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For a cycle of codimension 1 in a toric variety, its degree with respect to a nef toric divisor can be understood in terms of the mixed volume of the polytopes associated to the divisor and to the cycle. We prove here that an analogous combinatorial formula holds in the arithmetic setting: the global height of a 1-codimensional cycle with respect to a toric divisor equipped with a semipositive toric metric can be expressed in terms of mixed integrals of the v-adic roof functions associated to the metric and the Legendre–Fenchel dual of the v-adic Ronkin function of the Laurent polynomial of the cycle.

Introduction

The arithmetic intersection theory of toric varieties with respect to toric line bundles equipped with their canonical metric was first studied by Maillot [2000]. Later, the systematic extension of the toric dictionary to Arakelov geometry was carried out by Burgos Gil, Philippon and Sombra [Burgos Gil et al. 2014]. It turns out from their study that suitably metrized toric line bundles can be expressed in terms of families of concave functions on convex polytopes and that the height of the toric variety with respect to this choice is related to the integral of such functions. Their theory allows one to treat a large spectrum of height functions, namely the ones arising from toric line bundles equipped with toric metrics; this includes the canonical heights studied by Maillot and the Fubini–Study height. On the other hand, the techniques developed in [Burgos Gil et al. 2014] only apply to the computation of the height of toric subvarieties and do not solve, for instance, the problem of determining the height of a general cycle of codimension 1. This question was answered in a very special case by [Maillot 2000], where a relation between the canonical height of a hypersurface in a smooth projective toric variety and the Mahler measure of the corresponding

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polynomial was given. Other computations have been performed by Cassaigne and Maillot [2000] for the Fubini–Study height of hypersurfaces in projective spaces. Extending the techniques of [Burgos Gil et al. 2014], we give here a combinatorial formula for the height of a 1-codimensional cycle in a toric variety for a much more general choice of metrics.

For the sake of simplicity, we restrict for the moment to the case of an ambient proper toric variety X_Σ of dimension n over $\mathbb{Q}$, leaving the treatment of the case of an arbitrary base adelic field to the body of the paper. Let $\mathfrak{M}$ stand for the set of places of $\mathbb{Q}$. As usual in toric geometry, we denote by M the lattice of characters of the torus of X_Σ, by N its dual lattice, and by $M_\mathbb{R}$ and $N_\mathbb{R}$ the corresponding real vector spaces. We are interested in a combinatorial expression for the height of a cycle of codimension 1 in X_Σ with respect to a suitable choice of a metrized (Cartier) divisor. By the linearity of height functions, we can restrict to the case of an irreducible hypersurface Y. Moreover, since irreducible hypersurfaces in X_Σ not intersecting its dense open torus have to coincide with 1-codimensional toric orbits, whose heights have already been calculated in [Burgos Gil et al. 2014, Proposition 5.1.11], we can assume that the generic point of Y lies in the dense open orbit of X_Σ. Under this assumption, Y is described by an irreducible Laurent polynomial f with rational coefficients. Its Newton polytope $\mathrm{NP}(f)$ is a nonempty subset of $M_\mathbb{R}$ capturing enough information for the intersection-theoretical properties of Y. For instance, Proposition 5.2 implies that the degree of Y with respect to a toric divisor D on X_Σ generated by its global sections is given by

$$\deg_D(Y) = \mathrm{MV}_M(\Delta, \dots, \Delta, \mathrm{NP}(f)),$$

where Δ is the polytope in $M_\mathbb{R}$ associated to D and MV_M denotes the mixed volume of convex bodies in $M_\mathbb{R}$ with respect to a suitably normalized Haar measure.

The height of a cycle in X is the arithmetic counterpart of its degree with respect to a divisor D. Its definition requires as an extra datum the choice of an adelic semipositive metric on D; see Section 3 for a precise definition. To have a combinatorial description of heights in toric varieties, it is necessary to ask D to be a toric divisor (with associated polytope Δ) and the metric on it to be "toric invariant" in some sense. In such a situation, Burgos Gil, Philippon and Sombra have shown that a combinatorial description is possible, translating the additional information of the metric into an extra dimension on the convex geometrical side: an adelic semipositive toric metric on D is associated to a family $(\vartheta_v)_{v\in\mathfrak{M}}$ of continuous concave functions on Δ, called the *roof functions* of the metric, such that $\vartheta_v = 0$ for all but finitely many v. We show how the height of Y with respect to the adelic semipositive toric metrized divisor $\overline{D}$ can be expressed using such an extra-dimensional representation, in a spirit analogous to the formula for its degree mentioned above. The key idea consists in associating to the polynomial f defining Y, for every place v of $\mathbb{Q}$, a suitable function which we call the *v-adic Ronkin function* of f and denote by $\rho_{f,v}$. It is a concave function on $N_\mathbb{R}$ whose value at u can be interpreted as an average of $-\log|f|$ on the fiber of the tropicalization map over u. When v is archimedean, it is the Ronkin function studied by Passare and Rullgård among others, while for nonarchimedean places it coincides with the v-adic tropicalization of the polynomial f. Its Legendre–Fenchel dual $\rho_{f,v}^\vee$ is a concave function on $M_\mathbb{R}$ which is supported on the

Newton polytope of f. Recall now that Philippon and Sombra [2008a] introduced a polarized version of the integration of a concave function with bounded support, called the *mixed integral*. For the choice of a suitably normalized Haar measure on the vector space $M_{\mathbb{R}}$, it is a multilinear symmetric real-valued function MI_M taking as entries $n+1$ concave functions supported on convex bodies in $M_{\mathbb{R}}$.

Theorem 1. *The height of Y with respect to $\overline{D}$ is given by*

$$h_{\overline{D}}(Y) = \sum_{v \in \mathfrak{M}} \mathrm{MI}_M(\vartheta_v, \ldots, \vartheta_v, \rho_{f,v}^{\vee}).$$

Despite the complexity of the computation of the archimedean Ronkin function, the formula in the previous theorem clarifies the relation between the defining polynomial of an irreducible hypersurface and its height with respect to an adelic semipositive toric metrized divisor. It is easy to specialize it to the case of the canonical metric on D, where it reduces to the equality proved in [Maillot 2000], or of the Fubini–Study metric in the projective setting. We hope that a better understanding of the properties of mixed integrals and archimedean Ronkin functions could be used to deduce both lower and upper bounds for the height of Y. More importantly, our result asserts that the collection of the v-adic Ronkin functions of a hypersurface contains enough information to determine its height; we wonder whether other arithmetical properties of Y might be read in terms of such functions.

To show the stated result, we prove more precise formulas for the local height and the toric local height of a 1-codimensional cycle. We also show some new properties of mixed integrals and we propose a more uniform definition and study of v-adic Ronkin functions which is independent of whether the place v is archimedean or not. The obtained formulas for the height extend to the case of admissible adelic toric metrized divisors as alternated sums of mixed integrals, as in [Burgos Gil et al. 2014, Remark 5.1.10].

For an arbitrary adelic base field K, we remark that one needs to prove that the global height of Y with respect to an adelic semipositive toric metrized divisor is a finite sum and hence well-defined. This is automatic if K is a global field, because of [loc. cit., Proposition 1.5.14 and Theorem 4.9.3]. We show it here for an arbitrary adelic field K with product formula, in which case the formulas for the height stay true. In the more general setting of an adelic field K not satisfying the product formula, it is easy to verify that the same equality for the global height holds up to the sum by the *defect* of K; see [loc. cit., Definition 1.5.9]. Finally, the recent work [Gubler and Hertel 2017] suggests that similar statements might hold for a base M-field.

We now briefly summarize the content of each section.

In Section 1, we recall the tools from convex geometry which are needed throughout the paper: Legendre–Fenchel duality of concave functions, real Monge–Ampère measures and mixed integrals. In particular, we reinterpret the recursive formula for mixed integrals proved by Philippon and Sombra in terms of mixed real Monge–Ampère measures. We then make use of it to deduce two elementary, though useful, properties of such operators. Finally, we describe what happens when one of the functions appearing in the mixed integral is the indicator function of a line segment.

Section 2 deals with the key object of our work: v-adic Ronkin functions of Laurent polynomials, which are introduced and described after recalling the needed preliminaries in tropical and nonarchimedean geometry. In this context, the discussion of a notion of minimal boundaries allows one to treat the archimedean and nonarchimedean cases homogeneously.

In Section 3 we briefly recall the general adelic Arakelov framework and focus then on the results obtained by Burgos Gil, Philippon and Sombra in the toric setting. To keep the treatment of archimedean and nonarchimedean places on equal footing, we rephrase their description of the Chambert-Loir measure of semipositive toric metrized divisors in terms of minimal boundaries of tropical fibers.

As a needed step for the main proof, we combinatorially describe the Weil divisor of the rational function defined by a Laurent polynomial on a toric variety. This result can be of independent interest and has thus been set aside in Section 4.

Section 5 is dedicated to the proofs of our main results Theorems 5.9 and 5.12, which are formulas for the local height and the toric local height of cycles of codimension 1 in toric varieties. We then make use of them to prove the integrability statement and a formula for their global heights, with respect to the choice of adelic semipositive toric metrized divisors.

For binomial hypersurfaces, such a formula is compatible with the one deduced from [Burgos Gil et al. 2014]. This is shown in Section 6, where we also apply our results to some other particular cases. We provide convex geometrical formulas for the canonical height of 1-codimensional cycles, obtaining the quoted result by Maillot, and for the Fubini–Study height of a projective hypersurface. We also propose a new height function, the ρ-height, for which we give a compact formula. We do not know any application of such a height, which could be anyway worth studying.

Terminology and notation. A *variety* X is assumed to be a reduced and irreducible separated scheme of finite type over a field. By an *irreducible hypersurface* in it we mean a closed integral subscheme of codimension 1 in X. A *divisor* on X is a Cartier divisor, unless otherwise stated. Toric varieties are assumed to be normal; whenever the choice of the base field K is clear from the context, the notation X_Σ will refer to the toric variety over K associated to the fan Σ.

The term *measure* on a topological space stands for a signed Borel measure on it; in particular, measures admit a well-defined push-forward via continuous mappings. A measure which only takes nonnegative real values on Borel subsets is called a *positive measure*.

1. Preliminaries in convex geometry

This section is devoted to recalling notions from convex geometry that will be useful in the sequel. We follow the conventions and notation of [Burgos Gil et al. 2014, Chapter 2], referring to [Rockafellar 1970, §12] for a more complete treatment of the subject. We refer to these two sources for the proofs of the statements we make here.

For the whole section, let N be a lattice of rank n and $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual lattice. Denote by $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and by $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ the corresponding n-dimensional real vector spaces.

By a *polyhedron* in $N_{\mathbb{R}}$ we mean a convex subset of $N_{\mathbb{R}}$ obtained as the intersection of finitely many closed half-spaces $\{u \in N_{\mathbb{R}} : \langle x, u\rangle + c \geq 0\}$, with $x \in M_{\mathbb{R}}$ and $c \in \mathbb{R}$. If all the slopes x can be chosen in M, the polyhedron is said to be *rational*. A *polytope* is a bounded polyhedron. A polytope in $N_{\mathbb{R}}$ whose vertices all lie in N is called a *lattice polytope*; it is in particular a rational polytope. A compact convex subset of $N_{\mathbb{R}}$ is called a *convex body*.

1A. *Legendre–Fenchel duality.* A function $f : N_{\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ is said to be *concave* if it is not identically $-\infty$ and, for every $u_1, u_2 \in N_{\mathbb{R}}$ and for every $t \in [0, 1]$, one has the inequality

$$f(tu_1 + (1-t)u_2) \geq tf(u_1) + (1-t)f(u_2).$$

The *effective domain* of a concave function f is the set on which the function takes values different from $-\infty$ and it is denoted by $\mathrm{dom}(f)$: it is a convex subset of $N_{\mathbb{R}}$. A concave function is said to be *closed* if it is upper semicontinuous. Every concave function with closed effective domain, on which it is continuous, is closed. The *recession function* of a closed concave function f is the concave conical function which takes on $u \in N_{\mathbb{R}}$ the value

$$\mathrm{rec}(f)(u) := \lim_{\lambda\to\infty} \frac{f(v_0 + \lambda u)}{\lambda} \in \mathbb{R} \cup \{-\infty\}$$

for any $v_0 \in \mathrm{dom}(f)$; see [Rockafellar 1970, Theorem 8.5]. Finally, a concave function f with effective domain a polyhedron in $N_{\mathbb{R}}$ is *piecewise affine* if

$$f(u) = \min_{\alpha\in S}(\langle\alpha, u\rangle + c_\alpha)$$

for every $u \in \mathrm{dom}(f)$, with S a finite subset of $M_{\mathbb{R}}$ and $c_\alpha \in \mathbb{R}$ for every $\alpha \in S$.

To each concave function f on $N_{\mathbb{R}}$, one can associate its *Legendre–Fenchel dual*, which is the closed concave function $f^\vee$ on $M_{\mathbb{R}}$ defined as

$$f^\vee(x) := \inf_{u\in N_{\mathbb{R}}} (\langle x, u\rangle - f(u))$$

for every $x \in M_{\mathbb{R}}$; see [Burgos Gil et al. 2014, §2.2]. If f is closed, $(f^\vee)^\vee = f$. The effective domain of $f^\vee$ is a convex subset of $M_{\mathbb{R}}$, which one calls the *stability set of f* and denotes by $\mathrm{stab}(f)$.

The following example is classical and will play a role later on.

Example 1.1. Any nonempty convex body B in $M_{\mathbb{R}}$ induces a concave function Ψ_B on $N_{\mathbb{R}}$, called the *support function* of B and defined as

$$\Psi_B(u) := \min_{x\in B}\langle x, u\rangle$$

for every $u \in N_{\mathbb{R}}$. Its Legendre–Fenchel dual is the *indicator function ι_B of B*, which is the function taking the value 0 on B and $-\infty$ elsewhere. Hence, $\mathrm{dom}(\Psi_B) = N_{\mathbb{R}}$ and $\mathrm{stab}(\Psi_B) = B$. Notice that, whenever B is a polytope, Ψ_B is a conic piecewise affine concave function.

We also recall that there exist a number of operations that one can define on concave functions, in addition to the usual pointwise sum and scalar multiplication. Among these, the *sup-convolution* of two concave functions f and g on $N_{\mathbb{R}}$ with nondisjoint stability sets is defined as

$$(f \boxplus g)(v) := \sup_{u_1+u_2=v} (f(u_1)+g(u_2))$$

and the *right scalar multiplication* of f by $\lambda \in \mathbb{R}_{\geq 0}$ as

$$(f\lambda)(u) := \lambda f(u/\lambda).$$

Also, the *translate* of a concave function f on $N_{\mathbb{R}}$ by a point $u_0 \in N_{\mathbb{R}}$ is set to be

$$(\tau_{u_0} f)(u) := f(u-u_0).$$

These operations are dual, via Legendre–Fenchel duality, to the usual pointwise addition, scalar multiplication and sum by a linear function, respectively; see [Burgos Gil et al. 2014, Propositions 2.3.1 and 2.3.3].

1B. *Real Monge–Ampère measures.* For any closed concave function f on $N_{\mathbb{R}}$ with $\mathrm{dom}(f) = N_{\mathbb{R}}$ and for any Haar measure μ on $M_{\mathbb{R}}$, one can define a corresponding *real Monge–Ampère measure* $\mathcal{M}_\mu(f)$, as in [Burgos Gil et al. 2014, §2.7]. It is a measure on $N_{\mathbb{R}}$, of total mass $\mu(\mathrm{stab}(f))$, being supported on finitely many points if f is piecewise affine. The Monge–Ampère operator, associating to each closed concave function f with $\mathrm{dom}(f) = N_{\mathbb{R}}$ the corresponding measure $\mathcal{M}_\mu(f)$ on $N_{\mathbb{R}}$ is homogeneous of degree n with respect to pointwise scalar multiplication. It was shown in [Passare and Rullgård 2004] that such an operator admits a polarization: for $f_1, \ldots, f_n$ closed concave functions with effective domain $N_{\mathbb{R}}$, their *mixed real Monge–Ampère measure* is defined as the measure

$$\mathrm{M}\mathcal{M}_\mu(f_1, \ldots, f_n) := \sum_{k=1}^{n} (-1)^{n-k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathcal{M}_\mu(f_{i_1} + \cdots + f_{i_k}). \tag{1-1}$$

Notice that this definition differs from [Passare and Rullgård 2004, formula (14)] and [Burgos Gil et al. 2014, Definition 2.7.12] by a multiplicative constant. It follows from [Passare and Rullgård 2004, §5] that the measure in (1-1) is in fact a positive measure. The so-obtained mixed Monge–Ampère operator is, by definition, symmetric and multilinear in its entries (with respect to pointwise sum) and it satisfies

$$\mathrm{M}\mathcal{M}_\mu(f, \ldots, f) = n!\, \mathcal{M}_\mu(f)$$

for every closed concave function f. In particular, if $f_1, \ldots, f_n$ are closed concave functions with convex bodies as stability sets, their mixed real Monge–Ampère measure is a finite measure on $N_{\mathbb{R}}$ of total mass $\mathrm{MV}_\mu(\mathrm{stab}(f_1), \ldots, \mathrm{stab}(f_n))$. Here, MV_μ denotes the mixed volume of convex bodies with respect to the measure μ, normalized in such a way that $\mathrm{MV}_\mu(Q, \ldots, Q) = n!\, \mu(Q)$ for every convex body Q in $M_{\mathbb{R}}$.

