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The mean value of symmetric square L -functions

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We study the first moment of symmetric-square L -functions at the critical point in the weight aspect. Asymptotics with the best known error term $O(k^{-1/2})$ were obtained independently by Fomenko in 2003 and by Sun in 2013. We prove that there is an extra main term of size $k^{-1/2}$ in the asymptotic formula and show that the remainder term decays exponentially in k . The twisted first moment was evaluated asymptotically by Ng with the error bounded by $lk^{-1/2+\epsilon}$. We improve the error bound to $l^{5/6+\epsilon}k^{-1/2+\epsilon}$ unconditionally and to $l^{1/2+\epsilon}k^{-1/2}$ under the Lindelöf hypothesis for quadratic Dirichlet L -functions.

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1. Introduction

Asymptotic behavior of high moments of L -functions within different families can be predicted using random matrix theory [Conrey et al. 2005] or multiple Dirichlet series [Diaconu et al. 2003]. However, obtaining asymptotic formulas with sharp error bounds is a hard problem even in the case of small moments.

One of the most challenging families is symmetric square L -functions in weight aspect. Gelbart and Jacquet [1978] proved that these are L -function attached to $\mathrm{GL}(3)$ cusp forms.

Despite numerous efforts, even an upper bound for the second moment of symmetric square L -functions remains an open problem. See [Khan 2010, Conjecture 1.2].

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The first moment has been studied intensively during the last decades. See [Fomenko 2003; Khan 2007; Kohlen and Sengupta 2002; Lau 2002; Ng 2016; 2017; Sun 2013]. Nevertheless, even the best known asymptotic error estimates do not appear to be sharp.

The present paper aims to optimize error bounds in existing asymptotic formulas. With this goal, we prove an exact formula for the twisted first moment of symmetric square L -functions, and apply the Liouville–Green method (also called WKB approximation) to estimate remainder terms. This technique, originating from the theory of approximation of second-order differential equations, is quite unusual for analytic number theory, yet very effective. See, for example, [Balkanova and Frolenkov 2016; Zavorotny 1989].

2. Main results

Let $S_{2k}(1)$ denote the space of holomorphic cusp forms of weight $2k \geq 2$ with respect to the full modular group. Denote by H_{2k} the normalized Hecke basis for $S_{2k}(1)$. Every $f \in H_{2k}$ has a Fourier expansion of the form

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{k-1/2} \exp(2\pi i n z), \quad (2-1)$$

$$\lambda_f(1) = 1. \quad (2-2)$$

For $\Re s > 1$ the associated symmetric square L -function is given by

$$L(\text{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}. \quad (2-3)$$

Let $\Gamma(s)$ be the Gamma function and define

$$L_{\infty}(s) := \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-1}{2} + k\right) \Gamma\left(\frac{s}{2} + k\right). \quad (2-4)$$

Shimura [1975] showed that the completed L -function

$$\Lambda(\text{sym}^2 f, s) := L_{\infty}(s) L(\text{sym}^2 f, s)$$

is entire and satisfies the functional equation

$$\Lambda(\text{sym}^2 f, s) = \Lambda(\text{sym}^2 f, 1 - s). \quad (2-5)$$

Consider

$$M_1(l, s) := \sum_{f \in H_{2k}}^h \lambda_f(l^2) L(\text{sym}^2 f, s). \quad (2-6)$$

The superscript h in the formula above indicates that the expression in the sum is multiplied by the harmonic weight $\Gamma(2k-1)/((4\pi)^{2k-1} \langle f, f \rangle_1)$, where $\langle f, f \rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

Denote by γ the Euler constant and by $\psi(s)$ the logarithmic derivative of the Gamma function. Let ${}_2F_1(a, b, c; x)$ be the Gauss hypergeometric function and

$$\Phi_k(x) := \frac{\Gamma(k - \frac{1}{4})\Gamma(\frac{3}{4} - k)}{\Gamma(\frac{1}{2})} {}_2F_1(k - \frac{1}{4}, \frac{3}{4} - k, \frac{1}{2}; x), \quad (2-7)$$

$$\Psi_k(x) := x^k \frac{\Gamma(k - \frac{1}{4})\Gamma(k + \frac{1}{4})}{\Gamma(2k)} {}_2F_1(k - \frac{1}{4}, k + \frac{1}{4}, 2k; x). \quad (2-8)$$

We prove the following exact formula for the twisted first moment.

Theorem 2.1. *For any $l \geq 1$ one has*

$$\begin{aligned} M_1\left(l, \frac{1}{2}\right) &= \frac{1}{2\sqrt{l}} \left(-2 \log l - 3 \log 2\pi + \frac{\pi}{2} + 3\gamma + \psi\left(k - \frac{1}{4}\right) + \psi\left(k + \frac{1}{4}\right) \right) \\ &\quad + \frac{\sqrt{2\pi}(-1)^k \Gamma(k - \frac{1}{4})}{2\sqrt{l} \Gamma(k + \frac{1}{4})} \mathcal{L}_{-4l^2}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{l}} \sum_{1 \leq n < 2l} \mathcal{L}_{n^2 - 4l^2}\left(\frac{1}{2}\right) \Phi_k\left(\frac{n^2}{4l^2}\right) \\ &\quad + \frac{1}{l\sqrt{2}} \sum_{n > 2l} \mathcal{L}_{n^2 - 4l^2}\left(\frac{1}{2}\right) \sqrt{n} \Psi_k\left(\frac{4l^2}{n^2}\right), \end{aligned} \quad (2-9)$$

where

$$\mathcal{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{1}{q^s} \left(\sum_{\substack{1 \leq r \leq 2q \\ r^2 \equiv n \pmod{4q}}} 1 \right). \quad (2-10)$$

A similar formula, where the last two summands are expressed in terms of the Legendre function of the first kind, was established by a different method by Zagier [1977, Theorem 1]. Zagier's formula was applied by Kohlen and Sengupta [2002] to prove an upper bound for $M_1(1, \frac{1}{2})$, by Fomenko [2003] to obtain an asymptotic formula for $M_1(1, \frac{1}{2})$, and by Luo [2012] to estimate the second moment of $L(\text{sym}^2 f, \frac{1}{2})$ over short intervals.

The proof of Theorem 2.1 is quite simple and makes use of Petersson's trace formula and the functional equation for the Lerch zeta function.

When $l = 1$, exact formula (2-9) allows one to isolate the second main term of size $k^{-1/2}$ in the asymptotic formula so that the remainder term decays exponentially.

Corollary 2.2. *For some $c > 0$ one has*

$$\begin{aligned} M_1\left(1, \frac{1}{2}\right) &= \frac{1}{2} \left(\frac{\pi}{2} - 3 \log 2\pi + 3\gamma + \psi\left(k - \frac{1}{4}\right) + \psi\left(k + \frac{1}{4}\right) \right) + \frac{\sqrt{2\pi}(-1)^k \Gamma(k - \frac{1}{4})}{2 \Gamma(k + \frac{1}{4})} L\left(\frac{1}{2}, \chi_{-4}\right) \\ &\quad + \Phi_k\left(\frac{1}{4}\right) L\left(\frac{1}{2}, \chi_{-3}\right) + O\left(\frac{1}{\sqrt{k}} \exp(-ck)\right), \end{aligned} \quad (2-11)$$

where $L(\frac{1}{2}, \chi_D)$ is a Dirichlet L -function for the primitive quadratic character of conductor D and

$$\Phi_k\left(\frac{1}{4}\right) = -\frac{2^{3/2}\sqrt{\pi}}{3^{1/4}} \frac{1}{\sqrt{2k-1}} \sin \frac{\pi(2k-1)}{3} \left(1 + O\left(\frac{1}{k}\right)\right). \quad (2-12)$$

Remark. After posting the first version of this paper to the arXiv, the authors have been informed by Shenhui Liu that he has independently obtained an asymptotic formula similar to (2-11) by using an approximate functional equation. See [Liu 2017].

Corollary 2.2 improves the series of previously known results with the following error bounds:

- $k^{-0.008}$ [Lau 2002]
- $k^{-1/20}$ [Khan 2007]
- $k^{-1/2}$ [Fomenko 2003; Sun 2013].

Corollary 2.3. *For any $\epsilon > 0$, $l > 1$, one has*

$$M_1\left(l, \frac{1}{2}\right) = \frac{1}{2\sqrt{l}} \left(-2 \log l - 3 \log 2\pi + \frac{\pi}{2} + 4\gamma + \psi(1) + \psi\left(k - \frac{1}{4}\right) + \psi\left(k + \frac{1}{4}\right) \right) + O\left(\frac{l^{5/6+\epsilon}}{\sqrt{k}}\right). \quad (2-13)$$

Assuming the Lindelöf hypothesis for quadratic Dirichlet L -functions, the error term above can be replaced by $O(l^{1/2+\epsilon}k^{-1/2})$.

This improves the error bound $lk^{-1/2+\epsilon}$ proved by Ng (see [Ng 2016, Theorem 2.1.1] and [Ng 2017]).

3. Notation and tools

Let $e(x) = \exp(2\pi i x)$. For $v \in \mathbb{C}$ let

$$\tau_v(n) = \sum_{n_1 n_2 = n} \binom{n_1}{n_2}^v. \quad (3-1)$$

The classical Kloosterman sum is defined by

$$S(n, m; c) = \sum_{\substack{a \pmod{c} \\ (a, c) = 1}} e\left(\frac{an + a^*m}{c}\right), \quad aa^* \equiv 1 \pmod{c}.$$

Lemma 3.1 (Weil's bound [1948]). *One has*

$$|S(m, n; c)| \leq \tau_0(c) \sqrt{(m, n, c)} \sqrt{c}. \quad (3-2)$$

Let $J_\nu(x)$ be the Bessel function of the first kind.

