Algebra & Number Theory

Volume 12
2018
No. 1

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ALGEBRA AND NUMBER THEORY 12:1 (2018)
dx.doi.org/10.2140/ant.2018.12.99

Regular pairs of quadratic forms on odd-dimensional spaces in characteristic 2

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We describe a normal form for a smooth intersection of two quadrics in even-dimensional projective spaces over an arbitrary field of characteristic 2. We use this to obtain a description of the automorphism group of such a variety. As an application, we show that every quartic del Pezzo surface over a perfect field of characteristic 2 has a canonical rational point and, thus, is unirational.

1. Introduction

Let (q_0, q_1) be a pair of quadratic forms on a vector space E over a field $\Bbbk$. The common zeros of the pair define a subvariety $X = V(q_0, q_1)$ in the projective space $|E| = \mathbb{P}(E^\vee)$ of lines in E. We say the pair is *regular* if the variety X is smooth and of codimension 2 in $|E|$. One may also consider the pencil of quadrics spanned by $V(q_0)$ and $V(q_1)$; the pencil is *regular* if X is smooth.

When $\Bbbk = \mathbb{C}$, the field of complex numbers, a classical result due to A. Cauchy and C. Jacobi (see [Muth 2016]) states that any regular pair can be *simultaneously diagonalized*. When q_0 is nondegenerate, this means that we may find a basis $x_1, \dots, x_n$ in the dual space $E^\vee$ such that

$$q_0 = \sum_{i=1}^{n} x_i^2, \quad q_1 = \sum_{i=1}^{n} a_i x_i^2.$$

Here the coefficients $a_1, \dots, a_n$ are distinct roots of the discriminant polynomial $\delta(t) = \det(tM_0 - M_1)$, where M_0 and M_1 are symmetric matrices representing the polar symmetric bilinear forms associated to q_0 and q_1. When the pair is not necessarily regular, the classification was carried out by K. Weierstrass [Weierstrass 1868] by considering the elementary divisors of $tM_0 - M_1$. Over an arbitrary field $\Bbbk$ of characteristic not 2, pairs of quadratic forms were classified by L. Kronecker and L. Dickson. A modern exposition of their theory can be found in [Waterhouse 1976] (see also [Leep and Schueller 1999]).

Unsurprisingly, in characteristic $p = 2$ the situation is more complicated. The main reason for this is that one can no longer identify quadratic forms with symmetric bilinear forms. Even worse, when $n = \dim(E)$ is odd the determinant of $tM_0 - M_1$ is identically zero. One may still consider symmetric

Duncan was partially supported by National Science Foundation grant DMS 0943832 and National Security Agency grant H98230-16-1-0309.

MSC2010: primary 11E04; secondary 14C21, 14G17.

Keywords: quadratic forms, characteristic 2, quadrics.

and alternating bilinear forms in this case (see, for example, [Waterhouse 1977; Leep and Schueller 1999; Ishitsuka and Ito 2015]), but the connection with quadratic forms is more tenuous.

The geometry of the intersection X of two quadrics differs drastically depending on whether n is even or odd. Examples when n is even include quartic elliptic curves in $\mathbb{P}^3$ and quadratic line complexes in $\mathbb{P}^5$. When n is odd, the first interesting example is a quartic del Pezzo surface in $\mathbb{P}^4$ isomorphic to a blowup of 5 points in the projective plane.

When $n = 2m$ is even and $\Bbbk = \mathbb{C}$, one can associate to X its intermediate Jacobian $J(X)$. A theorem of Luc Gauthier and André Weil [Gauthier 1954–55] asserts that $J(X)$ is isomorphic to the Jacobian variety of a hyperelliptic curve C of genus $m-1$ with equation $y^2 + \delta(t) = 0$. Over an algebraically closed field of characteristic $p \neq 2$, the Jacobian is isomorphic to the variety of $(m-2)$-planes in X (see also [Wang 2013]). If $p = 2$ and n is even, one should use the Pfaffian $P(t)$ defined so that $P(t)^2 = \delta(t)$. Under the condition that the roots $\alpha_1, \ldots, \alpha_m$ of $P(t)$ are distinct, U. Bhosle [1990] provides a normal form

$$q_0 = \sum_{i=1}^{m} x_i x_{m+i}, \quad q_1 = \sum_{i=1}^{m} (a_i x_i^2 + \alpha_i x_i x_{m+i} + b_i x_{m+i}^2)$$

where $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ are in $\Bbbk$. The variety of $(m-2)$-planes is isomorphic to the Jacobian of a hyperelliptic curve in this case as well.

This paper is concerned with pairs of quadratic forms, and their corresponding pencils, in the case where $\Bbbk$ has characteristic 2 and $n = \dim(E) = 2m+1$ is odd. As mentioned above, the discriminant polynomial is identically zero in this case. Instead, we use the *half-discriminant* introduced by M. Kneser [2002], which we recall in Section 2. This is a homogeneous polynomial of degree n

$$\Delta(t_0, t_1) := \tfrac{1}{2} \operatorname{disc}(t_0 q_0 + t_1 q_1) = a_0 t_0^n + a_1 t_0^{n-1} t_1 + \ldots + a_n t_1^n \tag{1-1}$$

in $\Bbbk[t_0, t_1]$. This polynomial behaves like the usual discriminant when the characteristic is not 2: the pencil is regular if and only if the zero subscheme $V(\Delta)$ in $\mathbb{P}^1$ is smooth of dimension 0 (we give a geometric proof; an algebraic proof can be found in [Leep and Schueller 2002]). When $\Bbbk$ is algebraically closed, this simply means that $\Delta \not\equiv 0$ and $V(\Delta)$ consists of n distinct points.

Our first result is the following:

Theorem 1.1. *Let (q_0, q_1) be a regular pair of quadratic forms on a vector space E of dimension $n = 2m+1 \geq 3$ over a field $\Bbbk$ of characteristic 2. Then there exists a basis $(x_0, \ldots, x_m, y_0, \ldots, y_{m-1})$ in $E^\vee$ such that*

$$q_0 = \sum_{i=0}^{m} a_{2i} x_i^2 + \sum_{i=0}^{m-1} x_{i+1} y_i + \sum_{i=0}^{m-1} r_{2i+1} y_i^2, \quad q_1 = \sum_{i=0}^{m} a_{2i+1} x_i^2 + \sum_{i=0}^{m-1} x_i y_i + \sum_{i=0}^{m-1} r_{2i} y_i^2, \tag{1-2}$$

where the coefficients $a_0, \ldots, a_n$ are equal to those of the half-discriminant polynomial (1-1), and $r_0, \ldots, r_{n-2}$ are in $\Bbbk$.

To the authors' surprise, our normal form seems to be new even in the case $n = 5$ corresponding to quartic del Pezzo surfaces. At least in retrospect, it can be easily deduced from the classification of pairs

of alternate bilinear forms (see [Leep and Schueller 1999]). Nevertheless, the existence of the normal form has the following arithmetic consequence particular to the case of characteristic 2.

Corollary 1.2. *Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^{2m}$ defined over a perfect field $\Bbbk$ of characteristic 2. If $\dim(X) = 2$ (resp. > 2), then X is unirational (resp. rational) over $\Bbbk$.*

(Note that the unirationality of del Pezzo surfaces of degree 4 is already known for finite fields; see [Manin and Tsfasman 1986].)

Following [Reid 1972], a *generator* is a linear subspace of X of dimension m. Over the algebraic closure, there are exactly 2^{2m} generators. Following [Skorobogatov 2010], one says X is *quasisplit* if it contains a generator, and X is *split* if all 2^{2m} generators are defined over the base field.

Theorem 1.3. *A regular pair (q_0, q_1) has quasisplit base locus X if and only if the pair has a normal form such that $r_0 = \cdots = r_{n-2} = 0$. In particular, this is always possible when $\Bbbk$ is algebraically closed.*

If $a_n \neq 0$, we assign to the coefficients $(r_0, \dots, r_{n-2})$ an element r of the $\Bbbk$-algebra $A = \Bbbk[T]/(f(T))$ where $f(T)$ is the dehomogenization of Δ with respect to $T = t_1/t_0$. This element, which we call the *r-invariant*, determines the normal form (it can also be defined when $a_n = 0$, see Remark 4.8). An *isomorphism* between pairs (q_0, q_1), (q_0', q_1') of quadratic forms is an element $g \in \mathrm{GL}(E)$ such that

$$q_0'(v) = q_0(gv) \quad \text{and} \quad q_1'(v) = q_1(gv)$$

for all $v \in E$.

Theorem 1.4. *Suppose two regular pairs (q_0, q_1) and (q_0', q_1') have the same half-discriminant polynomial and have normal forms with invariants r and r', respectively. The pairs are isomorphic if and only if*

$$r \equiv r' \mod \Bbbk + \wp(A)$$

where $\wp(s) = s^2 + s$ is the Artin–Schreier map $A \to A$.

Denote by $\mathrm{Aut}(q_0, q_1)$ the group of automorphisms of the pair (q_0, q_1). When X is split, the group of automorphisms $\mathrm{Aut}(q_0, q_1)$ of a regular pair (q_0, q_1) is isomorphic to an elementary abelian 2-group of rank $2m$, which acts simply transitively on the set of generators of X. In general, the set of generators can be viewed as an $\mathrm{Aut}(q_0, q_1)$-torsor.

Theorem 1.5. *Let $\wp(s) = s^2 + s$ be the Artin–Schreier map $A \to A$. There is an isomorphism between the algebra $A/(\wp(A) + \Bbbk)$ and the Galois cohomology group $H^1(\Bbbk, \mathrm{Aut}(q_0, q_1))$, which takes the r-invariant to the class of the $\mathrm{Aut}(q_0, q_1)$-torsor of generators.*

This theorem provides a Galois-cohomological interpretation of Theorems 1.3 and 1.4 in the vein of [Skorobogatov 2010]. Namely, the isomorphism class of a regular pair is determined by its half-discriminant polynomial Δ and its torsor of generators. Similarly, the base locus X is determined up to isomorphism by a smooth subscheme $V(\Delta)$ of $\mathbb{P}^1$ and the torsor of generators; the variety X is quasisplit if and only if the torsor is trivial.

One of the more concrete motivations of this paper is to study the possible groups of automorphisms of a quartic del Pezzo surface over an algebraically closed field of characteristic 2 (in order to study the conjugacy classes of finite subgroups of the plane Cremona group over fields of positive characteristic).

Recall that a *reflection* is an involution of a vector space (or a projective space) which leaves pointwise-fixed a hyperplane. As a corollary of our main results about the classification of pairs of quadratic forms we obtain the following.

Theorem 1.6. *Let $X = V(q_0, q_1)$ be a smooth complete intersection of two quadrics in $\mathbb{P}^{2m}$ over an algebraically closed field $\Bbbk$ of characteristic 2. Then*

$$\mathrm{Aut}(X) = \mathrm{Aut}(q_0, q_1) \rtimes G,$$

where $\mathrm{Aut}(q_0, q_1)$ is generated by reflections in canonical bijection with the points of $V(\Delta)$ and G is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ which leaves invariant the points $V(\Delta)$.

This extends to characteristic 2 a classical result on automorphisms of quartic del Pezzo surfaces (see [Dolgachev 2012]). There is a natural action of the Weyl group $W(D_n) \cong 2^{2m} \rtimes \mathfrak{S}_n$ on the cohomology classes of the generators. The automorphism group $\mathrm{Aut}(X)$ is naturally a subgroup of the Weyl group $W(D_n)$ where $\mathrm{Aut}(q_0, q_1) \cong 2^{2m}$ and $G \subset \mathfrak{S}_n$. Thus, as with quartic del Pezzo surfaces, the automorphism group is completely determined by its action on the generators.

In writing this paper, the authors tried to resolve a tension between two likely audiences: geometers who only work over algebraically closed fields, and algebraists without a strong background in geometry. Hopefully, the paper will appeal to both audiences and we occasionally supply multiple proofs to facilitate this. For example, while Theorem 1.3 is an immediate consequence of Theorem 1.4, we provide a more direct geometric proof in Section 4B. Also, while Theorem 1.4 is used in the proof of Theorem 1.6, a geometric description of the reflection group R can be found in Section 7B.

The remainder of the paper is structured as follows. In Sections 2 and 3 we establish some preliminaries on pencils of quadratic forms in characteristic 2. In Section 4, we prove Theorem 1.1, Corollary 1.2, and Theorem 1.3. In Section 5, we define the r-invariant and prove Theorem 1.4. In Section 6, we discuss possible interpretations for what the r-invariant represents. In Section 7, we determine the automorphism group of a pair of quadratic forms and use this to prove Theorems 1.5 and 1.6. Finally, in Section 8, we make some observations about the cohomology of smooth intersections of smooth quadrics extending results from M. Reid's thesis [1972] to the case where the characteristic is 2.

2. Preliminaries on quadratic forms

Throughout, $\Bbbk$ is a field of characteristic 2 and E is a k-vector space of dimension n.

2A. ***Alternating bilinear forms.*** Recall that a bilinear form $b : E \times E \to \Bbbk$ is *alternating* if $b(v, v) = 0$ for all $v \in E$. We may view b as an element in $\bigwedge^2 E^\vee$. The *radical* $\mathrm{rad}(b)$ of an alternating bilinear form b is the kernel of the induced map $E \to E^\vee$. A form b is *nondegenerate* if its radical is trivial. The *corank*

of a bilinear form is $\dim(\mathrm{rad}(b))$. Every alternating bilinear form is an orthogonal sum of its radical and a nondegenerate alternating bilinear form.

