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Proper G_a-actions on $\mathbb{C}^4$ preserving a coordinate

Shulim Kaliman

Dedicated to my teachers, Vladimir Yakovlevich Lin and Evgeniy Alekseevich Gorin

We prove that the actions mentioned in the title are translations. We show also that for certain G_a-actions on affine fourfolds the categorical quotient of the action is automatically an affine algebraic variety and describe the geometric structure of such quotients.

Introduction

An algebraic action G_a of the additive group $\mathbb{C}_+$ of complex numbers on a complex algebraic variety X is free if it has no fixed points. When X is a Euclidean space $\mathbb{C}^n$ with a coordinate system $(x_1, \dots, x_n)$ the simplest example of such an action is a translation for which the action of an element $t \in \mathbb{C}_+$ is given by $(x_1, x_2, \dots, x_n) \mapsto (x_1 + t, x_2, \dots, x_n)$. It turns out that for $n \leq 3$ these notions are "essentially" the same. More precisely, when $n \leq 3$ every nontrivial free G_a-action on $\mathbb{C}^n$ in a suitable polynomial coordinate system is a translation[1] (see [Gutwirth 1961; Rentschler 1968] for $n = 2$ and [Kaliman 2004] for $n = 3$).

Starting with $n = 4$ the similar statement does not hold and the basic example of Winkelmann [1990] gives a triangular[2] free G_a-action which is not a translation. In his example the geometric quotient of the action is not Hausdorff while for a translation on $\mathbb{C}^n$, the geometric quotient is isomorphic to $\mathbb{C}^{n-1}$. Recently Dubouloz, Finston, and Jaradat [Dubouloz et al. 2014] proved that every triangular action on $\mathbb{C}^n$ which is proper (in particular, it is free and has a Hausdorff geometric quotient), is a translation in a suitable coordinate system. Note that every triangular action preserves at least one of the coordinates, and one of the aims of this paper is the following generalization of the Finston–Dubouloz–Jaradat result:

Theorem 0.1. *Every proper G_a-action on $\mathbb{C}^4$ that preserves a coordinate is a translation in a suitable polynomial coordinate system.*[3]

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[1] In fact, for $n \leq 3$ every connected one-dimensional unipotent algebraic subgroup of Cremona group of $\mathbb{C}^n$ is conjugate to such a translation [Popov 2015, Corollary 5].

[2] Recall that a G_a-action on $\mathbb{C}^n$ is triangular if in a suitable polynomial coordinate system it is of the form $(x_1, \dots, x_n) \mapsto (x_1, x_2 + tp_2(x_1), x_3 + tp_3(x_1, x_2), \dots, x_n + tp_n(x_1, \dots x_{n-1}))$, where each p_i is a polynomial. For $n \geq 3$ not every G_a-action on $\mathbb{C}^n$ is triangulable (i.e., triangular in a suitable polynomial coordinate system) [Bass 1984].

[3] Neena Gupta informed the author that she and S. M. Bhatwadekar had recently obtained an independent proof of Theorem 0.1.

Its proof involves investigation of categorical quotients of G_a-actions on $\mathbb{C}^4$. Namely, let X be an affine algebraic variety equipped with a G_a-action $\Phi : G_a \times X \to X$. Denote by $\mathbb{C}[X]^\Phi$ the subring of Φ-invariant regular functions in the ring $\mathbb{C}[X]$ of regular functions on X and by $\operatorname{Spec}\mathbb{C}[X]^\Phi$ (resp. $\operatorname{Spec}\mathbb{C}[X]$) the spectrum of $\mathbb{C}[X]^\Phi$ (resp. $\mathbb{C}[X]$). Then the natural embedding $\mathbb{C}[X]^\Phi \hookrightarrow \mathbb{C}[X]$ induces a map $\operatorname{Spec}\mathbb{C}[X] \to \operatorname{Spec}\mathbb{C}[X]^\Phi$. The fact that $\mathbb{C}[X]^\Phi$ is finitely generated is equivalent to the fact that $\operatorname{Spec}\mathbb{C}[X]^\Phi$ can be viewed as an affine algebraic variety denoted by $X//\Phi$. It is called the categorical quotient of the action and the map of the spectra yields the quotient morphism $\varrho : X \to X//\Phi$ in the category of affine algebraic varieties. If $\dim X \leq 3$ then $\mathbb{C}[X]^\Phi$ is always finitely generated by a theorem of Zariski [1954]. In higher dimensions this fact is not necessarily true by Nagata's counterexample to the fourteenth Hilbert problem. Furthermore, extending Nagata's counterexample, Daigle and Freudenburg [1999] showed for a G_a-action on $\mathbb{C}^n$, the ring of invariant functions may not be finitely generated starting from dimension $n \geq 5$. In dimension 4 the same authors showed that the ring of regular functions invariant with respect to a triangular G_a-action on $\mathbb{C}^4$ is automatically finitely generated [Daigle and Freudenburg 2001] and later Bhatwadekar and Daigle [2009] proved that it remains finitely generated if one considers instead of triangular G_a-actions the wider class of G_a-actions preserving a coordinate.[4] In this paper we establish a stronger fact contained in the next theorem together with a generalization of Theorem 0.1.

Theorem 0.2. *Let $\varphi : X \to B$ be a surjective morphism of a factorial affine algebraic G_a-variety X (i.e., X is equipped with some G_a-action Φ) of dimension 4 into a smooth affine curve B. Suppose also that*

- *the action preserves each fiber of φ;*
- *the generic fiber of φ is a three-dimensional variety Y (over the field K of rational functions on B) for which the ring of invariants of the G_a-action (induced by Φ) on Y is the polynomial ring $K[z, w]$;[5]*
- *for every $b \in B$ the fiber $X_b = \varphi^{-1}(b)$ admits a nonconstant morphism into a curve if and only if this curve is a polynomial one (i.e., the normalization of the curve is the line $\mathbb{C}$).*

Then

(1) *the ring of Φ-invariant functions is finitely generated and, thus, it can be viewed as the ring of regular functions on an affine algebraic variety $Q = X//\Phi$;*

(2) *there is an affine modification $\psi : Q \to B \times \mathbb{C}^2$ such that for some nonempty Zariski dense subset $B^* \subset B$ the restriction of ψ over B^* is an isomorphism and every singular fiber of ψ is of form $C \times \mathbb{C}$ where C is a polynomial curve.*

(3) *Furthermore, if one requires additionally that*
   - *Φ is proper and X is Cohen–Macaulay,*
   - *each fiber X_b is normal,*

[4] The author is grateful to Neena Gupta for drawing his attention to the paper of Bhatwadekar and Daigle.

[5] This assumption that the ring of invariants of the induced action is isomorphic to $K[z, w]$ can be replaced by the following: for a general $b \in B$ the categorical quotient of the restriction of Φ to the fiber $X_b = \varphi^{-1}(b)$ is isomorphic to $\mathbb{C}^2$ (see Theorem 5.7).

- *and the restriction Φ_b of Φ to X_b is a translation* (*in particular*, X_b *is naturally isomorphic to a direct product* $(X_b//\Phi_b) \times \mathbb{C}$),

then the quotient Q is locally trivial $\mathbb{C}^2$-bundle over B (*and in particular it is a vector bundle by* [Bass et al. 1976/77]), *X is naturally isomorphic to $Q \times \mathbb{C}$, and Φ is generated by a translation on the second factor of $X \simeq Q \times \mathbb{C}$.*

Let us emphasize that the fact that X is a direct product $Q \times \mathbb{C}$ is a rather rare event for regular G_a-actions while in the category of rational actions of a connected linear algebraic group G on an algebraic variety Y (over an algebraically closed field of any characteristic) this variety is automatically birationally isomorphic to the product of $\mathbb{P}^s$ and the rational quotient of Y with respect to a Borel subgroup of G (see [Matsumura 1963; Popov 2016]). It is also worth mentioning that the requirement that the restriction of Φ to any X_b is a translation in (3) can be can be replaced by some topological assumptions. For instance, if each fiber X_b is smooth and factorial with trivial second and third homology groups then $\Phi|_{X_b}$ is automatically a translation by [Kaliman 2004, Theorem 5.1]. In particular, this is true when X_b is a smooth contractible threefold since by the Gurjar theorem [1980, Theorem 1] (see also [Fujita 1982]) such a threefold is factorial.

This leads to the following application of Theorem 0.2.

Corollary 0.3. *Let $\varphi : X \to B$ be a surjective morphism of a smooth factorial affine algebraic fourfold X into a smooth curve B such that X is equipped with a proper G_a-action Φ preserving every fiber $X_b = \varphi^{-1}(b)$, $b \in B$ of φ. Suppose that each X_b is a smooth contractible threefold such that the restriction of Φ to X_b has the plane as the quotient.*[6]

Then there exits a categorical quotient of the action $Q = X//\Phi$ in the category of affine algebraic varieties. Furthermore, Q is a two-dimensional vector bundle over B and X is naturally isomorphic to $Q \times \mathbb{C}$ while Φ is generated by a translation on the second factor of $X \simeq Q \times \mathbb{C}$.

Indeed, it is straightforward that the assumptions of Corollary 0.3 imply all assumptions of Theorem 0.2 with a possible exception of the condition on the generic fiber of φ. But this last condition follows from a combination of the Kambayashi [Kambayashi 1975] and the Kraft–Russell [Kraft and Russell 2014] theorems (see Theorem 5.4 and Example 5.6 below for details).

When $X \simeq \mathbb{C}^4$, $B \simeq \mathbb{C}$, and φ is a coordinate function the assumptions of Corollary 0.3 are automatically true and we have Theorem 0.1.

Some of the results mentioned before (including Theorem 0.1) are extended in the last section of this paper where using the Lefschetz principle we show that similar facts (see, Theorems 9.11 and 9.12) remain valid if we consider varieties X and B not over the field of complex numbers $\mathbb{C}$ but over any (not necessarily algebraically closed) field of characteristic zero.

[6] For $X_b = \mathbb{C}^3$ the quotient of a nontrivial G_a-action is always isomorphic to $\mathbb{C}^2$ due to Miyanishi's theorem [1980] and this is also true for smooth contractible threefolds with negative logarithmic Kodaira dimension [Kaliman and Saveliev 2004] like the Russell cubic $\{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{C}^4_{x,y,z,t}$.

1. General facts

Theorem 1.1. *Let $\varphi : X \to Y$ be a birational morphism of irreducible affine algebraic varieties such that Y is normal and there is a subvariety Z of Y with codimension at least 2 for which the restriction of φ to $X \setminus \varphi^{-1}(Z)$ is a surjective morphism onto $Y \setminus Z$. Suppose also that for every point $y \in Y \setminus Z$ the preimage $\varphi^{-1}(y)$ is finite. Then $\varphi : X \to Y$ is an isomorphism.*

Proof. The Zariski Main theorem (e.g., see [Grothendieck 1964, Théorème 8.12.6] or [Hartshorne 1977, Chapter III, Corollary 11.4]) implies that the restriction of φ yields an isomorphism between $X \setminus \varphi^{-1}(Z)$ and $Y \setminus Z$. By the Hartogs theorem the composition of the inverse map $\varphi^{-1} : Y \setminus Z \to X \setminus \varphi^{-1}(Z)$ with the inclusion $X \setminus \varphi^{-1}(Z) \hookrightarrow X$ extends to a morphism $Y \to X$ and we are done. □

Corollary 1.2. *Let $\varphi : X \to Y$ be a morphism of irreducible affine algebraic varieties, and $E \subset X$ and $D \subset Y$ be irreducible divisors such that the restriction of φ yields an isomorphism $X \setminus E \to Y \setminus D$. Suppose also that Y is normal and $\varphi(E)$ is Zariski dense in D. Then $\varphi : X \to Y$ is an isomorphism.*

Proof. The dimension argument implies that for a general point y in D the preimage $\varphi^{-1}(y)$ is finite. Hence it is finite outside a proper closed subvariety Z of D and we are done by Theorem 1.1. □

The next fact and the modified proof of Theorem 1.4 below were suggested by the referee.

Proposition 1.3. *Let $\varrho : X \to Q$ be a morphism of irreducible affine algebraic varieties such that Q is normal and $R = Q \setminus \varrho(X)$ is of codimension at least 2 in Q. Then every regular function f on X which is constant on the general fibers of ϱ descends to a unique regular function g on Q.*

Proof. Let us show first that f is a lift of a rational function g on Q. That is, one needs to establish the regularity of g on a Zariski dense open subset Q_0 of $Q \setminus R$. We can suppose that the restriction of ϱ to $X_0 = \varrho^{-1}(Q_0)$ is flat by [Grothendieck 1964, Théorème 6.9.1] and, therefore, being surjective it is faithfully flat [Atiyah and Macdonald 1969, Ch. 3, Exercise 16]. Consider $Y = X_0 \times_{Q_0} X_0$. Then the two natural projections $Y \to X_0$ generate two homomorphisms $e_1 : \mathbb{C}[X_0] \to \mathbb{C}[Y]$ and $e_2 : \mathbb{C}[X_0] \to \mathbb{C}[Y]$. Since f is constant on general fibers of ϱ we see that $f|_{X_0} \in \operatorname{Ker}(e_1 - e_2)$. In combination with the faithful flatness of the natural morphism $\mathbb{C}[Q_0] \to \mathbb{C}[X_0]$ this implies that f must be a lift of a regular function $g|_{Q_0}$ [Fantechi et al. 2005, Lemma 2.61]. Hence g is a rational function on Q.

Assume that there exists a divisor T in Q such that a general point $t_0 \in T$ does not belong to the indeterminacy set of g and $g(t_0) = \infty$. We can suppose that also $t_0 \notin R$, i.e., $\varrho^{-1}(t_0) \neq \varnothing$. Hence for a germ of a curve C in X through $x_0 \in \varrho^{-1}(t_0)$ one has $f(c) \to f(x_0)$ as $c \in C$ approaches x_0. This implies that $g(\varrho(c)) \to f(x_0) \neq \infty$ as $\varrho(c)$ approaches t_0, a contradiction. That is, g is regular on Q outside a subvariety of codimension at least 2. Since Q is normal, the Hartogs theorem implies that g is regular on Q and we are done. □

Theorem 1.4. *Let $\varrho : X \to Q$ be a morphism of irreducible affine algebraic varieties and $\Phi : G_a \times X \to X$ be a nontrivial G_a-action on X which preserves each fiber of ϱ. Let Q be normal and P be a subvariety of Q with codimension at least 2 such that $Q \setminus P \subset \varrho(X)$. Suppose also that for every point $q \in Q \setminus P$*

the preimage $\varrho^{-1}(q)$ is a curve and for a general point $q \in Q \setminus P$ the preimage $\varrho^{-1}(q)$ is an irreducible curve. Then $\varrho : X \to Q$ is the categorical quotient morphism in the category of affine algebraic varieties. In particular, the subring of G_a-invariants in the ring of regular functions on X is finitely generated.

Proof. Note that by the assumption every general fiber of ϱ is nothing but an orbit of Φ. Thus every Φ-invariant function f is constant on the general fibers of ϱ. By Proposition 1.3, f is a lift of a regular function on Q. Hence the ring of regular functions on Q coincides with the ring $\mathbb{C}[X]^\Phi$ of invariants and we are done. □

Now the dimension argument as in the proof of Corollary 1.2 implies the following:

Corollary 1.5. *Let $\varrho : X \to Q$ be a morphism of irreducible affine algebraic varieties such that Q is normal and let $\Phi : G_a \times X \to X$ be a nontrivial G_a-action on X preserving every fiber of ϱ. Let $E \subset X$ and $D \subset Q$ be irreducible divisors such that $X \setminus E$ and $Q \setminus D$ are affine algebraic varieties for which $\varrho(X \setminus E) = Q \setminus D$ and $\varrho|_{X\setminus E)} : X \setminus E \to Q \setminus D$ is the categorical quotient morphism of the action $\Phi|_{X\setminus E}$. Suppose also that $\varrho(E)$ is Zariski dense in D.*

Then $\varrho : X \to Q$ is the categorical quotient morphism of Φ.

Proposition 1.6 (cf., [Kaliman 2004, Lemma 2.1]). *Let $\varrho : X \to Q$ be a dominant morphism of normal affine algebraic varieties. Suppose that the general fibers of ϱ are irreducible and there are no nonconstant invertible regular functions on such fibers. Suppose also that $Q \setminus \varrho(X)$ is of codimension at least 2 in Q. Then*

(1) *for every principal irreducible divisor D in X, which does not meet general fibers of ϱ, the closure of $\varrho(D)$ is the support of a principal irreducible divisor in Q;*

(2) *if X is a factorial variety so is Q;*

(3) *if X is factorial the preimage of any irreducible reduced divisor T in Q is an irreducible reduced divisor in X.*

Proof. In (1) D is the zero locus of a regular function f. Since f does not vanish on general fibers and they are irreducible we see that f is constant on each general fiber. By Proposition 1.3, $f = g \circ \varrho$ where $g \in \mathbb{C}[Q]$. Let us show that the zero locus of g coincides with the closure of $\varrho(D)$. Assume to the contrary that there is a divisor $F \subset g^{-1}(0) \setminus \varrho(D)$ in Q. Choose a rational function h on Q whose poles are contained in the closure T of F and a regular function e on Q that vanishes on $T \cap \varrho(D) = T \cap \varrho(X)$ but not at general points of T. Then for sufficiently large k the function $(e^k h) \circ \varrho$ is regular on X. Since it is constant on every general fiber of X we conclude by Proposition 1.3 that $e^k h$ is regular on Q contrary to the fact that this function has poles on T. Thus we have (1).

Let T be an irreducible reduced Weil divisor in Q and let D be an irreducible component of $\varrho^{-1}(T)$ that is a divisor in X (such a component exists because $Q \setminus \varrho(X)$ is of codimension at least 2 in Q). Under the assumption of (2), $D = f^*(0)$ for a regular function f on X since X is factorial. By (1), $T = \overline{\varrho(D)}$ and it coincides with the zeros of $g \in \mathbb{C}[X]$ where $f = g \circ \varrho$. Furthermore, if e is another regular function on Q that vanishes on T then it is divisible by g since $e \circ \varrho$ is divisible by f, and by Proposition 1.3,

$e \circ \varrho / g \circ \varrho$ is the lift of a regular function on Q. That is, $T = g^*(0)$. This implies that every irreducible element in $\mathbb{C}[Q]$ has a zero locus which is reduced irreducible and principal. In particular, this element is prime and, therefore, we have (2).

Assume that D_1 and D_2 are reduced irreducible components of $\varrho^{-1}(T)$. Since X is factorial $D_i = f_i^*(0)$. By the argument above $f_i = g_i \circ \varrho$ for regular functions g_i on Q. Furthermore g_i vanishes on T only and thus g_1/g_2 is an invertible regular function. This implies that $D_1 = D_2$ and we have (3). □

Corollary 1.7. *Let X be a normal affine algebraic variety and $\varrho : X \to Q$ be a quotient morphism of a nontrivial G_a-action on X (in the category of affine algebraic varieties). Then $Q \setminus \varrho(X)$ has codimension at least 2 in Q. Furthermore, if X is factorial so is Q.*

Proof. Note that Q is normal since X is. If $Q \setminus \varrho(X)$ contains a divisor F of Q then as in Statement (1) of Proposition 1.6 we can construct a rational function g on Q with poles on F whose lift to X is regular. By construction this lift is G_a-invariant, i.e., g must be regular. A contradiction. This implies the first statement while the second one follows from Proposition 1.6. □

2. Basic definitions and properties of affine modifications

Definition 2.1. Recall that an affine modification is any birational morphism $\sigma : \hat{X} \to X$ of affine algebraic varieties [Kaliman and Zaidenberg 1999]. In particular, there exists a divisor $D \subset X$ such that for $\hat{D} = \sigma^{-1}(D)$ the restriction of σ yields an isomorphism $\hat{X} \setminus \hat{D} \to X \setminus D$. There is some freedom in the choice of D and we can always suppose that D is a principal effective divisor given by zeros of a regular function $f \in A := \mathbb{C}[X]$. This is the case which we consider in the present paper. When such f is fixed we call D the divisor of modification, $\hat{D}$ is called the exceptional divisor of modification, and the closure Z of $\sigma(\hat{D})$ in X is called the center of modification. The advantage of a principal divisor D is that the algebra $\hat{A} = \mathbb{C}[\hat{X}]$ can be viewed as a subalgebra $A[I/f]$ in the field $\mathrm{Frac}(A)$ of fractions of A where I is an ideal in A (see [Kaliman and Zaidenberg 1999]).

It is worth mentioning that I is not determined uniquely, i.e., one can find another ideal $J \subset A$ for which $\hat{A} = A[J/f]$. We suppose further that I is the largest among such ideals J and we call I the ideal of the modification. (Treating A as a subalgebra of $\hat{A}$ one can see that I is the intersection of A and the principal ideal generated by f in $\hat{A}$.)

Notation 2.2. The symbols X, Z, D, $\hat{X}$, $\hat{D}$, I, f in this section have the same meaning as in Definition 2.1.

Example 2.3. Let I be generated by regular functions $f, g_1, \dots, g_n \in \mathbb{C}[X]$. Consider the closed subspace Y of $X \times \mathbb{C}^n_{v_1,\dots,v_n}$ given by the system of equations $f v_j = g_j$ $(j = 1, \dots, n)$ and the proper transform $\hat{X}$ of X under the natural projection $Y \to X$ (i.e., $\hat{X}$ is the only irreducible component of Y whose image in X under the projection is dense in X). Then the restriction $\sigma : \hat{X} \to X$ of the natural projection is our affine modification. Note that D (resp. $\hat{D}$) coincides with the zero locus of f (resp. $f \circ \sigma$) and the center Z is given by the equations $f = g_1 = \dots = g_n = 0$.

Remark 2.4. (1) The geometrical construction behind the modification in Example 2.3 is the following. Consider blowing up $\tau : \tilde{X} \to X$ of X with respect to the ideal sheaf generated by f and $g_1, \dots, g_n$. Delete from $\tilde{X}$ divisors on which the zero multiplicity of $f \circ \tau$ is more than the zero multiplicity of at least one of the functions $g_j \circ \tau$. The resulting variety is $\hat{X}$.

(2) Note that the replacement in Example 2.3 of functions $f, g_1, \dots, g_n$ by functions $hf, hg_1, \dots, hg_n$ respectively (where $h \in \mathbb{C}[X]$ is nonzero) does not change the modification $\sigma : \hat{X} \to X$. In order to avoid this ambiguity we have to fix f. We are not going to specify such f's for affine modifications considered below since the choice of f will be clear from the context in each particular case.

Definition 2.5. (1) Let the center of modification Z in Example 2.3 be a set-theoretical complete intersection in X given by the zeros $f = g_1 = \cdots = g_n = 0$ (i.e., Z coincides with the set of common zeros of these functions and the codimension of Z in X is $n+1$). Then we call such $\sigma : \hat{X} \to X$ a Davis modification. Its main property is that $\hat{D}$ is naturally isomorphic to $Z \times \mathbb{C}^n$ and that the support of $\hat{X}$ coincides with the support of Y.[7]

(2) Let Z be a strict complete intersection in X given by $f = g_1 = \cdots = g_n = 0$; that is, Z is not only a set-theoretical complete intersection but also the defining ideal of the (reduced) subvariety Z in X coincides with the ideal I generated by $f, g_1, \dots, g_n$. Then we call σ a simple modification. In this case $\hat{X}$ coincides with Y as a scheme. Note also that the zero multiplicity of $f \circ \sigma$ is 1 at general points of $\hat{D}$.

Here are some useful properties of simple modifications from [Kaliman 2002] which we shall need later.

Proposition 2.6. *Let $\sigma : \hat{X} \to X$ be a simple modification. Then*

(1) *$\hat{X}$ is smooth over points from $Z_{\mathrm{reg}} \cap D_{\mathrm{reg}} \cap X_{\mathrm{reg}}$*;

(2) *$\hat{X}$ is Cohen–Macauley provided X is Cohen–Macauley*;

(3) *furthermore, if in* (2) *X is normal and none of irreducible components of the center Z of σ is contained in the singularities of X or D then $\hat{X}$ is normal.*

3. Pseudoaffine modifications

From a geometrical point of view it is sometimes convenient for us to consider a neighborhood $U \subset X$ of some point $z \in Z$ in the standard topology (we call such U a Euclidean neighborhood) and the restriction $\sigma|_{\hat{U}} : \hat{U} \to U$ where $\hat{U}$ is any connected component of $\sigma^{-1}(U)$. Since U and $\hat{U}$ are not algebraic varieties but only complex spaces let us consider the analogue of affine modifications in the analytic setting.[8]

[7] Indeed, the preimage of Z in Y is naturally isomorphic to $Z \times \mathbb{C}^n$ and therefore has dimension $\dim X - 1$. On the other hand counting the number of equations defining Y in $X \times \mathbb{C}^n$ we see that every irreducible component of Y has dimension at least $\dim X$. This implies that Y is irreducible (since all irreducible components of Y but one are contained in the preimage of Z) and thus $\hat{D} \simeq Z \times \mathbb{C}^n$.

[8] One can adhere to the algebraic setting by viewing U as an étale neighborhood of z in X.

Definition 3.1. Let X be an irreducible complex Stein space and $\psi : X \to W$ be a meromorphic map into a projective algebraic variety W with a fixed ample divisor H such that ψ is holomorphic over $W \setminus H$. A minimal resolution $\pi : \tilde{X} \to X$ of indeterminacy points for ψ leads to a holomorphic map $\tilde{\psi} : \tilde{X} \to W$. Removing from $\tilde{X}$ the preimage of H we obtain $\hat{X}$ which, together with the natural projection $\sigma : \hat{X} \to X$, will be called a pseudoaffine modification. Consider the Weil divisor $\tilde{\psi}^*(H)$ and its pushforward (as a cycle) D by $\pi : \tilde{X} \to X$. Then D is the divisor of the modification, $\hat{D} = \sigma^{-1}(D) \subset \hat{X}$ is its exceptional divisor, and the closure Z of $\sigma(\hat{D})$ is its center. The restriction of σ induces, of course, a biholomorphism between $\hat{X} \setminus \hat{D}$ and $X \setminus D$. Note also that similarly to the algebraic setting $\sigma|_{\hat{D}} : \hat{D} \to \sigma(\hat{D})$ is, by construction, the restriction of the proper morphism $\pi|_T : T \to Z$ where T is the closure of $\hat{D}$ in $\tilde{X}$. The latter observation is important for the next remark.

Remark 3.2. Though we mostly omit explicit formulations it should be emphasized that practically all the facts valid for affine modifications have similar analytic analogues for pseudoaffine modifications (e.g., see Proposition 3.6 below).

Example 3.3. Let us switch to the analytic setting in Example 2.3 by assuming that X is a Stein variety, $f, g_1, \ldots, g_n$ are holomorphic functions on X. As before, Y is given in $X \times \mathbb{C}^n_{v_1,\ldots,v_n}$ by equations

$$f v_j = g_j, \quad j = 1, \ldots, n$$

and $\hat{X}$ is the proper transform of X under the natural projection $Y \to X$. Then the restriction $\sigma : \hat{X} \to X$ of the natural projection is a pseudoaffine modification with D (resp. $\hat{D}$) being the zero locus of f (resp. $f \circ \sigma$) and the center Z given by the equations $f = g_1 = \cdots = g_n = 0$.

Definition 3.4. If in Example 3.3 Z is a set-theoretical complete intersection in X given by the zeros

$$f = g_1 = \cdots = g_n = 0,$$

then we call σ a Davis pseudoaffine modification. If furthermore Z is a strict complete intersection given by these function then σ is a simple pseudoaffine modification.

Remark 3.5. (1) For any pair $U, \hat{U}$ as in the beginning of this section the restriction $\hat{U} \to U$ of a simple (resp. Davis) affine modification $\sigma : \hat{X} \to X$ is automatically a simple (resp. Davis) pseudoaffine modification.

(2) Note that when X, D, and Z are smooth for a simple pseudoaffine modification then for every point $z \in Z$ the collection $\{f, g_1, \ldots, g_n\}$ can be extended to a local coordinate system in a Euclidean neighborhood U of z in X. In particular, if $n = 1$ we can treat U as a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \ldots, w_{n-2})$ such that D is given in U by $u = 0$, Z by $u = v = 0$, and the preimage of U in $\hat{X}$ is viewed as a subvariety of $U \times \mathbb{C}_w$ given by the equation $uw = v$.

(3) Similarly to the algebraic setting the exceptional divisor of a Davis pseudoaffine modification is a the product of its center and a Euclidean space.

The following analytic version of Proposition 2.6 for simple pseudoaffine modifications remains valid with a verbatim proof.

Proposition 3.6. *Let $\sigma : \hat{X} \to X$ be a simple pseudoaffine modification of irreducible Stein spaces. Then*

(1) *$\hat{X}$ is smooth over smooth points from Z that are not contained in the singularities of D or X;*

(2) *$\hat{X}$ is Cohen–Macaulay provided X is Cohen–Macaulay;*

(3) *furthermore, if in* (2) *X is normal and none of irreducible components of the center Z of σ is contained in the singularities of X or D then $\hat{X}$ is normal.*

Lemma 3.7. *Let $\psi : Y \to X$ be a holomorphic map of irreducible normal Stein spaces such that for some principal effective reduced divisor $D \subset X$ and every $x \in X \setminus D$ the preimage $\psi^{-1}(x)$ is a curve. Suppose that the divisor $E = \psi^*(D)$ is reduced and irreducible, $\psi(E)$ is not contained in the singularities of X or D, and for a general point $y \in E$ the variety $\psi^{-1}(\psi(y))$ is a surface. Then for such a point $y \in Y$ (resp. for $z = \psi(y) \in X$), there exists a local analytic coordinate system $(u', v', v'', \bar{w}')$ on Y at y (resp. $(u, v, \bar{w})$ on X at z) where $\bar{w}' = (w'_1, \dots, w'_{n-2})$ (resp. $\bar{w} = (w_1, \dots, w_{n-2})$) for which the local coordinate form of ψ is given by*

$$(u', v', v'', \bar{w}') \mapsto (u, v, \bar{w}) = (u', (u')^l q(u', v', v'', \bar{w}'), \bar{w}'),$$

where $l \geq 1$ is the minimal zero multiplicity of the $(n \times n)$-minors in the Jacobi matrix of ψ at y and the function $q(0, v', v'', \bar{w}')$ depends on v' or v''.

Proof. Since y is a general point of E we see that it is a smooth point of E and X and by the assumption z is a smooth point of $Z = \overline{\psi(E)}$, D, and X. Thus locally D is given by $u = 0$ and Z by $u = v = 0$, where the holomorphic functions u and v can be included in a local coordinate system $(u, v, \bar{w})$ on X at z. If $u' = u \circ \psi$ then by the assumption E is given locally near y by $u' = 0$. Since $\dim E - \dim Z = 2$, by [Chirka 1989, Appendix, Theorem 2] there exists a local coordinate system $(v', v'', \bar{w}')$ on E such that the coordinate form of $\psi|_E : E \to D$ is given by $v = 0$ and $\bar{w} = \bar{w}'$. Extending functions v', v'', and $\bar{w}'$ holomorphically to a neighborhood of y in Y we get a local coordinate system $(u', v', v'', \bar{w}')$ in this neighborhood. Furthermore, the extension $\bar{w}'$ can be chosen as $\bar{w}' = \bar{w} \circ \psi$. Hence ψ is given locally by $u = u'$, $\bar{w} = \bar{w}'$, and $v = (u')^l q(u', v', v'', \bar{w}')$, where $q(0, v', v'', \bar{w}')$ is not identically zero. If $q(0, v', v'', \bar{w}')$ is independent of v' or v'', then replacing v by $v - u^l q(0, \bar{w})$, we increase l (which is at least 1 since otherwise $\psi(E)$ is not contained in Z). On the other hand l cannot be larger than the minimal zero multiplicity of the $(n \times n)$-minors in the Jacobi matrix of ψ at y. Hence we can suppose that $q(0, v', v'', \bar{w}')$ depends on v' or v'' in which case l is such a multiplicity. □

Proposition 3.8. *Let the assumptions of Lemma 3.7 hold. Suppose also that*

$$\longrightarrow X_m \xrightarrow{\sigma_m} X_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := X$$

is a sequence of simple pseudoaffine modifications such that

(i) *there exists a holomorphic map $\psi_m : Y \to X_m$ for which $\psi = \sigma_1 \circ \cdots \circ \sigma_m \circ \psi_m$;*

(ii) *for $\psi_i = \sigma_{i+1} \circ \cdots \circ \sigma_m \circ \psi_m$ the closure of $\psi_i(E)$ coincides with $Z_i \subset D_i$, where Z_i and D_i are the center and the divisor of σ_{i+1} respectively;*

(iii) *general points of Z_i are contained in the smooth parts of X_i and D_i.*[9]

Then such a sequence cannot be extended to the left indefinitely.

Proof. Let y be a general point in E such that $\psi_i(y) = z_i \in Z_i$ which implies that near these points Z_i, D_i, E, Y, X are smooth. By Lemma 3.7 we can consider local coordinate systems $(u', v', v'', \bar{w}')$ on Y at y and $(u_i, v_i, \bar{w}_i)$ on X_i at z_i such that ψ_i is given locally by $u_i = u'$, $\bar{w}_i = \bar{w}'$, and $v_i = (u')^l q(u', v', v'', \bar{w}')$. By Remark 3.5 (2) for a local coordinate system $(u_{i+1}, v_{i+1}, \bar{w}_{i+1})$ on X_{i+1} the coordinate form of σ_{i+1} is $(u_i, v_i, \bar{w}_i) = (u_{i+1}, u_{i+1}v_{i+1}, \bar{w}_{i+1})$. Hence a local form of ψ_{i+1} is $u_{i+1} = u'$, $\bar{w}_{i+1} = \bar{w}'$, and $v_{i+1} = (u')^{l_i-1} q(u', v', v'', \bar{w}')$. That is, in this construction $l_{i+1} = l_i - 1$. Since such powers cannot be negative we get the desired conclusion. □

Remark 3.9. In fact, we showed that m in Proposition 3.8 cannot exceed the minimal zero multiplicity l of the $(n \times n)$-minors in the Jacobi matrix of ψ from Lemma 3.7.

Similarly, we have the following.

Proposition 3.10. *Let $\sigma : \hat{X} \to X$ be a pseudoaffine modification with center Z, divisor D, and exceptional divisor $\hat{D}$. Suppose that none of components of Z is contained in $X_{\text{sing}} \cup D_{\text{sing}}$, and the image of every component of $\hat{D}$ is of codimension 1 in D. Let*

$$\longrightarrow X_n \xrightarrow{\sigma_n} X_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 := X$$

be a sequence of simple modifications with similar conditions on centers and such that σ factors through the composition of these simple modifications. Then such a sequence cannot be extended to the left indefinitely without violating the fact that σ factors through the composition.

Proof. For local analytic coordinate systems at general points $y \in \hat{D}$ and $z = \sigma(y) \in Z \subset X$, consider the zero multiplicity k of the Jacobian of σ at y (clearly this multiplicity is independent of the choice of local coordinate system and the choice of a general point y) and let k_i be the similar multiplicity for the modification $\sigma_i \circ \cdots \circ \sigma_1 : X_i \to X$. Since $\sigma_{i+1} : X_{i+1} \to X_i$ contracts the exceptional divisor in X_{i+1} we see that $k_{i+1} > k_i$. On the other hand if σ factors through $\sigma_i \circ \cdots \circ \sigma_1$ one must have $k_i \leq k$. This yields the desired conclusion. □

Definition 3.11. (1) Given two (pseudo)affine modifications $\sigma : \hat{X} \to X$ and $\delta : \tilde{X} \to X$ with the same center Z and divisor $D \subset X$, we say that σ dominates δ if it factors through δ. For instance, consider the normalization $\nu : \tilde{X}' \to \tilde{X}$ and let $\delta' = \delta \circ \nu : \tilde{X}' \to X$. Then δ is dominated by δ'.

(2) Let f, g, and h be holomorphic functions on a Stein manifold X and $\sigma : \hat{X} \to X$ be a simple pseudoaffine modification with a smooth divisor $D = f^*(0)$ and a smooth center Z given by a strict complete intersection $f = h = 0$. Suppose that Z is also a set-theoretical complete intersection given by $f^k = g = 0$ and $\tilde{X} \subset X \times \mathbb{C}_w$ is given by $f^k w = g$. Then we call the Davis modification $\delta : \tilde{X} \to X$ (induced by the natural projection) homogeneous (of degree k) if $f I^{k-1}$ and g generate the ideal I^k where I is the defining ideal of Z in the ring of holomorphic functions on X.

[9] Actually, (iii) follows automatically from the Proposition 3.6.

Lemma 3.12. *Let X be a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \dots, w_{n-2})$ and let $\sigma : \hat{X} \to X$ and I be as in* Definition 3.11 (2) *with $f = u$ and $h = v$. Suppose also that g and δ are as in* Definition 3.11 (2) *without δ being a priori homogeneous. Then δ is homogeneous if and only if and only if the function g is of the form $g = ev^k + a$, where a is in fI^{k-1} and e is an invertible holomorphic function.*

Proof. Since $f = u$ and I is generated by u and v one can see that fI^{k-1} is generated by functions $u^k, u^{k-1}v, \dots, uv^{k-1}$. That is, the ideal I^k is generated by fI^{k-1} and v^k and also by fI^{k-1} and g. This is possible if and only if $g = ev^k + a$, where $a \in fI^{k-1}$ and e is an invertible holomorphic function. □

Proposition 3.13. *Let $\sigma : \hat{X} \to X$ and $\delta : \tilde{X} \to X$ be as in* Definition 3.11 (2). *Then $\hat{X}$ is a normalization of $\tilde{X}$ and in particular σ dominates δ.*

Proof. Since normalization is a local operation, by Remark 3.5 (2), we can view X as a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \dots, w_{n-2})$ such that D is given by $u = 0$, Z by $u = v = 0$ (i.e., $\hat{X}$ can be viewed as the hypersurface in $X \times \mathbb{C}_w$ given by the equation $uw = v$). By Lemma 3.12 $g = ev^k + a$, where $a \in fI^{k-1}$ and e is an invertible holomorphic function. Changing e we can suppose that the Taylor series of a does not contain monomials divisible by v^k. Hence $u = g/f^k = g/u^k = ew^k + c$, where c is a polynomial in w of degree at most $k - 1$ whose coefficients are holomorphic functions on X. Thus w is integral over the ring of holomorphic functions on $\tilde{X}$ which concludes the proof. □

Remark 3.14. (1) Note that for a simple pseudoaffine modification from Definition 3.11 (2) with a given divisor D the function f is determined uniquely up to an invertible factor. This implies that for a given $k \geq 1$ the notion of a homogeneous modification of degree k from Definition 3.11 (2) is determined not as much by f and h but by D and the defining ideal I (or equivalently the center Z).

(2) In particular if $\delta : \tilde{X} \to X$ is a pseudoaffine modification with a divisor D and a center Z such that $\dim Z = \dim X - 2$ then for a smooth point z of Z that is not in $X_{\mathrm{sing}} \cup D_{\mathrm{sing}}$ we can say whether δ is locally homogeneous at z or not (where in the former case there exists a Euclidean neighborhood U of z in X such that for every connected component $\tilde{U}$ of $\delta^{-1}(U)$ the modification $\delta|_{\tilde{U}} : \tilde{U} \to U$ is homogeneous).

Proposition 3.15. *Let X be a normal irreducible Stein space which is Cohen–Macaulay, $\delta : \tilde{X} \to X$ be a Davis modification with an irreducible divisor D and a center Z of* $\operatorname{codim}_X Z = 2$ *such that none of the irreducible components of Z is contained in $X_{\mathrm{sing}} \cup D_{\mathrm{sing}}$. Suppose also that Z is a strict complete intersection of the form $f = h = 0$, where $D = f^*(0)$ and that at general points of Z this modification δ is locally homogeneous. Let $\sigma : \hat{X} \to X$ be a simple pseudoaffine modification associated with divisor D and center Z. Then $\hat{X}$ is a normalization of $\tilde{X}$ and thus σ dominates δ.*

Proof. First note that $\hat{X}$ is a normal Stein space by Proposition 3.6. Let z be a general point of Z, i.e., z is a smooth point of Z and it is not contained in some subvariety $P \subset Z$ of $\operatorname{codim}_Z P \geq 1$ such that $Z \cap (X_{\mathrm{sing}} \cup D_{\mathrm{sing}}) \subset P$. Then by Proposition 3.13 there exists a Euclidean neighborhood U of z in X such that $\sigma^{-1}(U)$ is a normalization of $\delta^{-1}(U)$. That is, we have a biholomorphism ψ between $\sigma^{-1}(X \setminus P)$ and a normalization of $\delta^{-1}(X \setminus P)$. Since P is of codimension at least 3 in X and δ is a Davis modification the codimension of $\delta^{-1}(P)$ in $\tilde{X}$ is at least of 2 (because by Remark 3.5(3) $\delta^{-1}(P) \simeq P \times \mathbb{C}$). Hence the Hartogs theorem implies that ψ extends to a biholomorphism between $\hat{X}$ and a normalization of $\tilde{X}$ which is the desired conclusion. $\square$

4. Semifinite modifications

Notation 4.1. Let T be a germ of an analytic set at the origin o of $\mathbb{C}^m$, $\mathrm{Hol}(T)$ be the ring of holomorphic functions on T, and $D = T \times \mathbb{C}$. We consider a hypersurface Z in D such that $\pi|_Z : Z \to T$ is finite where $\pi : D \to T$ is the natural projection. We also suppose that Z coincides (as a set) with the zeros of an analytic function h.

Lemma 4.2. *Let Notation 4.1 hold and w be a coordinate on the second factor of $D = T \times \mathbb{C}$. Then h can be chosen as a monic polynomial in w with coefficients from* $\mathrm{Hol}(T)$, *i.e.*, $h \in \mathrm{Hol}(T)[w]$.

Proof. Let $\pi^{-1}(o) \cap Z$ consist of points $z_1, \ldots, z_n$, where $z_i = (o, w_i) \in T \times \mathbb{C}$. By the Weierstrass preparation theorem, for every z_i there is a Euclidean neighborhood in D in which h coincides with $e_i h_i(w)$, where e_i is an invertible function and $h_i \in \mathrm{Hol}(T)[w]$ is a monic polynomial in $w - w_i$. Note that Z coincides with the zeros of the product $h_1 h_2 \cdots h_m$ because of the finiteness of $\pi|_Z : Z \to T$. Thus replacing h with the product $h_1 h_2 \cdots h_m$ we get the desired conclusion. $\square$

Lemma 4.3. *Let Notation 4.1 hold and the zero multiplicity of h at general points of Z be n. Then $g = h^{1/n}$ is a holomorphic function on D and in particular the defining ideal of Z in the ring* $\mathrm{Hol}(T)[w]$ *is the principal ideal generated by g.*

Proof. By Lemma 4.2 we can suppose that h is a monic polynomial in w. Consider the restriction of h to any fiber. It is a polynomial whose roots have multiplicities divisible by n (since for general fibers such multiplicities are exactly n). Thus the restriction of g to any fiber of $\pi : D \to T$ can be chosen as a nonzero monic polynomial in w. Therefore, choosing such a restriction on $\pi^{-1}(o)$ we define by continuity a unique branch of $h^{1/n}$ on D as g. Note that g is holomorphic outside a subset $K = Z \cap \pi^{-1}(T_{\mathrm{sing}})$ and because of the assumption on h we see that $g = w^k + r_{k-1} w^{k-1} + \cdots + r_1 w + r_0$, where each r_i is a continuous function on T holomorphic on $T \setminus K$. Hence the first statement will follow from the claim.

Claim. Let g be a (not a priori continuous) function on D, holomorphic outside a proper analytic subset K that does not contain any fiber of π. Suppose also that g is a polynomial in w. Then g is holomorphic on D.

Taking a smaller T, if necessary, one can suppose that the image K_0 of K under the natural projection $D \to \mathbb{C}_w$ is relatively compact. In particular, the function $r_k w_0^k + r_{k-1} w_0^{k-1} + \cdots + r_1 w_0 + r_0$ is holomorphic on T for every w_0 in $\mathbb{C} \setminus K_0$. Since we have an infinite number of such w_0's, every coefficient r_i is also holomorphic on T, which concludes the proof of the Claim.

To see that Z is a principal divisor in D consider any function $f \in \mathrm{Hol}(T)[w]$ vanishing on Z. Then the quotient f/g is again holomorphic on D by the Claim which yields the desired conclusion. $\square$

Lemma 4.4. *Let Notation 4.1 hold and T be singular. Then so is Z.*

Proof. Assume that Z is smooth at some point $z \in Z \cap \pi^{-1}(o)$. Let g be a function as in Lemma 4.3. Extend g to a function $\tilde{g} \in \operatorname{Hol}(\mathbb{C}^m, o)[w]$ whose zeros define an extension $\tilde{Z}$ of Z. Note that if the partial derivative of $\tilde{g}$ with respect to w is nonzero at z then by the implicit function theorem $\tilde{Z}$ is biholomorphic to $(\mathbb{C}^m, o)$ and therefore Z is locally biholomorphic to T which implies that Z is singular. Hence we assume that this derivative is zero.

Let I be the defining ideal of T in $\operatorname{Hol}(\mathbb{C}^m, o)$ and k be the dimension of T. Choose any $m-k$ elements from I and consider their Jacobi matrix (with respect to coordinates $z_1, \dots, z_m$ of $\mathbb{C}^m$). Since T is singular at o, any $(m-k)$-minor M of this matrix vanishes at o. By Lemma 4.3 the defining ideal L of Z in $(\mathbb{C}^m, o) \times \mathbb{C}_w$ is generated by I and $\tilde{g}$. Take any $m+1-k$ elements from L and consider their Jacobi matrix with respect to the coordinates $(w, z_1, \dots, z_m)$. It follows from the previous argument about the partial derivative of $\tilde{g}$ with respect to w and about the $(m-k)$-minor M that any $(m+1-k)$-minor of this new Jacobi matrix vanishes at o. This is contrary to the fact that Z is smooth at z which concludes the proof. □

Example 4.5. Lemma 4.4 is one of the central technical results in this paper. The assumption that Z coincides with zeros of a global analytic function is very important. Indeed, consider the case when T is a semicubic parabola $x^2 - y^3 = 0$ in $\mathbb{C}^2_{x,y}$. Then there is a closed immersion of $\mathbb{C}$ into $D = T \times \mathbb{C} \subset \mathbb{C}^3_{x,y,w}$ given by $t \mapsto (t^3, t^2, t)$ such that the image is a smooth Weil divisor whose projection to the singular T is finite. Lemma 4.4 is not applicable here because this Weil divisor is not $\mathbb{Q}$-Cartier.

Remark 4.6. If T is not unibranch at o then the argument is much easier and, furthermore, Z is not unibranch at any point z_0 above o as well. Indeed, let T_1 and T_2 be distinct irreducible components of T at o. Then D has irreducible branches $T_1 \times \mathbb{C}$ and $T_2 \times \mathbb{C}$ meeting along the line $L = o \times \mathbb{C}$. Since the zeros of h contain the point $z_0 \in L$ but not the line L itself the zeros Z_i of h in $T_i \times \mathbb{C}$ produce two different branches Z_1 and Z_2 of Z at z_0.

Proposition 4.7. *Let T be an affine algebraic variety, $D = T \times \mathbb{C}$, and Z be an algebraic hypersurface in D such that $\pi|_Z : Z \to T$ is finite where $\pi : D \to T$ is the natural projection. Suppose that for every point $t_0 \in T$ there is a Euclidean neighborhood $U \subset T$ such that $Z \cap \pi^{-1}(U)$ is an analytic $\mathbb{Q}$-principal divisor in $\pi^{-1}(U)$, i.e., for some natural $k > 0$ and a holomorphic function g on $\pi^{-1}(U)$ the divisor $kZ \cap \pi^{-1}(U)$ coincides with $g^*(0)$.*

Then Z is an effective reduced principal divisor in D.

Proof. Let w be a function on $D = T \times \mathbb{C}$ induced by a coordinate on the second factor. The field $\mathbb{C}(Z)$ of rational functions on Z is a finite separable extension of the field $\mathbb{C}(T)$ and it is generated by $w|_Z$ over $\mathbb{C}(T)$. Let $g(w) = w^n + r_{n-1}(t)w^{n-1} + \cdots + r_1(t)w + r_0(t)$ be the minimal monic polynomial for w over $\mathbb{C}(T)$. In particular $Z \cap \pi^{-1}(U)$ is given by the zeros of this rational function g. By Lemma 4.3 $Z \cap \pi^{-1}(U)$ is a principal divisor in $\pi^{-1}(U)$ which implies that the function g is holomorphic. Therefore g is regular (e.g., see [Kaliman 1991, Theorem 5]). This yields the desired conclusion. □

Definition 4.8. Let $\sigma : \hat{X} \to X$ be an affine modification of normal affine algebraic varieties such that X is Cohen–Macaulay and

(a) its divisor $D = f^*(0)$ (where $f \in \mathbb{C}[X]$) is a reduced principal divisor in X isomorphic to a direct product $D \simeq T \times \mathbb{C}$;

(b) the restriction $\pi|_Z : Z \to T$ of the natural projection $\pi : D \to T$ to the center Z of σ is finite;

(c) none of irreducible components of Z is contained in the singularities of X (note that for the singularities of D a similar fact also holds);

(d) for any point z of Z there is a Euclidean neighborhood U in X for which the restriction $\sigma|_{\hat{U}} : \hat{U} \to U$ of σ to $\hat{U} = \sigma^{-1}(U)$ factors through some Davis pseudoaffine modification $\delta : \tilde{U} \to U$ with the following property: the restriction of δ over a neighborhood $U' \subset U$ of any general point $z' \in Z \cap U$ is homogeneous of degree k (where k does not depend on different irreducible components of $Z \cap U$).

Then we call such a σ a semifinite affine modification.

Proposition 4.9. *Let $\sigma : \hat{X} \to X$ be a semifinite affine modification with $D \simeq T \times \mathbb{C}$ and Z as in Definition 4.8. Then σ dominates a simple modification* (*with the same center and divisor*) *and if T is singular so is the center Z.*

Proof. Let a Davis modification $\delta : \tilde{U} \to U$ from Definition 4.8 (d) be defined by the ideal (f^k, g). By condition (d), δ is locally homogeneous at general points of $Z \cap U$ which in combination with Lemma 3.12 implies that for some $k \geq 1$ the divisor $kZ \cap U$ coincides with $g^*(0) \cap U$. That is, we are under the assumptions of Proposition 4.7. Hence $Z = h^*(0)$, where $h \in \mathbb{C}[D]$. Thus extending h to a regular function on X (denoted by the same symbol) we see that Z is a strict complete intersection given by $f = h = 0$. These functions f and h induce a simple affine modification $\tau : X' \to X$ with divisor D and center Z such that X' is a normal affine algebraic variety by Proposition 2.6. Furthermore, $\hat{X}$ and X' are also normal as analytic sets by [Zariski and Samuel 1960, Chapter 13, Theorem 32]. By Proposition 3.15 $\tau^{-1}(U)$ is an analytic normalization of $\tilde{U}$. Since $\hat{X}$ is normal then the holomorphic map $\sigma^{-1}(U) \to \tilde{U}$ (induced by the domination of δ by σ) factors through the normalization of $\tilde{U}$. This yields a desired holomorphic map from $\hat{X}$ to X' which is a morphism since its restriction over $X \setminus D$ is an algebraic isomorphism. That is, σ dominates τ. The second statement follows from Lemma 4.4. □

5. Applications of Kambayashi's theorem

Notation 5.1. In this section $\varphi : X \to B$ will be a morphism of complex factorial affine algebraic varieties, Φ will be a G_a-action on X which preserves the fibers of φ. We do not assume a priori that the $\mathbb{C}[B]$-algebra $\mathbb{C}[X]^\Phi$ of Φ-invariant regular functions is finitely generated.

However, such an algebra can be viewed (and will be viewed) as a direct limit $\varinjlim A_\alpha$ of its finitely generated $\mathbb{C}[B]$-subalgebras A_α (with respect to the partial order generated by inclusions) where α belongs to some index set and each A_α can be treated as the ring $\mathbb{C}[Q_\alpha]$ of regular functions of some affine algebraic variety Q_α. Replacing A_α with its integral closure we suppose that each of these Q_α is normal (the fact that this transition preserves affineness is a standard result, e.g., [Eisenbud 1995, Theorem 4.14]). Furthermore, by the Rosenlicht Theorem (e.g., see [Vinberg and Popov 1989, Theorem 2.3]) Q_α can be chosen so that the morphism $\varrho_\alpha : X \to Q_\alpha$ induced by the natural embedding $A_\alpha \subset \mathbb{C}[X]$ separates general orbits of Φ. As in [Flenner et al. 2016] we introduce the following.

Definition 5.2. A normal affine variety Q_α as before will be called a *partial quotient* of X by Φ and the morphism $\varrho_\alpha : X \to Q_\alpha$ (separating general orbits of Φ) will be called *a partial quotient morphism*.

Remark 5.3. Note that including the coordinate functions of the morphism φ into a ring A_α we can always choose a partial quotient morphism that is constant on the fibers of φ.

Theorem 5.4. *Let $\varphi : X \to B$ and Φ be as in Notation 5.1 and the generic fiber of φ be a three-dimensional variety Y (over the field K of rational functions on B). Let $\tilde{K}$ be the algebraic closure K and $\tilde{Y}$ be the variety over $\tilde{K}$ obtained from Y by the field extension. Suppose that $\varrho : X \to Q$ is a partial quotient morphism such that φ factors through ϱ. Suppose also that the ring of invariants of the G_a-action on $\tilde{Y}$ (induced by Φ) is a polynomial ring in two variables over $\tilde{K}$. Then there is a morphism $\psi : Q \to B \times \mathbb{C}^2$ such that for some nonempty Zariski dense open subset $B^* \subset B$ the restriction of ψ over B^* is an isomorphism.*

Proof. By the Kambayashi theorem [1975] the ring of invariants of the induced action on Y is also a polynomial ring in two variables over K. Hence for some B^* as above and $X^* = \varphi^{-1}(B^*)$, the induced action on X^* has the categorical quotient isomorphic to $B^* \times \mathbb{C}^2$. Assume that $B \setminus B^*$ is a principal effective divisor, which can be done without loss of generality. Let f be a regular function on B with zero locus $B \setminus B^*$ (we denote the lifts of f to X or Q by the same symbol). For every regular function $h \in \mathbb{C}[X^*]$ there exists natural k for which $f^k h$ extends to a regular function on X. Hence for sufficiently large k_1 and k_2 the composition of the quotient morphism $\varrho_0 : X^* \to B^* \times \mathbb{C}^2_{u_1,u_2}$ with the isomorphism $\kappa : B^* \times \mathbb{C}^2_{u_1,u_2} \to B^* \times \mathbb{C}^2_{u_1,u_2}$ given by $(b, u_1, u_2) \mapsto (b, f^{k_1}u_1, f^{k_2}u_2)$ extends to a morphism $\tau : X \to B \times \mathbb{C}^2$ (note that $\tau|_{X^*} : X^* \to B^* \times \mathbb{C}^2$ can be viewed now as the quotient morphism). Similarly, since $f \in \mathbb{C}[X]$ and each $u_i \in \mathbb{C}[X^*]$ are Φ-invariant we see that $f^{k_1}u_1$ and $f^{k_2}u_2$ can be viewed as regular functions on the normal variety Q. Thus $\tau = \psi \circ \theta$, where $\theta : X \to Q$ and $\psi : Q \to B \times \mathbb{C}$ are morphisms. By the universal property of quotient morphisms, $\theta|_{X^*} : X^* \to Q \setminus f^{-1}(0)$ factors through $\tau|_{X^*}$, which implies that ψ is invertible over B^*. This yields the desired conclusion. □

Remark 5.5. By construction ψ is an affine modification. Another observation is that if the general fibers of φ do not admit nonconstant invertible functions then for every irreducible divisor T in B the variety $\varphi^{-1}(T)$ is reduced and irreducible by Proposition 1.6. Actually, in this case there is no need to assume that B is factorial since it follows automatically from the fact that X is factorial.