Lemma 3.2 (Petersson's trace formula [1932]). *For $2k \geq 12$ and integral $l, n \geq 1$, one has*

$$\sum_{f \in H_{2k}}^h \lambda_f(l) \lambda_f(n) = \delta_{l, n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(l, n; c)}{c} J_{2k-1}\left(\frac{4\pi \sqrt{ln}}{c}\right). \quad (3-3)$$

The Lerch zeta function

$$\zeta(\alpha, \beta, s) = \sum_{n+\alpha > 0} \frac{e(n\beta)}{(n+\alpha)^s} \quad (3-4)$$

was introduced by Lipschitz [1857] and was named after Lerch, who proved in 1887 the functional equation. The special case $\zeta(\alpha, 0, s)$ is called the Hurwitz zeta function.

Lemma 3.3 [Lerch 1887]. *One has*

$$\zeta(\alpha, 0, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left(-ie\left(\frac{s}{4}\right)\zeta(0, \alpha, 1-s) + ie\left(-\frac{s}{4}\right)\zeta(0, -\alpha, 1-s) \right). \quad (3-5)$$

4. Some properties of $\mathcal{L}_n(s)$

The main references for this section are [Bykovskii 1994; Soundararajan and Young 2013; Zagier 1977]. Function (2-10) can be written as follows

$$\mathcal{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n)}{q^s} = \sum_{q=1}^{\infty} \frac{\lambda_q(n)}{q^s}, \quad (4-1)$$

where

$$\rho_q(n) := \#\{x \pmod{2q} : x^2 \equiv n \pmod{4q}\}, \quad (4-2)$$

$$\lambda_q(n) := \sum_{q_1^2 q_2 q_3 = q} \mu(q_2) \rho_{q_3}(n). \quad (4-3)$$

For a fixed n , both $\rho_q(n)$ and $\lambda_q(n)$ are multiplicative functions of q . Furthermore, for $n \equiv 2, 3 \pmod{4}$ the function $\rho_q(n)$ is identically zero. Therefore, $\mathcal{L}_n(s)$ does not vanish only for $n \equiv 0, 1 \pmod{4}$. If $n = 0$ then

$$\mathcal{L}_n(s) = \zeta(2s - 1). \quad (4-4)$$

Otherwise, for $n = Dl^2$ with D fundamental discriminant we have

$$\mathcal{L}_n(s) = l^{1/2-s} T_l^{(D)}(s) L(s, \chi_D), \quad (4-5)$$

where $L(s, \chi_D)$ is a Dirichlet L -function for primitive quadratic character χ_D and

$$T_l^{(D)}(s) = \sum_{l_1 l_2 = l} \chi_D(l_1) \frac{\mu(l_1)}{\sqrt{l_1}} \tau_{s-1/2}(l_2). \quad (4-6)$$

The completed L -function

$$\mathcal{L}_n^*(s) = \left(\frac{\pi}{|n|}\right)^{-s/2} \Gamma\left(\frac{s}{2} + \frac{1}{4} - \frac{\text{sgn } n}{4}\right) \mathcal{L}_n(s) \quad (4-7)$$

satisfies the functional equation

$$\mathcal{L}_n^*(s) = \mathcal{L}_n^*(1-s). \quad (4-8)$$

Lemma 4.1. *One has*

$$\sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod{q}} S(l^2, c^2; q) e\left(\frac{nc}{q}\right) = \frac{1}{\zeta(2s)} \mathcal{L}_{n^2-4l^2}(s). \quad (4-9)$$

Proof. Consider

$$S := \sum_{c \pmod{q}} S(l^2, c^2; q) e\left(n \frac{c}{q}\right) = \sum_{c \pmod{q}} \sum_{\substack{a \pmod{q} \\ (a, q)=1}} e\left(\frac{ac^2 + a^*l^2 + nc}{q}\right),$$

where $aa^* \equiv 1 \pmod{q}$. Making the change of variables $c = c_1 a^*$, we have

$$\begin{aligned} S &= \sum_{\substack{a \pmod{q} \\ (a, q)=1}} \sum_{c_1 \pmod{q}} e\left(\frac{c_1^2 a^* + l^2 a^* + n c_1 a^*}{q}\right) \\ &= \sum_{c_1 \pmod{q}} S(0, c_1^2 + l^2 + n c_1; q) = \sum_{c \pmod{q}} \sum_{\substack{bd=q \\ b|c^2+l^2+nc}} \mu(d)b \\ &= \sum_{bd=q} \mu(d)b \sum_{\substack{c \pmod{q} \\ c^2+l^2+nc \equiv 0 \pmod{b}}} 1 = \sum_{bd=q} \mu(d)b \sum_{\substack{c \pmod{b} \\ c^2+l^2+nc \equiv 0 \pmod{b}}} \frac{q}{b}. \end{aligned}$$

The condition $c^2 + l^2 + nc \equiv 0 \pmod{b}$ is equivalent to $(2c + n)^2 + 4l^2 - n^2 \equiv 0 \pmod{4b}$. Hence

$$S = q \sum_{bd=q} \mu(d) \sum_{\substack{c \pmod{2b} \\ c^2 \equiv n^2 - 4l^2 \pmod{4b}}} 1 = q \sum_{bd=q} \mu(d) \rho_b(n^2 - 4l^2).$$

Consequently,

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod{q}} S(l^2, c^2; q) e\left(\frac{nc}{q}\right) &= \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{bd=q} \mu(d) \rho_b(n^2 - 4l^2) \\ &= \sum_{b=1}^{\infty} \frac{\rho_b(n^2 - 4l^2)}{b^s} \sum_{q=1}^{\infty} \frac{\mu(q)}{q^s} \\ &= \frac{1}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n^2 - 4l^2)}{q^s} = \frac{\mathcal{L}_{n^2-4l^2}(s)}{\zeta(2s)}. \quad \square \end{aligned}$$

Lemma 4.2. *Assume that $d \neq 0$. For any $\epsilon > 0$, one has*

$$\mathcal{L}_d\left(\frac{1}{2}\right) \ll d^{1/6+\epsilon}. \quad (4-10)$$

If the Lindelöf hypothesis for Dirichlet L -functions is true, then

$$\mathcal{L}_d\left(\frac{1}{2}\right) \ll d^\epsilon. \quad (4-11)$$

Proof. For $d = Dl^2$, where D is the fundamental discriminant, one has

$$\mathcal{L}_d\left(\frac{1}{2}\right) = T_l^{(D)}\left(\frac{1}{2}\right) L\left(\frac{1}{2}, \chi_D\right)$$

by equation (4-5). It follows from equality (4-6) that

$$T_l^{(D)}\left(\frac{1}{2}\right) \ll \sum_{l_1 l_2 = l} \frac{\sigma(l_2)}{\sqrt{l_1}} \ll \sum_{l_1 | l} \frac{(l/l_1)^\epsilon}{\sqrt{l_1}} \ll l^\epsilon.$$

By [Conrey and Iwaniec 2000, Corollary 1.5] for any $\epsilon > 0$ one has

$$L\left(\frac{1}{2}, \chi_D\right) \ll D^{1/6+\epsilon}.$$

This implies the required bounds for the function $\mathcal{L}_d\left(\frac{1}{2}\right)$. \square

5. Exact formula

Lemma 5.1. For $\Re s > \frac{3}{2}$ one has

$$M_1(l, s) = \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k}}{2l^{1-s}} \frac{\Gamma(k-s/2)}{\Gamma(k+s/2)} \mathcal{L}_{-4l^2}(s) + (2\pi)^s i^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \mathcal{L}_{n^2-4l^2}(s) I\left(\frac{n}{l}\right), \quad (5-1)$$

where

$$I(x) := \frac{1}{2\pi i} \int_{(\Delta)} \frac{\Gamma\left(k - \frac{1}{2} + \frac{1}{2}w\right)}{\Gamma\left(k + \frac{1}{2} - \frac{1}{2}w\right)} \Gamma(1-s-w) \sin\left(\pi \frac{s+w}{2}\right) x^w dw \quad (5-2)$$

with $1-2k < \Delta < 1-\Re s$.

Proof. By the Petersson trace formula

$$\begin{aligned} M_1(l, s) &= \zeta(2s) \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{f \in H_{2k}}^h \lambda_f(l^2) \lambda_f(n^2) \\ &= \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \sum_{n=1}^{\infty} \frac{S(l^2, n^2; q)}{n^s} J_{2k-1}\left(4\pi \frac{ln}{q}\right). \end{aligned}$$

The change of order of summation above is justified by the absolute convergence for $\Re s > \frac{3}{2}$, which follows from the standard estimates

$$S(l^2, n^2; q) \ll q^{1/2+\epsilon} (l^2, n^2, q)^{1/2} \quad \text{for any } \epsilon > 0$$

and

$$J_{2k-1}\left(4\pi \frac{ln}{q}\right) \ll \begin{cases} \left(\frac{ln}{q}\right)^{2k-1}, & q > ln \\ \left(\frac{ln}{q}\right)^{-1/2}, & q < ln. \end{cases}$$