Every nondegenerate alternating bilinear form has even dimension $n = 2m$ and has a *symplectic basis* $v_1, \dots, v_m, w_1, \dots, w_m$ satisfying the relations

$$b(w_i, w_j) = 0, \quad b(v_i, v_j) = 0, \quad b(w_i, v_j) = \delta_{ij},$$

for $1 \le i, j \le m$. A subspace W of b is *totally isotropic* if $b(v, w) = 0$ for all $v, w \in W$. Note that $v_1, \dots, v_m$ and $w_1, \dots, w_m$ are bases for complementary totally isotropic subspaces of E. Conversely, for any pair of complementary totally isotropic subspaces of E, there exists a basis for each subspace such that the union is a symplectic basis for E.

When n is even (resp. odd), $\mathrm{corank}(b)$ is even (resp. odd). Thus, an alternating bilinear form b on a vector space of odd dimension $n = 2m + 1$ is always degenerate.

Proposition 2.1. *Let b be an alternating bilinear form on a vector space E of odd dimension $n = 2m + 1$. Up to a choice of volume form for E, there is a canonical vector $\omega \in E$ which spans $\mathrm{rad}(b)$ if $\mathrm{corank}(b) = 1$ and is 0 otherwise. Choosing a basis of $e_1, \dots, e_n$ in E such that $e_1 \wedge \cdots \wedge e_n \mapsto 1$ under the volume form, we have*

$$\omega = (\mathrm{Pf}_1, \dots, \mathrm{Pf}_n), \tag{2-1}$$

where Pf_i is the Pfaffian of the principal submatrix of the matrix of b obtained by deleting the i-th row and the i-th column.

Proof. It is obvious that such a vector exists; the point is to construct it canonically and find an explicit formula for its coordinates.

Let us view $b \in \bigwedge^2 E^\vee$ as an element of the divided power algebra of $E^\vee$. We may thus consider the m-th divided power $b^{(m)} \in \bigwedge^{2m} E^\vee$. (If $\Bbbk$ was of characteristic zero, we would have the formula $b^{(m)} = (b \wedge \cdots \wedge b)/m!$.) Under the isomorphism $\bigwedge^{2m} E^\vee \cong E$ defined by the volume form, $b^{(m)}$ maps to a vector ω in E. We have $\omega = \sum_{i=1}^n \mathrm{Pf}_i \, e_i$ in coordinates. The rank of an alternating matrix is equal to the size of the largest nonzero Pfaffian of a principal submatrix of even size. If the rank is equal to $n - 1$, then the nullspace is generated by the vector (2-1). All of these well-known facts can be found in [Buchsbaum and Eisenbud 1977]. □

Note that without a choice of volume form, ω is well defined up to a multiplicative constant. Thus, the corresponding point in the projective space $|E|$ does not depend on this choice.

2B. ***Quadratic forms.*** Recall that a quadratic form q is an element of the symmetric square $S^2(E^\vee)$ of the dual space $E^\vee$. When $q \neq 0$, one obtains a quadric hypersurface $V(q)$ in the projective space $|E|$. The quadratic form is called *nondegenerate* if $V(q)$ is a smooth hypersurface.

Equivalently, q can be viewed as a function $E \to \Bbbk$ such that

(1) $q(cv) = c^2 q(v)$ for all $c \in \Bbbk$, $v \in E$, and

(2) $b(v, w) = q(v + w) - q(v) - q(w)$ is a symmetric bilinear form.

The bilinear form b is called the *associated polar bilinear form* and since $\mathrm{char}(k) = 2$, we observe that b is alternating. The *corank* of a quadratic form q is simply the corank of the associated bilinear form.

A nonzero vector $v \in E$ is called a *singular vector of* q if $q(v) = 0$ and $v \in \mathrm{rad}(b)$. In geometric language, this means the corresponding point $[v] = \Bbbk v$ in $|E|$ is a singular point of the quadric $V(q)$.

A quadratic form is *totally singular* if its associated bilinear form is trivial; this is equivalent to $V(q)$ being singular at every geometric point. A quadratic form q is diagonalizable if and only if it is totally singular. Moreover, if q is diagonalizable, then it is diagonal relative to any basis. A subspace W of E is *totally isotropic* (resp. *totally singular*) with respect to q if the restricted form $q|_W$ is trivial (resp. totally singular).

The *discriminant* $\mathrm{disc}(q)$ of a quadratic form is the determinant of the matrix of the polar bilinear form b. If n is even, then a quadratic form is nondegenerate if and only if $\mathrm{disc}(q) \neq 0$. When n is odd, $\mathrm{disc}(q)$ is always zero since the polar bilinear form has odd corank; thus, we consider another invariant.

We define the *half-discriminant* of q as

$$\tfrac{1}{2}\,\mathrm{disc}(q) := q(\omega)$$

where ω is the canonical vector from Proposition 2.1. In coordinates, if

$$q = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

for a basis $x_1, \ldots, x_n$ for $E^\vee$, then

$$\tfrac{1}{2}\,\mathrm{disc}(q) = \sum_{1 \le i \le j \le n} a_{ij}\,\mathrm{Pf}_i\,\mathrm{Pf}_j\,. \tag{2-2}$$

Since each expression Pf_i is a homogeneous polynomial of degree m in the coefficients $\{a_{ij}\}$, the half-discriminant is either zero or a homogeneous polynomial of degree $n = 2m + 1$ in the coefficients $\{a_{ij}\}$. Recall that ω is only defined up to a multiplicative constant which depends on the choice of volume form. If we do not make such a choice, then $\frac{1}{2}\,\mathrm{disc}(q)$ is only well defined modulo nonzero squares in $\Bbbk$.

Proposition 2.2. *For n odd, $\frac{1}{2}\,\mathrm{disc}(q) \neq 0$ if and only if $V(q)$ is smooth.*

Proof. If $\dim \mathrm{rad}(b) > 1$, then the associated projective subspace intersects the quadric $Q = V(q)$; in this case Q is not smooth, $\omega = 0$, and $\frac{1}{2}\,\mathrm{disc}(q) = 0$. Otherwise, ω spans $\mathrm{rad}(b)$ and Q is singular if and only if $q(\omega) = 0$. □

Remark 2.3. The half-discriminant was first introduced by Kneser. Our definition is equivalent to that of [Kneser 2002, p. 43]. Indeed, both polynomials define the same reduced hypersurface in the projective space of the coordinates a_{ij} (they vanish precisely when $V(q)$ is singular). Their degrees are both equal to n, hence they differ by a constant factor. It remains only to compare their values on a particular quadratic form to see that they are equal.

Remark 2.4. The half-discriminant was known to Grothendieck as the *corrected discriminant*, as suggested in his unpublished notes.[1] It is also known as the *half-determinant* (see [Leep and Schueller 2002]),

[1] See Appendix 17.16 of http://www.jmilne.org/math/Documents/EGA-V.pdf.

or the *determinant* (see Remark 13.8 of [Elman et al. 2008]). The notion of the half-discriminant of a quadratic form can be extended to hypersurfaces of arbitrary degree; see [Demazure 2012].

Example 2.5. If $n = 3$, we have

$$\tfrac{1}{2}\operatorname{disc}(q) = a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 + a_{12}a_{23}a_{13}.$$

The locus of zeros $V\big(\frac{1}{2}\operatorname{disc}(q)\big)$ in $|S^2E^\vee| \cong \mathbb{P}^5$ is a cubic hypersurface singular along the plane of double lines $a_{12} = a_{13} = a_{23} = 0$.

If $n = 5$, then

$$\tfrac{1}{2}\operatorname{disc}(q) = (a_{11}a_{23}^2a_{45}^2 + \cdots) + (a_{12}^2a_{34}a_{45}a_{35} + \cdots) + (a_{12}a_{23}a_{34}a_{45}a_{15} + \cdots).$$

3. Pencils of quadratic forms

For the rest of the paper, we assume that the $\Bbbk$-vector space E has dimension $n = 2m + 1$ for a positive integer m.

Let U be a 2-dimensional $\Bbbk$-vector space. A *pencil of quadratic forms* is an injective linear map

$$\mathsf{q} : U \to S^2E^\vee.$$

We use subscript notation q_u to denote the image of $u \in U$ in order to avoid confusion with the interpretation of a quadratic form as a map $E \to \Bbbk$. We will also identify q with its image considering a pencil as a two-dimensional linear subspace of $S^2E^\vee$. The pencil q defines a one-dimensional linear system of quadric hypersurfaces in $|E|$ (also called a pencil). The corresponding rational map $|E| \dashrightarrow |U^\vee|$ has base subscheme $\mathrm{Bs}(\mathsf{q})$ equal to the intersection of quadrics $V(\mathsf{q}_u)$ over all $u \in U$.

We may view the map q as an element of $U^\vee \otimes S^2E^\vee$. Selecting a basis (u_0, u_1) for U with the dual basis (t_0, t_1), we can view q as an element $t_0q_0 + t_1q_1 \in U^\vee \otimes S^2E^\vee$, where

$$q_0 = \mathsf{q}_{u_0}, \quad q_1 = \mathsf{q}_{u_1}$$

define a pair of quadratic forms (q_0, q_1) which form a basis of the pencil. We have $\mathrm{Bs}(\mathsf{q}) = V(q_0, q_1) = V(q_0) \cap V(q_1)$. Conversely, a pair of nonproportional quadratic forms gives rise to a pencil of quadratic forms q (along with a choice of basis for U).

Given a pencil of quadratic forms we obtain an *associated pencil of alternating bilinear forms*

$$\mathsf{b} : U \to \textstyle\bigwedge^2 E^\vee$$

by setting b_u as the associated bilinear form of q_u for each $u \in U$. Note that, unlike q, the map b may not be injective. We may view b as an element of $U^\vee \otimes \bigwedge^2 E^\vee$.

3A. ***The radical map and the radical subspace.*** Fix a volume form on E. For each alternating bilinear form b_u there exists a canonical vector ω as in Proposition 2.1. Thus, we obtain a function

$$\Omega : U \to E \tag{3-1}$$

which is defined by polynomials of degree m in view of (2-1). Thus, abusing notation, we may view Ω as an element of $E \otimes S^m U^\vee$. We call the map (3-1) the *radical map* of the pencil. As u varies over U, the $\Omega(u)$ span a subspace W of E which we call the *radical subspace* of E. Equivalently, Ω gives rise to a $\Bbbk$-linear map

$$(S^m U^\vee)^\vee \to E \tag{3-2}$$

whose image equals the radical subspace W. (Note that we should not identify $(S^m U^\vee)^\vee$ and $S^m U$ in characteristic 2).

We define a function $\Delta : U \to \Bbbk$ via the composition

$$\Delta(u) = \mathsf{q}_u(\Omega(u))$$

for $u \in U$. We call Δ the *half-discriminant of the pencil* q since $\Delta(u)$ is the half-discriminant of each q_u. Since q can be viewed as a linear combination of quadratic forms with linear coefficients in $U^\vee$ and Ω is defined by polynomials of degree m in $U^\vee$, Δ is an element of $S^n(U^\vee)$. After choosing a basis (t_0, t_1) in $U^\vee$, it can be identified with a homogeneous polynomial of degree $n = 2m + 1$ in t_0, t_1.

3B. ***Regular pencils.*** We say a pencil (or pair) is *regular* if the base locus $X = \mathrm{Bs}(\mathsf{q})$ is a smooth variety over $\Bbbk$.

Remark 3.1. We borrowed the term regular from the classical terminology of regular linear systems of quadric hypersurfaces. It should not be confused with the regularity of the base scheme $\mathrm{Bs}(\mathsf{q})$ since, over a nonperfect field $\Bbbk$, the latter could be a regular scheme but not smooth.

From Theorem 5.1 of [Leep and Schueller 2002], we know that a pencil is regular if and only if Δ is nonzero and has n distinct linear factors over the algebraic closure. In other words:

Theorem 3.2. *A pencil* q *is regular if and only if*, *over an algebraic closure of* $\Bbbk$, *there are exactly* n *degenerate quadrics in* $|\mathsf{q}|$.

We will need the following consequence of this theorem:

Corollary 3.3. *If* q *is a regular pencil*, *then every nonzero quadratic form in the pencil has corank* 1.

Proof. It suffices to assume k is algebraically closed. Suppose the corank of some nonzero q in q is > 1. Then the radical $\mathrm{rad}(b)$ of the associated bilinear form b is of dimension ≥ 3. This implies that $|\mathrm{rad}(b)|$ contains a plane intersecting X nontrivially. This shows that $V(q)$ has a singular point x in X. The base locus X is smooth, so the tangent space of X at x is of codimension 2. We have $X = V(q) \cap V(q')$ for some $q' \in \mathsf{q}$. Thus, the tangent spaces of $V(q)$ and $V(q')$ at x intersect along a subspace of codimension 2. This implies that $V(q)$ is smooth at x, a contradiction. □

3C. ***Alternate proof of Theorem 3.2.*** In the remainder of this section, we present a geometric proof that works in arbitrary characteristic which we believe is of independent interest. It is not necessary for the rest of the paper. Note that Theorem 3.2 is well known when the characteristic of $\Bbbk$ is odd. Unlike

the situation for the rest of the paper, in this section we assume $\Bbbk$ is algebraically closed of arbitrary characteristic and that the integer n is odd only if the characteristic is even.

Let D be the subvariety of singular quadrics in $S^2E^\vee$ (this is just defined by the discriminant or half-discriminant). Let D_0 be the open subvariety of D consisting of quadrics which have a unique isolated singular point (an open condition since it is determined by the rank of the matrix of the associated bilinear form).