Example 5.6. Let Notation 5.1 hold and every general fiber of φ be isomorphic to the same affine algebraic threefold V. Suppose that Φ induces an action on each general fiber of φ with the categorical quotient isomorphic to $\mathbb{C}^2$. Then we claim that the above conclusion about the partial quotient Q as a modification of $B \times \mathbb{C}^2$ over B is valid.

Indeed, by [Kraft and Russell 2014] there exists a Zariski dense open subset $B^* \subset B$ and an unramified covering $\hat{B}^* \to B^*$ such that $\hat{X}^* = X \times_{B^*} \hat{B}^*$ is naturally isomorphic to $\hat{B}^* \times V$ over $\hat{B}^*$. In particular, the induced action on the generic fiber of $\hat{X}^* \to \hat{B}^*$ has the ring of invariants isomorphic to the polynomial ring $\hat{K}[x_1, x_2]$ where $\hat{K}$ is the field of rational functions on $\hat{B}^*$. Note that this ring is obtained from the ring of invariants on the generic fiber Y of $X^* \to B^*$ via a field extension $[\hat{K} : K]$. Hence by the Kambayashi theorem the latter ring of invariants is $K[x_1, x_2]$ and we are under the assumption of Theorem 5.4.

In fact, as suggested by the referee, we can strengthen Theorem 5.4 and Example 5.6 as follows:

Theorem 5.7. *Let Notation 5.1 hold and let Φ induce an action on each general fiber of φ with the categorical quotient isomorphic to $\mathbb{C}^2$. Let Y and K be as in Theorem 5.4. Then the ring of invariants of the G_a-action on Y (induced by Φ) is a polynomial ring in two variables over K. In particular, for a partial quotient morphism $\varrho : X \to Q$ for which φ factors through ϱ (i.e., for some morphism $\kappa : Q \to B$ one has $\varphi = \kappa \circ \varrho$) there is a morphism $\psi : Q \to B \times \mathbb{C}^2$ over B such that for a nonempty Zariski dense open subset $B^* \subset B$ the restriction of ψ over B^* is an isomorphism.*

In preparation for the proof of this theorem we need the next (certainly well-known) fact.

Proposition 5.8. *Let $\kappa : Q \to B$ be a dominant morphism of algebraic varieties such that Q is normal. Then for a general point $b \in B$ the variety $\kappa^{-1}(b)$ is normal.*

Proof. Replacing B by a Zariski dense open subset (and Q by its preimage) we can suppose that κ is flat [Grothendieck 1964, Théorème 6.9.1]. Let ω be a generic point of B. Then the fiber of κ over ω is normal (e.g., see [Atiyah and Macdonald 1969, Proposition 5.13]) which is equivalent to the fact that this fiber is geometrically normal[10] since we work over a field of characteristic zero. On the other hand the set of (not necessarily closed) points in B for which the fibers over them are geometrically normal is open [Grothendieck 1966, Théorème 12.1.6]. Since this set is nonempty it contains general (closed) points of B and we are done. □

Proof of Theorem 5.7. Let Q_b be the fiber $\kappa^{-1}(b)$ over a general point $b \in B$ and $Q^b \simeq \mathbb{C}^2$ be the categorical quotient of the action $\Phi|_{\varphi^{-1}(b)}$. By the universal property of quotient morphisms one has a morphism $\psi_b : Q^b \to Q_b$. Since ϱ separates general orbits ψ_b must be birational.

Assume that for a curve $C_b \subset Q^b$ the image $\psi_b(C_b)$ is a point $q_b \in Q_b$. By Corollary 1.7 the preimage of C_b in $\varphi^{-1}(b)$ is a surface S_b, i.e., $\varrho(S_b) = q_b$. By [Shafarevich 1994, Chapter I, Section 6.3, Corollary] the closure T of the set $\{q \in Q \mid \dim \varrho^{-1}(q) = 2\}$ is a subvariety. Let L be the union of the one-dimensional components of T (it is nonempty since $q_b \in T$ for general b). Note that $\varrho^{-1}(L)$ is a closed Φ-stable threefold V. By the Hironaka flattening theorem [Hironaka 1975] there exists of a proper birational morphism $\pi : \hat{Q} \to Q$ such that for the irreducible component $\hat{X}$ of $X \times_Q \hat{Q}$ dominant over X the natural projection $\hat{\varrho} : \hat{X} \to \hat{Q}$ is flat, i.e., all of its fibers are one-dimensional or empty. This implies $\pi^{-1}(L)$ contains a surface $\hat{P} = \hat{\varrho}(\hat{V})$ where $\hat{V}$ is the proper transform of V in $\hat{X}$. Choose a curve $\hat{C} \subset \hat{P}$ whose image in B is dense and such that $\hat{\varrho}^{-1}(\hat{c}) \neq \varnothing$ for a general point $\hat{c} \in \hat{C}$. Let $\hat{R}$ be a closed

[10] Recall that a scheme Z over a field k is geometrically normal if it is normal and, furthermore, it remains normal under any field extension of k. In the case of a perfect field k (e.g., a field of characteristic zero) every normal scheme is automatically geometrically normal.

surface in $\hat{Q}$ that meets $\hat{P}$ along $\hat{C}$ and let R be the closure of $\pi(\hat{R})$ (note that R contains L). Then the closure of $\varrho^{-1}(R \setminus L)$ contains a Φ-stable threefold W that meets V over general points of B (since $\hat{\varrho}^{-1}(\hat{c}) \neq \varnothing$). Because X is factorial the threefold W is the zero locus of a regular function $h \in \mathbb{C}[X]$ which, by construction, does not vanish on general fibers of ϱ. That is, it is constant on general fibers and, therefore, Φ-invariant. By construction h is not constant on the surface $V \cap \varphi^{-1}(b)$ for general b (this follows from the fact that W meets $V \cap \varphi^{-1}(b)$ along a curve). Enlarging a set of generators of the ring $\mathbb{C}[Q]$ by h (and taking the integral closure to preserve the normality of Q assumed in Definition 5.2) we obtain another partial quotient for which the image of V is a surface. That is, the set T as before becomes at most finite and for a general $b \in B$ this curve C_b does not exist.

Hence the morphism $\psi_b : Q^b \to Q_b$ is quasifinite now in addition to being birational. Assume that it is not an embedding. Then by the Zariski main theorem Q_b is not normal for general $b \in B$, contrary to Proposition 5.8. Hence $\psi_b : Q^b \to Q_b$ is an embedding.

Let D_b be the complement to the image of Q^b in Q_b. Suppose that D_b is a curve for a general $b \in B$ (since Q_b and Q^b are affine the alternative is an empty D_b). The set $Q_b \setminus \varrho(\varphi^{-1}(b))$ consists of D_b and a finite set by Corollary 1.7. Since $\varrho(X)$ is a constructible set by [Hartshorne 1977, Chap. II, Exercise 3.19] we see that $Q \setminus \varrho(X)$ is an algebraic variety. Consider the irreducible components of this variety which are dominant over B and remove those of them that have dimension $\dim B$. Then we are left with $D := \bigcup_{b \in B} D_b$ because $Q_b \setminus (\varrho(\varphi^{-1}(b)) \cup D_b)$ is finite. That is, D is an algebraic variety and $Q \setminus D$ is a quasiaffine variety. The general fibers $Q_b \setminus D_b$ of the natural morphism $Q \setminus D \to B$ are isomorphic to $\mathbb{C}^2$. By [Kaliman and Zaidenberg 2001] for a Zariski dense open subset $B^* \subset B$ the variety $Q^* = \kappa^{-1}(B^*) \setminus D$ naturally isomorphic to $B^* \times \mathbb{C}^2$. Hence for $X^* = \varphi^{-1}(B^*)$ we get a partial quotient morphism $\varrho|_{X^*} : X^* \to B^* \times \mathbb{C}^2$.

Let $\varrho' : X^* \to Q'$ be another partial quotient morphism for $\Phi|_{X^*}$ into a normal variety Q' (over B^*) such that ϱ factors through ϱ', i.e., $\varrho = \theta \circ \varrho'$ for a morphism $\theta : Q' \to B^* \times \mathbb{C}^2$. Suppose that Q'_b is the fiber of the natural morphism $Q' \to B^*$ over a point $b \in B^*$, i.e., the restriction of θ yields a morphism $\theta_b : Q'_b \to Q_b \simeq Q^b \simeq \mathbb{C}^2$. By the universal property of quotient morphisms we have a natural morphism $Q^b \to Q'_b$, i.e., θ_b is invertible. Hence θ is bijective. By the Zariski main theorem, θ is an isomorphism. This implies that $\varrho|_{X^*} : X^* \to B^* \times \mathbb{C}^2$ is the categorical quotient morphism. In particular, for Y and K as in Theorem 5.4 the ring of invariants of the G_a-action on Y (induced by Φ) is a polynomial ring in two variables over K. Now the desired conclusion follows from Theorem 5.4. $\square$

6. Criterion for existence of an affine quotient

Notation 6.1. Let B be a unibranch germ of a smooth complex algebraic curve at point o (i.e., o is the zero locus of some $f \in \mathbb{C}[B]$) and $\varphi : X \to B$ be a morphism from a complex factorial affine algebraic variety X equipped with a G_a-action Φ which preserves each fiber of φ. We suppose also that $\varrho : X \to Q$ is a partial quotient morphism of the action Φ. Let $\psi : Q \to Q_0$ be a morphism over B into a smooth affine algebraic variety Q_0 over B and $\tau = \psi \circ \varrho$. The divisor in X (resp. Q, resp. Q_0) over o will be denoted by E (resp. D, resp. D_0). We suppose that the morphism $Q_0 \to B$ is smooth (and in particular D_0 is a smooth reduced divisor) and that ψ induces an isomorphism $Q \setminus D \to Q_0 \setminus D_0$. We suppose also that $E = \varphi^*(o)$ is reduced and denote by Z_0 the closure of $\tau(E)$ in D_0. We consider the Stein factorization (e.g., see [Hartshorne 1977, Chapter III, Corollary 11.5]) of a proper extension $\overline{E} \to Z_0$ of the morphism $\tau|_E : E \to Z_0$. Its restriction to E enables us to treat $\tau|_E : E \to Z_0$ as a composition of a surjective morphism $\lambda : E \to V$ with connected general fibers and a quasifinite morphism $\theta : V \to Z_0$.

Theorem 1.1 implies the following:

Lemma 6.2. *Let Notation 6.1 hold and Z_0 (and therefore $\psi(D)$) be Zariski dense in D_0. Then ψ is an isomorphism.*

Convention 6.3. With an exception of Corollary 6.10 we suppose throughout this section that Z_0 is a divisor in D_0 and furthermore

(i) the morphism $\theta : V \to Z_0$ from Notation 6.1 is in fact *finite*;

(ii) for every point $z_0 \in Z_0$ the preimage $\tau^{-1}(z_0)$ is a *surface*.

Remark 6.4. Note that Convention 6.3 holds automatically when E is isomorphic to $\mathbb{C}^3$ (under the assumption that Z_0 is a curve). Indeed, $\tau^{-1}(z_0)$ cannot be three-dimensional (otherwise it coincides with E and $\tau(E)$ is a point, not a curve). Then both V and Z_0 must be polynomial curves and a nonconstant morphism of polynomial curve is always finite. In fact, this argument works not only when $E \simeq \mathbb{C}^3$. It is enough to assume that there is no nonconstant morphism from E into any nonpolynomial curve.

Notation 6.5. Let Notation 6.1 hold. Our aim is to present $\psi : Q \to Q_0$ over B as a composition

$$Q =: Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

of simple affine modifications $\sigma_i : Q_i \to Q_{i-1}$ over B. Let $\psi_i = \sigma_1 \circ \cdots \circ \sigma_i : Q_i \to Q_0$ and suppose that ψ can be presented as $\psi = \psi_i \circ \delta_i$ for a modification $\delta_i : Q \to Q_i$. Let $\tau_i = \delta_i \circ \varrho$ and thus $\tau = \psi_i \circ \tau_i$. We suppose that the zero locus of $f \circ \psi_i$ is reduced and coincides with $D_i := \psi_i^{-1}(D_0)$ (which is nothing but the divisor of modification σ_{i+1}) while the center Z_i of σ_{i+1} coincides with $\tau_i(E)$. Note that under such assumptions every Q_i is normal and Cohen–Macaulay by Proposition 2.6 and induction starting from smooth Q_0 and D_0.

Lemma 6.6. *Suppose that for some i one has a sequence of simple affine modifications*

$$Q_i \xrightarrow{\sigma_i} Q_{i-1} \xrightarrow{\sigma_{i-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

such that $\psi : Q \to Q_0$ factors through $\psi_i : Q_i \to Q_0$ and, therefore, $\psi = \psi_i \circ \delta_i$. Let $\delta_i : Q \to Q_i$ be a semifinite affine modification. Then the sequence can be extended to a sequence of simple affine modifications

$$Q_{i+1} \xrightarrow{\sigma_{i+1}} Q_i \xrightarrow{\sigma_i} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

such that D_i (resp. Z_i) is the divisor (resp. center) of σ_{i+1} and $\psi : Q \to Q_0$ factors through $\psi_{i+1} : Q_{i+1} \to Q_0$, i.e., $\psi = \psi_{i+1} \circ \delta_{i+1}$.

Proof. The statement follows from Proposition 4.9. □

In order to establish when δ_i is semifinite we need the following technical fact.

Lemma 6.7. *For any point $x_0 \in E$ there is the analytic germ S at x_0 of an algebraic subvariety of X of dimension* $\dim X - 2$ *such that*

(i) *$S_0 := \tau(S)$ is a germ of an analytic hypersurface in Q_0 at $z_0 = \tau(x_0)$;*

(ii) *the map $\tau|_S : S \to S_0$ is finite;*

(iii) *for every general point $z \in Z_0$ near z_0 there is a local coordinate system $(u, v, \bar{w})$ on Q_0 at z for which D_0 is given locally by $u = 0$, Z_0 by $u = v = 0$, and every irreducible branch of the germ of S_0 at z by an equation of form $e := v - u^l d(u, \bar{w}) = 0$ where d is holomorphic and l is the zero multiplicity of the function $e \circ \tau$ at general points of E.*

Proof. Consider two regular functions h and e on E with the set P of common zeros such that near any general point P is locally a strict complete intersection given by $h = e = 0$ and for $z_0 = \tau(x_0)$ the set $P \cap \tau^{-1}(z_0)$ contains x_0 as an isolated component (which is possible since $\tau^{-1}(z_0)$ is a surface). Extend h and e regularly to X. Without loss of generality we can suppose that the set of common zeros of these extensions is an $(n-2)$-dimensional subvariety T of X where $n = \dim X$. Denote by S (resp. R) the analytic germ of T (resp. P) at x_0 and by T_0 the closure of $\tau(T)$ in Q. Let S_0 be the analytic germ of the hypersurface $T_0 \subset Q_0$ at z_0. For the restriction $\kappa : S \to T_0$ of τ the preimage $\kappa^{-1}(z_0) = x_0$ is a singleton. Hence [Grauert and Remmert 1979, Chapter 1, Section 3, Theorem 2] implies that the holomorphic map $\tau|_S : S \to S_0$ is finite. Thus we have (i) and (ii).

By construction S meets E transversely at a general point x of R. Let $z = \tau(x)$ and $(u', v', v'', \bar{w}')$ (resp. $(u, v, \bar{w})$) be a local analytic coordinate system at $x \in X$ (resp. $z \in Q_0$) as in Lemma 3.7. That is, locally E (resp. D_0, resp. Z_0) is given by $u' = 0$ (resp. $u = 0$, resp. $u = v = 0$) and τ is given by $u = u'$, $\bar{w} = \bar{w}'$, and $v = (u')^l q(u', v', v'', \bar{w}')$.

The finiteness of $\tau|_S$ implies that $\tau|_{S\cap E} : S \cap E \to S_0 \cap D_0$ is étale over the general point $z \in Z_0$. Thus $E \cap S$ is given locally by equations of form $v' = h_1(\bar{w}')$ and $v'' = h_2(\bar{w}')$, where h_1 and h_2 are holomorphic functions. Since S meets E transversely we also see that S must be given locally by equations of form $v' = h_1(\bar{w}') + u' g_1(u', v', v'', \bar{w}')$ and $v'' = h_2(\bar{w}') + u' g_2(u', v', v'', \bar{w}')$ where g_1 and g_2 are holomorphic functions. Furthermore, by the implicit function theorem these equations can be rewritten as $v' = h_1(\bar{w}') + u' \tilde{g}_1(u', \bar{w}')$ and $v'' = h_2(\bar{w}') + u' \tilde{g}_2(u', \bar{w}')$ for some other holomorphic functions $\tilde{g}_1$ and $\tilde{g}_2$. Plugging these expressions for v' and v'' with $u' = u$ and $\bar{w} = \bar{w}'$ into equation $v = (u')^l q(u', v', v'', \bar{w}')$ we get the desired form of equation $e = 0$ in (iii). Note that the function $e \circ \tau$ is given by

$$e \circ \tau(u', v', v'', \bar{w}') = (u')^l [q(u', v', v'', \bar{w}') h(u', \bar{w}')].$$

Since $q(0, v', v'', \bar{w}')$ depends on v' or v'' by Lemma 3.7, we see that the expression in the brackets is not identically zero on E, which shows that l is the zero multiplicity of $e \circ \tau$ at a general point of E. □

Lemma 6.8. *The morphism δ_i is always semifinite unless $Z_i = \tau_i(E)$ is dense in D_i.*

Proof. As we mentioned in Notation 6.5, the variety Q_i is normal and Cohen–Macaulay, and the divisor $D_i = (f \circ \psi_i)^*(0)$ is principal. Furthermore, by construction, $D_i = Z_{i-1} \times \mathbb{C}$ for $i \geq 1$. Thus we have condition (a) from Definition 4.8. The finite morphism $V \to Z_0$ (from Notation 6.1 and Convention 6.3) factors through maps $\theta_i : V \to Z_i$, $Z_i \to Z_{i-1}$ and $Z_{i-1} \to Z_0$ each of which must be, therefore, finite, i.e., we have condition (b) from Definition 4.8. Condition (c) follows from construction and Proposition 2.6. It remains to check condition (d).

Consider the following construction of an analytic germ of an algebraic variety $\mathcal{S}$ in X of $\dim \mathcal{S} = n-2$ where $\dim X = n$. Let M be the set of points in V above z_0 (by Convention 6.3 M is finite and nonempty). For every $y \in M$ choose any $x \in E$ above y. Consider the analytic germ $S(y)$ of an $(n-2)$-dimensional algebraic subvariety of X at x such that $S_0(y) := \tau(S(y))$ is an analytic hypersurface in Q_0 and the restriction of τ yields a finite map $S(y) \to S_0(y)$ (the existence of such an $S(y)$ is provided by Lemma 6.7).

Let $V(y)$ be the analytic germ of V at y and $k(y)$ be the degree of the finite morphism $E \cap S(y) \to V(y)$ induced by $\lambda : E \to V$ from Notation 6.1. In general $k(y)$ may depend on y but we allow $S(y)$ to be nonreduced and then, replacing each $S(y)$ (and, therefore, $S_0(y)$) with a multiple of it, we can suppose that $k(y) = k$ for every $y \in M$.[11] Since the variety Q_0 is smooth every $S_0(y)$ coincides with $h_{0y}^*(0)$ for some holomorphic function h_{0y}. By Lemma 6.7 for a local analytic coordinate system $(u, v, \bar{w})$ at a general point $z \in Z_0$ this function h_{0y} can be viewed as $(v - u^l d(u, \bar{w}))^k$ where l is the zero multiplicity of $(v - u^l d) \circ \tau$ on E.

Let $\mathcal{S} = \bigcup_{y \in M} S(y)$, $\mathcal{S}_0 = \bigcup_{y \in M} S_0(y)$, and $h_0 = \prod_{y \in M} h_{0y}$, i.e., $\mathcal{S}_0 = h_0^*(0)$. Then by construction $\tau|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_0$ is finite and $\mathcal{S}_0$ meets D_0 along the analytic germ of Z_0 at z_0. For m being the degree of the finite morphism $\theta : V \to Z_0$ from Convention 6.3, this implies that near point z the function h_0 can be viewed as a product of km factors of the form $v - u^l d(u, \bar{w})$. Hence the zero multiplicity of $h_0 \circ \tau$ at general points of E is klm.

Let $S_i(y) = \tau_i(S(y))$ and $\mathcal{S}_i = \tau_i(\mathcal{S}) = \bigcup_{y \in M} S_i(y)$. Note that the finiteness of $\tau|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_0$ implies that $\mathcal{S}_i$ is an analytic set in Q_i and $(\tau_i)|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_i$ is also finite. Furthermore, by construction $\mathcal{S}_i$ meets D_i along the union of analytic germs of Z_i at the points from $\psi_i^{-1}(z_0) \cap \mathcal{S}_i$. Assume by induction that for a germ of some holomorphic function h_i on Q_i the variety $\mathcal{S}_i$ coincides with $h_i^*(0)$ and the zero multiplicity of $h_i \circ \tau_i$ at general points of E is $kl_i m_i$ where m_i is the degree of the finite morphism $\theta_i : V \to Z_i$ and $l_i = l - i$ (note that l_i must be greater than zero since otherwise Z_i is dense in D_i).

Hence the function h_i / f^{km_i} has a holomorphic lift to the normal variety X. Furthermore, it is constant along each fiber of ϱ and it can be also pushed to a holomorphic function on the normal variety Q. This implies that δ_i factors through the Davis modification $\kappa : Q' \to Q_i$ of Q_i along D_i with ideal generated by f^{km_i} and h_i. By Lemma 6.7 for a general point z' of Z_i there is a Euclidean neighborhood with a local coordinate system $(u', v', \bar{w}')$ such that D_i is given locally by $u' = 0$, Z_i by $u' = v' = 0$, and up to an invertible factor h_i can be presented locally as a product of km_i factors of form $v' - u' d(u', \bar{w}')$. Hence

[11] We use here the following definition: if r is the degree of a finite morphism $W \to U$ of reduced analytic sets and W_m is the m-multiple of W then we put the degree of the morphism $W_m \to U$ equal to mr. Treating W_m as a union of m disjoint samples of W one can see that this extended notion of degree has all the properties of the standard degree.

by Lemma 3.12 κ is locally homogeneous at z'. Thus δ_i is semifinite and there is a simple modification $\sigma_{i+1} : Q_{i+1} \to Q_i$ as required in Lemma 6.6.

In order to finish induction it remains to prove the existence of a function h_{i+1}. Note that over z' the function $h_i \circ \sigma_{i+1}$ has zero multiplicity km_i at general points of $D_{i+1} \subset Q_{i+1}$. Thus $h_{i+1} = h_i \circ \sigma_{i+1}/f^{km_i}$ is a regular function on the normal variety Q_{i+1} that does not vanish at general points of D_{i+1}. In particular h_{i+1} vanishes only on the proper transform of $\mathcal{S}_i$ which is, by construction, $\mathcal{S}_{i+1}$. This is the desired function and we are done. ☐

Theorem 6.9. *Let Notation 6.1 and Convention 6.3 hold. Then $\mathbb{C}[X]^\Phi$ is finitely generated and $Q = X//\Phi$ coincides with the affine algebraic variety Q_n from Notation 6.5 for some n.*

Proof. By Proposition 3.8 the sequence

$$Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_i} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

of simple modifications from Notation 6.5 cannot be extended indefinitely to the left. Hence by Lemmas 6.6 and 6.8 we can suppose that for some n the image $\tau_n(E)$ is dense in D_n. Suppose that $\varrho : X \to Q$ is any partial quotient morphism over Q_n. Then $\delta_n(D)$ is dense in D_n since τ_n factors through δ_n. Then by Lemma 6.2 Q is isomorphic to Q_n.

Recall that $\mathbb{C}[X]^\Phi = \varinjlim \mathbb{C}[Q_\alpha]$, where Q_α is a partial quotient. Since every such a quotient over Q_n is Q_n itself we see that $\mathbb{C}[X]^\Phi = \mathbb{C}[Q_n]$, which yields the desired conclusion. ☐

Corollary 6.10. *Let X be four-dimensional and let Notation 6.1 hold without assuming Convention 6.3. Suppose that E admits a nonconstant morphism into a curve if and only if this curve is a polynomial one.*

Then Convention 6.3 holds and thus $\mathbb{C}[X]^\Phi$ is finitely generated. In particular Statement (1) *of Theorem 0.2 is true.*

Proof. By Remark 6.4, Convention 6.3 is true if Z_0 is not a point. Assume it is a point z_0. Since Q_0 and $D_0 = f^*(0)$ are smooth we can suppose that z_0 is locally a strict complete intersection given by $f = g_1 = g_2 = 0$. Note that $g_i \circ \tau$ vanishes on E. Hence $(g_i/f) \circ \tau$ is a regular G_a-invariant function on the normal variety X. This implies that $\psi : Q \to Q_0$ factors through $\sigma_1 : Q_1 \to Q_0$, where σ_1 is the simple modification along D_0 with the ideal generated by f, g_1 and g_2. By construction Q_1 is smooth and the exceptional divisor is $D_1 \simeq \mathbb{C}^2$. Thus we can replace the pair (Q_0, D_0) with (Q_1, D_1). If the center of ψ after this replacement is still a point we observe that it is again a strict complete intersection $f = h_1 = h_2 = 0$ with $h_i = (g_i/f) \circ \sigma_1$. That is, the zero multiplicity of the lift h_i (to X) on E is less than the one of $g_i \circ \tau$. Continue this procedure. As soon as one of these multiplicities becomes zero the variety Z_0 becomes at least a curve. Thus we are done. ☐

Another consequence of the equality $Q = Q_n$ in Theorem 6.9 is that the fiber of the morphism $Q \to B$ over o is $D_n \simeq Z_{n-1} \times \mathbb{C}$. By the assumption of Theorem 0.2, Z_{n-1} is a polynomial curve. Hence:

Corollary 6.11. *Statement* (2) *of Theorem 0.2 is true.*

7. Proper actions

Definition 7.1. Recall that an action of an algebraic group G on a variety X is proper if the morphism $G \times X \to X \times X$ that sends $(g, x) \in G \times X$ to $(x, g.x) \in X \times X$ is proper.

It is well known that every proper G_a-action is automatically free. Some geometrical properties of such actions are described below.

Proposition 7.2. *Let X be a normal affine algebraic variety equipped with a proper G_a-action Φ and let C_i be a sequence of general orbits of Φ. Suppose that $C \subset X$ is a closed curve such that for every $c \in C$ and every neighborhood $V \subset X$ of c (in the standard topology) there exists i_0 for which C_i meets V whenever $i \geq i_0$. Then C is an orbit of Φ, i.e., $C \simeq \mathbb{C}$ (in particular it is smooth and connected).*

Proof. Assume that C meets two disjoint orbits C' and C'' of Φ and choose points $c_i', c_i'' \in C_i$ such that $c_i' \to c' \in C' \cap C$ and $c_i'' \to c'' \in C'' \cap C$ as $i \to \infty$. Note that $c_i'' = t_i.c_i'$. This sequence of numbers $\{t_i\}$ in the group $\mathbb{C}_+$ goes to ∞. Indeed, otherwise switching to a subsequence we can suppose that $\lim_{i\to\infty} t_i = t$ and by continuity $c'' = t.c'$, which is impossible since the last two points are in distinct orbits. But this implies that the preimage (in $G \times X$) of a small neighborhood of the point $(c', c'') \in X \times X$ contains points of form $(t_i, c_i') \in \mathbb{C}_+ \times X$ going to infinity, contrary to properness.

Hence C meets only one orbit O of Φ. Hence $C \setminus O = \varnothing$, since otherwise C meets other orbits. Therefore, $C = O \cap C$ which implies that $C = O \simeq \mathbb{C}$ (in particular, Φ is free) and we are done. $\square$

Corollary 7.3. *Suppose that for some $q \in Q$ the fiber $\varrho^{-1}(q)$ is a curve. Then $\varrho^{-1}(q) \simeq \mathbb{C}$.*

Let us fix notation for the rest of this section.

Notation 7.4. We suppose that B is a unibranch germ of a smooth complex algebraic curve at point $o = f^*(0)$ (where $f \in \mathbb{C}[B]$) and $\varphi : X \to B$ is a morphism from a complex factorial affine algebraic variety X equipped with a G_a-action Φ which preserves each fiber of φ and such that the fiber $E = \varphi^*(o)$ is reduced.

We suppose also that Convention 6.3 is true for some partial quotient morphism $\varrho : X \to Q$ of the action Φ and a morphism $\psi : Q \to Q_0$ into a smooth affine algebraic variety Q_0 over B described in Notation 6.1. Recall that in this case by Theorem 6.9 $\mathbb{C}[X]^\Phi$ is finitely generated and $Q = X//\Phi$ coincides with an affine algebraic variety Q_n from Notation 6.5 for some n, and, particular, in that notation $\tau_n : X \to Q_n$ is the quotient morphism. Furthermore, let D_n be the divisor in Q_n over $o \in B$. Then it follows from the construction of Q_n that $D_n \simeq Z_{n-1} \times \mathbb{C}$ and from the proof of Theorem 6.9 that $\tau_n(E)$ is dense in D_n.

Proposition 7.5. *Let $\alpha : E \to R$ be the quotient morphism of the restriction of Φ to E. Then there is a natural affine modification $\beta : R \to D_n$ over Z_{n-1}.*

Proof. The density of $\tau_n(E)$ in D_n implies that for a general point $q \in D_n$ its preimage $\tau_n^{-1}(q)$ is a curve. By Corollary 7.3 this curve is isomorphic to $\mathbb{C}$.

By the universal property of quotient morphisms the map $\tau_n|_E : E \to D_n$ factors through α, i.e., there exists $\beta : R \to D_n$ for which $\tau_n|_E = \beta \circ \alpha$. Since τ_{n-1} (which maps E onto Z_{n-1}) factors through τ_n we see that β is a morphism over Z_{n-1}. Note that $\beta^{-1}(q)$ is a point since $\tau_n^{-1}(q)$ is a connected curve. Thus β is birational. Any birational morphism of affine algebraic varieties is an affine modification [Kaliman and Zaidenberg 1999] and we are done. □

Remark 7.6. Proposition 7.5 implies that β is an isomorphism over a Zariski dense open subset D_n^* of D_n. Furthermore removing a proper subvariety from D_n^* one can suppose that for every point in $\beta^{-1}(D_n^*)$ the fiber of α over this point is an orbit of Φ. Let $Q' = (Q \setminus D_n) \cup D_n^*$. Note that by construction every fiber of ϱ over point of Q_* is an orbit of Φ and the codimension of $Q \setminus Q' = D_n \setminus D_n^*$ in Q is at least 2.

Recall that when X is four-dimensional and there is no nonconstant morphism from E into any nonpolynomial curve then by Remark 6.4, Convention 6.3 holds, Z_{n-1} is a polynomial curve, and thus the normalization of $D_n = Z_{n-1} \times \mathbb{C}$ is $\mathbb{C}^2$. In particular, if E is normal then R is normal (being the quotient space of normal E) and we can mention the following interesting fact (which won't be used later).

Corollary 7.7. *Let the assumption of Corollary 6.10 hold and E be normal. Then R is an affine modification of $\mathbb{C}^2$. Furthermore, for a factorial E one has $R \simeq \mathbb{C}^2$ and normalization of morphism β is a birational morphism $\beta' : \mathbb{C}^2 \to \mathbb{C}^2$.*

Proof. Since β is a morphism over the curve Z_{n-1} we see that $\beta' : R \to \mathbb{C}^2$ is a morphism over the normalization $Z_{n-1}^{\mathrm{norm}} \simeq \mathbb{C}$ of Z_{n-1}. Note that by Corollary 1.7 R is factorial, being the quotient space of the factorial threefold E. This implies that all fibers of the $\mathbb{C}$-fibration $R \to Z_{n-1}^{\mathrm{norm}}$ are reduced and irreducible. Hence $R \simeq \mathbb{C}^2$ and we are done. □

Proposition 7.8. *Let β be as in Proposition 7.5 and suppose that one of the following conditions is satisfied*:

(a) *β is finite, E is smooth outside codimension* 2, *and every fiber of $\alpha : E \to R$ is a curve with a possible exception of a finite number of such fibers*;

(b) *β is quasifinite and the assumption of Theorem 0.2* (3) *holds* (*in particular*, *X is Cohen–Macaulay*, *Φ is proper*, *each fiber X_b of $\varphi : X \to B$* (*including E*) *is normal*, *and*, *being the image of E*, *the curve Z_{n-1} is a polynomial one*).

Then Z_{n-1} (*and therefore $D_n = Z_{n-1} \times \mathbb{C}$*) *is smooth and β is an isomorphism.*

Proof. The assumptions on α and β imply that there is a finite subset $M \subset D_n$ such that for every $q \in D_n \setminus M$ the preimage $\tau_n^{-1}(q)$ is a curve.

Recall that X is Cohen–Macaulay which implies that the morphism ϱ is faithfully flat over $Q \setminus M$ (e.g., see [Matsumura 1970, Chapter 2 (3.J) and Theorem 3] and [Eisenbud 1995, Theorem 18.6]). Hence Φ is locally a translation over $Q \setminus M$ by [Deveney et al. 1994, Theorem 2.8] (see also [Deveney and Finston 1999, Theorem 1.2]). This implies that Z_{n-1} is smooth since otherwise $D_n \setminus M$ and therefore

$(D_n \setminus M) \times \mathbb{C} \subset E$ are not smooth outside codimension 2. It remains to note that, being birational and finite, β is an isomorphism by the Zariski main theorem.

For the second statement, consider the normalization $\kappa : R^{\mathrm{norm}} \to D_n^{\mathrm{norm}} \simeq \mathbb{C}^2$ of β. Since β is birational by Proposition 7.5 and quasifinite by the assumption, the morphism κ is an embedding. Since both R^{norm} and D_n^{norm} are affine we see that $D_n^{\mathrm{norm}} \setminus \kappa(R^{\mathrm{norm}})$ is either empty or a Cartier divisor. Since $D_n^{\mathrm{norm}} \simeq \mathbb{C}^2$, such a divisor must be given by zeros of a regular function g. Then g admits a regular lift to the normal variety E that does not vanish. This contradicts the assumption that E does not admit nonconstant morphisms into a nonpolynomial curve. Thus $\kappa(R^{\mathrm{norm}}) = D_n^{\mathrm{norm}}$ and as before the assumption on α implies that there is a finite subset $M \subset D_n$ such that for every $q \in D_n \setminus M$ the preimage $\tau_n^{-1}(q)$ is a curve. Hence we can repeat the previous argument and get the desired conclusion. □

Remark 7.9. (1) Let Q' be as in Remark 7.6. Note that exactly the same argument as in Proposition 7.8 for the set $Q \setminus M$ implies that the action Φ over Q' is locally trivial[12] even without the assumption that β is quasifinite.

(2) Actually, following [Kaliman 2002] one can extract Sathaye's theorem [1983] from the argument in the proof of Proposition 7.8. Indeed, under the assumption of Sathaye's theorem, $D_n = \mathbb{C}^2$ and thus $Z_{n-1} \simeq \mathbb{C}$. Since Z_{n-1} is smooth, every polynomial curve Z_i is smooth by Lemma 4.4 and therefore $Z_i \simeq \mathbb{C}$. Hence every $D_i = Z_{i-1} \times \mathbb{C}$ is the plane and by the Abhyankar–Moh–Suzuki theorem Z_i can be viewed as a coordinate line in D_i. It is a straightforward fact that a modification $\sigma_{i+1} : Q_{i+1} \to Q_i$ with such center Z_i and divisor D_i leads to Q_{i+1} isomorphic to Q_i which implies that $Q_n \simeq B \times \mathbb{C}^2$.

(3) The assumption on properness of Φ is crucial in Proposition 7.8. In the absence of properness Winkelmann [1990] constructed a free action on $X = \mathbb{C}^4_{x_1,x_2,x_3,x_4}$ such that the quotient Q is isomorphic to the hypersurface in $\mathbb{C}^4_{x_1,u,v,w}$ given by

$$x_1 w = v^2 - u^2 - u^3.$$

Both X and Q are considered over the curve $B = \mathbb{C}_{x_1}$ and in particular the zero fiber D of the morphism $Q \to B$ is given by $v^2 - u^2 - u^3 = 0$ in $\mathbb{C}^3_{u,v,w}$. That is, D is not a plane but it is the product of $\mathbb{C}$ and Z_{n-1} where Z_{n-1} is a polynomial curve with one node as singularity (in accordance with Theorem 0.2 (2)).

8. Some facts about fibered products

In order to establish quasifiniteness required in Proposition 7.8 we need some auxiliary facts.

Notation 8.1. Let $\varrho : X \to Q$ be a dominant morphism of normal quasiprojective algebraic varieties and S be an irreducible closed subvariety of X such that for a Zariski dense open subset S' of S the restriction $\varrho|_{S'} : S' \to Q$ is quasifinite. Suppose that $k = \dim X - \dim S$ is the dimension of general fibers of ϱ.

[12] That is, $\varrho^{-1}(Q')$ can be covered by Φ-stable affine open subsets on each of which the action admits an equivariant trivialization.

Lemma 8.2. *Consider an irreducible germ $\Gamma \subset S$ of a general curve through $s \in S$ (in particular, $\Gamma \setminus s \subset S'$ and the dimension of $\varrho^{-1}(\varrho(\Gamma \setminus s))$ is $k+1$). Then the dimension of the fiber F in the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s))$ (in X) over s is k.*

Proof. By the semicontinuity theorem [Shafarevich 1994, Chapter I, Section 6.3, Corollary] the dimension of F is at least k. On the other hand it cannot be greater than k since otherwise it is not contained in the closure of the $(k+1)$-dimensional variety $\varrho^{-1}(\varrho(\Gamma \setminus s))$. □

Note that for a fixed s this fiber F may in general depend on the choice of Γ, i.e., $F = F(\Gamma)$.

Definition 8.3. Suppose that $F_0(\Gamma)$ is the component of $F(\Gamma)$ containing s. We say that the data in Notation 8.1 satisfies Condition (A) if for every $s \in S$ the set $\bigcup_\Gamma F_0(\Gamma)$ consists of a finite number of k-dimensional varieties.

Example 8.4. (1) If $X = S$ then Condition (A) holds automatically.

(2) Suppose that $\varrho : X \to Q$ is the quotient morphism of a proper G_a-action Φ and S is a hypersurface of X such that $\varrho(S)$ is Zariski dense in Q. Then for a general germ Γ as in Definition 8.3 and every point $s' \in \Gamma \setminus s$ the fiber $\varrho^{-1}(\varrho(s'))$ is a union of orbits of Φ. The set $\bigcup_\Gamma F_0(\Gamma)$ contains, of course, the orbit of Φ through s and by Proposition 7.2 it contains nothing else. Thus Condition (A) holds.

Proposition 8.5. *Let Notation 8.1 hold and Condition* (A) *be satisfied. Suppose that $\tilde{X}$ is the irreducible component of $X \times_Q S$ such that it contains $\tilde{S} = \{(s, s) \mid s \in S\}$. Denote by $\lambda : \tilde{X} \to S$ the restriction to $\tilde{X}$ of the natural projection.*

Then for every irreducible subvariety $P \subset S \setminus S'$ the dimension of $\lambda^{-1}(P)$ coincides with $m + k$ where $m = \dim P$.

Proof. Suppose that for $l \geq 0$ the subvariety $P(l) \subset P$ consists of points $s \in P$ for which $\dim \lambda^{-1}(s) = k+l$. Our aim is to show that $\dim P(l) \leq m - l$ and thus $\dim \lambda^{-1}(P) = m + k$.

Let $s \in S$, Γ, F be as in Definition 8.3 and let the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s)) \cap S$ meets F at points $s_1, \ldots, s_r$ (i.e., each irreducible component of F passes at least through one of these points). Note that for a fixed s this set $\{s_1, \ldots, s_r\}$ may in general depend on the choice of Γ, i.e., $s_i = s_i(\Gamma)$. Suppose that $M(s)$ is the union $\bigcup_i \bigcup_\Gamma s_i(\Gamma)$. By definition of $\tilde{X}$ for every $s' \in \Gamma \setminus s$ one has $\lambda^{-1}(s') \simeq \varrho^{-1}((\varrho(s'))$. Hence by Lemma 8.2 and Condition (A), the variety $\lambda^{-1}(s) = \bigcup_\Gamma F(\Gamma)$ has dimension at most $\dim M(s) + k$. In particular $P(l)$ is contained in the set

$$\{s \in P \mid \dim M(s) \geq l\}$$

and we need to estimate the dimension of this last set.

For this we need to use the Hironaka flattening theorem [1975] which implies the existence of a proper birational morphism $\hat{Q} \to Q$ such that for the union $\hat{S}$ of irreducible components of $S \times_Q \hat{Q}$ with the dominant projections to S the induced morphism $\hat{S} \to \hat{Q}$ is quasifinite. We have the commutative diagram

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\beta} & S \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\gamma} \\ \hat{Q} & \xrightarrow{\alpha} & Q \end{array}$$

where the morphisms α and β are birational and the morphism θ and $\gamma|_{S'}$ are quasifinite.

Let $P = \bigcup_i P_i$, where P_i are disjoint (not necessarily closed) irreducible varieties such that for any irreducible component V of $\beta^{-1}(P_i)$ all fibers of the morphism $\beta|_V : V \to P_i$ are of the same dimension. Varying i and V consider the collection $C_1, C_2, \dots$ of irreducible subvarieties of $\hat{Q}$ that are of the form $C_j = \overline{\theta(V)}$. Observe that, stratifying the varieties P_i's further, one can suppose that for indices $j \neq l$ the varieties C_j and C_l are either equal or disjoint. Denote by I the set of all pairs (i, j) for which there exist $V \subset \beta^{-1}(P_i)$ with $\theta(V) = C_j$ and put $C_{ij} = V$ (in particular $\dim C_j = \dim C_{ij}$ because of quasifiniteness of θ).

Let n_{ij} be the dimension of a fiber $F_{ij} \subset \beta^{-1}(s)$ of the morphism $\beta|_{C_{ij}} : C_{ij} \to P_i$. In particular $\dim P_i = \dim C_j - n_{ij} \leq m$. Hence for $n_j = \min_i \{n_{ij} | (i, j) \in I\}$ one has $\dim P_i \leq m + n_j - n_{ij}$ for any $(i, j) \in I$.

Let $s \in P_i$, i.e., $\beta^{-1}(s) = \bigcup_j F_{ij}$. For every where $(tj) \in I$ we put $F_{ij}^t = \theta^{-1}(\theta(F_{ij})) \cap C_{tj}$, i.e., $\dim F_{ij}^t = n_{ij}$ because of quasifiniteness. Note that

$$\dim \beta(F_{ij}^t) = \max(0, n_{ij} - n_{tj}) \leq n_{ij} - n_j \leq \max_j (n_{ij} - n_j).$$

Thus $l := \dim \beta(\bigcup_t F_{ij}^t) \leq \max_j (n_{ij} - n_j)$ while $\dim P_i \leq m - \max_j (n_{ij} - n_j) \leq m - l$.

Since we started with a point $s \in P_i$ for the desired inequality $\dim P(l) \leq m - l$ it suffices to show that $M(s) \subset \bigcup_j \beta(\bigcup_t F_{ij}^t)$ (since the last set is contained in P_i). But this follows from the fact that the proper transform of Γ (mentioned in the definition of $M(s)$) in $\hat{S}$ meets $\beta^{-1}(s)$ and therefore meets some F_{ij}. By quasifiniteness the proper transform (in $\hat{S}$) of every component in the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s)) \cap S$ meets some F_{ij}^t. This implies that this closure passes through a point in $\beta(F_{ij}^t)$ which yields $M(s) \subset \bigcup_j \beta(\bigcup_t F_{ij}^t)$ and we get $\dim \lambda^{-1}(P) \leq m + k$, i.e., $\dim \lambda^{-1}(P) \leq \dim \varrho^{-1}(\varrho(P))$. $\square$

Corollary 8.6. *If Condition* (A) *holds and* $\varrho^{-1}(\varrho(P))$ *is at least of codimension* 2 *in* X *then* $\lambda^{-1}(P)$ *has codimension at least* 2 *in* $\tilde{X}$.

9. Main theorems

Notation 9.1. Up to the proof of Theorem 0.2 below we adhere to the assumptions of Proposition 7.5 and, in particular, Notation 6.1. That is, B is the germ of a smooth algebraic curve at point $o = f^*(0)$, $\varphi : X \to B$ is a morphism of factorial varieties, Φ is a G_a-action on X, $Q = Q_n$ is the categorical quotient $X//\Phi$ in the category of affine algebraic varieties, $\tau_n = \varrho$, and $\varrho(E)$ is dense in $D_n =: D$. Since β is birational by Proposition 7.5, there is a Zariski dense open $R^* \subset R$ such that the restriction of $\beta|_{R^*} : R^* \to D \subset Q$ to R^* is an embedding.

Suppose also (as in Theorem 0.2 (3)) that the restriction of our proper G_a-action Φ to E is a translation. In particular, the quotient morphism $\alpha : E \to R$ admits a section whose image in E, by abuse of notation, will be denoted by R. The image of R^* in this section will keep notation R^* as well (i.e., $\varrho|_{R^*}: R^* \to D$ is an embedding).

The section R coincides with the zero locus of a regular function on E which has simple zeros at every point of R. Extend this function to a regular function h on X and consider the zero locus $h^{-1}(0) =: S \subset X$ of this extension.

Lemma 9.2. *The function h from Notation 9.1 can be perturbed so that for $S^* = S \setminus R$ and $Q^* = Q \setminus D$ the restriction $\varrho|_{S^*}: S^* \to Q^*$ is finite. In particular, for $S' = S^* \cup R^*$ the restriction $\varrho|_{S'}: S' \to Q$ is quasifinite.*

Proof. Recall that $X \setminus E \simeq Q^* \times \mathbb{C}$. Let w be a coordinate on the second factor, i.e., $h|_{X\setminus E}$ can be treated as a polynomial in w with coefficients in $\mathbb{C}[Q^*]$. Let n be the degree of this polynomial and f be as in Notation 6.1. Then for sufficiently large k the function $h+f^k w^{n+1}$ is regular on X and the restriction of $h+f^k w^{n+1}$ to E has only simple zeros on its zero locus R. This $h+f^k w^{n+1}$ yields the desired perturbation. □

Proposition 9.3. *Suppose that $\tilde{X}$ is the irreducible component of $X \times_Q S$ containing $\tilde{S} = \{(s, s) \mid s \in S\}$. Then $\tilde{X}$ is naturally isomorphic to $S \times \mathbb{C}$, with $\tilde{S}$ being the section of the natural projection $\lambda : \tilde{X} \to S$.*

Proof. Let S' be as in Lemma 9.2 and $s \in S'$. Note that $\varrho^{-1}(\varrho(s))$ is a disjoint union of orbits of Φ because of quasifiniteness of $\varrho|_{S'}$. One of these orbits O_s passes through s. By construction $\lambda^{-1}(\lambda(s)) \subset \varrho^{-1}(\varrho(s)) \times s$. Since $O_s \times s$ is the only component of $\varrho^{-1}(\varrho(s)) \times s$ that meets $\tilde{S}$ we see that $\lambda^{-1}(\lambda(s)) = O_s \times s$, which is an orbit of the free action $\tilde{\Phi}$ on $\tilde{X}$ induced by Φ (in particular $\tilde{\Phi}$ preserves the fibers of λ). The variety $\tilde{S}' = \{(s, s) \mid s \in S'\}$ is a section of $\tilde{\Phi}$ over S'. Hence $\lambda^{-1}(S')$ is naturally isomorphic to $S' \times \mathbb{C}$ because the action $\tilde{\Phi}$ is free. By construction $S \setminus S'$ is of codimension at least 2 in S. Hence the same is true for the subvariety $(S \setminus S') \times \mathbb{C}$ in $S \times \mathbb{C}$. By Corollary 8.6 and Example 8.4 (2) $\tilde{X} \setminus \lambda^{-1}(S')$ is of codimension at least 2 in $\tilde{X}$. By the Hartogs theorem the isomorphism $\lambda^{-1}(S') \simeq S' \times \mathbb{C}$ extends to an isomorphism $\tilde{X} \simeq S \times \mathbb{C}$ since both varieties are affine. This is the desired conclusion. □

Lemma 9.4. *For $\tilde{X}$ from Proposition 9.3 the restriction $\kappa : \tilde{X} \to X$ of the natural projection $X \times_Q S \to X$ is quasifinite.*

Proof. The restriction of κ to $\lambda^{-1}(S')$ is quasifinite because of quasifiniteness of $\varrho|_{S'}$. Since $\tilde{X} \setminus \lambda^{-1}(S') = (S \setminus S') \times \mathbb{C} = R \times \mathbb{C}$ it suffices to check that the restriction of κ yields a quasifinite morphism $R \times \mathbb{C} \to E$. The last map is an isomorphism since R is a section of $\Phi|_E$ and we are done. □

Notation 9.5. By the Grothendieck version of the Zariski main theorem (see [Grothendieck 1964, Théorème 8.12.6]) there is an embedding $\tilde{X} \to \hat{X}$ of $\tilde{X}$ into an algebraic variety $\hat{X}$ such that κ can be extended to a finite morphism $\chi : \hat{X} \to X$. Note that the finiteness implies that $\hat{X}$ is affine since X is. Furthermore, replacing if necessary $\hat{X}$, $\tilde{X}$, and S with their normalizations, we suppose that these varieties are normal.

Denote the lift of $f \in \mathbb{C}[B]$ from Notation 9.1 to X (resp. $\tilde{X}$, resp. $\hat{X}$) by the same letter f (resp. $\tilde{f}$, resp. $\hat{f}$). Let ν be the locally nilpotent vector field on X associated with the action Φ. Since $f \in \operatorname{Ker} \nu$ the field $f^k \nu$ is also locally nilpotent for every $k > 0$ and it is associated with a G_a-action Φ_k on X which has the same quotient morphism $\varrho : X \to Q$. The lift of ν to $\tilde{X}$ (resp. $\hat{X}$) will be denoted by $\tilde{\nu}$ (resp. $\hat{\nu}$). By construction $\tilde{\nu}$ (and, hence, $\tilde{f}^k \tilde{\nu}$) is locally nilpotent on $\tilde{X}$ and $\hat{\nu}$ is a rational vector field on $\hat{X}$. Since $\hat{X} \setminus \hat{f}^{-1}(0) \subset \tilde{X}$ we see that for sufficiently large k the field $\hat{f}^k \hat{\nu}$ is locally nilpotent on $\hat{X}$. We denote by $\hat{\Phi}_k$ the G_a-action associated with the last field.

Proposition 9.6. *The action $\hat{\Phi}_k$ admits a quotient morphism $\hat{\varrho} : \hat{X} \to \hat{Q}$ in the category of affine algebraic varieties such that the commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{\varrho}"] \arrow[d, "\chi"] & \hat{Q} \arrow[d, "\tau"] \\ X \arrow[r, "\varrho"] & Q \end{tikzcd}$$

holds, where τ is a finite morphism.

Proof. For every function $g \in \mathbb{C}[\tilde{X}]^{\tilde{\Phi}}$ and sufficiently large $m > 0$, the function $\tilde{f}^m g$ has a regular extension to $\hat{X}$, i.e., it can be viewed as a function from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$. Since by construction $\hat{X} \setminus \hat{f}^{-1}(0) = \tilde{X} \setminus \tilde{f}^{-1}(0) = \lambda^{-1}(S \setminus R)$, where $\lambda : \tilde{X} \to S$ is the quotient morphism of $\tilde{\Phi}$ (by Proposition 9.3) we see now that a finite number of functions from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ separate the orbits of $\hat{\Phi}_k$ contained in $\lambda^{-1}(S \setminus R)$. Consider a partial quotient morphism $\hat{\varrho} : \hat{X} \to \hat{Q}$ whose coordinates include these functions. Furthermore, since the lift of every function from $\mathbb{C}[Q]$ to $\hat{X}$ is a function from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ we can include the lifts of the coordinate functions of ϱ into the set of coordinate functions of $\hat{\varrho}$. Then the commutative diagram as before make sense.

Let us show that τ is finite. Indeed, by finiteness of χ every function g from $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ is integral over $\mathbb{C}[X]$. That is, g is a root of an irreducible monic polynomial $P(g) := g^n + a_{n-1}g^{n-1} + \cdots + a_1 g + a_0$, where each $a_i \in \mathbb{C}[X]$. Since g is $\hat{\Phi}_k$-invariant it satisfies also an equation $g^n + a_{n-1}^t g^{n-1} + \cdots + a_1^t g + a_0^t = 0$, where a_i^t is the result of the action of $t \in \mathbb{C}_+$ (induced by $\hat{\Phi}_k$) on a_i. However, the minimal polynomial P should be unique, which implies that a_i are $\hat{\Phi}_k$-invariant and thus $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ is integral over $\mathbb{C}[Q]$. That is, every function from $\mathbb{C}[\hat{Q}]$ is integral over $\mathbb{C}[Q]$ and hence τ is finite.

