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We investigate the structure of finite groups whose noncentral real class sizes have the same 2-part. In particular, we prove that such groups are solvable and have 2-length one. As a consequence, we show that a finite group is solvable if it has two real class sizes. This confirms a conjecture due to G. Navarro, L. Sanus and P. Tiep.

1. Introduction

Let G be a finite group. An element x of G is *real* if x and x^{-1} are conjugate in G. A conjugacy class x^G of G is real if x^G contains a real element. If x^G is a real class of G, then we call the size $|x^G|$ of x^G a *real class size*. We call $|x^G|$ a *noncentral real class size* if x is real and noncentral in G, i.e., x does not lie in the center $\mathbf{Z}(G)$ of G.

A classical result due to Burnside (see [Dornhoff 1971, Corollary 23.4]) states that a finite group is of odd order if and only if the identity element is the only real element. This result has been generalized by Chillag and Mann [1998], who showed that if a finite group G has only one real class size, or equivalently if every real element lies in $\mathbf{Z}(G)$, then G is isomorphic to a direct product of a 2-group and a group of odd order.

The main purpose of this paper is to prove the following.

Theorem A. *Let G be a finite group. If G has two real class sizes, then G is solvable.*

This confirms a conjecture due to G. Navarro, L. Sanus and P. Tiep. Theorem A is best possible in the sense that there are nonsolvable groups with exactly three real class sizes. In fact, the special linear group $\mathrm{SL}_2(q)$ of degree 2 over a finite field of size q, where $q \geq 7$ is a prime power and is congruent to -1 modulo 4, has three real class sizes, namely 1, $q(q-1)$ and $q(q+1)$ [Dornhoff 1971, Theorem 38.1], but $\mathrm{SL}_2(q)$ is nonsolvable.

The extremal condition as in Theorem A has been studied extensively in the literature for conjugacy classes as well as character degrees of finite groups. Itô [1953] has shown that if a finite group has only two class sizes, then it is nilpotent. The alternating group A_4 has two real class sizes, which are 1 and 3, but it is not nilpotent. So, we cannot replace solvability by nilpotency in Theorem A. Itô [1970] also showed that if a finite group has three class sizes, then it is solvable. Clearly, this cannot happen for real class sizes by our example in the previous paragraph.

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Recall that a character χ of a finite group G is *real-valued* if χ takes real values, or equivalently, χ coincides with its complex conjugate. Notice that the reality of conjugacy classes of a group can be read off from the character table of the group. This follows from the fact that an element $x \in G$ is real if and only if $\chi(x)$ is real for all complex irreducible characters χ of G [Dornhoff 1971, Lemma 23.2].

For the corresponding results in real-valued characters, Iwasaki [1980] and Moretó and Navarro [2008] have studied the structure of finite groups with two and three real-valued irreducible characters. They show that all these groups must be solvable and their Sylow 2-subgroups have a very restricted structure. By Brauer's lemma on character tables [Dornhoff 1971, Theorem 23.3], the number of real conjugacy classes and the number of real-valued irreducible characters of a finite group are the same. Thus the aforementioned results also give the structure of finite groups with at most three real conjugacy classes. For degrees of real-valued characters, Navarro, Sanus and Tiep [Navarro et al. 2009, Theorem B] proved that a finite group is solvable if it has at most three real-valued character degrees.

As already noted in [Navarro et al. 2009], any possible proof of Theorem A is complicated. Instead of giving a direct proof of Theorem A, we prove a much stronger result which implies Theorem A. For an integer $n \geq 1$ and a prime p, the p-part of n, denoted by n_p, is the largest power of p dividing n.

Theorem B. *Let G be a finite group. Suppose that all noncentral real class sizes of G have the same 2-part. Then G is solvable.*

In other words, if $|x^G|_2 = 2^a$ for all noncentral real elements $x \in G$, where $a \geq 0$ is a fixed integer, then G is solvable. In fact, we can say more about the structure of these groups.

Theorem C. *Let G be a finite group. Suppose that all noncentral real class sizes of G have the same 2-part. Then G has 2-length one.*

Recall that a group G is said to have 2-length one if there exist normal subgroups $N \leq K \leq G$ such that N and G/K have odd order and K/N is a 2-group. Theorem C confirms a conjecture proposed in [Tong-Viet 2013].

Several variations of Theorem B are simply not true. Indeed, if we weaken the hypothesis of Theorem B by assuming that $|x^G|_2$ is 1 or 2^a for all real elements $x \in G$, then G need not be solvable. For example, let $G = \mathrm{SL}_2(2^f)$ with $f \geq 2$. Then every element of G is real and $|x^G|_2 = 1$ or 2^f for all elements $x \in G$. We cannot restrict the hypothesis to only real elements of odd order as $\mathrm{SL}_2(7)$ has only one conjugacy class of noncentral real elements of odd order. Also, Theorem B does not hold for odd primes, at least for primes p with $p \equiv -1 \pmod 4$. In fact, we can take $G = \mathrm{PSL}_2(27)$ if $p = 3$ and $G = \mathrm{SL}_2(p)$ if $p \geq 7$. We can check that all noncentral real class sizes of G have the same p-part. There are also some examples for primes p with $p \equiv 1 \pmod 4$; for example, we can take $G = \mathrm{PSL}_3(3)$ if $p = 13$. (We are unable to find any example with $p = 5$.)

Theorems B and C can be considered as (weak) real conjugacy classes versions (only for even prime) of the famous Thompson's theorem on character degrees stating that if a prime p divides the degrees of all nonlinear irreducible complex characters of a finite group G, then G has a normal p-complement. The exact analog of Thompson's theorem does not hold for conjugacy classes. However, Casolo, Dolfi and

Jabara [Casolo et al. 2012] proved that for a fixed prime p, if all noncentral class sizes of a finite group G have the same p-part, then G is solvable and has a normal p-complement. Some real-valued characters versions of Thompson's theorem were obtained in [Navarro et al. 2009; Navarro and Tiep 2010].

In order to describe our strategy, we need some terminology and results from abstract group theory which can be found in Chapter 31 of [Aschbacher 2000]. For a finite group X, the *layer* of X, denoted by $\boldsymbol{E}(X)$, is the subgroup of X generated by all quasisimple subnormal subgroups (or *components*) of X. A finite group L is said to be *quasisimple* if L is perfect and $L/\boldsymbol{Z}(L)$ is a nonabelian simple group. The *generalized Fitting subgroup* of X, denoted by $\boldsymbol{F}^*(X)$, is the central product of $\boldsymbol{E}(X)$ and the Fitting subgroup $\boldsymbol{F}(X)$, the largest nilpotent normal subgroup of X. Bender's theorem states that $\boldsymbol{C}_X(\boldsymbol{F}^*(X)) \leq \boldsymbol{F}^*(X)$ (see, for example, [Aschbacher 2000, 31.13]).

Returning to our problem, for a finite group G, we denote by $\mathrm{Re}(G)$ the set of all real elements of G. Let G be a finite group with $|x^G|_2 = 2^a$ for all $x \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$, where $a \geq 0$ is a fixed integer. An important consequence of this hypothesis which is key to our proofs is that if $x \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ is a 2-element and $t \in G$ is a 2-element inverting x, then $\boldsymbol{C}_G(\langle x, t\rangle)$ has a normal Sylow 2-subgroup; in particular, if i is a noncentral involution of G, then $\boldsymbol{C}_G(i)$ has a normal Sylow 2-subgroup (Lemma 2.4).