Next, we use the Mellin–Barnes representation for the Bessel function

$$M_1(l, s) = \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \times \frac{1}{4\pi i} \int_{(\Delta)} \frac{\Gamma\left(k - \frac{1}{2} + \frac{1}{2}w\right)}{\Gamma\left(k + \frac{1}{2} - \frac{1}{2}w\right)} \sum_{n=1}^{\infty} \frac{S(l^2, n^2; q)}{n^{w+s}} \left(\frac{q}{2\pi l}\right)^w dw,$$

where $1 - 2k < \Delta < 0$. To guarantee the absolute convergence of the integral over w and the sums over q, n we require that

$$\max(1 - 2k, 1 - \Re s) < \Delta < -\frac{1}{2},$$

which is true for $\Re s > \frac{3}{2}$. Consider

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{S(l^2, n^2; q)}{n^{w+s}} &= \sum_{c \pmod{q}} \sum_{n \equiv c \pmod{q}} \frac{S(l^2, c^2; q)}{n^{w+s}} = \sum_{c \pmod{q}} S(l^2, c^2; q) \sum_{n=1}^{\infty} \frac{1}{(c+nq)^{w+s}} \\ &= \sum_{c \pmod{q}} \frac{S(l^2, c^2; q)}{q^{w+s}} \zeta\left(\frac{c}{q}, 0, w+s\right). \end{aligned}$$

Note that for all c , the Lerch zeta function has a simple pole at $w = 1 - s$ with residue one. The next step is to apply functional equation (3-5) for the Lerch zeta function which is only possible when $\Re(s+w) < 0$. Accordingly, we move the w -contour to the left up to $\Delta_1 := -s - \epsilon$, crossing a simple pole at $w = 1 - s$. Therefore,

$$\begin{aligned} M_1(l, s) &= \frac{\zeta(2s)}{l^s} + 2\pi i^{2k} \zeta(2s) \frac{\Gamma(k - \frac{1}{2}s)}{\Gamma(k + \frac{1}{2}s)} \sum_{q=1}^{\infty} \sum_{c \pmod{q}} \frac{S(l^2, c^2; q)}{2q^2} \left(\frac{q}{2\pi l}\right)^{1-s} + 2\pi i^{2k} \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q} \frac{1}{4\pi i} \\ &\quad \times \int_{(\Delta_1)} \frac{\Gamma(k - \frac{1}{2} + \frac{1}{2}w)}{\Gamma(k + \frac{1}{2} - \frac{1}{2}w)} \left(\frac{q}{2\pi l}\right)^w \sum_{c \pmod{q}} \frac{S(l^2, c^2; q)}{q^{s+w}} \zeta\left(\frac{c}{q}, 0; s+w\right) dw. \end{aligned}$$

Using the functional equation (3-5), we obtain

$$\begin{aligned} \sum_{c \pmod{q}} S(l^2, c^2; q) \zeta\left(\frac{c}{q}, 0; s+w\right) \\ = 2(2\pi)^{s+w-1} \Gamma(1-s-w) \sin\left(\pi \frac{s+w}{2}\right) \sum_{c \pmod{q}} S(l^2, c^2; q) \zeta\left(0, \frac{c}{q}; 1-s-w\right). \end{aligned}$$

Substituting this into $M_1(l, s)$ and opening the Lerch zeta function, one has

$$\begin{aligned} M_1(l, s) &= \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k}}{2l^{1-s}} \zeta(2s) \frac{\Gamma(k - \frac{1}{2}s)}{\Gamma(k + \frac{1}{2}s)} \sum_{q=1}^{\infty} \sum_{c \pmod{q}} \frac{S(l^2, c^2; q)}{q^{1+s}} \\ &\quad + (2\pi)^s i^{2k} \zeta(2s) \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod{q}} S(l^2, c^2; q) e\left(\frac{nc}{q}\right) I\left(\frac{n}{l}\right), \end{aligned}$$

where $I(x)$ is defined by equation (5-2). Finally, computing the sums over c and q using formula (4-9), we prove the lemma. \square

Lemma 5.2. *If $x \geq 2$, then*

$$I(x) = \frac{2^{2k} (-1)^k}{2^s \sqrt{\pi}} \cos\left(\frac{\pi s}{2}\right) x^{1-2k} \frac{\Gamma(k - \frac{1}{2}s) \Gamma(k + \frac{1}{2} - \frac{1}{2}s)}{\Gamma(2k)} \times {}_2F_1\left(k - \frac{s}{2}, k + \frac{1}{2} - \frac{s}{2}, 2k; \frac{4}{x^2}\right). \quad (5-3)$$

Proof. Moving the contour of integration in (5-2) to the left, we cross simple poles at $w = 1 - 2k - 2j$, $j = 0, 1, 2, \dots$. Therefore,

$$I(x) = 2(-1)^k \cos\left(\frac{\pi s}{2}\right) x^{1-2k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma(2k-s+2j)}{\Gamma(2k+j)} x^{-2j}.$$

By the duplication formula we have

$$\Gamma\left(2\left(k+j-\frac{1}{2}s\right)\right) = \frac{2^{2k-1-s+2j}}{\sqrt{\pi}} \Gamma\left(k+j-\frac{1}{2}s\right) \Gamma\left(k+j+\frac{1}{2}1-\frac{1}{2}s\right).$$

This yields

$$\begin{aligned} I(x) &= \frac{2(-1)^k}{\sqrt{\pi}} 2^{2k-1-s} \cos\left(\frac{\pi s}{2}\right) x^{1-2k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\Gamma\left(k-\frac{1}{2}s+j\right)}{\Gamma(2k+j)} \Gamma\left(k+\frac{1}{2}-\frac{s}{2}+j\right) \left(\frac{4}{x^2}\right)^j \\ &= \frac{2^{2k}(-1)^k}{2^s \sqrt{\pi}} \cos\left(\frac{\pi s}{2}\right) x^{1-2k} \frac{\Gamma\left(k-\frac{1}{2}s\right) \Gamma\left(k+\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma(2k)} {}_2F_1\left(k-\frac{s}{2}, k+\frac{1}{2}-\frac{s}{2}, 2k; \frac{4}{x^2}\right). \quad \square \end{aligned}$$

Lemma 5.3. *One has*

$$I(2) = \frac{2(-1)^k}{2^s \sqrt{\pi}} \cos\left(\frac{1}{2}\pi s\right) \frac{\Gamma\left(k-\frac{1}{2}s\right) \Gamma\left(k+\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(k+\frac{1}{2}s\right) \Gamma\left(k-\frac{1}{2}+\frac{1}{2}s\right)} \Gamma\left(s-\frac{1}{2}\right). \quad (5-4)$$

Proof. Letting $x = 2$ in (5-3) and applying [Olver et al. 2010, Equation 15.4.20], we find

$${}_2F_1\left(k-\frac{1}{2}s, k+\frac{1}{2}-\frac{1}{2}s, 2k; 1\right) = \frac{\Gamma(2k) \Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}s\right) \Gamma\left(k-\frac{1}{2}+\frac{1}{2}s\right)}.$$

The assertion follows. □

Lemma 5.4. *If $x < 2$, then*

$$I(x) = \frac{(-1)^k}{\sqrt{\pi}} \sin\left(\frac{1}{2}\pi s\right) x^{1-s} \frac{\Gamma\left(k-\frac{1}{2}s\right) \Gamma\left(1-k-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}\right)} {}_2F_1\left(k-\frac{1}{2}s, 1-k-\frac{1}{2}s, \frac{1}{2}; \frac{1}{4}x^2\right). \quad (5-5)$$

Proof. Moving the contour of integration in (5-2) to the right we cross simple poles at $w = 1 - s + j$, $j = 0, 1, 2, \dots$. Accordingly,

$$I(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{\Gamma\left(k-\frac{1}{2}s+\frac{1}{2}j\right)}{\Gamma\left(k+\frac{1}{2}s-\frac{1}{2}j\right)} \sin\left(\pi \frac{1+j}{2}\right) x^{1-s+j}.$$

Note that

$$\sin\left(\pi \frac{1+j}{2}\right) = \cos\left(\frac{\pi j}{2}\right) = \begin{cases} 0, & j \text{ is odd,} \\ (-1)^m, & j = 2m. \end{cases}$$

Thus

$$I(x) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \frac{\Gamma\left(k-\frac{1}{2}s+m\right)}{\Gamma\left(k+\frac{1}{2}s-m\right)} (-1)^m x^{1-s+2m}.$$

In order to express $I(x)$ in terms of the Gauss hypergeometric function we apply the duplication formula, obtaining

$$(2m)! = \Gamma(2(m + \frac{1}{2})) = \frac{1}{\sqrt{\pi}} 2^{2m} \Gamma(m + \frac{1}{2}) \Gamma(m + 1).$$

Furthermore, by Euler's reflection formula

$$\Gamma(k + \frac{1}{2}s - m) = \frac{\pi}{(-1)^{k-m} \sin(\frac{1}{2}\pi s) \Gamma(1 - k - \frac{1}{2}s + m)}.$$

Finally,

$$\begin{aligned} I(x) &= \frac{(-1)^k}{\sqrt{\pi}} \sin(\frac{1}{2}\pi s) x^{1-s} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\Gamma(k - \frac{1}{2}s + m)}{\Gamma(m + \frac{1}{2})} \Gamma(1 - k - \frac{1}{2}s + m) \left(\frac{x^2}{4}\right)^m \\ &= \frac{(-1)^k}{\sqrt{\pi}} \sin(\frac{1}{2}\pi s) x^{1-s} \frac{\Gamma(k - \frac{1}{2}s) \Gamma(1 - k - \frac{1}{2}s)}{\Gamma(\frac{1}{2})} {}_2F_1\left(k - \frac{1}{2}s, 1 - k - \frac{1}{2}s, \frac{1}{2}; \frac{1}{4}x^2\right). \quad \square \end{aligned}$$

Next, we substitute equations (5-3), (5-4), (5-5) into expression (5-1), proving the exact formula for the shifted first moment.