This leads to the commutative diagram

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d] & \hat{D} \arrow[d] \\ E \arrow[r] & D \end{tikzcd}$$

where $\hat{D} := \tau^{-1}(D)$, $\hat{E} = \hat{f}^{-1}(0)$, and the vertical morphisms are finite. Since for a general point $q \in D$ its preimage $\varrho^{-1}(q)$ is a curve in E the diagram implies that for a general point $\hat{q} \in \hat{D}$ its preimage $\hat{\varrho}^{-1}(\hat{q})$ is a curve in $\hat{E}$. Thus, by Theorem 1.4 $\hat{\varrho}$ is a quotient morphism which is the desired conclusion. $\square$

Lemma 9.7. *Let $Q' = Q^* \cup D^*$ be as in Remark 7.9. Then*

(i) *$\hat{X} \setminus (\varrho \circ \chi)^{-1}(Q')$ has codimension at least 2 in $\hat{X}$;*

(ii) *for every point in Q' there is a Zariski neighborhood $V \subset Q'$ such that $(\varrho \circ \chi)^{-1}(V)$ is naturally isomorphic to $\hat{V} \times \mathbb{C}$ where $\hat{V} = \tau^{-1}(V)$.*

Proof. Recall that $\varrho^{-1}(Q \setminus Q') = E \setminus (\varrho|_E)^{-1}(D^*)$ is of codimension at least 2 in X. Hence (i) is a consequence of finiteness of χ.

By Remark 7.9 Φ is a locally trivial over Q'. In particular, for every point in Q' there is a Zariski neighborhood $V \subset Q'$ such that $\varrho^{-1}(V)$ is naturally isomorphic to $V \times \mathbb{C}$ where the projection $\varrho^{-1}(V) \to V$ is the restriction of ϱ. Note that the restriction of the commutative diagram in Notation 9.5 yields another commutative diagram

$$\begin{tikzcd} \hat{W} \arrow[r] \arrow[d] & \hat{V} \arrow[d] \\ \varrho^{-1}(V) \arrow[r] & V \end{tikzcd}$$

where $\hat{W} = (\varrho \circ \chi)^{-1}(V)$. Hence we have $\hat{W} \simeq \hat{V} \times \mathbb{C}$ since $\varrho^{-1}(V) \simeq V \times \mathbb{C}$. This yields (ii). $\square$

Proposition 9.8. *Let $\hat{S}$ be the closure of $\tilde{S}$ in $\hat{X}$. Then $\hat{S}$ is a section of the quotient morphism $\hat{\varrho} : \hat{X} \to \hat{Q}$.*

Proof. If suffices to construct a $\hat{\Phi}_k$-equivariant morphism $\psi : \hat{X} \to \hat{S}$ from $\hat{X}$ to $\hat{S}$. Indeed, then by the universal properties of quotient morphisms this morphism factors through $\hat{Q}$ and, therefore, $\hat{\varrho}|_{\hat{S}} : \hat{S} \to \hat{Q}$ is invertible.

Choose as the restriction of such a ψ to $\tilde{X}$ the natural projection $\lambda : \tilde{X} = S \times \mathbb{C} \to S$. Note that for $\hat{V} \simeq V \times \mathbb{C}$ from Lemma 9.7(ii) λ agrees on $\tilde{X} \cap \hat{V}$ with the natural projection $\hat{V} \to V$. Hence we can extend ψ to $\tilde{X} \cup \chi^{-1}(Q')$. Now because of Lemma 9.7(i) and the Hartogs theorem this restriction extends to $\hat{X}$ and we are done. $\square$

Remark 9.9. The fact that the morphism $\alpha : E \to R$ has a section may be made weaker for Proposition 9.8. It suffices, say, to require that there exists is a reduced irreducible principal effective divisor $T = h^*(0)$ in E such that the restriction $\alpha|_T : T \to R$ is surjective and quasifinite. Then one can extend h to a regular function on X and the same proofs as before imply that the result remains valid.

Corollary 9.10. *The morphism $\beta : R \to D$ is quasifinite.*

Proof. Let $\tilde{R}$ be the preimage of $R \subset S$ in $\tilde{S}$. Note that $\tilde{R}$ is a subvariety of $\tilde{X} \subset \hat{X}$. On the other hand since $\hat{Q}$ is isomorphic to $\hat{S} \supset \tilde{S}$ by Proposition 9.8, we can treat $\tilde{R}$ as a subvariety of $\hat{Q}$. Furthermore, by construction the quotient morphism $\hat{\varrho}$ yields an automorphism of $\tilde{R}$. Hence the restriction of the morphism $\tau \circ \hat{\varrho} = \varrho \circ \chi$ to $\tilde{R} \subset \hat{X}$ yields a quasifinite morphism $\tilde{R} \to D$ (where τ, χ, $\hat{\varrho}$ are from Proposition 9.6). Since morphism $\varrho \circ \chi|_{\tilde{R}}$ factors through $\beta = \varrho|_R : R \to D$, the latter is also quasifinite and we are done. $\square$

Proof of Theorem 0.2. Note that Convention 6.3 is valid by Remark 6.4. Hence Theorem 6.9 and thus Proposition 7.5 is applicable. For the time being consider the case when B is a germ of a smooth curve at a point o and denote by E the fiber in X over o. Since $\Phi|_E$ is a translation in Theorem 0.2 (3), for every $r \in R$ the preimage $\alpha^{-1}(r)$ is a curve. Thus, taking into consideration Corollary 9.10, we see that the assumptions of Proposition 7.8 are valid. This implies that the polynomial curve Z_{n-1} is smooth and we have $Z_{n-1} \simeq \mathbb{C}$ and $D_n \simeq \mathbb{C}^2$. Hence the polynomial curve Z_{n-1} is smooth by Proposition 7.8 and we have $Z_{n-1} \simeq \mathbb{C}$ and $D_n \simeq \mathbb{C}^2$. By assumption D_n is also a smooth reduced fiber of the morphism $Q_n \to B$ and $Q_n \setminus D_n \simeq B^* \times \mathbb{C}^2$. Sathaye's theorem [1983] implies now that Q_n is naturally isomorphic to $B \times \mathbb{C}^2$. Similarly $X = Q_n \times \mathbb{C}$ when B is a germ of a curve.

Let us return now to the general case when B is a smooth affine curve. The local statement over germ of B at o implies the first global claim of Theorem 0.2 (3) while the second one is the consequence of the fact (see [Serre 1955]) that every affine manifold with a free G_a-action and an affine geometrical quotient is the direct product of that quotient and $\mathbb{C}$. Thus Statement (3) of Theorem 0.2 is valid while Statements (1) and (2) are true by Corollaries 6.10 and 6.11. Hence we are done with the proof of Theorem 0.2. □

General case of an affine algebraic variety X over a field $\mathbf{k}$ of characteristic zero. Recall that there is a one-to-one correspondence between the set of G_a-actions on an affine algebraic variety X and the set of locally nilpotent derivations (LNDs) on the algebra $\mathcal{B}$ of regular functions on X (e.g., see [Freudenburg 2006]). Suppose that ν is such an LND associated with a proper G_a-action Φ on X. Then condition that Φ is free (resp. a translation) is equivalent to the fact that the ideal generated by $\nu(\mathcal{B})$ coincides with $\mathcal{B}$ (resp. $\nu(\mathcal{B}) = \mathcal{B}$). To show the validity of Theorem 0.1 for any $\mathbf{k}$ of characteristic zero one needs to repeat the argument of Daigle from [Daigle and Kaliman 2009, Theorem 3.2].

Namely, if a fact is true for varieties over $\mathbb{C}$ it is true in the case when X is considered over a field which is a universal domain[13] [Eklof 1973]. Thus consider a field extension $\mathbf{k}'/\mathbf{k}$ where $\mathbf{k}'$ is a universal domain. Then ν extends to an LND ν' on $\mathcal{B}' = \mathbf{k}' \otimes_{\mathbf{k}} B$ associated with a G_a-action Φ'. Note that ν' is free since ν is. Similarly, Φ' is proper, since properness survives base extension [Hartshorne 1977, Corollary 4.8]. Thus under the assumptions of Theorem 0.1 (with $\mathbb{C}$ replaced by $\mathbf{k}'$) Φ' is a translation by the argument above, i.e., $\nu' : \mathcal{B}' \to \mathcal{B}'$ is surjective. Since ν' is obtained by applying the functor $\mathbf{k}' \otimes_{\mathbf{k}} -$ to ν and since $\mathbf{k}'$ is a faithfully flat k-module we see that $\nu : \mathcal{B} \to \mathcal{B}$ is also surjective. Therefore Φ is a translation and we have:

Theorem 9.11. *Let $\mathbf{k}$ be a field of characteristic zero and Φ be a proper G_a-action on the four-space $\mathbb{A}^4_{\mathbf{k}}$ preserving a coordinate. Then Φ is a translation in a suitable polynomial coordinate system.*

Similarly the argument before implies the following.

Theorem 9.12. *Let the assumptions of Theorem 0.2 hold with the only change that X and B are varieties over some field $\mathbf{k}$ of characteristic zero* (*and not over $\mathbb{C}$*). *Suppose also that Φ is proper and the restriction of Φ to every fiber is a translation* (*as in Theorem 0.2* (3)). *Then Φ is a translation.*

Conclusive Remark. In dimension 5 the analogue of Theorem 0.1 is not true by the second of Winkelmann's examples in [Winkelmann 1990]. This raises the question of whether the properness is the right condition for this type of problems. The author suspects that another condition may be more effective at least in dimension 4, which leads to the following question:

[13] Recall that a universal domain is an algebraically closed field containing $\mathbb{Q}$ such that it has an infinite transcendence degree over $\mathbb{Q}$.

Question. Suppose that the quotient of a free G_a-action Φ on $\mathbb{C}^4$ is isomorphic to $\mathbb{C}^3$. Is it true that Φ is a translation?

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Nonemptiness of Newton strata of Shimura varieties of Hodge type

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For a Shimura variety of Hodge type with hyperspecial level at a prime p, the Newton stratification on its special fiber at p is a stratification defined in terms of the isomorphism class of the rational Dieudonné module of parameterized abelian varieties endowed with a certain fixed set of Frobenius-invariant crystalline tensors ("$G_{\mathbb{Q}_p}$-isocrystal"). There has been a conjectural group-theoretic description of the F-isocrystals that are expected to show up in the special fiber. We confirm this conjecture. More precisely, for any $G_{\mathbb{Q}_p}$-isocrystal that is expected to appear (in a precise sense), we construct a special point whose reduction has associated F-isocrystal equal to the given one.

1. Introduction

Fix a prime $p > 0$, and for $g \geq 1$, let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g in characteristic p. Then the Newton stratification on $\mathcal{A}_g$ is the stratification such that each stratum consists of points $x = (A_x, \lambda_x)$ whose associated p-divisible group $(A_x[p^\infty], \lambda_x[p^\infty])$ with quasipolarization is quasi-isogenous to a fixed one. The rational Dieudonné module $\mathbb{D}(X)$ of a p-divisible group X over a perfect field k of characteristic $p > 0$ provides an equivalence of categories between the isogeny category of (quasipolarized) p-divisible groups and the category of (quasipolarized) F-isocrystals. An *F-isocrystal* (or simply *isocrystal*) over a perfect field k of characteristic p is a finite-dimensional $L(k)$-vector space M with a σ-linear bijective operator $\Phi : M \to M$, where $L(k) := \mathrm{Frac}(W(k))$ and σ is its Frobenius automorphism.

According to the Dieudonné–Manin classification, an isocrystal over an algebraically closed field is determined by the slope sequence of Φ. The latter combinatorial datum in turn can be faithfully represented by a lower-convex (piecewise-linear) polygon with integral break points lying in the first quadrant of $\mathbb{Z}^2$, which will be called the *Newton polygon* hereafter. In other words, the Newton polygon of $\mathbb{D}(A_x[p^\infty])$ determines the quasi-isogeny class of $A_x[p^\infty]$. The existence of a quasipolarization forces the Newton polygon of $\mathbb{D}(A_x[p^\infty])$ to be "symmetric". Then, it is a natural question to ask which symmetric Newton polygon can be the Newton polygon of a point of $\mathcal{A}_g$. In fact, the Newton polygons arising from points of $\mathcal{A}_g$ in the way just described meet two more restrictions. First, when the initial point is located at the origin, the end point is always $(2g, g) \in \mathbb{Z}^2$. Secondly, it "lies above" the *ordinary* Newton polygon, which by definition is the Newton polygon having two slopes $(0, 1)$ with the same multiplicity g. Then, it

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has long been known (after it was conjectured by Manin) that *any* Newton polygon subject to these two restraints is the Newton polygon of a point on $\mathcal{A}_g$; a proof can be found, e.g., in [Oort 2013, Corollary 7.8], which uses the Honda–Tate theory, and thus gives examples defined over finite fields.

The main result of this article is a generalization of this fact to more general moduli spaces of abelian varieties, namely to Shimura varieties of Hodge type. First, we explain how each point in characteristic $p > 0$ of a Shimura variety of Hodge type gives an F-isocrystal with certain additional structure. A detailed discussion will appear in Section 2.

Let (G, X) be a Shimura datum of Hodge type. This means that there exists an embedding of Shimura data $(G, X) \hookrightarrow (\mathrm{GSp}(W, \psi), \mathfrak{H}_g^{\pm})$ into a Siegel Shimura datum, in the sense that there exists an embedding $\rho_W : G \hookrightarrow \mathrm{GSp}(W, \psi)$ of $\mathbb{Q}$-groups which sends each morphism in X to a member of $\mathfrak{H}_g^{\pm}$; we fix such an embedding. For a compact open subgroup $K \subset G(\mathbb{A}_f)$ (which will be tacitly assumed sufficiently small), the associated (canonical model over the reflex field $E(G, X)$ of the) Shimura variety $\mathrm{Sh}_K(G, X)$ parameterizes polarized abelian varieties endowed with a fixed set of (absolute) Hodge cycles and a K-level structure, and carries a universal family of abelian varieties $A \to S_K := \mathrm{Sh}_K(G, X)$ equipped with a similar set of (absolute Hodge) tensors on $H^1_{\mathrm{dR}}(A/S_K)$. To study its reduction modulo p, we fix a prime $\wp$ of $E(G, X)$ and let $\mathcal{O} := \mathcal{O}_{(\wp)}$ be the localization at $\wp$ of the ring of integers of $E(G, X)$ with residue field $\kappa(\wp)$. To obtain an integral model over $\mathcal{O}$ with good reduction, we assume that $G_{\mathbb{Q}_p}$ is unramified, which is equivalent to the existence of a reductive group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$. We choose one such model $G_{\mathbb{Z}_p}$ and set $K_p := G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ (such compact open subgroups of $G(\mathbb{Q}_p)$ are called *hyperspecial*). For K, we take $K = K_p \times K^p$ for a (sufficiently small) compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$. Then, by [Vasiu 1999; Kisin 2010], it is known that there exists a smooth integral model $\mathcal{S}_K := \mathcal{S}_K(G, X)$ over $\mathcal{O}$ with generic fiber S_K, which is furthermore uniquely characterized by a "Neron-extension property". Also, this integral model carries a universal abelian scheme $\mathcal{A}$ over it by construction. Then each point on the reduction $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$ gives an F-isocrystal as follows.

Let z be a point of $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$ defined over an algebraically closed field k, assumed (for simplicity) to be of finite transcendence degree over $\overline{\mathbb{F}}_p$. By smoothness of $\mathcal{S}_K$ and the moduli interpretation of S_K, the Dieudonné module $\mathbb{D}(\mathcal{A}_z)$ is supplied with a set of Frobenius-invariant tensors $\{s_{\alpha,0,z}\}_{\alpha\in\mathcal{J}}$, and there exists an $L(k)$-isomorphism between the dual space $W^\vee \otimes L(k)$ and $\mathbb{D}(\mathcal{A}_z)$ which matches the G-invariant tensors on $W^\vee$ and the tensors $\{s_{\alpha,0,z}\}_{\alpha\in\mathcal{J}}$ on $\mathbb{D}(\mathcal{A}_z)$, which is thus canonically determined up to the action of $G_{L(k)}$ on $W^\vee_{L(k)}$ (Lemma 2.2.5). Choosing such an isomorphism and transporting the Frobenius operator Φ to $W^\vee_{L(k)}$, we get an element $b \in G(L(k))$ such that $\Phi = \rho_{W^\vee}(b)(\mathrm{id}_{W^\vee} \otimes \sigma)$, where $\rho_{W^\vee} : G \hookrightarrow \mathrm{GL}(W^\vee)$ is the contragredient representation of the symplectic representation ρ_W fixed in the beginning. Recall that two elements g_1, g_2 of $G(L(k))$ are called *σ-conjugate* if there exists $h \in G(L(k))$ with $g_2 = hg_1\sigma(h)^{-1}$. Then the σ-conjugacy class $[b]$ of b is independent of the choice of an isomorphism $W^\vee_{L(k)} \to \mathbb{D}(A_x)$ just explained, and the isomorphism class of the F-isocrystal $(\mathbb{D}(\mathcal{A}_z), \Phi)$ equipped with the Frobenius-invariant tensors $\{s_{\alpha,0,z}\}_{\alpha\in\mathcal{J}}$, called *$G_{\mathbb{Q}_p}$-isocrystal*, corresponds to a unique σ-conjugacy class of elements of $G(L(k))$. If $B(G_{\mathbb{Q}_p})$ denotes the set of the σ-conjugacy classes of elements of $G(L(k))$, there exists a subset $B(G_{\mathbb{Q}_p}, X)$ of $B(G_{\mathbb{Q}_p})$ which is expected to be the set of isocrystals with

$G_{\mathbb{Q}_p}$-structure coming from points on $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$. We remark that this is a subset of $B(G_{\mathbb{Q}_p})$ defined by certain two conditions that are analogous to those discussed above in the Siegel case, and it turns out that for any point of $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$, its associated $G_{\mathbb{Q}_p}$-isocrystal belongs to this subset $B(G_{\mathbb{Q}_p}, X)$.

For $[b] \in B(G_{\mathbb{Q}_p}, X)$, let $\mathcal{S}_{[b]}$ be the subset of points of $\mathcal{S}_K \otimes_{\mathcal{O}} \kappa(\wp)$ whose associated $G_{\mathbb{Q}_p}$-isocrystal is $[b]$. The main result of this paper is then that for every $[b] \in B(G_{\mathbb{Q}_p}, X)$, $\mathcal{S}_{[b]}$ is nonempty. Our proof consists of finding a special point whose reduction has F-isocrystal equal to a given one. We recall that a point $[h, g_f \cdot K] \in \mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is called *special* if $h \in X$ factors through $T_{\mathbb{R}}$ for some maximal $\mathbb{Q}$-torus T of G. Special points are known to be defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$ and extend over the valuation ring of an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for any reasonable integral model.

Theorem 1.0.1 (Theorem 3.2.1). *Let (G, X) be a Shimura datum and p a rational prime. Assume that $G_{\mathbb{Q}_p}$ is unramified. Fix a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$ and a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$; put $K = K_p \times K^p$. Choose a prime $\wp$ of $E(G, X)$ above p, and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing $\wp$.*

(1) *For every $[b] \in B(G_{\mathbb{Q}_p}, X)$, there exists a special Shimura subdatum $(T, h \in \mathrm{Hom}(\mathbb{S}, T_{\mathbb{R}}) \cap X)$ such that $T_{\mathbb{Q}_p}$ is unramified and the σ-conjugacy class of $\mu_h^{-1}(p) \in G(\mathbb{Q}_p^{\mathrm{ur}})$ equals $[b]$, and further, such that the (unique) hyperspecial subgroup of $T(\mathbb{Q}_p)$ is contained in K_p.*

(2) *If (G, X) is of Hodge type, for every special point $[h, g_f]_K \in \mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}})$ (defined for any $g_f \in G(\mathbb{A}_f)$), its reduction has F-isocrystal equal to $[b]$.*

The statement (1) here is in fact a combination of two statements (1) and (3) of Theorem 3.2.1.

The nonemptiness of Newton strata has been conjectured by Fargues [2004, Conjecture 3.1.1] and Rapoport [2005, Conjecture 7.1]. Some partial cases confirming this conjecture were known. We only mention previous works in two directions. For PEL-type Shimura varieties (which form a subclass of Hodge-type Shimura varieties), this conjecture (i.e., nonemptiness of *all* Newton strata in $B(G_{\mathbb{Q}_p}, X)$) was proved by C.-F. Yu [2005] in the Lie-type C cases, then by Viehmann and Wedhorn [2013, Theorem 1.6] in general, and by Kret [2012] for some simple groups of Lie type A or C. For a general Hodge-type Shimura variety, to the best of the author's knowledge, there are two partial results. First, Wortmann [2013] showed nonemptiness of the *μ-ordinary locus*, a generalization of the (usual) ordinary locus in $\mathcal{A}_g$. For *projective* Shimura varieties of Hodge type, Koskivirta [2016] proved the conjecture (nonemptiness of *all* Newton strata).

While all these previous results on the nonemptiness of Newton strata, except for that of Kret, are obtained by algebro-geometric methods in positive characteristic, which are sometimes of limited applicability for Hodge-type Shimura varieties, we point out that the first claim of our theorem above is a purely group-theoretic statement on Shimura *data* without any reference to the geometry of the Shimura variety. When the Shimura datum is of Hodge type, this group-theoretic statement carries the geometric meaning of the second claim (by Lemma 3.1.1). In this geometric interpretation of the group-theoretic statement of the first claim, one does not need a fine geometric property of the integral canonical model $\mathcal{S}_K$.

We also point out that our method of proof of the theorem, more precisely that of constructing a special point with certain prescribed properties, allows us to obtain some finer results. First, as is already apparent, we prove more than just nonemptiness of Newton strata; namely, we show existence of a special point whose reduction lies in any given Newton stratum. As another example, we also prove a generalization of a result of Wedhorn [1999, (1.6.3)] (the first statement of the next result in the PEL-type cases of Lie type A or C), using an argument different from the original one of Wedhorn (see Remark 3.2.5).

Corollary 1.0.2 (Corollary 3.2.3). *Let (G, X) be a Shimura datum of Hodge type. Suppose that $G_{\mathbb{Q}_p}$ is unramified and choose a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$. Let $\wp$ be a prime of $E(G, X)$ above p. Then the reduction $\mathcal{S}_{K_p}(G, X) \times \kappa(\wp)$ has nonempty ordinary locus if and only if $\wp$ has absolute height one (that is, $E(G, X)_{\wp} = \mathbb{Q}_p$). In this case, if the chosen embedding $G \hookrightarrow \mathrm{GSp}(W, \psi)$ induces an embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p})$ of $\mathbb{Z}_p$-group schemes, there exists a special point which is the Serre–Tate canonical lifting of its reduction.*

We remark (see 2.2.1) that the constructions of the integral canonical model by Vasiu and Kisin both require this condition on the embedding $G \hookrightarrow \mathrm{GSp}(W, \psi)$ to be satisfied. Finally, we remark that since our methods of proof of the above results are purely group-theoretic (based on Galois cohomology theory of algebraic groups), they are very likely to apply to more general Shimura varieties, particularly to Shimura varieties of abelian-type and/or with nonhyperspecial level. We do not pursue such ideas in this article, though.

This article is organized as follows. In Section 2, we review the construction of the canonical integral model $\mathcal{S}_K(G, X)$ of a Hodge-type Shimura variety $\mathrm{Sh}_K(G, X)$ with hyperspecial level at p, due to Vasiu and Kisin, and of the Newton stratification on its (good) reduction $\mathcal{S}_K(G, X) \otimes \kappa(\wp)$. We also recall the definitions of Newton map and Kottwitz map for reductive groups over p-adic fields, which are needed to define Newton stratification. Section 3 is devoted to the proof of the nonemptiness of Newton strata, by constructing a special point with prescribed F-isocrystal. For that, we generalize (in Hodge-type situation) a result of Kottwitz which identifies the $G_{\mathbb{Q}_p}$-isocrystal of the reduction of a special point in terms of the defining special Shimura datum. As a corollary of our method, we also obtain a criterion for when the good reduction of a Shimura variety with hyperspecial level has an ordinary point, and prove that in such cases, there always exists a special point which is the Serre–Tate canonical lifting of its reduction.

Notation. Throughout this paper, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ (so $\overline{\mathbb{Q}}$ has a privileged embedding into $\mathbb{C}$). For a (connected) reductive group G over a field, we let G^{sc} be the universal covering of its derived group G^{der}, and for a (general linear algebraic) group G, $Z(G)$ and G^{ad} denote its center and the adjoint group $G/Z(G)$, respectively.

2. Newton stratification on good reduction of Shimura varieties

In this section, we give an account of the construction of Newton stratification on the reduction of the integral canonical model of a Hodge-type Shimura variety with hyperspecial level.

First, we begin with a brief review of the Newton map and Kottwitz map, which are defined for connected reductive groups over p-adic fields. For a reductive group G over a local field F and a conjugacy class $\mathcal{C}$ of cocharacters of $G_{\overline{F}}$, we define a certain subset $B(G, \mathcal{C})$ of $B(G)$, the set of all G-isocrystals.

2.1. ***G-isocrystals and the set $B(G, \mathcal{C})$.*** In this subsection only, all algebraic groups considered will be defined over p-adic fields.

Let k be an algebraically closed field of characteristic p, and let $W(k)$ be its Witt vector ring and $K_0 = L(k)$ the fraction field of $W(k)$. Fix an algebraic closure $\overline{K_0}$ of K_0. Let F be a finite extension of $\mathbb{Q}_p$ in $\overline{K_0}$, and denote by L the composite of K_0 and F in $\overline{K_0}$; for our application to Shimura varieties, the most interesting case will be when $k = \overline{\mathbb{F}}_p$ and $F = \mathbb{Q}_p$, so that $L = L(\overline{\mathbb{F}}_p)$. Denote by σ the Frobenius automorphism of L/F (i.e., the automorphism of L fixing F and inducing the q-th power map on the residue field k of L, where $q \in \mathbb{N}$ is the cardinality of the residue field of F). Let G be a connected reductive group over F. Put $\mathrm{Gal}_F := \mathrm{Gal}(\overline{F}/F)$.

2.1.1. Let $B(G)$ be the set of σ-conjugacy classes in $G(L)$:

$$B(G) = G(L)/\sim,$$

where two elements x, y of $G(L)$ are σ-conjugated, denoted by $x \sim y$, if $x = gy\sigma(g)^{-1}$ for some $g \in G(L)$; the σ-conjugacy class of $b \in G(L)$ is denoted by $[b]$. An element of $B(G)$ is called an *(F-)isocrystal with G-structure*.

To relate this notion to the previously introduced notion of F-isocrystal (i.e., a finite-dimensional vector space U over L endowed with a σ-linear bijection $\Phi : U \to U$), let $\mathcal{REP}_F(G)$ be the category of finite-dimensional F-linear representations of G, and $\mathcal{CRYS}$ the category of F-isocrystals. Both are in a natural manner F-linear Tannakian categories (for $\mathcal{CRYS}$, see [Kottwitz 1985, §3]). An element $[b]$ of $B(G)$ can equivalently be considered as an F-linear exact (faithful) tensor functor

$$\mathcal{REP}_F(G) \to \mathcal{CRYS} : (\rho, V) \mapsto (V \otimes L, \rho(b) \cdot (\mathrm{id}_V \otimes \sigma)),$$

for any representative $b \in G(L)$ of $[b] \in B(G)$; see [Rapoport and Richartz 1996, Remark 3.4(i)]. If $[b_1] = [b_2]$, there exists a natural transformation between the corresponding tensor functors. Then any F-isocrystal (U, Φ) of height n (i.e., $\dim_L U = n$) gives rise to an F-isocrystal with GL_n-structure [Rapoport and Richartz 1996, Remarks 3.4(ii)]. So, $B(\mathrm{GL}_n)$ classifies the isomorphism classes of F-isocrystals (U, Φ) of height n.

2.1.2. Let $\mathbb{D}$ be the pro-algebraic torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. We define $\mathcal{N}(G)$ to be the set of σ-invariants in the set of conjugacy classes of homomorphisms $\mathbb{D}_L \to G_L$:

$$\mathcal{N}(G) = (\mathrm{Int}\, G(L) \setminus \mathrm{Hom}_L(\mathbb{D}, G))^{\langle \sigma \rangle},$$

where $\langle \sigma \rangle$ denotes the infinite cyclic group generated by σ (and is endowed with the discrete topology). For example, when $G = T$ is a torus, we have $\mathcal{N}(T) = X_*(T)_{\mathbb{Q}}^{\mathrm{Gal}_F}$.

If $T \subset G$ is a maximal F-torus with (absolute) Weyl group Ω, there is a natural identification

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/\Omega)^{\mathrm{Gal}_F},$$

where the Galois action on $X_*(T)_{\mathbb{Q}}/\Omega$ is the canonical action.

Kottwitz [1985, §4] (cf. [Rapoport and Richartz 1996, Theorem 1.8]) constructs a map

$$\nu = \nu_G : G(L) \to \mathrm{Hom}_L(\mathbb{D}, G)$$

which is functorial in G. We call the induced functorial map $\bar{\nu}$, defined on the category of connected reductive groups over F, the *Newton map*:

$$\bar{\nu} : B(\,\cdot\,) \to \mathcal{N}(\,\cdot\,); \qquad \bar{\nu}_G([b]) = \overline{\nu_G(b)}, \quad b \in [b].$$

Here, $b \in G(L)$ is a representative of $[b]$ and $\overline{\nu_G(b)}$ is the conjugacy class of $\nu_G(b)$.

When $G = \mathrm{GL}_n$, the Newton map sends an F-isocrystal (U, Φ) of height n to its Newton polygon, which is represented in $\mathcal{N}(\mathrm{GL}_n)$ by the corresponding slope homomorphism; see Example 1.10 of [Rapoport and Richartz 1996].

2.1.3. Let $\pi_1(G) = \pi_1(G, T) := X_*(T)/\sum_{\alpha \in R^*} \mathbb{Z}\alpha^\vee$ be the algebraic fundamental group à la Borovoi, where T is a maximal torus of $G_{\bar{F}}$ and $R^* \subset X^*(T)$ is the set of roots of (G, T). This is a Gal_F-module in a natural manner, and (as the notation suggests) is canonically attached to G only. The functor $G \mapsto \pi_1(G)$ is an exact functor from the category of connected reductive groups over F to the category of finitely generated discrete Gal_F-modules; see [Rapoport and Richartz 1996, 1.13]. Kottwitz [1990, §6] constructs a natural transformation

$$\kappa : B(\,\cdot\,) \to \pi_1(\,\cdot\,)_{\mathrm{Gal}_F} \tag{2.1.3.1}$$

of set-valued functors on the category of connected reductive groups over F, where $\pi_1(\,\cdot\,)_{\mathrm{Gal}_F}$ is the group of coinvariants for the canonical Galois action [Rapoport and Richartz 1996, 1.13]. For tori, for which $B(\,\cdot\,)$ has an obvious abelian group structure, this map is an isomorphism [Kottwitz 1985, (5.5.1)] (cf. [Rapoport and Richartz 1996, Theorem 1.15]).

2.1.4. In the following discussion, suppose given a $G(\bar{F})$-conjugacy class $\mathcal{C}$ of cocharacters into $G_{\bar{F}}$. We define a finite subset $B(G, \mathcal{C})$ of $B(G)$, following Kottwitz [1997, §6] (cf. [Rapoport 2005, §4]).

Choose a Borel pair (T, B) (i.e., T is a maximal $\bar{F}$-torus and B is a Borel subgroup of $G_{\bar{F}}$ containing T). Let $\mathcal{BR}(G) = (X^*(T), R^*, X_*(T), R_*, \Delta)$ be the associated based root datum and $\bar{C} \subset X_*(T)_{\mathbb{R}}$ the closed Weyl chamber associated with the root base $\Delta \subset R^*$.

Denoting by $\mu_{\bar{C}} = \mu_{\bar{C}}(G, \mathcal{C})$ the (unique) representative in $\bar{C}$ of $\mathcal{C}$, we set

$$\bar{\mu}(G, \mathcal{C}) := |\mathrm{Gal}_F \cdot \mu_{\bar{C}}|^{-1} \sum_{\mu' \in \mathrm{Gal}_F \cdot \mu_{\bar{C}}} \mu' \in \bar{C}.$$

Here, the action of Gal_F on $\bar{C}$ is the canonical action. The $G(\bar{F})$-conjugacy class of cocharacters containing $\bar{\mu}(G, \mathcal{C})$ depends only on the pair $(G, \mathcal{C})$ (i.e., is independent of the choice of a Borel pair (T, B)).

Let $\mu^\natural = \mu^\natural(\mathcal{C})$ denote the image of $\mu \in X_*(T)$ under the natural map

$$X_*(T) \to \pi_1(G)_{\mathrm{Gal}_F} = \Big(X_*(T) \Big/ \sum_{\alpha \in R^*} \mathbb{Z}\alpha^\vee\Big)_{\mathrm{Gal}_F}$$

for any cocharacter μ of $T_{\overline{F}}$ lying in the conjugacy class $\mathcal{C}$; $\mu^\natural$ depends only on the conjugacy class $\mathcal{C}$, and neither on the choice of a representative μ in $T_{\overline{F}}$ nor on that of T.

For a connected reductive group G over F and a $G(\overline{F})$-conjugacy class $\mathcal{C}$ of cocharacters into $G_{\overline{F}}$, we define

$$B(G, \mathcal{C}) := \{[b] \in B(G) \mid \kappa_G([b]) = \mu^\natural,\ \bar{\nu}_G([b]) \preceq \bar{\mu}(G, \mathcal{C})\},$$

where $\preceq$ is the natural partial order on the closed Weyl chamber $\overline{C}$ defined so that $\nu \preceq \nu'$ if $\nu' - \nu$ is a nonnegative linear combination with *real* coefficients of simple coroots in $X_*(T)_{\mathbb{R}}$ [Rapoport and Richartz 1996, Lemma 2.2]. Also, here we use the canonical identification of $X_*(T)_{\mathbb{Q}}/\Omega$ as a subset of $\overline{C}$.

One knows by [Kottwitz 1997, 4.13] that the map $(\bar{\nu}, \kappa) : B(G) \to \mathcal{N}(G) \times \pi_1(G)_{\mathrm{Gal}_F}$ is injective; hence $B(G, \mathcal{C})$ can be identified with a subset of $\mathcal{N}(G)$.

2.2. ***Integral canonical model and Newton stratification.*** Let (G, X) be a Shimura datum of Hodge type. We assume given an embedding $\rho : (G, X) \to (\mathrm{GSp}(W, \psi), \mathfrak{H}^\pm)$ of Shimura data (i.e., there exists an embedding $\rho_W : G \hookrightarrow \mathrm{GSp}(W, \psi)$ of $\mathbb{Q}$-algebraic groups which sends each morphism in X to a member of $\mathfrak{H}_g^\pm$). Consider a compact open subgroup K of $G(\mathbb{A}_f)$ of the form $K = K_p K^p$, where K_p and K^p are compact open subgroups of $G(\mathbb{Q}_p)$ and $G(\mathbb{A}_f^p)$, respectively. Choose a prime $\wp$ of $E(G, X)$ above p and let $\mathcal{O}_{(\wp)}$ be the localization at $\wp$ of the ring of integers $\mathcal{O}_{E(G,X)}$ of $E(G, X)$. We also use $\mathrm{Sh}_K(G, X)$ to denote the canonical model over $E(G, X)$ [Deligne 1979, 2.2].

2.2.1. When K_p is hyperspecial, Vasiu [1999] (cf. [Vasiu 2007a; 2007b]) and Kisin [2010] independently constructed a smooth integral model $\mathcal{S}_K(G, X)$ over $\mathcal{O}_{(\wp)}$ of the canonical model $\mathrm{Sh}_K(G, X)$. The projective limit $\mathcal{S}_{K_p}(G, X)$ of $\mathcal{S}_{K_p K^p}(G, X)$ over varying $\{K^p\}$ is characterized, as a scheme over $\mathcal{O}_{(\wp)}$, uniquely by the extension property: for any regular, formally smooth $\mathcal{O}_{(\wp)}$-scheme S, every $E(G, X)$-morphism $S_{E(G,X)} \to \mathrm{Sh}_{K_p}(G, X)$ extends uniquely over $\mathcal{O}_{(\wp)}$.

Remark 2.2.2. This definition of the extension property is due to Kisin [2010, Theorem 2.3.8]. It differs slightly from that of Vasiu, who uses, as test schemes, healthy regular schemes over $\mathcal{O}_{(\wp)}$ [Vasiu 1999, Section 3.2] instead of regular, formally smooth $\mathcal{O}_{(\wp)}$-schemes. But, when $G_{\mathbb{Q}_p}$ is unramified, every regular, formally smooth scheme over $\mathcal{O}_{(\wp)}$ is also healthy regular over $\mathcal{O}_{(\wp)}$, because $\mathcal{O}_{(\wp)}$ is unramified over $\mathbb{Z}_{(p)}$ [Milne 1994, Corollary 4.7(a)] and every regular, formally smooth scheme is healthy regular (if $p > 2$) over any d.v.r. unramified over $\mathbb{Z}_{(p)}$, according to a lemma of Faltings (see [Moonen 1998, Lemma 3.6]). Consequently, this difference (and related other minor differences) will not matter when the extension property is invoked for such schemes.

In our work, we need a more precise description of the integral canonical model. Let $G_{\mathbb{Z}_p}$ be a reductive group scheme over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$ and such that $G_{\mathbb{Z}_p}(\mathbb{Z}_p) = K_p$. By [Kisin 2010, (2.3)], there exists a lattice $W_{\mathbb{Z}}$ of W such that the embedding $\rho_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}_p})$ is induced by a closed embedding

$$G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p}),$$

where $W_{\mathbb{Z}_p} := W_{\mathbb{Z}} \otimes \mathbb{Z}_p$. For K^p stabilizing $W_{\widehat{\mathbb{Z}}} = W_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$, one can find a compact open subgroup $H = H^p H_p \subset \mathrm{GSp}(W, \psi)(\mathbb{A}_f)$ with $\rho(K) \subset H$ such that H stabilizes $W_{\widehat{\mathbb{Z}}}$ and also that $\rho : G \hookrightarrow \mathrm{GSp}(W, \psi)$ induces an embedding (of weakly canonical models of Shimura varieties)

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_H(\mathrm{GSp}(W_{\mathbb{Z}}, \psi), \mathfrak{H}^{\pm}) \otimes_{\mathbb{Q}} E(G, X).$$

For $d \in \mathbb{N}$ and sufficiently small H, let $\mathcal{A}_{g,d,H}$ denote the Mumford (fine) moduli scheme over $\mathbb{Z}[1/d]$ which parametrizes abelian schemes endowed with a polarization of degree d and a level H-structure. By replacing $W_{\mathbb{Z}}$ by a scalar multiple of it, we may and do assume that ψ is $\mathbb{Z}$-valued on $W_{\mathbb{Z}}$; let $d := [W_{\mathbb{Z}}^* : W_{\mathbb{Z}}]$ for the dual $W_{\mathbb{Z}}^* \subset W$. Then, by taking sufficiently small K^p (so that H^p is also so), we get an embedding of schemes

$$\mathrm{Sh}_H(\mathrm{GSp}(W_{\mathbb{Z}}, \psi), \mathfrak{H}^{\pm}) \hookrightarrow \mathcal{A}_{g,d,H}.$$

By construction, $\mathcal{S}_K(G, X)$ is the normalization of the Zariski closure of the image of the resulting embedding

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathcal{A}_{g,d,H} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\wp)}. \tag{2.2.2.1}$$

From now on, when we talk about the canonical integral model $\mathcal{S}_{K_pK^p}$ (with hyperspecial K_p), we will tacitly assume that K^p is small enough that the above conditions imposed are achieved.

Let $\mathcal{S}_K \otimes \kappa(\wp)$ be the reduction of the integral canonical model $\mathcal{S}_K = \mathcal{S}_K(G, X)$ at $\wp$. Next, we define the Newton stratification on this scheme, following [Rapoport 2005; Vasiu 2008] (cf. [Wortmann 2013]). We fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ which induces the chosen prime $\wp$ on $E(G, X)$; such choice amounts to fixing an embedding $E(G, X)_{\wp} \hookrightarrow \overline{\mathbb{Q}}_p$ together with an embedding ι_p of $E(G, X)$-algebras.

2.2.3. For a vector space V, let $\mathcal{T}(V)$ be the tensor space attached to V:

$$\mathcal{T}(V) = \bigoplus_{(s,t)\in\mathbb{N}^2} V^{\otimes s} \otimes (V^{\vee})^{\otimes t}.$$

An element of $\mathcal{T}(V)$ is called a *tensor on* V. Let $\{s_\alpha\}_{\alpha\in\mathcal{J}}$ be the set of tensors on $W_{\mathbb{Q}}$ fixed under the extended action of $G_{\mathbb{Q}} \hookrightarrow \mathrm{GL}(W)$ on $\mathcal{T}(W)$.

Let $(\pi : A \to \mathrm{Sh}_K, \lambda_A)$ be the pullback to $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ of the universal abelian scheme with a polarization of degree d over $\mathcal{A}_{g,d,H}$; as usual, we assume K^p to be sufficiently small for this to make sense. We have the local system $\mathcal{W} := R^1(\pi^{\mathrm{an}})_*\mathbb{Q}$ of $\mathbb{Q}$-vector spaces and the (analytic) vector bundle $\mathbb{W}_{\mathbb{C}}^{\mathrm{an}} := R^1(\pi^{\mathrm{an}})_*\Omega^{\bullet}_{A_{\mathbb{C}}/\mathrm{Sh}_K}$ with Gauss–Manin connection (which is an integrable connection with regular singularities). By Deligne's theorem, the latter is the analytification of a unique algebraic vector bundle $\mathbb{W}$ over Sh_K (in this case, the relative de Rham cohomology $H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K) := R^1\pi_*\Omega^{\bullet}_{A/\mathrm{Sh}_K}$) with integrable connection. Each tensor s_α defines a global section $s_{\alpha,\mathrm{B}}$ of the local system $\mathcal{T}(\mathcal{W})$ and also a global section $s_{\alpha,\mathrm{dR}}$ of the vector bundle $\mathcal{T}(H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K))$ [Kisin 2010, (2.2)].

Now, let $F \supset E(G, X)$ be an extension which can be embedded in $\mathbb{C}$; we fix an embedding $\sigma_\infty : \bar{F} \hookrightarrow \mathbb{C}$ which extends the given embedding $E(G, X) \hookrightarrow \mathbb{C}$. For $x \in \mathrm{Sh}_K(G, X)(F)$, let A_x be the corresponding abelian variety over F, and let $H^m_{\mathrm{dR}}(A_x/F)$ and $H^m_{\text{ét}}(A_x \otimes_F \bar{F}, \mathbb{Q}_p)$ denote the de Rham cohomology of A_x/F and the étale cohomology of $A_x \otimes_F \bar{F}$, respectively. For each tensor $s_\alpha \in \mathcal{T}(W)$, let $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ in $\mathcal{T}(H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}))$ denote the fiber of $s_{\alpha,\mathrm{B}}$ at $\sigma_\infty(x) \in \mathrm{Sh}_K(G, X)(\mathbb{C})$. By the moduli interpretation of the complex points of $\mathrm{Sh}_K(G, X)$ [Milne 1994, Proposition 3.9], there is an isomorphism $W^\vee \xrightarrow{\sim} H^1_{\mathrm{B}}(A_{\sigma_\infty(x)}, \mathbb{Q})$, uniquely determined up to action of $G(\mathbb{Q})$, where G acts on $W^\vee$ by the contragredient representation. Moreover, $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ is the image of s_α by any such isomorphism (in particular, it does not depend on the choice of such isomorphism). For each prime l, let $s_{\alpha,l,\sigma_\infty(x)}$ be the tensor in $\mathcal{T}(H^1_{\text{ét}}(A_x \otimes_F \bar{F}, \mathbb{Q}_l))$ which is the preimage of $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ under the canonical isomorphism

$$H^1_{\text{ét}}(A_x \otimes_F \bar{F}, \mathbb{Q}_l) \xrightarrow{\sim} H^1_{\text{ét}}(\sigma_\infty(A_x \otimes_F \bar{F}), \mathbb{Q}_l) = H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}) \otimes \mathbb{Q}_l, \tag{2.2.3.1}$$

where the first isomorphism σ_∞ is the proper base change isomorphism in étale cohomology. The image of $s_{\alpha,\mathrm{B},\sigma_\infty(x)}$ under the comparison isomorphism

$$H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}) \otimes \mathbb{C} = H^1_{\mathrm{dR}}(A_x \otimes_{F,\sigma_\infty} \mathbb{C}/\mathbb{C})$$

is the stalk $s_{\alpha,\mathrm{dR},\sigma_\infty(x)}$ at $\sigma_\infty(x)$ of the section $s_{\alpha,\mathrm{dR}}$ of $\mathcal{T}(H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K))$. As the algebraic vector bundle $H^1_{\mathrm{dR}}(A/\mathrm{Sh}_K)$ is defined over $E(G, X)$ [Kisin 2010, Corollary 2.2.2], so is $\sigma_\infty(s_{\alpha,\mathrm{dR},x})$, the base change via σ_∞ of the stalk $s_{\alpha,\mathrm{dR},x}$ of $s_{\alpha,\mathrm{dR}}$ at x. Set $H^m_{\mathbb{A}}(A_x) := H^m_{\mathrm{dR}}(A_x/F) \times \prod_l H^m_{\text{ét}}(A_x, \mathbb{Q}_l)$ and $\mathbb{Q}(1) := F \times \mathbb{A}_f(1)$, where $\mathbb{A}_f(1) := (\varprojlim_r \mu_r) \otimes_{\mathbb{Z}} \mathbb{Q}$ (as usual, $\mu_r := \{\zeta \in \bar{F} \mid \zeta^r = 1\}$), and let $\mathcal{T}(H^1_{\mathbb{A}}(A_x), \mathbb{Q}(1))$ be the tensor space generated by these $F \times \mathbb{A}_f$-modules $H^m_{\mathbb{A}}(A_x)$, $\mathbb{Q}(1)$ (and their duals) [Deligne et al. 1982, Chapter I, §1]. Then, the element

$$(s_{\alpha,\mathrm{dR},x}, s_{\alpha,\text{ét},\sigma_\infty(x)} := (s_{\alpha,l,\sigma_\infty(x)})_l)$$

of $\mathcal{T}(H^1_{\mathbb{A}}(A_x), \mathbb{Q}(1))$ is an absolute Hodge cycle on A_x [Deligne et al. 1982, Chapter I, §2].

2.2.4. Now we show that every geometric point $z \in \mathcal{S}_K \otimes \bar{\mathbb{F}}_p(k)$ (with $k = \bar{k}$, $\mathrm{char}\, k = p$) gives rise to a $G_{\mathbb{Q}_p}$-isocrystal over k, i.e., a σ-conjugacy class of elements in $G(L)$. Here, we assume that $L(k)$ can be embedded in $\mathbb{C}$; for example, this is the case if k has finite transcendence degree over $\bar{\mathbb{F}}_p$. The p-divisible group $(A_z[p^\infty], \lambda_{A_z}[p^\infty])$ with quasipolarization gives rise to an F-isocrystal over k equipped with a nondegenerate alternating pairing. More precisely, the crystalline cohomology $M := H^1_{\mathrm{cris}}(A_z/W(k))$ of A_z with quasipolarization λ_z is a quasipolarized F-crystal $(M, \phi, \langle\ ,\ \rangle)$: M is a free $W(k)$-module of rank $2 \dim A_z$, $\phi \in \mathrm{End}_{\mathbb{Z}_p}(M)$ is a σ-linear endomorphism such that $pM \subset \phi(M)$, and $\langle\ ,\ \rangle : M \times M \to W(k)$ is an alternating form with the property that $\langle \phi(v_1), \phi(v_2)\rangle = p\langle v_1, v_2\rangle^\sigma$ for $v_1, v_2 \in M$.

To proceed, we fix an embedding $E(G, X)_\wp \hookrightarrow L(k)$, and thus an identification $E(G, X)_\wp = L(\mathbb{F}_q)$ as well, where $\mathbb{F}_q$ is the residue field of $\mathcal{O}_{(\wp)}$. Then there exists a lift of $z : \mathrm{Spec}(k) \to \mathcal{S}_K \otimes_\kappa k$ to $\mathrm{Spec}(W(k))$, since $\mathcal{S}_K(G, X)$ is smooth over $\mathcal{O}_{(\wp)}$ and $\mathcal{O}_{(\wp)}$ is unramified over $\mathbb{Z}_{(p)}$ [Milne 1994, Corollary 4.7]. We choose one, say $x : \mathrm{Spec}(W(k)) \to \mathcal{S}_K$, and set $(A_x, \lambda_x) := x^*(A, \lambda_A)$.

Lemma 2.2.5. *For each $\alpha \in \mathcal{J}$, let $s_{\alpha,0,x} \in \mathcal{T}(M \otimes_{W(k)} L(k))$ denote the image of $s_{\alpha,\mathrm{dR},x}$ (the stalk at x of $s_{\alpha,\mathrm{dR}}$) under the canonical isomorphism $H^1_{\mathrm{dR}}(A_x/L(k)) = M \otimes_{W(k)} L(k)$.*

(1) *The tensor $s_{\alpha,0,x}$ is a* crystalline cycle, *namely $s_{\alpha,0,z}$ belongs to the F^0-filtration of $\mathcal{T}(M \otimes L(k))$ and is fixed under the Frobenius ϕ, where the filtration on $\mathcal{T}(M \otimes L(k))$ is the one transported from the Hodge filtration on $\mathcal{T}(H^1_{\mathrm{dR}}(A_x/L(k))$ by the canonical isomorphism $M \otimes L(k) = H^1_{\mathrm{dR}}(A_x/L(k))$.*

(2) *The triple*

$$(M \otimes F, \phi, (s_{\alpha,0,x})_{\alpha\in\mathcal{J}})$$

depends only on z, not on the lift x; let us write $s_{\alpha,0,z}$ for $s_{\alpha,0,x}$. There exists an $L(k)$-isomorphism

$$W^\vee \otimes_{\mathbb{Q}} L(k) \xrightarrow{\sim} M \otimes_{W(k)} L(k) \tag{2.2.5.1}$$

which maps s_α to $s_{\alpha,0,z}$ for every $\alpha \in \mathcal{J}$.

Proof. (1) This is well-known. Originally, this statement was proved by Blasius, Ogus, and Wintenberger independently (see [Blasius 1994, §5]) when x was defined over a number field, as a consequence of the fact that the Hodge cycles on an abelian variety over a number field are *de Rham* and the compatibility of the crystalline and de Rham comparison isomorphisms [Blasius 1994, 5.1(5)]. Then Vasiu [2008, Section 8] generalized it to arbitrary fields.

(2) As before, we choose an embedding $\sigma_\infty : \overline{L(k)} \hookrightarrow \mathbb{C}$ of $E(G,X)$-algebras. We have seen that for each $\alpha \in \mathcal{J}$, the image of s_α under the comparison isomorphism

$$W^\vee \otimes \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B}}(\sigma_\infty(A_x), \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_x/L(k)) \otimes_{L(k),\sigma_\infty} \mathbb{C} \tag{2.2.5.2}$$

is $\sigma_\infty(s_{\alpha,\mathrm{dR},x}) = s_{\alpha,\mathrm{dR},x} \otimes 1$ (a section of $\mathcal{T}(H^1_{\mathrm{dR}}(A_{x'}/L(k))) \otimes_{L(k),\sigma_\infty} \mathbb{C}$); recall that the first isomorphism is unique up to the action of $G(\mathbb{Q})$ on $W^\vee$. Next, for any two lifts x, x' over $W(k)$ of z, the composite of the canonical maps

$$H^1_{\mathrm{dR}}(A_x/L(k)) \xrightarrow{\sim} H^1_{\mathrm{cris}}(A_z/W(k)) \otimes L(k) \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_{x'}/L(k))$$

is given by parallel transport with respect to the Gauss–Manin connection [Berthelot and Ogus 1983, Remark 2.9]. As $\sigma_\infty(s_{\alpha,\mathrm{dR},y})$ for $y = x, x'$ are the stalks of a horizontal global section $s_{\alpha,\mathrm{B}}$ of the local system $\mathcal{T}(\mathcal{W}_{\mathbb{C}})$, we see that under the composite map at hand, $\sigma_\infty(s_{\alpha,\mathrm{dR},x})$ must map to $\sigma_\infty(s_{\alpha,\mathrm{dR},x'})$. This proves that the triple $(M \otimes L(k), \phi, (s_{\alpha,0,x})_{\alpha\in\mathcal{J}})$ is independent of the choice of lift x. Let us write $s_{\alpha,0,z}$ for $s_{\alpha,0,x}$ ($\alpha \in \mathcal{J}$).

Now, the functor defined on $L(k)$-algebras

$$R \mapsto \mathrm{Isom}_R((W^\vee_{L(k)} \otimes R, \{s_\alpha \otimes 1\}), (M \otimes R, \{s_{\alpha,0,z} \otimes 1\}))$$

of isomorphisms between $W^\vee \otimes_{\mathbb{Q}} L(k)$ and $M = H^1_{\mathrm{cris}}(A_z/L(k))$ taking s_α to $s_{\alpha,0,z}$ for every $\alpha \in \mathcal{J}$ is represented by a scheme which is nonempty since it has a $\mathbb{C}$-valued point, as was just seen, and thus is a torsor over $L(k)$ under $G_{L(k)}$. Since $H^1(L(k), G_{L(k)}) = \{0\}$ (Steinberg's theorem), this torsor is trivial, namely has an $L(k)$-valued point. $\square$

Therefore, when one chooses an isomorphism as in (2.2.5.1) and transports the σ-linear map ϕ to $W^\vee \otimes L(k)$, we obtain an element $b \in G(L(k))$ with

$$\phi = \rho_{W^\vee}(b)(\mathrm{id}_{W^\vee} \otimes \sigma),$$

where $\rho_{W^\vee} : G \hookrightarrow \mathrm{GL}(W^\vee)$ denotes the contragredient representation: indeed, both ϕ and $\mathrm{id}_{W^\vee} \otimes \sigma$ (thus $\rho_{W^\vee}(b)$ as well) fix each $s_\alpha \otimes 1 \in \mathcal{T}(W^\vee \otimes L(k))$. Although b depends on the choice of an isomorphism (2.2.5.1), its σ-conjugacy class $[b] \in B(G_{\mathbb{Q}_p})$ is independent of such choice. This shows that the F-isocrystal over k attached to z is an $G_{\mathbb{Q}_p}$-isocrystal. For an arbitrary point z of $\mathcal{S}_K(G, X) \otimes \kappa(\wp)$, we define the F-isocrystal attached to z to be the F-isocrystal of the geometric point in $\mathcal{S}_K(G, X)(k)$ induced from z for *any* algebraically closed field k containing the residue field $\kappa(z)$ of z. The resulting F-isocrystal does not depend on the choice of k'; see [Viehmann and Wedhorn 2013, §8; Rapoport and Richartz 1996, Lemma 1.3]. In this way, we obtain a (set-theoretic) map

$$\Theta : \mathcal{S}_K(G, X) \otimes \kappa(\wp) \to B(G_{\mathbb{Q}_p}).$$

By the same argument (applied to $\mathrm{Sh}(T, h)$ instead of $\mathrm{Sh}(G, X)$), we see that if $\sigma_\infty(x)$ is a special point, i.e., $\sigma_\infty(x) = [h, g_f] \in \mathrm{Sh}_K(G, X)(\sigma_\infty(F))$ for some $h \in X$ factoring through a maximal $\mathbb{Q}$-torus T of G, the $G_{\mathbb{Q}_p}$-isocrystal (attached to its reduction $z \in \mathcal{S}_K(G, X)$) has a representative in $T(L)$.