Let $K = \boldsymbol{O}^{2'}(G)$. Then K satisfies the same hypothesis as G does (Lemma 2.5). Let H be the quotient group $K/\boldsymbol{O}_{2'}(K)$. Then $H = \boldsymbol{O}^{2'}(H)$ and $\boldsymbol{O}_{2'}(H) = 1$ which implies that $\boldsymbol{F}(H) = \boldsymbol{O}_2(H)$ and $\boldsymbol{F}^*(H) = \boldsymbol{O}_2(H)\boldsymbol{E}(H)$. We consider two cases according to whether $\boldsymbol{E}(H)$ is trivial or not. When $\boldsymbol{E}(H) = 1$, we have $\boldsymbol{F}^*(H) = \boldsymbol{O}_2(H)$. Using [Aschbacher 2000, 31.16],

$$\boldsymbol{F}^*(\boldsymbol{N}_H(U)) = \boldsymbol{O}_2(\boldsymbol{N}_H(U))$$

for all 2-subgroups $U \leq H$. Using this, we can show that H is a 2-group and Theorem B holds in this case (Theorem 3.3). When $\boldsymbol{E}(H)$ is nontrivial, it must be a quasisimple group (Lemma 4.1) and actually it is isomorphic to $\mathrm{SL}_2(q)$ with $q \geq 5$ an odd prime power (Theorem 4.3). Using some ad hoc arguments, we finally show that this cannot occur, proving Theorem B. For Theorem C, we know that G is solvable by Theorem B. Now by Bender's theorem, we know that $\boldsymbol{F}^*(H) = \boldsymbol{O}_2(H)$ and thus H is a 2-group by Theorem 3.3. It follows that G has 2-length one as wanted.

Finally, we would like to point out some connections to other classical results in the literature. If G is a finite group which satisfies the hypothesis of Theorem B, then the centralizer of every noncentral involution of G has a normal Sylow 2-subgroup. Finite groups in which the centralizers of involutions are 2-closed (a group is 2-closed if it has a normal Sylow 2-subgroup) have been studied in [Suzuki 1965]. These are the so-called (C)-groups. On the other hand, the groups in which all involutions are central were classified by R. Griess [1978]. In view of these results, it would be interesting to investigate the structure of finite groups in which in the centralizers of all noncentral involutions are 2-closed.

The paper is organized as follows. In Section 2, we collect some properties of groups whose noncentral real class sizes have the same 2-part. We prove Theorem C in Section 3 and Theorem B in Section 4. In the last section, we classify all finite quasisimple groups that can appear as composition factors of the groups considered in Section 4.

2. 2-parts of noncentral real class sizes

In this section, we collect some important properties of real classes as well as draw some consequences on the structure of groups satisfying the hypothesis of Theorem B. Recall that $\mathrm{Re}(G)$ is the set of all real elements of a finite group G. We begin with some properties of real elements and real class sizes.

Fix a nontrivial real element $x \in G$ and set $\boldsymbol{C}^*_G(x) = \{g \in G \mid x^g \in \{x, x^{-1}\}\}$. Then $\boldsymbol{C}^*_G(x)$ is a subgroup of G containing $\boldsymbol{C}_G(x)$ as a normal subgroup. If $t \in G$ such that $x^t = x^{-1}$, then $x^{t^2} = x$ so $t^2 \in \boldsymbol{C}_G(x)$. Assume that x is not an involution. We see that $t \in \boldsymbol{C}^*_G(x) \setminus \boldsymbol{C}_G(x)$ and if $h \in \boldsymbol{C}^*_G(x) \setminus \boldsymbol{C}_G(x)$, then $x^h = x^{-1} = x^t$ so that $ht^{-1} \in \boldsymbol{C}_G(x)$, or equivalently $h \in \boldsymbol{C}_G(x)t$. Thus $\boldsymbol{C}_G(x)$ has index 2 in $\boldsymbol{C}^*_G(x)$ and $\boldsymbol{C}^*_G(x) = \boldsymbol{C}_G(x)\langle t\rangle$. Hence $|x^G|$ is even whenever x is a real element with $x^2 \neq 1$. In particular, $|x^G|$ is even if x is a nontrivial real element of odd order. Notice that if $x \in \mathrm{Re}(G) \cap \boldsymbol{Z}(G)$, then $x^2 = 1$.

Lemma 2.1. *Let G be a finite group and let $N \trianglelefteq G$.*

(1) *If $x \in G$ is real, then every power of x is also real.*

(2) *If $x^g = x^{-1}$ for some $x, g \in G$, then $x^t = x^{-1}$ for some 2-element $t \in G$.*

(3) *If $x \in \mathrm{Re}(G)$ and $|x^G|$ is odd, then $x^2 = 1$.*

(4) *If $|G : N|$ is odd, then $\mathrm{Re}(G) = \mathrm{Re}(N)$.*

Proof. Let $x \in G$ be a real element. Then $x^g = x^{-1}$ for some $g \in G$. If k is any integer, then

$$(x^k)^g = (x^g)^k = (x^{-1})^k = (x^k)^{-1},$$

so x^k is real which proves (1). Write $o(g) = 2^a m$ with $(2, m) = 1$ and let $t = g^m$. Then t is a 2-element and $x^t = x^{g^m} = x^g = x^{-1}$ as $g^2 \in \boldsymbol{C}_G(x)$ and m is odd. This proves (2). Part (3) follows from the discussion above. For part (4), suppose that $N \trianglelefteq G$ and $|G/N|$ is odd. Let $x \in \mathrm{Re}(G)$. Then there exists a 2-element $t \in G$ inverting x by (2). As $|G/N|$ is odd and t is a 2-element, $t \in N$ and thus $x^{-2} = t^{-1}(xtx^{-1}) \in N$. Again, as $|G/N|$ is odd, $x \in N$ and so $x \in \mathrm{Re}(N)$. □

The first two claims of the following lemma are well-known. The last claim follows from [Dolfi et al. 2008, Proposition 6.4], whose proof uses the Baer–Suzuki theorem.

Lemma 2.2. *Let G be a finite group and let $N \trianglelefteq G$.*

(1) *If $x \in N$, then $|x^N|$ divides $|x^G|$.*

(2) *If $Nx \in G/N$, then $|(Nx)^{G/N}|$ divides $|x^G|$.*

(3) *G has no nontrivial real element of odd order if and only if G has a normal Sylow 2-subgroup.*

We next study the structure of finite groups in which all real 2-elements are central. The proof of the following result is similar to that of [Isaacs and Navarro 2010, Corollary C].

Lemma 2.3. *Let G be a finite group. Assume that all real 2-elements of G lie in $\boldsymbol{Z}(G)$. Then G has a normal 2-complement.*

Proof. Let P be a 2-subgroup of G. Then $\mathrm{Re}(P) \subseteq \boldsymbol{Z}(G)$ by the hypothesis of the lemma. Let $Q \leq \boldsymbol{N}_G(P)$ be a $2'$-group. Since $\mathrm{Re}(P) \subseteq \boldsymbol{Z}(G)$, Q centralizes all real elements of P. Now [Isaacs and Navarro 2010, Theorem B] implies that Q centralizes P. Hence, $Q \leq \boldsymbol{C}_G(P)$. It follows that $\boldsymbol{N}_G(P)/\boldsymbol{C}_G(P)$ is a 2-group. As P is chosen arbitrarily, the Frobenius normal p-complement theorem [Aschbacher 2000, 39.4] implies that G has a normal 2-complement. □

Let G be a finite group with $|G|_2 = 2^{a+b}$, where $a, b \geq 0$ are fixed integers. Observe that the two conditions $|x^G|_2 = 2^a$ for all elements $x \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ and $|\boldsymbol{C}_G(x)|_2 = 2^b$ for all $x \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ are equivalent. For brevity, we say that a finite group G is an $\mathcal{R}(a,b)$*-group* if G satisfies one of the two equivalent conditions above.

Lemma 2.4. *Let G be an $\mathcal{R}(a,b)$-group, let $x \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ and let $t \in G$ be a 2-element such that $x^t = x^{-1}$.*

(a) *If $o(x) = 2m$ for some integer $m > 1$, then $x^m \in \boldsymbol{Z}(G)$.*

(b) *If x is a 2-element, then $\boldsymbol{C}_G(\langle x, t\rangle)$ is 2-closed. Moreover, if x is an involution of G, then $\boldsymbol{C}_G(x)$ is 2-closed.*

Proof. For (a), suppose that $o(x) = 2m$ with $m > 1$. Notice that x^m is an involution, $t \in \boldsymbol{C}^*_G(x) \setminus \boldsymbol{C}_G(x)$ and $|\boldsymbol{C}_G(x)|_2 = 2^b$. We have $(x^m)^t = (x^m)^{-1} = x^m$, so $t \in \boldsymbol{C}_G(x^m)$ and hence $\boldsymbol{C}_G(x) \leq \boldsymbol{C}^*_G(x) \leq \boldsymbol{C}_G(x^m)$. Since $x^2 \neq 1$, we have $|\boldsymbol{C}^*_G(x)|_2 = 2|\boldsymbol{C}_G(x)|_2 = 2^{b+1}$. Therefore $|\boldsymbol{C}_G(x^m)|_2 \geq |\boldsymbol{C}^*_G(x)|_2 > 2^b$. Since G is an $\mathcal{R}(a,b)$-group, this forces $x^m \in \boldsymbol{Z}(G)$.