Lemma 3.4. *Let L be a linear subspace of $S^2E^\vee$ of dimension $r > 0$. Then the tangent space of $\mathsf{D} \cap L$ at a point $q \in \mathsf{D}_0 \cap L$ with singular vector $v_0 \in E$ is the linear space $\{q \in L : q(v_0) = 0\}$.*

Proof. Let

$$\tilde{\mathsf{D}} = \{(q, v) \in S^2E^\vee \setminus \{0\} \times E \setminus \{0\} : [v] \in \mathrm{Sing}(V(q))\}.$$

Consider the maps $p_1 : \tilde{\mathsf{D}} \to S^2E^\vee$ and $p_2 : \tilde{\mathsf{D}} \to E$ obtained from the projections. For any $v \in E \setminus \{0\}$, let F_v be the closure of $p_1(p_2^{-1}(v))$ in $S^2E^\vee$. By choosing a basis in E such that $v = (1, 0, \ldots, 0)$, the subvariety F_v consists of quadratic forms that do not contain the first variable. Thus, F_v is a linear subspace of $S^2E^\vee$ of codimension n. In particular, $\tilde{\mathsf{D}}$ is a fibration over $E \setminus \{0\}$ with fibers isomorphic to linear spaces of the same dimension. It follows that $\tilde{D}$ is smooth and of dimension $n + \dim S^2E^\vee - n = \dim S^2E^\vee$.

The variety $p_2(p_1^{-1}(q))$ is, set-theoretically, the linear space of singular vectors. Passing to the map of the associated projective varieties $\pi : |\tilde{\mathsf{D}}| \to |S^2E^\vee|$, we expect that the map π induces a birational isomorphism with the subvariety $|\mathsf{D}_0|$ of quadrics with isolated singular point.

We will compute the differential of the map $p_1 : \tilde{\mathsf{D}} \to S^2E^\vee$ at a point (q_0, v_0) represented by a quadratic form q_0 with one-dimensional space of singular vectors generated by v_0.

Let $t_1, \ldots, t_n$ be coordinates in E and let A_{ij} be coordinates in $S^2E^\vee$ corresponding to the coefficients of a quadratic form. Then $\tilde{\mathsf{D}}$ is given by $n + 1$ equations:

$$F_k = \sum_{j=1}^{n} B_{kj} t_j = 0, \quad k = 1, \ldots, n, \tag{3-3}$$

$$F_{n+1} = \sum_{1 \le i \le j \le n} A_{ij} t_i t_j = 0, \tag{3-4}$$

where $B_{ii} = 2A_{ii}$ and $B_{ij} = B_{ji} = A_{ij}$ if $i < j$.

Let $q_0 = \sum_{1 \le i \le j \le n} a_{ij} t_i t_j$ and $v_0 = (c_1, \ldots, c_n)$. The embedded tangent space $T_{q_0, v_0}(\tilde{\mathsf{D}})$ of $\tilde{\mathsf{D}}$ is a subspace of $T_{q_0}(S^2E^\vee) \oplus T_{v_0}(E) = S^2E^\vee \oplus E$ given by a system of linear equations in variables t_i, A_{ij} with matrix of coefficients $[M_1 M_2](a_{ij}, c_i)$, where

$$M_1 = \left(\frac{\partial F_k}{\partial t_j} \right)_{\substack{1 \le k \le n+1 \\ 1 \le j \le n}} \quad \text{and} \quad M_2 = \left(\frac{\partial F_k}{\partial A_{ij}} \right)_{\substack{1 \le k \le n+1 \\ 1 \le i \le j \le n}}. \tag{3-5}$$

Computing the partial derivatives, we find that $T_{q_0, v_0}(\tilde{\mathsf{D}})$ consists of pairs (q, v) satisfying the conditions

$$b_0(v, w) + b(v_0, w) = 0, \quad b_0(v_0, v) + q(v_0) = q(v_0) = 0, \tag{3-6}$$

for all $w \in W$, where b and b_0 are the associated bilinear forms of q and q_0.

The kernel of the differential of the map $\tilde{\mathsf{D}} \to \mathsf{D},\ (q, v) \mapsto q$ obtained from the projection can be identified with the linear space of vectors $v \in E$ such that $b_0(v, w) = 0$ for all $w \in E$; this is the radical of the bilinear form b_0. The map

$$T_{(q_0,v_0)}\tilde{\mathsf{D}} \to \{q : q(v_0) = 0\},\ (q, v) \to q$$

is linear and has kernel $\operatorname{rad}(b_0)$. Since the tangent space of D_0 at q_0 is isomorphic to the quotient space $T_{(q_0,v_0)}\tilde{\mathsf{D}}/\operatorname{rad}(b_0)$, we obtain that

$$T_{q_0}\mathsf{D} = \{q \in S^2E^\vee : q(v_0) = 0\}.$$

Note that $\operatorname{rad}(b_0)$ is one-dimensional (this does not hold when n is even and $\Bbbk$ has characteristic 2). This is equal to the dimension of the fiber $p_1^{-1}(q_0)$. This shows that the differential of p_1 is of maximal rank, and hence π is an isomorphism over $|\mathsf{D}_0|$. Now let $L \subset S^2E^\vee$ be a linear nonzero subspace of $S^2E^\vee$, then the variety of quadratic forms in L with one-dimensional space of singular vectors is equal to $L \cap \mathsf{D}_0$. We have

$$T_{q_0}(L \cap \mathsf{D}) = T_{q_0}L \cap T_{q_0}\mathsf{D} = L \cap T_{q_0}\mathsf{D} = \{q \in L : q(v_0) = 0\}.$$

This proves the assertion. □

Proof of Theorem 3.2. We may assume $\Bbbk$ is algebraically closed. Consider $L := \mathsf{q}(U)$, the two-dimensional linear subspace of $S^2E^\vee$. First, we claim that x is a singular point of X if and only if x is a singular point of $Q = V(q)$ for some $q \in L$. The tangent space T_xX is the intersection of the tangent spaces $T_xQ \cap T_xQ'$, for any two distinct quadrics Q, Q' from L. If x is a singular point of Q, then $T_xQ = |E|$. Thus $\operatorname{codim} T_xX \leq 1$ and x is a singular point of X. If x is a singular point of X, then either $T_xQ = |E|$ for some quadric Q, or $T_xQ = T_xQ'$ for any two quadrics Q, Q' from L. In the first case, x is a singular point of Q. In the latter case, if b, b' are the corresponding bilinear forms, then the tangent spaces are defined by linear forms $b(v, -)$ and $b'(v, -)$ where $v \in E$ represents x. Since the linear forms are proportional by assumption, some nontrivial linear combination $(\lambda b + \mu b')(v, -)$ is trivial. The quadric corresponding to that linear combination is therefore singular at x. This establishes the claim.

As an immediate consequence, if a quadric Q in L has corank > 1, then X is not smooth; indeed, in this case, the singular locus of Q has nontrivial intersection with some other quadric in the pencil since it is positive dimensional.

Assume q is regular; in other words, that X is smooth. Suppose q_0 is a degenerate quadratic form in L with corresponding quadric Q_0. Since X is smooth, Q_0 has corank ≤ 1, so q_0 must be in D_0. Let $v_0 \in E$ represent the unique singular point $|v_0|$ of Q_0. Since $|v_0| \notin X$, the space

$$|\{q \in L : q(v_0) = 0\}|$$

is just the point $|q_0|$. By Lemma 3.4, we conclude that the tangent space of $|D \cap L|$ at the point $|q_0|$ is 0-dimensional. Thus $|L|$ and $|D|$ meet transversally at each intersection point and $|L \cap D|$ consists of exactly n distinct points as desired.

Now, suppose $|L \cap \mathsf{D}|$ consists of exactly n distinct points. This is equivalent to $|L|$ intersecting transversally the hypersurface $|\mathsf{D}|$. From the proof of Lemma 3.4, the differential of $\tilde{\mathsf{D}} \to \mathsf{D}$ at a point (q, v) has kernel equal to the radical of the associated bilinear form b. The dimension of this radical is minimal precisely for D_0; so D_0 is the smooth locus of D. Since $|L|$ intersects $|\mathsf{D}|$ transversally at every intersection point, in fact $|L \cap \mathsf{D}| = |L \cap \mathsf{D}_0|$. If $q_0 \in L \cap \mathsf{D}$, then, by Lemma 3.4, $|\{q \in L : q(v_0) = 0\}|$ is a single point. Thus, the unique singular point of the quadric Q_0 corresponding to q_0 does not lie in X. Since none of the singular points of the quadrics lie in X, we conclude from above that X is smooth. □

4. Normal forms

4A. ***Kronecker basis.*** In this section, we prove Theorem 1.1 and discuss some of its consequences. At least in retrospect, the theorem follows quite easily using known techniques regarding pairs of bilinear forms going back to Kronecker. Throughout, E is a vector space of dimension $n = 2m + 1 \geq 3$ over a field of characteristic 2.

Given a pair of alternating bilinear forms (b_0, b_1) on E, we say that a direct sum $E = E_1 \oplus E_2$ is *orthogonal* if it is orthogonal with respect to both b_0 and b_1. A pair of alternating bilinear forms is *nonsingular* if the corresponding determinant polynomial is nonzero. For a positive integer k, a *basic singular pair* is a pair of alternating bilinear forms (b_0, b_1) such that there exists a basis $\mathcal{B} = (w_0, \ldots, w_k, v_0, \ldots, v_{k-1})$ of the underlying vector space such that

$$\begin{aligned} b_0(w_i, w_j) &= 0, \quad b_0(v_i, v_j) = 0, \quad b_0(w_i, v_j) = \delta_{i(j+1)}, \\ b_1(w_i, w_j) &= 0, \quad b_1(v_i, v_j) = 0, \quad b_1(w_i, v_j) = \delta_{ij}, \end{aligned} \tag{4-1}$$

for all valid i, j. We call a basis $\mathcal{B}$ as above a *Kronecker basis*.

Theorem 4.1. *Let (b_0, b_1) be a pair of alternating bilinear forms on a vector space E. Then E can be written as an orthogonal direct sum of a nonsingular pair and a set of basic singular pairs.*

Theorem 4.1 is a special case of Theorem 3.3(1) of [Leep and Schueller 1999]. The statement and its proof are essentially due to Kronecker, although his version applies to symmetric bilinear forms in characteristic $\neq 2$; see Theorem 3.1 of [Waterhouse 1976]. The proof there carries over to the case of alternating bilinear forms in characteristic 2 as observed before the statement of Theorem 9 of [Waterhouse 1977].

Corollary 4.2. *Let (q_0, q_1) be a regular pair of quadratic forms and let (b_0, b_1) be the associated pair of alternating bilinear forms. Then (b_0, b_1) is a basic singular pair and the vector*

$$\Omega = \sum_{i=0}^{m} \lambda^{m-i} \mu^i w_i \tag{4-2}$$

spans $\operatorname{rad}(\lambda b_0 + \mu b_1)$, *where w_i are basis elements in a Kronecker basis (4-1).*

Proof. By a straightforward calculation (see Lemma 3.2 of [Leep and Schueller 1999]), for a basic singular pair (b'_0, b'_1) on a vector space of dimension $2k+1$, the radical of $\lambda b'_0 + \mu b'_1$ is spanned by the vector

$$\Omega' = \sum_{i=0}^{k} \lambda^{k-i} \mu^i w_i.$$

By Corollary 3.3, any nonzero bilinear form from the family $\lambda b_0 + \mu b_1$ has corank 1. Thus there is exactly one basic singular pair in the decomposition from Theorem 4.1 and its radical is given by $\Omega = \Omega'$. It remains only to establish that $k = m$, and thus there is no nonsingular summand in the decomposition.

The vector Ω can be viewed as a homogeneous polynomial function $U \to E$ of degree m. Thus, if $k < m$ then in any choice of basis the entries of Ω must all be divisible by a homogeneous polynomial $g(\lambda, \mu)$ of positive degree. But $\Delta(u) = \mathsf{q}_u(\Omega(u))$ for all $u \in U$, so the half-discriminant would then be divisible by $g(\lambda, \mu)^2$. This contradicts that the roots of Δ are distinct and so $k = m$ as desired. □

It follows that the map $\Omega : U \to E$ is injective and its image spans the same subspace W as that spanned by $w_0, \ldots, w_m$. Since Ω does not depend on any choice of coordinates, we see that W is canonical. Any $(m+1)$-dimensional totally isotropic subspace of an alternating bilinear form of rank $2m$ on a $(2m+1)$-dimensional space must contain the radical. Since W is the minimal space containing the radical of the associated bilinear form to every quadratic form in the pencil, we obtain the following[2]:

Corollary 4.3. *The space W spanned by $w_0, \ldots, w_m$ is canonical. It is the unique common $(m+1)$-dimensional totally singular subspace of all quadratic forms in the pencil.*

Note that choice of basis $w_0, \ldots, w_m$ is determined by the choice of basis u_0, u_1 in U (or, equivalently, the choice of pair (q_0, q_1) corresponding to the pencil $|\mathsf{q}|$). We also see that a choice of a basis u_0, u_1 in U and the canonical isomorphism W with $(S^m U^\vee)^\vee$ defines a basis in W equal to the image of the dual basis to the monomial basis $(t_0^m, t_0^{m-1}t_1, \ldots, t_1^m)$ in $S^m U^\vee$.