Finally, we remark that although our definition of the F-isocrystal attached to a point of $\mathcal{S}_K(G, X)(k)$ uses *cohomology* spaces, one can equally work with homology spaces, as adopted by other people (such as Viehmann and Wedhorn [2013]). This does not alter the definition of the map Θ.

2.2.6. With every Shimura datum (G, X), there is associated a natural $G(\mathbb{C})$-conjugacy class $c(G, X)$ of cocharacters of $G_{\mathbb{C}}$, namely that containing the Hodge cocharacter $\mu_h^{-1} := h^{-1}|_{\mathbb{G}_m}$, where $\mathbb{G}_m$ refers to the factor of $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} = \prod_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_m$ corresponding to the identity embedding of $\mathbb{C}$. Recall that we fixed an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the chosen prime $\wp$ on $E(G, X)$ and $\overline{\mathbb{Q}}$ is given as a subfield of $\mathbb{C}$. As is well-known, this choice allows us to consider $c(G, X)$ as a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}_p}$; we continue to denote it by $c(G, X)$.

Put $B(G_{\mathbb{Q}_p}, X) := B(G_{\mathbb{Q}_p}, c(G, X))$ (the subset of $B(G_{\mathbb{Q}_p})$ defined in 2.1.4 for $\mathcal{C} = c(G, X)$) and $\bar{\mu}(G_{\mathbb{Q}_p}, X) := \bar{\mu}(G_{\mathbb{Q}_p}, c(G, X))$.

Proposition 2.2.7. *Let (G, X) be a Shimura datum of Hodge type and $K = K_pK^p \subset G(\mathbb{A}_f)$ a compact open subgroup. Suppose that G is unramified over $\mathbb{Q}_p$ and $K_p \subset G(\mathbb{Q}_p)$ is hyperspecial.*

(1) *The image of $\Theta : \mathcal{S}_K(G, X) \otimes \kappa(\wp) \to B(G_{\mathbb{Q}_p})$ is contained in $B(G_{\mathbb{Q}_p}, X)$.*

(2) *For $[b] \in B(G_{\mathbb{Q}_p})$, the subset*

$$\mathcal{S}_{[b]} := \Theta^{-1}([b])$$

is locally closed inside $\mathcal{S}_K(G, X) \otimes \overline{\mathbb{F}}_p$.

For (1), see, e.g., [Viehmann 2015, Theorem 3.7], and for (2), [Vasiu 2011, 5.3.1].

Endowed with reduced induced subscheme structure, the subvarieties $\mathcal{S}_{[b]}$ of $\mathcal{S}_K(G, X) \otimes \overline{\mathbb{F}}_p$ are called the *Newton strata* of $\mathcal{S}_K(G, X)$.

Rapoport [2005, Conjecture 7.1] conjectures the following.

Conjecture 2.2.8. $$\mathrm{Im}(\Theta) = B(G_{\mathbb{Q}_p}, X).$$

Remark 2.2.9. (1) It is known (see [Rapoport and Richartz 1996; Chai 2000]) that with respect to the partial order $\preceq$ on $B(G_{\mathbb{Q}_p}, X)$, there exist a unique maximal element, called the *μ-ordinary* element, and a unique minimal element, called the *basic* element. The μ-ordinary element is just (the σ-conjugacy class in $B(G_{\mathbb{Q}_p}, X)$ with Newton point) $\bar{\mu}(G_{\mathbb{Q}_p}, X)$; it is clear from the definition that $\bar{\mu}(G_{\mathbb{Q}_p}, X) \in B(G_{\mathbb{Q}_p}, X)$. Previously, nonemptiness is known for these two special strata. For the μ-ordinary locus, this was first proved by Wedhorn [1999] in the PEL-type cases, using an equicharacteristic deformation argument (which is thus yet unavailable for Hodge type Shimura varieties) and by Wortmann [2013] for general Hodge-type cases, along the lines of [Viehmann and Wedhorn 2013] comparing the Newton stratification with the Ekedahl–Oort stratification. There was also a group-theoretic approach of [Bültel 2001]. Nonemptiness of the basic locus in the PEL-type cases was shown in [Fargues 2004].

(2) The Newton stratification on Shimura varieties has been a research topic of intensive study with constant progress and outputs. We refer the reader to the recent survey article [Viehmann 2015] on the current status of research (updating the reports by Rapoport [2003; 2005]). Here, we just mention one question directly related to our work, namely the dimension of Newton strata (the result on which either assumes or subsumes the nonemptiness of stratum). There is a conjectural formula for the (co)dimension of each Newton stratum [Chai 2000, Question 7.6; Rapoport 2005, p. 296]. This conjecture was inspired by the result of Oort [2001, Theorem 4.2] confirming it in the Siegel case, and was proved, among others, in the general PEL-type case by Hamacher [2015]; with our nonemptiness result, his recent work [Hamacher 2017] also establishes this formula in the general Hodge-type case. Also, the result of [Kret 2012], alluded to in the introduction, in fact establishes nonemptiness of Newton strata by proving this formula in the simple PEL-type cases of Lie type A or C; his result, however, depends on the resolution of the Langlands–Rapoport conjecture [Kisin 2017] and the stabilization of the twisted trace formula. There is also a work of Scholze and Shin [2013, Corollary 8.4] of similar flavor.

(3) We do not know yet if our definition of the $G_{\mathbb{Q}_p}$-isocrystal is a $G_{\mathbb{Q}_p}$-isocrystal on $\mathcal{S}_K \otimes \kappa(\wp)$ in the sense of [Rapoport and Richartz 1996, §3]. Due to this flaw, some basic properties on Newton stratification that are known for PEL-type Shimura varieties, such as the Grothendieck specialization theorem, are not yet established for Hodge type Shimura varieties. For some known properties, we refer to [Rapoport 2003; Vasiu 2008].

3. Proof of nonemptiness of Newton strata

We keep the notation from the previous section; in particular, we chose an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing $\wp$ on $E(G, X)$. In this section, we prove the main result, namely that for any $G_{\mathbb{Q}_p}$-isocrystal $[b]$ lying in $B(G_{\mathbb{Q}_p}, X)$, there exists a special point in $\mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}})$ such that the F-isocrystal of its reduction is $[b]$. For that, we give a generalization of a result of Kottwitz [1992, Lemma 13.1] (in the

PEL-cases) which identifies the $G_{\mathbb{Q}_p}$-isocrystal of the reduction of a special point in terms of the special Shimura subdatum defining the point.

3.1. ***$G_{\mathbb{Q}_p}$-isocrystal attached to special points.*** We recall that for a torus T over a nonarchimedean local field F, there exists a unique maximal compact subgroup of $T(F)$ (which is also open). In fact, T has a canonical integral form $\mathcal{T}$ over $\mathcal{O}_F$ such that for every finite extension F' of F, the maximal compact subgroup of $T(F')$ is $\mathcal{T}(\mathcal{O}_{F'})$. If E is a finite Galois extension of F splitting T, the maximal compact subgroup of $T(E) = \mathrm{Hom}(X^*(T), E^\times)$ ($\simeq (E^\times)^{\dim T}$) is $\mathcal{T}(\mathcal{O}_E) = \mathrm{Hom}(X^*(T), \mathcal{O}_E^\times)$ ($\simeq (\mathcal{O}_E^\times)^{\dim T}$); furthermore, we have $\mathcal{T}(\mathcal{O}_F) = \mathrm{Hom}_{\mathrm{Gal}(E/F)}(X^*(T), \mathcal{O}_E^\times)$. If T is unramified over F, $\mathcal{T}(\mathcal{O}_F)$ is the unique hyperspecial subgroup of $T(F)$ (see Theorem 1 and the discussion after Proposition 2 in §10.3 of [Voskresenskiĭ 1998]; also see [Tits 1979]). We will also write simply $T(\mathcal{O}_F)$ for $\mathcal{T}(\mathcal{O}_F)$.

Now, let $x = [h, k_f] \in \mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}})$ be a special point, namely $h \in \mathrm{Hom}(\mathbb{S}, T_{\mathbb{R}}) \cap X$ for a maximal $\mathbb{Q}$-torus T of G and $k_f \in G(\mathbb{A}_f)$. Then, $\iota_p(x) \in \mathrm{Sh}_K(G, X)(\overline{\mathbb{Q}}_p)$ extends (uniquely) to an $\mathcal{O}_{\iota_p}$-valued point of the integral canonical model $\mathcal{S}_K$, where $\mathcal{O}_{\iota_p}$ is the valuation ring of $\overline{\mathbb{Q}}$ defined by ι_p. This is due to the *construction* of $\mathcal{S}_K$ as the normalization of the Zariski closure of the image of a natural embedding $\mathrm{Sh}_K(G, X) \hookrightarrow \mathcal{A}_{g,d,H} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\wp)}$ (as provided by (2.2.2.1)). Indeed, the image of x under such an embedding is a polarized abelian variety of CM-type; thus, it extends to an $\mathcal{O}_{\iota_p}$-valued point of $\mathcal{A}_{g,d.H}$. Since the morphism $\mathcal{S}_K(G, X) \to \mathcal{A}_{g,d,H} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\wp)}$ of $\mathcal{O}_{(\wp)}$-schemes is finite (as the target is an excellent scheme), the claim follows by the valuative criterion of properness applied to a suitable normal model over $\mathcal{O}_{(\wp)}$ of $\mathrm{Sh}_{K'}(T, h)$ for $K' \subset T(\mathbb{A}_f) \cap K$ a compact open subgroup. Set $L := L(\overline{\mathbb{F}}_p)$.

Lemma 3.1.1. (1) *The $G_{\mathbb{Q}_p}$-isocrystal of the reduction z of $\iota_p(x) \in \mathrm{Sh}_K(G, X)(\mathcal{O}_{\iota_p})$ is the image of $-\underline{\mu_h} \in X_*(T_{\mathbb{Q}_p})_{\mathrm{Gal}_{\mathbb{Q}_p}}$ under the natural map $X_*(T_{\mathbb{Q}_p})_{\mathrm{Gal}_{\mathbb{Q}_p}} \xrightarrow{\sim} B(T_{\mathbb{Q}_p}) \to B(G_{\mathbb{Q}_p})$, where the first isomorphism is the inverse of the isomorphism $\kappa_{T_{\mathbb{Q}_p}} : B(T_{\mathbb{Q}_p}) \xrightarrow{\sim} X_*(T_{\mathbb{Q}_p})_{\mathrm{Gal}_{\mathbb{Q}_p}}$* (2.1.3.1).

(2) *If $T_{\mathbb{Q}_p}$ is unramified and the* (*unique*) *hyperspecial subgroup $T(\mathbb{Z}_p)$ of $T(\mathbb{Q}_p)$ is contained in K_p, then $\iota_p(x)$ is defined over an unramified extension E' of $E(G, X)_\wp$ and extends uniquely over $\mathcal{O}_{E'}$.*

It follows readily from [Kottwitz 1985, 2.5] that the $G_{\mathbb{Q}_p}$-isocrystal of the reduction z is represented by

$$\mathrm{Nm}_{E/E_0}(\mu(\pi_E))^{-1} \in T(L),$$

where $E = E(T, h)_{\mathfrak{p}}$ for the prime $\mathfrak{p}$ of the reflex field $E(T, h)$ induced by ι_p, $E_0 \subset E$ is the maximal unramified subextension of $\mathbb{Q}_p$, $\mathrm{Nm}_{E/E_0} : T(E) \to T(E_0)$ is the Norm map, and π_E is a uniformizer of E.

Proof. (1) This was proved by Kottwitz [1992, Lemma 13.1] in his PE(L)-type situation. In our general Hodge-type situation, we adapt his arguments; incidentally. We already observed that the F-isocrystal of z is represented by an element of $T(L)$. Before entering into a detailed proof, we briefly explain the strategy of Kottwitz's proof [1992, §12–13]. It consists of two parts. First, with any $\mathbb{Q}_p$-torus $(T', \mu' \in X_*(T'))$ endowed with a cocharacter, he constructs a certain T'-isocrystal, i.e., a $\mathbb{Q}_p$-linear exact tensor functor

$$\Xi_{T',\mu'} : \mathcal{REP}_{\mathbb{Q}_p}(T') \longrightarrow \mathcal{CRYS} \qquad (3.1.1.1)$$

(see Section 2.1) and identifies it (via the isomorphism $\kappa_{T'} : B(T') \xrightarrow{\sim} X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ (2.1.3.1)); this part of his argument applies to any torus over a p-adic field endowed with a cocharacter. Next, he specializes the pair (T', μ') to the one $(T_{\mathbb{Q}_p}, \mu_h)$ coming from a special Shimura subdatum (T, h) of $(\mathrm{GSp}(W, \psi), \mathfrak{H}^{\pm})$, and in this case obtains the geometric interpretation $\Xi_{T_{\mathbb{Q}_p}, \mu_h}(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L)$, where $W_{\mathbb{Q}_p}$ refers to the representation of $T_{\mathbb{Q}_p}$ given by the chosen embedding $T \hookrightarrow \mathrm{GSp}(W, \psi)$ and $H_1^{\mathrm{cris}}(A_z/L)$ is the F-isocrystal dual to $H^1_{\mathrm{cris}}(A_z/L)$ for the reduction z of $\iota_p(x)$ [Kottwitz 1992, Lemma 13.1]. Here, Kottwitz works with special Shimura-subdatum defined by a PEL-datum, but one easily checks that this statement continues to hold for any special Shimura subdatum of $(\mathrm{GSp}(W, \psi), \mathfrak{H}^{\pm})$ (we will justify this shortly). Finally, a standard argument of Tannakian theory gives an isomorphism

$$f(W_{\mathbb{Q}_p}) : H_1^{\mathrm{cris}}(A_z/L) \xrightarrow{\sim} W_{\mathbb{Q}_p} \otimes L$$

of L-vector spaces; this is obtained by identifying some two L-valued fiber functors of the Tannakian category $\mathcal{REP}_{\mathbb{Q}_p}(T')$, from which it follows (cf. Section 2.1) that the Frobenius automorphism $H_1^{\mathrm{cris}}(A_z/L)$ transfers to $b\sigma$ with $b \in T'(L)$ via such isomorphism. Our main job then consists in verifying that in a general Hodge-type situation, such an isomorphism $f(W_{\mathbb{Q}_p})$ still qualifies as an isomorphism of (2.2.5.1), namely that under the induced map $f(W_{\mathbb{Q}_p})^\vee : W_L^\vee \xrightarrow{\sim} H^1_{\mathrm{cris}}(A_z/L)$, for each $\alpha \in \mathcal{J}$, the tensor s_α goes over to the crystalline cycle $s_{\alpha,0,z}$ of Lemma 2.2.5(2).

To justify these claims in detail, we begin by recalling the construction of the functor $\Xi_{T_{\mathbb{Q}_p}, \mu_h}$ (3.1.1.1). Let $F \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{T}^F$ the category of tori over $\mathbb{Q}_p$ split by F. The functor $T \mapsto X_*(T)$ is an equivalence of categories between $\mathcal{T}^F$ and the category of $\mathrm{Gal}(F/\mathbb{Q}_p)$-modules that are free of finite rank as abelian groups. Then, for any object T' of $\mathcal{T}^F$ endowed with a cocharacter $\mu' \in X_*(T')$, one defines a $\mathbb{Q}_p$-linear exact tensor functor

$$\Xi_{T', \mu'} : \mathcal{REP}_{\mathbb{Q}_p}(T') \to \mathcal{CRYS} \tag{3.1.1.2}$$

as the composite of two $\mathbb{Q}_p$-linear exact tensor functors

$$\rho^*_{T', \mu'} : \mathcal{REP}_{\mathbb{Q}_p}(T') \to \mathcal{CR}(F), \qquad \mathbb{D} : \mathcal{CR}(F) \to \mathcal{CRYS},$$

where $\mathcal{CR}(F)$ is the neutral $\mathbb{Q}_p$-linear Tannakian category of finite-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}})$, where $F^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ is the maximal unramified subextension of F. Here, the first functor $\rho^*_{T', \mu'}$ is dual to a certain natural homomorphism

$$\rho_{T', \mu'} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to T'(\mathbb{Q}_p) \tag{3.1.1.3}$$

that is constructed from (T', μ') in [Kottwitz 1992, §12]. The second functor $\mathbb{D}$ is the Dieudonné functor in the Fontaine–Messing theory: for a finite unramified extension E ($\subset F^{\mathrm{ur}}$) of F and a crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$,

$$\mathbb{D}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)},$$

where B_{cris} is the crystalline period ring of [Fontaine and Messing 1987]. Then, in view of [Kottwitz 1985, 2.5], Lemma 12.1 of [Kottwitz 1992] says that, regarding $\Xi_{T',\mu'}$ as an element of $B(T')$, one has the equality

$$\Xi_{T',\mu'} = -\underline{\mu'}$$

under the isomorphism $\kappa_{T'} : B(T') \xrightarrow{\sim} X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ (2.1.3.1).

In the geometric situation where $(T', \mu') = (T_{\mathbb{Q}_p}, \mu_h)$, we claim that one has a canonical isomorphism of F-isocrystals

$$\Xi_{T_{\mathbb{Q}_p},\mu_h}(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L),$$

where $H_1^{\mathrm{cris}}(A_z/L)$ is the F-isocrystal dual to $H^1_{\mathrm{cris}}(A_z/L)$, i.e., the linear dual $\mathrm{Hom}_L(H^1_{\mathrm{cris}}(A_z/L), L)$ equipped with the Frobenius operator $\Phi_1(f)(v) := p^{-1}\cdot{}^{\sigma}f(Vv)$ for $f \in H_1^{\mathrm{cris}}(A_z/L)$ and $v \in H^1_{\mathrm{cris}}(A_z/L)$ (V being the Verschiebung operator on $H^1_{\mathrm{cris}}(A_z/L)$). In view of the comparison theorem of [Fontaine and Messing 1987] and [Faltings 1989], it is sufficient to show that the (crystalline) representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to T_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}_p})$ (3.1.1.3) is isomorphic, via some isomorphism $W^\vee_{\mathbb{Q}_p} \xrightarrow{\sim} H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)$, to the canonical Galois representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to \mathrm{GL}(H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p))$, where $H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)$ is $H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)^\vee$ (linear dual). The moduli interpretation of $\mathrm{Sh}_K(G, X)(\mathbb{C})$ provides an isomorphism $W^\vee \xrightarrow{\sim} H^1_{\mathrm{B}}(A_x, \mathbb{Q})$, uniquely determined up to action of $G(\mathbb{Q})$. There exists such an isomorphism under which the Hodge structure $h : \mathbb{C}^\times \to \mathrm{GL}(W)_{\mathbb{R}}$ corresponds to the defining Hodge structure $h_x : \mathbb{C}^\times \to \mathrm{GL}(H^1_{\mathrm{B}}(A_x, \mathbb{Q}))$ of A_x; we fix one, say θ. The induced isomorphism

$$W^\vee_{\mathbb{Q}_p} \xrightarrow{\sim} H^1_{\mathrm{B}}(A_x, \mathbb{Q}) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p) \tag{3.1.1.4}$$

will be what we want. Under such isomorphism, the Galois representation $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{ur}}) \to T_{\mathbb{Q}_p}$ (3.1.1.3) is identified with the one attached to the pair (T_x, h_x) by the same procedure in [Kottwitz 1992, §12], where $T_x := \mathrm{Int}(\theta)(T) \subset \mathrm{GL}(H^1_{\mathrm{B}}(A_x, \mathbb{Q}))$. Therefore, given these, the claim at hand follows readily from a result in the theory of complex multiplication of Shimura and Taniyama, which says that when $\iota_p(A_x)$ is defined over a finite extension F of $\mathbb{Q}_p$, the canonical Galois representation $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{GL}(H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p))$ composed with the isomorphism $F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ in local class field theory (normalized in such a way that the geometric Frobenius automorphisms map to uniformizers) is

$$r_{T_x,\mu_{h_x}}|_{E(T,h)^\times_{\mathfrak{p}}} \circ \mathrm{Nm}_{F/E(T,h)_{\mathfrak{p}}} : F^\times \to E(T,h)^\times_{\mathfrak{p}} \to (T_x)_{\mathbb{Q}_p},$$

where $E(T, h)$ is the reflex field of (T, μ_h) and $\mathfrak{p}$ is the place of $E(T, h)$ defined by $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (so that $E(T, h)_{\mathfrak{p}} \subset F$); see [Chai et al. 2014, A.2.5.3, A.2.5.8, (1), and A.2.4.5].

From $\Xi_{T_{\mathbb{Q}_p},\mu_h}$ (defined for $(T', F, \mu') = (T_{\mathbb{Q}_p}, F, \mu_h)$), and the obvious fiber functor $\mathcal{CRYS} \to \mathcal{VEC}_L : (U, \Phi) \mapsto U$, we get a fiber functor over L: $\omega_1 : \mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p}) \to \mathcal{VEC}_L$. By Steinberg's theorem ($H^1(L, T_L) = \{0\}$), this nonstandard fiber functor is isomorphic to the standard fiber functor $\omega_0 : \mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p}) \to \mathcal{VEC}_L : (\rho, V) \mapsto V \otimes_{\mathbb{Q}_p} L$; we fix such an isomorphism $f : \omega_1 \xrightarrow{\sim} \omega_0$. Then, we have seen that there exists $b \in T(L)$ such that

$$(H_1^{\mathrm{cris}}(A_z/L), \Phi_1) \simeq (W \otimes L, \rho_W(b)(\mathrm{id}_W \otimes \sigma)),$$

via the induced isomorphism $f(W_{\mathbb{Q}_p}) : \omega_1(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L) \xrightarrow{\sim} \omega_0(W_{\mathbb{Q}_p}) = W_{\mathbb{Q}_p} \otimes L$, and that $[b] = -\underline{\mu}' \in X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ under the isomorphism $\kappa_{T'} : B(T') \xrightarrow{\sim} X_*(T')_{\mathrm{Gal}_{\mathbb{Q}_p}}$ (2.1.3.1). Also note in passing that $H^1_{\mathrm{cris}}(A_x/L) \simeq (W^\vee \otimes L, \rho_{W^\vee}(b)(\mathrm{id}_{W^\vee} \otimes \sigma))$. Therefore, to prove statement (1), we only need to show that for any isomorphism $f : \omega_1 \xrightarrow{\sim} \omega_0$ of fiber functors, $f(W_{\mathbb{Q}_p}) : H_1^{\mathrm{cris}}(A_z/L) \xrightarrow{\sim} W_L$ qualifies as an isomorphism of (2.2.5.1) (i.e., an isomorphism compatible with tensors). We remark that in the PEL-type setting, the tensors involved were morphisms in (abelian) categories, so this was obvious. In a general Hodge-type case, however, the proof of this fact requires a nontrivial fact [Blasius 1994] that the Hodge cycles on abelian varieties over number fields are de Rham.

In more detail, we consider each tensor s_α as a morphism $s_\alpha : \mathbf{1} \to W^{\otimes}_{\mathbb{Q}_p}$ in $\mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p})$, where $\mathbf{1}$ is the trivial representation $\mathbb{Q}_p$ (which is an identity object of the tensor category $\mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p})$) and $W^{\otimes}_{\mathbb{Q}_p}$ denotes some specific object of $\mathcal{REP}_{\mathbb{Q}_p}(T_{\mathbb{Q}_p})$. Applying the functor $\Xi_{T_{\mathbb{Q}_p},\mu_h}$, we get a morphism $s'_\alpha : \Xi_{T_{\mathbb{Q}_p},\mu_h}(\mathbf{1}) \to \Xi_{T_{\mathbb{Q}_p},\mu_h}(W^{\otimes}_{\mathbb{Q}_p})$ in $\mathcal{CRYS}$. As $\Xi_{T_{\mathbb{Q}_p},\mu_h}(\mathbf{1})$ is the trivial F-isocrystal (L, σ), again this is equivalently regarded as a crystalline tensor on $\Xi_{T_{\mathbb{Q}_p},\mu_h}(W_{\mathbb{Q}_p}) = H_1^{\mathrm{cris}}(A_z/L)$. Moreover, as $f : \omega_1 \to \omega_0$ is an isomorphism of fiber functors, we have a commutative diagram of L-vector spaces:

$$\begin{tikzcd}
\omega_1(\mathbf{1}) \arrow[r, "s'_\alpha"] \arrow[d, "f(\mathbf{1})"', "\simeq"] & \omega_1(W^{\otimes}_{\mathbb{Q}_p}) \arrow[d, "\simeq"', "f(W^{\otimes}_{\mathbb{Q}_p})"] \\
\omega_0(\mathbf{1}) \arrow[r, "s_\alpha"] & \omega_0(W^{\otimes}_{\mathbb{Q}_p})
\end{tikzcd}$$

So, it remains to check that the crystalline tensor s'_α on $H^1_{\mathrm{cris}}(A_z/L)$ equals the crystalline cycle $s_{\alpha,0,z}$, i.e., the de Rham cycle $s_{\alpha,\mathrm{dR},x}$ on $H^1_{\mathrm{dR}}(A_x/L)$ (under the canonical isomorphism $H^1_{\mathrm{cris}}(A_z/L) = H^1_{\mathrm{dR}}(A_x/L)$). First, it follows from the previous discussion that the image of the tensor $s_\alpha : \mathbf{1} \to W^{\otimes}_{\mathbb{Q}_p}$ under the functor $\rho^*_{T_{\mathbb{Q}_p},\mu_h}$ is its image under our chosen identification $W^\vee_{\mathbb{Q}_p} \xrightarrow{\sim} H^1_{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)$ (3.1.1.4). As this identification is induced from an isomorphism $W^\vee \xrightarrow{\sim} H^1_{\mathrm{B}}(A_x, \mathbb{Q})$ provided by the moduli interpretation of $\mathrm{Sh}_{K_pK^p}(G, X)(\mathbb{C})$, the image tensor equals the tensor $s_{\alpha,p,\sigma_\infty(x)} : \mathbf{1} \to H_1^{\text{ét}}(\iota_p(A_x), \mathbb{Q}_p)^{\otimes}$. This is the p-adic component of the absolute Hodge cycle $(s_{\alpha,\mathrm{dR},x}, (s_{\alpha,l,\sigma_\infty(x)})_l)$. Then, since the étale and the de Rham component of an absolute Hodge cycle on an abelian variety over a number field match under the p-adic comparison isomorphism, from which the second functor $\mathbb{D}$ is induced [Blasius 1994, Theorem 0.3], the image of $s_{\alpha,p,\sigma_\infty(x)}$ under the functor $\mathbb{D}$ must be $s_{\alpha,\mathrm{dR},x} : \mathbf{1} \to H_1^{\mathrm{dR}}(A_x/L)^{\otimes}$. So, we have just proved that $s'_\alpha = \Xi_{\mu_h}(s_\alpha) = s_{\alpha,\mathrm{dR},x}$, and consequently statement (1).

(2) The statement on the field of definition is an easy consequence of the reciprocity law characterizing the canonical model of Shimura varieties [Deligne 1979, 2.2] and local class field theory. For any compact open subgroup K' contained in $T(\mathbb{A}_f) \cap K$, there is a natural map $\mathrm{Sh}_{K'}(T, h) \to \mathrm{Sh}_K(G, X) \otimes_{E(G,X)} E(T, h)$ of weakly canonical models of Shimura varieties [Deligne 1979, 2.2.5]. Hence, it suffices to show that for the compact open subgroup $K' = T(\mathbb{Z}_p) \times K'^p$ for any $K'^p \subset T(\mathbb{A}^p) \cap K^p$, the connected components of the finite scheme $\mathrm{Sh}_{K'}(T, h)$ over $E(T, h)$ are defined over an (abelian) extension E' of $E(T, h)$ such that the prime $\mathfrak{p}'$ of E' induced by ι_p is unramified over $\wp$. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$ (which factors through

the topological abelianization $\mathrm{Gal}(E(T,h)^{\mathrm{ab}}/E(T,h)))$ on the group $\mathrm{Sh}_{K'}(T,h)(\overline{\mathbb{Q}}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K'$ is given by (the adelic points of) the reciprocity map [Deligne 1979, 2.2]

$$r_{T,\mu_h} : \mathrm{Res}_{E(T,h)/\mathbb{Q}}(\mathbb{G}_{m\,E(T,h)}) \xrightarrow{\mathrm{Res}_{E(T,h)/\mathbb{Q}}(\mu_h)} \mathrm{Res}_{E(T,h)/\mathbb{Q}}(T_{E(G,h)}) \xrightarrow{N_{E(T,h)/\mathbb{Q}}} T \tag{3.1.1.5}$$

via class field theory: for $\rho \in \mathrm{Gal}(E(T,h)^{\mathrm{ab}}_{\mathfrak{p}}/E(T,h)_{\mathfrak{p}}) = E(T,h)^{\times}_{\mathfrak{p}}$ and $[t]_{K'} \in \mathrm{Sh}_{K'}(T,h)(\overline{E_{\mathfrak{p}}}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K'$, one has

$$\rho[t] = [t \cdot r_{T,\mu_h}(\rho)^{-1}],$$

where one uses the convention that under the identification $\mathrm{Gal}(E(T,h)^{\mathrm{ab}}_{\mathfrak{p}}/E(T,h)_{\mathfrak{p}}) = E(T,h)^{\times}_{\mathfrak{p}}$, the *geometric Frobenius* corresponds to a uniformizer of $E(T,h)_{\mathfrak{p}}$. Obviously, the image in $T(\mathbb{Q}_p)$ under the map r_{T,μ_h} of the unit group $(\mathcal{O}_{E(T,h)} \otimes \mathbb{Z}_p)^{\times}$ is contained in the maximal compact open subgroup $T(\mathbb{Z}_p)$. As $\mathrm{Gal}(E(T,h)^{\mathrm{ab}}_{\mathfrak{p}}/E(T,h)^{\mathrm{ur}}_{\mathfrak{p}}) = (\mathcal{O}_{E(T,h)})^{\times}_{\mathfrak{p}} \subset (\mathcal{O}_{E(T,h)} \otimes \mathbb{Z}_p)^{\times}$, this proves the claim. Then, by the extension property [Kisin 2010, 2.3.7] and Remark 2.2.2 of the integral canonical model, x extends to an $\mathcal{O}_F$-valued point of $\mathcal{S}_K$. □

Note that when the conditions of (2) hold, the "good reduction at ι_p" of the special point follows from an intrinsic property of the integral canonical model (i.e., the extension property), rather than its construction.

3.2. *Construction of special Shimura data with prescribed F-isocrystals.*

Theorem 3.2.1. *Let (G, X) be a Shimura datum (not necessarily of Hodge type) such that $G_{\mathbb{Q}_p}$ is unramified and $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ a hyperspecial subgroup of $G(\mathbb{Q}_p)$.*

(1) *For every $[b] \in B(G_{\mathbb{Q}_p}, X)$, there exists a special Shimura subdatum $(T, h : \mathbb{S} \to T_{\mathbb{R}})$ such that*

$$\bar{\nu}_{G_{\mathbb{Q}_p}}([b]) = -\frac{1}{[F:\mathbb{Q}_p]}\mathrm{Nm}_{F/\mathbb{Q}_p}(\mu_h).$$

Here, the right-hand side is the image of the corresponding quasicocharacter under the canonical map $X_*(T)^{\mathrm{Gal}_{\mathbb{Q}_p}}_{\mathbb{Q}} = \mathcal{N}(T_{\mathbb{Q}_p}) \to \mathcal{N}(G_{\mathbb{Q}_p})$ and $F \subset \overline{\mathbb{Q}}_p$ is the field of definition of $\mu \in X_*(T_{\mathbb{Q}_p})$.

(2) *If (G, X) is of Hodge type, for every $[b] \in B(G_{\mathbb{Q}_p}, X)$, there exists a special Shimura subdatum (T, h) such that for any $g_f \in G(\mathbb{A}_f)$, the reduction (at ι_p) of the special point $[h, g_f]_{K_p} \in \mathrm{Sh}_{K_p}(G, X)(\overline{\mathbb{Q}})$ has the F-isocrystal equal to $[b]$.*

(3) *Furthermore, we may find such a special Shimura subdatum (T, h) with the additional properties that $T_{\mathbb{Q}_p}$ is unramified and $T(\mathbb{Z}_p) \subset K_p$.*

The additional properties in statement (3) specify the prime-to-p isogeny class of the reduction of a special point defined by such (T, h), not just its $\mathbb{Q}$-isogeny class; such finer information has an amusing application (see Corollary 3.2.3).

Proof. (1) Our proof consists of two steps. Fix a rational prime $l \neq p$.

Step 1. We claim that for an *arbitrary* maximal $\mathbb{Q}_p$-torus T' of $G_{\mathbb{Q}_p}$, there exist a maximal $\mathbb{Q}$-torus T_0 of G such that $(T_0)_{\mathbb{Q}_v}$ is elliptic in $G_{\mathbb{Q}_v}$ for $v = \infty, l$ and $(T_0)_{\mathbb{Q}_p} = \mathrm{Int}(g_p)(T')$ for some $g_p \in G(\mathbb{Q}_p)$. Then, for such T_0, we will shortly find $\mu \in X_*(T_0)$ satisfying the following conditions:

$$\mu \in X_*(T_0) \cap c(G, X) \quad \text{and} \quad \bar{\nu}_{G_{\mathbb{Q}_p}}([b]) = \frac{1}{[F : \mathbb{Q}_p]} \mathrm{Nm}_{F/\mathbb{Q}_p}(\mu). \tag{3.2.1.1}$$

Here, in the equation on the right, F is any extension of $\mathbb{Q}_p$ splitting $(T_0)_{\mathbb{Q}_p}$; note that the quasicocharacter itself remains the same if we take F to be a field of definition of μ.

First, to find a maximal $\mathbb{Q}$-torus T_0 of G with the required properties, we consider the following three sets:

$X_v = \{x \in G^{\mathrm{sc}}(\mathbb{Q}_v) \mid x$ is regular semisimple and $\mathrm{Cent}_G(x)$ is elliptic in $G_{\mathbb{Q}_v}\}, \quad (v = \infty, l)$;

$X_p = \{x \in G^{\mathrm{sc}}(\mathbb{Q}_p) \mid x$ is regular semisimple and $\mathrm{Cent}_G(x) = \mathrm{Int}(g_p)(T')$ for some $g_p \in G(\mathbb{Q}_p)\}$.

These sets are all nonempty: for $v = \infty$, this follows from Deligne's condition on Shimura data [Deligne 1979, (2.1.1.2)], and for $v = l$, from [Platonov and Rapinchuk 1994, Theorem 6.21]. They are also open in the real, l-adic, and p-adic topology, respectively: for two sufficiently close regular semisimple elements, their centralizers are conjugated (cf. [Bültel 2001, proof of Lemma 3.1]). So, by the weak approximation theorem, there exists an element $x \in G^{\mathrm{sc}}(\mathbb{Q})$ which lies in all of them. Its centralizer $T_0 := \mathrm{Cent}_G(x)$ is then the desired maximal torus.

Secondly, existence of a cocharacter $\mu \in X_*(T_0)$ satisfying the conditions (3.2.1.1) is established in Lemma 5.11 of [Langlands and Rapoport 1987]. (See [Lee 2016, Proposition 4.2.4] for another proof.) We remark that this proof of Langlands–Rapoport uses, as the input condition, nonemptiness of the affine Deligne–Lusztig variety $X(c(G, X), b)_{K_p}$ attached to the tuple $(G_{\mathbb{Q}_p}, c(G, X), K_p, [b])$:

$$X(c(G, X), b)_{K_p} := \{g G_{\mathbb{Z}_p}(\mathcal{O}_L) \in G(L)/G_{\mathbb{Z}_p}(\mathcal{O}_L) \mid g^{-1} b\sigma(g) \in G_{\mathbb{Z}_p}(\mathcal{O}_L)\mu(p)G_{\mathbb{Z}_p}(\mathcal{O}_L)\},$$

where μ is any morphism $\mathbb{G}_{\mathrm{m},\mathcal{O}_L} \to G_{\mathbb{Z}_p} \otimes \mathcal{O}_L$ lying in $c(G, X)$. But [Wintenberger 2005] shows that $[b] \in B(G_{\mathbb{Q}_p}, c(G, X))$ implies such nonemptiness.

Step 2. Next, for any maximal $\mathbb{Q}$-torus T_0 of G, elliptic at $\mathbb{R}$, and each cocharacter $\mu \in X_*(T_0) \cap c(G, X)$, we claim that there exists $u \in G(\overline{\mathbb{Q}})$ such that

(i) $\mathrm{Int}\, u : (T_0)_{\overline{\mathbb{Q}}} \hookrightarrow G_{\overline{\mathbb{Q}}}$ is defined over $\mathbb{Q}$ and the $\mathbb{Q}$-torus $T := \mathrm{Int}\, u(T_0)$ is elliptic in G over $\mathbb{R}$,

(ii) $\mathrm{Int}\, u(\mu)$ equals μ_h^{-1} for some $h \in \mathrm{Hom}(\mathbb{S}, T_{\mathbb{R}}) \cap X$.[1]

Moreover, if $(T_0)_{\mathbb{Q}_l}$ is elliptic in $G_{\mathbb{Q}_l}$ for some $l \neq p$, there exist such $u \in G(\overline{\mathbb{Q}})$ and $h \in X$ which satisfy, in addition to these properties, that

(iii) there exists $y \in G(\mathbb{Q}_p)$ such that $(T_{\mathbb{Q}_p}, \mu_h^{-1}) = \mathrm{Int}\, y((T_0)_{\mathbb{Q}_p}, \mu)$.

Our proof is an adaptation of the argument of [Langlands and Rapoport 1987, Lemma 5.12]. Let T_0^{sc} be the inverse image of $(T_0 \cap G^{\mathrm{der}})^0$ under the canonical isogeny $G^{\mathrm{sc}} \to G^{\mathrm{der}}$, and pick any morphism $h_0 \in X$

[1] Recall our convention that $c(G, X)$ is the conjugacy class of cocharacters into $G_{\mathbb{C}}$ containing μ_h^{-1} for some $h \in X$.

factoring through $(T_0)_{\mathbb{R}}$, which exists since $(T_0)_{\mathbb{R}}$ is elliptic in $G_{\mathbb{R}}$ and any two elliptic maximal tori in any connected reductive real algebraic group H are conjugate under $H(\mathbb{R})$. Choose $w \in N_G(T_0)(\mathbb{C})$ such that $\mu = w(\mu_{h_0}^{-1})$; we may find such a w in $N_{G^{\mathrm{sc}}}(T_0^{\mathrm{sc}})(\mathbb{C})$ as the Weyl groups of G and G^{sc} are the same. This gives us a cocycle $\alpha^\infty \in Z^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G^{\mathrm{sc}}(\mathbb{C}))$ defined by

$$\alpha_\iota^\infty = w \cdot \iota(w^{-1})$$

for all $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. As a matter of fact, this has values in $T_0^{\mathrm{sc}}(\mathbb{C})$. Indeed, by [Shelstad 1979, Proposition 2.2], the automorphism $\mathrm{Int}(w^{-1})$ of $(T_0^{\mathrm{sc}})_{\mathbb{C}}$ is defined over $\mathbb{R}$, so

$$\mathrm{Int}(w^{-1})(\iota(t)) = \iota(\mathrm{Int}(w^{-1})t) = \mathrm{Int}(\iota(w^{-1}))(\iota(t))$$

for all $t \in T_0^{\mathrm{sc}}(\mathbb{C})$, i.e., $\iota(w)w^{-1} \in \mathrm{Cent}_{G^{\mathrm{sc}}}(T_0^{\mathrm{sc}})(\mathbb{C}) = T_0^{\mathrm{sc}}(\mathbb{C})$, and so is $w\iota(w^{-1}) = \iota(\iota(w)w^{-1})$. Then, by Lemma 7.16 of [Langlands 1983], there exists a global cocycle

$$\alpha \in Z^1(\mathbb{Q}, T_0^{\mathrm{sc}})$$

mapping to $\alpha^\infty \in H^1(\mathbb{R}, T_0^{\mathrm{sc}})$. When $(T_0)_{\mathbb{Q}_l}$ is elliptic in $G_{\mathbb{Q}_l}$ for some $l \neq p$, then the following lemma ensures the existence of a cocycle $\alpha \in Z^1(\mathbb{Q}, T_0^{\mathrm{sc}})$ mapping to $\alpha^\infty \in H^1(\mathbb{R}, T_0^{\mathrm{sc}})$ and further to zero in $H^1(\mathbb{Q}_p, T_0^{\mathrm{sc}})$. For a finitely generated abelian group A, let A_{tors} be the subgroup of its torsion elements.

Lemma 3.2.2. (1) *For a maximal $\mathbb{Q}$-torus T of G which is elliptic at $l \neq p$, the natural map*

$$(\pi_1(T^{\mathrm{sc}})_{\Gamma(l)})_{\mathrm{tors}} \to (\pi_1(T^{\mathrm{sc}})_{\Gamma})_{\mathrm{tors}}$$

is surjective, where we write $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma(l) = \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$.

(2) *There exists $\alpha \in Z^1(\mathbb{Q}, T_0^{\mathrm{sc}})$ mapping to $\alpha^\infty \in H^1(\mathbb{R}, T_0^{\mathrm{sc}})$ and to zero in $H^1(\mathbb{Q}_p, T_0^{\mathrm{sc}})$.*

Proof of lemma. First, we recall some (standard) notations: for a torus T over a field F, $\widehat{T}$ denotes the dual torus of T, and for an abelian topological group A, A^D denotes its dual group $\mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{C}^\times)$ (group of continuous homomorphisms).

(1) Since this map is induced (by restriction) by the natural surjection $\pi_1(T^{\mathrm{sc}})_{\Gamma(l)} \twoheadrightarrow \pi_1(T^{\mathrm{sc}})_{\Gamma}$, it suffices to prove that $\pi_1(T^{\mathrm{sc}})_{\Gamma(l)}$ is a torsion group. But, since $T^{\mathrm{sc}}_{\mathbb{Q}_l}$ is anisotropic, $\widehat{T^{\mathrm{sc}}}^{\Gamma(l)}$ is a finite group, and so is

$$\pi_1(T^{\mathrm{sc}})_{\Gamma(l)} = X^*(\widehat{T^{\mathrm{sc}}}^{\Gamma(l)}) = \mathrm{Hom}(\widehat{T^{\mathrm{sc}}}^{\Gamma(l)}, \mathbb{C}^\times).$$

(2) For any place v of $\mathbb{Q}$, there exists a canonical isomorphism

$$H^1(F, T^{\mathrm{sc}}) \xrightarrow{\sim} \mathrm{Hom}(\pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma(v)}), \mathbb{C}^\times) = ((\widehat{T^{\mathrm{sc}}}^{\Gamma(v)})^D)_{\mathrm{tors}} \cong (\pi_1(T^{\mathrm{sc}})_{\Gamma(v)})_{\mathrm{tors}}$$

[Kottwitz 1984, (3.3.1); Rapoport and Richartz 1996, 1.14], and a short exact sequence

$$H^1(\mathbb{Q}, T^{\mathrm{sc}}) \to H^1(\mathbb{Q}, T^{\mathrm{sc}}(\overline{\mathbb{A}})) := \bigoplus_v H^1(\mathbb{Q}_v, T^{\mathrm{sc}}) \xrightarrow{\theta} \pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma})^D = (\pi_1(T^{\mathrm{sc}})_{\Gamma})_{\mathrm{tors}}$$

[Kottwitz 1986, Proposition 2.6]. Here, θ is the composite

$$\bigoplus_v H^1(\mathbb{Q}_v, T^{\mathrm{sc}}) \xrightarrow{\sim} \bigoplus_v \pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma(v)})^D \to \pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma})^D,$$

of which the second map is identified with the natural map $\bigoplus_v (\pi_1(T^{\mathrm{sc}})_{\Gamma(v)})_{\mathrm{tors}} \twoheadrightarrow (\pi_1(T^{\mathrm{sc}})_{\Gamma})_{\mathrm{tors}}$ (thus, it equals the direct sum of the maps considered in the statement of (1)). Hence, by (1), there exists a class $\alpha^l \in H^1(\mathbb{Q}_l, T^{\mathrm{sc}})$ such that $\theta(\alpha^l) = -\theta(\alpha^\infty)$, and thus the element $(\beta^v)_v \in H^1(\mathbb{Q}, T^{\mathrm{sc}}(\overline{\mathbb{A}}))$ defined by the condition that $\beta^v = \alpha^v$ for $v = \infty, l$ and $\beta^v = 0$ for $v \neq l, \infty$ in $\pi_0(\widehat{T^{\mathrm{sc}}}^{\Gamma})^D$. By exactness of the sequence, we conclude existence of $\alpha \in Z^1(\mathbb{Q}, T^{\mathrm{sc}})$ whose cohomology class maps to $(\beta^v)_v$. □

Now, by changing α^∞ and w further, we may assume that α^∞ *is* the restriction of α to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then, using the fact that the restriction map $H^1(\mathbb{Q}, G^{\mathrm{sc}}) \to H^1(\mathbb{R}, G^{\mathrm{sc}})$ is injective (which follows from the Hasse principle [Platonov and Rapinchuk 1994, Theorem 6.6] and the fact that $H^1(\mathbb{Q}_v, G^{\mathrm{sc}}) = 0$ for any nonarchimedean place v [Platonov and Rapinchuk 1994, Theorem 6.4]), we see that α becomes trivial as a cocycle with values in $G^{\mathrm{sc}}(\overline{\mathbb{Q}})$. We summarize this discussion in the diagram:

$$\begin{array}{ccc} H^1(\mathbb{R}, T_0^{\mathrm{sc}}) & \longrightarrow & H^1(\mathbb{R}, G^{\mathrm{sc}}) \\ \uparrow & & \uparrow \\ H^1(\mathbb{Q}, T_0^{\mathrm{sc}}) & \longrightarrow & H^1(\mathbb{Q}, G^{\mathrm{sc}}) \end{array} \qquad \begin{array}{ccc} \alpha^\infty & \longmapsto & 0 \\ \uparrow & & \uparrow \\ \exists\, \alpha & \longmapsto & \alpha' \end{array} \quad \Rightarrow \quad \alpha' = 0.$$

So there exists $u \in G^{\mathrm{sc}}(\overline{\mathbb{Q}})$ such that for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\alpha_\rho = u^{-1}\rho(u).$$

Then the homomorphism $\mathrm{Int}\, u : (T_0)_{\overline{\mathbb{Q}}} \to G_{\overline{\mathbb{Q}}} : t \mapsto t' = utu^{-1}$ is defined over $\mathbb{Q}$; in particular, $T := uT_0u^{-1}$ is a torus defined over $\mathbb{Q}$, and as the restriction of $\mathrm{Int}\, u$ to $Z(G)$ is the identity, $T_{\mathbb{R}}$ is also elliptic in $G_{\mathbb{R}}$. Moreover, since $u^{-1}\iota(u) = w\iota(w^{-1})$ for $\iota \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, one has that $uw \in G^{\mathrm{sc}}(\mathbb{R})$ and

$$\mathrm{Int}\, u(\mu) = \mathrm{Int}(uw)\mu_{h_0}^{-1} = \mu_h^{-1}$$

for $h := \mathrm{Int}(uw)(h_0) \in X \cap \mathrm{Hom}(\mathbb{S}, T)$. This establishes the claims (i) and (ii). The proof of claim (iii) is similar. As the restriction of α to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is trivial, there exists $x \in T(\overline{\mathbb{Q}}_p)$ such that $x\rho(x^{-1}) = \alpha_\rho = u^{-1}\rho(u)$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, i.e., $ux \in G(\mathbb{Q}_p)$. But the homomorphism $\mathrm{Int}\, u : (T_0)_{\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ also equals $\mathrm{Int}\, u = \mathrm{Int}\, ux$. This proves (iii).

Now we complete the proof of Theorem 3.2.1(1). Let $[b] \in B(G_{\mathbb{Q}_p}, X)$. For the first statement, we take an arbitrary maximal $\mathbb{Q}$-torus T_0 of G that is elliptic over $\mathbb{R}$, and find a cocharacter μ of T_0 having the properties (3.2.1.1); we remark that Lemma 5.1 of [Langlands and Rapoport 1987], which was used to find such a μ_0, works for any maximal $\mathbb{Q}_p$-torus of $G_{\mathbb{Q}_p}$ (in particular, for $(T_0)_{\mathbb{Q}_p}$). From such a pair (T_0, μ), we get a special Shimura subdatum (T, h) with the properties (i) and (ii) of Step 2. This special Shimura subdatum (T, h) satisfies the condition of the first statement, since $\mathrm{Int}(u) : T_0 \to T$ is defined over $\mathbb{Q}$ (for $u \in G(\overline{\mathbb{Q}})$ as chosen in the proof) so that $\mathrm{Int}(u)(\mathrm{Nm}_{F/\mathbb{Q}_p}(\mu)) = \mathrm{Nm}_{F/\mathbb{Q}_p}(\mu_h^{-1})$.

(2) Note that the quasicocharacter $-\frac{1}{[F:\mathbb{Q}_p]}\mathrm{Nm}_{F/\mathbb{Q}_p}(\mu_h)$ equals the image of $-\underline{\mu_h}$ under the natural map $B(T_{\mathbb{Q}_p}) \cong X_*(T)_{\mathrm{Gal}_{\mathbb{Q}_p}} \to \mathcal{N}(T_{\mathbb{Q}_p}) = X_*(T)_{\mathbb{Q}}^{\mathrm{Gal}_{\mathbb{Q}_p}}$: use the fact that there exists a commutative diagram

$$\begin{tikzcd} B(G) \arrow[r, "\bar{\nu}_G"] \arrow[d, "\kappa_G"'] & \mathcal{N}(G) \arrow[d, "\delta_G"] \\ \pi_1(G)_{\mathrm{Gal}_F} \arrow[r, "\alpha_G"] & \pi_1(G)_{\mathbb{Q}}^{\mathrm{Gal}_F} \end{tikzcd}$$

(for any reductive group G over a p-adic field F). Here, the map α_G in the bottom is given by

$$\bar{\mu} \mapsto |\mathrm{Gal}_F \cdot \mu|^{-1} \sum_{\mu' \in \mathrm{Gal}_F \cdot \mu} \mu'$$

(cf. [Kottwitz 1985, 2.8; Rapoport and Richartz 1996, Theorem 1.15]). Therefore, since the map $(\bar{\nu}, \kappa) : B(G_{\mathbb{Q}_p}) \to \mathcal{N}(G_{\mathbb{Q}_p}) \times \pi_1(G)_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is injective [Kottwitz 1997, 4.13], the second statement follows from the first statement and Lemma 3.1.1.

(3) Finally, for the existence of a special Shimura subdatum (T, h) with the additional property on the position of the hyperspecial subgroup $T(\mathbb{Z}_p)$, we begin with an unramified maximal $\mathbb{Q}_p$-torus T' of $G_{\mathbb{Q}_p}$ such that the unique hyperspecial subgroup $T'(\mathbb{Z}_p)$ of $T'(\mathbb{Q}_p)$ is contained in K_p: such a torus exists by Lemma 3.2.4 below. Then, from such T', we can find (by Step 1) a pair (T_0, μ) having the properties (3.2.1.1) of Step 1 and, further, such that $(T_0)_{\mathbb{Q}_v}$ is elliptic for $v = \infty$, (some) $l \neq p$ and also that the unique hyperspecial subgroup of $T_0(\mathbb{Q}_p)$ is contained in $\mathrm{Int}(g_p)(K_p)$ for some $g_p \in G(\mathbb{Q}_p)$. For such (T_0, μ), again Step 2 produces a special Shimura datum (T_1, h_1) satisfying the properties (i)–(iii) of Step 2. Now, since $G(\mathbb{Q})$ is dense in $G(\mathbb{Q}_p)$ [Milne 1994, Lemma 4.10], there exists $g \in G(\mathbb{Q}) \cap K_p \cdot g_p^{-1}$. Then the new special Shimura datum $(T, h) := \mathrm{Int}(g)(T_1, h_1)$ still satisfies the condition in the first statement, and further, the unique hyperspecial subgroup of $T(\mathbb{Q}_p)$ is contained in K_p. □

Corollary 3.2.3. *Let (G, X) be a Shimura datum of Hodge type. Choose an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and let $\wp$ be the prime of $E(G, X)$ induced by ι_p. Suppose that $G_{\mathbb{Q}_p}$ is unramified and choose a hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$. Then the reduction $\mathcal{S}_{K_p}(G, X) \times \kappa(\wp)$ has nonempty ordinary locus if and only if $\wp$ has absolute height one* (*i.e.*, *$E(G, X)_\wp = \mathbb{Q}_p$*). *In this case, if $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p})$ is an embedding of $\mathbb{Z}_p$-group schemes* (see [Kisin 2010, (2.3.1)]), *there exists a special point which is the canonical lifting of its reduction.*

Recall from 2.2.1 that in the construction of the integral canonical model, an embedding $G \hookrightarrow \mathrm{GSp}(W, \psi)$ is chosen so as to satisfy the condition in the corollary.

Proof. The necessity of $E(G, X)_\wp = \mathbb{Q}_p$ was proved in [Bültel 2001, Section 3]. For sufficiency, we note that it suffices to find a special Shimura subdatum (T, h) such that

there exists a Borel subgroup B over $\mathbb{Q}_p$ of $G_{\mathbb{Q}_p}$ containing $T_{\mathbb{Q}_p}$ and such that $\mu_h \in X_*(T)$ lies in the closed Weyl chamber determined by $(T_{\mathbb{Q}_p}, B)$. (∗)

Indeed, then the canonical Galois action of $\mathrm{Gal}_{\mathbb{Q}_p}$ on $X_*(T)$ coincides with the naive Galois action; hence, one has that $E(G, X)_\wp = E(T, h)_\mathfrak{p}$, where $\mathfrak{p}$ is the prime of $E(T, h)$ induced by ι_p. Given this, the corollary then follows from [Bültel 2001, Lemma 2.2]: for a CM-abelian variety A over $\overline{\mathbb{Q}}$ ($\subset \mathbb{C}$) with CM-type (F, Φ), where F is a CM-algebra and Φ is a subset of $\mathrm{Hom}(F, \mathbb{C})$ with $\Phi \sqcup \iota \circ \Phi = \mathrm{Hom}(F, \mathbb{C})$, the reduction of A at $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ is an ordinary abelian variety if and only if the prime of the reflex field $E = E(F, \Phi)$ induced by ι_p is of absolute height one; when A is defined by a special Shimura datum (T, h), the reflex field $E(F, \Phi)$ equals the reflex field $E(T, h)$ defined earlier.