Remark 1.2. The integral structure on $M_{\mathbb{R}}$ coming from the subjacent lattice M gives a distinguished measure vol_M on $M_{\mathbb{R}}$, which is the unique Haar measure for which the volume of a fundamental domain of M is 1 (this does not depend on the choice of a basis of M). To lighten the notation, the corresponding (mixed) real Monge–Ampère operator will be denoted by $\mathrm{M}\mathcal{M}_M$.

1C. *Mixed integrals.* For any Haar measure μ on $M_{\mathbb{R}}$, the application mapping a compactly supported concave function g on $M_{\mathbb{R}}$ to its Lebesgue integral with respect to μ is homogeneous of degree $n+1$ with respect to right scalar multiplication. As a consequence of [Philippon and Sombra 2008a, Proposition 4.5], there exists a polarized operator: for $g_0, \dots, g_n$ concave functions on $M_{\mathbb{R}}$ with respective domains the convex bodies $Q_0, \dots, Q_n$, their *mixed integral* with respect to μ is defined as

$$\mathrm{MI}_{\mu}(g_0, \dots, g_n) := \sum_{k=0}^{n} (-1)^{n-k} \sum_{0 \le i_0 < \cdots < i_k \le n} \int_{Q_{i_0} + \cdots + Q_{i_k}} (g_{i_0} \boxplus \cdots \boxplus g_{i_k})\, d\mu.$$

This notion was introduced and studied in [Philippon and Sombra 2008a; 2008b]. The mixed integral operator is symmetric and multilinear in its entries (with respect to sup-convolution) and it satisfies

$$\mathrm{MI}_{\mu}(g, \dots, g) = (n+1)! \int_{Q} g\, d\mu$$

for every concave function g on a convex body Q. As for the mixed Monge–Ampère operator, we will denote by MI_M the mixed integral computed with respect to the Haar measure vol_M on $M_{\mathbb{R}}$.

The following proposition describes the behavior of mixed integrals with respect to translation of the entries.

Proposition 1.3. *Let g_i be a concave function defined on a convex body Q_i in $M_{\mathbb{R}}$ for $i = 0, \dots, n$. Let also $x_0 \in M_{\mathbb{R}}$. Then,*

$$\mathrm{MI}_{\mu}(\tau_{x_0} g_0, g_1, \dots, g_n) = \mathrm{MI}_{\mu}(g_0, \dots, g_n).$$

Proof. For any subset $\{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$ one has, directly by definition, that

$$((\tau_{x_0} g_0) \boxplus g_{i_1} \boxplus \cdots \boxplus g_{i_r})(x + x_0) = (g_0 \boxplus g_{i_1} \boxplus \cdots \boxplus g_{i_r})(x)$$

for every $x \in M_{\mathbb{R}}$. The change of variables formula implies then that the integrals appearing in the definitions of $\mathrm{MI}_{\mu}(\tau_{x_0} g_0, g_1, \dots, g_n)$ and of $\mathrm{MI}_{\mu}(g_0, \dots, g_n)$ are pairwise equal, from which the statement follows trivially. □

We now focus on a recursive formula for mixed integrals. For any closed convex subset C in $M_{\mathbb{R}}$, and for every $u \in N_{\mathbb{R}}$, one can consider the subset

$$C^u := \bigl\{ x \in C : \langle x, u \rangle = \inf_{y \in C} \langle y, u \rangle \bigr\}$$

of C. For $u \neq 0$, this subset is contained in an affine subspace of $M_{\mathbb{R}}$ of codimension 1 and parallel to $u^{\perp} := \{x \in M_{\mathbb{R}} : \langle x, u \rangle = 0\}$. It is immediate from the definition that for every pair of convex bodies C_1 and C_2 in $M_{\mathbb{R}}$, and for every $u \in N_{\mathbb{R}}$, one has $C_1^u + C_2^u = (C_1 + C_2)^u$, where the plus sign denotes the Minkowski sum of convex sets. When $u \in N_{\mathbb{Q}} \setminus \{0\}$, the intersection $M(u) := M \cap u^{\perp}$ is a lattice of rank $n-1$ spanning the linear space $u^{\perp}$ and hence it induces a normalized Haar measure $\mathrm{vol}_{M(u)}$ on $u^{\perp}$, as in Remark 1.2.

Remark 1.4. Let $B_1, \dots, B_n$ be n convex bodies in $M_{\mathbb{R}}$ and let $u \in N_{\mathbb{R}}$. The invariance under translation of $\mathrm{vol}_{M(u)}$ allows one to consistently consider the mixed volume of $B_1^u, \dots, B_n^u$ as convex bodies in $u^{\perp}$.

Similarly, let $g_0, \dots, g_n$ be concave functions defined on convex bodies $B_0, \dots, B_n$ in $M_{\mathbb{R}}$, respectively. For every $u \in N_{\mathbb{Q}} \setminus \{0\}$, Proposition 1.3 allows one to consider the mixed integral with respect to $\mathrm{vol}_{M(u)}$ of $g_0|_{B_0^u}, \dots, g_n|_{B_n^u}$, as functions defined on convex subsets of $u^{\perp}$.

For a concave function g with effective domain a polytope Q in $M_{\mathbb{R}}$, its *hypograph* is the closed convex set

$$\Gamma(g) := \{(x,t) : x \in Q, t \leq g(x)\} \subseteq M_{\mathbb{R}} \times \mathbb{R}.$$

With the pairing between $N_{\mathbb{R}} \times \mathbb{R}$ and $M_{\mathbb{R}} \times \mathbb{R}$ given by $\langle (x,t), (u,\lambda)\rangle := \langle x, u\rangle + t\lambda$ for every $(u,\lambda) \in N_{\mathbb{R}} \times \mathbb{R}$ and $(x,t) \in M_{\mathbb{R}} \times \mathbb{R}$, one can consider the subset $\Gamma(g)^{(u,\lambda)}$ of $\Gamma(g)$ for every $(u,\lambda) \in N_{\mathbb{R}} \times \mathbb{R}$. It is empty when $\lambda > 0$.

The mixed real Monge–Ampère measure of piecewise affine concave functions can be made explicit in terms of the hypographs of their Legendre–Fenchel duals. We denote by δ_v the Dirac measure supported on v.

Proposition 1.5. *For $i = 1, \dots, n$, let g_i be a piecewise affine concave function with effective domain a polytope $Q_i \subset M_{\mathbb{R}}$, and Γ_i the hypograph of g_i. Denote by $\pi : M_{\mathbb{R}} \times \mathbb{R} \to M_{\mathbb{R}}$ the projection onto the first factor. For any choice of a Haar measure μ on $M_{\mathbb{R}}$ one has*

$$\mathrm{M}\mathcal{M}_{\mu}(g_1^{\vee}, \dots, g_n^{\vee}) = \sum_{v \in N_{\mathbb{R}}} \mathrm{MV}_{\mu}(\pi(\Gamma_1^{(v,-1)}), \dots, \pi(\Gamma_n^{(v,-1)}))\, \delta_v,$$

and the sum is finite.

Proof. For a piecewise affine concave function g with bounded domain in $M_{\mathbb{R}}$, its Legendre–Fenchel dual is a piecewise affine concave function with domain $N_{\mathbb{R}}$ and [Burgos Gil et al. 2014, Proposition 2.7.4] affirms that

$$\mathcal{M}_{\mu}(g^{\vee}) = \sum_{v \in N_{\mathbb{R}}} \mu(v^*)\delta_v,$$

with $v^* = \{x \in M_{\mathbb{R}} : g^{\vee}(v) = \langle x, v\rangle - g(x)\}$. From the definition of the Legendre–Fenchel duality, one has hence that

$$v^* = \Big\{x \in M_{\mathbb{R}} : \langle x, v\rangle - g(x) = \min_{y \in M_{\mathbb{R}}} (\langle y, v\rangle - g(y))\Big\} = \{x \in M_{\mathbb{R}} : (x, g(x)) \in \Gamma(g)^{(v,-1)}\},$$

and so

$$\mathcal{M}_{\mu}(g^{\vee}) = \sum_{v \in N_{\mathbb{R}}} \mu(\pi(\Gamma(g)^{(v,-1)}))\delta_v.$$

The sum is moreover supported on finitely many $v \in N_{\mathbb{R}}$, corresponding to the directions of the finitely many exposed faces of $\Gamma(g)$.

By [Attouch and Wets 1989, §2], the relation

$$\Gamma(g_i \boxplus g_j) = \Gamma(g_i) + \Gamma(g_j)$$

on the hypographs of g_i and g_j holds for any $i, j \in \{1, \dots, n\}$. As a consequence, for every subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, [Burgos Gil et al. 2014, Proposition 2.3.1] and the linearity of π yield

$$\begin{aligned}\mathcal{M}_\mu(g_{i_1}^\vee + \cdots + g_{i_k}^\vee) &= \mathcal{M}_\mu((g_{i_1} \boxplus \cdots \boxplus g_{i_k})^\vee)\\ &= \sum_{v \in N_\mathbb{R}} \mu(\pi(\Gamma_{i_1}^{(v,-1)} + \cdots + \Gamma_{i_k}^{(v,-1)}))\delta_v\\ &= \sum_{v \in N_\mathbb{R}} \mu(\pi(\Gamma_{i_1}^{(v,-1)}) + \cdots + \pi(\Gamma_{i_k}^{(v,-1)}))\delta_v,\end{aligned}$$

and the sum is finite. The statement follows then from the definition of the mixed real Monge–Ampère measure, rearranging the terms. □

We can now prove a recursive formula relating the notions of the mixed real Monge–Ampère measure and the mixed integral of concave functions, via Legendre–Fenchel duality. A vector $u \in N$ is said to be *primitive* if it is nonzero and there is no other element $u' \in N$ such that $ku' = u$ for some positive integer k.

Theorem 1.6. *For $i = 0, \dots, n$, let g_i be a continuous concave function on a rational polytope Q_i in $M_\mathbb{R}$. Then*

$$\mathrm{MI}_M(g_0, \dots, g_n) = -\sum_{\substack{u \in N\\ \text{primitive}}} \Psi_{Q_0}(u)\,\mathrm{MI}_{M(u)}(g_1|_{Q_1^u}, \dots, g_n|_{Q_n^u}) - \int_{N_\mathbb{R}} g_0^\vee \, d\,\mathrm{M}\mathcal{M}_M(g_1^\vee, \dots, g_n^\vee),$$

the first sum being finite.

In particular, if g_i is a piecewise affine concave function on Q_i with hypograph Γ_i for any $i = 0, \dots, n$, denoting by $\pi : M_\mathbb{R} \times \mathbb{R} \to M_\mathbb{R}$ the projection onto the first factor, one has

$$\begin{aligned}&\mathrm{MI}_M(g_0, \dots, g_n)\\ &\quad= -\sum_{\substack{u \in N\\ \text{primitive}}} \Psi_{Q_0}(u)\,\mathrm{MI}_{M(u)}(g_1|_{Q_1^u}, \dots, g_n|_{Q_n^u}) - \sum_{v \in N_\mathbb{R}} g_0^\vee(v)\,\mathrm{MV}_M(\pi(\Gamma_1^{(v,-1)}), \dots, \pi(\Gamma_n^{(v,-1)})).\end{aligned}$$

Proof. By [Burgos Gil et al. 2014, Proposition 2.5.23(1)], any continuous concave function on a polytope can be approximated, with respect to uniform convergence, by a sequence of piecewise affine concave functions on the polytope itself. On the other hand, the Legendre–Fenchel duality and the real Monge–Ampère operator are continuous with respect to uniform limits of concave functions; see [Burgos Gil et al. 2014, Proposition 2.2.3] and [Rauch and Taylor 1977, §3], respectively. It is not difficult to show that the same holds for mixed integrals. Thanks to Proposition 1.5, it is hence enough to prove the formula in the particular case of $g_0, \dots, g_n$ being piecewise affine concave functions.

Let hence g_i be a concave piecewise affine function on the rational polytope Q_i in $M_\mathbb{R}$, and Γ_i its hypograph, for $i = 0, \dots, n$. The choice of a basis of N (and of the dual basis of M) endows $N_\mathbb{R}$ and $M_\mathbb{R}$ with a euclidean structure, allowing one to consider the sets

$$\mathbb{S}^{n-1} := \{w \in N_\mathbb{R} : \|w\| = 1\} \subseteq N_\mathbb{R}$$

and

$$\mathbb{S}^n_- := \{(v,t) \in N_\mathbb{R} \times \mathbb{R} : \|(v,t)\| = 1, t < 0\} \subseteq N_\mathbb{R} \times \mathbb{R}.$$

After a change of sign due to the use of different notation, [Philippon and Sombra 2008b, Proposition 8.5] affirms that

$$\mathrm{MI}_M(g_0,\dots,g_n) = -\sum_{w\in\mathbb{S}^{n-1}} \Psi_{Q_0}(w)\,\mathrm{MI}_{n-1}(g_1|_{Q_1^w},\dots,g_n|_{Q_n^w}) - \sum_{r\in\mathbb{S}^n_-} \Psi_{\Gamma_0}(r)\ \mathrm{MV}_n(\Gamma_1^r,\dots,\Gamma_n^r), \quad (1\text{-}2)$$

where, on the right-hand side, one refers to the mixed integral with respect to the measure obtained restricting vol_M to $w^\perp$ and to the mixed volume with respect to the restriction of $\mathrm{vol}_{M\oplus\mathbb{Z}}$ to $r^\perp$.

Concerning the first sum on the right-hand side of (1-2), if a term in the sum is different from zero, then there exists a subset $I \subset \{1,\dots,n\}$ such that the Minkowski sum of Q_i^w, with $i \in I$, is of dimension $n-1$; in particular, if one sets $Q := Q_1 + \cdots + Q_n$, then $Q^w = Q_1^w + \cdots + Q_n^w$ needs to be of dimension $n-1$. As a consequence, one can restrict the sum to the set of vectors $w \in \mathbb{S}^{n-1}$ for which Q^w is an $(n-1)$-dimensional face of Q. This set is included in the set of vectors of unitary length which are perpendicular to an $(n-1)$-dimensional face of Q; hence it is finite since Q is a polytope. Moreover, since Q is rational, the ray spanned by such a vector w contains a unique primitive vector $u \in N$. The linearity of Ψ_{Q_0} yields hence the equality

$$\sum_{w\in\mathbb{S}^{n-1}} \Psi_{Q_0}(w)\,\mathrm{MI}_{n-1}(g_1|_{Q_1^w},\dots,g_n|_{Q_n^w}) = \sum_{\substack{u\in N\\ \text{primitive}}} \frac{\Psi_{Q_0}(u)}{\|u\|}\,\mathrm{MI}_{n-1}(g_1|_{Q_1^u},\dots,g_n|_{Q_n^u}).$$

The fact that the restriction of vol_M to $u^\perp$ is equal to the measure $\mathrm{vol}_{M(u)}$ multiplied by $\|u\|$, see [Burgos Gil et al. 2014, proof of Corollary 2.7.10], allows one to conclude that the first sum in (1-2) coincides with the desired one.

Regarding the second sum in (1-2), there exists an obvious bijection between $\mathbb{S}^n_-$ and $N_\mathbb{R}$ given by associating to each $r \in \mathbb{S}^n_-$ the only vector $v \in N_\mathbb{R}$ such that $(v,-1)$ lies on the line spanned by r. Hence,

$$\sum_{r\in\mathbb{S}^n_-} \Psi_{\Gamma_0}(r)\ \mathrm{MV}_n(\Gamma_1^r,\dots,\Gamma_n^r) = \sum_{v\in N_\mathbb{R}} \frac{\Psi_{\Gamma_0}(v,-1)}{\|(v,-1)\|}\ \mathrm{MV}_n(\Gamma_1^{(v,-1)},\dots,\Gamma_n^{(v,-1)}).$$

Directly by the definition of Legendre–Fenchel duality, one has that $\Psi_{\Gamma_0}(v,-1) = g_0^\vee(v)$. The statement follows then from the fact that for every Borel set E in $(v,-1)^\perp$, the measure of E with respect to the restriction of $\mathrm{vol}_{M\oplus\mathbb{Z}}$ to $(v,-1)^\perp$ equals $\|(v,-1)\|\cdot\mathrm{vol}_M(\pi(E))$, again by [Burgos Gil et al. 2014, proof of Corollary 2.7.10]. □

Remark 1.7. For a rational polytope P of full dimension n in $M_\mathbb{R}$, every facet F of P, that is a face of dimension $n-1$, admits a distinguished orthogonal vector: it is the unique primitive vector $v_F \in N$ which satisfies $P^{v_F} = F$. Under the additional assumption that the Minkowski sum $Q := Q_1 + \cdots + Q_n$ is of dimension n in $M_\mathbb{R}$, the formula in Theorem 1.6 can be written as

$$\mathrm{MI}_M(g_0,\dots,g_n) = -\sum_F \Psi_{Q_0}(v_F)\,\mathrm{MI}_{M(v_F)}(g_1|_{Q_1^{v_F}},\dots,g_n|_{Q_n^{v_F}}) - \int_{N_\mathbb{R}} g_0^\vee\, d\,\mathrm{M}\mathcal{M}_M(g_1^\vee,\dots,g_n^\vee),$$

the first sum being over the finite set of facets of the polytope Q. Indeed, in such a situation the application $F \mapsto v_F$ realizes a bijection between the set of facets of Q and the set of primitive vectors $u \in N$ for which Q^u is an $(n-1)$-dimensional face of Q, which are the only vectors for which the term of the sum in the statement of the theorem does not vanish.

