collection (for definitions see Section 1B)

$$\mathscr{K}(D;x) \overset{\text{def}}{=} \left\{ \Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n_+ \mid Y_\bullet \text{ is an admissible flag centered at } x \right\}.$$

It turns out that, whenever X is a smooth projective surface, the above collection $\mathscr{K}(D;x)$ is independent of x away from a countable union of proper subvarieties.

We always work on smooth projective surfaces over the complex number field, unless otherwise mentioned. Our first main result is a version of the combinatorial characterization of torus-invariant ample/nef divisors on toric varieties valid on all surfaces. We go one step further and offer an analogous description in terms of infinitesimal Newton–Okounkov bodies.

Let us introduce some notation. For positive real number $\lambda, \lambda', \xi > 0$ we set

$$\Delta_{\lambda,\lambda'} \overset{\text{def}}{=} \{(t,y) \in \mathbb{R}^2_+ \mid \lambda' t + \lambda y \leqslant \lambda\lambda'\},$$
$$\Delta^{-1}_{\xi} \overset{\text{def}}{=} \{(t,y) \in \mathbb{R}^2_+ \mid 0 \leqslant t \leqslant \xi, 0 \leqslant y \leqslant t\}.$$

If $\lambda = \lambda'$, then denote by $\Delta_\lambda \overset{\text{def}}{=} \Delta_{\lambda,\lambda}$, which is the standard simplex of side length λ.

We offer the following convex geometric characterization of ampleness and nefness.

Theorem A (nefness and ampleness criteria). *Let X be a smooth projective surface, D a big $\mathbb{R}$-divisor on X, and $\pi : X' \to X$ the blow-up of $x \in X$ with exceptional divisor E. Then*:

(i) *D is nef $\Leftrightarrow$ for all $x \in X$ there is a flag (C,x) such that $(0,0) \in \Delta_{(C,x)}(D)$*
$\Leftrightarrow$ for all $x \in X$ there is a $z \in E$ such that $(0,0) \in \Delta_{(E,z)}(\pi^*(D))$.

(ii) *D is ample $\Leftrightarrow$ for all $x \in X$ there is a flag (C,x) and $\lambda > 0$ such that $\Delta_\lambda \subseteq \Delta_{(C,x)}(D)$*
$\Leftrightarrow$ for all $x \in X$ there is a $z \in E$ and $\xi > 0$ such that $\Delta^{-1}_\xi \subseteq \Delta_{(E,z)}(\pi^*(D))$.

Theorem A is a particular case of more general results. In Theorem 2.4 we prove the corresponding criteria for a point $x \in X$ not to be contained either in the negative or null locus of D in terms of Newton–Okounkov bodies defined on X. Furthermore, in Theorem 3.8 we connect these loci to the shape of infinitesimal Newton–Okounkov bodies, defined on the blow-up X'.

As mentioned above, the Newton–Okounkov body of a big $\mathbb{Q}$-divisor D encodes how all the sections of all powers of D vanish along a fixed flag. Conversely, it is a very exciting problem to find out exactly which points in the plane $\mathbb{R}^2 \supseteq \Delta_{Y_\bullet}(D)$ are given by valuations of sections, whether these points lie in the interior or the boundary of the Newton–Okounkov body. This is expressed by saying that a rational point of $\Delta_{Y_\bullet}(D)$ is "valuative". Finding valuative points in Newton–Okounkov bodies is a recurring theme of this article, some partial answers are summarized in the following corollary.

Corollary B (valuative points). *Let X be a smooth projective surface, D a big $\mathbb{Q}$-divisor, (C, x) a flag on X, and $\pi : X' \to X$ the blow-up of X at x with exceptional divisor E. Then*:

(i) *Any rational point in the interior of $\Delta_{(C,x)}(D)$ is valuative.*

(ii) *Suppose $\Delta_{\lambda,\lambda'} \subseteq \Delta_{(C,x)}(D)$ for some $\lambda, \lambda' > 0$. Then any rational point on the horizontal segment $[0, \lambda) \times \{0\}$ and the vertical one $\{0\} \times [0, \lambda')$ is valuative.*

(iii) *Suppose that $\Delta_\xi^{-1} \subseteq \Delta_{(E,z_0)}(\pi^*(D))$ for some $\xi > 0$ and $z_0 \in E$. Then any rational point on the diagonal segment $\{(t, t) \mid 0 \leqslant t < \xi\}$ and on the horizontal segment $[0, \xi) \times \{0\}$ is valuative.*

It is interesting to note that statement (ii) can be obtained via restricted volumes (see [Ein et al. 2009] for the basic theory). However, we present here a different proof for the surface case, that relies only on ideas of convex geometric nature and Theorem A. As for statement (iii), the rational points on the diagonal are valuative due to the fact that the exceptional divisor E is a rational curve.

As a consequence of Theorem A, all Newton–Okounkov bodies of an ample divisor A are bound to contain a standard simplex Δ_λ of some size. By choosing the curve in the flag to be very positive, one can make the size of the maximal such simplex as small as we wish. Therefore the exciting question to ask is how large it can become. This leads to the definition of the largest simplex constant:

$$\lambda(A; x) \stackrel{\text{def}}{=} \sup_{(C,x)} \sup\{\lambda > 0 \mid \Delta_\lambda \subseteq \Delta_{(C,x)}(A)\},$$

where the first supremum runs through all admissible flags centered at the point $x \in X$.

As expected of asymptotic invariants, the largest simplex constant is homogeneous in A, invariant with respect to numerical equivalence of divisors; moreover it is a superadditive function of A. In Proposition 4.7 we observe that $\epsilon(A; x) \geqslant \lambda(A; x)$, where the left-hand side denotes the classical Seshadri constant of A at x. We illustrate in Remark 4.9 that $\lambda(A; x) \neq \epsilon(A; x)$ in general.

Using Diophantine approximation we establish the uniform positivity of largest simplex constants assuming the rational polyhedrality of the nef cone.

Theorem C (positivity of the largest simplex constant). *Let X be a smooth projective surface with a rational polyhedral nef cone. Then there exists a strictly positive constant $\lambda(X) > 0$ such that*

$$\epsilon(A; x) \geqslant \lambda(A; x) \geqslant \lambda(X),$$

for any $x \in X$ and any ample Cartier divisor A on X.

Even further, we show the existence of strictly positive lower bound on Seshadri constants for any smooth projective variety X of any dimension, whenever the nef cone of X is rational polyhedral.

By turning our attention to the collection of infinitesimal Newton–Okounkov bodies we can take a closer look at Seshadri constants. As it happens, infinitesimal Newton–Okounkov bodies seem to capture the local positivity of divisors more precisely. Hence one can expect them to determine (moving) Seshadri constants as well. This is indeed the case, as we will immediately explain.

Here again, the key geometric invariant is the largest "inverted" standard simplex that fits inside an infinitesimal Newton–Okounkov body. With notation as above, if D is a big $\mathbb{R}$-divisor on X, $\pi : \widetilde{X} \to X$ the blow-up of X at x with exceptional divisor E, then we set

$$\xi(\pi^*(D); z) \stackrel{\text{def}}{=} \sup\{\xi \geqslant 0 \mid \Delta_\xi^{-1} \subseteq \Delta_{(E,z)}(\pi^*(D))\}.$$

It is not hard to see that this invariant does not depend on the choice of the point $z \in E$, as seen in Theorem 3.8. So, if we denote by $\xi(D; x) \stackrel{\text{def}}{=} \xi(\pi^*(D); z)$ for some $z \in E$, then the following theorem says that this invariant is actually the moving Seshadri constant of D at point x.

Theorem D (characterization of moving Seshadri constants). *Let D be a big $\mathbb{R}$-divisor on X. If $x \notin \mathrm{Neg}(D)$, then*

$$\epsilon(\|D\|; x) = \xi(D; x).$$

For the definition of $\mathrm{Neg}(D)$ see Section 1A. As a further application of infinitesimal Newton–Okounkov bodies, we translate ideas of Nakamaye and Cascini into the language of convex geometry to provide a new proof of the Ein–Lazarsfeld lower bound on Seshadri constants at very general points (see Corollary 3.17).

A few words about the organization of this paper. Section 1 hosts a quick recap of Newton–Okounkov bodies and Zariski decomposition, here several small new observations have been added that we will use repeatedly later on. Section 2 is devoted to our main results on Newton–Okounkov bodies on surfaces, while Section 3 is given over to the treatment of infinitesimal Newton–Okounkov bodies and their relation to moving Seshadri constants. In Section 4 we present various applications of the material developed so far.

1. Notation and introductory remarks

We introduce the notation that will be used throughout this paper, and give a brief introduction to Zariski decomposition and to the construction of Newton–Okounkov polygons for the sake of the reader. In addition, for the lack of a suitable reference, we include some remarks that are probably known to experts.

1A. ***Zariski decomposition.*** Let X be a smooth complex projective surface. As proven in [Fujita 1979] (see also [Bădescu 2001, Theorem 14.14] or [Zariski 1962] for the case of $\mathbb{Q}$-divisors), every pseudoeffective $\mathbb{R}$-divisor D on X has a *Zariski decomposition*, i.e., D can be written uniquely as a sum

$$D = P_D + N_D.$$

of $\mathbb{R}$-divisors, such that P_D is nef, N_D is either zero or an effective divisor with negative definite intersection matrix, and $(P_D \cdot N_D) = 0$. The divisor P_D is called the *positive part*, N_D the *negative part* of D. Note that P_D and N_D will be $\mathbb{Q}$-divisors whenever D is such. Furthermore, when D is a $\mathbb{Q}$-divisor, Zariski decomposition is an equality of divisors and not merely of numerical equivalence classes.

A crucial property of Zariski decomposition is that the positive part carries all the sections, more precisely (see [Lazarsfeld 2004, Proposition 2.3.21] for instance), assuming that mD, mP_D and mN_D are all integral, the natural inclusion map $H^0(X, \mathscr{O}_X(mP_D)) \to H^0(X, \mathscr{O}_X(mD))$, defined by the multiplication with the divisor mN_D, is an isomorphism.

Following [Bauer et al. 2004], one can associate to D the loci

$$\mathrm{Null}(D) \stackrel{\mathrm{def}}{=} \bigcup_{P_D.E=0} E \quad \text{and} \quad \mathrm{Neg}(D) \stackrel{\mathrm{def}}{=} \bigcup_{E\subseteq \mathrm{Supp}(N_D)} E,$$

where the unions are taken over irreducible curves on X. The orthogonality property of Zariski decomposition yields $\mathrm{Neg}(D) \subseteq \mathrm{Null}(D)$.

In higher dimensions, these correspond to the augmented and restricted base loci of D introduced in [Ein et al. 2006]. We do not rely on the higher dimensional definition of these loci, but it is nevertheless important to keep in mind that $\boldsymbol{B}_+(D) = \mathrm{Null}(D)$ and $\boldsymbol{B}_-(D) = \mathrm{Neg}(D)$ as observed in [Ein et al. 2006]. Vaguely speaking $\mathrm{Null}(D)$ consists of those points where D is locally not ample while $\mathrm{Neg}(D)$ is the locus of points where D is locally not nef.

In [Bauer et al. 2004] the main goal is to prove the variation of Zariski decomposition inside the big cone. Based on the description of Zariski chambers given there (see [Bauer et al. 2004, p. 214]), we prove a statement that can be seen as a more precise version of Kodaira's lemma.

Lemma 1.1. *Let P be a big and nef $\mathbb{R}$-divisor on X, and let $\mathrm{Null}(P) = E_1 \cup \cdots \cup E_r$. Then there exists a maximal-dimensional rational polyhedral cone $\mathscr{C} \subseteq \mathbb{R}^r_{\geqslant 0}$ such that $P + \alpha \cdot (E_1, \ldots, E_r)^T$ is ample for all $\alpha \in -\mathrm{int}\,\mathscr{C}$ of sufficiently small norm.*

Remark 1.2. Under the assumptions of Lemma 1.1, it is an easy consequence of the Hodge index theorem that the intersection matrix of the curves $E_1, \ldots, E_r$ is negative definite: take a nontrivial divisor $M = \sum_i a_i E_i$; notice that $(P \cdot M) = 0$. Thus, by Hodge, we see that $M^2 < 0$, since M is not numerically trivial. This last fact holds as the classes of the curves $E_1, \ldots, E_r$ are linearly independent in $\mathrm{N}^1(X)_{\mathbb{R}}$.

Proof. The claim is immediate if P is ample, hence we can assume that $P \in \partial\,\mathrm{Nef}(X) \cap \mathrm{Big}(X)$. By [Bauer et al. 2004, Corollary 1.3], there exists an open neighborhood $\mathscr{U}$ of P and the curves $C_1, \ldots, C_s$ such that the boundary of the Zariski chamber decomposition inside $\mathscr{U}$ is given by the hyperplanes $C_i^{\perp} \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$.

If $(P \cdot C_i) \neq 0$ for a curve C_i, then by shrinking $\mathscr{U}$ if need be, we can arrange that $\mathscr{U}$ lies on one side of $C_i^{\perp}$. Therefore, we can assume without loss of generality that

$$\mathscr{U} \cap \mathrm{Amp}(X) = E_1^{>0} \cap \cdots \cap E_r^{>0} \cap \mathscr{U}.$$

By possibly shrinking $\mathscr{U}$, we assume also that the self-intersection form is strictly positive on it.

The Nakai–Moishezon criterion then says that a divisor $P - \sum_{i=1}^{r} a_i E_i \in \mathscr{U}$ is ample if and only if

$$\left(P - \sum_{i=1}^{r} a_i E_i\right) \cdot E_j > 0 \quad \text{for every } 1 \leqslant j \leqslant r.$$

As $(P \cdot E_j) = 0$, by assumption, for all $1 \leqslant j \leqslant r$, then this is equivalent to

$$\sum_{i=1}^{r} a_i (E_i \cdot E_j) > 0 \quad \text{for all } 1 \leqslant j \leqslant r. \tag{1.2.1}$$

We are left with looking for a sufficient condition on $\alpha = (a_1, \dots, a_r) \in \mathbb{R}^r_{\geqslant 0}$ such that (1.2.1) holds; not surprisingly, this ends up being a linear algebra question.

Let A be the intersection matrix of the curves $E_1, \dots, E_r$. This matrix is negative definite by Remark 1.2. Then [Bauer et al. 2004, Lemma 4.1] shows that A^{-1} is a nonsingular matrix with nonpositive entries. In this notation, the statement (1.2.1) is then equivalent to asking that $A \cdot (a_1, \dots, a_r)^T$ has only strictly positive coefficients.

Let $e_1, \dots, e_r$ be the standard basis of $\mathbb{R}^r$ and $v_1 \stackrel{\text{def}}{=} A^{-1}e_1, \dots, v_r \stackrel{\text{def}}{=} A^{-1}e_r$. We claim that the cone $\mathscr{C}$ spanned by $v_1, \dots, v_r$ has the required property. Indeed, since the elements of A^{-1} are all nonpositive, then $-\mathscr{C} \subseteq \mathbb{R}^r_{\geqslant 0}$. Thus, every $v \in -\operatorname{int}\mathscr{C}$ has only positive coefficients. Furthermore, if $v = \sum_{i=1}^{r} a_i v_i \in \operatorname{int}\mathscr{C}$, then $Av = \sum_{i=1}^{r} a_i A(A^{-1}e_i) = \sum_{i=1}^{r} a_i e_i > 0$, hence (1.2.1) is satisfied. □

Remark 1.3. Let $x \in X$ be a point and $\pi : X' \to X$ be the blow-up of X at x. Suppose D is a pseudoeffective $\mathbb{R}$-divisor on X and $D = P_D + N_D$ its Zariski decomposition. If $x \notin \operatorname{Neg}(D)$, then $\pi^* D = \pi^* P_D + \pi^* N_D$ is the Zariski decomposition of $\pi^* D$. To see this, it suffices, by uniqueness of Zariski decomposition, to check that the right-hand side has the right properties. So, $\pi^* P_D$ remains nef, and $(\pi^* P_D \cdot \pi^* N_D) = (P_D \cdot N_D) = 0$. Since $x \notin \operatorname{Neg}(D)$, $\pi^* N_D$ equals the strict transform of N_D, and the respective intersection matrices agree.

1B. ***Newton–Okounkov polygons.*** For the general theory of Newton–Okounkov bodies the reader is referred to [Kaveh and Khovanskii 2012; Lazarsfeld and Mustaţă 2009] or the excellent expository work [Boucksom 2014]. Here we only summarize some surface-specific facts.

As before, let X be a smooth projective surface, and D be a big $\mathbb{Q}$-divisor on X. We say that the pair (C, x) is an *admissible flag* on X if $C \subseteq X$ is an irreducible curve and $x \in C$ is a smooth point. Then the *Newton–Okounkov body* associated to this date is defined via

$$\Delta_{(C,x)}(D) \stackrel{\text{def}}{=} \overline{\nu_{(C,x)}\big(\{D' \mid D' \sim_{\mathbb{Q}} D \text{ effective } \mathbb{Q}\text{-divisor}\}\big)},$$

where the rank-two valuation $\nu_{(C,x)}(D') = (\nu_1(D'), \nu_2(D'))$ is given by

$$\nu_1(D') = \operatorname{ord}_C(D') \quad \text{and} \quad \nu_2(D') = \operatorname{ord}_x((D' - \nu_1(D')C)|_C).$$

Making use of [Lazarsfeld and Mustaţă 2009, Proposition 4.1], one can replace $\mathbb{Q}$-linear equivalence by numerical equivalence (denoted by $\equiv$). If D is a big $\mathbb{R}$-divisor, one can also associate the Newton–Okounkov body $\Delta_{(C,x)}(D)$, where $\mathbb{Q}$-linear equivalence is replaced by numerical equivalence in the definition.

Drawing on Zariski decomposition on surfaces, [Lazarsfeld and Mustaţă 2009, Theorem 6.4] gives a practical description of Newton–Okounkov bodies in the surface case. Following [Lazarsfeld and Mustaţă 2009, Section 6], let ν be the coefficient of C in the negative part $N(D)$ and set

$$\mu = \mu(D; C) \overset{\text{def}}{=} \sup\{t > 0 \mid D - tC \text{ is big}\}.$$

Whenever there is no risk of confusion we will write $\mu(D)$.

For any $t \in [\nu, \mu]$ we set $D_t \overset{\text{def}}{=} D - tC$. Let $D_t = P_t + N_t$ be the Zariski decomposition of D_t. Consider the functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ defined as follows:

$$\alpha(t) \overset{\text{def}}{=} \operatorname{ord}_x(N_t|_C), \quad \beta(t) \overset{\text{def}}{=} \operatorname{ord}_x(N_t|_C) + P_t \cdot C.$$

Then Lazarsfeld and Mustaţă show that the Newton–Okounkov body is described as follows:

Theorem 1.4 [Lazarsfeld and Mustaţă 2009, Theorem 6.4]. *Let D be a big $\mathbb{R}$-divisor, and (C, x) an admissible flag on a smooth projective surface. Then*

$$\Delta_{(C,x)}(D) = \left\{(t, y) \in \mathbb{R}^2_+ \mid \nu \leqslant t \leqslant \mu, \alpha(t) \leqslant y \leqslant \beta(t)\right\}.$$

Remark 1.5. The Newton–Okounkov body $\Delta_{(C,x)}(D) \subseteq \mathbb{R}^2_+$ has been shown to be a polygon in [Küronya et al. 2012, Section 2]. The results of [Küronya et al. 2012] reveal further properties of $\Delta_{(C,x)}(D)$. The function $t \mapsto N_t$ is increasing on the interval $[\nu, \mu] \subseteq \mathbb{R}$, i.e., for any $\nu \leqslant t_1 \leqslant t_2 \leqslant \mu$ the difference $N_{t_2} - N_{t_1}$ is an effective divisor. This implies that a vertex of $\Delta_{(C,x)}(D)$ may only occur for those $t \in [\nu, \mu]$, where a new curve appears in $\operatorname{Neg}(D_t)$.

Remark 1.6. It was observed in [Bauer et al. 2004, Proposition 1.14] that Zariski decomposition is continuous inside the big cone but not in general when the limiting divisor class is only pseudoeffective. It turns out that there exists another important situation where continuity holds.

Let C be an irreducible curve on X and as before let $\mu(D) = \mu(D; C)$. From [Küronya et al. 2012, Proposition 2.1] and the ideas of the proof of Proposition 1.14 from [Bauer et al. 2004], it follows that

$$N_{D-tC} \to N_{D-\mu(D)C} \quad \text{and} \quad P_{D-tC} \to P_{D-\mu(D)C}$$

whenever $t \to \mu(D)$. If $x \in C$ is a smooth point, then this says in particular that the segment $\Delta_{(C,x)}(D) \cap (\mu(D) \times \mathbb{R})$ is computed from actual divisors on X, i.e., $P_{\mu(D)}$ and $N_{\mu(D)}$.

Remark 1.7. The above description of Newton–Okounkov polygons gives rise to the equality

$$\Delta_{(C,x)}(D)_{t \geqslant t_0} = \Delta_{(C,x)}(D - t_0 C) + (t_0, 0).$$

for any $t_0 \in \mathbb{R}$. In [Lazarsfeld and Mustaţă 2009, Theorem 4.24], this is proved under the additional assumption that $C \nsubseteq \mathrm{Supp}(\mathrm{Null}(D))$. As it turns out, this condition is not necessary.

Remark 1.8. Going back to the initial definition of Newton–Okounkov polygons, one remarks that the valuation map can be seen through local intersection numbers. Let D be a big $\mathbb{Q}$-divisor, (C, x) an admissible flag, and $D' \sim_{\mathbb{Q}} D$ an effective divisor. Note that $C \nsubseteq \mathrm{Supp}(D' - \nu_1(D')C)$, thus

$$\nu_2(D') = \big((D' - \nu_1(D)C).C\big)_x,$$

where the right-hand side is the local intersection number of the effective $\mathbb{Q}$-divisor $D - \nu_1(D)C$ and the curve C at the point x. As it is well known, one has

$$(C'.C'') = \sum_{P \in X} (C'.C'')_P,$$

for distinct irreducible curves C', $C'' \subseteq X$. In particular, one has the inequality

$$\big((D - \nu_1(D')C).C\big) \geqslant \nu_2(D')$$

as $D \equiv D'$. These ideas give rise to a somewhat different construction of Newton–Okounkov polygons in the surface case that is based on local intersection numbers.

Remark 1.9. For a Newton–Okounkov polygon $\Delta_{(C,x)}(D)$, the lengths of the vertical slices are independent of the point x. To see this, let

$$\Delta_{(C,x)}(D)_{t=\xi} \stackrel{\text{def}}{=} \Delta_{(C,x)}(D) \bigcap \{t = \xi\} \times \mathbb{R}$$

for any $\xi \in [\nu, \mu]$. Then

$$\mathrm{length}(\Delta_{(C,x)}(D)_{t=\xi}) = \beta(\xi) - \alpha(\xi) = (P_{D-\xi C} \cdot C),$$

hence the observation.

The following lemma helps reduce the problem of computing the Newton–Okounkov polygon of a divisor to the computation of the polygon of its positive part. This is implicitly contained in [Łuszcz-Świdecka and Schmitz 2014] for $\mathbb{Q}$-divisors; here we give a proof of the $\mathbb{R}$-divisor case.

Lemma 1.10. *Let D be a big $\mathbb{R}$-divisor on a smooth projective surface X and (C, x) an admissible flag on X. If $x \notin \mathrm{Neg}(D)$, then*

$$\Delta_{(C,x)}(D) = \Delta_{(C,x)}(P_D).$$

Proof. We first prove the statement for $\mathbb{Q}$-divisors and then use a continuity argument for the general case. If D is a big $\mathbb{Q}$-divisor, then by the homogeneity of Newton–Okounkov polygons and Zariski decomposition we can assume that D, P_D and N_D are all integral Cartier divisors. The multiplications maps

$$H^0(X, \mathscr{O}_X(mP_D)) \to H^0(X, \mathscr{O}_X(mD))$$

by mN_D are isomorphisms (see [Lazarsfeld 2004, Proposition 2.3.21]). Hence the definition of Newton–Okounkov polygons and the condition $x \notin \operatorname{Supp}(N_D)$ imply the statement.

For the general case, fix a norm $\|\cdot\|$ on the finite-dimensional vector space $\mathrm{N}^1(X)_{\mathbb{R}}$. Let D be a big $\mathbb{R}$-divisor, and $(A_n)_{n\in\mathbb{N}}$ a sequence of ample $\mathbb{R}$-divisors such that $\lim_{n\to\infty}(\|A_n\|)=0,\ A_{n+1}-A_n$ is an ample $\mathbb{Q}$-divisor, and $D+A_n$ is a $\mathbb{Q}$-divisor for any $n\in\mathbb{N}$. By the proof of [Anderson et al. 2014, Lemma 8][1], one has the equality

$$\Delta_{(C,x)}(D)=\bigcap_{n\in\mathbb{N}}^{\infty}\Delta_{(C,x)}(D+A_n).$$

This reduces the proof to the $\mathbb{Q}$-divisor case via the continuity of Zariski decomposition (see [Bauer et al. 2004, Proposition 1.14]). $\square$

2. Newton–Okounkov polygons and special loci associated to divisors

2A. *Local constancy of Newton–Okounkov polygons.* The goal of this subsection is to prove that the set of all Newton–Okounkov polygons, where the flags are taken to be centered at a very generic point x, is independent of the point.

Let X be a smooth projective surface, $x\in X$ a point and D a big $\mathbb{R}$-divisor on X. Denote by

$$\mathscr{K}(D,x)=\left\{\Delta\subseteq\mathbb{R}^2_+ \mid \exists (C,x),\ \text{an admissible flag, such that } \Delta=\Delta_{(C,x)}(D)\right\},$$

the set of all Newton–Okounkov polygons of D, where the flags are based at x. By [Küronya et al. 2012], this set is countable. Our goal is to figure out how this set varies with the point x.

Theorem 2.1. *With notation as above, there exists a subset $F=\bigcup_{m\in\mathbb{N}}F_m\subseteq X$ consisting of a countable union of Zariski-closed proper subsets $F_m\subsetneq X$ such that the set $\mathscr{K}(D,x)$ is independent of $x\in X\setminus F$.*

The proof relies on the following observation.

Lemma 2.2. *Let D be a big $\mathbb{R}$-divisor on X, $U\subseteq X$ be a subset with the following properties*:

(1) *U is disjoint from all negative curves on X.*

(2) *For every $x, x'\in U$ and (C,x) an admissible flag on X, there exists an irreducible curve C' such that (C',x') is again admissible, and $C'\equiv C$.*

Then $\mathscr{K}(D,x)$ is independent of $x\in U$.

Proof. Let $x,x'\in U$, and (C,x) as in the statement. Fix an admissible flag (C',x'), such that $C'\equiv C$. Observe that $\nu_{(C,x)}(D)=\nu_{(C',x')}(D)=0$, since both curves are not negative ones by (1). Furthermore, since $C\equiv C'$, we have $D-tC\equiv D-tC'$, and therefore $\mu_{(C,x)}(D)=\mu_{(C',x')}(D)$ as well.

Again, as U avoids all negative curves, $\alpha_{(C,x)}(t)=\alpha_{(C',x')}(t)=0$ for all $0\leqslant t\leqslant\mu(D)$. Finally, since Zariski decomposition respects numerical equivalence, $P_{D-tC}\equiv P_{D-tC'}$ for all $0\leqslant t\leqslant\mu_D(t)$, therefore $\beta_{(C,x)}(t)=\beta_{(C',x')}(t)$ for all $0\leqslant t\leqslant\mu(D)$, hence $\Delta_{(C,x)}(D)=\Delta_{(C',x')}(D)$, as required. $\square$

[1]The same statement works when each A_n is taken to be big and semiample. This will be used in Section 3.

Remark 2.3. Roughly speaking the main idea of the proof of Theorem 2.1 is that whenever $x \in X$ is a very general point then any cycle passing through this point can be deformed nontrivially in its numerical equivalence class. The source for this material is [Kollár 1995, Chapter 2].

To be more precise, [Kollár 1995, Proposition 2.5] says that there exists a countable union of proper closed subvarieties $F \subseteq X$ such that for any $x \in VG(X) \stackrel{\text{def}}{=} X \setminus F$ and any proper birational morphism $w_0 : C \to X$, where C is a smooth irreducible curve with $x \in \operatorname{Im}(w_0)$, there exists a topologically trivial family of normal cycles

$$\begin{array}{ccc} U & \xrightarrow[u]{} & X \\ \big\downarrow{\scriptstyle p} & & \\ T & & \end{array} \tag{2.3.2}$$

where T is irreducible, p is smooth (any fiber is smooth irreducible curve), u is dominant, for any $t \in T$ the morphism $u|_{U_t} : U_t = p^{-1}(t) \to X$ is a birational morphism, and there exists $t_0 \in T$, such that $u|_{U_{t_0}} : U_{t_0} \to X$ is the map w_0.

Proof of Theorem 2.1. First let us make the following definition. For every element $\gamma \in \mathrm{N}^1(X)$, the Néron–Severi group, let

$$\operatorname{Sing}(\gamma) \stackrel{\text{def}}{=} \big\{x \in X \mid x \in \operatorname{Sing}(C) \text{ for every curve } C \in \gamma \text{ with } x \in X\big\}.$$

Note that $\operatorname{Sing}(\gamma)$ is always a proper subvariety of X. Let $VG(X) \subseteq X$ be the very general subset from Remark 2.3; for every $\gamma \in \mathrm{N}^1(X)$, fix a topologically trivial family of normal cycles u_γ with fiber class γ (as in Remark 2.3), and set

$$\operatorname{Defect}(u_\gamma) \stackrel{\text{def}}{=} X \setminus \operatorname{Im}(u_\gamma).$$

Since u_γ is dominant, $\operatorname{Defect}(u_\gamma)$ is contained in a proper closed subvariety of X. Let

$$\mathscr{S} \stackrel{\text{def}}{=} \bigcup_{\gamma \in \mathrm{N}^1(X)} \operatorname{Sing}(\gamma) \quad \text{and} \quad \mathscr{D} \stackrel{\text{def}}{=} \bigcup_{\gamma \in \mathrm{N}^1(X)} \operatorname{Defect}(u_\gamma).$$

We will apply Lemma 2.2 with

$$\mathscr{U} \stackrel{\text{def}}{=} VG(X) \Big\backslash \bigg(\mathscr{S} \cup \mathscr{D} \cup \bigg(\bigcup_{C \text{ is a negative curve}} C\bigg)\bigg).$$

Observe that by construction $\mathscr{U}$ is disjoint from all negative curves on X.

Let $x, x' \in \mathscr{U}$, and (C, x) be an admissible flag. Then $x \notin \operatorname{Defect}(u_\gamma)$, hence u_γ has a fiber through x, whose image is C, and the same applies to x', let us call this curve C'. By construction C' is irreducible. Although C' might itself be singular at x', there will be a curve numerically equivalent to it which is smooth at x', as $x' \notin \operatorname{Sing}([C])$ by the construction of U. $\square$

2B. ***Null and negative loci versus Newton–Okounkov polygons.*** As mentioned above, the complements of the null or the negative loci describe the set of points on X where D is positive locally. The following theorem explains this philosophy in the language of Newton–Okounkov polygons.

Theorem 2.4. *Let X be a smooth projective surface, D be a big $\mathbb{R}$-divisor on X and $x \in X$ be an arbitrary point. Then*:

(i) *$x \notin \mathrm{Neg}(D)$ if and only if there exists an admissible flag (C, x) such that the Newton–Okounkov polygon $\Delta_{(C,x)}(D)$ contains the origin $(0,0) \in \mathbb{R}^2$.*

(ii) *$x \notin \mathrm{Null}(D)$ if and only if there exists an admissible flag (C, x) and a positive real number $\lambda > 0$ such that $\Delta_\lambda \subseteq \Delta_{(C,x)}(D)$.*

Remark 2.5. It is immediate to see that D is nef if and only if $\mathrm{Neg}(D) = \varnothing$, while D is ample if and only if $\mathrm{Null}(D) = \varnothing$. Thus the corresponding nefness and ampleness criteria of Theorem A follow immediately from Theorem 2.4.

Remark 2.6. Theorem 2.4 implies that whenever $(0,0) \in \Delta_{(C,x)}(D)$ for some admissible flag (C, x), then the same holds for all admissible flags centered at x. The analogous statement about points not contained in $\mathrm{Null}(D)$ is true as well.

Example 2.7. We discuss an example of a Newton–Okounkov polygon that does not contain vertical/horizontal edges emanating from the origin, but the origin is contained in it. Let $X = \mathrm{Bl}_P(\mathbb{P}^2)$ be the blow-up of the projective plane $\mathbb{P}^2$ at the point P, H the pull-back of $\mathscr{O}_{\mathbb{P}^2}(1)$, E the exceptional curve, and $x \in E$ be a point. Furthermore, let $\pi : X' \to X$ be the blow-up of X at a point $x \in E$. Denote by E_1 the exceptional divisor of π, by E_2 the proper transform of E on X', and E_3 the proper transform of the line in $\mathbb{P}^2$ passing through P with the tangent direction given by the point $x \in E$.

It is not hard to see that $(E_1^2) = (E_3^2) = -1$ and $(E_2^2) = -2$. Notice in addition that E_1 intersects both E_2 and E_3 at different points transversally, while the pair E_2 and E_3 does not intersect. The classes E_1, E_2 and E_3 generate the space $\mathrm{N}^1(X')_{\mathbb{R}}$. A quick computation gives $\pi^*(H) = 2E_1 + E_2 + E_3$. Thus

$$(\pi^*(D) - tE_1 \cdot E_2) = ((2-t)E_1 + E_2 + E_3 \cdot E_2) = -t,$$

hence $E_2 \subseteq \mathrm{Neg}(\pi^*(D) - E_1)$ for all $0 < t \ll 1$.

Let $\{y\} \stackrel{\text{def}}{=} E_1 \cap E_2$. Comparing with [Lazarsfeld and Mustaţă 2009, Theorem 6.4], note that $\alpha(t) > 0$ for any $0 < t \ll 1$ with respect to the flag (E_1, y). Furthermore, by Proposition 3.1 and the above, one sees easily that the Newton–Okounkov polygon $\Delta_{(E_1,y)}(\pi^*(H))$ does not contain either a horizontal or a vertical edge starting at the origin, but contains the origin. This convex polygon is what we call in the next section the infinitesimal Newton–Okounkov polygon of the divisor H on X at the point x.

It is interesting to note that X' has three negative curves E_1, E_2 and E_3, but the nef cone is minimally generated by four classes: $\pi^*(H)$, $\pi^*(H) + E_3$, $\pi^*(H) + E_1 + E_3$ and $E_1 + E_3$

Proof of Theorem 2.4. (i) By [Lazarsfeld and Mustaţă 2009, Theorem 6.4] we obtain the following sequence of equivalences:

$$\begin{aligned}(0,0) \in \Delta_{(C,x)}(D) &\Leftrightarrow \nu = 0 \quad \text{and} \quad \alpha(0) = 0\\ &\Leftrightarrow C \nsubseteq \mathrm{Neg}(D) \quad \text{and} \quad x \notin \mathrm{Neg}(D)\\ &\Leftrightarrow x \notin \mathrm{Neg}(D),\end{aligned}$$

which is what we wanted.

(ii) "$\Rightarrow$": Let us assume that $x \notin \mathrm{Null}(D)$. Since $\mathrm{Neg}(D) \subseteq \mathrm{Null}(D)$, this implies $x \notin \mathrm{Neg}(D)$. By Lemma 1.10 we can also assume without loss of generality that $D = P_D$, that is, D is big and nef. These conditions yield that $(D.C) > 0$ for any irreducible curve $C \subseteq X$ passing through x. In particular the convex polygon $\Delta_{(C,x)}(D)$ contains the origin $(0,0)$ and the vertical segment $\{0\} \times [0, (D.C)]$ with nonempty interior for any admissible flag (C, x).

By fixing an admissible flag (C, x), it remains then to show that the polygon $\Delta_{(C,x)}(D)$ contains a horizontal segment with nonempty interior starting at the origin. On the other hand, by the convexity of Newton–Okounkov polygons, statement (i), and Remark 1.7, in order to prove the latter condition, it is enough to show that there exists $t > 0$, such that $x \notin \mathrm{Neg}(D - tC)$.

By [Bauer et al. 2004, Theorem 1.1], the divisor D has an open neighborhood $\mathcal{U}$ inside the big cone, which intersects only finitely many Zariski chambers. Thus, there exist finitely many curves $\Gamma_1, \dots, \Gamma_s, \Gamma'_{s+1}, \dots, \Gamma'_r$ such that

$$\mathrm{Neg}(D - tC) \subseteq \left(\bigcup_{i=1}^{i=s} \Gamma_i\right) \cup \left(\bigcup_{j=s+1}^{j=r} \Gamma'_j\right), \quad \text{whenever } D - tC \in \mathcal{U},$$

where Γ_i are the curves containing x and Γ'_j are those which don't contain this point. By the above $\gamma_i \stackrel{\text{def}}{=} (D \cdot \Gamma_i) > 0$ and thus $\gamma \stackrel{\text{def}}{=} \min_{1 \leqslant i \leqslant s} \gamma_i > 0$.

Observe that by possibly shrinking $\mathcal{U}$, we can arrange that even its closure intersects only finitely many Zariski chambers and remains inside the big cone. Let

$$D - tC = P_t + \left(\sum_{i=1}^{s} a_i^t \Gamma_i + \sum_{j=s+1}^{r} b_j^t \Gamma'_j\right)$$

be the Zariski decomposition of $D - tC$. By [Küronya et al. 2012, Proposition 2.1], the functions a_i^t and b_j^t depend continuously on t as long as $D - tC \in \mathcal{U}$. Also, $a_i^0 = b_j^0 = 0$ for all i and j, since D is nef.

Consider now the divisor

$$D'_t \stackrel{\text{def}}{=} D - tC - \sum_{j=s+1}^{r} b_j^t \Gamma'_j = P_t + \sum_{i=1}^{s} a_i^t \Gamma_i.$$

Notice that the right-hand side is the Zariski decomposition of D'_t. For every $1 \leqslant i \leqslant s$ one has

$$(D'_t \cdot \Gamma_i) = (D \cdot \Gamma_i) - t(C \cdot \Gamma_i) - \left(\sum_{j=s+1}^{r} b_j^t \Gamma'_j\right) \cdot \Gamma_i \geqslant \gamma - t(C \cdot \Gamma_i) - \sum_{j=s+1}^{r} b_j^t (\Gamma'_j \cdot \Gamma_i).$$

By taking t sufficiently small and by the continuity of b_j^t as a function of t, then $(D'_t \cdot \Gamma_i) > 0$ for all $1 \leqslant i \leqslant s$. On the other hand these negative curves are the only candidates for components of $\mathrm{Neg}(D'_t)$.

Consequently, all the a_i^t's are zero, and

$$D - tC = P_t + \sum_{j=s+1}^{r} b_j^t \Gamma_j'$$

is the Zariski decomposition of $D - tC$. Since the curves Γ_j' are exactly the ones not containing x, we arrive at the desired conclusion $x \notin \mathrm{Neg}(D - tC)$ for small t.

"$\Leftarrow$": We consider first the case when D is a big and nef divisor, i.e., $D = P_D$ and $N_D = 0$. Suppose for a contradiction that there exists a curve $E \subseteq \mathrm{Null}(D)$ such that $x \in E$. If $C = E$, then, by the description of the Newton–Okounkov polygon of D, $\beta(0) = 0$, as $(D.E) = 0$. Thus,

$$\Delta_{(C,x)}(D) \cap (\{0\} \times \mathbb{R}) = (0, 0).$$

Consequently, $\Delta_{(C,x)}(D)$ cannot contain a small standard simplex.

If $C \neq E$, then $C \cdot E > 0$, as both C and E contain the point x. Therefor, $(D - tC) \cdot E < 0$ for any $t \ll 1$, This implies that $E \subseteq \mathrm{Supp}(N_t)$ and consequently that $\alpha(t) = \mathrm{ord}_x(N_t|_C) > 0$ for $t \ll 1$. Therefore,

$$\Delta_{(C,x)}(D) \cap (\mathbb{R} \times \{0\}) = (0, 0),$$

and again $\Delta_{(C,x)}(D)$ does not contain a standard simplex of any size. This leads to a contradiction to the existence of the curve E.

The general case, when D is big, follows immediately from Lemma 1.10 and the observation that the condition $x \notin \mathrm{Neg}(D)$ is implied by the equivalence in statement (i). □

2C. *Valuative points.* By the definition given by Lazarsfeld and Mustaţă, the Newton–Okounkov polygon of a big $\mathbb{Q}$-divisor D encodes how all the sections of all powers of D vanish along a fixed flag. Although not observed in [Lazarsfeld and Mustaţă 2009], the points that come from evaluating sections form a dense subset in the Newton–Okounkov polygon (hence, a posteriori there is no need for forming the convex hull in the construction).

Conversely, it is a very exciting problem to find out exactly which points in the plane are given by valuations of sections, whether these points lie in the interior or the boundary of the Newton–Okounkov polygon. To provide a partial answer, we start with the following definition:

Definition 2.8. Let D be a big $\mathbb{Q}$ ($\mathbb{R}$)-divisor and (C, x) an admissible flag on X. We say that a point $(t, y) \in \Delta_{(C,x)}(D) \cap \mathbb{Q}^2$ (or $(t, y) \in \Delta_{(C,x)}(D) \cap \mathbb{R}^2$ in the case of real divisors) is *a valuative point of D with respect to the flag* (C, x), if there exists an effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} D$ (an effective $\mathbb{R}$-divisor $D' \equiv D$) satisfying the property $\nu_{(C,x)}(D') = (t, y)$.

Remark 2.9. The fact that certain rational points in a Newton–Okounkov polygon are valuative is equivalent to the existence of sections with prescribed vanishing behavior along the given flag in a linear series $|mD|$ for $m \gg 0$.

Corollary 2.10. *Let D be a big $\mathbb{Q}$-divisor and (C, x) be an admissible flag on X. Then*:

(i) *Any rational point in* $\operatorname{int}(\Delta_{(C,x)}(D))$ *is a valuative point.*

(ii) *Suppose* $\Delta_{\lambda,\lambda'} \subseteq \Delta_{(C,x)}(D)$ *for some* $\lambda, \lambda' > 0$. *Then any rational point on the horizontal segment* $[0,\lambda) \times \{0\}$ *and the vertical one* $\{0\} \times [0,\lambda')$ *is a valuative point.*

Remark 2.11. Observe that Corollary 2.10 remains valid if we take D to be a big $\mathbb{R}$-divisor with the appropriate notion of a valuative point (in this case a point $\alpha \in \Delta_{Y_\bullet}(D)$ is valuative if there exists an effective $\mathbb{R}$-divisor $D' \sim mD$ for some $m \in \mathbb{N}$ and $\nu_{Y_\bullet}(D') = \alpha$.

Remark 2.12. For the vertical segment $\{0\} \times [0,\lambda')$, one can obtain the statement as a consequence of [Ein et al. 2009, Theorem 2.13], as illustrated in Example 2.13 of the same paper, along with the restriction theorem [Lazarsfeld and Mustaţă 2009, Theorem 4.24] for Newton–Okounkov bodies. Here we give a different proof for the surface case, relying only on ideas of a convex geometric nature arising from the theory developed so far.

Remark 2.13. Let A be an ample $\mathbb{Q}$-divisor on X, $C \subseteq X$ be a smooth rational curve, $x \in C$, and set $d = (A \cdot C)$. Then $(0, d)$, the highest vertex of the polygon $\Delta_{(C,x)}(A)$ on the y-axis, is also a valuative point. The argument follows from Serre vanishing and the rationality of C.

Since A is ample, $H^1(X, \mathscr{O}_X(mA - C)) = 0$ for all $m \gg 0$ by Serre vanishing. Therefore, the restriction maps $H^0(X, \mathscr{O}_X(mA)) \to H^0(C, \mathscr{O}_C(md))$ are all surjective. As C is a rational curve, there exists $\bar{s} \in H^0(C, \mathscr{O}_C(md))$ such that $\operatorname{mult}_x(\bar{s}) = md$. By the surjectivity of the restriction maps there exists a section $s \in H^0(X, \mathscr{O}_X(mA))$ whose image is $\bar{s}$. In particular, $\nu_{(C,x)}(s) = (0, d)$.

The proof shows that in fact all rational points on the edge of $\Delta_{(C,x)}(A)$ with vertices at $(0,0)$ and $(0,d)$ are valuative points. In this sense Corollary 2.10 serves as a local restriction theorem, and Newton–Okounkov polygons give us some elbow room to obtain local statements without having to rely on vanishing theorems.

As explained in [Anderson et al. 2014, Proposition 14], if C is not rational, then there exist line bundles L of degree $d > 0$ on C, so that no section $s \in H^0(C, mL)$ has $\operatorname{ord}_x(s) = md$ for any $m > 0$. Thus, the rationality of C is crucial.

If we take A to be big (and C still rational), then the highest vertex of the polygon $\Delta_{(C,x)}(D)$ on the y-axis is a valuative point if we assume additionally that $\operatorname{Null}(D) \cap C = \varnothing$. This can be deduced using the ideas from the proof of Corollary 3.15.

Proof of Corollary 2.10. (i) The following remark will be used repeatedly throughout this proof: given two valuative points $A = (t, y)$, $B = (t', y') \in \Delta_{(C,x)}(D)$, any rational point contained in the line segment $[AB]$, connecting the point A with B, is again a valuative point. This is due to the fact that $\nu_{(C,x)}$ is a valuation map.

Let $(t_0, y_0) \in \operatorname{int}(\Delta_{(C,x)}(D))$ be a point with rational coordinates. The idea is to show that there exist valuative points on the vertical line $t = t_0$ above and below (t_0, y_0). We verify the existence of a valuative point lying above (t_0, y_0), the other case being completely analogous. Consider the interior of the shape $\Delta_{(C,x)}(D) \cap \{(t, y) \mid y \geqslant y_0\}$, which is divided into two nonempty subsets by the line $t = t_0$. Since valuative points are dense in each subset, we can choose a point in each of them. The line segment connecting these two points intersects the line $t = t_0$ in a rational point that is above (t_0, y_0). Hence by the above observations this point is also valuative.

(ii) We check first the assertion on the horizontal line segment. Let $\delta \in [0, \lambda)$ be a rational number. By Remark 1.7, the polygon $\Delta_{(C,x)}(D - \delta C)$ contains a nonzero simplex. By Theorem 2.4 this latter condition implies that $x \notin \mathrm{Null}(D - \delta C)$. Since $\boldsymbol{B}(D - \delta C) \subseteq \mathrm{Null}(D - \delta C)$, then $x \notin \boldsymbol{B}(D - \delta C)$. Thus, the origin $(0, 0) \in \Delta_{(C,x)}(D - \delta C)$ is a valuative point with respect to the divisor $D - \delta C$. Using Remark 1.7, then the point $(\delta, 0) \in \Delta_{(C,x)}(D)$ is also valuative with respect now to D.

It remains to show that all rational points on the open line segment $\{0\} \times (0, \lambda')$ are valuative as well. To this end, observe by Remark 1.7 that

$$\Delta_{(C,x)}(D + \epsilon C)_{t \geqslant \epsilon} = (\epsilon, 0) + \Delta_{(C,x)}(D) \quad \text{for any } \epsilon > 0. \tag{2.13.3}$$

So, if we show that $\mathrm{vol}_X(D + \epsilon C) > \mathrm{vol}_X(D)$ for some rational $\epsilon > 0$, then there will exist a valuative point in the area $\Delta_{(C,x)}(D + \epsilon C) \cap (0, \epsilon) \times \mathbb{R}$. On the other hand this implies that the open line segment $\{\epsilon\} \times (0, \lambda')$ is inside $\Delta_{(C,x)}(D + \epsilon C)$. By statement (i), any rational point on this line segment is valuative for the $\mathbb{Q}$-divisor $D + \epsilon C$. Applying (2.13.3) again along with Lazarsfeld and Mustaţă's definition of Newton–Okounkov polygons yields that the same can be said about all the rational points on the vertical segment $\{0\} \times (0, \lambda')$ in the polygon $\Delta_{(C,x)}(D)$.

It remains to prove that $\mathrm{vol}_X(D + \epsilon C) > \mathrm{vol}_X(D)$ for some $0 < \epsilon \ll 1$. To this end, assume first that $D = P$ is big and nef. Since $\Delta_{(C,x)}(D)$ contains a standard simplex, then by Theorem 2.4 we know that $C \nsubseteq \mathrm{Null}(D)$. This latter condition implies that $(D \cdot C) > 0$. Thus, $D + \epsilon C$ is also nef for $\epsilon \ll 1$, and we have the following inequality

$$\mathrm{vol}_X(D + \epsilon C) = \frac{(D + \epsilon C)^2}{2} > \frac{D^2}{2} = \mathrm{vol}_X(D),$$

whenever $\epsilon \ll 1$, settling the claim for D big and nef. For the general case, let $D = P + N$ and $D + \epsilon C = P_\epsilon + N_\epsilon$ be the respective Zariski decompositions of D and $D + \epsilon C$. By the previous step we know that $P + \epsilon C$ is nef for all $0 < \epsilon \ll 1$, hence one can write $D + \epsilon C = (P + \epsilon C) + N$, and by the minimality of the Zariski decomposition we see that $N - N_\epsilon$ is effective. Consequently, $P_\epsilon - (P + \epsilon C)$ is also effective. Using the statement in the big and nef case, proved just above, we deduce that

$$\mathrm{vol}_X(D + \epsilon C) = \mathrm{vol}_X(P_\epsilon) \geqslant \mathrm{vol}_X(P + \epsilon C) > \mathrm{vol}_X(P) = \mathrm{vol}_X(D)$$

for all $0 < \epsilon \ll 1$. This also proves the corollary in the big case. □

3. Moving Seshadri constants and infinitesimal Newton–Okounkov polygons

The goal that we pursue in this section is to study the relationship between the positivity properties of a big divisor and the geometry of its Newton–Okounkov polygons that can be defined on the blow-up of a point. We show how moving Seshadri constants can be read off from these polygons, and study which of their boundary points are valuative.

As before, we assume X to be a smooth projective surface, D a big $\mathbb{Q}$ (or $\mathbb{R}$) divisor on X, and $x \in X$ a point. We denote by $\pi : X' \to X$ the blow-up of X at x with E the exceptional divisor. For any point $z \in E$, we call the polygon $\Delta_{(E,z)}(\pi^*(D))$ the *infinitesimal Newton–Okounkov polygon* of D attached to the admissible flag (E, z). This concept originates in [Lazarsfeld and Mustaţă 2009, Section 5] (note the deviation from the terminology of [Lazarsfeld and Mustaţă 2009]). The goal of this section is to explore the relationship between the positivity properties of the divisor D and the geometry of the polygons $\Delta_{(E,z)}(\pi^*(D))$.

3A. *Infinitesimal Newton–Okounkov polygons.* In this subsection, we study the basic properties of the infinitesimal Newton–Okounkov polygons. For a big $\mathbb{R}$-divisor D, write

$$\mu' = \mu'(D, x) = \mu(\pi^*(D), E) \stackrel{\text{def}}{=} \sup\{t > 0 \mid \pi^*(D) - tE \text{ is big}\}.$$

It follows from [Lazarsfeld and Mustaţă 2009, Theorem 6.4] and the definition of μ' that $\Delta_{(E,z)}(\pi^*(D)) \subseteq \mathbb{R}_+ \times [0, \mu']$.

Furthermore, for all $x \in X$, we associate to the divisor D the set of all infinitesimal Newton–Okounkov polygons rooted at x:

$$\mathscr{K}'(D, x) \stackrel{\text{def}}{=} \{\Delta \subseteq \mathbb{R}^2_+ \mid \exists z \in E, \text{ such that } \Delta = \Delta_{(E,z)}(\pi^*(D))\}.$$

Proposition 3.1. *With notation as above, we have:*

(i) *The polygon $\Delta_{(E,z)}(\pi^*(D))$ is contained in $\Delta^{-1}_{\mu'(D,x)}$ for any $z \in E$;*

(ii) *there exist finitely many points $z_1, \dots, z_k \in E$ such that $\Delta_{(E,z)}(\pi^*(D))$ is independent of $z \in E \setminus \{z_1, \dots, z_k\}$, with base the whole line segment $[0, \mu'] \times \{0\}$.*

Remark 3.2. The constant $\mu'(D'; x)$ can be computed on X. If $|V|$ is a linear series on X, define

$$\overline{\mathrm{mult}}_x(|V|) \stackrel{\text{def}}{=} \sup\{\mathrm{mult}_x(F) \mid F \in |V|\}.$$

If D is a big Cartier divisor, set

$$\overline{\mathrm{mult}}_x(\|D\|) \stackrel{\text{def}}{=} \limsup_{m\to\infty} \frac{\overline{\mathrm{mult}}_x(|pD|)}{p}.$$

Then $\mu' = \overline{\mathrm{mult}}_x(\|D\|)$, by simple properties of the multiplicity (cf. [Dumnicki et al. 2013, Proposition 3.2]). The same holds whenever D is a big $\mathbb{Q}$-divisor and by continuity (cf. [Ein et al. 2006, Theorem A]) the statement extends to $\mathbb{R}$-divisors.

Proof of Proposition 3.1. (i) Based on the second part of the proof of Lemma 1.10, it is not hard to see that it suffices to show the statement when D is merely a big Cartier divisor. It was pointed out above that $\Delta_{(E,z)}(\pi^*(D)) \subseteq \mathbb{R} \times [0, \mu'(D, x)]$. Thus, it remains to show that $\Delta_{(E,z)}(\pi^*(D))$ lies below the diagonal $y = t$.

By Zariski's main theorem (see [Hartshorne 1977, Theorem III.11.4]) one has the isomorphisms

$$H^0(X', \mathscr{O}_{X'}(m \cdot \pi^*(D))) \simeq H^0(X, \mathscr{O}_X(mD)) \quad \text{for all } m > 0.$$

Hence all the sections of $\pi^*(D)$ can be seen as pull-backs of sections from X. Let $D' \in |mD|$ for some $m > 0$. In order to end the proof it is sufficient to check the inequality

$$\mathrm{mult}_x(D') = \mathrm{ord}_E(\pi^*(D')) \geqslant \mathrm{ord}_z\big(\big(\pi^*(D') - \mathrm{mult}_x(D')E\big)|_E\big).$$

Here we use that the multiplicity of a tangent direction of a given curve at x cannot exceed the multiplicity at this point. This can in fact be checked locally: Let $\{u_1, u_2\}$ be a local system of parameters in a neighborhood $U \subseteq X$ of the point x. Then a section $s \in H^0(X, \mathscr{O}_X(mD))$ restricted to U can be written in terms of the local coordinates u_1 and u_2 as

$$s|_U = f_d(u_1, u_2) + f_{d+1}(u_1, u_2) + \cdots + f_{d+l}(u_1, u_2),$$

where each f_i is a homogeneous polynomial of degree i with $d = \mathrm{mult}_x(s)$. Since we are working over the complex numbers and the polynomial f_d is homogeneous, we can write it as follows

$$f_d(u_1, u_2) = (u_1 - \alpha_1 u_2)^{i_1} \cdot (u_1 - \alpha_2 u_2)^{i_2} \cdots (u_1 - \alpha_k u_2)^{i_k},$$

where $i_j \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, and $\sum_{j=1}^{j=k} i_j = d$.

Now, let

$$\pi|_U : U' \overset{\text{def}}{=} \{u_1 v_1 = u_2 v_2\} \subseteq U \times \mathbb{P}^1 \to U$$

be the blow-up of U at x. The form of the decomposition of f_d implies that it is enough to do the computations on the open subset $U'_1 = \{v_1 = 1\} \subseteq U'$. Then

$$\pi^*(s)|_{U'_1} = u_2^d \cdot \big(f_d(v_2, 1) + u_2 f_{d+1}(v_2, 1) + \cdots + u_2^l f_{d+l}(v_2, 1)\big) \overset{\text{def}}{=} u_2^d \cdot F(u_2, v_2).$$

The first term of the right-hand side yields $\mathrm{ord}_E(\pi^*(s)) = d$. The shape of the second one gives us

$$\nu_2(\pi^*(s)) = \mathrm{ord}_{[1:\alpha]}(F(0, v_2)) = \begin{cases} i_j & \text{if } \alpha = \alpha_j, \text{ for some } i = 1, \dots, l, \\ 0 & \text{otherwise.} \end{cases}$$

Since $d \geqslant i_j$ for all $1 \leqslant j \leqslant k$ by construction, we are done.

(ii) Let

$$D'_t \overset{\text{def}}{=} \pi^*(D) - tE = P'_t + N'_t$$

be the appropriate Zariski decomposition. Neither of the coefficient ν' of E in the negative part N'_0 and $\mu'(D, x)$ depend on the choice of $z \in E$, hence $\Delta_{(E,z)}(\pi^*(D)) \subseteq [\nu', \mu'] \times \mathbb{R}_+$ for any $z \in E$.

For any $\xi \in [\nu', \mu']$ the length of the vertical slice $\Delta_{(E,z)}(\pi^*(D))_{t=\xi}$ is independent of $z \in E$, by Remark 1.9. Thus, to finish up, it suffices to show that there exist finitely many points $z_1, \dots, z_k \in E$ such that $[\nu', \mu'] \times \{0\} \subseteq \Delta_{(E,z)}(\pi^*(D))$ for all $z \in E \setminus \{z_1, \dots, z_k\}$.

This, however, is a consequence of [Küronya et al. 2012, Proposition 2.1] which states that the function $t \in [\nu', \mu'] \to N'_t$ is increasing, i.e., if $\nu' \leqslant t_1 \leqslant t_2 \leqslant \mu'$ then the divisor $N'_{t_2} - N'_{t_1}$ is effective. In particular, the divisor $N'_{\mu'} - N'_t$ is effective for any $t \in [\nu', \mu']$. Consequently, the function α is identically zero on the whole interval $[\nu', \mu']$ whenever $x \notin \mathrm{Supp}(N'_{\mu'}) \cap E$, as stated. □

It now makes good sense to introduce the following definition:

Definition 3.3. With notation as above, for a big $\mathbb{R}$-divisor D on X, we call the polygon $\Delta_{(E,z)}(\pi^*(D))$, where the point $z \in E$ is chosen to be general, *the generic infinitesimal Newton–Okounkov polygon*[2] of D at x. We denote this polygon by $\Delta(D, x)$.

The set of polygons $\mathscr{K}'(D, x)$ as we have seen above is finite. Furthermore, by [Lazarsfeld and Mustaţă 2009, Theorem A], we also know that all the polygons in this set have the same area equal to $\mathrm{vol}_X(D)$. Whence it is natural to ask what other data remains invariant for all polygons in the finite set $\mathscr{K}'(D, x)$, thus giving rise to natural invariants of D and the point x.

Proposition 3.4. *The set of all t-coordinates of the vertices of the infinitesimal Newton–Okounkov polygon $\Delta_{(E,z)}(\pi^*(D))$ does not depend on z.*

Remark 3.5. As we shall see in Theorem 3.8 and Theorem 3.10 below, whenever $x \notin \mathrm{Null}(D)$, then the origin $(0, 0)$, (ϵ, ϵ) and $(\epsilon, 0)$ are all vertices of the polygon $\Delta_{(E,z)}(\pi^*(D))$ for any $z \in E$, where $\epsilon = \epsilon(\|D\|, x)$ is the moving Seshadri constant.

Proof. The main idea for the proof is that the function α, defining the lower bound of the Newton–Okounkov polygon, is increasing and concave-up and β, defining the upper bound, is concave-down. So, let $t_0 = \nu' < t_1 < \cdots < t_{k-1} < t_k = \mu'$ be the sequence of the t-coordinates of all the vertices of the generic infinitesimal Newton–Okounkov polygon $\Delta(D, x)$. By Proposition 3.1, these coordinates come from the vertices sitting on the upper boundary defined by the function β.

So, let Δ be another infinitesimal Newton–Okounkov polygon that is not equal to $\Delta(D, x)$. Suppose there is an intermediate point $t' \in (t_i, t_{i+1})$, for some $i = 1, \ldots, k$, which is the t-coordinate of some vertex on Δ. Assume first that this vertex is on the lower bound of this polygon. By Remark 1.9 we know that for any $t'' \in [\nu', \mu']$ the length of the vertical segment $\Delta_{(E,z)}(\pi^*(D))_{t=t''}$ does not depend on z. Furthermore the upper bound of $\Delta(D, x)$ is a straight segment in a neighborhood of the line $t = t'$. These two facts force the upper bound of Δ in a neighborhood of the vertex defining the coordinate t' to be concave up. This leads to a contradiction since the upper bound is actually concave down. Also, the vertex on Δ giving t' cannot be either on the upper bound of this polygon. This is due to the fact that the trapezoid in $\Delta(D, x) \cap [t_i, t_{i+1}] \times \mathbb{R}$ is transformed into the shape $\Delta \cap [t_i, t_{i+1}] \times \mathbb{R}$ by an affine transformation, thus inducing lines into lines.

The same reasoning implies that if t_i is the t-coordinate of some vertex on $\Delta(D, x)$ then this coordinate is also the t-coordinate for some vertex of Δ. □

[2] In [Lazarsfeld and Mustaţă 2009] this was originally named the infinitesimal Okounkov body.

3B. ***Local constancy of generic infinitesimal Newton–Okounkov polygons.*** In the previous subsection we have attached a generic infinitesimal Newton–Okounkov polygon $\Delta(D; x)$ to a big $\mathbb{R}$-divisor D and a point $x \in X$. In light of Theorem 2.1 it is then natural to study how $\Delta(D; x)$ varies when the point $x \in X$ moves around.

Theorem 3.6. *Let D be a big $\mathbb{R}$-divisor on a smooth projective surface X. Then there exists a subset $F' = \bigcup_{m\in\mathbb{N}} F'_m \subseteq X$ consisting of a countable union of Zariski-closed proper subsets $F'_m \subsetneq X$ such that the polygon $\Delta(D, x) \subseteq \mathbb{R}^2$ is independent of $x \in X \setminus F'$.*

Remark 3.7. Suppose A is an ample Cartier divisor on X. Proposition 4.2 below says that the polygon $\Delta(A, x)$, as explained in Theorem 3.6 does not depend on x for very general choices, is contained in an area determined by the global Seshadri constant $\epsilon(A) = \sup\{\epsilon(A, x) \mid x \in X\}$.

Proof of Theorem 3.6. By an argument similar to the second part of proof of Lemma 1.10, we can assume without loss of generality that D is a big Cartier divisor.

Denote by $p_1, p_2 : X \times X \to X$ the respective projections onto the first and the second factors, let $\Delta_X \subseteq X \times X$ be the diagonal. Write $\pi : Y \overset{\text{def}}{=} \mathrm{Bl}_{\Delta_X}(X \times X) \to X \times X$ for the blow-up along the diagonal with exceptional divisor $E_X \subseteq Y$, and projection morphisms $\pi_1, \pi_2 : Y \to X$.

We will study the family $\pi_1 : Y \to X$, which has the property that for $x \in X$ the fiber $\pi_1^{-1}(x) = \mathrm{Bl}_x(X)$ is the blow-up of X at x. Let $\mathscr{D} = \pi_2^*(D)$ and notice that $\mathscr{D}|_{\pi_1^{-1}(x)} = \pi_x^*(D)$, where $\pi_x = \pi|_{\pi^{-1}(x\times X)} : \mathrm{Bl}_x(X) \to X$. Consider the incomplete flag

$$Y_0 = Y \supseteq Y_1 = E_X \supseteq Y_2,$$

where Y_2 is defined as follows: because the diagonal in $X \times X$ is smooth, E_X is a projective bundle over Δ_X; now let Y_2 be an arbitrary section of $E_X \to \Delta_X$. It is worth noting here that E_X as a projective bundle might not have sections, but one can take an open subset $U \subseteq \Delta_X$ and work throughout on the family $\mathrm{Bl}_{\Delta_U}(U \times U) \to U$. We will explain the picture in the global case, when we assume that Y_2 is a section of π_1, as the local case follows by the same ideas. Denote by $E_x \overset{\text{def}}{=} E_X \cap \pi_1^{-1}(x)$, the exceptional divisor of the map π_x, and by $z_x = Y_2 \cap \pi_1^{-1}(x) \in E_x$.

Thus the goal is to understand the family of Newton–Okounkov polygons $\Delta_{(E_x,z_x)}(\pi_x^*(D))$ for $x \in X$. Applying [Lazarsfeld and Mustaţă 2009, Theorem 5.1] to the flat family $\pi_1 : Y \to X$, one deduces that there exists a countable family $F' = \cup_{m\in\mathbb{N}} F'_m \subseteq X$, where each $F'_m \subseteq X$ is a proper Zariski closed subvariety, satisfying the property that

$$\Delta_{(E_x,z_x)}(\pi_x^*(D)) \subseteq \mathbb{R}^2 \text{ is independent of } x \text{ for } x \in X \setminus F'.$$

Let $[\nu', \mu']$ be the support on the t-axis of the polygon $\Delta \overset{\text{def}}{=} \Delta_{(E_x,z_x)}(\pi_x^*(D))$ for $x \in X \setminus F'$. If $[\nu', \mu'] \times \{0\} \subseteq \Delta$, then, by Proposition 3.1, the set $\Delta_{(E_x,z_x)}(\pi_x^*(D))$ is the generic infinitesimal Newton–Okounkov polygon of D at x for any $x \in X \setminus F'$.

Otherwise, by Proposition 3.4, we know that the t-coordinates of the vertices of Δ are the t-coordinates of the generic infinitesimal Newton–Okounkov polygon $\Delta(D, x)$ for any $x \in X \setminus F'$.

Furthermore, by Remark 1.9 for any $t \in [\nu', \mu']$ the length of the vertical segment $\Delta \cap \{t\} \times \mathbb{R}$ is equal to the length of the vertical segment $\Delta(D, x)$ for any $x \in X \setminus F'$. The two latter facts imply that $\Delta(D, x) \subseteq \mathbb{R}^2$ is independent of $x \in X \setminus F'$, which concludes the proof. □

3C. *Infinitesimal Newton–Okounkov polygons and base loci.* In this subsection, we are trying to explore how certain ideas of Nakamaye [2003], connecting augmented base loci and blow-ups can be seen in the language of infinitesimal Newton–Okounkov polygons.

Returning to Example 2.7, we remark that the infinitesimal Newton–Okounkov polygon considered there does not contain a triangle of the form the Δ_ξ^{-1} for any $\xi > 0$. Notice also that the base point was taken to be contained in the null locus. These observations lead to our first goal, namely, to find conditions under which all infinitesimal Newton–Okounkov polygons contain a triangle Δ_ξ^{-1} for some $\xi > 0$. We shall see below that this information suffices to describe the complement of the null locus.

Furthermore, we discuss here how the points of the negative locus can be read from infinitesimal data. This kind of connection has not been looked at before and completes the picture that started in [Nakamaye 2003] in a clean way, at least in the case of surfaces.

Theorem 3.8. *Let D be a big $\mathbb{R}$-divisor on a smooth projective surface X. Then*:

(1) $x \notin \mathrm{Neg}(D)$ *if and only if* $(0, 0) \in \Delta_{(E,z)}(\pi^*(D))$ *for any* $z \in E$.

(2) $x \notin \mathrm{Null}(D)$ *if and only if there exists* $\xi > 0$ *such that* $\Delta_\xi^{-1} \subseteq \Delta_{(E,z)}(\pi^*(D))$ *for any* $z \in E$.

(3) *With notation as above*, *the function*

$$E \ni z \mapsto \xi(\pi^*(D); z)$$

is constant. We denote then $\xi(\pi^*(D); z)$ by $\xi(D; x)$. In particular, *it suffices to check the conditions above for just one point $z \in E$.*

Proof. (1) Based on the ideas from the second part of the proof of Lemma 1.10, it is not hard to see that it suffices to check the statement in the case when D is a big $\mathbb{Q}$-divisor.

Assume first that $x \notin \mathrm{Neg}(D)$, and let $D = P_D + N_D$ be the corresponding Zariski decomposition. Then by Remark 1.3 implies that $\pi^*(D) = \pi^*(P_D) + \pi^*(N_D)$ is the Zariski decomposition of π^*D. In particular, $\mathrm{Neg}(\pi^*(D)) \cap E = \varnothing$, and Theorem 2.4 yields $(0, 0) \in \Delta_{(E,z)}(\pi^*(D))$.

For the reverse implication suppose on the contrary that $x \in \mathrm{Neg}(D)$. By scaling we can assume that D, P_D, and N_D are all integral. Let $x \in C \subseteq X$ be an irreducible curve appearing in N_D with a strictly positive coefficient $a > 0$. By [Lazarsfeld 2004, Proposition 2.3.21] we know that for any natural number $m > 0$ and any effective divisor $D' \in |mD|$ there exists another effective divisor $P' \in |mP_D|$ for which $D' = P' + N_D$. Therefore, $\mathrm{mult}_C(D') \geqslant m \cdot a$.

On the other hand, as the polygon $\Delta_{(E,z)}(\pi^*(D))$ is the closure of all normalized valuation vectors of the effective divisors $\pi^*(D')$, for any $D' \in |mD|$ and any $m > 0$, we obtain by the above that

$$\nu_1(\pi^*(D')) = \mathrm{mult}_E(\pi^*(D')) \geqslant ma \cdot \mathrm{mult}_x(C).$$

Consequently, by the definition of the Newton–Okounkov polygons we obtain that $(0,0) \notin \Delta_{(E,z)}(\pi^*(D))$, contradicting our initial assumption.

(2) We start with the direct implication. Since $x \notin \mathrm{Null}(D)$, Theorem 2.4 implies that the polygon $\Delta_{(C,x)}(D)$ contains a small standard simplex for any flag (C, x).

Let $C_1, C_2 \subseteq X$ be two irreducible curves intersecting transversally at x (in particular x is a smooth point of both). Then there exists $\lambda > 0$, such that Δ_λ is contained both in $\Delta_{(C_1,x)}(D)$ and $\Delta_{(C_2,x)}(D)$. Now, for any given real number $0 < \xi < \lambda$, the point $(\xi, 0)$ is a valuative point of D with respect to both flags (C_1, x) and (C_2, x) according to Corollary 2.10, as pointed out in Remark 2.11. In particular, there exist effective $\mathbb{R}$-divisors D_1 and D_2, both numerically equivalent to D, such that $D_i = \xi C_i + D_i'$, where $x \notin \mathrm{Supp}(D_i')$, for any $i = 1, 2$. Since each C_i is smooth at x, we have $\pi^*(C_i) = \bar{C}_i + E$, where $\bar{C}_i$ is the proper transform of C_i. Set $y_i = \bar{C}_i \cap E$; as C_1 and C_2 intersect transversally at x, one has $z_1 \neq z_2$. Note also that each $\pi^*(D_i)$ contributes to the Newton–Okounkov polygon $\Delta_{(E,z)}(\pi^*(D))$. Therefore, Remark 1.8 yields that

$$\nu_{(E,y)}(\pi^*(D_i)) = \begin{cases} (\xi, 0) & \text{if } z \neq z_i, \\ (\xi, \xi) & \text{if } z = z_i \end{cases}$$

for any $i = 1, 2$. Since $z_1 \neq z_2$, we obtain that $\Delta_\xi^{-1} \subseteq \Delta_{(E,z)}(\pi^*(D))$ for any $z \in E$.

For the reverse implication, assume that there exists $\xi > 0$ with

$$\Delta_\xi^{-1} \subseteq \Delta_{(E,z)}(\pi^*(D)) \quad \text{for any } z \in E.$$

By Remark 1.7, the infinitesimal polygon $\Delta_{(E,z)}(\pi^*(D) - tE)$ contains a small simplex for any real number $0 < t < \xi$ and any $z \in E$. As a consequence, Theorem 2.4 gives that

$$\mathrm{Null}(\pi^*(D) - tE) \cap E = \varnothing \tag{3.8.4}$$

for any rational $0 < t \ll 1$.

We intend to reduce the problem to the case when $D = P_D$ is big and nef. As $(0,0) \in \Delta_{(E,z)}(\pi^*(D))$, then $x \notin \mathrm{Neg}(D)$. Thus, by Remark 1.3, we know that $\pi^*(D) = \pi^*(P_D) + \pi^*(N_D)$ is the Zariski decomposition of $\pi^*(D)$. This in turn implies that $\mathrm{Supp}(\pi^*(N_D)) \cap E = \varnothing$. Now, by Lemma 1.10, this yields that

$$\Delta_{(E,z)}(\pi^*(D)) = \Delta_{(E,z)}(\pi^*(P_D))$$

for any $z \in E$, which allows us to reduce the problem to the big and nef case.

Assume now that $D = P_D$ is big and nef; aiming at a contradiction suppose that there exists an irreducible curve $C \subseteq \mathrm{Null}(D)$ containing x. This implies that $(D \cdot C) = 0$. If we denote by $\bar{C} \stackrel{\text{def}}{=} \pi^*(C) - \mathrm{mult}_x(C)E$ the proper transform of C, then for all $0 < t \ll 1$ one has

$$\big((\pi^*(D) - tE) \cdot \bar{C}\big) = (D \cdot C) - t \cdot \mathrm{mult}_x(C) < 0.$$

The algorithm for finding Zariski decompositions then yields $\bar{C} \subseteq \mathrm{Neg}(\pi^*(D) - tE)$, hence $\bar{C} \subseteq \mathrm{Null}(\pi^*(D) - tE)$. Since $\bar{C} \cap E \neq \varnothing$, this contradicts (3.8.4), and we are done.

(3) To begin with, the length of the vertical segment $\Delta_{(E,z)}(\pi^*(D))_{t=\xi}$ is independent of $z \in E$ for any $\xi \in [0, \mu']$, by Remark 1.9. Furthermore,

$$\Delta_{(E,z)}(\pi^*(D))_{t=\xi} \subseteq \{\xi\} \times [0, \xi],$$

by Proposition 3.1. Hence whenever $\{\xi\} \times [0, \xi] \subseteq \Delta_{(E,z)}(\pi^*(D))$ for some $z \in E$, the same holds for all points of E. Since $x \notin \mathrm{Neg}(D)$, we have $(0, 0) \in \Delta_{(E,z)}(\pi^*(D))$ for any $z \in E$, via part (1) of the theorem. Therefore $\Delta_{\xi}^{-1} \subseteq \Delta_{(E,z)}(\pi^*(D))$ for all $z \in E$, assuming that the same property is known for just one point in E. Thus $\xi(\pi^*(D), z)$ does not depend on z. □

An interesting consequence of the above statement is the following criterion for a point not to be in the null locus of a big real divisor.

Corollary 3.9. *In the setting of Theorem 3.8, one has $x \notin \mathrm{Null}(D)$ if and only if there exist irreducible curves $C_1, C_2 \subseteq X$ that intersect transversally at x, and a positive number $\lambda > 0$ such that the horizontal segment $[0, \lambda] \times \{0\}$ is contained both in $\Delta_{(C_1,x)}(D)$ and $\Delta_{(C_2,x)}(D)$.*

Proof. Choose a positive real number $\lambda' \in (0, \lambda)$. Each $\Delta_{(C_i,x)}(D - \lambda' C_i)$ contains a small simplex, by Remark 1.7. By Corollary 2.10 the origin $(0, 0)$ is a valuative point in each of these polygons. Applying Remark 1.7 again, we see that the point $(\lambda', 0)$ is a valuative point in each $\Delta_{(C_i,x)}(D)$. Using this fact and the last part of the proof of the direct implication of Theorem 3.8, one deduces that the polygon $\Delta_{(E,z)}(\pi^*(D))$ contains the triangle $\Delta_{\lambda'}^{-1}$ for any $z \in E$. By Theorem 3.8, this implies that $z \notin \mathrm{Null}(D)$.

The reverse implication is an easy consequence of Theorem 2.4. □

3D. ***Moving Seshadri constants.*** It has been long known (and has been illustrated in the previous section in connection with Newton–Okounkov polygons) that many important positivity aspects can be observed infinitesimally. The philosophy dates back at least to Demailly's work [1992], where he introduces Seshadri constants in order to capture the local positivity of a divisor.

These ideas were further developed in [Nakamaye 2003], which introduced moving Seshadri constants, and then generalized to a large extent in [Ein et al. 2009]. In fact, one of the highlights of [Ein et al. 2009] is the description of the connection between augmented base loci and moving Seshadri constants.

The goal of the current subsection is to find a similar relation between infinitesimal Newton–Okounkov polygons and moving Seshadri constants.

Let D be a big and nef $\mathbb{Q}$-divisor on a smooth projective surface X, $x \in X$ a closed point. The *Seshadri constant* of D at x is defined to be the nonnegative real number

$$\epsilon(D; x) \stackrel{\mathrm{def}}{=} \inf_{x \in C \subseteq X} \left\{ \frac{(D.C)}{\mathrm{mult}_x(C)} \right\},$$

where the infimum is taken over all reduced irreducible curves $C \subseteq X$ passing through x. If $\pi : X' \to X$ denotes the blow-up of X at x, then

$$\epsilon(D; x) = \max\{\epsilon > 0 \mid \pi^*(D) - \epsilon E \text{ is nef}\}.$$

For basic properties of Seshadri constants and further references the reader is invited to consult [Lazarsfeld 2004, Section 5.1].

Moving Seshadri constants were initially introduced by Nakamaye [2003] with the purpose of encoding local positivity of the $\mathbb{Q}$-divisor D, when it is merely big. For nef divisors moving Seshadri constants agree with Seshadri constants as defined above. If $x \notin \mathrm{Null}(D)$, then the *moving Seshadri constant* of D at x is defined to be

$$\epsilon(\|D\|; x) \stackrel{\text{def}}{=} \sup_{f^*(D)=A+E} \epsilon(A, f^{-1}(x)),$$

where the supremum is taken over all birational morphisms $f : X'' \to X$ with X'' smooth which are isomorphisms over a neighborhood of x, and all decompositions $f^*(D) = A + E$, with A an ample $\mathbb{Q}$-divisor, E effective. If $x \in \mathrm{Null}(D)$, then we put $\epsilon(\|D\|, x) = 0$. This construction works well when D is a real divisor and by Theorem 6.2 from [Ein et al. 2009], the function $\epsilon(\|\cdot\|; x) : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{R}_+$ is continuous.

Suppose that D is a big $\mathbb{R}$-divisor on X and that $x \notin \mathrm{Neg}(D)$. By Theorem 3.8 one can introduce the following invariant for points $z \in E$:

$$\xi(\pi^*(D); z) \stackrel{\text{def}}{=} \sup\{\xi \geqslant 0 \mid \Delta_{\xi}^{-1} \subseteq \Delta_{(E,z)}(\pi^*(D))\}.$$

The main goal of this subsection is to prove the following theorem connecting moving Seshadri constants to infinitesimal Newton–Okounkov polygons.

Theorem 3.10. *Let D be a big $\mathbb{R}$-divisor on X. If $x \notin \mathrm{Neg}(D)$, then*

$$\epsilon(\|D\|; x) = \xi(\pi^*(D); z)$$

for any closed point $z \in E = \pi^{-1}(x)$.

Remark 3.11. Via Theorem 3.10 and Theorem 3.8, we have a very good (convex-geometric) explanation for why $\epsilon(\|D\|; x) = 0$ for a divisor D when

$$x \in \mathrm{Null}(D) \setminus \mathrm{Neg}(D).$$

Thus, one can redefine the moving Seshadri constant for a big $\mathbb{R}$-divisor D as follows

$$\epsilon(\|D\|; x) \stackrel{\text{def}}{=} \begin{cases} \xi(\pi^*(D), z) & \text{if } x \notin \mathrm{Null}(D) \setminus \mathrm{Neg}(D) \text{ and any } z \in E, \\ 0 & \text{if } x \in \mathrm{Neg}(D). \end{cases}$$

where we have set the Seshadri constant to be zero, whenever the point $x \in \mathrm{Neg}(D)$. This was done in order to have the moving Seshadri function $\epsilon(\|\cdot; x\|) : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{R}_+$ to be continuous as in Theorem 6.2 from [Ein et al. 2009].

It is also important to point out that Theorem 3.10 explains how this constant can be computed directly on the blow-up of X at x instead of taking into account all the blow-ups as we have seen in the original definition of moving Seshadri constant.

Remark 3.12. Our proof of Theorem 3.10 is self-contained in the sense that it does not make use of nontrivial material beside the content of this article. Admittedly, it could be streamlined by relying on the fact that moving Seshadri constants describe the asymptotic rate of growth of jet separation at x, but this requires the introduction of restricted volumes, and as such goes against our intentions.

First we move on to give a proof of Theorem 3.10 in the big and nef case.

Proposition 3.13. *Let P be a big and nef $\mathbb{R}$-divisor on X. Then $\epsilon(P, x) = \xi(P; x)$ for any $x \in X$.*

Proof. We verify first that $\epsilon(D; x) \leqslant \xi(D; x)$. By the definition of Seshadri constants it is enough to show that if $\pi^*(P) - tE$ is nef for all $0 \leqslant t \leqslant \epsilon$, then $\epsilon < \xi(\pi^*(P); z)$ for some $z \in E$.

Recall from Section 2A that

$$\Delta_{(E,z)}(\pi^*(P)) = \{(t, z) \in \mathbb{R}^2_+ \mid \nu \leqslant t \leqslant \mu, \alpha(t) \leqslant z \leqslant \beta(t)\},$$

where $\alpha(t) = \mathrm{ord}_z(N_t|_E)$, $\beta(t) = \alpha(t) + P_t \cdot E$, and $\pi^*(P) - tE = P_t + N_t$ is the appropriate Zariski decomposition. Note that $\pi^*(P) - tE$ is nef, and thus $N_t = 0$ for all $0 \leqslant t \leqslant \epsilon$. In particular, $\alpha(t) = 0$ and $\beta(t) = t$. Hence $\Delta^{-1}_\epsilon \subseteq \Delta_{(E,z)}(\pi^*(P))$ and consequently, $\epsilon \leqslant \xi(\pi^*(P); x)$.

For the reverse inequality, we show that if $\xi < \xi(P; x)$ then $\pi^*(P) - tE$ is nef for all $0 \leqslant t \leqslant \xi$. By Remark 1.7, then $(0, 0) \in \Delta_{(E,z)}(\pi^*(D) - tE)$ for any $t \in [0, \xi]$ and all $z \in E$. Thus, Theorem 2.4 yields

$$\mathrm{Neg}(\pi^*(P) - tE) \cap E = \varnothing \quad \text{for all } t \in [0, \xi]. \tag{3.13.5}$$

We prove that this condition forces $\pi^*(P) - tE$ to be nef. Let $\pi^*(P) - tE = P_t + \sum a_i E_i^t$ be its Zariski decomposition. Since P is nef, (3.13.5) implies that $(\pi^*(P) - tE) \cdot E_i^t \geqslant 0$ for all i. On the other hand, by the construction of Zariski decomposition we must have $(\pi^*(P) - tE) \cdot E_i^t < 0$ for some i. Thus, each $a_i = 0$ and $\pi^*(P) - tE$ is big and nef for each $t \in [0, \xi]$. This ends the proof. □

Proof of Theorem 3.10. By Proposition 3.13, it suffices to show that

$$\xi(D; x) = \sup_{f^*(D) = A + E} \{\xi(A, f^{-1}(x))\}, \quad \text{whenever } x \notin \mathrm{Neg}(D),$$

where the supremum is taken over all birational morphisms $f : X'' \to X$ with X'' smooth that are isomorphisms over a neighborhood of x, and all decompositions $f^*(D) = A + E$, with A an ample $\mathbb{R}$-divisor, E effective, and $x \notin \mathrm{Supp}(E)$.

For a given such map f it is not hard to see that $\xi(D, x) = \xi(f^*(D), f^{-1}(x))$ as a consequence of a stronger statement saying that $\mathscr{K}'(D, x) = \mathscr{K}'(f^*(D), f^{-1}(x))$, proved in Lemma 3.14 below.

Granting this, it only remains to show that if D is a big $\mathbb{R}$-divisor on X, then

$$\xi(D, x) = \sup\big\{\xi(A, x) \mid D = A + E,\ A \text{ ample},\ E \text{ effective and } x \notin \mathrm{Supp}(E)\big\}. \tag{3.13.6}$$

To this end, let $D = A + E$ be a decomposition as in (3.13.6), and let $\pi : X' \to X$ the blow-up of X at the point x. Since $x \notin \mathrm{Supp}(E)$, it follows quickly that

$$\Delta_{(E,z)}(\pi^*(A)) \subseteq \Delta_{(E,z)}(\pi^*(D))$$

for any point $z \in E$: namely, if $D' \equiv A$ is an effective $\mathbb{R}$-divisor, then $D' + E \equiv D$ is also $\mathbb{R}$-effective. Furthermore, since $x \notin \operatorname{Supp}(E)$, one has

$$\nu_{(E,z)}(\pi^*(D'+E)) = \nu_{(E,z)}(\pi^*(D')), \quad \text{for all } z \in E.$$

Using the definition of Lazarsfeld and Mustaţă for Newton–Okounkov polygons of $\mathbb{R}$-divisors, one obtains the inclusion of the polygons above. This proves the inequality "$\geqslant$" in (3.13.6).

For the inequality "$\leqslant$" in (3.13.6), let $D = P_D + N_D$ be the corresponding Zariski decomposition. Since $x \notin \operatorname{Supp}(N_D)$, then, by Remark 1.3, we know that $\pi^*(D) = \pi^*(P_D) + \pi^*(N_D)$ is the Zariski decomposition of $\pi^*(D)$. This condition also implies that $\operatorname{Supp}(\pi^*(N_D)) \cap E = \varnothing$. Thus, by Lemma 1.10, we have

$$\Delta_{(E,z)}(\pi^*(D)) = \Delta_{(E,z)}(\pi^*(P_D)) \quad \text{for any } z \in E.$$

This reduces our problem to case when $D = P$ is big and nef. However, by Lemma 1.1 there exists an effective divisor E, such that $P - \frac{1}{k}E$ is ample $\mathbb{R}$-divisor for any natural $k \gg 0$. Using this and the continuity property of the Newton–Okounkov polygons inside the big cone, the direct inequality takes places when $D = P$ is big and nef, which finishes the proof of the theorem. □

Lemma 3.14. *With notation as above*,

$$\mathscr{K}'(D, x) = \mathscr{K}'(f^*(D), f^{-1}(x)).$$

Proof. Applying the ideas from the last part of the proof of Lemma 1.10, i.e., Lemma 8 from [Anderson et al. 2014], where the classes A_n forming the limit are big and semiample, it is enough to show the statement in the case when D is a big $\mathbb{Q}$-divisor.

Now, by Zariski's main theorem pulling back sections defines the isomorphisms

$$H^0(X, \mathscr{O}_X(mD)) \simeq H^0(X'', \mathscr{O}_{X''}(mf^*(D)))$$

for all $m > 0$. As f is an isomorphism over a neighborhood of x, the computations of the infinitesimal Newton–Okounkov polygons on both sides of f can be done on two isomorphic neighborhoods containing the corresponding exceptional divisors. Thus, the two sets of polygons are equal. □

3E. ***Applications to questions about Seshadri constants.*** In this subsection we discuss some interesting applications to questions about Seshadri constants using the material above. We start with an observation regarding valuative points on the boundary of infinitesimal Newton–Okounkov polygons.

Corollary 3.15. *Let D be a big $\mathbb{Q}$-divisor on X. Fix a point $x \notin \operatorname{Null}(D)$ and suppose that*

$$\Delta_{\xi}^{-1} \subseteq \Delta_{(E,z_0)}(\pi^*(D))$$

for some $\xi > 0$ and $z_0 \in E$. Then any rational point on the diagonal segment $\{(t,t) \mid 0 \leqslant t < \xi\}$ and on the horizontal segment $[0,\xi) \times \{0\}$ is valuative.

Remark 3.16. It is somewhat surprising that the rational points on the diagonal segment are valuative. As it was pointed out in Remark 2.13, the reason is that the curve E is rational.

Proof. The statement for the horizontal line segment $(0,\xi)\times\{0\}$ can be obtained analogously as in the first part of the proof of Corollary 2.10. Furthermore, since $x\notin \mathrm{Null}(D)$, we have $x\notin \boldsymbol{B}(D)$ (where the latter is the asymptotic base locus associated to the divisor D). This implies that the origin $(0,0)$ is a valuative point.

For the points on the diagonal, let $t\in[0,\xi)$ be a rational number. Our goal is to prove that (t,t) is a valuative point. By Remark 1.7, it is enough to show that $(0,t)\in\Delta_{(E,z_0)}(\pi^*(D)-tE)$ is a valuative point for the divisor $\pi^*(D)-tE$.

By Theorem 3.10, we know that $\Delta_\xi^{-1}\subseteq\Delta_{(E,z)}(\pi^*(D))$ for all $z\in E$. Thus, $\Delta_{(E,z)}(\pi^*(D)-tE)$ contains a small simplex for any $z\in E$, if we make use of Remark 1.7. This implies via Theorem 2.4 that

$$\mathrm{Null}(\pi^*(D)-tE)\cap E=\varnothing. \tag{3.16.7}$$

In what follows we reduce the statement to the case of ample divisors. Let $\pi^*(D)-tE=P_t+N_t$ be the appropriate Zariski decomposition, and assume that all the divisors involved are integral. By (3.16.7), we know that $z_0\notin\mathrm{Supp}(N_t)$. Thus, by Lemma 1.10,

$$\Delta_{(E,z_0)}(\pi^*(D)-tE)=\Delta_{(E,z_0)}(P_t).$$

Recall that [Lazarsfeld 2004, Proposition 2.3.21] shows that the inclusion map

$$H^0\big(X',sO_{X'}(mP)\big)\to H^0\big(X',\mathscr{O}_{X'}(m(\pi^*(D)-tE))\big),$$

defined by the multiplication by the divisor mN_t, is an isomorphism. Hence, we reduced the problem to the case when $\pi^*(D)-tE=P_t$ is big and nef and the point of interest is $(0,t)=(0,(P_t.E))$.

By Remark 1.3, there exist irreducible curves $C_i\subseteq\mathrm{Null}(P_t)$ and rational numbers $\epsilon_i>0$ for $i=1,\dots,k$, such that $A_t\overset{\text{def}}{=}P_t-\sum_{i=1}^{i=k}\epsilon_iC_i$ is an ample $\mathbb{Q}$-divisor. By (3.16.7), we have that $C_i\cap E=\varnothing$ for all $i=1,\dots,k$. Thus the point $(0,t)=(0,(A_t.E))$ belongs to $\Delta_{(E,z_0)}(A_t)$, and it suffices to treat the case of ample $\mathbb{Q}$-divisors. However, this situation has already been discussed in Remark 2.13, which finishes the proof. □

As a consequence, we obtain criteria for finding lower bounds for Seshadri constants.

Corollary 3.17. *Let X be a smooth surface, $x\in X$ a point, A an ample $\mathbb{Q}$-divisor on X and $q>0$ a rational number. Then the following conditions are equivalent*:

(1) *The Seshadri constant $\epsilon(A,x)$ is $\geqslant q$.*

(2) *There exists a point $z\in E$ such that $\Delta_{(E,z)}(\pi^*(A))$ contains both the points $(q,0)$ and (q,q).*

(3) *For all $z\in E$, $(q,0)\in\Delta_{(E,z)}(\pi^*(A))$.*

(4) *For all $z\in E$, $(q,q)\in\Delta_{(E,z)}(\pi^*(A))$.*

(5) *There exists $z_1\neq z_2\in E$ such that the point $(q,q)\in\Delta_{(E,z_i)}(\pi^*(A))$ for any $i=1,2$.*

Remark 3.18. As $q \in \mathbb{Q}$, Corollary 3.15 implies that the conditions in the statement can be translated to ones about linear series on X itself. For example, the point $(q, 0)$ lies in $\Delta_{(E,z)}(\pi^*(A))$, provided there exists an effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} A$ for which $\mathrm{mult}_x(D') \geqslant q$ and the tangent direction of each branch of D' is distinct from the one defined by $z \in E$.

Similarly, the point $(q, q) \in \Delta_{(E,z)}(\pi^*(A))$ whenever there exists an effective $\mathbb{Q}$-divisor $D'' \sim_{\mathbb{Q}} A$ such that $\mathrm{mult}_x(D'') \geqslant q$ and the tangent direction of each branch of D'' is the same as the one defined by the point z.

Proof. For starters, we observe that (1) implies conditions (2)-(5) via Theorem 3.10. Therefore we are left with proving the reverse implications.

First, (2) implies (1) follows again from Theorem 3.10 and the fact that $(0, 0) \in \Delta_{(E,z)}(\pi^*(A))$, as A is ample. Notice that (4) implies (5) is immediate, and (3) implies (1) follows word by word from the second part of the proof of Proposition 3.13.

We are left to prove (5) $\Rightarrow$ (1). Fix $t \in (0, q)$ and the goal is to show that $\pi^*(A) - tE$ is nef. Let $\pi^*(A) - tE = P + N$ be the corresponding Zariski decomposition. Since both points $(0, 0)$ and (q, q) are contained in $\Delta_{(E,z_i)}(\pi^*(A))$ for any $i = 1, 2$, by convexity the point (t, t) is also contained in these polygons. By the formula for Newton–Okounkov polygons from Section 2A, this implies that

$$\mathrm{ord}_{z_i}(N|_E) + (P.E) = t \quad \text{for any } i = 1, 2.$$

On the other hand, $(\pi^*(A) - tE).E = (P + N).E = t$. In particular,

$$(N \cdot E) = \mathrm{ord}_{z_1}(N|_E) = \mathrm{ord}_{z_2}(N|_E).$$

By the same token as in Remark 1.8, the first equality implies that the effective divisors N and E intersect only at z_1, while the second one implies that they intersect only at z_2. Since $z_1 \neq z_2$, necessarily $N = 0$, i.e., $\pi^*(A) - tE$ is nef. $\square$

4. Applications

We present some applications to questions regarding Seshadri constants seen through the lenses of the theory of Newton–Okounkov polygons developed in the previous sections.

First, we give a new proof of a lower bound for very generic points by Ein and Lazarsfeld that relied originally on deformation theory; our argumentation is based on earlier work of Nakamaye. Second, based on Theorem 2.4, we introduce a new invariant that encodes the size of the largest simplex that can be included in some Newton–Okounkov polygon of a given divisor by varying the curve flag. We connect this invariant to the Seshadri constant. Lastly, using Diophantine approximation, we show that whenever the surface has a rational polyhedral nef cone, the global Seshadri constant at any point is strictly positive.

4A. ***Generic infinitesimal Newton–Okounkov polygon.*** Let A be an ample Cartier divisor on a smooth projective surface X. Ein and Lazarsfeld proved [1993] that $\epsilon(A, x) \geqslant 1$ for very general point $x \in X$. Later, Cascini and Nakamaye [2014], gave a different proof avoiding deformation theory based on ideas developed previously by Nakamaye [2005]. Here we translate the line of thought of Cascini and Nakamaye to the language of infinitesimal Newton–Okounkov polygons.

The main extra ingredient is the following observation of Nakamaye (see [Nakamaye 2005, Lemma 1.3]). As he points out, the result is an easy consequence of a statement about the smoothing divisors in families as seen in [Lazarsfeld 2004, Proposition 5.2.13]. This claim initially appears in [Nakamaye 2005], and it is used both in [Nakamaye 2005] and [Cascini and Nakamaye 2014] to establish lower bounds on Seshadri constants in higher dimensions.

Lemma 4.1. *Let $x \in X$ be a very general point and D be an effective integral divisor on X. Suppose $W \subseteq X$ is an irreducible curve passing through x. Let $\overline{W}$ be the proper transform of W through the blow-up $\pi : X' \to X$ of the point x. Also, define*

$$\alpha(W) = \inf_{\beta \in \mathbb{Q}} \{\overline{W} \subseteq \mathrm{Null}(\pi^*(D) - \beta E)\}.$$

Then $\mathrm{mult}_{\overline{W}}(\|\pi^*(D) - \beta E\|) \geqslant \beta - \alpha(W)$ *for all* $\beta \geqslant \alpha(W)$.

Lemma 4.1 forces the generic infinitesimal Newton–Okounkov polygon of very generic points to land in a certain area of the plane containing it, depending on the Seshadri constant.

Proposition 4.2. *Let A be an ample Cartier divisor on X and let $x \in X$ be a very general point. Then the following mutually exclusive cases can occur*:

(1) $\mu'(A,x) = \epsilon(A,x)$: *then* $\Delta(A,x) = \Delta^{-1}_{\epsilon(A,x)}$.

(2) $\mu'(A,x) > \epsilon(A,x)$: *then there exists an irreducible curve $C \subseteq X$ with $(A \cdot C) = p$ and* $\mathrm{mult}_x(C) = q$ *such that $\epsilon(A,x) = p/q$. Under these circumstances,*

  (a) *whenever $q \geqslant 2$, $\Delta(A,x) \subseteq \triangle_{ODB}$, where $O = (0,0)$, $D = (p/q, p/q)$ and $B = (p/(q-1), 0)$;*

  (b) *whenever $q = 1$, the polygon $\Delta(A,x)$ is contained in the area below the line $y = t$ and between the horizontal lines $y = 0$ and $y = \epsilon(A,x)$.*

Corollary 4.3. *Let A be an ample line bundle on a smooth projective surface. Then $\epsilon(A,x) \geqslant 1$ for very generic points $x \in X$.*

Proof. By the definition of Seshadri constants and Proposition 4.2, it suffices to consider the case $(2a)$. Thus, we know that $\Delta(A;x) \subseteq \triangle_{ODB}$, and as a consequence

$$\mathrm{area}(\Delta(A;x)) = \frac{A^2}{2} \leqslant \mathrm{area}(ODB) = \frac{p^2}{2q(q-1)}.$$

In particular, $\epsilon(A,x) \geqslant \sqrt{(A^2)(1-1/q)}$. Hence, if we assume $\epsilon(A,x) < 1$, then by the rationality of $\epsilon(A;x)$, we also have $\epsilon(A,x) \leqslant (q-1)/q$. Using the inequality between the areas, we arrive at $(A^2) < 1$, which stands in contradiction with the assumption that A is an ample Cartier divisor. □

Proof of Proposition 4.2. If $\epsilon(A,x) = \mu'(A,x)$, then automatically $\Delta = \Delta^{-1}_{\mu'(A,x)}$. Therefore we can assume without loss of generality that $\mu'(A,x) > \epsilon(A,x)$. In particular, there exists a curve $C \subseteq X$ with $(A \cdot C) = p$ and $\mathrm{mult}_x(C) = q$ such that $\epsilon(A,x) = p/q$.