For (b), assume that x is a real 2-element inverted by t. Let $J := \boldsymbol{C}_G(\langle x, t\rangle)$. We first claim that J has no nontrivial real element of odd order. By contradiction, suppose that there exist $y, s \in J$ with $y^s = y^{-1}$, where $o(y) > 1$ is odd. Observe that $[x, y] = [x, s] = [t, y] = [t, s] = 1$ and $x^t = x^{-1}$, $y^s = y^{-1}$, so

$$(xy)^{st} = x^{st}y^{st} = x^t y^{ts} = x^t y^s = x^{-1}y^{-1} = y^{-1}x^{-1} = (xy)^{-1}.$$

So $xy \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$. Since $(o(x), o(y)) = 1$, $\boldsymbol{C}_G(xy) = \boldsymbol{C}_G(x) \cap \boldsymbol{C}_G(y) = \boldsymbol{C}_A(y)$, where $A = \boldsymbol{C}_G(x) \geq J$. Let U be a Sylow 2-subgroup of $\boldsymbol{C}_G(xy)$. Then U is also a Sylow 2-subgroup of both $\boldsymbol{C}_G(x)$ and $\boldsymbol{C}_G(y)$ as $|\boldsymbol{C}_G(xy)|_2 = |\boldsymbol{C}_G(x)|_2 = |\boldsymbol{C}_G(y)|_2 = 2^b$ (noting $x, y, xy \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$). Thus $\boldsymbol{C}_A(y)$ contains a Sylow 2-subgroup of A. Therefore $|y^A|$ is odd and hence $y^2 = 1$ by Lemma 2.1(3). (Note that $y \in \mathrm{Re}(J) \subseteq \mathrm{Re}(A)$.) However, as $o(y)$ is odd, $y = 1$, which is a contradiction. Thus J has no nontrivial real element of odd order and so by Lemma 2.2(3) J has a normal Sylow 2-subgroup as required. Finally, if x is an involution, then we can choose $t = 1$ and the result follows. □

In the next lemma, we show that every normal subgroup of odd index of an $\mathcal{R}(a,b)$-group is also an $\mathcal{R}(a,b)$-group.

Lemma 2.5. *Suppose that G is an $\mathcal{R}(a,b)$-group. Let K be a normal subgroup of G of odd index. Then K is also an $\mathcal{R}(a,b)$-group.*

Proof. Let K be a normal subgroup of G of odd index. By Lemma 2.1(4), we have $\mathrm{Re}(G) = \mathrm{Re}(K)$. Let $x \in \mathrm{Re}(K) \setminus \boldsymbol{Z}(K)$ and let $C := \boldsymbol{C}_G(x)$. Let $P \in \mathrm{Syl}_2(C)$ and let $S \in \mathrm{Syl}_2(G)$ such that $P \leq S$. Since

$|G:K|$ is odd, we have $P \leq S \leq K$. In particular, $S \in \mathrm{Syl}_2(K)$. We have $P \leq K \cap C = \boldsymbol{C}_K(x) \leq C$. Thus P is also a Sylow 2-subgroup of $\boldsymbol{C}_K(x)$. Therefore $|C|_2 = |\boldsymbol{C}_K(x)|_2$ and hence K is an $\mathcal{R}(a,b)$-group. □

The first part of the following lemma is essentially Lemma 2.2 in [Guralnick et al. 2011] and its proof. For the reader's convenience, we include the proof here.

Lemma 2.6. *Let G be a finite group and let $N \trianglelefteq G$ be a normal subgroup of odd order. Let $xN \in G/N$ be a real element. Write $\overline{G} = G/N$.*

(1) *There exist $y \in G$ and a 2-element $t \in G$ such that $\bar{x} = \bar{y}$ and $y^t = y^{-1}$.*

(2) *If $\bar{x}$ is a 2-element, then y can be chosen to be a 2-element. Moreover, if $\bar{x}$ is an involution in $\overline{G}$, then y can be chosen to be an involution in G.*

(3) *In (2), we have $|\bar{x}^{\overline{G}}|_2 = |y^G|_2$ and $\boldsymbol{C}_{\overline{G}}(\langle \bar{x}, \bar{t}\rangle) = \overline{\boldsymbol{C}_G(\langle y, t\rangle)}$.*

Proof. Let xN be a real element of G/N. Then there exists a 2-element $tN \in G/N$ inverting xN by Lemma 2.1(2). By considering the 2-part of t, we can assume that t is a 2-element. The maps $z \mapsto z^{-1}$ and $z \mapsto z^t$ are commuting permutations of G having 2-power order and thus their product σ has 2-power order. Each of these two permutations maps xN to $x^{-1}N$ and $x^{-1}N$ to xN. So σ defines a permutation on xN. Since σ has 2-power order and $|xN| = |N|$ is odd, σ has a fixed point $y \in xN$. Thus $y = y^\sigma = (y^t)^{-1}$. So $y^t = y^{-1}$ as wanted. This proves (1).

Assume next that the order of xN in G/N is 2^k for some integer $k \geq 0$. By part (1), there exist elements $y, t \in G$ such that $xN = yN$ and $y^t = y^{-1}$. Since $|N|$ is odd, $o(y) = 2^k m$, where m is odd and $y^{2^k} \in N$. Since $(2^k, m) = 1$, there exist integers u, v such that $1 = um + v2^k$ and thus $yN = (y^{um}N)(y^{v2^k}N) = y^{um}N$. Clearly, y^{um} is a 2-element and is inverted by t. Replace y by y^{um}, the first claim of part (2) follows. Moreover, as $|N|$ is odd, if $y^2 \in N$ and y is a 2-element, then $y^2 = 1$. Hence the last claim of (2) follows.

Finally, for part (3), by [Isaacs 2008, Lemma 7.7], $\boldsymbol{C}_{\overline{G}}(\bar{x}) = \boldsymbol{C}_{\overline{G}}(\bar{y}) = \overline{\boldsymbol{C}_G(y)}$ since $(o(y), |N|) = 1$. If U is a Sylow 2-subgroup of $\boldsymbol{C}_G(y)$, then $\overline{U}$ is a Sylow 2-subgroup of $\overline{\boldsymbol{C}_G(y)}$ and $|U| = |\overline{U}|$ so $|\bar{y}^{\overline{G}}|_2 = |y^G|_2$. Finally, we have $\boldsymbol{C}_{\overline{G}}(\langle \bar{x}, \bar{t}\rangle) = \boldsymbol{C}_{\overline{G}}(\langle \bar{y}, \bar{t}\rangle) = \overline{\boldsymbol{C}_G(\langle y, t\rangle)}$, where the second equality follows from [Isaacs 2008, Lemma 7.7] again as $\langle y, t\rangle$ is a 2-group. □

In the next lemma, we determine some properties of the quotient group $G/\boldsymbol{O}_{2'}(G)$.