Remark 1.8. The statement of Theorem 1.6 can be reformulated in terms of Legendre–Fenchel duality. For $i = 0, \dots, n$, let f_i be a concave function on $N_{\mathbb{R}}$ with stability set a rational polytope Q_i in $M_{\mathbb{R}}$. Under the assumption that $Q_1 + \cdots + Q_n$ is of dimension n in $M_{\mathbb{R}}$, Remark 1.7 yields

$$\mathrm{MI}_M(f_0^\vee, \dots, f_n^\vee) = -\sum_F \Psi_{Q_0}(v_F)\,\mathrm{MI}_{M(v_F)}(f_1^\vee|_{Q_1^{v_F}}, \dots, f_n^\vee|_{Q_n^{v_F}}) - \int_{N_{\mathbb{R}}} f_0 \, d\,\mathrm{M}\mathcal{M}_M(f_1, \dots, f_n). \quad (1\text{-}3)$$

Indeed, it is sufficient to readily apply the previous theorem to the functions $f_0^\vee, \dots, f_n^\vee$, which are continuous on their domain and satisfy the equality $(f_i^\vee)^\vee = f_i$ for each $i = 0, \dots, n$ by concavity and closedness. It is easy to verify that the choice $f_0 = \cdots = f_n = f$ in (1-3) yields the formula in [Burgos Gil et al. 2014, Corollary 2.7.10].

We present now two applications of the recursive formula proved above. The first one concerns the computation of the mixed integral when all except one entry are indicator functions in the sense of Example 1.1.

Corollary 1.9. *Let $Q_1, \dots, Q_n$ be rational polytopes in $M_{\mathbb{R}}$ and f a concave function on $N_{\mathbb{R}}$ with stability set a rational polytope. Then*

$$\mathrm{MI}_M(\iota_{Q_1}, \dots, \iota_{Q_n}, f^\vee) = -\mathrm{MV}_M(Q_1, \dots, Q_n) \cdot f(0).$$

Proof. By symmetry, one can develop the recursive formula in Remark 1.8 with respect to $f^\vee$ to obtain

$$\mathrm{MI}_M(\iota_{Q_1}, \dots, \iota_{Q_n}, f^\vee) = -\int_{N_{\mathbb{R}}} f \, d\,\mathrm{M}\mathcal{M}_M(\iota_{Q_1}^\vee, \dots, \iota_{Q_n}^\vee),$$

the indicator functions $\iota_{Q_1}, \dots, \iota_{Q_n}$ being zero where defined. The duality in Example 1.1 and the fact that

$$\mathrm{M}\mathcal{M}_M(\Psi_{Q_1}, \dots, \Psi_{Q_n}) = \mathrm{MV}_M(Q_1, \dots, Q_n)\delta_0$$

because of Proposition 1.5 conclude the proof. □

The second application explains how the mixed integral behaves with respect to pointwise sum by a constant in one entry.

Corollary 1.10. *Let g_i be a concave function defined on a rational polytope $Q_i \subseteq M_{\mathbb{R}}$ for $i = 0, \dots, n$ and $c \in \mathbb{R}$. Then*

$$\mathrm{MI}_M(g_0, \dots, g_{n-1}, g_n + c) = \mathrm{MI}_M(g_0, \dots, g_n) + c \cdot \mathrm{MV}_M(Q_0, \dots, Q_{n-1}).$$

Proof. Denoting by $c\delta_0$ the concave function which has value c at 0 and $-\infty$ otherwise, it follows from the definitions that $g_n + c = g_n \boxplus c\delta_0$. The multilinearity of mixed integrals implies then that

$$\mathrm{MI}_M(g_0, \dots, g_{n-1}, g_n + c) = \mathrm{MI}_M(g_0, \dots, g_n) + \mathrm{MI}_M(g_0, \dots, g_{n-1}, c\delta_0).$$

Using the fact that $(c\delta_0)^\vee = -c$, the recursive formula in Theorem 1.6, developed with respect to $c\delta_0$, yields

$$\mathrm{MI}_M(g_0, \dots, g_{n-1}, c\delta_0) = \int_{N_\mathbb{R}} c \, d\,\mathrm{M}\mathcal{M}_M(g_0^\vee, \dots, g_{n-1}^\vee) = c \cdot \mathrm{M}\mathcal{M}_M(g_0^\vee, \dots, g_{n-1}^\vee)(N_\mathbb{R}).$$

The statement follows then from the fact that the total volume of the mixed Monge–Ampère measure of $g_0^\vee, \dots, g_{n-1}^\vee$ is equal to $\mathrm{MV}_M(\mathrm{dom}(g_0), \dots, \mathrm{dom}(g_{n-1}))$ by [Passare and Rullgård 2004, Proposition 3(iv)]. □

We conclude the section by proving a formula expressing the mixed integral of an $(n+1)$-tuple of concave functions on $M_\mathbb{R}$ where one of them is the indicator function of a line segment.

Let m be a primitive vector of M and consider the quotient $P := M/\mathbb{Z}m$. Since m is primitive, P is a lattice of rank $n-1$. By abuse of notation, let π denote both the projection from M to P and the induced linear map from $M_\mathbb{R}$ to $P_\mathbb{R}$. For each closed concave function g defined on a compact subset B of $M_\mathbb{R}$, let

$$\pi_* g : \pi(B) \to \mathbb{R}, \quad x \mapsto \max_{y \in \pi^{-1}(x)} g(y), \tag{1-4}$$

be the *direct image* of g by π. It is a well-defined closed concave function with domain a bounded subset of $P_\mathbb{R}$; see [Rockafellar 1970, Theorems 5.7 and 9.2]. Finally, for $x_1, x_2 \in M_\mathbb{R}$, denote by $\overline{x_1 x_2}$ the line segment in $M_\mathbb{R}$ with extremal points x_1 and x_2. The following lemma is a generalization of [Ewald 1996, Exercise 3, p. 128] and seems to be well known to experts, though we could not find an adequate reference in the literature for its proof.

Lemma 1.11. *In the above hypotheses and notation and for $n \geq 2$, let $Q_1, \dots, Q_{n-1}$ be polytopes in $M_\mathbb{R}$. Then,*

$$\mathrm{MV}_M(\overline{0m}, Q_1, \dots, Q_{n-1}) = \mathrm{MV}_P(\pi(Q_1), \dots, \pi(Q_{n-1})).$$

Proof. Since the vector m is primitive, it can be extended to a basis of the lattice M; see for instance [Lekkerkerker 1969, Theorem 5, p. 21]. We suppose fixed throughout the proof such a basis $(m_1, \dots, m_{n-1}, m)$ of M and the induced isomorphism $M_\mathbb{R} \simeq \mathbb{R}^n$; under this identification, the normalized volume vol_M corresponds to the Lebesgue measure vol_n on $\mathbb{R}^n$. Since $(\pi(m_1), \dots, \pi(m_{n-1}))$ is a basis of P, such a lattice is isomorphic to the span of $m_1, \dots, m_{n-1}$ in M and hence it is identified with the linear subspace $\mathbb{R}^{n-1} \times \{0\}$ of $\mathbb{R}^n$. Moreover, vol_P corresponds to the $(n-1)$-dimensional Lebesgue measure vol_{n-1} on $\mathbb{R}^{n-1} \times \{0\}$ and the map π to the vertical projection.

The claim reduces then to the particular case of a family of polytopes $Q_1, \dots, Q_{n-1}$ in $\mathbb{R}^n$, $m = (0, \dots, 0, 1)$ and π the vertical projection. Denoting by S the vertical segment of unitary length and rearranging the terms in the definition of the mixed volume given for instance in [Burgos Gil et al. 2014, Definition 2.7.14] one obtains

$$\mathrm{MV}_n(S, Q_1, \dots, Q_{n-1}) = \sum_{k=1}^{n-1} (-1)^{n-1-k} \sum_{1 \leq i_1 < \dots < i_k \leq n-1} (\mathrm{vol}_n(S + Q_{i_1} + \dots + Q_{i_k}) - \mathrm{vol}_n(Q_{i_1} + \dots + Q_{i_k}))$$

since the n-dimensional volume of a line segment vanishes for $n \geq 2$. To prove the claim it is hence enough to show that for each polytope Q in $\mathbb{R}^n$ the equality

$$\mathrm{vol}_n(S+Q) - \mathrm{vol}_n(Q) = \mathrm{vol}_{n-1}(\pi(Q))$$

holds. But $Q \subset S+Q$ and the difference of their volumes coincides with the integral over $\pi(Q) = \pi(S+Q)$ of the difference between the concave functions parametrizing the roof of the polytope $S+Q$ and Q respectively. Such a difference being constantly equal to 1 on $\pi(Q)$, the claim follows from the definition of the Lebesgue integral, concluding the proof. □

Proposition 1.12. *In the above hypotheses and notation, let g_i be a continuous concave function defined on a polytope Q_i in $M_{\mathbb{R}}$, for $i = 1, \dots, n$. Then,*

$$\mathrm{MI}_M(\iota_{\overline{0m}}, g_1, \dots, g_n) = \mathrm{MI}_P(\pi_* g_1, \dots, \pi_* g_n).$$

Proof. For $n = 1$, the claim follows from Corollary 1.9. Assume hence $n \geq 2$. Choose for each $i = 1, \dots, n$ a nonpositive real number γ_i such that $\gamma_i \leq \min_{x \in Q_i} g_i(x)$ and consider the convex body

$$Q_{g_i,\gamma_i} := \{(x,t) \in Q_i \times \mathbb{R} : \gamma_i \leq t \leq g_i(x)\}$$

in $M_{\mathbb{R}} \times \mathbb{R}$. The formula in [Philippon and Sombra 2008a, Proposition IV.5(d)] implies that

$$\mathrm{MI}_M(\iota_{\overline{0m}}, g_1, \dots, g_n) = \mathrm{MV}_{M \oplus \mathbb{Z}}(\overline{(0,0)(m,0)}, Q_{g_1,\gamma_1}, \dots, Q_{g_n,\gamma_n}) \\ + \sum_{i=1}^{n} \gamma_i \, \mathrm{MV}_M(\overline{0m}, Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n).$$

By the hypotheses on m, $(m, 0)$ is a nonzero primitive vector of the lattice $M \oplus \mathbb{Z}$. The map $\pi' := \pi \times \mathrm{id}_{\mathbb{Z}} : M \oplus \mathbb{Z} \to P \oplus \mathbb{Z}$ is a surjective group homomorphism, giving $(M \oplus \mathbb{Z})/\mathbb{Z}(m, 0) \simeq P \oplus \mathbb{Z}$. By Lemma 1.11,

$$\mathrm{MI}_M(\iota_{\overline{0m}}, g_1, \dots, g_n) = \mathrm{MV}_{P \oplus \mathbb{Z}}(\pi'(Q_{g_1,\gamma_1}), \dots, \pi'(Q_{g_n,\gamma_n})) \\ + \sum_{i=1}^{n} \gamma_i \, \mathrm{MV}_P(\pi(Q_1), \dots, \pi(Q_{i-1}), \pi(Q_{i+1}), \dots, \pi(Q_n)).$$

The statement follows hence from the equality in [Philippon and Sombra 2008a, Proposition IV.5(d)] applied to the concave functions $\pi_* g_1, \dots, \pi_* g_n$, the direct image of g_i by π being a concave function defined on $\pi(Q_i)$ and satisfying

$$\pi'(Q_{g_i,\gamma_i}) = \{(\pi(y), t) \in \pi(Q_i) \times \mathbb{R} : \gamma_i \leq t \leq g_i(y)\} \\ = \{(x, t) \in \pi(Q_i) \times \mathbb{R} : \gamma_i \leq t \leq (\pi_* g_i)(x)\}$$

for every $i = 1, \dots, n$. □

2. Ronkin functions

Fix for the whole section an algebraically closed complete field $(K, |\cdot|)$, that is, a pair of an algebraically closed field and an absolute value with respect to which the field is complete. Depending on the nature of the absolute value, $(K, |\cdot|)$ is said to be archimedean or nonarchimedean. As a consequence of the Gelfand–Mazur theorem, the only archimedean algebraically closed complete field is $\mathbb{C}$ endowed with a power of the usual absolute value. When the choice of the absolute value is clear from the context, an algebraically closed complete field will be simply denoted by K.

2A. *Tropicalization.* Given an affine variety $X = \operatorname{Spec} A$ over an algebraically closed complete field K, let $\iota : K \to A$ be the corresponding ring homomorphism. Berkovich's construction allows one to define a locally ringed space $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$, called the *analytification* of X; as a set,

$$X^{\mathrm{an}} := \{\|\cdot\|_x \text{ multiplicative seminorm on } A : \|\iota(k)\|_x = |k| \text{ for all } k \in K\}.$$

We refer to [Berkovich 1990, §1.5] (or also to [Burgos Gil et al. 2014, §1.2] for a more concise treatment) for the definition of the topology and the structure sheaf on X^{an}. By a gluing argument, one can then extend the definition to any variety over K.

Remark 2.1. When $K = \mathbb{C}$ endowed with the usual absolute value, the Gelfand–Mazur theorem implies that the locally ringed space produced by Berkovich's construction agrees with the standard complex analytification of a variety over $\operatorname{Spec} \mathbb{C}$.

Let $\mathbb{T}$ denote a split torus over K of dimension n and M its character lattice, in such a way that $\mathbb{T} = \operatorname{Spec} K[M]$. A basis of the K-algebra $K[M]$ will be denoted, as in [Fulton 1993, beginning of §1.3], by $(\chi^m)_{m\in M}$ and the elements of $K[M]$ will be called *Laurent polynomials* over K. Let N be the dual lattice of M and denote by $N_{\mathbb{R}} = N \otimes \mathbb{R}$ the associated real vector space.

Definition 2.2. The *tropicalization map* $\operatorname{trop} : \mathbb{T}^{\mathrm{an}} \to N_{\mathbb{R}} = \operatorname{Hom}(M, \mathbb{R})$ is the application defined by

$$(\operatorname{trop}(\|\cdot\|_x))(m) := -\log \|\chi^m\|_x$$

for every $\|\cdot\|_x \in \mathbb{T}^{\mathrm{an}}$.

The tropicalization map turns out to be a continuous application with respect to the Berkovich topology of $\mathbb{T}^{\mathrm{an}}$ and the Euclidean topology of $N_{\mathbb{R}}$. Moreover, in the archimedean case, the choice of a basis of M allows one to write such map in the more familiar form

$$\operatorname{trop}((z_1, \ldots, z_n)) = (-\log|z_1|, \ldots, -\log|z_n|).$$

Remark 2.3. The tropicalization map coincides with the valuation map val used by Burgos Gil, Philippon and Sombra [Burgos Gil et al. 2014, equation 4.1.2].

When the absolute value on K is nonarchimedean, one can construct a suitable section of trop. For each $u \in N_{\mathbb{R}}$, consider the map which associates to a Laurent polynomial $f = \sum c_m \chi^m$ the real value

$$\|f\|_{\kappa(u)} := \max_m |c_m| e^{-\langle m,u\rangle}.$$

One can verify that $\|\cdot\|_{\kappa(u)}$ is a multiplicative seminorm on $K[M]$ extending the absolute value on K, and hence it corresponds to a point $\kappa(u) \in \mathbb{T}^{\mathrm{an}}$, which one calls the *Gauss point* over u. The application

$\kappa : N_{\mathbb{R}} \to \mathbb{T}^{\mathrm{an}}$ defined by

$$\kappa : u \to \|\cdot\|_{\kappa(u)}$$

is proved to be a continuous section of trop; see for instance [Burgos Gil et al. 2014, Proposition-Definition 4.2.12], noting that κ coincides with $\theta_0 \circ \boldsymbol{e}$ in the cited reference.

It is easily seen that for any closed subvariety X of $\mathbb{T}$, there is a natural inclusion of sets $X^{\mathrm{an}} \subseteq \mathbb{T}^{\mathrm{an}}$, making the following definition meaningful.

Definition 2.4. Let f be a nonzero Laurent polynomial in $K[M]$ and $V(f)$ the associated closed subvariety of $\mathbb{T}$. The subset

$$\mathcal{A}_f := \mathrm{trop}(V(f)^{\mathrm{an}}) \subseteq N_{\mathbb{R}}$$

is called the *amoeba* of f.

The so-defined set has been widely studied in the literature. In the archimedean case, it coincides (up to a change of sign) with the notion of amoeba studied by Gelfand, Kapranov and Zelevinsky [1994] and by Passare and Rullgård [2004]. In the nonarchimedean case, the amoeba of a nonzero Laurent polynomial f coincides with the corner locus of the associated tropical polynomial f^{trop}; see [Einsiedler et al. 2006, Theorem 2.1.1].

2B. *Minimal boundaries.* To keep the treatment of the archimedean and nonarchimedean settings uniform, the following notion is crucial.

Definition 2.5. Let K be an algebraically closed complete field and A a K-algebra. For any set $\mathcal{S}$ of multiplicative seminorms on A extending the absolute value of K, a *boundary* of $\mathcal{S}$ is a subset $\mathcal{B} \subseteq \mathcal{S}$ such that

$$\max_{x \in \mathcal{S}} \|a\|_x = \max_{x \in \mathcal{B}} \|a\|_x$$

for every $a \in A$.

In other words, any boundary of $\mathcal{S}$ contains all of the information about the maximal values that the seminorms in $\mathcal{S}$ can attain. As a trivial example, $\mathcal{S}$ is a boundary of $\mathcal{S}$.