Let $\bar{C}$ be the proper transform of C on X'. The idea of the proof is to calculate the length of the vertical segment in the polygon $\Delta(A, x)$ at $t = t_0$ for any $t_0 \geqslant \epsilon(A, x)$:

$$\text{length}(\Delta(A)_{t=t_0}) = (P_{t_0} \cdot E) = t_0 - (N_{t_0} \cdot E),$$

where $\pi^*(A) - t_0 E = P_{t_0} + N_{t_0}$ is the corresponding Zariski decomposition. By Lemma 4.1, one can write $N_{t_0} = (t_0 - \epsilon(A, x))\bar{C} + N'_{t_0}$, where N'_{t_0} remains effective. This implies the following inequality

$$\text{length}(\Delta(A)_{t=t_0}) = t_0 - \big((t_0 - \epsilon(A, x))\bar{C} + N'_{t_0} \cdot E\big) \leqslant t_0 - (t_0 - \epsilon(A, x))q. \tag{4.3.8}$$

By Proposition 3.1, the vertical line segment $\Delta(A; x)_{t=t_0}$ starts on the t-axis at the point $(t_0, 0)$ for any $t_0 \geqslant 0$. Therefore, by (4.3.8), the polygon sits below the line

$$y = t_0 - (t_0 - \epsilon(A, x))q = (1 - q)t_0 + \epsilon(A, x)q.$$

When $q = 1$, this line is the horizontal line $y = \epsilon(A, x)$ and when $q \geqslant 2$, it is the line passing through the points $D = (p/q, p/q)$ and $B = (p/(q-1), 0)$. □

We conclude this subsection with a lower bound on Seshadri constants of quintic surfaces. While the bound might be known to experts, we include it here since it is an illustration of the use of infinitesimal Newton–Okounkov bodies.

Example 4.4. Inspired by the work of Nakamaye, we show that for any smooth quintic surface $X \subseteq \mathbb{P}^3$, if A is the line bundle defining the embedding, then we have $\epsilon(A; x) \geqslant 2$ for a very generic point $x \in X$.

The main ingredient is Proposition 4.2; suppose that $\epsilon(A; x) < 2$. Then there exists an irreducible curve $C \subseteq X$ containing the point x such that $\text{mult}_x(C) = q$, $(A \cdot C) = p$ and $\epsilon(A; x) = p/q$.

If $q = 1$, then $p = 1$ as well, and this implies that through a very general point of X there passes a line. This forces X to be uniruled, which is not the case for quintic surfaces.

Thus we can assume $q \geqslant 2$. Then the generic infinitesimal Newton–Okounkov polygon $\Delta(A, x)$ is contained in the triangle $\triangle_{ODB}$, where $O = (0, 0)$, $D = (p/q, p/q)$, and $B = (p/(q-1), 0)$ by Proposition 4.2. As a consequence we have the following inequality

$$\text{area}(\triangle_{ODB}) = \frac{p}{q} \cdot \frac{p}{q-1} \geqslant \text{area}(\Delta(A, x)) = 5.$$

If $q \geqslant 5$, then this yields $p \geqslant 2q$, which contradicts our initial assumption that $\epsilon(A; x) < 2$. On the other hand, if $2 \leqslant q \leqslant 4$, then we find that the list of remaining choices to tackle is

$$\frac{p}{q} = \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4},$$

since $p < 2q$. However, none of these pairs satisfy the area inequality above, hence we are done.

The same line of thought implies that whenever $\text{Pic}(X) = \mathbb{Z}A$, then $\epsilon(A; x) = 2$ for a very generic point $x \in X$ if and only if there is a curve $C \in |2A|$ with the property that $\text{mult}_x(C) = 5$.

4B. ***The largest simplex constant.*** It was established in Section 3 that all Newton–Okounkov polygons of ample line bundles contain a standard simplex of some size that depends on the choice of the flag. If the curve in the flag is chosen to be very positive, the sizes of these standard simplices can become arbitrarily small. Thus, the exciting question to ask is how large they can become.

Definition 4.5 (largest simplex constant). Let A be an ample $\mathbb{Q}$-divisor on X and let (C, x) be an admissible flag. We define

$$\lambda(A; C, x) \stackrel{\text{def}}{=} \sup\{\lambda \geqslant 0 \mid \Delta_\lambda \subseteq \Delta_{(C,x)}(A)\}.$$

The *largest simplex constant* of A at the point x is defined to be

$$\lambda(A; x) \stackrel{\text{def}}{=} \sup\bigl\{\lambda(A; C, x) \mid C \subseteq X \text{ is an irreducible curve that is smooth at } x\bigr\}.$$

Remark 4.6. Not unexpectedly, one can define the largest simplex constant for big divisors in general, assuming that the point x is not contained in the null locus of the divisor. All formal properties of $\lambda(A; x)$ go through almost verbatim, hence the details are left to the (interested) reader.

The goal of this subsection is to relate the largest simplex constant to Seshadri constants.

Proposition 4.7. *With the notation as above*, $\epsilon(A; x) \geqslant \lambda(A; x)$.

Remark 4.8. Proposition 4.7 implies that there is no uniform lower bound on the largest simplex constant holding at every point of every surface. This follows from the nonexistence of the analogous bound for Seshadri constants as seen in Miranda's example in [Lazarsfeld 2004, Example 5.2.1].

Remark 4.9. It is a natural question after Proposition 4.7 whether there are examples with $\lambda(A; x) \neq \epsilon(A; x)$. One such example is Mumford's fake projective plane (for the actual construction see [Mumford 1979]).

The surface X is of general type with ample canonical class K_X, $(K_X^2) = 9$, and geometric genus $p_g = H^0(X, \mathscr{O}_X(K_X)) = 0$. Since $\operatorname{Pic}(X) = \mathbb{Z}H$, these conditions imply that $H^0(X, \mathscr{O}_X(H)) = 0$. This means that whenever (C, x) is an admissible flag, we have $C \in |dH|$ with $d \geqslant 2$, hence clearly $\lambda(H, x) \leqslant \frac{1}{2}$ for any $x \in X$. On the other hand, we know by Corollary 4.3 that $\epsilon(H, x) \geqslant 1$ when $x \in X$ is a very general point.

Proof of Proposition 4.7. Theorem 2.4 yields that $\lambda \stackrel{\text{def}}{=} \lambda(A; C; x) > 0$ for any admissible flag (C, x). By fixing the flag (C, x), it is enough to show that $\epsilon(A, x) \geqslant \lambda$.

By Corollary 3.15 there exist sequences of real numbers ϵ_n^v and ϵ_n^h with both $\lambda - \epsilon_n^v$ and $\lambda - \epsilon_n^h$ rational, and sequences of effective $\mathbb{Q}$-divisors (D_n^v) and (D_n^h) with $D_n^v, D_n^h =_{\text{num}} D$ for any $n \in \mathbb{N}$ such that

$$\nu_{(C,x)}(D_n^v) = (0, \lambda - \epsilon_n^v) \quad \text{and} \quad \nu_{(C,x)}(D_n^h) = (\lambda - \epsilon_n^h, 0).$$

This yields $C \nsubseteq \operatorname{Supp}(D_n^v)$ for any $n \in \mathbb{N}$, and by Remark 1.8, we obtain that

$$(D \cdot C) = \frac{(D_n^v \cdot C)}{\operatorname{mult}_x(C)} \geqslant \lambda - \epsilon_n^v, \tag{4.9.9}$$

where we took into account that C is smooth at x. By looking at the valuation vector of D_n^h, we can write $D_n^h = (\lambda - \epsilon_n^h)C + N_n$, where N_n is effective and $\mathrm{mult}_x(N_n) = 0$. Thus, for any irreducible curve $F \neq C$ passing through x, we have that $F \not\subseteq \mathrm{Supp}(D_n^h)$. As a consequence we get the following string of inequalities

$$\frac{(D \cdot F)}{\mathrm{mult}_x(F)} = \frac{(D_n^h \cdot F)}{\mathrm{mult}_x(F)} \geqslant \frac{(\lambda - \epsilon_n^h)(C \cdot F)}{\mathrm{mult}_x(F)} \geqslant \lambda - \epsilon_n^h, \tag{4.9.10}$$

where the last inequality follows from the fact that $(C \cdot F) \geqslant \mathrm{mult}_x(F) \cdot \mathrm{mult}_x(C)$ whenever $F \neq C$.

Observing the definition of Seshadri constants, and taking the limit in both equations (4.9.9) and (4.9.10), we arrive at $\epsilon(D; x) \geqslant \lambda$, as required. □

4C. ***Diophantine approximation.*** Here we show via Diophantine approximation that the largest simplex constant of a surface is strictly positive whenever it has a rational polyhedral nef cone. It is important to note that the semigroup of ample line bundles of X is not necessarily finitely generated even if the nef cone is rational polyhedral: the lattice semigroup $\mathbb{N}^2 \cap \mathbb{R}^2_{>0}$ is one such example. Furthermore, the line bundles sitting on the boundary of the nef cone might not even have sections asymptotically as seen in examples provided in [Ottem 2015].

It was Nadel who first stressed the relevance of Diophantine approximation to local positivity issues (see [Ein et al. 1995]). This train of thought was further explored by Nakamaye. Very recently a deep connection between Diophantine approximation and Seshadri constants was established by McKinnon and Roth [2015].

Theorem 4.10. *Let X be an irreducible projective variety with a rational polyhedral nef cone. Then there exists a natural number $m > 0$ such that the linear series $|mA|$ is base-point free for any ample Cartier divisor A on X.*

Remark 4.11. Theorem C follows easily as a consequence of Theorem 4.10 and Proposition 4.7. Furthermore, the above theorem implies that whenever X is a smooth projective variety with a rational polyhedral nef cone, then there exists a strictly positive constant $\epsilon(X) > 0$ such that $\epsilon(A; x) \geqslant \epsilon(X)$ for any $x \in X$ and any ample Cartier divisor A on X. This is due to the fact that whenever B is an ample and base-point free divisor, then $\epsilon(B; x) \geqslant 1$ for any $x \in X$.

We will need the following statement during the proof.

Lemma 4.12 [Fujita 1983, Corollary 3]. *Let X be an irreducible projective variety. Then there exists a Cartier divisor B such that the divisor $B + P$ is base-point free for any nef Cartier divisor P on X.*

Remark 4.13. When X is a smooth projective variety, one can be more specific about the divisor B. By the Anghern–Siu theorem, the divisor $K_X + n(n+1)/2A + A + P$ defines a base-point free linear series for any ample A and nef P. Thus, B can be taken to be $K_X + n(n+1)/2A + A$ for instance.

In the general case, assuming that one does not need a specific B, then one obtains Lemma 4.12 by making use of Fujita's vanishing theorem and Castelnuovo–Mumford regularity as in [Fujita 1983] or [Lazarsfeld 2004, Theorem 2.3.9].

Proof of Theorem 4.10. Note first that in the language of cones, Lemma 4.12 says that there exists a nef divisor B so that any Cartier divisor whose class lands in the pointed cone $B+\mathrm{Nef}(X)_{\mathbb{R}}$, defines a base-point free linear series. In particular, the statement follows provided we can prove that there exists a constant $m>0$ such that $mA \in B+\mathrm{Nef}(X)_{\mathbb{R}}$ for any ample Cartier divisor A on X.

This reduces the problem to a question about convex cones. So, consider a bijective linear map $f : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathbb{R}^\rho$ whose matrix has integral entries, i.e., $f(L) \in \mathbb{Z}^\rho \subseteq \mathbb{R}^\rho$ for any class L given by some Cartier divisor on X. Let $\mathscr{C} = f(\mathrm{Nef}(X)_{\mathbb{R}})$ and $b = f(B)$. Then it suffices to check that there exists a natural number $m>0$ having the property that $m\xi \in b+\mathscr{C}$ for any $\xi \in \mathrm{int}(\mathscr{C}) \cap \mathbb{Z}^\rho$.

Let $H \subseteq \mathbb{R}^\rho$ be a hyperplane given by an equation with integral coefficients, that is, we assume that there exists a vector $u \in \mathbb{Z}^\rho$ such that $H = \{x \in \mathbb{R}^\rho \mid <x, u> = 0\}$. If $P \notin H$, then the distance from P to the hyperplane H is given by the formula

$$\mathrm{distance}(P, H) = \frac{|\langle P, u\rangle|}{\|u\|}.$$

If we ask for $P \in \mathbb{Z}^\rho$, then $|\langle P, u\rangle| \geqslant 1$, and in particular, $\mathrm{distance}(P, H) \geqslant 1/\|u\|$. This yields that there exists a constant $c>0$ such that $\mathrm{distance}(P, H) \geqslant c$ for any integral point $P \notin H$.

Going back to our setup, the conditions in the statement imply that the cone $\mathscr{C} \subseteq \mathbb{R}^\rho$ is rational polyhedral, i.e., the support hyperplanes for each face are given by an equation with integral coefficients. Thus there exists a constant $c>0$ such that

$$\mathrm{distance}(P, \partial\mathscr{C}) \geqslant c \quad \text{for any point } P \in \mathrm{int}(\mathscr{C}) \cap \mathbb{Z}^\rho,$$

where $\partial\mathscr{C}$ denotes the boundary in $\mathbb{R}^\rho$ of the cone $\mathscr{C}$.

Pick $P \in \mathrm{int}(\mathscr{C})$, and let Λ be the plane determined by b, P and the origin $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^\rho$. Let $\mathscr{C}_\Lambda = \mathscr{C} \cap \Lambda$. This is a cone in $\mathbb{R}^2$, thus it is generated by two rays $\mathbb{R}_+ l_1$ and $\mathbb{R}_+ l_2$, where both l_1 and l_2 can be taken to be rational vectors since the boundary $\partial\mathscr{C}$ is supported by rational hyperplanes, and the plane Λ is also defined by an equation with rational coefficients.

Furthermore, the set $(b+\mathscr{C}) \cap \Lambda$ is the cone $\mathbb{R}_+ l_1 + \mathbb{R}_+ l_2$ shifted by b. Without loss of generality, suppose that the ray $\mathbb{R}_+ OP$ intersects first the half line $b + \mathbb{R}_+ l_1$ at point D. Then, taking into account what was said above, it is enough to find $C>0$, that does not depend on the choice of the point P, so that the quotient $\|OD\|/\|OP\| < C$. Using similar triangles arguments, one has

$$\frac{\|OD\|}{\|OP\|} = \frac{\mathrm{distance}(D, l_1)}{\mathrm{distance}(P, l_1)} \leqslant \frac{\|OB\|}{c},$$

where the latter inequality follows from the Diophantine approximation statement we proved above. $\square$

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The mean value of symmetric square L-functions

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We study the first moment of symmetric-square L-functions at the critical point in the weight aspect. Asymptotics with the best known error term $O(k^{-1/2})$ were obtained independently by Fomenko in 2003 and by Sun in 2013. We prove that there is an extra main term of size $k^{-1/2}$ in the asymptotic formula and show that the remainder term decays exponentially in k. The twisted first moment was evaluated asymptotically by Ng with the error bounded by $lk^{-1/2+\epsilon}$. We improve the error bound to $l^{5/6+\epsilon}k^{-1/2+\epsilon}$ unconditionally and to $l^{1/2+\epsilon}k^{-1/2}$ under the Lindelöf hypothesis for quadratic Dirichlet L-functions.

1. Introduction

Asymptotic behavior of high moments of L-functions within different families can be predicted using random matrix theory [Conrey et al. 2005] or multiple Dirichlet series [Diaconu et al. 2003]. However, obtaining asymptotic formulas with sharp error bounds is a hard problem even in the case of small moments.

One of the most challenging families is symmetric square L-functions in weight aspect. Gelbart and Jacquet [1978] proved that these are L-function attached to GL(3) cusp forms.

Despite numerous efforts, even an upper bound for the second moment of symmetric square L-functions remains an open problem. See [Khan 2010, Conjecture 1.2].

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The first moment has been studied intensively during the last decades. See [Fomenko 2003; Khan 2007; Kohnen and Sengupta 2002; Lau 2002; Ng 2016; 2017; Sun 2013]. Nevertheless, even the best known asymptotic error estimates do not appear to be sharp.

The present paper aims to optimize error bounds in existing asymptotic formulas. With this goal, we prove an exact formula for the twisted first moment of symmetric square L-functions, and apply the Liouville–Green method (also called WKB approximation) to estimate remainder terms. This technique, originating from the theory of approximation of second-order differential equations, is quite unusual for analytic number theory, yet very effective. See, for example, [Balkanova and Frolenkov 2016; Zavorotny 1989].

2. Main results

Let $S_{2k}(1)$ denote the space of holomorphic cusp forms of weight $2k \geq 2$ with respect to the full modular group. Denote by H_{2k} the normalized Hecke basis for $S_{2k}(1)$. Every $f \in H_{2k}$ has a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{k-1/2} \exp(2\pi i n z), \tag{2-1}$$

$$\lambda_f(1) = 1. \tag{2-2}$$

For $\Re s > 1$ the associated symmetric square L-function is given by

$$L(\text{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}. \tag{2-3}$$

Let $\Gamma(s)$ be the Gamma function and define

$$L_\infty(s) := \pi^{-3s/2} \Gamma\Big(\frac{s+1}{2}\Big) \Gamma\Big(\frac{s-1}{2} + k\Big) \Gamma\Big(\frac{s}{2} + k\Big). \tag{2-4}$$

Shimura [1975] showed that the completed L-function

$$\Lambda(\text{sym}^2 f, s) := L_\infty(s) L(\text{sym}^2 f, s)$$

is entire and satisfies the functional equation

$$\Lambda(\text{sym}^2 f, s) = \Lambda(\text{sym}^2 f, 1-s). \tag{2-5}$$

Consider

$$M_1(l, s) := \sum_{f\in H_{2k}}^{h} \lambda_f(l^2) L(\text{sym}^2 f, s). \tag{2-6}$$

The superscript h in the formula above indicates that the expression in the sum is multiplied by the harmonic weight $\Gamma(2k-1)/((4\pi)^{2k-1}\langle f, f\rangle_1)$, where $\langle f, f\rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

Denote by γ the Euler constant and by $\psi(s)$ the logarithmic derivative of the Gamma function. Let ${}_2F_1(a,b,c;x)$ be the Gauss hypergeometric function and

$$\Phi_k(x):=\frac{\Gamma\left(k-\frac{1}{4}\right)\Gamma\left(\frac{3}{4}-k\right)}{\Gamma\left(\frac{1}{2}\right)}{}_2F_1\left(k-\tfrac{1}{4},\tfrac{3}{4}-k,\tfrac{1}{2};x\right), \tag{2-7}$$

$$\Psi_k(x):=x^k\frac{\Gamma\left(k-\frac{1}{4}\right)\Gamma\left(k+\frac{1}{4}\right)}{\Gamma(2k)}{}_2F_1\left(k-\tfrac{1}{4},k+\tfrac{1}{4},2k;x\right). \tag{2-8}$$

We prove the following exact formula for the twisted first moment.

Theorem 2.1. *For any $l\geq 1$ one has*

$$\begin{aligned}M_1\left(l,\frac{1}{2}\right)=\frac{1}{2\sqrt{l}}\left(-2\log l-3\log 2\pi+\frac{\pi}{2}+3\gamma+\psi\left(k-\frac{1}{4}\right)+\psi\left(k+\frac{1}{4}\right)\right)\\+\frac{\sqrt{2\pi}(-1)^k}{2\sqrt{l}}\frac{\Gamma\left(k-\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)}\mathscr{L}_{-4l^2}\left(\frac{1}{2}\right)+\frac{1}{\sqrt{l}}\sum_{1\leq n<2l}\mathscr{L}_{n^2-4l^2}\left(\frac{1}{2}\right)\Phi_k\left(\frac{n^2}{4l^2}\right)\\+\frac{1}{l\sqrt{2}}\sum_{n>2l}\mathscr{L}_{n^2-4l^2}\left(\frac{1}{2}\right)\sqrt{n}\Psi_k\left(\frac{4l^2}{n^2}\right),\end{aligned} \tag{2-9}$$

where

$$\mathscr{L}_n(s)=\frac{\zeta(2s)}{\zeta(s)}\sum_{q=1}^{\infty}\frac{1}{q^s}\Bigl(\sum_{\substack{1\leq r\leq 2q\\ r^2\equiv n \pmod{4q}}}1\Bigr). \tag{2-10}$$

A similar formula, where the last two summands are expressed in terms of the Legendre function of the first kind, was established by a different method by Zagier [1977, Theorem 1]. Zagier's formula was applied by Kohnen and Sengupta [2002] to prove an upper bound for $M_1\left(1,\frac{1}{2}\right)$, by Fomenko [2003] to obtain an asymptotic formula for $M_1\left(1,\frac{1}{2}\right)$, and by Luo [2012] to estimate the second moment of $L\left(\mathrm{sym}^2 f,\frac{1}{2}\right)$ over short intervals.

The proof of Theorem 2.1 is quite simple and makes use of Petersson's trace formula and the functional equation for the Lerch zeta function.

When $l=1$, exact formula (2-9) allows one to isolate the second main term of size $k^{-1/2}$ in the asymptotic formula so that the remainder term decays exponentially.

Corollary 2.2. *For some $c>0$ one has*

$$\begin{aligned}M_1\left(1,\frac{1}{2}\right)=\frac{1}{2}\left(\frac{\pi}{2}-3\log 2\pi+3\gamma+\psi\left(k-\frac{1}{4}\right)+\psi\left(k+\frac{1}{4}\right)\right)+\frac{\sqrt{2\pi}(-1)^k}{2}\frac{\Gamma\left(k-\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)}L\left(\frac{1}{2},\chi_{-4}\right)\\+\Phi_k\left(\frac{1}{4}\right)L\left(\frac{1}{2},\chi_{-3}\right)+O\left(\frac{1}{\sqrt{k}}\exp(-ck)\right),\end{aligned} \tag{2-11}$$

where $L\left(\frac{1}{2},\chi_D\right)$ is a Dirichlet L-function for the primitive quadratic character of conductor D and

$$\Phi_k\left(\frac{1}{4}\right)=-\frac{2^{3/2}\sqrt{\pi}}{3^{1/4}}\frac{1}{\sqrt{2k-1}}\sin\frac{\pi(2k-1)}{3}\left(1+O\left(\frac{1}{k}\right)\right). \tag{2-12}$$

Remark. After posting the first version of this paper to the arXiv, the authors have been informed by Shenhui Liu that he has independently obtained an asymptotic formula similar to (2-11) by using an approximate functional equation. See [Liu 2017].

Corollary 2.2 improves the series of previously known results with the following error bounds:

- $k^{-0.008}$ [Lau 2002]
- $k^{-1/20}$ [Khan 2007]
- $k^{-1/2}$ [Fomenko 2003; Sun 2013].

Corollary 2.3. *For any* $\epsilon > 0,\ l > 1$, *one has*

$$M_1\Big(l, \frac{1}{2}\Big) = \frac{1}{2\sqrt{l}}\Big(-2\log l - 3\log 2\pi + \frac{\pi}{2} + 4\gamma + \psi(1) + \psi\Big(k-\frac{1}{4}\Big) + \psi\Big(k+\frac{1}{4}\Big)\Big) + O\Big(\frac{l^{5/6+\epsilon}}{\sqrt{k}}\Big). \quad (2\text{-}13)$$

Assuming the Lindelöf hypothesis for quadratic Dirichlet L-functions, the error term above can be replaced by $O(l^{1/2+\epsilon}k^{-1/2})$.

This improves the error bound $lk^{-1/2+\epsilon}$ proved by Ng (see [Ng 2016, Theorem 2.1.1] and [Ng 2017]).

3. Notation and tools

Let $e(x) = \exp(2\pi i x)$. For $v \in \mathbb{C}$ let

$$\tau_v(n) = \sum_{n_1 n_2 = n} \Big(\frac{n_1}{n_2}\Big)^v. \tag{3-1}$$

The classical Kloosterman sum is defined by

$$S(n, m; c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} e\Big(\frac{an + a^* m}{c}\Big), \qquad aa^* \equiv 1 \pmod c.$$

Lemma 3.1 (Weil's bound [1948]). *One has*

$$|S(m, n; c)| \le \tau_0(c)\sqrt{(m, n, c)}\sqrt{c}. \tag{3-2}$$

Let $J_v(x)$ be the Bessel function of the first kind.

Lemma 3.2 (Petersson's trace formula [1932]). *For* $2k \ge 12$ *and integral* $l, n \ge 1$, *one has*

$$\sum_{f \in H_{2k}}^{h} \lambda_f(l)\lambda_f(n) = \delta_{l,n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(l, n; c)}{c} J_{2k-1}\Big(\frac{4\pi\sqrt{ln}}{c}\Big). \tag{3-3}$$

The Lerch zeta function

$$\zeta(\alpha, \beta, s) = \sum_{n+\alpha>0} \frac{e(n\beta)}{(n+\alpha)^s} \tag{3-4}$$

was introduced by Lipschitz [1857] and was named after Lerch, who proved in 1887 the functional equation. The special case $\zeta(\alpha, 0, s)$ is called the Hurwitz zeta function.

Lemma 3.3 [Lerch 1887]. *One has*

$$\zeta(\alpha, 0, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big(-ie\Big(\frac{s}{4}\Big)\zeta(0, \alpha, 1-s) + ie\Big(-\frac{s}{4}\Big)\zeta(0, -\alpha, 1-s)\Big). \tag{3-5}$$

4. Some properties of $\mathscr{L}_n(s)$

The main references for this section are [Bykovskiĭ 1994; Soundararajan and Young 2013; Zagier 1977]. Function (2-10) can be written as follows

$$\mathscr{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)}\sum_{q=1}^{\infty}\frac{\rho_q(n)}{q^s} = \sum_{q=1}^{\infty}\frac{\lambda_q(n)}{q^s}, \tag{4-1}$$

where

$$\rho_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}, \tag{4-2}$$

$$\lambda_q(n) := \sum_{q_1^2 q_2 q_3 = q} \mu(q_2)\rho_{q_3}(n). \tag{4-3}$$

For a fixed n, both $\rho_q(n)$ and $\lambda_q(n)$ are multiplicative functions of q. Furthermore, for $n \equiv 2, 3 \pmod 4$ the function $\rho_q(n)$ is identically zero. Therefore, $\mathscr{L}_n(s)$ does not vanish only for $n \equiv 0, 1 \pmod 4$. If $n = 0$ then

$$\mathscr{L}_n(s) = \zeta(2s-1). \tag{4-4}$$

Otherwise, for $n = Dl^2$ with D fundamental discriminant we have

$$\mathscr{L}_n(s) = l^{1/2-s}T_l^{(D)}(s)L(s, \chi_D), \tag{4-5}$$

where $L(s, \chi_D)$ is a Dirichlet L-function for primitive quadratic character χ_D and

$$T_l^{(D)}(s) = \sum_{l_1 l_2 = l}\chi_D(l_1)\frac{\mu(l_1)}{\sqrt{l_1}}\tau_{s-1/2}(l_2). \tag{4-6}$$

The completed L-function

$$\mathscr{L}_n^*(s) = \Big(\frac{\pi}{|n|}\Big)^{-s/2}\Gamma\Big(\frac{s}{2} + \frac{1}{4} - \frac{\operatorname{sgn} n}{4}\Big)\mathscr{L}_n(s) \tag{4-7}$$

satisfies the functional equation

$$\mathscr{L}_n^*(s) = \mathscr{L}_n^*(1-s). \tag{4-8}$$

Lemma 4.1. *One has*

$$\sum_{q=1}^{\infty}\frac{1}{q^{1+s}}\sum_{c \pmod q} S(l^2, c^2; q)e\Big(\frac{nc}{q}\Big) = \frac{1}{\zeta(2s)}\mathscr{L}_{n^2-4l^2}(s). \tag{4-9}$$

Proof. Consider

$$S := \sum_{c \pmod q} S(l^2, c^2; q) e\Big(n\frac{c}{q}\Big) = \sum_{c \pmod q} \sum_{\substack{a \pmod q \\ (a,q)=1}} e\Big(\frac{ac^2+a^*l^2+nc}{q}\Big),$$

where $aa^* \equiv 1 \pmod q$. Making the change of variables $c = c_1 a^*$, we have

$$\begin{aligned} S &= \sum_{\substack{a \pmod q \\ (a,q)=1}} \sum_{c_1 \pmod q} e\Big(\frac{c_1^2 a^* + l^2 a^* + nc_1 a^*}{q}\Big) \\ &= \sum_{c_1 \pmod q} S(0, c_1^2 + l^2 + nc_1; q) = \sum_{c \pmod q} \sum_{\substack{bd=q \\ b \mid c^2+l^2+nc}} \mu(d) b \\ &= \sum_{bd=q} \mu(d) b \sum_{\substack{c \pmod q \\ c^2+l^2+nc \equiv 0 \pmod b}} 1 = \sum_{bd=q} \mu(d) b \sum_{\substack{c \pmod b \\ c^2+l^2+nc \equiv 0 \pmod b}} \frac{q}{b}. \end{aligned}$$

The condition $c^2 + l^2 + nc \equiv 0 \pmod b$ is equivalent to $(2c+n)^2 + 4l^2 - n^2 \equiv 0 \pmod{4b}$. Hence

$$S = q \sum_{bd=q} \mu(d) \sum_{\substack{c \pmod{2b} \\ c^2 \equiv n^2 - 4l^2 \pmod{4b}}} 1 = q \sum_{bd=q} \mu(d) \rho_b(n^2 - 4l^2).$$

Consequently,

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod q} S(l^2, c^2; q) e\Big(\frac{nc}{q}\Big) &= \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{bd=q} \mu(d) \rho_b(n^2 - 4l^2) \\ &= \sum_{b=1}^{\infty} \frac{\rho_b(n^2-4l^2)}{b^s} \sum_{q=1}^{\infty} \frac{\mu(q)}{q^s} \\ &= \frac{1}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n^2-4l^2)}{q^s} = \frac{\mathscr{L}_{n^2-4l^2}(s)}{\zeta(2s)}. \end{aligned}$$ □

Lemma 4.2. *Assume that $d \neq 0$. For any $\epsilon > 0$, one has*

$$\mathscr{L}_d\big(\tfrac{1}{2}\big) \ll d^{1/6+\epsilon}. \tag{4-10}$$

If the Lindelöf hypothesis for Dirichlet L-functions is true, then

$$\mathscr{L}_d\big(\tfrac{1}{2}\big) \ll d^{\epsilon}. \tag{4-11}$$

Proof. For $d = Dl^2$, where D is the fundamental discriminant, one has

$$\mathscr{L}_d\big(\tfrac{1}{2}\big) = T_l^{(D)}\big(\tfrac{1}{2}\big) L\big(\tfrac{1}{2}, \chi_D\big)$$

by equation (4-5). It follows from equality (4-6) that

$$T_l^{(D)}\Big(\frac{1}{2}\Big) \ll \sum_{l_1 l_2 = l} \frac{\sigma(l_2)}{\sqrt{l_1}} \ll \sum_{l_1 | l} \frac{(l/l_1)^{\epsilon}}{\sqrt{l_1}} \ll l^{\epsilon}.$$

By [Conrey and Iwaniec 2000, Corollary 1.5] for any $\epsilon > 0$ one has

$$L\big(\tfrac{1}{2}, \chi_D\big) \ll D^{1/6+\epsilon}.$$

This implies the required bounds for the function $\mathscr{L}_d\big(\frac{1}{2}\big)$. ☐

5. Exact formula

Lemma 5.1. *For $\Re s > \frac{3}{2}$ one has*

$$M_1(l,s) = \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k}}{2l^{1-s}} \frac{\Gamma(k-s/2)}{\Gamma(k+s/2)} \mathscr{L}_{-4l^2}(s) + (2\pi)^s i^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \mathscr{L}_{n^2-4l^2}(s) I\Big(\frac{n}{l}\Big), \tag{5-1}$$

where

$$I(x) := \frac{1}{2\pi i} \int_{(\Delta)} \frac{\Gamma\big(k-\frac{1}{2}+\frac{1}{2}w\big)}{\Gamma\big(k+\frac{1}{2}-\frac{1}{2}w\big)} \Gamma(1-s-w) \sin\Big(\pi \frac{s+w}{2}\Big) x^w \, dw \tag{5-2}$$

with $1-2k < \Delta < 1-\Re s$.

Proof. By the Petersson trace formula

$$\begin{aligned} M_1(l,s) &= \zeta(2s) \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{f \in H_{2k}}^{h} \lambda_f(l^2) \lambda_f(n^2) \\ &= \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \sum_{n=1}^{\infty} \frac{S(l^2, n^2; q)}{n^s} J_{2k-1}\Big(4\pi \frac{ln}{q}\Big). \end{aligned}$$

The change of order of summation above is justified by the absolute convergence for $\Re s > \frac{3}{2}$, which follows from the standard estimates

$$S(l^2, n^2; q) \ll q^{1/2+\epsilon} (l^2, n^2, q)^{1/2} \quad \text{for any } \epsilon > 0$$

and

$$J_{2k-1}\Big(4\pi \frac{ln}{q}\Big) \ll \begin{cases} \Big(\dfrac{ln}{q}\Big)^{2k-1}, & q > ln \\ \Big(\dfrac{ln}{q}\Big)^{-1/2}, & q < ln. \end{cases}$$

Next, we use the Mellin–Barnes representation for the Bessel function

$$M_1(l,s) = \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \times \frac{1}{4\pi i} \int_{(\Delta)} \frac{\Gamma\big(k-\frac{1}{2}+\frac{1}{2}w\big)}{\Gamma\big(k+\frac{1}{2}-\frac{1}{2}w\big)} \sum_{n=1}^{\infty} \frac{S(l^2, n^2; q)}{n^{w+s}} \Big(\frac{q}{2\pi l}\Big)^w dw,$$

where $1-2k<\Delta<0$. To guarantee the absolute convergence of the integral over w and the sums over q, n we require that

$$\max(1-2k, 1-\Re s)<\Delta<-\tfrac{1}{2},$$

which is true for $\Re s>\frac{3}{2}$. Consider

$$\sum_{n=1}^{\infty}\frac{S(l^2,n^2;q)}{n^{w+s}}=\sum_{c \pmod q}\sum_{n\equiv c \pmod q}\frac{S(l^2,c^2;q)}{n^{w+s}}=\sum_{c \pmod q}S(l^2,c^2;q)\sum_{n=1}^{\infty}\frac{1}{(c+nq)^{w+s}}$$
$$=\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{q^{w+s}}\zeta\Big(\frac{c}{q},0,w+s\Big).$$

Note that for all c, the Lerch zeta function has a simple pole at $w=1-s$ with residue one. The next step is to apply functional equation (3-5) for the Lerch zeta function which is only possible when $\Re(s+w)<0$. Accordingly, we move the w-contour to the left up to $\Delta_1:=-s-\epsilon$, crossing a simple pole at $w=1-s$. Therefore,

$$M_1(l,s)=\frac{\zeta(2s)}{l^s}+2\pi i^{2k}\zeta(2s)\frac{\Gamma\big(k-\frac{1}{2}s\big)}{\Gamma\big(k+\frac{1}{2}s\big)}\sum_{q=1}^{\infty}\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{2q^2}\Big(\frac{q}{2\pi l}\Big)^{1-s}+2\pi i^{2k}\zeta(2s)\sum_{q=1}^{\infty}\frac{1}{q}\frac{1}{4\pi i}$$
$$\times\int_{(\Delta_1)}\frac{\Gamma\big(k-\frac{1}{2}+\frac{1}{2}w\big)}{\Gamma\big(k+\frac{1}{2}-\frac{1}{2}w\big)}\Big(\frac{q}{2\pi l}\Big)^{w}\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{q^{s+w}}\zeta\Big(\frac{c}{q},0;s+w\Big)\,dw.$$

Using the functional equation (3-5), we obtain

$$\sum_{c \pmod q}S(l^2,c^2;q)\zeta\Big(\frac{c}{q},0;s+w\Big)$$
$$=2(2\pi)^{s+w-1}\Gamma(1-s-w)\sin\Big(\pi\frac{s+w}{2}\Big)\sum_{c \pmod q}S(l^2,c^2;q)\zeta\Big(0,\frac{c}{q};1-s-w\Big).$$

Substituting this into $M_1(l,s)$ and opening the Lerch zeta function, one has

$$M_1(l,s)=\frac{\zeta(2s)}{l^s}+\frac{(2\pi)^s i^{2k}}{2l^{1-s}}\zeta(2s)\frac{\Gamma\big(k-\frac{1}{2}s\big)}{\Gamma\big(k+\frac{1}{2}s\big)}\sum_{q=1}^{\infty}\sum_{c \pmod q}\frac{S(l^2,c^2;q)}{q^{1+s}}$$
$$+(2\pi)^s i^{2k}\zeta(2s)\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\sum_{q=1}^{\infty}\frac{1}{q^{1+s}}\sum_{c \pmod q}S(l^2,c^2;q)e\Big(\frac{nc}{q}\Big)I\Big(\frac{n}{l}\Big),$$

where $I(x)$ is defined by equation (5-2). Finally, computing the sums over c and q using formula (4-9), we prove the lemma. □

Lemma 5.2. *If $x\geq 2$, then*

$$I(x)=\frac{2^{2k}(-1)^k}{2^s\sqrt{\pi}}\cos\Big(\frac{\pi s}{2}\Big)x^{1-2k}\frac{\Gamma\big(k-\frac{1}{2}s\big)\Gamma\big(k+\frac{1}{2}-\frac{1}{2}s\big)}{\Gamma(2k)}\times{}_2F_1\Big(k-\frac{s}{2},k+\frac{1}{2}-\frac{s}{2},2k;\frac{4}{x^2}\Big).\tag{5-3}$$

Proof. Moving the contour of integration in (5-2) to the left, we cross simple poles at $w = 1 - 2k - 2j$, $j = 0, 1, 2, \ldots$. Therefore,

$$I(x) = 2(-1)^k \cos\Big(\frac{\pi s}{2}\Big) x^{1-2k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma(2k - s + 2j)}{\Gamma(2k + j)} x^{-2j}.$$

By the duplication formula we have

$$\Gamma\big(2\big(k + j - \tfrac{1}{2}s\big)\big) = \frac{2^{2k-1-s+2j}}{\sqrt{\pi}} \Gamma\big(k + j - \tfrac{1}{2}s\big) \Gamma\big(k + j + \tfrac{1}{2}1 - \tfrac{1}{2}s\big).$$

This yields

$$\begin{aligned} I(x) &= \frac{2(-1)^k}{\sqrt{\pi}} 2^{2k-1-s} \cos\Big(\frac{\pi s}{2}\Big) x^{1-2k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma\big(k - \frac{1}{2}s + j\big)}{\Gamma(2k + j)} \Gamma\Big(k + \frac{1}{2} - \frac{s}{2} + j\Big) \Big(\frac{4}{x^2}\Big)^j \\ &= \frac{2^{2k}(-1)^k}{2^s \sqrt{\pi}} \cos\Big(\frac{\pi s}{2}\Big) x^{1-2k} \frac{\Gamma\big(k - \frac{1}{2}s\big) \Gamma\big(k + \frac{1}{2} - \frac{1}{2}s\big)}{\Gamma(2k)} {}_2F_1\Big(k - \frac{s}{2}, k + \frac{1}{2} - \frac{s}{2}, 2k; \frac{4}{x^2}\Big). \end{aligned}$$ □

Lemma 5.3. *One has*

$$I(2) = \frac{2(-1)^k}{2^s \sqrt{\pi}} \cos\big(\tfrac{1}{2}\pi s\big) \frac{\Gamma\big(k - \frac{1}{2}s\big) \Gamma\big(k + \frac{1}{2} - \frac{1}{2}s\big)}{\Gamma\big(k + \frac{1}{2}s\big) \Gamma\big(k - \frac{1}{2} + \frac{1}{2}s\big)} \Gamma\big(s - \tfrac{1}{2}\big). \tag{5-4}$$

Proof. Letting $x = 2$ in (5-3) and applying [Olver et al. 2010, Equation 15.4.20], we find

$${}_2F_1\big(k - \tfrac{1}{2}s, k + \tfrac{1}{2} - \tfrac{1}{2}s, 2k; 1\big) = \frac{\Gamma(2k) \Gamma\big(s - \frac{1}{2}\big)}{\Gamma\big(k + \frac{1}{2}s\big) \Gamma\big(k - \frac{1}{2} + \frac{1}{2}s\big)}.$$

The assertion follows. □

Lemma 5.4. *If $x < 2$, then*

$$I(x) = \frac{(-1)^k}{\sqrt{\pi}} \sin\big(\tfrac{1}{2}\pi s\big) x^{1-s} \frac{\Gamma\big(k - \frac{1}{2}s\big) \Gamma\big(1 - k - \frac{1}{2}s\big)}{\Gamma\big(\frac{1}{2}\big)} {}_2F_1\big(k - \tfrac{1}{2}s, 1 - k - \tfrac{1}{2}s, \tfrac{1}{2}; \tfrac{1}{4}x^2\big). \tag{5-5}$$

Proof. Moving the contour of integration in (5-2) to the right we cross simple poles at $w = 1 - s + j$, $j = 0, 1, 2, \ldots$. Accordingly,

$$I(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\Gamma\big(k - \frac{1}{2}s + \frac{1}{2}j\big)}{\Gamma\big(k + \frac{1}{2}s - \frac{1}{2}j\big)} \sin\Big(\pi \frac{1+j}{2}\Big) x^{1-s+j}.$$

Note that

$$\sin\Big(\pi \frac{1+j}{2}\Big) = \cos\Big(\frac{\pi j}{2}\Big) = \begin{cases} 0, & j \text{ is odd}, \\ (-1)^m, & j = 2m. \end{cases}$$

Thus

$$I(x) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \frac{\Gamma\big(k - \frac{1}{2}s + m\big)}{\Gamma\big(k + \frac{1}{2}s - m\big)} (-1)^m x^{1-s+2m}.$$

In order to express $I(x)$ in terms of the Gauss hypergeometric function we apply the duplication formula, obtaining

$$(2m)! = \Gamma\bigl(2\bigl(m+\tfrac{1}{2}\bigr)\bigr) = \frac{1}{\sqrt{\pi}} 2^{2m}\Gamma\bigl(m+\tfrac{1}{2}\bigr)\Gamma(m+1).$$

Furthermore, by Euler's reflection formula

$$\Gamma\bigl(k+\tfrac{1}{2}s-m\bigr) = \frac{\pi}{(-1)^{k-m}\sin\bigl(\frac{1}{2}\pi s\bigr)\Gamma\bigl(1-k-\frac{1}{2}s+m\bigr)}.$$

Finally,

$$\begin{aligned} I(x) &= \frac{(-1)^k}{\sqrt{\pi}}\sin\bigl(\tfrac{1}{2}\pi s\bigr)x^{1-s}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\Gamma\bigl(k-\frac{1}{2}s+m\bigr)}{\Gamma\bigl(m+\frac{1}{2}\bigr)}\Gamma\bigl(1-k-\tfrac{1}{2}s+m\bigr)\Bigl(\frac{x^2}{4}\Bigr)^m \\ &= \frac{(-1)^k}{\sqrt{\pi}}\sin\bigl(\tfrac{1}{2}\pi s\bigr)x^{1-s}\frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(1-k-\frac{1}{2}s\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)}{}_2F_1\bigl(k-\tfrac{1}{2}s,\,1-k-\tfrac{1}{2}s,\,\tfrac{1}{2};\,\tfrac{1}{4}x^2\bigr). \end{aligned}$$ □

Next, we substitute equations (5-3), (5-4), (5-5) into expression (5-1), proving the exact formula for the shifted first moment.

Theorem 5.5. *For any $l \geq 1$ and $2-2k < \Re s < 2k-1$, one has*

$$\begin{aligned} M_1(l,s) &= \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k}}{2l^{1-s}}\frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)}{\Gamma\bigl(k+\frac{1}{2}s\bigr)}\mathscr{L}_{-4l^2}(s) \\ &\quad + \frac{(2\pi)^s}{\sqrt{\pi}}\frac{\zeta(2s-1)}{l^{1-s}}\cos\bigl(\tfrac{1}{2}\pi s\bigr)\frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(k+\frac{1}{2}-\frac{1}{2}s\bigr)}{\Gamma\bigl(k+\frac{1}{2}s\bigr)\Gamma\bigl(k-\frac{1}{2}+\frac{1}{2}s\bigr)}\Gamma\bigl(s-\tfrac{1}{2}\bigr) \\ &\quad + \frac{(2\pi)^s\sin\bigl(\frac{1}{2}\pi s\bigr)}{\sqrt{\pi}l^{1-s}}\sum_{1\leq n<2l}\mathscr{L}_{n^2-4l^2}(s)\frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(1-k-\frac{1}{2}s\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)} \\ &\qquad\qquad \times {}_2F_1\Bigl(k-\frac{s}{2},\,1-k-\frac{s}{2},\,\frac{1}{2};\,\Bigl(\frac{n}{2l}\Bigr)^2\Bigr) \\ &\quad + \frac{2^{2k}\pi^s\cos\bigl(\frac{1}{2}\pi s\bigr)}{\sqrt{\pi}}\sum_{n>2l}\mathscr{L}_{n^2-4l^2}(s)\frac{\Gamma\bigl(k-\frac{1}{2}s\bigr)\Gamma\bigl(k+\frac{1}{2}-\frac{1}{2}s\bigr)}{\Gamma(2k)} \\ &\qquad\qquad \times \frac{1}{n^{1-s}}\Bigl(\frac{n}{l}\Bigr)^{1-2k}{}_2F_1\Bigl(k-\frac{s}{2},\,k+\frac{1}{2}-\frac{s}{2},\,2k;\,\Bigl(\frac{2l}{n}\Bigr)^2\Bigr). \end{aligned} \tag{5-6}$$

Note that in equation (5-6) only the first and the third summands have poles at $s=\frac{1}{2}$. Computing the limit as $s \to \frac{1}{2}$ we find that these poles cancel each other. This allows us to prove the exact formula for the first moment of symmetric square L-functions at the critical point given by Theorem 2.1.

6. Liouville–Green approximation of special functions

The aim of this section is to approximate the functions $\Phi_k(x)$ and $\Psi_k(x)$ appearing in exact formula (2-9). Both of these special functions can be expressed in terms of the Gauss hypergeometric function, whose asymptotic approximation has been a subject of intensive research during the last century. The works of Watson [1948], Jones [2001] and Farid Khwaja and Olde Daalhuis [2014; 2017] are the most relevant in our case. However, application of these results to approximation of $\Phi_k(x)$ and $\Psi_k(x)$ is not straightforward and involves certain technical difficulties due to either the region of validity of asymptotic expansion or the lack of asymptotic error estimates.

Therefore, we choose to follow the approach of [Olver 1974] and [Boyd and Dunster 1986]. Accordingly, we apply the Liouville–Green approximation directly to the functions $\Phi_k(x)$ and $\Psi_k(x)$. It turns out that these special functions have similar behavior to the ones occurring in the exact formula for the second moment of cusp form L-functions. See [Balkanova and Frolenkov 2016, Theorem 4.2].

Properties of Φ_k. Consider the function $\Phi_k(x)$ for $0 < x < 1$.

Lemma 6.1. *One has*

$$\Phi_k(x) = -\pi\left({}_2F_1\left(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1-\sqrt{x})\right) + {}_2F_1\left(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1+\sqrt{x})\right)\right). \tag{6-1}$$

Proof. Applying the quadratic transformation given by [Olver et al. 2010, Equation 15.8.27] we obtain

$$\begin{aligned}\Phi_k(x) = \Gamma\left(k-\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}-k\right)\Gamma\left(k+\tfrac{1}{4}\right)\Gamma\left(\tfrac{5}{4}-k\right)\frac{1}{2\Gamma^2\left(\frac{1}{2}\right)}&\\ \times\left({}_2F_1\left(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1-\sqrt{x})\right) + {}_2F_1\left(2k-\tfrac{1}{2}, \tfrac{3}{2}-2k, 1; \tfrac{1}{2}(1+\sqrt{x})\right)\right)&.\end{aligned}$$

Note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Euler's reflection formula yields

$$\Gamma\left(k-\tfrac{1}{4}\right)\Gamma\left(k+\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}-k\right)\Gamma\left(\tfrac{5}{4}-k\right) = -2\pi^2.$$

The assertion follows. □

Making the change of variables

$$m := 2k-\tfrac{1}{2}, \quad k \in \mathbb{N}, \tag{6-2}$$

$$y := \tfrac{1}{2}(1-\sqrt{x}), \quad 0 < y < \tfrac{1}{2}, \tag{6-3}$$

one has

$$\Phi_k(x) = -\pi({}_2F_1(m, 1-m, 1; y) + {}_2F_1(m, 1-m, 1; 1-y)). \tag{6-4}$$

At the point $y = 0$, the function ${}_2F_1(m, 1-m, 1; y)$ is recessive and the function ${}_2F_1(m, 1-m, 1; 1-y)$ is dominant. Therefore, further transformations are required to apply the Liouville–Green method to the second function. In particular, we show that ${}_2F_1(m, 1-m, 1; 1-y)$ has a similar shape to $\phi_k(x)$ studied in [Balkanova and Frolenkov 2016]. Note that the parameter m is now half-integral.

Lemma 6.2. *Let $m := 2k-\frac{1}{2}$, $k \in \mathbb{N}$. Then*

$$\begin{aligned}{}_2F_1(m, 1-m, 1; 1-y) = (-\log y + 2\psi(1) - 2\psi(m)) \times {}_2F_1(m, 1-m, 1; y)&\\ + \frac{1}{\pi}\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}\right){}_2F_1(a, b, c; y)\Big|_{\substack{a=m\\ b=1-m\\ c=1}}&.\end{aligned} \tag{6-5}$$

Proof. By [Beitman and Erdelyi 1953, Equation 33, p. 107] we have

$$
\begin{aligned}
{}_2F_1(m+u,1-m+u,1;1-y) = {}_2F_1(m+u,1-m+u,1+2u;y)\frac{\Gamma(1)\Gamma(-2u)}{\Gamma(1-m-u)\Gamma(m-u)}\\
+{}_2F_1(m-u,1-m-u,1-2u;y)\frac{\Gamma(1)\Gamma(2u)}{\Gamma(1-m+u)\Gamma(m+u)}y^{-2u}.
\end{aligned}
$$

Computing the limit as $u\to 0$, we prove the assertion. ☐

Lemma 6.3. *For $m=2k-\frac{1}{2},\ k\in\mathbb{N}$ one has*

$$
{}_2F_1\big(m,1-m,1;\tfrac{1}{2}\big) = -\frac{(-1)^k}{\sqrt{2\pi}}\frac{\Gamma\big(k-\frac14\big)}{\Gamma\big(k+\frac14\big)}, \tag{6-6}
$$

$$
\frac{d}{dx}\big({}_2F_1(m,1-m,1;x)\big)\Big|_{x=1/2} = \frac{4(-1)^k}{\sqrt{2\pi}}\frac{\Gamma\big(k+\frac14\big)}{\Gamma\big(k-\frac14\big)}. \tag{6-7}
$$

Proof. On the one hand, by equation (6-1) we have

$$
{}_2F_1\big(m,1-m,1;\tfrac{1}{2}\big) = -\tfrac{1}{2\pi}\Phi_k(0).
$$

On the other hand, equation (2-7) yields

$$
\Phi_k(0) = \frac{\Gamma\big(k-\frac14\big)\Gamma\big(\frac34-k\big)}{\Gamma\big(\frac12\big)} = (-1)^k\sqrt{2\pi}\,\frac{\Gamma\big(k-\frac14\big)}{\Gamma\big(k+\frac14\big)}.
$$

The last two equalities imply (6-6).

As a consequence of [Olver et al. 2010, Equation 15.8.25] we obtain

$$
\begin{aligned}
{}_2F_1(m,1-m,1;x) = \frac{\Gamma\big(\frac12\big)}{\Gamma\big(\frac12 m+\frac12\big)\Gamma\big(1-\frac12 m\big)}\times {}_2F_1\big(\tfrac12 m,\tfrac12-\tfrac12 m,\tfrac12;(1-2x)^2\big)\\
+(1-2x)\frac{\Gamma\big(-\frac12\big)}{\Gamma\big(\frac12 m\big)\Gamma\big(\frac12-\frac12 m\big)}{}_2F_1\big(\tfrac12 m+\tfrac12,1-\tfrac12 m,\tfrac32;(1-2x)^2\big).
\end{aligned}
$$

Then equality (6-7) follows by differentiating the last expression in x and setting $x=\frac12$. ☐

Lemma 6.4. *The functions*

$$
{}_2F_1(m,1-m,1;y) \quad \textit{and} \quad {}_2F_1(m,1-m,1;1-y)
$$

are solutions of differential equation

$$
y(1-y)F''(y)+(1-2y)F'(y)+m(m-1)F(y)=0. \tag{6-8}
$$

Proof. This follows from the differential equation for hypergeometric functions. ☐

Approximation of Φ_k. In order to find a Liouville–Green approximation for $\Phi_k(y)$, we use formula (6-4) and study separately each of the hypergeometric functions ${}_2F_1(m, 1-m, 1; y)$ and ${}_2F_1(m, 1-m, 1; 1-y)$. As shown in Lemma 6.4, these functions are solutions of differential equation (6-8) that was already approximated in [Balkanova and Frolenkov 2016, Section 5.2]. So our problem reduces to computation of the Liouville–Green constants C_Y and C_J in the approximation of ${}_2F_1(m, 1-m, 1; 1-y)$.

For the reader's convenience, we briefly recall the required results of [Balkanova and Frolenkov 2016, Section 5.2]. It follows from Lemma 6.4 that the functions

$$G_1(y) := {}_2F_1(m, 1-m, 1; y)\sqrt{y(1-y)}, \tag{6-9}$$

$$G_2(y) := {}_2F_1(m, 1-m, 1; 1-y)\sqrt{y(1-y)} \tag{6-10}$$

are solutions of the differential equation

$$G''(y) = (u^2 f(y) + g(y))G(y), \tag{6-11}$$

where

$$u := 2k-1, \quad f(y) := -\frac{1}{y(1-y)}, \tag{6-12}$$

$$g(y) := -\frac{1}{4y^2(1-y)^2} + \frac{1}{4y(1-y)}. \tag{6-13}$$

Making the change of variables

$$Z(y) := \frac{G(y)}{\alpha(y)}, \quad \alpha(y) := \frac{(y-y^2)^{1/4}}{2(\arcsin\sqrt{y})^{1/2}}, \tag{6-14}$$

$$\xi := 4\arcsin^2\sqrt{y}, \tag{6-15}$$

we transform equation (6-11) into the following shape

$$\frac{d^2Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2} + \frac{\psi(\xi)}{\xi}\right]Z = 0, \tag{6-16}$$

with

$$\psi(\xi) := \frac{1}{16\sin^2\sqrt{\xi}} - \frac{1}{16\xi}. \tag{6-17}$$

Removing the summand with $\psi(\xi)/\xi$ in equation (6-16), we have

$$\frac{d^2Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right]Z = 0. \tag{6-18}$$

The solutions of (6-18) are defined by

$$Z_C = \sqrt{\xi}\,C_0(u\sqrt{\xi}), \tag{6-19}$$

where C_i is either J or Y Bessel function of index i.

Then according to [Olver 1974, Chapter 12] solutions of original differential equation (6-16) can be found in the form

$$Z_C(\xi)=\sqrt{\xi}C_0(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{A(n;\xi)}{u^{2n}}-\frac{\xi}{u}C_1(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{B(n;\xi)}{u^{2n}}. \tag{6-20}$$

In order to determine coefficients $A(n;\xi)$, $B(n;\xi)$ we use differential equations (see [Gradshteyn and Ryzhik 2007, Equation 8.491(3)]) for functions

$$W(\xi):=\sqrt{\xi}C_0(u\sqrt{\xi}),\quad V(\xi):=\xi C_1(u\sqrt{\xi}) \tag{6-21}$$

and substitute (6-20) in equation (6-16). This yields

$$W(\xi)\sum_{n=0}^{\infty}\frac{C_n(\xi)}{u^{2n}}-V(\xi)\sum_{n=0}^{\infty}\frac{D_n(\xi)}{u^{2n-1}}=0, \tag{6-22}$$

where

$$C_n(\xi):=A''(n;\xi)+\frac{1}{\xi}A'(n;\xi)-\frac{\psi(\xi)}{\xi}A(n;\xi)-B'(n;\xi)-\frac{B(n;\xi)}{2\xi},$$
$$D_n(\xi):=B''(n-1;\xi)+\frac{1}{\xi}B'(n-1;\xi)-\frac{\psi(\xi)}{\xi}B(n-1;\xi)+\frac{1}{\xi}A'(n;\xi).$$

Letting $C_n(\xi)=D_n(\xi)=0$, we find the required recurrence relations

$$A(n;\xi)=-\xi B'(n-1;\xi)+\int_0^{\xi}\psi(x)B(n-1;x)\,dx+\lambda_n, \tag{6-23}$$

$$\sqrt{\xi}B(n;\xi)=\int_0^{\xi}\frac{1}{\sqrt{x}}\big(xA''(n;x)+A'(n;x)-\psi(x)A(n;x)\big)\,dx \tag{6-24}$$

for some real constants of integration λ_n.

Assume that $A(0;\xi)=1$. Then

$$B(0;\xi)=-\frac{1}{8\sqrt{\xi}}\bigg(\cot\sqrt{\xi}-\frac{1}{\sqrt{\xi}}\bigg), \tag{6-25}$$

$$A(1;\xi)=\frac{1}{8}\bigg(\frac{1}{\xi}-\frac{\cot\sqrt{\xi}}{2\sqrt{\xi}}-\frac{1}{2\sin^2\sqrt{\xi}}\bigg)-\frac{1}{128}\bigg(\cot\sqrt{\xi}-\frac{1}{\sqrt{\xi}}\bigg)^2+\lambda_1. \tag{6-26}$$

Furthermore, solutions (6-20) can be approximated by finite series using [Olver 1974, Theorem 4.1, p. 444] or [Boyd and Dunster 1986, Theorem 1].

Theorem 6.5. *Let $\xi_2=\frac{1}{4}\pi^2$. For each value of u and each nonnegative integer N, equation* (6-16) *has solutions $Z_Y(\xi)$, $Z_J(\xi)$ which are infinitely differentiable in ξ on interval* $(0,\xi_2)$, *and are given by*

$$Z_Y(\xi)=\sqrt{\xi}Y_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_Y(n;\xi)}{u^{2n}}-\frac{\xi}{u}Y_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_Y(n;\xi)}{u^{2n}}+\epsilon_{2N+1,1}(u,\xi), \tag{6-27}$$

$$Z_J(\xi)=\sqrt{\xi}J_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_J(n;\xi)}{u^{2n}}-\frac{\xi}{u}J_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_J(n;\xi)}{u^{2n}}+\epsilon_{2N+1,2}(u,\xi), \tag{6-28}$$

where

$$\epsilon_{2N+1,1}(u,\xi) \ll \frac{\sqrt{\xi}\,|Y_0(u\sqrt{\xi})|}{u^{2N+1}}\sqrt{\xi_2-\xi}, \tag{6-29}$$

$$\epsilon_{2N+1,2}(u,\xi) \ll \frac{\sqrt{\xi}\,|J_0(u\sqrt{\xi})|}{u^{2N+1}}\min(\sqrt{\xi},1) \tag{6-30}$$

and coefficients $(A_Y(n;\xi), B_Y(n;\xi))$, $(A_J(n;\xi), B_J(n;\xi))$ *are defined by* (6-23)–(6-24).

The functions $\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_1(\sin^2\sqrt{\xi}/2)$ and $Z_J(\xi)$ are recessive solutions of equation (6-16) as $\xi \to 0$. Therefore, there is c_0 such that

$$\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_1(\sin^2\sqrt{\xi}/2) = c_0 Z_J(\xi). \tag{6-31}$$

The value of the constant c_0 is determined by computing the limit of the left- and right-hand sides of equation (6-31) as $\xi \to 0$. On the one hand,

$$\lim_{\xi\to 0} {}_2F_1(k, 1-k, 1; \sin^2\sqrt{\xi}/2) = 1. \tag{6-32}$$

On the other hand,

$$Z_J(\xi) = \sqrt{\xi}\sum_{n=0}^{N}\frac{A_J(n;\xi)}{u^{2n}} + O(\xi) \quad \text{as } \xi \to 0. \tag{6-33}$$

Choosing $A_J(n;\xi)$ such that $A_J(0;0) = 1$ and $A_J(n;0) = 0$ for $n \geq 1$ we find that $c_0 = 1$.

To sum up, we have proved the following lemma:

Lemma 6.6. *Let* $\xi_2 = \frac{1}{4}\pi^2$. *For* $\xi \in (0,\xi_2)$, *one has*

$$\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_1\left(\sin^2\frac{\sqrt{\xi}}{2}\right) = Z_J(\xi), \tag{6-34}$$

where $Z_J(\xi)$ *is given by* (6-28).

This concludes the summary of results of [Balkanova and Frolenkov 2016, Section 5.2]. Our final goal is to compute $C_Y = C_Y(u)$ and $C_J = C_J(u)$ such that

$$\xi^{1/4}(\sin\sqrt{\xi})^{1/2}G_2\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\big) = C_Y Z_Y(\xi) + C_J Z_J(\xi). \tag{6-35}$$

Note that there exist c_1, c_2 such that

$$\begin{aligned} Z_Y(\xi) &= \big(c_1G_1\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\,\big) + c_2G_2\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\big)\big) \times \xi^{1/4}(\sin\sqrt{\xi})^{1/2} \\ &= c_1Z_J(\xi) + \xi^{1/4}\big(\sin\sqrt{\xi}\,\big)^{1/2}c_2G_2\big(\sin^2\tfrac{1}{2}\sqrt{\xi}\,\big). \end{aligned} \tag{6-36}$$

The last two equalities imply

$$C_Y = \frac{1}{c_2}, \quad C_J = -\frac{c_1}{c_2}. \tag{6-37}$$

Lemma 6.7. *One has*

$$2c_1=(-1)^{k+1}\sqrt{2\pi}\frac{\Gamma\left(k+\frac14\right)}{\Gamma\left(k-\frac14\right)}\frac{Z_Y(\xi_2)}{\xi_2^{1/4}}+(-1)^k\sqrt{2\pi}\frac{\Gamma\left(k-\frac14\right)}{\Gamma\left(k+\frac14\right)}\xi_2^{1/4}\left(Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}\right), \tag{6-38}$$

$$2c_2=(-1)^{k+1}\sqrt{2\pi}\frac{\Gamma\left(k+\frac14\right)}{\Gamma\left(k-\frac14\right)}\frac{Z_Y(\xi_2)}{\xi_2^{1/4}}-(-1)^k\sqrt{2\pi}\frac{\Gamma\left(k-\frac14\right)}{\Gamma\left(k+\frac14\right)}\xi_2^{1/4}\left(Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}\right). \tag{6-39}$$

Proof. To determine coefficients c_1, c_2 we consider the pair of equations

$$Z_Y(\xi_2)=\xi_2^{1/4}\left(c_1G_1\left(\tfrac12\right)+c_2G_2\left(\tfrac12\right)\right),\quad Z_Y'(\xi_2)=\frac{Z_Y(\xi_2)}{4\xi_2}+\frac{1}{4\xi_2^{1/4}}\left(c_1G_1'\left(\tfrac12\right)+c_2G_2'\left(\tfrac12\right)\right).$$

Note that

$$G_1\left(\tfrac12\right)=G_2\left(\tfrac12\right)\quad\text{and}\quad G_1'\left(\tfrac12\right)=-G_2'\left(\tfrac12\right).$$

Therefore,

$$(c_1+c_2)G_1\left(\tfrac12\right)=\frac{Z_Y(\xi_2)}{\xi_2^{1/4}},\quad (c_1-c_2)G_1'\left(\tfrac12\right)=4\xi_2^{1/4}\left(Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}\right).$$

The assertion follows by Lemma 6.3. ☐

Lemma 6.8. *For $\xi_2=\frac14\pi^2$, one has*

$$Z_Y(\xi_2)=\frac{(-1)^{k+1}}{\sqrt{2u}}\left(1+\frac{1}{u^2}\left(\lambda_1-\frac{1}{16}\right)+O\left(\frac{1}{u^3}\right)\right), \tag{6-40}$$

$$Z_Y'(\xi_2)=\frac{(-1)^{k+1}}{\sqrt{2u}}\left(\frac{u}{\pi}+\frac{1}{\pi^2}+\frac{2\lambda_1+\frac18}{2\pi u}+O\left(\frac{1}{u^2}\right)\right). \tag{6-41}$$

Proof. Applying [Boyd and Dunster 1986, Theorem 1] with $N=1$ we obtain

$$\epsilon_{3,1}(u;\xi_2)=0\quad\text{and}\quad\frac{\partial}{\partial\xi}\epsilon_{3,1}(u;\xi)\big|_{\xi=\xi_2}=0.$$

Therefore,

$$Z_Y(\xi_2)=\sqrt{\xi_2}Y_0(u\sqrt{\xi_2})\left(1+\frac{A_Y(1,\xi_2)}{u^2}\right)-\xi_2Y_1(u\sqrt{\xi_2})\frac{B_Y(0,\xi_2)}{u}$$

and

$$\begin{aligned}Z_Y'(\xi_2)=\sqrt{\xi_2}Y_0(u\sqrt{\xi_2})&\left(\frac{1}{2\xi_2}\left[1+\frac{A_Y(1;\xi_2)}{u^2}\right]+\frac{A_Y'(1;\xi_2)}{u^2}-\frac12B_Y(0;\xi_2)\right)\\&-\xi_2Y_1(u\sqrt{\xi_2})\left(\frac{u}{2\xi_2}\left[1+\frac{A_Y(1;\xi_2)}{u^2}\right]+\frac{1}{2\xi_2u}B_Y(0;\xi_2)+\frac1uB_Y'(0;\xi_2)\right).\end{aligned}$$

By means of the Hankel asymptotic expansion (see [Olver et al. 2010, Equation 10.17.1, 10.17.4] and [Gradshteyn and Ryzhik 2007, Equation 8.451(1,7,8)]) we evaluate

$$\sqrt{\xi_2}Y_0(u\sqrt{\xi_2}) = \frac{\pi}{2}Y_0\Big((2k-1)\frac{\pi}{2}\Big) = \frac{(-1)^{k+1}}{\sqrt{2u}}\bigg(\sum_{j=0}^{\infty}(-1)^j\frac{a_{2j}(0)}{(\pi u/2)^{2j}} + \sum_{j=0}^{\infty}(-1)^j\frac{a_{2j+1}(0)}{(\pi u/2)^{2j+1}}\bigg),$$

$$\xi_2Y_1(u\sqrt{\xi_2}) = \frac{\pi}{2}\frac{(-1)^k}{\sqrt{2u}}\bigg(\sum_{j=0}^{\infty}(-1)^j\frac{a_{2j}(1)}{(\pi u/2)^{2j}} - \sum_{j=0}^{\infty}(-1)^j\frac{a_{2j+1}(1)}{(\pi u/2)^{2j+1}}\bigg),$$

where

$$a_j(v) = \frac{\Gamma\big(v+j+\frac{1}{2}\big)}{2^j j!\,\Gamma\big(v-j+\frac{1}{2}\big)}.$$

This yields

$$Z_Y(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}}\bigg(1+\frac{a_1(0)}{\pi u/2} - \frac{a_2(0)}{(\pi u/2)^2} + O(u^{-3})\bigg)\bigg(1+\frac{A_Y(1;\xi_2)}{u^2}\bigg) - \frac{\pi}{2}\frac{(-1)^k}{\sqrt{2u}}\bigg(1-\frac{a_1(1)}{\pi u/2} - \frac{a_2(1)}{(\pi u/2)^2} + O(u^{-3})\bigg)\frac{B_Y(0,\xi_2)}{u}.$$

Simplifying the expression above, one has

$$Z_Y(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}}\bigg(1+\frac{1}{u}\Big(\frac{2}{\pi}a_1(0)+\frac{\pi}{2}B_Y(0,\xi_2)\Big) + \frac{1}{u^2}\Big(A_Y(1,\xi_2) - \frac{4}{\pi^2}a_2(0) - a_1(1)B_Y(0,\xi_2)\Big)\bigg) + O(u^{-7/2}).$$

Using formulas (6-25) and (6-26), we find

$$a_1(0) = -\frac{1}{8},\quad \frac{2}{\pi}a_1(0)+\frac{\pi}{2}B_Y(0,\xi_2) = 0,\quad A_Y(1,\xi_2) - \frac{4}{\pi^2}a_2(0) - a_1(1)B_Y(0,\xi_2) = \lambda_1 - \frac{1}{16}.$$

This gives equation (6-40).

Similarly, we obtain

$$Z_Y'(\xi_2) = \sqrt{\xi_2}Y_0(u\sqrt{\xi_2})\Big[\frac{7}{4\pi^2} + \frac{1}{u^2}\Big(\frac{1}{\pi^2}\Big(2\lambda_1 - \frac{1}{32}\Big) - \frac{15}{8\pi^4}\Big)\Big] - \xi_2Y_1(u\sqrt{\xi_2})\Big[u\frac{2}{\pi^2} + \frac{1}{u}\Big(\frac{1}{\pi^2}\Big(2\lambda_1+\frac{1}{8}\Big) - \frac{1}{16\pi^4}\Big)\Big].$$

Hankel's expansion for Bessel functions yields the following asymptotics:

$$Z_Y'(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}}\bigg[u\frac{1}{\pi} + \Big(\frac{7}{4\pi^2} - \frac{2}{\pi^2}a_1(1)\Big) + \frac{1}{u}\Big(\frac{7}{2\pi^3}a_1(0) + \frac{\pi}{2}\Big(\frac{2\lambda_1+\frac{1}{8}}{\pi^2} - \frac{1}{16\pi^4}\Big) - \frac{4}{\pi^3}a_2(1)\Big) + \frac{1}{u^2}\Big(-\frac{7}{\pi^4}a_2(0) - a_1(1)\Big(\frac{2\lambda_1+\frac{1}{8}}{\pi^2} - \frac{1}{16\pi^4}\Big) + \frac{8}{\pi^4}a_3(1)\Big) + O(u^{-3})\bigg].$$

Finally, substituting

$$a_1(0) = -\tfrac{1}{8},\quad a_1(1) = \tfrac{3}{8},\quad a_2(1) = -\tfrac{15}{128},$$

we prove equation (6-41). □

Corollary 6.9. *One has*

$$C_Y = 1 + O\Big(\frac{1}{k}\Big),\quad C_J = O\Big(\frac{1}{k^2}\Big). \tag{6-42}$$

Proof. By Lemma 6.8

$$Z_Y'(\xi_2)-\frac{Z_Y(\xi_2)}{4\xi_2}=\frac{(-1)^{k+1}}{\sqrt{2u}}\left(\frac{u}{\pi}+\frac{2\lambda_1+\frac{1}{8}}{2\pi u}+O(u^{-2})\right).$$

It follows from [Olver et al. 2010, Equation 5.11.13] that

$$\frac{\Gamma\left(k+\frac{1}{4}\right)}{\Gamma\left(k-\frac{1}{4}\right)}=k^{1/2}-\tfrac{1}{4}k^{-1/2}+O(k^{-3/2})\quad\text{and}\quad\frac{\Gamma\left(k-\frac{1}{4}\right)}{\Gamma\left(k+\frac{1}{4}\right)}=k^{-1/2}+O(k^{-3/2}).$$

Then Lemma 6.7 gives

$$c_1=O(k^{-2}),\quad c_2=1+O(k^{-1}).$$

Equations (6-37) yield the assertion. □

As a consequence of equation (6-35), Lemma 6.6, Theorem 6.5 and Corollary 6.9, we obtain the main result.

Theorem 6.10. *Let* $u=2k-1$, $\xi_2=\frac{1}{4}\pi^2$. *Then for* $\xi\in(0,\xi_2)$ *one has*

$$\Phi_k(\cos^2\sqrt{\xi})=\frac{-\pi}{\xi^{1/4}(\sin\sqrt{\xi})^{1/2}}[Z_J(\xi)+C_YZ_Y(\xi)+C_JZ_J(\xi)],\tag{6-43}$$

where Z_Y, Z_J *are given by* (6-27), (6-28) *and*

$$C_Y=1+O\left(\frac{1}{k}\right),\quad C_J=O\left(\frac{1}{k^2}\right).\tag{6-44}$$

Corollary 6.11. *We have*

$$\Phi_k\left(\frac{1}{4}\right)=-\frac{2^{3/2}\sqrt{\pi}}{3^{1/4}}\frac{1}{\sqrt{2k-1}}\sin\frac{\pi(2k-1)}{3}\left(1+O\left(\frac{1}{k}\right)\right).$$

Approximation of $\boldsymbol{\Psi_k}$***.*** Next, we find a Liouville–Green approximation for the function Ψ_k. With this goal, we follow the arguments of [Balkanova and Frolenkov 2016, Section 5.3] with minor changes. In particular, the differential equation for $\Psi_k(x)$ is slightly different, and, therefore, one must recompute various functions and constants appearing in the Liouville–Green approximation. We provide all details here to make the presentation self-contained.

Consider the function

$$y(x):=\sqrt{1-x}\,\Psi_k(x).\tag{6-45}$$

Let $u:=k-\frac{1}{2}$ and

$$f(x):=\frac{1}{x^2(1-x)},\quad g(x):=-\frac{1}{4x^2(1-x)^2}+\frac{3}{16x(1-x)}.\tag{6-46}$$

Lemma 6.12. *The function* $y=y(x)$ *is a solution of equation*

$$y''(x)-(u^2f(x)+g(x))y(x)=0.\tag{6-47}$$

Proof. Using the differential equation for the hypergeometric function, we find that $y = y(x)$ satisfies the following differential equation

$$y'' + \left(\frac{1-(2k-1)^2}{4x^2} + \frac{1}{4(1-x)^2} + \frac{\frac{5}{4}-(2k-1)^2}{4x(1-x)}\right)y = 0.$$

The assertion follows. □

Making the change

$$Z(x) := \frac{y(x)}{\alpha(x)}, \quad \alpha(x) := \frac{(x^2-x^3)^{1/4}}{2(\operatorname{artanh}\sqrt{1-x})^{1/2}} \tag{6-48}$$

and the substitution

$$\xi := 4\operatorname{artanh}^2\sqrt{1-x}, \tag{6-49}$$

we transform equation (6-47) to the type

$$\frac{d^2Z}{d\xi^2} + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2} - \frac{\psi(\xi)}{\xi}\right)Z = 0, \tag{6-50}$$

where

$$\psi(\xi) = \frac{1}{16}\left(\frac{1}{\xi} - \frac{1}{4\sinh^2\sqrt{\xi}/2}\right) \tag{6-51}$$

is an analytic function as $\xi \to 0$.

In order to find a Liouville–Green approximation to equation (6-50), we remove the term with $\psi(\xi)/\xi$ in (6-50). The resulting equation

$$Z'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi}\right)Z = 0 \tag{6-52}$$

has I and K Bessel functions as solutions (see [Olver et al. 2010, Equation 10.13.2]), namely

$$Z_L = \sqrt{\xi}L_0(u\sqrt{\xi}), \tag{6-53}$$

where

$$L_v(x) := \begin{cases} I_v(x) \\ e^{\pi i v}K_v(x). \end{cases} \tag{6-54}$$

This suggests that solutions of the original differential equation (6-50) can be written in the form (see [Olver 1974, Equation 2.09, Chapter 12])

$$Z_L = \sqrt{\xi}L_0(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{A(n;\xi)}{u^{2n}} + \frac{\xi}{u}L_1(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{B(n;\xi)}{u^{2n}}. \tag{6-55}$$

Lemma 6.13. *The coefficients $A(n;\xi)$ and $B(n;\xi)$ are given by*

$$\sqrt{\xi}B(n;\xi) = -\sqrt{\xi}A'(n;\xi) + \int_0^{\xi}\left(\psi(x)A(n;x) - \frac{1}{2}A'(n;x)\right)\frac{dx}{\sqrt{x}}, \tag{6-56}$$

$$A(n;\xi) = -\xi B'(n-1;\xi) + \int_0^{\xi}\psi(x)B(n-1;x)\,dx + \lambda_n \tag{6-57}$$

for some real constants of integration λ_n.

Proof. By [Olver et al. 2010, Equations 10.13.2, 10.13.5, 10.36, 10.29.2, 10.29.3] the functions

$$W(\xi) := \sqrt{\xi} L_0(u\sqrt{\xi}), \quad V(\xi) := \xi L_1(u\sqrt{\xi})$$

satisfy the relations

$$W'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right)W = 0, \quad V'' - \frac{1}{\xi}V' + \left(-\frac{u^2}{4\xi} + \frac{3}{4\xi^2}\right)V = 0, \quad V' = \frac{1}{2\xi}V + \frac{u}{2}W, \quad W' = \frac{1}{2\xi}W + \frac{u}{2\xi}V.$$

Using this and substituting solution (6-55) into equation (6-50), we obtain that

$$W(\xi)\sum_{n=0}^{\infty}\frac{C(n;\xi)}{u^{2n}} + V(\xi)\sum_{n=0}^{\infty}\frac{D(n;\xi)}{u^{2n+1}} = 0,$$

where

$$C(n;\xi) = A''(n;\xi) + \frac{A'(n;\xi)}{\xi} - \frac{\psi(\xi)}{\xi}A(n;\xi) + B'(n;\xi) + \frac{B(n;\xi)}{2\xi},$$
$$D(n;\xi) = B''(n-1;\xi) + \frac{B'(n-1;\xi)}{\xi} - \frac{\psi(\xi)}{\xi}B(n-1;\xi) + \frac{A'(n;\xi)}{\xi}.$$

Setting $C(n;\xi) = D(n;\xi) = 0$ we find the required recurrence relations. □

Let $A(0;\xi) = 1$. Then

$$B(0;\xi) = \frac{1}{16}\left(\frac{\coth\sqrt{\xi}}{\sqrt{\xi}} - \frac{2}{\xi}\right), \tag{6-58}$$

$$A(1;\xi) = -\frac{1}{32}\left(\frac{4}{\xi} - \frac{\coth\sqrt{\xi/4}}{\sqrt{\xi}} - \frac{1}{2\sinh^2\sqrt{\xi/4}}\right) + \frac{1}{512}\left(\coth\sqrt{\xi/4} - \frac{2}{\sqrt{\xi}}\right)^2 + \lambda_1. \tag{6-59}$$

Note that

$$\lim_{\xi\to\infty}\sqrt{\xi}B(0;\xi) = \tfrac{1}{16}, \quad \lim_{\xi\to\infty}A(1;\xi) = \tfrac{1}{512} + \lambda_1. \tag{6-60}$$

The variation of the function is given by

$$V_{a,b}(f(x)) := \int_a^b |(f(x))'|\,dx. \tag{6-61}$$

Lemma 6.14. *The function $V_{\xi,\infty}(\sqrt{x}B(1;x))$ is bounded. For $n > 1$, the function $V_{\xi,\infty}(\sqrt{x}B(n;x))$ converges.*

Proof. As a consequence of recurrence relation (6-56), we find

$$(\sqrt{x}B(1;x))' = O(x^{-1/2}) \quad \text{as } x\to 0 \quad \text{and} \quad (\sqrt{x}B(1;x))' = O(x^{-2}) \quad \text{as } x\to\infty.$$

Thus $V_{\xi,\infty}(\sqrt{x}B(1;x))$ is bounded. Note that

$$\psi^{(s)}(\xi) = O\left(\frac{1}{|\xi|^{s+1}}\right).$$

The convergence of variation for $n > 1$ then follows by [Olver 1974, Exercise 4.2, p. 445]. $\square$

Using Lemma 6.14, we can truncate the infinite summation in (6-55) up to N summands with a negligible error term. The value of N determines the quality of approximation: the error is smaller for larger N.

Lemma 6.15. *For each value of u and each nonnegative integer N, equation* (6-50) *has a solution $Z_K(\xi)$ which is infinitely differentiable in ξ on interval* $(0, \infty)$ *and is given by*

$$Z_K(\xi) = \sqrt{\xi} K_0(u\sqrt{\xi}) \sum_{n=0}^{N} \frac{A_K(n;\xi)}{u^{2n}} - \frac{\xi}{u} K_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_K(n;\xi)}{u^{2n}} + \epsilon_{2N+1,3}(u,\xi), \tag{6-62}$$

where

$$|\epsilon_{2N+1,3}(u,\xi)| \leq \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^{2N+1}} \times V_{\xi,\infty}(\sqrt{\xi} B_K(N;\xi)) \exp\Big(\frac{1}{u} V_{\xi,\infty}(\sqrt{\xi} B_K(0;\xi))\Big). \tag{6-63}$$

In particular, for $N = 1$

$$\epsilon_{3,3}(u,\xi) \ll \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^3} \min\Big(\sqrt{\xi}, \frac{1}{\xi}\Big). \tag{6-64}$$

As $\xi \to \infty$, the differential equation (6-50) has two recessive solutions, namely

$$Z(\xi) = \Psi_k\bigg(\frac{1}{\cosh^2 \sqrt{\xi}/2}\bigg)(\xi \sinh^2 \sqrt{\xi})^{1/4} \tag{6-65}$$

and $Z_K(\xi)$ given by (6-62). Thus there is $C_K = C_K(u)$ such that

$$\Psi_k\bigg(\frac{1}{\cosh^2 \sqrt{\xi}/2}\bigg)(\xi \sinh^2 \sqrt{\xi}\,)^{1/4} = C_K Z_K(\xi). \tag{6-66}$$

The last step is to compute $C_K = C_K(u)$.

Lemma 6.16. *One has*

$$C_K = 2 + O(k^{-1}). \tag{6-67}$$

Proof. To determine C_K, we compute the limit of the left- and right-hand sides of equation (6-66) as $\xi \to \infty$. This implies

$$C_K = \frac{\Gamma\big(k-\frac{1}{4}\big)\Gamma\big(k+\frac{1}{4}\big)}{\Gamma(2k)} \frac{2^{2k}\sqrt{u}}{\sqrt{\pi}} \bigg(\sum_{n=0}^{N} \frac{a_n}{u^{2n}} - \sum_{n=0}^{N-1} \frac{b_n}{u^{2n+1}}\bigg)^{-1},$$

where

$$a_n = \lim_{\xi\to\infty} A(n;\xi), \quad b_n = \lim_{\xi\to\infty} B(n;\xi)\sqrt{\xi}.$$

According to (6-60), we know that $a_0 = 1$, $a_1 = \frac{1}{512} + \lambda_1$, $b_0 = \frac{1}{16}$. Furthermore,

$$\frac{\Gamma\big(k-\frac{1}{4}\big)\Gamma\big(k+\frac{1}{4}\big)}{\Gamma(2k)} = \frac{\Gamma^2(k)}{\Gamma(2k)}\Big(1+O\Big(\frac{1}{k}\Big)\Big) = \frac{2\sqrt{\pi}}{\sqrt{k}2^{2k}}\Big(1+O\Big(\frac{1}{k}\Big)\Big).$$

The assertion follows. □

Finally, we obtain the main theorem.

Theorem 6.17. *For $\xi \in (0,\infty)$ the following equality holds*

$$\Psi_k\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)\left(\xi \sinh^2\sqrt{\xi}\right)^{1/4} = C_K Z_K(\xi), \tag{6-68}$$

where $Z_K(\xi)$ is defined by (6-62) *and $C_K = 2 + O(k^{-1})$.*

7. Asymptotic formula

Corollaries 2.2 and 2.3 are derived from Theorem 2.1 by estimating the last two summands in exact formula (2-9), as we now show.

Lemma 7.1. *For any $\epsilon > 0$,*

$$E_1(k,l) := \frac{1}{\sqrt{l}} \sum_{1\le n<2l} \mathscr{L}_{n^2-4l^2}\left(\frac{1}{2}\right)\Phi_k\left(\frac{n^2}{4l^2}\right) \ll \frac{l^{5/6+\epsilon}}{\sqrt{k}}. \tag{7-1}$$

Proof. Using the subconvexity bound (4-10) we obtain

$$E_1(k,l) \ll \frac{l^\epsilon}{\sqrt{l}} \sum_{1\le n<2l} (2l-n)^{1/6}(2l+n)^{1/6}\left|\Phi_k\left(\frac{n^2}{4l^2}\right)\right| \ll \frac{l^{1/6+\epsilon}}{\sqrt{l}} \sum_{1\le n<2l} n^{1/6}\left|\Phi_k\left(\left(1-\frac{n}{2l}\right)^2\right)\right|$$

Let $\xi = 4\left(\arcsin\sqrt{n/4l}\right)^2$, $u = 2k-1$. Then by Theorem 6.10 one has

$$\Phi_k\left(\left(1-\frac{n}{2l}\right)^2\right) \ll \frac{\sqrt{\xi}Y_0(u\sqrt{\xi})}{\left(\arcsin\sqrt{n/4l}\right)^{1/2}(n/l)^{1/4}} \ll \frac{\left(\arcsin\sqrt{n/4l}\right)^{1/2}}{(n/l)^{1/4}} Y_0(u\sqrt{\xi}).$$

If $l \ll k^2$, one has $u\sqrt{\xi} \gg 1$. Then the estimate for the Bessel function

$$Y_0(u\sqrt{\xi}) \ll \frac{1}{u^{1/2}\xi^{1/4}}$$

yields

$$\Phi_k\left(\left(1-\frac{n}{2l}\right)^2\right) \ll \frac{1}{k^{1/2}(n/l)^{1/4}}.$$

Consequently,

$$E_1(k,l) \ll \frac{l^{1/6+\epsilon}}{\sqrt{l}} \sum_{n<2l} \frac{n^{1/6}l^{1/4}}{k^{1/2}n^{1/4}} \ll \frac{l^{5/6+\epsilon}}{\sqrt{k}}.$$ □

Corollary 7.2. *If the Lindelöf hypothesis for Dirichlet L-functions is true, then*

$$E_1(k,l) \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}} \quad \textit{for any } \epsilon > 0. \tag{7-2}$$

Lemma 7.3. *For some $c>0$,*

$$E_2(k,l):=\frac{1}{l\sqrt{2}}\sum_{n>2l}\mathscr{L}_{n^2-4l^2}\Big(\frac{1}{2}\Big)\sqrt{n}\Psi_k\Big(\frac{4l^2}{n^2}\Big)\ll\frac{l^{-1/12}}{\sqrt{k}}\exp\Big(-\frac{ck}{\sqrt{l}}\Big). \tag{7-3}$$

Proof. It follows from the subconvexity bound (4-10) that

$$\begin{aligned}E_2(k,l)&\ll\frac{1}{l}\sum_{n>2l}(n^2-4l^2)^{1/6}\sqrt{n}\Big|\Psi_k\Big(\frac{4l^2}{n^2}\Big)\Big|\\&\ll\frac{1}{l}\int_{2l+1}^{\infty}x^{1/2}(x^2-4l^2)^{1/6}\Big|\Psi_k\Big(\frac{4l^2}{x^2}\Big)\Big|\,dx+l^{-1/3}\Big|\Psi_k\Big(\frac{4l^2}{(2l+1)^2}\Big)\Big|.\end{aligned}$$

Next, we make the change of variables $x=2l\cosh\big(\frac{1}{2}\sqrt{\xi}\big)$ and estimate $E_2(k,l)$ using Theorem 6.17 with $N=0$. Consider the first summand

$$\begin{aligned}E_{2,1}(k,l):=\frac{1}{l}\int_{2l+1}^{\infty}x^{1/2}(x^2-4l^2)^{1/6}\Big|\Psi_k\Big(\frac{4l^2}{x^2}\Big)\Big|\,dx&\ll l^{5/6}\int_{\xi_0}^{\infty}\Big(\sinh\frac{\sqrt{\xi}}{2}\Big)^{5/6}\frac{|Z_k(\xi)|}{\xi^{3/4}}\,d\xi\\&\ll l^{5/6}\int_{\xi_0}^{\infty}\Big(\sinh\frac{\sqrt{\xi}}{2}\Big)^{5/6}\frac{|K_0(u\sqrt{\xi})|}{\xi^{1/4}}\,d\xi,\end{aligned}$$

where $u=k-\frac{1}{2}$ and the limit of integration ξ_0 is defined by

$$\cosh\frac{\sqrt{\xi_0}}{2}=1+\frac{1}{2l}.$$

Making the change of variables $\sqrt{\xi}=t$, one has

$$t_0=4\operatorname{arcsinh}\frac{1}{\sqrt{4l}}.$$

Since $t_0\gg 1/\sqrt{l}$ and $ut\geq ut_0\gg k/\sqrt{l}\gg 1$, we estimate the Bessel function as follows

$$K_0(ut)\ll\frac{\exp(-ut)}{\sqrt{ut}}.$$

Finally,

$$\begin{aligned}E_{2,1}(k,l)&\ll l^{5/6}\int_{t_0}^{\infty}\Big(\sinh\frac{t}{2}\Big)^{5/6}\frac{|K_0(ut)|}{\sqrt{t}}t\,dt\ll\frac{l^{5/6}}{\sqrt{k}}\int_{t_0}^{\infty}\Big(\sinh\frac{t}{2}\Big)^{5/6}e^{-ut}\,dt\\&\ll\frac{l^{5/6}}{\sqrt{k}}\Big(\int_{t_0}^{1}t^{5/6}\exp(-ut)\,dt+\int_{1}^{\infty}\exp(-ut+5t/12)\,dt\Big)\ll\frac{l^{5/12}}{u\sqrt{k}}\exp(-ut_0).\end{aligned}$$

The second summand can be estimated similarly:

$$\begin{aligned}E_{2,2}(k,l):=l^{-1/3}\Big|\Psi_k\Big(\frac{4l^2}{(2l+1)^2}\Big)\Big|&\ll l^{-1/3}\frac{C_K|Z_k(\xi)|}{\xi^{1/4}(\sinh\sqrt{\xi})^{1/2}}\\&\ll l^{-1/3}l^{1/2}|Z_K(\xi)|\ll l^{1/6}\sqrt{\xi}|K_0(u\sqrt{\xi})|\ll\frac{\exp(-ut_0)}{k^{1/2}l^{1/12}}.\end{aligned}$$

To sum up,

$$E_2(k,l) \ll E_{2,1}(k,l) + E_{2,2}(k,l) \ll \frac{1}{l^{1/12}\sqrt{k}} \exp\left(\frac{-ck}{\sqrt{l}}\right)$$

for some $c > 0$. □

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F-signature and Hilbert–Kunz multiplicity: a combined approach and comparison

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We present a unified approach to the study of F-signature, Hilbert–Kunz multiplicity, and related limits governed by Frobenius and Cartier linear actions in positive characteristic commutative algebra. We introduce general techniques that give vastly simplified proofs of existence, semicontinuity, and positivity. Furthermore, we give an affirmative answer to a question of Watanabe and Yoshida allowing the F-signature to be viewed as the infimum of relative differences in the Hilbert–Kunz multiplicities of the cofinite ideals in a local ring.

1. Introduction

Throughout this paper, we shall assume all rings R are commutative and Noetherian with prime characteristic $p > 0$. Central to the study of such rings is the use of the Frobenius or p-th power endomorphism $F\colon R \to R$ defined by $r \mapsto r^p$ for all $r \in R$. Following the result of Kunz [1969] characterizing regularity by the flatness of Frobenius, it has long been understood that the behavior of Frobenius governs the singularities of such rings. In this article, we will be primarily concerned with two important numerical invariants that measure the failure of flatness for the iterated Frobenius: the Hilbert–Kunz multiplicity [Monsky 1983] and the F-signature [Smith and Van den Bergh 1997; Huneke and Leuschke 2002]. Our aim is to revisit a number of core results about these invariants — existence [Tucker 2012], semicontinuity [Smirnov 2016; Polstra 2015], and positivity [Hochster and Huneke 1994; Aberbach and Leuschke 2003] — and provide vastly simplified proofs, which in turn yield new and important results. In particular, we confirm the suspicion of Watanabe and Yoshida [2004, Question 1.10] allowing the F-signature to be viewed as the infimum of relative differences in the Hilbert–Kunz multiplicities of the cofinite ideals in a local ring.

For the sake of simplicity in introducing Hilbert–Kunz multiplicity and F-signature, assume that $(R, \mathfrak{m}, k)$ is a complete local domain of dimension d and $k = k^{1/p}$ is perfect. If $I \subseteq R$ is an ideal with finite colength $\ell_R(R/I) < \infty$, the Hilbert–Kunz multiplicity along I is defined by $e_{\mathrm{HK}}(I) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I^{[p^e]})$ where $I^{[p^e]} = (F^e(I))$ is the expansion of I along the e-iterated Frobenius. Note that, unlike for Hilbert–Samuel multiplicity, the function $e \mapsto \ell_R(R/I^{[p^e]})$ is far from polynomial in p^e [Han and Monsky 1993]. The existence of Hilbert–Kunz limits was shown by Monsky [1983], and

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it has recently been shown by Brenner [2013] that there exist irrational Hilbert–Kunz multiplicities. Of particular interest is the Hilbert–Kunz multiplicity along the maximal ideal $\mathfrak{m}$ denoted by $e_{\mathrm{HK}}(R) = e_{\mathrm{HK}}(\mathfrak{m})$, where it is known that $e_{\mathrm{HK}}(R) \geq 1$ with equality if and only if R is regular [Watanabe and Yoshida 2000, Theorem 1.5]. More generally, if the Hilbert–Kunz multiplicity of R is sufficiently small, then R is Gorenstein and strongly F-regular [Blickle and Enescu 2004, Proposition 2.5; Aberbach and Enescu 2008, Corollary 3.6]. Recently, it has been shown that the Hilbert–Kunz multiplicity determines an upper semicontinuous $\mathbb{R}$-valued function on ring spectra [Smirnov 2016].

Another useful perspective comes from viewing the Hilbert–Kunz function $\ell_R(R/\mathfrak{m}^{[p^e]}) = \mu_R(R^{1/p^e})$ as measuring the minimal number of generators of the finitely generated R-module R^{1/p^e}, the ring of p^e-th roots of R inside an algebraic closure of its fraction field. As the rank $\mathrm{rk}_R(R^{1/p^e})$ of this torsion-free R-module is p^{ed}, we see immediately that R^{1/p^e} is free — and hence F^e is flat and R is regular — if and only if $\mu_R(R^{1/p^e}) = p^{ed}$. Similarly, letting $a_e = \mathrm{frk}(R^{1/p^e})$ denote the largest rank of a free summand appearing in an R-module direct sum decomposition of R^{1/p^e}, we have that R^{1/p^e} is free if and only if $a_e = p^{ed}$. The limit $s(R) = \lim_{e\to\infty}(1/p^{ed})\,\mathrm{frk}(R^{1/p^e})$ was first studied in [Smith and Van den Bergh 1997] and revisited in [Huneke and Leuschke 2002], where it was coined the F-signature and shown to exist for Gorenstein rings; existence in full generality was first shown in [Tucker 2012]. Aberbach and Leuschke [2003] have shown that the positivity of the F-signature characterizes the notion of strong F-regularity introduced by Hochster and Huneke [1994] in their celebrated study of tight closure [1990]. In [Polstra 2015] the first author showed that the F-signature determines a lower semicontinuous $\mathbb{R}$-valued function on ring spectra; an unpublished and independent proof was simultaneously found and shown to experts by the second author, and has been incorporated into this article.

At the heart of this work is an elementary argument simultaneously proving the existence of Hilbert–Kunz multiplicity and F-signature (Theorem 3.2), derived from basic properties of the functions $\mu_R(\_)$ and $\mathrm{frk}_R(\_)$. Building upon the philosophy introduced in [Tucker 2012], we further attempt to carefully track the uniform constants controlling convergence in the proof and throughout this article. To that end, we rely heavily on the uniform bounds for Hilbert–Kunz functions over ring spectra (Theorem 3.3) shown in [Polstra 2015, Theorem 4.3]. In particular, this also allows us to give vastly simplified proofs of the upper semicontinuity of Hilbert–Kunz multiplicity (Theorem 3.4) and lower semicontinuity of the F-signature (Theorem 3.8).

The F-signature has also long been known to be closely related to the relative differences $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)$ in Hilbert–Kunz multiplicities for ideals $I \subseteq J$ with finite colength [Huneke and Leuschke 2002, Theorem 15] (see Lemma 6.1). The infimum among these differences was studied independently by Watanabe and Yoshida [2004], and the connection to F-signature was made by Yao [2006]. Moreover, Watanabe and Yoshida [2004, Question 1.10] expected and formally questioned if the infimum of the relative Hilbert–Kunz differences was equal to the F-signature. We confirm their suspicions in Section 6.

Theorem A (Corollary 6.5). *If $(R, \mathfrak{m}, k)$ is an F-finite local ring, then*

$$s(R) = \inf_{\substack{I\subseteq J\subseteq R,\ \ell_R(R/I)<\infty \\ I\neq J,\ \ell_R(R/J)<\infty}} \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} = \inf_{\substack{I\subseteq R,\ \ell_R(R/I)<\infty \\ x\in R,\ (I:x)=\mathfrak{m}}} e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I,x)).$$

Much interest in relative differences of Hilbert–Kunz multiplicities stems from the result of Hochster and Huneke [1990, Theorem 8.17] that such differences can be used to detect instances of tight closure. Together with the characterization of strong F-regularity in terms of the positivity of the F-signature [Aberbach and Leuschke 2003], these two results are at the core of the use of asymptotic Frobenius techniques to measure singularities. Building off of previous work of the second author [Blickle et al. 2012; Schwede and Tucker 2014; 2015], we present new and highly simplified proofs of these results. In particular, the first proof of Theorem 5.1 gives a readily computable lower bound for the F-signature of a strongly F-regular ring.

Using standard reduction to characteristic $p > 0$ techniques, the singularities governed by Frobenius in positive characteristic commutative algebra have been closely related to those appearing in complex algebraic geometry. This connection has motivated several important generalized settings for tight closure. Hara and Watanabe [2002] have defined tight closure for divisor pairs (compare to [Takagi 2004]), and Hara and Yoshida [2003] have done the same for ideals with a real coefficient; both of these works build on the important results of Smith [1997], Hara [1998], and Mehta and Srinivas [1997]. Previous work of the second author [Blickle et al. 2012; 2013] has extended the notion of F-signature to these settings and beyond, incorporating the notion of a Cartier subalgebra [Blickle and Schwede 2013]. We work to extend all of our results as far as possible to these settings, including new and simplified proofs of existence (Theorem 4.7), semicontinuity (Theorems 4.9, 4.11, and 4.12), and positivity (Corollaries 5.12, 5.7(ii), and 5.8(ii)); many of these results have not appeared previously. In particular, while Hilbert–Kunz theory of divisor and ideal pairs has not been introduced, we are able to give an analogue of relative Hilbert–Kunz differences in these settings (Corollaries 5.7(i) and 5.8(i)). Once more, the F-signatures can be viewed as a minimum among the relative Hilbert–Kunz differences (Corollaries 6.9 and 6.10).