Lemma 2.7. *Let G be an $\mathcal{R}(a,b)$-group and T a subgroup of G containing $\boldsymbol{O}_{2'}(G)$. Let $\overline{G} = G/\boldsymbol{O}_{2'}(G)$.*

(1) *If $\bar{x} \in \mathrm{Re}(\overline{T}) \setminus \boldsymbol{Z}(\overline{T})$ is a 2-element, then there exists a 2-element $\bar{t} \in \overline{T}$ inverting $\bar{x}$ and $\boldsymbol{C}_{\overline{G}}(\langle \bar{x}, \bar{t}\rangle)$ has a normal Sylow 2-subgroup.*

(2) *If $\bar{x} \in \mathrm{Re}(\overline{T}) \setminus \boldsymbol{Z}(\overline{T})$ is an involution, then $\boldsymbol{C}_{\overline{G}}(\bar{x})$ is 2-closed.*

(3) *If $\bar{z}, \bar{x} \in \mathrm{Re}(\overline{G}) \setminus \boldsymbol{Z}(\overline{G})$ and $\bar{x}$ is a 2-element, then $|\bar{z}^{\overline{G}}|_2 \leq |\bar{x}^{\overline{G}}|_2 = 2^a$ and $|\boldsymbol{C}_{\overline{G}}(\bar{x})|_2 = 2^b$.*

Proof. For part (1), let $\bar{x} \in \mathrm{Re}(\overline{T}) \setminus \boldsymbol{Z}(\overline{T})$ be a 2-element. By applying Lemma 2.6 for T, there exist 2-elements $y, t \in T$ such that $\bar{x} = \bar{y}$ and $y^t = y^{-1}$. Since $\bar{y} = \bar{x} \notin \boldsymbol{Z}(\overline{T})$, $y \notin \boldsymbol{Z}(T)$ and so $y \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$. By Lemma 2.4(b), $\boldsymbol{C}_G(\langle y, t\rangle)$ is 2-closed and thus by Lemma 2.6(3), $\boldsymbol{C}_{\overline{G}}(\langle \bar{x}, \bar{t}\rangle) = \boldsymbol{C}_{\overline{G}}(\langle \bar{y}, \bar{t}\rangle) = \overline{\boldsymbol{C}_G(\langle y, t\rangle)}$ is 2-closed.

Part (2) follows from Lemma 2.6(2) and part (1) above. For part (3), let $\bar{z}, \bar{x} \in \mathrm{Re}(\overline{G}) \setminus \boldsymbol{Z}(\overline{G})$, where $\bar{x}$ is a 2-element. By Lemma 2.6(1), there exists $w \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ with $\bar{w} = \bar{z}$. We have that $|\bar{w}^{\overline{G}}|_2 \leq |w^G|_2 = 2^a$ as $|\bar{w}^{\overline{G}}|$ divides $|w^G|$. By Lemma 2.6(3), $|\bar{x}^{\overline{G}}|_2 = |y^G|_2 = 2^a$, which implies that $|\boldsymbol{C}_{\overline{G}}(\bar{x})|_2 = 2^b$. The proof is now complete. □

3. Proof of Theorem C

In this section, we prove Theorem C assuming the solvability from Theorem B. Indeed, Theorem C follows immediately from Theorem 3.3. It is convenient to state the following.

Hypothesis A. Let G be a finite group with $\boldsymbol{O}_{2'}(G) = 1$ and $G = \boldsymbol{O}^{2'}(G)$. Let $a, b \geq 0$ be fixed integers. Assume the following hold.

(1) If $T \leq G$ and $x \in \mathrm{Re}(T) \setminus \boldsymbol{Z}(T)$ is a 2-element, then there exists a 2-element $t \in T$ inverting x and $\boldsymbol{C}_G(\langle x, t\rangle)$ is 2-closed.

(2) If $x \in G \setminus \boldsymbol{Z}(G)$ is an involution, then $\boldsymbol{C}_G(x)$ is 2-closed.

(3) If $x \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ is a 2-element, then $|x^G|_2 = 2^a$ and $|\boldsymbol{C}_G(x)|_2 = 2^b$.

(4) If $z \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$, then $|z^G|_2 \leq 2^a$ and $|\boldsymbol{C}_G(z)|_2 \geq 2^b$.

Using Lemmas 2.5 and 2.7, we show that if a finite group G satisfies the hypothesis of Theorem B, then a certain section of G satisfies the hypothesis above.

Lemma 3.1. *If G is an $\mathcal{R}(a, b)$-group, $K = \boldsymbol{O}^{2'}(G)$ and $H = K/\boldsymbol{O}_{2'}(K)$, then H satisfies Hypothesis A.*

Proof. Let G be an $\mathcal{R}(a, b)$-group and let $K = \boldsymbol{O}^{2'}(G)$. We know from Lemma 2.5 that K is also an $\mathcal{R}(a, b)$-group. Let $H = K/\boldsymbol{O}_{2'}(K)$. Then $\boldsymbol{O}^{2'}(H) = H$, $\boldsymbol{O}_{2'}(H) = 1$ and H satisfies the conclusion of Lemma 2.7. Therefore, H satisfies Hypothesis A. □

The following is an easy consequence of Lemma 2.3.

Lemma 3.2. *Let G be a finite group. Assume that $|\boldsymbol{C}_G(u)|_2 = 2^b$ for all 2-elements $u \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$. If $U \leq G$ is a 2-subgroup of G with $|U| \geq 2^b$, then $\boldsymbol{C}_G(U)$ has a normal 2-complement.*

Proof. Let $U \leq G$ be a 2-group with $|U| \geq 2^b$. Let $C = \boldsymbol{C}_G(U)$. Observe that if all 2-elements in $\mathrm{Re}(C)$ lie in $\boldsymbol{Z}(C)$, then C has a normal 2-complement by Lemma 2.3. Thus, by contradiction, assume that there exists a real 2-element $y \in \mathrm{Re}(C) \setminus \boldsymbol{Z}(C)$. Clearly $y \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ and so $|\boldsymbol{C}_G(y)|_2 = 2^b$. We see that $U \leq \boldsymbol{C}_G(y)$ as $y \in \boldsymbol{C}_G(U)$, so $\langle U, y\rangle$ is a 2-subgroup of $\boldsymbol{C}_G(y)$. Since $|\boldsymbol{C}_G(y)|_2 = 2^b$ and $|U| \geq 2^b$, $U \in \mathrm{Syl}_2(\boldsymbol{C}_G(y))$, which implies that $y \in U$. But this implies that $y \in \boldsymbol{Z}(C)$ since $[C, U] = 1$ and $y \in U$, a contradiction. □

We now prove the key result of this section.

Theorem 3.3. *Let G be a finite group satisfying Hypothesis A. If $\boldsymbol{F}^*(G) = \boldsymbol{O}_2(G)$, then G is a 2-group.*

Proof. Suppose that G satisfies Hypothesis A and $\boldsymbol{F}^*(G) = \boldsymbol{O}_2(G)$ but G is not a 2-group. Notice that $G = \boldsymbol{O}^{2'}(G)$. If G has no nontrivial real element of odd order, then G has normal Sylow 2-subgroup by Lemma 2.2(3). But then since $G = \boldsymbol{O}^{2'}(G)$, it must be a 2-group. Therefore, we may assume that there exists an element $z \in \mathrm{Re}(G) \setminus \boldsymbol{Z}(G)$ of odd order.

Let $V \in \mathrm{Syl}_2(\boldsymbol{C}_G(z))$. By Hypothesis A(4), $|V| \geq 2^b$. Clearly, by Hypothesis A(3), G satisfies the hypothesis of Lemma 3.2 and so $\boldsymbol{C}_G(V)$ has a normal 2-complement, say W. Then $W = \boldsymbol{O}_{2'}(\boldsymbol{C}_G(V))$ and $\boldsymbol{C}_G(V)/W$ is a 2-group. Now, we see that $W \trianglelefteq \boldsymbol{C}_G(V) \trianglelefteq \boldsymbol{N}_G(V)$ and W is characteristic in $\boldsymbol{C}_G(V)$ so $W \leq \boldsymbol{O}_{2'}(\boldsymbol{N}_G(V))$. However, by [Aschbacher 2000, 31.16], as $\boldsymbol{F}^*(G) = \boldsymbol{O}_2(G)$, we have $\boldsymbol{F}^*(\boldsymbol{N}_G(V)) = \boldsymbol{O}_2(\boldsymbol{N}_G(V))$, which forces $\boldsymbol{O}_{2'}(\boldsymbol{N}_G(V)) = 1$ (by Bender's theorem [Aschbacher 2000, 31.13]). Hence $W = 1$, so $\boldsymbol{C}_G(V)$ is a 2-group, which is impossible since $z \in \boldsymbol{C}_G(V)$ is a nontrivial element of odd order. This contradiction shows that G is a 2-group. □

Assuming the solvability from Theorem B, we can now prove Theorem C, which is included in the following.