Theorem 5.5. *For any $l \geq 1$ and $2 - 2k < \Re s < 2k - 1$, one has*

$$\begin{aligned} M_1(l, s) &= \frac{\zeta(2s)}{l^s} + \frac{(2\pi)^s i^{2k} \Gamma(k - \frac{1}{2}s)}{2l^{1-s} \Gamma(k + \frac{1}{2}s)} \mathcal{L}_{-4l^2}(s) \\ &+ \frac{(2\pi)^s \zeta(2s-1)}{\sqrt{\pi} l^{1-s}} \cos(\frac{1}{2}\pi s) \frac{\Gamma(k - \frac{1}{2}s) \Gamma(k + \frac{1}{2} - \frac{1}{2}s)}{\Gamma(k + \frac{1}{2}s) \Gamma(k - \frac{1}{2} + \frac{1}{2}s)} \Gamma(s - \frac{1}{2}) \\ &+ \frac{(2\pi)^s \sin(\frac{1}{2}\pi s)}{\sqrt{\pi} l^{1-s}} \sum_{1 \leq n < 2l} \mathcal{L}_{n^2-4l^2}(s) \frac{\Gamma(k - \frac{1}{2}s) \Gamma(1 - k - \frac{1}{2}s)}{\Gamma(\frac{1}{2})} \\ &\quad \times {}_2F_1\left(k - \frac{s}{2}, 1 - k - \frac{s}{2}, \frac{1}{2}; \left(\frac{n}{2l}\right)^2\right) \\ &+ \frac{2^{2k} \pi^s \cos(\frac{1}{2}\pi s)}{\sqrt{\pi}} \sum_{n > 2l} \mathcal{L}_{n^2-4l^2}(s) \frac{\Gamma(k - \frac{1}{2}s) \Gamma(k + \frac{1}{2} - \frac{1}{2}s)}{\Gamma(2k)} \\ &\quad \times \frac{1}{n^{1-s}} \left(\frac{n}{l}\right)^{1-2k} {}_2F_1\left(k - \frac{s}{2}, k + \frac{1}{2} - \frac{s}{2}, 2k; \left(\frac{2l}{n}\right)^2\right). \quad (5-6) \end{aligned}$$

Note that in equation (5-6) only the first and the third summands have poles at $s = \frac{1}{2}$. Computing the limit as $s \rightarrow \frac{1}{2}$ we find that these poles cancel each other. This allows us to prove the exact formula for the first moment of symmetric square L -functions at the critical point given by Theorem 2.1.

6. Liouville–Green approximation of special functions

The aim of this section is to approximate the functions $\Phi_k(x)$ and $\Psi_k(x)$ appearing in exact formula (2-9). Both of these special functions can be expressed in terms of the Gauss hypergeometric function, whose asymptotic approximation has been a subject of intensive research during the last century. The

works of Watson [1948], Jones [2001] and Farid Khwaja and Olde Daalhuis [2014; 2017] are the most relevant in our case. However, application of these results to approximation of $\Phi_k(x)$ and $\Psi_k(x)$ is not straightforward and involves certain technical difficulties due to either the region of validity of asymptotic expansion or the lack of asymptotic error estimates.

Therefore, we choose to follow the approach of [Olver 1974] and [Boyd and Dunster 1986]. Accordingly, we apply the Liouville–Green approximation directly to the functions $\Phi_k(x)$ and $\Psi_k(x)$. It turns out that these special functions have similar behavior to the ones occurring in the exact formula for the second moment of cusp form L -functions. See [Balkanova and Frolenkov 2016, Theorem 4.2].

Properties of Φ_k . Consider the function $\Phi_k(x)$ for $0 < x < 1$.

Lemma 6.1. *One has*

$$\Phi_k(x) = -\pi \left({}_2F_1\left(2k - \frac{1}{2}, \frac{3}{2} - 2k, 1; \frac{1}{2}(1 - \sqrt{x})\right) + {}_2F_1\left(2k - \frac{1}{2}, \frac{3}{2} - 2k, 1; \frac{1}{2}(1 + \sqrt{x})\right) \right). \quad (6-1)$$

Proof. Applying the quadratic transformation given by [Olver et al. 2010, Equation 15.8.27] we obtain

$$\begin{aligned} \Phi_k(x) &= \Gamma\left(k - \frac{1}{4}\right)\Gamma\left(\frac{3}{4} - k\right)\Gamma\left(k + \frac{1}{4}\right)\Gamma\left(\frac{5}{4} - k\right) \frac{1}{2\Gamma^2\left(\frac{1}{2}\right)} \\ &\quad \times \left({}_2F_1\left(2k - \frac{1}{2}, \frac{3}{2} - 2k, 1; \frac{1}{2}(1 - \sqrt{x})\right) + {}_2F_1\left(2k - \frac{1}{2}, \frac{3}{2} - 2k, 1; \frac{1}{2}(1 + \sqrt{x})\right) \right). \end{aligned}$$

Note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Euler’s reflection formula yields

$$\Gamma\left(k - \frac{1}{4}\right)\Gamma\left(k + \frac{1}{4}\right)\Gamma\left(\frac{3}{4} - k\right)\Gamma\left(\frac{5}{4} - k\right) = -2\pi^2.$$

The assertion follows. □

Making the change of variables

$$m := 2k - \frac{1}{2}, \quad k \in \mathbb{N}, \quad (6-2)$$

$$y := \frac{1}{2}(1 - \sqrt{x}), \quad 0 < y < \frac{1}{2}, \quad (6-3)$$

one has

$$\Phi_k(x) = -\pi \left({}_2F_1(m, 1 - m, 1; y) + {}_2F_1(m, 1 - m, 1; 1 - y) \right). \quad (6-4)$$

At the point $y = 0$, the function ${}_2F_1(m, 1 - m, 1; y)$ is recessive and the function ${}_2F_1(m, 1 - m, 1; 1 - y)$ is dominant. Therefore, further transformations are required to apply the Liouville–Green method to the second function. In particular, we show that ${}_2F_1(m, 1 - m, 1; 1 - y)$ has a similar shape to $\phi_k(x)$ studied in [Balkanova and Frolenkov 2016]. Note that the parameter m is now half-integral.

Lemma 6.2. *Let $m := 2k - \frac{1}{2}$, $k \in \mathbb{N}$. Then*

$$\begin{aligned} {}_2F_1(m, 1 - m, 1; 1 - y) &= (-\log y + 2\psi(1) - 2\psi(m)) \times {}_2F_1(m, 1 - m, 1; y) \\ &\quad + \frac{1}{\pi} \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c} \right) {}_2F_1(a, b, c; y) \Big|_{\substack{a=m \\ b=1-m \\ c=1}}. \end{aligned} \quad (6-5)$$

Proof. By [Beitman and Erdelyi 1953, Equation 33, p. 107] we have

$$\begin{aligned} {}_2F_1(m+u, 1-m+u, 1; 1-y) &= {}_2F_1(m+u, 1-m+u, 1+2u; y) \frac{\Gamma(1)\Gamma(-2u)}{\Gamma(1-m-u)\Gamma(m-u)} \\ &\quad + {}_2F_1(m-u, 1-m-u, 1-2u; y) \frac{\Gamma(1)\Gamma(2u)}{\Gamma(1-m+u)\Gamma(m+u)} y^{-2u}. \end{aligned}$$

Computing the limit as $u \rightarrow 0$, we prove the assertion. \square

Lemma 6.3. For $m = 2k - \frac{1}{2}$, $k \in \mathbb{N}$ one has

$${}_2F_1(m, 1-m, 1; \frac{1}{2}) = -\frac{(-1)^k \Gamma(k - \frac{1}{4})}{\sqrt{2\pi} \Gamma(k + \frac{1}{4})}, \quad (6-6)$$

$$\frac{d}{dx} ({}_2F_1(m, 1-m, 1; x)) \Big|_{x=1/2} = \frac{4(-1)^k \Gamma(k + \frac{1}{4})}{\sqrt{2\pi} \Gamma(k - \frac{1}{4})}. \quad (6-7)$$

Proof. On the one hand, by equation (6-1) we have

$${}_2F_1(m, 1-m, 1; \frac{1}{2}) = -\frac{1}{2\pi} \Phi_k(0).$$

On the other hand, equation (2-7) yields

$$\Phi_k(0) = \frac{\Gamma(k - \frac{1}{4}) \Gamma(\frac{3}{4} - k)}{\Gamma(\frac{1}{2})} = (-1)^k \sqrt{2\pi} \frac{\Gamma(k - \frac{1}{4})}{\Gamma(k + \frac{1}{4})}.$$

The last two equalities imply (6-6).

As a consequence of [Olver et al. 2010, Equation 15.8.25] we obtain

$$\begin{aligned} {}_2F_1(m, 1-m, 1; x) &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}m + \frac{1}{2})\Gamma(1 - \frac{1}{2}m)} \times {}_2F_1(\frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}; (1-2x)^2) \\ &\quad + (1-2x) \frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2} - \frac{1}{2}m)} {}_2F_1(\frac{1}{2}m + \frac{1}{2}, 1 - \frac{1}{2}m, \frac{3}{2}; (1-2x)^2). \end{aligned}$$

Then equality (6-7) follows by differentiating the last expression in x and setting $x = \frac{1}{2}$. \square

Lemma 6.4. The functions

$${}_2F_1(m, 1-m, 1; y) \quad \text{and} \quad {}_2F_1(m, 1-m, 1; 1-y)$$

are solutions of differential equation

$$y(1-y)F''(y) + (1-2y)F'(y) + m(m-1)F(y) = 0. \quad (6-8)$$

Proof. This follows from the differential equation for hypergeometric functions. \square

Approximation of Φ_k . In order to find a Liouville–Green approximation for $\Phi_k(y)$, we use formula (6-4) and study separately each of the hypergeometric functions ${}_2F_1(m, 1-m, 1; y)$ and ${}_2F_1(m, 1-m, 1; 1-y)$. As shown in Lemma 6.4, these functions are solutions of differential equation (6-8) that was already approximated in [Balkanova and Frolenkov 2016, Section 5.2]. So our problem reduces to computation of the Liouville–Green constants C_Y and C_J in the approximation of ${}_2F_1(m, 1-m, 1; 1-y)$.