To give an overview of our methods, for each $e \in \mathbb{N}$ let $I_e^{\mathrm{HK}} = \mathfrak{m}^{[p^e]}$ and $I_e^{F\text{-sig}} = (r \in R \mid \psi(r^{1/p^e}) \in \mathfrak{m}$ for all $\psi \in \mathrm{Hom}_R(R^{1/p^e}, R))$. As k is perfect, straightforward computations show that $(1/p^{ed})\mu(R^{1/p^e}) = (1/p^{ed})\ell(R/I_e^{\mathrm{HK}})$ and $(1/p^{ed})\,\mathrm{frk}(R^{1/p^e}) = (1/p^{ed})\ell(R/I_e^{F\text{-sig}})$, so that the Hilbert–Kunz multiplicity and the F-signature can be understood by studying properties of sequences of ideals $\{I_e\}_{e\in\mathbb{N}}$ satisfying properties similar to those of $\{I_e^{\mathrm{HK}}\}_{e\in\mathbb{N}}$ and $\{I_e^{F\text{-sig}}\}_{e\in\mathbb{N}}$.

Theorem B (Corollary 4.5, Theorem 5.5). *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, and consider a sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$ such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for each e. Then the limit $\eta = \lim_{e\to\infty}(1/p^{ed})\ell(R/I_e)$ exists provided one of the following conditions holds.*

(i) *There exists $0 \neq c \in R$ so that $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$.*

(ii) *There exists a nonzero $\psi \in \mathrm{Hom}_R(R^{1/p}, R)$ so that $\psi(I_{e+1}^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$.*

Moreover, *in case* (ii), *we have $\eta > 0$ if and only if $\bigcap_{e\in\mathbb{N}} I_e = 0$.*

It is easy to check that the sequences $\{I_e^{\mathrm{HK}}\}_{e\in\mathbb{N}}$ or $\{I_e^{F\text{-sig}}\}_{e\in\mathbb{N}}$ satisfy both conditions (i) and (ii) above (with *any* choice of c and ψ). Our results on positivity come from the stated criterion for sequences of ideals satisfying (ii), while the relative Hilbert–Kunz statements largely come through careful tracking of uniform constants bounding the growth rates for sequences satisfying (i).

The article is organized as follows. Section 2 recalls a number of preliminary statements about rings in positive characteristic; in particular, we review properties of F-finite rings, which will be the primary setting of this article. We also discuss elementary properties of the maximal free rank (see Lemma 2.1) which are used in the unified proof of the existence of the Hilbert–Kunz multiplicity and the F-signature provided in Section 3; new proofs of semicontinuity follow immediately thereafter. Section 4 is the technical heart of the paper, and generalizes the methods of the previous sections for Hilbert–Kunz multiplicity and F-signature alone to the various sequences of ideals satisfying the conditions in Theorem B above.

Having dispatched with the results on existence and semicontinuity, we turn in Section 5 to a discussion of positivity statements. In particular, two simple proofs of the positivity of the F-signature for strongly F-regular rings appear at the beginning of this section. The relative Hilbert–Kunz criteria for tight closure along a divisor or ideal pair appear at the end of this section, together with a discussion of the positivity of the F-splitting ratio. Section 6 is devoted to relating the F-signature with minimal relative Hilbert–Kunz differences giving a proof of Theorem A, and we conclude in Section 7 with a discussion of a number of related open questions.

2. Preliminaries

Throughout this paper, we shall assume all rings R are commutative with a unit, Noetherian, and have prime characteristic $p > 0$. If $\mathfrak{p} \in \operatorname{Spec}(R)$, we let $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ denote the corresponding residue field. A local ring is a triple $(R, \mathfrak{m}, k)$ where $\mathfrak{m}$ is the unique maximal ideal of the ring R and $k = k(\mathfrak{m}) = R/\mathfrak{m}$.

Maximal free rank. For any ring R and R-module M, $\ell_R(M)$ denotes the length of M, and $\mu_R(M)$ denotes the minimal number of generators of M. When ambiguity is unlikely, we freely omit the subscript R from these and similar notations. We define the *maximal free rank* $\operatorname{frk}_R(M)$ of M to be the maximal rank of a free R-module quotient of M. As a surjection onto a free module necessarily admits a section, $\operatorname{frk}_R(M)$ is also the maximal rank of a free direct summand of M. Equivalently, M admits a direct sum decomposition $M = R^{\oplus \operatorname{frk}_R(M)} \oplus N$ where N has no free direct summands — a property characterized by $\phi(N) \neq R$ for all $\phi \in \operatorname{Hom}_R(N, R)$. It is immediate that $\operatorname{frk}_R(M) \leq \mu_R(M)$ with equality if and only if M is a free R-module.

When R is a domain, we will use $\operatorname{rk}_R(M)$ to denote the torsion-free rank of M. It is easily checked that

$$\operatorname{frk}_R(M) \leq \operatorname{rk}_R(M) \leq \mu_R(M) \tag{1}$$

and the second inequality is strict when M is not free; assuming M is torsion-free but not free, the first inequality is strict as well. Notice that, whereas $\mu(\_)$ is subadditive on short exact sequences over local rings, this is not the case for $\operatorname{frk}(\_)$ and motivates the following lemma.

Lemma 2.1. *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(i) *If M_1 and M_2 are R-modules, then* $\operatorname{frk}(M_1 \oplus M_2) = \operatorname{frk}(M_1) + \operatorname{frk}(M_2)$.

(ii) *If M is an R-module with $M' \subseteq M$ a submodule and $M'' = M/M'$, then*

$$\mathrm{frk}(M'') \leq \mathrm{frk}(M) \leq \mathrm{frk}(M') + \mu(M'').$$

Proof. (i) If $M_1 = R^{\oplus \mathrm{frk}_R(M_1)} \oplus N_1$ and $M_2 = R^{\oplus \mathrm{frk}_R(M_2)} \oplus N_2$ where N_1, N_2 have no free direct summands, then $M_1 \oplus M_2 = R^{\oplus(\mathrm{frk}_R(M_1)+\mathrm{frk}_R(M_2))} \oplus (N_1 \oplus N_2)$. If $\phi \in \mathrm{Hom}_R(N_1 \oplus N_2, R)$, then $\phi(N_1), \phi(N_2) \subseteq \mathfrak{m}$ as N_1, N_2 have no free direct summands. It follows that $\phi(N_1 \oplus N_2) = \phi(N_1) + \phi(N_2) \subseteq \mathfrak{m}$ as well, showing $N_1 \oplus N_2$ has no free direct summands as desired.

(ii) For the first inequality, a surjection $M'' \to R^{\oplus \mathrm{frk}(M'')}$ induces another $M \to M/M' = M'' \to R^{\oplus \mathrm{frk}(M'')}$, showing $\mathrm{frk}(M'') \leq \mathrm{frk}(M)$. To show the final inequality, let n be the maximal rank of a mutual free direct summand of M and M'. In other words, simultaneously decompose $M = R^{\oplus n} \oplus N$ and $M' = R^{\oplus n} \oplus N'$ where $M' \subseteq M$ is given by equality on $R^{\oplus n}$ and an inclusion $N' \subseteq N$, and we have $\phi(N') \subseteq \mathfrak{m}$ for every $\phi\colon N \to R$. Taking a surjection $\Phi\colon N \to R^{\oplus \mathrm{frk}(N)}$, it follows that $\Phi(N') \subseteq \mathfrak{m}^{\oplus \mathrm{frk}(N)}$. Thus, Φ induces a surjection $M'' = N/N' \to k^{\oplus \mathrm{frk}(N)}$ and hence also $M''/\mathfrak{m}M'' \to k^{\oplus \mathrm{frk}(N)}$, which shows $\mu(M'') \geq \mathrm{frk}(N)$. This gives $\mathrm{frk}(M) = n + \mathrm{frk}(N) \leq \mathrm{frk}(M') + \mu(M'')$ as desired. □

For X any topological space, recall that a function $f : X \to \mathbb{R}$ is lower semicontinuous if and only if for any $\delta \in \mathbb{R}$ the set $\{x \in X \mid f(x) > \delta\}$ is open. Similarly, $f : X \to \mathbb{R}$ is upper semicontinuous if and only if for any $\delta \in \mathbb{R}$ the set $\{x \in X \mid f(x) < \delta\}$ is open.

Lemma 2.2. *If M is a finitely generated R-module, the function* $\mathrm{Spec}(R) \to \mathbb{R}$ *given by* $\mathfrak{p} \mapsto \mathrm{frk}_{R_\mathfrak{p}}(M_\mathfrak{p})$ *is lower semicontinuous. Similarly, the function* $\mathrm{Spec}(R) \to \mathbb{R}$ *given by* $\mathfrak{p} \mapsto \mu_{R_\mathfrak{p}}(M_\mathfrak{p})$ *is upper semicontinuous.*

Proof. For $\mathfrak{p} \in \mathrm{Spec}(R)$, let $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ denote the corresponding residue field. If $\mathrm{frk}_{R_\mathfrak{p}}(M_\mathfrak{p}) = n$, we can find a surjection $M_\mathfrak{p} \to R_\mathfrak{p}^{\oplus n}$. Without loss of generality, since M is finitely generated, we may assume this surjection is the localization of an R-module homomorphism $M \to R^{\oplus n}$. As it becomes surjective when localized at $\mathfrak{p}$, there is some $g \in R \setminus \mathfrak{p}$ so that $M_g \to R_g^{\oplus n}$ is surjective. Localizing at any $\mathfrak{q} \in \mathrm{Spec}(R)$ with $g \notin \mathfrak{q}$ yields a surjection $M_\mathfrak{q} \to R_\mathfrak{q}^{\oplus n}$, implying $\mathrm{frk}_{R_\mathfrak{q}}(M_\mathfrak{q}) \geq n = \mathrm{frk}_{R_\mathfrak{p}}(M_\mathfrak{p})$ for any $\mathfrak{q}$ in the Zariski open neighborhood $D(g) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid g \notin \mathfrak{q}\}$ of $\mathfrak{p}$ in $\mathrm{Spec}(R)$. If $\delta \in \mathbb{R}$ and $\mathrm{frk}_{R_\mathfrak{p}}(M_\mathfrak{p}) > \delta$, then so also $\mathrm{frk}_{R_\mathfrak{q}}(M_\mathfrak{q}) > \delta$ for any $\mathfrak{q} \in D(g)$. This shows the function $\mathrm{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \mathrm{frk}_R(M_\mathfrak{p})$ is lower semicontinuous. That the function $\mathrm{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \mu(M_\mathfrak{p})$ is upper semicontinuous proceeds in an analogous manner. □

***F*-finite rings.** The Frobenius or p-th power endomorphism $F\colon R \to R$ is defined by $r \mapsto r^p$ for all $r \in R$. Similarly, for $e \in \mathbb{N}$, we have $F^e\colon R \to R$ given by $r \mapsto r^{p^e}$. The expansion of an ideal I over F^e is denoted $I^{[p^e]} = (F^e(I))$. In case R is a domain, we let R^{1/p^e} denote the subring of p^e-th roots of R inside a fixed algebraic closure of the fraction field of R. By taking p^e-th roots, R^{1/p^e} is abstractly isomorphic to R, and the Frobenius map is identified with the ring extension $R \subseteq R^{1/p^e}$. More generally, for any R and any R-module M, we will write M^{1/p^e} for the R-module given by restriction of scalars for F^e; this is an exact functor on the category of R modules. We say that R is F-finite when R^{1/p^e} is a finitely generated R-module, which further implies that R is excellent [Kunz 1976, Theorem 2.5]. A complete local ring $(R, \mathfrak{m}, k)$ is F-finite if and only if its residue field k is F-finite. When R is F-finite and M is a finitely generated R-module, M^{1/p^e} is also a finitely generated R-module and

$$\ell_R(M^{1/p^e}) = [k^{1/p^e} : k] \cdot \ell_{R^{1/p^e}}(M^{1/p^e}) = [k^{1/p^e} : k] \cdot \ell_R(M).$$

In particular, note that

$$\mu_R(M^{1/p^e}) = \ell_R(M^{1/p^e} \otimes_R k) = \ell_R((M/\mathfrak{m}^{[p^e]}M)^{1/p^e}) = [k^{1/p^e} : k] \cdot \ell_R(M/\mathfrak{m}^{[p^e]}M). \tag{2}$$

Lemma 2.3. *Suppose $(R, \mathfrak{m}, k)$ is a complete local F-finite domain with coefficient field k and system of parameters $x_1, \dots, x_d$ chosen so that R is module finite and generically separable over the regular subring $A = k[\![x_1, \dots, x_d]\!]$. Then there exists $0 \neq c \in A$ so that $c \cdot R^{1/p^e} \subseteq R[A^{1/p^e}] = R \otimes_A A^{1/p^e}$ for all $e \in \mathbb{N}$.*

Proof. Note first that, by the Cohen–Gabber structure theorem, every complete local ring admits such a coefficient field and system of parameters; see [Kurano and Shimomoto 2015] for an elementary proof. Let $K = \operatorname{Frac}(A) \subseteq L = \operatorname{Frac}(R)$ be the corresponding extension of fraction fields. The separability assumption implies L and K^{1/p^e} are linearly disjoint over K, so that $L \otimes_K K^{1/p^e} = LK^{1/p^e}$ is reduced. Since A^{1/p^e} is a free A-module, $R \otimes_A A^{1/p^e}$ is a free R-module and injects into its localization $L \otimes_A A^{1/p^e} = L \otimes_K K^{1/p^e}$. Thus, $R \otimes_A A^{1/p^e}$ is also reduced, which gives the equality $R \otimes_A A^{1/p^e} = R[A^{1/p^e}]$. If $e_1, \dots, e_s \in R$ generate R as an A-module, it is easy to see $c = (\det(\operatorname{tr}(e_i e_j)))^2$ satisfies $c \cdot R^{1/p^e} \subseteq R[A^{1/p^e}]$ where tr is the trace map; see [Hochster and Huneke 2000, Lemma 4.3] for a proof and [Hochster and Huneke 1990, §6] for further discussion. □

The corollaries below follow immediately; see [Kunz 1976] for details.

Corollary 2.4. *If $(R, \mathfrak{m}, k)$ is an F-finite local domain of dimension d, then $\operatorname{rk}_R(R^{1/p^e}) = [k^{1/p^e} : k] \cdot p^{ed}$.*

Corollary 2.5. *If R is F-finite, for any two prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of R it follows that*

$$[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] = [k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] \cdot p^{e \dim(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})}.$$

If additionally R is locally equidimensional, we have

$$[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e \dim(R_\mathfrak{p})} = [k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] \cdot p^{e \dim(R_\mathfrak{q})}$$

and the function $\operatorname{Spec}(R) \to \mathbb{N}$ given by $\mathfrak{p} \mapsto [k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e \dim(R_\mathfrak{p})}$ is constant on the connected components of $\operatorname{Spec}(R)$.

Corollary 2.6. *If R is a locally equidimensional F-finite reduced ring with connected spectrum and $p^\gamma = [k(\mathfrak{p})^{1/p} : k(\mathfrak{p})] \cdot p^{\dim(R_\mathfrak{p})}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, there exist short exact sequences*

$$0 \longrightarrow R^{\oplus p^\gamma} \longrightarrow R^{1/p} \longrightarrow M \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow R^{1/p} \longrightarrow R^{\oplus p^\gamma} \longrightarrow N \longrightarrow 0 \tag{4}$$

so that $\dim(M_\mathfrak{p}) < \dim(R_\mathfrak{p})$ and $\dim(N_\mathfrak{p}) < \dim(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

3. F-signature and Hilbert–Kunz multiplicity revisited

The first goal of this section is to provide an elementary proof that simultaneously shows the existence of the F-signature and Hilbert–Kunz multiplicity for an F-finite local domain $(R, \mathfrak{m}, k)$. At the heart of all known proofs of existence is a basic observation on the growth rate of a Hilbert–Kunz function.

Lemma 3.1 [Monsky 1983, Lemma 1.1]. *Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Then there exists a positive constant $C(M, \mathfrak{m}) \in \mathbb{R}$ so that*

$$\ell(M/\mathfrak{m}^{[p^e]}M) \leq C(M, \mathfrak{m}) \cdot p^{e \dim(M)}$$

for each $e \in \mathbb{N}$.

Proof. If $\mathfrak{m}$ is generated by t elements, then $\mathfrak{m}^{tp^e} \subseteq \mathfrak{m}^{[p^e]}$ for each $e \in \mathbb{N}$. Thus, $\ell(M/\mathfrak{m}^{[p^e]}M) \leq \ell(M/\mathfrak{m}^{tp^e}M)$, which is eventually polynomial of degree $\dim(M)$ in the entry p^e. $\square$

Theorem 3.2 [Monsky 1983, Theorem 1.8; Tucker 2012, Theorem 4.9]. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite local domain and $\dim(R) = d$. Then the limits*

$$e_{\mathrm{HK}}(R) = \lim_{e\to\infty} \frac{\mu_R(R^{1/p^e})}{[k^{1/p^e} : k] \cdot p^{ed}}, \qquad s(R) = \lim_{e\to\infty} \frac{\mathrm{frk}_R(R^{1/p^e})}{[k^{1/p^e} : k] \cdot p^{ed}}$$

exist.

Proof. Let $p^\gamma = [k^{1/p} : k] \cdot p^d = \mathrm{rk}_R(R^{1/p^e})$ and $\nu(\_)$ denote either $\mu(\_)$ or $\mathrm{frk}(\_)$. Set $\eta_e = (1/p^{e\gamma})\nu_R(R^{1/p^e})$, and observe using Lemma 3.1 with (1) and (2) that the sequence $\{\eta_e\}_{e\in\mathbb{N}}$ is bounded above. Put $\eta^+ = \limsup_{e\to\infty} \eta_e$ and $\eta^- = \liminf_{e\to\infty} \eta_e$.

As in (3), fix a short exact sequence

$$0 \longrightarrow R^{\oplus p^\gamma} \longrightarrow R^{1/p} \longrightarrow M \longrightarrow 0$$

where $\dim(M) < d$. Applying the exact functors $(\_)^{1/p^e}$ gives the short exact sequences

$$0 \longrightarrow (R^{1/p^e})^{\oplus p^\gamma} \longrightarrow R^{1/p^{e+1}} \longrightarrow M^{1/p^e} \longrightarrow 0,$$

and Lemma 2.1 implies

$$\nu(R^{1/p^{e+1}}) \leq p^\gamma \cdot \nu(R^{1/p^e}) + \mu(M^{1/p^e})$$

for each $e \in \mathbb{N}$. By Lemma 3.1 and using (2), there exists a positive constant $C(M, \mathfrak{m}) \in \mathbb{R}$ with $\mu(M^{1/p^e}) \leq C(M, \mathfrak{m})p^{e \dim(M)} \leq C(M, \mathfrak{m})p^{e(d-1)}$. Dividing through by $p^{(e+1)d}$ and setting $C = C(M, \mathfrak{m})/p^d \in \mathbb{R}$ yields $\eta_{e+1} \leq \eta_e + C/p^e$. Iterating this inequality gives

$$\eta_{e+e'} \leq \eta_e + \frac{C}{p^e}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e'-1}}\right) \leq \eta_e + \frac{2C}{p^e}$$

for all $e, e' \in \mathbb{N}$. For each e, taking $\limsup_{e'\to\infty}$ then gives $\eta^+ \leq \eta_e + 2C/p^e$. Now taking $\liminf_{e\to\infty}$ gives $\eta^+ \leq \eta^-$ as desired. $\square$

Our next goal is to give a direct proof of the upper semicontinuity of Hilbert–Kunz multiplicity. The key observation is that the constant appearing in Lemma 3.1 can be taken uniformly on ring spectra.

Theorem 3.3 [Polstra 2015, Theorem 4.3]. *For any F-finite ring R and finitely generated module M, there exists a positive constant $C(M)$ such that for all $\mathfrak{p} \in \operatorname{Spec}(R)$*

$$\ell_{R_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}^{[p^e]}M_\mathfrak{p}) \le C(M)p^{e\dim(M_\mathfrak{p})}.$$

From the proof of Theorem 3.2 above, which is valid without change for a local equidimensional reduced ring, it follows that there is a positive constant $C \in \mathbb{R}$ such that

$$e_{\mathrm{HK}}(R_\mathfrak{p}) \le \frac{\mu_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\dim(R_\mathfrak{p})}} + \frac{2C}{p^e}, \qquad s(R_\mathfrak{p}) \le \frac{\operatorname{frk}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\dim(R_\mathfrak{p})}} + \frac{2C}{p^e}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all $e \in \mathbb{N}$.

Theorem 3.4 [Smirnov 2016, Main result]. *For any F-finite locally equidimensional reduced ring R, the function $e_{\mathrm{HK}}\colon \operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto e_{\mathrm{HK}}(R_\mathfrak{p})$ is upper semicontinuous.*

Proof. Restricting to a connected component, we may assume without loss of generality that $\operatorname{Spec}(R)$ is connected and hence $p^\gamma = [k(\mathfrak{p})^{1/p} : k(\mathfrak{p})] \cdot p^{\dim(R_\mathfrak{p})}$ is constant for all $\mathfrak{p} \in \operatorname{Spec}(R)$ by Corollary 2.5. If $\delta \in \mathbb{R}$ and $e_{\mathrm{HK}}(R_\mathfrak{q}) < \delta$ for some $\mathfrak{q} \in \operatorname{Spec}(R)$, then

$$\frac{\mu_{R_\mathfrak{q}}(R_\mathfrak{q}^{1/p^e})}{[k(\mathfrak{q})^{1/p^e} : k(\mathfrak{q})] \cdot p^{e\dim(R_\mathfrak{q})}} < \delta - \epsilon$$

for some $0 < \epsilon \ll 1$ and some $e \in \mathbb{N}$ with $2C/p^e < \epsilon$. By Lemma 2.2, the same holds true for all $\mathfrak{p}$ in a neighborhood of $\mathfrak{q}$. This yields

$$e_{\mathrm{HK}}(R_\mathfrak{p}) \le \frac{\mu_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\dim(R_\mathfrak{p})}} + \frac{2C}{p^e} < \delta - \epsilon + \frac{2C}{p^e} < \delta$$

as desired and completes the proof. □

To get a similar argument for lower semicontinuity of F-signature, we need to reverse the estimates arising in the proof of existence. To that end, we first record the following elementary lemma.

Lemma 3.5. *Let p be a prime number, $d \in \mathbb{N}$, and $\{\lambda_e\}_{e\in\mathbb{N}}$ be a sequence of real numbers so that $\{\lambda_e/p^{ed}\}_{e\in\mathbb{N}}$ is bounded.*

(i) *If there exists a positive constant $C \in \mathbb{R}$ so that $\lambda_{e+1}/p^{(e+1)d} \le \lambda_e/p^{ed} + C/p^e$ for all $e \in \mathbb{N}$, then the limit $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda - \lambda_e/p^{ed} \le 2C/p^e$ for all $e \in \mathbb{N}$.*

(ii) *If there exists a positive constant $C \in \mathbb{R}$ so that $\lambda_e/p^{ed} \le \lambda_{e+1}/p^{(e+1)d} + C/p^e$ for all $e \in \mathbb{N}$, then the limit $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda_e/p^{ed} - \lambda \le 2C/p^e$ for all $e \in \mathbb{N}$.*

(iii) *If there exists a positive constant $C \in \mathbb{R}$ so that $|\lambda_{e+1}/p^{(e+1)d} - \lambda_e/p^{ed}| \le C/p^e$ for all $e \in \mathbb{N}$, then the limit $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $|\lambda_e/p^{ed} - \lambda| \le 2C/p^e$ for all $e \in \mathbb{N}$. In particular, $\lambda_e = \lambda p^{ed} + O(p^{e(d-1)})$.*

Proof. Let $\lambda^+ = \limsup_{e\to\infty} \lambda_e/p^{ed}$ and $\lambda^- = \liminf_{e\to\infty} \lambda_e/p^{ed}$, which are finite as $\{\lambda_e/p^{ed}\}_{e\in\mathbb{N}}$ is bounded. Iterating the inequality in (i) yields

$$\frac{1}{p^{(e+e')d}}\lambda_{e+e'} \le \frac{1}{p^{ed}}\lambda_e + \frac{C}{p^e}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e'-1}}\right) \le \frac{1}{p^{ed}}\lambda_e + \frac{2C}{p^e}$$

for all $e, e' \in \mathbb{N}$. Taking $\limsup_{e'\to\infty}$ for each fixed $e \in \mathbb{N}$ gives $\lambda^+ \le \lambda_e/p^{ed} + 2C/p^e$, and then applying $\liminf_{e\to\infty}$ gives $\lambda^+ \le \lambda^-$ so that $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda - \lambda_e/p^{ed} \le 2C/p^e$ for all $e \in \mathbb{N}$. Similarly, iterating the inequality in (ii) yields

$$\frac{1}{p^{ed}}\lambda_e \le \frac{1}{p^{(e+e')d}}\lambda_{e+e'} + \frac{C}{p^e}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{e'-1}}\right) \le \frac{1}{p^{(e+e')d}}\lambda_{e+e'} + \frac{2C}{p^e}$$

for all $e, e' \in \mathbb{N}$. Taking $\liminf_{e'\to\infty}$ for each fixed $e \in \mathbb{N}$ gives $\lambda_e/p^{ed} \le \lambda^- + 2C/p^e$, and then applying $\limsup_{e\to\infty}$ gives $\lambda^+ \le \lambda^-$ so that $\lambda = \lim_{e\to\infty} \lambda_e/p^{ed}$ exists and $\lambda_e/p^{ed} - \lambda \le 2C/p^e$ for all $e \in \mathbb{N}$. The final statement (iii) follows immediately from a combination of (i) and (ii). □

Theorem 3.6. *If R is a locally equidimensional reduced F-finite ring of dimension d, there is a positive constant $C \in \mathbb{R}$ so that*

$$\left|\frac{\mu_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})]\cdot p^{e\dim R_\mathfrak{p}}} - e_{\mathrm{HK}}(R_\mathfrak{p})\right| \le \frac{C}{p^e}, \qquad \left|\frac{\mathrm{frk}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})]\cdot p^{e\dim R_\mathfrak{p}}} - s(R_\mathfrak{p})\right| \le \frac{C}{p^e}$$

for all $e \in \mathbb{N}$ and $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. Restricting to a connected component, we may assume without loss of generality that $\mathrm{Spec}(R)$ is connected and hence $p^\gamma = [k(\mathfrak{p})^{1/p} : k(\mathfrak{p})]\cdot p^{\dim(R_\mathfrak{p})}$ is constant for all $\mathfrak{p} \in \mathrm{Spec}(R)$ by Corollary 2.5.

The proofs of Theorems 3.2 and 3.4 above relied on a short exact sequence as in (3) of Corollary 2.6 and produced inequalities as in Lemma 3.5(i). Repeating those same arguments on a short exact sequence as in (4) of Corollary 2.6 yields inequalities as in Lemma 3.5(ii), which combine to give the desired result. □

Corollary 3.7. *If $(R, \mathfrak{m}, k)$ is an F-finite local equidimensional reduced ring of dimension d,*

$$\frac{\mu_R(R^{1/p^e})}{[k : k^{p^e}]} = e_{\mathrm{HK}}(R)p^{ed} + O(p^{e(d-1)}), \qquad \frac{\mathrm{frk}_R(R^{1/p^e})}{[k : k^{p^e}]} = s(R)p^{ed} + O(p^{e(d-1)}).$$

Theorem 3.8 [Polstra 2015, Theorem 5.7]. *For any F-finite locally equidimensional reduced ring R, the function $s\colon \mathrm{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto s(R_\mathfrak{p})$ is lower semicontinuous.*

Proof. From Theorem 3.6, there is a $C \in \mathbb{R}$ such that

$$\frac{\mathrm{frk}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e})}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})]\cdot p^{e\dim(R_\mathfrak{p})}} \le s(R_\mathfrak{p}) + \frac{C}{p^e}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and all $e \in \mathbb{N}$. The argument in Theorem 3.4 immediately gives the desired result. Alternatively, using the whole of Theorem 3.6, both semicontinuity statements follow from the fact that the uniform limit of upper [Kunz 1976, Corollary 3.4] or lower [Enescu and Yao 2011, Corollary 2.5] semicontinuous functions is upper or lower semicontinuous, respectively. □

4. Limits via Frobenius and Cartier linear maps

Background. Suppose that M is an R-module. Recall that an additive map $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M, M)$ is said to be Frobenius linear or *p-linear* if $\phi(rm) = r^p\phi(m)$ for all $r \in R$ and $m \in M$. Similarly, we say that $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M, M)$ is p^e-linear if $\phi(rm) = r^{p^e}m$ for all $r \in R$ and $m \in M$. The set of all p^e-linear maps on M is denoted $\mathscr{F}_e(M)$ and is naturally both a left and right R-module readily identified with $\mathrm{Hom}_R(M, M^{1/p^e})$. The composition of a p^{e_1}-linear map ϕ_1 and a p^{e_2}-linear map ϕ_2 yields a $p^{e_1+e_2}$-linear map $\phi_1 \circ \phi_2$, so that the ring of Frobenius linear operators $\mathscr{F}(M) = \bigoplus_{e\geq 0} \mathscr{F}_e(M)$ on M forms a noncommutative $\mathbb{N}$-graded ring with $R \to \mathscr{F}_0(M) = \mathrm{Hom}_R(M, M)$. In the case $M = R$, we have that every p^e-linear map $\phi \in \mathrm{Hom}_R(R, R^{1/p^e}) = R^{1/p^e}$ is a postmultiple of the Frobenius F^e by some $c^{1/p^e} \in R$.

Dual to p-linear maps are the Cartier linear or *p^{-1}-linear maps*; an additive map $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M, M)$ is said to be p^{-1}-linear if $\phi(r^pm) = r\phi(m)$ for all $r \in R$ and $m \in M$. Similarly, we say that $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M, M)$ is p^{-e}-linear if $\phi(r^{p^e}m) = rm$ for all $r \in R$ and $m \in M$. The set of all p^{-e}-linear maps on M is denoted $\mathscr{C}_e(M)$ and is naturally both a left and right R-module readily identified with $\mathrm{Hom}_R(M^{1/p^e}, M)$. The composition of a p^{-e_1}-linear map ϕ_1 and a p^{-e_2}-linear map ϕ_2 yields a $p^{-(e_1+e_2)}$-linear map $\phi_1 \circ \phi_2$, so that the ring of Cartier linear operators $\mathscr{C}(M) = \bigoplus_{e\geq 0} \mathscr{C}_e(M)$ on M forms a noncommutative $\mathbb{N}$-graded ring with $R \to \mathscr{C}_0(M) = \mathrm{Hom}_R(M, M)$.

If we have $\phi_1 \in \mathrm{Hom}_R(M^{1/p^{e_1}}, M)$ and $\phi_2 \in \mathrm{Hom}_R(M^{1/p^{e_2}}, M)$, we write $\phi_1 \cdot \phi_2$ for the composition $\phi_1 \circ (\phi_2)^{1/p^{e_1}}$, which coincides with their product when viewed as elements of $\mathscr{C}(M)$. In particular, given $\phi \in \mathrm{Hom}_R(M^{1/p}, M)$ we write $\phi^e \in \mathrm{Hom}_R(M^{1/p^e}, M)$ for the corresponding e-th iterate. If $\psi \in \mathrm{Hom}_R(M^{1/p^e}, M)$, we let $\psi(r^{1/p^e} \cdot \_)$ denote the R-linear map

$$M^{1/p^e} \xrightarrow{r^{1/p^e}} M^{1/p^e} \xrightarrow{\psi} R$$

given by premultiplying with $r^{1/p^e} \in R^{1/p^e}$.

Lemma 4.1. *Let R be an F-finite domain of dimension d with* $\mathrm{rk}_R(R^{1/p}) = p^\gamma$.

(i) *If $0 \neq c \in R$, then there exists a short exact sequence*

$$0 \longrightarrow R^{\oplus p^\gamma} \xrightarrow{\Phi} R^{1/p} \longrightarrow M \longrightarrow 0$$

so that $\dim(M) < R$ *and* $\Phi(R^{\oplus p^\gamma}) \subseteq (cR)^{1/p}$.

(ii) *Let $0 \neq \psi \in \mathrm{Hom}_R(R^{1/p}, R)$ be a nonzero map. There exists a short exact sequence*

$$0 \longrightarrow R^{1/p} \xrightarrow{\Psi} R^{\oplus p^\gamma} \longrightarrow N \longrightarrow 0$$

so that every component function of Ψ *is a premultiple of* ψ. *In other words, there exists* $r_1, \ldots, r_{p^\gamma} \in R$ *so that* $\Psi = (\psi(r_1^{1/p} \cdot \_), \ldots, \psi(r_{p^\gamma}^{1/p} \cdot \_))$.

Proof. For (i), start with any inclusion $R^{\oplus p^\gamma} \subseteq R^{1/p}$ with a torsion quotient as in (3) and postmultiply by $c^{1/p}$ on $R^{1/p}$. For (ii), note that $\mathrm{rk}_{R^{1/p}}(\mathrm{Hom}_R(R^{1/p}, R)) = 1$ so that there is some nonzero $z \in R$ with $z^{1/p} \cdot \mathrm{Hom}_R(R^{1/p}, R) \subseteq \psi \cdot R^{1/p}$. Start with an inclusion $R^{1/p} \subseteq R^{\oplus p^\gamma}$ with torsion quotient as in (4) and premultiply by $z^{1/p}$ on $R^{1/p}$ to achieve the desired sequence. □

If R is an F-finite normal domain and D is a (Weil) divisor on $X = \mathrm{Spec}(R)$, we use $R(D)$ to denote $\Gamma(X, \mathcal{O}_X(D))$. There is a well known correspondence between p^{-e}-linear maps and certain effective $\mathbb{Q}$-divisors. Fixing a canonical divisor K_R, standard duality arguments for finite extensions show that $\mathrm{Hom}_R(R^{1/p^e}, R) = (R((1-p^e)K_R))^{1/p^e}$ for each $e \in \mathbb{N}$. Thus, to each $0 \neq \phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ we can associate a divisor D_ϕ so that $D_\phi \sim_{\mathbb{Z}} (1-p^e)K_R$, and we set $\Delta_\phi = (1/(p^e-1))D_\phi$. If $\phi, \psi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, then $\Delta_\phi \geq \Delta_\psi$ if and only if $\phi(\_) = \psi(r^{1/p^e} \cdot \_)$ for some $0 \neq r \in R$, in which case $\Delta_\phi = \Delta_\psi + (1/(p^e-1))\,\mathrm{div}_R(r)$. Moreover, if $\phi_1 \in \mathrm{Hom}_R(R^{1/p^{e_1}}, R)$ and $\phi_2 \in \mathrm{Hom}_R(R^{1/p^{e_2}}, R)$ with $\phi = \phi_1 \cdot \phi_2 = \phi_1 \circ (\phi_2)^{1/p^{e_1}}$, then $\Delta_\phi = (1/(p^{e_1+e_2}-1))((p^{e_2}-1)\Delta_{\phi_2} + p^{e_2}(p^{e_1}-1)\Delta_{\phi_1})$. In particular, $\Delta_\phi = \Delta_{\phi^n}$ for all $e, n \in \mathbb{N}$ and all $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. If Δ is an effective $\mathbb{Q}$-divisor on $X = \mathrm{Spec}(R)$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, note that $\Delta_\phi \geq \Delta$ if and only if $(p^e-1)\Delta_\phi \geq \lceil (p^e-1)\Delta \rceil$ (as $(p^e-1)\Delta_\phi$ is integral), which is equivalent to asking that ϕ is in the image of the natural restriction mapping

$$\mathrm{Hom}_R(R(\lceil (p^e-1)\Delta \rceil)^{1/p^e}, R) \to \mathrm{Hom}_R(R^{1/p^e}, R).$$

See [Schwede and Tucker 2012, §4] or [2014, §2] for further details.

Lemma 4.2. *Let R be an F-finite domain and $0 \neq \psi \in \mathrm{Hom}_R(R^{1/p}, R)$. There exists an element $0 \neq z \in R$ so that, for any $e \in \mathbb{N}$ and any $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$, the map $z \cdot \phi(\_) = \phi(z \cdot \_) = \phi((z^{p^e})^{1/p^e} \cdot \_) = \psi^e(r^{1/p^e} \cdot \_)$ for some $r^{1/p^e} \in R^{1/p^e}$. In other words, $z\phi$ is a premultiple of ψ^e.*

Proof. Let $\bar{R}$ be the normalization of R in its fraction field, and take $0 \neq c \in R$ inside the conductor ideal $\mathfrak{c} = \mathrm{Ann}_R(\bar{R}/R)$. One can show [Blickle and Schwede 2013, Exercise 6.14] that every p^{-e}-linear map $\phi \colon R^{1/p^e} \to R$ extends uniquely to a p^{-e}-linear map on $\bar{R}$ compatible with $\mathfrak{c}$, which we denoted here by $\bar{\phi} \colon \bar{R}^{1/p^e} \to \bar{R}$. Let $0 \neq x \in R$ be such that $\mathrm{div}_{\bar{R}}(x) \geq \Delta_{\bar{\psi}}$, and put $z = cx$. Then $\Delta_{x\bar{\phi}} = \Delta_{\bar{\phi}} + (p^e/(p^e-1))\,\mathrm{div}_{\bar{R}}(x) \geq \Delta_{\bar{\psi}} = \Delta_{\bar{\psi^e}}$ so that $\bar{\phi}(x \cdot \_) = \psi^e(\bar{r}^{1/p^e} \cdot \_)$ for some $\bar{r} \in \bar{R}$. Letting $r^{1/p^e} = c\bar{r}^{1/p^e} \in R^{1/p^e}$ and restricting back to R, this gives $\phi(z \cdot \_) = \psi^e(r^{1/p^e} \cdot \_)$ as desired. □

Recall that a Cartier subalgebra on R is a graded subring $\mathcal{D} \subseteq \mathcal{C}(R)$ of the ring of Cartier linear operators on R containing $\mathcal{C}_0(R) = \mathrm{Hom}_R(R, R) = R$. Cartier subalgebras can be seen as a natural generalization of a number of commonly studied settings in positive characteristic commutative algebra. For instance, if R is an F-finite domain and $0 \neq \mathfrak{a} \subseteq R$ is an ideal and $t \in \mathbb{R}_{\geq 0}$, the Cartier subalgebra $\mathcal{C}^{\mathfrak{a}^t} = \bigoplus_{e \geq 0} \mathcal{C}_e^{\mathfrak{a}^t}$ where

$$\begin{aligned}\mathcal{C}_e^{\mathfrak{a}^t} &= \mathfrak{a}^{\lceil t(p^e-1) \rceil / p^e}\, \mathrm{Hom}_R(R^{1/p^e}, R) \\ &= \{\phi(x^{1/p^e} \cdot \_) \mid x \in \mathfrak{a}^{\lceil t(p^e-1) \rceil} \text{ and } \phi \in \mathrm{Hom}_R(R^{1/p^e}, R)\}\end{aligned}$$

recovers the framework of [Hara and Yoshida 2003]. Similarly, if R is an F-finite normal domain and Δ is an effective $\mathbb{Q}$-divisor on $\operatorname{Spec}(R)$, the Cartier subalgebra $\mathscr{C}^{(R,\Delta)} = \bigoplus_{e\geq 0} \mathscr{C}_e^{(R,\Delta)}$ where

$$\begin{aligned}\mathscr{C}_e^{(R,\Delta)} &= \{\phi \in \operatorname{Hom}_R(R^{1/p^e}, R) \mid \Delta_\phi \geq \Delta\}\\ &= \operatorname{im}(\operatorname{Hom}_R(R(\lceil (p^e-1)\Delta\rceil)^{1/p^e}, R) \to \operatorname{Hom}_R(R^{1/p^e}, R))\end{aligned}$$

recovers the setting of [Hara and Watanabe 2002; Takagi 2004]. For more information on Cartier subalgebras, see [Blickle and Schwede 2013]. We will mainly be interested in the generalization of F-signature to a Cartier subalgebra $\mathscr{D}$ introduced in [Blickle et al. 2012; 2013].

Existence and semicontinuity.

Theorem 4.3. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, and $\{I_e\}_{e\in\mathbb{N}}$ a sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e \in \mathbb{N}$.*

(i) *If there exists $0 \neq c \in R$ so that $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$, then $\eta = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists. Moreover, there exists a positive constant $C(c)$ depending only on $c \in \mathbb{R}$ with $\eta - (1/p^{ed})\ell_R(R/I_e) \leq C(c)/p^e$ for all $e \in \mathbb{N}$.*

(ii) *If there exists a nonzero R-linear map $\psi\colon R^{1/p} \to R$ so that $\psi(I_{e+1}^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$, then $\eta = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists. Moreover, there exists a positive constant $C(\psi) \in \mathbb{R}$ depending only on ψ such that $(1/p^{ed})\ell_R(R/I_e) - \eta \leq C(\psi)/p^e$ for all $e \in \mathbb{N}$.*

Remark 4.4. The conditions on ideal sequences $\{I_e\}_{e\in\mathbb{N}}$ in Theorem 4.3(i)–(ii) are far more symmetric when phrased in terms of p-linear and p^{-1}-linear maps. In (i), the requirement is simply that there exists a p-linear map $0 \neq \phi \in \mathscr{F}_1(R)$ on R such that $\phi(I_e) \subseteq I_{e+1}$ for each $e \in \mathbb{N}$. Similarly, for (ii) the requirement is that there exists a p^{-1}-linear map $0 \neq \phi \in \mathscr{C}_1(R)$ such that $\phi(I_{e+1}) \subseteq I_e$ for each $e \in \mathbb{N}$.

Proof of Theorem 4.3. For (i), put $\operatorname{rk}_R(R^{1/p}) = [k^{1/p} : k]\cdot p^d = p^\gamma$ and consider a short exact sequence

$$0 \longrightarrow R^{\oplus p^\gamma} \xrightarrow{\ \Phi\ } R^{1/p} \longrightarrow M \longrightarrow 0 \tag{5}$$

as in Lemma 4.1(i) with $\dim(M) < \dim(R)$ and $\Phi(R^{\oplus p^\gamma}) \subseteq (Rc)^{1/p}$. Note that the condition $cI_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$ can be restated as $I_e(Rc)^{1/p} \subseteq (I_{e+1})^{1/p}$, so that $\Phi(I_e^{\oplus p^\gamma}) = \Phi(I_e(R^{\oplus p^\gamma})) \subseteq I_e(Rc)^{1/p} \subseteq (I_{e+1})^{1/p}$. In particular, Φ induces a quotient map

$$\bar{\Phi}\colon (R/I_e)^{\oplus p^\gamma} \to (R/I_{e+1})^{1/p}$$

and it follows that

$$[k^{1/p} : k]\ell_R(R/I_{e+1}) = \ell_R((R/I_{e+1})^{1/p}) \leq p^\gamma \ell_R(R/I_e) + \ell_R(\operatorname{coker}(\bar{\Phi})).$$

Since $\mathfrak{m}^{[p^{e+1}]} \subseteq I_{e+1}$ and $\operatorname{coker}(\bar{\Phi})$ is a quotient of $(R/I_{e+1})^{1/p}$, we have $\mathfrak{m}^{[p^e]} \subseteq \operatorname{Ann}_R(\operatorname{coker}(\bar{\Phi}))$. But $\operatorname{coker}(\bar{\Phi})$ is also a quotient of $\operatorname{coker}(\Phi) = M$ and thus $M/\mathfrak{m}^{[p^e]}M$, so $\ell_R(\operatorname{coker}(\bar{\Phi})) \leq \ell_R(M/\mathfrak{m}^{[p^e]}M) \leq$

$C(M,\mathfrak{m})p^{e(d-1)}$ by Lemma 3.1. Dividing through by $[k^{1/p}:k]p^{(e+1)d}=p^{\gamma+ed}$ yields

$$\frac{1}{p^{(e+1)d}}\ell_R(R/I_{e+1})\le\frac{1}{p^{ed}}\ell_R(R/I_e)+\frac{C(M,\mathfrak{m})/p^{\gamma}}{p^e},$$

and the result now follows from Lemma 3.5 with $C(c)=2C(M,\mathfrak{m})/p^{\gamma}$ independent of the sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$ satisfying the condition in (i).

Similarly for (ii), consider a short exact sequence

$$0\longrightarrow R^{1/p}\xrightarrow{\ \Psi\ }R^{\oplus p^{\gamma}}\longrightarrow N\longrightarrow 0$$

as in Lemma 4.1(ii) with $\dim(N)<\dim(R)$ so that every component function of Ψ is a premultiple of ψ. It follows that $\Psi((I_{e+1})^{1/p})\subseteq I_e^{\oplus p^{\gamma}}$, so that Ψ induces a quotient map

$$\bar{\Psi}\colon (R/I_{e+1})^{1/p}\to(R/I_e)^{\oplus p^{\gamma}}$$

and thus

$$\begin{aligned}p^{\gamma}\ell_R(R/I_e)&\le\ell_R((R/I_{e+1})^{1/p})+\ell_R(\operatorname{coker}(\bar{\Psi}))\\&=[k^{1/p}:k]\ell_R(R/I_{e+1})+\ell_R(\operatorname{coker}(\bar{\Psi})).\end{aligned}$$

Since $\mathfrak{m}^{[p^e]}\subseteq I_e$ and $\operatorname{coker}(\bar{\Psi})$ is a quotient of $(R/I_e)^{\oplus p^{\gamma}}$, we have that $\mathfrak{m}^{[p^e]}\subseteq\operatorname{Ann}_R(\operatorname{coker}(\bar{\Psi}))$. But $\operatorname{coker}(\bar{\Psi})$ is also a quotient of $\operatorname{coker}(\Psi)=N$ and thus $N/\mathfrak{m}^{[p^e]}N$, so $\ell_R(\operatorname{coker}(\bar{\Psi}))\le\ell_R(N/\mathfrak{m}^{[p^e]}N)\le C(N,\mathfrak{m})p^{e(d-1)}$ by Lemma 3.1. Dividing through by $[k^{1/p}:k]p^{(e+1)d}=p^{\gamma+ed}$ yields

$$\frac{1}{p^{ed}}\ell_R(R/I_e)\le\frac{1}{p^{(e+1)d}}\ell_R(R/I_{e+1})+\frac{C(N,\mathfrak{m})/p^{\gamma}}{p^e},$$

and the result now follows from Lemma 3.5 with $C(\psi)=2C(N,\mathfrak{m})/p^{\gamma}$ independent of the sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$ satisfying the condition in (ii). □

Corollary 4.5. *Let $(R,\mathfrak{m},k)$ be an F-finite local domain of dimension d, and $\{I_e\}_{e\in\mathbb{N}}$ a sequence of ideals such that $\mathfrak{m}^{[p^e]}\subseteq I_e$ for all $e\in\mathbb{N}$. Suppose there exists $0\neq c\in R$ so that $cI_e^{[p]}\subseteq I_{e+1}$ for all $e\in\mathbb{N}$, and a nonzero R-linear map $\psi\colon R^{1/p}\to R$ so that $\psi(I_{e+1}^{1/p})\subseteq I_e$ for all $e\in\mathbb{N}$. Then $\eta=\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists, and there is a positive constant $C\in\mathbb{R}$ such that $|(1/p^{ed})\ell_R(R/I_e)-\eta|\le C/p^e$ for all $e\in\mathbb{N}$ and all such sequence of ideals $\{I_e\}_{e\in\mathbb{N}}$. In particular, $\ell_R(R/I_e)=\eta p^{ed}+O(p^{e(d-1)})$.*

Corollary 4.5 gives yet another perspective on the existence proofs for Hilbert–Kunz multiplicity and F-signature in terms of the properties of certain sequences of ideals. If $(R,\mathfrak{m},k)$ is a local domain and we set $I_e^{\mathrm{HK}}=\mathfrak{m}^{[p^e]}$ and $I_e^{F\text{-sig}}=(r\in R\mid\phi(r^{1/p^e})\in\mathfrak{m}\text{ for all }\phi\in\operatorname{Hom}_R(R^{1/p^e},R))$, it is easy to see $\mu_R(R^{1/p^e})=\ell_R((R/I_e^{\mathrm{HK}})^{1/p^e})$ and $\operatorname{frk}_R(R^{1/p^e})=\ell_R((R/I_e^{F\text{-sig}})^{1/p^e})$. It is shown in [Tucker 2012] that both sequences satisfy the conditions of Corollary 4.5; in fact, any choice of $0\neq c\in R$ and $0\neq\psi\in\operatorname{Hom}_R(R^{1/p},R)$ will suffice.

As in [Tucker 2012], in settings such as Theorem 4.3, we have opted to consider sequences of ideals $\{I_e\}_{e\in\mathbb{N}}$ with $\mathfrak{m}^{[p^e]}\subseteq I_e$. In light of Theorem 3.3, uniformity of constants over $\operatorname{Spec}(R)$ would seem more transparent for such sequences. Nonetheless, for any $\mathfrak{m}$-primary ideal J, one could consider sequences with $J^{[p^e]} \subseteq I_e$. One has $\mathfrak{m}^{[p^{e_0}]} \subseteq J$ for some $e_0 \in \mathbb{N}$, so that $\mathfrak{m}^{[p^{e+e_0}]} \subseteq J^{[p^e]} \subseteq I_e$ for such sequences. In particular, our techniques carry over to this setting after allowing for reindexing, and it is straightforward to recover Monsky's original limit existence result.

Corollary 4.6 [Monsky 1983]. *If $(R, \mathfrak{m}, k)$ is a local ring of dimension d and $J \subseteq R$ is an $\mathfrak{m}$-primary ideal, then*

$$e_{\mathrm{HK}}(R, J) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/J^{[p^e]})$$

exists and there is a positive constant $C \in \mathbb{R}$ such that

$$\left| e_{\mathrm{HK}}(R, J) - \frac{1}{p^{ed}} \ell_R(R/J^{[p^e]}) \right| < \frac{C}{p^e}$$

for all e.

Proof. The reindexing argument preceding the statement together with Theorem 4.3 yields the desired conclusion when R is an F-finite domain, which can be used to deduce the general case. For completeness, we include the reduction from [Monsky 1983]. First, passing to a faithfully flat unramified local extension, we may assume that R is complete with perfect residue field (and hence F-finite). Suppose $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are the minimal primes of R with dimension d. Take $e_0 \gg 0$ so that $(\sqrt{0})^{[p^{e_0}]} = 0$, and put $N_i = \ell_{R_{\mathfrak{p}_i}}(F_*^{e_0} R_{\mathfrak{p}_i})$ for $i = 1, \dots, s$. If

$$M = \bigoplus_{i=1}^{s} ((R/\mathfrak{p}_i)^{\oplus N_i}),$$

then M and $F_*^{e_0} R$ agree upon localization to any of the $\mathfrak{p}_i$, and it follows that there exist R-module homomorphisms $M \to F_*^{e_0} R$ and $F_*^{e_0} R \to M$ with cokernels of dimension strictly less than d. Applying $R/I^{[p^e]} \otimes \_$ and using Lemma 3.1 yields a positive constant C' so that

$$|\ell_R(R/I^{[p^{e+e_0}]}) - \ell_R(M/I^{[p^e]}M)| \le C' p^{e(d-1)}$$

for all $e \in \mathbb{N}$. Dividing through by $p^{(e+e_0)d}$ and using

$$\ell_R(M/I^{[p^e]}M) = \sum_{i=1}^{s} N_i \cdot \ell_R\left(\frac{R/\mathfrak{p}_i}{I^{[p^e]}R/\mathfrak{p}_i}\right),$$

the statement for R now follows from that of the $R/\mathfrak{p}_i$, which are F-finite complete domains. $\square$

Another useful modification of the setup of Theorem 4.3 proceeds in the following manner. Fixing $e_0 \in \mathbb{N}$, one can modify the condition on the sequences of ideals $\{I_e\}_{e\in\mathbb{N}}$ of Theorem 4.3(i) to require that $cI_e^{[p^{e_0}]} \subseteq I_{e+e_0}$ for all $e \in \mathbb{N}$. Similarly, in Theorem 4.3(ii), one can ask that there exists $0 \neq \psi \in \mathrm{Hom}_R(R^{1/p^{e_0}}, R)$ such that $\psi((I_{e+e_0})^{1/p^{e_0}}) \subseteq I_e$ for all $e \in \mathbb{N}$. In both cases, after reindexing, the same methods apply to yield analogous results.

For example, this last generalization is particularly relevant when working with an arbitrary Cartier subalgebra $\mathscr{D}$ on a local F-finite domain $(R, \mathfrak{m}, k)$. As $\mathscr{D}$ is closed under composition, it is easily verified that

$$\Gamma_{\mathscr{D}} = \{e \in \mathbb{N} \mid \mathscr{D}_e \neq 0\}$$

is a subsemigroup of $(\mathbb{N}, +)$, and is called the semigroup of $\mathscr{D}$. For $e \in \Gamma_{\mathscr{D}}$, one defines the e-th F-splitting number $a_e^{\mathscr{D}}$ of R along $\mathscr{D}$ to be the maximal number of copies of R with projection maps in $\mathscr{D}_e$ appearing in an R-module direct sum decomposition of R^{1/p^e}. In other words, $a_e^{\mathscr{D}}$ is the largest integer so that there is a surjection $R^{1/p^e} \to R^{\oplus a_e^{\mathscr{D}}}$ where the induced map $\operatorname{Hom}_R(R^{\oplus a_e^{\mathscr{D}}}, R) \to \operatorname{Hom}_R(R^{1/p^e}, R) = \mathscr{C}_e(R)$ has image inside of $\mathscr{D}_e$. See [Blickle et al. 2012, §3.1] for further details. Setting $I_e^{\mathscr{D}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \mathscr{D}_e)$, it is again easy to check that $a_e^{\mathscr{D}} = \ell_R((R/I_e^{\mathscr{D}})^{1/p^e})$, and the method of Theorem 4.3(ii) yields the following result.

Theorem 4.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d and $\mathscr{D}$ a Cartier subalgebra on R. Then the F-signature $s(R, \mathscr{D}) = \lim_{\substack{e\to\infty \\ e\in\Gamma_{\mathscr{D}}}}(1/([k^{1/p^e} : k] \cdot p^{ed}))a_e^{\mathscr{D}}$ of R along $\mathscr{D}$ exists, and there is a positive constant $C \in \mathbb{R}$ so that*

$$\frac{1}{[k^{1/p^e} : k] \cdot p^{ed}} a_e^{\mathscr{D}} \leq s(R, \mathscr{D}) + \frac{C}{p^e}$$

for all $e \in \Gamma_{\mathscr{D}}$.

Proof. Let $p^\gamma = \operatorname{rk}_R(R^{1/p}) = [k^{1/p} : k] \cdot p^d$, and take a set[1] of generators $e_1, \dots, e_s$ for the semigroup $\Gamma_{\mathscr{D}}$. Fixing $0 \neq \psi_i \in \mathscr{D}_{e_i}$ for each $i = 1, \dots, s$, we can find a short exact sequence of R-modules

$$0 \longrightarrow R^{1/p^{e_i}} \xrightarrow{\Psi_i} R^{\oplus p^{\gamma e_i}} \longrightarrow M_i \longrightarrow 0 \tag{6}$$

where $\dim(M_i) < d$ and every component function of Ψ_i is a premultiple of ψ_i as in Lemma 4.1(ii). Since the Cartier linear maps in $\mathscr{D}$ are closed under composition, it follows readily that $\psi_i((I_{e+e_i}^{\mathscr{D}})^{1/p^{e_i}}) \subseteq I_e^{\mathscr{D}}$ for any $e \in \Gamma_{\mathscr{D}}$. In particular, $\Psi_i((I_{e+e_i}^{\mathscr{D}})^{1/p^{e_i}}) \subseteq (I_e^{\mathscr{D}})^{\oplus p^{\gamma e_i}}$ and proceeding as in Theorem 4.3(ii), we see that

$$\frac{1}{p^{e\gamma}} a_e^{\mathscr{D}} \leq \frac{1}{p^{(e+e_i)\gamma}} a_{e+e_i}^{\mathscr{D}} + \frac{C'}{p^e}$$

for any $i = 1, \dots, s$ and $e \in \Gamma_{\mathscr{D}}$ where $C' = \max\{C(M_1, \mathfrak{m})/p^{e_1\gamma}, \dots, C(M_s, \mathfrak{m})/p^{e_s\gamma}\}$. It follows as in Lemma 3.5(ii) that $s(R, \mathscr{D}) = \lim_{\substack{e\to\infty \\ e\in\Gamma_{\mathscr{D}}}}(1/p^{e\gamma})a_e^{\mathscr{D}}$ exists and $s(R, \mathscr{D}) - (1/p^{e\gamma})a_e^{\mathscr{D}} \leq 2C'/p^e$ for all $e \in \Gamma_{\mathscr{D}}$ as desired. $\square$

Remark 4.8 (compare to [Polstra 2015, Condition (1)]). Consider a Cartier subalgebra $\mathscr{D}$ on an F-finite local domain $(R, \mathfrak{m}, k)$ with the following property: if $\phi \in \mathscr{D}_{e+1}$ for some $e \in \mathbb{N}$, then $\phi|_{R^{1/p^e}} \in \mathscr{D}_e$. It is easy to check $(I_e^{\mathscr{D}})^{[p]} \subseteq I_{e+1}^{\mathscr{D}}$ for all $e \in \mathbb{N} = \Gamma_{\mathscr{D}}$, and Theorem 4.3(i) gives that there is a positive constant $C \in \mathbb{R}$ so that

$$\left| s(R, \mathscr{D}) - \frac{1}{[k^{1/p^e} : k] \cdot p^{ed}} a_e^{\mathscr{D}} \right| \leq \frac{C}{p^e} \tag{7}$$

[1] Every subsemigroup of $\mathbb{N}$ is finitely generated [Grillet 2001, Proposition 4.1].

for all $e \in \mathbb{N}$. Arbitrary Cartier subalgebras will not satisfy such properties, and we know of no reason to expect (7) to hold in general. However, we will see below in Theorems 4.11 and 4.12 that the Cartier subalgebras constructed using ideals and divisors do satisfy (7) by showing they enjoy a perturbation of the above property.

The proof of Theorem 4.3 made use of positive constants arising from Lemma 3.1. However, in case R is not local and the p-linear map in (i) or p^{-1}-linear map in (ii) extend to R, one can make these constants spread uniformly over $\operatorname{Spec}(R)$ using Theorem 3.3 instead. In the case of Theorem 4.7, this immediately yields the following result.

Theorem 4.9. *Let R be an F-finite domain and $\mathscr{D}$ a Cartier subalgebra on R. There is a positive constant $C \in \mathbb{R}$ so that*

$$\frac{1}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\operatorname{ht}(\mathfrak{p})}} a_e^{\mathscr{D}_\mathfrak{p}} \leq s(R_\mathfrak{p}, \mathscr{D}_\mathfrak{p}) + \frac{C}{p^e}$$

for all $e \in \Gamma_\mathscr{D}$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$. Moreover, the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto s(R_\mathfrak{p}, \mathscr{D}_\mathfrak{p})$ is lower semicontinuous.

Note that, for the lower semicontinuity property, one can relax the above requirement that R is a domain — any F-finite ring will suffice. The F-signature function is identically zero off of the strongly F-regular locus of R, so that $s(R_\mathfrak{p}, \mathscr{D}_\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$ where $R_\mathfrak{p}$ is not a normal domain. Thus, it suffices to check lower semicontinuity on the normal locus $U \subseteq \operatorname{Spec}(R)$, which follows from the lower semicontinuity on each $\operatorname{Spec}(R_f)$ for each $f \in R$ where R_f is a normal domain.

Proof. Following along in the proof of Theorem 4.7, observe that sequences in (6) can be taken to be global and then localized to each $\mathfrak{p} \in \operatorname{Spec}(R)$. By Theorem 3.3, we may then take the positive constant $C = \max\{2C(M_1)/p^{e_1\gamma}, \dots, 2C(M_s)/p^{e_s\gamma}\}$ independent of $\mathfrak{p} \in \operatorname{Spec}(R)$. For the remainder, first note the argument of Lemma 2.2 readily adapts to show the function $\operatorname{Spec}(R) \to \mathbb{R}$ given by $\mathfrak{p} \mapsto \mathfrak{a}_e^{\mathscr{D}_\mathfrak{p}}$ is lower semicontinuous. The lower semicontinuity of the F-signatures thus follows from the argument given in Theorem 3.8. □

Remark 4.10. Analogous to the argument above, for an $\mathbb{R}$-valued function on $\operatorname{Spec}(R)$ governed locally by sequences of ideals as in Theorem 4.3(i), upper semicontinuity passes from the individual terms in the sequences to the limit as in Theorem 3.4. However, we are unaware of any such functions that are not directly related to Hilbert–Kunz multiplicity itself.

Theorem 4.11. *Let R be an F-finite domain, $\mathfrak{a} \subseteq R$ a nonzero ideal, $t \in \mathbb{R}_{\geq 0}$, and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the F-signature*

$$s(R_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}^t) = s(R_\mathfrak{p}, \mathscr{C}^{\mathfrak{a}_p^t}) = \lim_{e\to\infty} \frac{1}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\operatorname{ht}(\mathfrak{p})}} a_e^{\mathfrak{a}_\mathfrak{p}^t}$$

of $R_{\mathfrak{p}}$ along $\mathfrak{a}_{\mathfrak{p}}^t$ exists and determines a lower semicontinuous $\mathbb{R}$-valued function on $\operatorname{Spec}(R)$. *Moreover, there is a positive constant $C \in \mathbb{R}$ so that*

$$\left| s(R_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}^t) - \frac{a_e^{\mathfrak{a}_{\mathfrak{p}}^t}}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\mathrm{ht}(\mathfrak{p})}} \right| \leq \frac{C}{p^e} \tag{8}$$

for all $e \in \mathbb{N}$, all $\mathfrak{p} \in \operatorname{Spec}(R)$, and all $t \in \mathbb{R}_{\geq 0}$. In particular, $a_e^{\mathfrak{a}_{\mathfrak{p}}^t}/[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] = s(R_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}^t) p^{e\,\mathrm{ht}(\mathfrak{p})} + O(p^{e(\mathrm{ht}(\mathfrak{p})-1)})$.

Proof. The existence and semicontinuity statements follow immediately from Theorems 4.7 and 4.9 above. Let $f_1, \ldots, f_s$ be a set of generators for $\mathfrak{a}$. If $t > s$, then $a_e^{\mathfrak{a}_{\mathfrak{p}}^t} = 0$ for all $e \in \mathbb{N}$ and any $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus, we are free to assume $t \leq s$ going forward.

For a fixed $t \in \mathbb{R}_{\geq 0}$, one inequality in (8) also follows from Theorem 4.7 above; however, it remains to show that the positive constant C can be chosen independent of $t \in \mathbb{R}_{\geq 0}$. To that end, fix a choice of $0 \neq x \in \mathfrak{a}^{s(p-1)}$ and $\phi \in \operatorname{Hom}_R(R^{1/p}, R)$. Then $\psi(\_) = \phi(x^{1/p} \cdot \_) \in \mathscr{C}_1^{\mathfrak{a}^t}$ for any $t \leq s$. The desired independence follows from the observation that a short exact sequence as in Lemma 4.1(ii) can be used for (6) in the proof of Theorem 4.7 for any $t \leq s$.

To show the reverse inequality, choose an element $0 \neq c \in R$ that satisfies $c\mathfrak{a}^{\lceil t(p^{e+1}-1)\rceil} \subseteq (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{[p]}$ for all $e \in \mathbb{N}$ and any $t \in \mathbb{R}_{\geq 0}$; one can check that $c = f_1^p \cdots f_s^p$ will suffice. We will show that $c(I_e^{\mathfrak{a}_{\mathfrak{p}}^t})^{[p]} \subseteq I_{e+1}^{\mathfrak{a}_{\mathfrak{p}}^t}$ for all $t \in \mathbb{R}_{\geq 0}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$, after which the result follows by using a short exact sequence as in Lemma 4.1(i) for (5) in the proof of Theorem 4.3(i) for any $t \leq s$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ with $C(M_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) = C(M)$ from Theorem 3.3. Supposing that $\phi \in \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^{e+1}}, R_{\mathfrak{p}})$ and $y \in \mathfrak{a}_{\mathfrak{p}}^{\lceil t(p^{e+1}-1)\rceil}$, we must show $\phi((yc(I_e^{\mathfrak{a}_{\mathfrak{p}}^t})^{[p]})^{1/p^{e+1}}) \subseteq \mathfrak{p}R_{\mathfrak{p}}$. Write $cy = \sum_{i=1}^n g_i^p r_i$ where each $r_i \in R_{\mathfrak{p}}$ and $g_i \in \mathfrak{a}_{\mathfrak{p}}^{\lceil t(p^e-1)\rceil}$. Let $\phi_i \in \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{1/p^e}, R_{\mathfrak{p}})$ be the map defined by $\phi_i(z^{1/p^e}) = \phi(r_i^{1/p^{e+1}} z^{1/p^e})$ for all $z \in R_{\mathfrak{p}}$. If $x \in I_e^{\mathfrak{a}_{\mathfrak{p}}^t}$, then $\phi((ycx^p)^{1/p^{e+1}}) = \sum_{i=1}^n \phi(r_i^{1/p^{e+1}}(g_i x)^{1/p^e}) = \sum_{i=1}^n \phi_i((g_i x)^{1/p^e}) \in \mathfrak{p}R_{\mathfrak{p}}$ as $\phi_i(g_i^{1/p^e} \cdot \_) \in \mathscr{C}_e^{\mathfrak{a}_{\mathfrak{p}}^t}$ for $i = 1, \ldots, n$. □

Theorem 4.12. *Let R be an F-finite normal domain, Δ an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec}(R)$, *and $\mathfrak{p} \in$* $\operatorname{Spec}(R)$. *Then the F-signature*

$$s(R_{\mathfrak{p}}, \Delta) = s(R_{\mathfrak{p}}, \mathscr{C}^{(R_{\mathfrak{p}},\Delta)}) = \lim_{e\to\infty} \frac{1}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\mathrm{ht}(\mathfrak{p})}} a_e^{(R_{\mathfrak{p}},\Delta)}$$

of $R_{\mathfrak{p}}$ along Δ exists and determines a lower semicontinuous R-valued function on $\operatorname{Spec}(R)$. *Moreover, if $\bar{\Delta}$ is a fixed effective integral divisor on* $\operatorname{Spec}(R)$, *there exists a positive constant $C \in \mathbb{R}$ so that*

$$\left| s(R_{\mathfrak{p}}, \Delta) - \frac{a_e^{(R_{\mathfrak{p}},\Delta)}}{[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] \cdot p^{e\,\mathrm{ht}(\mathfrak{p})}} \right| \leq \frac{C}{p^e} \tag{9}$$

for all $e \in \mathbb{N}$, all $\mathfrak{p} \in \operatorname{Spec}(R)$, and all effective $\mathbb{Q}$-divisors Δ with $\Delta \leq \bar{\Delta}$. In particular, we have that $a_e^{(R_{\mathfrak{p}},\Delta)}/[k(\mathfrak{p})^{1/p^e} : k(\mathfrak{p})] = s(R_{\mathfrak{p}}, \Delta) p^{e\,\mathrm{ht}(\mathfrak{p})} + O(p^{e(d-1)})$.

Proof. The existence and semicontinuity statements follow immediately from Theorems 4.7 and 4.9 above. For a fixed Δ, one inequality in (9) also follows from Theorem 4.7 above; however, it remains to show that the positive constant C can be chosen independent of $\Delta \leq \bar{\Delta}$. To that end, fix a choice of $0 \neq \psi \in \mathscr{C}_1^{(R,\bar{\Delta})}$. Since $\mathscr{C}_1^{(R,\bar{\Delta})} \subseteq \mathscr{C}_1^{(R,\Delta)}$, the desired independence follows from the observation that a short exact sequence as in Lemma 4.1(ii) can be used for (6) in the proof of Theorem 4.7 for any $\Delta \leq \bar{\Delta}$.

For the reverse inequality, choose $0 \neq c \in R$ so that $\operatorname{div}_R(c) \geq p\bar{\Delta}$. We will show that $c(I_e^{(R_\mathfrak{p},\Delta)})^{[p]} \subseteq I_{e+1}^{(R_\mathfrak{p},\Delta)}$ for all $\Delta \leq \bar{\Delta}$, after which the result follows by using a short exact sequence as in Lemma 4.1(i) for (5) in the proof of Theorem 4.3(i) for any $\Delta \leq \bar{\Delta}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ with $C(M_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}) = C(M)$ from Theorem 3.3. Supposing $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\phi \in \mathscr{C}_{e+1}^{(R_\mathfrak{p},\Delta)}$, we must show $\phi((c(I_e^{(R_\mathfrak{p},\Delta)})^{[p]})^{1/p^{e+1}}) \subseteq \mathfrak{p}R_\mathfrak{p}$. Let $\psi \in \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^{e+1}}, R_\mathfrak{p})$ be the map given by $\psi(\_) = \phi(c^{1/p^{e+1}} \cdot \_)$, and $\psi_e = \psi|_{R_\mathfrak{p}^{1/p^e}} \in \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}^{1/p^e}, R_\mathfrak{p})$ be the restriction to $R_\mathfrak{p}^{1/p^e}$. It suffices to show that $\Delta_{\psi_e} \geq \Delta$, so that for any $x \in I_e^{(R_\mathfrak{p},\Delta)}$ we have $\phi((cx^p)^{1/p^{e+1}}) = \psi_e(x^{1/p^e}) \in \mathfrak{p}R_\mathfrak{p}$.

To that end, consider the inclusions for each $e \in \mathbb{N}$

$$\begin{array}{ccc} R^{1/p^e} & \subseteq & R^{1/p^{e+1}} \\ \cap\!\!| & & \cap\!\!| \\ (R(\lceil (p^e-1)\Delta \rceil))^{1/p^e} & \subseteq & (R(p\lceil (p^e-1)\Delta \rceil))^{1/p^{e+1}} \\ & & \cap\!\!| \\ & & (R(\lceil (p^{e+1}-1)\Delta \rceil + \operatorname{div}_R(c)))^{1/p^{e+1}} \end{array}$$

which hold because $\lceil (p^e-1)\Delta \rceil \leq (p^e-1)\Delta + \bar{\Delta}$ implies

$$\begin{aligned} p\lceil (p^e-1)\Delta \rceil &\leq (p^{e+1}-p)\Delta + p\bar{\Delta} \\ &\leq (p^{e+1}-1)\Delta + p\bar{\Delta} \\ &\leq \lceil (p^{e+1}-1)\Delta \rceil + \operatorname{div}_R(c). \end{aligned}$$

We know that ψ is the restriction to $R_\mathfrak{p}^{1/p^{e+1}}$ of a map in

$$\operatorname{Hom}_{R_\mathfrak{p}}((R_\mathfrak{p}(\lceil (p^{e+1}-1)\Delta \rceil + \operatorname{div}_R(c)))^{1/p^{e+1}}, R_\mathfrak{p}),$$

and by localizing the above inclusions at $\mathfrak{p}$, we see that $\psi_e = \psi|_{R_\mathfrak{p}^{1/p^e}}$ can be extended to a map in $\operatorname{Hom}_{R_\mathfrak{p}}((R_\mathfrak{p}(\lceil (p^e-1)\Delta)\rceil)^{1/p^e}, R_\mathfrak{p})$. It follows immediately that $\Delta_{\psi_e} \geq \Delta$. $\square$

As the above examples demonstrate, Theorem 4.3 can be used to show the existence of limits in a large number of important settings. Moreover, the following well known lemma allows similar techniques to be used more broadly still.

Lemma 4.13. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d and $0 \neq c \in R$. Suppose $\{I_e\}_{e\in\mathbb{N}}$ and $\{J_e\}_{e\in\mathbb{N}}$ are two sequences of ideals in R so that*

$$\mathfrak{m}^{[p^e]} \subseteq I_e \subseteq J_e \subseteq (I_e : c)$$

for all $e \in \mathbb{N}$. Then there exists a positive constant $C \in \mathbb{R}$ so that

$$\left| \frac{1}{p^{ed}} \ell_R(R/I_e) - \frac{1}{p^{ed}} \ell_R(R/J_e) \right| = \frac{1}{p^{ed}} \ell_R(J_e/I_e) \leq \frac{C}{p^e}$$

for all $e \in \mathbb{N}$ *and so* $\lim_{e\to\infty}((1/p^{ed})\ell_R(R/I_e)-(1/p^{ed})\ell_R(R/J_e))=0$. *Hence,* $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/J_e)$ *exists if and only if* $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ *exists, in which case they are equal.*

Proof. Since $(I_e : c)/I_e$ is the kernel and $R/(I_e, c)$ the cokernel of multiplication by c on R/I_e, their lengths are equal. Thus,

$$\frac{1}{p^{ed}}\ell_R(J_e/I_e) \le \frac{1}{p^{ed}}\ell_R((I_e : x)/I_e) = \frac{1}{p^{ed}}\ell_R(R/(I_e, c)) \le \frac{1}{p^{ed}}\ell_R(R/(\mathfrak{m}^{[p^e]}, c)) \le \frac{C}{p^e}$$

by applying Lemma 3.1 with $M = R/(c)$. □

Once again, the constant C in the above lemma depends only on the choice of $0 \neq c \in R$ and not on any particular sequence of ideals; moreover, in case R is not local, the constant C can be chosen uniformly over $\mathrm{Spec}(R)$ using Theorem 3.3.

5. Positivity

Positivity of the F-signature. Recall that an F-finite ring R is said to be strongly F-regular if and only if, for all $x \in R$ not contained in any minimal prime (e.g., $x \neq 0$ when R is a domain), there exist some $e \in \mathbb{N}$ and $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ with $\phi(x^{1/p^e}) = 1$. Strongly F-regular rings are always products of Cohen–Macaulay normal domains. Moreover, a strongly F-regular local domain remains so after completion.

Theorem 5.1 [Aberbach and Leuschke 2003, Main Result]. *Suppose* $(R, \mathfrak{m}, k)$ *is a local F-finite domain. Then R is strongly F-regular if and only if* $s(R) > 0$.

We will give two new proofs of Theorem 5.1. The first proof is notable in that it gives a readily computable lower bound for the F-signature.

First proof of Theorem 5.1. We will assume for simplicity in the exposition that $k = k^p$ is perfect — the proof is easily adapted to arbitrary k. Suppose first that R is not strongly F-regular, so that there exists some $0 \neq x \in R$ with $\phi(x^{1/p^e}) \in \mathfrak{m}$ for all $e \in \mathbb{N}$ and all $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$. Take a surjection $\Phi\colon R^{1/p^e} \to R^{\oplus \mathrm{frk}_R(R^{1/p^e})}$ of R-modules. It follows that $\Phi((xR)^{1/p^e}) \subseteq \mathfrak{m}^{\oplus \mathrm{frk}_R(R^{1/p^e})}$, so that Φ induces a surjection of R-modules $(R/xR)^{1/p^e} \to k^{\oplus \mathrm{frk}_R(R^{1/p^e})}$ and hence $\mathrm{frk}_R(R^{1/p^e}) \le \mu_R((R/xR)^{1/p^e}) \le C(R/xR, \mathfrak{m})p^{e(d-1)}$ using (2) and Lemma 3.1 for some positive constant $C(R/xR, \mathfrak{m}) \in \mathbb{R}$. Dividing through by p^{ed} and taking $\lim_{e\to\infty}$ then gives $s(R) = 0$. Thus, $s(R) > 0$ implies that R is strongly F-regular.

Conversely, suppose that R is strongly F-regular. The F-signature remains unchanged upon completion, so we may assume R is complete and choose a coefficient field k and system of parameters $x_1, \dots, x_d$ so that R is module finite and generically separable over the regular subring $A = k[\![x_1, \dots, x_d]\!]$. Take $0 \neq c \in A$ so that $cR^{1/p^e} \subseteq R[A^{1/p^e}] = R \otimes_A A^{1/p^e}$ as in Lemma 2.3 for all $e \in \mathbb{N}$. Since R is strongly F-regular, we can find $e_0 \in \mathbb{N}$ and $\phi \in \mathrm{Hom}_R(R^{1/p^{e_0}}, R)$ with $\phi(c^{1/p^{e_0}}) = 1$. We will show $s(R) \ge 1/p^{e_0 d} > 0$.

For any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$ and $e \in \mathbb{N}$, we write $\underline{x}^{\alpha/p^e}$ for $x_1^{\alpha_1/p^e} \cdots x_d^{\alpha_d/p^e}$. Let $\mathcal{I}_e = \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d \mid 0 \le \alpha_i < p^e \text{ for } i = 1, \dots, d\}$. The monomials $\underline{x}^{\alpha/p^e}$ for $\alpha \in \mathcal{I}_e$ are a free basis for A^{1/p^e} over A. As such, for each $\alpha \in \mathcal{I}_e$, we can find an A-linear map $\pi_\alpha\colon A^{1/p^e} \to A$ with $\pi_\alpha(\underline{x}^{\alpha/p^e}) = 1$ and $\pi_\alpha(\underline{x}^{\beta/p^e}) = 0$ for all $\alpha \neq \beta \in \mathcal{I}_e$. Applying $R \otimes_A \_$ gives an R-linear map $\tilde{\pi}_\alpha\colon R[A^{1/p^e}] \to R$ with $\tilde{\pi}_\alpha(\underline{x}^{\alpha/p^e}) = 1$ and $\tilde{\pi}_\alpha(\underline{x}^{\beta/p^e}) = 0$ for all $\alpha \neq \beta \in \mathcal{I}_e$. Multiplication by c gives an R-linear map $R^{1/p^e} \to R[A^{1/p^e}]$, and composing with $\tilde{\pi}_\alpha$ gives an R-linear map $\tilde{\pi}_\alpha(c \cdot \_)\colon R^{1/p^e} \to R$ so that $\tilde{\pi}_\alpha(c\underline{x}^{\alpha/p^e}) = c\tilde{\pi}_\alpha(\underline{x}^{\alpha/p^e}) = c$ and $\tilde{\pi}_\alpha(c\underline{x}^{\beta/p^e}) = c\tilde{\pi}_\alpha(\underline{x}^{\beta/p^e}) = 0$ for all $\alpha \neq \beta \in \mathcal{I}_e$ (since $c \in A$). Setting $\psi_\alpha(\_) = \phi \circ (\tilde{\pi}_\alpha(c \cdot \_))^{1/p^{e_0}}\colon R^{1/p^{e+e_0}} \to R$, we have $\psi_\alpha \in \operatorname{Hom}_R(R^{1/p^{e+e_0}}, R)$ with $\psi_\alpha(\underline{x}^{\alpha/p^{e+e_0}}) = 1$ and $\psi_\alpha(\underline{x}^{\beta/p^{e+e_0}}) = 0$ for all $\alpha \neq \beta \in \mathcal{I}_e$. It follows that $\Psi = \bigoplus_{\alpha \in \mathcal{I}_e} \psi_\alpha\colon R^{1/p^{e+e_0}} \to R^{\oplus p^{ed}}$ is an R-module surjection and hence $\operatorname{frk}_R(R^{1/p^{e+e_0}}) \ge p^{ed}$. Dividing through by $p^{(e+e_0)d}$ and taking $\lim_{e\to\infty}$ yields $s(R) \ge (1/p^{e_0 d}) > 0$ as desired. $\square$

For the second proof, we will need the following lemma, which should be compared with [Hochster and Huneke 1991, Theorem 3.3]. In the next subsection, we will show how to adapt this proof to arbitrary sequences of ideals as in Theorem 4.3(ii).

Lemma 5.2. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite Cohen–Macaulay domain of dimension d. There exists $N \in \mathbb{N}$ with the following property: for any $e \in \mathbb{N}$ and all $x \in R \setminus \mathfrak{m}^{[p^e]}$, there exists a map $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ with $\phi(x^{1/p^e}) \notin \mathfrak{m}^N$.*

Proof. Choosing a coefficient field k and system of parameters $x_1, \dots, x_d$, we have that R is a finitely generated free module over $A = k[[x_1, \dots, x_d]]$ [Bruns and Herzog 1993, Proposition 2.2.11]; let $\mathfrak{m}_A$ denote the maximal ideal of A. Fix a nonzero map $\tau \in \operatorname{Hom}_A(R, A)$. Since $\operatorname{Hom}_A(R, A)$ is a rank-one torsion-free R-module, we can find $0 \neq y \in \operatorname{Ann}_R(\operatorname{Hom}_A(R, A)/(R\tau))$; replacing by a nonzero multiple, we may further assume $0 \neq y \in A$. If $N', N \in \mathbb{N}$ are sufficiently large so that $y \notin \mathfrak{m}_A^{N'}$ and $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'}R$, we will show N satisfies the desired property.

Fix $e \in \mathbb{N}$ and $x \in R \setminus \mathfrak{m}^{[p^e]}$, so also $x \notin \mathfrak{m}_A^{[p^e]}R$. Since R^{1/p^e} is a free A-module and $x^{1/p^e} \notin \mathfrak{m}_A R^{1/p^e}$, x^{1/p^e} can be taken as part of a free basis and there exists an A-linear map $\chi\colon R^{1/p^e} \to A$ with $\chi(x^{1/p^e}) = 1$. If $\mathrm{Tr}\colon \operatorname{Hom}_A(R, A) \to A$ is the A-linear map given by evaluation at 1, there is a commutative diagram

$$\begin{tikzcd}
R^{1/p^e} \arrow[r, "\tilde{\chi}"] \arrow[drr, "\chi"'] & \operatorname{Hom}_A(R^{1/p^e}, A) \arrow[r] & \operatorname{Hom}_A(R, A) \arrow[r, "\cdot y"] \arrow[d, "\mathrm{Tr}"] & R\tau \arrow[r, "\simeq"] \arrow[d, "\mathrm{Tr}"] & R \arrow[d, "\tau"] \\
 & & A \arrow[r, "\cdot y"] & A \arrow[r, "="] & A
\end{tikzcd} \tag{10}$$

where $\tilde{\chi}$ is the R^{1/p^e}-linear map sending $1 \mapsto \chi$ and where $\operatorname{Hom}_A(R^{1/p^e}, A) \to \operatorname{Hom}_A(R, A)$ is the R-linear map given by restriction to $R \subseteq R^{1/p^e}$. Let $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ be the composition along the top row of this diagram, so that $y\chi = \tau \circ \phi$. We have $\tau(\phi(x^{1/p^e})) = y\chi(x^{1/p^e}) = y \notin \mathfrak{m}_A^{N'}$, and thus $\phi(x^{1/p^e}) \notin \mathfrak{m}_A^{N'}R$ as τ is A-linear. In particular, since $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'}R$, we have $\phi(x^{1/p^e}) \notin \mathfrak{m}^N$ as desired. $\square$

Second proof of Theorem 5.1. Suppose that R is strongly F-regular, and as such Cohen–Macaulay. As before, without loss of generality, we may assume R is complete. If $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$

for all $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R))$ and $\psi \in \mathrm{Hom}_R(R^{1/p^{e'}}, R)$, it is easy to check that $\psi((I_{e+e'}^{F\text{-sig}})^{1/p^{e'}}) \subseteq I_e^{F\text{-sig}}$ for all $e, e' \in \mathbb{N}$. Since R is F-split, we may conclude $I_{e+1}^{F\text{-sig}} \subseteq I_e^{F\text{-sig}}$ for all $e \in \mathbb{N}$; moreover, by the definition of strong F-regularity, we have $\bigcap_{e\in\mathbb{N}} I_e^{F\text{-sig}} = 0$. Thus, for N as in Lemma 5.2, Chevalley's lemma [1943, Lemma 7] gives $e_0 \in \mathbb{N}$ with $I_{e_0}^{F\text{-sig}} \subseteq \mathfrak{m}^N$. It follows from Lemma 5.2 that $I_{e+e_0}^{F\text{-sig}} \subseteq \mathfrak{m}^{[p^e]}$ for all $e \in \mathbb{N}$. Indeed for each $x \in R \setminus \mathfrak{m}^{[p^e]}$ there is some $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ with $\phi(x^{1/p^e}) \notin \mathfrak{m}^N$ and hence $\phi(x^{1/p^e}) \notin I_{e_0}^{F\text{-sig}}$; as $\phi((I_{e+e_0}^{F\text{-sig}})^{1/p^e}) \subseteq I_{e_0}^{F\text{-sig}}$ we must have $x \notin I_{e+e_0}^{F\text{-sig}}$. Thus, we compute

$$s(R) = \lim_{e\to\infty} \frac{1}{p^{(e+e_0)d}} \ell_R(R/I_{e+e_0}^{F\text{-sig}}) \geq \lim_{e\to\infty} \frac{1}{p^{(e+e_0)d}} \ell_R(R/\mathfrak{m}^{[p^e]}) = \frac{1}{p^{e_0 d}} e_{\mathrm{HK}}(R) \geq \frac{1}{p^{e_0 d}} > 0$$

as desired. □

Positivity via a Cartier linear map. In this section, we generalize the method of the second proof of Theorem 5.1 to examine the positivity of the limits appearing in Theorem 4.3(ii). The essential technique is to exploit the following generalization of Lemma 5.2, which should again be compared with [Hochster and Huneke 1991, Theorem 3.3].

Proposition 5.3. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d, and suppose that $0 \neq \psi \in \mathrm{Hom}_R(R^{1/p}, R)$. There exist $N \in \mathbb{N}$ and $0 \neq c \in R$ so that $\psi^e((xR)^{1/p^e}) \nsubseteq \mathfrak{m}^N$ for any $e \in \mathbb{N}$ and all $x \in R \setminus (\mathfrak{m}^{[p^e]} : c)$.*

Proof. Choose a coefficient field k and system of parameters $x_1, \dots, x_d$ so that R is module finite over the regular subring $A = k[\![x_1, \dots, x_d]\!]$. As R is a torsion-free A-module, choose a free A-submodule $G = A^{\oplus \mathrm{rank}_A(R)} \subseteq R$ of maximal rank and take $0 \neq c \in \mathrm{Ann}_A(R/G)$. In particular, $c^{1/p^e} R^{1/p^e} \subseteq G^{1/p^e} \subseteq R^{1/p^e}$ for all $e \in \mathbb{N}$.

Fixing a nonzero map $\tau \in \mathrm{Hom}_A(R, A)$, we can find $0 \neq y \in \mathrm{Ann}_R(\mathrm{Hom}_A(R, A)/(R\tau))$ as $\mathrm{Hom}_A(R, A)$ is a rank-one torsion-free R-module. Let $0 \neq z \in R$ be as in Lemma 4.2; replacing by nonzero multiples, we may further assume both $0 \neq y \in A$ and $0 \neq z \in A$. Let $N', N \in \mathbb{N}$ be sufficiently large so that $yz \notin \mathfrak{m}_A^{N'}$ and $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'} R$; we will show that the integer N and element $c = yz$ satisfy the desired property.

Suppose $e \in \mathbb{N}$ and $x \in R \setminus (\mathfrak{m}^{[p^e]} : c)$. Since $xc \notin \mathfrak{m}_A^{[p^e]} R \subseteq \mathfrak{m}^{[p^e]}$, we have $(xc)^{1/p^e} \notin \mathfrak{m}_A R^{1/p^e}$ and so also $(xc)^{1/p^e} \notin \mathfrak{m}_A G^{1/p^e} \subseteq \mathfrak{m}_A R^{1/p^e}$. As $G^{1/p^e} = (A^{1/p^e})^{\oplus \mathrm{rank}_A(R)}$ is a free A-module, there is a $\chi' \in \mathrm{Hom}_A(G^{1/p^e}, A)$ with $\chi'((xc)^{1/p^e}) = 1$. Let $\chi \in \mathrm{Hom}_A(R^{1/p^e}, A)$ be the composition

$$R^{1/p^e} \xrightarrow{\cdot c^{1/p^e}} G^{1/p^e} \xrightarrow{\chi'} A$$

which satisfies $\chi(x^{1/p^e}) = 1$. If $\mathrm{Tr}\colon \mathrm{Hom}_A(R, A) \to A$ is the A-linear map given by evaluation at 1, the same diagram (10) from Lemma 5.2 commutes where $\tilde{\chi}$ is the R^{1/p^e}-linear map sending $1 \mapsto \chi$ and $\mathrm{Hom}_A(R^{1/p^e}, A) \to \mathrm{Hom}_A(R, A)$ is the R-linear map given by restriction to $R \subseteq R^{1/p^e}$. Let $\phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$ be the composition along the top row of this diagram, so that $y\chi = \tau \circ \phi$. By Lemma 4.2, we can find $r \in R$ so that $z\phi(\_) = \psi^e(r^{1/p^e} \cdot \_)$. We compute

$$\tau(\psi^e((rx)^{1/p^e})) = \tau(z\phi(x^{1/p^e})) = z\tau(\phi(x^{1/p^e})) = yz\chi(x^{1/p^e}) = yz \notin \mathfrak{m}_A^{N'}$$

and conclude $\psi^e((rx)^{1/p^e}) \notin \mathfrak{m}_A^{N'} R$ as τ is A-linear. In particular, since $\mathfrak{m}^N \subseteq \mathfrak{m}_A^{N'} R$, we have that $\psi^e((xR)^{1/p^e}) \not\subseteq \mathfrak{m}^N$ as desired. □

In the proof of the main result of this section, we will need to use a well known result on the stabilization of the images of an iterated p^{-1}-linear map. Rooted in the work of Hartshorne and Speiser [1977, Proposition 1.11] and later generalized by Lyubeznik [1997], the following version is due to Gabber [2004].

Theorem 5.4 [Gabber 2004, Lemma 13.1; Blickle and Böckle 2011, Proposition 2.14]. *Let M be a finitely generated module over an F-finite ring R and $0 \neq \phi \in \operatorname{Hom}_R(M^{1/p}, M)$. Then the descending chain of R-modules*

$$M \supseteq \phi(M^{1/p}) \supseteq \phi^2(M^{1/p^2}) \supseteq \cdots \supseteq \phi^e(M^{1/p^e}) \supseteq \cdots$$

stabilizes for $e \gg 0$.

Theorem 5.5. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d and $\{I_e\}_{e\in\mathbb{N}}$ a sequence of ideals so that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e \in \mathbb{N}$. Suppose there exists a nonzero $\psi \in \operatorname{Hom}_R(R^{1/p}, R)$ so that $\psi((I_{e+1})^{1/p}) \subseteq I_e$ for all $e \in \mathbb{N}$. Then the limit $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ is positive if and only if $\bigcap_{e\in\mathbb{N}} I_e = 0$.*

Proof. By Lemma 3.1, if $0 \neq c \in \bigcap_{e\in\mathbb{N}} I_e$, then $\ell_R(R/I_e) \leq \ell_R(R/(\mathfrak{m}^{[p^e]}, c)) \leq C(R/(c), \mathfrak{m})p^{e(d-1)}$ for all $e \in \mathbb{N}$. It follows that $\bigcap_{e\in\mathbb{N}} I_e \neq 0$ implies that $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e) = 0$.

For the converse, suppose $\bigcap_{e\in\mathbb{N}} I_e = 0$. If $0 \neq \sigma = \psi^e(R^{1/p^e})$ for $e \gg 0$ is the stable image of ψ as in Theorem 5.4, we also have $\bigcap_{e\in\mathbb{N}}(I_e : \sigma) = 0$ as $\sigma\big(\bigcap_{e\in\mathbb{N}}(I_e : \sigma)\big) \subseteq \bigcap_{e\in\mathbb{N}} I_e = 0$ and R is a domain. In addition, it follows from $\psi(\sigma^{1/p}) = \sigma$ that $(I_{e+1} : \sigma) \subseteq (I_e : \sigma)$ for all $e \in \mathbb{N}$ as

$$(I_{e+1} : \sigma)\sigma = (I_{e+1} : \sigma)\psi(\sigma^{1/p}) = \psi(((I_{e+1} : \sigma)^{[p]}\sigma)^{1/p}) \subseteq \psi(((I_{e+1} : \sigma)\sigma)^{1/p}) \subseteq \psi(I_{e+1}^{1/p}) \subseteq I_e.$$

Take $N \in \mathbb{N}$ and $0 \neq c \in R$ as in Proposition 5.3. By Chevalley's lemma [1943, Lemma 7], there is some $e_0 \in \mathbb{N}$ with $I_{e_0} \subseteq (I_{e_0} : \sigma) \subseteq \mathfrak{m}^N$. It follows from Proposition 5.3 that $I_{e+e_0} \subseteq (\mathfrak{m}^{[p^e]} : c)$ for all $e \in \mathbb{N}$. Indeed for each $x \in R \setminus (\mathfrak{m}^{p^e} : c)$ we have $\psi^e((xR)^{1/p^e}) \not\subseteq \mathfrak{m}^N$ and hence $\psi^e((xR)^{1/p^e}) \not\subseteq I_{e_0}$; since $\psi^e((I_{e+e_0})^{1/p^e}) \subseteq I_{e_0}$ we must have $x \notin I_{e+e_0}$. Using Lemma 4.13, we compute

$$\begin{aligned}\lim_{e\to\infty} \frac{1}{p^{(e+e_0)d}}\ell_R(R/I_{e+e_0}) &\geq \frac{1}{p^{e_0 d}} \lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/(\mathfrak{m}^{[p^e]} : c)) \\ &= \frac{1}{p^{e_0 d}} \lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/\mathfrak{m}^{[p^e]}) \\ &= \frac{1}{p^{e_0 d}} e_{\mathrm{HK}}(R) \geq \frac{1}{p^{e_0 d}} > 0.\end{aligned}$$

□

Theorem 5.5 is quite powerful and allows one to show positivity of the limits appearing in Theorem 4.3(ii) in a number of different situations. One should view this result not only as a generalization of the work done by Aberbach and Leuschke [2003], but also the work of Hochster and Huneke [1990]. If R is a domain and $I \subseteq R$, recall that $x \in R$ is in the tight closure I^* of I if there exists an element $0 \neq c \in R$

such that $cx^{p^e} \in I^{[p^e]}$ for all $e \in \mathbb{N}$. The tight closure I^* is an ideal containing I, and $I^{**} = I$. A ring is said to be weakly F-regular if all ideals I of R are tightly closed, i.e., satisfy $I^* = I$. See [Huneke 1996] or [Hochster 2007] for further details.

Corollary 5.6 [Hochster and Huneke 1990, Theorem 8.17] (length criterion for tight closure). *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d. Suppose that $I \subseteq J$ is an inclusion of $\mathfrak{m}$-primary ideals in R. Then $I^* = J^*$ if and only if $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J)$.*

Proof. Applying the criterion to each term of a composition series of J/I, we may assume $J = (I, x)$ for some $x \in R$ with $(I : x) = \mathfrak{m}$. Consider the sequence of ideals $(I^{[p^e]} : x^{p^e})$ for each $e \in \mathbb{N}$. We have that $(I^{[p^e]} : x^{p^e})^{[p]} \subseteq (I^{[p^{e+1}]} : x^{p^{e+1}})$ for all $e \geq 0$, so that in particular $\mathfrak{m}^{[p^e]} \subseteq (I^{[p^e]} : x^{p^e})$ for each $e \in \mathbb{N}$. Moreover, for any nonzero $\psi \in \mathrm{Hom}_R(R^{1/p}, R)$ it is easy to check $\psi((I^{[p^{e+1}]} : x^{p^{e+1}})^{1/p}) \subseteq (I^{[p^e]} : x^{p^e})$. Since $(I, x)^{[p^e]}/I^{[p^e]} \simeq R/(I^{[p^e]} : x^{p^e})$, we see by Theorem 5.5 that

$$e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/(I^{[p^e]} : x^{p^e})) \tag{11}$$

is zero if and only if $\bigcap_{e\in\mathbb{N}} (I^{[p^e]} : x^{p^e}) \neq 0$, which is equivalent to $x \in I^*$ by definition. $\square$

Recall that there are also a number of well known generalizations of tight closure and strong F-regularity [Hara and Yoshida 2003; Hara and Watanabe 2002; Takagi 2004]. If R is an F-finite local domain, $\mathfrak{a}$ is a nonzero ideal of R, and $t \in \mathbb{R}_{\geq 0}$, one can speak of the $\mathfrak{a}^t$-tight closure $I^{*\mathfrak{a}^t}$ of an ideal $I \subseteq R$. By definition, if $x \in R$, then $x \in I^{*\mathfrak{a}^t}$ if and only if there exists $0 \neq c \in R$ with $c\mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e} \in I^{[p^e]}$ for all $e \in \mathbb{N}$. The pair $(R, \mathfrak{a}^t)$ is strongly F-regular provided for any $0 \neq x \in R$ there exists $e \in \mathbb{N}$ and $\phi \in \mathscr{C}_e^{\mathfrak{a}^t}$ with $\phi(x^{1/p^e}) = 1$. Similarly, if R is an F-finite normal local domain and Δ is an effective $\mathbb{Q}$-divisor on $\mathrm{Spec}(R)$, we have the Δ-tight closure $I^{*\Delta}$ of an ideal $I \subseteq R$. By definition, if $x \in R$, then $x \in I^{*\Delta}$ if and only if there exists $0 \neq c \in R$ with $cx^{p^e} \in I^{[p^e]}(R(\lceil (p^e-1)\Delta \rceil))$ for all $e \in \mathbb{N}$. The pair (R, Δ) is strongly F-regular provided for any $0 \neq x \in R$ there exists $e \in \mathbb{N}$ and $\phi \in \mathscr{C}_e^{(R,\Delta)}$ with $\phi(x^{1/p^e}) = 1$. As of yet, there does not exist a well formed theory of Hilbert–Kunz multiplicity for an ideal or divisor; nonetheless, one can use Theorem 5.5 to give analogues of Corollary 5.6 for these operations.