We are especially interested in the boundary of the fibers of the tropicalization map. Keeping the notation of the previous subsection, for a point $u \in N_{\mathbb{R}}$, it is immediate to show that

$$\mathrm{trop}^{-1}(u) = \{\|\cdot\|_x \in \mathbb{T}^{\mathrm{an}} : \|\chi^m\|_x = e^{-\langle m,u\rangle} \text{ for all } m \in M\}.$$

In the complex case, the existence of a unique minimal boundary for an algebra of functions on a compact space has been proved by Shilov. The following result, which equally holds in the nonarchimedean case, is well-known by experts.

Proposition 2.6. *The set* $\mathrm{trop}^{-1}(u)$ *has a unique minimal boundary. In the archimedean case, it coincides with the whole* $\mathrm{trop}^{-1}(u)$, *while in the nonarchimedean case it consists of the Gauss point over* u.

Proof. In the archimedean case, after having chosen a basis of M, one can treat $\mathbb{T}^{\mathrm{an}}$ as $(\mathbb{C}^*)^n$, by identifying a point $z = (z_1, \dots, z_n)$ of $(\mathbb{C}^*)^n$ with the seminorm $\|\cdot\|_z$ defined as

$$\|f\|_z = |f(z)|$$

for every $f \in K[T_1, \dots, T_n]$. Assuming that the coordinates of u in the basis dual to the chosen one are $(u_1, \dots, u_n)$, the set $\mathrm{trop}^{-1}(u)$ corresponds to the subset of $(\mathbb{C}^*)^n$ consisting of the points $(z_1, \dots, z_n)$ satisfying $-\log|z_i| = u_i$ for every $i = 1, \dots, n$. A point $\tilde{z}$ of $\mathrm{trop}^{-1}(u)$ is then of the form

$$\tilde{z} := (e^{-u_1 + i\theta_1}, \dots, e^{-u_n + i\theta_n}),$$

with $\theta_i \in [0, 2\pi)$ for every $i = 1, \dots, n$. It is easy to check that such a point is the only one in $\mathrm{trop}^{-1}(u)$ for which

$$\|(T_1 + e^{-u_1 + i\theta_1}) \cdots (T_n + e^{-u_n + i\theta_n})\|_z$$

is maximal. As a consequence, $\tilde{z}$ must belong to any boundary of $\mathrm{trop}^{-1}(u)$. This proves that the only boundary of $\mathrm{trop}^{-1}(u)$ is the set itself.

In the nonarchimedean case, for every $\|\cdot\|_x \in \mathrm{trop}^{-1}(u)$ and for every $f = \sum c_m \chi^m \in K[M]$ one has (by definition of nonarchimedean absolute value) that

$$\|f\|_x \le \max_m |c_m| \|\chi^m\|_x = \max_m |c_m| e^{-\langle m, u\rangle}.$$

This trivially implies that the Gauss norm $\|f\|_{\kappa(u)} = \max_m |c_m| e^{-\langle m,u\rangle}$ is a boundary for $\mathrm{trop}^{-1}(u)$. $\square$

For $u \in N_{\mathbb{R}}$, one denotes the unique minimal boundary of $\mathrm{trop}^{-1}(u)$ described in the previous proposition by $\mathcal{B}(u)$. If not otherwise mentioned, such a set is considered to be endowed with the topology induced from $\mathbb{T}^{\mathrm{an}}$. If one sets

$$\mathcal{B}_K := \begin{cases} (\mathbb{S}^1)^n & \text{if } K \text{ is archimedean,} \\ \{1\} & \text{otherwise,} \end{cases}$$

the boundary $\mathcal{B}(u)$ is homeomorphic to the compact group $\mathcal{B}_K$ for every $u \in N_{\mathbb{R}}$. This allows one to define the measure

$$\sigma_u := \mathrm{Haar}_{\mathcal{B}(u)}$$

on $\mathrm{trop}^{-1}(u)$, which is the Haar measure on the compact group $\mathcal{B}(u)$ normalized to have total mass 1; it is a finite measure on $\mathrm{trop}^{-1}(u)$, supported on $\mathcal{B}(u)$ and distributing homogeneously on this set. In the nonarchimedean case it coincides with the Dirac delta at the Gauss point over u.

There exists an embedding $\iota : N_{\mathbb{R}} \times \mathcal{B}_K \to \mathbb{T}^{\mathrm{an}}$ fitting in the commutative diagram

$$\begin{tikzcd} N_{\mathbb{R}} \times \mathcal{B}_K \arrow[r, hook, "\iota"] \arrow[d] & \mathbb{T}^{\mathrm{an}} \arrow[dl, "\mathrm{trop}"] \\ N_{\mathbb{R}} & \end{tikzcd} \tag{2-1}$$

with the vertical arrow being the projection onto the first factor. In the archimedean case, it is determined by the choice of a homeomorphism $(\mathbb{C}^*)^n \simeq N_{\mathbb{R}} \times (\mathbb{S}^1)^n$, while in the nonarchimedean case it coincides with the map $(u, 1) \mapsto \kappa(u)$. The image of ι is homeomorphic to $N_{\mathbb{R}} \times \mathcal{B}_K$ and it is a deformation retract of the analytic torus, coinciding with it if K is archimedean.

2C. *Ronkin functions.* The terminology and notation introduced in the previous subsection allow one to define Ronkin functions in the archimedean and nonarchimedean cases simultaneously.

Definition 2.7. Let f be a nonzero Laurent polynomial over K. The *Ronkin function* of f is the map $\rho_f : N_{\mathbb{R}} \to \mathbb{R}$ defined as

$$\rho_f(u) := \int_{\mathrm{trop}^{-1}(u)} -\log \|f\|_x \, d\sigma_u(x)$$

for every $u \in N_{\mathbb{R}}$.

The integral in the previous definition is finite. Indeed, logarithmic singularities are integrable in the archimedean case, and the Gauss norm of a nonzero Laurent polynomial is positive in the nonarchimedean case.

Remark 2.8. In the archimedean case, $\rho_f(u) = -N_f(-u)$, where N_f is the classical Ronkin function associated to a complex Laurent polynomial; refer for instance to [Passare and Rullgård 2004]. In the nonarchimedean case, it is easily checked that $\rho_f = f^{\mathrm{trop}}$, where the tropicalization of a Laurent polynomial $f = \sum c_m \chi^m$ is defined by

$$f^{\mathrm{trop}}(u) = \min_m(\langle m, u\rangle - \log |c_m|)$$

for every $u \in N_{\mathbb{R}}$.

The following property of the Ronkin function follows immediately from its definition.

Proposition 2.9. *For every pair of nonzero Laurent polynomials f and g, one has $\rho_{f \cdot g} = \rho_f + \rho_g$. For every $\lambda \in K$, moreover, $\rho_{\lambda \cdot f} = -\log |\lambda| + \rho_f$.*

Recall that for any Laurent polynomial $f = \sum_m c_m \chi^m$, we denote by $\mathrm{NP}(f)$ the *Newton polytope of f*, that is, the convex hull in $M_{\mathbb{R}}$ of the set $\{m \in M : c_m \neq 0\}$. Its *support function*, as defined in Example 1.1, is denoted by $\Psi_{\mathrm{NP}(f)}$.

Proposition 2.10. *Let f be a nonzero Laurent polynomial. Then*:

(1) *ρ_f is a continuous concave function on $N_{\mathbb{R}}$ (in particular it is closed) and it is affine on each connected component of the complement of the amoeba of f.*

(2) *$|\rho_f - \Psi_{\mathrm{NP}(f)}|$ is bounded on $N_{\mathbb{R}}$.*

(3) *The stability set of ρ_f coincides with* $\mathrm{NP}(f)$ *and* $\mathrm{rec}(\rho_f) = \Psi_{\mathrm{NP}(f)}$.

Proof. The statements in (1) are trivial in the nonarchimedean case. In the archimedean case, the concavity of ρ_f and its affinity outside the amoeba are shown in [Passare and Rullgård 2004, Theorem 1]. As a consequence of concavity, ρ_f is continuous on $N_{\mathbb{R}}$, and hence closed.

To prove (2), suppose that $f = \sum_m c_m \chi^m$ and let $\gamma_K(f)$ be the number of nonzero coefficients of f if K is archimedean, and 1 otherwise. For any $x \in \mathrm{trop}^{-1}(u)$, the inequality $\|f\|_x \leq \gamma_K(f) \cdot \max_m(|c_m| \|\chi^m\|_x)$ implies

$$-\log \|f\|_x \geq \min_m(\langle m, u\rangle - \log |c_m|) - \log \gamma_K(f) \geq \Psi_{\mathrm{NP}(f)}(u) - \max_m \log |c_m| - \log \gamma_K(f)$$

and hence

$$\rho_f(u) \geq \Psi_{\mathrm{NP}(f)}(u) - \log(\gamma_K(f) \max_m |c_m|)$$

for every $u \in N_{\mathbb{R}}$. For a reverse inequality, denote by $\mathcal{V}(f)$ the set of vertices of $\mathrm{NP}(f)$. Then, for every $m \in \mathcal{V}(f)$ the Ronkin function of f coincides with $\langle m, u\rangle - \log |c_m|$ in a nonempty open subset of $N_{\mathbb{R}}$ (this follows from [Passare and Rullgård 2004, Proposition 2] in the archimedean setting and from [Einsiedler et al. 2006, Corollary 2.1.2] in the nonarchimedean one). By the concavity of ρ_f, one deduces hence that

$$\rho_f(u) \leq \Psi_{\mathrm{NP}(f)}(u) - \log \min_{m \in \mathcal{V}(f)} |c_m|$$

for every $u \in N_{\mathbb{R}}$, concluding the proof of (2).

The statements in (3) follow directly from (2). Indeed, since $|\rho_f - \Psi_{\mathrm{NP}(f)}|$ is bounded, $\mathrm{stab}(\rho_f) = \mathrm{stab}(\Psi_{\mathrm{NP}(f)}) = \mathrm{NP}(f)$; the last equality follows then from [Rockafellar 1970, Theorem 13.3]. □

The calculation of the Ronkin function of a nonzero Laurent polynomial is typically very difficult. Anyway, an explicit expression for it is available in the following two simple situations.

Example 2.11. It follows from the definition that the Ronkin function of the monomial χ^m coincides with the linear function m on $N_{\mathbb{R}}$ for every $m \in M$.

Example 2.12. For any $m, m' \in M$ with $m \neq m'$, the Ronkin function of the binomial $f = \chi^m - \chi^{m'}$ coincides with the support function of the segment $\overline{mm'}$, that is,

$$\rho_f(u) = \min(\langle m, u\rangle, \langle m', u\rangle)$$

for every $u \in N_{\mathbb{R}}$. To prove this, note first that one can restrict to the case of the binomial $f = \chi^m - 1$ with $m \neq 0$ because of Proposition 2.9 and Example 2.11 and by factoring with a monomial. In the nonarchimedean case the statement follows immediately from Remark 2.8. In the archimedean one, the choice of a basis for M allows one to write

$$\rho_f(u) = -\frac{1}{(2\pi)^n} \int_{\theta_1, \ldots, \theta_n \in [0, 2\pi]} \log\left|e^{-m_1 u_1 + i m_1 \theta_1} \cdot \ldots \cdot e^{-m_n u_n + i m_n \theta_n} - 1\right| d\theta_1 \cdots d\theta_n,$$

with $m_1, \ldots, m_n$ being the coordinates of m in such a basis and $u_1, \ldots, u_n$ the coordinates of u in the dual one. Assuming that $m_1 > 0$, which is always possible since $m \neq 0$, Jensen's formula yields, for every $\theta_2, \ldots, \theta_n$,

$$\int_{\theta_1 \in [0, 2\pi]} \log\left|e^{-m_1 u_1 + i m_1 \theta_1} \cdot \ldots \cdot e^{-m_n u_n + i m_n \theta_n} - 1\right| d\theta_1 = -2\pi \sum_{j=1}^{k} \log \frac{|\alpha_j|}{e^{-m_1 u_1}} \tag{2-2}$$

with $\alpha_1, \dots, \alpha_k$ being the zeros of the univariate polynomial

$$(e^{-m_2u_2+im_2\theta_2} \cdot \dots \cdot e^{-m_nu_n+im_n\theta_n})T - 1$$

lying inside the closed disk of radius $e^{-m_1u_1}$, repeated according to multiplicity. The only complex zero of the above polynomial has modulus $e^{m_2u_2+\dots+m_nu_n}$; the integral in (2-2) is then zero if $m_1u_1+\dots+m_nu_n > 0$; otherwise it equals $-2\pi(m_1u_1+\dots+m_nu_n)$. It follows that

$$\rho_f(u) = \min(m_1u_1+\dots+m_nu_n, 0),$$

and hence the claim.

3. Heights of toric varieties

A well-suited framework to develop Arakelov geometry is provided by the study of varieties over adelic fields. In this setting, local and global heights of cycles of arbitrary dimension can be defined following [Zhang 1995; Gubler 1998; Chambert-Loir 2006]. A more general approach involving M-fields has been suggested in [Gubler 1997]. Even if the theory is often phrased in terms of line bundles, we adopt here the equivalent point of view of divisors, which turns out to be more convenient in the toric case; see for instance [Burgos Gil et al. 2015].

3A. *Adelic fields.* By a *place* on a field K, we mean an equivalence class of absolute values on K, which could be either archimedean or nonarchimedean. Whenever $\mathfrak{M}$ is a collection of places on K, the subset of archimedean places in $\mathfrak{M}$ is denoted by $\mathfrak{M}_\infty$.

Definition 3.1. Let K be a field. A family of places $\mathfrak{M}$ on K is said to be *adelic* if it satisfies the following properties:

(1) For every $v \in \mathfrak{M} \setminus \mathfrak{M}_\infty$, one (and hence all) absolute value in the class of v is associated to a nontrivial discrete valuation.

(2) For each $\alpha \in K^*$, the set of places v for which $|\alpha|_v \neq 1$ for any $|\cdot|_v \in v$ is finite.

It is clear that the two conditions of the previous definition do not depend on the choice of the representative of the class v.

Definition 3.2. An *adelic field* is a field K together with an adelic family of places $\mathfrak{M}$ on K and a choice of an absolute value $|\cdot|_v$ and of a real positive number n_v for each place $v \in \mathfrak{M}$. An adelic field $(K, (|\cdot|_v, n_v)_{v\in\mathfrak{M}})$ is said to satisfy the *product formula* if for every $\alpha \in K^*$

$$\sum_{v\in\mathfrak{M}} n_v \log|\alpha|_v = 0.$$

Whenever there is no ambiguity on its adelic structure, an adelic field will be simply denoted by K.

The following property is an easy, though fundamental, consequence of the definition.

Lemma 3.3. *Any adelic field K only admits finitely many archimedean places.*

Proof. Since an absolute value on K is nonarchimedean if and only if it is bounded on the image of $\mathbb{Z}$ in K, a field with positive characteristic has no archimedean absolute values. Suppose hence that K has characteristic zero. In this case it contains a copy of $\mathbb{Q}$ and any archimedean absolute value $|\cdot|_v$ on K restricts to an archimedean absolute value on $\mathbb{Q}$. By Ostrowski's theorem, one has $|2|_v > 1$. The second axiom in Definition 3.1 then allows one to conclude the claim. □

For an adelic field K and a finite field extension F of K, there exists a canonical way of endowing F with the structure of an adelic field; see [Gubler 1997, Remark 2.5] and [Martínez and Sombra 2018, §3] for the detailed construction. With this induced adelic structure, F satisfies the product formula whenever K does.

Example 3.4. The archetypal example of an adelic field satisfying the product formula is given by the field $\mathbb{Q}$, together with the collection of all its nontrivial places, the standard normalized absolute value for each of them and weights equal to 1.

More generally, any *global field*, that is, a number field or the function field of a smooth projective curve over a field k with the structure described in [Burgos Gil et al. 2014, Example 1.5.4], is an adelic field satisfying the product formula.

3B. ***Local and global heights.*** Let K be an adelic field satisfying the product formula and X a proper variety of dimension n over K. For every place $v \in \mathfrak{M}$, denote by K_v the completion of K with respect to $|\cdot|_v$ and by $\mathbb{C}_v$ the completion of an algebraic closure of K_v with respect to the unique extension of the absolute value. It is a well-known fact that $\mathbb{C}_v$ is algebraically closed; moreover, $\mathbb{C}_v$ comes with an absolute value that one denotes, with abuse of notation, by $|\cdot|_v$. The pair $(\mathbb{C}_v, |\cdot|_v)$ is hence an algebraically closed complete field as in Section 2.

The base change $X_{\mathbb{C}_v}$ is a scheme of finite type over $\operatorname{Spec}\mathbb{C}_v$ to which one associates its Berkovich analytification $(X_v^{\mathrm{an}}, \mathscr{O}_{X_v^{\mathrm{an}}})$, whose underlying topological space is compact because of the properness of X. To stress its dependence on the choice of the place v, X_v^{an} is called the *v-adic analytification* of X. Similarly, one can consider the base change X_{K_v} of X over $\operatorname{Spec} K_v$ and consider its Berkovich analytification $X_{K_v}^{\mathrm{an}}$. The two spaces are related by the isomorphism

$$X_{K_v}^{\mathrm{an}} \simeq X_v^{\mathrm{an}} / \operatorname{Gal}(\overline{K}_v^{\mathrm{sep}}/K_v),$$

as shown in [Berkovich 1990, Proposition 1.3.5]. Moreover, there exists a surjective morphism of locally ringed space $\pi_v : X_v^{\mathrm{an}} \to X_{\mathbb{C}_v}$.