Now, pick an arbitrary maximal $\mathbb{Q}_p$-torus T' of $G_{\mathbb{Q}_p}$ containing a maximal $\mathbb{Q}_p$-split torus. As $G_{\mathbb{Q}_p}$ is quasisplit, there exists a Borel subgroup B' defined over $\mathbb{Q}_p$ which contains T'. Let μ' be the cocharacter in $c(G, X)$ factoring through T' and lying in the closed Weyl chamber determined by (T', B'). Let (T, h) be a special Shimura subdatum produced from (T', μ') as in the proof of Theorem 3.2.1 and such that $(T_{\mathbb{Q}_p}, \mu_h)$ is conjugate to (T', μ') under $G(\mathbb{Q}_p)$. Then, for such (T, μ_h), the condition $(*)$ continues to hold, which proves the claim.

For the second statement, there exists a special Shimura subdatum (T, h) as in the last statement of Theorem 3.2.1. Then, as μ_h is a morphism of $\mathbb{Z}_p$-group schemes $\mathbb{G}_\mathrm{m} \to G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(W_{\mathbb{Z}_p})$, any special point $[h, 1 \times g_f^p] \in \mathrm{Sh}_{K_p K^p}(G, X)(\overline{\mathbb{Q}})$ (with arbitrary $g_f^p \in G(\mathbb{A}_f^p)$) is the canonical lifting of its reduction, according to [Milne 2006, Proposition 9.24]. □

Lemma 3.2.4. *Let F be a p-adic field with ring of integers $\mathcal{O}_F$. For any reductive group scheme $H_{\mathcal{O}_F}$ over $\mathcal{O}_F$, there exists a reductive $\mathcal{O}_F$-subgroup scheme $S_{\mathcal{O}_F}$ of $H_{\mathcal{O}_F}$ whose generic fiber is a maximal F-split torus of $H := H_{\mathcal{O}_F} \otimes F$. Its centralizer $T'_{\mathcal{O}_F} := \mathrm{Cent}_{H_{\mathcal{O}_F}}(S_{\mathcal{O}_F})$ is a smooth, closed subscheme of $H_{\mathcal{O}_F}$.*

Note that as H is quasisplit, the generic fiber T' of $T'_{\mathcal{O}_F}$ is a maximal F-torus of H.

Proof. Let $\mathcal{B}(H, F)$ denote the Bruhat–Tits building of H over F. Choose a maximal F-split F-torus S of H such that the hyperspecial point x fixed by $K_p := H_{\mathcal{O}_F}(\mathcal{O}_F)$ is contained in the apartment associated with S. Let $S_{\mathcal{O}_F}$ be the (unique) reductive $\mathcal{O}_F$-group scheme with generic fiber S. Then, $S_{\mathcal{O}_F}$ is a closed subscheme of $H_{\mathcal{O}_F}$. Indeed, we first observe that $S_{\mathcal{O}_F}(\mathcal{O}_L)$ maps to $H_{\mathcal{O}_F}(\mathcal{O}_L)$ under the embedding $S \hookrightarrow H$, where L is the completion of the maximal unramified extension (in an algebraic closure $\overline{F}$) of F and $\mathcal{O}_L$ is the ring of integers of L. This follows from these two facts: first, the hyperspecial point x of $\mathcal{B}(H, F)$ is also a hyperspecial point of $\mathcal{B}(H, L)$ with the associated $\mathcal{O}_L$-group scheme being $(H_{\mathcal{O}_F})_{\mathcal{O}_L}$ [Tits 1979, 3.4, 3.8], and secondly, for a maximal L-split torus S_1 of H containing S and defined over F (which exists by [Bruhat and Tits 1984, 5.1.12]), $x \in \mathcal{B}(H, L)$ lies in the apartment corresponding to S_1 [Tits 1979, 1.10]. So, x is fixed under the maximal bounded subgroup $S_1(\mathcal{O}_L)$ of $S_1(L)$, and thus under $S_{\mathcal{O}_F}(\mathcal{O}_L)$ as well. Then, according to [Vasiu 1999, 3.1.2.1], this implies that the F-embedding $S \hookrightarrow H$ extends to an $\mathcal{O}_F$-embedding $S_{\mathcal{O}_F} \hookrightarrow H_{\mathcal{O}_F}$. The fact that $T'_{\mathcal{O}_F} := \mathrm{Cent}_{H_{\mathcal{O}_F}}(S_{\mathcal{O}_F})$ is a smooth closed group $\mathcal{O}_F$-subscheme of $H_{\mathcal{O}_F}$ is well-known [SGA 3_{II} 1970, XI, Corollaire 5.3; Conrad 2014, 2.2]. Moreover, one knows that it represents the functor on $\mathcal{O}_F$-schemes

$$R' \mapsto \{g \in H(R') \mid g(S_{\mathcal{O}_F})_{R'} g^{-1} = (S_{\mathcal{O}_F})_{R'}\},$$

and its generic fiber $T' := \mathrm{Cent}_H(S)$ is a maximal torus of H which splits over L, since H is quasisplit [Tits 1979, 1.10]. $\square$

Remark 3.2.5. (1) In fact, for any special Shimura subdatum (T, h) with the property $(*)$ in the proof, every special point of $\mathrm{Sh}_{K_p}(G, X)$ defined by it has μ-ordinary reduction at ι_p. So, we get a more direct proof of nonemptiness of μ-ordinary locus (without invoking Lemma 5.11 of [Langlands and Rapoport 1987] used in the proof of Theorem 3.2.1).

(2) This corollary was proved in the PEL-type cases of Lie type A or C by Wedhorn [1999]. His argument was to first establish nonemptiness of the μ-ordinary locus, and then to check that the μ-ordinary locus equals the (usual) ordinary locus when $E(G, X)_{\wp} = \mathbb{Q}_p$ (in the PEL-type cases of Lie type A or C). But to check the second statement, he relied on a case-by-case analysis. In contrast, we observe that in view of (1), the equality of the μ-ordinary Newton point and the ordinary Newton point when $E(G, X)_{\wp} = \mathbb{Q}_p$ is simply a consequence of the existence of a special Shimura subdatum (T, h) with the property $(*)$ in the proof (since the reduction of such a special point attains simultaneously the two Newton points when $E(G, X)_{\wp} = \mathbb{Q}_p$).

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Towards Boij–Söderberg theory for Grassmannians: the case of square matrices

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We characterize the cone of GL-equivariant Betti tables of Cohen–Macaulay modules of codimension 1, up to rational multiple, over the coordinate ring of square matrices. This result serves as the base case for "Boij–Söderberg theory for Grassmannians," with the goal of characterizing the cones of GL_k-equivariant Betti tables of modules over the coordinate ring of $k \times n$ matrices, and, dually, cohomology tables of vector bundles on the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$. The proof uses Hall's theorem on perfect matchings in bipartite graphs to compute the extremal rays of the cone, and constructs the corresponding equivariant free resolutions by applying Weyman's geometric technique to certain graded pure complexes of Eisenbud–Fløystad–Weyman.

1. Introduction

1A. ***Ordinary and equivariant Boij–Söderberg theory.*** Let M be a finitely generated $\mathbb{Z}$-graded module over a polynomial ring $A = \mathbb{C}[x_1, \dots, x_n]$. The *Betti table* of M counts the number of generators in each degree of a minimal free resolution of M. More precisely, if M has a graded minimal free resolution of the form

$$M \leftarrow \bigoplus_{d \in \mathbb{Z}} A(-d)^{\oplus \beta_{0d}} \leftarrow \cdots \leftarrow \bigoplus_{d \in \mathbb{Z}} A(-d)^{\beta_{nd}} \leftarrow 0,$$

the Betti table of M is the collection of numbers β_{ij}. Equivalently, β_{ij} is the dimension of the degree-j part of the graded module $\mathrm{Tor}_i^A(M, \mathbb{C})$.

Boij–Söderberg theory (initiated in [Boij and Söderberg 2008]) seeks to characterize the possible Betti tables of graded modules over polynomial rings, with the key insight that it is easier to study these tables only up to positive scalar multiple. The theory has been broadly successful: while the earliest results concerned Betti tables of Cohen–Macaulay modules (stratified by their codimension) [Eisenbud et al. 2011; Eisenbud and Schreyer 2009], the theory was extended to all modules [Boij and Söderberg 2012], to certain modules over multigraded and toric rings [Berkesch et al. 2012; Eisenbud and Erman 2017], and more [Kummini and Sam 2015; Gheorghita and Sam 2016]. For some surveys, see [Fløystad 2012; Fløystad et al. 2016].

In fact, the classification is surprisingly simple. We say a Betti table is *pure* if, for each i, exactly one β_{ij} is nonzero, i.e., each step of the minimal free resolution is concentrated in a single degree. For each

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increasing sequence of integers $d_0 < d_1 < \cdots < d_k$, there is a Cohen–Macaulay A-module whose Betti table is pure with the β_{id_i}'s as the only nonzero entries. Moreover, the resulting Betti table is unique up to a rational multiple, and any purported Betti table is a positive rational multiple of the Betti table of an actual module if and only if it can be written as a positive $\mathbb{Q}$-linear combination of pure tables [Eisenbud and Schreyer 2009]. If we bound the degrees that can occur in a Betti table — that is, bound the j's for which β_{ij} may be nonzero — the Betti tables that fit within these bounds form a rational polyhedral cone. The pure tables then form the extremal rays of this cone.

A key feature of the theory is the discovery that the cone of Betti tables is dual to another cone, consisting of *cohomology tables* of vector bundles and sheaves on projective space. Given such a sheaf $\mathcal{F}$, its cohomology table is the table of numbers $\gamma_{ij} = h^i(\mathcal{F} \otimes \mathcal{O}(j))$. There is a family of nonnegative bilinear pairings between Betti tables of modules and cohomology tables of sheaves, and the inequalities that cut out the cone of Betti tables can all be realized explicitly in terms of this pairing. Consequently, Betti tables yield numerical constraints on the possible cohomology tables on $\mathbb{P}^n$, and vice versa. Recent work of Eisenbud and Erman [2017] has categorified this pairing, realizing it through a functorial pairing between the underlying algebraic objects.

This paper is the beginning of an attempt to generalize this story to GL_k-equivariant modules over a polynomial ring (all GL_k-modules are required to be algebraic representations). Write $R = \mathbb{C}[x_{11}, x_{12}, \ldots, x_{kn}]$, the coordinate ring of the affine space of $k \times n$ matrices with the left GL_k action. In this setting, as we will see in Section 2, a minimal free resolution of a finitely generated equivariant R-module comes with an action of GL_k, so in forming our Betti tables we can ask which representations appear at each step of the resolution rather than just which degrees. Specifically, by analogy with the ordinary case, we wish to understand:

(i) the cone $\mathrm{BS}_{k,n}$ of GL_k-*equivariant Betti tables* of modules supported on the locus of rank-deficient matrices in $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, and

(ii) the cone $\mathrm{ES}_{k,n}$ of GL-*cohomology tables* of vector bundles on the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$.

Both of these constructions will be defined more precisely in Section 2.

Remark 1.1. The case $k = 1$ is the ordinary Boij–Söderberg theory of graded modules and vector bundles on projective space since an algebraic action of GL_1 is equivalent to a choice of $\mathbb{Z}$-grading (see Remark 2.1). We should point out that in this case, [Sam and Weyman 2011] studies a GL_n-equivariant analogue of Boij–Söderberg theory using a Schur-positive analogue of convex cones. In [Sam and Weyman 2011], the equivariant Betti table records characters and the Boij–Söderberg cone is defined to be closed under "Schur positive rational functions" while in the current work, the equivariant Betti table records multiplicities and the cone has an action of the positive rational numbers instead.

In a later paper, we establish a nonnegative pairing between these tables, extending the pairing of Eisenbud–Schreyer; the hope is that the cones are dual, as they are in ordinary Boij–Söderberg theory. On the algebraic side, we will restrict to Cohen–Macaulay modules supported everywhere along the rank-deficient locus.

A fundamental base case in ordinary Boij–Söderberg theory is to understand Betti tables of torsion graded modules over a polynomial ring $\mathbb{C}[t]$ in one variable. The categorified pairing of Eisenbud–Erman effectively outputs such a module (actually a complex of such modules); as such, the structure of these tables, while very simple, controls the structure of the general Boij–Söderberg cone and its dual. It is relatively straightforward to write down the inequalities that cut out the corresponding cone, and every inequality cutting out the larger cone of Betti tables comes from pulling back one of these through the pairing mentioned earlier.

This paper is concerned with the corresponding base case, namely, the cone of equivariant modules over the coordinate ring of *square* matrices. This case looks simple at first glance: the modules have codimension 1, and the corresponding Grassmannian is just a point, so there is no dual picture involving vector bundles. Unlike in the graded setting, however, the equivariant base case is already both combinatorially and algebraically interesting. Our main result is the following description of this cone:

Theorem 1.2. *In the square matrix case, the supporting hyperplanes of the equivariant Boij–Söderberg cone* $\mathrm{BS}_{k,k}$ *correspond to* ***antichains*** *in the extended Young's lattice* $\mathbb{Y}_{\pm}$ *of weakly decreasing integer sequences. Its extremal rays correspond to* ***pure resolutions*** *and are indexed by* ***comparable pairs*** *of weakly decreasing integer sequences*, $\lambda^{(0)} \subsetneq \lambda^{(1)}$.

For a more precise version of this statement, see Theorem 3.8.

We will exhibit a free resolution to realize each extremal ray, but the construction is nontrivial and relies on existing results of Eisenbud–Fløystad–Weyman from ordinary Boij–Söderberg theory. The proof presented here also depends crucially on the Borel–Weil–Bott theorem, so we do not know if our results hold in positive characteristic.

We expect the description of the cone in Theorem 1.2 to control the structure of the equivariant Boij–Söderberg cone in the general case. In particular, the generalized Eisenbud–Schreyer pairing will map the larger cones $\mathrm{BS}_{k,n}$ and $\mathrm{ES}_{k,n}$ to the square-matrix cone. We sketch this construction in Section 2.

1B. ***Structure of the paper.*** The paper is structured as follows: In Section 2, we introduce the relevant notions, namely, equivariant Betti tables and (briefly) GL-cohomology tables for sheaves on Grassmannians. In Section 3, we describe the combinatorics of the equivariant Boij–Söderberg cone for square matrices. In Section 4, we show that each extremal ray is realizable, using Weyman's geometric technique.

2. Setup

Throughout, let V, W be vector spaces over $\mathbb{C}$ of dimensions k, n respectively, with $k \leq n$. Starting in Section 3, we will assume $n = k$.

2A. ***Background.*** We will only consider algebraic representations of $\mathrm{GL}(V)$. A good introduction to these notions is [Fulton 1996]. We will also refer the reader to [Sam and Snowden 2012, §3] for a succinct summary (with references) of what we'll need about the representation theory of the general linear group.

The irreducible (algebraic) representations of $\mathrm{GL}(V)$ are indexed by weakly decreasing integer sequences $\lambda = (\lambda_1, \dots, \lambda_k)$, where $k = \dim(V)$. We write $\mathbb{S}_\lambda(V)$ for the corresponding representation, called a *Schur functor*. If λ has all nonnegative parts, we write $\lambda \geq 0$ and say λ is a *partition*. We often represent partitions by their Young diagrams:

$$\lambda = (3, 1) \longleftrightarrow \lambda = \ydiagram{3,1}.$$

We partially order partitions and integer sequences by containment:

$$\lambda \subseteq \mu \quad \text{if } \lambda_i \leq \mu_i \text{ for all } i.$$

We write $\mathbb{Y}$ for the poset of all partitions with this ordering, called *Young's lattice*. We write $\mathbb{Y}_\pm$ for the set of all weakly decreasing integer sequences; we call it the *extended Young's lattice*.

If λ is a partition, $\mathbb{S}_\lambda(V)$ is functorial for linear transformations $V \to W$. If λ has negative parts, $\mathbb{S}_\lambda$ is only functorial for isomorphisms $V \xrightarrow{\sim} W$. If $\lambda = (d, 0, \dots, 0)$, then $\mathbb{S}_\lambda(V) = \mathrm{Sym}^d(V)$ and if $\lambda = 1^d = (1, \dots, 1, 0, \dots, 0)$ with d 1's, then $\mathbb{S}_\lambda(V) = \bigwedge^d(V)$. If $\dim V = k$, we'll write $\det(V)$ for the one-dimensional representation $\bigwedge^k(V) = \mathbb{S}_{1^k}(V)$. We write $K_\lambda(k)$ for the dimension of $\mathbb{S}_\lambda(\mathbb{C}^k)$.

We may always twist a representation by powers of the determinant:

$$\det(V)^{\otimes a} \otimes \mathbb{S}_{\lambda_1,\dots,\lambda_k}(V) = \mathbb{S}_{\lambda_1+a,\dots,\lambda_k+a}(V)$$

for any integer $a \in \mathbb{Z}$. This operation is invertible and can sometimes be used to reduce to considering the case when λ is a partition.

By semisimplicity, any tensor product of Schur functors is isomorphic to a direct sum of Schur functors with some multiplicities:

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) \cong \bigoplus_\nu \mathbb{S}_\nu(V)^{\oplus c^\nu_{\lambda,\mu}}.$$

The $c^\nu_{\lambda,\mu}$ are the *Littlewood–Richardson coefficients*; we won't need to know how they are computed in general, though we will use that if $c^\nu_{\lambda,\mu} \neq 0$ and λ is a partition, then $\mu \subseteq \nu$ (and similarly, if μ is a partition, then $\lambda \subseteq \nu$). Also, by symmetry of tensor products, we have $c^\nu_{\lambda,\mu} = c^\nu_{\mu,\lambda}$. An important special case is *Pieri's rule* when $\lambda = (d)$. In this case, $c^\nu_{(d),\mu} \leq 1$ and is nonzero if and only if $\mu \subseteq \nu$ and the complement of μ in ν is a horizontal strip, i.e., does not have more than 1 box in any column. This is equivalent to the interlacing inequalities $\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots$.

If R is a $\mathbb{C}$-algebra with an action of $\mathrm{GL}(V)$, and S is any $\mathrm{GL}(V)$-representation, then $S \otimes_{\mathbb{C}} R$ is an *equivariant free R-module*; it has the universal property

$$\mathrm{Hom}_{\mathrm{GL}(V),R}(S \otimes_{\mathbb{C}} R, M) \cong \mathrm{Hom}_{\mathrm{GL}(V)}(S, M)$$

for all equivariant R-modules M. The basic examples will be the modules $\mathbb{S}_\lambda(V) \otimes R$.

Remark 2.1 (Gradings and GL_1). If $\dim(V) = 1$, the notion of $\mathrm{GL}(V)$-equivariant ring or module is identical to "($\mathbb{Z}$-)graded." In particular, in this case $V^{\otimes d} \otimes R \cong R(-d)$, the rank-1 free module generated in degree d.

In general, the modules $\mathbb{S}_\lambda(V)\otimes R$ are the equivariant analogues of the twisted graded modules $R(-d)$. The analogous notion to "$\mathbb{N}$-graded ring" is that, as a $\mathrm{GL}(V)$-representation, R should contain only those $\mathbb{S}_\lambda$ with nonnegative parts.

2B. ***Equivariant modules and Betti tables.*** Fix two vector spaces V and W with $k = \dim(V) \le \dim(W) = n$. Let X be the affine variety $\mathrm{Hom}(V, W)$, with coordinate ring

$$R = \mathcal{O}_X = \mathrm{Sym}(\mathrm{Hom}(V, W)^*) = \mathrm{Sym}(V \otimes W^*) \cong \mathbb{C}\left[x_{ij} : \begin{array}{c} 1 \le i \le k, \\ 1 \le j \le n \end{array}\right].$$

The ring R has actions of $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$. Its structure as a representation is given by the Cauchy identity (see [Sam and Snowden 2012, (3.13)] for example),

$$R \cong \bigoplus_{\nu \ge 0} \mathbb{S}_\nu(V) \otimes \mathbb{S}_\nu(W^*). \tag{2-1}$$

We are primarily interested in the $\mathrm{GL}(V)$-action, though we will use both actions when we construct resolutions in Section 4.

The *rank-deficient locus* $\{T : \ker(T) \neq 0\} \subset X$ is an irreducible subvariety of codimension $n-k+1$. Its prime ideal P_k is generated by the $\binom{n}{k}$ maximal minors of the $k \times n$ matrix (x_{ij}). When $k = n$, P_k is a principal ideal, generated by the determinant.

Note that the maximal ideal $\mathfrak{m} = (x_{ij})$ of the origin in X and the ideal P_k are $\mathrm{GL}(V)$- and $\mathrm{GL}(W)$-equivariant.

Let M be a finitely generated $\mathrm{GL}(V)$-equivariant R-module. The module $\mathrm{Tor}^i_R(R/\mathfrak{m}, M)$ naturally has the structure of a finite-dimensional $\mathrm{GL}(V)$-representation. We define the *equivariant Betti number* $\beta_{i,\lambda}(M)$ as the multiplicity of the Schur functor $\mathbb{S}_\lambda(V)$ in this Tor module, i.e.,

$$\mathrm{Tor}^i_R(R/\mathfrak{m}, M) \;\cong\; \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\oplus \beta_{i,\lambda}(M)} \quad \text{(as } \mathrm{GL}(V)\text{-representations)}.$$

It is convenient also to define the (equivariant) *rank Betti number* $\widetilde{\beta_{i,\lambda}}(M)$ as the dimension of the λ-isotypic component, that is,

$$\widetilde{\beta_{i,\lambda}} := \beta_{i,\lambda} \cdot \dim_{\mathbb{C}}(\mathbb{S}_\lambda(V)) = \beta_{i,\lambda} \cdot K_\lambda(k).$$

By semisimplicity of $\mathrm{GL}(V)$-representations, it is easy to see that any minimal free resolution of M can be made equivariant, so we may instead define $\beta_{i,\lambda}$ as the multiplicity of the equivariant free module $\mathbb{S}_\lambda(V) \otimes R$ in the i-th step of an equivariant minimal free resolution of M:

$$M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0, \quad \text{where } F_i = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{i,\lambda}(M)} \otimes R,$$

and likewise $\widetilde{\beta_{i,\lambda}}$ is the rank of the corresponding summand as an R-module.

Remark 2.2. Both definitions $\beta_{i,\lambda}$, $\widetilde{\beta_{i,\lambda}}$ are useful. The Betti number is needed for the pairing with vector bundles, but the rank Betti number is more relevant to the square matrix case and will play the more significant role in this paper.

2B1. *Betti tables and cones.* Let

$$\mathbb{B}_{k,n} = \bigoplus_{i=0}^{n-k+1} \bigoplus_{\lambda} \mathbb{Q}_{i,\lambda}$$

be a direct sum of copies of $\mathbb{Q}$, indexed by homological degree i and partition λ. We think of an element of $\mathbb{B}_{k,n}$ as an abstract Betti table, that is, a choice of $\beta_{i,\lambda}$ for each i and λ. Similarly, we write $\widetilde{\mathbb{B}}_{k,n}$ for the space of abstract rank Betti tables $(\widetilde{\beta_{i,\lambda}})$.

We define the *equivariant Boij–Söderberg cones* $\mathrm{BS}_{k,n} \subseteq \mathbb{B}_{k,n}$, $\widetilde{\mathrm{BS}}_{k,n} \subseteq \widetilde{\mathbb{B}}_{k,n}$ as the positive linear span of all (multiplicity or rank) Betti tables of finitely generated Cohen–Macaulay modules M supported on the rank-deficient locus Spec $R/P_k \subset X$. (That is, M for which $\sqrt{\mathrm{ann}(M)} = P_k$.)

2C. GL-*cohomology tables for Grassmannians.* Let $\mathrm{Gr}(k, W)$ denote the Grassmannian variety of k-dimensional subspaces in W, with tautological exact sequence of vector bundles

$$0 \to \mathcal{S} \to W \to \mathcal{Q} \to 0,$$

where W denotes the trivial rank-n vector bundle and

$$\mathcal{S} = \{(x, U) \in W \times \mathrm{Gr}(k, W) : x \in U\}.$$

Let $\mathcal{E}$ be any coherent sheaf on $\mathrm{Gr}(k, W)$. We define the GL-*cohomology table* $\gamma_{i,\lambda}(\mathcal{E})$ by

$$\gamma_{i,\lambda}(\mathcal{E}) := \dim_{\mathbb{C}} \mathrm{H}^i(\mathcal{E} \otimes \mathbb{S}_\lambda(\mathcal{S})).$$

We let $\mathrm{ES}_{k,n} \subset \bigoplus_i \prod_\lambda \mathbb{Q}_{i,\lambda}$ be the positive span of such tables.

Remark 2.3. Note that if $k = 1$ then $\mathcal{S} = \mathcal{O}(-1)$ on the projective space $\mathbb{P}(W)$. Since this is a line bundle, λ can have only one row, say $\lambda = (j)$. Then

$$\mathbb{S}_\lambda(\mathcal{S}) = \mathrm{Sym}^j(\mathcal{O}(-1)) = \mathcal{O}(-j),$$

so $\gamma_{i,\lambda}(\mathcal{E}) = \gamma_{i,-j}(\mathcal{E})$ is the usual cohomology table of $\mathcal{E}$ with respect to $\mathcal{O}(1)$. In general, the GL-cohomology table contains more information than the usual cohomology table with respect to twists by $\mathcal{O}(1)$; in particular, it determines the class of $\mathcal{E}$ in K-theory $\mathrm{K}(\mathrm{Gr}(k, W))$, while the usual table only determines the K-class of $i_*(\mathcal{E})$, where $i : \mathrm{Gr}(k, W) \hookrightarrow \mathbb{P}(\bigwedge^k(W))$ is the Plücker embedding.

2D. *The numerical pairing.* We briefly discuss the pairing between equivariant Betti tables and GL-cohomology tables. For details, see [Ford and Levinson 2016]. Let $B = (\beta_{i,\lambda})$ be an equivariant Betti table and $\Gamma = (\gamma_{i,\lambda})$ a GL-cohomology table. We define a *rank* table $\widetilde{\Phi}(B, \Gamma) = (\widetilde{\phi_{i,\lambda}}(B, \Gamma))$, for $i \in \mathbb{Z}$, by

$$\widetilde{\phi_{i,\lambda}}(B, \Gamma) = \sum_{p-q=i} \beta_{p,\lambda} \cdot \gamma_{q,\lambda}.$$

In this definition we do not assume any bounds on i, so it is convenient to define the *derived Boij–Söderberg cone* $\mathrm{BS}^D_{k,n} \subset \bigoplus_{i\in\mathbb{Z}} \bigoplus_\lambda \mathbb{Q}_{i,\lambda}$ as the positive linear span of Betti tables of minimal free equivariant complexes $F^\bullet = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{\bullet,\lambda}} \otimes R$ with homology modules supported in the rank-deficient locus.

Theorem 2.4 [Ford and Levinson 2016, Theorem 1.13]. *The map $\widetilde{\Phi}$ defines a pairing of cones*,

$$\widetilde{\Phi} : \mathrm{BS}^D_{k,n} \times \mathrm{ES}_{k,n} \to \widetilde{\mathrm{BS}}^D_{k,k}.$$

Consequently, any nonnegative linear functional on the cone $\widetilde{\mathrm{BS}}^D_{k,k}$ determines a nonnegative bilinear pairing between equivariant Betti tables and GL-cohomology tables. The extended cone $\widetilde{\mathrm{BS}}^D_{k,k}$ has extremal rays and facets closely resembling those of $\widetilde{\mathrm{BS}}_{k,k}$. See [Ford and Levinson 2016, Section 4.2] for an explicit description.

3. The Boij–Söderberg cone on square matrices

We now assume V, W are vector spaces of the same dimension k, and we describe the cone $\widetilde{\mathrm{BS}}_{k,k} \subset \widetilde{\mathbb{B}}_{k,k}$. In particular, we would like to know both the extremal rays and the equations of the supporting hyperplanes. The modules M of interest are Cohen–Macaulay of codimension 1, so their minimal free resolutions are just injective maps $F_1 \hookrightarrow F_0$ of equivariant free modules. For $i = 0, 1$, we put

$$F_i = \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\beta_{i,\lambda}} \otimes R,$$

and define $\widetilde{\beta_{i,\lambda}} = \beta_{i,\lambda} \cdot K_\lambda(k)$ as in Section 2.

The first observation is that, since M is a torsion module, we must have

$$\operatorname{rank} F_0 = \operatorname{rank} F_1, \quad \text{that is,} \quad \sum_\lambda \widetilde{\beta_{0,\lambda}}(M) = \sum_\lambda \widetilde{\beta_{1,\lambda}}(M). \tag{3-1}$$

Conversely, any injective map of free modules of this form must have a torsion cokernel, which is then Cohen–Macaulay of codimension 1. We will see that the rank condition is the only *linear* constraint on Betti tables, that is, the cone spans this entire linear subspace.

3A. ***Antichain inequalities.*** The maps of any minimal complex have positive degree. More precisely, we have the following:

Lemma 3.1 (Sequences contract under minimal maps). *Let*

$$f : \mathbb{S}_\mu(V) \otimes R \to \mathbb{S}_\lambda(V) \otimes R$$

be any nonzero map. If $\mu = \lambda$, then f is an isomorphism. Otherwise, $\mu \supsetneq \lambda$ and f is minimal.

Proof. This follows from the universal property of equivariant free modules,

$$\mathrm{Hom}_{\mathrm{GL}(V),R}(\mathbb{S}_\mu(V) \otimes R, \mathbb{S}_\lambda(V) \otimes R) \cong \mathrm{Hom}_{\mathrm{GL}(V)}(\mathbb{S}_\mu(V), \mathbb{S}_\lambda(V) \otimes R).$$

We apply the Cauchy identity (2-1) for R as a $\mathrm{GL}(V)$-representation. We see that

$$\mathrm{Hom}_{\mathrm{GL}(V)}(\mathbb{S}_\mu(V), \mathbb{S}_\lambda(V)\otimes R) \cong \bigoplus_{\nu\geq 0} \mathrm{Hom}_{\mathrm{GL}(V)}\big(\mathbb{S}_\mu(V), \mathbb{S}_\lambda(V)\otimes \mathbb{S}_\nu(V)\big)\otimes \mathbb{S}_\nu(W^*).$$

By the Littlewood–Richardson rule, if $\mu \not\supseteq \lambda$, every summand is 0. If $\mu = \lambda$, the only nonzero summand comes from $\nu = \varnothing$; we see that the corresponding map is an isomorphism (if nonzero). Finally, if $\mu \supsetneq \lambda$, there is at least one ν for which the corresponding summand is nonzero and any such ν must satisfy $|\nu| = |\mu| - |\lambda| > 0$, so the corresponding map of R-modules has strictly positive degree (equal to $|\mu|$), hence is a minimal map. $\square$

Remark 3.2. Because the ring R involves W^*, not W, the analogous computation shows that the sequence labeling W *expands* under a minimal map: that is, a nonzero $\mathrm{GL}(W)$-equivariant map $\mathbb{S}_\mu(W)\otimes R \to \mathbb{S}_\lambda(W)\otimes R$ exists if and only if $\mu\subseteq\lambda$ (and is minimal if and only if $\mu\neq\lambda$).

In particular, for any fixed μ, a minimal injective map $F_1 \hookrightarrow F_0$ of free modules must inject the summands $\lambda\subseteq\mu$ of F_1 into the summands $\lambda\subsetneq\mu$ of F_0, and so

$$\sum_{\lambda\subsetneq\mu} \widetilde{\beta_{0,\lambda}} \geq \sum_{\lambda\subseteq\mu} \widetilde{\beta_{1,\lambda}}, \tag{3-2}$$

which gives us some of the inequalities our Betti tables need to satisfy. But in fact these inequalities are not enough. For example, for any pair of partitions α, β, the summands of F_1 given by

$$\{\lambda : \lambda\subseteq\alpha \text{ or } \lambda\subseteq\beta\}$$

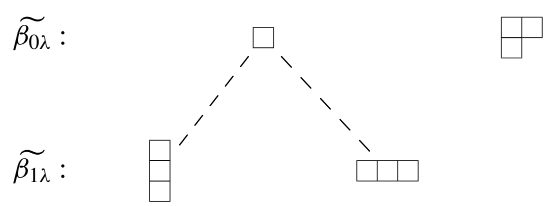
must inject into the summands of F_0 given by

$$\{\lambda : \lambda\subsetneq\alpha \text{ or } \lambda\subsetneq\beta\}.$$

This gives the additional, nonredundant condition

$$\sum_{\lambda\subsetneq\alpha \text{ or } \lambda\subsetneq\beta} \widetilde{\beta_{0,\lambda}} \geq \sum_{\lambda\subseteq\alpha \text{ or } \lambda\subsetneq\beta} \widetilde{\beta_{1,\lambda}}.$$

Example 3.3. The following example illustrates that the inequalities (3-2) are not sufficient. Consider the following rank Betti table, with all entries equal to 1 (shown transposed, with dashed lines indicating containment of partitions):



It is evident that this cannot be the Betti table of a torsion module, since nothing maps to the $\ydiagram{2,1}$ summand. Likewise, the table violates the inequality for the pair $\ydiagram{3}$, $\ydiagram{1,1,1}$. For any single partition μ, however, the inequality (3-2) is satisfied. (Note that if μ contains both $\ydiagram{3}$ and $\ydiagram{1,1,1}$, then it strictly contains $\ydiagram{2,1}$.)

The complete set of inequalities is as follows. Recall that if P is a poset, $I \subseteq P$ is an *order ideal* if $x \in I$ and $y \leq x$ implies that $y \in I$, i.e., I is a downwards-closed subset. We define the *interior* of I to be the subset

$$I^\circ = \{x \in P : x < y \text{ for some } y \in I\}$$

of elements strictly contained in I. The maximal elements $I \setminus I^\circ$ of I form an *antichain*, that is, they are pairwise incomparable. We have the following:

Lemma 3.4 (Antichain inequalities). *Let $\mathbb{Y}_\pm$ be the extended Young's lattice and $I \subseteq \mathbb{Y}_\pm$ an order ideal. Let $(\beta_{i,\lambda}) \in \mathrm{BS}_{k,k}$ be a Betti table. Then*

$$\sum_{\lambda \in I^\circ} \widetilde{\beta_{0,\lambda}} \geq \sum_{\lambda \in I} \widetilde{\beta_{1,\lambda}}. \tag{3-3}$$

Proof. Follows from the above discussion. □

Remark 3.5 (Inequalities for upwards-closed sets). It is also the case that, for any *upwards*-closed subset $U \subseteq \mathbb{Y}_\pm$, we have a "dual" inequality

$$\sum_{\lambda \in U} \widetilde{\beta_{0,\lambda}} \leq \sum_{\lambda \in U_\circ} \widetilde{\beta_{1,\lambda}}, \tag{3-4}$$

where $U_\circ = \{\lambda \in U : \lambda \geq \mu \text{ for some } \mu \in U\}$ is its upwards-interior.

Algebraically, this corresponds to the following observation: let $(F_1)_{U_\circ}$, $(F_0)_U$ be the summands corresponding to $U_\circ$, U. The projection $F_0 \twoheadrightarrow (F_0)_U$ vanishes on the images of the non-$U_\circ$ summands of F_1, so we have a commutative diagram

$$\begin{tikzcd} F_1 \arrow[r, "f"] \arrow[d] & F_0 \arrow[d] \\ (F_1)_{U_\circ} \arrow[r, "\bar{f}"] & (F_0)_U \end{tikzcd}$$

It follows that $\operatorname{coker}(f) \to \operatorname{coker}(\bar{f})$ is surjective, hence $\operatorname{coker}(\bar{f})$ is also torsion (since $\operatorname{coker}(f)$ is). Consequently, we obtain the desired inequality $\operatorname{rank}(F_1)_{U_\circ} \geq \operatorname{rank}(F_0)_U$.

Alternatively, we may deduce (3-4) by subtracting the inequality (3-3) with $I = \mathbb{Y}_\pm \setminus (U^\circ)$ from the rank Equation (3-1), and observing that

$$(P \setminus (U_\circ))^\circ \subseteq P \setminus U$$

holds in any poset P. (Note that the complement of an upwards-closed set is downwards-closed.) In particular, given the rank Equation (3-1), the "upwards-facing" and "downwards-facing" inequalities collectively cut out the same cone.

3B. ***Extremal rays and pure diagrams.*** We can construct a very simple Betti table by letting $\lambda^{(0)} \subsetneq \lambda^{(1)}$ be any pair of distinct, comparable partitions. Let $\widetilde{\beta_{0,\lambda^{(0)}}} = \widetilde{\beta_{1,\lambda^{(1)}}} = 1$ and let all other entries of the Betti table be 0. We call the resulting table $\widetilde{P}(\lambda^{(0)}, \lambda^{(1)})$ a *pure table of type* $(\lambda^{(0)}, \lambda^{(1)})$, and any resolution corresponding to a positive multiple of this table a *pure resolution*.

Note that a pure table clearly satisfies all of the antichain inequalities (3-3), as well as the linear constraint (3-1). Moreover, since a Betti table must have at least two nonzero entries, a pure table cannot be written as a nontrivial positive combination of other tables. Any realizable pure table therefore generates an extremal ray of $\widetilde{\mathrm{BS}}_{k,k}$.

We will show in Theorem 4.1 that every pure table has a realizable scalar multiple. Consequently, *every* pure table generates an extremal ray of the Boij–Söderberg cone $\widetilde{\mathrm{BS}}_{k,k}$.

We now show that, *assuming* Theorem 4.1, every extremal ray is of this form.

Theorem 3.6 (Extremal rays). *Every realizable rank Betti table is a positive $\mathbb{Z}$-linear combination of pure rank tables.*

The proof uses Hall's Theorem on perfect matchings in bipartite graphs. Recall that a *perfect matching* on a graph G is a subset $E' \subseteq E$ of the edges of G, such that every vertex of G occurs on exactly one edge from E'. We recall the statement of Hall's matching theorem (see [Lovász and Plummer 1986, Theorem 1.1.3] for a proof):

Theorem 3.7 (Hall). *Let G be a bipartite graph with left vertices L, right vertices R and edges E. For a collection of vertices S, let $\Gamma(S)$ be the set of neighboring vertices to S.*

Assume $|L| = |R|$. Then G has a perfect matching if and only if $|\Gamma(S)| \geq |S|$ for all subsets $S \subseteq R$.

Proof of Theorem 3.6. Let $(\widetilde{\beta_{i,\lambda}}) \in \widetilde{\mathrm{BS}}_{k,k}$ be a realizable rank Betti table; by rescaling, we may assume all the entries are integers. We define a bipartite graph $G = (L, R, E)$ as follows: For each λ, L (resp. R) will have $\widetilde{\beta_{0,\lambda}}$ vertices (resp. $\widetilde{\beta_{1,\lambda}}$) labeled λ. Every vertex labeled λ in L is connected to every vertex labeled μ in R whenever $\lambda \subsetneq \mu$. By the rank condition (3-1), G satisfies $|L| = |R|$.

Observe that a perfect matching on G decomposes $(\beta_{i,\lambda})$ as a $\mathbb{Z}$-linear combination of pure tables: each edge $(\lambda^{(0)} \leftarrow \lambda^{(1)})$ in the matching corresponds to a pure table $\widetilde{P}(\lambda^{(0)}, \lambda^{(1)})$. Thus, it suffices to show that G has a perfect matching.

We apply Hall's theorem (Theorem 3.7). Let $S \subseteq R$. Observe that if S contains a vertex labeled μ, then without loss of generality, we may assume S contains every vertex labeled μ and, in addition, every vertex labeled μ' with $\mu' \subseteq \mu$, since adding these vertices makes S larger but does not change $\Gamma(S)$.

Let I be the order ideal generated by the set of vertex labels appearing in S. We see that $|S| = \sum_{\lambda \in I} \widetilde{\beta_{1,\lambda}}$ and $|\Gamma(S)| = \sum_{\lambda \in I^\circ} \widetilde{\beta_{0,\lambda}}$, so the condition $|\Gamma(S)| \geq |S|$ is precisely the antichain inequality (3-3) for I. □

Thus, assuming Theorem 4.1, we have shown:

Theorem 3.8 (Combinatorial description of $\widetilde{\mathrm{BS}}_{k,k}$). *The cone $\widetilde{\mathrm{BS}}_{k,k} \subset \widetilde{\mathbb{B}}_{k,k}$ is cut out by the rank equation* (3-1), *the antichain inequalities* (3-3) *and the conditions $\widetilde{\beta}_{i,\lambda} \geq 0$. Its extremal rays are exactly the rays spanned by the pure tables corresponding to all pairs $\lambda^{(0)} \subsetneq \lambda^{(1)}$ of comparable elements of $\mathbb{Y}_\pm$.*

Remark 3.9 (Decomposing Betti tables). There are efficient algorithms for computing perfect matchings of graphs; see, e.g., [Lovász and Plummer 1986, §1.2]. A standard proof of Hall's theorem implicitly uses the following algorithm, which is inefficient but conceptually clear. Let $\widetilde{B} \in \widetilde{\mathrm{BS}}_{k,k}$ be a Betti table.

Case 1: Suppose every antichain inequality is strict. Choose any pure table $\widetilde{P}(\lambda^{(0)}, \lambda^{(1)})$ whose entries occur with nonzero values in $\widetilde{B}$. Then

$$\widetilde{B}_{\mathrm{rest}} = \widetilde{B} - \widetilde{P}(\lambda^{(0)}, \lambda^{(1)}) \in \widetilde{\mathrm{BS}}_{k,k}.$$

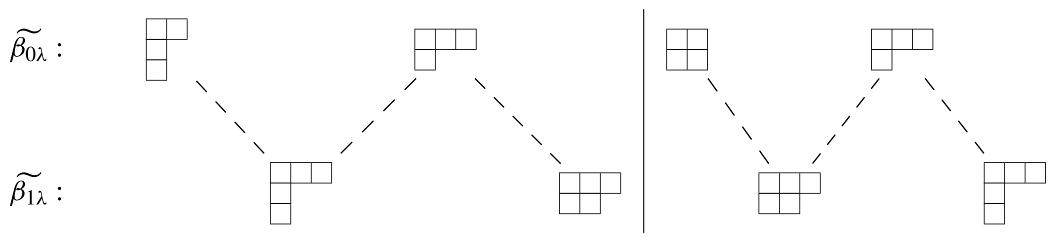
Continue the algorithm on $\widetilde{B}_{\mathrm{rest}}$.

Case 2: Suppose, instead, there exists an antichain I for which (3-3) is an equality. Write

$$\widetilde{B} = \widetilde{B}_I + \widetilde{B}_{\mathrm{rest}},$$

where $\widetilde{B}_I$ contains all the entries involved in the equality ($\widetilde{\beta_{0,\lambda}}$ for $\lambda \in I^\circ$ and $\widetilde{\beta_{1,\lambda}}$ for $\lambda \in I$). Then both $\widetilde{B}_I \in \widetilde{\mathrm{BS}}_{k,k}$ and $\widetilde{B}_{\mathrm{rest}} \in \widetilde{\mathrm{BS}}_{k,k}$; continue the algorithm separately for each.

We contrast the algorithm above with the usual algorithm [Eisenbud and Schreyer 2009, §1] for decomposing graded Betti tables. For graded tables, the decomposition is "greedy" and deterministic. It relies on a partial ordering on pure graded Betti tables, which induces a decomposition of the Boij–Söderberg cone as a simplicial fan. Unfortunately, the natural choices of partial ordering on the equivariant pure tables $P(\lambda^{(0)}, \lambda^{(1)})$ do not yield valid greedy decomposition algorithms. For example, suppose the graph G of Theorem 3.6 consists of a single long path. Compare the following two examples:



In both cases, G has a unique perfect matching, but whether an edge is used depends on its placement along the path, not just on the partitions labeling its vertices. Hence, an algorithm that (for instance) greedily selects the lexicographically largest pair $(\lambda^{(0)}, \lambda^{(1)})$ will fail: in both cases, the lex-largest $\lambda^{(0)}$ is $= (3, 1)$ and its lex-largest neighbor is $\lambda^{(1)} = = (3, 2)$. This leads to the (unique) correct matching on the first graph, but fails on the graph to the right.

Similarly, if the graph structure of G is a cycle, then G has two perfect matchings, so a deterministic algorithm must have a way of selecting one.

Finally, unlike in the graded case, we do not know a good simplicial decomposition of $\widetilde{\mathrm{BS}}_{k,k}$; it would be interesting to find such a structure.

4. Constructing Pure Resolutions

The main theorem of this section is as follows.

Theorem 4.1. *For any partitions $\lambda^{(0)} \subsetneq \lambda^{(1)}$, there exists a torsion*, $\mathrm{GL}(V)$*-equivariant R-module M with minimal free resolution*

$$M \leftarrow \mathbb{S}_{\lambda^{(0)}}(V)^{c_0} \otimes R \leftarrow \mathbb{S}_{\lambda^{(1)}}(V)^{c_1} \otimes R \leftarrow 0,$$

for some positive integers c_0, c_1.

We first consider a pair of partitions differing by a box. The same argument works somewhat more generally (see Remark 4.8), but we restrict to this case for notational simplicity.

By the Pieri rule, there is a unique $\mathrm{GL}(V)\times\mathrm{GL}(W)$-equivariant R-linear map

$$\mathbb{S}_{\lambda^{(0)}}(V) \otimes \mathbb{S}_{\lambda^{(1)}}(W) \otimes R \leftarrow \mathbb{S}_{\lambda^{(1)}}(V) \otimes \mathbb{S}_{\lambda^{(0)}}(W) \otimes R. \tag{4-1}$$

Theorem 4.2. *The biequivariant map* (4-1) *is injective.*

We postpone the proof to Section 4A and now explain how it implies Theorem 4.1.

Proof of Theorem 4.1. Let $|\lambda^{(1)}| - |\lambda^{(0)}| = r$. Choose a chain of partitions

$$\lambda^{(1)} = \alpha^{(r)} \supsetneq \alpha^{(r-1)} \supsetneq \cdots \supsetneq \alpha^{(0)} = \lambda^{(0)}, \qquad \text{with } |\alpha^{(i)}| = |\lambda^{(0)}| + i \quad \text{for all } i.$$

By Theorem 4.2, for $i = 1, \dots, r$, there exists a sequence of biequivariant, linear injections

$$f_i : \mathbb{S}_{\alpha^{(i)}}(V) \otimes \mathbb{S}_{\alpha^{(i-1)}}(W) \otimes R \hookrightarrow \mathbb{S}_{\alpha^{(i-1)}}(V) \otimes \mathbb{S}_{\alpha^{(i)}}(W) \otimes R.$$

Let g be the composite map

$$\begin{array}{c}
F_1 = \mathbb{S}_{\alpha^{(r)}}(V) \otimes \mathbb{S}_{\alpha^{(r-1)}}(W) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(W) \otimes \mathbb{S}_{\alpha^{(0)}}(W) \otimes R \\
\Big\downarrow {\scriptstyle f_r \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}} \\
\mathbb{S}_{\alpha^{(r)}}(W) \otimes \mathbb{S}_{\alpha^{(r-1)}}(V) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(W) \otimes \mathbb{S}_{\alpha^{(0)}}(W) \otimes R \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes f_{r-1} \otimes \cdots \otimes \mathrm{id}} \\
\vdots \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \cdots \otimes f_2 \otimes \mathrm{id}} \\
\mathbb{S}_{\alpha^{(r)}}(W) \otimes \mathbb{S}_{\alpha^{(r-1)}}(W) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(V) \otimes \mathbb{S}_{\alpha^{(0)}}(W) \otimes R \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes f_1} \\
F_0 = \mathbb{S}_{\alpha^{(r)}}(W) \otimes \mathbb{S}_{\alpha^{(r-1)}}(W) \otimes \cdots \otimes \mathbb{S}_{\alpha^{(1)}}(W) \otimes \mathbb{S}_{\alpha^{(0)}}(V) \otimes R
\end{array}$$

Clearly g is again injective. Since $\mathrm{rank}(F_1) = \mathrm{rank}(F_0)$, we are done. □

4A. ***Proof of Theorem 4.2.*** To cut down on indices, we write λ for the smaller partition and μ for the larger. We put

$$\begin{aligned}
\mu &= (\mu_1, \dots, \mu_k), && \text{and we assume } \mu_r > \mu_{r+1}, \\
\lambda &= \mu, && \text{except for } \lambda_r = \mu_r - 1.
\end{aligned}$$

The proof relies on the Borel–Weil–Bott theorem and a construction of Eisenbud–Fløystad–Weyman. We first review these results, then give an informal summary of the argument, and finally give a proof of the theorem.

4A1. *Borel–Weil–Bott and Eisenbud–Fløystad–Weyman.* On $\mathbb{P}(W^*)$, we have the short exact sequence

$$0 \to \mathcal{S} \to W \to \mathcal{O}(1) \to 0.$$

Note that we are using W, not W^*.

Given a permutation σ, define $\ell(\sigma) = \#\{i < j : \sigma(i) > \sigma(j)\}$, the number of inversions.

Theorem 4.3 (Borel–Weil–Bott, [Weyman 2003, Corollary 4.1.9]). *Let $\beta = (\beta_1, \dots, \beta_{k-1})$ be weakly decreasing and let $d \in \mathbb{Z}$. The cohomology of $\mathbb{S}_\beta(\mathcal{S})(d)$ is determined as follows. Write*

$$(d, \beta_1, \dots, \beta_{k-1}) - (0, 1, \dots, k-1) = (a_1, \dots, a_k).$$

(1) *If $a_i = a_j$ for some $i \neq j$, every cohomology group of $\mathbb{S}_\beta(\mathcal{S})(d)$ vanishes.*

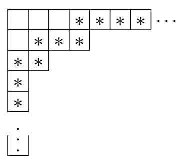
(2) *Otherwise, a unique permutation σ sorts the a_i into decreasing order, $a_{\sigma(1)} > a_{\sigma(2)} > \cdots > a_{\sigma(k)}$. Put $\lambda = (a_{\sigma(1)}, \dots, a_{\sigma(k)}) + (0, 1, \dots, k-1)$. Then*

$$\mathrm{H}^{\ell(\sigma)}\big(\mathbb{S}_\beta(\mathcal{S})(d)\big) = \mathbb{S}_\lambda(W),$$

and $\mathrm{H}^i(\mathbb{S}_\beta(\mathcal{S})(d)) = 0$ for $i \neq \ell(\sigma)$.

We will also use the following result, on the existence of certain equivariant graded free resolutions.

First, for a partition λ, we say (i, j) is an *outer border square* if $(i, j) \notin \lambda$ and $(i-1, j-1) \in \lambda$ (or $i = 1$ or $j = 1$), as in the $*$'s below, for $\lambda = (3, 1)$:



Let α be a partition with k parts, and let $\alpha' \supsetneq \alpha$ be obtained by adding at least one border square in row 1, and all possible border squares in rows $2, \dots, k$. Let $\alpha^{(0)} = \alpha$, and for $i = 1, \dots, k$, let $\alpha^{(i)}$ be obtained by adding the chosen border squares only in rows $1, \dots, i$.

Theorem 4.4 [Eisenbud et al. 2011, Theorem 3.2]. *Let E be a k-dimensional complex vector space and $R = \mathrm{Sym}(E)$ its symmetric algebra. There is a finite, $\mathrm{GL}(E)$-equivariant R-module M whose equivariant minimal free resolution is, with $\alpha^{(i)}$ defined as above,*

$$F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_k \leftarrow 0, \quad F_i = \mathbb{S}_{\alpha^{(i)}}(E) \otimes R.$$

Since the construction is equivariant, it works in families:

Theorem 4.5. *Let X be a complex variety and $\mathcal{E}$ a rank k vector bundle over X. Let $\mathcal{E}^* \to X$ be the dual bundle. There is a sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{E}^*}$-modules with a locally free resolution*

$$\mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_k \leftarrow 0, \quad \mathcal{F}_i = \mathbb{S}_{\alpha^{(i)}}(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{E}^*}.$$

This follows by applying the Eisenbud–Fløystad–Weyman (EFW) construction to the sheaf of algebras $\mathcal{O}_{\mathcal{E}^*} = \operatorname{Sym}(\mathcal{E})$. The resolved sheaf $\mathcal{M}$ is locally given by M above. Note that $\mathcal{M}$ is coherent as an $\mathcal{O}_X$-module, though we will not need this.

Remark 4.6. The construction we presented is also a direct corollary of a special case Kostant's version of the Borel–Weil–Bott theorem, for example see [Erman and Sam 2017, §6] for some discussion and references. We expect that other cases of Kostant's theorem are relevant for constructing complexes in the nonsquare matrix case.

4A2. *Informal summary of the argument.* We have fixed $\lambda \subsetneq \mu$, a pair of partitions differing by a box. There is a unique EFW complex with, in one step, a linear differential of the form

$$\mathbb{S}_\mu(E) \otimes \operatorname{Sym}(E) \to \mathbb{S}_\lambda(E) \otimes \operatorname{Sym}(E).$$

The rest of the complex is uniquely determined by this pair of shapes, and is functorial in E. We "sheafify" the complex, lifting it to a complex of modules over the algebra $\operatorname{Sym}(V \otimes \mathcal{O}(-1))$ on $\mathbb{P}(W^*)$, with terms of the form

$$\mathcal{O}(-d_i) \otimes \mathbb{S}_\alpha(V) \otimes \operatorname{Sym}(V \otimes \mathcal{O}(-1)),$$

We twist so that the 0th term has degree $d = \mu_1$, base change along the flat extension $\operatorname{Sym}(V \otimes \mathcal{O}(-1)) \hookrightarrow \operatorname{Sym}(V \otimes W^*)$, and finally tensor through by the vector bundle $\mathbb{S}_\beta(\mathcal{S})$, where $\mathcal{S} \subset W \times \mathbb{P}(W^*)$ is the tautological rank-$(k-1)$ subbundle, and β is chosen so that all the terms of the resulting complex *except* the desired pair have no cohomology. (That is, $\mathbb{S}_\beta(\mathcal{S})$ has supernatural cohomology with roots at each of the other d_i's.) Finally, we obtain the desired map from the hypercohomology spectral sequence for the complex.

Example 4.7. Let $k = 4$ and let $\lambda = (6, 1, 1, 0)$, $\mu = (6, 2, 1, 0) = \ytableaushort{\ \ \ \ \ \ ,\ \star,\ }$ (the added box is starred). Working on $\mathbb{P}(W^*)$, the corresponding locally free resolution of sheaves (with the twisting degrees indicated) is, after twisting and base-changing,

$$\ydiagram{1,1,1}(6) \leftarrow \ydiagram{6,1,1}(1) \xleftarrow{\star} \ydiagram{6,2,1} \leftarrow \ydiagram{6,2,2}(-1) \leftarrow \ydiagram{6,2,2,2}(-3),$$

where $\alpha(d)$ stands for the sheaf $\mathcal{O}(d) \otimes \mathbb{S}_\alpha(V) \otimes \operatorname{Sym}(V \otimes W^*)$ on $\mathbb{P}(W^*)$. The desired linear differential is marked with a $\star$.