Corollary 5.7 (length criterion for $\mathfrak{a}^t$-tight closure). *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d, $\mathfrak{a}$ a nonzero ideal of R, and $t \in \mathbb{R}_{\geq 0}$.*

(i) *For any $\mathfrak{m}$-primary ideal I and $x \in R$, $\lim_{e\to\infty} (1/p^{ed})\ell_R(R/(I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e}))$ exists; moreover, this limit equals zero if and only if $x \in I^{*\mathfrak{a}^t}$.*

(ii) *The F-signature $s(R, \mathfrak{a}^t) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e^{\mathfrak{a}^t})$ of R along $\mathfrak{a}^t$ is positive if and only if $(R, \mathfrak{a}^t)$ is strongly F-regular.*

Proof. For (i), take $0 \neq \psi \in \mathscr{C}_1^{\mathfrak{a}^t} = (\mathfrak{a}^{\lceil t(p-1)\rceil})^{1/p} \cdot \mathrm{Hom}_R(R^{1/p}, R)$, and consider the sequence of ideals $(I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e})$ for $e \in \mathbb{N}$. It is easy to check $\psi((I^{[p^{e+1}]} : \mathfrak{a}^{\lceil t(p^{e+1}-1)\rceil} x^{p^{e+1}})^{1/p}) \subseteq (I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e})$ for all $e \in \mathbb{N}$. If $e_0 \in \mathbb{N}$ is sufficiently large that $\mathfrak{m}^{[p^{e_0}]} \subseteq I$, then we have $\mathfrak{m}^{[p^{e+e_0}]} \subseteq (I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e})$ for all $e \in \mathbb{N}$. After shifting the index, Theorem 4.3(ii) and Theorem 5.5 apply, giving the existence of the limit and showing that it equals zero if and only if $\bigcap_{e\in\mathbb{N}}(I^{[p^e]} : \mathfrak{a}^{\lceil t(p^e-1)\rceil}x^{p^e}) \neq 0$, which is equivalent to $x \in I^{*\mathfrak{a}^t}$ by definition. To see (ii), simply note once more that $\psi((I_{e+1}^{\mathfrak{a}^t})^{1/p}) \subseteq I_e^{\mathfrak{a}^t}$ and apply Theorem 5.5. □

Corollary 5.8 (length criterion for Δ-tight closure). *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite normal domain of dimension d, and Δ an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec}(R)$.

(i) *For any $\mathfrak{m}$-primary ideal $I \subseteq R$ and $x \in R$,* $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/(I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e}))$ *exists; moreover, this limit equals zero if and only if $x \in I^{*\Delta}$.*

(ii) *The F-signature $s(R, \Delta) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e^{(R,\Delta)})$ of R along Δ is positive if and only if (R, Δ) is strongly F-regular.*

Proof. Let $0 \neq \psi \in \mathscr{C}_1^{\Delta} = \operatorname{im}(\operatorname{Hom}_R((R(\lceil(p-1)\Delta\rceil))^{1/p}, R) \to \operatorname{Hom}_R(R^{1/p}, R))$. That is, identifying ψ with its unique extension to a p^{-1}-linear map on $K = \operatorname{Frac}(R)$, we have that $\psi((R(\lceil(p-1)\Delta\rceil))^{1/p}) \subseteq R$. Twisting by $\lceil(p^e-1)\Delta\rceil$ and using the fact that $p\lceil(p^e-1)\Delta\rceil + \lceil(p-1)\Delta\rceil \geq \lceil(p^{e+1}-1)\Delta\rceil$, observe $\psi((R(\lceil(p^{e+1}-1)\Delta\rceil))^{1/p}) \subseteq R(\lceil(p^e-1)\Delta\rceil)$ for all $e \in \mathbb{N}$.

For (i), consider the sequence of ideals $(I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e})$ for $e \in \mathbb{N}$. It is easy to check $\psi((I^{[p^{e+1}]}R(\lceil(p^{e+1}-1)\Delta\rceil) : x^{p^{e+1}})^{1/p}) \subseteq (I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e})$ for all $e \in \mathbb{N}$. If $e_0 \in \mathbb{N}$ is sufficiently large that $\mathfrak{m}^{[p^{e_0}]} \subseteq I$, then we have $\mathfrak{m}^{[p^{e+e_0}]} \subseteq (I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e})$ for all $e \in \mathbb{N}$. After shifting the index, Theorems 4.3(ii) and 5.5 apply, giving the existence of the limit and showing that it equals zero if and only if $\bigcap_{e\in\mathbb{N}}(I^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : x^{p^e}) \neq 0$, which is equivalent to $x \in I^{*\Delta}$ by definition. For (ii), simply note once more that $\psi((I_{e+1}^{(R,\Delta)})^{1/p}) \subseteq I_e^{(R,\Delta)}$ and apply Theorem 5.5. □

Remark 5.9. The analogue of Theorem 5.5 for sequences of ideals satisfying Theorem 4.3(i) instead of (ii) is false. Here is an easy counterexample. Let $(R, \mathfrak{m}, k)$ be a complete regular local ring of dimension d, and consider the sequence of ideals $I_e = \mathfrak{m}^{[p^{\lfloor e/2\rfloor}]}$ for $e \in \mathbb{N}$. Then $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $I_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$ and certainly $\bigcap_{e\in\mathbb{N}} I_e = 0$. However, we have $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e) = \lim_{e\to\infty} 1/p^{(e-\lfloor e/2\rfloor)d} = 0$.

To see yet another application of Theorem 5.5, suppose that $(R, \mathfrak{m}, k)$ is an F-split F-finite local ring. If $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$, it is straightforward to check that $\mathscr{P} = \bigcap_{e\in\mathbb{N}} I_e^{F\text{-sig}}$ is a prime ideal [Tucker 2012, Lemma 4.7], coined the F-splitting prime by Aberbach and Enescu [2005]. Furthermore, they suspected that the dimension of the F-splitting prime governed the growth rate of $\operatorname{frk}_R(R^{1/p^e})$ when R is not strongly F-regular; this observation was verified in joint work of the second author with Blickle and Schwede.

Theorem 5.10 [Blickle et al. 2012]. *Suppose $(R, \mathfrak{m}, k)$ is an F-split F-finite local ring and $\mathscr{P}$ is the F-splitting prime. If $n = \dim(R/\mathscr{P})$, the limit $r_F(R) = \lim_{e\to\infty} 1/(([k^{1/p^e} : k] \cdot p^{en}))\operatorname{frk}_R(R^{1/p^e})$ exists and is positive, called the F-splitting ratio. Moreover,* $(1/[k^{1/p^e} : k])\operatorname{frk}_R(R^{1/p^e}) = r_F(R)p^{en} + O(p^{e(n-1)})$.

Proof. Without loss of generality, we may assume R is complete. If $I_e^{F\text{-sig}} = \langle r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)\rangle$, we have $\mathfrak{m}^{[p^e]} \subseteq I_e^{F\text{-sig}}$ and $(I_e^{F\text{-sig}})^{[p]} \subseteq I_{e+1}^{F\text{-sig}}$ for all $e \in \mathbb{N}$. Fixing a surjective map $\psi \in \operatorname{Hom}_R(R^{1/p}, R)$, we have $\psi((I_{e+1}^{F\text{-sig}})^{1/p}) \subseteq I_e^{F\text{-sig}}$. In particular, this implies

$\psi(\mathcal{P}^{1/p}) \subseteq \mathcal{P}$ and so ψ induces a map $\bar{\psi} \in \operatorname{Hom}_{\bar{R}}(\bar{R}^{1/p}, \bar{R})$ where $\bar{R} = R/\mathcal{P}$. Note that $\bar{\psi}$ is still surjective and hence nonzero. Thus, passing to the sequence of ideals $\bar{I_e} = I_e^{F\text{-sig}}\bar{R}$, we have $\mathfrak{m}^{[p^e]}\bar{R} \subseteq \bar{I_e}$, $\bar{I_e}^{[p]} \subseteq \overline{I_{e+1}}$, and $\bar{\psi}((\overline{I_{e+1}})^{1/p}) \subseteq \bar{I_e}$ for all $e \in \mathbb{N}$. Moreover, $\bigcap_{e\in\mathbb{N}} \bar{I_e} = 0$ in $\bar{R}$. The result now follows immediately from Corollary 4.5 together with Theorem 5.5, using that $\operatorname{frk}_R(R^{1/p^e})/[k^{1/p^e} : k] = \ell_R(R/I_e^{F\text{-sig}}) = \ell_{\bar{R}}(\bar{R}/\bar{I_e})$ for all $e \in \mathbb{N}$. □

Remark 5.11. It is straightforward to generalize the notions of F-splitting prime and F-splitting ratio to arbitrary Cartier subalgebras; see [Blickle et al. 2012] for further details. In all cases, the argument from Theorem 5.10 applies and greatly simplifies the proofs. In particular, the methods of the proofs of Theorems 4.7 and 5.5 immediately give an alternative proof of the following result. Recall that a Cartier subalgebra $\mathcal{D}$ on a domain R is said to be strongly F-regular provided that for any $0 \neq x \in R$ there exists an $e \in \mathbb{N}$ and $\phi \in \mathcal{D}_e$ with $\phi(x^{1/p^e}) = 1$. This definition is compatible with the generalizations of strong F-regularity to ideal and divisor pairs discussed prior to Corollary 5.7 as well.

Corollary 5.12 [Blickle et al. 2012, Theorem 3.18]. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d and $\mathcal{D}$ a Cartier subalgebra on R. Then the F-signature $s(R, \mathcal{D}) = \lim_{\substack{e\to\infty \\ e\in\Gamma_{\mathcal{D}}}}(1/([k^{1/p^e} : k] \cdot p^{ed}))a_e^{\mathcal{D}}$ of R along $\mathcal{D}$ is positive if and only if $(R, \mathcal{D})$ is strongly F-regular.*

As in the proof of Theorem 4.7, certain reindexing arguments are required in order to deduce Corollary 5.12 from Theorem 5.5. More generally, it is straightforward to modify Theorems 4.3(ii) and 5.5 for sequences of $\mathfrak{m}$-primary ideals governed by a nonzero p^{-e_0}-linear map for some $e_0 \in \mathbb{N}$. Rather than requiring the reader to trace through the various arguments above, we provide a simpler direct proof.

Corollary 5.13. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d and $e_0 \in \mathbb{N}$. Suppose $\{I_n\}_{n\in\mathbb{N}}$ is a sequence of ideals with $\mathfrak{m}^{[p^{ne_0}]} \subseteq I_n$ for all $n \in \mathbb{N}$, and there is a nonzero $\psi \in \operatorname{Hom}_R(R^{1/p^{e_0}}, R)$ so that $\psi((I_{n+1})^{1/p^{e_0}}) \subseteq I_n$ for all $n \in \mathbb{N}$. Then the limit $\lim_{n\to\infty}(1/p^{ne_0d})\ell_R(R/I_n)$ exists, and it is positive if and only if $\bigcap_{n\in\mathbb{N}} I_n = 0$.*

Proof. Replacing with a premultiple and using Lemma 4.2, we may assume there is a nonzero $\phi \in \operatorname{Hom}_R(R^{1/p}, R)$ so that $\psi = \phi^{e_0}$. Define a new sequence of ideals $\{J_e\}_{e\in\mathbb{N}}$ by

$$J_e = \phi^r((I_{\lceil e/e_0\rceil})^{1/p^r}) + \mathfrak{m}^{[p^e]} \quad \text{where } r = \lceil e/e_0\rceil e_0 - e$$

so that $J_{ne_0} = I_n$ for all $n \in \mathbb{N}$. By construction, we have that $\phi(J_{e+1}^{1/p}) \subseteq J_e$ for all $e \in \mathbb{N}$ so that Theorem 4.3(ii) applies directly to show the limits

$$\lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/J_e) = \lim_{n\to\infty} \frac{1}{p^{ne_0d}}\ell_R(R/J_{ne_0}) = \lim_{n\to\infty} \frac{1}{p^{ne_0d}}\ell_R(R/I_n)$$

exist. For the positivity statement, if $\bigcap_{n\in\mathbb{N}} I_n = 0$, then so too $\bigcap_{e\in\mathbb{N}} J_e = 0$ as $\bigcap_{e\in\mathbb{N}} J_e \subseteq \bigcap_{n\in\mathbb{N}} I_n$. Thus, Theorem 5.5 gives that the limits above are positive. Else, if $0 \neq c \in \bigcap_{n\in\mathbb{N}} I_n$, then $\ell_R(R/I_n) \leq \ell_R(R/(\mathfrak{m}^{[p^{ne_0}]}, c)) \leq C(R/(c), \mathfrak{m})p^{ne_0(d-1)}$ for all $n \in \mathbb{N}$ by Lemma 3.1, so $\lim_{n\to\infty}(1/p^{ne_0d})\ell_R(R/I_n)$ vanishes as desired. □

6. F-signature and minimal relative Hilbert–Kunz multiplicity

Our next aim is to realize the F-signature as the infimum of relative differences in the Hilbert–Kunz multiplicities of the cofinite ideals in a local ring (Corollary 6.5). After first bounding such differences from below by the F-signature (Lemma 6.1), we will make use of approximately Gorenstein sequences to find differences arbitrarily close to the F-signature. The crucial step and main technical result is Theorem 6.3, which uses the uniformity of the constants tracked above to swap limits between iterations of Frobenius and progression in an approximately Gorenstein sequence.

The remainder of the section is reserved for constructions of explicit sequences of relative Hilbert–Kunz differences that approach the F-signature (Corollary 6.6); we also analyze when the infimum is known to be achieved (Corollary 6.8). Generalizations to divisor and ideal pairs are also given (Corollaries 6.9 and 6.10).

Lemma 6.1. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite local ring and $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$ for $e \in \mathbb{N}$. Then*

$$I_e^{F\text{-sig}} \supseteq \sum_{\substack{I \subseteq J \subseteq R \\ 0 < \ell_R(J/I) < \infty}} (I^{[p^e]} : J^{[p^e]}) \supseteq \sum_{\substack{I \subseteq R,\, \ell_R(R/I) < \infty \\ x \in R,\, (I:x) = \mathfrak{m}}} (I^{[p^e]} : x^{p^e}) \tag{12}$$

and we have

$$s(R) \leq \inf_{\substack{I \subseteq J \subseteq R,\, \ell_R(R/I) < \infty \\ I \neq J,\, \ell_R(R/J) < \infty}} \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} \leq \inf_{\substack{I \subseteq R,\, \ell_R(R/I) < \infty \\ x \in R,\, (I:x) = \mathfrak{m}}} e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)). \tag{13}$$

Proof. If $I \subsetneq J$ is a proper inclusion of ideals with $\ell_R(J/I) < \infty$ and $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, we have that $\phi((I^{[p^e]} : J^{[p^e]})^{1/p^e}) \subseteq (I : J) \subseteq \mathfrak{m}$. It follows that

$$\sum_{\substack{I \subseteq J \subseteq R \\ 0 < \ell_R(J/I) < \infty}} (I^{[p^e]} : J^{[p^e]}) \subseteq I_e^{F\text{-sig}}.$$

Since the sum on the right is over a smaller set of proper inclusions, (12) follows immediately.

For (13), if $I \subseteq R$ is $\mathfrak{m}$-primary and $x \in R$ with $(I : x) = \mathfrak{m}$, then $(I^{[p^e]} : x^{p^e}) \subseteq I_e^{F\text{-sig}}$ for all $e \in \mathbb{N}$ implies $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)) \geq s(R)$ using (11). Moreover, if $I \subsetneq J$ is a proper inclusion of $\mathfrak{m}$-primary ideals, summing up this inequality for each factor in a composition series of J/I shows $\ell_R(J/I)s(R) \leq (e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J))$. The inequalities in (13) now follow immediately, noting again that the infimum on the right is over a smaller set of proper inclusions. □

Recall that a local ring $(R, \mathfrak{m}, k)$ is said to be approximately Gorenstein if there exists a descending chain of irreducible ideals $\{J_t\}_{t \in \mathbb{N}}$ cofinal with powers of the maximal ideal. In particular, each R/J_t is a zero-dimensional Gorenstein local ring and has a one-dimensional socle. By [Hochster 1977, Theorem 1.6], a reduced excellent local ring is always approximately Gorenstein. It is easy to check that equality holds throughout (12) for such rings.

Lemma 6.2. *Suppose that $(R, \mathfrak{m}, k)$ is an approximately Gorenstein F-finite local ring, and $\{J_t\}_{t\in\mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$. If $\delta_t \in R$ generates the socle of R/J_t, then for all $e \in \mathbb{N}$ we have $(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e})$ and*

$$\begin{aligned} I_e^{F\text{-sig}} &= \langle r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m} \textit{ for all } \phi \in \operatorname{Hom}_R(R^{1/p^e}, R)\rangle \\ &= \sum_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) \\ &= \bigcup_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) \\ &= (J_{t_e}^{[p^e]} : \delta_{t_e}^{p^e}) \quad \textit{for all } t_e \gg 0 \textit{ sufficiently large.} \end{aligned}$$

Moreover, R is weakly F-regular if and only if $J_t^* = J_t$ is tightly closed for all $t \in \mathbb{N}$.

Proof. Since each R/J_t is an Artinian Gorenstein local ring, we have $\operatorname{Ann}_E(J_t) \simeq R/J_t$ where $E = E_R(k)$ is the injective hull of the residue field of R. Thus, we may view $E = E_R(k) = \varinjlim R/J_t$ as the direct limit of inclusions $R/J_t \to R/J_{t+1}$ mapping (the class of) $\delta_t \mapsto \delta_{t+1}$. In particular, after applying $\_ \otimes_R R^{1/p^e}$, one sees that $(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e})$ for all $t \in \mathbb{N}$. It follows immediately that

$$\sum_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) = \bigcup_{t\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e}) = (J_{t_e}^{[p^e]} : \delta_{t_e}^{p^e}) \quad \text{for all } t_e \gg 0.$$

An inclusion $R \to M$ to a finitely generated R-module M determined by $1 \mapsto m$ splits if and only if $E \to E \otimes_R M$ remains injective, which is equivalent to $\delta_t m \notin J_t M$ for all $t \in \mathbb{N}$. See [Hochster 2007, p. 155] for further details. In particular, if $x \in R$, applying this splitting criterion to the map $R \to R^{1/p^e}$ with $1 \mapsto x^{1/p^e}$ gives that $x \in R \setminus I_e^{F\text{-sig}}$ if and only if $x \in R \setminus (J_t^{[p^e]} : \delta_t^{p^e})$ for all $t \in \mathbb{N}$, and so we have that $I_e^{F\text{-sig}} = (J_{t_e}^{[p^e]} : \delta_{t_e}^{p^e})$ for $t_e \gg 0$.

Lastly, suppose there is an ideal $I \subseteq R$ that is not tightly closed. Then $I = \bigcap_{n\in\mathbb{N}} (I + \mathfrak{m}^n)$ is an intersection of $\mathfrak{m}$-primary ideals. The arbitrary intersection of tightly closed ideals is tightly closed [Hochster and Huneke 1990, Proposition 4.1(b)]. Hence, we may replace I with an $\mathfrak{m}$-primary ideal which is not tightly closed and choose $x \in I^*$ with $\mathfrak{m} = (I : x)$. Since R/I injects into a direct sum of copies of E, we can find an R-module homomorphism $R/I \to E$ so that $x + I$ has nonzero image in E and hence must generate the socle $k \subseteq E$. Using that $E = \varinjlim R/J_t$, we may assume $R/I \to R/J_t$ and $x + I \mapsto \delta_t + J_t$ for some $t \in \mathbb{N}$. For each $e \in \mathbb{N}$, applying $\_ \otimes_R R^{1/p^e}$ and viewing as an R^{1/p^e}-module gives $R/I^{[p^e]} \to R/J_t^{[p^e]}$ where $x^{p^e} + I^{[p^e]} \mapsto \delta_t^{p^e} + J_t^{[p^e]}$. In particular, it follows that $(I^{[p^e]} : x^{p^e}) \subseteq (J_t^{[p^e]} : \delta_t^{p^e})$. Thus, $\bigcap_{e\in\mathbb{N}} (I^{[p^e]} : x^{p^e}) \subseteq \bigcap_{e\in\mathbb{N}} (J_t^{[p^e]} : \delta_t^{p^e})$ is not contained in any minimal prime of R; it follows that $\delta_t \in J_t^*$ and hence J_t is not tightly closed (compare to [Hochster and Huneke 1990, Proposition 8.23(f)]). ☐

In the next result, we show how to make use of the uniformity of constants from Theorem 4.3 together with approximately Gorenstein sequences in order to compare F-signature and Hilbert–Kunz multiplicity. As an application, we answer a question posed by Watanabe and Yoshida [2004, Question 1.10].

Theorem 6.3. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite domain of dimension d, and fix $0 \neq c \in R$. Suppose we are given sequences of ideals $\{I_{t,e}\}_{t,e\in\mathbb{N}}$ satisfying $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}$, $c(I_{t,e}^{[p]}) \subseteq I_{t,e+1}$, and $I_{t,e} \subseteq I_{t+1,e}$ for all $t, e \in \mathbb{N}$. Then*

$$\lim_{e\to\infty}\lim_{t\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_{t,e}) = \lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_e) = \lim_{t\to\infty}\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_{t,e})$$

where $I_e = \sum_{t\in\mathbb{N}} I_{t,e}$.

Proof. For each fixed $t \in \mathbb{N}$, as $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}$ and $c(I_{t,e}^{[p]}) \subseteq I_{t,e+1}$ for $e \in \mathbb{N}$, Theorem 4.3(i) guarantees the existence of $\eta_t = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_{t,e})$ and provides a uniform positive constant $C \in \mathbb{R}$ so that

$$\eta_t \leq \frac{1}{p^{ed}}\ell_R(R/I_{t,e}) + \frac{C}{p^e} \tag{14}$$

for all $t, e \in \mathbb{N}$. The sequence $\{I_e\}_{e\in\mathbb{N}}$ inherits the properties $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $c(I_e^{[p]}) \subseteq I_{e+1}$ for $e \in \mathbb{N}$; hence, $\lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ exists as well. Since $I_{t,e} \subseteq I_e$ and so $\ell_R(R/I_e) \leq \ell_R(R/I_{t,e})$ for all $t, e \in \mathbb{N}$, applying $\lim_{e\to\infty}$ gives

$$\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_e) \leq \eta_t \tag{15}$$

for all $t \in \mathbb{N}$.

Since $I_{t,e} \subseteq I_{t+1,e}$ is increasing in t for fixed e, it follows that $\eta_t \geq \eta_{t+1} \geq 0$ for all $t \in \mathbb{N}$ and hence $\lim_{t\to\infty}\eta_t$ exists. We also have $I_e = I_{t_e,e}$ for $t_e \gg 0$, so that $\lim_{e\to\infty}\lim_{t\to\infty}(1/p^{ed})\ell_R(R/I_{t,e}) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$. Applying $\lim_{t\to\infty}$ to (14) and (15) gives

$$\lim_{e\to\infty}\frac{1}{p^{ed}}\ell_R(R/I_e) \leq \lim_{t\to\infty}\eta_t \leq \frac{1}{p^{ed}}\ell_R(R/I_e) + \frac{C}{p^e}$$

for all $e \in \mathbb{N}$. Further taking $\lim_{e\to\infty}$ gives $\lim_{t\to\infty}\eta_t = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e)$ and completes the proof. $\square$

Theorem 6.4. *Let $(R, \mathfrak{m}, k)$ be an approximately Gorenstein F-finite local ring of dimension d. Suppose $\{J_t\}_{t\in\mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$, and $\delta_t \in R$ generates the socle of R/J_t. Then $s(R) = \lim_{t\to\infty} e_{\mathrm{HK}}(J_t) - e_{\mathrm{HK}}((J_t, \delta_t))$.*

Proof. Both invariants are unchanged after completion, so we may assume R is complete. Suppose first that R is not weakly F-regular, so that $\delta_t \in J_t^*$ for some $t \in \mathbb{N}$ and $(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e}) \subseteq \cdots \subseteq (J_{t+t'}^{[p^e]} : \delta_{t+t'}^{p^e}) \subseteq \cdots$ for all $e, t' \in \mathbb{N}$ by Lemma 6.2. If $c \in \bigcap_{e\in\mathbb{N}}(J_t^{[p^e]} : \delta_t^{p^e}) \subseteq \bigcap_{e\in\mathbb{N}}(J_{t+t'}^{[p^e]} : \delta_{t+t'}^{p^e})$ is not in any minimal prime, then we have $0 \leq e_{\mathrm{HK}}(J_{t+t'}) - e_{\mathrm{HK}}((J_{t+t'}, \delta_{t+t'})) \leq \lim_{e\to\infty}(1/p^{ed})\ell_R(R/(c, \mathfrak{m}^{[p^e]})) = 0$ using (11) from Corollary 5.6 and applying Lemma 3.1 with $M = R/(c)$. Using Lemma 6.1, we have that $s(R) = \lim_{t'\to\infty} e_{\mathrm{HK}}(J_{t+t'}) - e_{\mathrm{HK}}((J_{t+t'}, \delta_{t+t'})) = 0$ as desired.

Thus, we assume for the remainder that R is weakly F-regular and hence a domain. Consider the sequences of ideals $I_{t,e} = (J_t^{[p^e]} : \delta_t^{p^e})$ for $t, e \in \mathbb{N}$. We check

$$\begin{aligned}
\mathfrak{m}^{[p^e]} &= (J_t : \delta_t)^{[p^e]} \subseteq (J_t^{[p^e]} : \delta_t^{p^e}) = I_{t,e},\\
I_{t,e}^{[p]} &= (J_t : \delta_t)^{[p^e]} \subseteq (J_t^{[p^e]} : \delta_t^{p^e})^{[p]} \subseteq (J_t^{[p^{e+1}]} : \delta_t^{p^{e+1}}) = I_{t,e+1},\\
I_{t,e} &= (J_t^{[p^e]} : \delta_t^{p^e}) \subseteq (J_{t+1}^{[p^e]} : \delta_{t+1}^{p^e}) = I_{t+1,e},
\end{aligned}$$

so Theorem 6.3 applies with $c = 1$. Using Lemma 6.2 and (11) from Corollary 5.6, we conclude

$$\begin{aligned}
s(R) = \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/I_e^{F\text{-sig}}) &= \lim_{e\to\infty}\lim_{t\to\infty} \frac{1}{p^{ed}} \ell_R(R/I_{t,e})\\
&= \lim_{t\to\infty}\lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/I_{t,e}) = \lim_{t\to\infty} e_{\mathrm{HK}}(J_t) - e_{\mathrm{HK}}((J_t, \delta_t)).
\end{aligned}$$ □

Theorem 6.4 provides a positive answer to [Watanabe and Yoshida 2004, Question 1.10].

Corollary 6.5. *If $(R, \mathfrak{m}, k)$ is an F-finite local ring, then*

$$s(R) = \inf_{\substack{I\subseteq J\subseteq R,\, \ell_R(R/I)<\infty\\ I\neq J,\, \ell_R(R/J)<\infty}} \frac{e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J)}{\ell_R(J/I)} = \inf_{\substack{I\subseteq R,\, \ell_R(R/I)<\infty\\ x\in R,\, (I:x)=\mathfrak{m}}} e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x)).$$

Proof. If R is not reduced, we can find some $0 \neq x \in R$ with $x^p = 0$. For $n \gg 0$, we have $x \notin \mathfrak{m}^n$ and can find an ideal I with $\mathfrak{m}^n \subseteq I \subseteq (\mathfrak{m}^n, x)$ where $(I : x) = \mathfrak{m}$. Since $x^{p^e} = 0$ for all $e \in \mathbb{N}$, we have $I^{[p^e]} = (I, x)^{[p^e]}$, and thus $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}((I, x))$. It follows from Lemma 6.1 that $s(R) = 0$ and equality holds throughout (13). Thus, we may assume R is reduced. By [Hochster 1977, Theorem 1.7], R is approximately Gorenstein and Theorem 6.4 implies equality holds throughout (13) as desired. □

When $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite domain of dimension d, one can make Theorem 6.4 more explicit still. Recall that a canonical ideal $J \subseteq R$ is an ideal such that J is isomorphic to a canonical module ω_R, which exists as R is assumed complete. A canonical ideal J is necessarily unmixed with height 1, and moreover, R/J will be Gorenstein of dimension $d - 1$ [Bruns and Herzog 1993, Proposition 3.3.18]. When R is also normal, fixing a canonical ideal is equivalent to fixing a choice of effective anticanonical divisor.

Corollary 6.6. *Suppose that $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite domain of dimension d, and J is a canonical ideal of R. Let $x_1 \in J$ and $x_2, \dots, x_d \in R$ be chosen so that $x_1, \dots, x_d$ give a system of parameters for R, and suppose $\delta \in R$ generates the socle of $R/(J, x_2, \dots, x_d)$. Then*

$$s(R) = \lim_{t\to\infty} e_{\mathrm{HK}}((J, x_2^t, \dots, x_d^t)) - e_{\mathrm{HK}}((J, x_2^t, \dots, x_d^t, (x_2^{t-1}\cdots x_d^{t-1}\delta))).$$

Proof. Note that $x_1^{t-1}J$ is yet another canonical ideal for any $t \in \mathbb{N}$, and $x_2^t, \dots, x_d^t$ give a system of parameters for $R/(x_1^{t-1}J)$. Thus, the sequence $J_t = (x_1^{t-1}J, x_2^t, \dots, x_d^t)$ gives a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$. It is easy to check that $\delta_t = x_1^{t-1}x_2^{t-1}\cdots x_d^{t-1}\delta$ generates the socle of R/J_t using that $x_1, \dots, x_d$ form a regular sequence, which further implies $(J_t^{p^e} : \delta_t^{p^e}) = ((J, x_2^t, \dots, x_d^t)^{[p^e]} : (x_2^{t-1} \cdots x_d^{t-1}\delta)^{p^e})$ for all $t, e \in \mathbb{N}$. Thus, Theorem 6.4 gives

$$\begin{aligned} s(R) &= \lim_{t\to\infty} \frac{1}{p^{ed}} \ell_R(R/(J_t^{p^e} : \delta_t^{p^e})) \\ &= \lim_{t\to\infty} e_{\mathrm{HK}}((x_1^{t-1}J, x_2^t, \dots, x_d^t)) - e_{\mathrm{HK}}((x_1^{t-1}J, x_2^t, \dots, x_d^t, (x_1^{t-1}x_2^{t-1} \cdots x_d^{t-1}\delta))) \\ &= \lim_{t\to\infty} \frac{1}{p^{ed}} \ell_R(R/((J, x_2^t, \dots, x_d^t)^{[p^e]} : (x_2^{t-1} \cdots x_d^{t-1}\delta)^{p^e})) \\ &= \lim_{t\to\infty} e_{\mathrm{HK}}((J, x_2^t, \dots, x_d^t)) - e_{\mathrm{HK}}((J, x_2^t, \dots, x_d^t, (x_2^{t-1} \cdots x_d^{t-1}\delta))) \end{aligned}$$

using the relation in (11) once more. $\square$

One can push the above analysis further still. In the notation of the previous proof, when R is normal x_2 can be chosen so that R_{x_2} is Gorenstein and J_{x_2} is principal. This allows one to remove the exponent t on x_2 in the limit above using (i) from the subsequent lemma. Following the methods of [Aberbach 2002] (compare to [Aberbach and Enescu 2006; MacCrimmon 1996; Yao 2006]), we present a complete treatment in Corollary 6.8 below.

Lemma 6.7. *Suppose that $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite normal domain of dimension d, and D an effective Weil divisor on* $\mathrm{Spec}(R)$. *Put $J = R(-D) \subseteq R$ so that $J^{(n)} = R(-nD)$ for $n \in \mathbb{N}$. Let $x_1 \in J$ and $x_2, \dots, x_d \in R$ be chosen so that $x_1, \dots, x_d$ give a system of parameters for R, and fix $e \in \mathbb{N}$.*

(i) *If $x_2 J \subseteq a_2 R$ for some $a_2 \in J$, there exists $b_2 \in J$ so that $a_2, x_2 + b_2, x_3, \dots, x_d$ give a system of parameters for R. Moreover, for any nonnegative integers $N_2, \dots, N_d$ with $N_2 \geq 2$, we have that*

$$\begin{aligned} ((J^{(p^e)}, x_2^{N_2p^e}, x_3^{N_3p^e}, \dots, x_d^{N_dp^e}) : x_2^{(N_2-1)p^e}) &= ((J^{[p^e]}, x_2^{N_2p^e}, x_3^{N_3p^e}, \dots, x_d^{N_dp^e}) : x_2^{(N_2-1)p^e}) \\ &= ((J^{[p^e]}, x_2^{2p^e}, x_3^{N_3p^e}, \dots, x_d^{N_dp^e}) : x_2^{p^e}). \end{aligned}$$

(ii) *Suppose $x_d^n J^{(n)} \subseteq a_d R$ for some $n \in \mathbb{N}$ and $a_d \in J^{(n)}$. Then there exists $b_d \in J$ such that $a_d, x_2, \dots, x_{d-1}, x_d + b_d$ give a system of parameters for R. Moreover, for any nonnegative integers $N_2, \dots, N_d$ with $N_d \geq 2$, we have that*

$$((J^{(p^e)}, x_2^{N_2p^e}, \dots, x_{d-1}^{N_{d-1}p^e}, x_d^{N_dp^e}) : x_d^{(N_d-1)p^e}) \subseteq ((J^{(p^e)}, x_2^{N_2p^e}, \dots, x_{d-1}^{N_{d-1}p^e}, x_d^{2p^e}) : x_1^n x_d^{p^e})$$

Proof. For (i), note first a_2 is a nonzero divisor on $R/(x_3, \dots, x_d)$ as its multiple $x_1x_2 \in x_2J$ is a nonzero divisor. Hence, $\dim(R/(x_3, \dots, x_d, a_2)) = d - 1$. Since $\ell_R(R/(J, x_2, \dots, x_d)) \leq \ell_R(R/(x_1, \dots, x_d)) < \infty$, we know that (x_2, J) is not contained in any minimal prime ideal of $(x_3, \dots, x_d, a_2)$. In particular, using standard prime avoidance arguments, one can find $b_2 \in J$ so that $x_2 + b_2$ also avoids all of the minimal primes of $(x_3, \dots, x_d, a_2)$; it follows that the sequence $x_3, \dots, x_d, a_2, x_2 + b_2$ is a system of parameters for R.

Since $R(-\mathrm{div}(x_2) - D) = x_2 J \subseteq a_2 R = R(-\mathrm{div}(a_2))$, we have that $D + \mathrm{div}(x_2) \geq \mathrm{div}(a_2)$ and hence also $p^e D + \mathrm{div}(x_2^{p^e}) \geq \mathrm{div}(a_2^{p^e})$; it follows that $x_2^{p^e} J^{(p^e)} \subseteq a_2^{p^e} R$. The ideals inclusions $\supseteq$ in (i) are clear, so we need only check the reverse. Let $I = (x_3^{N_3 p^e}, \dots, x_d^{N_d p^e})$, and suppose that we have $cx_2^{(N_2-1)p^e} \in (J^{(p^e)}, x_2^{N_2 p^e}, I)$ for some $c \in R$. We can find $r_2 \in R$ with $(c - r_2 x_2^{p^e}) x_2^{(N_2-1)p^e} \in (J^{(p^e)}, I)$. Since $b_2 \in J$, we have $b^{p^e} \in J^{(p^e)}$ and so $(c - r_2 x_2^{p^e})(x_2 + b_2)^{(N_2-1)p^e} \in (J^{(p^e)}, I)$. Multiplying through by $x_2^{p^e}$ and using that $x_2^{p^e} J^{(p^e)} \subseteq a_2^{p^e}$ gives $(cx_2^{p^e} - r_2 x_2^{2p^e})(x_2 + b_2)^{(N_2-1)p^e} \in (a_2^{p^e}, I)$. Using that $a_2, x_2 + b_2, x_3, \dots, x_d$ give a system of parameters for R, we see $(cx_2^{p^e} - r_2 x_2^{2p^e}) \in (a_2^{p^e}, I) \subseteq (J^{[p^e]}, I)$ and conclude $cx_2^{p^e} \in (J^{[p^e]}, x_2^{2p^e}, I)$ as desired.

Statement (ii) proceeds similarly. We have that a_d is a nonzero divisor on $R/(x_2, \dots, x_{d-1})$ as its multiple $x_d^n x_1^n$ is a nonzero divisor, and again (x_d, J) is not contained in any minimal prime of $(a_d, x_2, \dots, x_{d-1})$. We can find $b_d \in J$ so that $x_d + b_d$ is not in any minimal prime of $(a_d, x_2, \dots, x_{d-1})$ and thus $a_d, x_2, \dots, x_{d-1}, x_d + b_d$ is a system of parameters for R. From $x_d^n J^{(n)} \subseteq a_d R$, we have $nD + \operatorname{div}(x_d^n) \geq \operatorname{div}(a_d)$ and it follows that $x_d^{p^e} J^{(p^e)} \subseteq a_d^{\lfloor p^e/n \rfloor} R$. To see the final inclusion, let $I = (x_2^{N_2 p^e}, \dots, x_{d-1}^{N_{d-1} p^e})$ and suppose $cx_d^{(N_d-1)p^e} \in (J^{(p^e)}, I, x_d^{N_d p^e})$ for some $c \in R$. We can find $r_d \in R$ so that $(c - r_d x_d^{p^e}) x_d^{(N_d-1)p^e} \in (J^{(p^e)}, I)$. As $b_d \in J$ so $b_d^{p^e} \in J^{(p^e)}$, hence also $(c - r_d x_d^{p^e})(x_d + b_d)^{(N_d-1)p^e} \in (J^{(p^e)}, I)$. Multiplying through by $x_d^{p^e}$ gives $(cx_d^{p^e} - r_d x_d^{2p^e})(x_d + b_d)^{(N_d-1)p^e} \in (a_d^{\lfloor p^e/n \rfloor}, I)$. Using that $a_d, x_2, \dots, x_{d-1}, x_d + b_d$ are a system of parameters, we see that $(cx_d^{p^e} - r_d x_d^{2p^e}) \in (a_d^{\lfloor p^e/n \rfloor}, I)$. Multiplying through by x_1^n and using that $a_d^{\lfloor p^e/n \rfloor} x_1^n \in (J^{(n)})^{\lfloor p^e/n \rfloor + 1} \subseteq J^{(p^e)}$ gives $(cx_1^n x_d^{p^e} - r_d x_1^n x_d^{2p^e}) \in (J^{(p^e)}, I)$ and in particular $cx_1^n x_d^{p^e} \in (J^{(p^e)}, I, x_d^{2p^e})$ as desired. $\square$

Corollary 6.8. *Suppose $(R, \mathfrak{m}, k)$ is a complete Cohen–Macaulay local F-finite domain of dimension d.*

(i) *If R is Gorenstein and $x_1, \dots, x_d$ are any system of parameters for R and $\delta \in R$ generates the socle of $R/(x_1, \dots, x_d)$, then $I_e^{F\text{-sig}} = ((x_1, \dots, x_d)^{[p^e]} : \delta^{p^e})$ for all $e \in \mathbb{N}$ and $s(R) = e_{\mathrm{HK}}((x_1, \dots, x_d)) - e_{\mathrm{HK}}((x_1, \dots, x_d, \delta))$* [Huneke and Leuschke 2002].

(ii) *If the punctured spectrum $\operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ is $\mathbb{Q}$-Gorenstein, there exists an $\mathfrak{m}$-primary ideal I and $x \in R$ with $(I : x) = \mathfrak{m}$ so that $s(R) = e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}((I, x))$ and moreover $I_e^{F\text{-sig}} \subseteq ((I^{[p^e]})^* : x^{p^e})$ for all $e \in \mathbb{N}$. In particular, if R is weakly F-regular, $I_e^{F\text{-sig}} = (I^{[p^e]} : x^{p^e})$ for all $e \in \mathbb{N}$.*

Proof. (i) Since R is Gorenstein, we may take $J = (x_1)$ to be the canonical ideal above in Corollary 6.6. For fixed $e \in \mathbb{N}$, we have that $I_e^{F\text{-sig}} = ((x_1^{tp^e}, \dots, x_d^{tp^e}) : (x_1^{(t-1)p^e} \cdots x_d^{(t-1)p^e} \delta^{p^e}))$ for all $t \gg 0$. However, since $x_1, \dots, x_d$ are a regular sequence on R, it follows that $((x_1^{tp^e}, \dots, x_d^{tp^e}) : (x_1^{(t-1)p^e} \cdots x_d^{(t-1)p^e} \delta^{p^e})) = ((x_1^{p^e}, \dots, x_d^{p^e}) : \delta^{p^e})$ for any $t \in \mathbb{N}$. The final statement follows again using the relation in (11) once more.

For (ii), we begin with the following construction. Let J be a canonical ideal of R, and choose $0 \neq x_1 \in J$. Let U_2 be the complement of the minimal primes of x_1, which are all height one and contain the set of minimal primes of J. We have that $U_2^{-1} J$ is principal (as $U_2^{-1} R$ is a semilocal Dedekind domain) and can choose $x_2 \in U_2$ and $a_2 \in J$ so that $J_{x_2} = a_2 R_{x_2}$ and $x_2 J \subseteq a_2 R$.

As $R_{\mathfrak{p}}$ is $\mathbb{Q}$-Gorenstein for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, we may choose $n \in \mathbb{N}$ so that $J_{\mathfrak{p}}^{(n)}$ is principal for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Assuming $x_1, \dots, x_{i-1}$ have already been chosen, let U_i be the complement of the set of minimal primes of $x_1, \dots, x_{i-1}$. Again, $U_i^{-1} J^{(n)}$ is principal (as $U_i^{-1} R$ is a semilocal domain, every locally principal ideal is principal by [Kaplansky 1974, Theorem 60]), so we can choose $x_i \in U_i$ and $a_i \in J^{(n)}$ so that $J_{x_i}^{(n)} = a_i R_{x_i}$ and $x_i^n J^{(n)} \subseteq a_i R$. Continuing in this manner gives a system of parameters

$x_1, \dots, x_d$ of R with $x_1 \in J$, and a sequence of elements $a_3, \dots, a_d \in J^{(n)}$ so that $x_i^n J^{(n)} \subseteq a_i R$ for each $i = 3, \dots, d$. For fixed $e \in \mathbb{N}$ and $t \gg 0$, we have from Corollary 6.6 and repeated application of Lemma 6.7(ii) for $x_d, \dots, x_3$ and then Lemma 6.7(i) for x_2 that

$$\begin{aligned} I_e^{F\text{-sig}} &= ((J^{[p^e]}, x_2^{tp^e}, \dots, x_{d-1}^{tp^e}, x_d^{tp^e}) : (x_2^{(t-1)p^e} \cdots x_{d-1}^{(t-1)p^e} x_d^{(t-1)p^e} \delta^{p^e})) \\ &\subseteq ((J^{(p^e)}, x_2^{tp^e}, \dots, x_{d-1}^{tp^e}, x_d^{tp^e}) : (x_2^{(t-1)p^e} \cdots x_{d-1}^{(t-1)p^e} x_d^{(t-1)p^e} \delta^{p^e})) \\ &\subseteq ((J^{(p^e)}, x_2^{tp^e}, \dots, x_{d-1}^{tp^e}, x_d^{2p^e}) : (x_1^n x_2^{(t-1)p^e} \cdots x_{d-1}^{(t-1)p^e} x_d^{p^e} \delta^{p^e})) \\ &\;\;\vdots \\ &\subseteq ((J^{(p^e)}, x_2^{tp^e}, x_3^{2p^e}, \dots, x_d^{2p^e}) : (x_1^{(d-2)n} x_2^{(t-1)p^e} x_3^{p^e} \cdots x_d^{p^e} \delta^{p^e})) \\ &= ((J^{[p^e]}, x_2^{2p^e}, x_3^{2p^e}, \dots, x_d^{2p^e}) : (x_1^{(d-2)n} x_2^{p^e} x_3^{p^e} \cdots x_d^{p^e} \delta^{p^e})) \\ &= (((J^{[p^e]}, x_2^{2p^e}, \dots, x_d^{2p^e}) : (x_2^{p^e} \cdots x_d^{p^e} \delta^{p^e})) : x_1^{(d-2)n}), \end{aligned} \tag{16}$$

so setting $I = (J, x_2^2, \dots, x_d^2)$ and $x = x_2 \cdots x_d \delta$ gives $s(R) = e_{\text{HK}}(I) - e_{\text{HK}}((I, x))$ using Lemma 4.13 (with $I_e = ((J^{[p^e]}, x_2^{2p^e}, \dots, x_d^{2p^e}) : (x_2^{p^e} \cdots x_d^{p^e} \delta^{p^e}))$, $J_e = I_e^{F\text{-sig}}$, and $c = x_1^{(d-2)n}$).

For the final statement, since $(I_e^{F\text{-sig}})^{[p^{e'}]} \subseteq I_{e+e'}^{F\text{-sig}}$ for all $e, e' \in \mathbb{N}$, it follows from (16) that

$$0 \neq x_1^{(d-2)n} \in \bigcap_{e' \in \mathbb{N}} (I^{[p^{e+e'}]} : (x^{p^e} I_e^{F\text{-sig}})^{[p^{e'}]})$$

so that $x^{p^e} I_e^{F\text{-sig}} \subseteq (I^{[p^e]})^*$ or $I_e^{F\text{-sig}} \subseteq ((I^{[p^e]})^* : x^{p^e})$ as claimed. In particular, if R is weakly F-regular, we have that $I_e^{F\text{-sig}} \subseteq ((I^{[p^e]}) : x^{p^e})$ and equality follows from Lemma 6.1. □

One can also use Theorem 6.3 to show that the length criteria in Corollaries 5.7 and 5.8 also determine the F-signatures in those settings.

Corollary 6.9. *Let $(R, \mathfrak{m}, k)$ be a complete F-finite local domain of dimension d. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $\xi \in \mathbb{R}_{\geq 0}$. Suppose that $\{J_t\}_{t \in \mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$, and $\delta_t \in R$ generates the socle of R/J_t. Then*

$$s(R, \mathfrak{a}^\xi) = \lim_{t \to \infty} \lim_{e \to \infty} (1/p^{ed}) \ell_R \cdot (R/(J_t^{[p^e]} : \mathfrak{a}^{\lceil \xi(p^e-1) \rceil} \delta_t^{p^e})).$$

Proof. It is straightforward to check for each $e \in \mathbb{N}$ that

$$\begin{aligned} I_e^{\mathfrak{a}^\xi} &= (x \in R \mid \phi(x^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \mathscr{C}_e^{\mathfrak{a}^\xi}) \\ &= (x \in R \mid (xy)^{1/p^e} R \subseteq R^{1/p^e} \text{ is not split for any } y \in \mathfrak{a}^{\lceil \xi(p^e-1) \rceil}), \end{aligned}$$

and it follows that $I_e^{\mathfrak{a}^\xi} = \sum_{t \in \mathbb{N}} I_{t,e}^{\mathfrak{a}^\xi}$ where $I_{t,e}^{\mathfrak{a}^\xi} = (J_t^{[p^e]} : \delta_t^{p^e} \mathfrak{a}^{\lceil (p^e-1)\xi \rceil})$ for $t, e \in \mathbb{N}$. Choosing an element $0 \neq c \in R$ such that $c\mathfrak{a}^{\lceil \xi(p^{e+1}-1) \rceil} \subseteq (\mathfrak{a}^{\lceil \xi(p^e-1) \rceil})^{[p]}$ for all $e \in \mathbb{N}$ as in the proof of Theorem 4.11, we have that $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}^{\mathfrak{a}^\xi}$, $c((I_{t,e}^{\mathfrak{a}^\xi})^{[p]}) \subseteq I_{t,e+1}^{\mathfrak{a}^\xi}$, and $I_{t,e}^{\mathfrak{a}^\xi} \subseteq I_{t+1,e}^{\mathfrak{a}^\xi}$ for all $t, e \in \mathbb{N}$. The result now follows immediately from Theorem 6.3 as we know $s(R, \mathfrak{a}^\xi) = \lim_{e \to \infty} (1/p^{ed}) \ell_R(R/I_e^{\mathfrak{a}^\xi})$. □

Corollary 6.10. *Let $(R, \mathfrak{m}, k)$ be a complete F-finite local normal domain of dimension d. Suppose $\{J_t\}_{t\in\mathbb{N}}$ is a descending chain of irreducible ideals cofinal with the powers of $\mathfrak{m}$, and $\delta_t \in R$ generates the socle of R/J_t. If Δ is an effective $\mathbb{Q}$-divisor on* $\operatorname{Spec}(R)$, *then* $s(R, \Delta) = \lim_{t\to\infty} \lim_{e\to\infty}(1/p^{ed})\ell_R \cdot (R/(J_t^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : \delta_t^{p^e}))$.

Proof. Again it is straightforward to check for each $e \in \mathbb{N}$ that

$$\begin{aligned} I_e^{(R,\Delta)} &= (x \in R \mid \phi(x^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \mathscr{C}_e^{(R,\Delta)}) \\ &= (x \in R \mid (x)^{1/p^e}R \subseteq (R(\lceil(p^e-1)\Delta\rceil))^{1/p^e} \text{ is not split}), \end{aligned}$$

and it follows that $I_e^{(R,\Delta)} = \sum_{t\in\mathbb{N}} I_{t,e}^{(R,\Delta)}$ where $I_{t,e}^{(R,\Delta)} = (J_t^{[p^e]}R(\lceil(p^e-1)\Delta\rceil) : \delta_t^{p^e})$ for $t, e \in \mathbb{N}$. Choosing an element $0 \neq c \in R$ so that $\operatorname{div}_R(c) \geq p\lceil\Delta\rceil$ as in the proof of Theorem 4.12, we have that $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}^{(R,\Delta)}$, $c((I_{t,e}^{(R,\Delta)})^{[p]}) \subseteq I_{t,e+1}^{(R,\Delta)}$, and $I_{t,e}^{(R,\Delta)} \subseteq I_{t+1,e}^{\mathfrak{a}^{\xi}}$ for all $t, e \in \mathbb{N}$. The result now follows immediately from Theorem 6.3 as we know $s(R, \Delta) = \lim_{e\to\infty}(1/p^{ed})\ell_R(R/I_e^{(R,\Delta)})$. ☐

Lastly, note that Theorem 6.3 can also be applied outside the context of descending chains of irreducible ideals, and gives yet another perspective on the original proof of the existence of the F-signature.

Corollary 6.11 [Tucker 2012, Proof of Theorem 4.9]. *Let $(R, \mathfrak{m}, k)$ be a complete F-finite local domain of dimension d and $I_e^{F\text{-sig}} = (r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R))$. Then $s(R) = \lim_{t\to\infty}(1/p^{td})e_{\mathrm{HK}}(I_t^{F\text{-sig}})$.*

Proof. Let

$$I_{t,e} = \begin{cases} (I_t^{F\text{-sig}})^{[p^{e-t}]}, & t < e, \\ I_e^{F\text{-sig}}, & t \geq e. \end{cases}$$

It is easily checked that $\mathfrak{m}^{[p^e]} \subseteq I_{t,e}$, $I_{t,e}^{[p]} \subseteq I_{t,e+1}$, and $I_{t,e} \subseteq I_{t+1,e}$ for all $t, e \in \mathbb{N}$ using that $\mathfrak{m}^{[p^e]} \subseteq I_e^{F\text{-sig}}$ and $(I_e^{F\text{-sig}})^{[p]} \subseteq I_{e+1}^{F\text{-sig}}$ for all $e \in \mathbb{N}$. The result now follows immediately from Theorem 6.3, as we have

$$\begin{aligned} s(R) &= \lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(I_e^{F\text{-sig}}) = \lim_{e\to\infty}\lim_{t\to\infty} \frac{1}{p^{ed}}\ell_R(R/I_{t,e}) = \lim_{t\to\infty}\lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/I_{t,e}) \\ &= \lim_{t\to\infty}\lim_{e\to\infty} \frac{1}{p^{ed}}\ell_R(R/(I_t^{F\text{-sig}})^{[p^{e-t}]}) = \lim_{t\to\infty} \frac{1}{p^{td}}e_{\mathrm{HK}}(I_t^{F\text{-sig}}) \end{aligned}$$

as desired. ☐

7. Open questions

In this section, we collect together some information on the important questions left unanswered in this article. We have seen that Hilbert–Kunz multiplicity and the F-signature enjoy semicontinuity properties for F-finite rings; it is not difficult to extend these results to rings which are essentially of finite type over an excellent local ring. More generally, however, this raises the following question.

Question 7.1. *If R is an excellent domain that is not F-finite or essentially of finite type over an excellent local ring, do the Hilbert–Kunz multiplicity and F-signature determine semicontinuous $\mathbb{R}$-valued functions on* $\operatorname{Spec}(R)$?

In the case of F-signature, a positive answer would imply the openness of the strongly F-regular locus for such a ring. Note that this question would seem closely related to the existence of (locally and completely stable) test elements for excellent domains, which also remains unanswered.

Perhaps the most important question left open in this article regarding the relationship between Hilbert–Kunz multiplicity and F-signature is the following, which (in light of the results of the previous section) is attributed to Watanabe and Yoshida [2004].

Question 7.2. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite normal domain. Do there exist $\mathfrak{m}$-primary ideals $I \subsetneq J$ so that $e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(J) = s(R)$?*

For any ring such that Question 7.2 has a positive answer, it that weak F-regularity is equivalent to strong F-regularity by the length criterion for tight closure [Hochster and Huneke 1994, Theorem 8.17] (Corollary 5.6 above).

We see from Corollary 6.8 that Question 7.2 is true provided R is $\mathbb{Q}$-Gorenstein on the punctured spectrum. When in addition R is weakly F-regular, the stronger condition below is satisfied as well.

Question 7.3. *Let $(R, \mathfrak{m}, k)$ be a complete local F-finite normal domain. Do there exist $\mathfrak{m}$-primary ideals $I \subsetneq J$ so that*

$$\frac{\operatorname{frk}_R(R^{1/p^e})}{[k^{1/p^e} : k]} = \frac{\ell_R(R/I^{[p^e]}) - \ell_R(R/J^{[p^e]})}{\ell_R(J/I)}$$

for all $e \in \mathbb{N}$?

The importance of Question 7.3 stems from the observation that it allows one to apply the results of Huneke, McDermott, and Monsky [Huneke et al. 2004]; for an $\mathfrak{m}$-primary ideal $I \subseteq R$, they show the existence of a constant $\alpha(I) \in \mathbb{R}$ so that $\ell(R/I^{[p^e]}) = e_{\mathrm{HK}}(I)p^{ed} + \alpha(I)p^{e(d-1)} + O(p^{e(d-2)})$. In other words, Hilbert–Kunz functions in normal local F-finite domains are polynomial in p^e to an extra degree. In particular, for any ring such that Question 7.3 has a positive answer, so also does the following question.

Question 7.4. *If $(R, \mathfrak{m}, k)$ is a complete local F-finite normal domain, when does there exist a positive constant $\alpha(R) \in \mathbb{R}$ so that $\operatorname{frk}_R(R^{1/p^e})/[k^{1/p^e} : k] = s(R)p^{ed} + \alpha(R)p^{e(d-1)} + O(p^{e(d-2)})$?*

Finally, we have tried to emphasize the applicability of our techniques to the settings of divisor and ideal pairs throughout, and the questions above are readily generalized to those settings. Moreover, it may well be the case that answers to the questions above require the use of such pairs (particularly to remove Gorenstein or $\mathbb{Q}$-Gorenstein hypotheses). In this direction, and in view of [Schwede 2011], one could imagine a positive and constructive answer to the question below could prove quite useful.

Question 7.5. *Suppose $(R, \mathfrak{m}, k)$ is an F-finite local normal domain and $X = \operatorname{Spec}(R)$. For each $\epsilon > 0$, does there exist an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index prime to p and so that $s(R) - s(R, \Delta) < \epsilon$? In case R is not local, can Δ be chosen globally to have this property locally over $\operatorname{Spec}(R)$?*

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Regular pairs of quadratic forms on odd-dimensional spaces in characteristic 2

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We describe a normal form for a smooth intersection of two quadrics in even-dimensional projective spaces over an arbitrary field of characteristic 2. We use this to obtain a description of the automorphism group of such a variety. As an application, we show that every quartic del Pezzo surface over a perfect field of characteristic 2 has a canonical rational point and, thus, is unirational.

1. Introduction

Let (q_0, q_1) be a pair of quadratic forms on a vector space E over a field $\Bbbk$. The common zeros of the pair define a subvariety $X = V(q_0, q_1)$ in the projective space $|E| = \mathbb{P}(E^\vee)$ of lines in E. We say the pair is *regular* if the variety X is smooth and of codimension 2 in $|E|$. One may also consider the pencil of quadrics spanned by $V(q_0)$ and $V(q_1)$; the pencil is *regular* if X is smooth.

When $\Bbbk = \mathbb{C}$, the field of complex numbers, a classical result due to A. Cauchy and C. Jacobi (see [Muth 2016]) states that any regular pair can be *simultaneously diagonalized*. When q_0 is nondegenerate, this means that we may find a basis $x_1, \dots, x_n$ in the dual space $E^\vee$ such that

$$q_0 = \sum_{i=1}^{n} x_i^2, \quad q_1 = \sum_{i=1}^{n} a_i x_i^2.$$

Here the coefficients $a_1, \dots, a_n$ are distinct roots of the discriminant polynomial $\delta(t) = \det(tM_0 - M_1)$, where M_0 and M_1 are symmetric matrices representing the polar symmetric bilinear forms associated to q_0 and q_1. When the pair is not necessarily regular, the classification was carried out by K. Weierstrass [Weierstrass 1868] by considering the elementary divisors of $tM_0 - M_1$. Over an arbitrary field $\Bbbk$ of characteristic not 2, pairs of quadratic forms were classified by L. Kronecker and L. Dickson. A modern exposition of their theory can be found in [Waterhouse 1976] (see also [Leep and Schueller 1999]).

Unsurprisingly, in characteristic $p = 2$ the situation is more complicated. The main reason for this is that one can no longer identify quadratic forms with symmetric bilinear forms. Even worse, when $n = \dim(E)$ is odd the determinant of $tM_0 - M_1$ is identically zero. One may still consider symmetric

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and alternating bilinear forms in this case (see, for example, [Waterhouse 1977; Leep and Schueller 1999; Ishitsuka and Ito 2015]), but the connection with quadratic forms is more tenuous.

The geometry of the intersection X of two quadrics differs drastically depending on whether n is even or odd. Examples when n is even include quartic elliptic curves in $\mathbb{P}^3$ and quadratic line complexes in $\mathbb{P}^5$. When n is odd, the first interesting example is a quartic del Pezzo surface in $\mathbb{P}^4$ isomorphic to a blowup of 5 points in the projective plane.

When $n = 2m$ is even and $\Bbbk = \mathbb{C}$, one can associate to X its intermediate Jacobian $J(X)$. A theorem of Luc Gauthier and André Weil [Gauthier 1954–55] asserts that $J(X)$ is isomorphic to the Jacobian variety of a hyperelliptic curve C of genus $m-1$ with equation $y^2 + \delta(t) = 0$. Over an algebraically closed field of characteristic $p \neq 2$, the Jacobian is isomorphic to the variety of $(m-2)$-planes in X (see also [Wang 2013]). If $p = 2$ and n is even, one should use the Pfaffian $P(t)$ defined so that $P(t)^2 = \delta(t)$. Under the condition that the roots $\alpha_1, \ldots, \alpha_m$ of $P(t)$ are distinct, U. Bhosle [1990] provides a normal form

$$q_0 = \sum_{i=1}^{m} x_i x_{m+i}, \quad q_1 = \sum_{i=1}^{m} (a_i x_i^2 + \alpha_i x_i x_{m+i} + b_i x_{m+i}^2)$$

where $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ are in $\Bbbk$. The variety of $(m-2)$-planes is isomorphic to the Jacobian of a hyperelliptic curve in this case as well.

This paper is concerned with pairs of quadratic forms, and their corresponding pencils, in the case where $\Bbbk$ has characteristic 2 and $n = \dim(E) = 2m+1$ is odd. As mentioned above, the discriminant polynomial is identically zero in this case. Instead, we use the *half-discriminant* introduced by M. Kneser [2002], which we recall in Section 2. This is a homogeneous polynomial of degree n

$$\Delta(t_0, t_1) := \tfrac{1}{2}\operatorname{disc}(t_0 q_0 + t_1 q_1) = a_0 t_0^n + a_1 t_0^{n-1} t_1 + \ldots + a_n t_1^n \tag{1-1}$$

in $\Bbbk[t_0, t_1]$. This polynomial behaves like the usual discriminant when the characteristic is not 2: the pencil is regular if and only if the zero subscheme $V(\Delta)$ in $\mathbb{P}^1$ is smooth of dimension 0 (we give a geometric proof; an algebraic proof can be found in [Leep and Schueller 2002]). When $\Bbbk$ is algebraically closed, this simply means that $\Delta \not\equiv 0$ and $V(\Delta)$ consists of n distinct points.

Our first result is the following:

Theorem 1.1. *Let (q_0, q_1) be a regular pair of quadratic forms on a vector space E of dimension $n = 2m+1 \geq 3$ over a field $\Bbbk$ of characteristic 2. Then there exists a basis $(x_0, \ldots, x_m, y_0, \ldots, y_{m-1})$ in $E^\vee$ such that*

$$q_0 = \sum_{i=0}^{m} a_{2i} x_i^2 + \sum_{i=0}^{m-1} x_{i+1} y_i + \sum_{i=0}^{m-1} r_{2i+1} y_i^2, \quad q_1 = \sum_{i=0}^{m} a_{2i+1} x_i^2 + \sum_{i=0}^{m-1} x_i y_i + \sum_{i=0}^{m-1} r_{2i} y_i^2, \tag{1-2}$$

where the coefficients $a_0, \ldots, a_n$ are equal to those of the half-discriminant polynomial (1-1), *and $r_0, \ldots, r_{n-2}$ are in $\Bbbk$.*

To the authors' surprise, our normal form seems to be new even in the case $n = 5$ corresponding to quartic del Pezzo surfaces. At least in retrospect, it can be easily deduced from the classification of pairs of alternate bilinear forms (see [Leep and Schueller 1999]). Nevertheless, the existence of the normal form has the following arithmetic consequence particular to the case of characteristic 2.

Corollary 1.2. *Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^{2m}$ defined over a perfect field $\Bbbk$ of characteristic* 2. *If* $\dim(X) = 2$ (*resp.* > 2), *then X is unirational* (*resp. rational*) *over $\Bbbk$.*

(Note that the unirationality of del Pezzo surfaces of degree 4 is already known for finite fields; see [Manin and Tsfasman 1986].)

Following [Reid 1972], a *generator* is a linear subspace of X of dimension m. Over the algebraic closure, there are exactly 2^{2m} generators. Following [Skorobogatov 2010], one says X is *quasisplit* if it contains a generator, and X is *split* if all 2^{2m} generators are defined over the base field.

Theorem 1.3. *A regular pair* (q_0, q_1) *has quasisplit base locus X if and only if the pair has a normal form such that $r_0 = \cdots = r_{n-2} = 0$. In particular, this is always possible when $\Bbbk$ is algebraically closed.*

If $a_n \neq 0$, we assign to the coefficients $(r_0, \ldots, r_{n-2})$ an element r of the $\Bbbk$-algebra $A = \Bbbk[T]/(f(T))$ where $f(T)$ is the dehomogenization of Δ with respect to $T = t_1/t_0$. This element, which we call the *r-invariant*, determines the normal form (it can also be defined when $a_n = 0$, see Remark 4.8). An *isomorphism* between pairs (q_0, q_1), (q_0', q_1') of quadratic forms is an element $g \in \mathrm{GL}(E)$ such that

$$q_0'(v) = q_0(gv) \quad \text{and} \quad q_1'(v) = q_1(gv)$$

for all $v \in E$.

Theorem 1.4. *Suppose two regular pairs* (q_0, q_1) *and* (q_0', q_1') *have the same half-discriminant polynomial and have normal forms with invariants r and r', respectively. The pairs are isomorphic if and only if*

$$r \equiv r' \mod \Bbbk + \wp(A)$$

where $\wp(s) = s^2 + s$ is the Artin–Schreier map $A \to A$.

Denote by $\mathrm{Aut}(q_0, q_1)$ the group of automorphisms of the pair (q_0, q_1). When X is split, the group of automorphisms $\mathrm{Aut}(q_0, q_1)$ of a regular pair (q_0, q_1) is isomorphic to an elementary abelian 2-group of rank $2m$, which acts simply transitively on the set of generators of X. In general, the set of generators can be viewed as an $\mathrm{Aut}(q_0, q_1)$-torsor.

Theorem 1.5. *Let $\wp(s) = s^2 + s$ be the Artin–Schreier map $A \to A$. There is an isomorphism between the algebra $A/(\wp(A) + \Bbbk)$ and the Galois cohomology group* $H^1(\Bbbk, \mathrm{Aut}(q_0, q_1))$, *which takes the r-invariant to the class of the* $\mathrm{Aut}(q_0, q_1)$*-torsor of generators.*

This theorem provides a Galois-cohomological interpretation of Theorems 1.3 and 1.4 in the vein of [Skorobogatov 2010]. Namely, the isomorphism class of a regular pair is determined by its half-discriminant polynomial Δ and its torsor of generators. Similarly, the base locus X is determined up to isomorphism by a smooth subscheme $V(\Delta)$ of $\mathbb{P}^1$ and the torsor of generators; the variety X is quasisplit if and only if the torsor is trivial.

One of the more concrete motivations of this paper is to study the possible groups of automorphisms of a quartic del Pezzo surface over an algebraically closed field of characteristic 2 (in order to study the conjugacy classes of finite subgroups of the plane Cremona group over fields of positive characteristic).

Recall that a *reflection* is an involution of a vector space (or a projective space) which leaves pointwise-fixed a hyperplane. As a corollary of our main results about the classification of pairs of quadratic forms we obtain the following.

Theorem 1.6. *Let $X = V(q_0, q_1)$ be a smooth complete intersection of two quadrics in $\mathbb{P}^{2m}$ over an algebraically closed field $\Bbbk$ of characteristic 2. Then*

$$\mathrm{Aut}(X) = \mathrm{Aut}(q_0, q_1) \rtimes G,$$

where $\mathrm{Aut}(q_0, q_1)$ is generated by reflections in canonical bijection with the points of $V(\Delta)$ and G is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ which leaves invariant the points $V(\Delta)$.

This extends to characteristic 2 a classical result on automorphisms of quartic del Pezzo surfaces (see [Dolgachev 2012]). There is a natural action of the Weyl group $W(D_n) \cong 2^{2m} \rtimes \mathfrak{S}_n$ on the cohomology classes of the generators. The automorphism group $\mathrm{Aut}(X)$ is naturally a subgroup of the Weyl group $W(D_n)$ where $\mathrm{Aut}(q_0, q_1) \cong 2^{2m}$ and $G \subset \mathfrak{S}_n$. Thus, as with quartic del Pezzo surfaces, the automorphism group is completely determined by its action on the generators.

In writing this paper, the authors tried to resolve a tension between two likely audiences: geometers who only work over algebraically closed fields, and algebraists without a strong background in geometry. Hopefully, the paper will appeal to both audiences and we occasionally supply multiple proofs to facilitate this. For example, while Theorem 1.3 is an immediate consequence of Theorem 1.4, we provide a more direct geometric proof in Section 4B. Also, while Theorem 1.4 is used in the proof of Theorem 1.6, a geometric description of the reflection group R can be found in Section 7B.

The remainder of the paper is structured as follows. In Sections 2 and 3 we establish some preliminaries on pencils of quadratic forms in characteristic 2. In Section 4, we prove Theorem 1.1, Corollary 1.2, and Theorem 1.3. In Section 5, we define the r-invariant and prove Theorem 1.4. In Section 6, we discuss possible interpretations for what the r-invariant represents. In Section 7, we determine the automorphism group of a pair of quadratic forms and use this to prove Theorems 1.5 and 1.6. Finally, in Section 8, we make some observations about the cohomology of smooth intersections of smooth quadrics extending results from M. Reid's thesis [1972] to the case where the characteristic is 2.

2. Preliminaries on quadratic forms

Throughout, $\Bbbk$ is a field of characteristic 2 and E is a k-vector space of dimension n.

2A. ***Alternating bilinear forms.*** Recall that a bilinear form $b : E \times E \to \Bbbk$ is *alternating* if $b(v, v) = 0$ for all $v \in E$. We may view b as an element in $\bigwedge^2 E^\vee$. The *radical* $\mathrm{rad}(b)$ of an alternating bilinear form b is the kernel of the induced map $E \to E^\vee$. A form b is *nondegenerate* if its radical is trivial. The *corank* of a bilinear form is $\dim(\mathrm{rad}(b))$. Every alternating bilinear form is an orthogonal sum of its radical and a nondegenerate alternating bilinear form.

Every nondegenerate alternating bilinear form has even dimension $n = 2m$ and has a *symplectic basis* $v_1, \ldots, v_m, w_1, \ldots, w_m$ satisfying the relations

$$b(w_i, w_j) = 0, \quad b(v_i, v_j) = 0, \quad b(w_i, v_j) = \delta_{ij},$$

for $1 \le i, j \le m$. A subspace W of b is *totally isotropic* if $b(v, w) = 0$ for all $v, w \in W$. Note that $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are bases for complementary totally isotropic subspaces of E. Conversely, for any pair of complementary totally isotropic subspaces of E, there exists a basis for each subspace such that the union is a symplectic basis for E.

When n is even (resp. odd), $\mathrm{corank}(b)$ is even (resp. odd). Thus, an alternating bilinear form b on a vector space of odd dimension $n = 2m + 1$ is always degenerate.

Proposition 2.1. *Let b be an alternating bilinear form on a vector space E of odd dimension $n = 2m + 1$. Up to a choice of volume form for E, there is a canonical vector $\omega \in E$ which spans* $\mathrm{rad}(b)$ *if* $\mathrm{corank}(b) = 1$ *and is* 0 *otherwise. Choosing a basis of $e_1, \ldots, e_n$ in E such that $e_1 \wedge \cdots \wedge e_n \mapsto 1$ under the volume form, we have*

$$\omega = (\mathrm{Pf}_1, \ldots, \mathrm{Pf}_n), \tag{2-1}$$

where Pf_i *is the Pfaffian of the principal submatrix of the matrix of b obtained by deleting the i-th row and the i-th column.*

Proof. It is obvious that such a vector exists; the point is to construct it canonically and find an explicit formula for its coordinates.

Let us view $b \in \bigwedge^2 E^\vee$ as an element of the divided power algebra of $E^\vee$. We may thus consider the m-th divided power $b^{(m)} \in \bigwedge^{2m} E^\vee$. (If $\Bbbk$ was of characteristic zero, we would have the formula $b^{(m)} = (b \wedge \cdots \wedge b)/m!$.) Under the isomorphism $\bigwedge^{2m} E^\vee \cong E$ defined by the volume form, $b^{(m)}$ maps to a vector ω in E. We have $\omega = \sum_{i=1}^n \mathrm{Pf}_i \, e_i$ in coordinates. The rank of an alternating matrix is equal to the size of the largest nonzero Pfaffian of a principal submatrix of even size. If the rank is equal to $n - 1$, then the nullspace is generated by the vector (2-1). All of these well-known facts can be found in [Buchsbaum and Eisenbud 1977]. □

Note that without a choice of volume form, ω is well defined up to a multiplicative constant. Thus, the corresponding point in the projective space $|E|$ does not depend on this choice.

2B. ***Quadratic forms.*** Recall that a quadratic form q is an element of the symmetric square $S^2(E^\vee)$ of the dual space $E^\vee$. When $q \neq 0$, one obtains a quadric hypersurface $V(q)$ in the projective space $|E|$. The quadratic form is called *nondegenerate* if $V(q)$ is a smooth hypersurface.

Equivalently, q can be viewed as a function $E \to \Bbbk$ such that

(1) $q(cv) = c^2 q(v)$ for all $c \in \Bbbk$, $v \in E$, and

(2) $b(v, w) = q(v + w) - q(v) - q(w)$ is a symmetric bilinear form.

The bilinear form b is called the *associated polar bilinear form* and since $\mathrm{char}(k) = 2$, we observe that b is alternating. The *corank* of a quadratic form q is simply the corank of the associated bilinear form.

A nonzero vector $v \in E$ is called a *singular vector of* q if $q(v) = 0$ and $v \in \mathrm{rad}(b)$. In geometric language, this means the corresponding point $[v] = \Bbbk v$ in $|E|$ is a singular point of the quadric $V(q)$.

A quadratic form is *totally singular* if its associated bilinear form is trivial; this is equivalent to $V(q)$ being singular at every geometric point. A quadratic form q is diagonalizable if and only if it is totally singular. Moreover, if q is diagonalizable, then it is diagonal relative to any basis. A subspace W of E is *totally isotropic* (resp. *totally singular*) with respect to q if the restricted form $q|_W$ is trivial (resp. totally singular).

The *discriminant* $\mathrm{disc}(q)$ of a quadratic form is the determinant of the matrix of the polar bilinear form b. If n is even, then a quadratic form is nondegenerate if and only if $\mathrm{disc}(q) \neq 0$. When n is odd, $\mathrm{disc}(q)$ is always zero since the polar bilinear form has odd corank; thus, we consider another invariant.

We define the *half-discriminant* of q as

$$\tfrac{1}{2}\,\mathrm{disc}(q) := q(\omega)$$

where ω is the canonical vector from Proposition 2.1. In coordinates, if

$$q = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

for a basis $x_1, \ldots, x_n$ for $E^\vee$, then

$$\tfrac{1}{2}\,\mathrm{disc}(q) = \sum_{1 \le i \le j \le n} a_{ij}\,\mathrm{Pf}_i\,\mathrm{Pf}_j\,. \tag{2-2}$$

Since each expression Pf_i is a homogeneous polynomial of degree m in the coefficients $\{a_{ij}\}$, the half-discriminant is either zero or a homogeneous polynomial of degree $n = 2m + 1$ in the coefficients $\{a_{ij}\}$. Recall that ω is only defined up to a multiplicative constant which depends on the choice of volume form. If we do not make such a choice, then $\frac{1}{2}\,\mathrm{disc}(q)$ is only well defined modulo nonzero squares in $\Bbbk$.

Proposition 2.2. *For n odd, $\frac{1}{2}\,\mathrm{disc}(q) \neq 0$ if and only if $V(q)$ is smooth.*

Proof. If $\dim \mathrm{rad}(b) > 1$, then the associated projective subspace intersects the quadric $Q = V(q)$; in this case Q is not smooth, $\omega = 0$, and $\frac{1}{2}\,\mathrm{disc}(q) = 0$. Otherwise, ω spans $\mathrm{rad}(b)$ and Q is singular if and only if $q(\omega) = 0$. □

Remark 2.3. The half-discriminant was first introduced by Kneser. Our definition is equivalent to that of [Kneser 2002, p. 43]. Indeed, both polynomials define the same reduced hypersurface in the projective space of the coordinates a_{ij} (they vanish precisely when $V(q)$ is singular). Their degrees are both equal to n, hence they differ by a constant factor. It remains only to compare their values on a particular quadratic form to see that they are equal.

Remark 2.4. The half-discriminant was known to Grothendieck as the *corrected discriminant*, as suggested in his unpublished notes.[1] It is also known as the *half-determinant* (see [Leep and Schueller 2002]),

[1] See Appendix 17.16 of http://www.jmilne.org/math/Documents/EGA-V.pdf.

or the *determinant* (see Remark 13.8 of [Elman et al. 2008]). The notion of the half-discriminant of a quadratic form can be extended to hypersurfaces of arbitrary degree; see [Demazure 2012].

Example 2.5. If $n = 3$, we have

$$\tfrac{1}{2}\,\mathrm{disc}(q) = a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 + a_{12}a_{23}a_{13}.$$

The locus of zeros $V\big(\tfrac{1}{2}\,\mathrm{disc}(q)\big)$ in $|S^2E^\vee| \cong \mathbb{P}^5$ is a cubic hypersurface singular along the plane of double lines $a_{12} = a_{13} = a_{23} = 0$.

If $n = 5$, then

$$\tfrac{1}{2}\,\mathrm{disc}(q) = (a_{11}a_{23}^2a_{45}^2 + \cdots) + (a_{12}^2a_{34}a_{45}a_{35} + \cdots) + (a_{12}a_{23}a_{34}a_{45}a_{15} + \cdots).$$

3. Pencils of quadratic forms

For the rest of the paper, we assume that the $\Bbbk$-vector space E has dimension $n = 2m + 1$ for a positive integer m.

Let U be a 2-dimensional $\Bbbk$-vector space. A *pencil of quadratic forms* is an injective linear map

$$\mathsf{q} : U \to S^2E^\vee.$$

We use subscript notation q_u to denote the image of $u \in U$ in order to avoid confusion with the interpretation of a quadratic form as a map $E \to \Bbbk$. We will also identify q with its image considering a pencil as a two-dimensional linear subspace of $S^2E^\vee$. The pencil q defines a one-dimensional linear system of quadric hypersurfaces in $|E|$ (also called a pencil). The corresponding rational map $|E| \dashrightarrow |U^\vee|$ has base subscheme $\mathrm{Bs}(\mathsf{q})$ equal to the intersection of quadrics $V(\mathsf{q}_u)$ over all $u \in U$.

We may view the map q as an element of $U^\vee \otimes S^2E^\vee$. Selecting a basis (u_0, u_1) for U with the dual basis (t_0, t_1), we can view q as an element $t_0q_0 + t_1q_1 \in U^\vee \otimes S^2E^\vee$, where

$$q_0 = \mathsf{q}_{u_0}, \quad q_1 = \mathsf{q}_{u_1}$$

define a pair of quadratic forms (q_0, q_1) which form a basis of the pencil. We have $\mathrm{Bs}(\mathsf{q}) = V(q_0, q_1) = V(q_0) \cap V(q_1)$. Conversely, a pair of nonproportional quadratic forms gives rise to a pencil of quadratic forms q (along with a choice of basis for U).

Given a pencil of quadratic forms we obtain an *associated pencil of alternating bilinear forms*

$$\mathsf{b} : U \to \textstyle\bigwedge^2 E^\vee$$

by setting b_u as the associated bilinear form of q_u for each $u \in U$. Note that, unlike q, the map b may not be injective. We may view b as an element of $U^\vee \otimes \bigwedge^2 E^\vee$.

3A. ***The radical map and the radical subspace.*** Fix a volume form on E. For each alternating bilinear form b_u there exists a canonical vector ω as in Proposition 2.1. Thus, we obtain a function

$$\Omega : U \to E \tag{3-1}$$

which is defined by polynomials of degree m in view of (2-1). Thus, abusing notation, we may view Ω as an element of $E \otimes S^m U^\vee$. We call the map (3-1) the *radical map* of the pencil. As u varies over U, the $\Omega(u)$ span a subspace W of E which we call the *radical subspace* of E. Equivalently, Ω gives rise to a $\Bbbk$-linear map

$$(S^m U^\vee)^\vee \to E \tag{3-2}$$

whose image equals the radical subspace W. (Note that we should not identify $(S^m U^\vee)^\vee$ and $S^m U$ in characteristic 2).

We define a function $\Delta : U \to \Bbbk$ via the composition

$$\Delta(u) = \mathsf{q}_u(\Omega(u))$$

for $u \in U$. We call Δ the *half-discriminant of the pencil* q since $\Delta(u)$ is the half-discriminant of each q_u. Since q can be viewed as a linear combination of quadratic forms with linear coefficients in $U^\vee$ and Ω is defined by polynomials of degree m in $U^\vee$, Δ is an element of $S^n(U^\vee)$. After choosing a basis (t_0, t_1) in $U^\vee$, it can be identified with a homogeneous polynomial of degree $n = 2m + 1$ in t_0, t_1.

3B. *Regular pencils.* We say a pencil (or pair) is *regular* if the base locus $X = \mathrm{Bs}(\mathsf{q})$ is a smooth variety over $\Bbbk$.

Remark 3.1. We borrowed the term regular from the classical terminology of regular linear systems of quadric hypersurfaces. It should not be confused with the regularity of the base scheme $\mathrm{Bs}(\mathsf{q})$ since, over a nonperfect field $\Bbbk$, the latter could be a regular scheme but not smooth.

From Theorem 5.1 of [Leep and Schueller 2002], we know that a pencil is regular if and only if Δ is nonzero and has n distinct linear factors over the algebraic closure. In other words:

Theorem 3.2. *A pencil* q *is regular if and only if, over an algebraic closure of* $\Bbbk$, *there are exactly* n *degenerate quadrics in* $|\mathsf{q}|$.

We will need the following consequence of this theorem:

Corollary 3.3. *If* q *is a regular pencil, then every nonzero quadratic form in the pencil has corank* 1.

Proof. It suffices to assume k is algebraically closed. Suppose the corank of some nonzero q in q is > 1. Then the radical $\mathrm{rad}(b)$ of the associated bilinear form b is of dimension ≥ 3. This implies that $|\mathrm{rad}(b)|$ contains a plane intersecting X nontrivially. This shows that $V(q)$ has a singular point x in X. The base locus X is smooth, so the tangent space of X at x is of codimension 2. We have $X = V(q) \cap V(q')$ for some $q' \in \mathsf{q}$. Thus, the tangent spaces of $V(q)$ and $V(q')$ at x intersect along a subspace of codimension 2. This implies that $V(q)$ is smooth at x, a contradiction. □

3C. *Alternate proof of Theorem 3.2.* In the remainder of this section, we present a geometric proof that works in arbitrary characteristic which we believe is of independent interest. It is not necessary for the rest of the paper. Note that Theorem 3.2 is well known when the characteristic of $\Bbbk$ is odd. Unlike the situation for the rest of the paper, in this section we assume $\Bbbk$ is algebraically closed of arbitrary characteristic and that the integer n is odd only if the characteristic is even.

Let D be the subvariety of singular quadrics in $S^2E^\vee$ (this is just defined by the discriminant or half-discriminant). Let D_0 be the open subvariety of D consisting of quadrics which have a unique isolated singular point (an open condition since it is determined by the rank of the matrix of the associated bilinear form).

Lemma 3.4. *Let L be a linear subspace of $S^2E^\vee$ of dimension $r > 0$. Then the tangent space of $\mathsf{D} \cap L$ at a point $q \in \mathsf{D}_0 \cap L$ with singular vector $v_0 \in E$ is the linear space $\{q \in L : q(v_0) = 0\}$.*

Proof. Let

$$\tilde{\mathsf{D}} = \{(q, v) \in S^2E^\vee \setminus \{0\} \times E \setminus \{0\} : [v] \in \mathrm{Sing}(V(q))\}.$$

Consider the maps $p_1 : \tilde{\mathsf{D}} \to S^2E^\vee$ and $p_2 : \tilde{\mathsf{D}} \to E$ obtained from the projections. For any $v \in E \setminus \{0\}$, let F_v be the closure of $p_1(p_2^{-1}(v))$ in $S^2E^\vee$. By choosing a basis in E such that $v = (1, 0, \dots, 0)$, the subvariety F_v consists of quadratic forms that do not contain the first variable. Thus, F_v is a linear subspace of $S^2E^\vee$ of codimension n. In particular, $\tilde{\mathsf{D}}$ is a fibration over $E \setminus \{0\}$ with fibers isomorphic to linear spaces of the same dimension. It follows that $\tilde{D}$ is smooth and of dimension $n + \dim S^2E^\vee - n = \dim S^2E^\vee$.

The variety $p_2(p_1^{-1}(q))$ is, set-theoretically, the linear space of singular vectors. Passing to the map of the associated projective varieties $\pi : |\tilde{\mathsf{D}}| \to |S^2E^\vee|$, we expect that the map π induces a birational isomorphism with the subvariety $|\mathsf{D}_0|$ of quadrics with isolated singular point.

We will compute the differential of the map $p_1 : \tilde{\mathsf{D}} \to S^2E^\vee$ at a point (q_0, v_0) represented by a quadratic form q_0 with one-dimensional space of singular vectors generated by v_0.

Let $t_1, \dots, t_n$ be coordinates in E and let A_{ij} be coordinates in $S^2E^\vee$ corresponding to the coefficients of a quadratic form. Then $\tilde{\mathsf{D}}$ is given by $n + 1$ equations:

$$F_k = \sum_{j=1}^{n} B_{kj} t_j = 0, \quad k = 1, \dots, n, \tag{3-3}$$

$$F_{n+1} = \sum_{1 \le i \le j \le n} A_{ij} t_i t_j = 0, \tag{3-4}$$

where $B_{ii} = 2A_{ii}$ and $B_{ij} = B_{ji} = A_{ij}$ if $i < j$.

Let $q_0 = \sum_{1 \le i \le j \le n} a_{ij} t_i t_j$ and $v_0 = (c_1, \dots, c_n)$. The embedded tangent space $T_{q_0,v_0}(\tilde{\mathsf{D}})$ of $\tilde{\mathsf{D}}$ is a subspace of $T_{q_0}(S^2E^\vee) \oplus T_{v_0}(E) = S^2E^\vee \oplus E$ given by a system of linear equations in variables t_i, A_{ij} with matrix of coefficients $[M_1 M_2](a_{ij}, c_i)$, where

$$M_1 = \left(\frac{\partial F_k}{\partial t_j}\right)_{\substack{1 \le k \le n+1 \\ 1 \le j \le n}} \quad \text{and} \quad M_2 = \left(\frac{\partial F_k}{\partial A_{ij}}\right)_{\substack{1 \le k \le n+1 \\ 1 \le i \le j \le n}}. \tag{3-5}$$

Computing the partial derivatives, we find that $T_{q_0,v_0}(\tilde{\mathsf{D}})$ consists of pairs (q, v) satisfying the conditions

$$b_0(v, w) + b(v_0, w) = 0, \quad b_0(v_0, v) + q(v_0) = q(v_0) = 0, \tag{3-6}$$

for all $w \in W$, where b and b_0 are the associated bilinear forms of q and q_0.

The kernel of the differential of the map $\tilde{\mathsf{D}} \to \mathsf{D},\ (q, v) \mapsto q$ obtained from the projection can be identified with the linear space of vectors $v \in E$ such that $b_0(v, w) = 0$ for all $w \in E$; this is the radical of the bilinear form b_0. The map

$$T_{(q_0,v_0)}\tilde{\mathsf{D}} \to \{q : q(v_0) = 0\}, (q, v) \to q$$

is linear and has kernel $\operatorname{rad}(b_0)$. Since the tangent space of D_0 at q_0 is isomorphic to the quotient space $T_{(q_0,v_0)}\tilde{\mathsf{D}}/\operatorname{rad}(b_0)$, we obtain that

$$T_{q_0}\mathsf{D} = \{q \in S^2E^\vee : q(v_0) = 0\}.$$

Note that $\operatorname{rad}(b_0)$ is one-dimensional (this does not hold when n is even and $\Bbbk$ has characteristic 2). This is equal to the dimension of the fiber $p_1^{-1}(q_0)$. This shows that the differential of p_1 is of maximal rank, and hence π is an isomorphism over $|\mathsf{D}_0|$. Now let $L \subset S^2E^\vee$ be a linear nonzero subspace of $S^2E^\vee$, then the variety of quadratic forms in L with one-dimensional space of singular vectors is equal to $L \cap \mathsf{D}_0$. We have

$$T_{q_0}(L \cap \mathsf{D}) = T_{q_0}L \cap T_{q_0}\mathsf{D} = L \cap T_{q_0}\mathsf{D} = \{q \in L : q(v_0) = 0\}.$$

This proves the assertion. □

Proof of Theorem 3.2. We may assume $\Bbbk$ is algebraically closed. Consider $L := \mathsf{q}(U)$, the two-dimensional linear subspace of $S^2E^\vee$. First, we claim that x is a singular point of X if and only if x is a singular point of $Q = V(q)$ for some $q \in L$. The tangent space T_xX is the intersection of the tangent spaces $T_xQ \cap T_xQ'$, for any two distinct quadrics Q, Q' from L. If x is a singular point of Q, then $T_xQ = |E|$. Thus $\operatorname{codim} T_xX \leq 1$ and x is a singular point of X. If x is a singular point of X, then either $T_xQ = |E|$ for some quadric Q, or $T_xQ = T_xQ'$ for any two quadrics Q, Q' from L. In the first case, x is a singular point of Q. In the latter case, if b, b' are the corresponding bilinear forms, then the tangent spaces are defined by linear forms $b(v, -)$ and $b'(v, -)$ where $v \in E$ represents x. Since the linear forms are proportional by assumption, some nontrivial linear combination $(\lambda b + \mu b')(v, -)$ is trivial. The quadric corresponding to that linear combination is therefore singular at x. This establishes the claim.

As an immediate consequence, if a quadric Q in L has corank > 1, then X is not smooth; indeed, in this case, the singular locus of Q has nontrivial intersection with some other quadric in the pencil since it is positive dimensional.

Assume q is regular; in other words, that X is smooth. Suppose q_0 is a degenerate quadratic form in L with corresponding quadric Q_0. Since X is smooth, Q_0 has corank ≤ 1, so q_0 must be in D_0. Let $v_0 \in E$ represent the unique singular point $|v_0|$ of Q_0. Since $|v_0| \notin X$, the space

$$|\{q \in L : q(v_0) = 0\}|$$

is just the point $|q_0|$. By Lemma 3.4, we conclude that the tangent space of $|D \cap L|$ at the point $|q_0|$ is 0-dimensional. Thus $|L|$ and $|D|$ meet transversally at each intersection point and $|L \cap D|$ consists of exactly n distinct points as desired.

Now, suppose $|L \cap \mathsf{D}|$ consists of exactly n distinct points. This is equivalent to $|L|$ intersecting transversally the hypersurface $|\mathsf{D}|$. From the proof of Lemma 3.4, the differential of $\tilde{\mathsf{D}} \to \mathsf{D}$ at a point (q, v) has kernel equal to the radical of the associated bilinear form b. The dimension of this radical is minimal precisely for D_0; so D_0 is the smooth locus of D. Since $|L|$ intersects $|\mathsf{D}|$ transversally at every intersection point, in fact $|L \cap \mathsf{D}| = |L \cap \mathsf{D}_0|$. If $q_0 \in L \cap \mathsf{D}$, then, by Lemma 3.4, $|\{q \in L : q(v_0) = 0\}|$ is a single point. Thus, the unique singular point of the quadric Q_0 corresponding to q_0 does not lie in X. Since none of the singular points of the quadrics lie in X, we conclude from above that X is smooth. □

4. Normal forms

4A. ***Kronecker basis.*** In this section, we prove Theorem 1.1 and discuss some of its consequences. At least in retrospect, the theorem follows quite easily using known techniques regarding pairs of bilinear forms going back to Kronecker. Throughout, E is a vector space of dimension $n = 2m + 1 \geq 3$ over a field of characteristic 2.

Given a pair of alternating bilinear forms (b_0, b_1) on E, we say that a direct sum $E = E_1 \oplus E_2$ is *orthogonal* if it is orthogonal with respect to both b_0 and b_1. A pair of alternating bilinear forms is *nonsingular* if the corresponding determinant polynomial is nonzero. For a positive integer k, a *basic singular pair* is a pair of alternating bilinear forms (b_0, b_1) such that there exists a basis $\mathcal{B} = (w_0, \ldots, w_k, v_0, \ldots, v_{k-1})$ of the underlying vector space such that

$$\begin{aligned} b_0(w_i, w_j) &= 0, & b_0(v_i, v_j) &= 0, & b_0(w_i, v_j) &= \delta_{i(j+1)}, \\ b_1(w_i, w_j) &= 0, & b_1(v_i, v_j) &= 0, & b_1(w_i, v_j) &= \delta_{ij}, \end{aligned} \tag{4-1}$$

for all valid i, j. We call a basis $\mathcal{B}$ as above a *Kronecker basis.*

Theorem 4.1. *Let (b_0, b_1) be a pair of alternating bilinear forms on a vector space E. Then E can be written as an orthogonal direct sum of a nonsingular pair and a set of basic singular pairs.*

Theorem 4.1 is a special case of Theorem 3.3(1) of [Leep and Schueller 1999]. The statement and its proof are essentially due to Kronecker, although his version applies to symmetric bilinear forms in characteristic $\neq 2$; see Theorem 3.1 of [Waterhouse 1976]. The proof there carries over to the case of alternating bilinear forms in characteristic 2 as observed before the statement of Theorem 9 of [Waterhouse 1977].

Corollary 4.2. *Let (q_0, q_1) be a regular pair of quadratic forms and let (b_0, b_1) be the associated pair of alternating bilinear forms. Then (b_0, b_1) is a basic singular pair and the vector*

$$\Omega = \sum_{i=0}^{m} \lambda^{m-i} \mu^i w_i \tag{4-2}$$

spans $\operatorname{rad}(\lambda b_0 + \mu b_1)$, *where w_i are basis elements in a Kronecker basis* (4-1).

Proof. By a straightforward calculation (see Lemma 3.2 of [Leep and Schueller 1999]), for a basic singular pair (b'_0, b'_1) on a vector space of dimension $2k+1$, the radical of $\lambda b'_0 + \mu b'_1$ is spanned by the vector

$$\Omega' = \sum_{i=0}^{k} \lambda^{k-i}\mu^i w_i.$$

By Corollary 3.3, any nonzero bilinear form from the family $\lambda b_0 + \mu b_1$ has corank 1. Thus there is exactly one basic singular pair in the decomposition from Theorem 4.1 and its radical is given by $\Omega = \Omega'$. It remains only to establish that $k = m$, and thus there is no nonsingular summand in the decomposition.

The vector Ω can be viewed as a homogeneous polynomial function $U \to E$ of degree m. Thus, if $k < m$ then in any choice of basis the entries of Ω must all be divisible by a homogeneous polynomial $g(\lambda, \mu)$ of positive degree. But $\Delta(u) = \mathsf{q}_u(\Omega(u))$ for all $u \in U$, so the half-discriminant would then be divisible by $g(\lambda, \mu)^2$. This contradicts that the roots of Δ are distinct and so $k = m$ as desired. □

It follows that the map $\Omega : U \to E$ is injective and its image spans the same subspace W as that spanned by $w_0, \dots, w_m$. Since Ω does not depend on any choice of coordinates, we see that W is canonical. Any $(m+1)$-dimensional totally isotropic subspace of an alternating bilinear form of rank $2m$ on a $(2m+1)$-dimensional space must contain the radical. Since W is the minimal space containing the radical of the associated bilinear form to every quadratic form in the pencil, we obtain the following[2]:

Corollary 4.3. *The space W spanned by $w_0, \dots, w_m$ is canonical. It is the unique common $(m+1)$-dimensional totally singular subspace of all quadratic forms in the pencil.*

Note that choice of basis $w_0, \dots, w_m$ is determined by the choice of basis u_0, u_1 in U (or, equivalently, the choice of pair (q_0, q_1) corresponding to the pencil $|\mathsf{q}|$). We also see that a choice of a basis u_0, u_1 in U and the canonical isomorphism W with $(S^m U^\vee)^\vee$ defines a basis in W equal to the image of the dual basis to the monomial basis $(t_0^m, t_0^{m-1}t_1, \dots, t_1^m)$ in $S^m U^\vee$.

Another consequence of the Kronecker theorem is that the map $\Omega : U \to E$ is equal to the composition of the map $U \to (S^m U^\vee)^\vee \to E$, where the second map is injective. This implies that the image of U is equal to the affine cone over a *Veronese curve* R_m of degree m in $|W| \cong |(S^m U^\vee)^\vee| \cong \mathbb{P}^m$. Recall that a Veronese curve in $\mathbb{P}^m$ is a smooth rational curve of degree m equal to the image of $\mathbb{P}^1$ under a map given by linearly independent homogeneous polynomials of degree m. By choosing a monomial basis $t_0^{m-i}t_1^i$ of such polynomials, it is projectively equivalent to a curve R_m in $\mathbb{P}^m$ given by equations expressing the condition that

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & \cdots & x_{m-1} \\ x_1 & x_2 & \cdots & x_m \end{pmatrix} = 1. \tag{4-3}$$

For example, when $m = 2$, this is a smooth conic in $\mathbb{P}^2$. In a coordinate-free way, the Veronese curve is the image of $\mathbb{P}^1 = |U|$ in $|(S^m U^\vee)^\vee|$ under the complete linear system $|\mathcal{O}_{|U|}(m)| = |S^m U^\vee|$. The image of the scheme $V(\Delta)$ of roots of the half-discriminant Δ of the pencil under the map $\Omega : |U| \to |W|$ is a

[2]The existence of such a subspace for any pencil of alternating forms is a known fact; see the Lemma in [Beauville 2005].

0-dimensional closed subscheme Z of the Veronese curve R_m of length n. Over an algebraically closed extension of $\Bbbk$ it becomes a union of n distinct points.

We now prove the main theorem:

Proof of Theorem 1.1. Choose a Kronecker basis $w_0, \dots, w_m, v_0, \dots, v_{m-1}$ and the corresponding coordinates $x_0, \dots, x_m, y_0, \dots, y_{m-1}$ in E. Let W be the span of $w_0, \dots, w_m$ and let L be the span of $v_0, \dots, v_{m-1}$. Thus $E = W \oplus L$ is a direct sum of totally singular subspaces for all quadrics in the pencil $|\mathsf{q}|$ (since they are totally isotropic for the associated alternating bilinear forms). This implies that the restrictions of q_0, q_1 to W (resp. L) is equal to a linear combination of squares of x_i (resp. y_i). So, we have reduced q_0, q_1 to the expressions from the Theorem. It remains to prove the assertion about the coefficients $\{a_i\}$. We use the expression of Ω computed above to find the half-discriminant. Let u_0, u_1 be the basis of U corresponding to q_0, q_1 and t_0, t_1 be the dual basis. Then

$$\begin{aligned}\Delta(t_0, t_1) &= (t_0 q_0 + t_1 q_1)(t_0^m, t_0^{m-1} t_1, \dots, t_1^m, 0, \dots, 0)\\ &= t_0 \Big(\sum_{i=0}^{m} a_{2i} t_0^{2(m-i)} t_1^{2i} \Big) + t_1 \Big(\sum_{i=0}^{m} a_{2i+1} t_0^{2(m-i)} t_1^{2i} \Big)\\ &= \sum_{i=0}^{2m+1} a_i t_0^{2m+1-i} t_1^i\end{aligned}$$

as desired. □

We say that a regular pair of quadratic forms is in *normal form* if it is written in coordinates corresponding to a Kronecker basis in E.

Note that, given a specific choice of pair (q_0, q_1), the basis $w_0, \dots, w_m$ in a Kronecker basis is canonical. However, given only a pencil $|\mathsf{q}|$, only the vector space W spanned by $w_0, \dots, w_m$ is canonical. This has some strong consequences which we discuss now. In the following X denotes the base scheme $\mathrm{Bs}(\mathsf{q})$ of a regular pencil of quadrics $|\mathsf{q}|$ in $|E|$.