Another consequence of the Kronecker theorem is that the map $\Omega : U \to E$ is equal to the composition of the map $U \to (S^m U^\vee)^\vee \to E$, where the second map is injective. This implies that the image of U is equal to the affine cone over a *Veronese curve* R_m of degree m in $|W| \cong |(S^m U^\vee)^\vee| \cong \mathbb{P}^m$. Recall that a Veronese curve in $\mathbb{P}^m$ is a smooth rational curve of degree m equal to the image of $\mathbb{P}^1$ under a map given by linearly independent homogeneous polynomials of degree m. By choosing a monomial basis $t_0^{m-i}t_1^i$ of such polynomials, it is projectively equivalent to a curve R_m in $\mathbb{P}^m$ given by equations expressing the condition that

$$\operatorname{rank} \begin{pmatrix} x_0 & x_1 & \cdots & x_{m-1} \\ x_1 & x_2 & \cdots & x_m \end{pmatrix} = 1. \tag{4-3}$$

For example, when $m = 2$, this is a smooth conic in $\mathbb{P}^2$. In a coordinate-free way, the Veronese curve is the image of $\mathbb{P}^1 = |U|$ in $|(S^m U^\vee)^\vee|$ under the complete linear system $|\mathcal{O}_{|U|}(m)| = |S^m U^\vee|$. The image of the scheme $V(\Delta)$ of roots of the half-discriminant Δ of the pencil under the map $\Omega : |U| \to |W|$ is a

[2]The existence of such a subspace for any pencil of alternating forms is a known fact; see the Lemma in [Beauville 2005].

0-dimensional closed subscheme Z of the Veronese curve R_m of length n. Over an algebraically closed extension of $\Bbbk$ it becomes a union of n distinct points.

We now prove the main theorem:

Proof of Theorem 1.1. Choose a Kronecker basis $w_0, \dots, w_m, v_0, \dots, v_{m-1}$ and the corresponding coordinates $x_0, \dots, x_m, y_0, \dots, y_{m-1}$ in E. Let W be the span of $w_0, \dots, w_m$ and let L be the span of $v_0, \dots, v_{m-1}$. Thus $E = W \oplus L$ is a direct sum of totally singular subspaces for all quadrics in the pencil $|\mathsf{q}|$ (since they are totally isotropic for the associated alternating bilinear forms). This implies that the restrictions of q_0, q_1 to W (resp. L) is equal to a linear combination of squares of x_i (resp. y_i). So, we have reduced q_0, q_1 to the expressions from the Theorem. It remains to prove the assertion about the coefficients $\{a_i\}$. We use the expression of Ω computed above to find the half-discriminant. Let u_0, u_1 be the basis of U corresponding to q_0, q_1 and t_0, t_1 be the dual basis. Then

$$\begin{aligned}\Delta(t_0, t_1) &= (t_0 q_0 + t_1 q_1)(t_0^m, t_0^{m-1} t_1, \dots, t_1^m, 0, \dots, 0) \\ &= t_0 \left(\sum_{i=0}^{m} a_{2i} t_0^{2(m-i)} t_1^{2i} \right) + t_1 \left(\sum_{i=0}^{m} a_{2i+1} t_0^{2(m-i)} t_1^{2i} \right) \\ &= \sum_{i=0}^{2m+1} a_i t_0^{2m+1-i} t_1^i\end{aligned}$$

as desired. □

We say that a regular pair of quadratic forms is in *normal form* if it is written in coordinates corresponding to a Kronecker basis in E.

Note that, given a specific choice of pair (q_0, q_1), the basis $w_0, \dots, w_m$ in a Kronecker basis is canonical. However, given only a pencil $|\mathsf{q}|$, only the vector space W spanned by $w_0, \dots, w_m$ is canonical. This has some strong consequences which we discuss now. In the following X denotes the base scheme $\mathrm{Bs}(\mathsf{q})$ of a regular pencil of quadrics $|\mathsf{q}|$ in $|E|$.

Theorem 4.4. *Assume $m \geq 2$. If $\Bbbk$ is a perfect field, then the variety X contains a canonical $(m-2)$-plane Π defined over $\Bbbk$. In particular, a del Pezzo surface of degree 4 over a perfect field of characteristic 2 has a canonical point.*

Proof. Note that the ideal of the subscheme $X \cap |W|$ is generated by

$$q_0|_W = \sum_{i=0}^{m} a_{2i} x_i^2, \quad q_1|_W = \sum_{i=0}^{m} a_{2i+1} x_i^2$$

in projective coordinates for $|W|$. Over a perfect field, the radical ideal is $\langle l_0, l_1 \rangle$ where $l_0^2 = q_0|_W$ and $l_1^2 = q_1|_W$. If $l_0 = c l_1$ for some $c \in \Bbbk$, then $a_{2i} = c^2 a_{2i+1}$ for all i in $0, \dots, m$. But this implies that

$$\Delta(T, 1) = (T + c^2) g(T^2)$$

for some polynomial $g \in \Bbbk[T]$, contradicting the separability of $\Delta(T, 1)$. Thus the subspace $\{l_0 = l_1 = 0\}$ in X has codimension 2 in $|W|$ as desired. □

The vector space L spanned by $v_0, \dots, v_{m-1}$ is not canonical; in fact, we have the following.

Lemma 4.5. *Let (q_0, q_1) be a regular pair of quadratic forms and (b_0, b_1) the associated bilinear forms. Let W be the canonical subspace spanned by $(w_0, \dots, w_m)$ in a Kronecker basis. Suppose L is a totally isotropic subspace for both b_0 and b_1 and that $E = W \oplus L$. Then there exists a basis $(v_0, \dots, v_{m-1})$ in L that, together with the canonical basis $(w_0, \dots, w_m)$ in W, forms a Kronecker basis in E.*

Proof. Since b_1 has a 1-dimensional radical spanned by w_m, the alternating bilinear form induced on the quotient space $\bar{E} = E/\langle w_m \rangle$ is a direct sum of m hyperbolic planes. Since $w_m \in W$, the images of W and L in $\bar{E}$ are complementary totally isotropic subspaces of $\bar{E}$; and thus provide a pair of complementary maximal totally isotropic subspaces. Thus, we may find a basis $v_0, \dots, v_{m-1}$ for L such that it satisfies the desired equations for b_1 from (4-1).

We have established all desired equations from (4-1) except for $b_0(w_i, v_j) = \delta_{i(j+1)}$. We know that some Kronecker basis $(w_0, \dots, w_m, v'_0, \dots, v'_{m-1})$ exists with corresponding splitting $E = W \oplus L'$. For any $v \in L$ we can write $v = w + v'$ for $w \in W$, $v' \in L'$. For all $i = 0, \dots, m$, we have $b_0(w_i, v) = b_0(w_i, v')$ since $b_0(w_i, w) = 0$; and similarly for b_1. From (4-1), we have that $b_1(w_i, v'_j) = \delta_{ij} = b_0(w_{i+1}, v'_j)$ for all valid i, j. From this we conclude that $b_1(w_i, v) = b_0(w_{i+1}, v)$ for any $v \in L$. We conclude that $b_0(w_i, v_j) = b_1(w_{i-1}, v_j) = \delta_{i(j+1)}$ for all i, j except when $i = 0$. This last case follows from the fact that w_0 is in the radical of b_0. □

We have identified the radical subspace W with $(S^m U^\vee)^\vee$. The next proposition tells that we can identify any complement L as in Lemma 4.5 with the space $S^{m-1}U^\vee$.

Proposition 4.6. *Let (q_0, q_1) be a regular pair of quadratic forms on E. Up to a choice of volume form on E, there is a canonical isomorphism $\iota : U \to U^\vee$ and a canonical isomorphism $L \cong S^{m-1}(U^\vee)$, such that*

$$\mathsf{b}_u(v, w) = \psi_1(\iota(u) f_2) + \psi_2(\iota(u) f_1), \tag{4-4}$$

where $v = (\psi_1, f_1)$ and $w = (\psi_2, f_2)$ are elements of $W \oplus L = (S^m U^\vee)^\vee \oplus S^{m-1}U^\vee$ and we note that $\iota(u) f_i$ can be viewed as an element of $S^m(U^\vee)$ via the multiplication map $U^\vee \otimes S^{m-1}U^\vee \to S^m U^\vee$.

Proof. Let u_0, u_1 be the basis for U such that $\mathsf{q}_{u_0} = q_0$ and $\mathsf{q}_{u_1} = q_1$, and let t_0, t_1 be the dual basis. The map ι is defined to satisfy $t_0 = \iota(u_1)$ and $t_1 = \iota(u_0)$. We use the canonical isomorphism between W and $(S^m U^\vee)^\vee$ which we deduced from formula (4-2); more concretely: $w_i \mapsto (t_0^{m-i} t_1^i)^\vee$ (this is where we use the volume form on W). Consider the isomorphism $L \cong S^{m-1}(U^\vee)$ given by $v_i \mapsto t_0^{(m-1)-i} t_1^i$ for i in $0, \dots, m-1$. A direct comparison of (4-4) and (4-1) in coordinates for U, W, and L establishes the result. □

Proposition 4.6 allows us to compare normal forms for different choices of pairs in the same pencil.

Lemma 4.7. *Suppose $u, u' \in U$ are linearly independent and let $(w_0, \dots, w_m, v_0, \dots, v_{m-1})$ be a Kronecker basis for the pair $(q_u, q_{u'})$. For any $g \in \mathrm{GL}(U)$, there exists an $h \in \mathrm{GL}(W) \times \mathrm{GL}(L)$ such that $(h(w_0), \dots, h(v_{m-1}))$ is a Kronecker basis for $(\mathsf{q}_{g(u)}, \mathsf{q}_{g(u')})$.*

Proof. There exists some $c \in \Bbbk$ such that $\iota(gu) = c(g^\vee)^{-1}\iota(u)$ for all $u \in U$. Using the isomorphisms from Proposition 4.6, consider

$$h = ((S^m g^\vee)^\vee, c^{-1}S^{m-1}(g^\vee)^{-1})$$

in $\mathrm{GL}(W) \times \mathrm{GL}(L)$. Using the shorthand notation $g_* = (S^k g^\vee)^\vee$ and $g^* = S^{k-1}(g^\vee)$ for all positive integers k, we find

$$\begin{aligned}(h\psi)(\iota(gu)(hf)) &= (g_*\psi)((\iota(gu))(c^{-1}(g^*)^{-1}f)) \\ &= \psi(g^*((g^*)^{-1}\iota(u))((g^*)^{-1}f)) \\ &= \psi(g^*(g^*)^{-1}(\iota(u)f)) = \psi(\iota(u)f)\end{aligned}$$

for all $\psi \in W \cong (S^m U^\vee)^\vee$, $f \in L \cong S^{m-1}(U^\vee)$, and $u \in U$. From this we conclude that

$$\mathsf{b}_{gu}(hv, hw) = \mathsf{b}_u(v, w)$$

for all $v, w \in E$ and $u \in U$. Thus, $(\mathsf{b}_{gu}, \mathsf{b}_{gu'})$ is in Kronecker normal form relative to the basis $h(w_0), \ldots, h(v_{m-1})$. □

Remark 4.8. It is frequently convenient to assume that $a_n \neq 0$ in the half-discriminant polynomial or, equivalently, that the quadratic form q_1 is nondegenerate. Lemma 4.7 shows that this is often a harmless assumption.

4B. ***The quasisplit case.*** Following the classical terminology in geometry of ruled varieties, a *generator* of the variety X is a linear subspace of X of maximal possible dimension (over the algebraic closure). In the spirit of [Skorobogatov 2010], we say that X is *quasisplit* if $X = \mathrm{Bs}(\mathsf{q})$ has a generator defined over $\Bbbk$. We say X is *split* if all generators of X are defined over $\Bbbk$. We shall see in Corollary 7.5 below that, when $\Bbbk$ is algebraically closed, every X has exactly 2^{2m} generators. (For two general quadrics, this also follows by Example 14.7.15 of [Fulton 1998].)

Theorem 4.9. *A generator of X has dimension $m-1$. If $\Bbbk$ is algebraically closed, then X contains a generator.*

Proof. Assume $\Bbbk$ is algebraically closed. Let $F_r(X)$ be the subscheme of the Grassmannian $G_r(\mathbb{P}^{n-1})$ of r-dimensional subspaces in $\mathbb{P}^{n-1}$ which are contained in X. From [Debarre and Manivel 1998, Theorem 2.1], we know $F_r(X)$ is nonempty if and only if $r \leq m-1$. Thus, X has a generator (which is of dimension $m-1$). □

Remark 4.10. We shall see below that all X are quasisplit precisely when $\Bbbk$ is closed under separable quadratic extensions.

This brings us to the main theorem of this section. Since X is always quasisplit over an algebraically closed field, Theorem 1.3 follows immediately from the following.

Theorem 4.11. *X is quasisplit if and only if there exists a normal form for the pencil q with $r_1 = \ldots = r_{2m} = 0$.*

Proof. Note that a generator Λ is equal to $|L|$, where L is totally isotropic for all quadrics in the pencil. When $r_i = 0$ in the normal form defined by a Kronecker basis $(w_0, \dots, w_m, v_0, \dots, v_{m-1})$, the span of $v_0, \dots, v_{m-1}$ provides such a subspace. Conversely, given such a subspace L, Lemma 4.5 provides the desired normal form under the condition that L is complementary to W.

Thus, the difficulty is proving that $|L| \cap |W| = \varnothing$ (in other words, that $L \cap W = 0$). We may assume without loss of generality that $\Bbbk$ is algebraically closed.