Remark 3.5. By Ostrowski's theorem and the Gelfand–Mazur theorem, if v is an archimedean absolute value on K, then $\mathbb{C}_v$ is isometric to the field $\mathbb{C}$ endowed with a power of the usual absolute value. In this case, the Berkovich space $(X_v^{\mathrm{an}}, \mathscr{O}_{X_v^{\mathrm{an}}})$ is isomorphic to the usual complex analytification of $X_{\mathbb{C}}$.

For any line bundle L on X, its v-adic analytification is the analytic line bundle

$$L_v^{\mathrm{an}} := \pi_v^* L_{\mathbb{C}_v}$$

on X_v^{an}. Continuous metrics on L_v^{an} are defined as in [Chambert-Loir 2011, §1.1.1], independently of the nature of the place v. Relevant classes of metrics on L_v^{an} are *smooth* metrics in the archimedean case and *algebraic* (or, equivalently, formal, see [Gubler and Künnemann 2017, Proposition 8.13]) metrics when v is nonarchimedean; see for example [Chambert-Loir 2011, §1] and [Gubler and Künnemann 2017, 8.8 and 8.12] for the precise definitions. A divisor D on X together with a continuous $\mathrm{Gal}(\overline{K}_v^{\mathrm{sep}}/K_v)$-invariant metric $\|\cdot\|_v$ on the analytic line bundle $\mathscr{O}(D)_v^{\mathrm{an}}$ is called a *v-adic metrized divisor* and it is denoted by $\overline{D}_v$ or also by $(\mathscr{O}(D), \|\cdot\|_v)$. Sums and pull-backs of v-adic metrized divisors can be defined as in [Chambert-Loir 2011, §1.2].

A v-adic metrized divisor $\overline{D}_v$ is said to be *semipositive* if the corresponding metric can be approximated by semipositive smooth (when v is archimedean) or algebraic (when v is nonarchimedean) semipositive metrics in the sense of [Burgos Gil et al. 2014, §1.4]. For any d-dimensional subvariety Y of X and for any d-tuple of v-adic semipositive metrized divisors $\overline{D}_{0,v}, \dots, \overline{D}_{d-1,v}$ on X, there exists a positive measure

$$c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{d-1,v}) \wedge \delta_Y \tag{3-1}$$

on X_v^{an}, which was first introduced in [Chambert-Loir 2006, Définition 2.4 and Proposition 2.7(b)] in the nonarchimedean setting and extended in [Gubler 2007, §3.8] under weaker assumptions. The suggestive notation for the measure in (3-1) is compatible with the wedge product of first Chern forms in the smooth archimedean case, while it is justified by the recent advances in the theory of forms and currents on Berkovich spaces otherwise, as shown in [Chambert-Loir and Ducros 2012, §6.9] and [Gubler and Künnemann 2017, Theorem 10.5].

Recall also that for a d-dimensional cycle Z in X and a family $(D_0, s_0), \dots, (D_d, s_d)$ of divisors on X with rational sections of the associated line bundles, one says that $s_0, \dots, s_d$ *meet Z properly* if for every $J \subseteq \{0, \dots, d\}$, each irreducible component of $|Z| \cap \bigcap_{i\in J} |\mathrm{div}(s_i)|$ has dimension $d - \#J$, where $|\cdot|$ denotes here the support of a cycle.

Definition 3.6. Let Z be a d-dimensional cycle in X and $(\overline{D}_{0,v}, s_0), \dots, (\overline{D}_{d,v}, s_d)$ a collection of v-adic semipositive metrized divisors on X with rational sections of the corresponding line bundles, with $s_0, \dots, s_d$ meeting Z properly. The *v-adic local height of Z in X with respect to* $(\overline{D}_{i,v}, s_i)$ for $i = 0, \dots, d$ is defined, linearly in its irreducible components, by the recursive formula

$$\begin{aligned} &h_{\overline{D}_{0,v},\dots,\overline{D}_{d,v}}(Z; s_0, \dots, s_d) \\ &\quad := h_{\overline{D}_{0,v},\dots,\overline{D}_{d-1,v}}(Z \cdot \mathrm{div}(s_d); s_0, \dots, s_{d-1}) - \int_{X_v^{\mathrm{an}}} \log \|s_d\|_{d,v}\, c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{d-1,v}) \wedge \delta_Z, \end{aligned}$$

where $\|\cdot\|_{d,v}$ denotes the metric of $\overline{D}_{d,v}$ and one sets the height of the zero cycle to be zero.

The integrals appearing in the previous definition are well-defined, as shown in [Chambert-Loir and Thuillier 2009, Théorème 4.1] in both the archimedean and nonarchimedean settings, and in [Gubler and Hertel 2017, Theorem 1.4.3] for the case of nonarchimedean valuations which are not necessarily discrete. The v-adic local height function is moreover symmetric and multilinear with respect to sums of metrized divisors with rational sections of the associated line bundles; see [Gubler 2003, Proposition 3.4 and Remark 9.3].

The adelic structure on the field K allows one to define a *semipositive metrized divisor* $\overline{D}$ on X by the choice, for every place $v \in \mathfrak{M}$, of a continuous semipositive metric on $\mathscr{O}(D)_v^{\mathrm{an}}$. This global definition induces a notion of a v-adic local height function at each place of K. Some care has to be taken when defining global heights as sums of such v-adic local heights, since they do not need to be well-defined in general.

Definition 3.7. A d-dimensional irreducible subvariety Y of X is said to be *integrable* with respect to the choice of $d+1$ semipositive metrized divisors $\overline{D}_0, \dots, \overline{D}_d$ if there exists a birational proper map $\varphi : Y' \to Y$, with Y' projective, and sections s_i of $\varphi^*\mathscr{O}(D_i)$ for each $i = 0, \dots, d$, meeting Y' properly, such that the v-adic local height

$$h_{\varphi^*\overline{D}_{0,v},\dots,\varphi^*\overline{D}_{d,v}}(Y'; s_0, \dots, s_d)$$

is zero for all but finitely many places $v \in \mathfrak{M}$. A d-dimensional cycle is said to be *integrable* if each of its irreducible components is. If Y is an integrable d-dimensional irreducible subvariety, the *global height of Y in X with respect to $\overline{D}_0, \dots, \overline{D}_d$* is defined as

$$h_{\overline{D}_0,\dots,\overline{D}_d}(Y) := \sum_{v \in \mathfrak{M}} n_v \, h_{\varphi^*\overline{D}_{0,v},\dots,\varphi^*\overline{D}_{d,v}}(Y'; s_0, \dots, s_d).$$

The *global height* of integrable cycles is defined by linearity.

The previous definition does not depend on the choice of the projective resolution Y' of Y nor of the sections $s_0, \dots, s_d$, as a consequence of [Gubler 2003, Proposition 3.6 and Remark 9.3], [Burgos Gil et al. 2014, Theorem 1.4.17(3)] and the product formula on K. As its local counterparts, the global height is symmetric and multilinear with respect to sums of metrized divisors. Moreover, it is well-behaved under proper transformations, in the sense of the next proposition.

Proposition 3.8. *Let $\varphi : X' \to X$ be a dominant morphism of proper varieties over K, $\overline{D}_0, \dots, \overline{D}_d$ semipositive metrized divisors over X and Z' a d-dimensional cycle in X'. The cycle $\varphi_* Z'$ is integrable with respect to $\overline{D}_0, \dots, \overline{D}_d$ if and only if Z' is integrable with respect to $\varphi^*\overline{D}_0, \dots, \varphi^*\overline{D}_d$ and in this case*

$$h_{\overline{D}_0,\dots,\overline{D}_d}(\varphi_* Z') = h_{\varphi^*\overline{D}_0,\dots,\varphi^*\overline{D}_d}(Z').$$

Proof. The statement about integrability is [Burgos Gil et al. 2014, Proposition 1.5.8(2)], while the equality of the global heights follows from the same property on local heights, as proved in [Gubler 2003, Proposition 3.6 and Remark 9.3] in the more general context of pseudodivisors. □

3C. *Heights on toric varieties.* In this section the basic constructions and results of the arithmetic geometry of toric varieties are recalled, following the treatment of [Burgos Gil et al. 2014]. Let hence X_Σ be a proper toric variety of dimension n over an adelic field K, with torus $\mathbb{T}$ and dense open orbit X_0. Denote by N and M the character and cocharacter groups of $\mathbb{T}$ and by $N_\mathbb{R}$ and $M_\mathbb{R}$ the associated (reciprocally dual) real vector spaces. The toric variety X_Σ is associated with a complete fan Σ in $N_\mathbb{R}$, whose collection of k-dimensional cones is written $\Sigma^{(k)}$. For any $\tau \in \Sigma^{(1)}$, denote by v_τ its minimal nonzero integral vector and by $V(\tau)$ its associated orbit closure, as in [Fulton 1993, §3.1].

Divisors on X_Σ which are invariant under the torus action are called *toric divisors* and admit a nice combinatorial description, as follows. By [Kempf et al. 1973, §I.2, Theorem 9], there exists a bijection between the set of toric Cartier divisors on X_Σ and the set of *virtual support functions* on Σ, that is, piecewise linear real-valued functions on the support of Σ, with integral slope on each cone of Σ. The toric Cartier divisor constructed from the virtual support function Ψ is denoted by D_Ψ and it defines a distinguished rational section s_Ψ of $\mathscr{O}(D_\Psi)$ satisfying $\operatorname{div}(s_\Psi) = D_\Psi$. The corresponding Weil divisor is given by

$$[D_\Psi] = \sum_{\tau \in \Sigma^{(1)}} -\Psi(v_\tau) V(\tau). \tag{3-2}$$

In particular, the rational section s_Ψ is regular and nowhere-vanishing on X_0. A toric divisor D_Ψ also determines a polyhedron

$$\Delta_\Psi := \{x \in M_\mathbb{R} : x - \Psi \geq 0\}$$

in $M_\mathbb{R}$, which is bounded because of the properness of X_Σ, see [Fulton 1993, Proposition on p. 67], and coincides with the stability set of Ψ if Ψ is concave. Many algebro-geometric properties of a toric divisor are read from its associated virtual support function: for instance, D_Ψ is generated by global sections if and only if Ψ is concave, and it is ample if and only if Ψ is concave and restricts to different linear functions on different maximal cones of Σ.

Regarding the arithmetic part of the toric dictionary, let D_Ψ be a toric divisor on X_Σ, with associated virtual support function Ψ. The continuous metrics on $\mathscr{O}(D_\Psi)$ admitting a combinatorial description are the ones which are invariant under the action of a certain compact torus; see [Burgos Gil et al. 2014, §4.2] for more details about this notion. In concrete terms, a continuous metric $\|\cdot\|_v$ on $\mathscr{O}(D_\Psi)_v^{\mathrm{an}}$ is called a *v-adic toric metric* if the map

$$(X_0)_v^{\mathrm{an}} \to \mathbb{R}, \quad p \mapsto \|s_\Psi(p)\|_v,$$

is constant along the fibers of the v-adic tropicalization map $\operatorname{trop}_v : (X_0)_v^{\mathrm{an}} \to N_\mathbb{R}$ introduced in Definition 2.2. A toric divisor D together with a v-adic toric metric on $\mathscr{O}(D)$ is called a *v-adic toric metrized divisor*. To a v-adic toric metrized divisor $\overline{D}_v$ one can associate the real-valued map $\psi_{\overline{D}_v}$ on $N_\mathbb{R}$ satisfying the equality

$$\psi_{\overline{D}_v} \circ \operatorname{trop}_v = \log \|s_D\|_v \tag{3-3}$$

on the analytic torus $(X_0)_v^{\mathrm{an}}$, where s_D is the distinguished rational section of $\mathscr{O}(D)$.

The map $\psi_{\overline{D}_v}$, which will be referred to as the *metric function* of $\overline{D}_v$, was introduced by Burgos Gil, Philippon and Sombra in their study of Arakelov geometry of toric varieties to encode many arithmetic properties of $\overline{D}_v$; see [Burgos Gil et al. 2014, Chapter 4]. For instance, it is smooth in the archimedean case if the metric is smooth, while in the nonarchimedean setting it is rational piecewise affine if the metric is algebraic; see [Burgos Gil et al. 2014, Theorem 4.5.10(1)] and [Gubler and Hertel 2017, Proposition 2.5.5]. Also, the semipositivity of $\overline{D}_v$ is translated into the concavity of its corresponding metric function.

Theorem 3.9. *Let D be the toric divisor associated to the virtual support function Ψ. The assignment $\|\cdot\|_v \mapsto \psi_{\overline{D}_v}$ is a bijection between the space of v-adic semipositive toric metrics on $\mathscr{O}(D)_v^{\mathrm{an}}$ and the space of concave functions ψ on $N_{\mathbb{R}}$ such that $|\psi - \Psi|$ is bounded.*

Proof. This is [Burgos Gil et al. 2014, Theorem 4.8.1(1)]. The extension to the general nonarchimedean case is [Gubler and Hertel 2017, Theorem 2.5.8]. □

If $\overline{D}_v$ is a v-adic semipositive toric metrized divisor, the Legendre–Fenchel dual of the metric function of $\overline{D}_v$ is called the *roof function* of $\overline{D}_v$ and denoted by $\vartheta_{\overline{D}_v}$: it is a concave function on $M_{\mathbb{R}}$ with effective domain the polytope Δ_Ψ.

A toric divisor admits a v-adic semipositive metric if and only if it is generated by global sections, as proved in [Burgos Gil et al. 2014, Corollary 4.8.5]. For such divisors, moreover, there exists a distinguished choice of a v-adic semipositive metric.

Definition 3.10. Let D be a toric divisor generated by global sections, and Ψ its associated virtual support function. The *v-adic canonical metric* on D is the semipositive toric metric on $\mathscr{O}(D)_v^{\mathrm{an}}$ corresponding to Ψ in the bijection of Theorem 3.9.

In the nonarchimedean case, the canonical metric on D coincides with the algebraic metric induced by the canonical model of X_Σ and D; see [Burgos Gil et al. 2014, Example 4.5.4].

For semipositive v-adic toric metrized divisors, the measure in (3-1) can be expressed in terms of the associated metric functions. Indeed, recall from Section 2B that there exists an embedding

$$\iota_v : N_{\mathbb{R}} \times \mathcal{B}_{\mathbb{C}_v} \to X_{0,v}^{\mathrm{an}}$$

which fits into the commutative diagram (2-1), and denote by $\mathrm{Haar}_{\mathcal{B}_{\mathbb{C}_v}}$ the Haar measure on $\mathcal{B}_{\mathbb{C}_v}$ normalized to have total mass 1.

Theorem 3.11. *For $i = 0, \dots, n-1$, let $\overline{D}_{i,v}$ be a semipositive v-adic toric metrized divisor on X_Σ, Ψ_i the virtual support function associated to D_i and $\psi_{i,v}$ the metric function of $\overline{D}_{i,v}$. Then, the positive measure*

$$c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{n-1,v}) \wedge \delta_{X_\Sigma}$$

is zero outside $X_{0,v}^{\mathrm{an}}$ and

$$c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{n-1,v}) \wedge \delta_{X_\Sigma}|_{X_{0,v}^{\mathrm{an}}} = (\iota_v)_*(\mathrm{M}\mathcal{M}_M(\psi_{0,v}, \dots, \psi_{n-1,v}) \times \mathrm{Haar}_{\mathcal{B}_{\mathbb{C}_v}}).$$

In particular,

$$(\mathrm{trop}_v)_*(c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{n-1,v}) \wedge \delta_{X_\Sigma}|_{X_{0,v}^{\mathrm{an}}}) = \mathrm{M}\mathcal{M}_M(\psi_{0,v}, \dots, \psi_{n-1,v})$$

as measures on $N_{\mathbb{R}}$.

Proof. The first statement follows from [Burgos Gil et al. 2014, Theorem 1.4.10(1)] and [Gubler and Hertel 2017, Corollary 1.4.5]. The expression for the measure in the archimedean and the discrete nonarchimedean case is obtained from [Burgos Gil et al. 2014, Theorem 4.8.11] and multilinearity; the general nonarchimedean case is deduced from [Gubler and Hertel 2017, Theorem 2.5.10]. The last assertion is an easy consequence of the commutativity of the diagram (2-1). □

Moving to the global case, a (*semipositive*) *toric metric* on a toric divisor D is a choice, for each place $v \in \mathfrak{M}$, of a (semipositive) v-adic toric metric on the line bundle $\mathscr{O}(D)$. The toric divisor D together with a (semipositive) toric metric is called a (*semipositive*) *toric metrized divisor* and it is denoted by $\overline{D}$. From the point of view of convex geometry, the semipositive toric metrized divisor $\overline{D}$ is completely described by the collection $(\psi_v)_{v\in\mathfrak{M}}$ of its metric functions or, equivalently, by the collection $(\vartheta_v)_{v\in\mathfrak{M}}$ of its roof functions.

A notion of well-behaving toric metrics was defined in [Burgos Gil et al. 2014, Definition 4.9.1].

Definition 3.12. A toric metric $(\|\cdot\|_v)_{v\in\mathfrak{M}}$ on a toric divisor is said to be *adelic* if for all but finitely many $v \in \mathfrak{M}$ the v-adic toric metric $\|\cdot\|_v$ is the canonical one, in the sense of Definition 3.10.

In convex terms, a toric metric on the toric divisor D associated to the virtual support function Ψ is adelic if and only if the family $(\psi_v)_v$ of its metric functions satisfies $\psi_v = \Psi$ for all but finitely many $v \in \mathfrak{M}$.

It follows from [Burgos Gil et al. 2014, Theorem 5.2.4] that any toric subvariety of X_Σ is integrable with respect to the choice of adelic semipositive toric metrized divisors. In particular, one can compute the global height of the n-dimensional cycle X_Σ with respect to such choices.