Theorem 4.4. *Assume $m \geq 2$. If $\Bbbk$ is a perfect field, then the variety X contains a canonical $(m-2)$-plane Π defined over $\Bbbk$. In particular, a del Pezzo surface of degree 4 over a perfect field of characteristic 2 has a canonical point.*

Proof. Note that the ideal of the subscheme $X \cap |W|$ is generated by

$$q_0|_W = \sum_{i=0}^{m} a_{2i} x_i^2, \quad q_1|_W = \sum_{i=0}^{m} a_{2i+1} x_i^2$$

in projective coordinates for $|W|$. Over a perfect field, the radical ideal is $\langle l_0, l_1 \rangle$ where $l_0^2 = q_0|_W$ and $l_1^2 = q_1|_W$. If $l_0 = c l_1$ for some $c \in \Bbbk$, then $a_{2i} = c^2 a_{2i+1}$ for all i in $0, \dots, m$. But this implies that

$$\Delta(T, 1) = (T + c^2) g(T^2)$$

for some polynomial $g \in \Bbbk[T]$, contradicting the separability of $\Delta(T, 1)$. Thus the subspace $\{l_0 = l_1 = 0\}$ in X has codimension 2 in $|W|$ as desired. □

The vector space L spanned by $v_0, \dots, v_{m-1}$ is not canonical; in fact, we have the following.

Lemma 4.5. *Let (q_0, q_1) be a regular pair of quadratic forms and (b_0, b_1) the associated bilinear forms. Let W be the canonical subspace spanned by $(w_0, \dots, w_m)$ in a Kronecker basis. Suppose L is a totally isotropic subspace for both b_0 and b_1 and that $E = W \oplus L$. Then there exists a basis $(v_0, \dots, v_{m-1})$ in L that, together with the canonical basis $(w_0, \dots, w_m)$ in W, forms a Kronecker basis in E.*

Proof. Since b_1 has a 1-dimensional radical spanned by w_m, the alternating bilinear form induced on the quotient space $\bar{E} = E/\langle w_m \rangle$ is a direct sum of m hyperbolic planes. Since $w_m \in W$, the images of W and L in $\bar{E}$ are complementary totally isotropic subspaces of $\bar{E}$; and thus provide a pair of complementary maximal totally isotropic subspaces. Thus, we may find a basis $v_0, \dots, v_{m-1}$ for L such that it satisfies the desired equations for b_1 from (4-1).

We have established all desired equations from (4-1) except for $b_0(w_i, v_j) = \delta_{i(j+1)}$. We know that some Kronecker basis $(w_0, \dots, w_m, v'_0, \dots, v'_{m-1})$ exists with corresponding splitting $E = W \oplus L'$. For any $v \in L$ we can write $v = w + v'$ for $w \in W,\ v' \in L'$. For all $i = 0, \dots, m$, we have $b_0(w_i, v) = b_0(w_i, v')$ since $b_0(w_i, w) = 0$; and similarly for b_1. From (4-1), we have that $b_1(w_i, v'_j) = \delta_{ij} = b_0(w_{i+1}, v'_j)$ for all valid i, j. From this we conclude that $b_1(w_i, v) = b_0(w_{i+1}, v)$ for any $v \in L$. We conclude that $b_0(w_i, v_j) = b_1(w_{i-1}, v_j) = \delta_{i(j+1)}$ for all i, j except when $i = 0$. This last case follows from the fact that w_0 is in the radical of b_0. □

We have identified the radical subspace W with $(S^m U^\vee)^\vee$. The next proposition tells that we can identify any complement L as in Lemma 4.5 with the space $S^{m-1}U^\vee$.

Proposition 4.6. *Let (q_0, q_1) be a regular pair of quadratic forms on E. Up to a choice of volume form on E, there is a canonical isomorphism $\iota : U \to U^\vee$ and a canonical isomorphism $L \cong S^{m-1}(U^\vee)$, such that*

$$\mathsf{b}_u(v, w) = \psi_1(\iota(u) f_2) + \psi_2(\iota(u) f_1), \tag{4-4}$$

where $v = (\psi_1, f_1)$ and $w = (\psi_2, f_2)$ are elements of $W \oplus L = (S^m U^\vee)^\vee \oplus S^{m-1}U^\vee$ and we note that $\iota(u) f_i$ can be viewed as an element of $S^m(U^\vee)$ via the multiplication map $U^\vee \otimes S^{m-1}U^\vee \to S^m U^\vee$.

Proof. Let u_0, u_1 be the basis for U such that $\mathsf{q}_{u_0} = q_0$ and $\mathsf{q}_{u_1} = q_1$, and let t_0, t_1 be the dual basis. The map ι is defined to satisfy $t_0 = \iota(u_1)$ and $t_1 = \iota(u_0)$. We use the canonical isomorphism between W and $(S^m U^\vee)^\vee$ which we deduced from formula (4-2); more concretely: $w_i \mapsto (t_0^{m-i} t_1^i)^\vee$ (this is where we use the volume form on W). Consider the isomorphism $L \cong S^{m-1}(U^\vee)$ given by $v_i \mapsto t_0^{(m-1)-i} t_1^i$ for i in $0, \dots, m-1$. A direct comparison of (4-4) and (4-1) in coordinates for U, W, and L establishes the result. □

Proposition 4.6 allows us to compare normal forms for different choices of pairs in the same pencil.

Lemma 4.7. *Suppose $u, u' \in U$ are linearly independent and let $(w_0, \dots, w_m, v_0, \dots, v_{m-1})$ be a Kronecker basis for the pair $(q_u, q_{u'})$. For any $g \in \mathrm{GL}(U)$, there exists an $h \in \mathrm{GL}(W) \times \mathrm{GL}(L)$ such that $(h(w_0), \dots, h(v_{m-1}))$ is a Kronecker basis for $(\mathsf{q}_{g(u)}, \mathsf{q}_{g(u')})$.*

Proof. There exists some $c \in \Bbbk$ such that $\iota(gu) = c(g^\vee)^{-1}\iota(u)$ for all $u \in U$. Using the isomorphisms from Proposition 4.6, consider

$$h = ((S^m g^\vee)^\vee, c^{-1} S^{m-1}(g^\vee)^{-1})$$

in $\mathrm{GL}(W) \times \mathrm{GL}(L)$. Using the shorthand notation $g_* = (S^k g^\vee)^\vee$ and $g^* = S^{k-1}(g^\vee)$ for all positive integers k, we find

$$\begin{aligned}(h\psi)(\iota(gu)(hf)) &= (g_*\psi)((\iota(gu))(c^{-1}(g^*)^{-1}f)) \\ &= \psi(g^*((g^*)^{-1}\iota(u))((g^*)^{-1}f)) \\ &= \psi(g^*(g^*)^{-1}(\iota(u)f)) = \psi(\iota(u)f)\end{aligned}$$

for all $\psi \in W \cong (S^m U^\vee)^\vee$, $f \in L \cong S^{m-1}(U^\vee)$, and $u \in U$. From this we conclude that

$$\mathsf{b}_{gu}(hv, hw) = \mathsf{b}_u(v, w)$$

for all $v, w \in E$ and $u \in U$. Thus, $(\mathsf{b}_{gu}, \mathsf{b}_{gu'})$ is in Kronecker normal form relative to the basis $h(w_0), \ldots, h(v_{m-1})$. □

Remark 4.8. It is frequently convenient to assume that $a_n \neq 0$ in the half-discriminant polynomial or, equivalently, that the quadratic form q_1 is nondegenerate. Lemma 4.7 shows that this is often a harmless assumption.

4B. ***The quasisplit case.*** Following the classical terminology in geometry of ruled varieties, a *generator* of the variety X is a linear subspace of X of maximal possible dimension (over the algebraic closure). In the spirit of [Skorobogatov 2010], we say that X is *quasisplit* if $X = \mathrm{Bs}(\mathsf{q})$ has a generator defined over $\Bbbk$. We say X is *split* if all generators of X are defined over $\Bbbk$. We shall see in Corollary 7.5 below that, when $\Bbbk$ is algebraically closed, every X has exactly 2^{2m} generators. (For two general quadrics, this also follows by Example 14.7.15 of [Fulton 1998].)

Theorem 4.9. *A generator of X has dimension $m-1$. If $\Bbbk$ is algebraically closed, then X contains a generator.*

Proof. Assume $\Bbbk$ is algebraically closed. Let $F_r(X)$ be the subscheme of the Grassmannian $G_r(\mathbb{P}^{n-1})$ of r-dimensional subspaces in $\mathbb{P}^{n-1}$ which are contained in X. From [Debarre and Manivel 1998, Theorem 2.1], we know $F_r(X)$ is nonempty if and only if $r \leq m-1$. Thus, X has a generator (which is of dimension $m-1$). □

Remark 4.10. We shall see below that all X are quasisplit precisely when $\Bbbk$ is closed under separable quadratic extensions.

This brings us to the main theorem of this section. Since X is always quasisplit over an algebraically closed field, Theorem 1.3 follows immediately from the following.

Theorem 4.11. *X is quasisplit if and only if there exists a normal form for the pencil q with $r_1 = \ldots = r_{2m} = 0$.*

Proof. Note that a generator Λ is equal to $|L|$, where L is totally isotropic for all quadrics in the pencil. When $r_i = 0$ in the normal form defined by a Kronecker basis $(w_0, \dots, w_m, v_0, \dots, v_{m-1})$, the span of $v_0, \dots, v_{m-1}$ provides such a subspace. Conversely, given such a subspace L, Lemma 4.5 provides the desired normal form under the condition that L is complementary to W.

Thus, the difficulty is proving that $|L| \cap |W| = \varnothing$ (in other words, that $L \cap W = 0$). We may assume without loss of generality that $\Bbbk$ is algebraically closed.

Let $\Pi = |W| \cap X$ be the canonical subspace from Theorem 4.4. Let $S = |L| \cap |W| = |L| \cap \Pi$. Suppose $\dim S = k \geq 0$. Let P be the span of $|W|$ and $|L|$, its dimension is equal to $2m - k - 1$. Let $Q = V(q)$ be a quadric from the pencil. The restriction of Q to $|W|$ is a linear subspace Y of dimension $m - 1$ (taken with multiplicity 2). The quadric Q also contains the linear subspace $|L|$ of the same dimension. They intersect along the k-dimensional subspace S. We claim that Q must be singular along S.

To see this, choose coordinates $x_0, \dots, x_{2m-k-1}$ for P in such a way that $|L|$ is given by $x_m = \cdots = x_{2m-k-1} = 0$ and $|W|$ is given by $x_0 = \cdots = x_{m-k-2} = 0$, so that S is given by $x_i = 0$, $i \neq m-k-1, \dots, m-1$. The restriction $Q|_P$ of Q to P must have an equation of the form

$$L(x_m, \dots, x_{2m-k-1})^2 + \sum_{i=0}^{m-k-2} x_i M_i(x_m, \dots, x_{2m-k-1}) = 0,$$

where L and $M_1, \dots, M_{m-k-2}$ are linear homogeneous polynomials. Since the polynomial defining this equation does not contain the variables x_i, $i \neq m-k-1, \dots, m-1$, the quadric is singular along the subspace S. In other words, the subspace P is tangent to the quadric Q along S. Thus any hyperplane containing P is tangent to Q along S. Since S and P do not depend on Q, we can find a hyperplane tangent to any Q along S. Since the tangent space of X at any point is of codimension 2, and it is equal to the intersection of tangent hyperplanes of quadrics in the pencil, we find a contradiction with the assumption that X is smooth. □

The proof of the preceding theorem has the following useful consequence:

Corollary 4.12. *If Λ is a generator of X, then there is a Kronecker normal form with decomposition $E = W \oplus L$ such that $|L| = \Lambda$.*

4C. ***Applications to rationality.*** This canonical subspace Π from Theorem 4.4 allows us to prove Corollary 1.2 from the introduction:

Proof of Corollary 1.2. First we consider the case $m = 2$, where we are dealing with quartic del Pezzo surfaces in $|E| \cong \mathbb{P}^4$. The canonical $(m-2)$-plane Π is thus a canonical point. It is known that any del Pezzo surface of degree ≥ 3 with a rational point is unirational over the base field (see Theorem 3.5.1 of [Manin and Tsfasman 1986]).

Now we consider the case where $m > 2$. Consider the projection $p : X \setminus \Pi \to \mathbb{P}^{m+1}$ with center at Π. Since Π is defined over $\Bbbk$, the projection is defined over $\Bbbk$. The morphism p extends to a morphism $p' : X_\Pi \to \mathbb{P}^{m+1}$ where X_Π is the blowup of X at center Π.

Let x be a point in $X \setminus \Pi$. Let F_x be the closure of $p^{-1}(p(x))$ in $\mathbb{P}^{n-1}$. We claim that F_x is the span $\langle x, \Pi \cap T_x X\rangle$ where $T_x X$ is the embedded tangent space to X at x. Let $v \in E$ be a vector such that $x = [v]$ and let $e_0, \ldots, e_{m-2}$ be a basis of Π. The subvariety F_x is equal to the closure of the intersection of $X \setminus \Pi$ and the span $\langle \Pi, x\rangle$ of Π and x. The span $\langle \Pi, x\rangle$ is equal to $\left[tv + \sum_{i=0}^{m-2} s_i e_i\right]$, where $[s_0, \ldots, s_{m-2}, t]$ are projective coordinates in $\langle \Pi, x\rangle$. Plugging in the equations $q_0 = q_1 = 0$ of X we get, for $k = 0, 1$, that

$$q_k\left(tv + \sum_{i=0}^{m-2} s_i e_i\right) = t^2 q_k(v) + q_k\left(\sum_{i=0}^{m-2} s_i e_i\right) + t \sum_{i=0}^{m-2} s_i b_k(e_i, v) = t \sum_{i=0}^{m-2} s_i b_k(e_i, v).$$

This shows that F_x is the span $\langle x, \Pi \cap T_x(X)\rangle$ as claimed.

First, assume that $\Bbbk$ is algebraically closed. We know that if $\Pi \cap T_x(X)$ is nonempty then $\dim \Pi \cap T_x(X)$ is $m - 2 - r(x)$, where $r(x)$ is the rank of the matrix

$$\begin{pmatrix} b_0(e_1, v) & \cdots & b_0(e_{m-2}, v) \\ b_1(e_1, v) & \cdots & b_1(e_{m-2}, v) \end{pmatrix}.$$

Let C be the set of points $x \in X$ where $r(x) < 2$: these are the points such that, for some quadric Q given by $\lambda q_0 + \mu q_1 = 0$, the associated bilinear form b satisfies $b(w, v) = 0$ for all $w \in E$ representing $[w] \in \Pi$. Suppose C is equal to all of X. By Theorem 4.9, the space X contains a $(m - 1)$-dimensional projective subspace Λ. Choose a point x from Λ. Note that the tangent space $T_x(Q)$ is defined by the fact that $b(w, v) = 0$ for all $w \in E$. Thus $T_x(Q)$ contains the $(m - 1)$-dimensional projective subspace Λ and the $(m-2)$-dimensional projective subspace Π. Since $\Lambda \cap \Pi = \varnothing$, we obtain that $T_x(Q) \cap Q$ contains a projective subspace of dimension $2m - 2$. However, by Corollary 3.3, the corank of any quadric in the pencil is equal to 1. Hence Q cannot contain projective subspaces of dimension greater than $m - 1$. This contradicts C being the whole space. Thus, for a general point $x \in X$, we have $r(x) = 2$.

Consequently, $\Pi \cap T_x(X)$ is empty if $m \leq 3$, or of dimension $\geq m - 4$ if $m \geq 4$. This implies that $F_x = \langle x, \Pi \cap T_x(X)\rangle$ consists of x if $m \leq 3$ and $\dim(F_x) = m - 3$ if $m \geq 4$. If $m = 2$, X is a del Pezzo surface of degree 4 and the projection p is a birational map onto a cubic surface in $\mathbb{P}^3$. If $m = 3$, we obtain that the projection p is a birational map onto $\mathbb{P}^4$. If $m > 3$, we obtain that the general fiber $(p')^{-1}(p(x))$ of the projection map $p' : X_\Pi \to \mathbb{P}^{m+1}$ is isomorphic to the blowup of the $(m-3)$-dimensional subspace $\langle x, \Pi \cap T_x(X)\rangle$ along the $(m-4)$-dimensional subspace $\Pi \cap T_x(X)$. Thus the general fiber is a projective space of dimension $m - 3$. Thus, X is birationally isomorphic to a projective $(m - 3)$-bundle over $\mathbb{P}^{m+1}$.

Now, assume $\Bbbk$ is any perfect field. Since the projection is a $\Bbbk$-rational map, we immediately get that X is rational if $m = 3$. It remains to consider $m > 3$. By construction, X_Π is a subvariety of a projective bundle $\mathbb{P}(E)$ over $\mathbb{P}^{m+1}$. Moreover, the general fibers of $X_\Pi \to \mathbb{P}^{m+1}$ are linear subspaces of the fibers of $\mathbb{P}(E) \to \mathbb{P}^{m+1}$. Thus, over some open subscheme S of $\mathbb{P}^{m+1}$, the subvariety X_Π restricts to a projective bundle over S (not just a Severi–Brauer scheme over S). Thus X_Π is rational. □

Y. Prokhorov asked us for a geometric interpretation of the canonical point Π in the plane model of a del Pezzo surface of degree 4. We consider this question in Remark 8.4.

5. Isomorphisms of normal forms

The main goal of this section is to prove Theorem 1.4. In order to do this, we need to study isomorphisms between pairs of quadratic forms in Kronecker normal form. The associated bilinear form of any such pair is always the same, so any isomorphism of pairs of quadratic forms is an automorphism of a basic singular pair of alternating bilinear forms.

Thus, our first task is to compute the automorphisms of a basic singular pair of alternating bilinear forms. Next, we collect some results about the finite dimensional algebra A from the introduction. We then use this algebra to determine which automorphisms of basic singular pairs correspond to which isomorphisms of pairs of quadratic forms.

5A. ***Automorphisms of basic singular pairs.*** Let (b_0, b_1) be a basic singular pair of alternating bilinear forms (for example, obtained from a regular pair of quadratic forms), and let $w_0, \dots, w_m, v_0, \dots, v_{m-1}$ be a Kronecker basis for the underlying vector space E of dimension $n = 2m+1$.

An *automorphism* of the pair (b_0, b_1) is an element $g \in \mathrm{GL}(E)$ such that

$$b_0(gv, gv) = b_0(v, v) \quad \text{and} \quad b_1(gv, gv) = b_1(v, v)$$

for any $v \in V$.

Lemma 5.1. *The group of automorphisms* $\mathrm{Aut}(b_0, b_1)$ *of the pair* (b_0, b_1) *consists of elements* $g \in \mathrm{GL}(E)$ *of the form*

$$g(w_i) = w_i, \quad g(v_i) = v_i + \sum_{k=0}^{m} s_{i+k} w_k, \tag{5-1}$$

where $s_0, \dots, s_{n-2}$ *are any elements in* $\Bbbk$.

Explicitly, the automorphisms have the form

$$g = \begin{pmatrix} I_{m+1} & S \\ 0_{m,m+1} & I_m \end{pmatrix}, \quad \text{where } S = \begin{pmatrix} s_0 & s_1 & \cdots & s_{m-1} \\ s_1 & s_2 & \cdots & s_m \\ \vdots & \vdots & \vdots & \vdots \\ s_m & s_{m+1} & \cdots & s_{2m-1} \end{pmatrix}$$

is a *catalecticant matrix* $\mathrm{Cat}_{m-1}(s_0, \dots, s_{2m-1})$ (see [Dolgachev 2012, 1.4.1]).

Proof. One checks directly that the given elements are automorphisms of (b_0, b_1) via (4-1). It remains to check that these are the only automorphisms.

Let g be an automorphism of (b_0, b_1). Note that g must fix each w_i since they are canonical. It remains to consider $v_0, \dots, v_{m-1}$. Thus

$$g(v_i) = h(v_i) + \sum_{k=0}^{m} l_{ik} w_k$$

for some $h \in \mathrm{GL}(L)$ and elements l_{ik} in $\Bbbk$. Since

$$\delta_{ij} = b_1(gv_i, gw_j) = b_1(hv_i, w_j)$$

we conclude that h is the identity. Since

$$0 = b_1(gv_i, gv_j) = \sum_{k=0}^{m} l_{jk} b_1(v_i, w_k) + \sum_{k=0}^{m} l_{ik} b_1(w_k, v_j) = l_{ij} + l_{ji}$$

we conclude that $l_{ij} = l_{ji}$. Similarly, using b_0 we conclude that $l_{j(i+1)} = l_{i(j+1)}$. We obtain that $l_{i(j+1)} = l_{(i+1)j}$ and thus the values of l_{ij} depend only on the sum of their indices: $l_{ik} = s_{i+k}$. ☐

5B. ***Results on $\Bbbk$-algebras.*** Here we collect some facts about finite $\Bbbk$-algebras which we will need later. All are standard results except for Lemma 5.3.

Consider a separable polynomial

$$f(T) = a_n T^n + \cdots + a_1 T + a_0$$

in $\Bbbk[T]$ of degree n. Consider the $\Bbbk$-algebra $A = \Bbbk[T]/(f(T))$ and let t be the image of T under the map $\Bbbk[T] \to A$.

Note that since f is a separable polynomial, we may also write A as a direct sum $A = A_1 \oplus \cdots \oplus A_l$ where each A_i is a separable field extension of $\Bbbk$. Indeed, we have isomorphisms $A_i \cong \Bbbk[T]/(f_i(T)) \cong A/(f_i(t))$ where f_i is an irreducible polynomial dividing f.

Define polynomials $g_1, \dots, g_l$ in $\Bbbk[T]$ via $f(T) = f_i(T) g_i(T)$. Since the polynomials g_i are coprime, we may write

$$1 = \epsilon_1 + \cdots + \epsilon_l \tag{5-2}$$

where $\epsilon_i \in (g_i(t))$ and they satisfy relations $\epsilon_i \epsilon_j = 0$ when $i \neq j$, and $\epsilon_i^2 = \epsilon_i$. In other words, $\epsilon_1, \dots, \epsilon_l$ form an orthogonal set of idempotents in A.

Multiplication by an idempotent ϵ_i defines a homomorphism from A into the corresponding summand A_i. Specifically, if h is any polynomial in $\Bbbk[T]$ we have

$$\epsilon_i h(t) \equiv h(t) \mod (f_i(t)) \tag{5-3}$$

in $A_i \cong A/(f_i(t))$. If f_i is of degree 1 with root α_i, then multiplication by ϵ_i corresponds to the $\Bbbk$-algebra homomorphism $A \to A_i \simeq \Bbbk$ determined by $t \mapsto \alpha_i$.

For any $a \in A$, the linear map $x \mapsto ax$ is an endomorphism of the vector space A over $\Bbbk$, we denote by $\mathrm{Tr}_{A/\Bbbk}(a)$ its trace. The formula

$$\langle a, b \rangle = \mathrm{Tr}_{A/\Bbbk}(ab)$$

defines a symmetric bilinear form on A. Its restriction to each summand A_i is the usual trace for a separable extension of fields. Since A is separable, the trace form is nondegenerate. In particular, the natural homomorphism

$$A \to A^\vee, \quad a \mapsto (x \mapsto \mathrm{Tr}_{A/\Bbbk}(ax))$$

of vector spaces over $\Bbbk$ is a bijection.

If f splits completely into linear factors (for example, if $\Bbbk$ is separably closed), then $A \cong \Bbbk^n$. If $\alpha_1, \dots, \alpha_n$ are the roots of f, then a canonical isomorphism $A \cong \Bbbk^n$ is defined via

$$h(t) \mapsto (h(\alpha_1), \dots, h(\alpha_n))$$

for any polynomial h in $k[T]$. In this case, the trace can be computed as

$$\mathrm{Tr}_{A/\Bbbk}(h(t)) = \sum_{i=1}^{n} h(\alpha_i).$$

By passing to a separable closure, one can compute the trace even when f does not split into linear factors over the original field.

Consider a basis of A given by the elements

$$d_i = a_{i+1} + a_{i+2}t + \cdots + a_n t^{n-1-i} \quad \text{for } i = 0, \dots, n-1. \tag{5-4}$$

Using the recurrence relation $d_i = t d_{i+1} + a_{i+1}$, we get the identity in $A[X]$

$$f(X) = (X - t)(d_0 + d_1 X + \cdots + d_{n-1} X^{n-1}). \tag{5-5}$$

Note that d_{n-1} spans the "constant" subalgebra $\Bbbk \subset A$.

Let $f'(T)$ denote the formal derivative of $f(T)$. Since $f(T)$ is a separable polynomial, $f'(T)$ is coprime to $f(T)$ and thus $f'(t)$ is an invertible element in A.

The following proposition shows that the elements $d_i / f'(t)$ form the dual basis of the basis $1, t, \dots, t^{n-1}$ with respect to the trace form.

Proposition 5.2. $$\mathrm{Tr}_{A/\Bbbk}\Big(d_i \frac{t^j}{f'(t)}\Big) = \delta_{ij} \quad \textit{for } i, j \textit{ in } 0, \dots, n-1.$$

Proof. In the case when A is a field, this can be found in, for example, [Lang 1994, Proposition III.1.2]. We give the proof here since it is short and uses formulas that we will need later. We may assume without loss of generality that $\Bbbk$ is separably closed.

Let $\alpha_1, \dots, \alpha_n$ be the roots of f. We use the following *Euler's identity*:

$$X^k - \sum_{i=1}^{n} \frac{f(X)}{X - \alpha_i} \frac{\alpha_i^k}{f'(\alpha_i)} = 0, \quad k = 0, \dots, n-1 \tag{5-6}$$

in the ring $A[\![X]\!]$. To see this, one uses that $f'(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j)$, then checks that the expression on the left-hand side is a polynomial of degree $< n$ and has n roots $\alpha_1, \dots, \alpha_n$, hence it must be zero.

One can extend the trace $\mathrm{Tr}_{A/\Bbbk}$ to a $\Bbbk$-linear function $A[\![X]\!] \to \Bbbk[\![X]\!]$ by applying the trace function to each coefficient of X^i. Thus, using (5-5), we obtain

$$\mathrm{Tr}_{A/\Bbbk}\Big(\frac{f(X)}{X - t} \frac{t^k}{f'(t)}\Big) = \mathrm{Tr}_{A/\Bbbk}\bigg(\Big(\sum_{i=0}^{n-1} d_i X^i\Big) \frac{t^k}{f'(t)}\bigg) = X^k,$$

for every k in $0, \dots, n-1$. Comparing the coefficients, we get the assertion. $\square$

We need an explicit formula for squaring an element with respect to the basis $\{d_0, \dots, d_{n-1}\}$.

Lemma 5.3. *Given the equation*

$$r_0 d_0 + \cdots + r_{n-1} d_{n-1} = (s_0 d_0 + \cdots + s_{n-1} d_{n-1})^2$$

for some coefficients $s_0, \dots, s_{n-1}$ *in* $\Bbbk$, *we have*

$$r_k = \sum_{j=0}^{n-1} s_j^2 a_{2j+1-k}$$

for all k *in* $0, \dots, n-1$. (*We assume by convention* $a_i = 0$ *when* $i < 0$ *or* $i > n$.)

Proof. Using Proposition 5.2, it suffices to show that

$$\mathrm{Tr}_{A/\Bbbk}\Big(d_j^2 \frac{t^k}{f'(t)}\Big) = a_{2j+1-k}$$

for all j, k in $0, \dots, n-1$.

Rearranging Euler's equation (5-6) we obtain

$$\frac{X^k}{f(X)} = \sum_{i=1}^{n} \frac{1}{X - \alpha_i} \frac{\alpha_i^k}{f'(\alpha_i)}.$$

Now, differentiating with respect to X, we have

$$\begin{aligned} \frac{kX^{k-1} f(X) - X^k f'(X)}{f(X)^2} &= \sum_{i=1}^{n} \frac{-1}{(X - \alpha_i)^2} \frac{\alpha_i^k}{f'(\alpha_i)} \\ \frac{\partial}{\partial X}(X^k f(X)) &= \sum_{i=1}^{n} \Big(\frac{f(X)}{X - \alpha_i}\Big)^2 \frac{\alpha_i^k}{f'(\alpha_i)}. \end{aligned}$$

Using (5-5), we conclude

$$\frac{\partial}{\partial X}\Big(\sum_{i=0}^{n} a_i X^{k+i}\Big) = \mathrm{Tr}_{A/\Bbbk}\Big(\Big(\sum_{j=0}^{n-1} d_j^2 X^{2j}\Big) \frac{t^k}{f'(t)}\Big), \quad \sum_j a_{2j+1-k} X^{2j} = \sum_{j=0}^{n-1} \mathrm{Tr}_{A/\Bbbk}\Big(d_j^2 \frac{t^k}{f'(t)}\Big) X^{2j}$$

as desired. Note that the assumption that $\Bbbk$ has characteristic 2 is essential! □

5C. ***Isomorphisms of pairs of quadratic forms.*** Let (q_0, q_1) be a regular pair of quadratic forms in normal form with respect to a Kronecker basis $\mathcal{B} = (w_0, \dots, w_m, v_0, \dots, v_{m-1})$ for E. Note that since $w_0, \dots, w_m$ are canonical, the polynomial Δ and its coefficients $a_0, \dots, a_n$ are the same regardless of the choice of basis $\mathcal{B}$. However, the coefficients $r_0, \dots, r_{n-2}$ from Theorem 1.1 are more subtle.

Assume that $a_n \neq 0$ in Δ or, equivalently, that q_1 is nondegenerate (see Remark 4.8). Define

$$f(T) := \Delta(1, T) = a_0 + \cdots + a_n T^n.$$

Setting $A = \Bbbk[T]/(f(T))$, we have an algebra as in the previous section.

The r-*invariant* of the normal form is the element $r \in A$ given by

$$r = r_0 d_0 + \cdots + r_{n-2} d_{n-2} \tag{5-7}$$

where $d_0, \ldots, d_{n-1}$ is the basis for A defined in (5-4). Note that the basis element d_{n-1} spanning $\Bbbk$ does not appear in the expression (5-7), so the set of possible r-invariants are in bijection with the quotient space $A/\Bbbk$.

For $g \in \mathrm{GL}(E)$, let (q_0', q_1') be the pair given by

$$q_0'(v) = q_0(gv) \quad \text{and} \quad q_1'(v) = q_1(gv)$$

for $v \in E$. Note that the half-discriminant polynomial of (q_0', q_1') remains the same. Let r' be the r-invariant of the new normal form.

We assume, without loss of generality, that (q_0', q_1') is also in Kronecker normal form with respect to the basis $\mathcal{B}$. Consequently, both pairs have the same pair (b_0, b_1) of associated bilinear forms and we may take $g \in \mathrm{Aut}(b_0, b_1)$.

Any $g \in \mathrm{Aut}(b_0, b_1)$ is determined by the elements $s_0, \ldots, s_{n-2}$ from (5-1). We construct an element

$$s = s_0 d_0 + \cdots + s_{n-2} d_{n-2} \tag{5-8}$$

in A as with the r-invariant above.

Reversing the process, one checks that this gives rise to a group homomorphism

$$\phi : A \to \mathrm{Aut}(b_0, b_1) \subset \mathrm{GL}(E), \tag{5-9}$$

where A is viewed as an additive group. Note that the kernel of ϕ is the subgroup $\Bbbk$ since d_{n-1} spans $\Bbbk$.

Theorem 1.4 is a consequence of the following.

Theorem 5.4. *If $g = \phi(s)$ for some $s \in A$, then*

$$r' \equiv r + \wp(s) \mod \Bbbk,$$

where $\wp : A \to A$ is the Artin–Schreier map $s \mapsto s^2 + s$.

Proof. Recall that $g(w_i) = w_i$ and $g(v_i) = v_i + \sum_{k=0}^{m} s_{i+k} w_k$. We need to compute $q_j(gv_i)$ for $j = 0, 1$ and $i = 0, \ldots, m-1$. We find

$$\begin{aligned} q_0(gv_i) &= q_0\left(v_i + \sum_{k=0}^{m} s_{i+k} w_k\right) \\ &= q_0(v_i) + b_0\left(v_i, \sum_{k=0}^{m} s_{i+k} w_k\right) + q_0\left(\sum_{k=0}^{m} s_{i+k} w_k\right) \\ &= r_{2i+1} + s_{2i+1} + \sum_{k=0}^{m} s_{i+k}^2 a_{2k}, \end{aligned}$$

and similarly that

$$q_1(gv_i) = r_{2i} + s_{2i} + \sum_{j=i}^{m+i} s_j^2 a_{2j+1-2i} \ .$$

By Lemma 5.3, we conclude that the new invariant is $r + s + s^2$ as desired. □

6. Relations to the Arf invariant

The following theorem was suggested to us by A. Efimov.

Recall that any nondegenerate quadratic form in $2m$ variables over a field $\Bbbk$ of characteristic 2 can be reduced to the form

$$q = \sum_{i=1}^{m} a_i x_i^2 + x_i y_i + b_i y_i^2$$

for some basis $x_1, \ldots, x_m, y_1, \ldots, y_m$. The *Arf invariant* of q is

$$\mathrm{Arf}(q) = \sum_{i=1}^{m} a_i b_i \in \Bbbk/\wp(\Bbbk).$$

It is independent of a choice of a canonical form from above.

Let (q_0, q_1) be a regular pair of quadratic forms in normal form with invariants Δ and r. We assume that $a_n \neq 0$ and define $A = \Bbbk[T]/(f(T))$ with t the image of T as in previous sections. The pair gives rise to a quadratic form q_A on the module $E_A = E \otimes_{\Bbbk} A$ by defining

$$q_A = q_0 + t q_1$$

as an element $S^2(E_A^\vee)$. Since A is a sum of fields $A_1, \ldots A_l$, we may define the Arf invariant of q_A to be the sum of the Arf invariants of the restrictions q_{A_i}.

Theorem 6.1. *The form q_A is an orthogonal sum of a 1-dimensional trivial form and a nondegenerate $2m$-dimensional form whose Arf invariant is r modulo $\wp(A) + \Bbbk$.*

Proof. We select a new basis for $E_A = E \otimes_{\Bbbk} A$ as follows:

$$\begin{cases} w_i' = \sum_{k=i}^{m} w_i \otimes t^{k-i} & \text{for } i \text{ in } 0, \ldots, m, \\ v_i' = v_i \otimes 1 & \text{for } i \text{ in } 0, \ldots, m-1 \end{cases}$$

and find that

$$b_A(w_i', w_j') = 0, \quad b_A(v_i', v_j') = 0, \quad b_A(v_i', w_j') = \delta_{(i+1)j},$$

where b_A is the bilinear form associated to q_A. In particular, w_0' is the image of Ω in E_A and we see that it spans a 1-dimensional form where $q_A(w_0') = 0$.

The remaining vectors span a subspace which decomposes into an orthogonal sum of 2-dimensional subspaces $\langle w_{i+1}', v_i' \rangle$ on which q_A is isomorphic to the quadratic form

$$q_A(w_{i+1}')x^2 + xy + q_A(v_i')y^2$$

for appropriate coordinates x and y.

Note that

$$q_A(w'_{i+1}) = \sum_{k=i}^{m}(a_{2i}+a_{2i+1}t)t^{2(k-i)} = d_{2i+1}$$

while

$$q_A(v'_i) = r_{2i}t + r_{2i+1}.$$

The Arf invariant of the $2m$-dimensional subspace is thus

$$\sum_{i=0}^{m-1} q_A(w'_{i+1})q_A(v'_i) = \sum_{i=0}^{m-1}(r_{2i}t+r_{2i+1})d_{2i+1} = \left(\sum_{i=0}^{n-2} r_i d_i\right) + \left(\sum_{i=0}^{m-1} r_{2i}a_{2i+1}\right),$$

which is equal to $r \mod \Bbbk$. □

7. Automorphisms of pencils of quadratic forms

7A. ***Arithmetic description of automorphisms of a pair.*** Let A be the $\Bbbk$-algebra from the previous two sections. Since the characteristic is 2, the set of idempotents $\mathrm{Idem}(A)$ form a subgroup of the additive group of A. Recall that an idempotent a in A is an element such that $a^2 = a$. Consequently, the group $\mathrm{Idem}(A)$ can be characterized as the kernel of the Artin–Schreier map

$$\wp : A \to A, \quad a \mapsto a^2 + a.$$

Let $\mathrm{Aut}(q_0, q_1)$ be the set of automorphisms of the pair (q_0, q_1); in other words, the subgroup of $\mathrm{GL}(E)$ whose elements induce an isomorphism of the pair with itself. From Theorem 5.4, we see that every automorphism comes from an idempotent via (5-1). Conversely, the only nontrivial idempotent which gives rise to a trivial automorphism is the multiplicative identity $1 \in \Bbbk \subset A$. Thus we have the following:

Theorem 7.1. *There is a canonical isomorphism*

$$\mathrm{Aut}(q_0, q_1) \cong \mathrm{Idem}(A)/\langle 1\rangle,$$

where $\langle 1\rangle$ is the additive subgroup of A generated by the unit element $1 \in A$.

The group $\mathrm{Idem}(A)$ is generated by the orthogonal set of idempotents $\epsilon_1, \dots, \epsilon_l$ from (5-2). Since the additive subgroup of A is commutative and every element has order 2, we see that $\mathrm{Idem}(A)$ is an elementary abelian 2-group of order 2^l.

Consequently, $\mathrm{Aut}(q_0, q_1)$ is an elementary abelian 2-group of order 2^{l-1} where l is the number of irreducible factors in $f(T)$. When A is a field (equivalent to $f(T)$ being irreducible), then the only idempotent is 1 and thus $\mathrm{Aut}(q_0, q_1)$ is trivial. At the other extreme, when $f(T)$ splits completely, $\mathrm{Aut}(q_0, q_1)$ has order 2^{2m} and we will see below that the idempotents $\epsilon_1, \dots, \epsilon_n$ correspond to reflections.

$\mathrm{Idem}(A)$, respectively $\mathrm{Aut}(q_0, q_1)$, can be viewed as the set of $\Bbbk$-points of a finite étale group scheme $\underline{\mathrm{Idem}}(A)$, respectively $\underline{\mathrm{Aut}}(q_0, q_1)$, over $\Bbbk$. The group scheme $\underline{\mathrm{Idem}}(A)$ is simply the Weil restriction of scalars $\mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)$ where $\mathbb{Z}/2\mathbb{Z}$ is the constant group scheme of order 2. The group scheme $\underline{\mathrm{Aut}}(q_0, q_1)$ is the quotient in the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A) \to \underline{\mathrm{Aut}}(q_0, q_1) \to 0, \tag{7-1}$$

where ι is the natural monomorphism.

Proposition 7.2. *There is a canonical isomorphism between $H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1))$, the Galois cohomology group, and the group $A/(\wp(A) + \Bbbk)$.*

Proof. This follows the same reasoning as §1 of [Skorobogatov 2010]. The composition

$$H^2(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \to H^2(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)) = H^2(A, \mathbb{Z}/2\mathbb{Z})$$

factors through the restriction-corestriction map, which is multiplication by the odd number $\dim_{\Bbbk}(A)$. Since $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)$ have an even number of elements over a separable closure, the cohomology groups are 2-primary. Thus the map $H^2(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \to H^2(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A))$ is injective. Thus from (7-1), the group $H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1)$ is the cokernel of the map

$$H^1(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \to H^1(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)).$$

The characteristic is 2, thus $H^1(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \cong \Bbbk/\wp(\Bbbk)$ and

$$H^1(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)) = H^1(A, (\mathbb{Z}/2\mathbb{Z})_A)) \cong A/\wp(A),$$

so the result follows. □

Remark 7.3. The above proposition could be used to recover a weak form of Theorem 5.4. Of course, this argument would be circular in our presentation since Theorem 5.4 was used to determine the automorphism group.

7B. ***Geometric description of automorphisms of a pair.*** In this subsection we assume that $\Bbbk$ is algebraically closed.

Let $x_1, \dots, x_n$ be vectors in U representing the points $\overline{x_1}, \dots, \overline{x_n}$ in the zero locus of $V(\Delta)$ on $|U|$. Recall that $V(\mathsf{q}_{x_1}), \dots, V(\mathsf{q}_{x_n})$ are precisely the singular quadrics of the pencil $|\mathsf{q}|$.

Fix i in $1, \dots, n$. Since $V(\mathsf{q}_{x_i})$ is of corank 1, it contains a unique singular point $\overline{z_i} \in |E|$ represented by the vector $z_i = \Omega(x_i)$ in E.

For any $u \in U$ such that $|u| \neq |x_i|$ in $|U|$ consider the following linear automorphism of E

$$\rho_i(v) = v + \frac{\mathsf{b}_u(z_i, v)}{\mathsf{q}_u(z_i)} z_i, \tag{7-2}$$

where $v \in E$. A priori, the formula for ρ_i depends on the choice of $u \in U$. However, one can check that ρ_i does not depend on the choice of u as long as it is not a multiple of x_i (use that $\mathsf{q}_{x_i}(z_i) = 0$ and $\mathsf{b}_{x_i}(z_i, v) = 0$ for any choice of $v \in E$).

Theorem 7.4. *The automorphism group* $\mathrm{Aut}(q_0, q_1)$ *is an elementary abelian* 2*-group of order* 2^{2m} *generated by the reflections* $\rho_1, \dots, \rho_n$ *subject to the relation* $\rho_1 \cdots \rho_n = 1$.

Proof. It is clear that the reflections $\{\rho_i\}$ are automorphisms, that they are of order 2, and that they commute. It remains to show that there are no other automorphisms and to show that they satisfy the desired relations and no others. We will show that the reflections $\rho_1, \dots, \rho_n$ correspond to a set of orthogonal idempotents of A. The result then follows by Theorem 7.1.

The remainder of the proof is simply a direct calculation. In view of Remark 4.8, we may assume without loss of generality that q_0 is nondegenerate and each vector x_i has coordinates $(\alpha_i, 1)$, where α_i is a root of $f(T)$. Let ϵ_i be the idempotent corresponding to α_i; in other words, multiplication by ϵ_i gives rise to a map $A \to \Bbbk$ such that $t \mapsto \alpha_i$ as in (5-3).

We will establish that $\phi(\epsilon_i) = \rho_i$, where $\phi : A \to \mathrm{GL}(E)$ is the map (5-9). To do this, we determine the values of $s_0, \dots, s_{n-2}$ in (5-1). From (5-8), these are the first $n-1$ coordinates of ϵ_i in the basis $d_0, \dots, d_{n-1}$ of A defined in (5-4). Thus, by Proposition 5.2, we have

$$s_j = \mathrm{Tr}_{A/\Bbbk}\left(\epsilon_i \frac{t^i}{f'(t)}\right) = \frac{\alpha_i^j}{f'(\alpha_i)}$$

for each j in $0, \dots, n-2$.

Now, note that $z_i = \sum_{i=0}^m \alpha^i w_i$. Since $f'(T) = \sum_{k=0}^m a_{2i+1} T^{2i}$, we have $f'(\alpha_i) = q_1(z_i)$. We have $b_1(v_j, z_i) = \alpha_i^j$ for j in $0, \dots, m-1$, where $w_0, \dots, w_m, v_0, \dots, v_{m-1}$ is the Kronecker basis for E.

Putting these observations together, we see that

$$\phi(\epsilon_i)(v_j) - v_j = \sum_{k=0}^m s_{j+k} w_k = \frac{\alpha_i^j}{f'(\alpha_i)} \sum_{k=0}^m \alpha^k w_k = \frac{b_1(z_i, v_j)}{q_1(z_i)} z_i$$

for all j in $0, \dots, m-1$. Since, additionally, $\phi(\epsilon_i)(w_j) = \rho(w_j) = w_j$ for j in $0, \dots, m$, we conclude that $\phi(\epsilon_i)(v) = \rho(v)$ for all $v \in E$. □

With this description of the automorphism group, we have:

Corollary 7.5. *There are exactly* 2^{2m} *generators permuted simply transitively by the group* $\mathrm{Aut}(q_0, q_1)$.

Proof. By Corollary 4.12, the generators are in bijective correspondence with subspaces L occurring in a Kronecker normal form such that L is totally isotropic with respect to every quadratic form in the pencil. The group $\mathrm{Aut}(q_0, q_1)$ permutes such spaces. The subspace L is never invariant under a nontrivial automorphism as in (5-1); thus the action is simply transitive. The number of generators is 2^{2m} since that is the order of the group. □

From the above corollary, we prove Theorem 1.5 from the introduction:

Proof of Theorem 1.5. The set of generators of X corresponds to a smooth 0-dimensional subscheme of an appropriate Grassmannian. In view of Corollary 7.5, over a nonclosed field this is an $\underline{\mathrm{Aut}}(q_0, q_1)$-torsor. From Proposition 7.2 we have that $A/(\wp(A) + \Bbbk) \cong H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1))$. Thus the r-invariant in A naturally gives rise to a class in $H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1)$. □

7C. ***Automorphisms of the base locus.*** The goal of this section is to prove Theorem 1.6, which describes the automorphisms of the base scheme $\mathrm{Bs}(\mathsf{q}) = V(q_0, q_1)$ in $\mathbb{P}^{n-1}$.

Lemma 7.6. *Two smooth complete intersections of pairs of quadrics in $\mathbb{P}^{2m}$ are isomorphic if and only if there is an element of PGL_{2m+1} inducing the isomorphism.*

Proof. If $m = 1$ then the two varieties are smooth subschemes of $\mathbb{P}^2$, of dimension 0 of degree 4; the result is clear in this case. Otherwise, when $m \geq 2$, the adjunction formula gives that the canonical sheaf ω_X is isomorphic to $\mathcal{O}_X(-n+4)$. Since $n = 2m+1 \geq 5$, the anticanonical divisor class $-K_X$ is a positive multiple of the hyperplane section. Hence the embedding of X in $\mathbb{P}^{2m}$ comes from a multiple of the anticanonical sheaf. Any isomorphism between the varieties induces isomorphisms of the corresponding anticanonical sheaves, and the result follows. □

Theorem 1.6 follows immediately from the following.

Theorem 7.7. *Assume X is quasisplit. The automorphism group of X,* $\mathrm{Aut}(X)$, *is isomorphic to* $\mathrm{Aut}(q_0, q_1) \rtimes G$ *where G is the subgroup of* $\mathrm{PGL}(U)$ *which leaves invariant the scheme $V(\Delta)$ of zeros of the half-discriminant.*

Proof. By Lemma 7.6, any automorphism of X can be represented by an element of $\mathrm{PGL}(E)$.

Recall that $|W|$ is a canonical subspace and must be invariant under any automorphism of X. From Proposition 4.6, W is canonically isomorphic to $S^m(U^\vee)^\vee$. Thus an automorphism $h \in \mathrm{PGL}(E)$ fixes $|W|$ pointwise if and only if it fixes $|U|$ pointwise. Thus there is a group homomorphism $\pi : \mathrm{Aut}(X) \to \mathrm{PGL}(U)$ whose kernel is precisely the group $\mathrm{Aut}(q_1, q_2)$ of automorphisms of the pair (q_1, q_2).

Note that since the zeroes of Δ correspond to the singular quadrics of the pencil, all automorphisms must leave invariant the zeroes of Δ. Thus the image of π is contained in G. It remains to show that any element $g \in \mathrm{GL}(U)$ representing an element of G can be lifted to $\mathrm{Aut}(X)$.

Since X is quasisplit, we have a normal form for the pair (q_{u_1}, q_{u_2}) where $r = 0$. We may scale our representative $g \in \mathrm{GL}(U)$ such that $g^*\Delta = \Delta$. By Lemma 4.7, there is an $h \in \mathrm{GL}(W) \times \mathrm{GL}(L)$ so that $(\mathsf{q}_{gu_0}, \mathsf{q}_{gu_1})$ has a normal form with respect to the basis $h(v_1), \dots, h(w_m)$. Since $g^*\Delta = \Delta$, we have $\mathsf{q}_{g(u_i)}(w_j) = \mathsf{q}_{u_i}(w_j)$ for $i = 0, 1$ and $j = 0, \dots, m$. Since L is totally isotropic, $\mathsf{q}_{g(u_i)}(v_j) = \mathsf{q}_{u_i}(v_j) = 0$ for $i = 0, 1$ and $j = 0, \dots, m-1$. Thus the normal forms are equal and g is an automorphism as desired. □

Remark 7.8. Note in the last step of the proof, we required that $r = 0$ in order to lift automorphisms from $g \in \mathrm{PGL}(U)$ to $\mathrm{Aut}(X)$. Thus, when X is not quasisplit, the automorphism group may not surject onto G.

8. Cohomology of intersection of two quadrics in $\mathbb{P}^{2m}$

Let X be a smooth intersection of two quadrics in $\mathbb{P}^{2m}$ over an algebraically closed field of characteristic $p \geq 0$. Recall that in the case $m = 2$ when X is a del Pezzo surface of degree 4, the Picard group $\mathrm{Pic}(X)$ of algebraic 2-cycles on X is a free abelian group of rank 6 and the cycle homomorphism $\mathrm{Pic}(X)_{\mathbb{Z}_\ell} \to H^2(X, \mathbb{Z}_\ell)$ to the ℓ-adic cohomology is an isomorphism. It is also compatible with the intersection pairing $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$ and the cup product $H^2(X, \mathbb{Z}_\ell) \times H^2(X, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell$ on the cohomology. This well-known result follows from the fact that X is isomorphic to the blowup of 5 points $p_1, \dots, p_5$ in the projective plane $\mathbb{P}^2$, no three of which are collinear. In the anticanonical model, the exceptional curves of the blowup are 5 disjoint lines on X. The remaining eleven lines come from the proper transforms of lines $\overline{p_i, p_j}$ and the conic passing through the five points.

Let $e_1, \dots, e_5$ be the classes of the exceptional curves, and e_0 be the class of the preimage of a line in the plane under the blowup. Then $\mathrm{Pic}(X)$ is freely generated by $e_0, \dots, e_5$ and the canonical class of X is equal to $-3e_0 + e_1 + \cdots + e_5$. Its orthogonal complement in $\mathrm{Pic}(X)$ is isomorphic to the negative root lattice of type D_5. The group of automorphisms of X is faithfully represented in the Weyl group of this lattice and is a subgroup to $2^4 \rtimes \mathfrak{S}_5$, where $\mathfrak{S}_5$ denotes the symmetric group on 5 letters.

In this section we extend these well known facts to the case of arbitrary m (for $\mathrm{char}(\Bbbk) \neq 2$ see [Reid 1972]). Throughout, X will be the base locus of a regular pencil q of quadrics and P will denote the projectivized radical subspace $|W|$ of $|E|$.

Fix a generator $\Lambda = |L|$ in X. Let $X_\Lambda \to X$ be the blowup of X with center at Λ. Consider the projection map $p_\Lambda : X_\Lambda \to \mathbb{P}^m$ from Λ. We have $E = W \oplus L$ so we may take p_Λ to be the projectivization of the projection $E \to W$ and thus identify $\mathbb{P}^m$ with $\mathsf{P} = |W|$. For any point $x \in X \setminus \Lambda$, $p_\Lambda(x) = \langle \Lambda, x\rangle \cap |W|$. The base locus of the restriction of the pencil q to the subspace $\langle \Lambda, x\rangle \cong \mathbb{P}^m$ contains Λ. Thus the residual pencil consists of hyperplanes in $\langle \Lambda, x\rangle$ containing x. Its base locus is a codimension 2 linear subspace of $\langle \Lambda, x\rangle$ containing x unless one of the quadrics contains $\langle \Lambda, x\rangle$ and hence the two quadrics intersect along a hyperplane Λ' in $\langle \Lambda, x\rangle$. If Λ' exists, then it is a generator in X intersecting Λ along a hyperplane. Note that $\Lambda' = \rho_i(\Lambda)$ for some reflection $\rho_i \in \mathrm{Aut}(q_0, q_1)$ and $\Lambda' \cap \Lambda$ is the intersection of the fixed hyperplane F_i of ρ_i with Λ'.

The image of each such generator under the projection map p_Λ is the point $y_i = [v_i]$ in $|W|$ which is the singular point of one of the singular quadrics in the pencil. In particular, we found $2m+1$ generators that do not intersect each other outside Λ but each intersects Λ along a hyperplane. Their proper transforms on the blowup are disjoint subvarieties each isomorphic to $\mathbb{P}^{m-1}$. This gives the following birational picture of X.

Let $\mathbb{P}^{2m}_\Lambda \to \mathbb{P}^{2m}$ be the blowup of $\mathbb{P}^{2m}$ along Λ. The projection from Λ defines an isomorphism from $\mathbb{P}^{2m}_\Lambda$ to the projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^m}^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^m}(-1)$.

Proposition 8.1. *X_Λ is isomorphic over $\mathsf{P} \cong \mathbb{P}^m$ to a closed subvariety of the projective $(m+1)$-bundle $\mathbb{P}(\mathcal{E})$ over P. Let $\Sigma = \{y_1, \dots, y_{2m+1}\}$ be the images in $|W|$ of the vertices of singular quadrics in the pencil. If $x \notin \Sigma$ (resp. $x \in \Sigma$), then the fiber $p_\Lambda^{-1}(x)$ is a codimension 2 (resp. codimension 1) linear subspace in the fiber of the projective bundle.*

The inclusion $X_\Lambda \hookrightarrow \mathbb{P}(\mathcal{E})$ is given by a surjective homomorphism of coherent sheaves $\mathcal{E} \to \mathcal{F}$ such that $X_\Lambda \cong \mathbb{P}(\mathcal{F}) := \mathrm{Proj}\, S^\bullet(\mathcal{F})$. The set Σ is equal to the singular set $\mathrm{Sing}(\mathcal{F})$, the set of points $x \in \mathbb{P}^m$ such that $\dim_\Bbbk(\mathcal{F}(x)) > \mathrm{rank}(\mathcal{F})$. In our case $\mathrm{rank}(\mathcal{F}) = m-1$ and $\dim_\Bbbk \mathcal{F}(x) = m$ for $x \in \mathrm{Sing}(\mathcal{F})$.

Let $U = \mathbb{P}^m \setminus \Sigma$ and $j : U \hookrightarrow \mathbb{P}^m$ be the open inclusion. Since $\operatorname{codim} \Sigma \geq 2$, the sheaf $j_* j^* \mathcal{F}$ is a locally free sheaf of rank $m-1$ and $\mathcal{F}$ is its subsheaf; in particular, it is a reflexive sheaf [Hartshorne 1980]. Thus, we obtain

Proposition 8.2. $$X_\Lambda \cong \mathbb{P}(\mathcal{F}),$$

where $\mathcal{F}$ is a reflexive sheaf of rank $m-1$ with $\operatorname{Sing}(\mathcal{F}) = \Sigma$. *In particular, X is birationally equivalent to a projective $(m-2)$-bundle over $\mathbb{P}^m$.*

Example 8.3. Assume $m = 2$, then p_Λ is the projection of the quartic del Pezzo surface X from a line. It is isomorphic to the blowup $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is the ideal sheaf $\mathcal{I}_\Sigma$ of the set Σ of five points in the plane.

Remark 8.4. Recall that, over an algebraically closed field, a del Pezzo surface of degree 4 is obtained by blowing up 5 general points in the plane $\mathbb{P}^2$ which lie on a unique conic $C = V(q)$, where q is a nondegenerate quadratic form. Prokhorov asked us for a geometric interpretation in the plane model of the canonical point Π from Theorem 4.4.

A natural guess is that it is the *strange point* of the conic. It has the property that any line through this point is tangent to C. This turns out to be false. In fact, our conic depends only on the pencil of the associated alternate bilinear forms. In a Kronecker basis it can be given by equation $x_1x_3 + x_2^2 = 0$. The equation of Π depends on the coefficients $(a_0, \dots, a_5)$ of the half-discriminant, and coincides with the strange point only in the case when $a_2a_5 = a_3a_4$, $a_0a_3 = a_1a_2$.

The following is well known when the characteristic is not 2; see Chapter 3 of [Reid 1972] for a proof over $\mathbb{C}$.

Proposition 8.5. *Let $A_{m-1}(X)$ be the Chow group of algebraic $(m-1)$-cycles on X equipped with a structure of a quadratic lattice with respect to the intersection of algebraic cycles on X. Then*

$$A_{m-1}(X) \cong \mathbb{Z}^{2m+2}.$$

Let $A_{m-1}(X)_0$ denote the orthogonal complement of K_X^{m-1} in $A_{m-1}(X)$. Then it is isomorphic to the roots lattice of type D_{2m+1} taken with the sign $(-1)^{m-1}$.

Proof. Let $U = \mathbb{P}^m \setminus \Sigma$, let $Y_\Sigma = p_\Lambda^{-1}(\Sigma)$, and let $X_U = p_\Lambda^{-1}(U)$. By Proposition 1.8 from [Fulton 1998], we have the following exact sequence of Chow groups A_k of algebraic k-cycles:

$$A_k(Y_\Sigma) \to A_k(X_\Lambda) \to A_k(X_U) \to 0.$$

For a vector bundle $\mathcal{E}$ of rank $e+1$ over a base Z, the Chow groups of the projective bundle $\mathbb{P}(\mathcal{E})$ are well known. By Theorem 3.3 of the same work, we have

$$A_k(\mathbb{P}(\mathcal{E})) = \bigoplus_{i=0}^{e} A_{k-e+i}(Z).$$

It follows that

$$A_{m-1}(X_U) \cong \mathbb{Z}^{m-1}, \quad A_{m-1}(Y_\Sigma) \cong \mathbb{Z}^{2m+1}.$$

The homomorphism $A_{m-1}(Y_\Sigma) \to A_{m-1}(X_\Sigma)$ is injective because Y_Σ is the union of disjoint subvarieties of dimension $m-1$. This shows that $A_{m-1}(X_\Lambda) \cong \mathbb{Z}^{3m}$. It remains to use [Fulton 1998], Proposition 6.7(e) that computes the cohomology of the blowup. We have $A_{m-1}(X_\Lambda) = A_{m-1}(X) \oplus \mathbb{Z}^{m-2}$. This gives $A_{m-1}(X) \cong \mathbb{Z}^{2m+2}$.

The rest of the arguments can be borrowed from Chapter 3 of [Reid 1972] since they do not depend on the characteristic of the ground field. We include this for completeness sake. First, we fix one generator Λ and use Corollary 7.5 to index generators by subsets of the set $[1, 2m+1] = \{1, 2, \dots, 2m+1\}$ modulo taking the complementary subset. Here $\Lambda_\varnothing := \Lambda$ is the generator corresponding to the empty set, and each $\Lambda_{\{i\}} := \rho_i(\Lambda)$ is the generator corresponding to the image of Λ under the reflection ρ_i.

We equip $A_{m-1}(X)$ with a structure of a nondegenerate quadratic lattice with respect to the intersection pairing $A_{m-1}(X) \times A_{m-1}(X) \to \mathbb{Z}$. We have

$$[\Lambda_I] \cdot [\Lambda_J] = (-1)^r \Big(\Big\lfloor \frac{r}{2} \Big\rfloor + 1 \Big),$$

where $r = \dim \Lambda_I \cap \Lambda_J$. In particular,

$$[\Lambda_I]^2 = (-1)^{m-1} \Big(\Big\lfloor \frac{m-1}{2} \Big\rfloor + 1 \Big) \tag{8-1}$$

and, if $\dim \Lambda_I \cap \Lambda_J = m-3$,

$$([\Lambda_I] - [\Lambda_J])^2 = 2(-1)^{m-1}. \tag{8-2}$$

Let η be the class of a hyperplane section of X. We have

$$\eta^{m-1} \cdot [\Lambda_I] = 1.$$

Since $K_X = (3-2m)\eta$, we have $K_X^{m-1} = (3-2m)^{m-1}\eta^{m-1}$, hence

$$A_{m-1}(X)_0 := (K_X^{m-1})^\perp = (\eta^{m-1})^\perp.$$

We may assume that the generators Λ_i, $i = 1, \dots, 2m+1$, are the preimages of the points $y_i \in \Sigma$ under the projection map $p_\Lambda : X_\Lambda \to |W|$. Let $e_i = [\Lambda_i]$ be their classes in $A_{m-1}(X)$. Let

$$e_0 = \eta^{m-1} - [\Lambda_\varnothing]$$

(note that in the case $m = 2$ it coincides with the class of the pre-image of a line in $\mathbb{P}^2$). One checks that the classes $e_0, e_1, \dots, e_{2m+1}$ freely generate $A_{m-1}(X)$ and the classes

$$\alpha_0 = -e_0 + [\Lambda_\varnothing] + e_{2m} + e_{2m+1}, \quad \alpha_i = e_i - e_{i+1},\ i = 1, \dots 2m,$$

freely generate $A_{m-1}(X)_0$. In fact they form a root basis of type D_{2m+1} in $A_{m-1}(X)_0$ (taken with the sign $(-1)^{m-1}$). □

Recall that the Weyl group $W(D_n)$ of the root lattice of type D_n generated by reflections in simple roots α_i is isomorphic to $2^{n-1} \rtimes \mathfrak{S}_n$. The group $\mathrm{Aut}(X)$ acts naturally on $A_{m-1}(X)$ leaving K_X invariant.

This defines a homomorphism

$$\rho : \mathrm{Aut}(X) \to \mathrm{Or}(A_{m-1}(X)_0) \cong \mathrm{Or}(\mathsf{D}_n)$$

to the orthogonal group of the quadratic lattice D_n defined by the Cartan matrix of the root system of type D_n. The image is contained in the subgroup of $\mathrm{Or}(\mathsf{D}_n)$ of isometries that can be lifted to $A_{m-1}(X)$. It follows from the theory of quadratic lattices that this subgroup is of index 2 and coincides with the Weyl group $W(D_n)$. Thus, we have defined a homomorphism

$$\rho : \mathrm{Aut}(X) \to W(D_n) \cong 2^{n-1} \rtimes \mathfrak{S}_n.$$

Theorem 8.6. *The homomorphism ρ is injective and sends the subgroup* $\mathrm{Aut}(q_0, q_1)$ *to the subgroup* 2^{n-1} *of* $W(D_n)$.

Proof. An element g of the kernel fixes all generators. Assume $p \neq 2$. We know that any generator Λ intersects $n = 2m+1$ other generators that intersect Λ along a hyperplane. We know from the description of the projection map $p_\Lambda : X_\Lambda \to |W|$ given in the beginning of this section that Λ intersects n generators Λ_i that are projected to the singular points $y_i \in |W|$ of n singular fibers from the pencil. Thus g in its action in $|W|$ fixes these points, and since they generate $|W|$ fixes $|W|$ pointwise. Thus the projection of g to G from Theorem 7.7 is the identity, hence g is the identity. □

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Graded Steinberg algebras and their representations

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We study the category of left unital graded modules over the Steinberg algebra of a graded ample Hausdorff groupoid. In the first part of the paper, we show that this category is isomorphic to the category of unital left modules over the Steinberg algebra of the skew-product groupoid arising from the grading. To do this, we show that the Steinberg algebra of the skew product is graded isomorphic to a natural generalisation of the Cohen–Montgomery smash product of the Steinberg algebra of the underlying groupoid with the grading group. In the second part of the paper, we study the minimal (that is, irreducible) representations in the category of graded modules of a Steinberg algebra, and establish a connection between the annihilator ideals of these minimal representations, and effectiveness of the groupoid.

Specialising our results, we produce a representation of the monoid of graded finitely generated projective modules over a Leavitt path algebra. We deduce that the lattice of order-ideals in the K_0-group of the Leavitt path algebra is isomorphic to the lattice of graded ideals of the algebra. We also investigate the graded monoid for Kumjian–Pask algebras of row-finite k-graphs with no sources. We prove that these algebras are graded von Neumann regular rings, and record some structural consequences of this.

1. Introduction

There has long been a trend of "algebraisation" of concepts from operator theory into algebra. This trend seems to have started with von Neumann and Kaplansky and their students Berberian and Rickart to see what properties in operator algebra theory arise naturally from discrete underlying structures [Kaplansky 1968]. As Berberian [1972] puts it, "if all the functional analysis is stripped away ... what remains should stand firmly as a substantial piece of algebra, completely accessible through algebraic avenues".

In the last decade, Leavitt path algebras [Abrams and Aranda Pino 2005; Ara et al. 2007] were introduced as an algebraisation of graph C^*-algebras [Kumjian et al. 1997; Raeburn 2005] and in particular Cuntz–Krieger algebras. Later, Kumjian–Pask algebras [Aranda Pino et al. 2013] arose as an algebraisation of higher-rank graph C^*-algebras [Kumjian and Pask 2000]. Quite recently Steinberg algebras were introduced in [Steinberg 2010; Clark et al. 2014] as an algebraisation of the groupoid C^*-algebras first studied by Renault [1980]. Groupoid C^*-algebras include all graph C^*-algebras and higher-rank graph C^*-algebras, and Steinberg algebras include Leavitt and Kumjian–Pask algebras as

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well as inverse semigroup algebras. More generally, groupoid C^*-algebras provide a model for inverse-semigroup C^*-algebras, and the corresponding inverse-semigroup algebras are the Steinberg algebras of the corresponding groupoids. All of these classes of algebras have been attracting significant attention, with particular interest in whether K-theoretic data can be used to classify various classes of Leavitt path algebras, inspired by the Kirchberg–Phillips classification theorem for C^*-algebras [Phillips 2000].

In this note we study graded representations of Steinberg algebras. For a Γ-graded groupoid $\mathcal{G}$, (i.e., a groupoid $\mathcal{G}$ with a cocycle map $c : \mathcal{G} \to \Gamma$) Renault [1980, Theorem 5.7] proved that if Γ is a discrete abelian group with Pontryagin dual $\widehat{\Gamma}$, then the C^*-algebra $C^*(\mathcal{G} \times_c \Gamma)$ of the skew-product groupoid is isomorphic to a crossed-product C^*-algebra $C^*(\mathcal{G}) \times \widehat{\Gamma}$. Kumjian and Pask [1999] used Renault's results to show that if there is a free action of a group Γ on a graph E, then the crossed product of graph C^*-algebra by the induced action is strongly Morita equivalent to $C^*(E/\Gamma)$, where E/Γ is the quotient graph.

Parallelling Renault's work, we first consider the Steinberg algebras of skew-product groupoids (for arbitrary discrete groups Γ). We extend Cohen and Montgomery's definition of the smash product of a graded ring by the grading group (introduced and studied in their seminal paper [Cohen and Montgomery 1984]) to the setting of nonunital rings. We then prove that the Steinberg algebra of the skew-product groupoid is isomorphic to the corresponding smash product. This allows us to relate the category of graded modules of the algebra to the category of modules of its smash product. Specialising to Leavitt path algebras, the smash product by the integers arising from the canonical grading yields an ultramatricial algebra. This allows us to give a presentation of the monoid of graded finitely generated projective modules for Leavitt path algebras of arbitrary graphs. In particular, we prove that this monoid is cancellative. The group completion of this monoid is called the graded Grothendieck group, K_0^{gr}, which is a crucial invariant in study of Leavitt path algebras. It is conjectured [Hazrat 2016, §3.9] that the graded Grothendieck group is a complete invariant for Leavitt path algebras. We study the lattice of order ideals of K_0^{gr} and establish a lattice isomorphism between order ideals of K_0^{gr} and graded ideals of Leavitt path algebras.

We then apply the smash product to Kumjian–Pask algebras $\mathrm{KP}_K(\Lambda)$. Unlike Leavitt path algebras, Kumjian–Pask algebras of arbitrary higher rank graphs are poorly understood, so we restrict our attention to row finite k-graphs with no sources. We show that the smash product of $\mathrm{KP}_K(\Lambda)$ by $\mathbb{Z}^k$ is also an ultramatricial algebra. This allows us to show that $\mathrm{KP}_K(\Lambda)$ is a graded von Neumann regular ring and, as in the case of Leavitt path algebras, its graded monoid is cancellative. Several very interesting properties of Kumjian–Pask algebras follow as a consequence of general results for graded von Neumann regular rings.

We then proceed with a systematic study of the irreducible representations of Steinberg algebras. Chen [2015] used infinite paths in a graph E to construct an irreducible representation of the Leavitt path algebra E. These representations were further explored in a series of papers [Abrams et al. 2015; Ara and Rangaswamy 2014; 2015; Hazrat and Rangaswamy 2016; Rangaswamy 2016]. The infinite path representations of Kumjian–Pask algebras were also defined in [Aranda Pino et al. 2013]. In the setting of a groupoid $\mathcal{G}$, the infinite path space becomes the unit space of the groupoid. For any invariant subset W of the unit space, the free module RW with basis W is a representation of the Steinberg algebra $A_R(\mathcal{G})$

[Brown et al. 2014]. These representations were used to construct nontrivial ideals of the Steinberg algebra, and ultimately to characterise simplicity.

For the Γ-graded groupoid $\mathcal{G}$, we introduce what we call Γ-aperiodic invariant subsets of the unit space of the groupoid $\mathcal{G}$. We obtain graded (irreducible) representations of the Steinberg algebra via these Γ-aperiodic invariant subsets. We then describe the annihilator ideals of these graded representations and establish a connection between these annihilator ideals and effectiveness of the groupoid. Specialising to the case of Leavitt and Kumjian–Pask algebras we obtain new results about representations of these algebras.

The paper is organised as follows. In Section 2, we recall the background we need on graded ring theory, and then introduce the smash product $A \# \Gamma$ of an arbitrary Γ-graded ring A, possibly without unit. We establish an isomorphism of categories between the category of unital left $A \# \Gamma$-modules and the category of unital left Γ-graded A-modules. This theory is used in Section 3, where we consider the Steinberg algebra associated to a Γ-graded ample groupoid $\mathcal{G}$. We prove that the Steinberg algebra of the skew-product of $\mathcal{G} \times_c \Gamma$ is graded isomorphic to the smash product of $A_R(\mathcal{G})$ with the group Γ.

In Section 4 we collect the facts we need to study the monoid of graded rings with graded local units. In Section 5 and Section 6, we apply the isomorphism of categories in Section 2 and the graded isomorphism of Steinberg algebras (Theorem 3.4) on the setting of Leavitt path algebras and Kumjian–Pask algebras. Although Kumjian–Pask algebras are a generalisation of Leavitt path algebras, we treat these classes separately as we are able to study Leavitt path algebras associated to any arbitrary graph, whereas for Kumjian–Pask algebras we consider only row-finite k-graphs with no sources, as the general case is much more complicated [Raeburn et al. 2004; Sims 2006]. We describe the monoids of graded finitely generated projective modules over Leavitt path algebras and Kumjian–Pask algebras, and obtain a new description of their lattices of graded ideals. In Section 7, we turn our attention to the irreducible representations of Steinberg algebras. We consider what we call Γ-aperiodic invariant subset of the groupoid $\mathcal{G}$ and construct graded simple $A_R(\mathcal{G})$-modules. This covers, as a special case, previous work done in the setting of Leavitt path algebras, and gives new results in the setting of Kumjian–Pask algebras. We describe the annihilator ideals of the graded modules over a Steinberg algebra and prove that these ideals reflect the effectiveness of the groupoid.

2. Graded rings and smash products

2A. *Graded rings.* Let Γ be a group with identity ε. A ring A (possibly without unit) is called a Γ-*graded ring* if $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ such that each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The group A_γ is called the γ-*homogeneous component* of A. When it is clear from context that a ring A is graded by group Γ, we simply say that A is a *graded ring*. If A is an algebra over a ring R, then A is called a *graded algebra* if A is a graded ring and A_γ is a R-submodule for any $\gamma \in \Gamma$. A Γ-graded ring $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$is called *strongly graded* if $A_\gamma A_\delta = A_{\gamma\delta}$ for all γ, δ in Γ.

The elements of $\bigcup_{\gamma\in\Gamma} A_\gamma$ in a graded ring A are called *homogeneous elements* of A. The nonzero elements of A_γ are called *homogeneous of degree* γ and we write $\deg(a) = \gamma$ for $a \in A_\gamma\backslash\{0\}$. The set

$\Gamma_A = \{\gamma \in \Gamma \mid A_\gamma \neq 0\}$ is called the *support* of A. We say that a Γ-graded ring A is *trivially graded* if the support of A is the trivial group $\{\varepsilon\}$ — that is, $A_\varepsilon = A$, so $A_\gamma = 0$ for $\gamma \in \Gamma \backslash \{\varepsilon\}$. Any ring admits a trivial grading by any group. If A is a Γ-graded ring and $s \in A$, then we write s_α, $\alpha \in \Gamma$ for the unique elements $s_\alpha \in A_\alpha$ such that $s = \sum_{\alpha \in \Gamma} s_\alpha$. Note that $\{\alpha \in \Gamma \mid s_\alpha \neq 0\}$ is finite for every $s \in A$.

We say a Γ-graded ring A has *graded local units* if for any finite set of homogeneous elements $\{x_1, \ldots, x_n\} \subseteq A$, there exists a homogeneous idempotent $e \in A$ such that $\{x_1, \ldots, x_n\} \subseteq eAe$. Equivalently, A has graded local units, if A_ε has local units and $A_\varepsilon A_\gamma = A_\gamma A_\varepsilon = A_\gamma$ for every $\gamma \in \Gamma$.

Let M be a left A-module. We say M is unital if $AM = M$ and it is Γ-*graded* if there is a decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ such that $A_\alpha M_\gamma \subseteq M_{\alpha\gamma}$ for all $\alpha, \gamma \in \Gamma$. We denote by A-Mod the category of unital left A-modules and by A-Gr the category of Γ-graded unital left A-modules with morphisms the A-module homomorphisms that preserve grading.

For a graded left A-module M, we define the α-*shifted* graded left A-module $M(\alpha)$ as

$$M(\alpha) = \bigoplus_{\gamma \in \Gamma} M(\alpha)_\gamma, \tag{2-1}$$

where $M(\alpha)_\gamma = M_{\gamma\alpha}$. That is, as an ungraded module, $M(\alpha)$ is a copy of M, but the grading is shifted by α. For $\alpha \in \Gamma$, the *shift functor*

$$\mathcal{T}_\alpha : A\text{-Gr} \to A\text{-Gr}, \quad M \mapsto M(\alpha)$$

is an isomorphism with the property $\mathcal{T}_\alpha \mathcal{T}_\beta = \mathcal{T}_{\alpha\beta}$ for $\alpha, \beta \in \Gamma$.

2B. ***Smash products.*** Let A be a Γ-graded unital R-algebra where Γ is a finite group. In an influential paper, Cohen and Montgomery [1984] introduced the smash product associated to A, denoted by $A\#R[\Gamma]^*$. They proved two main theorems, duality for actions and coactions, which related the smash product to the ring A. In turn, these theorems relate the graded structure of A to nongraded properties of A. The construction has been extended to the case of infinite groups (see for example [Beattie 1988; Liu and Van Oystaeyen 1988; Năstăsescu and Van Oystaeyen 2004, §7]). We need to adopt the construction of smash products for algebras with local units as the main algebras we will be concerned with are Steinberg algebras which are not necessarily unital but have local units. The main theorem of Section 3 shows that the Steinberg algebra of the skew-product of a groupoid by a group can be represented using the smash product construction (Theorem 3.4).

We start with a general definition of smash product for any ring.

Definition 2.1. For a Γ-graded ring A (possibly without unit), the *smash product* ring $A \# \Gamma$ is defined as the set of all formal sums $\sum_{\gamma \in \Gamma} r^{(\gamma)} p_\gamma$, where $r^{(\gamma)} \in A$ and p_γ are symbols. Addition is defined component-wise and multiplication is defined by linear extension of the rule $(rp_\alpha)(sp_\beta) = rs_{\alpha\beta^{-1}} p_\beta$, where $r, s \in A$ and $\alpha, \beta \in \Gamma$.

It is routine to check that $A \# \Gamma$ is a ring. We emphasise that the symbols p_γ do not belong to $A \# \Gamma$; however if the ring A has unit, then we regard the p_γ as elements of $A \# \Gamma$ by identifying $1_A p_\gamma$ with p_γ.

Each p_γ is then an idempotent element of $A \# \Gamma$. In this case $A \# \Gamma$ coincides with the ring $A \# \Gamma^*$ of [Beattie 1988]. If Γ is finite, then $A \# \Gamma$ is the same as the smash product $A \# k[\Gamma]^*$ of [Cohen and Montgomery 1984]. Note that $A \# \Gamma$ is always a Γ-graded ring with

$$(A \# \Gamma)_\gamma = \sum_{\alpha \in \Gamma} A_\gamma p_\alpha. \tag{2-2}$$

Next we define a shift functor on $A \# \Gamma$-Mod. This functor will coincide with the shift functor on A-Gr (see Proposition 2.5). This does not seem to be exploited in the literature and will be crucial in our study of K-theory of Leavitt path algebras (Section 5C).

For each $\alpha \in \Gamma$, there is an algebra automorphism

$$\mathcal{S}^\alpha : A \# \Gamma \to A \# \Gamma,$$

such that $\mathcal{S}^\alpha(sp_\beta) = sp_{\beta\alpha}$, for $sp_\beta \in A \# \Gamma$ with $s \in A$ and $\beta \in \Gamma$. We sometimes call $\mathcal{S}^\alpha$ the *shift map* associated to α. For $M \in A \# \Gamma$-Mod and $\alpha \in \Gamma$, we obtain a shifted $A \# \Gamma$-module $\mathcal{S}^\alpha_* M$ obtained by setting $\mathcal{S}^\alpha_* M := M$ as a group, and defining the left action by $a \cdot_{\mathcal{S}^\alpha_* M} m := \mathcal{S}^\alpha(a) \cdot_M m$. For $\alpha \in \Gamma$, the *shift functor*

$$\tilde{\mathcal{S}}^\alpha : A \# \Gamma\text{-Mod} \to A \# \Gamma\text{-Mod}, \quad M \mapsto \mathcal{S}^\alpha_* M,$$

is an isomorphism satisfying $\tilde{\mathcal{S}}_\alpha \tilde{\mathcal{S}}_\beta = \tilde{\mathcal{S}}_{\alpha\beta}$ for $\alpha, \beta \in \Gamma$.

If A is a unital ring then $A \# \Gamma$ has local units [Beattie 1988, Proposition 2.3]. We extend this to rings with graded local units.

Lemma 2.2. *Let A be a Γ-graded ring with graded local units. Then the ring $A \# \Gamma$ has graded local units.*

Proof. Take a finite subset $X = \{x_1, x_2, \dots, x_n\} \subseteq A \# \Gamma$ such that all x_i are homogeneous elements. Since homogeneous elements of $A \# \Gamma$ are sums of elements of the form rp_α for $r \in A$ a homogeneous element and $\alpha \in \Gamma$, we may assume that $x_i = r_i p_{\alpha_i}$, $1 \le i \le n$, where $r_i \in A$ are homogeneous of degree γ_i and $\alpha_i \in \Gamma$. Since A has graded local units, there exists a homogeneous idempotent $e \in A$ such that $er_i = r_i e = r_i$ for all i. Consider the finite set

$$Y = \big\{\gamma \in \Gamma \mid \gamma = \alpha_i \text{ or } \gamma = \gamma_i \alpha_i \quad \text{for } 1 \le i \le n\big\},$$

and let $w = \sum_{\gamma \in Y} ep_\gamma$. Since the idempotent $e \in A$ is homogeneous, w is a homogeneous element of $A \# \Gamma$. It is easy now to check that $w^2 = w$ and $wx_i = x_i = x_i w$ for all i. □

As we will see in Sections 5 and 6, smash products of Leavitt path algebras or of Kumjian–Pask algebras are ultramatricial algebras, which are very well-behaved. This allows us to obtain results about the path algebras via their smash product. For example, ultramatricial algebras are von Neumann regular rings. The following lemma allows us to exploit this property (see Theorems 6.4 and 6.5). Recall that a graded ring is called *graded von Neumann regular* if for any homogeneous element a, there is an element b such that $aba = a$.

Lemma 2.3. *Let A be a Γ-graded ring (possibly without unit). Then $A \# \Gamma$ is graded von Neumann regular if and only if A is graded von Neumann regular.*

Proof. Suppose $A \# \Gamma$ is graded regular and $a \in A_\gamma$, for some $\gamma \in G$. Since $ap_e \in (A \# \Gamma)_\gamma$ (see (2-2)), there is an element $\sum_{\alpha\in\Gamma} b_{\gamma_\alpha} p_\alpha \in (A \# \Gamma)_{\gamma^{-1}}$ with $\deg(b_{\gamma_\alpha}) = \gamma^{-1}$, $\alpha \in \Gamma$, such that

$$ap_e\Big(\sum_{\alpha\in\Gamma} b_{\gamma_\alpha} p_\alpha\Big)ap_e = ap_e.$$

This identity reduces to $ab_{\gamma_\gamma} ap_e = ap_e$. Thus $ab_{\gamma_\gamma} a = a$. This shows that A is graded regular.

Conversely, suppose A is graded regular and $x := \sum_{\alpha\in\Gamma} a_{\gamma_\alpha} p_\alpha \in (A\#\Gamma)_\gamma$. By (2-2) we have $\deg(a_{\gamma_\alpha}) = \gamma$, $\alpha \in \Gamma$. Then there are $b_{\gamma_\alpha^{-1}} \in A_{\gamma^{-1}}$ such that $a_{\gamma_\alpha} b_{\gamma_\alpha^{-1}} a_{\gamma_\alpha} = a_{\gamma_\alpha}$, for $\alpha \in \Gamma$. Consider the element $y := \sum_{\alpha\in\Gamma} b_{\gamma_\alpha^{-1}} p_{\gamma\alpha} \in (A \# \Gamma)_{\gamma^{-1}}$. One can then check that $xyx = x$. Thus $A \# \Gamma$ is graded regular. □

2C. *An isomorphism of module categories.* In this section we first prove that, for a Γ-graded ring A with graded local units, there is an isomorphism between the categories $A \# \Gamma$-Mod and A-Gr (Proposition 2.5). This is a generalisation of [Cohen and Montgomery 1984, Theorem 2.2; Beattie 1988, Theorem 2.6]. We check that the isomorphism respects the shifting in these categories. This in turn translates the shifting of modules in the category of graded modules to an action of the group on the category of modules for the smash-product. Since graded Steinberg algebras have graded local units, using this result and Theorem 3.4, we obtain a shift preserving isomorphism

$$A_R(\mathcal{G} \times_c \Gamma)\text{-Mod} \cong A_R(\mathcal{G})\text{-Gr}.$$

In Section 5 we will use this in the setting of Leavitt path algebras to establish an isomorphism between the category of graded modules of $L_R(E)$ and the category of modules of $L_R(\bar{E})$, where $\bar{E}$ is the covering graph of E (Section 5B). This yields a presentation of the monoid of graded finitely generated projective modules of a Leavitt path algebra.

We start with the following fact, which extends [Beattie 1988, Corollary 2.4] to rings with local units.

Lemma 2.4. *Let A be a Γ-graded ring with a set of graded local units E. A left $A \# \Gamma$-module M is unital if and only if for every finite subset F of M, there exists $w = \sum_{i=1}^n up_{\gamma_i}$ with $\gamma_i \in \Gamma$, and $u \in E$ such that $wx = x$ for all $x \in F$.*

Proof. Suppose that M is unital. Then each $m \in F$ may be written as $m = \sum_{n\in G_m} y_n n$ for some finite $G_m \subseteq M$ and choice of scalars $\{y_n \mid n \in G_m\} \subseteq A \# \Gamma$. Let $T := \bigcup_{m\in F} G_m$. By Lemma 2.2, there exists a finite set Y of Γ such that $w = \sum_{\gamma\in Y} up_\gamma$ satisfies $wy = y$ for all $y \in T$. So $wm = m$ for all $m \in F$.

Conversely, for $m \in M$, take $F = \{m\}$. Then there exists w such that $m = wm \in (A \# \Gamma)M$; that is, $(A \# \Gamma)M = M$. □

Proposition 2.5. *Let A be a Γ-graded ring with graded local units. Then there is an isomorphism of categories $\psi : A\text{-Gr} \to A \# \Gamma\text{-Mod}$ such that the following diagram commutes for every $\alpha \in \Gamma$.*

$$\begin{array}{ccc} A\text{-Gr} & \xrightarrow{\psi} & A\,\#\,\Gamma\text{-Mod} \\ \Big\downarrow{\scriptstyle \mathcal{T}_\alpha} & & \Big\downarrow{\scriptstyle \tilde{\mathcal{S}}^\alpha} \\ A\text{-Gr} & \xrightarrow{\psi} & A\,\#\,\Gamma\text{-Mod} \end{array} \tag{2-3}$$

Proof. We first define a functor $\phi : A\,\#\,\Gamma\text{-Mod} \to A\text{-Gr}$ as follows. Fix a set E of graded local units for A. Let M be a unital left $A\,\#\,\Gamma$-module. We view M as a Γ-graded left A-module M' as follows. For each $\gamma \in \Gamma$, define

$$M'_\gamma := \sum_{u \in E} u p_\gamma M.$$

We first show that for $\alpha \in \Gamma$, we have $M'_\alpha \cap \sum_{\gamma \in \Gamma, \gamma \neq \alpha} M'_\gamma = \{0\}$. Suppose this is not the case, so there exist finite index sets F and $\{F'_\gamma \mid \gamma \in \Gamma\}$ (only finitely many nonempty), elements $\{u_i \mid i \in F\}$ and $\{v_{\gamma,j} \mid \gamma \in \Gamma \text{ and } j \in F'_\gamma\}$ in E, and elements $\{m_i \mid i \in F\}$ and $\{n_{\gamma,j} \mid \gamma \in \Gamma \text{ and } j \in F'_\gamma\}$ such that

$$x = \sum_{i \in F} u_i p_\alpha m_i = \sum_{\gamma \in \Gamma,\, \gamma \neq \alpha} \sum_{j \in F'_\gamma} v_{\gamma,j} p_\gamma n_{\gamma,j},$$

Fix $e \in E$ such that $eu_i = u_i = u_i e$ for all $i \in F$. Using that the u_i are homogeneous elements of trivial degree at the second equality, we have

$$ep_\alpha x = \sum_{i \in F} (ep_\alpha u_i p_\alpha) m_i = \sum_{i \in F} eu_i p_\alpha m_i = x.$$

We also have

$$ep_\alpha x = \sum_{\gamma \in \Gamma \setminus \{\alpha\}} \sum_{j \in F'_\gamma} ep_\alpha v_{\gamma,j} p_\gamma n_{\gamma,j} = 0.$$

Hence $x = 0$.

For $r \in A_\gamma$ and $m \in M'_\alpha$, define $rm := rp_\alpha m$. This determines a left A-action on M'_α. For $u \in E$ satisfying $ur = r = ru$, we have

$$up_{\gamma\alpha} rm = (up_{\gamma\alpha} r p_\alpha) m = urp_\alpha m = rm.$$

Hence $rm \in M'_{\gamma\alpha}$. One can easily check the associativity of the A-action. Using Lemma 2.4 we see that $M = M'$ as sets. We claim that M' is a unital A-module. For $m \in M'_\gamma$, we write $m = \sum_{u \in E'} up_\gamma m_u$, where $E' \subseteq E$ is a finite set and $m_u \in M$. Since u is a homogeneous idempotent,

$$u(up_\gamma m_u) = up_\gamma (up_\gamma m_u) = up_\gamma m_u.$$

Thus $u(up_\gamma m_u) = up_\gamma m_u \in AM'$ implies that $m \in AM'$ showing that $M' = AM'$. We can therefore define

$$\phi : \mathrm{Obj}(A\,\#\,\Gamma\text{-Mod}) \to \mathrm{Obj}(A\text{-Gr}),$$

by $\phi(M) = M'$.

To define ϕ on morphisms, fix a morphism f in $A \# \Gamma$-Mod. For $m = \sum_{\gamma\in\Gamma} m_\gamma \in M'$ such that $m_\gamma = \sum_{u\in F_\gamma} up_\gamma m_u$ with F_γ a finite subset of E, we define $f' : M' \to N'$ by

$$f'(m_\gamma) = f\biggl(\sum_{u\in F_\gamma} up_\gamma m_u\biggr) = \sum_{u\in F_\gamma} up_\gamma f(m_u) = f(m)_\gamma. \tag{2-4}$$

To see that f' is an A-module homomorphism, fix $m \in M'_\gamma$ and $r \in A$. Since $f(m) \in M'_\gamma$, we have

$$f'(rm) = f(rp_\gamma m) = rp_\gamma f(m) = rf'(m).$$

The definition (2-4) shows that it preserves the gradings. That is, f' is a Γ-graded A-module homomorphism. So we can define ϕ on morphisms by $\phi(f) = f'$. It is routine to check that ϕ is a functor.

Next we define a functor $\psi : A\text{-Gr} \to A \# \Gamma\text{-Mod}$ as follows. Let $N = \bigoplus_{\gamma\in\Gamma} N_\gamma$ be a Γ-graded unital left A-module. Let N'' be a copy of N as a group. Fix $n \in N$, and write $n = \sum_{\gamma\in\Gamma} n_\gamma$. Fix $r \in A$ and $\alpha \in \Gamma$, and define

$$(rp_\alpha)n = rn_\alpha.$$

It is straightforward to check that this determines an associative left $A \# \Gamma$-action on N''. We claim that N'' is a unital $A \# \Gamma$-module. To see this, fix $n \in N''$. Since $AN = N$, we can express $n = \sum_{i=1}^{l} r_i n_i$, with the n_i homogeneous in N and the $r_i \in A$, and we can then write each r_i as $r_i = \sum_{\beta\in\Gamma} r_{i,\beta}$ as a sum of homogeneous elements $r_{i,\beta} \in A_\beta$. For any $\gamma \in \Gamma$,

$$n_\gamma = \sum_{i,\beta} r_{i,\beta}(n_i)_{\beta^{-1}\gamma} = \sum_{i,\beta}^{l} (r_{i,\beta} p_{\beta^{-1}\gamma}) n_i \in (A \# \Gamma)N''.$$

So we can define

$$\psi : \mathrm{Obj}(A\text{-Gr}) \to \mathrm{Obj}(A \# \Gamma\text{-Mod}),$$

by $\psi(N) = N''$. Since $\psi(N) = N''$ is just a copy of N as a module, we can define ψ on morphisms simply as the identity map; that is, if $f : M \to N$ is a homomorphism of graded A-modules, then for $m \in M$ we write m'' for the same element regarded as an element of M'', and we have $\psi(f)(m'') = f(m)''$. Again, it is straightforward to check that ψ is a functor.

To prove that $\psi\circ\phi = \mathrm{Id}_{A\#\Gamma\text{-Mod}}$ and $\phi\circ\psi = \mathrm{Id}_{A\text{-Gr}}$, it suffices to show that $(M')'' = M$ for $M \in A\#\Gamma$-Mod and $(N'')' = N$ for $N \in A$-Gr; but this is straightforward from the definitions.

To prove the commutativity of the diagram in (2-3), it suffices to show that the $A \# \Gamma$-actions on $(\psi \circ \mathcal{T}_\alpha)(N) = N(\alpha)''$ and $(\tilde{\mathcal{S}}^\alpha \circ \psi)(N) = N''(\alpha)$ coincide for any $N \in A$-Gr. Take any $n \in N$ and $sp_\beta \in A \# \Gamma$ with $s \in A$ and $\beta \in \Gamma$. For $n \in N''(\alpha)$ and a typical spanning element sp_β of $A \# \Gamma$, we have $(sp_\beta)n = (sp_{\beta\alpha})n = sn_{\beta\alpha}$. On the other hand, for the same n regarded as an element of N'', and the same $sp_\beta \in A\#\Gamma$, we have $(sp_\beta)n = sn'_\beta = sn_{\beta\alpha}$. Since $N(\alpha)_\beta = N_{\beta\alpha}$ by definition, this completes the proof. $\square$

3. The Steinberg algebra of the skew-product

In this section, we consider the skew-product of an ample groupoid $\mathcal{G}$ carrying a grading by a discrete group Γ. We prove that the Steinberg algebra of the skew-product is graded isomorphic to the smash product by Γ of the Steinberg algebra associated to $\mathcal{G}$. This result will be used in Section 5 to study the category of graded modules over Leavitt path algebras and give a representation of the graded finitely generated projective modules.

3A. ***Graded groupoids.*** A groupoid is a small category in which every morphism is invertible. It can also be viewed as a generalisation of a group which has partial binary operation. Let $\mathcal{G}$ be a groupoid. If $x \in \mathcal{G}$, $d(x) = x^{-1}x$ is the *domain* of x and $r(x) = xx^{-1}$ is its *range*. The pair (x, y) is composable if and only if $r(y) = d(x)$. The set $\mathcal{G}^{(0)} := d(\mathcal{G}) = r(\mathcal{G})$ is called the *unit space* of $\mathcal{G}$. Elements of $\mathcal{G}^{(0)}$ are units in the sense that $xd(x) = x$ and $r(x)x = x$ for all $x \in \mathcal{G}$. For $U, V \in \mathcal{G}$, we define

$$UV = \big\{\alpha\beta \mid \alpha \in U,\ \beta \in V \text{ and } r(\beta) = d(\alpha)\big\}.$$

A topological groupoid is a groupoid endowed with a topology under which the inverse map is continuous, and such that composition is continuous with respect to the relative topology on $\mathcal{G}^{(2)} := \{(\beta, \gamma) \in \mathcal{G} \times \mathcal{G} \mid d(\beta) = r(\gamma)\}$ inherited from $\mathcal{G} \times \mathcal{G}$. An *étale* groupoid is a topological groupoid $\mathcal{G}$ such that the domain map d is a local homeomorphism. In this case, the range map r is also a local homeomorphism. An *open bisection* of $\mathcal{G}$ is an open subset $U \subseteq \mathcal{G}$ such that $d|_U$ and $r|_U$ are homeomorphisms onto an open subset of $\mathcal{G}^{(0)}$. We say that an étale groupoid $\mathcal{G}$ is *ample* if there is a basis consisting of compact open bisections for its topology.

Let Γ be a discrete group and $\mathcal{G}$ a topological groupoid. A Γ-grading of $\mathcal{G}$ is a continuous function $c : \mathcal{G} \to \Gamma$ such that $c(\alpha)c(\beta) = c(\alpha\beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$; such a function c is called a *cocycle* on $\mathcal{G}$. In this paper, we shall also refer to c as the *degree map* on $\mathcal{G}$. Observe that $\mathcal{G}$ decomposes as a topological disjoint union $\bigsqcup_{\gamma\in\Gamma} c^{-1}(\gamma)$ of subsets satisfying $c^{-1}(\beta)c^{-1}(\gamma) \subseteq c^{-1}(\beta\gamma)$. We say that $\mathcal{G}$ is *strongly graded* if $c^{-1}(\beta)c^{-1}(\gamma) = c^{-1}(\beta\gamma)$ for all β, γ. For $\gamma \in \Gamma$, we say that $X \subseteq \mathcal{G}$ is γ-graded if $X \subseteq c^{-1}(\gamma)$. We always have $\mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$, so $\mathcal{G}^{(0)}$ is ε-graded. We write $B^{\mathrm{co}}_\gamma(\mathcal{G})$ for the collection of all γ-graded compact open bisections of $\mathcal{G}$ and

$$B^{\mathrm{co}}_*(\mathcal{G}) = \bigcup_{\gamma\in\Gamma} B^{\mathrm{co}}_\gamma(\mathcal{G}).$$

Throughout this note we only consider Γ-graded ample Hausdorff groupoids.

3B. ***Steinberg algebras.*** Steinberg algebras were introduced in [Steinberg 2010] in the context of discrete inverse semigroup algebras and independently in [Clark et al. 2014] as a model for Leavitt path algebras. We recall the notion of the Steinberg algebra as a universal algebra generated by certain compact open subsets of an ample Hausdorff groupoid.

Definition 3.1. Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid and $B^{\mathrm{co}}_*(\mathcal{G}) = \bigcup_{\gamma\in\Gamma} B^{\mathrm{co}}_\gamma(\mathcal{G})$ the collection of all graded compact open bisections. Given a commutative ring R with identity, the *Steinberg R-algebra* associated to $\mathcal{G}$, denoted $A_R(\mathcal{G})$, is the algebra generated by the set $\{t_B \mid B \in B_*^{\mathrm{co}}(\mathcal{G})\}$ with coefficients in R, subject to

(R1) $t_\varnothing = 0$;

(R2) $t_B t_D = t_{BD}$ for all $B, D \in B_*^{\mathrm{co}}(\mathcal{G})$; and

(R3) $t_B + t_D = t_{B\cup D}$, whenever B and D are disjoint elements of $B_\gamma^{\mathrm{co}}(\mathcal{G})$ for some $\gamma \in \Gamma$ such that $B \cup D$ is a bisection.

Every element $f \in A_R(\mathcal{G})$ can be expressed as $f = \sum_{U\in F} a_U t_U$, where F is a finite subset of elements of $B_*^{\mathrm{co}}(\mathcal{G})$. It was proved in [Clark and Edie-Michell 2015, Proposition 2.3] (see also [Clark et al. 2014, Theorem 3.10]) that the Steinberg algebra defined above is isomorphic to the following construction:

$$A_R(\mathcal{G}) = \mathrm{span}\big\{1_U \mid U \text{ is a compact open bisection of } \mathcal{G}\big\},$$

where $1_U : \mathcal{G} \to R$ denotes the characteristic function on U. Equivalently, if we give R the discrete topology, then continuous functions from $\mathcal{G}$ to R are exactly locally constant functions from $\mathcal{G}$ to R, and so $A_R(\mathcal{G}) = C_c(\mathcal{G}, R)$, the space of compactly supported continuous functions from $\mathcal{G}$ to R. Addition is point-wise and multiplication is given by convolution

$$(f * g)(\gamma) = \sum_{\{\alpha\beta=\gamma\}} f(\alpha)g(\beta).$$

It is useful to note that

$$1_U * 1_V = 1_{UV}$$

for compact open bisections U and V (see [Steinberg 2010, Proposition 4.5(3)]) and the isomorphism between the two constructions is given by $t_U \mapsto 1_U$ on the generators. By [Clark and Edie-Michell 2015, Lemma 2.2; Clark et al. 2014, Lemma 3.5], every element $f \in A_R(\mathcal{G})$ can be expressed as

$$f = \sum_{U\in F} a_U 1_U, \tag{3-1}$$

where F is a finite subset of mutually disjoint elements of $B_*^{\mathrm{co}}(\mathcal{G})$.