Let $\Pi = |W| \cap X$ be the canonical subspace from Theorem 4.4. Let $S = |L| \cap |W| = |L| \cap \Pi$. Suppose $\dim S = k \geq 0$. Let P be the span of $|W|$ and $|L|$, its dimension is equal to $2m - k - 1$. Let $Q = V(q)$ be a quadric from the pencil. The restriction of Q to $|W|$ is a linear subspace Y of dimension $m - 1$ (taken with multiplicity 2). The quadric Q also contains the linear subspace $|L|$ of the same dimension. They intersect along the k-dimensional subspace S. We claim that Q must be singular along S.

To see this, choose coordinates $x_0, \dots, x_{2m-k-1}$ for P in such a way that $|L|$ is given by $x_m = \cdots = x_{2m-k-1} = 0$ and $|W|$ is given by $x_0 = \cdots = x_{m-k-2} = 0$, so that S is given by $x_i = 0$, $i \neq m-k-1, \dots, m-1$. The restriction $Q|_P$ of Q to P must have an equation of the form

$$L(x_m, \dots, x_{2m-k-1})^2 + \sum_{i=0}^{m-k-2} x_i M_i(x_m, \dots, x_{2m-k-1}) = 0,$$

where L and $M_1, \dots, M_{m-k-2}$ are linear homogeneous polynomials. Since the polynomial defining this equation does not contain the variables x_i, $i \neq m-k-1, \dots, m-1$, the quadric is singular along the subspace S. In other words, the subspace P is tangent to the quadric Q along S. Thus any hyperplane containing P is tangent to Q along S. Since S and P do not depend on Q, we can find a hyperplane tangent to any Q along S. Since the tangent space of X at any point is of codimension 2, and it is equal to the intersection of tangent hyperplanes of quadrics in the pencil, we find a contradiction with the assumption that X is smooth. □

The proof of the preceding theorem has the following useful consequence:

Corollary 4.12. *If Λ is a generator of X, then there is a Kronecker normal form with decomposition $E = W \oplus L$ such that $|L| = \Lambda$.*

4C. ***Applications to rationality.*** This canonical subspace Π from Theorem 4.4 allows us to prove Corollary 1.2 from the introduction:

Proof of Corollary 1.2. First we consider the case $m = 2$, where we are dealing with quartic del Pezzo surfaces in $|E| \cong \mathbb{P}^4$. The canonical $(m-2)$-plane Π is thus a canonical point. It is known that any del Pezzo surface of degree ≥ 3 with a rational point is unirational over the base field (see Theorem 3.5.1 of [Manin and Tsfasman 1986]).

Now we consider the case where $m > 2$. Consider the projection $p : X \setminus \Pi \to \mathbb{P}^{m+1}$ with center at Π. Since Π is defined over $\Bbbk$, the projection is defined over $\Bbbk$. The morphism p extends to a morphism $p' : X_\Pi \to \mathbb{P}^{m+1}$ where X_Π is the blowup of X at center Π.

Let x be a point in $X \setminus \Pi$. Let F_x be the closure of $p^{-1}(p(x))$ in $\mathbb{P}^{n-1}$. We claim that F_x is the span $\langle x, \Pi \cap T_x X\rangle$ where $T_x X$ is the embedded tangent space to X at x. Let $v \in E$ be a vector such that $x = [v]$ and let $e_0, \ldots, e_{m-2}$ be a basis of Π. The subvariety F_x is equal to the closure of the intersection of $X \setminus \Pi$ and the span $\langle \Pi, x\rangle$ of Π and x. The span $\langle \Pi, x\rangle$ is equal to $\left[tv + \sum_{i=0}^{m-2} s_i e_i\right]$, where $[s_0, \ldots, s_{m-2}, t]$ are projective coordinates in $\langle \Pi, x\rangle$. Plugging in the equations $q_0 = q_1 = 0$ of X we get, for $k = 0, 1$, that

$$q_k\left(tv + \sum_{i=0}^{m-2} s_i e_i\right) = t^2 q_k(v) + q_k\left(\sum_{i=0}^{m-2} s_i e_i\right) + t \sum_{i=0}^{m-2} s_i b_k(e_i, v) = t \sum_{i=0}^{m-2} s_i b_k(e_i, v).$$

This shows that F_x is the span $\langle x, \Pi \cap T_x(X)\rangle$ as claimed.

First, assume that $\Bbbk$ is algebraically closed. We know that if $\Pi \cap T_x(X)$ is nonempty then $\dim \Pi \cap T_x(X)$ is $m - 2 - r(x)$, where $r(x)$ is the rank of the matrix

$$\begin{pmatrix} b_0(e_1, v) & \cdots & b_0(e_{m-2}, v) \\ b_1(e_1, v) & \cdots & b_1(e_{m-2}, v) \end{pmatrix}.$$

Let C be the set of points $x \in X$ where $r(x) < 2$: these are the points such that, for some quadric Q given by $\lambda q_0 + \mu q_1 = 0$, the associated bilinear form b satisfies $b(w, v) = 0$ for all $w \in E$ representing $[w] \in \Pi$. Suppose C is equal to all of X. By Theorem 4.9, the space X contains a $(m-1)$-dimensional projective subspace Λ. Choose a point x from Λ. Note that the tangent space $T_x(Q)$ is defined by the fact that $b(w, v) = 0$ for all $w \in E$. Thus $T_x(Q)$ contains the $(m-1)$-dimensional projective subspace Λ and the $(m-2)$-dimensional projective subspace Π. Since $\Lambda \cap \Pi = \varnothing$, we obtain that $T_x(Q) \cap Q$ contains a projective subspace of dimension $2m-2$. However, by Corollary 3.3, the corank of any quadric in the pencil is equal to 1. Hence Q cannot contain projective subspaces of dimension greater than $m-1$. This contradicts C being the whole space. Thus, for a general point $x \in X$, we have $r(x) = 2$.

Consequently, $\Pi \cap T_x(X)$ is empty if $m \leq 3$, or of dimension $\geq m-4$ if $m \geq 4$. This implies that $F_x = \langle x, \Pi \cap T_x(X)\rangle$ consists of x if $m \leq 3$ and $\dim(F_x) = m-3$ if $m \geq 4$. If $m = 2$, X is a del Pezzo surface of degree 4 and the projection p is a birational map onto a cubic surface in $\mathbb{P}^3$. If $m = 3$, we obtain that the projection p is a birational map onto $\mathbb{P}^4$. If $m > 3$, we obtain that the general fiber $(p')^{-1}(p(x))$ of the projection map $p' : X_\Pi \to \mathbb{P}^{m+1}$ is isomorphic to the blowup of the $(m-3)$-dimensional subspace $\langle x, \Pi \cap T_x(X)\rangle$ along the $(m-4)$-dimensional subspace $\Pi \cap T_x(X)$. Thus the general fiber is a projective space of dimension $m-3$. Thus, X is birationally isomorphic to a projective $(m-3)$-bundle over $\mathbb{P}^{m+1}$.

Now, assume $\Bbbk$ is any perfect field. Since the projection is a $\Bbbk$-rational map, we immediately get that X is rational if $m = 3$. It remains to consider $m > 3$. By construction, X_Π is a subvariety of a projective bundle $\mathbb{P}(E)$ over $\mathbb{P}^{m+1}$. Moreover, the general fibers of $X_\Pi \to \mathbb{P}^{m+1}$ are linear subspaces of the fibers of $\mathbb{P}(E) \to \mathbb{P}^{m+1}$. Thus, over some open subscheme S of $\mathbb{P}^{m+1}$, the subvariety X_Π restricts to a projective bundle over S (not just a Severi–Brauer scheme over S). Thus X_Π is rational. □

Y. Prokhorov asked us for a geometric interpretation of the canonical point Π in the plane model of a del Pezzo surface of degree 4. We consider this question in Remark 8.4.

5. Isomorphisms of normal forms

The main goal of this section is to prove Theorem 1.4. In order to do this, we need to study isomorphisms between pairs of quadratic forms in Kronecker normal form. The associated bilinear form of any such pair is always the same, so any isomorphism of pairs of quadratic forms is an automorphism of a basic singular pair of alternating bilinear forms.

Thus, our first task is to compute the automorphisms of a basic singular pair of alternating bilinear forms. Next, we collect some results about the finite dimensional algebra A from the introduction. We then use this algebra to determine which automorphisms of basic singular pairs correspond to which isomorphisms of pairs of quadratic forms.

5A. ***Automorphisms of basic singular pairs.*** Let (b_0, b_1) be a basic singular pair of alternating bilinear forms (for example, obtained from a regular pair of quadratic forms), and let $w_0, \dots, w_m, v_0, \dots, v_{m-1}$ be a Kronecker basis for the underlying vector space E of dimension $n = 2m + 1$.

An *automorphism* of the pair (b_0, b_1) is an element $g \in \mathrm{GL}(E)$ such that

$$b_0(gv, gv) = b_0(v, v) \quad \text{and} \quad b_1(gv, gv) = b_1(v, v)$$

for any $v \in V$.

Lemma 5.1. *The group of automorphisms* $\mathrm{Aut}(b_0, b_1)$ *of the pair* (b_0, b_1) *consists of elements* $g \in \mathrm{GL}(E)$ *of the form*

$$g(w_i) = w_i, \quad g(v_i) = v_i + \sum_{k=0}^{m} s_{i+k} w_k, \tag{5-1}$$

where $s_0, \dots, s_{n-2}$ *are any elements in* $\Bbbk$.

Explicitly, the automorphisms have the form

$$g = \begin{pmatrix} I_{m+1} & S \\ 0_{m,m+1} & I_m \end{pmatrix}, \quad \text{where } S = \begin{pmatrix} s_0 & s_1 & \cdots & s_{m-1} \\ s_1 & s_2 & \cdots & s_m \\ \vdots & \vdots & \vdots & \vdots \\ s_m & s_{m+1} & \cdots & s_{2m-1} \end{pmatrix}$$

is a *catalecticant matrix* $\mathrm{Cat}_{m-1}(s_0, \dots, s_{2m-1})$ (see [Dolgachev 2012, 1.4.1]).

Proof. One checks directly that the given elements are automorphisms of (b_0, b_1) via (4-1). It remains to check that these are the only automorphisms.

Let g be an automorphism of (b_0, b_1). Note that g must fix each w_i since they are canonical. It remains to consider $v_0, \dots, v_{m-1}$. Thus

$$g(v_i) = h(v_i) + \sum_{k=0}^{m} l_{ik} w_k$$

for some $h \in \mathrm{GL}(L)$ and elements l_{ik} in $\Bbbk$. Since

$$\delta_{ij} = b_1(gv_i, gw_j) = b_1(hv_i, w_j)$$

we conclude that h is the identity. Since

$$0 = b_1(gv_i, gv_j) = \sum_{k=0}^{m} l_{jk} b_1(v_i, w_k) + \sum_{k=0}^{m} l_{ik} b_1(w_k, v_j) = l_{ij} + l_{ji}$$

we conclude that $l_{ij} = l_{ji}$. Similarly, using b_0 we conclude that $l_{j(i+1)} = l_{i(j+1)}$. We obtain that $l_{i(j+1)} = l_{(i+1)j}$ and thus the values of l_{ij} depend only on the sum of their indices: $l_{ik} = s_{i+k}$. □

5B. ***Results on*** **$\Bbbk$*-algebras.*** Here we collect some facts about finite $\Bbbk$-algebras which we will need later. All are standard results except for Lemma 5.3.

Consider a separable polynomial

$$f(T) = a_n T^n + \cdots + a_1 T + a_0$$

in $\Bbbk[T]$ of degree n. Consider the $\Bbbk$-algebra $A = \Bbbk[T]/(f(T))$ and let t be the image of T under the map $\Bbbk[T] \to A$.

Note that since f is a separable polynomial, we may also write A as a direct sum $A = A_1 \oplus \cdots \oplus A_l$ where each A_i is a separable field extension of $\Bbbk$. Indeed, we have isomorphisms $A_i \cong \Bbbk[T]/(f_i(T)) \cong A/(f_i(t))$ where f_i is an irreducible polynomial dividing f.

Define polynomials $g_1, \ldots, g_l$ in $\Bbbk[T]$ via $f(T) = f_i(T) g_i(T)$. Since the polynomials g_i are coprime, we may write

$$1 = \epsilon_1 + \cdots + \epsilon_l \tag{5-2}$$

where $\epsilon_i \in (g_i(t))$ and they satisfy relations $\epsilon_i \epsilon_j = 0$ when $i \neq j$, and $\epsilon_i^2 = \epsilon_i$. In other words, $\epsilon_1, \ldots, \epsilon_l$ form an orthogonal set of idempotents in A.

Multiplication by an idempotent ϵ_i defines a homomorphism from A into the corresponding summand A_i. Specifically, if h is any polynomial in $\Bbbk[T]$ we have

$$\epsilon_i h(t) \equiv h(t) \mod (f_i(t)) \tag{5-3}$$

in $A_i \cong A/(f_i(t))$. If f_i is of degree 1 with root α_i, then multiplication by ϵ_i corresponds to the $\Bbbk$-algebra homomorphism $A \to A_i \simeq \Bbbk$ determined by $t \mapsto \alpha_i$.

For any $a \in A$, the linear map $x \mapsto ax$ is an endomorphism of the vector space A over $\Bbbk$, we denote by $\mathrm{Tr}_{A/\Bbbk}(a)$ its trace. The formula

$$\langle a, b \rangle = \mathrm{Tr}_{A/\Bbbk}(ab)$$

defines a symmetric bilinear form on A. Its restriction to each summand A_i is the usual trace for a separable extension of fields. Since A is separable, the trace form is nondegenerate. In particular, the natural homomorphism

$$A \to A^\vee, \quad a \mapsto (x \mapsto \mathrm{Tr}_{A/\Bbbk}(ax))$$

of vector spaces over $\Bbbk$ is a bijection.