Theorem 3.4. *Let G be an $\mathcal{R}(a, b)$-group. Then $\boldsymbol{O}^{2'}(G)$ has a normal 2-complement N. So G has 2-length one. Moreover, the 2-group $\boldsymbol{O}^{2'}(G)/N$ has at most two real class sizes.*

Proof. By Theorem B, we assume that G is solvable. Let $K = \boldsymbol{O}^{2'}(G)$ and $H := K/\boldsymbol{O}_{2'}(K)$. Then H satisfies Hypothesis A by Lemma 3.1. As H is solvable and $\boldsymbol{O}_{2'}(H) = 1$, $\boldsymbol{F}^*(H) = \boldsymbol{F}(H) = \boldsymbol{O}_2(H)$. By Theorem 3.3, H is a 2-group and thus K has a normal 2-complement $N = \boldsymbol{O}_{2'}(K)$. So G has 2-length one. Finally, as H is a 2-group which satisfies Hypothesis A, for any $x \in \mathrm{Re}(H)$, we have $|x^H| = 1$ or 2^a. □

4. Proof of Theorem B

We prove Theorem B in this section. We first prove some reduction results.

Lemma 4.1. *Let G be a finite nonsolvable group satisfying Hypothesis A. Let $\overline{G} = G/\boldsymbol{O}_2(G)$. Then $L = \boldsymbol{E}(G)$ is a quasisimple group whose center is a 2-group and $\overline{G}$ is an almost simple group with socle $\bar{L} \cong L/\boldsymbol{Z}(L)$.*

Proof. Since G satisfies Hypothesis A, $G = \boldsymbol{O}^{2'}(G)$ and $\boldsymbol{O}_{2'}(G) = 1$. We have $\boldsymbol{F}^*(G) = \boldsymbol{O}_2(G)\boldsymbol{E}(G)$. As G is nonsolvable, $\boldsymbol{E}(G)$ is nontrivial by Theorem 3.3. Let L be a component of G, that is, L is a perfect quasisimple subnormal subgroup of G. Recall that $\boldsymbol{E}(G)$ is generated by all components of G.

We first claim that $L = \boldsymbol{E}(G)$. Suppose by contradiction that G has another component, say $L_1 \neq L$. Since L is nonsolvable, by Lemma 2.3 there exists a real 2-element $x \in \mathrm{Re}(L) \setminus \boldsymbol{Z}(L)$. By Hypothesis A(1), there exists a 2-element $t \in L$ such that $x^t = x^{-1}$ and $\boldsymbol{C}_G(\langle x, t\rangle)$ has a normal Sylow 2-subgroup, so it is solvable. However, by [Aschbacher 2000, 31.5], $[L_1, L] = 1$, and since $\langle x, t\rangle \leq L$, we have $L_1 \leq \boldsymbol{C}_G(\langle x, t\rangle)$, which is impossible. Thus L is the unique component of G and so $L = \boldsymbol{E}(G)$. As $\boldsymbol{O}_{2'}(G) = 1$, $\boldsymbol{Z}(L)$ is a 2-group.

Let $C = \boldsymbol{C}_G(L)$. We claim that $C = \boldsymbol{O}_2(G)$, which implies that $\overline{G}$ is almost simple with socle $\bar{L}$. In fact, we show that every real 2-element of C lies in $\boldsymbol{Z}(C)$ and so by Lemma 2.3, C has a normal 2-complement which is $\boldsymbol{O}_{2'}(C)$. Since $L \trianglelefteq G$, $C \trianglelefteq G$ and so $\boldsymbol{O}_{2'}(C) \leq \boldsymbol{O}_{2'}(G) = 1$. Hence C is a 2-group and thus $C \leq \boldsymbol{O}_2(G)$. Furthermore, by [Aschbacher 2000, 31.12], $[\boldsymbol{O}_2(G), L] = 1$ so $\boldsymbol{O}_2(G) \leq C$. Therefore, $C = \boldsymbol{O}_2(G)$ as wanted.

To finish the proof, suppose by contradiction that there exists a real 2-element $y \in \mathrm{Re}(C)$ which is not in $\boldsymbol{Z}(C)$. By Hypothesis A(1), there exists a 2-element $t \in C$ inverting y, and $\boldsymbol{C}_G(\langle y, t\rangle)$ has a normal Sylow 2-subgroup. As $[L, C] = 1$ and $\langle y, t\rangle \leq C$, we have $L \leq \boldsymbol{C}_G(\langle y, t\rangle)$, which is impossible. This completes our proof. □

In order to classify all the possible finite quasisimple groups which can appear as $\boldsymbol{E}(G)$ in the previous lemma, we need the following.

Lemma 4.2. *Let G be a finite nonsolvable group satisfying Hypothesis A, and let $L = \boldsymbol{E}(G)$ and $\overline{G} = G/\boldsymbol{O}_2(G)$. If $x, z \in \mathrm{Re}(L) \setminus \boldsymbol{Z}(L)$ and x is a 2-element, then*

$$|z^L|_2 \leq |z^G|_2 \leq 2^a = |x^G|_2 \leq |\overline{G} : \bar{L}|_2 \cdot |x^L|_2 \leq |\mathrm{Out}(\bar{L})|_2 \cdot |x^L|_2. \tag{4-1}$$

Proof. Notice that $L = \boldsymbol{E}(G)$ is a quasisimple group by Lemma 4.1, and by [Aschbacher 2000, 31.12], $[\boldsymbol{O}_2(G), L] = 1$. Let $x, z \in \mathrm{Re}(L) \setminus \boldsymbol{Z}(L)$, where x is a 2-element. By Hypothesis A(3) and (4), $|z^G|_2 \leq 2^a = |x^G|_2$. Since $L \trianglelefteq G$, $|z^L|_2 \leq |z^G|_2 \leq 2^a$. For the remaining inequalities, observe that $\boldsymbol{O}_2(G) \leq \boldsymbol{C}_G(x)$ and

$$|x^G| = |G : L\boldsymbol{C}_G(x)| \cdot |L\boldsymbol{C}_G(x) : \boldsymbol{C}_G(x)|.$$

We have $|L\boldsymbol{C}_G(x) : \boldsymbol{C}_G(x)| = |L : \boldsymbol{C}_L(x)| = |x^L|$ and

$$|G : L\boldsymbol{C}_G(x)| = |G : L\boldsymbol{O}_2(G)| \,/\, |\boldsymbol{C}_G(x) : \boldsymbol{C}_G(x) \cap L\boldsymbol{O}_2(G)|.$$

As $|G : L\boldsymbol{O}_2(G)| = |\overline{G} : \bar{L}|$ divides $|\mathrm{Out}(\bar{L})|$, by taking the 2-parts, we obtain

$$|x^G|_2 \leq |\overline{G} : \bar{L}|_2 \cdot |x^L|_2 \leq |\mathrm{Out}(\bar{L})|_2 \cdot |x^L|_2.$$

The proof is now complete. □

Using the classification of finite simple groups, we can show that L in the previous lemma is isomorphic to a special linear group $\mathrm{SL}_2(q)$, where $q \geq 5$ is an odd prime power. We defer the proof of the following theorem until next section.

Theorem 4.3. *Let L be a quasisimple group with center Z and let $S = L/Z$. Suppose that Z is a 2-group and the following conditions hold*:

(1) *If i is a noncentral involution of L, then $\boldsymbol{C}_L(i)$ is 2-closed.*

(2) *If $x, z \in L$ are noncentral real elements and x is a 2-element, then $|z^L|_2 \leq |\mathrm{Out}(S)|_2 \cdot |x^L|_2$.*

Then $L \cong \mathrm{SL}_2(q)$ with $q \geq 5$ odd.

Let p be a prime. If $n \geq 1$ is an integer, then the p-adic valuation of n, denoted by $\nu_p(n)$, is the highest exponent ν such that p^ν divides n. Hence $n_p = p^{\nu_p(n)}$. Notice that $\nu_p(xy) = \nu_p(x) + \nu_p(y)$ for all integers $x, y \geq 1$.

The following number theoretic result is obvious.