For the reader’s convenience, we briefly recall the required results of [Balkanova and Frolenkov 2016, Section 5.2]. It follows from Lemma 6.4 that the functions

$$G_1(y) := {}_2F_1(m, 1-m, 1; y)\sqrt{y(1-y)}, \quad (6-9)$$

$$G_2(y) := {}_2F_1(m, 1-m, 1; 1-y)\sqrt{y(1-y)} \quad (6-10)$$

are solutions of the differential equation

$$G''(y) = (u^2 f(y) + g(y))G(y), \quad (6-11)$$

where

$$u := 2k - 1, \quad f(y) := -\frac{1}{y(1-y)}, \quad (6-12)$$

$$g(y) := -\frac{1}{4y^2(1-y)^2} + \frac{1}{4y(1-y)}. \quad (6-13)$$

Making the change of variables

$$Z(y) := \frac{G(y)}{\alpha(y)}, \quad \alpha(y) := \frac{(y-y^2)^{1/4}}{2(\arcsin \sqrt{y})^{1/2}}, \quad (6-14)$$

$$\xi := 4 \arcsin^2 \sqrt{y}, \quad (6-15)$$

we transform equation (6-11) into the following shape

$$\frac{d^2 Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2} + \frac{\psi(\xi)}{\xi} \right] Z = 0, \quad (6-16)$$

with

$$\psi(\xi) := \frac{1}{16 \sin^2 \sqrt{\xi}} - \frac{1}{16\xi}. \quad (6-17)$$

Removing the summand with $\psi(\xi)/\xi$ in equation (6-16), we have

$$\frac{d^2 Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2} \right] Z = 0. \quad (6-18)$$

The solutions of (6-18) are defined by

$$Z_C = \sqrt{\xi} C_0(u\sqrt{\xi}), \quad (6-19)$$

where C_i is either J or Y Bessel function of index i .

Then according to [Olver 1974, Chapter 12] solutions of original differential equation (6-16) can be found in the form

$$Z_C(\xi) = \sqrt{\xi} C_0(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{A(n; \xi)}{u^{2n}} - \frac{\xi}{u} C_1(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{B(n; \xi)}{u^{2n}}. \quad (6-20)$$

In order to determine coefficients $A(n; \xi)$, $B(n; \xi)$ we use differential equations (see [Gradshteyn and Ryzhik 2007, Equation 8.491(3)]) for functions

$$W(\xi) := \sqrt{\xi} C_0(u\sqrt{\xi}), \quad V(\xi) := \xi C_1(u\sqrt{\xi}) \quad (6-21)$$

and substitute (6-20) in equation (6-16). This yields

$$W(\xi) \sum_{n=0}^{\infty} \frac{C_n(\xi)}{u^{2n}} - V(\xi) \sum_{n=0}^{\infty} \frac{D_n(\xi)}{u^{2n-1}} = 0, \quad (6-22)$$

where

$$C_n(\xi) := A''(n; \xi) + \frac{1}{\xi} A'(n; \xi) - \frac{\psi(\xi)}{\xi} A(n; \xi) - B'(n; \xi) - \frac{B(n; \xi)}{2\xi},$$

$$D_n(\xi) := B''(n-1; \xi) + \frac{1}{\xi} B'(n-1; \xi) - \frac{\psi(\xi)}{\xi} B(n-1; \xi) + \frac{1}{\xi} A'(n; \xi).$$

Letting $C_n(\xi) = D_n(\xi) = 0$, we find the required recurrence relations

$$A(n; \xi) = -\xi B'(n-1; \xi) + \int_0^\xi \psi(x) B(n-1; x) dx + \lambda_n, \quad (6-23)$$

$$\sqrt{\xi} B(n; \xi) = \int_0^\xi \frac{1}{\sqrt{x}} (x A''(n; x) + A'(n; x) - \psi(x) A(n; x)) dx \quad (6-24)$$

for some real constants of integration λ_n .

Assume that $A(0; \xi) = 1$. Then

$$B(0; \xi) = -\frac{1}{8\sqrt{\xi}} \left(\cot \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right), \quad (6-25)$$

$$A(1; \xi) = \frac{1}{8} \left(\frac{1}{\xi} - \frac{\cot \sqrt{\xi}}{2\sqrt{\xi}} - \frac{1}{2 \sin^2 \sqrt{\xi}} \right) - \frac{1}{128} \left(\cot \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right)^2 + \lambda_1. \quad (6-26)$$

Furthermore, solutions (6-20) can be approximated by finite series using [Olver 1974, Theorem 4.1, p. 444] or [Boyd and Dunster 1986, Theorem 1].

Theorem 6.5. *Let $\xi_2 = \frac{1}{4}\pi^2$. For each value of u and each nonnegative integer N , equation (6-16) has solutions $Z_Y(\xi)$, $Z_J(\xi)$ which are infinitely differentiable in ξ on interval $(0, \xi_2)$, and are given by*

$$Z_Y(\xi) = \sqrt{\xi} Y_0(u\sqrt{\xi}) \sum_{n=0}^N \frac{A_Y(n; \xi)}{u^{2n}} - \frac{\xi}{u} Y_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_Y(n; \xi)}{u^{2n}} + \epsilon_{2N+1,1}(u, \xi), \quad (6-27)$$

$$Z_J(\xi) = \sqrt{\xi} J_0(u\sqrt{\xi}) \sum_{n=0}^N \frac{A_J(n; \xi)}{u^{2n}} - \frac{\xi}{u} J_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_J(n; \xi)}{u^{2n}} + \epsilon_{2N+1,2}(u, \xi), \quad (6-28)$$

where

$$\epsilon_{2N+1,1}(u, \xi) \ll \frac{\sqrt{\xi} |Y_0(u\sqrt{\xi})|}{u^{2N+1}} \sqrt{\xi_2 - \xi}, \quad (6-29)$$

$$\epsilon_{2N+1,2}(u, \xi) \ll \frac{\sqrt{\xi} |J_0(u\sqrt{\xi})|}{u^{2N+1}} \min(\sqrt{\xi}, 1) \quad (6-30)$$

and coefficients $(A_Y(n; \xi), B_Y(n; \xi)), (A_J(n; \xi), B_J(n; \xi))$ are defined by (6-23)–(6-24).

The functions $\xi^{1/4}(\sin \sqrt{\xi})^{1/2} G_1(\sin^2 \sqrt{\xi}/2)$ and $Z_J(\xi)$ are recessive solutions of equation (6-16) as $\xi \rightarrow 0$. Therefore, there is c_0 such that

$$\xi^{1/4}(\sin \sqrt{\xi})^{1/2} G_1(\sin^2 \sqrt{\xi}/2) = c_0 Z_J(\xi). \quad (6-31)$$

The value of the constant c_0 is determined by computing the limit of the left- and right-hand sides of equation (6-31) as $\xi \rightarrow 0$. On the one hand,

$$\lim_{\xi \rightarrow 0} {}_2F_1(k, 1-k, 1; \sin^2 \sqrt{\xi}/2) = 1. \quad (6-32)$$

On the other hand,

$$Z_J(\xi) = \sqrt{\xi} \sum_{n=0}^N \frac{A_J(n; \xi)}{u^{2n}} + O(\xi) \quad \text{as } \xi \rightarrow 0. \quad (6-33)$$

Choosing $A_J(n; \xi)$ such that $A_J(0; 0) = 1$ and $A_J(n; 0) = 0$ for $n \geq 1$ we find that $c_0 = 1$.

To sum up, we have proved the following lemma:

Lemma 6.6. *Let $\xi_2 = \frac{1}{4}\pi^2$. For $\xi \in (0, \xi_2)$, one has*

$$\xi^{1/4}(\sin \sqrt{\xi})^{1/2} G_1\left(\sin^2 \frac{\sqrt{\xi}}{2}\right) = Z_J(\xi), \quad (6-34)$$

where $Z_J(\xi)$ is given by (6-28).

This concludes the summary of results of [Balkanova and Frolenkov 2016, Section 5.2]. Our final goal is to compute $C_Y = C_Y(u)$ and $C_J = C_J(u)$ such that

$$\xi^{1/4}(\sin \sqrt{\xi})^{1/2} G_2(\sin^2 \frac{1}{2}\sqrt{\xi}) = C_Y Z_Y(\xi) + C_J Z_J(\xi). \quad (6-35)$$

Note that there exist c_1, c_2 such that

$$\begin{aligned} Z_Y(\xi) &= (c_1 G_1(\sin^2 \frac{1}{2}\sqrt{\xi}) + c_2 G_2(\sin^2 \frac{1}{2}\sqrt{\xi})) \times \xi^{1/4}(\sin \sqrt{\xi})^{1/2} \\ &= c_1 Z_J(\xi) + \xi^{1/4}(\sin \sqrt{\xi})^{1/2} c_2 G_2(\sin^2 \frac{1}{2}\sqrt{\xi}). \end{aligned} \quad (6-36)$$