If f splits completely into linear factors (for example, if $\Bbbk$ is separably closed), then $A \cong \Bbbk^n$. If $\alpha_1, \dots, \alpha_n$ are the roots of f, then a canonical isomorphism $A \cong \Bbbk^n$ is defined via

$$h(t) \mapsto (h(\alpha_1), \dots, h(\alpha_n))$$

for any polynomial h in $k[T]$. In this case, the trace can be computed as

$$\mathrm{Tr}_{A/\Bbbk}(h(t)) = \sum_{i=1}^{n} h(\alpha_i).$$

By passing to a separable closure, one can compute the trace even when f does not split into linear factors over the original field.

Consider a basis of A given by the elements

$$d_i = a_{i+1} + a_{i+2}t + \cdots + a_n t^{n-1-i} \quad \text{for } i = 0, \dots, n-1. \tag{5-4}$$

Using the recurrence relation $d_i = td_{i+1} + a_{i+1}$, we get the identity in $A[X]$

$$f(X) = (X - t)(d_0 + d_1 X + \cdots + d_{n-1}X^{n-1}). \tag{5-5}$$

Note that d_{n-1} spans the "constant" subalgebra $\Bbbk \subset A$.

Let $f'(T)$ denote the formal derivative of $f(T)$. Since $f(T)$ is a separable polynomial, $f'(T)$ is coprime to $f(T)$ and thus $f'(t)$ is an invertible element in A.

The following proposition shows that the elements $d_i / f'(t)$ form the dual basis of the basis $1, t, \dots, t^{n-1}$ with respect to the trace form.

Proposition 5.2. $\mathrm{Tr}_{A/\Bbbk}\left(d_i \dfrac{t^j}{f'(t)}\right) = \delta_{ij}$ *for* i, j *in* $0, \dots, n-1$.

Proof. In the case when A is a field, this can be found in, for example, [Lang 1994, Proposition III.1.2]. We give the proof here since it is short and uses formulas that we will need later. We may assume without loss of generality that $\Bbbk$ is separably closed.

Let $\alpha_1, \dots, \alpha_n$ be the roots of f. We use the following *Euler's identity*:

$$X^k - \sum_{i=1}^{n} \frac{f(X)}{X - \alpha_i} \frac{\alpha_i^k}{f'(\alpha_i)} = 0, \quad k = 0, \dots, n-1 \tag{5-6}$$

in the ring $A[\![X]\!]$. To see this, one uses that $f'(\alpha_i) = \prod_{j \neq i}(\alpha_i - \alpha_j)$, then checks that the expression on the left-hand side is a polynomial of degree $< n$ and has n roots $\alpha_1, \dots, \alpha_n$, hence it must be zero.

One can extend the trace $\mathrm{Tr}_{A/\Bbbk}$ to a $\Bbbk$-linear function $A[\![X]\!] \to \Bbbk[\![X]\!]$ by applying the trace function to each coefficient of X^i. Thus, using (5-5), we obtain

$$\mathrm{Tr}_{A/\Bbbk}\left(\frac{f(X)}{X - t}\frac{t^k}{f'(t)}\right) = \mathrm{Tr}_{A/\Bbbk}\left(\left(\sum_{i=0}^{n-1} d_i X^i\right)\frac{t^k}{f'(t)}\right) = X^k,$$

for every k in $0, \dots, n-1$. Comparing the coefficients, we get the assertion. □

We need an explicit formula for squaring an element with respect to the basis $\{d_0, \dots, d_{n-1}\}$.

Lemma 5.3. *Given the equation*

$$r_0 d_0 + \cdots + r_{n-1} d_{n-1} = (s_0 d_0 + \cdots + s_{n-1} d_{n-1})^2$$

for some coefficients $s_0, \dots, s_{n-1}$ in $\Bbbk$, we have

$$r_k = \sum_{j=0}^{n-1} s_j^2 a_{2j+1-k}$$

for all k in $0, \dots, n-1$. (We assume by convention $a_i = 0$ when $i < 0$ or $i > n$.)

Proof. Using Proposition 5.2, it suffices to show that

$$\mathrm{Tr}_{A/\Bbbk}\left(d_j^2 \frac{t^k}{f'(t)}\right) = a_{2j+1-k}$$

for all j, k in $0, \dots, n-1$.

Rearranging Euler's equation (5-6) we obtain

$$\frac{X^k}{f(X)} = \sum_{i=1}^{n} \frac{1}{X - \alpha_i} \frac{\alpha_i^k}{f'(\alpha_i)}.$$

Now, differentiating with respect to X, we have

$$\frac{kX^{k-1} f(X) - X^k f'(X)}{f(X)^2} = \sum_{i=1}^{n} \frac{-1}{(X-\alpha_i)^2} \frac{\alpha_i^k}{f'(\alpha_i)}$$

$$\frac{\partial}{\partial X}(X^k f(X)) = \sum_{i=1}^{n} \left(\frac{f(X)}{X - \alpha_i}\right)^2 \frac{\alpha_i^k}{f'(\alpha_i)}.$$

Using (5-5), we conclude

$$\frac{\partial}{\partial X}\left(\sum_{i=0}^{n} a_i X^{k+i}\right) = \mathrm{Tr}_{A/\Bbbk}\left(\left(\sum_{j=0}^{n-1} d_j^2 X^{2j}\right) \frac{t^k}{f'(t)}\right), \quad \sum_j a_{2j+1-k} X^{2j} = \sum_{j=0}^{n-1} \mathrm{Tr}_{A/\Bbbk}\left(d_j^2 \frac{t^k}{f'(t)}\right) X^{2j}$$

as desired. Note that the assumption that $\Bbbk$ has characteristic 2 is essential! □

5C. ***Isomorphisms of pairs of quadratic forms.*** Let (q_0, q_1) be a regular pair of quadratic forms in normal form with respect to a Kronecker basis $\mathcal{B} = (w_0, \dots, w_m, v_0, \dots, v_{m-1})$ for E. Note that since $w_0, \dots, w_m$ are canonical, the polynomial Δ and its coefficients $a_0, \dots, a_n$ are the same regardless of the choice of basis $\mathcal{B}$. However, the coefficients $r_0, \dots, r_{n-2}$ from Theorem 1.1 are more subtle.

Assume that $a_n \neq 0$ in Δ or, equivalently, that q_1 is nondegenerate (see Remark 4.8). Define

$$f(T) := \Delta(1, T) = a_0 + \cdots + a_n T^n.$$

Setting $A = \Bbbk[T]/(f(T))$, we have an algebra as in the previous section.

The *r-invariant* of the normal form is the element $r \in A$ given by

$$r = r_0 d_0 + \cdots + r_{n-2} d_{n-2} \tag{5-7}$$

where $d_0, \ldots, d_{n-1}$ is the basis for A defined in (5-4). Note that the basis element d_{n-1} spanning $\Bbbk$ does not appear in the expression (5-7), so the set of possible r-invariants are in bijection with the quotient space $A/\Bbbk$.

For $g \in \mathrm{GL}(E)$, let (q_0', q_1') be the pair given by

$$q_0'(v) = q_0(gv) \quad \text{and} \quad q_1'(v) = q_1(gv)$$

for $v \in E$. Note that the half-discriminant polynomial of (q_0', q_1') remains the same. Let r' be the r-invariant of the new normal form.

We assume, without loss of generality, that (q_0', q_1') is also in Kronecker normal form with respect to the basis $\mathcal{B}$. Consequently, both pairs have the same pair (b_0, b_1) of associated bilinear forms and we may take $g \in \mathrm{Aut}(b_0, b_1)$.

Any $g \in \mathrm{Aut}(b_0, b_1)$ is determined by the elements $s_0, \ldots, s_{n-2}$ from (5-1). We construct an element

$$s = s_0 d_0 + \cdots + s_{n-2} d_{n-2} \tag{5-8}$$

in A as with the r-invariant above.

Reversing the process, one checks that this gives rise to a group homomorphism

$$\phi : A \to \mathrm{Aut}(b_0, b_1) \subset \mathrm{GL}(E), \tag{5-9}$$

where A is viewed as an additive group. Note that the kernel of ϕ is the subgroup $\Bbbk$ since d_{n-1} spans $\Bbbk$.

Theorem 1.4 is a consequence of the following.

Theorem 5.4. *If $g = \phi(s)$ for some $s \in A$, then*

$$r' \equiv r + \wp(s) \mod \Bbbk,$$

where $\wp : A \to A$ is the Artin–Schreier map $s \mapsto s^2 + s$.

Proof. Recall that $g(w_i) = w_i$ and $g(v_i) = v_i + \sum_{k=0}^{m} s_{i+k} w_k$. We need to compute $q_j(gv_i)$ for $j = 0, 1$ and $i = 0, \ldots, m-1$. We find

$$\begin{aligned} q_0(gv_i) &= q_0\left(v_i + \sum_{k=0}^{m} s_{i+k} w_k\right) \\ &= q_0(v_i) + b_0\left(v_i, \sum_{k=0}^{m} s_{i+k} w_k\right) + q_0\left(\sum_{k=0}^{m} s_{i+k} w_k\right) \\ &= r_{2i+1} + s_{2i+1} + \sum_{k=0}^{m} s_{i+k}^2 a_{2k}, \end{aligned}$$

and similarly that

$$q_1(gv_i) = r_{2i} + s_{2i} + \sum_{j=i}^{m+i} s_j^2 a_{2j+1-2i} \ .$$

By Lemma 5.3, we conclude that the new invariant is $r + s + s^2$ as desired. □

6. Relations to the Arf invariant

The following theorem was suggested to us by A. Efimov.

Recall that any nondegenerate quadratic form in $2m$ variables over a field $\Bbbk$ of characteristic 2 can be reduced to the form

$$q = \sum_{i=1}^{m} a_i x_i^2 + x_i y_i + b_i y_i^2$$

for some basis $x_1, \ldots, x_m, y_1, \ldots, y_m$. The *Arf invariant* of q is

$$\mathrm{Arf}(q) = \sum_{i=1}^{m} a_i b_i \in \Bbbk/\wp(\Bbbk).$$

It is independent of a choice of a canonical form from above.

Let (q_0, q_1) be a regular pair of quadratic forms in normal form with invariants Δ and r. We assume that $a_n \neq 0$ and define $A = \Bbbk[T]/(f(T))$ with t the image of T as in previous sections. The pair gives rise to a quadratic form q_A on the module $E_A = E \otimes_{\Bbbk} A$ by defining

$$q_A = q_0 + tq_1$$

as an element $S^2(E_A^\vee)$. Since A is a sum of fields $A_1, \ldots A_l$, we may define the Arf invariant of q_A to be the sum of the Arf invariants of the restrictions q_{A_i}.

Theorem 6.1. *The form q_A is an orthogonal sum of a 1-dimensional trivial form and a nondegenerate $2m$-dimensional form whose Arf invariant is r modulo $\wp(A) + \Bbbk$.*

Proof. We select a new basis for $E_A = E \otimes_{\Bbbk} A$ as follows:

$$\begin{cases} w_i' = \sum_{k=i}^{m} w_i \otimes t^{k-i} & \text{for } i \text{ in } 0, \ldots, m, \\ v_i' = v_i \otimes 1 & \text{for } i \text{ in } 0, \ldots, m-1 \end{cases}$$

and find that

$$b_A(w_i', w_j') = 0, \quad b_A(v_i', v_j') = 0, \quad b_A(v_i', w_j') = \delta_{(i+1)j},$$

where b_A is the bilinear form associated to q_A. In particular, w_0' is the image of Ω in E_A and we see that it spans a 1-dimensional form where $q_A(w_0') = 0$.

The remaining vectors span a subspace which decomposes into an orthogonal sum of 2-dimensional subspaces $\langle w_{i+1}', v_i' \rangle$ on which q_A is isomorphic to the quadratic form

$$q_A(w_{i+1}')x^2 + xy + q_A(v_i')y^2$$

for appropriate coordinates x and y.

Note that

$$q_A(w'_{i+1}) = \sum_{k=i}^{m} (a_{2i} + a_{2i+1}t)t^{2(k-i)} = d_{2i+1}$$

while

$$q_A(v'_i) = r_{2i}t + r_{2i+1}.$$

The Arf invariant of the $2m$-dimensional subspace is thus

$$\sum_{i=0}^{m-1} q_A(w'_{i+1})q_A(v'_i) = \sum_{i=0}^{m-1} (r_{2i}t + r_{2i+1})d_{2i+1} = \left(\sum_{i=0}^{n-2} r_i d_i\right) + \left(\sum_{i=0}^{m-1} r_{2i}a_{2i+1}\right),$$

which is equal to $r \mod \Bbbk$. □

7. Automorphisms of pencils of quadratic forms

7A. ***Arithmetic description of automorphisms of a pair.*** Let A be the $\Bbbk$-algebra from the previous two sections. Since the characteristic is 2, the set of idempotents $\mathrm{Idem}(A)$ form a subgroup of the additive group of A. Recall that an idempotent a in A is an element such that $a^2 = a$. Consequently, the group $\mathrm{Idem}(A)$ can be characterized as the kernel of the Artin–Schreier map

$$\wp : A \to A, \quad a \mapsto a^2 + a.$$

Let $\mathrm{Aut}(q_0, q_1)$ be the set of automorphisms of the pair (q_0, q_1); in other words, the subgroup of $\mathrm{GL}(E)$ whose elements induce an isomorphism of the pair with itself. From Theorem 5.4, we see that every automorphism comes from an idempotent via (5-1). Conversely, the only nontrivial idempotent which gives rise to a trivial automorphism is the multiplicative identity $1 \in \Bbbk \subset A$. Thus we have the following:

Theorem 7.1. *There is a canonical isomorphism*

$$\mathrm{Aut}(q_0, q_1) \cong \mathrm{Idem}(A)/\langle 1 \rangle,$$

where $\langle 1 \rangle$ *is the additive subgroup of* A *generated by the unit element* $1 \in A$.