Lemma 4.4. *Let $m, k \geq 1$ be integers, where $m \geq 3$ is odd. Then $\nu_2\big(m^{2^k} - 1\big) \geq k + 2$.*

Proof. We proceed by induction on $k \geq 1$. Suppose that $m \equiv \epsilon \pmod 4$, where $\epsilon = \pm 1$. For the base case, assume $k = 1$. Clearly, $m - \epsilon$ is divisible by 4 while $m + \epsilon$ is divisible by 2. So $\nu_2(m^2 - 1) \geq 3 = k + 2$.

Assume that $\nu_2\big(m^{2^t} - 1\big) \geq t + 2$ for some integer $t \geq 1$. We have

$$m^{2^{t+1}} - 1 = \big(m^{2^t} - 1\big)\big(m^{2^t} + 1\big).$$

Since m is odd, $\nu_2\big(m^{2^t} + 1\big) \geq 1$ and $\nu_2\big(m^{2^t} - 1\big) \geq t + 2$ by the induction hypothesis. Therefore,

$$\nu_2\big(m^{2^{t+1}} - 1\big) = \nu_2\big(m^{2^t} + 1\big) + \nu_2\big(m^{2^t} - 1\big) \geq 1 + (t+2) = (t+1) + 2.$$

By induction, the lemma follows. □

We are now ready to prove Theorem B, which we restate here.

Theorem 4.5. *If G is an $\mathcal{R}(a, b)$-group, then G is solvable.*

Proof. Let G be a counterexample to the theorem of minimal order. Then G is nonsolvable and by Lemma 3.1, H satisfies Hypothesis A, where $H = K/\boldsymbol{O}_{2'}(K)$ and $K = \boldsymbol{O}^{2'}(G)$. As G is nonsolvable, H is nonsolvable. Let $L = \boldsymbol{E}(H)$. By Lemmas 4.1, 4.2 and Hypothesis A(2), L is a quasisimple group satisfying the hypothesis of Theorem 4.3, so $L \cong \mathrm{SL}_2(q)$ with $q \geq 5$ an odd prime power. Moreover, $\boldsymbol{O}^{2'}(\overline{H}) = \overline{H}$ and $\overline{H}$ is almost simple with socle $\bar{L}$, where $\overline{H} = H/\boldsymbol{O}_2(H)$.

Write $q = p^f \geq 5$, where $p > 2$ is a prime and $f \geq 1$. It is well-known that

$$\mathrm{Out}(\mathrm{PSL}_2(q)) \cong \langle\delta\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_f,$$

where δ is a diagonal automorphism of order 2 and φ is a field automorphism of order f of $\mathrm{PSL}_2(p^f)$. Assume $q \equiv \eta \pmod 4$ with $\eta = \pm 1$. We have $k := \nu_2(q - \eta) \geq 2$, $\nu_2(q + \eta) = 1$ and $|L|_2 = (q^2 - 1)_2 = 2^{k+1}$.

Now L has two noncentral real elements z and x (which lie in the cyclic subgroups of L of order $q + \eta$ and $q - \eta$) of order $(q + \eta)/2$ and $2^k > 2$, respectively. It is easy to check that $|x^L|_2 = 2$ and $|z^L|_2 = 2^k$. Moreover, z^L is invariant under the diagonal automorphisms of L. As $\overline{H}/\bar{L}$ is a subgroup of $\mathrm{Out}(\bar{L}) \cong \mathbb{Z}_2 \times \mathbb{Z}_f$ with $\boldsymbol{O}^{2'}(\overline{H}/\bar{L}) = \overline{H}/\bar{L}$, it follows that $\overline{H}/\bar{L}$ is an abelian 2-group of order at most 2^{c+1}, where $c = \nu_2(f)$.

(a) Assume that f is odd. Then $\mathrm{PSL}_2(q) \cong \bar{L} \trianglelefteq \overline{H} \leq \mathrm{PGL}_2(q)$.

If $\overline{H} \cong \mathrm{PSL}_2(q)$, then $H = L\boldsymbol{O}_2(H)$. In this case, we see that $|z^H|_2 = |z^L|_2 = 2^k > 2 = |x^L|_2 = |x^H|_2$, violating (4-1).

Assume that $\overline{H} \cong \mathrm{PGL}_2(q)$. From the character table of $\mathrm{PGL}_2(q)$ (see [Steinberg 1951, Table III]), $\overline{H} \cong \mathrm{PGL}_2(q)$ contains a real element $\bar{y}$ of order p (labeled by A_2) with $|\bar{y}^{\overline{H}}| = q^2 - 1$. Since $o(\bar{y}) = p$ is odd, by [Guralnick et al. 2011, Lemma 2.2] there exists a real element $w \in H$ of order p such that $\bar{w} = \bar{y}$. Now $|w^H|_2 \geq |\bar{y}^{\overline{H}}|_2 = 2^{k+1} > 4 = |\mathrm{Out}(\bar{L})|_2 \cdot |x^L|_2$, violating (4-1).

(b) Assume f is even. We have that $q \equiv 1 \pmod 4$ and since p is odd, by Lemma 4.4 we have $k = \nu_2(q-1) \geq c+2$. Recall that $c = \nu_2(f)$.

From [Dornhoff 1971, Theorem 38.1], $L \cong \mathrm{SL}_2(q)$ has a real element

$$y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

of order p (labeled by c). The image $\bar{y}$ of y in $\bar{L} \cong \mathrm{PSL}_2(q)$ is also a real element of order p and $|y^L| = |\bar{y}^{\bar{L}}| = (q^2-1)/2$. Hence $|y^H|_2 \geq |\bar{y}^{\bar{L}}|_2 = 2^k$ as $|\mathrm{PSL}_2(q)|_2 = 2^k$.

Assume that $|\overline{H}/\bar{L}|_2 \leq 2^c$. We have $|\overline{H} : \bar{L}|_2 \cdot |x^L|_2 \leq 2^{c+1} < 2^{c+2} \leq 2^k \leq |y^H|_2$, contradicting (4-1).

Finally, we assume that $|\overline{H}/\bar{L}|_2 = 2^{c+1}$ and so $\overline{H}/\bar{L} = \langle \delta \rangle \times \langle \varphi^m \rangle$ with $m = f/2^c$. We know that $\bar{y} \in \bar{L}$ is φ-invariant but not δ-invariant. Thus $|\bar{y}^{\overline{H}}|_2 = 2^{k+1}$ and hence $|y^H|_2 \geq |\bar{y}^{\overline{H}}|_2 = 2^{k+1} \geq 2^{c+3}$. Now $|\mathrm{Out}(\bar{L})|_2 \cdot |x^L|_2 = 2^{c+2} < 2^{c+3} \leq |y^H|_2$, which violates (4-1) again. The proof is now complete. □

5. Quasisimple groups

The main purpose of this section is to prove Theorem 4.3. We first prove some easy results which will be needed in our classification.

For a prime p and a group X, the p-rank of X, denoted by $r_p(X)$, is the maximum rank of an elementary abelian p-subgroup of X. Recall that the p-rank of an elementary abelian p-group of order p^k is k. The following easy result should be known.

Lemma 5.1. *Let X be a finite group and let Z be a subgroup of $\mathbf{Z}(X)$. Let $i \in X$ be a noncentral involution. Let r be the 2-rank of Z and let $\overline{X} = X/Z$. Let $T \leq X$ be the full inverse image of $\mathbf{C}_{\overline{X}}(\bar{i})$. Then $T/\mathbf{C}_X(i)$ is an elementary abelian 2-group of order at most 2^r and $|i^X|_2 \leq 2^r \cdot |\bar{i}^{\overline{X}}|_2$.*

Proof. Let $g \in T$. Then $i^g = iz$ for some $z \in Z$. As $i^2 = 1$ and $iz = zi$, we have $z^2 = 1$. Now $\mathbf{C}_X(i)^g = \mathbf{C}_X(i^g) = \mathbf{C}_X(iz) = \mathbf{C}_X(i)$, so $\mathbf{C}_X(i) \trianglelefteq T$. Moreover, $i^{g^2} = (iz)^g = i^g z = i$, and hence $g^2 \in \mathbf{C}_X(i)$. Thus $T/\mathbf{C}_X(i)$ is an elementary abelian 2-group. Let $\Omega = \{x \in Z : x^2 = 1\}$. Clearly Ω is an elementary abelian 2-subgroup of X of order 2^r. For each $h \in T$, there exists $z \in \Omega$ such that $i^h = iz$. Moreover, if $i^h = i^k$ for some $h, k \in T$, then $h\mathbf{C}_X(i) = k\mathbf{C}_X(i)$. Thus there is an injective map from the set of left cosets of $\mathbf{C}_X(i)$ in T to Ω and hence $|T : \mathbf{C}_X(i)| \leq 2^r$. Now

$$|i^X|_2 = |X : T|_2 \cdot |T : \mathbf{C}_X(i)|_2 = |\bar{i}^{\overline{X}}|_2 \cdot |T : \mathbf{C}_X(i)| \leq 2^r \cdot |\bar{i}^{\overline{X}}|_2.$$

The proof is complete. □

We next determine all quasisimple groups which have an involution i whose centralizer is solvable. Notice the isomorphisms $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$, $A_5 \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$, $A_6 \cong \mathrm{PSL}_2(9)$ and $A_8 \cong \mathrm{PSL}_4(2)$.