Theorem 3.13. *Let $\overline{D}_0, \dots, \overline{D}_n$ be toric divisors over X_Σ, equipped with adelic semipositive toric metrics. Then*

$$h_{\overline{D}_0,\dots,\overline{D}_n}(X_\Sigma) = \sum_{v\in\mathfrak{M}} n_v \operatorname{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n,v}),$$

where $\vartheta_{i,v}$ is the roof function of $\overline{D}_{i,v}$ for every $i = 0, \dots, n$ and $v \in \mathfrak{M}$.

Proof. This is [Burgos Gil et al. 2014, Theorem 5.2.5]. □

4. Divisors of rational functions

The present section focuses on the combinatorial description of the Weil divisor on a toric variety of the rational function coming from a Laurent polynomial. This result will be used in the proof of the main theorems in the next section.

To fix notation, let X_Σ be a proper smooth toric variety of dimension n over a field K, M the character lattice of its torus $\mathbb{T}$ and $N_\mathbb{R}$ the associated dual vector space. The toric variety X_Σ has a dense open orbit X_0 isomorphic to $\mathbb{T}$ and hence the function field of X_Σ coincides with $K(M)$. In particular, any Laurent polynomial $f = \sum c_m \chi^m$ gives rise to a rational function on X_Σ, which is regular and coincides with f on X_0. To avoid confusion, one will denote by $\tilde{f}$ such a rational function.

Recall from [Cox et al. 2011, Theorem 3.1.19(a)] that the toric variety X_Σ is smooth if and only if each cone of Σ is a smooth cone, that is, it is generated by a part of a basis of the lattice N. For each cone τ in Σ of dimension 1, denote by v_τ its minimal nonzero integral vector, which generates $\tau \cap N$ as a monoid. If σ is a smooth cone of dimension n in $N_\mathbb{R}$, the collection $(v_\tau)_\tau$, with τ ranging in the set of 1-dimensional faces of σ, is a basis of N and hence gives a dual basis $(v_\tau^\vee)_\tau$ of the lattice M.

Lemma 4.1. *Let σ be a strongly convex polyhedral rational cone in $N_\mathbb{R}$. For every face τ of σ of dimension 1, the orbit closure $V(\tau)$ in the affine toric variety X_σ is the subvariety corresponding to the prime ideal*

$$\mathfrak{p} = (\chi^m : m \in \sigma^\vee \cap M, m \notin \tau^\perp)$$

of $\mathscr{O}(X_\sigma) = K[\sigma^\vee \cap M]$. Moreover, if σ is smooth and of maximal dimension in $N_\mathbb{R}$, then $\mathfrak{p}$ is principal and generated by $\chi^{v_\tau^\vee}$.

Proof. Recall for example from [Fulton 1993, §3.1] that the orbit closure $V(\tau)$ is the toric variety $\operatorname{Spec} K[\sigma^\vee \cap \tau^\perp \cap M]$ and can be embedded in $X_\sigma = \operatorname{Spec} K[\sigma^\vee \cap M]$ via the surjection of rings sending χ^m to itself if $m \in \tau^\perp$, and to 0 otherwise. Then $V(\tau)$ is seen as the subvariety of X_σ corresponding to the kernel of such homomorphism, that is,

$$\mathfrak{p} = \bigoplus_{\substack{m \in \sigma^\vee \cap M \\ m \notin \tau^\perp}} K\chi^m = (\chi^m : m \in \sigma^\vee \cap M, m \notin \tau^\perp),$$

proving the first statement.

Suppose now that σ is smooth and of dimension n in $N_\mathbb{R}$; denote by $v_1, v_2, \dots, v_n$ the basis of N given by the minimal integral vectors of the rays of σ, with the assumption that $v_1 = v_\tau$. By definition,

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n.$$

As a result, denoting by $(v_i^\vee)_{i=1,\dots,n}$ the basis of M dual to $(v_i)_{i=1,\dots,n}$, one has that

$$\langle v_i^\vee, u \rangle = \lambda_i \geq 0$$

for every $i \in \{1, \dots, n\}$ and for every $u = \sum_i \lambda_i v_i \in \sigma$. In particular, $v_\tau^\vee \in \sigma^\vee$. It is easy to check that $v_\tau^\vee$ is integrally valued on each element of N and hence it belongs to M. It follows then from $\langle v_\tau^\vee, v_\tau \rangle = 1$ that

$$(\chi^{v_\tau^\vee}) \subseteq \mathfrak{p}.$$

For the reverse inclusion, consider $m \in \sigma^\vee \cap M$ with $m \notin \tau^\perp$. By assumption, $\langle m, v_\tau \rangle \in \mathbb{Z}$ and $\langle m, v_\tau \rangle \geq 0$; moreover, since $m \notin \tau^\perp$, one has $\langle m, v_\tau \rangle \geq 1$. For each $u = \sum_i \lambda_i v_i \in \sigma$ one has

$$\langle m - v_\tau^\vee, u \rangle = \lambda_1 \langle m - v_\tau^\vee, v_\tau \rangle + \sum_{i \geq 2} \lambda_i \langle m, v_i \rangle \geq \lambda_1 (\langle m, v_\tau \rangle - 1) \geq 0.$$

As a result, $m - v_\tau^\vee \in \sigma^\vee \cap M$ and hence $\chi^m = \chi^{v_\tau^\vee} \cdot \chi^{m - v_\tau^\vee} \in (\chi^{v_\tau^\vee})$, completing the proof. □

Remark 4.2. The last statement of the previous lemma is not true for a general strongly convex polyhedral rational cone σ of maximal dimension in N. For example, if σ has more than n faces of dimension 1, the divisor class group of X_σ, which is generated by the classes of the orbit closures associated to the rays, turns out to be nontrivial, as a consequence of [Fulton 1993, Proposition on p. 63].

For a nonzero Laurent polynomial $f \in K[M]$, the subset $V(f)$ of zeros of f in X_0 is a closed subscheme of the dense open orbit. Its closure in X_Σ is a closed subscheme of X_Σ, denoted by $\overline{V(f)}$. Taking into account multiplicities, one can consider the associated Weil divisor $[\overline{V(f)}]$. It is the zero cycle when f is a monomial.

Theorem 4.3. *Let f be a nonzero Laurent polynomial and $\tilde{f}$ the rational function on X_Σ arising from f. Then,*

$$\operatorname{div}(\tilde{f}) = [\overline{V(f)}] + \sum_{\tau \in \Sigma^{(1)}} \Psi_{\mathrm{NP}(f)}(v_\tau) V(\tau),$$

where $\Psi_{\mathrm{NP}(f)}$ denotes the support function of the Newton polytope of f. In particular, $[\overline{V(f)}]$ is rationally equivalent to the cycle $-\sum_{\tau\in\Sigma^{(1)}} \Psi_{\mathrm{NP}(f)}(v_\tau)V(\tau)$ on X_Σ.

Proof. By [Fulton 1993, formula on p. 55], the irreducible components of $X_\Sigma \setminus X_0$ are exactly the orbit closures $V(\tau)$, with τ ranging in the set of 1-dimensional cones of Σ. Since moreover the restriction of $\tilde{f}$ to X_0 is the regular function f, it follows from the classical theory of divisors that

$$\operatorname{div}(\tilde{f}) = [\overline{V(f)}] + \sum_{\tau} \nu_\tau(\tilde{f}) V(\tau),$$

where $\nu_\tau(\tilde{f}) \in \mathbb{Z}$ is the order of vanishing of $\tilde{f}$ along $V(\tau)$. The statement of the theorem then follows from the fact that, for every $\tau \in \Sigma^{(1)}$, such an order equals $\Psi_{\mathrm{NP}(f)}(v_\tau)$.

This claim can be proved locally; fix a ray $\tau \in \Sigma^{(1)}$ and let σ be any maximal-dimensional cone of Σ containing τ. The fan being complete and consisting of smooth cones, such a σ exists and the minimal integral vectors $v_1, \dots, v_n$ of its rays are a basis of N. Assume moreover that $v_1 = v_\tau$ and, for simplicity, denote by $R := K[\sigma^\vee \cap M]$ the ring of regular functions over X_σ. The order of vanishing of $\tilde{f}$ along $V(\tau)$ is computed as the valuation of $\tilde{f}$ determined by the valuation ring $R_\mathfrak{p}$, the localization of R at the prime ideal $\mathfrak{p}$ corresponding to the subvariety $V(\tau)$ in X_σ. By Lemma 4.1, since the cone σ is smooth and maximal-dimensional, one has that $\mathfrak{p} = (\chi^{v_\tau^\vee})$. The maximal ideal $\mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$ is hence the principal ideal generated by $\chi^{v_\tau^\vee}$.

Suppose first that $\tilde{f} = \sum_m c_m \chi^m$ lies in R, that is, every m appearing in $\tilde{f}$ belongs to $\sigma^\vee \cap M$. By definition of the valuation in $R_\mathfrak{p}$,

$$\begin{aligned}\nu_\tau(\tilde{f}) &= \max\{l \in \mathbb{N} : \tilde{f} \in (\mathfrak{p}R_\mathfrak{p})^l\} = \max\,\{l \in \mathbb{N} : \tilde{f} \in (\chi^{l v_\tau^\vee})\} \\ &= \max\,\{l \in \mathbb{N} : \chi^{m - l v_\tau^\vee} \in R_\mathfrak{p} \text{ for all } m \text{ with } c_m \neq 0\}.\end{aligned}$$

The condition $\chi^{m - l v_\tau^\vee} \in R_\mathfrak{p}$ is equivalent to the fact that $\langle m, v_\tau\rangle \geq l$. Indeed, if the first is true, then

$$\langle m, v_\tau\rangle - l = \langle m - l v_\tau^\vee, v_\tau\rangle \geq 0.$$

Conversely, for each $u = \sum_i \lambda_i v_i \in \sigma$ one has

$$\langle m - l v_\tau^\vee, u\rangle = \lambda_1(\langle m, v_\tau\rangle - l) + \sum_{i\geq 2} \lambda_i \langle m, v_i\rangle \geq 0,$$

and so $m - l v_\tau^\vee \in \sigma^\vee \cap M$. As a consequence,

$$\begin{aligned} \nu_\tau(\tilde{f}) &= \max\{l \in \mathbb{N} : \langle m, v_\tau\rangle \geq l \text{ for all } m \text{ with } c_m \neq 0\} \\ &= \min\{\langle m, v_\tau\rangle : m \text{ with } c_m \neq 0\} = \Psi_{\mathrm{NP}(f)}(v_\tau). \end{aligned}$$

For a general $\tilde{f} = \sum_m c_m \chi^m$, the fact that $\sigma^\vee$ has dimension n in $M_\mathbb{R}$ (σ is indeed strongly convex) ensures that there exists a big enough vector $m_0 \in \sigma^\vee \cap M$ for which $m + m_0 \in \sigma^\vee \cap M$ for each m such that $c_m \neq 0$. Hence

$$\tilde{f} = \frac{\sum_m c_m \chi^{m+m_0}}{\chi^{m_0}},$$

with both the numerator and the denominator belonging to R. Applying the result for such elements one deduces

$$\nu_\tau(\tilde{f}) = \nu_\tau\left(\sum_m c_m \chi^{m+m_0}\right) - \nu_\tau(\chi^{m_0}) = \Psi_{\mathrm{NP}(f)+m_0}(v_\tau) - \langle m_0, v_\tau\rangle = \Psi_{\mathrm{NP}(f)}(v_\tau),$$

concluding the proof. □

5. Local and global heights of hypersurfaces

Fix for the whole section an adelic field $(K, (|\cdot|_v, n_v)_{v\in\mathfrak{M}})$ satisfying the product formula. Let X_Σ be a proper toric variety over K, of dimension n, with torus $\mathbb{T} = \operatorname{Spec} K[M]$ and dense open orbit X_0. For an effective cycle Z on X_Σ of pure codimension 1, whose prime components intersect X_0, we present a series of results concerning its integrability and its local and global height with respect to suitable choices of metrized divisors on X_Σ.

5A. ***Degrees.*** With the notation and assumptions given above, the effective cycle Z can be written as

$$Z = \sum_{i=1}^{r} \ell_i Y_i$$

for prime divisors $Y_1, \ldots, Y_r$ intersecting X_0. For every $i = 1, \ldots, r$, the closed irreducible subvariety of X_0 obtained as the intersection between Y_i and X_0 is associated to a prime ideal of height 1 in $K[M]$, which is principal since $K[M]$ is a unique factorization domain; denote by f_i an irreducible Laurent polynomial generating such an ideal. The Laurent polynomial $f = f_1^{\ell_1} \cdot \cdots \cdot f_r^{\ell_r}$ is called a *defining polynomial* for the cycle Z and is uniquely defined up to multiplication by an invertible element of $K[M]$, which means by a monomial. Moreover,

$$[\overline{V(f)}] = Z; \tag{5-1}$$

that is, the cycle associated to the closure of the subscheme $V(f)$ in X_Σ agrees with Z. Let

$$\Psi_f := \Psi_{\mathrm{NP}(f)}$$

be the support function, in the sense of Example 1.1, of the Newton polytope $\mathrm{NP}(f)$ of f; it is a piecewise linear function on $N_\mathbb{R}$. It is not necessarily a virtual support function on the fan Σ, but it can always be made such after a suitable refinement of the fan.

Lemma 5.1. *For any proper toric variety X_Σ there exist a smooth projective toric variety $X_{\Sigma'}$ with fan Σ' in $N_\mathbb{R}$ and a proper toric morphism $\pi : X_{\Sigma'} \to X_\Sigma$ satisfying*:

(1) *π restricts to the identity on the dense open orbit of $X_{\Sigma'}$ and X_Σ.*

(2) *Ψ_f is a virtual support function on Σ'.*

Proof. One can always refine the complete fan Σ to a fan Σ' in such a way that Ψ_f is a virtual support function on Σ'. After possibly refining again, one can suppose that Σ' is the fan of a projective toric variety (because of the toric Chow lemma, see [Cox et al. 2011, Theorem 6.1.18]) and that each of its cones is generated by a part of a basis of N; see [Fulton 1993, §2.6]. The associated toric variety $X_{\Sigma'}$ is smooth, projective and it satisfies (2). Finally, since Σ' is a refinement of Σ, the toric morphism π given by [Cox et al. 2011, Theorem 3.3.4] is proper and restricts to the identity on the dense open orbit of $X_{\Sigma'}$. □

The previous lemma, together with the fact that intersection-theoretical properties are stable under birational transformations, allows one to compute the degree of a cycle of codimension 1 in a toric variety with respect to a family of toric divisors generated by global sections.

Proposition 5.2. *Let $D_{\Psi_1}, \dots, D_{\Psi_{n-1}}$ be toric divisors on X_Σ generated by global sections and Z an effective cycle on X_Σ of pure codimension 1 and prime components intersecting X_0, with defining polynomial f. Then*

$$\deg_{D_{\Psi_1},\dots,D_{\Psi_{n-1}}}(Z) = \mathrm{MV}_M(\Delta_{\Psi_1}, \dots, \Delta_{\Psi_{n-1}}, \mathrm{NP}(f)),$$

where MV_M denotes the mixed volume function associated to the measure vol_M (see Remark 1.2) and Δ_{Ψ_i} the polytope associated to the toric divisor D_{Ψ_i} for each $i = 1, \dots, n-1$.

Proof. Consider the smooth projective toric variety $X_{\Sigma'}$ and the proper toric morphism $\pi : X_{\Sigma'} \to X_\Sigma$ given by Lemma 5.1. Since the support function Ψ_f is a virtual support function on Σ', one can consider the corresponding toric divisor D_f on $X_{\Sigma'}$ and the associated distinguished rational section s_f of $\mathscr{O}(D_f)$. The product $\tilde{f} s_f$ is a rational section of $\mathscr{O}(D_f)$, with associated Weil divisor satisfying

$$\pi_* \operatorname{div}(\tilde{f} s_f) = \pi_*(\operatorname{div}(\tilde{f}) + \operatorname{div}(s_f)) = Z$$

by Theorem 4.3, (3-2), (5-1) and the definition of π. The projection formula in [Fulton 1998, Proposition 2.3(c)] and Bézout's theorem yield

$$\deg_{D_{\Psi_1},\dots,D_{\Psi_{n-1}}}(Z) = \deg_{\pi^* D_{\Psi_1},\dots,\pi^* D_{\Psi_{n-1}}, D_f}(X_{\Sigma'}).$$

Since the function Ψ_f is concave, D_f is generated by global sections. Moreover, the virtual support function associated to the toric divisor $\pi^* D_{\Psi_i}$ on $X_{\Sigma'}$ agrees with Ψ_i for every $i = 1, \dots, n-1$. The combinatorial description in [Oda 1988, Proposition 2.10] of the degree of a toric variety with respect to toric divisor generated by global sections then concludes the proof. $\square$

Remark 5.3. By [Fulton 1993, formula on page 55], the irreducible components of $X_\Sigma \setminus X_0$ are the orbit closures $V(\tau)$, with τ ranging in the set of 1-dimensional cones of Σ. It follows that if Z is a prime divisor of X_Σ not intersecting X_0, it coincides with $V(\tau)$ for some $\tau \in \Sigma^{(1)}$. In such a case, the degree of Z with respect to a collection $D_{\Psi_1}, \dots, D_{\Psi_{n-1}}$ of toric divisors on X_Σ generated by global sections is given by

$$\deg_{D_{\Psi_1}, \dots, D_{\Psi_{n-1}}}(V(\tau)) = \mathrm{MV}_{M(v_\tau)}(\Delta_{\Psi_1}^{v_\tau}, \dots, \Delta_{\Psi_{n-1}}^{v_\tau}),$$

where v_τ is the minimal nonzero integral vector of τ; see [Burgos Gil et al. 2014, formulae (3.4.1) and (3.4.4)].