The group $\mathrm{Idem}(A)$ is generated by the orthogonal set of idempotents $\epsilon_1, \ldots, \epsilon_l$ from (5-2). Since the additive subgroup of A is commutative and every element has order 2, we see that $\mathrm{Idem}(A)$ is an elementary abelian 2-group of order 2^l.

Consequently, $\mathrm{Aut}(q_0, q_1)$ is an elementary abelian 2-group of order 2^{l-1} where l is the number of irreducible factors in $f(T)$. When A is a field (equivalent to $f(T)$ being irreducible), then the only idempotent is 1 and thus $\mathrm{Aut}(q_0, q_1)$ is trivial. At the other extreme, when $f(T)$ splits completely, $\mathrm{Aut}(q_0, q_1)$ has order 2^{2m} and we will see below that the idempotents $\epsilon_1, \ldots, \epsilon_n$ correspond to reflections.

$\mathrm{Idem}(A)$, respectively $\mathrm{Aut}(q_0, q_1)$, can be viewed as the set of $\Bbbk$-points of a finite étale group scheme $\underline{\mathrm{Idem}}(A)$, respectively $\underline{\mathrm{Aut}}(q_0, q_1)$, over $\Bbbk$. The group scheme $\underline{\mathrm{Idem}}(A)$ is simply the Weil restriction

of scalars $\mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)$ where $\mathbb{Z}/2\mathbb{Z}$ is the constant group scheme of order 2. The group scheme $\underline{\mathrm{Aut}}(q_0, q_1)$ is the quotient in the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A) \to \underline{\mathrm{Aut}}(q_0, q_1) \to 0, \tag{7-1}$$

where ι is the natural monomorphism.

Proposition 7.2. *There is a canonical isomorphism between $H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1))$, the Galois cohomology group, and the group $A/(\wp(A) + \Bbbk)$.*

Proof. This follows the same reasoning as §1 of [Skorobogatov 2010]. The composition

$$H^2(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \to H^2(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)) = H^2(A, \mathbb{Z}/2\mathbb{Z})$$

factors through the restriction-corestriction map, which is multiplication by the odd number $\dim_{\Bbbk}(A)$. Since $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)$ have an even number of elements over a separable closure, the cohomology groups are 2-primary. Thus the map $H^2(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \to H^2(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A))$ is injective. Thus from (7-1), the group $H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1)$ is the cokernel of the map

$$H^1(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \to H^1(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)).$$

The characteristic is 2, thus $H^1(\Bbbk, \mathbb{Z}/2\mathbb{Z}) \cong \Bbbk/\wp(\Bbbk)$ and

$$H^1(\Bbbk, \mathbb{R}_{A/\Bbbk}((\mathbb{Z}/2\mathbb{Z})_A)) = H^1(A, (\mathbb{Z}/2\mathbb{Z})_A)) \cong A/\wp(A),$$

so the result follows. □

Remark 7.3. The above proposition could be used to recover a weak form of Theorem 5.4. Of course, this argument would be circular in our presentation since Theorem 5.4 was used to determine the automorphism group.

7B. ***Geometric description of automorphisms of a pair.*** In this subsection we assume that $\Bbbk$ is algebraically closed.

Let $x_1, \dots, x_n$ be vectors in U representing the points $\overline{x_1}, \dots, \overline{x_n}$ in the zero locus of $V(\Delta)$ on $|U|$. Recall that $V(\mathsf{q}_{x_1}), \dots, V(\mathsf{q}_{x_n})$ are precisely the singular quadrics of the pencil $|\mathsf{q}|$.

Fix i in $1, \dots, n$. Since $V(\mathsf{q}_{x_i})$ is of corank 1, it contains a unique singular point $\overline{z_i} \in |E|$ represented by the vector $z_i = \Omega(x_i)$ in E.

For any $u \in U$ such that $|u| \neq |x_i|$ in $|U|$ consider the following linear automorphism of E

$$\rho_i(v) = v + \frac{\mathsf{b}_u(z_i, v)}{\mathsf{q}_u(z_i)} z_i, \tag{7-2}$$

where $v \in E$. A priori, the formula for ρ_i depends on the choice of $u \in U$. However, one can check that ρ_i does not depend on the choice of u as long as it is not a multiple of x_i (use that $\mathsf{q}_{x_i}(z_i) = 0$ and $\mathsf{b}_{x_i}(z_i, v) = 0$ for any choice of $v \in E$).

Theorem 7.4. *The automorphism group* $\mathrm{Aut}(q_0, q_1)$ *is an elementary abelian* 2*-group of order* 2^{2m} *generated by the reflections* $\rho_1, \dots, \rho_n$ *subject to the relation* $\rho_1 \cdots \rho_n = 1$.

Proof. It is clear that the reflections $\{\rho_i\}$ are automorphisms, that they are of order 2, and that they commute. It remains to show that there are no other automorphisms and to show that they satisfy the desired relations and no others. We will show that the reflections $\rho_1, \dots, \rho_n$ correspond to a set of orthogonal idempotents of A. The result then follows by Theorem 7.1.

The remainder of the proof is simply a direct calculation. In view of Remark 4.8, we may assume without loss of generality that q_0 is nondegenerate and each vector x_i has coordinates $(\alpha_i, 1)$, where α_i is a root of $f(T)$. Let ϵ_i be the idempotent corresponding to α_i; in other words, multiplication by ϵ_i gives rise to a map $A \to \Bbbk$ such that $t \mapsto \alpha_i$ as in (5-3).

We will establish that $\phi(\epsilon_i) = \rho_i$, where $\phi : A \to \mathrm{GL}(E)$ is the map (5-9). To do this, we determine the values of $s_0, \dots, s_{n-2}$ in (5-1). From (5-8), these are the first $n-1$ coordinates of ϵ_i in the basis $d_0, \dots, d_{n-1}$ of A defined in (5-4). Thus, by Proposition 5.2, we have

$$s_j = \mathrm{Tr}_{A/\Bbbk}\left(\epsilon_i \frac{t^i}{f'(t)}\right) = \frac{\alpha_i^j}{f'(\alpha_i)}$$

for each j in $0, \dots, n-2$.

Now, note that $z_i = \sum_{i=0}^m \alpha^i w_i$. Since $f'(T) = \sum_{k=0}^m a_{2i+1} T^{2i}$, we have $f'(\alpha_i) = q_1(z_i)$. We have $b_1(v_j, z_i) = \alpha_i^j$ for j in $0, \dots, m-1$, where $w_0, \dots, w_m, v_0, \dots, v_{m-1}$ is the Kronecker basis for E.

Putting these observations together, we see that

$$\phi(\epsilon_i)(v_j) - v_j = \sum_{k=0}^m s_{j+k} w_k = \frac{\alpha_i^j}{f'(\alpha_i)} \sum_{k=0}^m \alpha^k w_k = \frac{b_1(z_i, v_j)}{q_1(z_i)} z_i$$

for all j in $0, \dots, m-1$. Since, additionally, $\phi(\epsilon_i)(w_j) = \rho(w_j) = w_j$ for j in $0, \dots, m$, we conclude that $\phi(\epsilon_i)(v) = \rho(v)$ for all $v \in E$. □

With this description of the automorphism group, we have:

Corollary 7.5. *There are exactly* 2^{2m} *generators permuted simply transitively by the group* $\mathrm{Aut}(q_0, q_1)$.

Proof. By Corollary 4.12, the generators are in bijective correspondence with subspaces L occurring in a Kronecker normal form such that L is totally isotropic with respect to every quadratic form in the pencil. The group $\mathrm{Aut}(q_0, q_1)$ permutes such spaces. The subspace L is never invariant under a nontrivial automorphism as in (5-1); thus the action is simply transitive. The number of generators is 2^{2m} since that is the order of the group. □

From the above corollary, we prove Theorem 1.5 from the introduction:

Proof of Theorem 1.5. The set of generators of X corresponds to a smooth 0-dimensional subscheme of an appropriate Grassmannian. In view of Corollary 7.5, over a nonclosed field this is an $\underline{\mathrm{Aut}}(q_0, q_1)$-torsor. From Proposition 7.2 we have that $A/(\wp(A) + \Bbbk) \cong H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1))$. Thus the r-invariant in A naturally gives rise to a class in $H^1(\Bbbk, \underline{\mathrm{Aut}}(q_0, q_1))$. □

7C. ***Automorphisms of the base locus.*** The goal of this section is to prove Theorem 1.6, which describes the automorphisms of the base scheme $\mathrm{Bs}(\mathsf{q}) = V(q_0, q_1)$ in $\mathbb{P}^{n-1}$.

Lemma 7.6. *Two smooth complete intersections of pairs of quadrics in $\mathbb{P}^{2m}$ are isomorphic if and only if there is an element of PGL_{2m+1} inducing the isomorphism.*

Proof. If $m = 1$ then the two varieties are smooth subschemes of $\mathbb{P}^2$, of dimension 0 of degree 4; the result is clear in this case. Otherwise, when $m \geq 2$, the adjunction formula gives that the canonical sheaf ω_X is isomorphic to $\mathcal{O}_X(-n+4)$. Since $n = 2m+1 \geq 5$, the anticanonical divisor class $-K_X$ is a positive multiple of the hyperplane section. Hence the embedding of X in $\mathbb{P}^{2m}$ comes from a multiple of the anticanonical sheaf. Any isomorphism between the varieties induces isomorphisms of the corresponding anticanonical sheaves, and the result follows. □

Theorem 1.6 follows immediately from the following.

Theorem 7.7. *Assume X is quasisplit. The automorphism group of X, $\mathrm{Aut}(X)$, is isomorphic to $\mathrm{Aut}(q_0, q_1) \rtimes G$ where G is the subgroup of $\mathrm{PGL}(U)$ which leaves invariant the scheme $V(\Delta)$ of zeros of the half-discriminant.*

Proof. By Lemma 7.6, any automorphism of X can be represented by an element of $\mathrm{PGL}(E)$.

Recall that $|W|$ is a canonical subspace and must be invariant under any automorphism of X. From Proposition 4.6, W is canonically isomorphic to $S^m(U^\vee)^\vee$. Thus an automorphism $h \in \mathrm{PGL}(E)$ fixes $|W|$ pointwise if and only if it fixes $|U|$ pointwise. Thus there is a group homomorphism $\pi : \mathrm{Aut}(X) \to \mathrm{PGL}(U)$ whose kernel is precisely the group $\mathrm{Aut}(q_1, q_2)$ of automorphisms of the pair (q_1, q_2).

Note that since the zeroes of Δ correspond to the singular quadrics of the pencil, all automorphisms must leave invariant the zeroes of Δ. Thus the image of π is contained in G. It remains to show that any element $g \in \mathrm{GL}(U)$ representing an element of G can be lifted to $\mathrm{Aut}(X)$.

Since X is quasisplit, we have a normal form for the pair (q_{u_1}, q_{u_2}) where $r = 0$. We may scale our representative $g \in \mathrm{GL}(U)$ such that $g^*\Delta = \Delta$. By Lemma 4.7, there is an $h \in \mathrm{GL}(W) \times \mathrm{GL}(L)$ so that $(\mathsf{q}_{gu_0}, \mathsf{q}_{gu_1})$ has a normal form with respect to the basis $h(v_1), \ldots, h(w_m)$. Since $g^*\Delta = \Delta$, we have $\mathsf{q}_{g(u_i)}(w_j) = \mathsf{q}_{u_i}(w_j)$ for $i = 0, 1$ and $j = 0, \ldots, m$. Since L is totally isotropic, $\mathsf{q}_{g(u_i)}(v_j) = \mathsf{q}_{u_i}(v_j) = 0$ for $i = 0, 1$ and $j = 0, \ldots, m-1$. Thus the normal forms are equal and g is an automorphism as desired. □

Remark 7.8. Note in the last step of the proof, we required that $r = 0$ in order to lift automorphisms from $g \in \mathrm{PGL}(U)$ to $\mathrm{Aut}(X)$. Thus, when X is not quasisplit, the automorphism group may not surject onto G.

8. Cohomology of intersection of two quadrics in $\mathbb{P}^{2m}$

Let X be a smooth intersection of two quadrics in $\mathbb{P}^{2m}$ over an algebraically closed field of characteristic $p \geq 0$. Recall that in the case $m = 2$ when X is a del Pezzo surface of degree 4, the Picard group $\mathrm{Pic}(X)$ of algebraic 2-cycles on X is a free abelian group of rank 6 and the cycle homomorphism $\mathrm{Pic}(X)_{\mathbb{Z}_\ell} \to H^2(X, \mathbb{Z}_\ell)$ to the ℓ-adic cohomology is an isomorphism. It is also compatible with the

intersection pairing $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$ and the cup product $H^2(X, \mathbb{Z}_\ell) \times H^2(X, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell$ on the cohomology. This well-known result follows from the fact that X is isomorphic to the blowup of 5 points $p_1, \dots, p_5$ in the projective plane $\mathbb{P}^2$, no three of which are collinear. In the anticanonical model, the exceptional curves of the blowup are 5 disjoint lines on X. The remaining eleven lines come from the proper transforms of lines $\overline{p_i, p_j}$ and the conic passing through the five points.