Lemma 5.2. *Let L be a quasisimple group and let $S = L/\boldsymbol{Z}(L)$. Suppose that L has an involution i such that $\boldsymbol{C}_L(i)$ is solvable. Then one of the following holds.*

(i) $L \cong \mathrm{M}_{11}$ *or* $S \cong \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{Fi}_{22}$.

(ii) $L \cong A_n$ $(5 \leq n \leq 12)$, $2 \cdot A_n$ $(8 \leq n \leq 12)$ *or* $3 \cdot A_n$ $(6 \leq n \leq 7)$.

(iii) $S \cong \mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{Sp}_4(q), {}^2\mathrm{B}_2(q)$ *with* $q = 2^f$.

(iv) $S \cong \mathrm{PSL}_n(2)$ $(n = 4, 5, 6)$, $\mathrm{PSU}_n(2)$ $(4 \leq n \leq 9)$, $\mathrm{Sp}_{2n}(2)$ $(3 \leq n \leq 5)$, $\Omega^{\pm}_{2n}(2)$ $(4 \leq n \leq 5)$, ${}^3\mathrm{D}_4(2), {}^2\mathrm{F}_4(2)', \mathrm{F}_4(2), {}^2\mathrm{E}_6(2)$.

(v) $S \cong \mathrm{PSL}_3(3), \mathrm{PSL}_4(3), \mathrm{PSU}_3(3), \mathrm{PSU}_4(3), \mathrm{PSp}_4(3), \Omega_7(3), \mathrm{P}\Omega^+_8(3), \mathrm{G}_2(3)$.

(vi) $L \cong \mathrm{PSL}_2(q)$ *with q odd.*

Proof. Let $\bar{L} = L/\boldsymbol{Z}(L)$. Since $\boldsymbol{C}_L(i)$ is solvable, $i \notin \boldsymbol{Z}(L)$ and $\boldsymbol{C}_{\bar{L}}(\bar{i})$ is solvable by Lemma 5.1. Thus S is a nonabelian simple group with a solvable involution centralizer.

(1) Assume that S is a sporadic simple group. The information on the centralizers of involutions of L can be read off from Tables 5.3(a)–(z) in [Gorenstein et al. 1998]. It follows that $L \cong \mathrm{M}_{11}$ or $S \cong \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{Fi}_{22}$.

(2) Assume $S \cong A_n$ with $n \geq 5$. We know that every involution j of $S \cong A_n$ is a product of r disjoint transpositions, where r is even. Let $s = n - 2r$. By [Gorenstein et al. 1998, Proposition 5.2.8] $\boldsymbol{C}_S(j) \cong (H_1 \times H_2)\langle t \rangle$, where $H_2 \cong A_s$ and $H_1 \cong R_1 L_1$ with $L_1 \cong \mathrm{S}_r$ and $R_1 \cong C_2^{r-1}$. Hence if r or s is at least 5, then $\boldsymbol{C}_S(j)$ is nonsolvable. Thus both r and s are at most 4 and hence $n \leq 12$. Therefore, if $L = A_n$ with $n \geq 5$, then $5 \leq n \leq 12$. If $L \cong 2 \cdot A_n$, then $n \leq 12$ by our observation above. However, $2 \cdot A_n$ has a noncentral involution only when $n \geq 8$. Thus if $L \cong 2 \cdot A_n$, then $8 \leq n \leq 12$. For $n = 6, 7$, only $3 \cdot A_n$ has a solvable involution centralizer.

(3) Assume S is a finite simple group of Lie type. The centralizers of involutions of L are determined in [Aschbacher and Seitz 1976] and Tables 4.5.1 and 4.5.2 in [Gorenstein et al. 1998]. From these results, it is easy to get all the possibilities for L. (See also [Liebeck and O'Brien 2007, Lemma 3.4].) □

We use the convention that $\mathrm{PSL}^{\epsilon}_n(q)$ is $\mathrm{PSL}_n(q)$ if $\epsilon = +$ and $\mathrm{PSU}_n(q)$ if $\epsilon = -$. A similar convention applies to $\mathrm{SL}^{\epsilon}_n(q)$. We now prove the main result of this section.

Theorem 5.3. *Let L be a quasisimple group with center Z and let $S = L/Z$. Suppose that Z is a 2-group and the following conditions hold*:

(1) *If i is a noncentral involution of L, then $\boldsymbol{C}_L(i)$ is 2-closed.*

(2) *If $x, z \in \mathrm{Re}(L) \setminus \boldsymbol{Z}(L)$ are noncentral real elements, where x is a 2-element, then*

$$|z^L|_2 \leq |\mathrm{Out}(S)|_2 \cdot |x^L|_2.$$

Then $L \cong \mathrm{SL}_2(q)$ with $q \geq 5$ odd.

Proof. We consider the following cases.

Case 1. All involutions of L are central. By the main theorem in [Griess 1978], $L \cong \mathrm{SL}_2(q)$ with $q \geq 5$ odd or $2 \cdot A_7$. If the first case holds, then we are done. So, assume that $L \cong 2 \cdot A_7$. We have $|\mathrm{Out}(S)|_2 = 2$. Using [Conway et al. 1985], we can check that L has two real elements z and x of order 3 and 4, respectively, with $|z^L| = 280$ and $|x^L| = 210$. Clearly $|z^L|_2 = 2^3 > |\mathrm{Out}(S)|_2 \cdot |x^L|_2 = 2^2$, violating condition (2). So this case cannot occur.

Case 2. $\boldsymbol{Z}(L)$ is trivial. Then L is a nonabelian simple group. Let $x \in L$ be a 2-central involution of L, i.e., x is an involution that lies in the center of some Sylow 2-subgroup of L. We have $|x^L|_2 = 1$ so (2) implies $|z^L|_2 \leq |\mathrm{Out}(L)|_2$ for all noncentral real elements $z \in L$.

The centralizer of every noncentral involution of L is 2-closed by the hypothesis. Since L is simple, it follows that the centralizer of every involution of L is 2-closed. Now [Suzuki 1965, Theorem 1] yields that L is isomorphic to one of the following groups: A_6, $\mathrm{PSL}_2(p)$ with p a Fermat or a Mersenne prime, $\mathrm{PSL}_2(2^f)$ with $f \geq 2$, $\mathrm{PSL}_3(q)$, $\mathrm{PSU}_3(q)$ with $q = 2^f$ or ${}^2\mathrm{B}_2(2^{2f+1})$ with $f \geq 1$.

Assume that $L \cong {}^2\mathrm{B}_2(2^{2f+1})$ with $f \geq 1$. It follows from Propositions 3 and 16 in [Suzuki 1962] that L has a real element z of order $2^f - 1$ with $|\boldsymbol{C}_L(z)| = 2^f - 1$ and so $|z^L|_2 = 2^{2(2f+1)} \geq 64$. Since $|\mathrm{Out}(L)| = 2f + 1$ is odd, $|z^L|_2 > |\mathrm{Out}(L)|_2$, and hence this case cannot occur.