We put $\beta = (7, 1, 0)$ and tensor through by $\mathbb{S}_\beta(\mathcal{S})$ (note that $\mathcal{S}$ has rank 3). Observe that $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology when $d \in \{6, -1, -3\}$, but that

$$\mathrm{H}^1\big(\mathbb{S}_\beta(\mathcal{S})(1)\big) = \mathbb{S}_{621}(W), \quad \mathrm{H}^1\big(\mathbb{S}_\beta(\mathcal{S})\big) = \mathbb{S}_{611}(W).$$

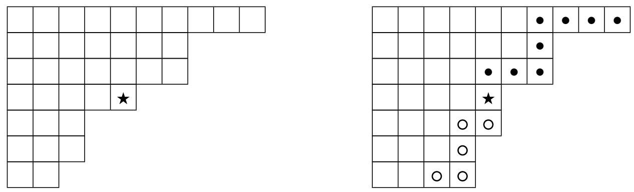


Figure 1. Left: The partition μ; the starred box in the 4th row is removed to form λ. Right: The outer strip is formed by connecting the *inner border strip* ($\bullet$) in rows 1 to $r-1$ to the *outer border strip* ($\circ$) outside rows $r+1, \dots, k$. The empty squares form $\alpha^{(0)}$; then $\alpha^{(i)}$ is obtained by adding all marked squares ($\bullet$, $\star$, $\circ$) up to row i. Note that $\alpha^{(3)} = \lambda$ and $\alpha^{(4)} = \mu$.

Consequently, the E_1 page of the hypercohomology spectral sequence only has the terms

$$\mathbb{S}_{611}(V) \otimes \mathbb{S}_{621}(W) \otimes R \leftarrow \mathbb{S}_{621}(V) \otimes \mathbb{S}_{611}(W) \otimes R,$$

in the second and third columns. (The left term is along the main diagonal.) The spectral sequence, run the other way, collapses with $\mathrm{H}^i(\mathcal{M})$ in the first column, where $\mathcal{M}$ is the sheaf resolved by this complex. We see that the only nonvanishing term can be $\mathrm{H}^0(\mathcal{M})$, giving an exact sequence of R-modules

$$0 \leftarrow \mathrm{H}^0(\mathcal{M}) \leftarrow \mathbb{S}_{611}(V) \otimes \mathbb{S}_{621}(W) \otimes R \leftarrow \mathbb{S}_{621}(V) \otimes \mathbb{S}_{611}(W) \otimes R \leftarrow 0.$$

Remark 4.8. There are two easy ways to generalize the construction that we have sketched above. First, in the map marked $\star$ above, there is no reason to assume that the two partitions differ by a single box, and the same construction allows them to differ by multiple boxes as long as they are in the same row. In this case, the Pieri rule still implies that the map (4-1) is unique up to scalar.

Second, in the above example we chose β so that $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology for all d besides the twists appearing in the target and domain of a single differential (in this case, the one marked $\star$). Alternatively, we could choose β so that $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology for all but two of the terms in the complex (not necessarily consecutive terms). The end result is also a map of the form (4-1) where $\lambda^{(0)}$ and $\lambda^{(1)}$ differ by a connected border strip. In general the map (4-1) is not unique up to scalar, however.

4A3. *Combinatorial setup.* We define shapes $\alpha^{(i)}$, $i = 0, \dots, k$, as follows. Consider the squares formed by

- the inner border strip of μ inside rows $1, \dots, r-1$,
- the rightmost square in row r of μ,
- the outer border strip of μ outside rows $r+1, \dots, k$.

(See Figure 1.) Then $\alpha = \alpha^{(0)}$ is obtained by deleting all these squares, and $\alpha^{(i)}$ is obtained by including those squares in rows $1, \dots, i$. Clearly, $\alpha^{(r)} = \mu$ and $\alpha^{(r-1)} = \lambda$.

Let e_i be the number of border squares in row i, so

$$e_i = \begin{cases} 1+\mu_i-\mu_{i+1} & \text{for } i=1,\ldots,r-1,\\ 1 & \text{for } i=r,\\ 1+\lambda_{i-1}-\lambda_i & \text{for } i=r+1,\ldots,k.\end{cases}$$

We define a modified (and negated) partial sum,

$$d_i := \mu_1-(e_1+\cdots+e_i) = \begin{cases} \mu_{i+1}-i & \text{if } i=0,\ldots,r-1,\\ \mu_r-r & \text{if } i=r,\\ \mu_i-i+1 & \text{if } i=r+1,\ldots,k.\end{cases}$$

Finally, we write $\beta=(\beta_1,\ldots,\beta_{k-1})$ for the unique partition such that, on $\mathbb{P}(W^*)$, the vector bundle $\mathbb{S}_\beta(\mathcal{S})(d)$ has no cohomology for each d_i, $i\in\{0,\ldots,k\}\setminus\{r-1,r\}$, where $\mathcal{S}\subset W\times\mathbb{P}(W^*)$ is the tautological rank-$(k-1)$ subbundle. By Borel–Weil–Bott, this determines β uniquely by

$$\beta-(1,\ldots,k-1)=(d_0,\ldots,d_{r-2},d_{r+1},\ldots,d_k).$$

With these choices, we check:

Lemma 4.9. *For $d=d_r$, the only nonvanishing cohomology of $\mathbb{S}_\beta(\mathcal{S})(d)$ on $\mathbb{P}(W^*)$ is $\mathrm{H}^{r-1}=\mathbb{S}_\mu(W)$. For $d=d_{r-1}$, the only nonvanishing cohomology is $\mathrm{H}^{r-1}=\mathbb{S}_\lambda(W)$.*

Proof. We apply Borel–Weil–Bott: we have to sort

$$(d,\beta_1,\ldots,\beta_k)-(0,1,\ldots,k-1)=(d,d_0,\ldots,d_{r-2},d_{r+1},\ldots,d_k).$$

For $d=d_{r-1}$ or d_r, sorting takes $r-1$ swaps, so in both cases H^{r-1} is nonvanishing. To see that the cohomology group is $\mathbb{S}_\mu(W)$ for d_{r-1} and $\mathbb{S}_\lambda(W)$ for d_r, we must check that

$$\mu=(d_0,\ldots,d_{r-2},d_{r-1},d_{r+1},\ldots,d_k)+(0,1,\ldots,k-1),$$
$$\lambda=(d_0,\ldots,d_{r-2},d_r,d_{r+1},\ldots,d_k)+(0,1,\ldots,k-1).$$

These are clear from the computation above. □

4A4. *The proof of Theorem 4.2.* Let $\alpha^{(i)}$ and d_i be defined as above. Consider the projective space $\mathbb{P}(W^*)$, with tautological line bundle $\mathcal{O}(-1)\subset W^*$ and rank-$(k-1)$ bundle $\mathcal{S}\subset W$. Set $\xi:=V\otimes\mathcal{O}(-1)$. By Theorem 4.5, we have an exact complex

$$F_0\leftarrow\cdots\leftarrow F_i\leftarrow F_{i+1}\leftarrow\cdots,\quad \text{where } F_i=\mathbb{S}_{\alpha^{(i)}}(\xi)\otimes\mathrm{Sym}(\xi).$$

Note that

$$\mathbb{S}_\lambda(\xi)=\mathbb{S}_\lambda(V)\otimes\mathcal{O}(-|\lambda|).$$

For legibility, we write $\mathcal{O}(-\lambda)$ for $\mathcal{O}(-|\lambda|)$. Thus, we have a locally free resolution

$$\mathbb{S}_{\alpha^{(0)}}(V)\otimes\mathcal{O}(-\alpha^{(0)})\otimes\mathrm{Sym}(\xi)\leftarrow\cdots\leftarrow\mathbb{S}_{\alpha^{(i)}}(V)\otimes\mathcal{O}(-\alpha^{(i)})\otimes\mathrm{Sym}(\xi)\leftarrow\cdots$$

of sheaves of $\mathrm{Sym}(\xi)$-modules.

Next, let $\mathscr{R} = \mathcal{O}_{\mathbb{P}(W^*)} \otimes \mathrm{Sym}(V \otimes W^*)$. Observe that $\mathrm{Sym}(\xi) \hookrightarrow \mathscr{R}$ is a flat ring extension (locally it is an inclusion of polynomial rings). Now base change to $\mathscr{R}$, which preserves exactness. Finally, we tensor by $\mathbb{S}_\beta(\mathcal{S}) \otimes \mathcal{O}(\alpha^{(0)} + \mu_1)$. Our final complex has terms

$$\mathbb{S}_{\alpha^{(i)}}(V) \otimes \mathbb{S}_\beta(\mathcal{S})(d_i) \otimes \mathscr{R}.$$

Let $\mathcal{M}$ be the sheaf resolved by the complex.

We run the hypercohomology spectral sequence. Running the horizontal maps first, we see that the sequence collapses on the E_2 page with $\mathrm{H}^i(\mathcal{M})$ in the leftmost column. Running the sequence the other way, the E_1 page has terms

$$\mathrm{H}^q\big(\mathbb{S}_{\alpha^{(p)}}(V) \otimes \mathbb{S}_\beta(\mathcal{S})(d_p) \otimes \mathscr{R}\big) = \mathbb{S}_{\alpha^{(p)}}(V) \otimes \mathrm{H}^q\big(\mathbb{S}_\beta(\mathcal{S})(d_p)\big) \otimes \mathscr{R}.$$

(We emphasize that $\mathbb{S}_{\alpha^{(p)}}(V)$ and $\mathscr{R}$ are trivial bundles.) By construction, the middle factor is zero unless $p = r-1, r$, where by Lemma 4.9 the nonvanishing cohomology is H^{r-1}, with

$$\mathrm{H}^{r-1}\big(\mathbb{S}_\beta(\mathcal{S})(d_{r-1})\big) = \mathbb{S}_\mu(W), \quad \mathrm{H}^{r-1}\big(\mathbb{S}_\beta(\mathcal{S})(d_r)\big) = \mathbb{S}_\lambda(W).$$

In particular, the E_1 page contains only the map

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W) \otimes R \leftarrow \mathbb{S}_\mu(V) \otimes \mathbb{S}_\lambda(W) \otimes R,$$

with the left term located on the main diagonal. We see that $\mathrm{H}^i(\mathcal{M}) = 0$ for $i > 0$ and that the map above is a resolution of $\mathrm{H}^0(\mathcal{M})$ by free R-modules.

4B. ***A stronger version of Theorem 4.1.*** Use the notation of Theorem 4.1. Since M is torsion, the ranks must agree,

$$c_0 K_{\lambda^{(0)}}(k) = c_1 K_{\lambda^{(1)}}(k).$$

A straightforward choice of c_0, c_1 is to take $c_0 = K_{\lambda^{(1)}}(k)$ and $c_1 = K_{\lambda^{(0)}}(k)$, and to look for a "small" resolution of the form

$$M \leftarrow \mathbb{S}_{\lambda^{(0)}}(V) \otimes \mathbb{S}_{\lambda^{(1)}}(W) \otimes R \leftarrow \mathbb{S}_{\lambda^{(1)}}(V) \otimes \mathbb{S}_{\lambda^{(0)}}(W) \otimes R \leftarrow 0, \tag{4-2}$$

equivariant for both $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$. Theorem 4.2 proves this conjecture when $|\lambda^{(1)}| = |\lambda^{(0)}| + 1$ and one can also do the case when $\lambda^{(1)}$ is obtained by adding a connected border strip to $\lambda^{(0)}$ using Remark 4.8. In general, we do not know if this particular form of resolution exists, though we conjecture that it does:

Conjecture 4.10. *A small pure resolution* (4-2) *exists, for any pair* $\lambda^{(0)} \subsetneq \lambda^{(1)}$.

We finish by establishing one more situation where Conjecture 4.10 is true:

Proposition 4.11. *Suppose there exists d such that $(\lambda^{(0)})_i \le d \le (\lambda^{(1)})_j$ for all i, j.* (*Equivalently, assume $(\lambda^{(0)})_1 \le (\lambda^{(1)})_k$.*) *Then a small pure resolution* (4-2) *exists.*

Proof. We construct the map geometrically. After twisting down by d, we may suppose instead $\lambda^{(0)} \le 0 \le \lambda^{(1)}$. Write $\lambda^{(0)} = -\mu^R$ for some partition $\mu \ge 0$. On $X = \mathrm{Hom}(V, W)$, there is a canonical, biequivariant map of vector bundles $\mathcal{T} : V \times X \to W \times X$, which is an isomorphism away from the determinant locus. When $\lambda \ge 0$, the Schur functor $\mathbb{S}_\lambda(V)$ is functorial for linear transformations of V (when λ has negative parts, $\mathbb{S}_\lambda$ is only functorial for isomorphisms), so there is an induced map

$$\mathbb{S}_{\lambda^{(1)}}(\mathcal{T}) : \mathbb{S}_{\lambda^{(1)}}(V) \times X \to \mathbb{S}_{\lambda^{(1)}}(W) \times X,$$

and, from the dual bundles, a second induced map

$$\mathbb{S}_\mu(\mathcal{T}^*) : \mathbb{S}_\mu(W^*) \times X \to \mathbb{S}_\mu(V^*) \times X.$$

Let $g = \mathbb{S}_{\lambda_{(1)}}(\mathcal{T}) \otimes \mathbb{S}_\mu(\mathcal{T}^*)$. Note that g is generically an isomorphism of vector bundles, so the corresponding map of R-modules is injective:

$$g : \mathbb{S}_{\lambda^{(1)}}(V) \otimes \mathbb{S}_\mu(W^*) \otimes R \to \mathbb{S}_\mu(V^*) \otimes \mathbb{S}_{\lambda^{(1)}}(W) \otimes R.$$

Finally, we note that there is a canonical isomorphism of representations $\mathbb{S}_\mu(E^*) \cong \mathbb{S}_{-\mu^R}(E)$ for any vector space E. Apply this to the free R-modules above to get the desired map. □

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Chebyshev's bias for products of k primes

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For any $k \geq 1$, we study the distribution of the difference between the number of integers $n \leq x$ with $\omega(n) = k$ or $\Omega(n) = k$ in two different arithmetic progressions, where $\omega(n)$ is the number of distinct prime factors of n and $\Omega(n)$ is the number of prime factors of n counted with multiplicity. Under some reasonable assumptions, we show that, if k is odd, the integers with $\Omega(n) = k$ have preference for quadratic nonresidue classes; and if k is even, such integers have preference for quadratic residue classes. This result confirms a conjecture of Richard Hudson. However, the integers with $\omega(n) = k$ always have preference for quadratic residue classes. Moreover, as k increases, the biases become smaller and smaller for both of the two cases.

1. Introduction and statement of results

First, we consider products of k primes in arithmetic progressions. Let

$$\pi_k(x; q, a) = |\{n \leq x : \omega(n) = k,\ n \equiv a \mod q\}|,$$

and

$$N_k(x; q, a) = |\{n \leq x : \Omega(n) = k,\ n \equiv a \mod q\}|,$$

where $\omega(n)$ is the number of distinct prime divisors of n, and $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. For example, when $k = 1$, $N_1(x; q, a)$ is the number of primes $\pi(x; q, a)$ in the arithmetic progression $a \mod q$; and $\pi_1(x; q, a)$ counts the number of prime powers $p^l \leq x$ for all $l \geq 1$ in the arithmetic progression $a \mod q$.

Dirichlet [1837] showed that, for any a and q with $(a, q) = 1$, there are infinitely many primes in the arithmetic progression $a \mod q$. Moreover, for any $(a, q) = 1$,

$$\pi(x; q, a) \sim \frac{x}{\phi(q) \log x},$$

where ϕ is Euler's totient function [Davenport 2000]. Analogous asymptotic formulas are available for products of k primes. Landau [1909] showed that, for each fixed integer $k \geq 1$,

$$N_k(x) := |\{n \leq x : \Omega(n) = k\}| \sim \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

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The same asymptotic is also true for the function $\pi_k(x) := |\{n \le x : \omega(n) = k\}|$. For more precise formulas, see [Tenenbaum 1995, II.6, Theorems 4 and 5]. Using similar methods as in [Tenenbaum 1995; Davenport 2000], one can show that, for any fixed residue class $a \bmod q$ with $(a, q) = 1$,

$$N_k(x; q, a) \sim \pi_k(x; q, a) \sim \frac{1}{\phi(q)} \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}.$$

For the case of counting primes ($\Omega(n) = 1$), Chebyshev [1853] observed that there seem to be more primes in the progression $3 \bmod 4$ than in the progression $1 \bmod 4$. That is, it appears that $\pi(x; 4, 3) \ge \pi(x; 4, 1)$. In general, for any $a \not\equiv b \bmod q$ and $(a, q) = (b, q) = 1$, one can study the behavior of the functions

$$\Delta_{\omega_k}(x; q, a, b) := \pi_k(x; q, a) - \pi_k(x; q, b),$$
$$\Delta_{\Omega_k}(x; q, a, b) := N_k(x; q, a) - N_k(x; q, b).$$

Denote $\Delta(x; q, a, b) := \Delta_{\Omega_1}(x; q, a, b)$. Littlewood [1914] proved that $\Delta(x; 4, 3, 1)$ changes sign infinitely often. Actually, $\Delta(x; 4, 3, 1)$ is negative for the first time at $x = 26,861$ [Leech 1957]. Knapowski and Turán published a series of papers starting with [1962] about the sign changes and extreme values of the functions $\Delta(x; q, a, b)$. And such problems are colloquially known today as "prime race problems". Irregularities in the distribution, that is, a tendency for $\Delta(x; q, a, b)$ to be of one sign is known as "Chebyshev's bias". For a nice survey of such works, see [Ford and Konyagin 2002; Granville and Martin 2006].

Chebyshev's bias can be well understood in the sense of *logarithmic density*. We say a set S of positive integers has *logarithmic density*, if the following limit exists:

$$\delta(S) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n \le x \\ n \in S}} \frac{1}{n}.$$

Let $\delta_{f_k}(q; a, b) = \delta(P_{f_k}(q; a, b))$, where $P_{f_k}(q; a, b)$ is the set of integers with $\Delta_{f_k}(n; q, a, b) > 0$, and $f = \Omega$ or ω. In order to study the Chebyshev's bias and the existence of the logarithmic density, we need the following assumptions:

(1) The *extended Riemann hypothesis* ($\mathrm{ERH_q}$) for Dirichlet L-functions modulo q.

(2) The *linear independence conjecture* ($\mathrm{LI_q}$), the imaginary parts of the zeros of all Dirichlet L-functions modulo q are linearly independent over $\mathbb{Q}$.

Under these two assumptions, Rubinstein and Sarnak [1994] showed that, for Chebyshev's bias for primes ($\Omega(n) = 1$), the logarithmic density $\delta_{\Omega_1}(q; a, b)$ exists, and in particular, $\delta_{\Omega_1}(4; 3, 1) \approx 0.996$ which indicates a strong bias for primes in the arithmetic progression $3 \bmod 4$. Recently, using the same assumptions, Ford and Sneed [2010] studied the Chebyshev's bias for products of two primes with $\Omega(n) = 2$ by transforming this problem into manipulations of some double integrals. They connected $\Delta_{\Omega_2}(x; q, a, b)$ with $\Delta(x; q, a, b)$, and showed that $\delta_{\Omega_2}(q; a, b)$ exists and the bias is in the opposite direction to the case of primes, in particular, $\delta_{\Omega_2}(4; 3, 1) \approx 0.10572$ which indicates a strong bias for the arithmetic progression $1 \bmod 4$.

By orthogonality of Dirichlet characters, we have

$$\Delta_{\Omega_k}(x; q, a, b) = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \bmod q} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{n \le x \\ \Omega(n)=k}} \chi(n), \tag{1-1}$$

and

$$\Delta_{\omega_k}(x; q, a, b) = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \bmod q} (\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\substack{n \le x \\ \omega(n)=k}} \chi(n). \tag{1-2}$$

The inner sums over n are usually analyzed using analytic methods. Neither the method of Rubinstein and Sarnak [1994] nor the method of Ford and Sneed [2010] readily generalizes to handle the cases of more prime factors ($k \ge 3$). From the point of view of L-functions, the most natural sum to consider is

$$\sum_{\substack{n_1 \cdots n_k \le x \\ n_1 \cdots n_k \equiv a \bmod q}} \Lambda(n_1) \cdots \Lambda(n_k). \tag{1-3}$$

However, estimates for $\Delta_{\Omega_k}(x; q, a, b)$ or $\Delta_{\omega_k}(x; q, a, b)$ cannot be readily recovered from such an analogue by partial summation. Ford and Sneed [2010] overcome this obstacle in the case $k = 2$ by means of the 2-dimensional integral

$$\int_0^\infty \int_0^\infty \sum_{p_1 p_2 \le x} \frac{\chi(p_1 p_2) \log p_1 \log p_2}{p_1^{u_1} p_2^{u_2}} \, du_1 \, du_2.$$

Analysis of an analogous k-dimensional integral leads to an explosion of cases, depending on the relative sizes of the variables u_j, and becomes increasingly messy as k increases.

We take an entirely different approach, working directly with the unweighted sums. We express the associated Dirichlet series in terms of products of the logarithms of Dirichlet L-functions, then apply Perron's formula, and use Hankel contours to avoid the zeros of $L(s, \chi)$ and the point $s = \frac{1}{2}$. Using the same assumptions (1) and (2), we show that, for any $k \ge 1$, both

$$\delta_{\Omega_k}(q; a, b) \quad \text{and} \quad \delta_{\omega_k}(q; a, b)$$

exist. Moreover, we show that, as k increases, if a is a quadratic nonresidue and b is a quadratic residue, the bias oscillates with respect to the parity of k for the case $\Omega(n) = k$, but $\delta_{\omega_k}(q; a, b)$ increases from below $\frac{1}{2}$ monotonically.

For some of our results, we need only a much weaker substitute for condition $\mathrm{LI_q}$, which we call the *simplicity hypothesis* ($\mathrm{SH_q}$): $\forall \chi \neq \chi_0 \bmod q$, $L\big(\frac{1}{2}, \chi\big) \neq 0$ and the zeros of $L(s, \chi)$ are simple. Let

$$N(q, a) := \#\{u \bmod q : u^2 \equiv a \bmod q\}.$$

Then, using the weaker assumptions $\mathrm{SH_q}$ and $\mathrm{ERH_q}$, we prove the following theorems.

Theorem 1. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k \geq 1$, *and fixed large* T_0,

$$\Delta_{\Omega_k}(x;q,a,b) = \frac{1}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\left\{\frac{(-1)^k}{\phi(q)}\sum_{\chi\neq\chi_0}(\bar{\chi}(a)-\bar{\chi}(b))\sum_{\substack{|\gamma_\chi|\leq T_0\\ L\left(\frac{1}{2}+i\gamma_\chi,\chi\right)=0}}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}\right.$$
$$\left. + \frac{(-1)^k}{2^{k-1}}\frac{N(q,a)-N(q,b)}{\phi(q)} + \Sigma_k(x;q,a,b,T_0)\right\},$$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y |\Sigma_k(e^y;q,a,b,T_0)|^2\,dy \ll \frac{\log^2 T_0}{T_0}.$$

Since $\Delta_{\Omega_1}(x;q,a,b) = \Delta(x;q,a,b)$, we get the following corollary.

Corollary 1.1. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k \geq 2$,

$$\frac{\Delta_{\Omega_k}(x;q,a,b)\log x}{\sqrt{x}(\log\log x)^{k-1}}$$
$$= \frac{(-1)^{k+1}}{(k-1)!}\left(1-\frac{1}{2^{k-1}}\right)\frac{N(q,a)-N(q,b)}{\phi(q)} + \frac{(-1)^{k+1}}{(k-1)!}\frac{\Delta(x;q,a,b)\log x}{\sqrt{x}} + \Sigma_k'(x;q,a,b),$$

where, as $Y \to \infty$,

$$\frac{1}{Y}\int_1^Y |\Sigma_k'(e^y;q,a,b)|^2\,dy = o(1).$$

In the above theorem, the constant $(-1)^k/(2^{k-1})\cdot(N(q,a)-N(q,b))/(\phi(q))$ represents the bias in the distribution of products of k primes counted with multiplicity. Richard Hudson conjectured that, as k increases, the bias would change directions according to the parity of k. Our result above confirms his conjecture (under $\mathrm{ERH_q}$ and $\mathrm{SH_q}$). Figures 1.1 and 1.2 show the graphs corresponding to $(q,a,b) = (4,3,1)$ for $2\log x/(\sqrt{x}(\log\log x)^2)\cdot\Delta_{\Omega_3}(x;4,3,1)$ and $6\log x/(\sqrt{x}(\log\log x)^3)\cdot\Delta_{\Omega_4}(x;4,3,1)$, plotted on a logarithmic scale from $x = 10^3$ to $x = 10^8$. In these graphs, the functions do not appear to be oscillating around $\frac{1}{4}$ and $-\frac{1}{8}$ respectively as predicted in our theorem. This is caused by some terms of order $1/\log\log x$ and even lower order terms, and $\log\log 10^8 \approx 2.91347$ and $1/\log\log 10^8 \approx 0.343233$. However, we can still observe the expected direction of the bias through these graphs.

For the distribution of products of k primes counted without multiplicity, we have the following theorem. In this case, the bias will be determined by the constant $(N(q,a)-N(q,b))/(2^{k-1}\phi(q))$ in the theorem below.

Theorem 2. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k \geq 1$, *and fixed large* T_0,

$$\Delta_{\omega_k}(x;q,a,b) = \frac{1}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\left\{\frac{(-1)^k}{\phi(q)}\sum_{\chi\neq\chi_0}(\bar{\chi}(a)-\bar{\chi}(b))\sum_{\substack{|\gamma_\chi|\leq T_0\\ L\left(\frac{1}{2}+i\gamma_\chi,\chi\right)=0}}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}\right.$$
$$\left. + \frac{N(q,a)-N(q,b)}{2^{k-1}\phi(q)} + \widetilde{\Sigma}_k(x;q,a,b,T_0)\right\},$$

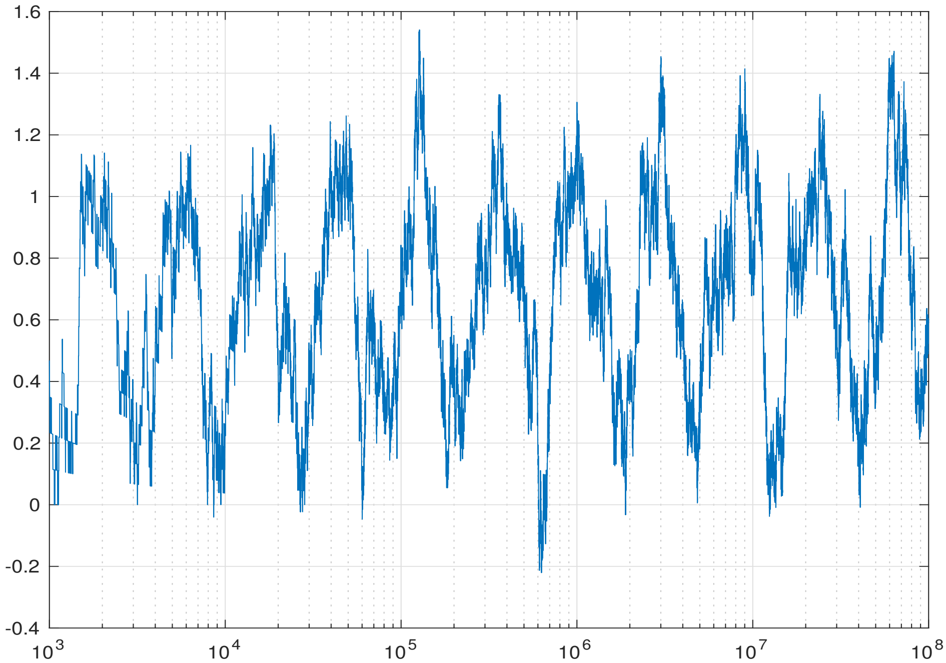

Figure 1.1. $\dfrac{2\log x}{\sqrt{x}(\log\log x)^2}\Delta_{\Omega_3}(x;4,3,1)$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y|\widetilde{\Sigma}_k(e^y;q,a,b,T_0)|^2\,dy \ll \frac{\log^2 T_0}{T_0}.$$

Corollary 2.1. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{SH_q}$. *Then, for any fixed* $k\geq 1$,

$$\frac{\Delta_{\omega_k}(x;q,a,b)\log x}{\sqrt{x}(\log\log x)^{k-1}} = \left(\frac{1}{2^{k-1}}+(-1)^{k+1}\right)\frac{N(q,a)-N(q,b)}{(k-1)!\phi(q)}+\frac{(-1)^{k+1}}{(k-1)!}\frac{\Delta(x;q,a,b)\log x}{\sqrt{x}}+\widetilde{\Sigma}'_k(x;q,a,b),$$

where, as $Y\to\infty$,

$$\frac{1}{Y}\int_1^Y|\widetilde{\Sigma}'_k(e^y;q,a,b)|^2\,dy = o(1).$$

For the distribution of $\Delta(x;q,a,b)$, Rubinstein and Sarnak [1994] showed the following theorem. This is the version from [Ford and Sneed 2010].

Theorem RS. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{LI_q}$. *For any* $a\not\equiv b \bmod q$ *and* $(a,q)=(b,q)=1$, *the function*

$$\frac{u\Delta(e^u;q,a,b)}{e^{u/2}}$$

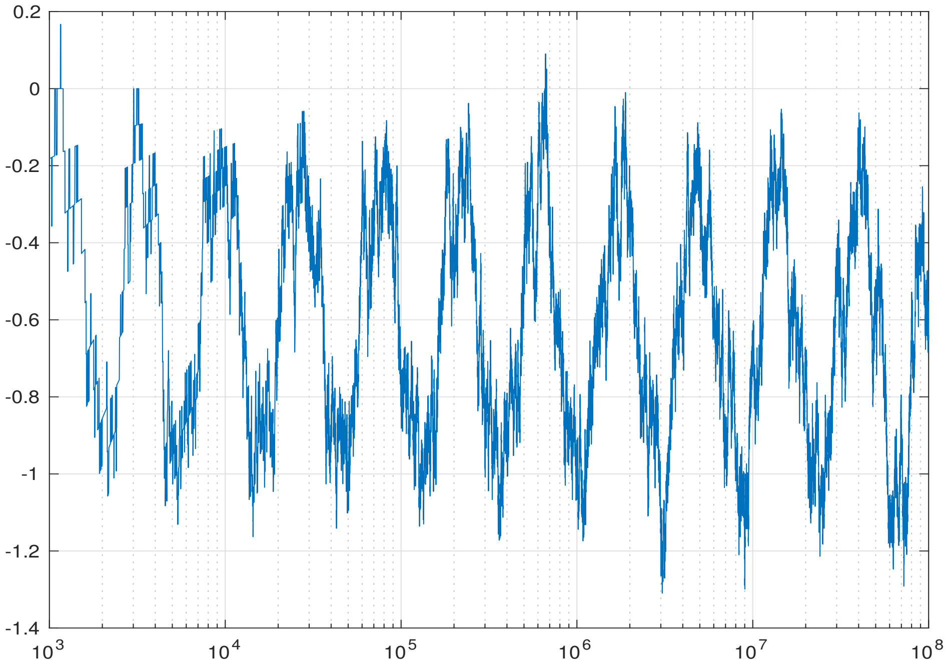

Figure 1.2. $\dfrac{6\log x}{\sqrt{x}(\log\log x)^3}\Delta_{\Omega_4}(x;4,3,1)$

has a probabilistic distribution. This distribution has mean $(N(q,b)-N(q,a))/\phi(q)$, *is symmetric with respect to its mean, and has a continuous density function.*

Corollaries 1.1, 2.1, and Theorem RS imply the following result.

Theorem 3. *Let* $a \not\equiv b \bmod q$ *and* $(a,q)=(b,q)=1$. *Assuming* $\mathrm{ERH_q}$ *and* $\mathrm{LI_q}$, *for any* $k \ge 1$, $\delta_{\Omega_k}(q;a,b)$ *and* $\delta_{\omega_k}(q;a,b)$ *exist. More precisely, if a and b are both quadratic residues or both quadratic nonresidues, then* $\delta_{\Omega_k}(q;a,b)=\delta_{\omega_k}(q;a,b)=\frac{1}{2}$. *Moreover, if a is a quadratic nonresidue and b is a quadratic residue, then, for any* $k \ge 1$,

$$1-\delta_{\Omega_{2k-1}}(q;a,b) < \delta_{\Omega_{2k}}(q;a,b) < \tfrac{1}{2} < \delta_{\Omega_{2k+1}}(q;a,b) < 1-\delta_{\Omega_{2k}}(q;a,b),$$
$$\delta_{\omega_k}(q;a,b) < \delta_{\omega_{k+1}}(q;a,b) < \tfrac{1}{2},$$
$$\delta_{\Omega_{2k}}(q;a,b) = \delta_{\omega_{2k}}(q;a,b), \quad \text{and} \quad \delta_{\Omega_{2k-1}}(q;a,b)+\delta_{\omega_{2k-1}}(q;a,b)=1.$$

Remark. The above results confirm a conjecture of Richard Hudson proposed years ago in his communications with Ford. Borrowing the methods from [Rubinstein and Sarnak 1994, Section 4], we are able to calculate $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$ precisely for special values of q, a, and b. In particular, we record in Table 1.1 the logarithmic densities up to products of 10 primes for two cases: $q=3$, $a=2$, $b=1$, and $q=4$, $a=3$, $b=1$.

k	$\delta_{\Omega_k}(3;2,1)$	$\delta_{\omega_k}(3;2,1)$	$\delta_{\Omega_k}(4;3,1)$	$\delta_{\omega_k}(4;3,1)$
1	0.99906†	0.00094	0.9959†	0.0041
2	0.069629	0.069629	0.10572‡	0.10572
3	0.766925	0.233075	0.730311	0.269689
4	0.35829	0.35829	0.380029	0.380029
5	0.571953	0.428047	0.380029	0.380029
6	0.463884	0.463884	0.469616	0.469616
7	0.518075	0.481925	0.515202	0.484798
8	0.49096	0.49096	0.492398	0.492398
9	0.50452	0.49548	0.503801	0.496199
10	0.49774	0.49774	0.498099	0.498099

Table 1.1. For $q = 3$, $a = 2$ and $b = 1$ (left-hand side) and $q = 4$, $a = 3$ and $b = 1$ (right-hand side). † [Rubinstein and Sarnak 1994] ‡ [Ford and Sneed 2010]

For fixed q and large k, we give asymptotic formulas for $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$.

Theorem 4. *Assume* $\mathrm{ERH_q}$ *and* $\mathrm{LI_q}$. *Let $A(q)$ be the number of real characters* mod q. *Let a be a quadratic nonresidue and b be a quadratic residue, and $(a,q)=(b,q)=1$. Then, for any nonnegative integer K, and any $\epsilon > 0$,*

$$\delta_{\Omega_k}(q;a,b)=\frac{1}{2}+\frac{(-1)^{k-1}}{2\pi}\sum_{j=0}^{K}\left(\frac{1}{2^{k-1}}\right)^{2j+1}\frac{(-1)^j A(q)^{2j+1}C_j(q;a,b)}{(2j+1)!}+O_{q,K,\epsilon}\left(\frac{1}{(2^{k-1})^{2K+3-\epsilon}}\right), \tag{1-4}$$

$$\delta_{\omega_k}(q;a,b)=\frac{1}{2}-\frac{1}{2\pi}\sum_{j=0}^{K}\left(\frac{1}{2^{k-1}}\right)^{2j+1}\frac{(-1)^j A(q)^{2j+1}C_j(q;a,b)}{(2j+1)!}+O_{q,K,\epsilon}\left(\frac{1}{(2^{k-1})^{2K+3-\epsilon}}\right), \tag{1-5}$$

where $C_j(q;a,b)$ is some constant depending on j, q, a, and b. In particular, for $K=0$,

$$\delta_{\Omega_k}(q;a,b)=\frac{1}{2}+(-1)^{k-1}\frac{A(q)C_0(q;a,b)}{2^k\pi}+O_{q,\epsilon}\left(\frac{1}{(2^k)^{3-\epsilon}}\right),$$
$$\delta_{\omega_k}(q;a,b)=\frac{1}{2}-\frac{A(q)C_0(q;a,b)}{2^k\pi}+O_{q,\epsilon}\left(\frac{1}{(2^k)^{3-\epsilon}}\right).$$

Remark. We have a formula for $C_j(q;a,b)$,

$$C_j(q;a,b)=\int_{-\infty}^{\infty}x^{2j}\,\Phi_{q;a,b}(x)\,dx,$$

where

$$\Phi_{q;a,b}(z)=\prod_{\chi\neq\chi_0}\prod_{\substack{\gamma_\chi>0\\ L\left(\frac{1}{2}+i\gamma_\chi\right)=0}}J_0\left(\frac{2|\chi(a)-\chi(b)|z}{\sqrt{\frac{1}{4}+\gamma_\chi^2}}\right),$$

and $J_0(z)$ is the Bessel function,

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{(m!)^2}.$$

Numerically, $C_0(3; 2, 1) \approx 3.66043$ and $C_0(4; 3, 1) \approx 3.08214$. When q is large, using the method in [Fiorilli and Martin 2013, Section 2], we can find asymptotic formulas for $C_j(q; a, b)$,

$$C_j(q; a, b) = \frac{(2j-1)!!\sqrt{2\pi}}{V(q; a, b)^{j+1/2}} + O_j\left(\frac{1}{V(q; a, b)^{j+3/2}}\right),$$

where $(2j-1)!! = (2j-1)(2j-3)\cdots 3\cdot 1$, $(-1)!! = 1$, and

$$V(q; a, b) = \sum_{\chi \bmod q} |\chi(b) - \chi(a)|^2 \sum_{\substack{\gamma_\chi \in \mathbb{R} \\ L\left(\frac{1}{2}+i\gamma_\chi, \chi\right)=0}} \frac{1}{\frac{1}{4} + \gamma_\chi^2}.$$

By Proposition 3.6 in [Fiorilli and Martin 2013], under $\mathrm{ERH_q}$, $V(q; a, b) \sim 2\phi(q)\log q$.

2. Formulas for the associated Dirichlet series and origin of the bias

Let χ be a nonprincipal Dirichlet character, and denote

$$F_{f_k}(s, \chi) := \sum_{f(n)=k} \frac{\chi(n)}{n^s},$$

where $f = \Omega$ or ω. The formulas for $F_{f_k}(s, \chi)$ are needed to analyze the character sums in (1-1) and (1-2). The purpose of this section is to express $F_{f_k}(s, \chi)$ in terms of Dirichlet L-functions, and to explain the source of the biases in the functions $\Delta_{\Omega_k}(x; q, a, b)$ and $\Delta_{\omega_k}(x; q, a, b)$.

Throughout the paper, the notation $\log z$ will always denote the *principal branch* of the logarithm of a complex number z.

2A. *Symmetric functions.* Let $x_1, x_2, \ldots$ be an infinite collection of indeterminates. We say a formal power series $P(x_1, x_2, \ldots)$ with bounded degree is a *symmetric function* if it is invariant under all finite permutations of the variables $x_1, x_2, \ldots$.

The n-th *elementary symmetric function* $e_n = e_n(x_1, x_2, \ldots)$ is defined by the generating function $\sum_{n=0}^{\infty} e_n z^n = \prod_{i=1}^{\infty}(1 + x_i z)$. Thus, e_n is the sum of all square-free monomials of degree n. Similarly, the n-th *homogeneous symmetric function* $h_n = h_n(x_1, x_2, \ldots)$ is defined by the generating function $\sum_{n=0}^{\infty} h_n z^n = \prod_{i=1}^{\infty} 1/(1 - x_i z)$. We see that, h_n is the sum of all possible monomials of degree n. And the n-th *power symmetric function* $p_n = p_n(x_1, x_2, \ldots)$ is defined to be $p_n = x_1^n + x_2^n + \cdots$.

The following result is due to Newton or Girard (see [Macdonald 1995, Chapter 1, (2.11) and (2.11'), page 23] or [Mendes and Remmel 2015, Chapter 2, Theorems 2.8 and 2.9]).

Lemma 5. *For any integer $k \geq 1$,*

$$kh_k = \sum_{n=1}^{k} h_{k-n} p_n, \tag{2-1}$$

$$ke_k = \sum_{n=1}^{k} (-1)^{n-1} e_{k-n} p_n. \tag{2-2}$$

2B. *Formula for $F_{\Omega_k}(s, \chi)$.* For $\Re(s) > 1$, we define

$$F(s, \chi) := \sum_{p} \frac{\chi(p)}{p^s},$$

the sum being over all primes p. Since

$$\log L(s, \chi) = \sum_{m=1}^{\infty} \sum_{p} \frac{\chi(p^m)}{mp^{ms}}, \tag{2-3}$$

we then have

$$F(s, \chi) = \log L(s, \chi) - \tfrac{1}{2} \log L(2s, \chi^2) + G(s), \tag{2-4}$$

where $G(s)$ is absolutely convergent for $\Re(s) \geq \sigma_0$ for any fixed $\sigma_0 > \frac{1}{3}$. Henceforth, σ_0 will be a fixed abscissa $> \frac{1}{3}$, say $\sigma_0 = 0.34$. Because $L(s, \chi)$ is an entire function for nonprincipal characters χ, formula (2-4) provides an analytic continuation of $F(s, \chi)$ to any simply connected domain within the half-plane $\{s : \Re(s) \geq \sigma_0\}$ which avoids the zeros of $L(s, \chi)$ and the zeros and possible pole of $L(2s, \chi^2)$.

For any complex number s with $\Re(s) \geq \sigma_0 > \frac{1}{3}$, let $x_p = \chi(p)/p^s$ if p is a prime, 0 otherwise. Then, by (2-1) in Lemma 5, we have the following relation

$$kF_{\Omega_k}(s, \chi) = \sum_{n=1}^{k} F_{\Omega_{k-n}}(s, \chi) F(ns, \chi^n). \tag{2-5}$$

For example, for $k = 1$, $F_{\Omega_1}(s, \chi) = F(s, \chi)$. For $k = 2$,

$$2F_{\Omega_2}(s, \chi) = F^2(s, \chi) + F(2s, \chi^2).$$

For $k = 3$,

$$\begin{aligned} 3!F_{\Omega_3}(s, \chi) &= 2F_{\Omega_2}(s, \chi)F(s, \chi) + 2F(s, \chi)F(2s, \chi^2) + 2F(3s, \chi^3) \\ &= F^3(s, \chi) + 3F(s, \chi)F(2s, \chi^2) + 2F(3s, \chi^3). \end{aligned}$$

For $k = 4$,

$$\begin{aligned} 4!F_{\Omega_4}(s, \chi) &= 3!F_{\Omega_3}(s, \chi)F(s, \chi) + 3!F_{\Omega_2}(s, \chi)F(2s, \chi^2) + 3!F(s, \chi)F(3s, \chi^2) + 3!F(4s, \chi^4) \\ &= F^4(s, \chi) + 6F^2(s, \chi)F(2s, \chi^2) + 8F(s, \chi)F(3s, \chi^3) + 6F(4s, \chi^4) + 3F^2(2s, \chi^2). \end{aligned}$$

For any integer $l \geq 1$, we define the set

$$S_{m,l}^{(k)} := \{(n_1, \cdots, n_l) \mid n_1 + \cdots + n_l = k - m,\ 2 \leq n_1 \leq n_2 \leq \cdots \leq n_l,\ n_j \in \mathbb{N}(1 \leq j \leq l)\}$$

Let $S_m^{(k)} = \bigcup_{l\geq 1} S_{m,l}^{(k)}$. Thus any element of $S_m^{(k)}$ is a partition of $k-m$ with each part ≥ 2. For any $\boldsymbol{n} = (n_1, n_2, \cdots, n_l) \in S_m^{(k)}$, denote

$$F(\boldsymbol{n}s, \chi) := \prod_{j=1}^{l} F(n_j s, \chi^{n_j}).$$

Hence, by (2-5) and induction on k, we deduce the following result.

Lemma 6. *For $k = 1$, $F_{\Omega_1}(s, \chi) = F(s, \chi)$. For any $k \geq 2$, we have*

$$k! F_{\Omega_k}(s, \chi) = F^k(s, \chi) + \sum_{m=0}^{k-2} F^m(s, \chi) F_{\boldsymbol{n_m}}(s, \chi), \tag{2-6}$$

where $F_{\boldsymbol{n_m}}(s, \chi) = \sum_{\boldsymbol{n}\in S_m^{(k)}} a_m^{(k)}(\boldsymbol{n}) F(\boldsymbol{n}s, \chi)$ for some $a_m^{(k)}(\boldsymbol{n}) \in \mathbb{N}$.

2C. *Formula for $F_{\omega_k}(s, \chi)$.* By definition, we have

$$F_{\omega_k}(s, \chi) = \sum_{p_1<p_2<\cdots<p_k} \prod_{n=1}^{k} \biggl(\sum_{j=1}^{\infty} \frac{\chi(p_n^j)}{p_n^j} \biggr).$$

Denote

$$\tilde{F}(s, \chi) := \sum_p \biggl(\frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \biggr),$$

and for any $u \in \mathbb{N}^+$,

$$\tilde{F}(s, \chi; u) := \sum_p \biggl(\frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots \biggr)^u = \sum_p \sum_{j=u}^{\infty} \biggl(D_u(j) \frac{\chi(p^j)}{p^{js}} \biggr),$$

where $D_u(j) = \binom{j-1}{u-1}$ is the number of ways of writing j as sum of u ordered positive integers.

By (2-3), we have

$$\tilde{F}(s, \chi) = \tilde{F}(s, \chi; 1) = \sum_p \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^{js}} = \log L(s, \chi) + \tfrac{1}{2} \log L(2s, \chi^2) + \tilde{G}_1(s) \tag{2-7}$$

and

$$\tilde{F}(s, \chi; 2) = \sum_p \sum_{j=2}^{\infty} (j-1) \frac{\chi(p^j)}{p^{js}} = \log L(2s, \chi^2) + \tilde{G}_2(s), \tag{2-8}$$

where $\tilde{G}_1(s)$ and $\tilde{G}_2(s)$ are absolutely convergent for $\Re(s) \geq \sigma_0$. Formula (2-7) provides an analytic continuation of $\tilde{F}(s, \chi)$ to any simply connected domain within the half-plane $\{s : \Re(s) \geq \sigma_0\}$ which avoids the zeros of $L(s, \chi)$ and the zeros and possible pole of $L(2s, \chi^2)$. Moreover, for any fixed $u \geq 3$, $\tilde{F}(s, \chi; u)$ is absolutely convergent for $\Re(s) \geq \sigma_0$.

For any complex number s with $\Re(s) \geq \sigma_0$, take $x_p = \sum_{j=1}^{\infty} \frac{\chi(p^j)}{p^{js}}$ if p is a prime, 0 otherwise. Then by (2-2) in Lemma 5, we get the following formula,

$$kF_{\omega_k}(s,\chi) = F_{\omega_{k-1}}(s,\chi)\tilde{F}(s,\chi) - \sum_{n=2}^{k}(-1)^n F_{\omega_{k-n}}(s,\chi)\tilde{F}(s,\chi;n). \tag{2-9}$$

For example, for $k=1$, $F_{\omega_1}(s,\chi) = \tilde{F}(s,\chi)$. For $k=2$,

$$2F_{\omega_2}(s,\chi) = \tilde{F}^2(s,\chi) - \tilde{F}(s,\chi;2).$$

For $k=3$,

$$\begin{aligned} 3!F_{\omega_3}(s,\chi) &= 2F_{\omega_2}(s,\chi)\tilde{F}(s,\chi) - 2F_{\omega_1}(s,\chi)\tilde{F}(s,\chi;2) + 2\tilde{F}(s,\chi;3) \\ &= \tilde{F}^3(s,\chi) - 3\tilde{F}(s,\chi)\tilde{F}(s,\chi;2) + 2\tilde{F}(s,\chi;3). \end{aligned}$$

For $k=4$,

$$\begin{aligned} 4!F_{\omega_4}(s,\chi) &= 3!F_{\omega_3}(s,\chi)\tilde{F}(s,\chi) - 3!F_{\omega_2}(s,\chi)\tilde{F}(s,\chi;2) + 3!\tilde{F}(s,\chi)\tilde{F}(s,\chi;3) - 3!\tilde{F}(s,\chi;4) \\ &= \tilde{F}^4(s,\chi) - 6\tilde{F}^2(s,\chi)\tilde{F}(s,\chi;2) + 8\tilde{F}(s,\chi)\tilde{F}(s,\chi;3) - 6\tilde{F}(s,\chi;4) + 3\tilde{F}^2(s,\chi;2). \end{aligned}$$

Hence, by (2-9) and induction on k, we get the following result.

Lemma 7. *For $k=1$, $F_{\omega_1}(s,\chi) = \tilde{F}(s,\chi)$. For any $k \geq 2$, we have*

$$k!F_{\omega_k}(s,\chi) = \tilde{F}^k(s,\chi) + \sum_{m=0}^{k-2} \tilde{F}^m(s,\chi)\tilde{F}_{\boldsymbol{n_m}}(s,\chi), \tag{2-10}$$

where $\tilde{F}_{\boldsymbol{n_m}}(s,\chi) = \sum_{\boldsymbol{n}\in S_m^{(k)}} b_m^{(k)}(\boldsymbol{n})\tilde{F}(\boldsymbol{n}s,\chi)$ for some $b_m^{(k)}(\boldsymbol{n}) \in \mathbb{Z}$, and for any $\boldsymbol{n} = (n_1,\ldots,n_l) \in S_m^{(k)}$, $\tilde{F}(\boldsymbol{n}s,\chi) := \prod_{j=1}^{l} \tilde{F}(s,\chi;n_j)$.

2D. *Origin of the bias.* In this section, we heuristically explain the origin of the bias in our theorems.

(1) *Analytical aspect.* In order to get formulas for $\Delta_{\Omega_k}(x;q,a,b)$ and $\Delta_{\omega_k}(x;q,a,b)$, our strategy is to apply Perron's formula to the associated Dirichlet series $F_{\Omega_k}(s,\chi)$ and $F_{\omega_k}(s,\chi)$, then we choose special contours to avoid the singularities of these Dirichlet series. See Section 3 for the details.

First, we have a look at the case of counting primes in arithmetic progressions. If we only count primes, by (2-4), we have

$$F_{\Omega_1}(s,\chi) = F(s,\chi) = \sum_p \frac{\chi(p)}{p^s} = \log L(s,\chi) - \tfrac{1}{2}\log L(2s,\chi^2) + G(s).$$

The main contributions for $\Delta_{\Omega_1}(x;q,a,b)$ are from the first two terms,

$$\log L(s,\chi) - \tfrac{1}{2}\log L(2s,\chi^2).$$

The first term $\log L(s,\chi)$ counts all the primes with weight 1 and prime squares with weight $\frac{1}{2}$. The higher order powers of primes are negligible since they only contribute $O(x^{1/3})$. The singularities of $\log L(s,\chi)$, i.e., the zeros of $L(s,\chi)$, on the critical line contribute the oscillating terms in our result. In our proof, we use special Hankel contours to avoid the singularities of $\log L(s,\chi)$ and extract these oscillating terms (Lemma 12). See Sections 3 and 4 for the details of how to handle these singularities. The second term $-\frac{1}{2}\log L(2s,\chi^2)$ counts the prime squares with weight $-\frac{1}{2}$ and contributes the bias term. When χ is a real character, the point $s=\frac{1}{2}$ is a pole of $L(2s,\chi^2)$, and hence the integration of $-\frac{1}{2}\log L(2s,\chi^2)$ over the Hankel contour around $s=\frac{1}{2}$ contributes a bias term with order of magnitude $\sqrt{x}/\log x$. Using the orthogonality of Dirichlet characters, and the formula $\sum_{\chi \text{ real}}(\bar{\chi}(a)-\bar{\chi}(b)) = N(q,a)-N(q,b)$, we get the expected size of the bias.

Another natural and convenient function to consider is $-L'(s,\chi)/L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\Lambda(n)/n^s$, which is much easier to analyze than $\log L(s,\chi)$. This weighted form counts each prime p and its powers with weight $\log p$. Similar to $\log L(s,\chi)$, all the singularities of the function $-L'(s,\chi)/L(s,\chi)$ on the critical line are the nontrivial zeros of $L(s,\chi)$ and thus there is no bias for this weighted counting function

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)-\sum_{\substack{n\le x\\ n\equiv b \bmod q}}\Lambda(n).$$

Thus, partial summation is used to extract the sum

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\frac{\Lambda(n)}{\log n}-\sum_{\substack{n\le x\\ n\equiv b \bmod q}}\frac{\Lambda(n)}{\log n}$$

from the above weighted form, which is possible because $\log n$ is a smooth function. However, there is no way to do this with the analogue (1-3) to recover the unweighted counting function $\Delta_{\Omega_k}(x;q,a,b)$ or $\Delta_{\omega_k}(x;q,a,b)$.

If we count all the prime powers with the same weight 1, by (2-7), we have

$$F_{\omega_1}(s,\chi)=\tilde{F}(s,\chi)=\log L(s,\chi)+\tfrac{1}{2}\log L(2s,\chi^2)+\tilde{G}_1(s).$$

In this case, the bias is from the second term $\frac{1}{2}\log L(2s,\chi^2)$ for real character χ which counts the prime squares with positive weight $\frac{1}{2}$. This is why the bias is opposite to the case of counting only primes.

For the general case, when we derive the formula for $\Delta_{\Omega_k}(x;q,a,b)$ using analytic methods, by (2-6) in Lemma 6, the main contributions for $F_{\Omega_k}(s,\chi)$ will be from $\frac{1}{k!}F^k(s,\chi)$, which is essentially

$$\frac{1}{k!}\big(\log L(s,\chi)-\tfrac{1}{2}\log L(2s,\chi^2)\big)^k.$$

In the expansion of the above formula, the term $\frac{1}{k!}\log^k L(s,\chi)$ contributes the oscillating terms (see (4-9) and (4-13))

$$\frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\sum_{L\left(\frac{1}{2}+i\gamma_\chi,\chi\right)=0}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}.$$

When χ is real, the term

$$\frac{1}{k!}\big(-\tfrac{1}{2}\log L(2s,\chi^2)\big)^k=\frac{(-1)^k}{k!2^k}(\log L(2s,\chi^2))^k$$

contributes a bias term (see (4-10) and (4-14))

$$\frac{1}{(k-1)!}\frac{(-1)^k}{2^{k-1}}\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}.$$

Then summing over all the real characters, we get the expected bias term in our formula for $\Delta_{\Omega_k}(x;q,a,b)$. The factor $(-1)^k/2^{k-1}$ explains why the bias has different directions depending on the parity of k and why the bias decreases as k increases. Other terms with factors of the form $\log^{k-j} L(s,\chi)\log^j(2s,\chi^2)$ for $1\le j\le k-1$ only contribute oscillating terms with lower orders of $\log\log x$ which can be put into the error term in our formula (see Lemma 14).