Remark 5.4. The reduction to the case of a smooth projective toric variety employed in the proof of Proposition 5.2 equally works when computing the local height of the cycle Z with respect to a family of v-adic semipositive toric metrized divisors $\overline{D}_{0,v}, \dots, \overline{D}_{n-1,v}$. Indeed, let f be a defining polynomial for Z, and $X_{\Sigma'}$ and π be as in the statement of Lemma 5.1. For every family of rational sections $s_0, \dots, s_{n-1}$ of $\mathscr{O}(D_0), \dots, \mathscr{O}(D_{n-1})$ respectively for which the following local heights are well-defined, the local arithmetic projection formula in [Burgos Gil et al. 2014, Theorem 1.4.17(2)] asserts that

$$h_{\overline{D}_{0,v}, \dots, \overline{D}_{n-1,v}}(Z; s_0, \dots, s_{n-1}) = h_{\pi^*\overline{D}_{0,v}, \dots, \pi^*\overline{D}_{n-1,v}}(Z'; \pi^* s_0, \dots, \pi^* s_{n-1}),$$

where Z' is the cycle in $X_{\Sigma'}$ associated to the subscheme obtained as the closure of $V(f)$ and has hence f as a defining polynomial. Because of [Burgos Gil et al. 2014, Proposition 4.3.19], the pull-back of $\overline{D}_{i,v}$ via π is a v-adic semipositive toric metrized divisor on $X_{\Sigma'}$ whose metric function coincides with the one of $\overline{D}_{i,v}$ for every $i = 0, \dots, n-1$ and $v \in \mathfrak{M}$. It follows that any combinatorial formula for the local height of Z' in $X_{\Sigma'}$ with respect to $\pi^*\overline{D}_{0,v}, \dots, \pi^*\overline{D}_{n-1,v}$ only involving the defining polynomial of Z and the metric functions of the metrized divisors equally holds for the local height of Z in X_Σ with respect to $\overline{D}_{0,v}, \dots, \overline{D}_{n-1,v}$. Similarly, the reduction step can be adopted when dealing with the integrability and the global height of Z, because of Proposition 3.8.

5B. *Local heights.* Let f be a defining polynomial for the cycle Z and, as in the previous subsection, denote by Ψ_f the support function of its Newton polytope. Under the assumption that Ψ_f is a virtual support function on the fan of X_Σ, it determines a toric divisor D_f on X_Σ together with a distinguished rational section s_f of $\mathscr{O}(D_f)$, as in Section 3C.

Definition 5.5. In the notation above, and for a place $v \in \mathfrak{M}$, the *v-adic Ronkin metric* on D_f is the v-adic semipositive toric metric on $\mathscr{O}(D_f)_v^{\mathrm{an}}$ corresponding to the v-adic Ronkin function $\rho_{f,v}$ via Theorem 3.9.

The previous definition makes sense since, for every $v \in \mathfrak{M}$, the v-adic Ronkin function of f is concave on $N_\mathbb{R}$ and has bounded difference from Ψ_f because of Proposition 2.10. If not otherwise specified, $\overline{D}_{f,v}$

will denote the divisor D_f equipped with the v-adic Ronkin metric $\|\cdot\|_{f,v}$ defined above. By definition,

$$\log \|s_f\|_{f,v} = \rho_{f,v} \circ \mathrm{trop}_v \tag{5-2}$$

on $X_{0,v}^{\mathrm{an}}$. To lighten the notation, we will drop the subscript v whenever the choice of the place is clear from the context.

Proposition 5.6. *Let f and g be two nonzero Laurent polynomials and assume that Ψ_f and Ψ_g are virtual support functions on the fan of X_Σ. Then,*

$$\overline{D}_f + \overline{D}_g = \overline{D}_{f\cdot g}.$$

Proof. The equality $\mathrm{NP}(f\cdot g) = \mathrm{NP}(f) + \mathrm{NP}(g)$ implies that $\Psi_{f\cdot g} = \Psi_f + \Psi_g$. In particular, $\Psi_{f\cdot g}$ is a virtual support function on the fan Σ and then defines a toric divisor $D_{f\cdot g}$ on X_Σ which satisfies $D_{f\cdot g} = D_f + D_g$ because of [Fulton 1993, §3.4]. The statement follows now from [Burgos Gil et al. 2014, Proposition 4.3.14(1)] and Proposition 2.9. ☐

The key property of the Ronkin metric is given in the following proposition.

Proposition 5.7. *Let X_Σ be a smooth projective toric variety and Z an effective cycle on X_Σ of pure codimension 1 and prime components intersecting X_0. Let f be a defining polynomial for Z and $\tilde{f}$ the associated rational function on X_Σ. Assume moreover that Ψ_f is a virtual support function on the fan Σ. For a fixed place v of K, let $\overline{D}_0, \dots, \overline{D}_{n-1}$ be toric divisors on X_Σ, equipped with v-adic semipositive toric metrics. Then*

$$h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z; s_0, \dots, s_{n-1}) = h_{\overline{D}_0,\dots,\overline{D}_{n-1},\overline{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f}s_f) \tag{5-3}$$

for every choice of rational sections $s_0, \dots, s_{n-1}$ of $\mathscr{O}(D_0), \dots, \mathscr{O}(D_{n-1})$ respectively with $\mathrm{div}(s_0), \dots, \mathrm{div}(s_{n-1})$, *$Z$ intersecting properly.*

Proof. The product $\tilde{f}s_f$ is a rational section of $\mathscr{O}(D_f)$ on X_Σ with associated Weil divisor

$$\mathrm{div}(\tilde{f}s_f) = \mathrm{div}(\tilde{f}) + \mathrm{div}(s_f) = Z$$

by Theorem 4.3, (3-2) and (5-1). Hence, the sections $s_0, \dots, s_{n-1}, \tilde{f}s_f$ meet X_Σ properly and the right-hand side term in (5-3) is well-defined.

Definition 3.6 states that

$$\begin{aligned} &h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z; s_0, \dots, s_{n-1}) \\ &\qquad = h_{\overline{D}_0,\dots,\overline{D}_{n-1},\overline{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f}s_f) + \int_{X_\Sigma^{\mathrm{an}}} \log \|\tilde{f}s_f\|_f \, c_1(\overline{D}_0) \wedge \dots \wedge c_1(\overline{D}_{n-1}), \end{aligned}$$

and thus the proposition follows from the vanishing of the integral on the right-hand side. Indeed, thanks to Theorem 3.11, such an integral is supported on the analytification of the dense open orbit of X_Σ, where the rational function $\tilde{f}$ coincides with the regular function f. Together with the definition of the Ronkin metric in (5-2), this yields

$$\int_{X_\Sigma^{\mathrm{an}}} \log \|\tilde{f} s_f\|_f \; c_1(\overline{D}_0)\wedge\cdots\wedge c_1(\overline{D}_{n-1}) \\ = \int_{X_0^{\mathrm{an}}} \log|f| \; c_1(\overline{D}_0)\wedge\cdots\wedge c_1(\overline{D}_{n-1}) + \int_{X_0^{\mathrm{an}}} (\rho_{f,v}\circ \mathrm{trop}_v)\; c_1(\overline{D}_0)\wedge\cdots\wedge c_1(\overline{D}_{n-1}).$$

For every $i=0,\dots,n-1$, denote by ψ_i the metric function of $\overline{D}_i$. The tropicalization map being continuous, the change of variables formula and Theorem 3.11 imply on the one hand that

$$\int_{X_0^{\mathrm{an}}} (\rho_{f,v}\circ \mathrm{trop}_v)\; c_1(\overline{D}_0)\wedge\cdots\wedge c_1(\overline{D}_{n-1}) = \int_{N_\mathbb{R}} \rho_{f,v}\, d\,\mathrm{M}\mathcal{M}_M(\psi_0,\dots,\psi_{n-1}).$$

On the other hand, Theorem 3.11, together with the change of variables formula and Fubini's theorem, gives

$$\int_{X_0^{\mathrm{an}}} \log|f| \; c_1(\overline{D}_0)\wedge\cdots\wedge c_1(\overline{D}_{n-1}) = \int_{N_\mathbb{R}} \left(\int_{\mathcal{B}_{\mathbb{C}_v}} (\log|f|\circ\iota_v)\, d\,\mathrm{Haar}_{\mathcal{B}_{\mathbb{C}_v}}\right) d\,\mathrm{M}\mathcal{M}_M(\psi_0,\dots,\psi_{n-1}).$$

The definition of the maps ι_v and $\rho_{f,v}$ ensures that the inner integral coincides with the opposite of the v-adic Ronkin function of f, concluding the proof. $\square$

5C. *Toric local heights.* Recall from Definition 3.10 that any toric divisor generated by global sections admits a distinguished v-adic semipositive toric metric, the canonical metric. This allows one to define a local height with respect to toric divisors that is independent of the choice of the sections. Such a notion was introduced in [Burgos Gil et al. 2014, §5.1] as a key step in the proof of the formula for the global height of a toric variety.

Definition 5.8. For a place v of K, let $\overline{D}_0,\dots,\overline{D}_d$ be toric divisors on X_Σ, endowed with v-adic semipositive toric metrics. Denote by $\overline{D}_0^{\mathrm{can}},\dots,\overline{D}_d^{\mathrm{can}}$ the same divisors equipped with their v-adic canonical metric. Let Y be an irreducible d-dimensional subvariety of X_Σ and $\varphi: Y'\to Y$ a birational morphism, with Y' projective. The *v-adic toric local height* of Y with respect to $\overline{D}_0,\dots,\overline{D}_d$ is defined as

$$h^{\mathrm{tor}}_{\overline{D}_0,\dots,\overline{D}_d}(Y) := h_{\varphi^*\overline{D}_0,\dots,\varphi^*\overline{D}_d}(Y'; s_0,\dots,s_d) - h_{\varphi^*\overline{D}_0^{\mathrm{can}},\dots,\varphi^*\overline{D}_d^{\mathrm{can}}}(Y'; s_0,\dots,s_d),$$

where s_i is a rational section of $\varphi^*\mathscr{O}(D_i)$ for every $i=0,\dots,d$ and $s_0,\dots,s_d$ meet Y' properly. The definition extends by linearity to any cycle of dimension d.

The toric local height of a cycle neither depends on the choice of the sections $s_0,\dots,s_d$, nor on the birational model Y' of Y because of [Burgos Gil et al. 2014, Theorem 1.4.17(2) and (3)]. Moreover, the definition is nonempty: Chow's lemma provides Y with a projective birational model, while the moving lemma ensures the existence of rational sections meeting Y' properly.

We prove here a formula for the toric local height of an effective cycle Z on X_Σ of pure codimension 1 and prime components intersecting X_0.

Theorem 5.9. *Let X_Σ be a proper toric variety, Z an effective cycle on X_Σ of pure codimension* 1 *and prime components intersecting X_0. For a place v of K, let $\overline{D}_0, \dots, \overline{D}_{n-1}$ be toric divisors on X_Σ, equipped with v-adic semipositive toric metrics. Then*

$$h^{\mathrm{tor}}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = \mathrm{MI}_M(\vartheta_0, \dots, \vartheta_{n-1}, \rho_{f,v}^\vee) + \deg_{D_0,\dots,D_{n-1}}(X_\Sigma) \cdot \rho_{f,v}(0),$$

where f is a defining polynomial for Z and ϑ_i is the roof function of $\overline{D}_i$ for $i = 0, \dots, n-1$.

Proof. Because of Remark 5.4, one can assume that X_Σ is a smooth projective toric variety on whose fan Ψ_f is a virtual support function. Thanks to the moving lemma, one can choose rational sections $s_0, \dots, s_{n-1}$ of $\mathscr{O}(D_0), \dots, \mathscr{O}(D_{n-1})$ respectively such that $\mathrm{div}(s_0), \dots, \mathrm{div}(s_{n-1}), Z$ intersect properly. Proposition 5.7 implies then that

$$h^{\mathrm{tor}}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = h_{\overline{D}_0,\dots,\overline{D}_{n-1},\overline{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f} s_f) - h_{\overline{D}_0^{\mathrm{can}},\dots,\overline{D}_{n-1}^{\mathrm{can}},\overline{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f} s_f).$$

By adding and subtracting the quantity

$$h_{\overline{D}_0^{\mathrm{can}},\dots,\overline{D}_{n-1}^{\mathrm{can}},\overline{D}_f^{\mathrm{can}}}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f} s_f)$$

on the right-hand side, one obtains that

$$h^{\mathrm{tor}}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = h^{\mathrm{tor}}_{\overline{D}_0,\dots,\overline{D}_{n-1},\overline{D}_f}(X_\Sigma) - h^{\mathrm{tor}}_{\overline{D}_0^{\mathrm{can}},\dots,\overline{D}_{n-1}^{\mathrm{can}},\overline{D}_f}(X_\Sigma).$$

Denote by Ψ_i the virtual support function on Σ associated to the toric divisor D_i for every $i = 0, \dots, n-1$. Thanks to [Burgos Gil et al. 2014, Corollary 5.1.9], the previous equality yields

$$h^{\mathrm{tor}}_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = \mathrm{MI}_M(\vartheta_0, \dots, \vartheta_{n-1}, \rho_{f,v}^\vee) - \mathrm{MI}_M(\Psi_0^\vee, \dots, \Psi_{n-1}^\vee, \rho_{f,v}^\vee).$$

Since they admit by hypothesis a semipositive toric metric, the toric divisors $D_0, \dots, D_{n-1}$ are generated by global sections. For every $i = 0, \dots, n-1$, the function Ψ_i is hence concave and conic and so it is the support function of the polytope $\Delta_i := \mathrm{stab}(\Psi_i) \subseteq M_\mathbb{R}$. The statement of the theorem follows from a combination of Example 1.1, Corollary 1.9 and the combinatorial expression for the degree of a toric variety with respect to toric divisors generated by their global sections; see for example [Oda 1988, Proposition 2.10]. □

5D. *Global heights.* To state the result concerning the global case, recall that for a defining polynomial f for Z, the support function of its Newton polytope is denoted by Ψ_f; whenever it is linear on each cone of a complete fan Σ, it defines a toric divisor D_f on the toric variety X_Σ, together with a distinguished rational section s_f of $\mathscr{O}(D_f)$.

Definition 5.10. In the above assumptions, the *Ronkin metric* on D_f is the choice, for every place $v \in \mathfrak{M}$, of the v-adic Ronkin metric on D_f defined in Definition 5.5.

Unless otherwise stated, $\overline{D}_f$ will denote the toric divisor D_f equipped with its Ronkin metric. By definition, it is a semipositive toric metrized divisor.

Lemma 5.11. *The Ronkin metric on D_f is adelic.*

Proof. For a nonarchimedean place $v \in \mathfrak{M}$, the function $\rho_{f,v}$ coincides with the tropicalization of the Laurent polynomial f, as claimed in Remark 2.8. The fact that f has finitely many nonzero coefficients and the second axiom in Definition 3.1 imply that $\rho_{f,v} = \Psi_f$ for all but finitely many nonarchimedean places. The statement follows then from Lemma 3.3. □

The definition of such a toric metrized divisor and the study of the local height of Z in Section 5B allow one to answer the question of the integrability of Z and to give a formula for its global height, implying Theorem 1 in the Introduction.

Theorem 5.12. *Let X_Σ be a proper toric variety and Z an effective cycle on X_Σ of pure codimension 1 and prime components intersecting X_0. Let $\overline{D}_0, \dots, \overline{D}_{n-1}$ be toric divisors on X_Σ, equipped with adelic semipositive toric metrics. Then, Z is integrable with respect to $\overline{D}_0, \dots, \overline{D}_{n-1}$ and its global height is given by*

$$h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = \sum_{v\in\mathfrak{M}} n_v \operatorname{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-1,v}, \rho_{f,v}^\vee),$$

where f is a defining polynomial for Z and $\vartheta_{i,v}$ is the roof function of $\overline{D}_{i,v}$ for every $i = 0, \dots, n-1$ and $v \in \mathfrak{M}$.

Proof. Because of Remark 5.4, one can assume that X_Σ is a smooth projective toric variety on whose fan Ψ_f is a virtual support function. Let hence $s_0, \dots, s_{n-1}$ be rational sections of $\mathscr{O}(D_0), \dots, \mathscr{O}(D_{n-1})$ respectively such that $\operatorname{div}(s_0), \dots, \operatorname{div}(s_{n-1}), Z$ intersect properly. Because of Proposition 5.7, the v-adic local height of Z with respect to the above choice of sections is given by

$$h_{\overline{D}_{0,v},\dots,\overline{D}_{n-1,v}}(Z; s_0, \dots, s_{n-1}) = h_{\overline{D}_{0,v},\dots,\overline{D}_{n-1,v},\overline{D}_{f,v}}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f} s_f). \tag{5-4}$$

Because of Lemma 5.11, each member of the family $\overline{D}_0, \dots, \overline{D}_{n-1}, \overline{D}_f$ is an adelic semipositive toric metrized divisor on X_Σ. As a consequence of the first assertion in [Burgos Gil et al. 2014, Proposition 5.2.4], X_Σ is integrable with respect to such a choice of metrized divisors and hence [loc. cit., Proposition 1.5.8(1)] allows one to conclude that

$$h_{\overline{D}_{0,v},\dots,\overline{D}_{n-1,v},\overline{D}_{f,v}}(X_\Sigma; s_0, \dots, s_{n-1}, \tilde{f} s_f) = 0$$

for all but finitely many places $v \in \mathfrak{M}$. Comparing with (5-4), one deduces that Z is integrable with respect to $\overline{D}_0, \dots, \overline{D}_{n-1}$.