The last two equalities imply

$$C_Y = \frac{1}{c_2}, \quad C_J = -\frac{c_1}{c_2}. \quad (6-37)$$

Lemma 6.7. *One has*

$$2c_1 = (-1)^{k+1} \sqrt{2\pi} \frac{\Gamma(k + \frac{1}{4})}{\Gamma(k - \frac{1}{4})} \frac{Z_Y(\xi_2)}{\xi_2^{1/4}} + (-1)^k \sqrt{2\pi} \frac{\Gamma(k - \frac{1}{4})}{\Gamma(k + \frac{1}{4})} \xi_2^{1/4} \left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2} \right), \quad (6-38)$$

$$2c_2 = (-1)^{k+1} \sqrt{2\pi} \frac{\Gamma(k + \frac{1}{4})}{\Gamma(k - \frac{1}{4})} \frac{Z_Y(\xi_2)}{\xi_2^{1/4}} - (-1)^k \sqrt{2\pi} \frac{\Gamma(k - \frac{1}{4})}{\Gamma(k + \frac{1}{4})} \xi_2^{1/4} \left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2} \right). \quad (6-39)$$

Proof. To determine coefficients c_1, c_2 we consider the pair of equations

$$Z_Y(\xi_2) = \xi_2^{1/4} (c_1 G_1(\frac{1}{2}) + c_2 G_2(\frac{1}{2})), \quad Z'_Y(\xi_2) = \frac{Z_Y(\xi_2)}{4\xi_2} + \frac{1}{4\xi_2^{1/4}} (c_1 G'_1(\frac{1}{2}) + c_2 G'_2(\frac{1}{2})).$$

Note that

$$G_1(\frac{1}{2}) = G_2(\frac{1}{2}) \quad \text{and} \quad G'_1(\frac{1}{2}) = -G'_2(\frac{1}{2}).$$

Therefore,

$$(c_1 + c_2) G_1(\frac{1}{2}) = \frac{Z_Y(\xi_2)}{\xi_2^{1/4}}, \quad (c_1 - c_2) G'_1(\frac{1}{2}) = 4\xi_2^{1/4} \left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2} \right).$$

The assertion follows by Lemma 6.3. □

Lemma 6.8. *For $\xi_2 = \frac{1}{4}\pi^2$, one has*

$$Z_Y(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}} \left(1 + \frac{1}{u^2} \left(\lambda_1 - \frac{1}{16} \right) + O\left(\frac{1}{u^3}\right) \right), \quad (6-40)$$

$$Z'_Y(\xi_2) = \frac{(-1)^{k+1}}{\sqrt{2u}} \left(\frac{u}{\pi} + \frac{1}{\pi^2} + \frac{2\lambda_1 + \frac{1}{8}}{2\pi u} + O\left(\frac{1}{u^2}\right) \right). \quad (6-41)$$

Proof. Applying [Boyd and Dunster 1986, Theorem 1] with $N = 1$ we obtain

$$\epsilon_{3,1}(u; \xi_2) = 0 \quad \text{and} \quad \frac{\partial}{\partial \xi} \epsilon_{3,1}(u; \xi) \Big|_{\xi=\xi_2} = 0.$$

Therefore,

$$Z_Y(\xi_2) = \sqrt{\xi_2} Y_0(u\sqrt{\xi_2}) \left(1 + \frac{A_Y(1; \xi_2)}{u^2} \right) - \xi_2 Y_1(u\sqrt{\xi_2}) \frac{B_Y(0; \xi_2)}{u}$$

and

$$\begin{aligned} Z'_Y(\xi_2) = & \sqrt{\xi_2} Y_0(u\sqrt{\xi_2}) \left(\frac{1}{2\xi_2} \left[1 + \frac{A_Y(1; \xi_2)}{u^2} \right] + \frac{A'_Y(1; \xi_2)}{u^2} - \frac{1}{2} B_Y(0; \xi_2) \right) \\ & - \xi_2 Y_1(u\sqrt{\xi_2}) \left(\frac{u}{2\xi_2} \left[1 + \frac{A_Y(1; \xi_2)}{u^2} \right] + \frac{1}{2\xi_2 u} B_Y(0; \xi_2) + \frac{1}{u} B'_Y(0; \xi_2) \right). \end{aligned}$$

By means of the Hankel asymptotic expansion (see [Olver et al. 2010, Equation 10.17.1, 10.17.4] and [Gradshteyn and Ryzhik 2007, Equation 8.451(1,7,8)]) we evaluate

$$\begin{aligned}\sqrt{\xi_2}Y_0(u\sqrt{\xi_2}) &= \frac{\pi}{2}Y_0\left((2k-1)\frac{\pi}{2}\right) = \frac{(-1)^{k+1}}{\sqrt{2u}}\left(\sum_{j=0}^{\infty}(-1)^j\frac{a_{2j}(0)}{(\pi u/2)^{2j}} + \sum_{j=0}^{\infty}(-1)^j\frac{a_{2j+1}(0)}{(\pi u/2)^{2j+1}}\right), \\ \xi_2Y_1(u\sqrt{\xi_2}) &= \frac{\pi}{2}\frac{(-1)^k}{\sqrt{2u}}\left(\sum_{j=0}^{\infty}(-1)^j\frac{a_{2j}(1)}{(\pi u/2)^{2j}} - \sum_{j=0}^{\infty}(-1)^j\frac{a_{2j+1}(1)}{(\pi u/2)^{2j+1}}\right),\end{aligned}$$

where

$$a_j(v) = \frac{\Gamma(v+j+\frac{1}{2})}{2^j j! \Gamma(v-j+\frac{1}{2})}.$$

This yields

$$\begin{aligned}Z_Y(\xi_2) &= \frac{(-1)^{k+1}}{\sqrt{2u}}\left(1 + \frac{a_1(0)}{\pi u/2} - \frac{a_2(0)}{(\pi u/2)^2} + O(u^{-3})\right)\left(1 + \frac{A_Y(1; \xi_2)}{u^2}\right) \\ &\quad - \frac{\pi}{2}\frac{(-1)^k}{\sqrt{2u}}\left(1 - \frac{a_1(1)}{\pi u/2} - \frac{a_2(1)}{(\pi u/2)^2} + O(u^{-3})\right)\frac{B_Y(0, \xi_2)}{u}.\end{aligned}$$

Simplifying the expression above, one has

$$\begin{aligned}Z_Y(\xi_2) &= \frac{(-1)^{k+1}}{\sqrt{2u}}\left(1 + \frac{1}{u}\left(\frac{2}{\pi}a_1(0) + \frac{\pi}{2}B_Y(0, \xi_2)\right) + \frac{1}{u^2}\left(A_Y(1, \xi_2) - \frac{4}{\pi^2}a_2(0) - a_1(1)B_Y(0, \xi_2)\right)\right) \\ &\quad + O(u^{-7/2}).\end{aligned}$$

Using formulas (6-25) and (6-26), we find

$$a_1(0) = -\frac{1}{8}, \quad \frac{2}{\pi}a_1(0) + \frac{\pi}{2}B_Y(0, \xi_2) = 0, \quad A_Y(1, \xi_2) - \frac{4}{\pi^2}a_2(0) - a_1(1)B_Y(0, \xi_2) = \lambda_1 - \frac{1}{16}.$$

This gives equation (6-40).

Similarly, we obtain

$$\begin{aligned}Z'_Y(\xi_2) &= \sqrt{\xi_2}Y_0(u\sqrt{\xi_2})\left[\frac{7}{4\pi^2} + \frac{1}{u^2}\left(\frac{1}{\pi^2}\left(2\lambda_1 - \frac{1}{32}\right) - \frac{15}{8\pi^4}\right)\right] \\ &\quad - \xi_2Y_1(u\sqrt{\xi_2})\left[u\frac{2}{\pi^2} + \frac{1}{u}\left(\frac{1}{\pi^2}\left(2\lambda_1 + \frac{1}{8}\right) - \frac{1}{16\pi^4}\right)\right].\end{aligned}$$

Hankel's expansion for Bessel functions yields the following asymptotics:

$$\begin{aligned}Z'_Y(\xi_2) &= \frac{(-1)^{k+1}}{\sqrt{2u}}\left[u\frac{1}{\pi} + \left(\frac{7}{4\pi^2} - \frac{2}{\pi^2}a_1(1)\right) + \frac{1}{u}\left(\frac{7}{2\pi^3}a_1(0) + \frac{\pi}{2}\left(\frac{2\lambda_1 + \frac{1}{8}}{\pi^2} - \frac{1}{16\pi^4}\right) - \frac{4}{\pi^3}a_2(1)\right)\right. \\ &\quad \left. + \frac{1}{u^2}\left(-\frac{7}{\pi^4}a_2(0) - a_1(1)\left(\frac{2\lambda_1 + \frac{1}{8}}{\pi^2} - \frac{1}{16\pi^4}\right) + \frac{8}{\pi^4}a_3(1)\right) + O(u^{-3})\right].\end{aligned}$$

Finally, substituting

$$a_1(0) = -\frac{1}{8}, \quad a_1(1) = \frac{3}{8}, \quad a_2(1) = -\frac{15}{128},$$

we prove equation (6-41). □

Corollary 6.9. *One has*

$$C_Y = 1 + O\left(\frac{1}{k}\right), \quad C_J = O\left(\frac{1}{k^2}\right). \quad (6-42)$$

Proof. By Lemma 6.8

$$Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2} = \frac{(-1)^{k+1}}{\sqrt{2u}} \left(\frac{u}{\pi} + \frac{2\lambda_1 + \frac{1}{8}}{2\pi u} + O(u^{-2}) \right).$$

It follows from [Olver et al. 2010, Equation 5.11.13] that

$$\frac{\Gamma(k + \frac{1}{4})}{\Gamma(k - \frac{1}{4})} = k^{1/2} - \frac{1}{4}k^{-1/2} + O(k^{-3/2}) \quad \text{and} \quad \frac{\Gamma(k - \frac{1}{4})}{\Gamma(k + \frac{1}{4})} = k^{-1/2} + O(k^{-3/2}).$$

Then Lemma 6.7 gives

$$c_1 = O(k^{-2}), \quad c_2 = 1 + O(k^{-1}).$$

Equations (6-37) yield the assertion. \square

As a consequence of equation (6-35), Lemma 6.6, Theorem 6.5 and Corollary 6.9, we obtain the main result.