Similarly, for the case of $\Delta_{\omega_k}(x;q,a,b)$, by (2-10) in Lemma 7, the main contributions for $F_{\omega_k}(s,\chi)$ are from

$$\frac{1}{k!}\tilde{F}^k(s,\chi)=\frac{1}{k!}\big(\log L(s,\chi)+\tfrac{1}{2}\log L(2s,\chi^2)+\tilde{G}_1(s)\big)^k.$$

The main terms are from the contributions of the terms $\frac{1}{k!}\log^k L(s,\chi)$ and $\frac{1}{k!}\big(\frac{1}{2}\log L(2s,\chi^2)\big)^k$. Thus, the main oscillating terms are the same as that of $\Delta_{\Omega_k}(x;q,a,b)$, and the bias term has the same size without direction change.

Through the above analysis, we see that the biases are mainly affected by the powers of $\pm\frac{1}{2}\log L(2s,\chi^2)$ for real characters which count the products of prime squares.

(2) *Combinatorial aspect.* Instead of giving precise prediction of the size of the bias as above, here we use a simpler combinatorial intuition to roughly explain the behavior of the bias. We borrowed this combinatorial explanation from Hudson [1980].

Pick a large number X. Let S_1 be the set of primes $p\equiv 1 \bmod 4$ up to X, and S_2 be the set of primes $p\equiv 3 \bmod 4$ up to X. Using these primes, we generate the set $V^{(2)}:=\{pq : p,q\in S_1\cup S_2,\ p \text{ and } q \text{ can be the same}\}$.

Let $V_1^{(2)}:=\{n\in V^{(2)} : n\equiv 1 \bmod 4\}$, and $V_2^{(2)}:=\{n\in V^{(2)} : n\equiv 3 \bmod 4\}$. Then, the integers in $V_1^{(2)}$ come from either products of two primes from S_1 or products of two primes from S_2. The integers in $V_2^{(2)}$ are the product of two primes pq with $p\in S_1$ and $q\in S_2$. Thus,

$$|V_1^{(2)}|=\binom{|S_1|}{2}+|S_1|+\binom{|S_2|}{2}+|S_2|=\frac{|S_1|^2+|S_2|^2}{2}+\frac{|S_1|+|S_2|}{2},$$

and

$$|V_2^{(2)}|=|S_1|\cdot|S_2|.$$

It is clear that $|V_1^{(2)}|>|V_2^{(2)}|$. Note that $\frac{1}{2}(|S_1|+|S_2|)$ counts the squares of primes with weight $\frac{1}{2}$ which makes a crucial difference between $V_1^{(2)}$ and $V_2^{(2)}$.

Let $V_1^{(0)}=\{1\}$ and $V_2^{(0)}=\varnothing$. For any $k\ge 1$, denote

$$V_1^{(k)}:=\{n=p_1\cdots p_k : p_j\in S_1\cup S_2 \text{ for all } 1\le j\le k,\ n\equiv 1 \bmod 4\},$$
$$V_2^{(k)}:=\{n=p_1\cdots p_k : p_j\in S_1\cup S_2 \text{ for all } 1\le j\le k,\ n\equiv 3 \bmod 4\},$$

where the p_j can be the same. Note that $V_1^{(1)}=S_1$ and $V_2^{(1)}=S_2$.

We give inductive formulas for $|V_1^{(k)}|$ and $|V_2^{(k)}|$. The elements of $V_1^{(k)}$ and $V_2^{(k)}$ are generated by integers of the form $p^j n_{k-j}$ for $p \in S_1$ or S_2 and $n_{k-j} \in V_1^{(k-j)}$ or $V_2^{(k-j)}$ $(1 \le j \le k)$. By (2-1) in Lemma 5, we have

$$k|V_1^{(k)}| = \underbrace{(|V_1^{(k-1)}|\cdot|S_1|+|V_2^{(k-1)}|\cdot|S_2|)}_{pn_{k-1}} + \underbrace{|V_1^{(k-2)}|(|S_1|+|S_2|)}_{p^2n_{k-2}} + \underbrace{(|V_1^{(k-3)}|\cdot|S_1|+|V_2^{(k-3)}|\cdot|S_2|)}_{p^3n_{k-3}} + \cdots$$

and

$$k|V_2^{(k)}| = \underbrace{(|V_2^{(k-1)}|\cdot|S_1|+|V_1^{(k-1)}|\cdot|S_2|)}_{pn_{k-1}} + \underbrace{|V_2^{(k-2)}|(|S_1|+|S_2|)}_{p^2n_{k-2}} + \underbrace{(|V_2^{(k-3)}|\cdot|S_1|+|V_1^{(k-3)}|\cdot|S_2|)}_{p^3n_{k-3}} + \cdots .$$

Thus,

$$\begin{aligned}
&k(|V_1^{(k)}|-|V_2^{(k)}|)\\
&\quad= (|V_1^{(k-1)}|\cdot|S_1|+|V_2^{(k-1)}|\cdot|S_2|) - (|V_2^{(k-1)}|\cdot|S_1|+|V_1^{(k-1)}|\cdot|S_2|) + (|V_1^{(k-2)}|-|V_2^{(k-2)}|)\\
&\qquad\quad (|S_1|+|S_2|) + (|V_1^{(k-3)}|\cdot|S_1|+|V_2^{(k-3)}|\cdot|S_2|) - (|V_2^{(k-3)}|\cdot|S_1|+|V_1^{(k-3)}|\cdot|S_2|) + \cdots\\
&\quad= (|V_1^{(k-1)}|-|V_2^{(k-1)}|)(|S_1|-|S_2|) + (|V_1^{(k-2)}|-|V_2^{(k-2)}|)(|S_1|+|S_2|)\\
&\qquad\quad + (|V_1^{(k-3)}|-|V_2^{(k-3)}|)(|S_1|-|S_2|) + \cdots .
\end{aligned} \tag{2-11}$$

For $1 \le j \le k-1$, suppose $|V_1^{(j)}| < |V_2^{(j)}|$ for odd j and $|V_1^{(j)}| > |V_2^{(j)}|$ for even j. Therefore, by (2-11) and induction, we deduce that $|V_1^{(k)}| < |V_2^{(k)}|$ for odd k and $|V_1^{(k)}| > |V_2^{(k)}|$ for even k. This provides us a heuristic explanation for the bias oscillation of $\Delta_{\Omega_k}(x; 4, 3, 1)$.

This heuristic also works for the quadratic residues and nonresidues modulo q whenever q has a primitive root. Because in this case, all the residues form a cyclic group (we thank the referee for pointing this out). When q has no primitive root, one should consider the group structure of the residue classes if we want to give a similar heuristic as above.

For example, for $q = 8$, we know that $1 \bmod 8$ is the only quadratic residue, and

$$3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \bmod 8, \quad 3\cdot 5 \equiv 7 \bmod 8, \quad 5\cdot 7 \equiv 3 \bmod 8, \quad \text{and} \quad 3\cdot 7 \equiv 5 \bmod 8. \tag{2-12}$$

If we define $V_j^{(1)} := \{p \equiv j \bmod 8,\, p \le X\}$ for $j = 1, 3, 5$, and 7. For any $k \ge 2$, let

$$V_j^{(k)} := \{n = p_1 \cdots p_k \equiv j \bmod 8,\, p_i \in V_1^{(1)} \cup V_3^{(1)} \cup V_5^{(1)} \cup V_7^{(1)},\, 1 \le i \le k\}, \quad \text{for } j = 1, 3, 5, 7.$$

Then similar to $q = 4$, using (2-12) and Lemma 5, we have

$$\begin{aligned}
k|V_1^{(k)}| &= \underbrace{\big(|V_1^{(k-1)}|\cdot|V_1^{(1)}| + |V_3^{(k-1)}|\cdot|V_3^{(1)}| + |V_5^{(k-1)}|\cdot|V_5^{(1)}| + |V_7^{(k-1)}|\cdot|V_7^{(1)}|\big)}_{pn_{k-1}}\\
&\qquad + \underbrace{|V_1^{(k-2)}|\big(|V_1^{(1)}| + |V_3^{(1)}| + |V_5^{(1)}| + |V_7^{(1)}|\big)}_{p^2n_{k-2}} + \cdots
\end{aligned}$$

$$k|V_3^{(k)}| = \underbrace{\big(|V_3^{(k-1)}|\cdot|V_1^{(1)}| + |V_1^{(k-1)}|\cdot|V_3^{(1)}| + |V_5^{(k-1)}|\cdot|V_7^{(1)}| + |V_7^{(k-1)}|\cdot|V_5^{(1)}|\big)}_{pn_{k-1}} + \underbrace{|V_3^{(k-2)}|\big(|V_1^{(1)}| + |V_3^{(1)}| + |V_5^{(1)}| + |V_7^{(1)}|\big)}_{p^2 n_{k-2}} + \cdots$$

and similarly for $k|V_5^{(k)}|$ and $k|V_7^{(k)}|$. For $1 \le l \le k-1$, suppose $|V_1^{(l)}| < |V_3^{(l)}| \sim |V_5^{(l)}| \sim |V_7^{(l)}|$ for odd l, by the formulas for $k|V_j^{(k)}|$ ($j = 1, 3, 5, 7$) and induction on k, we will derive the expected bias phenomenon.

3. Contour integral representation

In this section, we express the inner sums in (1-1) and (1-2) as integrals over truncated Hankel contours (see Lemma 10 below).

Let

$$\psi_{f_k}(x, \chi) := \sum_{\substack{n \le x \\ f(n)=k}} \chi(n),$$

where $f = \Omega$ or ω. By Perron's formula [Karatsuba 1993, Chapter V, Theorem 1] we have the following lemma.

Lemma 8. *For any $T \ge 2$,*

$$\psi_{f_k}(x, \chi) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F_{f_k}(s, \chi)\frac{x^s}{s}\,ds + O\bigg(\frac{x\log x}{T} + 1\bigg),$$

where $c = 1 + 1/\log x$, and $f = \Omega$ or ω.

Starting from Lemma 8, we will shift the contour to the left, in a way which avoids the singularities of the integrand. We will then require estimates of the integrand along the various parts of the new contour.

Lemma 9. *Assume* $\mathrm{ERH_q}$. *Then, for any $0 < \delta < \frac{1}{6}$ and for all $\chi \ne \chi_0 \bmod q$, there exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ satisfying $n \le T_n \le n+1$ such that, for $T \in \mathcal{T}$,*

$$F_{f_k}(\sigma + iT) = O(\log^k T), \quad \big(\tfrac{1}{2} - \delta < \sigma < 2\big)$$

where $f = \Omega$ or ω.

Proof. Using the similar method as in [Titchmarsh 1986, Theorem 14.16], one can show that, for any $\epsilon > 0$ and for all $\chi \ne \chi_0 \bmod q$, there exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ satisfying $n \le T_n \le n+1$ such that, $T_n^{-\epsilon} \ll |L(\sigma + iT_n, \chi)| \ll T_n^{\delta+\epsilon}$), $\big(\frac{1}{2} - \delta < \sigma < 2\big)$. Hence, by formulas (2-4), (2-6), (2-7), (2-8), and (2-10), we get the conclusion of this lemma. □

Let ρ be a zero of $L(s, \chi)$, Δ_ρ be the distance of ρ to the nearest other zero, and $D_\gamma := \min_{T \in \mathcal{T}}(|\gamma - T|)$. For each zero ρ, and $X > 0$, let $\mathcal{H}(\rho, X)$ denote the truncated Hankel contour surrounding the point $s = \rho$ with radius $0 < r_\rho \le \min\big(\frac{1}{x}, \frac{1}{3}\Delta_\rho, \frac{1}{2}D_\gamma, \frac{1}{3}\big|\rho - \frac{1}{2}\big|\big)$, which includes the circle $|s - \rho| = r_\rho$ excluding the

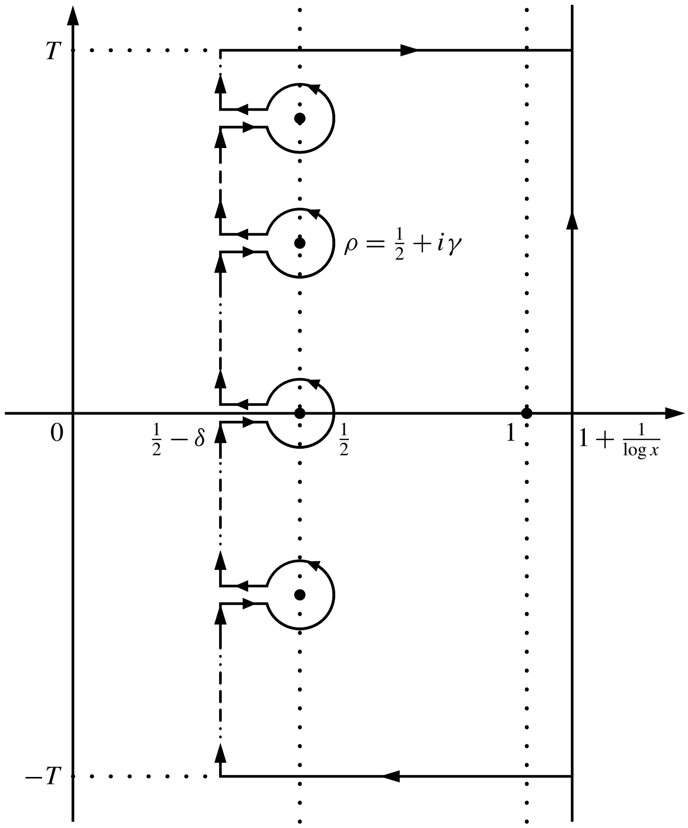

Figure 3.1. Integration contour

point $s = \rho - r_\rho$, and the half-line $(\rho - X, \rho - r]$ traced twice with arguments $+\pi$ and $-\pi$ respectively. Let Δ_0 be the distance of $\frac{1}{2}$ to the nearest zero. Let $\mathscr{H}(\frac{1}{2}, X)$ denote the corresponding truncated Hankel contour surrounding $s = \frac{1}{2}$ with radius $r_0 = \min\left(\frac{1}{x}, \frac{\Delta_0}{3}\right)$.

Take $\delta = \frac{1}{10}$. By Lemma 8, we pull the contour to the left to the line $\Re(s) = \frac{1}{2} - \delta$ using the truncated Hankel contour $\mathscr{H}(\rho, \delta)$ to avoid the zeros of $L(s, \chi)$ and using $\mathscr{H}\left(\frac{1}{2}, \delta\right)$ to avoid the point $s = \frac{1}{2}$. See Figure 3.1.

Then we have the following lemma.

Lemma 10. *Assume* $\mathrm{ERH_q}$, *and* $L\left(\frac{1}{2}, \chi\right) \neq 0$ $(\chi \neq \chi_0)$. *Then, for any fixed* $k \geq 1$, *and* $T \in \mathscr{T}$,

$$\psi_{f_k}(x, \chi) = \sum_{|\gamma| \leq T} \frac{1}{2\pi i} \int_{\mathscr{H}(\rho,\delta)} F_{f_k}(s, \chi) \frac{x^s}{s}\, ds + a(\chi) \frac{1}{2\pi i} \int_{\mathscr{H}\left(\frac{1}{2},\delta\right)} F_{f_k}(s, \chi) \frac{x^s}{s}\, ds \\ + O\left(\frac{x \log x}{T} + \frac{x (\log T)^k}{T} + x^{1/2-\delta} (\log T)^{k+1}\right),$$

where $a(\chi) = 1$ *if* χ *is real*, 0 *otherwise*, *and* $f = \omega$ *or* Ω.

Proof. By formulas (2-6) and (2-10), if χ is not real, $s = \frac{1}{2}$ is not a singularity of $F_{f_k}(s, \chi)$. Hence the second term is zero if χ is not real. By Lemma 9, the integral on the horizontal line is

$$\ll (\log T)^k \int_{\frac{1}{2}-\delta}^{c} \frac{x^\sigma}{|\sigma + iT|}\, d\sigma \ll \frac{x^c (\log T)^k}{T} \ll \frac{x(\log T)^k}{T}. \tag{3-1}$$

Under the assumption $\mathrm{ERH_q}$, the integral on the vertical line $\Re(s) = \frac{1}{2} - \delta$ is

$$\ll \int_{-T}^{T} \frac{x^{1/2-\delta} \log^k(|t|+2)}{\left|\frac{1}{2} - \delta + it\right|}\, dt \ll x^{1/2-\delta}(\log T)^{k+1}. \tag{3-2}$$

By (3-1), (3-2), and Lemma 8, we get the desired error term in this lemma. □

4. Proof of the main theorems

In this section, we give the full proof of Theorems 1 and 2, by quoting Lemmas 12, 13, 14, and 15 of which the proofs will appear in later sections.

Proof of Theorems 1 and 2. Let γ be the imaginary part of a zero of $L(s, \chi)$ in the critical strip. We have the following lemma.

Lemma 11 [Ford and Sneed 2010, Lemma 2.2]. *Let χ be a Dirichlet character modulo q. Let $N(T, \chi)$ denote the number of zeros of $L(s, \chi)$ with $0 < \Re(s) < 1$ and $|\Im(s)| < T$. Then*

(1) $N(T, \chi) = O(T \log(qT))$ *for* $T \ge 1$,

(2) $N(T, \chi) - N(T-1, \chi) = O(\log(qT))$ *for* $T \ge 1$,

(3) *uniformly for* $s = \sigma + it$ *and* $\sigma \ge -1$,

$$\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{|\gamma - t| < 1} \frac{1}{s - \rho} + O(\log q(|t|+2)). \tag{4-1}$$

For simplicity, we denote

$$\frac{1}{\Gamma_j(u)} := \left[\frac{d^j}{dz^j}\left(\frac{1}{\Gamma(z)}\right)\right]_{z=u}.$$

The following lemma is the starting lemma to give us the bias terms and oscillating terms in our main theorems. This lemma may have independent use, we will give the proof in Section 8.

Lemma 12. *Let $\mathcal{H}(a, \delta)$ be the truncated Hankel contour surrounding a complex number a ($\Re(a) > 2\delta$) with radius $0 < r \ll \frac{1}{x}$. Then, for any integer $k \ge 1$,*

$$\frac{1}{2\pi i} \int_{\mathcal{H}(a,\delta)} \log^k(s-a) \frac{x^s}{s}\, ds = \frac{(-1)^k x^a}{a \log x} \left\{ k(\log\log x)^{k-1} + \sum_{j=2}^{k} \binom{k}{j} (\log\log x)^{k-j} \frac{1}{\Gamma_j(0)} \right\} + O_k\left(\frac{|x^{a-\delta/3}|}{|a|}\right)$$
$$+ O_k\left(\frac{|x^a|}{|a|^2 \log^2 x} (\log\log x)^{k-1}\right) + O_k\left(\frac{|x^a|}{|a|^2 |\Re(a) - \delta|} \frac{(\log\log x)^{k-1}}{(\log x)^3}\right).$$

Remark. By (5-3) in the proof of Lemma 16, one can easily show that

$$\left|\frac{1}{\Gamma_j(0)}\right| \ll \Gamma(j+1). \tag{4-2}$$

By Lemma 10, we need to examine the integration over the truncated Hankel contours $\mathcal{H}(\rho,\delta)$ and $\mathcal{H}\big(\frac{1}{2},\delta\big)$. By (2-4) and (2-7), and the assumptions of our theorems, on each truncated Hankel contour $\mathcal{H}(\rho,\delta)$, we integrate the formula (4-1) in Lemma 11 to obtain

$$F(s,\chi) = \log(s-\rho) + H_\rho(s), \tag{4-3}$$

$$\tilde{F}(s,\chi) = \log(s-\rho) + \tilde{H}_\rho(s), \tag{4-4}$$

where

$$H_\rho(s) = \sum_{0<|\gamma'-\gamma|\le 1} \log(s-\rho') + O(\log|\gamma|),$$

$$\tilde{H}_\rho(s) = \sum_{0<|\gamma'-\gamma|\le 1} \log(s-\rho') + O(\log|\gamma|).$$

If χ is real, $s=\frac{1}{2}$ is a pole of $L(2s,\chi^2)$. So, by (2-4) and (2-7), on the truncated Hankel contour $\mathcal{H}\big(\frac{1}{2},\delta\big)$, for a real character χ, we write

$$F(s,\chi) = \tfrac{1}{2}\log\big(s-\tfrac{1}{2}\big) + H_B(s), \tag{4-5}$$

$$\tilde{F}(s,\chi) = -\tfrac{1}{2}\log\big(s-\tfrac{1}{2}\big) + \tilde{H}_B(s), \tag{4-6}$$

where $H_B(s)=O(1)$ and $\tilde{H}_B(s)=O(1)$.

Denote

$$I_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} k!F_{\Omega_k}(s,\chi)\frac{x^s}{s}\,ds, \quad I_B(x) := \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} k!F_{\Omega_k}(s,\chi)\frac{x^s}{s}\,ds,$$

and

$$\tilde{I}_\rho(x) := \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} k!F_{\omega_k}(s,\chi)\frac{x^s}{s}\,ds, \quad \tilde{I}_B(x) := \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac{1}{2},\delta\right)} k!F_{\omega_k}(s,\chi)\frac{x^s}{s}\,ds.$$

We define a function $T(x)$ as follows: for $T_{n'}\in\mathcal{T}$ satisfying $e^{2^{n+1}} \le T_{n'} \le e^{2^{n+1}}+1$, let $T(x)=T_{n'}$ for $e^{2^n} \le x \le e^{2^{n+1}}$. In particular, we have

$$x \le T(x) \le 2x^2, \quad (x \ge e^2).$$

Thus, by Lemma 10, for $T=T(x)$,

$$\psi_{\Omega_k}(x,\chi) = \frac{1}{k!}\sum_{|\gamma|\le T} I_\rho(x) + \frac{a(\chi)}{k!} I_B(x) + O(x^{1/2-\delta/2}), \tag{4-7}$$

$$\psi_{\omega_k}(x,\chi) = \frac{1}{k!}\sum_{|\gamma|\le T} \tilde{I}_\rho(x) + \frac{a(\chi)}{k!} \tilde{I}_B(x) + O(x^{1/2-\delta/2}). \tag{4-8}$$

We will see later that $\sum_{|\gamma|\le T} I_\rho(x)$ and $\sum_{|\gamma|\le T} \tilde{I}_\rho(x)$ will contribute the oscillating terms, i.e., the summation over zeros, in our theorems, and $I_B(x)$ and $\tilde{I}_B(x)$ will contribute the bias terms.

Next, we want to find the main contributions for $I_\rho(x)$, $I_B(x)$, $\tilde{I}_\rho(x)$, and $\tilde{I}_B(x)$. By (2-6) and (4-3), we have

$$\begin{aligned} I_\rho(x) &= \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{j=1}^{k}\binom{k}{j}(\log(s-\rho))^{k-j}(H_\rho(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{m=0}^{k-2} F^m(s,\chi)F_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: I_{M_\rho}(x) + E_{M_\rho}(x) + E_{R_\rho}(x), \end{aligned} \tag{4-9}$$

and by (2-6) and (4-5),

$$\begin{aligned} I_B(x) &= \frac{1}{2^k}\frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \left(\log\left(s-\tfrac12\right)\right)^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{j=1}^{k}\binom{k}{j}\left(\tfrac12\log\left(s-\tfrac12\right)\right)^{k-j}(H_B(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{m=0}^{k-2} F^m(s,\chi)F_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: B_M(x) + E_B(x) + E_R(x). \end{aligned} \tag{4-10}$$

Here, $I_{M_\rho}(x)$ and $B_M(x)$ will make main contributions to $I_\rho(x)$ and $I_B(x)$, respectively. Similarly, by (2-10) and (4-4), we have

$$\begin{aligned} \tilde{I}_\rho(x) &= \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{j=1}^{k}\binom{k}{j}(\log(s-\rho))^{k-j}(\tilde{H}_\rho(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}(\rho,\delta)} \sum_{m=0}^{k-2} \tilde{F}^m(s,\chi)\tilde{F}_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: \tilde{I}_{M_\rho}(x) + \tilde{E}_{M_\rho}(x) + \tilde{E}_{R_\rho}(x), \end{aligned} \tag{4-11}$$

and by (2-10) and (4-6),

$$\begin{aligned} \tilde{I}_B(x) &= \frac{(-1)^k}{2^k}\frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \left(\log\left(s-\tfrac12\right)\right)^k \frac{x^s}{s}\,ds + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{j=1}^{k}\binom{k}{j}\left(-\tfrac12\log\left(s-\tfrac12\right)\right)^{k-j}(\tilde{H}_B(s))^j\frac{x^s}{s}\,ds \\ &\qquad + \frac{1}{2\pi i}\int_{\mathcal{H}\left(\frac12,\delta\right)} \sum_{m=0}^{k-2} \tilde{F}^m(s,\chi)\tilde{F}_{\boldsymbol{n_m}}(s,\chi)\frac{x^s}{s}\,ds \\ &=: \tilde{B}_M(x) + \tilde{E}_B(x) + \tilde{E}_R(x). \end{aligned} \tag{4-12}$$

Here, $\tilde{I}_{M_\rho}(x)$ and $\tilde{B}_M(x)$ will make main contributions to $\tilde{I}_\rho(x)$ and $\tilde{I}_B(x)$, respectively.

Applying Lemma 12, we have

$$I_{M_\rho}(x)=\tilde{I}_{M_\rho}(x)=\frac{(-1)^k\sqrt{x}}{\log x}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}\left\{k(\log\log x)^{k-1}+\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}+O_k\left(\frac{1}{|\gamma|^2}\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right)+O_k\left(\frac{x^{1/2-\delta/3}}{|\gamma|}\right),\tag{4-13}$$

$$B_M(x)=\frac{(-1)^k\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1}+\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right)+O_k(x^{1/2-\delta/3}),\tag{4-14}$$

and

$$\tilde{B}_M(x)=(-1)^kB_M(x).\tag{4-15}$$

For the bias terms, by (4-10), (4-12), (4-14), and (4-15), we have

$$I_B(x)=\frac{(-1)^k\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1}+\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}+E_B(x)+E_R(x)+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right)+O_k(x^{1/2-\delta/3}),\tag{4-16}$$

and

$$\tilde{I}_B(x)=\frac{\sqrt{x}}{2^{k-1}\log x}\left\{k(\log\log x)^{k-1}+\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\}+\tilde{E}_B(x)+\tilde{E}_R(x)+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-1}}{(\log x)^2}\right)+O_k(x^{1/2-\delta/3}).\tag{4-17}$$

We will prove the following result in Section 5.

Lemma 13. *For the bias terms*,

$$I_B(x)=\frac{(-1)^kk}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right),$$
$$\tilde{I}_B(x)=\frac{k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right).$$

Then for the oscillating terms, by (4-9), (4-11), and (4-13), and Lemma 11, for $T = T(x)$,

$$\sum_{|\gamma|\le T} I_\rho(x) = \frac{(-1)^k k\sqrt{x}(\log\log x)^{k-1}}{\log x} \sum_{|\gamma|\le T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \frac{(-1)^k\sqrt{x}}{\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{(\log\log x)^{k-j}}{\Gamma_j(0)}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \sum_{|\gamma|\le T} E_{M_\rho}(x) + \sum_{|\gamma|\le T} E_{R_\rho}(x) + O_k\biggl(\frac{\sqrt{x}(\log\log x)^{k-1}}{\log^2 x}\biggr), \tag{4-18}$$

and

$$\sum_{|\gamma|\le T} \tilde{I}_\rho(x) = \frac{(-1)^k k\sqrt{x}(\log\log x)^{k-1}}{\log x} \sum_{|\gamma|\le T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \frac{(-1)^k\sqrt{x}}{\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{(\log\log x)^{k-j}}{\Gamma_j(0)}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \sum_{|\gamma|\le T} \tilde{E}_{M_\rho}(x) + \sum_{|\gamma|\le T} \tilde{E}_{R_\rho}(x) + O_k\biggl(\frac{\sqrt{x}(\log\log x)^{k-1}}{\log^2 x}\biggr). \tag{4-19}$$

The first terms in the above formulas are the main oscillating terms in our theorems. We will show in Section 6 that the other terms are small in average. For $T = T(x)$, denote

$$\Sigma_1(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} E_{M_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{M_\rho}(x), \tag{4-20}$$

$$\Sigma_2(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} E_{R_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} E'_{R_\rho}(x), \tag{4-21}$$

where $E'_{M_\rho}(x) = E_{M_\rho}(x)/x^\rho$, and $E'_{R_\rho}(x) = E_{R_\rho}(x)/x^\rho$. Similarly, denote

$$\widetilde{\Sigma}_1(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} \tilde{E}_{M_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} \tilde{E}'_{M_\rho}(x), \tag{4-22}$$

$$\widetilde{\Sigma}_2(x;\chi) := \frac{\log x}{\sqrt{x}} \sum_{|\gamma|\le T} \tilde{E}_{R_\rho}(x) = \log x \sum_{|\gamma|\le T} x^{i\gamma} \tilde{E}'_{R_\rho}(x), \tag{4-23}$$

where $\tilde{E}'_{M_\rho}(x) = \tilde{E}_{M_\rho}(x)/x^\rho$ and $\tilde{E}'_{R_\rho}(x) = \tilde{E}_{R_\rho}(x)/x^\rho$.

Then we have the following lemma (see Sections 6A and 6B for the proof).

Lemma 14. *For the error terms from the Hankel contours around zeros, we have*

$$\int_2^Y (|\Sigma_1(e^y;\chi)|^2 + |\Sigma_2(e^y;\chi)|^2)\,dy = o(Y(\log Y)^{2k-2}),$$
$$\int_2^Y (|\widetilde{\Sigma}_1(e^y;\chi)|^2 + |\widetilde{\Sigma}_2(e^y;\chi)|^2)\,dy = o(Y(\log Y)^{2k-2}).$$

Moreover, we also need to bound the lower order sum

$$S_1(x;\chi) := (-1)^k \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}, \tag{4-24}$$

and the error from the truncation by a fixed large T_0,

$$S_2(x, T_0; \chi) := \sum_{|\gamma|\le T} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} - \sum_{|\gamma|\le T_0} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}. \tag{4-25}$$

Then we have the following result (See Section 6C for the proof).

Lemma 15. *For the lower order sum and error from the truncation, we have*

$$\int_2^Y |S_1(e^y; \chi)|^2\, dy = o(Y(\log Y)^{2k-2}),$$

and for fixed large T_0,

$$\int_2^Y |S_2(e^y, T_0; \chi)|^2\, dy \ll Y\frac{\log^2 T_0}{T_0} + \log Y\frac{\log^3 T_0}{T_0} + \log^5 T_0.$$

Combining Lemmas 13, 14, and 15 with (4-7), (4-8), (4-18), and (4-19), we get, for fixed large T_0,

$$\psi_{\Omega_k}(x,\chi) = \frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}\biggl(\sum_{|\gamma|\le T_0}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \Sigma(x, T_0; \chi)\biggr) + a(\chi)\frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}, \tag{4-26}$$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y |\Sigma(e^y, T_0; \chi)|^2\, dy \ll \frac{\log^2 T_0}{T_0}.$$

Also,

$$\psi_{\omega_k}(x,\chi) = \frac{(-1)^k}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}\biggl(\sum_{|\gamma|\le T_0}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \widetilde{\Sigma}(x, T_0; \chi)\biggr) + a(\chi)\frac{1}{(k-1)!}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}, \tag{4-27}$$

where

$$\limsup_{Y\to\infty}\frac{1}{Y}\int_1^Y |\widetilde{\Sigma}(e^y, T_0; \chi)|^2\, dy \ll \frac{\log^2 T_0}{T_0}.$$

Note that $\sum_{\chi\ne\chi_0}(\bar{\chi}(a)-\bar{\chi}(b))a(\chi) = N(q,a)-N(q,b)$. Hence, combining (4-26) and (4-27) with (1-1) and (1-2), we get the conclusions of Theorem 1 and Theorem 2. Now we finish the proof of Theorems 1 and 2 modulo the proofs of Lemmas 12, 13, 14, and 15. □

5. The bias terms

In this section, we examine the bias terms and give the proof of Lemma 13.

5A. ***Estimates on the horizontal line.*** In order to examine the corresponding integration on the horizontal line in the Hankel contour, we prove the following estimate which we will use many times later to analyze the error terms in our theorems.

Lemma 16. *For any integers $k \geq 1$ and $m \geq 0$, we have*

$$\int_0^{\delta} |(\log \sigma - i\pi)^k - (\log \sigma + i\pi)^k| \sigma^m x^{-\sigma} \, d\sigma \ll_{m,k} \frac{(\log \log x)^{k-1}}{(\log x)^{m+1}}.$$

Proof. Let I represent the integral in the lemma. Then, we have

$$I \leq 2 \sum_{j=1}^{k} \binom{k}{j} \pi^j \int_0^{\delta} |\log \sigma|^{k-j} \sigma^m x^{-\sigma} \, d\sigma \ll_k \sum_{j=1}^{k} \int_0^{\delta} |\log \sigma|^{k-j} \sigma^m x^{-\sigma} \, d\sigma \tag{5-1}$$

Using a change of variable, $\sigma \log x = t$, we have

$$\begin{aligned}
\int_0^{\delta} |\log \sigma|^{k-j} \sigma^m x^{-\sigma} \, d\sigma &\leq \frac{1}{(\log x)^{m+1}} \int_0^{\delta \log x} |\log t - \log \log x|^{k-j} t^m e^{-t} \, dt \\
&\leq \frac{1}{(\log x)^{m+1}} \sum_{l=0}^{k-j} \binom{k-j}{l} (\log \log x)^{k-j-l} \int_0^{\delta \log x} |\log t|^l t^m e^{-t} \, dt \\
&\ll_k \frac{1}{(\log x)^{m+1}} \sum_{l=0}^{k-j} (\log \log x)^{k-j-l} \int_0^{\delta \log x} |\log t|^l t^m e^{-t} \, dt. \qquad (5\text{-}2)
\end{aligned}$$

Next, we estimate

$$\int_0^{\delta \log x} |\log t|^l t^m e^{-t} \, dt \leq \left(\int_0^1 + \int_1^{\infty} \right) |\log t|^l t^m e^{-t} \, dt =: I_{l_1} + I_{l_2}. \tag{5-3}$$

For the first integral in (5-3),

$$I_{l_1} = \int_0^1 |\log t|^l t^m e^{-t} \, dt \leq \int_0^1 |\log t|^l \, dt \overset{t \to 1/e^t}{=} \int_0^{\infty} \frac{t^l}{e^t} \, dt = \Gamma(l+1).$$

For the second integral in (5-3),

$$I_{l_2} = \int_1^{\infty} \frac{t^m (\log t)^l}{e^t} \, dt \overset{t \to e^t}{=} \int_0^{\infty} \frac{t^l}{e^{e^t - (m+1)t}} \, dt \ll_m \Gamma(l+1). \tag{5-4}$$

Then, by (5-2)-(5-4), we have

$$\int_0^{\delta} |\log \sigma|^{k-j} \sigma^m x^{-\sigma} \, d\sigma \ll_k \frac{1}{(\log x)^{m+1}} \sum_{l=0}^{k-j} (\log \log x)^{k-j-l} O_{m,l}(1) \ll_{m,k} \frac{(\log \log x)^{k-j}}{(\log x)^{m+1}}. \tag{5-5}$$

Thus, by (5-1),

$$I \ll_{m,k} \frac{(\log \log x)^{k-1}}{(\log x)^{m+1}}.$$

Hence, we get the conclusion of this lemma. □

5B. ***The bias terms.*** We have the following estimate for the integral over the truncated Hankel contour $\mathscr{H}\big(\frac{1}{2},\delta\big)$.

Lemma 17. *Assume the function $f(s)=O(1)$ on $\mathscr{H}\big(\frac{1}{2},\delta\big)$. Then, for any integer $m\ge 0$,*

$$\left|\int_{\mathscr{H}\left(\frac{1}{2},\delta\right)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^m f(s)\frac{x^s}{s}\,ds\right| \ll_m \frac{\sqrt{x}(\log\log x)^{m-1}}{\log x}.$$

Proof. Since the left-hand side is 0 when $m=0$, we assume $m\ge 1$ in the following proof. By Lemma 16, we have

$$\begin{aligned}\left|\int_{\mathscr{H}\left(\frac{1}{2},\delta\right)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^m f(s)\frac{x^s}{s}\,ds\right| &\le \left|\int_{r_0}^{\delta}\big((\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m\big)f\big(\tfrac{1}{2}-\sigma\big)\frac{x^{1/2-\sigma}}{\frac{1}{2}-\sigma}\,d\sigma\right| \\ &\qquad + O\left(\int_{-\pi}^{\pi}\frac{(\log 1/r_0+\pi)^m x^{1/2+r_0}}{\frac{1}{2}-r_0}r_0\,d\alpha\right) \\ &\ll \sqrt{x}\left(\int_0^{\delta}|(\log\sigma-i\pi)^m-(\log\sigma+i\pi)^m|x^{-\sigma}\,d\sigma+\frac{(\log x+\pi)^m}{x}\right) \\ &\ll_m \frac{\sqrt{x}(\log\log x)^{m-1}}{\log x}. \end{aligned} \tag{5-6}$$

This completes the proof of this lemma. □

In the following, we prove the asymptotic formulas for the bias terms.

Proof of Lemma 13. Since $H_B(s)=O(1)$, by (4-10), (4-12), and Lemma 17,

$$\begin{aligned}|E_B(x)| &\ll \sum_{j=1}^{k}\left|\int_{\mathscr{H}\left(\frac{1}{2},\delta\right)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^{k-j}(H_B(s))^j\frac{x^s}{s}\,ds\right| \\ &\ll \frac{\sqrt{x}}{\log x}\sum_{j=1}^{k}(\log\log x)^{k-j-1} \\ &\ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}. \end{aligned} \tag{5-7}$$

Similarly,

$$\begin{aligned}|\tilde{E}_B(x)| &\ll \sum_{j=1}^{k}\left|\int_{\mathscr{H}\left(\frac{1}{2},\delta\right)}\big(\log\big(s-\tfrac{1}{2}\big)\big)^{k-j}(\tilde{H}_B(s))^j\frac{x^s}{s}\,ds\right| \\ &\ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}. \end{aligned} \tag{5-8}$$

In the following, we estimate $E_R(x)$ in (4-10) and $\tilde{E}_R(x)$ in (4-12). If χ is not real, $E_R(x)=\tilde{E}_R(x)=0$. If χ is real, by (2-4), on $\mathscr{H}\big(\frac{1}{2},\delta\big)$, we write

$$F(2s,\chi^2)=-\log\big(s-\tfrac{1}{2}\big)+H_2(s). \tag{5-9}$$

On $\mathscr{H}\bigl(\frac{1}{2},\delta\bigr)$, $|H_2(s)|=O(1)$. By (2-6), we have

$$|E_R(x)| \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \biggl| \int_{\mathscr{H}\left(\frac{1}{2},\delta\right)} F^m(s,\chi) F(\boldsymbol{n}s,\chi) \frac{x^s}{s}\, ds \biggr|. \tag{5-10}$$

For each $0\le m\le k-2$, we write

$$F^m(s,\chi)F(\boldsymbol{n}s,\chi) = F^m(s,\chi)F^{m'}(2s,\chi^2)G_{\boldsymbol{n}}(s),$$

where $m+2m'\le k$, and $G_{\boldsymbol{n}}(s)=O(1)$ on $\mathscr{H}(\frac{1}{2},\delta)$. Thus, by (4-5), (5-9), and Lemma 17,

$$\begin{aligned}
&\biggl| \int_{\mathscr{H}\left(\frac{1}{2},\delta\right)} F^m(s,\chi) F(\boldsymbol{n}s,\chi) \frac{x^s}{s}\, ds \biggr| \\
&\qquad \ll \biggl| \int_{\mathscr{H}\left(\frac{1}{2},\delta\right)} \bigl(\log\bigl(s-\tfrac{1}{2}\bigr)+H_B(s)\bigr)^m \bigl(\log\bigl(s-\tfrac{1}{2}\bigr)-H_2(s)\bigr)^{m'} G_{\boldsymbol{n}}(s)\frac{x^s}{s}\, ds \biggr| \\
&\qquad \ll \sum_{j_1=0}^{m}\sum_{j_2=0}^{m'} \biggl| \int_{\mathscr{H}\left(\frac{1}{2},\delta\right)} \bigl(\log\bigl(s-\tfrac{1}{2}\bigr)\bigr)^{m+m'-j_1-j_2} (H_B(s))^{j_1}(H_2(s))^{j_2} G_{\boldsymbol{n}}(s)\frac{x^s}{s}\, ds \biggr| \\
&\qquad \ll \sum_{j_1=0}^{m}\sum_{j_2=0}^{m'} \frac{\sqrt{x}}{\log x}(\log\log x)^{m+m'-j_1-j_2-1} \\
&\qquad \ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}.
\end{aligned} \tag{5-11}$$

In the last step, we used the conditions $0\le m\le k-2$ and $m+2m'\le k$.

Combining (5-10) and (5-11), we deduce that

$$|E_R(x)| \ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}. \tag{5-12}$$

Similarly, if χ is real, by (2-8), we write

$$\tilde{F}(s,\chi;2) = -\log\bigl(s-\tfrac{1}{2}\bigr)+\tilde{H}_2(s), \tag{5-13}$$

where $\tilde{H}_2(s)=O(1)$ on $\mathscr{H}\bigl(\frac{1}{2},\delta\bigr)$. Using a similar argument as above, by (4-6), (5-13), and Lemma 17, we have

$$|\tilde{E}_R(x)| \ll \sum_{m=0}^{k-2} \sum_{\boldsymbol{n}\in S_m^{(k)}} \biggl| \int_{\mathscr{H}\left(\frac{1}{2},\delta\right)} \tilde{F}^m(s,\chi) \tilde{F}(\boldsymbol{n}s,\chi) \frac{x^s}{s}\, ds \biggr| \ll_k \frac{\sqrt{x}}{\log x}(\log\log x)^{k-2}. \tag{5-14}$$

By (4-10), (4-14), (5-7), and (5-12), we get

$$I_B(x) = \frac{(-1)^k\sqrt{x}}{2^{k-1}\log x}\biggl\{ k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)} \biggr\} + O_k\biggl(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\biggr).$$

Then, by (4-2),

$$\left|\sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right| \ll_k (\log\log x)^{k-2}.$$

Hence,

$$I_B(x)=\frac{(-1)^k k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right). \tag{5-15}$$

Similarly, by (4-12), (4-15), (5-8), and (5-14), we have

$$\tilde{I}_B(x)=\frac{k}{2^{k-1}}\frac{\sqrt{x}}{\log x}(\log\log x)^{k-1}+O_k\left(\frac{\sqrt{x}(\log\log x)^{k-2}}{\log x}\right). \tag{5-16}$$

This completes the proof of Lemma 13. □

6. Average order of the error terms

In Section 6A and Section 6B, we examine the error terms from the Hankel contours around zeros and give the proof of Lemma 14. In Section 6C, we examine the lower order sum and the error from the truncation, and give the proof of Lemma 15.

6A. ***Error terms from the Hankel contours around zeros.*** In this section, we give the proof of Lemma 14. The following lemma gives an average estimate for the integral over Hankel contours around zeros, which is the key lemma for our proof.

Lemma 18. *Let ρ be a zero of $L(s,\chi)$. Assume the function $g(s) \ll (\log|\gamma|)^c$ on $\mathcal{H}(\rho,\delta)$ for some constant $c \ge 0$, and*

$$H_\rho(s)=\sum_{0<|\gamma'-\gamma|\le 1}\log(s-\rho')+O(\log|\gamma|) \quad \text{on } \mathcal{H}(\rho,\delta). \tag{6-1}$$

For any integers $m, n \ge 0$, denote

$$E(x;\rho):=\int_{\mathcal{H}(\rho,\delta)}(\log(s-\rho))^m(H_\rho(s))^n g(s)\frac{x^{s-\rho}}{s}\,ds.$$

Then, for $T=T(x)$, we have

$$\int_2^Y\left|y\sum_{|\gamma|\le T(e^y)}e^{i\gamma y}E(e^y;\rho)\right|^2 dy=o(Y(\log Y)^{2m+2n-2}).$$

We will give the proof of Lemma 18 in next subsection. We use it in this section to prove Lemma 14 first.

Proof of Lemma 14. By (4-20), we have

$$|\Sigma_1(x;\chi)|^2=\left|\log x\sum_{|\gamma|\le T}x^{i\gamma}E'_{M_\rho}(x)\right|^2 \ll \sum_{j=1}^{k}\left|\log x\sum_{|\gamma|\le T}x^{i\gamma}E_{\rho,j}(x)\right|^2, \tag{6-2}$$

where
$$E_{\rho,j}(x)=\int_{\mathscr{H}(\rho,\delta)}(\log(s-\rho))^{k-j}(H_\rho(s))^j\frac{x^{s-\rho}}{s}\,ds.$$
By Lemma 18, take $m=k-j$, $n=j$, and $g(s)\equiv 1$, (i.e., $c=0$),
$$\int_2^Y\Big|y\sum_{|\gamma|\le T(e^y)}e^{i\gamma y}E_{\rho,j}(e^y)\Big|^2\,dy=o(Y(\log Y)^{2k-2}).$$
Thus,
$$\int_2^Y|\Sigma_1(e^y;\chi)|^2\,dy=o(Y(\log Y)^{2k-2}). \tag{6-3}$$

By definition (4-9) and (4-21), we have
$$\begin{aligned}|\Sigma_2(x;\chi)|^2&\ll\sum_{m=0}^{k-2}\sum_{\boldsymbol{n}\in S_m^{(k)}}\Big|\log x\sum_{|\gamma|\le T}x^{i\gamma}\int_{\mathscr{H}(\rho,\delta)}F^m(s,\chi)F(\boldsymbol{n}s,\chi)\frac{x^{s-\rho}}{s}\,ds\Big|^2\\&\ll\sum_{m=0}^{k-2}\sum_{\boldsymbol{n}\in S_m^{(k)}}\sum_{j=0}^{m}\Big|\log x\sum_{|\gamma|\le T}x^{i\gamma}E_{m,j}(x,\chi;\boldsymbol{n})\Big|^2,\end{aligned}\tag{6-4}$$
where
$$E_{m,j}(x,\chi;\boldsymbol{n})=\int_{\mathscr{H}(\rho,\delta)}(\log(s-\rho))^{m-j}(H_\rho(s))^jF(\boldsymbol{n}s,\chi)\frac{x^{s-\rho}}{s}\,ds.$$
Since on $\mathscr{H}(\rho,\delta)$, we know $F(\boldsymbol{n}s,\chi)=O((\log|\gamma|)^{\frac{1}{2}(k-m)})$, by Lemma 18, we get
$$\int_2^Y\Big|y\sum_{|\gamma|\le T(e^y)}e^{i\gamma y}E_{m,j}(e^y,\chi;\boldsymbol{n})\Big|^2\,dy=o(Y(\log Y)^{2m-2}).$$
Hence, by (6-4), we deduce that
$$\int_2^Y|\Sigma_2(e^y;\chi)|^2\,dy=o(Y(\log Y)^{2k-2}). \tag{6-5}$$
Combining (6-3) and (6-5), we get the first formula in Lemma 14.

For $\widetilde{\Sigma}_1(x;\chi)$ and $\widetilde{\Sigma}_2(x,\chi)$, by (4-22), using a similar argument with Lemma 18,
$$\int_2^Y|\widetilde{\Sigma}_1(e^y;\chi)|^2\,dy\ll\sum_{j=1}^{k}\int_2^Y\Big|y\sum_{|\gamma|\le T(e^y)}e^{i\gamma y}\tilde{E}_{\rho,j}(e^y)\Big|^2\,dy=o(Y(\log Y)^{2k-2}),\tag{6-6}$$
where
$$\tilde{E}_{\rho,j}(x)=\int_{\mathscr{H}(\rho,\delta)}(\log(s-\rho))^{k-j}(\tilde{H}_\rho(s))^j\frac{x^{s-\rho}}{s}\,ds.$$

Similarly, by (4-23) and Lemma 18,
$$\int_2^Y|\widetilde{\Sigma}_2(e^y;\chi)|^2\,dy\ll\sum_{m=0}^{k-2}\sum_{\boldsymbol{n}\in S_m^{(k)}}\sum_{j=0}^{m}\int_2^Y\Big|y\sum_{|\gamma|\le T(e^y)}e^{i\gamma y}\tilde{E}_{m,j}(e^y,\chi;\boldsymbol{n})\Big|^2\,dy=o(Y(\log Y)^{2k-2}),\tag{6-7}$$

where

$$\tilde{E}_{m,j}(x,\chi;\boldsymbol{n}) = \int_{\mathcal{H}(\rho,\delta)} (\log(s-\rho))^{m-j} (\tilde{H}_\rho(s))^j \tilde{F}(\boldsymbol{n}s,\chi) \frac{x^{s-\rho}}{s}\, ds.$$

Combining (6-6) and (6-7), we get the second formula in Lemma 14. □

6B. ***Estimates for integral over Hankel contours around zeros.*** We need the following results to finish the proof of Lemma 18.

Lemma 19 [Ford and Sneed 2010, Lemma 2.4]. *Assume* $L\left(\frac{1}{2},\chi\right)\neq 0$. *For* $A\geq 0$ *and real* $l\geq 0$,

$$\sum_{\substack{|\gamma_1|,|\gamma_2|\geq A\\ |\gamma_1-\gamma_2|\geq 1}} \frac{\log^l(|\gamma_1|+3)\log^l(|\gamma_2|+3)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \ll_l \frac{(\log(A+3))^{2l+3}}{A+1}.$$

Lemma 20. *For any integers* N, $j\geq 1$, *and* $0<|\delta_n|\leq 1$, *we have*

$$\int_0^\delta \left|\sum_{n=1}^N \log(\sigma+i\delta_n)\right|^j x^{-\sigma}\, d\sigma \ll_j \frac{1}{\log x}\left\{\min\left(N\log\log x, \log\frac{1}{\Delta_N}\right)+N\pi\right\}^j,$$

where $\Delta_N=\prod_{n=1}^N|\delta_n|$.

Proof. Let I denote the integral in the lemma. We consider two cases: $\Delta_N\geq\left(\frac{1}{\log x}\right)^N$ and $\Delta_N<\left(\frac{1}{\log x}\right)^N$.

(1) If $\Delta_N\geq(1/\log x)^N$, we have

$$I\ll\left(\log\frac{1}{\Delta_N}+N\pi\right)^j\int_0^\delta x^{-\sigma}\,d\sigma\ll\frac{1}{\log x}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j\ll\frac{1}{\log x}(N\log\log x+N\pi)^j. \quad (6\text{-}8)$$

(2) If $\Delta_N<(1/\log x)^N$, we write

$$I=\left(\int_0^{(\Delta_N)^{1/N}}+\int_{(\Delta_N)^{1/N}}^{1/\log x}+\int_{1/\log x}^{\delta}\right)\left|\sum_{n=1}^N\log(\sigma+i\delta_n)\right|^j x^{-\sigma}\,d\sigma =: I_1+I_2+I_3. \quad (6\text{-}9)$$

First, we estimate I_1,

$$I_1\ll\left(\log\frac{1}{\Delta_N}+N\pi\right)^j\int_0^{(\Delta_N)^{1/N}}x^{-\sigma}\,d\sigma\ll(\Delta_N)^{1/N}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j. \quad (6\text{-}10)$$

For $0<t<1$, consider the function $f(t)=t^{1/N}\left(\log\frac{1}{t}+N\pi\right)^j$. Since the critical point of $f(t)$ is $t=e^{N(\pi-1)}>1$, by (6-10), we have

$$I_1\ll f\left(\frac{1}{(\log x)^N}\right)=\frac{1}{\log x}(N\log\log x+N\pi)^j\ll\frac{1}{\log x}\left(\log\frac{1}{\Delta_N}+N\pi\right)^j. \quad (6\text{-}11)$$

Next, we estimate I_3. Using the change of variable $\sigma \log x = t$, we get

$$\begin{aligned}
I_3 &\ll \int_{1/\log x}^{\delta} \left(N \log \frac{1}{\sigma} + N\pi\right)^j x^{-\sigma}\, d\sigma \\
&= \frac{1}{\log x} \int_1^{\delta \log x} (N \log\log x - N \log t + N\pi)^j e^{-t}\, dt \\
&= \frac{N^j}{\log x} \sum_{l=0}^{j} \binom{j}{l} (\log\log x + \pi)^{j-l} \int_1^{\delta \log x} (-\log t)^l e^{-t}\, dt \\
&\ll_j \frac{N^j}{\log x} \sum_{l=0}^{j} (\log\log x + \pi)^{j-l} \int_1^{\infty} \frac{t^l}{e^t}\, dt \\
&\ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \\
&\ll \frac{1}{\log x} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j.
\end{aligned} \tag{6-12}$$

For I_2, similar to I_3, using the change of variable $\sigma \log x = t$, we get

$$\begin{aligned}
I_2 &\ll \int_{(\Delta_N)^{1/N}}^{1/\log x} \left(N \log \frac{1}{\sigma} + N\pi\right)^j x^{-\sigma}\, d\sigma \\
&= \frac{1}{\log x} \int_{(\Delta_N)^{1/N} \log x}^{1} (N \log\log x - N \log t + N\pi)^j e^{-t}\, dt \\
&= \frac{N^j}{\log x} \sum_{l=0}^{j} \binom{j}{l} (\log\log x + \pi)^{j-l} \int_{(\Delta_N)^{1/N} \log x}^{1} (-\log t)^l e^{-t}\, dt \qquad \left(t \to \frac{1}{e^t}\right) \\
&\ll_j \frac{N^j}{\log x} \sum_{l=0}^{j} (\log\log x + \pi)^{j-l} \int_0^{\infty} \frac{t^l}{e^t}\, dt \\
&\ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \\
&\ll \frac{1}{\log x} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j.
\end{aligned} \tag{6-13}$$

Combining (6-11), (6-12), (6-13), with (6-9), we get

$$I \ll_j \frac{(N \log\log x + N\pi)^j}{\log x} \ll_j \frac{1}{\log x} \left(\log \frac{1}{\Delta_N} + N\pi\right)^j. \tag{6-14}$$

By (6-8) and (6-14), we get the conclusion of this lemma. □

In the following, we use the above lemmas to prove Lemma 18.