Recall from [Clark and Sims 2015, Lemma 3.1] that if $c : \mathcal{G} \to \Gamma$ is a cocycle into a discrete group Γ, then the Steinberg algebra $A_R(\mathcal{G})$ is a Γ-graded algebra with homogeneous components

$$A_R(\mathcal{G})_\gamma = \{f \in A_R(\mathcal{G}) \mid \mathrm{supp}(f) \subseteq c^{-1}(\gamma)\}.$$

The family of all idempotent elements of $A_R(\mathcal{G}^{(0)})$ is a set of local units for $A_R(\mathcal{G})$ ([Clark et al. 2017, Lemma 2.6]). Here, $A_R(\mathcal{G}^{(0)}) \subseteq A_R(\mathcal{G})$ is a subalgebra. Since $\mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$ is trivially graded, $A_R(\mathcal{G})$ is a graded algebra with graded local units. Note that any ample Hausdorff groupoid admits the trivial cocycle from $\mathcal{G}$ to the trivial group $\{\varepsilon\}$, which gives rise to a trivial grading on $A_R(\mathcal{G})$.

3C. ***Skew-products.*** Let $\mathcal{G}$ be an ample Hausdorff groupoid, Γ a discrete group, and $c : \mathcal{G} \to \Gamma$ a cocycle. Then $\mathcal{G}$ admits a basis $\mathcal{B}$ of compact open bisections. Replacing $\mathcal{B}$ with $\mathcal{B}' = \{U \cap c^{-1}(\gamma) \mid U \in \mathcal{B}, \gamma \in \Gamma\}$, we obtain a basis of compact open homogeneous bisections.

To a Γ-graded groupoid $\mathcal{G}$ one can associate a groupoid called the skew-product of $\mathcal{G}$ by Γ. The aim of this section is to relate the Steinberg algebra of the skew-product groupoid to the Steinberg algebra of $\mathcal{G}$. We recall the notion of skew-product of a groupoid (see [Renault 1980, Definition 1.6]).

Definition 3.2. Let $\mathcal{G}$ be an ample Hausdorff groupoid, Γ a discrete group and $c : \mathcal{G} \to \Gamma$ a cocycle. The *skew-product* of $\mathcal{G}$ by Γ is the groupoid $\mathcal{G} \times_c \Gamma$ such that (x, α) and (y, β) are composable if x and y are composable and $\alpha = c(y)\beta$. The composition is then given by $(x, c(y)\beta)(y, \beta) = (xy, \beta)$ with the inverse $(x, \alpha)^{-1} = (x^{-1}, c(x)\alpha)$.

Note that our convention for the composition of the skew-product here is slightly different from that in [Renault 1980, Definition 1.6]. The two determine isomorphic groupoids, but when we establish the isomorphism of Theorem 3.4, the composition formula given here will be more obviously compatible with the multiplication in the smash product.

Lemma 3.3. *Let $\mathcal{G}$ be a Γ-graded ample groupoid. Then the skew-product $\mathcal{G} \times_c \Gamma$ is a Γ-graded ample groupoid under the product topology on $\mathcal{G} \times \Gamma$ and with degree map $\tilde{c}(x, \gamma) := c(x)$.*

Proof. We can directly check that under the product topology on $\mathcal{G} \times \Gamma$, the inverse and composition of the skew-product $\mathcal{G} \times_c \Gamma$ are continuous making it a topological groupoid. Since the domain map $d : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism, the domain map (also denoted d) from $\mathcal{G} \times_c \Gamma$ to $\mathcal{G}^{(0)} \times \Gamma$ is $d \times \mathrm{id}_\Gamma$ so restricts to a homeomorphism on $U \times \Gamma$ for any set U on which d is a homeomorphism. So $d : \mathcal{G} \times_c \Gamma \to (\mathcal{G} \times_c \Gamma)^{(0)}$ is a local homeomorphism. Since the inverse map is clearly a homeomorphism, it follows that the range map is also a local homeomorphism.

If $\mathcal{B}$ is a basis of compact open bisections for $\mathcal{G}$, then $\{B \times \{\gamma\} \mid B \in \mathcal{B} \text{ and } \gamma \in \Gamma\}$ is a basis of compact open bisections for the topology on $\mathcal{G} \times_c \Gamma$. Since composition on $\mathcal{G} \times_c \Gamma$ agrees with composition in $\mathcal{G}$ in the first coordinate, it is clear that $\tilde{c}$ is a cocycle. □

The Steinberg algebra $A_R(\mathcal{G} \times_c \Gamma)$ associated to $\mathcal{G} \times_c \Gamma$ is a Γ-graded algebra, with homogeneous components

$$A_R(\mathcal{G} \times_c \Gamma)_\gamma = \big\{ f \in A_R(\mathcal{G} \times_c \Gamma) \mid \operatorname{supp}(f) \subseteq c^{-1}(\gamma) \times \Gamma \big\},$$

for $\gamma \in \Gamma$.

We are in a position to state the main result of this section.

Theorem 3.4. *Let $\mathcal{G}$ be a Γ-graded ample, Hausdorff groupoid and R a unital commutative ring. Then there is an isomorphism of Γ-graded algebras $A_R(\mathcal{G} \times_c \Gamma) \cong A_R(\mathcal{G}) \# \Gamma$, assigning $1_{U \times \{\alpha\}}$ to $1_U p_\alpha$ for each compact open bisection U of $\mathcal{G}$ and $\alpha \in \Gamma$.*

Proof. We first define a representation $\{t_U \mid U \in B_*^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)\}$ in the algebra $A_R(\mathcal{G}) \# \Gamma$ (see Definition 3.1). If U is a graded compact open bisection of $\mathcal{G} \times_c \Gamma$, say $U \subseteq \tilde{c}^{-1}(\alpha)$, then for each $\gamma \in \Gamma$, the set $U \cap \mathcal{G} \times \{\gamma\}$ is a compact open bisection. Since these are mutually disjoint and U is compact, there are finitely many (distinct) $\gamma_1, \dots, \gamma_l \in \Gamma$ such that $U = \bigsqcup_{i=1}^{l} U \cap \mathcal{G} \times \{\gamma_i\}$. Each $U \cap \mathcal{G} \times \{\gamma_i\}$ has the form $U_i \times \{\gamma_i\}$ where $U_i \subseteq U$ is compact open. The U_i have mutually disjoint sources because the domain map on $\mathcal{G} \times_c \Gamma$ is just $d \times \mathrm{id}$, and U is a bisection. So each $U_i \in B_\alpha^{\mathrm{co}}(\mathcal{G})$, and $U = \bigsqcup_{i=1}^{l} U_i \times \{\gamma_i\}$. Using this decomposition, we define

$$t_U = \sum_{i=1}^{l} 1_{U_i} p_{\gamma_i}.$$

We show that these elements t_U satisfy (R1)–(R3). Certainly if $U = \varnothing$, then $t_U = 0$, giving (R1). For (R2), take $V \in B_\beta^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)$, and decompose $V = \bigcup_{j=1}^{m} V_j \times \{\gamma_j'\}$ as above. Then

$$\begin{aligned} t_U t_V &= \sum_{i=1}^{l} 1_{U_i} p_{\gamma_i} \sum_{j=1}^{m} 1_{V_j} p_{\gamma_j'} = \sum_{i=1}^{l} \sum_{j=1}^{m} 1_{U_i} p_{\gamma_i} 1_{V_j} p_{\gamma_j'} = \sum_{j=1}^{m} \sum_{i=1}^{l} 1_{U_i} p_{\gamma_i} 1_{V_j} p_{\gamma_j'} \\ &= \sum_{j=1}^{m} \sum_{\{1 \le i \le l \,|\, \gamma_i (\gamma_j')^{-1} = \beta\}} 1_{U_i V_j} p_{\gamma_j'}. \end{aligned} \tag{3-2}$$

On the other hand, by the composition of the skew-product $\mathcal{G} \times_c \Gamma$, we have

$$\begin{aligned} UV &= \bigcup_{i=1}^{l} \bigcup_{j=1}^{m} U_i \times \{\gamma_i\} \cdot V_j \times \{\gamma_j'\} = \bigcup_{j=1}^{m} \bigcup_{i=1}^{l} U_i \times \{\gamma_i\} \cdot V_j \times \{\gamma_j'\} \\ &= \bigcup_{j=1}^{m} \bigcup_{\{1 \le i \le l \,|\, \gamma_i = \beta \gamma_j'\}} U_i V_j \times \{\gamma_j'\}. \end{aligned}$$

For each $1 \le j \le m$, there exists at most one $1 \le i \le l$ such that $\gamma_i = \beta\gamma_j'$ and $U_i V_j \in B_{\alpha\beta}^{\mathrm{co}}(\mathcal{G})$. It follows that $t_{UV} = \sum_{j=1}^{m} \sum_{\{1 \le i \le l \,|\, \gamma_i (\gamma_j')^{-1} = \beta\}} 1_{U_i V_j} p_{\gamma_j'}$. Comparing this with (3-2), we obtain $t_U t_V = t_{UV}$.

To check (R3), suppose that U and V are disjoint elements of $B_\omega^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)$ for some $\omega \in \Gamma$ such that $U \cup V$ is a bisection of $\mathcal{G} \times_c \Gamma$. Write them as $U = \bigcup_{i=1}^{l} U_i \times \{\gamma_i\}$ and $V = \bigcup_{j=1}^{m} V_i \times \{\gamma_j'\}$ as above. We have

$$t_U + t_V = \sum_{i=1}^{l} 1_{U_i} p_{\gamma_i} + \sum_{j=1}^{m} 1_{V_j} p_{\gamma_j'}.$$

On the other hand

$$U \cup V = \Bigl(\bigcup_{i=1}^{l} U_i \times \{\gamma_i\}\Bigr) \cup \Bigl(\bigcup_{j=1}^{m} V_i \times \{\gamma_j'\}\Bigr).$$

If $\gamma_i = \gamma_j'$, then $U_i \times \{\gamma_i\} \cup V_j \times \{\gamma_j'\} = (U_i \cup V_j) \times \{\gamma_i\}$. Since U and V are disjoint and $U \cup V$ is a bisection, we deduce that $r(U_i) \cap r(V_j) = \varnothing = d(U_i) \cap d(V_j)$ so that $U_i \cup V_j$ is a bisection. So

$$t_{U_i \times \{\gamma_i\} \cup V_j \times \{\gamma_j'\}} = t_{U_i \cup V_j \times \{\gamma_i\}} = 1_{U_i \cup V_j} p_{\gamma_i} = 1_{U_i} p_{\gamma_i} + 1_{V_j} p_{\gamma_i} = 1_{U_i} p_{\gamma_i} + 1_{V_j} p_{\gamma_j'}.$$

This shows that after combining pairs where $\gamma_i = \gamma_j'$ as above, we obtain $t_U + t_V = t_{U \cup V}$.

By the universality of Steinberg algebras, we have an R-homomorphism,

$$\phi : A_R(\mathcal{G} \times_c \Gamma) \to A_R(\mathcal{G}) \# \Gamma$$

such that $\phi(1_{U\times\{\alpha\}}) = 1_U p_\alpha$ for each compact open bisection U of $\mathcal{G}$ and $\alpha \in \Gamma$. From the definition of ϕ, it is evident that ϕ preserves the grading. Hence, ϕ is a homomorphism of Γ-graded algebras.

Next we prove that ϕ is an isomorphism. For any element $ap_\gamma \in A_R(\mathcal{G}) \# \Gamma$ with $a \in A_R(\mathcal{G})$ and $\gamma \in \Gamma$, there is a finite index set T, elements $\{r_i \mid i \in T\}$ of R, and compact open bisections $K_i \in B_*^{\mathrm{co}}(\mathcal{G})$ such that

$$ap_\gamma = \sum_{i\in T} r_i 1_{K_i} p_\gamma = \sum_{i\in T} r_i \phi(1_{K_i\times\{\gamma\}}) \in \operatorname{Im}\phi.$$

So ϕ is surjective. It remains to prove that ϕ is injective. Take an element $x \in A_R(\mathcal{G} \times_c \Gamma)$ such that $\phi(x) = 0$. Since ϕ is graded, we can assume that x is homogeneous, say $x \in A_R(\mathcal{G} \times_c \Gamma)_\gamma$. By (3-1), there is a finite set F, mutually disjoint $B_i \in B_\gamma^{\mathrm{co}}(\mathcal{G} \times_c \Gamma)$ indexed by $i \in F$ and coefficients $r_i \in R$ indexed by $i \in F$ such that

$$x = \sum_{i\in F} r_i 1_{B_i}.$$

For each B_i, we write $B_i = \bigcup_{k\in F_i} B_{ik} \times \{\delta_{ik}\}$ such that F_i is a finite set and the δ_{ik} indexed by $k \in F_i$ are distinct. Set

$$\Delta = \{\delta_{ik} \mid i \in F, k \in F_i\}.$$

For each $\delta \in \Delta$, let $F_\delta \subseteq F$ be the collection $F_\delta = \big\{i \in F \mid \delta \in \{\delta_{ik} \mid k \in F_i\}\big\}$. Then

$$\phi(x) = \sum_{i\in F} r_i \phi(1_{B_i}) = \sum_{i\in F}\sum_{k\in F_i} r_i 1_{B_{ik}} p_{\delta_{ik}} = \sum_{\delta\in\Delta}\sum_{i\in F_\delta} r_i 1_{B_{i,k(\delta)}} p_\delta = 0.$$

For any $\delta \in \Delta$, we obtain $\sum_{i\in F_\delta} r_i 1_{B_{i,k(\delta)}} = 0$. Since the B_i are mutually disjoint, for any element $g \in \mathcal{G}$, we have

$$\Bigg(\sum_{i\in F_\delta} r_i 1_{B_{i,k(\delta)}}\Bigg)(g) = \begin{cases} r_i & \text{if } g \in B_{i,k(\delta)} \text{ for some } i \in F_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then $r_i = 0$ for any $i \in F_\delta$, giving $x = 0$. □

3D. ***C*-algebras and crossed-products.*** In the groupoid-C^*-algebra literature, it is well-known that if $\mathcal{G}$ is a Γ-graded groupoid, and Γ is abelian, then the C^*-algebra $C^*(\mathcal{G} \times \Gamma)$ of the skew-product groupoid is isomorphic to the crossed product C^*-algebra $C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$, where α^c is the action of the Pontryagin dual $\widehat{\Gamma}$ such that $\alpha^c_\chi(f)(g) = \chi(c(g))f(g)$ for $f \in C_c(\mathcal{G})$, $\chi \in \widehat{\Gamma}$, and $g \in \mathcal{G}$. This extends to nonabelian Γ via the theory of C^*-algebraic coactions.

In this subsection, we reconcile this result with Theorem 3.4 by showing that there is a natural embedding of $A_{\mathbb{C}}(\mathcal{G}) \# \Gamma$ into $C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ when Γ is abelian.

Lemma 3.5. *Suppose that Γ is a discrete abelian group and that $\mathcal{G}$ is a Γ-graded groupoid with grading cocycle $c : \mathcal{G} \to \Gamma$. For $a \in A_{\mathbb{C}}(\mathcal{G})$ and $\gamma \in \Gamma$, define $a \cdot \hat{\gamma} \in C(\widehat{\Gamma}, C^*(\mathcal{G})) \subseteq C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ by*

$$(a \cdot \hat{\gamma})(\chi) = \chi(\gamma)a.$$

Then there is a homomorphism $A_{\mathbb{C}}(\mathcal{G}) \# \Gamma \hookrightarrow C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ that carries ap_γ to $a \cdot \hat{\gamma}$.

Proof. The multiplication in the crossed-product C^*-algebra is given on elements of $C(\widehat{\Gamma}, C^*(\mathcal{G}))$ by $(F * G)(\chi) = \int_{\widehat{\Gamma}} F(\rho)\alpha^c_\rho(G(\rho^{-1}\chi))\, d\mu(\rho)$, where μ is Haar measure on $\widehat{\Gamma}$.

The action of $\widehat{\Gamma}$ induces a Γ-grading of $C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ such that for $a \in C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ and $\gamma \in \Gamma$, the corresponding homogeneous component a_γ of a is given by

$$a_\gamma = \int_{\widehat{\Gamma}} \overline{\chi(\gamma)}\alpha^c_\chi(a)\, d\mu(\chi).$$

There is certainly a linear map $i : A_{\mathbb{C}}(\mathcal{G}) \# \Gamma \to C^*(\mathcal{G}) \times_{\alpha^c} \widehat{\Gamma}$ satisfying $i(ap_\gamma) = a \cdot \hat{\gamma}$; we just have to check that it is multiplicative. For this, fix $a, b \in A_{\mathbb{C}}(\mathcal{G})$ and $\gamma, \beta \in \Gamma$ and $\chi \in \widehat{\Gamma}$, and calculate

$$\begin{aligned}\big(i(ap_\gamma)i(bp_\beta)\big)(\chi) &= \int_{\widehat{\Gamma}} i(ap_\gamma)(\rho)\alpha^c_\rho(i(bp_\beta)(\rho^{-1}\chi))\, d\mu(\rho) = \int_{\widehat{\Gamma}} a \cdot \hat{\gamma}(\rho)\alpha^c_\rho(b\hat{\beta}(\rho^{-1}\chi))\, d\mu(\rho)\\ &= \int_{\widehat{\Gamma}} \rho(\gamma)a(\rho^{-1}\chi)(\beta)\alpha^c_\rho(b)\, d\mu(\rho) = \chi(\beta)a \int_{\widehat{\Gamma}} \overline{\rho(\gamma^{-1}\beta)}\alpha^c_\rho(b)\, d\mu(\rho)\\ &= \chi(\beta)ab_{\gamma^{-1}\beta} = (ab_{\gamma^{-1}\beta}) \cdot \hat{\beta} = i(ab_{\gamma^{-1}\beta}p_\beta) = i(ap_\gamma bp_\beta).\end{aligned}$$

So i is multiplicative as required. □

4. Nonstable graded K-theory

For a unital ring A, we denote by $\mathcal{V}(A)$ the abelian monoid of isomorphism classes of finitely generated projective left A-modules under direct sum. In general for an abelian monoid M and elements $x, y \in M$, we write $x \le y$ if $y = x + z$ for some $z \in M$. An element $d \in M$ is called *distinguished* (or an *order unit*) if for any $x \in M$, we have $x \le nd$ for some $n \in \mathbb{N}$. A monoid is called *conical*, if $x + y = 0$ implies $x = y = 0$. Clearly $\mathcal{V}(A)$ is conical with a distinguished element $[A]$. For a finitely generated conical abelian monoid M containing a distinguished element d, Bergman constructed a "universal" K-algebra B (here K is a field) for which there is an isomorphism $\phi : \mathcal{V}(B) \to M$, such that $\phi([B]) \to d$ ([Bergman 1974, Theorem 6.2]).

For a (finite) directed graph E, one defines an abelian monoid M_E generated by the vertices, identifying a vertex with the sum of vertices connected to it by edges (see Section 5C). The Bergman universal algebra associated to this monoid (with the sum of vertices as a distinguished element) is the Leavitt path algebra $L_K(E)$ associated to the graph E, i.e., $\mathcal{V}(L_K(E)) \cong M_E$. Leavitt path algebras of directed graphs have been studied intensively since their introduction [Abrams and Aranda Pino 2005; Ara et al. 2007]. The classification of such algebras is still a major open topic and one would like to find a complete invariant for such algebras. Due to the success of K-theory in the classification of graph C^*-algebras [Phillips 2000], one would hope that the Grothendieck group K_0 with relevant ingredients might act as a complete invariant for Leavitt path algebras; particularly since $K_0(L_K(E))$ is the group completion of $\mathcal{V}(L_K(E))$. However, unless the graph consists of only cycles with no exit, $\mathcal{V}(L_K(E))$ is not a cancellative monoid (Lemma 5.5) and thus $\mathcal{V}(L_K(E)) \to K_0(L_K(E))$ is not injective, reflecting that K_0 might not capture all the properties of $L_K(E)$.

For a graded ring A one can consider the abelian monoid of isomorphism classes of graded finitely generated projective modules denoted by $\mathcal{V}^{\mathrm{gr}}(A)$. Since a Leavitt path algebra has a canonical $\mathbb{Z}$-graded structure, one can consider $\mathcal{V}^{\mathrm{gr}}(L_K(E))$. One of the main aims of this paper is to show that the graded monoid carries substantial information about the algebra.

In Sections 5 and 6 we will use the results on smash products obtained in Section 3 to study the graded monoid of Leavitt path algebras and Kumjian–Pask algebras. In this section we collect the facts we need on the graded monoid of a graded ring with graded local units.

4A. ***The monoid of a graded ring with graded local units.*** For a ring A with unit, the monoid $\mathcal{V}(A)$ is defined as the set of isomorphism classes $[P]$ of finitely generated projective A-modules P, with addition given by $[P]+[Q]=[P\oplus Q]$.

For a nonunital ring A, we consider a unital ring $\tilde{A}$ containing A as a two-sided ideal and define

$$\mathcal{V}(A)=\big\{[P]\mid P \text{ is a finitely generated projective } \tilde{A}\text{-module and } P=AP\big\}. \tag{4-1}$$

This construction does not depend on the choice of $\tilde{A}$, as can be seen from the following alternative description: $\mathcal{V}(A)$ is the set of equivalence classes of idempotents in $M_\infty(A)$, where $e\sim f$ in $M_\infty(A)$ if and only if there are $x, y\in M_\infty(A)$ such that $e=xy$ and $f=yx$ ([Menal and Moncasi 1987, p. 296]).

When A has local units,

$$\mathcal{V}(A)=\big\{[P]\mid P \text{ is a finitely generated projective } A\text{-module in } A\text{-Mod}\big\}. \tag{4-2}$$

To see this, recall that the *unitisation ring* $\tilde{A}$ of a ring A is a copy of $\mathbb{Z}\times A$ with componentwise addition, and with multiplication given by

$$(n,a)(m,b)=(nm, ma+nb+ab) \quad \text{for all } n,m\in\mathbb{Z} \text{ and } a,b\in A.$$

The forgetful functor provides a category isomorphism from $\tilde{A}$-Mod to the category of arbitrary left A-modules [Faith 1973, Proposition 8.29B]. Any A-module N can be viewed as a $\tilde{A}$-module via $(m,b)x=mx+bx$ for $(m,b)\in\tilde{A}$ and $x\in N$. By [Ara and Goodearl 2012, Lemma 10.2], the projective objects in A-Mod are precisely those which are projective as $\tilde{A}$-modules; that is, the projective $\tilde{A}$-modules P such that $AP=P$. Any finitely generated $\tilde{A}$-module M with $AM=M$ is a finitely generated A-module. In fact, suppose that M is generated as an $\tilde{A}$-module by $x_1,\dots,x_n$. Since $AM=M$, each x_i can be written as $x_i=\sum_{j=1}^{t_i} b_j x_{ij}$ for some $b_j\in A$ and $x_{ij}\in M$. Now any $m\in M$ can be written

$$m=\sum_{i=1}^{n} a_i x_i=\sum_{i=1}^{n}\sum_{j=1}^{t_i} a_i b_j x_{ij}$$

So $\{x_{ij} \mid 1 \le i \le n \text{ and } 1 \le j \le t_i\}$ generates M as an A-module. Clearly any finitely generated A-module is a finitely generated $\tilde{A}$-module. So the definitions of $\mathcal{V}(A)$ in (4-1) and (4-2) coincide.

We need a graded version of (4-2) as this presentation will be used to study the monoid associated to the Leavitt path algebras of arbitrary graphs.

Recall that for a group Γ and a Γ-graded ring A with unit, the monoid $\mathcal{V}^{\mathrm{gr}}(A)$ consists of isomorphism classes $[P]$ of graded finitely generated projective A-modules with the direct sum $[P]+[Q]=[P\oplus Q]$ as the addition operation.

For a nonunital graded ring A, a similar construction as in (4-1) can be carried over to the graded setting (see [Hazrat 2016, §3.5]). Let $\tilde{A}$ be a Γ-graded ring with identity such that A is a graded two-sided ideal of A. For example, consider $\tilde{A}=\mathbb{Z}\times A$. Then $\tilde{A}$ is Γ-graded with

$$\tilde{A}_0=\mathbb{Z}\times A_0, \qquad \text{and} \qquad \tilde{A}_\gamma = 0\times A_\gamma \quad \text{for } \gamma\neq 0.$$

Define

$$\mathcal{V}^{\mathrm{gr}}(A)=\big\{[P] \mid P \text{ is a graded finitely generated projective } \tilde{A}\text{-module and } AP=P\big\}, \tag{4-3}$$

where $[P]$ is the class of graded $\tilde{A}$-modules, graded isomorphic to P, and addition is defined via direct sum. Then $\mathcal{V}^{\mathrm{gr}}(A)$ is isomorphic to the monoid of equivalence classes of graded idempotent matrices over A [Hazrat 2016, p. 146].

Let A be a Γ-graded ring with graded local units. We will show that

$$\mathcal{V}^{\mathrm{gr}}(A)=\big\{[P] \mid P \text{ is a graded finitely generated projective } A\text{-module in } A\text{-Gr}\big\}. \tag{4-4}$$

For this we need to relate the graded projective modules to modules which are projective. A graded A-module P in A-Gr is called a graded projective A-module if for any epimorphism $\pi: M\to N$ of graded A-modules in A-Gr and any morphism $f: P\to N$ of graded A-modules in A-Gr, there exists a morphism $h: P\to M$ of graded A-modules such that $\pi\circ h=f$.

In the case of unital rings, a module is graded projective if and only if it is graded and projective [Hazrat 2016, Proposition 1.2.15]. We need a similar statement in the setting of rings with local units.

Lemma 4.1. *Let A be a Γ-graded ring with graded local units and P a graded unital left A-module. Then P is a graded projective left A-module in A-*Gr *if and only if P is a graded left A-module which is projective in A-*Mod*.*

Proof. First suppose that P is a graded projective A-module in A-Gr. It suffices to prove that P is projective in A-Mod. For any homogeneous element $p\in P$ of degree δ_p, there exists a homogeneous idempotent $e_p\in A$ such that $e_p p=p$. Let $\bigoplus_{p\in P^h} Ae_p(-\delta_p)$ be the direct sum of graded A-modules where $\deg(e_p)=\delta_p$ and P^h is the set of homogeneous elements of P. Then there exists a surjective graded A-module homomorphism

$$f:\bigoplus_{p\in P^h} Ae_p(-\delta_p)\to P$$

such that $f(ae_p) = ae_p p = ap$ for $a \in Ae_p$. Since P is graded projective, there exists a graded A-module homomorphism $g : P \to \bigoplus_{p \in P^h} Ae_p(-\delta_p)$ such that $fg = \mathrm{Id}_P$. Forgetting the grading, P is a direct summand of $\bigoplus_{p \in P^h} Ae_p$ as an A-module. By [Wisbauer 1991, §49.2(3)], $\bigoplus_{p \in P^h} Ae_p$ is projective in A-Mod. So P is projective in A-Mod.

Conversely, suppose that P is a graded and projective A-module. Let $\pi : M \to N$ be an epimorphism of graded A-modules in A-Gr and $f : P \to N$ a morphism of graded A-modules in A-Gr. We first claim that any epimorphism $\pi : M \to N$ of graded A-modules in A-Gr is surjective. To prove the claim, write A^h for the set of all homogeneous elements of A. Let $X = \{x \in N \mid A^h x \subseteq \pi(M)\} \subseteq N$ (compare [Goodearl 2009, §5.3]). Then X is a graded submodule of N. We denote by $q : N \to N/X$ the quotient map. Then $q \circ \pi = 0$. Hence, $q = 0$, giving $N = X$. It follows that $N = \pi(M)$. So the epimorphism $\pi : M \to N$ of graded A-modules in A-Gr is surjective. Forgetting the grading, $\pi : M \to N$ is a surjective morphism of A-modules in A-Mod. Since P is projective in A-Mod, there exists $h : P \to M$ such that $\pi \circ h = f$. By [Hazrat 2016, Lemma 1.2.14], there exists a morphism $h' : P \to M$ of graded A-modules such that $\pi \circ h' = f$. Thus, P is a graded projective left A-module in A-Gr. □

Thus for a Γ-graded ring A with graded local units, combining Lemma 4.1 with [Ara and Goodearl 2012, Lemma 10.2] (i.e., projective objects in A-Mod are precisely those that are projective as $\tilde{A}$-modules), P is a graded finitely generated projective $\tilde{A}$-module with $AP = P$ if and only if P is a finitely generated A-module which is graded projective in A-Gr. This shows that the definitions of $\mathcal{V}^{\mathrm{gr}}(A)$ by (4-3) and (4-4) coincide.

5. Application: Leavitt path algebras

In this section we study the monoid $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ of the Leavitt path algebra of a graph E (4-4). Using the results on smash products of Steinberg algebras obtained in Section 3, we give a presentation for this monoid in line with M_E (see Section 5C). Using this presentation we show that $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ is a cancellative monoid and thus the natural map $\mathcal{V}^{\mathrm{gr}}(L_K(E)) \to K_0^{\mathrm{gr}}(L_K(E))$ is injective (Corollary 5.8). It follows that there is a lattice correspondence between the graded ideals of $L_K(E)$ and the graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$ (Theorem 5.11). This is further evidence for the conjecture that the graded Grothendieck group K_0^{gr} may be a complete invariant for Leavitt path algebras [Hazrat 2013b].

5A. ***Leavitt path algebras modelled as Steinberg algebras.*** We briefly recall the definition of Leavitt path algebras and establish notation.

A directed graph E is a tuple (E^0, E^1, r, s), where E^0 and E^1 are sets and r, s are maps from E^1 to E^0. We think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. We use the convention that a (finite) path p in E is a sequence $p = \alpha_1 \alpha_2 \cdots \alpha_n$ of edges α_i in E such that $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \le i \le n-1$. We define $s(p) = s(\alpha_1)$, and $r(p) = r(\alpha_n)$. If $s(p) = r(p)$, then p is said to be closed. If p is closed and $s(\alpha_i) \neq s(\alpha_j)$ for $i \neq j$, then p is called a cycle. An edge α is an exit of a path $p = \alpha_1 \cdots \alpha_n$ if there exists i such that $s(\alpha) = s(\alpha_i)$ and $\alpha \neq \alpha_i$. A graph E is called acyclic if there is no closed path in E.

A directed graph E is said to be *row-finite* if for each vertex $u \in E^0$, there are at most finitely many edges in $s^{-1}(u)$. A vertex u for which $s^{-1}(u)$ is empty is called a *sink*, whereas $u \in E^0$ is called an *infinite emitter* if $s^{-1}(u)$ is infinite. If $u \in E^0$ is neither a sink nor an infinite emitter, then it is called a *regular vertex*.

Definition 5.1. Let E be a directed graph and R a commutative ring with unit. The *Leavitt path algebra* $L_R(E)$ of E is the R-algebra generated by the set $\{v \mid v \in E^0\} \cup \{e \mid e \in E^1\} \cup \{e^* \mid e \in E^1\}$ subject to the following relations:

(1) $uv = \delta_{u,v} v$ for every $u, v \in E^0$;

(2) $s(e)e = er(e) = e$ for all $e \in E^1$;

(3) $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$;

(4) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$; and

(5) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every regular vertex $v \in E^0$.

Let Γ be a group with identity ε, and let $w : E^1 \to \Gamma$ be a function. Extend w to vertices and ghost edges by defining $w(v) = \varepsilon$ for $v \in E^0$ and $w(e^*) = w(e)^{-1}$ for $e \in E^1$. The relations in Definition 5.1 are compatible with w, so there is a Γ-grading on $L_R(E)$ such that $e \in L_R(E)_{w(e)}$ and $e^* \in L_R(E)_{w(e)^{-1}}$, for all $e \in E^1$, and $v \in L_R(E)_\varepsilon$, for all $v \in E^0$. The set of all finite sums of distinct elements of E^0 is a set of graded local units for $L_R(E)$ [Abrams and Aranda Pino 2005, Lemma 1.6]. Furthermore, $L_R(E)$ is unital if and only if E^0 is finite.

Leavitt path algebras associated to arbitrary graphs can be realised as Steinberg algebras. We recall from [Clark and Sims 2015, Example 2.1] the construction of the groupoid $\mathcal{G}_E$ from an arbitrary graph E, which was introduced in [Kumjian et al. 1997] for row-finite graphs and generalised to arbitrary graphs in [Paterson 2002]. We then realise the Leavitt path algebra $L_R(E)$ as the Steinberg algebra $A_R(\mathcal{G})$. This allows us to apply Theorem 3.4 to the setting of Leavitt path algebras.

Let $E = (E^0, E^1, r, s)$ be a directed graph. We denote by E^∞ the set of infinite paths in E and by E^* the set of finite paths in E. Set

$$X := E^\infty \cup \{\mu \in E^* \mid r(\mu) \text{ is not a regular vertex}\}.$$

Let

$$\mathcal{G}_E := \big\{(\alpha x, |\alpha| - |\beta|, \beta x) \mid \alpha, \beta \in E^*, x \in X, r(\alpha) = r(\beta) = s(x)\big\}.$$

We view each $(x, k, y) \in \mathcal{G}_E$ as a morphism with range x and source y. The formulas $(x, k, y)(y, l, z) = (x, k+l, z)$ and $(x, k, y)^{-1} = (y, -k, x)$ define composition and inverse maps on $\mathcal{G}_E$ making it a groupoid with $\mathcal{G}_E^{(0)} = \{(x, 0, x) \mid x \in X\}$ which we identify with the set X.

Next, we describe a topology on $\mathcal{G}_E$. For $\mu \in E^*$ define

$$Z(\mu) = \{\mu x \mid x \in X, r(\mu) = s(x)\} \subseteq X.$$

For $\mu \in E^*$ and a finite $F \subseteq s^{-1}(r(\mu))$, define

$$Z(\mu \setminus F) = Z(\mu) \setminus \bigcup_{\alpha \in F} Z(\mu\alpha).$$

The sets $Z(\mu \setminus F)$ constitute a basis of compact open sets for a locally compact Hausdorff topology on $X = \mathcal{G}_E^{(0)}$ (see [Webster 2014, Theorem 2.1]).

For $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, and for a finite $F \subseteq E^*$ such that $r(\mu) = s(\alpha)$ for $\alpha \in F$, we define

$$Z(\mu, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \mid x \in X, r(\mu) = s(x)\},$$

and then

$$Z((\mu, \nu) \setminus F) = Z(\mu, \nu) \setminus \bigcup_{\alpha \in F} Z(\mu\alpha, \nu\alpha).$$

The sets $Z((\mu, \nu) \setminus F)$ constitute a basis of compact open bisections for a topology under which $\mathcal{G}_E$ is a Hausdorff ample groupoid. By [Clark and Sims 2015, Example 3.2], the map

$$\pi_E : L_R(E) \to A_R(\mathcal{G}_E) \tag{5-1}$$

defined by $\pi_E\big(\mu\nu^* - \sum_{\alpha \in F} \mu\alpha\alpha^*\nu^*\big) = 1_{Z((\mu,\nu)\setminus F)}$ extends to a $\mathbb{Z}$-graded algebra isomorphism. We observe that the isomorphism of algebras in (5-1) satisfies

$$\pi_E(v) = 1_{Z(v)}, \quad \pi_E(e) = 1_{Z(e, r(e))}, \quad \pi_E(e^*) = 1_{Z(r(e), e)}, \tag{5-2}$$

for each $v \in E^0$ and $e \in E^1$.

If $w : E^1 \to \Gamma$ is a function, we extend w to E^* by defining $w(v) = 0$ for $v \in E^0$, and $w(\alpha_1 \cdots \alpha_n) = w(\alpha_1) \cdots w(\alpha_n)$. Thus $L_R(E)$ is a Γ-graded ring. On the other hand, defining $\widetilde{w} : \mathcal{G}_E \to \Gamma$ by

$$\widetilde{w}(\alpha x, |\alpha| - |\beta|, \beta x) = w(\alpha)w(\beta)^{-1}, \tag{5-3}$$

gives a cocycle ([Kumjian and Pask 1999, Lemma 2.3]) and thus $A_R(\mathcal{G})$ is a Γ-graded ring as well. A quick inspection of isomorphism (5-1) shows that π_E respects the Γ-grading.

5B. ***Covering of a graph.*** In this section we show that the smash product of a Leavitt path algebra is graded isomorphic to the Leavitt path algebra of its covering graph. We briefly recall the concept of skew product or covering of a graph (see [Green 1983, §2; Kumjian and Pask 1999, Definition 2.1]).

Let Γ be a group and $w : E^1 \to \Gamma$ a function. As in [Green 1983, §2], the *covering graph* $\overline{E}$ of E with respect to w is given by

$$\begin{aligned} \overline{E}^0 &= \{v_\alpha \mid v \in E^0 \text{ and } \alpha \in \Gamma\}, & \overline{E}^1 &= \{e_\alpha \mid e \in E^1 \text{ and } \alpha \in \Gamma\}, \\ s(e_\alpha) &= s(e)_\alpha, & r(e_\alpha) &= r(e)_{w(e)^{-1}\alpha}. \end{aligned} \tag{5-4}$$

Example 5.2. Let E be a graph and define $w : E^1 \to \mathbb{Z}$ by $w(e) = 1$ for all $e \in E^1$. Then $\bar{E}$ (sometimes denoted $E \times_1 \mathbb{Z}$) is given by

$$\begin{aligned} \bar{E}^0 &= \{v_n \mid v \in E^0 \text{ and } n \in \mathbb{Z}\}, & \bar{E}^1 &= \bigl\{e_n \mid e \in E^1 \text{ and } n \in \mathbb{Z}\bigr\}, \\ s(e_n) &= s(e)_n, & r(e_n) &= r(e)_{n-1}. \end{aligned}$$

As examples, consider the following graphs

$$E: \begin{tikzcd} u \arrow[loop left, "e"] \arrow[r, bend left, "f"] & v \arrow[l, bend left, "g"] \end{tikzcd} \qquad F: \begin{tikzcd} u \arrow[loop above, "f"] \arrow[loop right, "e"] \end{tikzcd}$$

Then

$$\bar{E}: \begin{tikzcd} \cdots u_1 \arrow[r, "e_1"] \arrow[dr, "f_1" near end] & u_0 \arrow[r, "e_0"] \arrow[dr, "f_0" near end] & u_{-1} \arrow[r, "e_{-1}"] \arrow[dr, "f_{-1}" near end] & \cdots \\ \cdots v_1 \arrow[ur, "g_1" near start] & v_0 \arrow[ur, "g_0" near start] & v_{-1} \arrow[ur, "g_{-1}" near start] & \cdots \end{tikzcd}$$

and

$$\bar{F}: \begin{tikzcd} \cdots u_1 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "e_1"'] & u_0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "e_0"'] & u_{-1} \arrow[r, bend left, "f_{-1}"] \arrow[r, bend right, "e_{-1}"'] & \cdots \end{tikzcd}$$

Since $\widetilde{w} : \mathcal{G}_E \to \Gamma$ is a cocycle (5-3), by Lemma 3.3 the skew-product groupoid $\mathcal{G}_E \times \Gamma$ is a Γ-graded ample groupoid. For each (possibly infinite) path $x = e^1 e^2 e^3 \cdots$ of E and each $\gamma \in \Gamma$, there is a path x_γ of $\bar{E}$ given by

$$x_\gamma = e^1_\gamma e^2_{w(e^1)^{-1}\gamma} e^3_{w(e^1e^2)^{-1}\gamma} \cdots . \tag{5-5}$$

There is an isomorphism

$$f : \mathcal{G}_E \times \Gamma \to \mathcal{G}_{\bar{E}} \tag{5-6}$$

of groupoids such that $f((x, k, y), \gamma) = (x_{\widetilde{w}(x,k,y)\gamma}, k, y_\gamma)$ (see [Kumjian and Pask 1999, Theorem 2.4]). The grading of the skew-product $\mathcal{G}_E \times \Gamma$ induces a grading of $\mathcal{G}_{\bar{E}}$, and the isomorphism f respects the gradings of the two groupoids, and so induces a Γ-graded isomorphism of Steinberg algebras

$$\tilde{f} : A_R(\mathcal{G}_E \times \Gamma) \to A_R(\mathcal{G}_{\bar{E}}).$$

Set $g = \tilde{f}^{-1} : A_R(\mathcal{G}_{\bar{E}}) \to A_R(\mathcal{G}_E \times \Gamma)$. Then

$$\begin{aligned} g(1_{Z(v_\gamma)}) &= 1_{Z(v)\times\{\gamma\}} && \text{for } v \in E^0 \text{ and } \gamma \in \Gamma, \\ g(1_{Z(e_\alpha, r(e)_{w(e)^{-1}\alpha})}) &= 1_{Z(e,r(e))\times\{w(e)^{-1}\alpha\}} && \text{for } e \in E^1 \text{ and } \alpha \in \Gamma, \\ g(1_{Z(r(e)_{w(e)^{-1}\alpha}, e_\alpha)}) &= 1_{Z(r(e),e)\times\{\alpha\}} && \text{for } e \in E^1 \text{ and } \alpha \in \Gamma. \end{aligned} \tag{5-7}$$

Let $\phi : A_R(\mathcal{G}_E \times \Gamma) \to A_R(\mathcal{G}_E) \# \Gamma$ be the isomorphism of Theorem 3.4, let $g : A_R(\mathcal{G}_{\bar{E}}) \to A_R(\mathcal{G}_E \times \Gamma)$ be the isomorphism (5-7), let $\pi_E : L_R(E) \to A_R(\mathcal{G}_E)$ and $\pi_{\bar{E}} : L_R(\bar{E}) \to A_R(\mathcal{G}_{\bar{E}})$ be as in (5-1), and let $\widetilde{\pi}_E : L_R(E)\#\Gamma \to A_R(\mathcal{G}_E)\#\Gamma$ be given by $\widetilde{\pi}_E(xp_\gamma) = \pi_E(x)p_\gamma$, for $x \in L_R(E)$ and $\gamma \in \Gamma$. Define $\phi' := \widetilde{\pi}_E^{-1} \circ \phi \circ g \circ \pi_{\overline{E}}$. Then we have the following commuting diagram of Γ-graded isomorphisms:

$$\begin{tikzcd} L_R(\overline{E}) \arrow[r, "\phi'"] \arrow[d, "\pi_{\overline{E}}"] & L_R(E)\#\Gamma \arrow[d, "\widetilde{\pi}_E"] \\ A_R(\mathcal{G}_{\overline{E}}) \arrow[r, "\phi\circ g"] & A_R(\mathcal{G}_E)\#\Gamma \end{tikzcd} \tag{5-8}$$

Recall from (5-6) that $\mathcal{G}_{\overline{E}}$ is Γ-graded. Then the Steinberg algebra $A_R(\mathcal{G}_{\overline{E}})$ is Γ-graded. By the algebra isomorphism $\pi_{\overline{E}} : L_R(\overline{E}) \cong A_R(\mathcal{G}_{\overline{E}})$, $L_R(\overline{E})$ has a Γ-grading such that the grading map $w' : \overline{E}^1 \to \Gamma$ is given by $w'(e_\alpha) = w(e)$, for $e \in E^1$ and $\alpha \in \Gamma$.

Corollary 5.3. *The map $\phi' : L_R(\overline{E}) \to L_R(E)\#\Gamma$ is an isomorphism of Γ-graded algebras such that $\phi'(v_\beta) = vp_\beta$, $\phi'(e_\alpha) = ep_{w(e)^{-1}\alpha}$ and $\phi'(e_\alpha^*) = e^*p_\alpha$, for $v \in E^0$, $e \in E^1$, and $\alpha, \beta \in \Gamma$.*

Proof. Since all the homomorphisms in the diagram (5-8) preserve gradings of algebras, the map $\phi' : L_R(\overline{E}) \to L_R(E)\#\Gamma$ is an isomorphism of Γ-graded algebras. For each vertex $v_\gamma \in \overline{E}^0$ and each edge $e_\alpha \in \overline{E}^1$, we have

$$\begin{aligned}
\phi'(v_\gamma) &= (\widetilde{\pi}_E^{-1} \circ \phi \circ g)(1_{Z(v_\gamma)}) = (\widetilde{\pi}_E^{-1} \circ \phi)(1_{Z(v)\times\{\gamma\}}) = \widetilde{\pi}_E^{-1}(1_{Z(v)}p_\gamma) = vp_\gamma,\\
\phi'(e_\alpha) &= (\widetilde{\pi}_E^{-1} \circ \phi \circ g)(1_{Z(e_\alpha, r(e)_{w(e)^{-1}\alpha})}) = (\widetilde{\pi}_E^{-1} \circ \phi)(1_{Z(e,r(e))\times\{w(e)^{-1}\alpha\}}) = \widetilde{\pi}_E^{-1}(1_{Z(e,r(e))}p_{w(e)^{-1}\alpha})\\
&= ep_{w(e)^{-1}\alpha},\\
\phi'(e_\alpha^*) &= (\widetilde{\pi}_E^{-1} \circ \phi \circ g)(1_{Z(r(e)_{w(e)^{-1}\alpha}, e_\alpha)}) = (\widetilde{\pi}_E^{-1} \circ \phi)(1_{Z(r(e),e)\times\{\alpha\}}) = \widetilde{\pi}_E^{-1}(1_{Z(r(e),e)}p_\alpha) = e^*p_\alpha. \qquad \square
\end{aligned}$$

Kumjian and Pask [1999] show that given a free action of a group Γ on a graph E, the crossed product $C^*(E) \times \Gamma$ by the induced action is strongly Morita equivalent to $C^*(E/\Gamma)$, where E/Γ is the quotient graph and obtained an isomorphism similar to Corollary 5.3 for graph C^*-algebras. Corollary 5.3 shows that this isomorphism already occurs on the algebraic level (see Section 3D), so the following diagram commutes:

$$\begin{tikzcd} L_{\mathbb{C}}(\overline{E}) \arrow[r] \arrow[d] & L_{\mathbb{C}}(E)\#\Gamma \arrow[d] \\ C^*(\overline{E}) \arrow[r] & C^*(E)\times\Gamma \end{tikzcd}$$

Remark 5.4. Green [1983] showed that the theory of coverings of graphs with relations and the theory of graded algebras are essentially the same. For a Γ-graded path algebra A, Green constructed a covering of the quiver of A and showed that the category of representations of the covering satisfying a certain set of relations is equivalent to the category of finite dimensional graded A-modules.

For any graph E and a function $w : E^1 \to \Gamma$, we consider the smash product of a quotient algebra of the path algebra of E with the group Γ. Let K be a field, E a graph and $w : E^1 \to \Gamma$ a weight map. Denote by KE the path algebra of E. A *relation in E* is a K-linear combination $\sum_i k_i q_i$ with q_i paths in

E having the same source and range. Let r be a set of relations in E and $\langle r\rangle$ the two sided ideal of KE generated by r. Set

$$\mathcal{A}_r(E) = KE/\langle r\rangle.$$

We denote by $\bar{r}$ the lifting of r in $\bar{E}$ (see (5-4)): For each finite path $p = e^1e^2\cdots e^n$ in E and $\gamma \in \Gamma$, there is a path p^γ of $\bar{E}$ given by

$$p^\gamma = e^1_{\prod_{i=1}^n w(e^i)\gamma} \cdots e^{n-1}_{\prod_{i=n-1}^n w(e^i)\gamma} e^n_{w(e^n)\gamma},$$

similar to (5-5). Then for each relation $\sum_i k_i q_i \in r$ and each $\gamma \in \Gamma$, we have

$$\textstyle\sum_i k_i q_i^\gamma \in \bar{r}.$$

Now set

$$\mathcal{A}_{\bar{r}}(\bar{E}) = K\bar{E}/\langle \bar{r}\rangle.$$

We prove that $\mathcal{A}_{\bar{r}}(\bar{E}) \cong \mathcal{A}_r(E)\#\Gamma$. Define $h : K\bar{E} \to \mathcal{A}_r(E)\#\Gamma$ by $h(v_\gamma) = vp_\gamma$ and $h(e_\alpha) = ep_{w(e)^{-1}\alpha}$, for $v \in E^0$, $e \in E^1$ and $\alpha, \gamma \in \Gamma$. Since h annihilates the relations $\bar{r}$, it induces a homomorphism

$$\bar{h} : \mathcal{A}_{\bar{r}}(\bar{E}) \to \mathcal{A}_r(E)\#\Gamma.$$

We show that $\bar{h}$ is an isomorphism. For injectivity, suppose that $x = \sum_{i=1}^m \lambda_i \xi_i \in \mathcal{A}_{\bar{r}}(\bar{E})$ with $\lambda_i \in K$ and ξ_i pairwise distinct paths in $\bar{E}$. Each ξ_i has the form of $(\xi_i')^{\alpha_i}$ for some $\xi_i' \in E^*$ and $\alpha_i \in \Gamma$. If $\bar{h}(x) = 0$, then $h(x) = \sum_{i=1}^m \lambda_i \xi_i' p_{\alpha_i} = 0$. Suppose that the α_i are not distinct; so by rearranging, we can assume that $\alpha_1 = \cdots = \alpha_k$ for some $k \le m$. Then $\sum_{i=1}^k \lambda_i \xi_i' = 0$ in $\mathcal{A}_r(E)$. Observe that $\sum_{i=1}^k \lambda_i \xi_i' = 0$ in $\mathcal{A}_r(E)$ implies $\sum_{i=1}^k \lambda_i \xi_i = 0$ in $\mathcal{A}_{\bar{r}}(\bar{E})$. Hence $x = 0$, implying $\bar{h}$ is injective. For surjectivity, fix η in E^* and $\gamma \in \Gamma$. Then $h(\eta^\gamma) = \eta p_\gamma$ by definition. Since the elements $\{\eta p_\gamma \mid \eta \in E^*, \gamma \in \Gamma\}$ span $\mathcal{A}_r(E)\#\Gamma$, we deduce that $\bar{h}$ is surjective. Thus $\bar{h}$ is an isomorphism as claimed.

5C. *The monoid* $\mathcal{V}^{\mathrm{gr}}(L_K(E))$. In this subsection, we consider the Leavitt path algebra $L_K(E)$ over a field K. Ara, Moreno and Pardo [Ara et al. 2007] showed that for a Leavitt path algebra associated to the graph E, the monoid $\mathcal{V}(L_K(E))$ is entirely determined by elementary graph-theoretic data. Specifically, for a row-finite graph E, we define M_E to be the abelian monoid generated by E^0 subject to

$$v = \sum_{e \in s^{-1}(v)} r(e), \tag{5-9}$$

for every $v \in E^0$ that is not a sink. Theorem 3.5 of [Ara et al. 2007] says that $\mathcal{V}(L_K(E)) \cong M_E$.

There is an explicit description [Ara et al. 2007, §4] of the congruence on the free abelian monoid given by the defining relations of M_E. Let F be the free abelian monoid on the set E^0. The nonzero elements of F can be written in a unique form up to permutation as $\sum_{i=1}^n v_i$, where $v_i \in E^0$. Define a binary relation $\to_1$ on $F \setminus \{0\}$ by $\sum_{i=1}^n v_i \to_1 \sum_{i \ne j} v_i + \sum_{e \in s^{-1}(v_j)} r(e)$, whenever $j \in \{1, \ldots, n\}$ and v_j is not a sink. Let $\to$ be the transitive and reflexive closure of $\to_1$ on $F \setminus \{0\}$ and $\sim$ the congruence on F generated by the relation $\to$. Then $M_E = F/\sim$.

Ara and Goodearl [2012, Theorem 4.3] defined analogous monoids $M(E, C, S)$ and constructed natural isomorphisms $M(E, C, S) \cong \mathcal{V}(\mathrm{CL}_K(E, C, S))$ for arbitrary separated graphs. The nonseparated case reduces to that of ordinary Leavitt path algebras, and extends the result of [Ara et al. 2007] to non-row-finite graphs.

Following [Ara and Goodearl 2012; 2015], we recall the definition of M_E when E is not necessarily row-finite. In [Ara and Goodearl 2015, §4.1] the generators $v \in E^0$ of the abelian monoid M_E for E are supplemented by generators q_Z as Z runs through all nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. The relations are

(1) $v = \sum_{e\in s^{-1}(v)} r(e)$ for all regular vertices $v \in E^0$;

(2) $v = \sum_{e\in Z} r(e) + q_Z$ for all infinite emitters $v \in E^0$; and

(3) $q_{Z_1} = \sum_{e\in Z_2\setminus Z_1} r(e) + q_{Z_2}$ for all nonempty finite sets $Z_1 \subseteq Z_2 \subseteq s^{-1}(v)$, where $v \in E^0$ is an infinite emitter.

An abelian monoid M is *cancellative* if it satisfies full cancellation, namely, $x + z = y + z$ implies $x = y$, for any $x, y, z \in M$. In order to prove that the *graded* monoid associated to any Leavitt path algebra is cancellative (Corollary 5.8), we will need to know that the monoid associated to Leavitt path algebras of acyclic graphs are cancellative.

Lemma 5.5. *Let E be an arbitrary graph. The monoid M_E is cancellative if and only if no cycle in E has an exit. In particular, if E is acyclic, then M_E is cancellative.*

Proof. We first claim that M_E is cancellative for any row-finite acyclic graph E. By [Ara et al. 2007, Lemma 3.1], the row-finite graph E is a direct limit of a directed system of its finite complete subgraphs $\{E_i\}_{i\in X}$. In turn, the monoid M_E is the direct limit of $\{M_{E_i}\}_{i\in X}$ ([Ara et al. 2007, Lemma 3.4]). We claim that M_E is cancellative. Let $x + u = y + u$ in M_E, where x, y, u are sum of vertices in E. By [Ara et al. 2007, Lemma 4.3], there exists $b \in F$ (sum of vertices in E) such that $x + u \to b$ and $y + u \to b$. Observe that vertices involved in this transformations are finite. Thus there is a finite graph E_i such that all these vertices are in E_i. It follows that in M_{E_i} we have $x+u \to b$ and $y+u \to b$. Thus $x+u = y+u$ in M_{E_i}. Since the subgraph E_i of E is finite and acyclic, M_{E_i} is a direct sum of copies of $\mathbb{N}$ (as $L_K(E_i)$ is a semisimple ring) and thus is cancellative. So $x = y$ in M_{E_i} and so the same in M_E. Hence, M_E is cancellative.

We now show that it suffices to consider the case where E is a row-finite graph in which no cycle has an exit. To see this, let E be any graph, and let E_d be its Drinen–Tomforde desingularisation [Drinen and Tomforde 2005], which is row-finite. Then $L_K(E)$ and $L_K(E_d)$ are Morita equivalent, and so $M_E \cong M_{E_d}$ [Abrams and Aranda Pino 2008, Theorem 5.6]. So M_E has cancellation if and only if M_{E_d} has cancellation. Since no cycle in E has an exit if and only if E_d has the same property, it therefore suffices to prove the result for row-finite graph E in which no cycle has an exit.

Finally, we show that for any row-finite graph E in which no cycle has an exit, the monoid M_E is cancellative. For this, fix a set $C \subseteq E^1$ such that C contains exactly one edge from every cycle in E [Sims 2010]. Let F be the subgraph of E obtained by removing all the edges in C. We claim that

$M_F \cong M_E$. To see this, observe that they have the same generating set $F^0 = E^0$, and the generating relation $\xrightarrow{F}_1$ is contained in $\xrightarrow{E}_1$. So it suffices to show that $\xrightarrow{E}_1 \subseteq \xrightarrow{F}$. For this, note that for $v \in E^0$, we have $s_E^{-1}(v) = s_F^{-1}(v)$ unless $v = s(e)$ for some $e \in C$, in which case $s_E^{-1}(v) = \{e\}$ and $s_F^{-1}(v) = \varnothing$. So it suffices to show that for $e \in C$, we have $s(e) \xrightarrow{F} r(e)$. Let $p = e\alpha_2\alpha_3 \cdots \alpha_n$ be the cycle in E containing e. Then

$$r(e) \xrightarrow{F}_1 s(\alpha_2) \xrightarrow{F}_1 r(\alpha_2) \xrightarrow{F}_1 s(\alpha_3) \xrightarrow{F}_1 r(\alpha_3) \xrightarrow{F}_1 \cdots \xrightarrow{F}_1 s(\alpha_n) \xrightarrow{F}_1 r(\alpha_n) \xrightarrow{F}_1 s(e).$$

So $M_F \cong M_E$ as claimed. So the preceding paragraphs show that M_E is cancellative.

Now suppose that E has a cycle with an exit; say $p = \alpha_1 \cdots \alpha_n$ has an exit α. Without loss of generality, $s(\alpha) = s(\alpha_n)$ and $\alpha \neq \alpha_n$. Write

$$s(p)E^{\leq n} = \big\{q \in E^* \mid s(q) = s(p), \text{ and } |q| = n \text{ or } |q| < n \text{ and } r(q) \text{ is not regular}\big\}.$$

Let $p' := \alpha_1 \cdots \alpha_{n-1}\alpha$ and $X := s(p)E^{\leq n} \setminus \{p, p'\}$. A simple induction shows that

$$s(p) \to \sum_{q \in s(\alpha)E^{\leq n}} r(q) = r(p) + r(p') + \sum_{q \in X} r(q) = s(p) + r(p') + \sum_{q \in X} r(q).$$

Since $r(p') \neq 0$ in M_E, it follows that M_E does not have cancellation. □

In order to compute the monoid $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ for an arbitrary graph E, we define an abelian monoid M_E^{gr} such that the generators $\{a_v(\gamma) \mid v \in E^0, \gamma \in \Gamma\}$ are supplemented by generators $b_Z(\gamma)$ as $\gamma \in \Gamma$ and Z runs through all nonempty finite subsets of $s^{-1}(u)$ for infinite emitters $u \in E^0$. The relations are:

(1) $a_v(\gamma) = \sum_{e \in s^{-1}(v)} a_{r(e)}(w(e)^{-1}\gamma)$ for all regular vertices $v \in E^0$ and $\gamma \in \Gamma$;

(2) $a_u(\gamma) = \sum_{e \in Z} a_{r(e)}(w(e)^{-1}\gamma) + b_Z(\gamma)$ for all $\gamma \in \Gamma$, infinite emitters $u \in E^0$ and nonempty finite subsets $Z \subseteq s^{-1}(u)$;

(3) $b_{Z_1}(\gamma) = \sum_{e \in Z_2 \setminus Z_1} a_{r(e)}(w(e)^{-1}\gamma) + b_{Z_2}(\gamma)$ for all $\gamma \in \Gamma$, infinite emitters $u \in E^0$ and nonempty finite subsets $Z_1 \subseteq Z_2 \subseteq s^{-1}(u)$.

The group Γ acts on the monoid M_E^{gr} as follows. For any $\beta \in \Gamma$,

$$\beta \cdot a_v(\gamma) = a_v(\beta\gamma) \quad \text{and} \quad \beta \cdot b_Z(\gamma) = b_Z(\beta\gamma). \tag{5-10}$$

There is a surjective monoid homomorphism $\pi : M_E^{\mathrm{gr}} \to M_E$ such that $\pi(a_v(\gamma)) = v$ and $\pi(b_Z(\gamma)) = q_Z$ for $v \in E^0$ and nonempty finite subset $Z \subset s^{-1}(u)$, where u is an infinite emitter. π is Γ-equivariant in the sense that $\pi(\beta \cdot x) = \pi(x)$ for all $\beta \in \Gamma$ and $x \in M_E^{\mathrm{gr}}$.

Recall the covering graph $\overline{E}$ from Section 5B. Let $L_K(\overline{E})$-Mod be the category of unital left $L_K(\overline{E})$-modules and $L_K(E)$-Gr the category of graded unital left $L_K(E)$-modules. The isomorphism $\phi' : L_K(\overline{E}) \xrightarrow{\sim} L_K(E) \# \Gamma$ of Corollary 5.3 and Proposition 2.5 yield an isomorphism of categories

$$\Phi : L_K(E)\text{-Gr} \to L_K(\overline{E})\text{-Mod}. \tag{5-11}$$

Lemma 5.6. *Let E be an arbitrary graph, Γ a group and $w : E^1 \to \Gamma$ a function.*

(1) *Fix a path η in E, and $\beta \in \Gamma$, and let $\bar\eta = \eta_{\beta^{-1}}$ be the path in $\overline{E}$ defined at* (5-5). *Then*

$$\Phi\big((L_K(E)\eta\eta^*)(\beta)\big) \cong L_K(\overline{E})\bar\eta\bar\eta^*.$$

In particular, $\Phi((L_K(E)v)(\beta)) \cong L_K(\overline{E})v_{\beta^{-1}}$.

(2) *Let $u \in E^0$ be an infinite emitter, and let $Z \subseteq s_E^{-1}(u)$ be a nonempty finite set. Fix $\beta \in \Gamma$, and let $\overline{Z} = \{e_{\beta^{-1}} \mid e \in Z\}$. Then $u_{\beta^{-1}}$ is an infinite emitter in $\overline{E}$ and $\overline{Z}$ is a nonempty finite subset of $s_{\overline{E}}^{-1}(u_{\beta^{-1}})$. Moreover, $\Phi(L_K(E)(u-\sum_{e\in Z} ee^*)(\beta)) \cong L_K(\overline{E})(u_{\beta^{-1}} - \sum_{f\in\overline{Z}} ff^*)$.*

Proof. We prove (1). By the isomorphism of algebras in Corollary 5.3, we have

$$L_K(\overline{E})\bar\eta\bar\eta^* \cong (L_K(E)\,\#\,\Gamma)\eta\eta^* p_{\beta^{-1}}.$$

We claim that $f : \Phi((L_K(E)\eta\eta^*)(\beta)) \to (L_K(E)\#\Gamma)\eta\eta^* p_{\beta^{-1}}$ given by $f(y) = yp_{\beta^{-1}}$ is an isomorphism of left $L_K(\overline{E})$-modules. It is clearly a group isomorphism. To see that it is an $L_K(\overline{E})$-module morphism, note that $(rp_\gamma)y = ry_\gamma$ for $y \in (L_K(E)\eta\eta^*)(\beta)$ and y_γ a homogeneous element of degree γ. We have $y \in L_K(E)_{\gamma\beta}\eta\eta^*$, yielding $f((rp_\gamma)y) = ry_\gamma p_{\beta-1} = (rp_\gamma)(yp_{\beta^{-1}}) = rp_\gamma f(y)$. The proof for (2) is similar. □

Recall from Section 2B that there is a shift functor $\tilde{\mathcal{S}}_\alpha$ on $L_K(E)\,\#\,\Gamma$-Mod for each $\alpha \in \Gamma$. So the isomorphism $\phi' : L_K(\overline{E}) \xrightarrow{\sim} L_K(E)\#\Gamma$ of Corollary 5.3 yields a shift functor $\mathcal{T}_\alpha$ on $L_K(\overline{E})$-Mod. This in turn induces a homomorphism $\mathcal{T}_\alpha : \mathcal{V}(L_K(\overline{E})) \to \mathcal{V}(L_K(\overline{E}))$, giving a Γ-action on the monoid $\mathcal{V}(L_K(\overline{E}))$.

Fix $v_\gamma \in \overline{E}^0$, an infinite emitter $u_\beta \in \overline{E}^0$, and a finite $Z \subseteq s_{\overline{E}}^{-1}(u_\beta)$. Write $Z\cdot\alpha^{-1} = \{e_{\beta\alpha^{-1}} \mid e_\beta \in Z\}$. We claim that

$$\mathcal{T}_\alpha([L_K(\overline{E})v_\gamma]) = [L_K(\overline{E})v_{\gamma\alpha^{-1}}] \quad\text{and}\quad \mathcal{T}_\alpha([L_K(\overline{E})(u_\beta - \sum_{e\in Z} ee^*)]) = [L_K(\overline{E})(u_{\beta\alpha^{-1}} - \sum_{f\in Z\cdot a^{-1}} ff^*)]. \tag{5-12}$$

To establish the first equality in (5-12), we use Lemma 5.6 to see that

$$\Phi(L_K(E)v(\gamma^{-1})) = L_K(\overline{E})v_\gamma \quad\text{and}\quad \Phi(L_K(E)v(\alpha\gamma^{-1})) = L_K(\overline{E})v_{\gamma\alpha^{-1}}.$$

Using the commutative diagram (2-3) at the second equality, we see that

$$\begin{aligned}\mathcal{T}_\alpha(L_K(\overline{E})v_\gamma) &= (\mathcal{T}_\alpha\circ\Phi)(L_K(E)v(\gamma^{-1})) = (\Phi\circ\mathcal{T}_\alpha)(L_K(E)v(\gamma^{-1})) = \Phi(L_K(E)v(\alpha\gamma^{-1}))\\ &= L_K(\overline{E})v_{\gamma\alpha^{-1}}.\end{aligned}$$

The proof for the second equality in (5-12) is similar.

The group Γ acts on the monoid $M_{\overline{E}}$ as follows. Again fix $v_\gamma \in \overline{E}^0$, an infinite emitter $u_\beta \in \overline{E}^0$, and a finite $Z \subseteq s_{\overline{E}}^{-1}(u_\beta)$, and write $Z\cdot\alpha^{-1} = \{e_{\beta\alpha^{-1}} \mid e_\beta \in Z\}$. Then

$$\alpha\cdot v_\gamma = v_{\gamma\alpha^{-1}} \quad\text{and}\quad \alpha\cdot q_Z = q_{Z\cdot\alpha^{-1}}. \tag{5-13}$$

Proposition 5.7. *Let E be an arbitrary graph, K a field, Γ a group and $w : E^1 \to \Gamma$ a function. Let $\mathcal{A} = L_K(E)$ and $\bar{\mathcal{A}} = L_K(\overline{E})$. Then the monoid $\mathcal{V}^{\mathrm{gr}}(\mathcal{A})$ is generated by $[\mathcal{A}v(\alpha)]$ and $[\mathcal{A}(u-\sum_{e\in Z} ee^*)(\beta)]$, where $v \in E^0$, $\alpha, \beta \in \Gamma$ and Z runs through all nonempty finite subsets of $s^{-1}(u)$ for infinite emitters*

$u \in E^0$. Given an infinite emitter $u \in E^0$, a finite nonempty set $Z \subseteq s^{-1}(u)$, and $\beta \in \Gamma$, write $Z_{\beta^{-1}} := \{e_{\beta^{-1}} : e \in Z\} \subseteq s_{\bar{E}}^{-1}(u_{\beta^{-1}})$. Then there are Γ-module isomorphisms

$$\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) \cong \mathcal{V}(\bar{\mathcal{A}}) \cong M_{\bar{E}} \cong M_E^{\mathrm{gr}}, \tag{5-14}$$

that satisfy

$$[\mathcal{A}v(\alpha)] \mapsto [\bar{\mathcal{A}}v_{\alpha^{-1}}] \mapsto [v_{\alpha^{-1}}] \mapsto [a_v(\alpha)],$$

for all $v \in E^0$ and $\alpha \in \Gamma$, and

$$\Big[\mathcal{A}(u - \sum_{e \in Z} ee^*)(\beta)\Big] \mapsto \Big[\bar{\mathcal{A}}(u_{\beta^{-1}} - \sum_{\bar{e} \in Z_{\beta-1}} \bar{e}\bar{e}^*)\Big] \mapsto [q_{Z_{\beta^{-1}}}] \mapsto [b_Z(\beta)],$$

for every infinite emitter u, finite nonempty $Z \subseteq s^{-1}(u)$, and $\beta \in \Gamma$.

Proof. Let P be a graded finitely generated projective left $\mathcal{A}$-module. We claim that the isomorphism $\Phi : \mathcal{A}\text{-Gr} \to \bar{\mathcal{A}}\text{-Mod}$ in (5-11) preserves the finitely generated projective objects. Since Φ is an isomorphism of categories, $\Phi(P)$ is projective. Observe that P has finite number of homogeneous generators $x_1, \dots, x_n$ of degree γ_i. By the $\bar{\mathcal{A}}$-action of $\Phi(P)$, we have the following equalities:

(1) if $v \in E^0$ and $\gamma \in \Gamma$, then

$$v_\gamma x_i = v p_\gamma x_i = \begin{cases} v x_i & \text{if } \gamma_i = \gamma, \\ 0 & \text{otherwise;} \end{cases} \tag{5-15}$$

(2) if $e : u \to v \in E^1$, $w(e) = \beta$ and $\gamma \in \Gamma$, then

$$e_\gamma x_i = e p_{\beta^{-1}\gamma} x_i = \begin{cases} e x_i & \text{if } \gamma_i = \beta^{-1}\gamma, \\ 0 & \text{otherwise; and} \end{cases} \tag{5-16}$$

(3) if $e : u \to v \in E^1$, $w(e) = \beta$ and $\gamma \in \Gamma$, then

$$e^*_\gamma x_i = e^* p_\gamma x_i = \begin{cases} e^* x_i & \text{if } \gamma_i = \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{5-17}$$

So for $y \in \Phi(P)$, we can express $y = \sum_{i=1}^n r_i x_i$ for some $r_i \in \mathcal{A}$. Fix $i \le n$ and paths η, τ in E satisfying $r(\eta) = r(\tau)$. Then (5-15), (5-16), and (5-17) give

$$\tau\eta^* x_i = \tau_{w(\tau)w(\eta)^{-1}\gamma_i} (\eta_{\gamma_i})^* x_i. \tag{5-18}$$

Since $y = \sum_{i=1}^n r_i x_i = \sum_{i=1}^n \sum_{h \in \Gamma} r_{i,h} x_i$ with $r_{i,h}$ a homogeneous element of degree h, Equation (5-18) gives $y \in \bar{\mathcal{A}}(\Phi(P))$. Thus $\Phi(P)$ is a finitely generated projective $\bar{\mathcal{A}}$-module.

By (4-2) and (4-4), there exists a homomorphism $\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) \to \mathcal{V}(\bar{\mathcal{A}})$ sending $[P]$ to $[\Phi(P)]$ for a graded finitely generated projective left $\mathcal{A}$-module P. Applying [Ara and Goodearl 2012, Theorem 4.3] for the nonseparated case, we obtain the second monoid isomorphism $\mathcal{V}(\bar{\mathcal{A}}) \xrightarrow{\sim} M_{\bar{E}}$ in (5-14). Then for each graded finitely generated projective left $\mathcal{A}$-module P, the module $\Phi(P)$ in $\bar{\mathcal{A}}$-Mod is generated by the elements $\bar{\mathcal{A}}v_\alpha$ and $\bar{\mathcal{A}}(u_\beta - \sum_{e \in Z'} ee^*)$ that it contains. Combining this with Lemma 5.6 gives the first isomorphism of monoids. The last monoid isomorphism $M_{\bar{E}} \cong M_E^{\mathrm{gr}}$ follows directly by their definitions. By (5-10), (5-12) and (5-13), the monoid isomorphisms in (5-14) are Γ-module isomorphisms. □

Recall the following classification conjecture [Abrams 2015; Ara and Pardo 2014; Hazrat 2013b]. Let E and F be finite graphs. Then there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $\phi : K_0^{\mathrm{gr}}(L_K(E)) \to K_0^{\mathrm{gr}}(L_K(F))$ if and only if $L_K(E)$ is graded Morita equivalent to $L_K(F)$. Furthermore, if $\phi([L_K(E)] = [L_K(F)]$ then $L_K(E) \cong_{\mathrm{gr}} L_K(F)$.

Note that $K_0(L_K(E))$ and $K_0^{\mathrm{gr}}(L_K(E))$ are the group completions of $\mathcal{V}(L_K(E))$ and $\mathcal{V}^{\mathrm{gr}}(L_K(E))$, respectively. Let $\Gamma = \mathbb{Z}$ and let $w : E^1 \to \mathbb{Z}$ be the function assigning 1 to each edge. Then Proposition 5.7 implies that there is an order preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K_0^{\mathrm{gr}}(L_K(E)) \cong K_0(L_K(\bar{E}))$, thus relating the study of a Leavitt path algebra over an arbitrary graph to the case of acyclic graphs (see Example 5.2).

The following corollary is the first evidence that $K_0^{\mathrm{gr}}(L_K(E))$ preserves all the information of the graded monoid.

Corollary 5.8. *Let E be an arbitrary graph. Consider $L_K(E)$ as a graded ring with the grading determined by the function $w : E^1 \to \mathbb{Z}$ such that $w(e) = 1$ for all e. Then $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ is cancellative.*

Proof. By Proposition 5.7, we have $\mathcal{V}^{\mathrm{gr}}(L_K(E)) \cong M_{\bar{E}}$. Since $\bar{E} = E \times \mathbb{Z}$ is an acyclic graph, the monoid $M_{\bar{E}}$ is cancellative by Lemma 5.5. Hence $\mathcal{V}^{\mathrm{gr}}(L_K(E))$ is cancellative. □

For the next result we need to recall the notion of order-ideals of a monoid. An *order-ideal* of a monoid M is a submonoid I of M such that $x + y \in I$ implies $x, y \in I$. Equivalently, an order-ideal is a submonoid I of M that is hereditary in the sense that $x \leq y$ and $y \in I$ implies $x \in I$. The set $\mathcal{L}(M)$ of order-ideals of M forms a (complete) lattice (see [Ara et al. 2007, §5]). Given a subgroup I of $K_0^{\mathrm{gr}}(A)$, we write $I^+ = I \cap K_0^{\mathrm{gr}}(A)^+$. We say that I is a *graded ordered ideal* if I is closed under the action of $\mathbb{Z}[x, x^{-1}]$, $I = I^+ - I^+$, and I^+ is an order-ideal.

Let E be a graph. Recall that a subset $H \subseteq E^0$ is said to be hereditary if for any $e \in E^1$ we have that $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E^0$ is called saturated if whenever $0 < |s^{-1}(v)| < \infty$, then $\{r(e) \mid e \in E^1 \text{ and } s(e) = v\} \subseteq H$ implies $v \in H$. If H is a hereditary subset, a breaking vertex of H is a vertex $v \in E^0 \setminus H$ such that $|s^{-1}(v)| = \infty$ but $0 < |s^{-1}(v) \setminus r^{-1}(H)| < \infty$. We write $B_H := \{v \in E^0 \setminus H \mid v \text{ is a breaking vertex of } H\}$. We call (H, S) an *admissible pair* in E^0 if H is a saturated hereditary subset of E^0 and $S \subseteq B_H$.

Let E be a row-finite graph. Isomorphisms between the lattice of saturated hereditary subsets of E^0, the lattice $\mathcal{L}(M_E)$, and the lattice of graded ideals of $L_K(E)$ were established in [Ara et al. 2007, Theorem 5.3]. Tomforde [2007, Theorem 5.7] used the admissible pairs (H, S) of vertices to parametrise the graded ideals of $L_K(E)$ for a graph E which is not row-finite. In analogy, Ara and Goodearl [2012] proved that the lattice of those ideals of Cohn–Leavitt algebras $\mathrm{CL}_K(E, C, S)$ generated by idempotents is isomorphic to a certain lattice $\mathcal{A}_{C,S}$ of admissible pairs (H, G), where $H \subseteq E^0$ and $G \subseteq C$ (see [Ara and Goodearl 2012, Definition 6.5] for the precise definition). There is also a lattice isomorphism between

$\mathcal{A}_{C,S}$ and the lattice $\mathcal{L}(M(E, C, S))$ of order-ideals of $M(E, C, S)$. Specialising to the nonseparated graph E, there is a lattice isomorphism

$$\mathcal{H} \cong \mathcal{L}(M_E) \tag{5-19}$$

between the lattice $\mathcal{H}$ of admissible pairs (H, S) of E^0 and the lattice $\mathcal{L}(M_E)$ of order-ideals of the monoid M_E.

Let E be a finite graph with no sinks. There is a one-to-one correspondence [Hazrat 2013a, Theorem 12] between the set of hereditary and saturated subsets of E^0 and the set of graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$. The main theorem of this section describes a one-to-one correspondence between the set of admissible pairs (H, S) of vertices and the set of graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$ for an arbitrary graph E. To prove it, we first need to extend [Ara et al. 2007, Lemma 4.3] to arbitrary graphs. This may also be useful in other situations.

Lemma 5.9. *Let E be an arbitrary graph and denote by F the free abelian group generated by $E^0 \cup \{q_Z\}$, where Z ranges over all the nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. Let $\sim$ be the congruence on F such that $F/{\sim} = M_E$. Let $\to_1$ be the relation on F defined by $v+\alpha \to_1 \sum_{e\in s^{-1}(v)} r(e)+\alpha$ if v is a regular vertex in E, $v+\alpha \to_1 r(z)+q_{\{z\}}+\alpha$ if $v \in E^0$ is an infinite emitter and $z \in s^{-1}(v)$, and also $q_Z+\alpha \to_1 r(z)+q_{Z\cup\{z\}}+\alpha$, if Z is a nonempty finite subset of $s^{-1}(v)$ for an infinite emitter v and $z \in s^{-1}(v) \setminus Z$. Let $\to$ be the transitive and reflexive closure of $\to_1$. Then $\alpha \sim \beta$ in F if and only if there is a $\gamma \in F$ such that $\alpha \to \gamma$ and $\beta \to \gamma$.*

Proof. As in [Ara and Goodearl 2015, Alternative proof of Theorem 4.1], we write $M_E = \lim M(E', C', T')$, where E' ranges over all the finite complete subgraphs of E and

$$C' = \{s_{E'}^{-1}(v) \mid v \in (E')^0,\ |s_{E'}^{-1}(v)| > 0\}, \quad T' = \{s_{E'}^{-1}(v) \in C' \mid v \in (E')^0,\ 0 < |s_E^{-1}(v)| < \infty\}.$$

Applying [Ara and Goodearl 2012, Construction 5.3], we get that $M(E', C', T') = M_{\widetilde{E}}$ for some finite graph $\widetilde{E}$. The vertices of $\widetilde{E}$ are the vertices of E and the elements of the form q_Z, where $Z \in C' \setminus T'$, and there is a new edge $e_Z : v \to q_Z$ if the source of Z is v. If $\alpha \sim \beta$ in F, then $[\alpha] = [\beta]$ in M_E, and so there is (E', C', T') as above such that $[\alpha] = [\beta]$ in $M(E', C', T')$. But since $M(E', C', T') = M_{\widetilde{E}}$, and $\widetilde{E}$ is finite, we conclude from [Ara et al. 2007, Lemma 4.3] that there is an element γ in the free monoid on $(E')^0 \cup \{q_Z \mid Z \in C' \setminus T'\}$ such that $\alpha \to \gamma$ and $\beta \to \gamma$. This implies that $\alpha \to \gamma$ and $\beta \to \gamma$ in F. $\square$

Lemma 5.10. *Let E be an arbitrary graph and K a field. Consider $L_K(E)$ as a graded ring with the grading determined by the function $w : E^1 \to \mathbb{Z}$ such that $w(e) = 1$ for all e. Let $\mathcal{L}^c(M_E^{\mathrm{gr}})$ be the set of order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action. Let $\pi : M_E^{\mathrm{gr}} \to M_E$ be the canonical surjective homomorphism. Then the map $\phi\colon \mathcal{L}(M_E) \to \mathcal{L}^c(M_E^{\mathrm{gr}})$ defined by $\phi(I) = \pi^{-1}(I)$ is a lattice isomorphism.*

Proof. It is easy to show that the map ϕ is well-defined. The key to show the result is to prove the equality $\pi^{-1}(\pi(J)) = J$ for any $J \in L^c(M_E^{\mathrm{gr}})$. The inclusion $J \subseteq \pi^{-1}(\pi(J))$ is obvious. To show the reverse inclusion $\pi^{-1}(\pi(J)) \subseteq J$, denote by F the free abelian group on $E^0 \cup \{q_Z\}$, where Z ranges over all the nonempty finite subsets of $s^{-1}(v)$ for infinite emitters v. Take $z \in \pi^{-1}(\pi(J))$. Then there is $y \in J$ such that $\pi(z) = \pi(y)$. Now write

$$z = \sum_i a_{v_i}(\gamma_i) + \sum_j b_{Z_j}(\lambda_j), \quad y = \sum_i a_{v_i'}(\gamma_i') + \sum_j b_{Z_j'}(\lambda_j').$$

Then we have $\sum_i v_i + \sum_j q_{Z_j} = \pi(z) = \pi(y) = \sum_i v_i' + \sum_j q_{Z_j'}$. By Lemma 5.9, there is $x = \sum_i w_i + \sum_j q_{W_j}$ such that $\pi(z) \to x$ and $\pi(y) \to x$ in F. Now using the same changes than in the paths $\pi(y) \to x$ and $\pi(z) \to x$, but lifted to M_E^{gr}, we obtain that $y = \sum_i a_{w_i}(\eta_i) + \sum_j b_{W_j}(\nu_j)$ in M_E^{gr} and $z = \sum_i a_{w_i}(\eta_i') + \sum_j b_{W_j}(\nu_j')$ in M_E^{gr}. But now $y \in J$ and J is an order ideal of M_E^{gr}, so it follows that $a_{w_i}(\eta_i) \in J$ for all i and $b_{W_j}(\nu_j) \in J$ for all j. Using that J is invariant, we obtain $a_{w_i}(\eta_i') \in J$ for all i and $b_{W_j}(\nu_j') \in J$ for all j. Thus $z = \sum_i a_{w_i}(\eta_i') + \sum_j b_{W_j}(\nu_j') \in J$ and we conclude the proof.

Now using that $J = \pi^{-1}(\pi(J))$, we can easily show that $\pi(J)$ is an order-ideal of M_E and that the map ϕ is bijective, with $\phi^{-1}(J) = \pi(J)$. □

We can now state the main theorem of this section, which indicates that the graded K_0-group captures the lattice structure of graded ideals of a Leavitt path algebra.

Theorem 5.11. *Let E be an arbitrary graph and K a field. Consider $L_K(E)$ as a graded ring with the grading determined by the function $w : E^1 \to \mathbb{Z}$ such that $w(e) = 1$ for all e. Then there is a one-to-one correspondence between the admissible pairs of E^0 and the graded ordered ideals of $K_0^{\mathrm{gr}}(L_K(E))$.*

Proof. Let $\mathcal{H}$ be the set of all admissible pairs of E^0 and $\mathcal{L}(K_0^{\mathrm{gr}}(\mathcal{A}))$ the set of all graded ordered ideals of $K_0^{\mathrm{gr}}(\mathcal{A})$, where $\mathcal{A} = L_K(E)$. We first claim that there is a one-to-one correspondence between the order-ideals of M_E and order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action. Let $\mathcal{L}^c(M_E^{\mathrm{gr}})$ be the set of order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action.

The map $\phi : \mathcal{L}(M_E) \to \mathcal{L}^c(M_E^{\mathrm{gr}})$ has been defined in Lemma 5.10, where it is proved that it is a lattice isomorphism.

By Corollary 5.8, we have an injective homomorphism $\mathcal{V}^{\mathrm{gr}}(\mathcal{A}) \to K_0^{\mathrm{gr}}(\mathcal{A})$. By Proposition 5.7, there is a one-to-one correspondence between the order-ideals of M_E^{gr} which are closed under the $\mathbb{Z}$-action and the graded ordered ideals of $K_0^{\mathrm{gr}}(\mathcal{A})$. Finally by (5-19), we have lattice isomorphisms

$$\mathcal{H} \cong \mathcal{L}(M_E) \cong \mathcal{L}^c(M_E^{\mathrm{gr}}) \cong \mathcal{L}(K_0^{\mathrm{gr}}(\mathcal{A})).$$ □

6. Application: Kumjian–Pask algebras

In this section we will use our result on smash products (Theorem 3.4) to study the structure of Kumjian–Pask algebras [Aranda Pino et al. 2013] and their graded K-groups. We will see that the graded K_0-group remains a useful invariant for studying Kumjian–Pask algebras. We deal exclusively with row-finite k-graphs with no sources: our analysis for arbitrary graphs relied on constructions like desingularisation that are not available in general for k-graphs. We briefly recall the definition of Kumjian–Pask algebras and establish our notation. We follow the conventions used in the literature of this topic (in particular the paths are written from right to left).

Recall that a *graph of rank k* or k-graph is a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the following factorisation property: if $\lambda \in \Lambda$ and $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$. We say that Λ is *row finite* if $r^{-1}(v) \cap d^{-1}(n)$, abbreviated $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$; we say that Λ has *no sources* if each $v\Lambda^n$ is nonempty.

An important example is the k-graph Ω_k defined as a set by $\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$ with $d(m, n) = n - m,\ \Omega_k^0 = \mathbb{N}^k,\ r(m, n) = m,\ s(m, n) = n$ and $(m, n)(n, p) = (m, p)$.

Definition 6.1. Let Λ be a row-finite k-graph without sources and K a field. The *Kumjian–Pask K-algebra* of Λ is the K-algebra $\mathrm{KP}_K(\Lambda)$ generated by $\Lambda \cup \Lambda^*$ subject to the relations

(KP1) $\{v \in \Lambda^0\}$ is a family of mutually orthogonal idempotents satisfying $v = v^*$,

(KP2) for all $\lambda, \mu \in \Lambda$ with $r(\mu) = s(\lambda)$, we have

$$\lambda\mu = \lambda \circ \mu, \quad \mu^*\lambda^* = (\lambda \circ \mu)^*, \quad r(\lambda)\lambda = \lambda = \lambda s(\lambda), \quad s(\lambda)\lambda^* = \lambda^* = \lambda^* r(\lambda),$$

(KP3) for all $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$, we have

$$\lambda^*\mu = \delta_{\lambda,\mu} s(\lambda),$$

(KP4) for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k \setminus \{0\}$, we have

$$v = \sum_{\lambda \in v\Lambda^n} \lambda\lambda^*.$$

Let Λ be a row-finite k-graph without sources and $\mathrm{KP}_K(\Lambda)$ the Kumjian–Pask algebra of Λ. Following [Kumjian and Pask 2000, §2], an infinite path in Λ is a degree-preserving functor $x : \Omega_k \to \Lambda$. Denote the set of all infinite paths by Λ^∞. We define the relation of *tail equivalence* on the space of infinite path Λ^∞ as follows: for $x, y \in \Lambda^\infty$, we say x is tail equivalent to y, denoted, $x \sim y$, if $x(n, \infty) = y(m, \infty)$, for some $n, m \in \mathbb{N}^k$. This is an equivalence relation. For $x \in \Lambda^\infty$, we denote by $[x]$ the equivalence class of x, i.e., the set of all infinite paths which are tail equivalent to x. An infinite path x is called *aperiodic* if $x(n, \infty) = x(m, \infty),\ n, m \in \mathbb{N}^k$, implies $n = m$.

We can form the skew-product k-graph, or covering graph, $\bar{\Lambda} = \Lambda \times_d \mathbb{Z}^k$ which is equal as a set to $\Lambda \times \mathbb{Z}^k$, has degree map given by $\bar{d}(\lambda, n) = d(\lambda)$, range and source maps $r(\lambda, n) = (r(\lambda), n)$ and $s(\lambda, n) = (s(\lambda), n + d(\lambda))$ and composition given by $(\lambda, n)(\mu, n + d(\lambda)) = (\lambda\mu, n)$.

As in the theory of Leavitt path algebras, one can model Kumjian–Pask algebras as Steinberg algebras via the infinite-path groupoid of the k-graph (see [Clark and Pangalela 2017, Proposition 5.4]). For the k-graph Λ,

$$\mathcal{G}_\Lambda = \big\{(x, l - m, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid x(l, \infty) = y(m, \infty)\big\}.$$

Define range and source maps $r, s : \mathcal{G}_\Lambda \to \Lambda^\infty$ by $r(x, n, y) = x$ and $s(x, n, y) = y$. The multiplication and inverse are given for $(x, n, y), (y, l, z) \in \mathcal{G}_\Lambda$, by $(x, n, y)(y, l, z) = (x, n + l, z)$ and $(x, n, y)^{-1} = (y, -n, x)$. $\mathcal{G}_\Lambda$ is a groupoid with $\Lambda^\infty = \mathcal{G}_\Lambda^{(0)}$ under the identification $x \mapsto (x, 0, x)$. For

$\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, let $Z(\mu, \nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) \mid x \in \Lambda^\infty, x(0) = s(\mu)\}$. Then the sets $Z(\mu, \nu)$ comprise a basis of compact open sets for an ample Hausdorff topology on $\mathcal{G}_\Lambda$. There is a cocycle $c : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ given by $c(x, m, y) = m$.

For the skew-product k-graph $\bar{\Lambda} = \Lambda \times_d \mathbb{Z}^k$, we have $\mathcal{G}_{\bar{\Lambda}} \cong \mathcal{G}_\Lambda \times_c \mathbb{Z}^k$ (see [Kumjian and Pask 2000, Theorem 5.2]). Thus specialising Theorem 3.4 to this setting, we have

$$\mathrm{KP}_K(\bar{\Lambda}) \cong \mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k. \tag{6-1}$$

We will show that $\mathrm{KP}_K(\bar{\Lambda})$ is an ultramatricial algebra.

Given a set X and a ring R, $M_X(R)$ denotes the collection of finitely supported $X \times X$ matrices with values in R; that is, $M_X(R)$ consists of finitely supported functions from $X \times X$ to R such that the multiplication is given by $(ab)(x, y) = \sum_{z \in X} a(x, z)b(z, y)$.

Lemma 6.2. *For $n \in \mathbb{Z}^k$ define $B_n \subseteq \mathrm{KP}_K(\bar{\Lambda})$ by*

$$B_n = \mathrm{span}_K\big\{(\lambda, n - d(\lambda))(\mu, n - d(\mu))^* \mid \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\big\}.$$

Then B_n is a subalgebra of $\mathrm{KP}_K(\bar{\Lambda})$ and there is an isomorphism $B_n \cong \bigoplus_{v \in \Lambda^0} M_{\Lambda v}(K)$ that carries $(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*$ to the matrix unit $\boldsymbol{e}_{\lambda,\mu}$.

Proof. For the first statement we just have to show that for any $\lambda, \mu, \eta, \zeta \in \Lambda$ we have

$$(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*(\eta, n - d(\eta))(\zeta, n - d(\zeta))^* \in B_n.$$

This follows from the argument of [Kumjian and Pask 2000, Lemma 5.4]. To wit, we have

$$(\mu, n - d(\mu))^*(\eta, n - d(\eta)) = 0 \quad \text{unless } r(\mu, n - d(\mu)) = r(\eta, n - d(\eta)),$$

which in turn forces $d(\mu) = d(\eta)$. But then $\bar{d}(\mu, n - d(\mu)) = \bar{d}(\eta, n - d(\eta))$, and then the Cuntz–Krieger relation forces $(\mu, n - d(\mu))^*(\eta, n - d(\eta)) = \delta_{\mu,\eta}(s(\mu), n)$. Hence

$$(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*(\eta, n - d(\eta))(\zeta, n - d(\zeta))^* = \delta_{\mu,\eta}(\lambda, n - d(\lambda))(\zeta, n - d(\zeta))^* \in B_n.$$

For each $v \in \Lambda^0$, $M_{\bar{\Lambda}(v,n)}(K) \cong M_{\Lambda v}(K)$. So the elements $(\lambda, n - d(\lambda))(\mu, n - d(\mu))^*$ satisfy the same multiplication formula as the matrix units $\boldsymbol{e}_{\lambda,\mu}$ in $\bigoplus_{v \in \Lambda^0} M_{\Lambda v}(K)$. Hence the uniqueness of the latter shows that there is an isomorphism as claimed. □

Lemma 6.3. *For $m \le n \in \mathbb{Z}^k$, we have $B_m \subseteq B_n$, and in particular for each $v \in \Lambda^0$, we have $(v, m) = \sum_{\alpha \in v\Lambda^{n-m}} (\alpha, m)(\alpha, m)^*$.*

Proof. Again, this follows from the proof of [Kumjian and Pask 2000, Lemma 5.4]. We just apply the Cuntz–Krieger relation, using at the first equality that Λ has no sources:

$$\begin{aligned}(\lambda, m - d(\lambda))(\mu, m - d(\mu))^* &= (\lambda, m - d(\lambda))\Big(\sum_{\alpha \in s(\lambda)\Lambda^{n-m}} (\alpha, m)(\alpha, m)^*\Big)(\mu, m - d(\mu))^* \\ &= \sum_{\alpha \in s(\lambda)\Lambda^{n-m}} (\lambda\alpha, m - d(\lambda))(\mu\alpha, m - d(\mu))^* \in B_n.\end{aligned}$$

This gives the first assertion, and the second follows by taking $\lambda = \mu = v$. □

Theorem 6.4. *Let Λ be a row-finite k-graph with no sources and K a field. Then the Kumjian–Pask algebra $\mathrm{KP}_K(\Lambda)$ is a graded von Neumann regular ring.*

Proof. Lemma 2.3 shows that $\mathrm{KP}_K(\Lambda)$ is graded regular if and only if $\mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k$ is graded regular. By (6-1) $\mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k \cong \mathrm{KP}_K(\bar{\Lambda})$ and the latter is an ultramatricial algebra by Lemma 6.3. Since ultramatricial algebras are regular, the theorem follows. □

Since $\mathrm{KP}_K(\Lambda)$ is graded von Neumann regular, we immediately obtain the following statements.

Theorem 6.5. *Let Λ be a row-finite k-graph with no sources and K a field. Then the Kumjian–Pask algebra $A = \mathrm{KP}_K(\Lambda)$ has the following properties*:

(1) *any finitely generated right* (*left*) *graded ideal of A is generated by one homogeneous idempotent*;

(2) *any graded right* (*left*) *ideal of A is idempotent*;

(3) *any graded ideal is graded semiprime*;

(4) $J(A) = J^{\mathrm{gr}}(A) = 0$; *and*

(5) *there is a one-to-one correspondence between the graded right* (*left*) *ideals of A and the right* (*left*) *ideals of A_0.*

Proof. All the assertions are the properties of a graded von Neumann regular ring [Hazrat 2016, §1.1.9], so the result follows from Theorem 6.4. □

For the next result, given a k-graph Λ, and given $m \le n \in \mathbb{Z}^k$, we define $\phi_{m,n} : \mathbb{N}\Lambda^0 \to \mathbb{N}\Lambda^0$ by $\phi_{m,n}(v) = \sum_{w \in \Lambda^0} |v\Lambda^{n-m}w| w$. Here, $\mathbb{N}\Lambda^0$ is the abelian monoid freely generated by Λ^0, and $\phi_{m,n}$ is the unique monoid homomorphism determined by the above rule.

Corollary 6.6. *Let Λ be a row-finite k-graph with no sources and K a field. There is an isomorphism*

$$\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda})) \cong \varinjlim_{\mathbb{Z}^k} \big(\mathbb{N}\Lambda^0, \phi_{m,n}\big)$$

that carries $[(v,n)]$ to the copy of v in the n-th copy of $\mathbb{N}\Lambda^0$. Furthermore, the monoid $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$ is cancellative.

Proof. It is standard that there is an isomorphism $\mathcal{V}\big(\bigoplus_{v \in \Lambda^0} M_{\Lambda v}(K)\big) \cong \mathbb{N}\Lambda^0$ that takes $\boldsymbol{e}_{\lambda,\lambda}$ to $s(\lambda)$ for all λ. So Lemma 6.2 implies that there is an isomorphism $\mathcal{V}(B_n) \to \mathbb{N}\Lambda^0$ that carries $[(\lambda, n-d(\lambda))(\lambda, n-d(\lambda))^*]$ to $s(\lambda)$ for all λ. Let S_n be a copy $\mathbb{N}\Lambda^0 \times \{n\}$ of the monoid $\mathbb{N}\Lambda^0$ (so $(a,n)+(b,n) = (a+b,n)$ in S_n). Lemma 6.3 shows that these isomorphisms of monoids carry the inclusions $B_m \hookrightarrow B_n$ to the maps $(v,m) \mapsto \sum_{\lambda \in v\Lambda^{n-m}} (s(\lambda), n)$, which is precisely given by the formula $\phi_{m,n}$ for $m \le n \in \mathbb{Z}^k$. Since the monoid of a direct limit is the direct limit of the monoids of the approximating algebras, we have an isomorphism $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda})) \cong \varinjlim_{\mathbb{Z}^k} S_n$, which sends $[(v,n)]$ to $(v,n) \in S_n$.

Suppose that $x+z = y+z$ in $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$. By the isomorphism $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda})) \cong \varinjlim_{\mathbb{Z}^k} S_n$, there exist images x', y', z' of x, y, z, respectively, in $S_{n_0} = \mathbb{N}\Lambda^0 \times \{n_0\}$ for some $n_0 \in \mathbb{Z}^k$ such that $x'+z' = y'+z'$. The monoid $\mathbb{N}\Lambda^0$ is cancellative, so $\mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$ is too. □

Corollary 6.7. *Let Λ be a row-finite k-graph with no sources and K a field. Then*

$$\mathcal{V}^{\mathrm{gr}}(\mathrm{KP}_K(\Lambda)) \cong \varinjlim_{\mathbb{Z}^k}(\mathbb{N}\Lambda^0, \phi_{m,n}).$$

Proof. Recall from (6-1) that $\mathrm{KP}_K(\bar{\Lambda}) \cong \mathrm{KP}_K(\Lambda) \# \mathbb{Z}^k$. Specialising Proposition 2.5 to Kumjian–Pask algebras, we have the isomorphism of categories $\Psi : \mathrm{KP}_K(\Lambda)\text{-Gr} \xrightarrow{\sim} \mathrm{KP}_K(\bar{\Lambda})\text{-Mod}$. We argue as in the directed-graph situation that Ψ preserves finitely generated projective objects. By (4-2) and (4-4), we have $\mathcal{V}^{\mathrm{gr}}(\mathrm{KP}_K(\Lambda)) \cong \mathcal{V}(\mathrm{KP}_K(\bar{\Lambda}))$. □

7. The graded representations of the Steinberg algebra

In this section, for a Γ-graded groupoid $\mathcal{G}$ and its associated Steinberg algebra $A_R(\mathcal{G})$, we construct graded simple $A_R(\mathcal{G})$-modules. Specialising our results to the trivial grading, we obtain irreducible representations of (ungraded) Steinberg algebras. We determine the ideals arising from these representations and prove that these ideals relate to the effectiveness or otherwise of the groupoid.

7A. *Representations of a Steinberg algebra.* Let $\mathcal{G}$ be an ample Hausdorff groupoid, let Γ be a discrete group with identity ε, and let $c : \mathcal{G} \to \Gamma$ be a cocycle. A subset U of the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is *invariant* if $d(\gamma) \in U$ implies $r(\gamma) \in U$; equivalently,

$$r(d^{-1}(U)) = U = d(r^{-1}(U)).$$

Given an element $u \in \mathcal{G}^{(0)}$, we denote by $[u]$ the smallest invariant subset of $\mathcal{G}^{(0)}$ which contains u. Then

$$r(d^{-1}(u)) = [u] = d(r^{-1}(u)).$$

That is, for any $v \in [u]$, there exists $x \in \mathcal{G}$ such that $d(x) = u$ and $r(x) = v$; equivalently, for any $w \in [u]$, there exists $y \in \mathcal{G}$ such that $d(y) = w$ and $r(y) = u$. Thus for any $v, w \in [u]$, there exists $x \in \mathcal{G}$ such that $d(x) = v$ and $r(x) = w$. We call $[u]$ an *orbit*. Observe that an invariant subset $U \subseteq \mathcal{G}^{(0)}$ is an orbit if and only if for any $v, w \in U$, there exists $x \in \mathcal{G}$ such that $d(x) = v$ and $r(x) = w$.