Assume $L \cong \mathrm{PSL}_2(p)$ with p a Fermat or a Mersenne prime. We have $|\mathrm{Out}(L)| = 2$ and L possesses a real element z of odd order $(p+\delta)/2$, where $p \equiv \delta \pmod 4$ and $|z^L|_2 = |L|_2 \geq 4$. As $|z^L|_2 > 2 = |\mathrm{Out}(L)|_2$, this case cannot happen.

If $L \cong A_6$, then $|\mathrm{Out}(L)| = 2^2$. However, A_6 has a real element z of order 3 with $|z^L| = 40$ and thus $|z^L|_2 = 8 > |\mathrm{Out}(S)|_2 = 4$. Thus this case cannot happen.

Next, if $L \cong \mathrm{PSL}_2(2^f)$ with $f \geq 2$, then L has a real element z of order $2^f - 1$ with $|z^L| = 2^f(2^f + 1)$. Clearly $|z^L|_2 = 2^f > f \geq |\mathrm{Out}(L)|_2$ as $|\mathrm{Out}(L)| = f$.

Finally, assume $L \cong \mathrm{PSL}_3^{\epsilon}(2^f)$. We have $f \geq 2$ as $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, where $7 = 2^3 - 1$ is a Mersenne prime and $\mathrm{PSU}_3(2)$ is not simple. In both cases, $|\mathrm{Out}(L)| = 2df$ with $d = (3, 2^f - \epsilon 1)$ so $|\mathrm{Out}(L)|_2 = 2^{1+\nu_2(f)}$. The quasisimple group $X = \mathrm{SL}_3^{\epsilon}(2^f)$ possesses real elements h of order $2^f + \epsilon 1$ with $|\boldsymbol{C}_X(h)| = 4^f - 1$ and g of order $2^f - \epsilon 1$ with $|\boldsymbol{C}_X(g)| = (2^f - \epsilon 1)^2$ [Guralnick et al. 2011, Lemma 4.4(3)]. Now let $y \in \{g, h\}$ be an element with $o(y)$ relatively prime to $d = (3, 2^f - \epsilon 1)$ and let z be the image of y in $L = \mathrm{PSL}_3^{\epsilon}(2^f) \cong X/\boldsymbol{Z}(X)$. Then $|\boldsymbol{C}_L(z)| = |\boldsymbol{C}_X(y)/\boldsymbol{Z}(X)|$ (by [Isaacs 2008, Lemma 7.7]) is odd, so $|z^L|_2 = |L|_2 = 2^{3f}$. Since $f \geq 2$, $2^{3f} > 2^{2f} \geq 2^{1+\nu_2(f)}$. Hence these cases cannot occur.

Case 3. $\boldsymbol{Z}(L)$ is nontrivial and L has a noncentral involution. Let j be a noncentral involution of L. By condition (1), $\boldsymbol{C}_L(j)$ is 2-closed and thus it is solvable. Hence L is one of the quasisimple groups in Lemma 5.2. Moreover, as $\boldsymbol{Z}(L)$ is a nontrivial 2-group, the Schur multiplier $\mathrm{M}(S)$ of $S \cong L/\boldsymbol{Z}(L)$ is of even order. It follows that we only need to consider the following cases.

(i) $S \cong \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{Fi}_{22}, A_n$ $(8 \leq n \leq 12)$, $\mathrm{PSL}_3(4)$, ${}^2\mathrm{B}_2(8)$.

(ii) $S \cong \mathrm{PSU}_n(2)$ $(n = 4, 6)$, $\mathrm{Sp}_6(2)$, $\Omega_8^+(2)$, $\mathrm{F}_4(2)$, ${}^2\mathrm{E}_6(2)$.

(iii) $S \cong \mathrm{PSU}_4(3)$, $\mathrm{PSp}_4(3)$, $\Omega_7(3)$, $\mathrm{P}\Omega_8^+(3)$.

We show that these cases cannot occur by showing that condition (2) does not hold. Clearly $\bar{j} \in \bar{L}$ is a noncentral involution and by Lemma 5.1, $|j^L|_2 \le 2^{r_2(\mathbf{Z}(L))} \cdot |\bar{j}^{\bar{L}}|_2$, where $r_2(\mathbf{Z}(L))$ is the 2-rank of $\mathbf{Z}(L)$. Let $\widetilde{S}$ be the perfect central extension of S such that $\widetilde{S}/\mathbf{Z}(\widetilde{S}) \cong S$ and $|\mathbf{Z}(\widetilde{S})| = |\mathrm{M}(S)|$. From [Conway et al. 1985], we see that $\mathrm{M}(S)$ can be written as a direct product of at most two (possibly trivial) cyclic groups, so $r_2(\mathbf{Z}(L)) \le r_2(\mathrm{M}(S)) \le 2$. Let $e_2(S) = \max\{\nu_2(|x^S|) : x \text{ is an involution in } S\}$. Then for each noncentral involution $j \in L$, we have $\nu_2(|j^L|) \le r_2(\mathrm{M}(S)) + e_2(S)$.

Let $z \in S$ be a nontrivial real element of odd order. There exists $y \in \mathrm{Re}(L) \setminus \mathbf{Z}(L)$ of odd order such that z is the image of y in $L/\mathbf{Z}(L) \cong S$ (see [Navarro et al. 2009, Lemma 3.2] or [Guralnick et al. 2011, Lemma 2.2]) and by applying [Isaacs 2008, Lemma 7.7], $|z^S| = |y^L|$ (noting $(o(y), |\mathbf{Z}(L)|) = 1$). Hence to show that condition (2) does not hold, it suffices to find a nontrivial real element $z \in S$ of odd order such that the following inequality holds:

$$\nu_2(|z^S|) > \nu_2(|\mathrm{Out}(S)|) + r_2(\mathrm{M}(S)) + e_2(S). \tag{5-1}$$

For each simple group S in (i)–(iii) above, we list in Table 1 the invariant $e_2(S)$, the largest 2-part of the sizes of conjugacy classes of involutions of S in the second column, the Atlas class name and the 2-part of the conjugacy class of odd order real element $z \in S$, the order of the outer automorphism group

S	$e_2(S)$	z	$\nu_2(\lvert z^S\rvert)$	$\lvert\mathrm{Out}(S)\rvert$	$\mathrm{M}(S)$
M_{12}	2	$3a$	5	2	$\mathbb{Z}_2$
M_{22}	0	$5a$	7	2	$\mathbb{Z}_{12}$
Fi_{22}	1	$3d$	17	2	$\mathbb{Z}_6$
A_8	1	$5a$	6	2	$\mathbb{Z}_2$
A_9	1	$3b$	4	2	$\mathbb{Z}_2$
A_{10}	1	$3c$	7	2	$\mathbb{Z}_2$
A_{11}	1	$3c$	6	2	$\mathbb{Z}_2$
A_{12}	0	$3d$	7	2	$\mathbb{Z}_2$
$\mathrm{PSL}_3(4)$	0	$3a$	6	12	$\mathbb{Z}_4 \times \mathbb{Z}_{12}$
${}^2\mathrm{B}_2(8)$	0	$5a$	6	3	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{PSU}_4(2)$	1	$5a$	6	2	$\mathbb{Z}_2$
$\mathrm{PSU}_6(2)$	3	$3c$	12	6	$\mathbb{Z}_2 \times \mathbb{Z}_6$
$\mathrm{Sp}_6(2)$	2	$3c$	7	1	$\mathbb{Z}_2$
$\Omega_8^+(2)$	2	$3d$	9	6	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{F}_4(2)$	4	$3c$	18	2	$\mathbb{Z}_2$
${}^2\mathrm{E}_6(2)$	5	$3c$	27	6	$\mathbb{Z}_2 \times \mathbb{Z}_6$
$\mathrm{PSU}_4(3)$	0	$3d$	7	8	$\mathbb{Z}_3 \times \mathbb{Z}_{12}$
$\Omega_7(3)$	0	$3g$	8	2	$\mathbb{Z}_6$
$\mathrm{P}\Omega_8^+(3)$	2	$3m$	11	24	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Table 1. Some small simple groups.

Out(S) and in the last column the Schur multiplier M(S). The information in this table can be read off from the character table of S using [Conway et al. 1985].

From Table 1, we can check that (5-1) holds, which completes our proof. □

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