Theorem 6.10. *Let $u = 2k - 1$, $\xi_2 = \frac{1}{4}\pi^2$. Then for $\xi \in (0, \xi_2)$ one has*

$$\Phi_k(\cos^2 \sqrt{\xi}) = \frac{-\pi}{\xi^{1/4}(\sin \sqrt{\xi})^{1/2}} [Z_J(\xi) + C_Y Z_Y(\xi) + C_J Z_J(\xi)], \quad (6-43)$$

where Z_Y, Z_J are given by (6-27), (6-28) and

$$C_Y = 1 + O\left(\frac{1}{k}\right), \quad C_J = O\left(\frac{1}{k^2}\right). \quad (6-44)$$

Corollary 6.11. *We have*

$$\Phi_k\left(\frac{1}{4}\right) = -\frac{2^{3/2}\sqrt{\pi}}{3^{1/4}} \frac{1}{\sqrt{2k-1}} \sin \frac{\pi(2k-1)}{3} \left(1 + O\left(\frac{1}{k}\right)\right).$$

Approximation of Ψ_k . Next, we find a Liouville–Green approximation for the function Ψ_k . With this goal, we follow the arguments of [Balkanova and Frolenkov 2016, Section 5.3] with minor changes. In particular, the differential equation for $\Psi_k(x)$ is slightly different, and, therefore, one must recompute various functions and constants appearing in the Liouville–Green approximation. We provide all details here to make the presentation self-contained.

Consider the function

$$y(x) := \sqrt{1-x} \Psi_k(x). \quad (6-45)$$

Let $u := k - \frac{1}{2}$ and

$$f(x) := \frac{1}{x^2(1-x)}, \quad g(x) := -\frac{1}{4x^2(1-x)^2} + \frac{3}{16x(1-x)}. \quad (6-46)$$

Lemma 6.12. *The function $y = y(x)$ is a solution of equation*

$$y''(x) - (u^2 f(x) + g(x))y(x) = 0. \quad (6-47)$$

Proof. Using the differential equation for the hypergeometric function, we find that $y = y(x)$ satisfies the following differential equation

$$y'' + \left(\frac{1 - (2k - 1)^2}{4x^2} + \frac{1}{4(1 - x)^2} + \frac{\frac{5}{4} - (2k - 1)^2}{4x(1 - x)} \right) y = 0.$$

The assertion follows. \square

Making the change

$$Z(x) := \frac{y(x)}{\alpha(x)}, \quad \alpha(x) := \frac{(x^2 - x^3)^{1/4}}{2(\operatorname{artanh} \sqrt{1 - x})^{1/2}} \quad (6-48)$$

and the substitution

$$\xi := 4 \operatorname{artanh}^2 \sqrt{1 - x}, \quad (6-49)$$

we transform equation (6-47) to the type

$$\frac{d^2 Z}{d\xi^2} + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2} - \frac{\psi(\xi)}{\xi} \right) Z = 0, \quad (6-50)$$

where

$$\psi(\xi) = \frac{1}{16} \left(\frac{1}{\xi} - \frac{1}{4 \sinh^2 \sqrt{\xi}/2} \right) \quad (6-51)$$

is an analytic function as $\xi \rightarrow 0$.

In order to find a Liouville–Green approximation to equation (6-50), we remove the term with $\psi(\xi)/\xi$ in (6-50). The resulting equation

$$Z'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi} \right) Z = 0 \quad (6-52)$$

has I and K Bessel functions as solutions (see [Olver et al. 2010, Equation 10.13.2]), namely

$$Z_L = \sqrt{\xi} L_0(u\sqrt{\xi}), \quad (6-53)$$

where

$$L_\nu(x) := \begin{cases} I_\nu(x) \\ e^{\pi i \nu} K_\nu(x). \end{cases} \quad (6-54)$$

This suggests that solutions of the original differential equation (6-50) can be written in the form (see [Olver 1974, Equation 2.09, Chapter 12])

$$Z_L = \sqrt{\xi} L_0(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{A(n; \xi)}{u^{2n}} + \frac{\xi}{u} L_1(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{B(n; \xi)}{u^{2n}}. \quad (6-55)$$

Lemma 6.13. *The coefficients $A(n; \xi)$ and $B(n; \xi)$ are given by*

$$\sqrt{\xi} B(n; \xi) = -\sqrt{\xi} A'(n; \xi) + \int_0^\xi \left(\psi(x) A(n; x) - \frac{1}{2} A'(n; x) \right) \frac{dx}{\sqrt{x}}, \quad (6-56)$$

$$A(n; \xi) = -\xi B'(n - 1; \xi) + \int_0^\xi \psi(x) B(n - 1; x) dx + \lambda_n \quad (6-57)$$

for some real constants of integration λ_n .

Proof. By [Olver et al. 2010, Equations 10.13.2, 10.13.5, 10.36, 10.29.2, 10.29.3] the functions

$$W(\xi) := \sqrt{\xi} L_0(u\sqrt{\xi}), \quad V(\xi) := \xi L_1(u\sqrt{\xi})$$

satisfy the relations

$$W'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right)W = 0, \quad V'' - \frac{1}{\xi}V' + \left(-\frac{u^2}{4\xi} + \frac{3}{4\xi^2}\right)V = 0, \quad V' = \frac{1}{2\xi}V + \frac{u}{2}W, \quad W' = \frac{1}{2\xi}W + \frac{u}{2\xi}V.$$

Using this and substituting solution (6-55) into equation (6-50), we obtain that

$$W(\xi) \sum_{n=0}^{\infty} \frac{C(n; \xi)}{u^{2n}} + V(\xi) \sum_{n=0}^{\infty} \frac{D(n; \xi)}{u^{2n+1}} = 0,$$

where

$$C(n; \xi) = A''(n; \xi) + \frac{A'(n; \xi)}{\xi} - \frac{\psi(\xi)}{\xi} A(n; \xi) + B'(n; \xi) + \frac{B(n; \xi)}{2\xi},$$

$$D(n; \xi) = B''(n-1; \xi) + \frac{B'(n-1; \xi)}{\xi} - \frac{\psi(\xi)}{\xi} B(n-1; \xi) + \frac{A'(n; \xi)}{\xi}.$$

Setting $C(n; \xi) = D(n; \xi) = 0$ we find the required recurrence relations. \square

Let $A(0; \xi) = 1$. Then

$$B(0; \xi) = \frac{1}{16} \left(\frac{\coth \sqrt{\xi}}{\sqrt{\xi}} - \frac{2}{\xi} \right), \quad (6-58)$$

$$A(1; \xi) = -\frac{1}{32} \left(\frac{4}{\xi} - \frac{\coth \sqrt{\xi/4}}{\sqrt{\xi}} - \frac{1}{2 \sinh^2 \sqrt{\xi/4}} \right) + \frac{1}{512} \left(\coth \sqrt{\xi/4} - \frac{2}{\sqrt{\xi}} \right)^2 + \lambda_1. \quad (6-59)$$

Note that

$$\lim_{\xi \rightarrow \infty} \sqrt{\xi} B(0; \xi) = \frac{1}{16}, \quad \lim_{\xi \rightarrow \infty} A(1; \xi) = \frac{1}{512} + \lambda_1. \quad (6-60)$$

The variation of the function is given by

$$V_{a,b}(f(x)) := \int_a^b |(f(x))'| dx. \quad (6-61)$$

Lemma 6.14. *The function $V_{\xi, \infty}(\sqrt{x}B(1; x))$ is bounded. For $n > 1$, the function $V_{\xi, \infty}(\sqrt{x}B(n; x))$ converges.*

Proof. As a consequence of recurrence relation (6-56), we find

$$(\sqrt{x}B(1; x))' = O(x^{-1/2}) \quad \text{as } x \rightarrow 0 \quad \text{and} \quad (\sqrt{x}B(1; x))' = O(x^{-2}) \quad \text{as } x \rightarrow \infty.$$

Thus $V_{\xi, \infty}(\sqrt{x}B(1; x))$ is bounded. Note that

$$\psi^{(s)}(\xi) = O\left(\frac{1}{|\xi|^{s+1}}\right).$$

The convergence of variation for $n > 1$ then follows by [Olver 1974, Exercise 4.2, p. 445]. \square

Using Lemma 6.14, we can truncate the infinite summation in (6-55) up to N summands with a negligible error term. The value of N determines the quality of approximation: the error is smaller for larger N .

Lemma 6.15. *For each value of u and each nonnegative integer N , equation (6-50) has a solution $Z_K(\xi)$ which is infinitely differentiable in ξ on interval $(0, \infty)$ and is given by*

$$Z_K(\xi) = \sqrt{\xi} K_0(u\sqrt{\xi}) \sum_{n=0}^N \frac{A_K(n; \xi)}{u^{2n}} - \frac{\xi}{u} K_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_K(n; \xi)}{u^{2n}} + \epsilon_{2N+1,3}(u, \xi), \quad (6-62)$$

where

$$|\epsilon_{2N+1,3}(u, \xi)| \leq \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^{2N+1}} \times V_{\xi, \infty}(\sqrt{\xi} B_K(N; \xi)) \exp\left(\frac{1}{u} V_{\xi, \infty}(\sqrt{\xi} B_K(0; \xi))\right). \quad (6-63)$$

In particular, for $N = 1$

$$\epsilon_{3,3}(u, \xi) \ll \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^3} \min\left(\sqrt{\xi}, \frac{1}{\xi}\right). \quad (6-64)$$

As $\xi \rightarrow \infty$, the differential equation (6-50) has two recessive solutions, namely

$$Z(\xi) = \Psi_k\left(\frac{1}{\cosh^2 \sqrt{\xi}/2}\right) (\xi \sinh^2 \sqrt{\xi})^{1/4} \quad (6-65)$$

and $Z_K(\xi)$ given by (6-62). Thus there is $C_K = C_K(u)$ such that

$$\Psi_k\left(\frac{1}{\cosh^2 \sqrt{\xi}/2}\right) (\xi \sinh^2 \sqrt{\xi})^{1/4} = C_K Z_K(\xi). \quad (6-66)$$

The last step is to compute $C_K = C_K(u)$.