Lemma 7.1. *Let $u_1, u_2, \ldots, u_n$ be pairwise distinct elements of $\mathcal{G}^{(0)}$ with $n \geq 2$. Then there exist disjoint compact open bisections $B_i \subseteq \mathcal{G}^{(0)}$ such that $u_i \in B_i$ for each $i = 1, \ldots, n$.*

Proof. Since $\mathcal{G}^{(0)}$ is a Hausdorff space, there exist disjoint open subsets X_i of $\mathcal{G}^{(0)}$ such that $u_i \in X_i$ for all i. Since $\mathcal{G}$ is ample, we can choose compact open bisections $B_i \subseteq X_i$ such that $u_i \in B_i$ for all i. □

The *isotropy group* at a unit u of $\mathcal{G}$ is the group $\mathrm{Iso}(u) = \{\gamma \in \mathcal{G} \mid d(\gamma) = r(\gamma) = u\}$. A unit $u \in \mathcal{G}^{(0)}$ is called Γ-*aperiodic* if $\mathrm{Iso}(u) \subseteq c^{-1}(\varepsilon)$, otherwise u is called Γ-*periodic*. For an invariant subset $W \subseteq \mathcal{G}^{(0)}$, we denote by W_{ap} the collection of Γ-aperiodic elements of W and by W_{p} the collection of Γ-periodic elements of W. Then

$$W = W_{\mathrm{ap}} \bigsqcup W_{\mathrm{p}}.$$

If $W = W_{\mathrm{ap}}$, we say that W is Γ-*aperiodic*; If $W = W_{\mathrm{p}}$, we say that W is Γ-*periodic*.

Remark 7.2. Let E be a directed graph. Let $\mathcal{G}_E$ be the associated graph groupoid and $c : \mathcal{G}_E \to \mathbb{Z}$ the canonical cocycle $c(x, m, y) = m$. It was shown in [Kumjian et al. 1997] that $c^{-1}(0)$ is a principal groupoid, in the sense that $\mathrm{Iso}(c^{-1}(0)) = \mathcal{G}_E^{(0)}$. Hence $x \in \mathcal{G}_E^{(0)} = E^\infty$ is $\mathbb{Z}$-aperiodic if and only if $\mathrm{Iso}(x) = \{x\}$. It is standard that $\mathrm{Iso}(x) = \{x\}$ if and only if $x \neq \mu\lambda^\infty$ for any cycle λ in E. So x is $\mathbb{Z}$-aperiodic if and only if $x \neq \mu\lambda^\infty$ for any cycle λ.

Lemma 7.3. *Let $W \subseteq \mathcal{G}^{(0)}$ be an invariant subset. Then W_{ap} and W_{p} are both invariant subsets of $\mathcal{G}^{(0)}$.*

Proof. For $x \in \mathcal{G}$, let $u = d(x)$ and $v = r(x)$. Suppose that $u \in W_{\mathrm{ap}}$. If $c(y) \neq \varepsilon$ for some $y \in \mathrm{Iso}(v)$, then $x^{-1}yx \in \mathrm{Iso}(u)$ and $\varepsilon \neq c(y) = c(x)c(x^{-1}yx)c(x)^{-1}$, forcing $c(x^{-1}yx) \neq \varepsilon$, a contradiction. Hence, $v = r(x)$ is Γ-aperiodic. Since W is invariant, we have $v \in W_{\mathrm{ap}}$. So W_{ap} is invariant. Since $W = W_{\mathrm{ap}} \sqcup W_{\mathrm{p}}$, it follows that W_{p} is also invariant. □

By the proof of Lemma 7.3, $u \in \mathcal{G}^{(0)}$ is Γ-aperiodic if and only if its orbit $[u]$ is Γ-aperiodic.

Example 7.4. In this example we construct a $\mathbb{Z}$-aperiodic invariant subset which is neither open nor closed in $\mathcal{G}^{(0)}$. Let E be the following directed graph.

$$\begin{tikzcd} 1 \arrow[loop above, "\beta"] \arrow[loop below, "\alpha"] \arrow[r, "\lambda"] & 2 \arrow[loop above, "\gamma"] \arrow[loop below, "\delta"] \end{tikzcd}$$

Let u be the infinite path $\alpha\beta\alpha^2\beta\alpha^3\beta\cdots$. Then u is an element in $\mathcal{G}_E^{(0)}$. The orbit $[u]$ consists of all infinite paths tail equivalent to u. So $\alpha^n u \in [u]$ for all $n \in \mathbb{N}$. The sequence $\alpha^n u$ converges to α^∞, which does not belong to $[u]$. So $[u]$ is not closed. Similarly, the points $u_n := \alpha\beta\alpha^2\beta\cdots\alpha^n\beta\alpha^\infty$ all belong to $\mathcal{G}^{(0)} \setminus [u]$, but $u_n \to u$, so $[u]$ is not open. In particular, neither $[u]$ nor its complement is the invariant subset of $\mathcal{G}^{(0)}$ corresponding to any saturated hereditary subset of E^0.

We will employ Γ-aperiodic invariant subsets of $\mathcal{G}^{(0)}$ to obtain graded representations for the Steinberg algebra $A_R(\mathcal{G})$. For any invariant subset $U \subseteq \mathcal{G}^{(0)}$ and a unital commutative ring R, we denote by RU the free R-module with basis U. For every compact open bisection $B \subseteq \mathcal{G}$, there is a function $f_B : \mathcal{G}^{(0)} \to RU$ which has support contained in $d(B) \cap U$ and $f_B(d(\gamma)) = r(\gamma)$ for all $\gamma \in B \cap d^{-1}(U)$. There is a unique representation $\pi_U : A_R(\mathcal{G}) \to \mathrm{End}_R(RU)$ such that

$$\pi_U(1_B)(u) = f_B(u), \tag{7-1}$$

for every compact open bisection B and $u \in U$. This representation makes RU an $A_R(\mathcal{G})$-module (see [Brown et al. 2014, Proposition 4.3]). An $A_R(\mathcal{G})$-submodule $V \subseteq RU$ is called a *basic submodule* of RU if whenever $r \in R \setminus \{0\}$ and $ru \in V$, we have $u \in V$. We say an $A_R(\mathcal{G})$-module is *basic simple* if it has no nontrivial basic submodules.

We can state one of the main results of this section.

Theorem 7.5. *Let U be an invariant subset of $\mathcal{G}^{(0)}$. Then U is a Γ-aperiodic orbit if and only if RU is a graded basic simple $A_R(\mathcal{G})$-module. Furthermore, RU is a graded basic simple $A_R(\mathcal{G})$-module if and only if it is graded and basic simple.*

Proof. Suppose that $u \in \mathcal{G}^{(0)}$ satisfies $U = [u]$, and that $[u]$ is a Γ-aperiodic orbit. We first show that $R[u]$ is a Γ-graded $A_R(\mathcal{G})$-module. For any $\gamma \in \Gamma$, set

$$[u]_\gamma = \{v \in [u] \mid \text{ there exists } x \in \mathcal{G} \text{ such that } c(x) = \gamma,\ d(x) = u \text{ and } r(x) = v\}.$$

We claim that $[u]_\gamma \cap [u]_{\gamma'} \neq \varnothing$ implies $\gamma = \gamma'$. Indeed, if $v \in [u]_\gamma \cap [u]_{\gamma'}$, then there exist $x \in c^{-1}(\gamma)$ and $y \in c^{-1}(\gamma')$ such that $d(x) = d(y) = u$ and $r(x) = r(y) = v$. Now $x^{-1}y \in \operatorname{Iso}(u)$. Since u is Γ-aperiodic this forces $\gamma^{-1}\gamma' = c(x^{-1}y) = \varepsilon$, and so $\gamma = \gamma'$. This gives a partition $[u] = \bigsqcup_{\gamma \in \Gamma} [u]_\gamma$. Therefore $A_R(\mathcal{G})$-module $R[u]$ has a decomposition of R-modules

$$R[u] = \bigoplus_{\gamma \in \Gamma} (R[u])_\gamma,$$

where $(R[u])_\gamma$ is a free R-module with basis $[u]_\gamma$.

We show that $A_R(\mathcal{G})_\alpha \cdot (R[u])_\gamma \subseteq (R[u])_{\alpha\gamma}$, for $\alpha, \gamma \in \Gamma$. Fix $v \in [u]_\gamma$ and $B \in B^{\mathrm{co}}_\alpha(\mathcal{G})$. We use $\cdot$ to denote the action of $A_R(\mathcal{G})$ on RU. We have

$$1_B \cdot v = \begin{cases} r(b) & \text{if } b \in B \text{ satisfies } d(b) = v, \\ 0 & \text{if } v \notin d(B). \end{cases}$$

Clearly $0 \in (R[u])_{\alpha\gamma}$, so suppose that $b \in B$ satisfies $d(b) = v$. Since $v \in [u]_\gamma$, there exists $x \in \mathcal{G}$ such that $c(x) = \gamma$, $d(x) = u$, and $r(x) = v$. Now $d(bx) = u$, $r(bx) = r(b)$, and $c(bx) = c(b)c(x) = \alpha\gamma$. So $r(b) \in [u]_{\alpha\gamma}$. Since elements of the form 1_B where $B \in B^{\mathrm{co}}_\alpha(\mathcal{G})$ span $A_R(\mathcal{G})_\alpha$, we deduce that $A_R(\mathcal{G})_\alpha \cdot (R[u])_\gamma \subseteq (R[u])_{\alpha\gamma}$ as claimed.

Next we show that $R[u]$ is a basic simple $A_R(\mathcal{G})$-module. Suppose that $V \neq 0$ is a basic $A_R(\mathcal{G})$-submodule of $R[u]$. Take a nonzero element $x \in V$. Fix nonzero elements $r_i \in R$ and pairwise distinct $u_i \in [u]$ such that $x = \sum_{i=1}^m r_i u_i$. By Lemma 7.1, there exist disjoint compact open bisections $B_i \subseteq \mathcal{G}^{(0)}$ such that $u_i \in B_i$ for all $i = 1, \dots, m$. Now

$$1_{B_1} \cdot x = 1_{B_1} \cdot \sum_{i=1}^m r_i u_i = \sum_{i=1}^m r_i (1_{B_1} \cdot u_i) = r_1 f_{B_1}(u_1).$$

Thus $u_1 = f_{B_1}(u_1) \in V$, because V is a basic submodule. Fix $v \in [u]$ and choose $x \in \mathcal{G}$ such that $d(x) = u_1$ and $r(x) = v$. Fix a compact open bisection D containing x. Then $1_D \cdot u_1 = f_D(u_1) = r(x) = v \in V$, giving $V = R[u]$. Thus $R[u]$ is basic simple, and consequently graded basic simple.

For the converse suppose that RU is a graded basic simple $A_R(\mathcal{G})$-module. We first show that U is Γ-aperiodic. Let $u \in U$. We claim that there exists $r \in R \setminus \{0\}$ such that ru is a homogeneous element of RU. To see this, express $u = \sum_{i=1}^l h_i$, where $h_i \neq u$ are homogeneous elements. For each i, express $h_i = \sum_{j=1}^{s_i} \lambda_{ij} u_{ij}$ with $\lambda_{ij} \in R \setminus \{0\}$ and the $u_{ij} \in U$ pairwise distinct. We first show that $u \in \{u_{ij} \mid i = 1, \dots, l;\ j = 1, \dots, s_i\}$; for if not, then Lemma 7.1 gives compact open bisections B, B_{ij} such that $u \in B$ and $u \notin B_{ij}$ for all i, j. So $1_B \cdot u \neq 0$, whereas

$$1_B \cdot u = 1_B \cdot \sum_{i=1}^l h_i = 1_B \cdot \sum_{i=1}^l \sum_{j=1}^{s_i} \lambda_{ij} u_{ij} = \sum_{i=1}^l \sum_{j=1}^{s_i} \lambda_{ij} 1_B \cdot u_{ij} = 0.$$

This is a contradiction. So $u = u_{ij}$ for some i, j as claimed; without loss of generality, $u = u_{11}$. Hence $h_1 = \lambda_{11} u + \sum_{j=2}^{s_1} \lambda_{1j} u_{1j}$. There exist compact open bisections B', $B'_{1j} \subseteq \mathcal{G}^{(0)} \subseteq c^{-1}(\varepsilon)$ such that $u \in B'$ but $u \notin B'_{1j}$ for $j \neq 1$. Hence $r := \lambda_{11}$ belongs to $R \setminus \{0\}$, and

$$ru = \lambda_{11} 1_{B'} \cdot u = 1_{B'} \cdot h_1$$

is homogeneous as claimed. Now suppose that u is not Γ-aperiodic. Then there exists $x \in \mathrm{Iso}(u)$ with $c(x) \neq \varepsilon$. Fix $D \in B^{\mathrm{co}}_{c(x)}(\mathcal{G})$ containing x. Then $1_D \cdot ru = r 1_D \cdot u = ru$ is homogeneous. Thus $1_D \in A_R(\mathcal{G})_\varepsilon$, forcing $c(x) = \varepsilon$. This is a contradiction. Thus U is Γ-aperiodic.

For the last part of the theorem we prove that U is an orbit. If not then there exist $u, v \in \mathcal{G}^{(0)}$ with $[u] \cap [v] = \varnothing$ and $[u] \sqcup [v] \subseteq U$. Hence $R[u] \subseteq RU \setminus R[v]$ is a nontrivial proper graded basic submodule of RU by the first part of the theorem. This is a contradiction. So U is an orbit. The last statement of the theorem follows from the first part of the proof. □

Corollary 7.6. *Let $\mathcal{G}$ be an ample Hausdorff groupoid, and U be an invariant subset of $\mathcal{G}^{(0)}$. Then U is an orbit of $\mathcal{G}^{(0)}$ if and only if RU is a basic simple $A_R(\mathcal{G})$-module.*

Proof. Apply Theorem 7.5 with $c : \mathcal{G} \to \{\varepsilon\}$ the trivial grading. □

Specialising Theorem 7.5 to the case of Leavitt path algebras we obtain irreducible representations for these algebras.

Let K be a field. For an infinite path p in a graph E, Chen constructed the left $L_K(E)$-module $\mathcal{F}_{[p]}$ of the space of infinite paths tail-equivalent to p and proved that it is an irreducible representation of the Leavitt path algebra (see [Chen 2015, Theorem 3.3]). These were subsequently called Chen simple modules and further studied in [Abrams et al. 2015; Ara and Rangaswamy 2014; 2015; Hazrat and Rangaswamy 2016; Rangaswamy 2016]. In the groupoid setting, the infinite path p is an element in $\mathcal{G}_E^{(0)}$. Thus q belongs to the orbit $[p]$ if and only if q is tail-equivalent to p. Applying Corollary 7.6, we immediately obtain that $K[p] = \mathcal{F}_{[p]}$ is an irreducible representation of the Leavitt path algebra. Furthermore, by Theorem 7.5, p is an aperiodic infinite path (irrational path) if and only if $\mathcal{F}_{[p]}$ is a graded module (see [Hazrat and Rangaswamy 2016, Proposition 3.6]).

Recall from [Chen 2015, Theorem 3.3] that $\mathrm{End}_{L_K(E)}(\mathcal{F}_{[p]}) \cong K$. We claim that $\mathrm{End}_{A_R(\mathcal{G})}(R[u]) \cong R$ for $u \in \mathcal{G}_E^{(0)}$. Indeed, let $f : R[u] \to R[u]$ be a nonzero homomorphism of $A_R(\mathcal{G})$-modules. Then $\mathrm{Ker} f$ is a basic submodule of $R[u]$. Since $R[u]$ is basic simple, we deduce that f is injective. For $v \in [u]$, we write $f(v) = \sum_{i=1}^{n} r_i v_i$ with $0 \neq r_i \in R$ and v_i are distinct. We prove that $n = 1$ and $v = v_1$. For if not, then we may assume that $v \neq v_1$. By Lemma 7.1, there exist disjoint compact open bisections $B, B_1 \subseteq \mathcal{G}^{(0)}$ such that $v \in B$, $v_1 \in B_1$ and $v_i \notin B_1$ for $i \neq 1$. Then $1_{B_1} \cdot f(v) = f(1_{B_1} \cdot v) = 0$. But, $1_{B_1} \cdot f(v) = 1_{B_1} \cdot \sum_{i=1}^{n} r_i v_i = r_1 v_1$ which is a contradiction.

Likewise, Theorem 7.5 specialises to k-graph groupoids, giving new information about Kumjian–Pask algebras.

Corollary 7.7. *Let Λ be a row-finite k-graph without sources and $\mathrm{KP}_K(\Lambda)$ the Kumjian–Pask algebra of Λ. Then*:

(1) *for an infinite path* $x \in \Lambda^\infty$, $K[x]$ *is a simple left* $\mathrm{KP}_K(\Lambda)$*-module*;

(2) *for* $x, y \in \Lambda^\infty$, *we have* $K[x] \cong K[y]$ *if and only if* $x \sim y$; *and*

(3) *for* $x \in \Lambda^\infty$, $K[x]$ *is a graded module if and only if* x *is an aperiodic path.*

Proof. For (1), the equivalence class of x is the orbit of $\mathcal{G}_\Lambda^{(0)}$ which contains x. By (7-1) and Corollary 7.6, the statement follows directly. For (2), let $\phi : F([x]) \to F([y])$ be an isomorphism. Write $\phi(x) = \sum_{i=1}^{l} r_i y_i$, where $y_i \sim y$ are all distinct. If $x = y_i$, for some i, then by transitivity of $\sim$, $x \sim y$ and we are done. Otherwise one can choose $n \in \mathbb{N}^k$ such that all $y_i(0,n)$ and $x(0,n)$ are distinct. Setting $a = y_1(0,n)$, we have $0 = \phi(a^*x) = a^*\phi(x) = y_1(n,\infty)$, which is not possible unless $x = y_1$ and $l = 1$. This gives that $x \sim y$. The converse is clear. The statement (3) follows immediately by Theorem 7.5. □

7B. ***The annihilator ideals and effectiveness of groupoids.*** In this section, we describe the annihilator ideals of the graded modules over a Steinberg algebra and prove that these ideals reflect the effectiveness of the groupoid.

As in previous sections, we assume that $\mathcal{G}$ is a Γ-graded ample Hausdorff groupoid which has a basis of graded compact open bisections. Let R be a commutative ring with identity and $A_R(\mathcal{G})$ the Γ-graded Steinberg algebra associated to $\mathcal{G}$.

Let $W \subseteq \mathcal{G}^{(0)}$ be an invariant subset. We write $\mathcal{G}_W := d^{-1}(W)$ which coincides with the restriction $\mathcal{G}|_W = \{x \in \mathcal{G} \mid d(x) \in W, r(x) \in W\}$. Notice that $\mathcal{G}_W$ is a groupoid with unit space W.

Observe that the interior W° of an invariant subset W is invariant. Indeed, $r(d^{-1}(W^\circ))$ is an open subset of $\mathcal{G}^{(0)}$, since W° is an open subset of $\mathcal{G}^{(0)}$. Since W is invariant, $r(d^{-1}(W^\circ)) \subseteq W$. Thus $r(d^{-1}(W^\circ)) \subseteq W^\circ$. It follows that the closure W^- of W is also an invariant subset of $\mathcal{G}^{(0)}$, since $W^- = \mathcal{G}^{(0)} \setminus (\mathcal{G}^{(0)} \setminus W)^\circ$.

Recall from (7-1) that

$$\pi_W : A_R(\mathcal{G}) \to \mathrm{End}_R(RW)$$

makes RW an $A_R(\mathcal{G})$-module.

Lemma 7.8. *Let* $W \subseteq \mathcal{G}^{(0)}$ *be an invariant subset of the unit space of* $\mathcal{G}$, *and let* $U = (\mathcal{G}^{(0)} \setminus W)^\circ$. *Then*

$$A_R(\mathcal{G}_U) \subseteq \mathrm{Ann}_{A_R(\mathcal{G})}(RW).$$

Proof. For any $f \in A_R(\mathcal{G}_U)$, we write $f = \sum_{k=1}^{m} r_k 1_{B_k}$ with $B_k \subseteq \mathcal{G}_U$ compact open bisections of $\mathcal{G}$ and $r_k \in R$ nonzero scalars. Since $d(B_k) \subseteq U$, we have $d(B_k) \cap W = \varnothing$. Thus $f \cdot w = 0$ for any $w \in W$, and hence $f \in \mathrm{Ann}_{A_R(\mathcal{G})}(RW)$. □

From now on, $W \subseteq \mathcal{G}^{(0)}$ is a Γ-aperiodic invariant subset. We have

$$W = \bigcup_{u \in W} [u].$$

Of course, two elements of W may belong to the same orbit.

Recall from Theorem 7.5 that if $u \in \mathcal{G}^{(0)}$ is Γ-aperiodic, then $R[u]$ is a Γ-graded $A_R(\mathcal{G})$-module. Therefore RW is a Γ-graded $A_R(\mathcal{G})$-module. In order to construct graded representations for $A_R(\mathcal{G})$, we need to consider the "closed" subgroups of $\operatorname{End}_R(FW)$ defined in (7-1). Namely, we consider the subgroup $\operatorname{END}_R(RW) = \bigoplus_{\gamma\in\Gamma} \operatorname{Hom}_R(RW, RW)_\gamma$, where each component $\operatorname{Hom}_R(RW, RW)_\gamma$ consists of R-maps of degree γ.

Then the map

$$\pi_W : A_R(\mathcal{G}) \to \operatorname{END}_R(RW) \tag{7-2}$$

given by the $A_R(\mathcal{G})$-module action is a homomorphism of Γ-graded algebras. To prove that π_W preserves the grading, fix $\alpha \in \Gamma$ and $B \in B^{\mathrm{co}}_\alpha(\mathcal{G})$. Take $u \in W$ and $v \in [u]$. Fix $x \in \mathcal{G}$ with $d(x) = u$ and $r(x) = v$, and put $\beta = c(x)$ so that $v \in [u]_\beta$. Then

$$\pi_W(1_B)(v) = \begin{cases} r(\gamma) & \text{if } v = d(\gamma) \text{ for some } \gamma \in B,\\ 0 & \text{otherwise.}\end{cases}$$

Since $c(\gamma x) = \alpha\beta$, we obtain $\pi_W(1_B) \in \operatorname{Hom}_R(RW, RW)_\alpha$.

Recall that an ample Hausdorff groupoid $\mathcal{G}$ is *effective* if $\operatorname{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$, where $\operatorname{Iso}(\mathcal{G}) = \bigsqcup_{u\in\mathcal{G}^{(0)}} \operatorname{Iso}(u)$. It follows that $\mathcal{G}$ is effective if and only if for any nonempty $B \in B^{\mathrm{co}}_*(\mathcal{G})$ with $B \cap \mathcal{G}^{(0)} = \varnothing$, we have $B \not\subseteq \operatorname{Iso}(\mathcal{G})$ (see [Brown et al. 2014, Lemma 3.1] for other equivalent conditions).

We need the following graded uniqueness theorem for Steinberg algebras.

Lemma 7.9 [Clark and Edie-Michell 2015, Theorem 3.4]. *Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid such that $c^{-1}(\varepsilon)$ is effective. If $\pi : A_R(\mathcal{G}) \to A$ is a graded R-algebra homomorphism with* $\operatorname{Ker}(\pi) \neq 0$ *then there is a compact open subset $B \subseteq \mathcal{G}^{(0)}$ and $r \in R \setminus \{0\}$ such that $\pi(r1_B) = 0$.*

The following key lemma will be used to determine the annihilator ideal of the $A_R(\mathcal{G})$-module RW. This is a generalisation of [Brown et al. 2014, Proposition 4.4] adapted to the graded setting. Recall that if $\mathcal{G}$ is a graded groupoid with grading given by the cocycle $c : \mathcal{G} \to \Gamma$, then $c^{-1}(\varepsilon)$ is a (trivially graded) clopen subgroupoid of $\mathcal{G}$.

Lemma 7.10. *Let $W \subseteq \mathcal{G}^{(0)}$ be a Γ-aperiodic invariant subset and $\pi_W : A_R(\mathcal{G}) \to \operatorname{END}_R(RW)$ the homomorphism of Γ-graded algebras given in* (7-2). *Then π_W is injective if and only if W is dense in $\mathcal{G}^{(0)}$ and $c^{-1}(\varepsilon)$ is effective.*

Proof. Suppose π_W is injective and there exists an open subset K of $\mathcal{G}^{(0)}$ such that $K \cap W = \varnothing$. We have $K = \bigcup_i B_i$, where B_i are compact open bisections of $\mathcal{G}$. So $B_i \cap W = \varnothing$ for each i, giving $\pi_W(1_{B_i}) = 0$, a contradiction. Thus for any open subset K of $\mathcal{G}^{(0)}$, $K \cap W \neq \varnothing$. Therefore W is dense in $\mathcal{G}^{(0)}$.

Suppose now that $c^{-1}(\varepsilon)$ is not effective. Then there exists a nonempty compact open bisection $B \subseteq c^{-1}(\varepsilon) \setminus \mathcal{G}^{(0)}$ such that $d(b) = r(b)$ for all $b \in B$. We have that $d(B) \neq B$ and that B is a compact open bisection of $\mathcal{G}$. Thus $1_B - 1_{d(B)} \in \operatorname{Ker}(\pi_W)$. This is a contradiction. Hence, $c^{-1}(\varepsilon)$ is effective.

For the converse, Lemma 7.9 implies that it suffices to prove that for any compact open subset $B \subseteq \mathcal{G}^{(0)}$ and $r \in R \setminus \{0\}$, $\pi_W(r1_B) \neq 0$. Since W is dense in $\mathcal{G}^{(0)}$, we have $B \cap W \neq \varnothing$. There exists $w \in B \cap W$ such that $\pi_W(r1_B)(w) \neq 0$, proving $\pi_W(r1_B) \neq 0$. □

If the group Γ is trivial, then by Lemma 7.10, for an invariant subset $W \subseteq \mathcal{G}^{(0)}$, the homomorphism $\pi_W : A_R(\mathcal{G}) \to \operatorname{End}_R(RW)$ is injective if and only if W is dense in $\mathcal{G}^{(0)}$ and the groupoid $\mathcal{G}$ is effective.

The following is the main result of this section.

Theorem 7.11. *Let $\mathcal{G}$ be a Γ-graded ample Hausdorff groupoid, R a commutative ring with identity and $A_R(\mathcal{G})$ the Steinberg algebra associated to $\mathcal{G}$. The following statements are equivalent*:

(i) *Let $W \subseteq \mathcal{G}^{(0)}$ be a Γ-aperiodic invariant subset and W^- the closure of W. Then the groupoid $(c|_{\mathcal{G}_{W^-}})^{-1}(\varepsilon)$ is effective*;

(ii) *For any Γ-aperiodic invariant subset $W \subseteq \mathcal{G}^{(0)}$,*

$$\operatorname{Ann}_{A_R(\mathcal{G})}(RW) = A_R(\mathcal{G}_U),$$

where $U = (\mathcal{G}^{(0)} \setminus W)^\circ$ is the interior of the invariant subset $\mathcal{G}^{(0)} \setminus W$.

Proof. (i) $\Rightarrow$ (ii). Let $W \subseteq \mathcal{G}^{(0)}$ be a Γ-aperiodic invariant subset. By Theorem 7.5, RW is a graded $A_R(\mathcal{G})$-module. By Lemma 7.8, we have $A_R(\mathcal{G}_U) \subseteq \operatorname{Ann}_{A_R(\mathcal{G})}(RW)$ with $U = (\mathcal{G}^{(0)} \setminus W)^\circ$. It follows that RW is an $A_R(\mathcal{G})/A_R(\mathcal{G}_U)$-module. By [Clark et al. 2016, Lemma 3.6], we have an exact sequence of canonical ring homomorphisms

$$0 \to A_R(\mathcal{G}_U) \to A_R(\mathcal{G}) \to A_R(\mathcal{G}_D) \to 0,$$

where $D = \mathcal{G}^{(0)} \setminus U$. The homomorphisms are induced by extensions from $\mathcal{G}_U$ to $\mathcal{G}$ and restrictions from $\mathcal{G}$ to $\mathcal{G}_D$, respectively. One can easily check that the homomorphisms are graded. It therefore follows that the quotient algebra $A_R(\mathcal{G})/A_R(\mathcal{G}_U)$ is graded isomorphic to $A_R(\mathcal{G}_D)$. It follows that RW is a Γ-graded $A_R(\mathcal{G}_D)$-module (this also follows from Theorem 7.5). We denote by $\hat{\pi}_W : A_R(\mathcal{G}_D) \to \operatorname{END}_R(RW)$ the induced graded homomorphism. Observe that $(\mathcal{G}_D)^{(0)} = D$ is the closure of W. Thus by Lemma 7.10, the homomorphism $\hat{\pi}_W$ is injective. This implies that RW is a faithful $A_R(\mathcal{G}_D)$-module. Hence, the annihilator ideal of RW as an $A_R(\mathcal{G})$-module is $A_R(\mathcal{G}_U)$.

(ii) $\Leftarrow$ (i). Let D denote the closure of W in $\mathcal{G}^{(0)}$. Then RW is a faithful $A_R(\mathcal{G}_D)$-module. So the result follows from Lemma 7.10. $\square$

Recall that a groupoid $\mathcal{G}$ is *strongly effective* if for every nonempty closed invariant subset D of $\mathcal{G}^{(0)}$, the groupoid $\mathcal{G}_D$ is effective.

Remark 7.12. (1) If $c^{-1}(\varepsilon)$ is strongly effective, then Theorem 7.11(i) holds. In fact, a closed invariant subset D of the unit space of $\mathcal{G}$ is in particular a closed $c^{-1}(\varepsilon)$-invariant subset of $\mathcal{G}^{(0)}$. We have $c^{-1}(\varepsilon)_D = c^{-1}(\varepsilon) \cap \mathcal{G}_D = (c|_{\mathcal{G}_D})^{-1}(\varepsilon)$. Hence, Theorem 7.11(i) follows directly. Example 7.13 below, on the other hand, shows that Theorem 7.11(i) does not imply that $c^{-1}(\varepsilon)$ is strongly effective.

(2) Resume the notation of Example 7.4, so $u = \alpha\beta\alpha^2\beta \cdots \in E^\infty$. Let D be the closure of the $\mathbb{Z}$-aperiodic invariant subset $[u] \subseteq \mathcal{G}_E^{(0)}$. As we saw in that example, D is not itself $\mathbb{Z}$-aperiodic, because it contains α^∞.

Example 7.13. It is easy to construct examples of Γ-graded groupoids with no Γ-aperiodic points. For example, let X be the Cantor set. Regard $\mathcal{G} = X \times \mathbb{Z}^2$ as a groupoid with unit space $X \times \{0\}$ identified with X by setting $r(x, m) = x = d(x, m)$ and defining composition and inverses by $(x, n)(x, m) = (x, m+n)$ and $(x, m)^{-1} = (x, -m)$. The map $c : \mathcal{G} \to \mathbb{Z}$ given by $c(x, (m_1, m_2)) = m_1$ is a cocycle. We have $c^{-1}(0) = X \times (\{0\} \times \mathbb{Z})$, which is not effective (for example $X \times \{(0, 1)\}$ is a compact open bisection contained in the isotropy subgroupoid of $c^{-1}(0)$). Moreover, $\mathcal{G}^{(0)}$ has no $\mathbb{Z}$-aperiodic points because $\{u\} \times (\mathbb{Z} \times \{0\}) \subseteq \mathrm{Iso}(u) \setminus c^{-1}(0)$ for all $u \in \mathcal{G}^{(0)}$; so every $u \in \mathcal{G}^{(0)}$ is $\mathbb{Z}$-periodic.

Applying Theorem 7.11 to the trivial grading, we obtain a new characterisation of strong effectiveness.

Corollary 7.14. *Let $\mathcal{G}$ be an ample Hausdorff groupoid, and R be a commutative ring with identity. Then $\mathcal{G}$ is strongly effective if and only if for any invariant subset W of $\mathcal{G}^{(0)}$, the annihilator of the $A_R(\mathcal{G})$-module RW is $A_R(\mathcal{G}_U)$, where $U = (\mathcal{G}^{(0)} \setminus W)^\circ$.*

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Governing singularities of symmetric orbit closures

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We develop *interval pattern avoidance* and *Mars–Springer ideals* to study singularities of symmetric orbit closures in a flag variety. This paper focuses on the case of the Levi subgroup $\mathrm{GL}_p \times \mathrm{GL}_q$ acting on the classical flag variety. We prove that all reasonable singularity properties can be classified in terms of interval patterns of *clans*.

1. Introduction

1A. ***Overview.*** Let G/B be a generalized flag variety where G is a complex, reductive algebraic group and B is a choice of Borel subgroup. A subgroup K of G is *symmetric* if $K = G^\theta$ is the fixed point subgroup for an involutive automorphism θ of G. Such a subgroup is *spherical*, which means that its action on G/B by left translations has finitely many orbits.

The study of orbits of a symmetric subgroup on the flag variety was initiated in [Lusztig and Vogan 1983; Vogan 1983], where the singularities of closures of the orbits were related to characters of particular infinite-dimensional representations of a certain real form $G_{\mathbb{R}}$ of G. Since then, there has been a stream of results on the combinatorics and geometry of these orbit closures. Notably, R. W. Richardson and T. A. Springer [Richardson and Springer 1990] gave a description of the partial order given by inclusions of orbit closures, and M. Brion [2001] studied general properties of their singularities, showing that, in many cases, including the one addressed in this paper, all these orbit closures are normal and Cohen–Macaulay with rational singularities. One might also hope that the study of singularities on closures of symmetric subgroup orbits would lead to better understanding of the general relationship between the combinatorics associated to spherical varieties and singularities of orbit closures on them; N. Perrin has written a survey of this topic [Perrin 2014].

We initiate a combinatorial approach, backed by explicit commutative algebra computations, to the study of the singularities of these orbit closures. This paper considers the case of the symmetric subgroup $K = \mathrm{GL}_p \times \mathrm{GL}_q$ of block diagonal matrices in GL_n, where $n = p + q$. Here θ is defined by

$$\theta(M) = I_{p,q} M I_{p,q}, \tag{1}$$

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where $I_{p,q}$ is the diagonal matrix with diagonal consisting of 1 p times followed by -1 q times. In this case, $G_{\mathbb{R}} = U(p,q)$ is the *indefinite unitary group* of signature (p,q).

For this symmetric subgroup K, the finitely many K-orbits $\mathcal{O}_\gamma$ and their closures Y_γ can be parametrized by (p,q)-*clans* γ [Matsuki and Ōshima 1990, Theorem 4.1] (see also [Yamamoto 1997, Theorem 2.2.8]). These clans are partial matchings of vertices $\{1, 2, \ldots, n\}$, where each unmatched vertex is assigned a sign of $+$ or $-$; the difference in the number of $+$ and $-$ signs must be $p-q$. We represent clans by length n strings in $\mathbb{N} \cup \{+,-\}$, with pairs of equal numbers indicating a matching. Let $\mathtt{Clans}_{p,q}$ denote the set of all such clans. For example, three clans from $\mathtt{Clans}_{7,3}$ are represented by the strings:

$$1{+}{+}{+}2{+}1{-}2{+}, \quad {+}1{-}{+}{+}1{+}2{+}2, \quad \text{and} \quad {+}{+}{-}{+}{+}{+}{-}{+}{+}{-}.$$

Note that strings that differ by a permutation of the natural numbers, such as 1212 and 2121, represent the same clan.

One inspiration for this work is W. M. McGovern's characterization [2009] of singular K-orbit closures in terms of *pattern avoidance* of clans. Suppose $\gamma \in \mathtt{Clans}_{p,q}$ and $\theta \in \mathtt{Clans}_{r,s}$. Then $\theta = \theta_1 \cdots \theta_{r+s}$ is said to (*pattern*) *contain* $\gamma = \gamma_1 \cdots \gamma_{p+q}$ if there are indices $i_1 < i_2 < \cdots < i_{p+q}$ such that:

(1) if $\gamma_j = \pm$ then $\theta_{i_j} = \gamma_j$; and

(2) if $\gamma_k = \gamma_\ell$ then $\theta_{i_k} = \theta_{i_\ell}$.

For example, the clan $\gamma = 1{+}{+}{-}{-}1$ contains the pattern $\theta = 1{+}{+}{-}1$, taking $(i_1, \ldots, i_5)$ to be either $(1,2,3,4,6)$ or $(1,2,3,5,6)$.

Say that θ *(pattern) avoids* γ if θ does not contain γ. The main theorem of [McGovern 2009] asserts that Y_γ is smooth if and only if γ avoids the patterns

$$1{+}{-}1, \quad 1{-}{+}1, \quad 1212, \quad 1{+}221, \quad 1{-}221, \quad 122{+}1, \quad 122{-}1, \quad 122331.$$

On the other hand, in [Woo and Wyser 2015, Section 3.3] it is noted that $Y_{1{+}{+}{-}1}$ is non-Gorenstein, while $Y_{1{+}{+}{-}{-}1}$ is Gorenstein, even though $1{+}{+}{-}{-}1$ pattern contains $1{+}{+}{-}1$. Therefore, a more general notion will sometimes be required to characterize which K-orbits satisfy a particular singularity property.

Suppose that $\mathcal{P}$ is any *singularity mildness property*, by which we mean a local property of varieties that holds on open subsets and is stable under smooth morphisms. Many singularity properties, such as being Gorenstein, being a local complete intersection (lci), being factorial, having Cohen–Macaulay rank $\leq k$, or having Hilbert–Samuel multiplicity $\leq k$, satisfy these conditions. For such a $\mathcal{P}$, consider two related problems:

(I) Which K-orbit closures Y_γ are globally $\mathcal{P}$?

(II) What is the non-$\mathcal{P}$-locus of Y_γ?

This paper gives a universal combinatorial language, *interval pattern avoidance* of clans, to answer these questions for any singularity mildness property, at the cost of potentially requiring an infinite number of patterns. This language is also useful for collecting and analyzing data and partial results. We present explicit equations for computing whether a property holds at a specific orbit $\mathcal{O}_\alpha$ on an orbit closure Y_γ. Note that all points of $\mathcal{O}_\alpha \subseteq Y_\gamma$ are locally isomorphic to one another, since the K-action can be used to move any point of $\mathcal{O}_\alpha$ isomorphically to any other point. Since the non-$\mathcal{P}$-locus is closed, it is a union of K-orbit closures. Consequently, for any given clan γ, (II) can be answered by finding a finite set of clans $\{\alpha\}$, namely those indexing the irreducible components of the non-$\mathcal{P}$-locus. (I) asks if this set is nonempty.

This situation parallels that for Schubert varieties. In that setting, the first and third authors introduced interval pattern avoidance for permutations, showing that it provides a common perspective to study all reasonable singularity measures [Woo and Yong 2006; 2008; Woo 2010]. This paper gives the first analogue of those results for K-orbit closures.

Some properties $\mathcal{P}$ hold globally on every Y_γ. For those cases, the above questions are unnecessary. For example, this is true when $\mathcal{P} =$ "normal" and $\mathcal{P} =$ "Cohen–Macaulay" in the case $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$ of this paper. (This is not the case for all symmetric pairs (G, K).)

As is explained in [McGovern 2009], the above answer to (I) for $\mathcal{P} =$ "smooth" also is the answer for the property $\mathcal{P} =$ "rationally smooth". (Recall rational smoothness means that the local intersection cohomology of Y_γ (at a point of $\mathcal{O}_\alpha$) is trivial.) However, the answer to question (II), which asks for a combinatorial description of the (rationally) singular locus, is unsolved except in some special cases [Woo and Wyser 2015]. Actually, it is unknown whether the singular locus and rationally singular locus coincide for all orbit closures (but see Conjecture 7.5).

For most finer singularity mildness properties, answers to both (I) and (II) are unknown. Perhaps the most famous such property comes from the *Kazhdan–Lusztig–Vogan* (KLV) polynomials $P_{\gamma,\alpha}(q)$. These polynomials are the link to representation theory that originally motivated the study of K-orbit closures. For $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$, the KLV polynomial $P_{\gamma,\alpha}(q) \in \mathbb{Z}_{\geq 0}[q]$ is the Poincaré polynomial for the local intersection cohomology of Y_γ at any point of the orbit $\mathcal{O}_\alpha$ [Lusztig and Vogan 1983]. Rational smoothness of Y_γ along $\mathcal{O}_\alpha$ is hence equivalent to the equality $P_{\gamma,\alpha}(q) = 1$. More generally, for any fixed $k > 1$, the property $\mathcal{P} =$"$P_{\gamma,\alpha}(1) \leq k$" behaves as a singularity mildness property on K-orbit closures by recent work of McGovern [2015], but his proof uses representation theory, and the algebraic geometry is not well-understood.

1B. ***Main ideas.*** Each K-orbit closure Y_γ is a union of K-orbits $\mathcal{O}_\alpha$; let $\leq$ denote the *Bruhat (closure) order* on K-orbit closures on GL_n / B. (This means $\alpha \leq \gamma$ if and only if $Y_\alpha \subseteq Y_\gamma$). Also, let $[\alpha, \gamma]$ and $[\beta, \theta]$ be intervals in Bruhat order on $\texttt{Clans}_{p,q}$ and $\texttt{Clans}_{r,s}$, respectively.

Define $[\beta, \theta]$ to *interval pattern contain* $[\alpha, \gamma]$ (or equivalently $[\alpha, \gamma]$ to *interval pattern embed* into $[\beta, \theta]$), and write $[\alpha, \gamma] \hookrightarrow [\beta, \theta]$ if:

(a) there are indices

$$I : i_1 < i_2 < \cdots < i_{p+q}$$

which commonly witness the containment of γ into θ and α into β;

(b) θ and β agree outside of these indices; and

(c) $\ell(\theta) - \ell(\beta) = \ell(\gamma) - \ell(\alpha)$, where if $\gamma = \gamma_1\gamma_2 \dots \gamma_{p+q}$ then

$$\ell(\gamma) := \sum_{\gamma_i = \gamma_j \in \mathbb{N}} j - i - \#\{\gamma_s = \gamma_t \mid s < i < t < j\}.$$

Notice β is determined by α, γ, θ, and the set of indices I. In particular, β is the unique clan $\Phi(\alpha)$ that agrees with α on I and agrees with θ on $\{1, \dots, n\} \setminus I$. Thus, we define a clan θ to *interval pattern contain* $[\alpha, \gamma]$ if $[\alpha, \gamma]$ is contained in $[\Phi(\alpha), \theta]$. Similarly, we can speak of θ avoiding a list of intervals.

Example 1.1. Let $[\alpha, \gamma] = [+--+, 1212]$ and $[\beta, \theta] = [1+--+1, 123231]$. Then one can check $[\beta, \theta]$ interval contains $[\alpha, \gamma]$, using the middle four positions.

Example 1.2. Let $[\alpha, \gamma] = [+-, 11]$ and $[\beta, \theta] = [\underline{+}+\underline{-}, \underline{1}+\underline{1}]$. If $[\beta, \theta]$ contains $[\alpha, \gamma]$ it must be using the underlined positions. However, $\ell(\gamma) - \ell(\alpha) = 1$ while $\ell(\theta) - \ell(\beta) = 2$. Thus $[\beta, \theta]$ does not contain $[\alpha, \gamma]$.

Define

$$\mathcal{C} := \{[\alpha, \gamma] \mid \alpha \leq \gamma \text{ in some } \mathtt{Clans}_{p,q}\} \subseteq \mathtt{Clans} \times \mathtt{Clans},$$

where

$$\mathtt{Clans} = \bigcup_{p,q} \mathtt{Clans}_{p,q}.$$

Declare $\preceq_{\mathcal{C}}$ to be the poset relation on $\mathcal{C}$ generated by

- $[\alpha, \gamma] \preceq_{\mathcal{C}} [\beta, \theta]$ if $[\beta, \theta]$ interval pattern contains $[\alpha, \gamma]$, and
- $[\alpha, \gamma] \preceq_{\mathcal{C}} [\alpha', \gamma]$ if $\alpha' \leq \alpha$.

For any poset $(\mathcal{S}, \preceq)$, an *upper order ideal* is a subset $\mathcal{I}$ of $\mathcal{S}$ having the property that whenever $x \in \mathcal{I}$, we also have $y \in \mathcal{I}$ for all $y \geq x$. The following theorem provides a basic language to express answers to (I) and (II).

Theorem 1.3. *Let $\mathcal{P}$ be a singularity mildness property* (see Definition 2.1).

(I) *The set of intervals $[\alpha, \gamma] \in \mathcal{C}$ such that $\mathcal{P}$ fails on each point of $\mathcal{O}_\alpha \subseteq Y_\gamma$ is an upper order ideal in $(\mathcal{C}, \preceq_{\mathcal{C}})$.*

(II) *The set of clans γ such that $\mathcal{P}$ holds at all points of Y_γ are those that avoid all the intervals $[\alpha_i, \gamma_i]$ constituting some (possibly infinite) set $\mathcal{A}_{\mathcal{P}} \subseteq \mathcal{C}$.*

For simplicity, we work over $\mathbb{C}$, but our results are valid over any field of characteristic not 2, with the caveat that even for the same $\mathcal{P}$, the set $\mathcal{A}_{\mathcal{P}}$ may be field dependent.

Although stated in combinatorial language, as we will see, Theorem 1.3 follows from a geometric result, Theorem 4.6, which establishes a local isomorphism between certain "slices" of the orbit closures. Rather than working directly with the K-orbit closures on G/B, it is easier to establish this isomorphism using particular slices of B-orbit closures on G/K given by J. G. M. Mars and Springer [1998].

The space $G/K = \mathrm{GL}_n/(\mathrm{GL}_p \times \mathrm{GL}_q)$ is the configuration space of all splittings of $\mathbb{C}^n$ as a direct sum $V_1 \oplus V_2$ of subspaces, with $\dim(V_1) = p$ and $\dim(V_2) = q$. Indeed, $G = \mathrm{GL}_n(\mathbb{C})$ acts transitively on the set of all such splittings, and $K = \mathrm{GL}_p \times \mathrm{GL}_q$ is the stabilizer of the "standard" splitting of $\mathbb{C}^n$ as

$$\langle e_1, \dots, e_p \rangle \oplus \langle e_{p+1}, \dots, e_n \rangle,$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{C}^n$. A point $gK/K \in G/K$, with $g \in \mathrm{GL}_n(\mathbb{C})$ an invertible $n \times n$ matrix, is identified with the splitting whose p-dimensional component is the span of the first p columns of g and whose q-dimensional component is the span of the last q columns of g.

B-orbits on G/K are in bijection with K-orbits on G/B (both orbit sets being in bijection with the $B \times K$-orbits on G). In addition, this bijection preserves all mildness properties. This is because the orbit closures correspond via the two locally trivial fibrations $G \to G/K$ and $G \to G/B$, each of which has smooth fiber. The B-orbit on G/K corresponding to the clan γ will be denoted by Q_γ, and its closure will be denoted by W_γ. For any singularity mildness property $\mathcal{P}$, this discussion implies:

Observation 1.4. *Y_γ is $\mathcal{P}$ along $\mathcal{O}_\alpha$ if and only if W_γ is $\mathcal{P}$ along Q_α.*

In [Wyser and Yong 2014], the second and third authors considered certain open affine subsets of K-orbit closures on G/B which they called *patches*. That work also introduced *patch ideals* of equations which set-theoretically cut out the patches. Experimental computation using patch ideals led to several of the conjectures which appear in [Wyser and Yong 2014; Woo and Wyser 2015].

In contrast, the Mars–Springer slices are not open affine pieces of the orbit closures. However, as we will see, they are essentially as good, in that they carry all of the local information that we are interested in. They are analogues of *Kazhdan–Lusztig varieties*, which play a similar role in the study of Schubert varieties.

In this paper we introduce *Mars–Springer ideals* (see Section 4) with explicit equations that set-theoretically cut out the Mars–Springer slices. These equations are conjectured to also be scheme-theoretically correct; see Conjecture 7.1. These are the K-orbit versions of *Kazhdan–Lusztig ideals*, which define the aforementioned Kazhdan–Lusztig varieties. The latter ideals have been of use in both computational and theoretical analysis of Schubert varieties (see, for example, [Woo and Yong 2012; Ulfarsson and Woo 2013]). In the same vein, we mention a practical advantage of the Mars–Springer ideal over the patch ideal. Gröbner basis calculations with the former are several times faster than those of the latter, as fewer variables are involved.

1C. *Organization.* In Section 2, we present some preliminaries. In particular, we more precisely define "mildness properties," giving examples and establishing some basic facts that we will need. We next recall the attractive slices of Mars and Springer [1998]. Finally, we describe the quasiprojective variety structure of G/K for the case $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$ of this paper. In Section 3, we give explicit affine coordinates for the Mars–Springer slice (Theorem 4.3). Section 4 defines the *Mars–Springer variety* and its ideal. It culminates with Theorem 4.6, which asserts that certain Mars–Springer varieties are isomorphic to others. This theorem is the key to our proof of Theorem 1.3. In order to prove Theorem 4.6, we develop combinatorics of interval embeddings using earlier work of the second author [Wyser 2016] on the Bruhat order on clans; this is Section 5. We then give our proofs of Theorem 4.6 and Theorem 1.3 in Section 6. We conclude with problems and conjectures in Section 7.

2. Preliminaries

2A. ***Singularity mildness properties.*** We define the class of local properties that we are interested in, with specific examples.

Definition 2.1. Suppose that $\mathcal{P}$ is a local property of algebraic varieties, meaning that $\mathcal{P}$ is verified at a point w of an algebraic variety W based solely on the local ring $\mathcal{O}_{W,w}$. We say that $\mathcal{P}$ is a *singularity mildness property* (or simply a *mildness property*) if $\mathcal{P}$ is

(1) open, meaning that the $\mathcal{P}$-locus of any variety is open; and

(2) stable under smooth morphisms, i.e., if X and Y are any varieties, and $f : X \to Y$ is any smooth morphism between them, then for any $x \in X$, X is $\mathcal{P}$ at x if and only if Y is $\mathcal{P}$ at $f(x)$.

For any smooth variety S and for any variety X, the projection map $S \times X \to X$ is a smooth morphism. Thus:

Observation 2.2. *If $\mathcal{P}$ is a mildness property, S is a smooth variety, and X is any variety, then for any $x \in X$ and for any $s \in S$, X is $\mathcal{P}$ at x if and only if $S \times X$ is $\mathcal{P}$ at (s, x).*

The next lemma gives a (nonexhaustive) list of examples of mildness properties $\mathcal{P}$.

Lemma 2.3. *Examples of mildness properties $\mathcal{P}$ include*:

(i) *reducedness*;

(ii) *normality*;

(iii) *smoothness*;

(iv) *lci-ness*;

(v) *Gorensteinness.*

Proof. That these properties are open (on varieties) is well-known. That they are stable under smooth morphisms follows from results of [Matsumura 1986, §23]. Indeed, let X and Y be varieties, with $f : X \to Y$ a smooth morphism between them. Let $x \in X$ be given, and let $y = f(x)$. Now, f is flat, and also the fiber F_y over y is smooth over the residue field $\kappa(y)$, hence regular. Thus $\mathcal{O}_{F_y,x}$ is regular, hence also lci and Gorenstein. Now (iii) and (v) follow from Theorems 23.7 and 23.4 of [Matsumura 1986, §23], respectively, while (iv) is mentioned in the remark following Theorem 23.6 of [loc. cit.]; as indicated there, further details can be found in [Avramov 1975].

For (i) and (ii), we appeal to [Matsumura 1986, Corollary to Theorem 23.9]. Note that this result requires that the fiber ring corresponding to any prime ideal of $\mathcal{O}_{Y,y}$ be reduced (resp. normal), rather than just the fiber ring of the maximal ideal. In our case, all such fiber rings are once again regular (hence both reduced and normal). Indeed, if p is any prime of $\mathcal{O}_{Y,y}$, then p corresponds to an irreducible closed subvariety Y' of Y. The inverse image scheme $f^{-1}(Y')$ is smooth over Y', and the fiber ring $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(p)$ is the generic fiber of $f^{-1}(Y') \to Y'$. □

2B. *Mars–Springer slices.* Let W be any irreducible variety with an action of a Borel group B, let $w \in W$ be given, and let $Q = B \cdot w$.

Definition 2.4 [Mars and Springer 1998]. An *attractive slice* to Q at w in W is a locally closed subset S of W containing w such that:

(1) the restriction of the action map $B \times S \to W$ is a smooth morphism;

(2) $\dim(S) + \dim(Q) = \dim(W)$; and

(3) there exists a map $\lambda : \mathbb{G}_m \to B$ such that
  (a) S is stable under $\mathrm{Im}(\lambda)$;
  (b) w is a fixed point of $\mathrm{Im}(\lambda)$; and
  (c) the action map $\mathbb{G}_m \times S \to S$ extends to a morphism $\mathbb{A}^1 \times S \to S$ which sends $\{0\} \times S$ to w.

The following result justifies our use of slices in studying singularities, since taking slices preserves the local properties that we are interested in.

Lemma 2.5. (Using the notation of Definition 2.4.) *Given a mildness property $\mathcal{P}$, the variety W is $\mathcal{P}$ at w if and only if S is $\mathcal{P}$ at w.*

Proof. Indeed, the action map $B \times S \to W$ is smooth, so W is $\mathcal{P}$ at w if and only if $B \times S$ is $\mathcal{P}$ at $(1, w)$, which is the case if and only if S is $\mathcal{P}$ at w, by Observation 2.2. □

Let G be a complex algebraic group with an involutive automorphism θ, with $K = G^\theta$ the corresponding symmetric subgroup. Suppose that $T \subseteq B$ are a θ-stable maximal torus and Borel subgroup of G, respectively. Let N be the normalizer of T in G, and let $\mathcal{W} = N/T$ be the Weyl group.

Given a B-orbit Q on G/K, we now define the *Mars–Springer slice* $S_{\bar{x}}$ to Q at a specially chosen point $\bar{x} \in Q$. Letting U^- be the unipotent radical of the Borel subgroup opposite to B, this slice is of the form

$$S_{\bar{x}} := (U^- \cap \psi(U^-)) \cdot \bar{x}, \tag{2}$$

where ψ is a certain involution on G.

To be precise, consider the set

$$\mathcal{V} := \{x \in G \mid x(\theta x)^{-1} \in N\}.$$

In (2), we pick any

$$\bar{x} := xK/K \in Q \quad \text{with } x \in \mathcal{V};$$

such a choice exists by [Richardson and Springer 1990, Theorem 1.3]. Let η be the image in $\mathcal{W}$ of $x\theta(x)^{-1} \in N$. The element $x\theta(x)^{-1} \in N$ may depend upon the choice of $\bar{x}$, but η, its class modulo T, does not. Then in (2) we take

$$\psi := c_\eta \circ \theta,$$

where c_η denotes conjugation by η. With these choices, it is shown in [Mars and Springer 1998, Section 6.4] that $S_{\bar{x}}$, as defined in (2), is indeed an attractive slice to Q at $\bar{x}$ in G/K, in the sense of Definition 2.4.

The following lemma is more or less formal; we include a proof for completeness:

Lemma 2.6. *With notation as above, if W is any B-orbit closure on G/K containing Q, then*:

(i) *the scheme-theoretic intersection $W \cap S_{\bar{x}}$ is reduced, and*

(ii) *$W \cap S_{\bar{x}}$ is an attractive slice to Q at $\bar{x}$ in W.*

Proof. It is easy to see that the diagram

$$\begin{tikzcd} B \times (W \cap S_{\bar{x}}) \arrow[r] \arrow[d, hook] & W \arrow[d, hook] \\ B \times S_{\bar{x}} \arrow[r] & G/K \end{tikzcd}$$

is Cartesian. Therefore, since the action map along the bottom is smooth, the (restricted) action map along the top is also smooth, since smooth morphisms are stable under base extension. Since W is reduced and reducedness is a mildness property by Lemma 2.3, $B \times (W \cap S_{\bar{x}})$ is reduced, and thus $W \cap S_{\bar{x}}$ is reduced, by Observation 2.2. This proves (i).

For (ii), the preceding observation regarding the smoothness of the map along the top of the above diagram verifies Definition 2.4(1) for $S = W \cap S_{\bar{x}}$. This map furthermore has the same relative dimension as the one along the bottom, which implies that

$$\dim(G/K) - \dim(W) = \dim(S_{\bar{x}}) - \dim(W \cap S_{\bar{x}}).$$

Since $S_{\bar{x}}$ is an attractive slice to Q in G/K, we know that

$$\dim(S_{\bar{x}}) + \dim(Q) = \dim(G/K),$$

and combining these two equations gives

$$\dim(Q) + \dim(W \cap S_{\bar{x}}) = \dim(W),$$

as required by Definition 2.4(2).

That $W \cap S_{\bar{x}}$ satisfies Definition 2.4(3) is obvious: Let $\lambda : \mathbb{G}_m \to B$ be a one-parameter subgroup having properties (a) and (b) relative to the slice $S_{\bar{x}}$. Then clearly $W \cap S_{\bar{x}}$ is stable under $\mathrm{Im}(\lambda)$, $\bar{x}$ is still a fixed point of $\mathrm{Im}(\lambda)$, and the action map $\mathbb{G}_m \times (W \cap S_{\bar{x}}) \to W \cap S_{\bar{x}}$ still extends to a map $\mathbb{A}^1 \times (W \cap S_{\bar{x}}) \to W \cap S_{\bar{x}}$ sending $\{0\} \times (W \cap S_{\bar{x}})$ to $\bar{x}$ (simply restrict the original extension to the smaller subset). $\square$

Corollary 2.7. *For $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$, let clans α and γ be given, with $\alpha \leq \gamma$ in Bruhat order. Let X_α denote the Mars–Springer slice to Q_α at a particular point $x_\alpha \in Q_\alpha$ in G/K. Then the slice $W_\gamma \cap X_\alpha$ is irreducible.*

Proof. It is observed in [Brion 2001] that all B-orbit closures on G/K for this particular case are "multiplicity-free," which implies by [Brion 2001, Theorem 5] that W_γ has rational singularities. In particular, it is normal. Since normality is a mildness property by Lemma 2.3, $W_\gamma \cap X_\alpha$ is also normal. Now, the "attractive" condition of slices (part (3) of Definition 2.4) ensures that $W_\gamma \cap X_\alpha$ is connected because x_α must lie in every connected component of it. Being both normal and connected, $W_\gamma \cap X_\alpha$ must be irreducible. □

Remark. The result of Corollary 2.7 in fact holds for any symmetric pair (G, K) such that all of the B-orbit closures on G/K are multiplicity-free. Other such cases include $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$ and $(\mathrm{SO}_{2n}, \mathrm{GL}_n)$, as noted in [Brion 2001]. However, not all symmetric pairs (G, K) have this property. For those that do not, B-orbit closures on G/K can be nonnormal, and the Mars–Springer slices can in fact be reducible. One example is (GL_n, O_n). □

2C. $\mathbf{GL}_{p+q}/(\mathbf{GL}_p \times \mathbf{GL}_q)$ *as a quasiprojective variety.* We now discuss the structure of $G/K = \mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$ as a quasiprojective variety and its affine patches. The statements of this section could be extracted from standard definitions and results, but to the best of our knowledge, this has never been made explicit in the literature, so we take the opportunity to do so here.

As stated in the Introduction, G/K is the configuration space whose points correspond to splittings of $\mathbb{C}^n$ as a direct sum $V_1 \oplus V_2$ of subspaces, with $\dim(V_1) = p$ and $\dim(V_2) = q$. Hence it is natural to identify G/K with a subset of the product of Grassmannians $\mathrm{Gr}(p, \mathbb{C}^n) \times \mathrm{Gr}(q, \mathbb{C}^n)$. We can realize $\mathrm{Gr}(p, \mathbb{C}^n) \times \mathrm{Gr}(q, \mathbb{C}^n)$ as a subvariety of $\mathbb{P}^{\binom{n}{p}-1} \times \mathbb{P}^{\binom{n}{q}-1}$ using the Plücker embedding into each factor. The Segre embedding then realizes this product as a subvariety of $\mathbb{P}^{\binom{n}{p}\binom{n}{q}-1}$, so $\mathrm{Gr}(p, \mathbb{C}^n) \times \mathrm{Gr}(q, \mathbb{C}^n)$ is a projective variety.

We let p_S and q_T denote the Plücker coordinates on $\mathrm{Gr}(p, \mathbb{C}^n)$ and $\mathrm{Gr}(q, \mathbb{C}^n)$ respectively, where S and T respectively range over all subsets of $\{1, \dots, n\}$ of size p (respectively q).

Not every pair V_1 and V_2 of subspaces of $\mathbb{C}^n$ gives a splitting. In order for V_1 and V_2 to form a direct sum, a basis for V_1 must be linearly independent of a basis of V_2. Let M be a matrix whose first p columns are a basis for V_1 and whose last q columns are a basis for V_2. If we take the Laplace expansion of $\det M$ using the first p columns and identify determinants of submatrices with Plücker coordinates, then we get

$$\det M = \sum_{S \subset \{1,\dots,n\},\, \#S = p} (-1)^S p_S q_{\bar{S}}, \tag{3}$$

where $\bar{S} := \{1, \dots, n\} \setminus S$ and $(-1)^S := (-1)^{\left(\sum_{s \in S} s\right) - \binom{p+1}{2}}$. Hence we can identify G/K with the *open* subset of $\mathrm{Gr}(p, \mathbb{C}^n) \times \mathrm{Gr}(q, \mathbb{C}^n)$ where (3) is nonzero. This gives G/K the structure of a quasiprojective variety.

The product $\mathbb{P}^{\binom{n}{p}-1} \times \mathbb{P}^{\binom{n}{q}-1}$ of projective spaces has a covering by affine patches $\{U_{S,T} := U_S \times U_T\}$, where U_S is the patch where p_S is nonzero and U_T is the patch where q_T is nonzero. Thus $\mathrm{Gr}(p, \mathbb{C}^n) \times \mathrm{Gr}(q, \mathbb{C}^n)$ is covered by these affine patches. As explained in [Fulton 1997, Section 9.1], any subspace $V_1 \in \mathrm{Gr}(p, \mathbb{C}^n)$ having Plücker coordinate $p_S(V_1) \neq 0$ has a basis $\left\{e_s + \sum_{r \notin S} a_{s,r}(V_1)e_r\right\}_{s\in S}$ where e_i denotes the i-th standard basis vector, and the functions $a_{s,r}$ are local coordinates on the patch $U_S \cap \mathrm{Gr}(p, \mathbb{C}^n)$. Over this patch, there is an algebraic section ϕ_S of the projection map $M^0_{n\times p} \to \mathrm{Gr}(p, \mathbb{C}^n)$, where $M^0_{n\times p}$ denotes the $n \times p$ matrices of full rank p. The section ϕ sends the subspace V_1 to the matrix whose columns are precisely the aforementioned basis.

Likewise, any subspace $V_2 \in \mathrm{Gr}(q, \mathbb{C}^n)$ having Plücker coordinate $q_T(V_2) \neq 0$ has a basis

$$\left\{e_t + \sum_{r\notin T} b_{t,r}(V_2)e_r\right\}_{t\in T},$$

and the $b_{t,r}$ are local coordinates on the patch $U_T \cap \mathrm{Gr}(q, \mathbb{C}^n)$. Over this patch, there is a similar section ϕ_T which sends V_2 to the full rank $n \times q$ matrix whose columns are this basis.

Taking the products of the sections ϕ_S and ϕ_T, over $U_{S,T} \cap \bigl(\mathrm{Gr}(p, \mathbb{C}^n) \times \mathrm{Gr}(q, \mathbb{C}^n)\bigr)$ we have an algebraic map $\phi_{S,T}$ from G/K into $M^0_{n\times p} \times M^0_{n\times q}$, which naturally embeds into $M_{n\times n}$ by simple concatenation of matrices. Over G/K, this map takes values in $G = \mathrm{GL}_n$ and gives an algebraic section $\phi_{S,T}$ of the projection map $G \to G/K$.

3. Affine coordinates for the Mars–Springer slice

We provide explicit coordinates for the Mars–Springer slice for the case that

$$(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q).$$

3A. *The affine space S_α.* We define an affine space of matrices associated to each clan α. First, let $w_\alpha \in S_n$ (in one line notation) be defined as follows. From left to right, 1 through p are assigned to the $+$'s and left ends of matchings. Assign $\{p + 1, \dots, n\}$ as we read the clan from left to right. When we encounter a $-$, we assign the smallest unused number, and when we encounter the *left* end of a matching, we immediately assign the smallest unused number from $\{p + 1, \dots, n\}$ to the corresponding *right* end of the matching.

Example 3.1. If $\alpha = 122133 \in \mathtt{Clans}_{3,3}$ then $w_\alpha = 125436$.

We construct, in stages, the generic matrix $M_\alpha(z)$ of S_α. First, for $i = 1, \dots, n$:

(O.1) If $\alpha_i = \pm$, then row i has a 1 in position $w_\alpha(i)$.

(O.2) If $(i < j)$ is a matching of α, then row i has a 1 in positions $w_\alpha(i)$ and $w_\alpha(j)$.

(O.3) If $(j < i)$ is a matching of α, then row i has a -1 in position $w_\alpha(j)$ and a 1 in position $w_\alpha(i)$.

A *pivot* is the northmost 1 in each column. (Note that w_α is chosen precisely so that the pivots in the first p columns, and in the last q columns, occur northwest to southeast.) In the first p columns only, set to 0 all entries

(Z.1) in the same row as a pivot;

(Z.2) above a pivot (and in that pivot's column);

(Z.3) between the pivot 1 and the corresponding -1 of a column (for all columns to which this applies); or

(Z.4) right of a -1.

In the rightmost q columns, set to 0 all entries

(Z.5) in the same row as a pivot;

(Z.6) above a pivot (and in that pivot's column);

(Z.7) between the pivot 1 and the corresponding 1 below it in the same column (for all columns to which this applies); or

(Z.8) right of a 1 which is the second 1 in its column.

The remaining entries z_{ij} of $M_\alpha(z)$ are arbitrary, with an exception for each pair of matchings of α which are "in the pattern 1212". For each such pair of matchings $(i < k)$, $(j < \ell)$ with $i < j < k < \ell$:

(Z.9) position $(\ell, w_\alpha(i))$ has entry $z_{\ell, w_\alpha(i)}$ (as usual) whereas its "partner" position $(\ell, w_\alpha(k))$ has entry $-z_{\ell, w_\alpha(i)}$.

Example 3.2. Let $\alpha = 1{+}12{-}2 \in \mathtt{Clans}_{3,3}$. Then $w_\alpha = 124365$. The 1's and -1's are first placed into a 6×6 matrix, followed by placement of 0's as follows:

$$\begin{pmatrix} 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & 1 & \cdot \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & 0 & 0 & 0 & 1 \\ \cdot & \cdot & -1 & \cdot & 1 & 0 \end{pmatrix}.$$

Since α has no 1212-patterns, the remaining unspecialized entries of the matrix are arbitrary. Thus S_{1+12-2} is an affine space of dimension 5 (the codimension of Q_{1+12-2} in GL_6/K), with its generic entry being the matrix

$$M_{1+12-2}(z) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ z_{5,1} & z_{5,2} & 0 & 0 & 0 & 1 \\ z_{6,1} & z_{6,2} & -1 & z_{6,4} & 1 & 0 \end{pmatrix}.$$

Example 3.3. Now, consider $\alpha = 1{+}21{-}2$. Here $w_\alpha = 123465$. The placement of 1's and -1's, and then 0's, is achieved by:

$$\begin{pmatrix} 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot & 1 & \cdot \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 0 & 1 \\ \cdot & \cdot & -1 & \cdot & 1 & 0 \end{pmatrix}.$$

The 1212-pattern in positions $1 < 3 < 4 < 6$ dictates that the unspecialized positions $(6, w_\alpha(1)) = (6, 1)$ and $(6, w_\alpha(4)) = (6, 4)$ are negatives of one another. Thus S_{1+21-2} is an affine space of dimension 4 (the codimension of Q_{1+21-2} in GL_6/K), and

$$M_{1+21-2}(z) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ z_{5,1} & z_{5,2} & 0 & 0 & 0 & 1 \\ z_{6,1} & z_{6,2} & -1 & -z_{6,1} & 1 & 0 \end{pmatrix}.$$

Example 3.4. Each 1212-pattern gives rise to a separate identification of coordinates. For instance, when $\alpha = 123123$, the reader can verify that

$$M_{123123}(z) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ z_{5,1} & -1 & 0 & -z_{5,1} & 1 & 0 \\ z_{6,1} & z_{6,2} & -1 & -z_{6,1} & -z_{6,2} & 1 \end{pmatrix}.$$

The following result will be used in Section 4.

Lemma 3.5. *$M_\alpha(z)$ invertible over $\mathbb{Q}[z]$.*

Proof. By construction, the matrix is invertible over the field of rational functions $\mathbb{Q}(z)$. The lemma would follow if we could prove that the determinant is a constant (which will be necessarily nonzero). We argue by induction on $n = p' + q'$ that $\det M_{\alpha'}(z) \in \mathbb{Q}$ for $\alpha' \in \mathtt{Clans}_{p',q'}$. The base case $n = 1, 2$ is easy to check.

Suppose $\gamma \in \mathtt{Clans}_{p,q}$ where $n = p + q > 2$. If $\gamma_1 = +$ consider the submatrix of $M_\alpha(z)$ obtained by striking the first row and column. One checks that this submatrix M' is equal to $M_{\alpha'}(z)$, where α' is α with the first $+$ removed. By induction, $\det M' \in \mathbb{Q}$. Then by cofactor expansion along the first row, since the unique nonzero entry is a 1 in position $(1, 1)$, we have that $\det M = \det M' \in \mathbb{Q}$, as desired.

A similar argument may be used if $\alpha_1 = -$.

Finally, suppose α_1 is the left end of a matching. Let α_r be the right end of said matching. Observe that the submatrix M' obtained by deleting the first row and first column of $M = M_\alpha$ is the matrix $M_{\alpha'}$ where α' is obtained by deleting α_1 and replacing α_r by $-$. Thus by induction $\det M' \in \mathbb{Q}$.

Similarly, consider the submatrix M'' obtained by deleting the first row and $w(r)$-th column. We observe that by

- cyclically reordering rows $1, 2, \ldots, w(r) - 1$ of M''; and then
- multiplying the first row by -1,

we obtain the matrix $M_{\alpha''}$ where α'' is obtained from α by deleting α_r and replacing α_1 by $+$. Hence, by cofactor expansion along the first row of M, we obtain that $\det M = \det M' \pm \det M'' \in \mathbb{Q}$, as desired. □

3B. ***Identification of S_α with the Mars–Springer slice.*** Let B be the Borel subgroup of upper triangular matrices in GL_n, and let T be the maximal torus of diagonal matrices of GL_n. Recall from Section 2 that in order to define the slice, we must choose a point $x_\alpha = xK/K \in Q_\alpha$ such that $x\theta(x)^{-1} \in N$, where N is the normalizer of T, and where θ is the involution (1). Here, we choose x to be the origin of S_α, meaning the matrix where all z-variables are set to 0. The fact that $x_\alpha = xK/K$ is actually a point of Q_α follows from [Yamamoto 1997, Section 2.2-2], while the fact that $x\theta(x)^{-1} \in N$ is easy to check. Denote by X_α the Mars–Springer slice to Q_α at x_α in G/K, so

$$X_\alpha := (U^- \cap \psi(U^-)) \cdot x_\alpha = ((U^- \cap \psi(U^-))x)/K.$$

Theorem 3.6. *We have an isomorphism $X_\alpha \cong S_\alpha$.*

Proof. Recall from Section 2B that the map $\psi : G \to G$ is defined by $c_\eta \circ \theta$, where $\eta = x\theta(x)^{-1} \pmod T$.

Given a clan γ, let $\mathcal{I}(\alpha)$ denote the *underlying involution* for α; this is the permutation in S_n such that $\mathcal{I}(\alpha)(j) = j$ if $\alpha_j = \pm$, while, if $(i < j)$ is a matching of α, $\mathcal{I}(\alpha)(i) = j$ and $\mathcal{I}(\alpha)(j) = i$. The map $\mathcal{I}$ is the concrete realization for $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$ of the map defined by Richardson and Springer [1990] associating a twisted involution (which is an involution in this case) to every orbit.

Claim 3.7. *As an element of $\mathcal{W} = S_n$, we have $\eta = \mathcal{I}(\alpha)$.*

Proof. The matrix computation $x\theta(x)^{-1} = xI_{p,q}x^{-1}I_{p,q}$ is straightforward, and results in a monomial matrix representing the involution described above. □

Claim 3.8. *The group $U^- \cap \psi(U^-)$ consists of the unipotent matrices having 1's on the diagonal, 0's above the diagonal, 0's below the diagonal in all positions (i, j) with $i > j$ such that $\eta(i) < \eta(j)$, and arbitrary entries in all other positions below the diagonal.*

Proof. U^- is the set of lower triangular unipotent $n \times n$ matrices. U^- is θ-stable, so θ maps it isomorphically to itself, meaning that $\psi(U^-) = c_\eta(U^-)$. By Claim 3.7, $\eta = \eta^{-1}$. Thus conjugation by η sends the affine coordinate at position (i, j) $(i > j)$ to position $(\eta(i), \eta(j))$. The claim follows. □

Denote by S'_α the set of $n \times n$ matrices that satisfy (O.1), (O.2), (O.3), (Z.2), (Z.4), (Z.6), (Z.8), (Z.9), and the following condition:

(Z.10) If $(i < j)$ is any matching of α, and $i < k < j$, then:

- if k is the left end of a matching $(k < \ell)$ with $j < \ell$, then the entries at positions $(k, w_\alpha(i))$ and $(k, w_\alpha(j))$ are equal;
- otherwise, the entries at positions $(k, w_\alpha(i))$ and $(k, w_\alpha(j))$ are both zero.

Claim 3.9.
$$S'_\alpha = (U^- \cap \psi(U^-))x.$$

Proof. By a direct but tedious matrix computation using Claim 3.8, one checks that $(U^- \cap \psi(U^-))x \subseteq S'_\alpha$. The space $(U^- \cap \psi(U^-))x$ is evidently an affine space of dimension $\binom{n}{2} - \ell(\eta)$, where $\ell(\cdot)$ denotes Coxeter length. The space S'_α is also visibly an affine space, and a straightforward combinatorial argument shows that its dimension is also $\binom{n}{2} - \ell(\eta)$. Thus the aforementioned containment is in fact an equality. $\square$

Claim 3.10.
$$S_\alpha \subseteq S'_\alpha.$$

Proof. This follows from the fact that (Z.3) and (Z.7) together imply (Z.10). $\square$

The right action of K on G is by column operations which separately act on the first p columns and the last q columns of any matrix in G. Given a matrix $M' \in S'_\alpha$, we now describe an algorithm for column reducing M' via this K-action to another matrix M of a particular form.

Algorithm 3.11. Begin with the first p columns of M'. For each row $i = 1, 2, \ldots, n$, determine whether there is a pivot in row i. If not, move to the next row. If so, then this pivot is at $(i, w_\alpha(i))$. Now start with column $j = 1$, and move right toward column $j = w_\alpha(i) - 1$. For each such j, if entry (i, j) is zero, move to the next column. Otherwise, entry (i, j) is $\lambda_{i,j} \neq 0$. Replace column C_j with $C_j - \lambda_{i,j} \cdot C_i$. Repeat the same procedure using the last q columns.

Let M be the output of Algorithm 3.11.

Claim 3.12. *The first p columns and the last q columns of M are in reduced column echelon form, meaning that all entries in the same row as a pivot (except for the pivot) are zero. M is the unique matrix in the K-orbit of M' with this property.*

Proof. Consider first the leftmost p columns. Algorithm 3.11 brings to zero any entry in the same row as a pivot at $(i, w_\alpha(i))$ and to the left of $(i, w_\alpha(i))$. Since M' satisfies (Z.2), any entry to the right of a pivot is already zero: there is a pivot in every column, and by our choice of w_α, any pivot right of column $w_\alpha(i)$ occurs in a row $j > i$.

For the rightmost q columns, one argues identically, using the fact that M' satisfies (Z.6).

Uniqueness of M follows from the uniqueness of reduced column echelon form (for each of the two submatrices of M). $\square$

Claim 3.13. *The matrix M is an element of S_α.*

Proof. By Claim 3.12, M satisfies (Z.1) and (Z.5). We show that each step performed by Algorithm 3.11 preserves properties (O.1), (O.2), (O.3), (Z.2), (Z.4), (Z.6), and (Z.8), all of which are satisfied by M'. Hence M also possesses these properties. We also show that once Algorithm 3.11 is complete, the resulting matrix also satisfies (Z.3), (Z.7), and (Z.9). This will show that M possesses all the properties (O.1)–(O.3) and (Z.1)–(Z.9).

(O.1): Suppose that $c_i = +$. There is a pivot 1 at $(i, w_\alpha(i))$. Any step of Algorithm 3.11 affecting column $C_{w_\alpha(i)}$ is performed to make zero an entry $\lambda \neq 0$ at $(j, w_\alpha(i))$ for $j > i$ (using the pivot in row j). Such a step replaces column $C_{w_\alpha(i)}$ by $C_{w_\alpha(i)} - \lambda \cdot C_{w_\alpha(j)}$. Since $j > i$, by (Z.2), the entry at $(i, w_\alpha(j))$ is zero, so the 1 at $(i, w_\alpha(i))$ remains a 1 after this column operation.

When $c_i = -$, one argues identically using the last q columns and (Z.6). Thus (O.1) is preserved by all steps of our algorithm, as desired.

(O.2): The argument is identical to that for (O.1).

(O.3): Suppose $(i < j)$ is a matching. First consider the -1 at $(j, w_\alpha(i))$. Since entries to the right of this -1 are zero by (Z.4), the -1 is clearly unchanged by steps of the algorithm. Similarly, the 1 at $(j, w_\alpha(j))$ is unchanged, by (Z.8).

(Z.2): Given a zero entry above a pivot 1, any entry to its right is also zero by (Z.2) since such an entry is above a different pivot 1. Thus any step of the algorithm leaves the first of these two zero entries unchanged, as needed.

(Z.3): Suppose $(i < j)$ is a matching. There are a 1 and a -1 in positions $(i, w_\alpha(i))$ and $(j, w_\alpha(i))$, respectively. Let $i < k < j$. We must check that M is zero at $(k, w_\alpha(i))$. There are two cases based on (Z.10):

Case 1: (The $(k, w_\alpha(i))$ entry of M' is arbitrary.) This case occurs if and only if α_k is the left end of a matching $(k < \ell)$ such that $i < k < j < \ell$. Thus there is a pivot 1 in row k at position $(k, w_\alpha(k))$. Since M satisfies (Z.1), entry $(k, w_\alpha(i))$ of M must be zero.

Case 2: (The $(k, w_\alpha(i))$ entry of M' is required to be zero.) Consider a step of Algorithm 3.11 altering column $C_{w_\alpha(i)}$. It does so by replacing $C_{w_\alpha(i)}$ with $C_{w_\alpha(i)} - \lambda \cdot C_{w_\alpha(\ell)}$ for ℓ such that $\ell > i$ and $w_\alpha(\ell) > w_\alpha(i)$. This step is done if and only if there is a pivot 1 at $(\ell, w_\alpha(\ell))$ and a nonzero entry λ in position $(\ell, w_\alpha(i))$. We examine the effect of this step on entry $(k, w_\alpha(i))$.

If $\ell > k$, then entry $(k, w_\alpha(\ell))$ is zero by (Z.2), so entry $(k, w_\alpha(i))$ remains zero after the step in question. So suppose that $i < \ell < k$. If entry $(\ell, w_\alpha(i))$ is $\lambda \neq 0$, then by (Z.10) this must be because α_ℓ is the left end of a matching $(\ell < r)$ such that $i < \ell < j < r$. If the entry at $(k, w_\alpha(\ell))$ is zero, then the entry at $(k, w_\alpha(i))$ remains zero after our step, and we are done. So suppose it is not. Then by (Z.10) again, α_k is the left end of a matching $(k < s)$ with $\ell < k < r < s$. But then $i < k < j < s$, contradicting our assumption that (Z.10) required entry $(k, w_\alpha(i))$ of M' to be zero. Thus this situation cannot occur.

(Z.4): Suppose that M' has a -1 at position (i, j) with $1 \leq j \leq p$. Since M' satisfies (Z.4), any entry in a position (i, k) with $j < k \leq p$ is zero at the outset. Moreover, again by (Z.4), any entry in a position (i, ℓ) with $k < \ell \leq p$ is also zero at the outset. Thus all steps of Algorithm 3.11 leave the 0 in position (i, k) unchanged.

(Z.6): We argue as with (Z.2) above, but using the rightmost q columns.

(Z.7): We argue as with (Z.3).

(Z.8): We argue as with (Z.4).

(Z.9): Suppose α has a 1212 pattern consisting of matchings $(i < j)$ and $(k < \ell)$ with $i < k < j < \ell$. Then by (Z.10), in positions $(k, w_\alpha(i))$ and $(k, w_\alpha(j))$ of M', we have arbitrary but equal entries; assume that both entries are equal to λ_1. Moreover, by (Z.9), in positions $(\ell, w_\alpha(i))$ and $(\ell, w_\alpha(j))$, we have arbitrary entries that are negatives of one another. Suppose that the entry at $(\ell, w_\alpha(i))$ is λ_2, and that the entry at $(\ell, w_\alpha(j))$ is $-\lambda_2$. Consider two steps performed by Algorithm 3.11:

(i) Replace column $C_{w_\alpha(i)}$ by $C_{w_\alpha(i)} - \lambda_1 \cdot C_{w_\alpha(k)}$ to make the entry at $(k, w_\alpha(i))$ zero.

(ii) Replace column $C_{w_\alpha(j)}$ by $C_{w_\alpha(j)} - \lambda_1 \cdot C_{w_\alpha(\ell)}$ to make the entry at $(k, w_\alpha(j))$ zero.

Step (i) replaces the λ_2 at $(\ell, w_\alpha(i))$ by $\lambda_2 + \lambda_1$ (since there is a -1 in position $(\ell, w_\alpha(k))$). Meanwhile, step (ii) replaces the $-\lambda_2$ at $(\ell, w_\alpha(j))$ by $-\lambda_2 - \lambda_1$ (since there is a 1 in position $(\ell, w_\alpha(\ell))$). Thus after (i) and (ii), (Z.9) holds for the 1212 pattern $i < k < j < \ell$.

To conclude (Z.9) holds, it remains to show:

Subclaim 3.14. *Steps* (*i*) *and* (*ii*) *are the only steps of Algorithm 3.11 that can change either of the entries* $(\ell, w_\alpha(i))$ *or* $(\ell, w_\alpha(j))$.

Proof. Suppose there is a pivot 1 (among the first p columns) at $(r, w_\alpha(r))$, and suppose that there is a nonzero entry at $(r, w_\alpha(i))$. There are two possibilities. First, suppose $w_\alpha(r) > w_\alpha(k)$. In this case, $(\ell, w_\alpha(r))$ must be zero by (Z.4). Thus a column operation involving columns $w_\alpha(i)$ and $w_\alpha(r)$ leaves $(\ell, w_\alpha(i))$ unchanged, as desired. Otherwise, $w_\alpha(i) < w_\alpha(r) < w_\alpha(k)$. Then $i < r < j$, so by (Z.10) the entry at $(r, w_\alpha(i))$ can be nonzero only if there is a 1212 pattern formed by matchings $(i < j)$ and $(r < s)$ with $i < r < j < s$. However, this situation is covered by the argument preceding this subclaim. A similar analysis of the last q columns (using either (Z.8) or (Z.10)) shows that there are no steps of our algorithm that alter position $(\ell, w_\alpha(j))$. ☐

This completes the proof of Claim 3.13. ☐

By Claim 3.13, we can now define a map

$$\phi : X_\alpha \to S_\alpha$$

which sends the point $M' \cdot K$ of X_α to the matrix M, retaining the notation above. Its inverse, as a set map, is the restriction of the natural projection $\pi : G \to G/K$ to S_α. (Note that this restriction does take values in $X_\alpha = S'_\alpha/K$ by Claim 3.10.) Abusing notation, we denote this restriction simply by π. Then letting M' and M be as above, it is indeed clear that

$$\pi(\phi(M' \cdot K)) = \pi(\phi(M \cdot K)) = \pi(M) = M \cdot K = M' \cdot K$$

and that

$$\phi(\pi(M)) = \phi(M \cdot K) = M,$$

so that π and ϕ are inverse as set maps.

It remains to check that ϕ and π are actually morphisms of algebraic varieties. Clearly π is a morphism, since it is a restriction of the natural map $G \to G/K$. The map ϕ, on the other hand, is the restriction of an algebraic section of π to X_α. Indeed, it takes a splitting $V_1 \oplus V_2$ and produces a matrix M whose first p columns are a basis for V_1 and whose last q columns are a basis for V_2. These two bases are of the form described in Section 2C when we take S to be the positions of the $+$'s and left endpoints of α and T to be the position of the $-$'s and left endpoints of α. Thus $X_\alpha \subseteq (U_S \times U_T) \cap G/K$, and indeed the map ϕ is the restriction to X_α of the section $\phi_{S,T} : (U_S \times U_T) \cap G/K \to G$ described in Section 2C. $\square$

4. The Mars–Springer variety and ideal

For $\gamma \in \texttt{Clans}_{p,q}$, define the following collections of nonnegative integers [Wyser 2016]:

(1) $\gamma(i;+) :=$ number of matchings and $+$ signs among the first i positions of γ (for $i = 1, \dots, n$);

(2) $\gamma(i;-) :=$ number of matchings and $-$ signs among the first i positions of γ (for $i = 1, \dots, n$); and

(3) $\gamma(i;j) :=$ number of matchings whose left endpoint is weakly left of i and whose right endpoint is strictly right of j (for $1 \le i < j \le n$).

Given any invertible $n \times n$ matrix M and any pair of indices $i < j$, we define an auxiliary $n \times (i + j)$ matrix $M^{[i;j]}$ as follows. The first i columns of $M^{[i;j]}$ are formed by taking the first i columns of M^{-1} and zeroing out all entries in the last q rows. The last j columns of $M^{[i;j]}$ are the first j columns of M^{-1}, unaltered.

Example 4.1. Consider $\alpha = 1{+}{-}{-}{+}1 \in \texttt{Clans}_{3,3}$. Then

$$M_{1\;\;--\;\;1}(z) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & z_{3,2} & 0 & 0 & 1 & 0 \\ 0 & z_{4,2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & z_{5,5} & z_{5,6} \\ -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

as the reader can verify. The inverse of this matrix is

$$M_{1+--+1}(z)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & z_{3,2}z_{5,5} + z_{4,2}z_{5,6} & -z_{5,5} & -z_{5,6} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -z_{3,2} & 1 & 0 & 0 & 0 \\ 0 & -z_{4,2} & 0 & 1 & 0 & 0 \end{pmatrix}.$$

To obtain the auxiliary matrix $M_{1+--+1}(z)^{[2;4]}$, one takes the first 2 columns of $M_{1+--+1}(z)^{-1}$, zeroes out their last $q = 3$ rows, and then concatenates to this the first 4 columns of $M_{1+--+1}(z)^{-1}$, unaltered.

The result is

$$M_{1+--+1}(z)^{[2;4]}=\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & z_{3,2}z_{5,5}+z_{4,2}z_{5,6} & 0 & z_{3,2}z_{5,5}+z_{4,2}z_{5,6} & -z_{5,5} & -z_{5,6}\\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & -z_{3,2} & 1 & 0\\ 0 & 0 & 0 & -z_{4,2} & 0 & 1\end{pmatrix}.$$

Now define the *Mars–Springer variety* $\mathcal{N}_{\gamma,\alpha}$ to be the reduced subscheme of S_α whose points are the matrices M satisfying all of the following rank conditions:

(R.1) the rank of the southwest $(n-i)\times p$ submatrix of M is at most $p-\gamma(i;+)$ (for $i=1,\dots,n$);

(R.2) the rank of the southeast $(n-i)\times q$ submatrix of M is at most $q-\gamma(i;-)$ (for $i=1,\dots,n$); and

(R.3) the rank of $M^{[i;j]}$ is at most $j+\gamma(i;j)$ (for all $1\le i<j\le n$).

Note that $\mathcal{N}_{\gamma,\alpha}$ is clearly defined *set-theoretically* by a certain collection of determinants. Indeed, define the *Mars–Springer ideal* $\mathcal{I}_{\gamma,\alpha}$ to be the ideal of $\mathbb{C}[z]$ generated by:

(1) all minors of the southwest $(n-i)\times p$ submatrix of $M_\alpha(z)$ of size $p-\gamma(i;+)+1$ (for $i=1,\dots,n$);

(2) all minors of the southeast $(n-i)\times q$ submatrix of $M_\alpha(z)$ of size $q-\gamma(i;-)+1$ (for $i=1,\dots,n$); and

(3) all minors of $M_\alpha(z)^{[i;j]}$ of size $j+\gamma(i;j)+1$ (for all $1\le i<j\le n$).

(By Lemma 3.5, the minors of (3) above are in fact polynomials.)

Example 4.2. Consider the ideal $\mathcal{I}_{123231,1+--+1}$. One can check that, with the exception of the condition that the rank of $M^{[2;4]}$ be at most 5, the conditions defining $\mathcal{N}_{123231,1+--+1}$ are all actually nonconditions. Thus the ideal $\mathcal{I}_{123231,1+--+1}$ is generated by the determinant of the 6×6 matrix $M_{1+--+1}(z)^{[2;4]}$ computed above. One checks that this determinant equals (up to a scalar multiple) $z_{3,2}z_{5,5}+z_{4,2}z_{5,6}$. Thus

$$\mathcal{I}_{123231,1+--+1}=\langle z_{3,2}z_{5,5}+z_{4,2}z_{5,6}\rangle,$$

which defines a singular quadric hypersurface.

Theorem 4.3. *$\mathcal{N}_{\gamma,\alpha}\cong W_\gamma\cap X_\alpha$, the Mars–Springer slice to Q_α at x_α in W_γ.*

Proof. Recall that points of G/K are splittings $\mathcal{S}=V_1\oplus V_2$ of $\mathbb{C}^{p+q}$ where $\dim(V_1)=p$ and $\dim(V_2)=q$. The B-orbit closure $W_\gamma\subseteq G/K$ has a description due to the second author, which we now state. Define $\pi_{\mathcal{S}}:\mathbb{C}^n\to V_1$ to be the projection onto the first summand of $\mathcal{S}$ with kernel V_2, the second summand of $\mathcal{S}$.

Theorem 4.4 [Wyser 2016]. *Let $E_\bullet$ be the standard coordinate flag on $\mathbb{C}^n$, with E_i the span of the first i standard basis vectors. Then $W_\gamma\subseteq G/K$ is precisely the set of splittings $\mathcal{S}=V_1\oplus V_2$ satisfying all of the following incidence conditions with $E_\bullet$:*

(C.1) $\dim(V_1\cap E_i)\ge\gamma(i;+)$ *for $i=1,\dots,n$;*

(C.2) $\dim(V_2 \cap E_i) \geq \gamma(i; -)$ *for* $i = 1, \ldots, n$;

(C.3) $\dim(\pi_{\mathcal{S}}(E_i) + E_j) \leq j + \gamma(i; j)$ *for* $1 \leq i < j \leq n$.

Let $\phi : X_\alpha \to S_\alpha$ be the isomorphism from Theorem 3.6. It suffices to show that the image of

$$\phi|_{W_\gamma \cap X_\alpha} : (W_\gamma \cap X_\alpha) \to S_\alpha$$

is precisely $\mathcal{N}_{\gamma,\alpha}$.

The map ϕ is a bijection, taking a splitting $\mathcal{S} = V_1 \oplus V_2 \in X_\alpha$ to a matrix $M \in S_\alpha$, the span of whose first p columns is V_1, and the span of whose last q columns is V_2. Therefore it suffices to show that under this bijection, $\mathcal{S}$ satisfies (C.1)–(C.3) if and only if $M = \phi(\mathcal{S})$ satisfies (R.1)–(R.3).

The equivalence of (R.1) to (C.1) and of (R.2) to (C.2) is immediate.

To see the equivalence of (C.3) to (R.3) we argue as follows. Condition (C.3) states that

$$\dim(\pi_{\mathcal{S}}(E_i) + E_j) \leq j + \gamma(i; j).$$

If $M \in S_\alpha$ represents a splitting $\mathcal{S}$, then the matrix for $\pi_{\mathcal{S}}$ can be written as MPM^{-1}, where P is the diagonal matrix with p 1's on the diagonal followed by q 0's. Hence $\pi_{\mathcal{S}}(E_i) + E_j$ is the span of the first i columns of MPM^{-1} and the first j columns of the identity matrix I. Therefore,

$$\mathcal{S} \text{ satisfies (C.3)} \iff \operatorname{rank}(M^{\overline{[i;j]}}) \leq j + \gamma(i; j), \tag{4}$$

where $M^{\overline{[i;j]}}$ is the $n \times (i + j)$ matrix whose first i columns are the first i columns of MPM^{-1} and whose last j columns are the first j columns of I.

Since M^{-1} is invertible,

$$\operatorname{rank}(M^{[i;j]}) = \operatorname{rank}(M^{-1}M^{\overline{[i;j]}}) = \operatorname{rank}(M^{\overline{[i;j]}}). \tag{5}$$

Combining (4) and (5) we conclude

$$\begin{aligned} \mathcal{S} \text{ satisfies (C.3)} &\iff \operatorname{rank}(M^{[i;j]}) \leq j + \gamma(i; j) \quad \text{for all } i, j \text{ with } 1 \leq i < j \leq n \\ &\iff M \text{ satisfies (R.3)}, \end{aligned}$$

as desired. □

Corollary 4.5. *For any clans $\alpha \leq \gamma$, the Mars–Springer variety $\mathcal{N}_{\gamma,\alpha}$ is irreducible.*

Proof. This is immediate by Theorem 4.3 combined with Corollary 2.7. □

The key to our proof of Theorem 1.3 is:

Theorem 4.6 (isomorphism theorem). *If $[\beta, \theta]$ interval pattern contains $[\alpha, \gamma]$, then*

$$\mathcal{N}_{\theta,\beta} \cong \mathcal{N}_{\gamma,\alpha}.$$

This will be proved in Section 6.

5. Combinatorics of interval embeddings

Denote Bruhat order on clans by $<$. Let $\gamma \lessdot \tau$ denote a covering relation, so $\gamma \lessdot \tau$ if $\gamma < \tau$ and there is no clan β such that $\gamma < \beta < \tau$. Equivalently, $\gamma \lessdot \tau$ if $\gamma < \tau$ and $\ell(\tau) - \ell(\gamma) = 1$.

We need the following result of the second author describing the covering relations $\lessdot$:

Theorem 5.1 [Wyser 2016]. *Let $\gamma, \tau \in \texttt{Clans}_{p,q}$ such that $\gamma < \tau$. Then there exists $\gamma' \in \texttt{Clans}_{p,q}$ such that $\gamma < \gamma' \leq \tau$, where γ' is obtained from γ by one of the following operations on patterns in γ*:

(T.1) $+- \mapsto 11$,

(T.2) $-+ \mapsto 11$,

(T.3) $11+ \mapsto 1{+}1$,

(T.4) $11- \mapsto 1{-}1$,

(T.5) $+11 \mapsto 1{+}1$,

(T.6) $-11 \mapsto 1{-}1$,

(T.7) $1122 \mapsto 1212$,

(T.8) $1122 \mapsto 1{+}{-}1$,

(T.9) $1122 \mapsto 1{-}{+}1$,

(T.10) $1212 \mapsto 1221$.

Any ordered pair $\gamma \mapsto \gamma'$ of clans obtained by one of the operations (T.1)–(T.10) is called a *transposition*. Thus:

Corollary 5.2 [Wyser 2016]. *$\gamma \lessdot \gamma'$ if and only if $\gamma \mapsto \gamma'$ is a transposition such that $\ell(\gamma') = \ell(\gamma) + 1$.*

For any clan γ, denote by M_γ the set of matchings of γ. Call a matching $(a < b) \in M_\gamma$ with $a < i < b < j$ *incoming* to $(i < j) \in M_\gamma$. For $(i < j) \in M_\gamma$, let

$$I(i, j, \gamma) := \#\{(a < b) \in M_\gamma \mid (a < b) \text{ is incoming to } (i < j)\},$$

and let

$$C(i, j, \gamma) := j - i - I(i, j, \gamma).$$

Then by [Yamamoto 1997],

$$\ell(\gamma) := \sum_{(a<b) \in M_\gamma} C(a, b, \gamma).$$

We will need the following combinatorial fact:

Proposition 5.3. *Suppose $\gamma \mapsto \gamma'$ is a transposition. Then $\ell(\gamma) < \ell(\gamma')$.*

Proof. For brevity, we prove the hardest case (T.7). Arguments for the remaining cases of (T.1)–(T.10) are similar in nature.

Suppose $\gamma \mapsto \gamma'$ via (T.7). Say the 1122 pattern occurs at positions $i < j < k < \ell$. In order to compare $\ell(\gamma)$ to $\ell(\gamma')$, we consider other matchings $(a < b)$ and compare $I(a, b, \gamma')$ with $I(a, b, \gamma)$, $I(i, k, \gamma')$ with $I(i, j, \gamma)$, and $I(j, \ell, \gamma')$ with $I(k, \ell, \gamma)$.

There are 15 different possible orientations of a matching $(a < b)$ relative to $(i < j)$ and $(k < \ell)$ to consider.

Case 1: (The configuration $a < i < j < b < k < \ell$.) Here

$$I(a, b, \gamma') = I(a, b, \gamma), \quad I(j, \ell, \gamma') = I(k, \ell, \gamma) + 1,$$

and

$$I(i, k, \gamma') = I(i, j, \gamma) + 1.$$

Indeed, $(a < b)$ was incoming to neither $(i < j)$ nor $(k < \ell)$ in γ, but it is incoming to both $(i < k)$ and $(j < \ell)$ in γ'. So each such matching $(a < b)$ accounts for a decrease in length of 2 as we transpose from γ to γ'.