From the same equality, the global height of Z is seen to satisfy

$$h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = h_{\overline{D}_0,\dots,\overline{D}_{n-1},\overline{D}_f}(X_\Sigma).$$

The formula for the global height of Z follows then from Theorem 3.13. □

Remark 5.13. As in Remark 5.3, if Z is an irreducible hypersurface on X_Σ not intersecting X_0 it coincides with $V(\tau)$ for a 1-dimensional cone τ of the fan Σ. In such a case, Z is integrable with respect to a family of adelic semipositive toric metrized divisors $\overline{D}_0, \dots, \overline{D}_{n-1}$ on X_Σ and its global height is given by

$$h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(V(\tau)) = \sum_{v\in\mathfrak{M}} n_v \operatorname{MI}_{M(v_\tau)}(\vartheta_{0,v}|_{\Delta_0^{v_\tau}}, \dots, \vartheta_{n-1,v}|_{\Delta_{n-1}^{v_\tau}});$$

see [Burgos Gil et al. 2014, Propositions 5.1.11 and 5.2.4]. In the previous formula, Δ_i is the polytope associated to the divisor D_i and $\vartheta_{i,v}$ is the roof function of $\overline{D}_{i,v}$ for every $i = 0, \dots, n-1$ and $v \in \mathfrak{M}$, while v_τ is the minimal nonzero integral vector of τ.

Remark 5.14. Local and global heights of cycles are symmetric and multilinear with respect to sums of semipositive metrized divisors, provided that all terms are defined. The formulas obtained for 1-codimensional cycles in toric varieties are consistent with these properties, the sum of semipositive toric metrized divisors corresponding to the sup-convolution of the associated roof functions, as shown in [Burgos Gil et al. 2014, Proposition 4.3.14(1)].

6. Examples

For a fixed adelic field $(K, (|\cdot|_v, n_v)_{v\in\mathfrak{M}})$ satisfying the product formula and a proper toric variety X_Σ over K, of dimension n, with torus $\mathbb{T} = \operatorname{Spec} K[M]$ and dense open orbit X_0, we apply in this section the formula in Theorem 5.12 to four particular cases. In the first one, we focus on specific hypersurfaces of X_Σ, while in the following three we make relevant choices of the metrized divisors.

6A. ***Binomial hypersurfaces.*** For a primitive vector m in M one can consider the Laurent binomial $f = \chi^m - 1$; it is irreducible in $K[M]$, as can be verified by considering its Newton polytope. Hence the closure Z in X_Σ of the subvariety $V(f)$ of the torus $\operatorname{Spec} K[M]$ is an irreducible hypersurface of X_Σ with defining polynomial f.

Let $\overline{D}_0, \dots, \overline{D}_{n-1}$ be toric divisors on X_Σ, equipped with adelic semipositive toric metrics, with $\vartheta_{i,v}$ the roof function of $\overline{D}_{i,v}$ for every $i = 0, \dots, n-1$ and $v \in \mathfrak{M}$. By Example 2.12, $\rho_{f,v}$ coincides for every $v \in \mathfrak{M}$ with the support function of the segment $\overline{0m}$ in $M_\mathbb{R}$. The formula in Theorem 5.12 implies then that Z is integrable with respect to $\overline{D}_0, \dots, \overline{D}_{n-1}$ and that its global height is given by

$$h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = \sum_{v\in\mathfrak{M}} n_v \operatorname{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-1,v}, \iota_{\overline{0m}}),$$

because of Example 1.1. Considering, as at the end of Section 1C, the quotient lattice $P := M/\mathbb{Z}m$ and the associated projection $\pi : M \to P$, Proposition 1.12 allows one to deduce

$$h_{\overline{D}_0,\dots,\overline{D}_{n-1}}(Z) = \sum_{v\in\mathfrak{M}} n_v \operatorname{MI}_P(\pi_*\vartheta_{0,v}, \dots, \pi_*\vartheta_{n-1,v}), \tag{6-1}$$

with $\pi_*\vartheta_{i,v}$ denoting the direct image of $\vartheta_{i,v}$ by π for every $i = 0, \dots, n-1$ and $v \in \mathfrak{M}$; see (1-4).

Remark 6.1. Let Q be the dual lattice of $P = M/\mathbb{Z}m$. The projection $\pi : M \to P$ induces an injective dual map $Q \to N$, with image $m^\perp \cap N$. By identifying Q with such an image, which is a saturated sublattice of N, one can consider the restriction of the fan Σ to $Q_{\mathbb{R}}$; its corresponding toric variety X_{Σ_Q} is proper and has torus Spec $K[P]$. It also comes with a toric morphism $\varphi : X_{\Sigma_Q} \to X_\Sigma$, whose restriction to the dense open orbit coincides with the closed immersion of split tori Spec $K[P] \to$ Spec $K[M]$ given by the surjection $\pi : M \to P$; see [Burgos Gil et al. 2014, pp. 81–83]. Finally, the push-forward of the cycle X_{Σ_Q} by φ is the cycle Z associated to the hypersurface defined by $\chi^m - 1$. Indeed, the image of φ coincides by properness with the closure in X_Σ of the image of Spec $K[P] \to$ Spec $K[M]$, which is an irreducible $(n-1)$-dimensional subscheme of Spec $K[M]$ contained in $V(\chi^m - 1)$ as $\chi^{\pi(m)} - 1 = 0$.

Hence, equality (6-1) can also be obtained from Theorem 3.13, the arithmetic projection formula and the fact that $\pi_* \vartheta_{i,v}$ is the roof function of the pull-back of $\bar{D}_{i,v}$ via π for every $i = 0, \dots, n-1$ and $v \in \mathfrak{M}$, because of [loc. cit., Propositions 4.3.19 and 2.3.8(3)].

6B. ***The canonical height.*** A toric divisor D on X_Σ generated by global sections admits by Definition 3.10 a distinguished semipositive toric metric at any place. The metrized divisor obtained by the choice of such a family of v-adic canonical metrics is denoted by $\bar{D}^{\mathrm{can}}$; it is an adelic semipositive toric metrized divisor.

For a cycle Z of dimension d in X_Σ, the *canonical global height* of Z with respect to a family $D_0, \dots, D_d$ of toric divisors on X_Σ generated by global sections is defined to be its global height with respect to $\bar{D}_0^{\mathrm{can}}, \dots, \bar{D}_d^{\mathrm{can}}$ and it is also denoted by $h^{\mathrm{can}}_{D_0,\dots,D_d}(Z)$. The machinery developed in the previous sections allows one to express the canonical global height of an effective cycle on X_Σ of pure codimension 1 via convex geometry.

Proposition 6.2. *Let Z be an effective cycle on X_Σ of pure codimension 1 and prime components intersecting X_0 and $D_0, \dots, D_{n-1}$ a family of toric divisors on X_Σ generated by global sections. The canonical global height of Z with respect to $D_0, \dots, D_{n-1}$ is given by*

$$h^{\mathrm{can}}_{D_0,\dots,D_{n-1}}(Z) = -\deg_{D_0,\dots,D_{n-1}}(X_\Sigma) \cdot \sum_{v \in \mathfrak{M}} n_v \rho_{f,v}(0)$$

for any choice of a defining polynomial f for Z.

Proof. Denoting by Ψ_i, for any $i = 0, \dots, n-1$, the function associated to D_i, the property of being globally generated implies that Ψ_i is the support function of the lattice polytope $\Delta_i := \mathrm{stab}(\Psi_i) \subseteq M_{\mathbb{R}}$. The roof function of $\bar{D}^{\mathrm{can}}_{i,v}$ is hence ι_{Δ_i} for every $i = 0, \dots, n-1$ and for every $v \in \mathfrak{M}$, because of Example 1.1. It follows from Theorem 5.12 and Corollary 1.9 that

$$h^{\mathrm{can}}_{D_0,\dots,D_{n-1}}(Z) = -\mathrm{MV}_M(\Delta_0, \dots, \Delta_{n-1}) \cdot \sum_{v \in \mathfrak{M}} n_v \rho_{f,v}(0),$$

with f any defining polynomial for Z. To conclude, recall that the degree of X_Σ with respect to $D_0, \dots, D_{n-1}$ is given by the mixed volume of the associated polytopes, as proved in [Oda 1988, Proposition 2.10]. □

The case of the base field $\mathbb{Q}$ with the adelic structure described in Example 3.4 is particularly interesting for arithmetic purposes. For a Laurent polynomial f in n variables and complex coefficients, one defines its (*logarithmic*) *Mahler measure* to be

$$m(f) := \frac{1}{(2\pi)^n} \int_{\theta_1,\dots,\theta_n \in [0,2\pi]} \log |f(e^{i\theta_1}, \dots, e^{i\theta_n})| \, d\theta_1 \cdots d\theta_n.$$

Such a quantity is notoriously difficult to compute and is sometimes related to special values of L-functions; see [Smyth 1981; Deninger 1997; Boyd 1998; Lalín 2008].

Maillot [2000, Proposition 7.2.1] expressed the canonical height of a hypersurface in a toric variety over $\mathbb{Q}$ in terms of the Mahler measure of the associated section. While its proof relies on the study of the arithmetic Chow ring of the ambient toric variety, we here deduce his result from Proposition 6.2.

Corollary 6.3 (Maillot). *In the hypotheses and notation of Proposition 6.2, assume moreover that the base adelic field is $\mathbb{Q}$ with its usual adelic structure. Let f be a defining polynomial for Z having as coefficients integers with greatest common divisor* 1. *Then,*

$$h^{\mathrm{can}}_{D_0,\dots,D_{n-1}}(Z) = \deg_{D_0,\dots,D_{n-1}}(X_\Sigma) \cdot m(f).$$

Proof. Let f be a defining polynomial for Z satisfying the assumptions. Because of Remark 2.8, for every nonarchimedean place v of $\mathbb{Q}$

$$\rho_{f,v}(0) = f^{\mathrm{trop}}(0) = 0.$$

At the unique archimedean place v of $\mathbb{Q}$ one has by definition that $\rho_{f,v}(0) = -m(f)$. The statement follows then directly from Proposition 6.2. □

6C. *The ρ-height.* The strategy adopted in the previous section to prove the main results of the paper suggests the introduction of a distinguished height function. Let Z be an effective cycle on X_Σ of pure codimension 1 and prime components intersecting X_0 and assume that the support function of the Newton polytope of a defining polynomial for Z is a virtual support function on the fan Σ. By Lemma 5.1, this is always the case up to a birational toric transformation. In this setting, the choice of a defining polynomial f for Z determines a toric divisor D_f on X_Σ and a distinguished toric metric on it, the Ronkin metric, as introduced in Definition 5.10. The so-obtained metrized divisor, which is denoted by $\overline{D}_f$, is an adelic semipositive toric metrized divisor by Lemma 5.11.

Definition 6.4. In the above hypotheses and notation, the *ρ-height* of Z, denoted by $h_\rho(Z)$, is defined as its global height with respect to $\overline{D}_f, \dots, \overline{D}_f$ for a choice of a defining polynomial f for Z.

As shown below, the ρ-height of Z is independent of the choice of the defining polynomial f. Even if it is not clear whether such a height has a significant geometrical interpretation or arithmetical application, it is straightforward to give a combinatorial formula for it.

Proposition 6.5. *In the above hypotheses and notation, the ρ-height of Z is given by*

$$h_\rho(Z) = (n+1)! \sum_{v \in \mathfrak{M}} n_v \int_{\mathrm{NP}(f)} \rho^\vee_{f,v} \, d\,\mathrm{vol}_M,$$

where f is a defining polynomial for Z and NP(f) *is its Newton polytope.*

Proof. The statement follows trivially from Theorem 5.12, Proposition 2.10(3) and the properties of mixed integrals. $\square$

Remark 6.6. The equality in Proposition 6.5 shows that the ρ-height of Z does not depend on the choice of a defining polynomial for it. Indeed, if f' is another such polynomial, it must satisfy $f' = c \cdot \chi^m \cdot f$ for some nonzero monomial $c \cdot \chi^m \in K[M]$. For every $v \in \mathfrak{M}$, one has then that

$$\rho_{f',v} = -\log|c|_v + m + \rho_{f,v}$$

by Proposition 2.9 and Example 2.11. The stated independence follows hence from the relation

$$\rho^{\vee}_{f',v} = \tau_m \rho^{\vee}_{f,v} + \log|c|_v$$

obtained using [Burgos Gil et al. 2014, Proposition 2.3.3] and from the product formula on K.

It is significant to stress that the formula in Proposition 6.5, though compact, is difficult to evaluate because of the complexity of the archimedean Ronkin function.

6D. *The Fubini–Study height.* As a last example, consider the ambient toric variety X_Σ to be the n-dimensional projective space over K. Denote by D_∞ the toric divisor on $\mathbb{P}^n_K$ whose associated Weil divisor is the hyperplane at infinity; the corresponding sheaf is the universal line bundle $\mathscr{O}(1)$ on $\mathbb{P}^n_K$. If not otherwise specified, the notation $\overline{D}_\infty$ will refer to D_∞ equipped with the Fubini–Study metric at archimedean places, see [Burgos Gil et al. 2014, Example 1.1.2], and the canonical one at nonarchimedean places, in the sense of Definition 3.10. It turns out that $\overline{D}_\infty$ is an adelic semipositive toric metrized divisor. Thanks to Theorem 5.12, any effective cycle Z on $\mathbb{P}^n_K$ of pure codimension 1 is then integrable with respect to $\overline{D}_\infty, \dots, \overline{D}_\infty$ and the corresponding global height

$$h_{\mathrm{FS}}(Z) := h_{\overline{D}_\infty,\dots,\overline{D}_\infty}(Z)$$

is called the *Fubini–Study height* of Z.

Remark 6.7. The Fubini–Study height defined here coincides with the one introduced in [Faltings 1991] and studied in [Philippon 1995]. Examples of the computation of such height for projective hypersurfaces can be found in [Cassaigne and Maillot 2000].

Specializing Theorem 5.12, one can write the Fubini–Study height of a projective hypersurface in terms of convex geometry. To do so, denote by $\mathfrak{M}_\infty$ the collection of archimedean places of K, which is a finite set by Lemma 3.3. After fixing an isomorphism $M \simeq \mathbb{Z}^n$, consider the standard simplex

$$\Delta^n := \{(x_1, \dots, x_n) : x_1 + \dots + x_n \le 1, x_i \ge 0 \text{ for all } i = 1, \dots, n\}$$

in $M_\mathbb{R} \simeq \mathbb{R}^n$ and, agreeing that $x_0 := 1 - \sum_{i=1}^n x_i$, set the function $\vartheta_{\mathrm{FS}} : \Delta^n \to \mathbb{R}$ to be

$$\vartheta_{\mathrm{FS}}(x) := -\frac{1}{2}\sum_{i=0}^{n} x_i \log x_i,$$

which is defined on the boundary of Δ^n by continuity.

Proposition 6.8. *Let Z be an effective cycle on $\mathbb{P}^n_K$ of pure codimension* 1 *and prime components intersecting X_0. The Fubini–Study height of Z is given by*

$$h_{\mathrm{FS}}(Z) = \sum_{v \in \mathfrak{M}_\infty} n_v \, \mathrm{MI}_M(\vartheta_{\mathrm{FS}}, \dots, \vartheta_{\mathrm{FS}}, \rho^\vee_{f,v}) - \sum_{v \in \mathfrak{M} \setminus \mathfrak{M}_\infty} n_v \rho_{f,v}(0),$$

where f is a defining polynomial for Z.

Proof. The roof functions of the metrized divisor $\overline{D}_\infty$ are given by the function ϑ_{FS} at archimedean places, as remarked in [Burgos Gil et al. 2014, Examples 2.4.3 and 4.3.9(2)] and by the indicator function of Δ^n at nonarchimedean places, by [loc. cit., Example 4.3.9(1)] and Example 1.1. The statement follows then from Theorem 5.12 and Corollary 1.9, together with the fact that $\mathrm{MV}_M(\Delta^n, \dots, \Delta^n) = 1$ because of the conventions introduced in Section 1B and Remark 1.2. □

The nonarchimedean contributions to the Fubini–Study height are easily computable, since for every Laurent polynomial f with set of coefficients $\Gamma(f)$,

$$-\rho_{f,v}(0) = \log \max_{c \in \Gamma(f)} |c|_v$$

if $v \in \mathfrak{M} \setminus \mathfrak{M}_\infty$. From this equality one easily obtains the following special case.

Corollary 6.9. *Assume the base adelic field to be $\mathbb{Q}$ with its usual adelic structure. The Fubini–Study height of an effective cycle Z on $\mathbb{P}^n_{\mathbb{Q}}$ of pure codimension* 1 *and prime components intersecting X_0 is given by*

$$h_{\mathrm{FS}}(Z) = \mathrm{MI}_M(\vartheta_{\mathrm{FS}}, \dots, \vartheta_{\mathrm{FS}}, \rho^\vee_{f,\infty}),$$

where f is a defining polynomial for Z whose coefficients are integers with greatest common divisor 1.

Because of the presence of an archimedean Ronkin function, the formula in Corollary 6.9 appears arduous to evaluate. It would anyway be interesting to use it to study arithmetical properties of projective hypersurfaces or recover similar results to the ones obtained in [Cassaigne and Maillot 2000].

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