Lemma 6.16. *One has*

$$C_K = 2 + O(k^{-1}). \quad (6-67)$$

Proof. To determine C_K , we compute the limit of the left- and right-hand sides of equation (6-66) as $\xi \rightarrow \infty$. This implies

$$C_K = \frac{\Gamma(k - \frac{1}{4}) \Gamma(k + \frac{1}{4})}{\Gamma(2k)} \frac{2^{2k} \sqrt{u}}{\sqrt{\pi}} \left(\sum_{n=0}^N \frac{a_n}{u^{2n}} - \sum_{n=0}^{N-1} \frac{b_n}{u^{2n+1}} \right)^{-1},$$

where

$$a_n = \lim_{\xi \rightarrow \infty} A(n; \xi), \quad b_n = \lim_{\xi \rightarrow \infty} B(n; \xi) \sqrt{\xi}.$$

According to (6-60), we know that $a_0 = 1$, $a_1 = \frac{1}{512} + \lambda_1$, $b_0 = \frac{1}{16}$. Furthermore,

$$\frac{\Gamma(k - \frac{1}{4}) \Gamma(k + \frac{1}{4})}{\Gamma(2k)} = \frac{\Gamma^2(k)}{\Gamma(2k)} \left(1 + O\left(\frac{1}{k}\right)\right) = \frac{2\sqrt{\pi}}{\sqrt{k} 2^{2k}} \left(1 + O\left(\frac{1}{k}\right)\right).$$

The assertion follows. □

Finally, we obtain the main theorem.

Theorem 6.17. *For $\xi \in (0, \infty)$ the following equality holds*

$$\Psi_k \left(\frac{1}{\cosh^2 \sqrt{\xi}/2} \right) (\xi \sinh^2 \sqrt{\xi})^{1/4} = C_K Z_K(\xi), \quad (6-68)$$

where $Z_K(\xi)$ is defined by (6-62) and $C_K = 2 + O(k^{-1})$.

7. Asymptotic formula

Corollaries 2.2 and 2.3 are derived from Theorem 2.1 by estimating the last two summands in exact formula (2-9), as we now show.

Lemma 7.1. *For any $\epsilon > 0$,*

$$E_1(k, l) := \frac{1}{\sqrt{l}} \sum_{1 \leq n < 2l} \mathcal{L}_{n^2-4l^2} \left(\frac{1}{2} \right) \Phi_k \left(\frac{n^2}{4l^2} \right) \ll \frac{l^{5/6+\epsilon}}{\sqrt{k}}. \quad (7-1)$$

Proof. Using the subconvexity bound (4-10) we obtain

$$E_1(k, l) \ll \frac{l^\epsilon}{\sqrt{l}} \sum_{1 \leq n < 2l} (2l-n)^{1/6} (2l+n)^{1/6} \left| \Phi_k \left(\frac{n^2}{4l^2} \right) \right| \ll \frac{l^{1/6+\epsilon}}{\sqrt{l}} \sum_{1 \leq n < 2l} n^{1/6} \left| \Phi_k \left(\left(1 - \frac{n}{2l} \right)^2 \right) \right|$$

Let $\xi = 4(\arcsin \sqrt{n/4l})^2$, $u = 2k - 1$. Then by Theorem 6.10 one has

$$\Phi_k \left(\left(1 - \frac{n}{2l} \right)^2 \right) \ll \frac{\sqrt{\xi} Y_0(u\sqrt{\xi})}{(\arcsin \sqrt{n/4l})^{1/2} (n/l)^{1/4}} \ll \frac{(\arcsin \sqrt{n/4l})^{1/2}}{(n/l)^{1/4}} Y_0(u\sqrt{\xi}).$$

If $l \ll k^2$, one has $u\sqrt{\xi} \gg 1$. Then the estimate for the Bessel function

$$Y_0(u\sqrt{\xi}) \ll \frac{1}{u^{1/2} \xi^{1/4}}$$

yields

$$\Phi_k \left(\left(1 - \frac{n}{2l} \right)^2 \right) \ll \frac{1}{k^{1/2} (n/l)^{1/4}}.$$

Consequently,

$$E_1(k, l) \ll \frac{l^{1/6+\epsilon}}{\sqrt{l}} \sum_{n < 2l} \frac{n^{1/6} l^{1/4}}{k^{1/2} n^{1/4}} \ll \frac{l^{5/6+\epsilon}}{\sqrt{k}}. \quad \square$$

Corollary 7.2. *If the Lindelöf hypothesis for Dirichlet L-functions is true, then*

$$E_1(k, l) \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}} \quad \text{for any } \epsilon > 0. \quad (7-2)$$

Lemma 7.3. For some $c > 0$,

$$E_2(k, l) := \frac{1}{l\sqrt{2}} \sum_{n>2l} \mathcal{L}_{n^2-4l^2} \left(\frac{1}{2} \right) \sqrt{n} \Psi_k \left(\frac{4l^2}{n^2} \right) \ll \frac{l^{-1/12}}{\sqrt{k}} \exp \left(-\frac{ck}{\sqrt{l}} \right). \quad (7-3)$$

Proof. It follows from the subconvexity bound (4-10) that

$$\begin{aligned} E_2(k, l) &\ll \frac{1}{l} \sum_{n>2l} (n^2 - 4l^2)^{1/6} \sqrt{n} \left| \Psi_k \left(\frac{4l^2}{n^2} \right) \right| \\ &\ll \frac{1}{l} \int_{2l+1}^{\infty} x^{1/2} (x^2 - 4l^2)^{1/6} \left| \Psi_k \left(\frac{4l^2}{x^2} \right) \right| dx + l^{-1/3} \left| \Psi_k \left(\frac{4l^2}{(2l+1)^2} \right) \right|. \end{aligned}$$

Next, we make the change of variables $x = 2l \cosh \left(\frac{1}{2} \sqrt{\xi} \right)$ and estimate $E_2(k, l)$ using Theorem 6.17 with $N = 0$. Consider the first summand

$$\begin{aligned} E_{2,1}(k, l) &:= \frac{1}{l} \int_{2l+1}^{\infty} x^{1/2} (x^2 - 4l^2)^{1/6} \left| \Psi_k \left(\frac{4l^2}{x^2} \right) \right| dx \ll l^{5/6} \int_{\xi_0}^{\infty} \left(\sinh \frac{\sqrt{\xi}}{2} \right)^{5/6} \frac{|Z_k(\xi)|}{\xi^{3/4}} d\xi \\ &\ll l^{5/6} \int_{\xi_0}^{\infty} \left(\sinh \frac{\sqrt{\xi}}{2} \right)^{5/6} \frac{|K_0(u\sqrt{\xi})|}{\xi^{1/4}} d\xi, \end{aligned}$$

where $u = k - \frac{1}{2}$ and the limit of integration ξ_0 is defined by

$$\cosh \frac{\sqrt{\xi_0}}{2} = 1 + \frac{1}{2l}.$$

Making the change of variables $\sqrt{\xi} = t$, one has

$$t_0 = 4 \operatorname{arcsinh} \frac{1}{\sqrt{4l}}.$$

Since $t_0 \gg 1/\sqrt{l}$ and $ut \geq ut_0 \gg k/\sqrt{l} \gg 1$, we estimate the Bessel function as follows

$$K_0(ut) \ll \frac{\exp(-ut)}{\sqrt{ut}}.$$

Finally,

$$\begin{aligned} E_{2,1}(k, l) &\ll l^{5/6} \int_{t_0}^{\infty} \left(\sinh \frac{t}{2} \right)^{5/6} \frac{|K_0(ut)|}{\sqrt{t}} t dt \ll \frac{l^{5/6}}{\sqrt{k}} \int_{t_0}^{\infty} \left(\sinh \frac{t}{2} \right)^{5/6} e^{-ut} dt \\ &\ll \frac{l^{5/6}}{\sqrt{k}} \left(\int_{t_0}^1 t^{5/6} \exp(-ut) dt + \int_1^{\infty} \exp(-ut + 5t/12) dt \right) \ll \frac{l^{5/12}}{u\sqrt{k}} \exp(-ut_0). \end{aligned}$$

The second summand can be estimated similarly:

$$\begin{aligned} E_{2,2}(k, l) &:= l^{-1/3} \left| \Psi_k \left(\frac{4l^2}{(2l+1)^2} \right) \right| \ll l^{-1/3} \frac{C_K |Z_k(\xi)|}{\xi^{1/4} (\sinh \sqrt{\xi})^{1/2}} \\ &\ll l^{-1/3} l^{1/2} |Z_k(\xi)| \ll l^{1/6} \sqrt{\xi} |K_0(u\sqrt{\xi})| \ll \frac{\exp(-ut_0)}{k^{1/2} l^{1/12}}. \end{aligned}$$

To sum up,

$$E_2(k, l) \ll E_{2,1}(k, l) + E_{2,2}(k, l) \ll \frac{1}{l^{1/12}\sqrt{k}} \exp\left(\frac{-ck}{\sqrt{l}}\right)$$

for some $c > 0$. □

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