Case 2: (The configuration $i < j < a < k < \ell < b$.) Now,

$$I(a, b, \gamma') = I(a, b, \gamma) + 2.$$

Indeed, in γ neither $(i < j)$ nor $(k < \ell)$ are incoming to $(a < b)$, but in γ' both $(i < k)$ and $(j < \ell)$ are incoming to it. On the other hand, the move in this configuration does not introduce any new incoming matchings to $(i < k)$ or $(j < \ell)$. So each matching $(a < b)$ in this configuration also accounts for a decrease in length of 2 as we transpose from γ to γ'.

Case 3: (The configuration $i < a < j < b < k < \ell$.) When we transpose from γ to γ', $(a < b)$ loses an incoming matching: $(i < j)$ is incoming to $(a < b)$ in γ, but $(i < k)$ is not incoming to $(a < b)$ in γ'. However, $(j < \ell)$ gains an incoming matching compared to $(k < \ell)$, as $(a < b)$ is not incoming to $(k < \ell)$ in γ, but it is incoming to $(j < \ell)$ in γ'. Thus while the transposition causes a change in the number of matchings incoming to $(a < b)$ and in the number of matchings to which $(a < b)$ is incoming, the net effect is zero.

By arguing similarly to Case 3 for the remaining 12 configurations, one sees that Case 1 and Case 2 are the only two which result in a net change in length coming from changes in the number of incoming matchings.

The only change in the number of incoming matchings which we have not accounted for is that after the move, $(j < \ell)$ has also added the incoming matching $(i < k)$ in γ', whereas $(k < \ell)$ did not have the incoming matching $(i < j)$ in γ. This causes an additional decrease by 1 in length as we perform this transposition.

Finally, we must account for the change in total lengths of matchings caused by the transposition. This is $(k - i) + (\ell - j) - ((j - i) + (\ell - k)) = 2(k - j)$.

Putting this all together, we conclude that

$$\ell(\gamma') - \ell(\gamma) = 2(k-j) - 1 - 2\big(\#\{(a<b) \in M_\gamma \mid a<i<j<b<k<\ell\} + \#\{(a<b) \in M_\gamma \mid i<j<a<k<\ell<b\}\big). \quad (6)$$

Now, the sum in parentheses is clearly at most $k-j-1$, which implies that the right-hand side is at least 1. Thus $\ell(\gamma') > \ell(\gamma)$, as desired. □

Say $[\alpha,\gamma]$ *merely embeds* into $[\beta,\theta]$ if the pair of intervals satisfies the definition of an interval pattern embedding, *except* possibly for the length requirement

$$\ell(\gamma) - \ell(\alpha) = \ell(\theta) - \ell(\beta).$$

Given any α, we define $\Phi(\alpha)$ to be the unique clan such that $[\alpha,\gamma]$ merely embeds in $[\Phi(\alpha),\theta]$, so $\Phi(\alpha)$ agrees with α on the set of embedding indices I and agrees with θ on $\{1,\dots,n\}\setminus I$.

Theorem 5.4. *Assume* $[\alpha,\gamma]$ *merely embeds into* $[\Phi(\alpha),\theta]$.

(I) $\ell(\gamma) - \ell(\alpha) \le \ell(\theta) - \ell(\Phi(\alpha))$.

(II) *Further suppose* $[\alpha,\gamma]$ *interval embeds into* $[\Phi(\alpha),\theta]$. *If* $\alpha \lessdot \alpha' \le \gamma$ *then*

$$\Phi(\alpha) \lessdot \Phi(\alpha') \le \theta.$$

Proof. (I): Pick a chain of covering transpositions

$$\alpha = \alpha^{(0)} \mapsto \alpha^{(1)} \mapsto \cdots \mapsto \alpha^{(m)} = \gamma.$$

This induces a chain from $\Phi(\alpha)$ to θ by using the same transpositions (relative to the embedding). By Proposition 5.3, each transposition in the latter chain increases the length by at least one, from which the statement follows.

(II): There exists a covering transposition from α to α', followed by a chain of covering transpositions from α' to γ. Thus $\Phi(\alpha)$ is at least related to $\Phi(\alpha')$ by the same transposition (relative to the embedding), and the chain from α' to γ induces a chain $\Phi(\alpha') \to \cdots \to \theta$. Hence we have

$$\Phi(\alpha) < \Phi(\alpha') \le \theta.$$

It remains to show

$$\Phi(\alpha') \lessdot \Phi(\alpha).$$

By Proposition 5.3, we have

$$\ell(\Phi(\alpha')) - \ell(\Phi(\alpha)) \ge 1.$$

We are done unless

$$\ell(\Phi(\alpha')) - \ell(\Phi(\alpha)) \ge 2,$$

so assume this. Let

$$k := \ell(\gamma) - \ell(\alpha) = \ell(\theta) - \ell(\Phi(\alpha)).$$

Since $[\alpha', \gamma]$ merely embeds into $[\Phi(\alpha'), \theta]$, we obtain a contradiction:

$$\begin{aligned} k-1 = \ell(\gamma) - \ell(\alpha') &\leq \ell(\theta) - \ell(\Phi(\alpha')) \\ &= \ell(\theta) - \ell(\Phi(\alpha)) - (\ell(\Phi(\alpha')) - \ell(\Phi(\alpha))) \\ &\leq k-2, \end{aligned}$$

where for the first inequality we have used (I). □

Corollary 5.5. *If $[\alpha, \gamma] \hookrightarrow [\Phi(\alpha), \theta]$ and $\alpha \lessdot \alpha' \leq \gamma$, then $[\alpha', \gamma] \hookrightarrow [\Phi(\alpha'), \theta]$.*

Proof. By Theorem 5.4(II),

$$\Phi(\alpha) \lessdot \Phi(\alpha') \leq \theta.$$

Hence $[\alpha', \gamma]$ and $[\Phi(\alpha'), \theta]$ have the same length difference, as desired. □

Corollary 5.6. *If θ avoids $[\alpha', \gamma]$ then θ avoids $[\alpha, \gamma]$ for all $\alpha \leq \alpha'$.*

Proof. This is the contrapositive of Corollary 5.5, combined with induction on $\ell(\alpha') - \ell(\alpha)$. □

For the next lemma, assume that $[\alpha, \gamma]$ interval embeds into $[\beta, \theta]$, let I be the set of indices for the embedding, and let $J = \{1, \ldots, n\} \setminus I$ be the set of indices not involved in the embedding.

Lemma 5.7. (I) *If β_j (and hence θ_j) is a sign for some $j \in J$, then*

$$\beta(j; +) = \theta(j; +) \quad \textit{and} \quad \beta(j; -) = \theta(j; -).$$

(II) *If β_a and β_b are a matched pair for some $a, b \in J$, then*

$$\beta(a; b) = \theta(a; b).$$

Example 5.8. Let

$$[\alpha, \gamma] = [1212, 1221] \subseteq \mathtt{Clans}_{2,2} \text{ and } [\beta, \theta] = [1{+}212, 1{+}221] \subseteq \mathtt{Clans}_{3,2}.$$

Both are intervals of length 1, and hence $[\alpha, \gamma] \hookrightarrow [\beta, \theta]$. Here

$$n = 5, \quad I = \{1, 3, 4, 5\}, \quad \text{and} \quad J = \{2\}.$$

This is an instance of Lemma 5.7(I) since $\beta_2 = \theta_2 = +$, $\beta(2; +) = \theta(2; +) = 1$, and $\beta(2; -) = \theta(2; -) = 0$.

Example 5.9. Recall Example 1.1, where

$$[\alpha, \gamma] = [{+}{-}{-}{+}, 1212] \quad \text{and} \quad [\beta, \theta] = [1{+}{-}{-}{+}1, 123231].$$

Here

$$n = 6, \quad I = \{2, 3, 4, 5\}, \quad \text{and} \quad J = \{1, 6\}.$$

This is an instance of Lemma 5.7(II) since $a = 1$ and $b = 6$ are matched and $\beta(1; 6) = \theta(1; 6) = 0$.

Proof of Lemma 5.7. The proof is by induction on

$$k = \ell(\gamma) - \ell(\alpha) = \ell(\theta) - \ell(\beta).$$

The base case, $k = 0$, when $\theta = \beta$, is trivial, so we may assume $k \geq 1$. Suppose that

$$\alpha \lessdot \alpha' \leq \gamma.$$

By Corollary 5.5,

$$[\alpha', \gamma] \hookrightarrow [\beta', \theta], \quad \text{where } \beta' = \Phi(\alpha').$$

For the statement (I), by induction,

$$\beta'(j; \pm) = \theta(j; \pm).$$

Next, clearly

$$[\alpha, \alpha'] \hookrightarrow [\beta, \beta'].$$

If we know that

$$\beta(j; \pm) = \beta'(j; \pm),$$

then we are done. Hence we are reduced to the case $k = 1$. The statement (II) similarly reduces to this situation.

Therefore we prove both claims when $k = 1$. To do this, one must analyze each possible covering transposition $\beta \lessdot \beta'$. As with the proof of Proposition 5.3, for brevity we give the details only for the hardest case (T.7). The other cases are similar.

Suppose the 1122 pattern of β occurs at

$$i < j < k < \ell.$$

From the proof of Proposition 5.3, we recall from (6) that $\ell(\beta') - \ell(\beta) = 1$ if and only if each position between j and k is either the right endpoint of a matching $(a < b)$ with

$$a < i < j < b < k < \ell$$

or the left endpoint of a matching $(a < b)$ with

$$i < j < a < k < \ell < b.$$

For (I), let $m \in J$ be such that $\beta_m = \beta'_m$ are the same sign. If $m < j$ or $m > k$, then

$$\beta(m; \pm) = \beta'(m; \pm).$$

Since $\beta \lessdot \beta'$, by the observation of the previous paragraph, we cannot have $j < m < k$.

For (II), for $(a < b) \in M_\beta$, one checks that $\beta(a; b) = \beta'(a; b)$ unless $(a < b)$ is in one of the three configurations

$$i < a < j < b < k < \ell, \quad i < j < a < b < k < \ell, \quad \text{or} \quad i < j < a < k < b < \ell.$$

Since $\beta \lessdot \beta'$, none of these configurations are possible, again by the above observation. $\square$

6. Proof of Theorems 4.6 and 1.3

Assume that $[\alpha, \gamma]$ interval embeds into $[\beta, \theta]$ and that J is the set of indices *not* involved in this embedding. Let $w := w_\beta$ be the permutation associated to the clan β as described in Section 3A.

Proposition 6.1. *For any point $M \in \mathcal{N}_{\theta,\beta}$ and $j \in J$, all the entries in the j-th row and the $w(j)$-th column of M are* 0 *except for the required* ± 1 *specified by* (O.1), (O.2) *or* (O.3).

Assuming Proposition 6.1, whose proof is deferred to Section 6C, we prove Theorems 4.6 and 1.3 in Sections 6A and 6B.

6A. ***Proof of Theorem 4.6:*** Define a map

$$\Psi : \mathcal{N}_{\theta,\beta} \to S_\alpha$$

by deleting the rows j and columns $w(j)$ for $j \in J$. Proposition 6.1 says that Ψ is injective as a map of sets. Hence it is injective as a map of varieties, since it is the restriction of a linear map.

If we can show that

$$\operatorname{Im}(\Psi) \subseteq \mathcal{N}_{\gamma,\alpha}, \tag{7}$$

we are done. Indeed, this containment is then an equality since $\mathcal{N}_{\gamma,\alpha}$ is irreducible (by Corollary 4.5) and since

$$\dim(\mathcal{N}_{\theta,\beta}) = \ell(\theta) - \ell(\beta) = \ell(\gamma) - \ell(\alpha) = \dim(\mathcal{N}_{\gamma,\alpha}),$$

where the middle equality is by the interval embedding hypothesis.

To prove (7), we need to show $\Psi(m)$ satisfies (R.1), (R.2) and (R.3) whenever $m \in \mathcal{N}_{\theta,\beta}$. By (strong) induction on $\#J \geq 0$, we reduce to the case where θ is obtained from γ (or, equivalently, β is obtained from α) by adding a single $+$, a single $-$, or a single matching. (The base case $\#J = 0$ is trivial.)

Case 1: (θ is obtained from γ by adding a single $+$.) Let us suppose that γ and α are (p, q)-clans of length n, so that θ and β are $(p+1, q)$-clans of length $n+1$. Let us further suppose that the $+$ is added between the $(\ell-1)$-th and ℓ-th characters of α and γ. Thus

$$J = \{\ell\}, \quad \text{and} \quad \beta_\ell = \theta_\ell = +.$$

This $+$ in β corresponds to the ℓ-th row R and $w(\ell)$-th column C of a matrix $m \in \mathcal{N}_{\theta,\beta}$; C is among the first p columns of m. (It is R and C which are deleted by the map Ψ.) The column C has a single 1, which is the only nonzero entry in C and in R.

(R.1): For $i = 1, \ldots, n$, let R_i^+ denote the rank of the southwest $(n-i) \times p$ submatrix of $\Psi(m)$. For $j = 1, \ldots, n+1$, let $\mathcal{R}_j^+$ denote the rank of the southwest $(n+1-j) \times (p+1)$ submatrix of m. We want to show that

$$R_i^+ \leq p - \gamma(i;+) \quad \text{for } i = 1, \ldots, n,$$

knowing that

$$\mathcal{R}_j^+ \leq p + 1 - \theta(j; +) \quad \text{for } j = 1, \ldots, n+1.$$

There are two cases: Either $i < \ell$, or $i \geq \ell$.

If $i < \ell$, then we know that

$$\mathcal{R}_i^+ \leq p + 1 - \theta(i; +).$$

Furthermore, we know that $\mathcal{R}_i^+ = R_i^+ + 1$ since the row R is included among the last $n + 1 - i$ rows of m and hence contributes 1 to the rank of the southwest $(n + 1 - i) \times (p + 1)$ submatrix of m relative to the rank of the southwest $(n - i) \times p$ submatrix of $\Psi(m)$. Finally, we know that

$$\theta(i; +) = \gamma(i; +)$$

since θ and γ are the same up to position i. Putting these facts together gives us that

$$R_i^+ = \mathcal{R}_i^+ - 1 \leq p + 1 - \theta(i; +) - 1 = p - \gamma(i; +),$$

the desired conclusion.

On the other hand, if $i \geq \ell$, then we have that

$$\mathcal{R}_{i+1}^+ \leq p + 1 - \theta(i + 1; +).$$

We also have that

$$\mathcal{R}_{i+1}^+ = R_i^+$$

since the row R is now not among the last $n + 1 - (i + 1)$ rows of m and hence does not contribute 1 to the rank of the southwest $(n + 1 - (i + 1)) \times (p + 1)$ submatrix of m relative to the rank of the southwest $(n - i) \times p$ submatrix of $\Psi(m)$. Finally, we also clearly have that

$$\theta(i + 1; +) = \gamma(i; +) + 1.$$

Putting these facts together, we see that

$$R_i^+ = \mathcal{R}_{i+1}^+ \leq p + 1 - \theta(i + 1; +) = p + 1 - (\gamma(i; +) + 1) = p - \gamma(i; +),$$

as desired.

(R.2): Now, we let R_i^- denote the rank of the southeast $(n - i) \times q$ submatrix of $\Psi(m)$ for $i = 1, \ldots, n$, and we let $\mathcal{R}_j^-$ denote the rank of the southeast $(n + 1 - j) \times q$ submatrix of m. We want to show that

$$R_i^- \leq q - \gamma(i; -) \quad \text{for } i = 1, \ldots, n,$$

knowing that

$$\mathcal{R}_j^- \leq q - \theta(j; -) \quad \text{for } j = 1, \ldots, n+1.$$

Again, we consider the cases $i < \ell$ and $i \geq \ell$.

In this case, the map Ψ simply deletes the row R of 0's from the submatrix of m formed by the last q columns, which has no effect on the rank whether this row is included among the last $n+1-i$ rows or not. Thus for $i < \ell$, we know that

$$\mathcal{R}_i^- \leq q - \theta(i; -), \quad \mathcal{R}_i^- = R_i^-, \quad \text{and} \quad \gamma(i; -) = \theta(i; -),$$

so

$$R_i^- \leq q - \gamma(i; -),$$

as desired.

When $i \geq \ell$, we have

$$\mathcal{R}_{i+1}^- \leq q - \theta(i+1; -), \quad \mathcal{R}_{i+1}^- = R_i^-, \quad \text{and} \quad \gamma(i; -) = \theta(i+1; -).$$

Thus

$$R_i^- \leq q - \gamma(i; -),$$

again as desired.

(R.3): We now must consider the ranks of the (i, j)-auxiliary matrices for all possible $i < j$. Given $i < j$, we denote by $\mathrm{aux}(\Psi(m), i, j)$ the $n \times (i+j)$ auxiliary matrix formed from $\Psi(m)$, and we denote by $R_{i,j}$ the rank of this matrix. Similarly, we denote by $\mathrm{aux}(m, i, j)$ the $(n+1) \times (i+j)$ auxiliary matrix formed from m and by $\mathcal{R}_{i,j}$ the rank of this matrix. We want to see that

$$R_{i,j} \leq j + \gamma(i; j) \quad \text{for all } 1 \leq i < j \leq n,$$

given that

$$\mathcal{R}_{i,j} \leq j + \theta(i; j) \quad \text{for all } 1 \leq i < j \leq n+1.$$

Since m is obtained from $\Psi(m)$ by adding the row ℓ and column $w(\ell)$, containing a 1 in position $(\ell, w(\ell))$ and zeros elsewhere, m^{-1} is obtained from $\Psi(m)^{-1}$ by adding the row $w(\ell)$ and the column ℓ, containing a 1 in position $(w(\ell), \ell)$ and zeros elsewhere. Since the auxiliary matrices are built from m^{-1} and $\Psi(m)^{-1}$, our analysis relies primarily on this observation.

Now, there are three cases to consider:

$$i < j < \ell, \quad i < \ell \leq j, \quad \text{and } \ell \leq i < j.$$

When $i < j < \ell$, we compare $\mathrm{aux}(\Psi(m), i, j)$ to $\mathrm{aux}(m, i, j)$. Note that since the lone 1 added to $\Psi(m)^{-1}$ to form m^{-1} is located in column $\ell > j$, it does not appear in the latter auxiliary matrix; therefore, $\mathrm{aux}(m, i, j)$ is obtained from $\mathrm{aux}(\Psi(m), i, j)$ by simply adding a single row of zeros. Hence

$$R_{i,j} = \mathcal{R}_{i,j},$$

and since

$$\mathcal{R}_{i,j} \leq j + \theta(i; j), \quad \text{and} \quad \gamma(i; j) = \theta(i; j),$$

it follows that

$$R_{i,j} \leq j + \gamma(i; j),$$

as desired.

Now consider the case where $i < \ell \le j$. In this case, we compare $\mathrm{aux}(\Psi(m), i, j)$ to $\mathrm{aux}(m, i, j+1)$. Here, the latter matrix is obtained from the former by adding a single row and column, with a lone 1 at the intersection of this row and column, and 0's elsewhere. (Note that the 1 does in fact appear since it is in row $w(\ell) \le p$, by the definition of w.) Thus

$$\mathcal{R}_{i,j+1} = R_{i,j} + 1, \quad \text{and clearly} \quad \theta(i;\, j+1) = \gamma(i;\, j).$$

So since

$$\mathcal{R}_{i,j+1} \le j + 1 + \theta(i;\, j+1),$$

we have

$$R_{i,j} = \mathcal{R}_{i,j+1} - 1 \le j + 1 + \theta(i;\, j+1) - 1 = j + \gamma(i;\, j),$$

as desired.

Finally, consider the case $\ell \le i < j$. Here we compare $\mathrm{aux}(\Psi(m), i, j)$ to $\mathrm{aux}(m, i+1, j+1)$. The latter matrix is obtained from the former this time by adding one row and two columns, with the row containing two 1's, one at its intersection with each of the two columns. The row and the two columns have zeros in all other entries. Note that this adds 1 (*not* 2) to the rank, so

$$\mathcal{R}_{i+1,j+1} = R_{i+1,j+1} + 1.$$

Note further that

$$\gamma(i;\, j) = \theta(i+1;\, j+1),$$

so since

$$\mathcal{R}_{i+1;j+1} \le j + 1 + \theta(i+1;\, j+1),$$

we have

$$R_{i+1,j+1} = \mathcal{R}_{i+1,j+1} - 1 \le j + 1 + \theta(i+1;\, j+1) - 1 = j + \gamma(i;\, j),$$

as required.

Case 2: (θ is obtained from γ by adding a single $-$.) Here the arguments are similar to those of Case 1; we simply interchange signs and the roles of the last q columns and the first p columns in the previous argument. We omit the details.

Case 3: (θ is obtained from γ by adding a single matching.) Suppose that the $(p+1, q+1)$-clan θ is obtained from γ by adding to the (p, q)-clan γ a single matching, the left endpoint of which is added between positions $\ell - 1$ and ℓ, and the right endpoint of which is added between positions $\ell' - 1$ and ℓ'. Thus $J = \{\ell, \ell'\}$.

(R.1): As above, let R_i^+ denote the rank of the southwest $(n-i) \times p$ submatrix of $\Psi(m)$, and let $\mathcal{R}_j^+$ denote the rank of the southwest $(n+2-j) \times (p+1)$ submatrix of m. Now the submatrix of m formed by its first $p+1$ columns differs from the submatrix of $\Psi(m)$ formed by its first p columns in that the former has two extra rows and one extra column. The northernmost extra row (row ℓ) contains a 1 in column $w(\ell)$ and 0's elsewhere, and the southernmost extra row (row ℓ') contains a -1 in column $w(\ell)$ and 0's elsewhere. Besides the aforementioned 1 and -1, column $w(\ell)$ has all of its other entries zero.

We want to see that

$$R_i^+ \le p - \gamma(i;+) \quad \text{for all } i$$

given that

$$\mathcal{R}_j^+ \le p + 1 - \theta(j;+) \quad \text{for all } j.$$

We consider three cases:

$$i < \ell, \quad \ell \le i < \ell', \quad \text{and} \quad \ell' \le i.$$

If $i < \ell'$, then we know that

$$\mathcal{R}_i^+ \le p + 1 - \theta(i;+) \quad \text{and that} \quad \theta(i;+) = \gamma(i;+).$$

Also,

$$\mathcal{R}_i^+ = R_i^+ + 1$$

since the southwest $(n+2-i) \times (p+1)$ submatrix of m contains both the aforementioned extra rows relative to the southwest $(n-i) \times p$ submatrix of $\Psi(m)$, which causes the rank of the former to be 1 (*not* 2) higher than the rank of the latter (since the 1 and -1 occur in the same column). Thus

$$R_i^+ = \mathcal{R}_i^+ - 1 \le p + 1 - \theta(i;+) - 1 = p - \gamma(i;+),$$

as required.

Now, if $\ell \le i < \ell'$, then we know that

$$\mathcal{R}_{i+1}^+ \le p + 1 - \theta(i+1;+) \quad \text{and that} \quad \theta(i+1;+) = \gamma(i;+).$$

Note here that

$$\mathcal{R}_{i+1}^+ = R_i^+ + 1,$$

because the southwest $(n+2-(i+1)) \times (p+1)$ submatrix of m still contains one of the two extra rows (namely row ℓ') relative to the southwest $(n-i) \times p$ submatrix of $\Psi(m)$; this causes the rank of the former to be 1 larger than the rank of the latter. Then

$$R_i^+ = \mathcal{R}_{i+1}^+ - 1 \le p + 1 - \theta(i+1;+) - 1 = p - \gamma(i;+).$$

Finally, if $\ell' \le i$, then we know that

$$\mathcal{R}_{i+2} \le p + 1 - \theta(i+2;+) \quad \text{and that} \quad \theta(i+2;+) = \gamma(i;+) + 1.$$

Here,

$$\mathcal{R}_{i+2}^+ = R_i^+$$

because the southwest $(n+2-(i+2)) \times (p+1)$ submatrix of m now does not contain either of the additional rows, so it differs from the southwest $(n-i) \times p$ submatrix of $\Psi(m)$ only in that it has an extra column of zeros. Thus we have

$$R_i^+ = \mathcal{R}_{i+2}^+ \leq p+1-\theta(i+2;+) = p+1-(\gamma(i;+)+1) = p-\gamma(i;+),$$

as required.

(R.2): The argument is virtually identical to that for (R.1).

(R.3): We again must consider the ranks of the (i, j)-auxiliary matrices for all possible $i < j$. As above, given $i < j$, we denote by $\operatorname{aux}(\Psi(m), i, j)$ the $n \times (i+j)$ auxiliary matrix formed from $\Psi(m)$, and by $R_{i,j}$ the rank of this matrix. We denote by $\operatorname{aux}(m, i, j)$ the $(n+2) \times (i+j)$ auxiliary matrix formed from m, and by $\mathcal{R}_{i,j}$ the rank of this matrix. We want to see that

$$R_{i,j} \leq j+\gamma(i;j) \quad \text{for all } 1 \leq i < j \leq n$$

given that

$$\mathcal{R}_{i,j} \leq j+\theta(i;j) \quad \text{for all } 1 \leq i < j \leq n+2.$$

Recall that m is obtained from $\Psi(m)$ by adding rows ℓ and ℓ' and columns $w(\ell)$ and $w(\ell')$, with 1 in positions $(\ell, w(\ell))$, $(\ell, w(\ell'))$, and $(\ell', w(\ell'))$, a -1 in position $(\ell', w(\ell))$, and 0's elsewhere. Thus m^{-1} is obtained from $\Psi(m)^{-1}$ by adding rows $w(\ell)$ and $w(\ell')$ and columns ℓ and ℓ', with $\frac{1}{2}$ in positions $(w(\ell), \ell)$, $(w(\ell'), \ell)$, and $(w(\ell'), \ell')$, a $-\frac{1}{2}$ in position $(w(\ell), \ell')$, and 0's elsewhere. The analysis below relies primarily on this observation.

There are several cases to consider, depending upon the values of $i < j$ relative to ℓ and ℓ'. Given $i < j$, we define $i_\theta < j_\theta$ respectively to be the positions in θ of the i-th and j-th characters of γ relative to the embedding. So for a given i, there are three possibilities:

(1) $i_\theta < \ell$, in which case $i_\theta = i$;

(2) $\ell < i_\theta < \ell'$, in which case $i_\theta = i+1$; or

(3) $\ell' < i_\theta$, in which case $i_\theta = i+2$.

Now, we consider the following cases:

($i_\theta < j_\theta < \ell$): In this case,

$$i_\theta = i, \quad j_\theta = j, \quad \text{and} \quad \gamma(i;j) = \theta(i_\theta; j_\theta).$$

Furthermore, $\operatorname{aux}(m, i_\theta, j_\theta)$ differs from $\operatorname{aux}(\Psi(m), i, j)$ only by the addition of two rows of zeros, so

$$\mathcal{R}_{i_\theta, j_\theta} = R_{i,j}.$$

Thus

$$R_{i,j} = \mathcal{R}_{i_\theta, j_\theta} \leq j_\theta + \theta(i_\theta, j_\theta) = j + \gamma(i;j),$$

as required.

($i_\theta < \ell < j_\theta < \ell'$): Then

$$i_\theta = i, \quad j_\theta = j+1, \quad \text{and} \quad \gamma(i;\, j) = \theta(i_\theta,\, j_\theta).$$

Here, $\operatorname{aux}(m, i_\theta, j_\theta)$ differs from $\operatorname{aux}(\Psi(m), i, j)$ by the addition of two rows and one column. The column contains two $\frac{1}{2}$'s, each of which is contained in one of the two added rows. All other entries of this column and these two rows are zero. Thus

$$\mathcal{R}_{i_\theta, j_\theta} = R_{i,j} + 1.$$

Then

$$R_{i,j} = \mathcal{R}_{i_\theta, j_\theta} - 1 \leq j_\theta + \theta(i_\theta;\, j_\theta) - 1 = j + 1 + \gamma(i;\, j) - 1 = j + \gamma(i;\, j).$$

($i_\theta < \ell < \ell' < j_\theta$): Then

$$i_\theta = i, \quad j_\theta = j+2, \quad \text{and} \quad \gamma(i;\, j) = \theta(i_\theta,\, j_\theta).$$

Here, $\operatorname{aux}(m, i_\theta, j_\theta)$ differs from $\operatorname{aux}(\Psi(m), i, j)$ by the addition of two rows and two columns. The first column contains two $\frac{1}{2}$'s, each of which is contained in one of the two added rows. The second column contains a $-\frac{1}{2}$ and a $\frac{1}{2}$, again with each of these entries occurring in the two added rows. All other entries of the added rows and columns are zero. Thus

$$\mathcal{R}_{i_\theta, j_\theta} = R_{i,j} + 2.$$

Then

$$R_{i,j} = \mathcal{R}_{i_\theta, j_\theta} - 2 \leq j_\theta + \theta(i_\theta;\, j_\theta) - 2 = j + 2 + \gamma(i;\, j) - 2 = j + \gamma(i;\, j).$$

($\ell < i_\theta < j_\theta < \ell'$): Then

$$i_\theta = i+1, \quad j_\theta = j+1, \quad \text{and} \quad \gamma(i;\, j) + 1 = \theta(i_\theta,\, j_\theta).$$

Here, $\operatorname{aux}(m, i_\theta, j_\theta)$ differs from $\operatorname{aux}(\Psi(m), i, j)$ by the addition of two rows and two columns. The first column contains a single $\frac{1}{2}$ in the northmost added row. The second column contains two $\frac{1}{2}$'s, with each of these entries occurring in the two added rows. All other entries of the added rows and columns are zero. Thus

$$\mathcal{R}_{i_\theta, j_\theta} = R_{i,j} + 2.$$

Then

$$R_{i,j} = \mathcal{R}_{i_\theta, j_\theta} - 2 \leq j_\theta + \theta(i_\theta;\, j_\theta) - 2 = j + 1 + \gamma(i;\, j) + 1 - 2 = j + \gamma(i;\, j).$$

($\ell < i_\theta < \ell' < j_\theta$): Then

$$i_\theta = i+1, \quad j_\theta = j+2, \quad \text{and} \quad \gamma(i;\, j) = \theta(i_\theta,\, j_\theta).$$

Here, $\operatorname{aux}(m, i_\theta, j_\theta)$ differs from $\operatorname{aux}(\Psi(m), i, j)$ by the addition of two rows and three columns; one checks that again we have

$$\mathcal{R}_{i_\theta, j_\theta} = R_{i,j} + 2.$$

Thus

$$R_{i,j} = \mathcal{R}_{i_\theta, j_\theta} - 2 \leq j_\theta + \theta(i_\theta;\, j_\theta) - 2 = j + 2 + \gamma(i;\, j) - 2 = j + \gamma(i;\, j).$$

$(\ell < \ell' < i_\theta < j_\theta)$: Then

$$i_\theta = i + 2, \quad j_\theta = j + 2, \quad \text{and} \quad \gamma(i;\, j) = \theta(i_\theta,\, j_\theta).$$

Here, $\operatorname{aux}(m, i_\theta, j_\theta)$ differs from $\operatorname{aux}(\Psi(m), i, j)$ by the addition of two rows and four columns; again, it is the case that

$$\mathcal{R}_{i_\theta, j_\theta} = R_{i,j} + 2.$$

Then

$$R_{i,j} = \mathcal{R}_{i_\theta, j_\theta} - 2 \leq j_\theta + \theta(i_\theta;\, j_\theta) - 2 = j + 2 + \gamma(i;\, j) - 2 = j + \gamma(i;\, j).$$

This completes the proof of Theorem 4.6, having assumed Proposition 6.1. □

6B. ***Proof of Theorem 1.3:*** (I): Let $[\alpha, \gamma] \in \mathcal{C}$ be such that Y_γ is non-$\mathcal{P}$ along $\mathcal{O}_\alpha$. Now suppose

$$[\alpha, \gamma] \preceq_{\mathcal{C}} [\beta, \theta]. \tag{8}$$

Clearly, we may assume this is a covering relation, of which there are two kinds.

The first possibility is that $[\beta, \theta]$ interval pattern contains $[\alpha, \gamma]$. We know from Observation 1.4 that W_γ is non-$\mathcal{P}$ along Q_α, so in particular, W_γ is non-$\mathcal{P}$ at x_α. Thus $\mathcal{N}_{\gamma,\alpha}$ is non-$\mathcal{P}$ at the origin, since (non-)$\mathcal{P}$ is stable under taking slices by Lemma 2.5. Then $\mathcal{N}_{\theta,\beta}$ is non-$\mathcal{P}$ at the origin as well, by Theorem 4.6. (Note that the isomorphism Ψ of Theorem 4.6 carries the origin on $\mathcal{N}_{\theta,\beta}$ to the origin on $\mathcal{N}_{\gamma,\alpha}$.) Thus W_θ is non-$\mathcal{P}$ at x_β, again by slice-stability of (non-)$\mathcal{P}$, and then in fact W_θ is non-$\mathcal{P}$ along Q_β, by homogeneity. Using Observation 1.4 once more, we conclude that Y_θ is non-$\mathcal{P}$ along $\mathcal{O}_\beta$.

The other possibility is that $\theta = \gamma$ and $\beta \preceq \alpha$. Then $\mathcal{O}_\beta \subseteq \overline{\mathcal{O}_\alpha} \subseteq Y_\gamma$, by the definition of the closure order. Since Y_γ is non-$\mathcal{P}$ along $\mathcal{O}_\alpha$ and non-$\mathcal{P}$ is a closed property, Y_γ is non-$\mathcal{P}$ on $\overline{\mathcal{O}_\alpha}$, and hence on $\mathcal{O}_\beta$, as desired.

(II): This is the contrapositive of (I). □

6C. ***Proof of Proposition 6.1.*** Recall that $w := w_\beta$ is the permutation associated to the clan β, as defined at the beginning of Section 3A. Denote by $\mathcal{I} = \mathcal{I}(\beta)$ the underlying involution of β, as defined prior to Claim 3.7. This means $\mathcal{I}$ is the permutation that fixes j if β_j is a sign and interchanges i and j if $(i < j)$ is a matching of β.

We split the proof into three main cases (A, B and C) depending on the value of θ_j.

6C1. *Case A:* $(\theta_j = +)$. We prove Case A via two claims.

Claim 6.2. *The entries of column* $w(j)$ (*except the* 1 *in row* j *required by* (O.1)) *vanish.*

Proof. By Lemma 5.7(I), $\theta(j;+) = \beta(j;+)$. Define

$$r := p - \theta(j;+) = p - \beta(j;+). \tag{9}$$

Since $\beta(n;+)=p$, it follows from the definitions that

$$r=\#\big\{k \mid k>j,\ \text{and } \beta_k=+ \text{ or } \beta_k \text{ is the right end of a matching}\big\}. \tag{10}$$

Let $k_1<\cdots<k_r$ be the r indices of the set given in (10). Also define

$$c_i=\begin{cases} w(k_i) & \text{if } \beta_{k_i}=+,\\ w(\mathcal{I}(k_i)) & \text{if } \beta_k \text{ is the right end of a matching,}\end{cases}$$

for $i=1,\ldots,r$.

Subclaim 6.3. (I) *$c_i\le p$ for $1\le i\le r$.*

(II) *Column c_i of M has a fixed ± 1 in row k_i. Besides any fixed ± 1 in this row, any of the leftmost p columns of M strictly right of column c_i has zero in row k_i.*

Proof. First suppose $\beta_{k_i}=+$. Then by definition of w,

$$c_i=w(k_i)\le p,$$

proving (I) in this case. Case (II) holds by (O.1) and (Z.1) combined.

Otherwise, β_{k_i} is the right end of a matching. For (I), here $\mathcal{I}(k_i)$ is the left end of that matching, and

$$c_i=w(\mathcal{I}(k_i))\le p,$$

again by the definition of w. Now, by (O.2), columns $w(\mathcal{I}(k_i))$ and $w(k_i)$ of M are assigned -1 and 1 respectively in row k_i. Then (II) holds by (Z.4). This completes the proof of Subclaim 6.3. □

Now, let $\vec{v}_i$ denote the vector consisting of the last $n-j$ entries in column c_i. By Subclaim 6.3(II), it follows that $\{\vec{v}_1,\ldots,\vec{v}_r\}$ is a linearly independent set. Since $M\in\mathcal{N}_{\theta,\beta}$, by (R.1) (with $i=j$), the southwest $(n-j)\times p$ submatrix M° of M has rank at most r (see (9)). Thus

$$\vec{v}_1,\ldots,\vec{v}_r \text{ is a basis of } \operatorname{colspace}(M^\circ). \tag{11}$$

Since we assume $\theta_j=+$, by (Z.2), all entries of column $w(j)$ strictly north of row j are zero. Thus it remains to show the vector $\vec{v}$ consisting of the last $n-j$ entries in column $w(j)\le p$ is the zero vector. By (11) we have

$$\vec{v}\in\operatorname{Span}(\vec{v}_1,\ldots,\vec{v}_r). \tag{12}$$

Now, for $i=1,\ldots,r$, let

$$k_i'=\begin{cases}\mathcal{I}(k_i) & \text{if } \mathcal{I}(k_i)>j,\\ k_i & \text{otherwise.}\end{cases}$$

Notice that k_i' is either the row of a pivot, or it is a row containing a -1 which is located southwest of entry $(j,w(j))$. Thus either by (Z.1) in the former case, or by (Z.4) in the latter, $\vec{v}$ has a 0 in row k_i' for all i. In view of (12) and Subclaim 6.3(II), we see that $\vec{v}=\vec{0}$, as desired. This completes the proof of Claim 6.2. □

Claim 6.4. *The entries of row j (except the* 1 *in column $w(j)$ required by (O.1)) vanish.*

Proof. By (Z.1), the claim holds for the leftmost p columns of M. Thus, it remains to check the conclusion for the rightmost q columns $p+1, \dots, n$.

Let M° be the southeast $n-(j-1) \times q$ submatrix of M. By (R.2) for $i = j-1$, the rank of M° is at most

$$q - \theta(j-1; -) = q - \theta(j; -) = q - \beta(j; -) =: r,$$

where the second equality is by Lemma 5.7. As in the proof of Claim 6.2, there are r positions $k_1, \dots, k_r > j$ with β_{k_i} being either a $-$ or the right end of a matching. Each index corresponds to a 1 in row $k_i > j$ in one of the rightmost q columns of M. More precisely, if β_{k_i} is a $-$, this 1 is the pivot of row k_i, and if β_{k_i} is the right end of a matching, the 1 is the second 1 in its column. By (Z.5) and (Z.8) respectively, each of these 1's have all 0's to their right (and in the same row). None of these 1's appear in the same column, by (O.1) and (O.3). So, these r column vectors $\vec{v}_1, \dots, \vec{v}_r$ of M° are linearly independent, and therefore

$$\vec{v}_1, \dots, \vec{v}_r \text{ is a basis of colspace}(M^\circ). \tag{13}$$

In row j of M, the entry in each $\vec{v}_i$ is zero, by (Z.6) or (Z.7), since each such position is above either a pivot 1 or is between two 1's. So if any entry in row j among the last q were nonzero, its column in M° would be linearly independent of $\{\vec{v}_1, \dots, \vec{v}_r\}$, contradicting (13). The claim therefore holds. ☐

6C2. *Case B:* ($\theta_j = -$). The argument is nearly identical to that of Case A; we omit the details.

6C3. *Case C:* ($j, j' \in J$ (*with* $j < j'$) *is a matched pair*). We start with an observation we will use repeatedly:

Claim 6.5. *If* $(k < k')$ *is a matching of* β, *then*

$$1 \le a < w(k) \Longrightarrow w^{-1}(a) < k,$$

and

$$p+1 \le a < w(k') \Longrightarrow \mathcal{I}(w^{-1}(a)) < k.$$

Proof. Since k is the left end of a matching, $w(k) \le p$. If $1 \le a < w(k) \le p$, then by definition, a is the label assigned by w to either a $+$ or the left end of a matching appearing to the left of k, so $w^{-1}(a) < k$.

Similarly, since k' is the right end of a matching, we have $p+1 \le w(k') \le n$. If $p+1 \le a < w(k')$, then a is assigned by w to either a $-$ or the right end f' of a matching $(f < f')$. By definition of w, in the former case, the "$-$" must appear left of k, so that $w^{-1}(a) < k$. Since $\mathcal{I}(w^{-1}(a)) = w^{-1}(a)$ in this case, we have $\mathcal{I}(w^{-1}(a)) < k$, as claimed. In the latter case, where f' occurs at position $w^{-1}(a)$ and f at position $\mathcal{I}(w^{-1}(a))$, again by the definition of w, we must have that f is left of k, so $\mathcal{I}(w^{-1}(a)) < k$. ☐

Denote by $\vec{v}_i$ the i-th column vector of M.

Since M is invertible, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{C}^n$. Thus, for any $1 \le k \le n$, we have a linear dependence relation

$$\vec{e}_k + \sum_{a=p+1}^{n} \lambda_{k,a} \vec{v}_a = \sum_{a=1}^{p} \lambda_{k,a} \vec{v}_a \tag{14}$$

for some scalars $\lambda_{k,1}, \ldots, \lambda_{k,n} \in \mathbb{C}$.

Claim 6.6. *Given any k such that $\beta_k = +, -$, or the left end of a matching, we have*

$$\lambda_{k,w(b)} = \lambda_{k,w(\mathcal{I}(b))} = 0 \quad \textit{for any } b < k.$$

If β_k is the right end of a matching, then

$$\lambda_{k,w(b)} = \lambda_{k,w(\mathcal{I}(b))} = 0 \quad \textit{for any } b < \mathcal{I}(k).$$

Proof. Fix k. We prove both assertions by a common induction on $b < \min\{k, \mathcal{I}(k)\}$.

In the base case, when $b = 1$, β_1 cannot be the right end of a matching. So we check the remaining three possibilities in turn.

If $\beta_1 = +$, then the first coordinate of $\vec{v}_a$ is 1 for $a = 1$ and is 0 otherwise. This follows from the definition of w together with (O.1), (Z.1), and (Z.6). Since $k > b = 1$, the first row of (14) thus says that $0 = \lambda_{k,1}$. However,

$$1 = w(1) = w(\mathcal{I}(1))$$

in this case, so we are done. The argument when $\beta_1 = -$ is similar.

Now, suppose that β_1 is the left end of a matching. Then the right end of this matching is at position $\mathcal{I}(1)$. From w's definition, along with (O.2), (Z.1), and (Z.5), the first row of (14) asserts

$$0 + \lambda_{k,p+1} = \lambda_{k,1}.$$

Meanwhile, by (O.3), (Z.4), and (Z.8), row $\mathcal{I}(1)$ reads

$$0 + \lambda_{k,p+1} = -\lambda_{k,1}.$$

Putting these together, we have

$$\lambda_{k,w(1)} = \lambda_{k,1} = \lambda_{k,p+1} = \lambda_{k,w(\mathcal{I}(1))} = 0,$$

as desired.

For the inductive step, now suppose that $b > 1$, and that the claims hold for indices less than b.

Case 1: ($\beta_b = +$.) By (Z.1) in row b, all the entries among the first p columns are 0 except for a 1 in column $w(b)$. Therefore, the b-th row of (14) is

$$0 + \sum_{a=p+1}^{n} \lambda_{k,a} z_{b,a} = \lambda_{k,w(b)}. \tag{15}$$

If

$$\mathcal{I}(w^{-1}(a)) > b,$$

then $z_{b,a} = 0$ by (Z.6), since this entry is above the pivot in column a. If on the other hand

$$\mathcal{I}(w^{-1}(a)) < b,$$

then $\lambda_{k,a} = 0$ by the inductive hypothesis. Hence $\lambda_{k,w(b)} = 0$. Since $w(\mathcal{I}(b)) = w(b)$ in this case, we have $\lambda_{k,w(\mathcal{I}(b))} = 0$ as well.

Case 2: ($\beta_b = -$.) This is similar to Case 1.

Case 3: (β_b is left end of a matching.) Let $b' = \mathcal{I}(b)$ be the right end of the matching. Let

$$c = w(b) \quad \text{and} \quad c' = w(b').$$

In row b, (14) states that

$$\lambda_{k,c'} = \lambda_{k,c}, \tag{16}$$

since all other entries in row b are 0, by (Z.1) and (Z.5).

Now, note that if β_k is a $+$, a $-$, or the left end of a matching, then we clearly have that $k \neq b'$. If β_k is the right end of a matching, then since $b < \mathcal{I}(k)$, we again have that $k \neq b'$. Thus in row b', (14) states that

$$0 + \sum_{a=p+1}^{c'-1} \lambda_{k,a} z_{b',a} + \lambda_{k,c'} = \sum_{a=1}^{c-1} \lambda_{k,a} z_{b',a} - \lambda_{k,c}, \tag{17}$$

by (Z.4) and (Z.8).

Now,

$$\mathcal{I}(w^{-1}(a)) < w^{-1}(c) = b$$

for a in the first sum, whereas

$$w^{-1}(a) < w^{-1}(c) = b$$

for a in the second sum. Hence by induction, $\lambda_{k,a} = 0$ for $1 \leq a \leq c-1$ and $p+1 \leq a \leq c'-1$. So (17) reduces to

$$\lambda_{k,c'} = -\lambda_{k,c}.$$

Combining this with (16) gives

$$\lambda_{k,c} = \lambda_{k,c'} = 0,$$

as desired.

Case 4: (β_b is the right end of a matching.) The left end of the matching is at position $b' = \mathcal{I}(b)$. By Case 3,

$$\lambda_{k,w(b')} = \lambda_{k,w(\mathcal{I}(b'))} = 0,$$

which is the same as

$$\lambda_{k,w(b)} = \lambda_{k,w(\mathcal{I}(b))} = 0,$$

as required. □

Corollary 6.7. *If $(k < k')$ is a matching of β, then*

$$\lambda_{k,a} = \lambda_{k',a} = 0 \quad \textit{for } a = 1, \ldots, w(k)-1 \textit{ and for } a = p+1, \ldots, w(k')-1.$$

Proof. By Claim 6.5, for $1 \leq a < w(k)$, we have $w^{-1}(a) < k$, whereas for $p+1 \leq a < w(k')$, we have $\mathcal{I}(w^{-1}(a)) < k$. In either event, $\lambda_{k,a} = \lambda_{k',a} = 0$ by Claim 6.6. $\square$

Claim 6.8. *If $\beta_k = +$, then $\lambda_{k,w(k)} = 1$. If $\beta_k = -$, then $\lambda_{k,w(k)} = -1$.*

Proof. First suppose $\beta_k = +$. Consider row k of (14). By (Z.1) and (O.1), this equation is

$$1 + \sum_{a=p+1}^{n} \lambda_{k,a} z_{k,a} = \lambda_{k,w(k)}.$$

If $w^{-1}(a) < k$, then $\lambda_{k,a} = 0$ by Claim 6.6. On the other hand, if $w^{-1}(a) > k$, then $z_{k,a} = 0$ by (Z.6) or (Z.7). Hence the equation reduces to $1 = \lambda_{k,w(k)}$, as desired.

Now, if $\beta_k = -$, then row k of (14) is

$$1 + \lambda_{k,w(k)} = \sum_{a=1}^{p} \lambda_{k,a} z_{k,a}.$$

Again, if $w^{-1}(a) < k$, then $\lambda_{k,a} = 0$ by Claim 6.6. If $w^{-1}(a) > k$, then $z_{k,a} = 0$ by (Z.2). Hence the equation reduces to $1 + \lambda_{k,w(k)} = 0$, whence $\lambda_{k,w(k)} = -1$ as claimed. $\square$

Claim 6.9. *Suppose $(k < k')$ is a matching of β. Then*

$$\lambda_{k,w(k)} = \tfrac{1}{2} \quad \textit{and} \quad \lambda_{k,w(k')} = -\tfrac{1}{2},$$

and

$$\lambda_{k',w(k)} = \lambda_{k',w(k')} = -\tfrac{1}{2}.$$

Proof. First consider row k of (14). By (O.2), (Z.1), and (Z.5), it reads

$$1 + \lambda_{k,w(k')} = \lambda_{k,w(k)}. \tag{18}$$

Now consider row k' of (14). By (O.3), (Z.4), and (Z.8), it reads

$$\sum_{a=p+1}^{w(k')-1} \lambda_{k,a} z_{k',a} + \lambda_{k,w(k')} = \sum_{a=1}^{w(k)-1} \lambda_{k,a} z_{k',a} - \lambda_{k,w(k)}. \tag{19}$$

Applying Corollary 6.7 to (19) reduces it to

$$\lambda_{k,w(k')} = -\lambda_{k,w(k)}. \tag{20}$$

Solving (18) and (20) simultaneously gives $\lambda_{k,w(k)} = \frac{1}{2}$ and $\lambda_{k,w(k')} = -\frac{1}{2}$, as claimed.

Similarly, now consider (14), but with k replaced by k'; looking at row k gives

$$\lambda_{k',w(k')} = \lambda_{k',w(k)}. \tag{21}$$

If we examine row k', we obtain

$$1 + \lambda_{k',w(k')} = -\lambda_{k',w(k)}. \tag{22}$$

Solving (22) and (21) simultaneously gives the remainder of the claim. $\square$

Claim 6.10. *If* $(k < k')$ *is a matching of* β, *then*

$$\lambda_{k,w(b)} = \lambda_{k,w(\mathcal{I}(b))} = \lambda_{k',w(b)} = \lambda_{k',w(\mathcal{I}(b))} = 0$$

for $b = k+1, \ldots, k'-1$.

Proof. We only give the proof of the claim that

$$\lambda_{k,w(b)} = \lambda_{k,w(\mathcal{I}(b))} = 0.$$

The proof that

$$\lambda_{k',w(b)} = \lambda_{k',w(\mathcal{I}(b))} = 0$$

is identical.

We induct on b. For b in the appropriate range, the inductive hypothesis is that the claim holds for all a satisfying $k < a < b$.

Case 1: ($\beta_b = +$.) In this case, row b of (14) reads

$$0 + \sum_{a=p+1}^{n} \lambda_{k,a} z_{b,a} = \lambda_{k,w(b)}. \tag{23}$$

Now, if $\mathcal{I}(w^{-1}(a)) > b$, then $z_{b,a} = 0$ by (Z.6), since this entry is above the pivot in column a. Otherwise, either

$$\mathcal{I}(w^{-1}(a)) < k, \quad \mathcal{I}(w^{-1}(a)) = k, \quad \text{or} \quad k < \mathcal{I}(w^{-1}(a)) < b.$$

In the first case, we have that $\lambda_{k,a} = 0$ by Claim 6.6. In the last case, we have that $\lambda_{k,a} = 0$ by the inductive hypothesis. In the middle case, we have that $z_{b,a} = 0$ by (Z.7), since this entry is between the two 1's in positions (k,a) and (k',a). Thus the left-hand side of (23) is actually zero, implying that $\lambda_{k,w(b)} = \lambda_{k,w(\mathcal{I}(b))} = 0$, as claimed.

Case 2: ($\beta_b = -$.) This is very similar to Case 1; we omit the details.

Case 3: (β_b is the left end of a matching.) Let the right end of this matching be $\beta_{b'}$. Let $c = w(b)$, and let $c' = w(b')$.

By (Z.1) and (Z.5), in row b, (14) states that

$$\lambda_{k,c} = \lambda_{k,c'}. \tag{24}$$

By (Z.4), (Z.8) and (O.3), in row b', we have

$$0 + \sum_{a=p+1}^{c'-1} \lambda_{k,a} z_{b',a} + \lambda_{k,c'} = \sum_{a=1}^{c-1} \lambda_{k,a} z_{b',a} - \lambda_{k,c} \tag{25}$$

Subcase 3.1: ($b' < k'$.) In fact each term of the sum on the left-hand side of (25) vanishes for one of three reasons:

(L.3.1.1) $\mathcal{I}(w^{-1}(a)) < k$: $\lambda_{k,a} = 0$ by Claim 6.6.

(L.3.1.2) $\mathcal{I}(w^{-1}(a)) = k$: $z_{b',a} = 0$ by (Z.7), since in column $a = w(k')$, the entry in row b' is located between two 1's in rows k and k', since we have assumed that $b' < k'$.

(L.3.1.3) $\mathcal{I}(w^{-1}(a)) > k$: $\lambda_{k,a} = 0$ by the inductive hypothesis, since $p+1 \leq a < w(b')$ implies that $\mathcal{I}(w^{-1}(a)) < b$ by Claim 6.5.

Similarly, each term of the sum on the right hand side of (25) vanishes for one of these three reasons:

(R.3.1.1) $w^{-1}(a) < k$: $\lambda_{k,a} = 0$ by Claim 6.6.

(R.3.1.2) $w^{-1}(a) = k$: $z_{b',a} = 0$ by (Z.3), since in column $a = w(k)$, the entry in row b' is located between a 1 in row k and a -1 in row k', again because we have assumed that $b' < k'$.

(R.3.1.3) $w^{-1}(a) > k$: $\lambda_{k,a} = 0$ by the inductive hypothesis, since $1 \leq a < w(b)$ implies that $w^{-1}(a) < b$ by Claim 6.5.

Thus by (L.3.1.1)–(L.3.1.3) and (R.3.1.1)–(R.3.1.3) combined, (25) reduces to $\lambda_{k,c'} = -\lambda_{k,c}$. The simultaneous solution of this with (24) is $\lambda_{k,c} = \lambda_{k,c'} = 0$, as desired.

Subcase 3.2: ($b' > k'$.) The two matchings ($k < k'$) and ($b < b'$) form a 1212 pattern. We analyze each term of the sum on the left-hand side of (25):

(L.3.2.1) $\mathcal{I}(w^{-1}(a)) < k$: $\lambda_{k,a} = 0$ by Claim 6.6.

(L.3.2.2) $\mathcal{I}(w^{-1}(a)) = k$: We cannot conclude that $z_{b',a} = 0$. We do know by Claim 6.9 that $\lambda_{k,a} = \lambda_{k,w(k')} = -\frac{1}{2}$, so all we can say is that this particular term is equal to $-\frac{1}{2}z_{b',w(k')}$.

(L.3.2.3) $\mathcal{I}(w^{-1}(a)) > k$: $\lambda_{k,a} = 0$ by the inductive hypothesis, as in (L.3.1.3).

For the right hand side of (25), we see:

(R.3.2.1) $w^{-1}(a) < k$: $\lambda_{k,a} = 0$ by Claim 6.6.

(R.3.2.2) $w^{-1}(a) = k$: We do not know that $z_{b',a} = 0$, but we do at least know by Claim 6.9 that $\lambda_{k,a} = \lambda_{k,w(k)} = \frac{1}{2}$. Thus this term is equal to $\frac{1}{2}z_{b',w(k)}$.

(R.3.2.3) $w^{-1}(a) > k$: $\lambda_{k,a} = 0$ by the inductive hypothesis, as in (R.3.1.3).

In view of the 1212 pattern occurring in positions $k < b < k' < b'$, by (Z.9),

$$z_{b',w(k')} = -z_{b',w(k)}.$$

Thus the terms from (L.3.2.2) and (R.3.2.2) may be canceled in (25). Therefore (25) reduces to $\lambda_{k,c'} = -\lambda_{k,c}$. The simultaneous solution of this with (24) is

$$\lambda_{k,c} = \lambda_{k,c'} = 0,$$

as required.

Case 4: (β_b is the right end of a matching.) If the left end of this matching occurs prior to position k, we are done by Claim 6.6. Otherwise it occurs after position k, and we may apply Case 3. $\square$

Let

$$\pi_M : \mathbb{C}^n \to V_1$$

be the projection onto

$$V_1 := \langle \vec{v}_1, \ldots, \vec{v}_p \rangle$$

with kernel

$$V_2 := \langle \vec{v}_{p+1}, \ldots, \vec{v}_n \rangle.$$

Applying π_M to both sides of (14) gives

$$\pi_M(\vec{e}_k) = \sum_{a=1}^{p} \lambda_{k,a} \vec{v}_a. \tag{26}$$

Claim 6.11. *Suppose $(k < k')$ is a matching of β. Then*

(I) *$\pi_M(\vec{e}_k)$ is equal to $-1/2$ in row k'.*

(II) *If ℓ is the left end of a matching with $\ell > k$, then $\pi_M(\vec{e}_\ell)$ has entry 0 in row k'.*

(III) *$\pi_M(\vec{e}_k)$ is zero in row ℓ for any $\ell < k'$ except $\ell = k$.*

Proof. (I) We have

$$\pi_M(\vec{e}_k) = \sum_{a=1}^{p} \lambda_{k,a} \vec{v}_a = \sum_{a=w(k)}^{p} \lambda_{k,a} \vec{v}_a = \tfrac{1}{2} \vec{v}_{w(k)} + \sum_{a=w(k)+1}^{p} \lambda_{k,a} \vec{v}_a. \tag{27}$$

The first equality is (26). The second equality is by Corollary 6.7. The third equality applies Claim 6.9. Now, by (O.3), $z_{k',w(k)} = -1$, and by (Z.4), $z_{k',a} = 0$ for $w(k) < a \le p$. Therefore row k' of (27) is clearly $-\frac{1}{2}$, as claimed.

(II) By (Z.4) and (26), entry k' of $\pi_M(\vec{e}_\ell)$ is $\sum_{a=1}^{w(k)} \lambda_{\ell,a} z_{k',a}$. But if $a \le w(k)$, we have $w^{-1}(a) \le k < \ell$ by Claim 6.5. Thus by Claim 6.6, all $\lambda_{\ell,a}$ in this sum are zero.

(III) $\pi_M(\vec{e}_k)$ in row ℓ is $\sum_{a=1}^{p} \lambda_{k,a} z_{\ell,a}$ However, $\lambda_{k,a} = 0$ if $w^{-1}(a) < k'$ and $w^{-1}(a) \ne k$, either by Claim 6.6 (if $w^{-1}(a) < k$) or Claim 6.10 (if $k < w^{-1}(a) < k'$).

Now suppose $w^{-1}(a) = k$, so that $a = w(k)$. Then there is a 1 in position (k, a) and a -1 in position (k', a). If $\ell < k$, then $z_{\ell,a} = 0$ by (Z.2), while if $k < \ell < k'$, we have $z_{\ell,a} = 0$ by (Z.3).

Finally, if $w^{-1}(a) > k'$, then since $1 \le a \le p$ (and hence β has either a $+$ or a left endpoint at position $w^{-1}(a)$), there is a pivot at position $(w^{-1}(a), a)$. Then since $\ell < k'$, we have $z_{\ell,a} = 0$ by (Z.2). So in fact every term of the sum is zero. □

Set

$$r := \theta(j; j').$$

By Lemma 5.7,

$$r = \beta(j; j') = \#\{(k < k') \text{ a matching of } \beta \mid k < j < j' < k'\}.$$

Let the r matchings of the above set be

$$(k_1 < k_1'), (k_2 < k_2'), \dots, (k_r < k_r'), \quad \text{with } k_1 < k_2 < \cdots < k_r < j.$$

Claim 6.12. (I) *The set*

$$\{\pi_M(\vec{e}_{k_1}), \pi_M(\vec{e}_{k_2}), \dots, \pi_M(\vec{e}_{k_r})\} \quad (\textit{modulo } E_{j'})$$

forms a basis of $\pi_M(E_j) + E_{j'}/E_{j'}$.

(II) $\pi_M(\vec{e}_j) = \vec{0}$ *as an element of* $\pi_M(E_j) + E_{j'}/E_{j'}$. *In other words, for any* $t > j'$, $\pi_M(\vec{e}_j)$ *is* 0 *in row* t.

Proof. (I) Suppose in the quotient $\pi_M(E_j) + E_{j'}/E_{j'}$ that we have a dependence relation of the form

$$\sum_{i=1}^{r} \lambda_i (\pi_M(\vec{e}_{k_i}) + E_{j'}) = \sum_{i=1}^{r} \lambda_i \pi_M(\vec{e}_{k_i}) + E_{j'} = 0 \quad (\text{modulo } E_{j'}).$$

Then $\sum_{i=1}^{r} \lambda_i \pi_M(\vec{e}_{k_i}) \in E_{j'}$, meaning that for any index $\ell > j'$, the vector $\sum_{i=1}^{r} \lambda_i \pi_M(\vec{e}_{k_i})$ has entry 0 in position ℓ. In particular, this vector has entry 0 in positions $k_1', \dots, k_r'$. Then applying parts (I) and (II) of Claim 6.11, a triangularity argument gives that all the scalars λ_i are equal to zero. Hence the vectors $\pi_M(\vec{e}_{k_1}), \dots, \pi_M(\vec{e}_{k_r})$ descend to a linearly independent set in the quotient $\pi_M(E_j) + E_{j'}/E_{j'}$.

Then by (C.3) and the discussion following the statement of Theorem 4.4, we know that

$$\dim(\pi_M(E_j) + E_{j'}) \le j' + \theta(j; j') = j' + \beta(j; j') = j' + r.$$

Then $\dim(\pi_M(E_j) + E_{j'}/E_{j'}) \le r$, so in fact the linearly independent set $\{\pi_M(\vec{e}_{k_1}), \dots, \pi_M(\vec{e}_{k_r})\}$ is a basis for $\pi_M(E_j) + E_{j'}/E_{j'}$, as claimed.

(II) From Part (I), we have that $\pi_M(\vec{e}_j)$, viewed as an element of $\pi_M(E_j) + E_{j'}/E_{j'}$, is a linear combination of $\{\pi_M(\vec{e}_{k_1}), \pi_M(\vec{e}_{k_2}), \dots, \pi_M(\vec{e}_{k_r})\}$, or equivalently,

$$\pi_M(\vec{e}_j) + E_{j'} = \sum_{i=1}^{r} \lambda_i \pi_M(\vec{e}_{k_i}) + E_{j'}$$

for some scalars $\lambda_1, \dots, \lambda_r$. We want to show that $\lambda_i = 0$ for each i. Since we know that $k_i < j < k_i'$ for $1 \le i \le r$, we have by Claim 6.11(II) that $\pi_M(\vec{e}_j)$ has entry 0 in row $k_i' > j'$ for $1 \le i \le r$. We also know by Claim 6.11(I) that $\pi_M(\vec{e}_{k_i})$ is nonzero in position k_i' for $1 \le i \le r$. Combining these two facts gives that each $\lambda_i = 0$, and hence we have $\pi_M(\vec{e}_j) = 0$ in $\pi_M(E_j) + E_{j'}/E_{j'}$, as desired. Since we thus have $\pi_M(\vec{e}_j) \in E_{j'}$ the remaining assertion also follows. □

We also record the following related claim for later use:

Claim 6.13. (I) *The set*

$$\{\pi_M(\vec{e}_{k_1}), \pi_M(\vec{e}_{k_2}), \dots, \pi_M(\vec{e}_{k_r})\} \quad (\textit{modulo } E_{j'-1})$$

forms a basis of $(\pi_M(E_{j-1}) + E_{j'-1})/E_{j'-1}$.

(II) $\pi_M(\vec{v}) = 0$ *in row* j' *for any vector* $\vec{v}$.

Proof. (I) Since $(j < j')$ is a matching, we have

$$\theta(j-1; j'-1) = \theta(j; j') = r,$$

since a matching $(k < k')$ satisfies

$$k \leq j < j' < k' \iff k \leq j-1 < j'-1 < k'.$$

Hence the claim follows by exactly the same argument as for Claim 6.12(I).

(II) By (I), it suffices to prove this for $\vec{e}_{k_i}$ for $i = 1, \ldots, r$. Since $j' < k_i'$ by definition, this follows from Claim 6.11(III). □

Now, by (O.2),

$$z_{j,w(j)} = 1 \quad \text{and} \quad z_{j,w(j')} = 1.$$

By (O.3),

$$z_{j',w(j)} = -1 \quad \text{and} \quad z_{j',w(j')} = 1.$$

By (Z.2) and (Z.3),

$$z_{b,w(j)} = 0 \quad \text{if } b < j \text{ or } j < b < j'.$$

Finally, by (Z.6) and (Z.7),

$$z_{b,w(j')} = 0 \quad \text{if } b < j \text{ or } j < b < j'.$$

Hence the first assertion of our next claim is what remains to obtain the desired conclusion of Proposition 6.1 about columns $w(j)$ and $w(j')$. The second assertion is a technical strengthening for the induction argument we give.

Claim 6.14 (Case C: column zeroness). $z_{b,w(j)} = z_{b,w(j')} = 0$ *and* $\lambda_{j,w(b)} = \lambda_{j,w(\mathcal{I}(b))} = 0$ *for all* $b > j'$.

Proof. Our argument is by induction on $b > j'$. The inductive hypothesis is that the claims hold for all q satisfying $j' < q < b$. Our inductive step is to prove that this implies that the claims hold for b. Our argument depends upon the value of β_b.

($\beta_b = +$): The expression for $\pi_M(\vec{e}_j)$ in row b (see (26)) is

$$\sum_{a=1}^{p} \lambda_{j,a} z_{b,a}. \tag{28}$$

However, by (O.1) and (Z.1) we know

$$z_{b,w(b)} = 1 \quad \text{and} \quad z_{b,t} = 0 \qquad \text{for} \quad 1 \leq t \leq p \quad \text{and} \quad t \neq w(b).$$

Thus, in particular we obtain the claim's assertion that $z_{b,w(j)} = 0$. Moreover, (28) reduces to $\lambda_{j,w(b)}$. By Claim 6.12(II), all entries of $\pi_M(\vec{e}_j)$ beyond position j' are zero. Since we assume $b > j'$, we therefore conclude that $\lambda_{j,w(b)} = 0$. Since $\beta_b = +$, by definition we have

$$w(\mathcal{I}(b)) = w(b),$$

so

$$\lambda_{j,w(\mathcal{I}(b))} = \lambda_{j,w(b)} = 0,$$

proving the claim's second assertion in this case.

It remains to show that $z_{b,w(j')} = 0$. For this, we use (14), taking $k = j$, which in row b gives

$$0 + \sum_{a=p+1}^{n} \lambda_{j,a} z_{b,a} = \sum_{a=1}^{p} \lambda_{j,a} z_{b,a} = \lambda_{j,w(b)} = 0, \tag{29}$$

where the second and third equalities were shown in the previous paragraph.

We wish to isolate the $\lambda_{j,w(j')} z_{b,w(j')}$ term on the left-hand side of (29) by showing all other terms there vanish.

Subclaim 6.15. *For all $a \neq w(j')$ on the left-hand side of* (29), *either $\lambda_{j,a} = 0$ or $z_{b,a} = 0$.*

Proof of Subclaim 6.15. Given $p+1 \leq a \leq n$, we prove the subclaim by case analysis on the position of the pivot in column a.

Since $\beta_b = +$ and $p+1 \leq a \leq n$, the pivot of M in column a cannot be in row b. This leaves two other cases.

First, if the pivot of M in column a is in a row strictly north of row b, then this pivot occurs in row $\mathcal{I}(w^{-1}(a)) < b$. Note that we cannot have $\mathcal{I}(w^{-1}(a)) = j'$, since this would imply that $a = w(j)$, contradicting the fact that $p+1 \leq a \leq n$ while $1 \leq w(j) \leq p$. Furthermore, we cannot have $\mathcal{I}(w^{-1}(a)) = j$, since this implies that $a = w(j')$, and we are excluding this case from consideration. Thus we need only consider the cases where $\mathcal{I}(w^{-1}(a)) < j$, where $j < \mathcal{I}(w^{-1}(a) < j'$, and where $j' < \mathcal{I}(w^{-1}(a)) < b$. In the first case, we have that $\lambda_{j,a} = \lambda_{j,w(\mathcal{I}(\mathcal{I}(w^{-1}(a))))} = 0$ by Claim 6.6. In the second case, we have $\lambda_{j,a} = 0$ by Claim 6.10. And in the third case, we have that $\lambda_{j,a} = 0$ by the inductive hypothesis.

The second possibility is that the pivot of M in column a is in a row strictly south of row b. But in this case (Z.6) implies that $z_{b,a} = 0$, so we are done.

This completes the proof of Subclaim 6.15. □

By Claim 6.9, we have $\lambda_{j,w(j')} = -\frac{1}{2}$. Hence, by Subclaim 6.15, (29) reduces to

$$\lambda_{j,w(j')} z_{b,w(j')} = -\tfrac{1}{2} z_{b,w(j')} = 0.$$

Therefore, $z_{b,w(j')} = 0$, as needed.

(β has a $-$ at b): The reasoning here is very similar to the previous case. We omit the details.

(β has the left end of a matching at b): By (O.2) in row b of M, there are 1's in positions $w(b)$ and $w(\mathcal{I}(b))$. In addition, by (Z.1) and (Z.5), M has 0's elsewhere in row b. Thus we obtain the claim's assertion that

$$z_{b,w(j)} = z_{b,w(j')} = 0.$$

Using this, taking $k = j$ in (14) and examining row b, we see that

$$\lambda_{j,w(b)} = \lambda_{j,w(\mathcal{I}(b))}. \tag{30}$$

Claim 6.14's hypothesis that $b > j'$, combined with Claim 6.12(II), indicates that $\lambda_{j,w(b)} = 0$. Hence, by (30) we obtain the required $\lambda_{j,w(b)} = \lambda_{j,w(\mathcal{I}(b))} = 0$.

(β has the right end of a matching at b and its left end is at $\mathcal{I}(b) < j$): Since this subcase assumes $\mathcal{I}(b) < j$, by Claim 6.6, we have that

$$\lambda_{j,w(\mathcal{I}(b))} = \lambda_{j,w(b)} = 0,$$

as desired.

By definition of w,

$$w(\mathcal{I}(b)) < w(j) \leq p \quad \text{and} \quad p+1 \leq w(b) < w(j'). \tag{31}$$

Now, $z_{b,w(\mathcal{I}(b))} = -1$ by (O.3), and column $w(j)$ is right of column $w(\mathcal{I}(b))$ by the first inequality of (31), so by (Z.4), we know that $z_{b,w(j)} = 0$. Similarly, $z_{b,w(b)} = 1$ by (O.3), and using the last inequality of (31) combined with (Z.8), we see that $z_{b,w(j')} = 0$ as well.

(β has the right end of a matching at b and its left end $\mathcal{I}(b)$ satisfies $j < \mathcal{I}(b) < j'$): By Claim 6.10 we know $\lambda_{j,w(b)} = \lambda_{j,w(\mathcal{I}(b))} = 0$. Here, the matchings ($\mathcal{I}(b) < b$) and ($j < j'$) are in a 1212 pattern, so by (Z.9),

$$z_{b,w(j)} = -z_{b,w(j')}.$$

Thus we only need to see that $z_{b,w(j)} = 0$. For this, consider row b of (26), where we take $k = j$. Since $b > j'$, using Claim 6.12(II), (O.3), and (Z.4), we see that

$$\left(\sum_{a=1}^{w(\mathcal{I}(b))-1} \lambda_{j,a} z_{b,a}\right) - \lambda_{j,w(\mathcal{I}(b))} = 0. \tag{32}$$

It was argued already, in the proof of Claim 6.10 (see (R.3.2.1)–(R.3.2.3)), that the left-hand side of (32) reduces to $\frac{1}{2} z_{b,w(j)} - \lambda_{j,w(\mathcal{I}(b))}$, and we have noted above that $\lambda_{j,w(\mathcal{I}(b))} = 0$. So we have $\frac{1}{2} z_{b,w(j)} = 0$, whence $z_{b,w(j)} = 0$, as desired.

(β has the right end of a matching at b and its left end $\mathcal{I}(b)$ satisfies $j' < \mathcal{I}(b)$): Consider row $\mathcal{I}(b)$ of (14), where we take $k = j$. Since we assume $j' < \mathcal{I}(b)$, by (O.2), (Z.1) and (Z.5), (14) reads

$$\lambda_{j,w(b)} = \lambda_{j,w(\mathcal{I}(b))}. \tag{33}$$

By Claim 6.12(II) and the assumption $j' < \mathcal{I}(b)$, we see that $\lambda_{j,w(\mathcal{I}(b))} = 0$, and so by (33),

$$\lambda_{j,w(b)} = 0. \tag{34}$$

Let us show $z_{b,w(j)} = 0$. For this, we first analyze row b of (26), taking $k = j$. By (O.3), (Z.8) and Claim 6.12(II), this row is

$$\sum_{a=1}^{w(\mathcal{I}(b))-1} \lambda_{j,a} z_{b,a} - \lambda_{j,w(\mathcal{I}(b))} = 0.$$

Since by (33) and (34) we know $\lambda_{j,w(\mathcal{I}(b))} = 0$, this equation reduces to

$$\sum_{a=1}^{w(\mathcal{I}(b))-1} \lambda_{j,a} z_{b,a} = 0. \tag{35}$$

By Claim 6.5 (where $k = \mathcal{I}(b)$), all a indexing the summation in (35) satisfy

$$w^{-1}(a) < \mathcal{I}(b) < b.$$

Therefore, all $\lambda_{j,a}$ except for $\lambda_{j,w(j)}$ vanish, either by Claim 6.6, Claim 6.10, or the inductive hypothesis. In view of Claim 6.9,

$$\lambda_{j,w(j)} z_{b,w(j)} = \tfrac{1}{2} z_{b,w(j)}$$

does appear in the left-hand side of (35). Thus (35) actually states

$$\tfrac{1}{2} z_{b,w(j)} = 0,$$

whence $z_{b,w(j)} = 0$.

We now show $z_{b,w(j')} = 0$. For this, consider row b of (14) (taking $k = j$), which says that

$$0 + \sum_{a=p+1}^{w(b)-1} \lambda_{j,a} z_{b,a} + \lambda_{j,w(b)} = \sum_{a=1}^{w(\mathcal{I}(b))-1} \lambda_{j,a} z_{b,a} - \lambda_{j,w(\mathcal{I}(b))},$$

or equivalently, by (33) and (34),

$$\sum_{a=p+1}^{w(b)-1} \lambda_{j,a} z_{b,a} = \sum_{a=1}^{w(\mathcal{I}(b))-1} \lambda_{j,a} z_{b,a}.$$

By (35), the right hand side of this equation is zero, so we are reduced to

$$\sum_{a=p+1}^{w(b)-1} \lambda_{j,a} z_{b,a} = 0. \tag{36}$$

By Claim 6.5, all a indexing the sum satisfy $\mathcal{I}(w^{-1}(a)) < b$. Thus, all $\lambda_{j,a}$ vanish with the exception of $\lambda_{j,w(j')}$, either by Claim 6.6, Claim 6.10, or the inductive hypothesis. Thus (36) reduces to

$$\lambda_{j,w(j')} z_{b,w(j')} = -\tfrac{1}{2} z_{b,w(j')} = 0,$$

by Claim 6.9. Hence $z_{b,w(j')} = 0$.

This concludes the proof of Claim 6.14. $\square$

Now, we turn to rows j and j'. By (Z.1) and (Z.5), every entry in row j other than $z_{j,w(j)} = z_{j,w(j')} = 1$ is 0. Furthermore, by (Z.4) and (Z.8),

$$z_{j',a} = 0 \qquad \text{except when} \quad 1 \le a \le w(j) \quad \text{or} \quad p+1 \le a \le w(j').$$

Thus what remains to complete the proof of Proposition 6.1 is the following:

Claim 6.16 (Case C: row zeroness). $z_{j',a} = 0$ *for all* a *satisfying either* $1 \le a < w(j)$ *or* $p+1 \le a < w(j')$.

Proof. By Claim 6.5, it suffices to prove

$$z_{j',w(b)} = 0 \quad \text{and} \quad z_{j',w(\mathcal{I}(b))} = 0 \qquad \text{for all b<j.}$$

We prove this by *reverse* induction on b. Our inductive hypothesis is that

$$z_{j',w(c)} = z_{j',w(\mathcal{I}(c))} = 0 \quad \text{for all } c \text{ with } b < c < j.$$

Now, in view of (O.3), (Z.1), and (Z.5), row j of Equation (14) (taking $k = b$) gives

$$\lambda_{b,w(j)} = \lambda_{b,w(j')}. \tag{37}$$

Now consider row j' of (14), again taking $k = b$. By (O.3), (Z.4) and (Z.8),

$$\sum_{a=1}^{w(j)-1} \lambda_{b,a} z_{j',a} - \lambda_{b,w(j)} = \sum_{a=p+1}^{w(j')-1} \lambda_{b,a} z_{j',a} + \lambda_{b,w(j')}. \tag{38}$$

Hence, by Claim 6.13(II), we have

$$\sum_{a=1}^{w(j)-1} \lambda_{b,a} z_{j',a} - \lambda_{b,w(j)} = 0, \tag{39}$$

so by (38),

$$\sum_{a=p+1}^{w(j')-1} \lambda_{b,a} z_{j',a} + \lambda_{b,w(j')} = 0 \tag{40}$$

as well.

We now consider multiple cases depending on the value of β_b.

($\beta_b = +$): By Claim 6.5, for a indexing the sum in (40), we have $\mathcal{I}(w^{-1}(a)) < j$. Now, if $\mathcal{I}(w^{-1}(a)) < b$, then $\lambda_{b,a} = 0$ by Claim 6.6. If $b < \mathcal{I}(w^{-1}(a)) < j$, then $z_{j',a} = 0$ by the inductive hypothesis. Note that $\mathcal{I}(w^{-1}(a)) = b$ is not possible, since this would imply that $a = w(b)$, contradicting the fact that $p+1 \le a < w(j')$ while $1 \le w(b) \le p$. Thus (40) reduces to $\lambda_{b,w(j')} = 0$. Then by (37), $\lambda_{b,w(j)} = 0$ also. Then Equation (39) gives

$$\sum_{a=1}^{w(j)-1} \lambda_{b,a} z_{j',a} = 0. \tag{41}$$

Similarly to the previous paragraph,

$$\lambda_{b,a} = 0 \quad \text{if } w^{-1}(a) < b, \qquad \text{and} \qquad z_{j',a} = 0 \quad \text{if } w^{-1}(a) > b.$$

Hence (41) reduces to $\lambda_{b,w(b)} z_{j',w(b)} = 0$. Since $\lambda_{b,w(b)} \neq 0$ by Claim 6.8, we have $z_{j',w(b)} = 0$. Since $\beta_b = +$, $\mathcal{I}(b) = b$, so $z_{j',w(\mathcal{I}(b))} = 0$, as desired.

($\beta_b = -$): This is similar to the previous case, except the roles of Equations (39) and (40) are switched.

(β_b is a right endpoint): First, consider (39). Since $1 \le a < w(j)$, by Claim 6.5 we have that $w^{-1}(a) < j$. So consider the cases $w^{-1}(a) < \mathcal{I}(b)$, $w^{-1}(a) = \mathcal{I}(b)$, $\mathcal{I}(b) < w^{-1}(a) < b$, and $b < w^{-1}(a) < j$. (Note that $w^{-1}(a) = b$ is not possible, since $1 \le a < w(j) \le p$, while $p+1 \le w(b) \le n$.) In the first case, we have that $\lambda_{b,a} = 0$ by Claim 6.6. In the second case, we simply have that $a = w(\mathcal{I}(b))$. In the third case, we have that $\lambda_{b,a} = 0$ by Claim 6.10. Finally, in the fourth case, we have that $z_{j',a} = 0$ by the inductive hypothesis. Thus by Claim 6.9, (39) reduces to

$$-\tfrac{1}{2} z_{j',w(\mathcal{I}(b))} - \lambda_{b,w(j)} = 0. \tag{42}$$

Now, consider (40). Since $p+1 \le a < w(j')$, Claim 6.5 says that $\mathcal{I}(w^{-1}(a)) < j$. So consider the cases $\mathcal{I}(w^{-1}(a)) < \mathcal{I}(b)$, $\mathcal{I}(w^{-1}(a)) = \mathcal{I}(b)$, $\mathcal{I}(b) < \mathcal{I}(w^{-1}(a)) < b$, and $b < \mathcal{I}(w^{-1}(a)) < j$. (As above, $\mathcal{I}(w^{-1}(a)) = b$ cannot occur, since this would imply that $a = w(\mathcal{I}(b))$, but $p+1 \le a < w(j')$ while $1 \le w(\mathcal{I}(b)) < w(j) \le p$.) In the first case, $\lambda_{b,a} = 0$ by Claim 6.6. In the second case, we simply have that $a = w(b)$. In the third case, $\lambda_{b,a} = 0$ by Claim 6.10. Lastly, in the fourth case, we have that $z_{j',a} = 0$ by induction. Thus using Claim 6.9 again, (40) reduces to

$$-\tfrac{1}{2} z_{j',w(b)} + \lambda_{b,w(j')} = 0. \tag{43}$$

We can make arguments similar to the preceding ones when examining rows j and j' of (14), but taking $k = \mathcal{I}(b)$ instead. Doing so, analogously to (37), we obtain

$$\lambda_{\mathcal{I}(b),w(j)} = \lambda_{\mathcal{I}(b),w(j')}. \tag{44}$$

Analogously to (42) and (43), we obtain

$$\tfrac{1}{2} z_{j',w(\mathcal{I}(b))} - \lambda_{\mathcal{I}(b),w(j)} = 0 \tag{45}$$

and

$$-\tfrac{1}{2} z_{j',w(b)} + \lambda_{\mathcal{I}(b),w(j')} = 0. \tag{46}$$

The only simultaneous solution to (37), (44), (42), (43), (45), and (46) is

$$z_{j',w(\mathcal{I}(b))} = z_{j',w(b)} = \lambda_{\mathcal{I}(b),w(j)} = \lambda_{\mathcal{I}(b),w(j')} = \lambda_{b,w(j)} = \lambda_{b,w(j')} = 0.$$

(β_b is a left endpoint, and its right end satisfies $j' < \mathcal{I}(b)$): We have $z_{j',w(b)} = z_{j',w(\mathcal{I}(b))} = 0$ by (Z.3) and (Z.7) applied to columns $w(b)$ and $w(\mathcal{I}(b))$ respectively.

(β_b is a left endpoint, and its right end satisfies $j < \mathcal{I}(b) < j'$): Since the matchings $(b < \mathcal{I}(b))$ and $(j < j')$ are in a 1212 pattern, by (Z.9),

$$z_{j',w(b)} = -z_{j',w(\mathcal{I}(b))}. \tag{47}$$

By precisely the arguments that preceded (42) and (43), we see that (39) reduces to

$$\lambda_{b,w(b)} z_{j',w(b)} - \lambda_{b,w(j)} = 0,$$

while (40) reduces to

$$\lambda_{b,w(\mathcal{I}(b))} z_{j',w(\mathcal{I}(b))} + \lambda_{b,w(j')} = 0.$$

Using Claim 6.9, (37), and (47), we have

$$\tfrac{1}{2} z_{j',w(b)} - \lambda_{b,w(j)} = 0 \quad \text{and} \quad \tfrac{1}{2} z_{j',w(b)} + \lambda_{b,w(j)} = 0.$$

This implies $z_{j',w(b)} = \lambda_{b,w(j)} = 0$. Then we also have $z_{j',w(\mathcal{I}(b))} = 0$ by (47), as required.

(β_b is a left endpoint, and its right end satisfies $\mathcal{I}(b) < j$): In this case, the necessary claims are already proved by induction, since $b < \mathcal{I}(b) < j$.

This completes the proof of Claim 6.16, and hence the proof of Proposition 6.1. □

7. Conjectures and problems

Conjecture 7.1. *$\mathcal{I}_{\gamma,\alpha}$ is a radical ideal.*

Conjecture 7.1 has been verified using Macaulay 2 through $p+q=6$. This conjecture would follow from a solution to:

Problem 7.2. *Find a Gröbner basis for $\mathcal{I}_{\gamma,\alpha}$ with square-free lead terms.*

The analogous problem for *Kazhdan–Lusztig ideals* was solved in [Woo and Yong 2012]; it was shown that the defining generators of the ideal form a Gröbner basis. In contrast, we have:

Example 7.3. Let $\alpha = -1221+$ and $\gamma = 12-+12$. The defining generators of $\mathcal{I}_{\gamma,\alpha}$ are

$$z_{4,4}z_{6,6} + z_{5,4}z_{6,5} - z_{6,4}, \quad -\tfrac{1}{8} z_{4,4}z_{6,6} - \tfrac{1}{8} z_{5,4}z_{6,5} + \tfrac{1}{4} z_{6,4}.$$

Thus $z_{6,4} \in \mathcal{I}_{\gamma,\alpha}$, and hence $z_{4,4}z_{6,4} + z_{5,4}z_{6,5} \in \mathcal{I}_{\gamma,\alpha}$. Under any term order, the initial ideal must contain $z_{6,4}$ and either $z_{4,4}z_{6,4}$ or $z_{5,4}z_{6,5}$. However, at most one of these monomials can be realized as a lead term of the defining generators (for a fixed choice of term order). Hence the defining generators cannot be a Gröbner basis under any term order.

The *maximal singular locus* $\texttt{Maxsing}(Y_\gamma)$ is the set of K-orbits $\mathcal{O}_\alpha$ such that Y_γ is singular along $\mathcal{O}_\alpha$ and $\mathcal{O}_\alpha$ is maximal in Bruhat order with respect to this property.

Whereas the theorem of McGovern [2009] recalled in the introduction combinatorially characterizes which Y_γ are singular, the following question is open:

Problem 7.4. *Give a combinatorial description of* $\texttt{Maxsing}(Y_\gamma)$.

In [Woo and Wyser 2015], a solution to Problem 7.4 is given for clans γ which are "1212-avoiding". This gives a complete solution for the case $K = \mathrm{GL}_{n-1} \times \mathrm{GL}_1$ (since all $(n-1, 1)$-clans are 1212-avoiding). For the case of $K = \mathrm{GL}_{n-2} \times \mathrm{GL}_2$, we have a conjectural solution, but it is lengthy to state, so we omit it here.

The orbit closure Y_γ is *rationally smooth* along $\mathcal{O}_\alpha$ if the Kazhdan–Lusztig–Vogan polynomial $P_{\gamma,\alpha}(q)$ equals 1, and Y_γ is (globally) rationally smooth if $P_{\gamma,\alpha}(q) = 1$ for every $\alpha \leq \gamma$. It is known [McGovern 2009] that Y_γ is rationally smooth if and only if Y_γ is smooth. Using data from the `ATLAS` project, we have verified the following conjecture for $(p, q) = (2, 2), (3, 2)$:

Conjecture 7.5. *For $\alpha \leq \gamma$, Y_γ is rationally smooth on $\mathcal{O}_\alpha$ if and only if Y_γ is smooth on $\mathcal{O}_\alpha$.*

Let $(R, \mathfrak{m})$ be the local ring of a point p in a projective variety X. The *associated graded ring* is

$$\mathrm{gr}_{\mathfrak{m}} R = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}.$$

Conjecture 7.6. *Let $(R, \mathfrak{m})$ be the local ring associated to any point $p \in \mathcal{O}_\alpha \subseteq Y_\gamma$. Then $\mathrm{gr}_{\mathfrak{m}} R$ is Cohen–Macaulay and reduced. Moreover, it is Gorenstein whenever Y_γ is Gorenstein along $\mathcal{O}_\alpha$.*

This has been checked for $(p, q) = (2, 2), (3, 2), (3, 3), (4, 3)$. This conjecture does not follow from Conjecture 7.1, since reducedness, Cohen–Macaulayness, and Gorensteinness may be lost on degenerating to the associated graded ring.

The *projectivized tangent cone is* $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{m}} R)$. The *(Hilbert–Samuel) multiplicity* of $p \in X$ is

$$\mathrm{mult}_p(X) = \deg(\mathrm{Proj}(\mathrm{gr}_{\mathfrak{m}} R)).$$

This statistic provides singularity information. Specifically, $\mathrm{mult}_p(X) = 1$ if and only if p is a smooth point of X. Let $\mathrm{mult}_{\gamma,\alpha}$ be $\mathrm{mult}_p(Y_\gamma)$ for any point $p \in \mathcal{O}_\alpha$.

Define

$$\mathrm{maxmult}(Y_\gamma) = \max_{\alpha \leq \gamma} \mathrm{mult}_{\gamma,\alpha}\,.$$

Since multiplicity is a semicontinuous numerical invariant, $\mathrm{maxmult}(Y_\gamma) = \mathrm{mult}_{\gamma,\alpha}$ for some matchless $\alpha \leq \gamma$ (so $\mathcal{O}_\alpha$ is a closed orbit). A search for a combinatorial rule for $\mathrm{mult}_{\gamma,\alpha}$ might be partially guided by a solution to the following problem:

Problem 7.7. *Is the maximum value of* $\mathrm{maxmult}(Y_\gamma)$ *for* $\gamma \in \mathtt{Clans}_{p,q}$ *achieved at* $\gamma_{\max} = 1{+}^{p-1}{-}^{q-1}1 \in \mathtt{Clans}_{p,q}$?

We have checked that the answer to Problem 7.7 is affirmative $p + q \leq 6$. Note the maximizer in Problem 7.7 need not be unique. For instance, when $p = q = 2$, the maximum is achieved at $\gamma = 1212$, $\gamma = 1{+}{-}1$, and $\gamma = 1{-}{+}1$, with $\mathrm{maxmult}(Y_\gamma) = 2$.

Example 7.8 (Analogue of Lusztig's conjecture is false). Note that $P_{1+-1,++--}(q) = q + 1$, whereas $P_{1212,++--}(q) = 1$. However, $[{+}{+}{-}{-}, 1{+}{-}1] \cong [{+}{+}{-}{-}, 1212]$ as posets (see Figure 1). Thus, the KLV polynomial is not an invariant of the poset in Bruhat order. This contrasts with the situation for Schubert

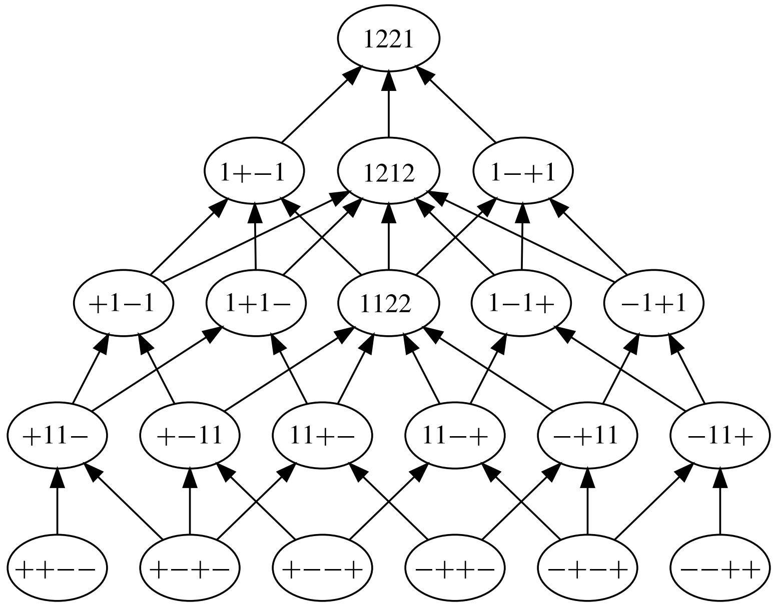

Figure 1. Bruhat order on $(\mathrm{GL}(4, \mathbb{C}), \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}))$.

varieties, where a conjecture attributed to Lusztig asserts that the Kazhdan–Lusztig polynomial is a poset invariant; see for example [Brenti 2002/04]. Similarly, $\mathrm{mult}_{1+-1,++--} = 2$ whereas $\mathrm{mult}_{1212,++--} = 1$.

The *Cohen–Macaulay type* of a local Cohen–Macaulay ring R is $\dim_{\mathbb{k}} \mathrm{Ext}_R^{\dim R}(\mathbb{k}, R)$. A ring R is *Gorenstein* if its Cohen–Macaulay type is 1.

Define the *maximal non-Gorenstein locus* $\mathtt{MaxnonGor}(Y_\gamma) = \{\mathcal{O}_\alpha\}$ to be the set of K-orbits $\mathcal{O}_\alpha \subseteq Y_\gamma$ such that Y_γ is non-Gorenstein at any point of $\mathcal{O}_\alpha$ and $\mathcal{O}_\alpha$ is maximal in Bruhat order with respect to this property.

Conjecture 7.9. $$\mathtt{MaxnonGor}(Y_\gamma) \subseteq \mathtt{Maxsing}(Y_\gamma).$$

If Y_γ is non-Gorenstein along $\mathcal{O}_\alpha \in \mathtt{Maxsing}(Y_\gamma)$ then $\mathcal{O}_\alpha \in \mathtt{MaxnonGor}(Y_\gamma)$. However, if Y_γ is Gorenstein along some $\mathcal{O}_\alpha \in \mathtt{Maxsing}(Y_\gamma)$, there a priori may be $\mathcal{O}_\beta \in \mathtt{MaxnonGor}(Y_\gamma)$ with $\beta < \alpha$. Conjecture 7.9 asserts that this does not occur. This conjecture is an analogue of [Woo and Yong 2008, Conjecture 6.7].

We verified Conjecture 7.9 for all (p, q) with $p + q \leq 7$ and for many cases of $(p, q) = (4, 4)$. In Table 1, we give the singularity data $(\gamma, \ell(\gamma), \mathtt{Maxsing}(Y_\gamma), \mathtt{MaxnonGor}(Y_\gamma))$ for $p = 3$, $q = 2$ for all the clans γ where Y_γ is singular.

Problem 7.10. *When is Y_γ (globally) Gorenstein?*

An answer to this question has been given by the first two authors in [Woo and Wyser 2015] in the event that γ is 1212-avoiding. As mentioned in the introduction, it is shown in [loc. cit.] that Gorensteinness cannot be characterized by ordinary pattern avoidance in general. However, for small q, it apparently can.

γ	$\ell(\gamma)$	$\mathtt{Maxsing}(Y_\gamma)$	$\mathtt{MaxnonGor}(Y_\gamma)$
122+1	5	$\{1{-}1{+}{+}\}$	$\{\}$
1+221	5	$\{{+}{+}1{-}1\}$	$\{\}$
12+12	5	$\{{-}{+}{+}{+}{-}, {+}1{-}1{+}\}$	$\{\}$
1+212	4	$\{{+}{+}{-}{-}{+}, 11{+}{+}{-}\}$	$\{{+}{+}{-}{-}{+}\}$
1−++1	4	$\{{-}{-}{+}{+}{+}\}$	$\{{-}{-}{+}{+}{+}\}$
121+2	4	$\{{+}{-}{-}{+}{+}, {-}{+}{+}11\}$	$\{{+}{-}{-}{+}{+}\}$
1+−+1	4	$\{11{-}{+}{+}, {+}{+}{-}11\}$	$\{\}$
1++−1	4	$\{{+}{+}{+}{-}{-}\}$	$\{{+}{+}{+}{-}{-}\}$
1+−1+	3	$\{{+}{+}{-}{-}{+}\}$	$\{\}$
+1+−1	3	$\{{+}{+}{+}{-}{-}\}$	$\{\}$
+1−+1	3	$\{{+}{-}{-}{+}{+}\}$	$\{\}$
1212+	3	$\{{+}{-}{-}{+}{+}, {-}{+}{+}{-}{+}\}$	$\{\}$
+1212	3	$\{{+}{-}{+}{+}{-}, {+}{+}{-}{-}{+}\}$	$\{\}$
1−+1+	3	$\{{-}{-}{+}{+}{+}\}$	$\{\}$

Table 1. The singularity data for $p = 3$, $q = 2$ for all the clans γ where Y_γ is singular.

When $q = 1$, all orbit closures are smooth (hence Gorenstein). Any $(p, 1)$-clan ($p \geq 1$) has at most one matching, and if it is has a matching, it can have no $-$ signs. Thus it necessarily avoids all of McGovern's singular patterns recalled in the introduction.

Next consider the case $q = 2$. In this case, the Gorensteinness criterion of [Woo and Wyser 2015] is easily seen to amount to the following ordinary pattern avoidance criterion.

Fact [Woo and Wyser 2015, Proposition 3.4.2]. *If γ is a* 1212*-avoiding $(p, 2)$-clan with $p \geq 2$, then Y_γ is Gorenstein if and only if γ avoids* 1++−1, 1−++1, 1++221, *and* 122++1.

Experimentally, it seems that when $q = 2$, Gorensteinness can be characterized by ordinary pattern avoidance even if we allow 1212-including clans.

Conjecture 7.11. *If γ is any $(p, 2)$-clan with $p \geq 2$, then Y_γ is Gorenstein if and only if γ avoids* $1{+}{+}{-}1$, $1 - + + 1$, $1 + {+}221$, $122 + {+}1$, $1 + 212$, *and* $121 + 2$.

This conjecture holds (by computation) for $n \leq 7$. The example recalled in the introduction says that there is no ordinary pattern avoidance characterization when $q > 2$.

Recall that a local ring R is said to be a *local complete intersection* (lci) if it is the quotient of a regular local ring by an ideal generated by a regular sequence. A variety is lci if each of its local rings are lci. Every smooth variety is lci, while every lci variety is Gorenstein (and hence Cohen–Macaulay).

At present, we do not have a characterization of which Y_γ are lci in general. An ordinary pattern avoidance criterion is given in [Woo and Wyser 2015] for 1212-avoiding clans. For general p and q, there are many non-lci patterns that must be avoided. However, when $q = 2$, the list of non-lci patterns that can actually occur is much smaller.

Fact [Woo and Wyser 2015, Proposition 3.3.3]. *If γ is a* 1212*-avoiding* $(p, 2)$*-clan with* $p \geq 2$, *then* Y_γ *is lci if and only if* γ *avoids* $1{+}{+}{-}1$, $1{-}{+}{+}1$, $1{+}{+}221$, *and* $122{+}{+}1$. *Thus* Y_γ *is lci if and only if it is Gorenstein.*

It is conjectured in [Woo and Wyser 2015] that ordinary pattern avoidance can be used to characterize lci-ness in general, whether γ is 1212-avoiding or not. The complete list of non-lci patterns is not known, even conjecturally. However, for $q = 2$, we have:

Conjecture 7.12. *If* γ *is any* $(p, 2)$*-clan with* $p \geq 2$, *then* Y_γ *is lci if and only if* γ *avoids* $1{+}{+}{-}1$, $1{-}{+}{+}1$, $1{+}{+}221$, $122{+}{+}1$, $1{+}212$, *and* $121{+}2$. *Equivalently, we conjecture that* Y_γ *is lci if and only if it is Gorenstein.*

This conjecture has been checked for $n \leq 7$. The equivalence of lci-ness and Gorensteinness breaks down when $q > 2$; for example, Y_{122331} is Gorenstein but not lci.

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