Let $e_1, \dots, e_5$ be the classes of the exceptional curves, and e_0 be the class of the preimage of a line in the plane under the blowup. Then $\mathrm{Pic}(X)$ is freely generated by $e_0, \dots, e_5$ and the canonical class of X is equal to $-3e_0 + e_1 + \cdots + e_5$. Its orthogonal complement in $\mathrm{Pic}(X)$ is isomorphic to the negative root lattice of type D_5. The group of automorphisms of X is faithfully represented in the Weyl group of this lattice and is a subgroup to $2^4 \rtimes \mathfrak{S}_5$, where $\mathfrak{S}_5$ denotes the symmetric group on 5 letters.

In this section we extend these well known facts to the case of arbitrary m (for $\mathrm{char}(\Bbbk) \neq 2$ see [Reid 1972]). Throughout, X will be the base locus of a regular pencil q of quadrics and P will denote the projectivized radical subspace $|W|$ of $|E|$.

Fix a generator $\Lambda = |L|$ in X. Let $X_\Lambda \to X$ be the blowup of X with center at Λ. Consider the projection map $p_\Lambda : X_\Lambda \to \mathbb{P}^m$ from Λ. We have $E = W \oplus L$ so we may take p_Λ to be the projectivization of the projection $E \to W$ and thus identify $\mathbb{P}^m$ with $\mathsf{P} = |W|$. For any point $x \in X \setminus \Lambda$, $p_\Lambda(x) = \langle \Lambda, x \rangle \cap |W|$. The base locus of the restriction of the pencil q to the subspace $\langle \Lambda, x \rangle \cong \mathbb{P}^m$ contains Λ. Thus the residual pencil consists of hyperplanes in $\langle \Lambda, x \rangle$ containing x. Its base locus is a codimension 2 linear subspace of $\langle \Lambda, x \rangle$ containing x unless one of the quadrics contains $\langle \Lambda, x \rangle$ and hence the two quadrics intersect along a hyperplane Λ' in $\langle \Lambda, x \rangle$. If Λ' exists, then it is a generator in X intersecting Λ along a hyperplane. Note that $\Lambda' = \rho_i(\Lambda)$ for some reflection $\rho_i \in \mathrm{Aut}(q_0, q_1)$ and $\Lambda' \cap \Lambda$ is the intersection of the fixed hyperplane F_i of ρ_i with Λ'.

The image of each such generator under the projection map p_Λ is the point $y_i = [v_i]$ in $|W|$ which is the singular point of one of the singular quadrics in the pencil. In particular, we found $2m + 1$ generators that do not intersect each other outside Λ but each intersects Λ along a hyperplane. Their proper transforms on the blowup are disjoint subvarieties each isomorphic to $\mathbb{P}^{m-1}$. This gives the following birational picture of X.

Let $\mathbb{P}^{2m}_\Lambda \to \mathbb{P}^{2m}$ be the blowup of $\mathbb{P}^{2m}$ along Λ. The projection from Λ defines an isomorphism from $\mathbb{P}^{2m}_\Lambda$ to the projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^m}^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^m}(-1)$.

Proposition 8.1. *X_Λ is isomorphic over $\mathsf{P} \cong \mathbb{P}^m$ to a closed subvariety of the projective $(m+1)$-bundle $\mathbb{P}(\mathcal{E})$ over P. Let $\Sigma = \{y_1, \dots, y_{2m+1}\}$ be the images in $|W|$ of the vertices of singular quadrics in the pencil. If $x \notin \Sigma$ (resp. $x \in \Sigma$), then the fiber $p_\Lambda^{-1}(x)$ is a codimension 2 (resp. codimension 1) linear subspace in the fiber of the projective bundle.*

The inclusion $X_\Lambda \hookrightarrow \mathbb{P}(\mathcal{E})$ is given by a surjective homomorphism of coherent sheaves $\mathcal{E} \to \mathcal{F}$ such that $X_\Lambda \cong \mathbb{P}(\mathcal{F}) := \mathrm{Proj}\, S^\bullet(\mathcal{F})$. The set Σ is equal to the singular set $\mathrm{Sing}(\mathcal{F})$, the set of points $x \in \mathbb{P}^m$ such that $\dim_{\Bbbk}(\mathcal{F}(x)) > \mathrm{rank}(\mathcal{F})$. In our case $\mathrm{rank}(\mathcal{F}) = m - 1$ and $\dim_{\Bbbk} \mathcal{F}(x) = m$ for $x \in \mathrm{Sing}(\mathcal{F})$.

Let $U = \mathbb{P}^m \setminus \Sigma$ and $j : U \hookrightarrow \mathbb{P}^m$ be the open inclusion. Since $\operatorname{codim} \Sigma \geq 2$, the sheaf $j_* j^* \mathcal{F}$ is a locally free sheaf of rank $m-1$ and $\mathcal{F}$ is its subsheaf; in particular, it is a reflexive sheaf [Hartshorne 1980]. Thus, we obtain

Proposition 8.2. $$X_\Lambda \cong \mathbb{P}(\mathcal{F}),$$

where $\mathcal{F}$ is a reflexive sheaf of rank $m-1$ with $\operatorname{Sing}(\mathcal{F}) = \Sigma$. In particular, X is birationally equivalent to a projective $(m-2)$-bundle over $\mathbb{P}^m$.

Example 8.3. Assume $m = 2$, then p_Λ is the projection of the quartic del Pezzo surface X from a line. It is isomorphic to the blowup $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is the ideal sheaf $\mathcal{I}_\Sigma$ of the set Σ of five points in the plane.

Remark 8.4. Recall that, over an algebraically closed field, a del Pezzo surface of degree 4 is obtained by blowing up 5 general points in the plane $\mathbb{P}^2$ which lie on a unique conic $C = V(q)$, where q is a nondegenerate quadratic form. Prokhorov asked us for a geometric interpretation in the plane model of the canonical point Π from Theorem 4.4.

A natural guess is that it is the *strange point* of the conic. It has the property that any line through this point is tangent to C. This turns out to be false. In fact, our conic depends only on the pencil of the associated alternate bilinear forms. In a Kronecker basis it can be given by equation $x_1 x_3 + x_2^2 = 0$. The equation of Π depends on the coefficients $(a_0, \ldots, a_5)$ of the half-discriminant, and coincides with the strange point only in the case when $a_2 a_5 = a_3 a_4$, $a_0 a_3 = a_1 a_2$.

The following is well known when the characteristic is not 2; see Chapter 3 of [Reid 1972] for a proof over $\mathbb{C}$.

Proposition 8.5. *Let $A_{m-1}(X)$ be the Chow group of algebraic $(m-1)$-cycles on X equipped with a structure of a quadratic lattice with respect to the intersection of algebraic cycles on X. Then*

$$A_{m-1}(X) \cong \mathbb{Z}^{2m+2}.$$

Let $A_{m-1}(X)_0$ denote the orthogonal complement of K_X^{m-1} in $A_{m-1}(X)$. Then it is isomorphic to the roots lattice of type D_{2m+1} taken with the sign $(-1)^{m-1}$.

Proof. Let $U = \mathbb{P}^m \setminus \Sigma$, let $Y_\Sigma = p_\Lambda^{-1}(\Sigma)$, and let $X_U = p_\Lambda^{-1}(U)$. By Proposition 1.8 from [Fulton 1998], we have the following exact sequence of Chow groups A_k of algebraic k-cycles:

$$A_k(Y_\Sigma) \to A_k(X_\Lambda) \to A_k(X_U) \to 0.$$

For a vector bundle $\mathcal{E}$ of rank $e+1$ over a base Z, the Chow groups of the projective bundle $\mathbb{P}(\mathcal{E})$ are well known. By Theorem 3.3 of the same work, we have

$$A_k(\mathbb{P}(\mathcal{E})) = \bigoplus_{i=0}^{e} A_{k-e+i}(Z).$$

It follows that

$$A_{m-1}(X_U) \cong \mathbb{Z}^{m-1}, \quad A_{m-1}(Y_\Sigma) \cong \mathbb{Z}^{2m+1}.$$

The homomorphism $A_{m-1}(Y_\Sigma) \to A_{m-1}(X_\Sigma)$ is injective because Y_Σ is the union of disjoint subvarieties of dimension $m-1$. This shows that $A_{m-1}(X_\Lambda) \cong \mathbb{Z}^{3m}$. It remains to use [Fulton 1998], Proposition 6.7(e) that computes the cohomology of the blowup. We have $A_{m-1}(X_\Lambda) = A_{m-1}(X) \oplus \mathbb{Z}^{m-2}$. This gives $A_{m-1}(X) \cong \mathbb{Z}^{2m+2}$.

The rest of the arguments can be borrowed from Chapter 3 of [Reid 1972] since they do not depend on the characteristic of the ground field. We include this for completeness sake. First, we fix one generator Λ and use Corollary 7.5 to index generators by subsets of the set $[1, 2m+1] = \{1, 2, \dots, 2m+1\}$ modulo taking the complementary subset. Here $\Lambda_\varnothing := \Lambda$ is the generator corresponding to the empty set, and each $\Lambda_{\{i\}} := \rho_i(\Lambda)$ is the generator corresponding to the image of Λ under the reflection ρ_i.

We equip $A_{m-1}(X)$ with a structure of a nondegenerate quadratic lattice with respect to the intersection pairing $A_{m-1}(X) \times A_{m-1}(X) \to \mathbb{Z}$. We have

$$[\Lambda_I] \cdot [\Lambda_J] = (-1)^r \Big(\Big\lfloor \frac{r}{2} \Big\rfloor + 1 \Big),$$

where $r = \dim \Lambda_I \cap \Lambda_J$. In particular,

$$[\Lambda_I]^2 = (-1)^{m-1} \Big(\Big\lfloor \frac{m-1}{2} \Big\rfloor + 1 \Big) \tag{8-1}$$

and, if $\dim \Lambda_I \cap \Lambda_J = m-3$,

$$([\Lambda_I] - [\Lambda_J])^2 = 2(-1)^{m-1}. \tag{8-2}$$

Let η be the class of a hyperplane section of X. We have

$$\eta^{m-1} \cdot [\Lambda_I] = 1.$$

Since $K_X = (3-2m)\eta$, we have $K_X^{m-1} = (3-2m)^{m-1}\eta^{m-1}$, hence

$$A_{m-1}(X)_0 := (K_X^{m-1})^\perp = (\eta^{m-1})^\perp.$$

We may assume that the generators Λ_i, $i = 1, \dots, 2m+1$, are the preimages of the points $y_i \in \Sigma$ under the projection map $p_\Lambda : X_\Lambda \to |W|$. Let $e_i = [\Lambda_i]$ be their classes in $A_{m-1}(X)$. Let

$$e_0 = \eta^{m-1} - [\Lambda_\varnothing]$$

(note that in the case $m = 2$ it coincides with the class of the pre-image of a line in $\mathbb{P}^2$). One checks that the classes $e_0, e_1, \dots, e_{2m+1}$ freely generate $A_{m-1}(X)$ and the classes

$$\alpha_0 = -e_0 + [\Lambda_\varnothing] + e_{2m} + e_{2m+1}, \quad \alpha_i = e_i - e_{i+1},\ i = 1, \dots 2m,$$

freely generate $A_{m-1}(X)_0$. In fact they form a root basis of type D_{2m+1} in $A_{m-1}(X)_0$ (taken with the sign $(-1)^{m-1}$). □

Recall that the Weyl group $W(D_n)$ of the root lattice of type D_n generated by reflections in simple roots α_i is isomorphic to $2^{n-1} \rtimes \mathfrak{S}_n$. The group $\mathrm{Aut}(X)$ acts naturally on $A_{m-1}(X)$ leaving K_X invariant.

This defines a homomorphism

$$\rho : \mathrm{Aut}(X) \to \mathrm{Or}(A_{m-1}(X)_0) \cong \mathrm{Or}(\mathsf{D}_n)$$

to the orthogonal group of the quadratic lattice D_n defined by the Cartan matrix of the root system of type D_n. The image is contained in the subgroup of $\mathrm{Or}(\mathsf{D}_n)$ of isometries that can be lifted to $A_{m-1}(X)$. It follows from the theory of quadratic lattices that this subgroup is of index 2 and coincides with the Weyl group $W(D_n)$. Thus, we have defined a homomorphism

$$\rho : \mathrm{Aut}(X) \to W(D_n) \cong 2^{n-1} \rtimes \mathfrak{S}_n.$$

Theorem 8.6. *The homomorphism ρ is injective and sends the subgroup $\mathrm{Aut}(q_0, q_1)$ to the subgroup 2^{n-1} of $W(D_n)$.*

Proof. An element g of the kernel fixes all generators. Assume $p \neq 2$. We know that any generator Λ intersects $n = 2m+1$ other generators that intersect Λ along a hyperplane. We know from the description of the projection map $p_\Lambda : X_\Lambda \to |W|$ given in the beginning of this section that Λ intersects n generators Λ_i that are projected to the singular points $y_i \in |W|$ of n singular fibers from the pencil. Thus g in its action in $|W|$ fixes these points, and since they generate $|W|$ fixes $|W|$ pointwise. Thus the projection of g to G from Theorem 7.7 is the identity, hence g is the identity. □

Acknowledgments

Duncan would like to thank D. Leep for helpful conversations and A. Efimov for suggesting the interpretation via the Arf invariant used in Section 6. Both authors thank A. Skorobogatov and an anonymous referee for comments.

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Communicated by Raman Parimala
Received 2017-02-01 Revised 2017-09-18 Accepted 2017-11-08

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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Algebra & Number Theory

Volume 12 No. 1 2018