Proof of Lemma 18. If $m=0$, $E(x;\rho)=0$ and hence the integral is 0. In the following, we assume $m\ge 1$. Let Γ_ρ represent the circle in the Hankel contour $\mathscr{H}(\rho,\delta)$. Then,

$$\begin{aligned}
E(x;\rho) &= \int_{\mathscr{H}(\rho,\delta)} (\log(s-\rho))^m (H_\rho(s))^n g(s)\frac{x^{s-\rho}}{s}\,ds \\
&= \int_{r_\rho}^{\delta} \bigl((\log\sigma - i\pi)^m - (\log\sigma + i\pi)^m\bigr)\bigl(H_\rho\bigl(\tfrac12-\sigma+i\gamma\bigr)\bigr)^n g\bigl(\tfrac12-\sigma+i\gamma\bigr)\frac{x^{-\sigma}}{\frac12-\sigma+i\gamma}\,d\sigma \\
&\qquad\qquad + \int_{\Gamma_\rho} (\log(s-\rho))^m (H_\rho(s))^n g(s)\frac{x^{s-\rho}}{s}\,ds \\
&=: E_h(x;\rho)+E_r(x;\rho). \qquad (6\text{-}15)
\end{aligned}$$

For the second integral in (6-15), since $r_\rho \le \frac{1}{x}$, by Lemma 11,

$$\begin{aligned}
|E_r(x;\rho)| &\ll \frac{(\log|\gamma|)^c r_\rho x^{r_\rho}}{|\gamma|}\left(\log\frac{1}{r_\rho}+\pi\right)^m \left(\sum_{0<|\gamma-\gamma'|\le 1}\log\left(\frac{1}{|\gamma'-\gamma|-r_\rho}\right)+O(\log|\gamma|)\right)^n \\
&\ll \frac{(\log|\gamma|)^c r_\rho x^{r_\rho}}{|\gamma|}\left(\log\frac{1}{r_\rho}+\pi\right)^m (\log|\gamma|)^n\left(\log\left(\frac{1}{r_\rho}\right)+O(1)\right)^n \\
&\ll \frac{(\log|\gamma|)^{n+c}}{|\gamma|}\frac{(\log(1/r_\rho)+\pi)^{m+n}}{1/r_\rho} \ll \frac{(\log|\gamma|)^{n+c}}{|\gamma|}\frac{1}{x^{1-\epsilon}}. \qquad (6\text{-}16)
\end{aligned}$$

Denote

$$\Sigma(x;\mathrm{g}) := \Bigl|\sum_{|\gamma|\le T} x^{i\gamma}E(x;\rho)\Bigr|^2 \ll \Bigl|\sum_{|\gamma|\le T} x^{i\gamma}E_h(x;\rho)\Bigr|^2 + \Bigl|\sum_{|\gamma|\le T} x^{i\gamma}E_r(x;\rho)\Bigr|^2. \qquad (6\text{-}17)$$

By (6-16), and $T(x)\ll x^2$, we get

$$\Bigl|\sum_{|\gamma|\le T} x^{i\gamma}E_r(x;\rho)\Bigr|^2 \ll \frac{1}{x^{2-\epsilon}}\left(\sum_{|\gamma|\le T(x)}\frac{(\log|\gamma|)^{n+c}}{|\gamma|}\right)^2 \ll \frac{1}{x^{2-\epsilon}}. \qquad (6\text{-}18)$$

For the first sum in (6-17),

$$\Bigl|\sum_{|\gamma|\le T} x^{i\gamma}E_h(x;\rho)\Bigr|^2 = \Biggl(\sum_{\substack{|\gamma_1-\gamma_2|\le 1\\ |\gamma_1|,|\gamma_2|\le T}} + \sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\le T}}\Biggr) x^{i(\gamma_1-\gamma_2)}E_h(x;\rho_1)E_h(x;\bar\rho_2) =: \Sigma_1(x;\mathrm{g})+\Sigma_2(x;\mathrm{g}).$$

By (6-15),

$$|E_h(x;\rho)| \ll \frac{(\log|\gamma|)^c}{|\gamma|}\sum_{j=1}^{m}\int_0^\delta |\log\sigma|^{m-j}\bigl|H_\rho\bigl(\tfrac12-\sigma+i\gamma\bigr)\bigr|^n x^{-\sigma}\,d\sigma. \qquad (6\text{-}19)$$

Let

$$S_j(x) := \int_0^\delta |\log\sigma|^{m-j} \left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^n x^{-\sigma}\,d\sigma$$
$$\leq \left(\int_0^\delta |\log\sigma|^{2(m-j)} x^{-\sigma}\,d\sigma\right)^{\frac{1}{2}} \left(\int_0^\delta \left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^{2n} x^{-\sigma}\,d\sigma\right)^{\frac{1}{2}}. \tag{6-20}$$

By (5-5) in the proof of Lemma 16,

$$\int_0^\delta |\log\sigma|^{2(m-j)} x^{-\sigma}\,d\sigma \ll \frac{(\log\log x)^{2(m-j)}}{\log x}. \tag{6-21}$$

By condition (6-1) and the Cauchy–Schwarz inequality,

$$\left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^{2n} \ll \left|\sum_{0<|\gamma'-\gamma|\leq 1} \log(\sigma+i(\gamma'-\gamma))\right|^{2n} + (\log|\gamma|)^{2n}.$$

Then, by Lemma 20,

$$\int_0^\delta \left|H_\rho\left(\tfrac{1}{2}-\sigma+i\gamma\right)\right|^{2n} x^{-\sigma}\,d\sigma \ll \frac{(M_\gamma(x))^{2n} + (\log|\gamma|)^{2n}}{\log x}, \tag{6-22}$$

where $M_\gamma(x) = \min(N(\gamma)\log\log x, \log 1/\Delta_{N(\gamma)})$, $N(\gamma)$ is the number of zeros γ' in the range $0 < |\gamma'-\gamma| \leq 1$, and $\Delta_{N(\gamma)} = \prod_{0<|\gamma'-\gamma|\leq 1} |\gamma'-\gamma|$.

Thus, by (6-21) and (6-22),

$$S_j(x) \ll \frac{(\log\log x)^{m-j}}{\log x}((M_\gamma(x))^n + (\log|\gamma|)^n).$$

Substituting this into (6-19), we get

$$|E_h(x;\rho)| \ll \frac{(\log|\gamma|)^c}{|\gamma|} \sum_{j=1}^{m} \frac{(\log\log x)^{m-j}}{\log x}((M_\gamma(x))^n + (\log|\gamma|)^n)$$
$$\ll \frac{(\log|\gamma|)^c}{|\gamma|} \frac{(\log\log x)^{m-1}}{\log x}((M_\gamma(x))^n + (\log|\gamma|)^n). \tag{6-23}$$

Then, by Lemma 11, we have

$$|\Sigma_1(x;\mathrm{g})| \ll \sum_{|\gamma|\leq T} \log(|\gamma|)(\max_{|\gamma'-\gamma|<1} |E_h(x;\rho')|)^2$$
$$\ll \frac{(\log\log x)^{2(m-1)}}{\log^2 x} \sum_\gamma \frac{(\log|\gamma|)^{2c}}{|\gamma|^2}((M_\gamma(x))^{2n} + (\log|\gamma|)^{2n})$$
$$= \frac{(\log\log x)^{2m+2n-2}}{\log^2 x} o(1). \tag{6-24}$$

Thus, for each positive integer l,

$$\int_{2^l}^{2^{l+1}} \Sigma_1(e^y;\mathrm{g})\,dy = o\left(\frac{l^{2m+2n-2}}{2^l}\right). \tag{6-25}$$

In the following, we examine $\Sigma_2(x; \mathrm{g})$. By (6-15),

$$\Sigma_2(x; \mathrm{g}) = \sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\le T}} x^{i(\gamma_1-\gamma_2)} E_h(x; \rho_1) E_h(x; \bar\rho_2). \tag{6-26}$$

For $e^{2^l} \le x \le e^{2^{l+1}}$, $T = T(x) = T_{l'}$ is a constant, and so we define

$$J(x; \mathrm{g}) := \sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\le T_{l'}}} x^{i(\gamma_1-\gamma_2)} \int_{r_{\rho_1}}^{\delta} \int_{r_{\bar\rho_2}}^{\delta} R_{\rho_1}(\sigma_1; x) R_{\bar\rho_2}(\sigma_2; x) \frac{d\sigma_1\, d\sigma_2}{i(\gamma_1-\gamma_2)-(\sigma_1+\sigma_2)}, \tag{6-27}$$

where

$$R_\rho(\sigma; x) = \big((\log\sigma - i\pi)^m - (\log\sigma + i\pi)^m\big) H_\rho^n\big(\tfrac12 - \sigma + i\gamma\big) \frac{g\big(\frac12 - \sigma + i\gamma\big) x^{-\sigma}}{\frac12 - \sigma + i\gamma}.$$

Thus,

$$\int_{e^{2^l}}^{e^{2^{l+1}}} \sum_{\substack{|\gamma_1-\gamma_2|>1\\ |\gamma_1|,|\gamma_2|\le T_{l'}}} x^{i(\gamma_1-\gamma_2)} E_h(x; \rho_1) E_h(x; \bar\rho_2) \frac{dx}{x} = J(e^{2^{l+1}}; \mathrm{g}) - J(e^{2^l}; \mathrm{g}). \tag{6-28}$$

By (6-27), (6-19), and (6-23), and Lemma 19, for $e^{2^l} \le x \le e^{2^{l+1}}$

$$\begin{aligned}
|J(x; \mathrm{g})| &\ll \sum_{|\gamma_1-\gamma_2|>1} \frac{(\log|\gamma_1|)^c (\log|\gamma_2|)^c}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \left(\frac{(\log\log x)^{m-1}}{\log x}\right)^2 ((M_{\gamma_1}(x))^n + (\log|\gamma_1|)^n)((M_{\gamma_2}(x))^n + (\log|\gamma_2|)^n) \\
&\ll \frac{(\log\log x)^{2m+2n-2}}{\log^2 x} \sum_{|\gamma_1-\gamma_2|>1} \frac{(\log|\gamma_1|)^{n+c} (\log|\gamma_2|)^{n+c}}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \\
&\ll \frac{(\log\log x)^{2m+2n-2}}{\log^2 x}.
\end{aligned} \tag{6-29}$$

Hence, by (6-26), (6-28), and (6-29), we get, for any positive integer l,

$$\int_{2^l}^{2^{l+1}} \Sigma_2(e^y; \mathrm{g})\, dy = o\left(\frac{l^{2m+2n-2}}{2^l}\right). \tag{6-30}$$

Therefore, by (6-18), (6-25) and (6-30),

$$\int_2^Y \left| y \sum_{|\gamma|\le T(e^y)} e^{i\gamma y} E(e^y; \rho)\right|^2 dy \ll \sum_{l \le \frac{\log Y}{\log 2}+1} 2^{2l} \int_{2^l}^{2^{l+1}} \Sigma(e^y; \mathrm{g})\, dy \tag{6-31}$$

$$\ll 1 + \sum_{l \le \frac{\log Y}{\log 2}+1} 2^{2l} \int_{2^l}^{2^{l+1}} (\Sigma_1(e^y; \mathrm{g}) + \Sigma_2(e^y; \mathrm{g}))\, dy \tag{6-32}$$

$$= o(Y(\log Y)^{2m+2n-2}). \tag{6-33}$$

This completes the proof of Lemma 18. □

6C. ***Lower order sum and error from the truncation.*** In this section, we examine the lower order sum and the error from the truncation by a fixed large T_0, and give the proof of Lemma 15.

Proof of Lemma 15. For the lower order sum, by (4-24), we have

$$\int_2^Y |S_1(e^y;\chi)|^2\,dy \ll \sum_{j=2}^{k} (\log Y)^{2k-2j} \int_2^Y \Bigl| \sum_{|\gamma|\le T(e^y)} \frac{e^{i\gamma y}}{\frac12+i\gamma} \Bigr|^2\,dy.$$

For the inner integral, by Lemma 11 and Lemma 19, and the definition of $T=T(x)$,

$$\begin{aligned}\int_2^Y \Bigl| \sum_{|\gamma|\le T(e^y)} \frac{e^{i\gamma y}}{\frac12+i\gamma} \Bigr|^2\,dy &\le \sum_{l\le \frac{\log Y}{\log 2}+1} \int_{2^l}^{2^{l+1}} \biggl(\sum_{\substack{|\gamma_1-\gamma_2|\le 1\\ |\gamma_1|,|\gamma_2|\le T_{l'}}} + \sum_{\substack{|\gamma_1-\gamma_2|> 1\\ |\gamma_1|,|\gamma_2|\le T_{l'}}} \biggr) \frac{e^{i(\gamma_1-\gamma_2)y}}{\bigl(\frac12+i\gamma_1\bigr)\bigl(\frac12-i\gamma_2\bigr)}\,dy \\ &\ll \sum_{l\le \frac{\log Y}{\log 2}+1} \biggl(2^l \sum_{\gamma} \frac{\log|\gamma|}{|\gamma|^2} + \sum_{\gamma_1,\gamma_2} \frac{1}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \biggr) \ll Y.\end{aligned}$$

Thus,

$$\int_2^Y |S_1(e^y;\chi)|^2\,dy \ll \sum_{j=2}^{k} Y(\log Y)^{2k-2j} = o(Y(\log Y)^{2k-2})).$$

Next, we examine $S_2(x,T_0;\chi)$. For fixed T_0, let X_0 be the largest x such that $T=T(x)\le T_0$. Since $x\le T(x)\le 2x^2$, $\log X_0 \asymp \log T_0$. By Lemma 11 and Lemma 19,

$$\begin{aligned}\int_2^Y |S_2(e^y,T_0;\chi)|^2\,dy &\le \int_2^{\log X_0} \Bigl| \sum_{|\gamma|\le T_0} \frac{1}{|\gamma|} \Bigr|^2\,dy + \int_{\log X_0}^{Y} \Bigl| \sum_{T_0\le|\gamma|\le T(e^y)} \frac{e^{i\gamma y}}{\frac12+i\gamma} \Bigr|^2\,dy \\ &\ll \log^5 T_0 + \sum_{\frac{\log\log X_0}{\log 2}\le l\le \frac{\log Y}{\log 2}+1} \int_{2^l}^{2^{l+1}} \biggl(\sum_{\substack{|\gamma_1-\gamma_2|\le 1\\ T_0\le|\gamma_1|,|\gamma_2|\le T_{l'}}} + \sum_{\substack{|\gamma_1-\gamma_2|> 1\\ T_0\le|\gamma_1|,|\gamma_2|\le T_{l'}}} \biggr) \frac{e^{i(\gamma_1-\gamma_2)y}}{\bigl(\frac12+i\gamma_1\bigr)\bigl(\frac12-i\gamma_2\bigr)}\,dy \\ &\ll \log^5 T_0 + \sum_{\frac{\log\log X_0}{\log 2}\le l\le \frac{\log Y}{\log 2}+1} \biggl(2^l \sum_{|\gamma|\ge T_0} \frac{\log|\gamma|}{|\gamma|^2} + \sum_{|\gamma_1|,|\gamma_2|\ge T_0} \frac{1}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \biggr) \\ &\ll Y\frac{\log^2 T_0}{T_0} + \log Y \frac{\log^3 T_0}{T_0} + \log^5 T_0.\end{aligned}$$

This completes the proof of this lemma. □

7. Asymptotic formulas for the logarithmic densities

In this section, we give the proof of Theorem 4.

Proof of Theorem 4. For large q, Fiorilli and Martin [2013] gave an asymptotic formula for $\delta_{\Omega_1}(q;a,b)$. Lamzouri [2013] also derived such an asymptotic formula using another method. Here, we want to derive asymptotic formulas for $\delta_{\Omega_k}(q;a,b)$ and $\delta_{\omega_k}(q;a,b)$ for fixed q and large k.

Let a be a quadratic nonresidue $\bmod q$ and b be a quadratic residue $\bmod q$, and $(a,q)=(b,q)=1$. Letting $\lambda_k = 1/2^{k-1}$, similar to formula (2.10) of [Fiorilli and Martin 2013], we have, under $\mathrm{ERH_q}$ and $\mathrm{LI_q}$,

$$\delta_{\Omega_k}(q;a,b) = \frac{1}{2} + \frac{(-1)^k}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\lambda_k(N(q;a)-N(q;b))x}{x}\Phi_{q;a,b}(x)\,dx.$$

Noting that $N(q,a)-N(q,b) = -A(q)$,

$$\delta_{\Omega_k}(q;a,b) = \frac{1}{2} + \frac{(-1)^{k-1}}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx. \tag{7-1}$$

For any $\epsilon > 0$,

$$\int_{-\infty}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx = \left(\int_{-\infty}^{1/\lambda_k^\epsilon} + \int_{-1/\lambda_k^\epsilon}^{1/\lambda_k^\epsilon} + \int_{1/\lambda_k^\epsilon}^{\infty}\right)\frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx. \tag{7-2}$$

By Proposition 2.17 in [Fiorilli and Martin 2013], $|\Phi_{q;a,b}(t)| \le e^{-0.0454\phi(q)t}$ for $t \ge 200$. So for large enough k,

$$\int_{1/\lambda_k^\epsilon}^{\infty} \frac{\sin(\lambda_k A(q)x)}{x}\Phi_{q;a,b}(x)\,dx \ll \lambda_k \int_{1/\lambda_k^\epsilon}^{\infty} e^{-0.0454\phi(q)x}\,dx \ll_{q,J,\epsilon} \lambda_k^J, \quad \text{for any } J>0. \tag{7-3}$$

The integral over $x \le -1/\lambda_k^\epsilon$ is also bounded by λ_k^J.

By Lemma 2.22 in [Fiorilli and Martin 2013], for each nonnegative integer K and real number $C>1$, we have, uniformly for $|z| \le C$,

$$\frac{\sin z}{z} = \sum_{j=0}^{K}(-1)^j \frac{z^{2j}}{(2j+1)!} + O_{C,K}(|z|^{2K+2}).$$

Thus, the second integral in (7-2) is equal to

$$\begin{aligned}\lambda_k A(q)\int_{-1/\lambda_k^\epsilon}^{1/\lambda_k^\epsilon} &\frac{\sin(\lambda_k A(q)x)}{\lambda_k A(q)x}\Phi_{q;a,b}(x)\,dx \\ &= \sum_{j=0}^{K}\lambda_k^{2j+1}\frac{(-1)^j A(q)^{2j+1}}{(2j+1)!}\int_{-1/\lambda_k^\epsilon}^{1/\lambda_k^\epsilon} x^{2j}\Phi_{q;a,b}(x)\,dx + O_{q,K}(\lambda_k^{2K+3-\epsilon}) \\ &= \sum_{j=0}^{K}\lambda_k^{2j+1}\frac{(-1)^j A(q)^{2j+1}}{(2j+1)!}\int_{-\infty}^{\infty} x^{2j}\Phi_{q;a,b}(x)\,dx + O_{q,K,\epsilon}(\lambda_k^{2K+3-\epsilon}).\end{aligned} \tag{7-4}$$

Combining (7-1), (7-3), and (7-4), we get the asymptotic formula (1-4) for $\delta_{\Omega_k}(q;a,b)$. Similarly, or by the results in Theorem 3, we have the asymptotic formula (1-5) for $\delta_{\omega_k}(q;a,b)$. □

8. The source of main terms and proof of Lemma 12

In this section, we give the proof of the main lemma we used for extracting out the bias terms and oscillating terms from the integrals over Hankel contours.

Let $\mathcal{H}(0, X)$ be the truncated Hankel contour surrounding 0 with radius r. Lau and Wu [2002] proved the following lemma.

Lemma 21 [Lau and Wu 2002, Lemma 5]. *For $X > 1$, $z \in \mathbb{C}$ and $j \in \mathbb{Z}^+$, we have*

$$\frac{1}{2\pi i}\int_{\mathcal{H}(0,X)} w^{-z}(\log w)^j e^w\,dw = (-1)^j \frac{d^j}{dz^j}\left(\frac{1}{\Gamma(z)}\right) + E_{j,z}(X),$$

where

$$|E_{j,z}(X)| \le \frac{e^{\pi|\Im(z)|}}{2\pi}\int_X^\infty \frac{(\log t+\pi)^j}{t^{\Re(z)}e^t}\,dt.$$

Proof of Lemma 12. We have the equality

$$\frac{1}{s} = \frac{1}{a} + \frac{a-s}{a^2} + \frac{(a-s)^2}{a^2 s}.$$

With the above equality, we write the integral in the lemma as

$$\frac{1}{2\pi i}\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\left(\frac{1}{a} + \frac{a-s}{a^2} + \frac{(a-s)^2}{a^2 s}\right)x^s\,ds =: I_1 + I_2 + I_3.$$

For I_3, using Lemma 16, we get

$$\begin{aligned}
&\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{(a-s)^2}{a^2 s}x^s\,ds \\
&\le \left|\int_r^\delta \big((\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k\big)\sigma^2 x^{-\sigma}\frac{x^a}{a^2(a-\sigma)}\,d\sigma\right| + \int_{-\pi}^{\pi} x^{\Re(a)+r}\big(\log\tfrac{1}{r}+\pi\big)^k \frac{r^2}{|a|^2|\Re(a)-r|}r, d\alpha \\
&\ll \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\left(\int_0^\delta |(\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k|\sigma^2 x^{-\sigma}\,d\sigma + \frac{(\log\frac{1}{r}+\pi)^k}{(1/r)^3}\right) \\
&\ll_k \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\left(\frac{(\log\log x)^{k-1}}{(\log x)^3} + \frac{1}{x^{3-\epsilon}}\right) \\
&\ll_k \frac{|x^a|}{|a|^2|\Re(a)-\delta|}\frac{(\log\log x)^{k-1}}{(\log x)^3}.
\end{aligned} \tag{8-1}$$

We estimate I_2 similarly. By Lemma 16,

$$\begin{aligned}
&\int_{\mathcal{H}(a,\delta)} \log^k(s-a)\frac{a-s}{a^2}x^s\,ds \\
&\qquad \le \left|\int_r^\delta ((\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k)\sigma x^{-\sigma}\frac{x^a}{a^2}\,d\sigma\right| + \int_{-\pi}^{\pi} x^{\Re(a)+r}\big(\log\tfrac{1}{r}+\pi\big)^k \frac{r}{|a|^2}r\,d\alpha \\
&\qquad \ll \frac{|x^a|}{|a|^2}\left(\int_0^\delta |(\log\sigma - i\pi)^k - (\log\sigma + i\pi)^k|\sigma x^{-\sigma}\,d\sigma + \frac{\big(\log\frac{1}{r}+\pi\big)^k}{(1/r)^2}\right) \\
&\qquad \ll_k \frac{|x^a|}{|a|^2}\left(\frac{(\log\log x)^{k-1}}{(\log x)^2} + \frac{1}{x^{2-\epsilon}}\right) \ll_k \frac{|x^a|}{|a|^2}\frac{(\log\log x)^{k-1}}{(\log x)^2}.
\end{aligned} \tag{8-2}$$

For I_1, using change of variable $(s-a)\log x = w$, by Lemma 21, we get

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi i}\frac{1}{\log x}\int_{\mathcal{H}(0,\delta\log x)} (\log w - \log\log x)^k \frac{x^a e^w}{a}\,dw \\
&= \frac{x^a}{a\log x}(-1)^k(\log\log x)^k \frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} e^w\,dw \\
&\quad + (-1)^{k-1}k\frac{x^a}{a\log x}(\log\log x)^{k-1}\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} e^w\log w\,dw \\
&\quad + \frac{x^a}{a\log x}\sum_{j=2}^{k}\binom{k}{j}\frac{1}{2\pi i}\int_{\mathcal{H}(0,\delta\log x)} (-\log\log x)^{k-j}(\log w)^j e^w\,dw \\
&= \frac{(-1)^k x^a}{a\log x}\left\{k(\log\log x)^{k-1} + \sum_{j=2}^{k}\binom{k}{j}(\log\log x)^{k-j}\frac{1}{\Gamma_j(0)}\right\} \\
&\quad + \frac{x^a}{a\log x}\sum_{j=1}^{k}\binom{k}{j}E_{j,0}(\delta\log x)(-\log\log x)^{k-j}.
\end{aligned}
\tag{8-3}
$$

By Lemma 21,

$$
|E_{j,0}(\delta\log x)| \le \frac{1}{2\pi}\int_{\delta\log x}^{\infty}\frac{(\log t+\pi)^j}{e^t}\,dt \ll_j e^{-\frac{1}{2}\delta\log x}\int_{\frac{1}{2}\delta\log x}^{\infty}\frac{(\log t)^j}{e^{t/2}}\,dt \ll_j x^{-\delta/2}.
$$

Hence, we get

$$
\left|\frac{x^a}{a\log x}\sum_{j=1}^{k}\binom{k}{j}E_{j,0}(\delta\log x)(-\log\log x)^{k-j}\right| \ll_k \frac{x^{\Re(a)}}{|a|\log x}\sum_{j=1}^{k}x^{-\delta/2}(\log\log x)^{k-j} \ll_k \frac{|x^{a-\delta/3}|}{|a|}. \tag{8-4}
$$

Combining (8-1), (8-2), (8-3), and (8-4), we get the conclusion of Lemma 12. □

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D-groups and the Dixmier–Moeglin equivalence

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A differential-algebraic geometric analogue of the Dixmier–Moeglin equivalence is articulated, and proven to hold for D-groups over the constants. The model theory of differentially closed fields of characteristic zero, in particular the notion of analysability in the constants, plays a central role. As an application it is shown that if R is a commutative affine Hopf algebra over a field of characteristic zero, and A is an Ore extension to which the Hopf algebra structure extends, then A satisfies the classical Dixmier–Moeglin equivalence. Along the way it is shown that all such A are Hopf Ore extensions in the sense of Brown et al., "Connected Hopf algebras and iterated Ore extensions", *J. Pure Appl. Algebra* **219**:6 (2015).

1. Introduction

This article is about an analogue of the Dixmier–Moeglin equivalence for differential-algebraic geometry. (The immediate motivation is an application to the classical noncommutative Dixmier–Moeglin problem, which we will describe later in this introduction.) The main objects of study here are D-varieties. An introduction to this category is given in Section 2A, but let us at least recall here that a D-variety (over the constants) is an algebraic variety V over a field k of characteristic zero, equipped with a regular section to the tangent bundle $s : V \to TV$. A D-subvariety is an algebraic subvariety W for which the restriction $s\restriction_W$ is a section to the tangent bundle of W. There are natural notions of D-morphism and D-rational map. For convenience, let us assume that k is algebraically closed. We are interested in the following properties of an irreducible D-subvariety $W \subseteq V$ over k:

- *δ-local-closedness*. There is a maximum proper D-subvariety of W over k.
- *δ-primitivity*. There is a k-point of W that is not contained in any proper D-subvariety of W over k.

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- *δ-rationality.* There is no nonconstant rational map from (W, s) to $(\mathbb{A}^1, 0)$ over k, where 0 denotes the zero section to the tangent bundle of the affine line.

The question is, for which ambient D-varieties (V, s) are these three properties equivalent for all D-subvarieties? It is not hard to see, and is spelled out in the proof of Corollary 2.14 below, that in general

$$\delta\text{-local-closedness} \implies \delta\text{-primitivity} \implies \delta\text{-rationality}.$$

In earlier work [Bell et al. 2017a], together with Stéphane Launois, we used the model theory of the Manin kernel to produce (in any dimension ≥ 3) a D-variety which is itself δ-rational but not δ-locally-closed. Here we focus on positive results. The main one, which appears as Corollary 2.17 below, is the following:

Theorem A. *Suppose (G, s) is a D-group over the constants — that is, G is an algebraic group and $s : G \to TG$ is a homomorphism of algebraic groups. Then for any D-subvariety of (G, s), δ-rationality implies δ-local-closedness. In particular, for every D-subvariety of (G, s), δ-rationality, δ-primitivity, and δ-local-closedness are equivalent properties.*

The proof of Theorem A relies on the model theory of differentially closed fields. In model-theoretic parlance, the point is that δ-rationality of (V, s) is equivalent to the generic type of the corresponding Kolchin closed set being weakly orthogonal to the constants, while δ-local-closedness means that the type is isolated. One context in which one can prove (using model-theoretic binding groups, for example) that weak orthogonality to the constants implies isolation is when the type in question is analysable in the constants. We give a geometric explanation of analysability in Section 2E in terms of what we call *compound isotriviality* of D-varieties. The reader can look there for a precise definition, but suffice it to say that a compound isotrivial D-variety is one that admits a finite sequence of fibrations where at each stage the fibres are isomorphic (possibly over a differential field extension of the base) to D-varieties where the section is the zero section. We show that for compound isotrivial D-varieties, δ-rationality implies δ-local-closedness (Proposition 2.13). Then we show, using known results about the structure of differential-algebraic groups, that every D-subvariety of a D-group over the constants is compound isotrivial (Proposition 2.16). Theorem A follows.

It turns out that for our intended application, namely Theorem B2 appearing later in this introduction, we need Theorem A to work for D-varieties that are slightly more general than D-groups. Given an affine algebraic group G, we may as well assume that $G \subseteq \mathrm{GL}_n$, so that a regular section to the tangent bundle is then of the form $s = (\mathrm{id}, \bar{s})$ where $\bar{s} : G \to \mathrm{Mat}_n$. It is not hard to check, from how the algebraic group structure is defined on the tangent bundle, that $s : G \to TG$ being a homomorphism is equivalent to the identity $\bar{s}(gh) = \bar{s}(g)h + g\bar{s}(h)$, where $g, h \in G$ are matrices and all addition and multiplication is matrix addition and multiplication. Now suppose we are given a homomorphism to the multiplicative group, $a : G \to \mathbb{G}_m$. By an *a-twisted D-group* we mean a D-variety (G, s) where G is an affine algebraic group and $s = (\mathrm{id}, \bar{s})$ satisfies the identity $\bar{s}(gh) = \bar{s}(g)h + a(g)g\bar{s}(h)$. So an a-twisted D-group is a D-group exactly when $a = 1$. We are able to show that D-subvarieties of a-twisted D-groups are also compound isotrivial. This yields the following generalisation of Theorem A: *For any D-subvariety of an*

a-twisted D-group over the constants, δ-rationality, δ-primitivity, and δ-local-closedness are equivalent properties. The passage from D-groups to a-twisted D-groups turns out to be technically quite difficult, and is done in Section 3.

It is worth pointing out that we have been intentionally ambiguous about the field of definitions in the statements of Theorem A and its a-twisted generalisation. The reason for this is that the results actually hold true for D-subvarieties of (G, s) that are defined over differential field extensions of the base field k. To make this precise one has to give a more general definition of D-variety using prolongations rather than tangent bundles, and we have decided to delay this to the main body of the article. While the final conclusion we are interested in is about D-subvarieties over k, this possibility of passing to base extensions is an important part of the inductive arguments involved.

Now for the application to noncommutative algebra, to which Section 4 is dedicated. The classical Dixmier–Moeglin equivalence (DME) is about prime ideals in a noetherian associative algebra over a field of characteristic zero; it asserts the equivalence between three properties of such prime ideals: primitivity (a representation-theoretic property), local-closedness (a geometric property), and rationality (an algebraic property). Precise definitions are given at the beginning of Section 4. We are interested in the question of when the DME holds for skew polynomial rings $R[x;\delta]$ over finitely generated commutative integral differential k-algebras (R, δ). Recall that $R[x;\delta]$ is the noncommutative polynomial ring in x over R where $xr = rx + \delta(r)$. This question is not vacuous since examples of such skew polynomial rings failing the DME were given in [Bell et al. 2017a]; indeed, these were the first counterexamples to the DME of finite Gelfand–Kirillov dimension. The connection to D-varieties should be clear: $R = k[V]$ for some irreducible algebraic variety V, and the k-linear derivation δ induces a regular section $s : V \to TV$. So the study of such (R, δ) is precisely the same thing as the study of D-varieties. We are able to prove (this is Proposition 4.6 below) that $R[x;\delta]$ satisfies the DME if δ-rationality implies δ-locally-closedness for all D-subvarieties of the D-variety (V, s) associated to (R, δ). Theorem A therefore answers our question in the special case of differential Hopf algebras.

Theorem B1. *If (R, δ) is a finitely generated commutative integral differential Hopf k-algebra*, *then $R[x;\delta]$ satisfies the* DME.

Being a differential Hopf algebra means that R has the structure of a Hopf algebra and that δ commutes with the coproduct; this is equivalent to saying that (R, δ) comes from an affine D-group (G, s).

More generally than skew polynomial rings, we consider Ore extensions. Suppose R is a finitely generated commutative integral k-algebra, σ is a k-algebra automorphism of R, and δ is a k-linear σ-derivation of R — meaning that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$. Recall that the Ore extension $R[x;\sigma,\delta]$ is the noncommutative polynomial ring in the variable x over R where $xr = \sigma(r)x + \delta(r)$. So when $\sigma = \mathrm{id}$ we are in the skew polynomial case discussed above. What about the DME for $R[x;\sigma,\delta]$?

Theorem B2. *Suppose R is a finitely generated commutative integral Hopf k-algebra. If an Ore extension $R[x;\sigma,\delta]$ admits a Hopf algebra structure extending that on R, then $R[x;\sigma,\delta]$ satisfies the* DME.

That Theorem B1 is a special case of Theorem B2 uses the (known) fact that one can always extend the Hopf structure on a differential algebra R to the skew polynomial ring extension $R[x;\delta]$, namely by the coproduct induced by $\Delta(x) = x \otimes 1 + 1 \otimes x$. Theorem B2 appears as Theorem 4.1 below. Its proof goes via a reduction to the case when $\sigma = \mathrm{id}$ and then an application of the stronger a-twisted version of Theorem A discussed above. Both of these steps use the work of Brown et al. [2015] on Hopf Ore extensions. One obstacle is that while their results hold for much more general R than we are considering, they are conditional on the coproduct of the variable x in the Ore extension taking the special form

$$\Delta(x) = a \otimes x + x \otimes b + v(x \otimes x) + w,$$

where $a, b \in R$ and $v, w \in R \otimes_k R$. This is part of their definition of a Hopf Ore extension, though they speculate about its necessity. We prove that when k is algebraically closed, after a linear change of variable, $\Delta(x)$ always has the above form. This is Theorem 4.2 below, and may be of independent interest:

Theorem C. *Suppose k is algebraically closed and R is a finitely generated commutative integral Hopf k-algebra. If an Ore extension $R[x;\sigma,\delta]$ admits a Hopf algebra structure extending that of R then, after a linear change of the variable x,*

$$\Delta(x) = a \otimes x + x \otimes b + w$$

for some $a, b \in R$, each of which is either 0 *or group-like, and some $w \in R \otimes_k R$. In particular, $R[x;\sigma,\delta]$ is a Hopf Ore extension of R.*

It has been conjectured [Bell and Leung 2014] that all finitely generated complex noetherian Hopf algebras of finite Gelfand–Kirillov dimension satisfy the DME. Theorem B2 verifies a special case. To make more significant progress on this conjecture one would like to pass from Hopf Ore extensions to iterated Hopf Ore extensions. As of now, this appears to be beyond the scope of the techniques used here.

Throughout this paper, by an *affine* k-algebra we mean a finitely generated commutative k-algebra that is an integral domain.

2. The δ-DME for D-groups over the constants

In this section we prove Theorem A of the introduction. After some preliminaries, we articulate in Section 2C the differential-algebraic geometric analogue of the DME suggested in the introduction, and call it the δ-DME. A sufficient condition for this to hold in terms of the model-theoretic notion of analysability to the constants is given in Section 2E, and then applied to show that D-groups over the constants satisfy the δ-DME in Section 2F. In a final section we reformulate δ-DME algebraically, as a statement about commutative differential Hopf algebras, thereby preparing the stage for the application to the classical DME in Section 4.

2A. ***Preliminaries on D-varieties.*** Suppose k is a field of characteristic zero equipped with a derivation δ. In this section we review the notion of a D-variety over k. Several more detailed expositions can be found in the literature, for instance [Buium 1992], which introduced the notion, and also [Kowalski and Pillay 2006, §2].

We first need to recall what prolongations are. If $V \subseteq \mathbb{A}^n$ is an affine algebraic variety over k, then by the δ*-prolongation* of V is meant the algebraic variety $\tau V \subseteq \mathbb{A}^{2n}$ over k whose defining equations are

$$P(X_1, \dots, X_n) = 0,$$
$$P^\delta(X_1, \dots, X_n) + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(X_1, \dots, X_n) \cdot X_i = 0,$$

for each $P \in I(V) \subset k[X_1, \dots, X_n]$. Here P^δ denotes the polynomial obtained by applying δ to all the coefficients of P. The projection onto the first n coordinates gives us a surjective morphism $\pi : \tau V \to V$.

Note that if K is any δ-field extension of k, and $a \in V(K)$, then

$$\nabla(a) := (a, \delta a) \in \tau V(K).$$

If V is defined over the constant field of (k, δ) then τV is nothing other than the tangent bundle TV. In general, τV is a torsor for the tangent bundle; for each $a \in V$ the fibre $\tau_a V$ is an affine translate of the tangent space $T_a V$. In particular, if V is smooth and irreducible then so is τV.

Taking prolongations is a functor which acts on morphisms $f : V \to W$ by acting on their graphs. It preserves the following properties of a morphism: being étale, being a closed embedding, and being smooth. The functor τ acts naturally on rational maps also; this is because for U a Zariski open subset of an irreducible variety V, $\tau V \restriction_U = \tau(U)$ is Zariski open in $\tau(V)$. Moreover, prolongations commute with base extension to δ-field extensions.

We have restricted our attention here to the affine case merely for concreteness. The prolongation construction extends to abstract varieties by patching over an affine cover in a natural and canonical way.

A *D-variety over k* is an algebraic variety V over k equipped with a regular section $s : V \to \tau V$ over k. An example is when V is defined over the constants and $s : V \to TV$ is the zero section.

If V is affine then a D-variety structure on V is nothing other than an extension of δ to the co-ordinate ring $k[V]$. Indeed, if $s : V \to \tau V$ is given by $s(X) = (X, s_1(X), \dots, s_n(X))$ in variables $X = (X_1, \dots, X_n)$, then we can extend δ to $k[X]$ by $X_j \mapsto s_j(X)$, and this will induce a derivation on $k[V]$. Conversely, given an extension of δ to $k[V]$, and choosing $s_j(X)$ to be such that $\delta(X_j + I(V)) = s_j(X) + I(V)$, we get that $s := (\mathrm{id}, s_1, \dots, s_n) : V \to \tau V$ is a regular section.

A *D-subvariety* of (V, s) is a closed algebraic subvariety $W \subseteq V$, over a possibly larger δ-field K, such that $s(W) \subseteq \tau W$. In principle one should talk about the base extension of V to K before talking about subvarieties *over K*, but as prolongations commute with base extension, and following standard model-theoretic practices, we allow D-subvarieties to be defined over arbitrary δ-field extensions unless explicitly stated otherwise.

A D-variety (V, s) over k is said to be *k-irreducible* if V is k-irreducible as an algebraic variety. In this case s induces on $k(V)$ the structure of a δ-field extending k. A D-variety (V, s) is called *irreducible* if V is absolutely irreducible. In general, every irreducible component of V is a D-subvariety over k^{alg} and these are called the *irreducible components* of (V, s).

A *morphism* of D-varieties $(V,s) \to (W,t)$ is a morphism of algebraic varieties $f : V \to W$ such that

$$\begin{array}{ccc} \tau V & \xrightarrow{\tau f} & \tau W \\ {\scriptstyle s}\uparrow & & \uparrow \\ V & \xrightarrow{f} & W_t \end{array}$$

commutes. It is not hard to verify that the pull-back of a D-variety, and the Zariski closure of the image of a D-variety, under a D-morphism, are again D-varieties.

In the same way, we can talk about *rational maps* between D-varieties. A useful fact is that if U is a nonempty Zariski open subset of V, then the prolongation of U is the restriction of τV to U, and so $(U, s\restriction_U)$ is a D-variety in its own right. So a rational map on V is a D-rational map if it is a D-morphism when restricted to the Zariski open subset on which it is defined.

A *D-constant* rational function on a D-variety (V,s) over k is a rational map over k from (V,s) to $(\mathbb{A}^1, 0)$, where 0 denotes the zero section to the tangent bundle of the affine line. In the case when (V,s) is k-irreducible, they correspond precisely to the δ-constants of $(k(V), \delta)$.

2B. ***Differentially closed fields and the Kolchin topology.*** Underlying our approach to the study of D-varieties is the model theory of *existentially closed* δ-fields (of characteristic zero). These are δ-fields K with the property that any finite sequence of δ-polynomial equations and inequations over K which have a solution in some δ-field extension, already have a solution in K. The class of existentially closed δ-fields of characteristic zero is axiomatisable in first-order logic, and its theory is denoted by DCF_0. We work in a fixed model of this theory, an existentially closed δ-field K. In particular, K is algebraically closed. We let K^δ denote the field of constants of K; it is an algebraically closed field that is *pure* in the sense that the structure induced on it by DCF_0 is simply that given by the language of rings.

Suppose (V,s) is a D-variety over K. Let $x \in V(K)$. Note that $\{x\}$ is a D-subvariety if and only if $\nabla(x) = s(x)$, recalling that $\nabla : V(K) \to \tau V(K)$ is given by $x \mapsto (x, \delta x)$. We call such points *D-points*, and denote the set of all D-points in $V(K)$ by $(V,s)^\sharp(K)$. It is an example, the main example we will encounter, of a *Kolchin closed* subset of $V(K)$. In general a Kolchin closed subset of the K-points of an algebraic variety is one that is defined Zariski-locally by the vanishing of δ-polynomials. Note that when (V,s) is defined over the constants and s is the zero section, $(V,s)^\sharp(K) = V(K^\delta)$.

One of the main consequences of working in an existentially closed δ-field is that $(V,s)^\sharp(K)$ is Zariski-dense in $V(K)$. In particular, for any subvariety $W \subseteq V$, we have that W is a D-subvariety if and only if $W \cap (V,s)^\sharp(K)$ is Zariski dense in $W(K)$.

Suppose (V,s) is k-irreducible for some δ-subfield k. If we allow ourselves to pass to a larger existentially closed δ-field, then we can always find a *k-generic D-point of* (V,s), that is, a D-point $x \in (V,s)^\sharp(K)$ that is Zariski-generic over k in V. Note that such a point is also Kolchin-generic in $(V,s)^\sharp(K)$ over k in the sense that it is not contained in any proper Kolchin closed subset defined over k.

In order to ensure the existence of generic D-points without having to pass to larger δ-fields, it is convenient to assume that K is already sufficiently large, namely *saturated*. This means that if k is a

δ-subfield of strictly smaller cardinality than K, and $\mathcal{F}$ is a collection of Kolchin constructible sets over k every finite subcollection of which has a nonempty intersection, then $\mathcal{F}$ has a nonempty intersection.

2C. ***An analogue of the* DME *for D-varieties.*** We fix from now on a saturated existentially closed δ-field K of sufficiently great cardinality. So K serves as a universal domain for δ-algebraic geometry (and hence, in particular, algebraic geometry). We also fix a small δ-subfield $k \subset K$ that will serve as the field of coefficients.

Definition 2.1 (δ-DME for D-varieties). Suppose (V, s) is a D-variety over k. We say that (V, s) *satisfies the* δ-DME *over* k, if for every k-irreducible D-subvariety $W \subseteq V$, the following are equivalent:

(i) (W, s) is *δ-locally-closed*: it has a maximum proper D-subvariety over k.

(ii) (W, s) is *δ-primitive*: there exists a point $p \in W(k^{\text{alg}})$ that is not contained in any proper D-subvariety of W over k.

(iii) (W, s) is *δ-rational*: $k(W)^\delta \subseteq k^{\text{alg}}$.

Remark 2.2. As the model-theorist will notice, and as we will prove in the next section, W being δ-locally closed means that the Kolchin generic type p of $(W, s)^\sharp(K)$ over k is isolated. The model-theoretic meaning of δ-rationality is that p is weakly orthogonal to the constants. On the other hand, it is not clear how to express a priori the δ-primitivity of W as a model-theoretic property of p.

Without additional assumptions on k there is no hope for the δ-DME to be satisfied. For example, there are positive-dimensional δ-rational D-varieties over any k, but if k is differentially closed then the only δ-locally closed D-varieties over k are zero-dimensional. This is because every D-variety over a differentially closed field k has a Zariski dense set of D-points over k, and so a D-subvariety over k containing all of them could not be proper. We are interested, however, in the case when k is very much not differentially closed; namely, when δ is trivial on k.

Proposition 2.3. *For any k-irreducible D-variety, δ-local-closedness implies δ-primitivity. Moreover, if $k \subseteq K^\delta$ then δ-primitivity implies δ-rationality.*

Proof. Let (W, s) be a k-irreducible D-variety.

Suppose (W, s) is δ-locally-closed, and denote by A the maximum proper D-subvariety of W over k. Then $p \in W(k^{\text{alg}}) \setminus A(k^{\text{alg}})$ witnesses δ-primitivity. This proves the first assertion.

Now suppose that $k \subseteq K^\delta$, (W, s) is δ-primitive, and $p \in W(k^{\text{alg}})$ is not contained in any proper D-subvariety over k. Suppose $f \in k(W)$ is a δ-constant. We want to show that $f \in k^{\text{alg}}$. We view it as a rational map of D-varieties, $f : (W, s) \to (\mathbb{A}^1, 0)$, and suppose for now that it is defined at p. So $f(p) \in \mathbb{A}^1(k^{\text{alg}})$. Because of our additional assumption that $k \subseteq K^\delta$, and hence $k^{\text{alg}} \subseteq K^\delta$, we have that $f(p)$ is a D-point of $(\mathbb{A}^1, 0)$. Now, let Λ be the orbit of $f(p)$ under the action of the absolute Galois group of k. Then Λ is a finite D-subvariety of $(\mathbb{A}^1, 0)$ over k. Hence the Zariski closure of $f^{-1}(\Lambda)$ is a D-subvariety of W over k that contains p. It follows that $f^{-1}(\Lambda) = W$. So f is k^{alg}-valued on all of W. We have shown that every element of $k(W)^\delta$ that is defined at p is in k^{alg}.

We have still to deal with the possibility that f is not defined at p. In that case, writing $f = \frac{a}{b}$ with $a, b \in k[W]$, we must have $b(p) = 0$. The fact that $\delta f = 0$ implies by the quotient rule that $b\delta a = a\delta b$. So either $\delta a = \delta b = 0$ or $f = \frac{a}{b} = \frac{\delta a}{\delta b}$. Since a and b are defined at p, if $\delta a = \delta b = 0$ then $a, b \in k^{\mathrm{alg}}$ by the previous paragraph, and hence $f \in k^{\mathrm{alg}}$. If, on the other hand, $f = \frac{\delta a}{\delta b}$, then we iterate the argument with $(\delta a, \delta b)$ in place of (a, b). What we get in the end is that either $f \in k^{\mathrm{alg}}$ or $f = \frac{\delta^\ell a}{\delta^\ell b}$ for all $\ell \geq 0$. We claim the latter is impossible. Indeed, it would imply that $\delta^\ell b(p) = 0$ for all ℓ, and so p is contained in the D-subvariety $V(I)$, where I is the δ-ideal of $k[W]$ generated by b. But the assumption on p would then imply that $V(I) = W$, contradicting the fact that $b \neq 0$. So $f \in k^{\mathrm{alg}}$, as desired. □

So the question becomes:

Question 2.4. Under the assumption that δ is trivial on k, for which D-varieties does δ-rationality imply δ-local-closedness?

Question 2.4 should be, we think, of general interest in differential-algebraic geometry. In [Bell et al. 2017a] it was pointed out that Manin kernels can be used to construct, in all Krull dimensions at least three, examples that were δ-rational but not δ-locally closed. Let us point out that in dimension ≤ 2 the answer is affirmative:

Proposition 2.5. *If $k \subseteq K^\delta$ then every D-variety over k of dimension ≤ 2 satisfies the δ-*DME *over k.*

Proof. Suppose (V, s) is a D-variety over k of dimension at most 2. By Proposition 2.3 it suffices to show that if $W \subseteq V$ is a k-irreducible δ-rational D-subvariety over k then it has a maximum proper D-subvariety over k. We may assume that $\dim W > 0$. Now, it is a known fact that δ-rationality implies the existence of only finitely many D-subvarieties of codimension one over k. Indeed, this is a theorem of Hrushovski[1]; see [Bell et al. 2017a, Theorem 6.1] and [Freitag and Moosa 2017, Theorem 4.2] for published generalisations. So it remains to consider the zero-dimensional D-subvarieties of W over k. But as $k \subseteq K^\delta$, the union of these is contained in the Zariski closure X of $(W, s)^\sharp(K) \cap W(K^\delta)$. Note that s restricts to the zero section on X, and hence X is a D-subvariety of W over k that must be proper by δ-rationality of (W, s). So the union of X and the finitely many codimension one D-subvarieties of W form the maximal proper D-subvariety over k. □

We will give a sufficient condition for δ-rationality to imply δ-local-closedness, and hence for the δ-DME, having to do with analysability to the constants in the model theory of differentially closed fields. We will then use this condition to prove the δ-DME for D-groups over the constants.

2D. ***Maximum D-subvarieties.*** Here we look closer at which D-varieties over k have a proper D-subvariety over k that contains all other proper D-subvarieties over k. This is something that never happens in the pure algebraic geometry setting: every variety over k has a Zariski dense set of k^{alg}-points, and each k^{alg}-point is contained in a finite subvariety defined over k. So a k-irreducible variety cannot have a maximum proper subvariety over k. In the enriched context of D-varieties there will be many

[1] It is Proposition 2.3 in his unpublished and untitled manuscript dated 1995 on “how to deduce the $\aleph_0$-categoricity of degree one strongly minimal sets in DCF_0 from Jouanolou’s work”.

D-points over a differential closure of k, say $\tilde{k}$, but a $\tilde{k}$-point need not live in a proper D-subvariety defined over k. So D-points are not an obstacle to the existence of a maximum proper D-subvariety. In fact, as the following lemma points out, the existence of a maximum proper D-subvariety is a natural property to consider from both the Kolchin topological and model-theoretic points of view.

We continue to work in our sufficiently saturated differentially closed field (K, δ), and fix a δ-subfield k of coefficients.

Lemma 2.6. *Suppose (V, s) is a k-irreducible D-variety. The following are equivalent.*

(i) *(V, s) is δ-locally closed.*

(ii) *(V, s) has finitely many maximal proper k-irreducible D-subvarieties.*

(iii) *$\Lambda := (V, s)^\sharp(K) \setminus \bigcup\{(W, s)^\sharp(K) : W \subsetneq V$ is a D-subvariety over $k\}$ is Kolchin constructible.*

(iv) *The Kolchin generic type of $(V, s)^\sharp(K)$ over k is an isolated type.*

Proof. (i)$\Rightarrow$(ii). Let W be the maximum proper D-subvariety over k. The k-irreducible components of W are D-subvarieties of V; see for example [Kaplansky 1976, Theorem 2.1]. Every proper k-irreducible D-subvariety of V is contained in one of these components. So the maximal proper k-irreducible D-subvarieties of V are precisely the k-irreducible components of W.

(ii)$\Rightarrow$(iii). Let $W_1, \dots, W_\ell$ be the maximal proper k-irreducible D-subvarieties of V. Then

$$\bigcup\{(W, s)^\sharp(K) : W \subsetneq V \text{ is a } D\text{-subvariety over } k\} = (W_1, s)^\sharp(K) \cup \cdots \cup (W_\ell, s)^\sharp(K).$$

(iii)$\Rightarrow$(iv). The Kolchin generic type of $(V, s)^\sharp(K)$ over k is the complete type $p(X)$ in DCF_0 axiomatised by the formulas saying that "$X \in (V, s)^\sharp(K)$", and, for each proper Kolchin closed subset A of $(V, s)^\sharp(K)$ over k, the formula "$X \notin A$". Note that as $(V, s)^\sharp(K)$ is defined by $\nabla(X) = s(X)$, the occurrences of each δX in the defining equations of A can be replaced by polynomials, so that $A = A^{\mathrm{Zar}} \cap (V, s)^\sharp(K)$, where A^{Zar} denotes the Zariski closure of A in V. When A is over k, we have that A^{Zar} is a D-subvariety of V over k. It follows that the set of realisations of p is precisely Λ, so that Λ being Kolchin constructible implies that p is axiomatised by a single formula, that is, it is isolated.

(iv)$\Rightarrow$(i). Let Λ be as in statement (iii). As we have seen, this is the set of realisations of the Kolchin generic type of $(V, s)^\sharp(K)$ over k. The latter being isolated implies, by quantifier elimination, that Λ is Kolchin constructible. By saturation, this in turn implies that

$$A := \bigcup\{(W, s)^\sharp(K) : W \subsetneq V \text{ is a } D\text{-subvariety over } k\}$$

is a finite union, and hence is itself a proper Kolchin closed subset over k. Then A^{Zar} is the maximum proper D-subvariety over k. $\square$

The following lemma will be useful in showing that certain D-varieties satisfy the equivalent conditions of Lemma 2.6.

Lemma 2.7. *Suppose $f : (V, s) \to (W, t)$ is a dominant D-rational map of k-irreducible D-varieties over k. The following are equivalent.*

(i) *(V, s) has a maximum proper D-subvariety over k.*

(ii) *(W, t) has a maximum proper D-subvariety over k, and for some (equivalently, every) k-generic D-point η of W, the fibre $V_\eta := f^{-1}(\eta)^{\mathrm{Zar}}$ has a maximum proper D-subvariety over $k(\eta)$.*

Proof. We show how this follows easily from basic properties of isolated types, leaving it to the reader to make the straightforward, but rather unwieldy, translation into an algebro-geometric argument if desired.

(i)$\Rightarrow$(ii). Suppose $a \in (V, s)^\sharp(K)$ is a k-generic D-point. Since f is a dominant D-rational map, $f(a) \in (W, T)^\sharp(K)$ is also k-generic. By characterisation (iv) of the previous lemma, $\mathrm{tp}(a/k)$ is isolated. As $f(a)$ is in the definable closure of a over k, it follows that $\mathrm{tp}(f(a)/k)$ is isolated. Hence, (W, t) has a maximum proper D-subvariety over k.

Now fix $\eta \in (W, t)^\sharp(K)$ a k-generic D-point, and let a be a $k(\eta)$-generic D-point of the fibre V_η. Then a is k-generic in (V, t), and hence $\mathrm{tp}(a/k)$ is isolated. It follows that the extension $\mathrm{tp}(a/k(\eta))$ is also isolated. So V_η has a maximum proper D-subvariety over $k(\eta)$.

(ii)$\Rightarrow$(i). Fix $\eta \in (W, t)^\sharp(K)$ a k-generic D-point such that V_η has a maximum proper D-subvariety over $k(\eta)$. Let a be a $k(\eta)$-generic D-point of V_η. So $\mathrm{tp}(a/k(\eta))$ and $\mathrm{tp}(\eta/k)$ are both isolated, implying that $\mathrm{tp}(a/k)$ is isolated. Since $a \in (V, s)^\sharp(K)$ is k-generic, condition (i) follows. ☐

2E. *Compound isotriviality.* Our sufficient condition for δ-rationality to imply δ-local-closedness will come from looking at isotrivial D-varieties.

Definition 2.8. An irreducible D-variety (V, s) over k is said to be *isotrivial* if there is some δ-field extension $F \supseteq k$ such that (V, s) is D-birationally equivalent over F to a D-variety of the form $(W, 0)$, where W is defined over the constants F^δ and 0 is the zero section. We say that a possibly reducible D-variety is isotrivial if every irreducible component is.

This definition comes from model theory: it is a geometric translation of the statement that the Kolchin generic type of $(V, s)^\sharp(K)$ over k is K^δ-internal. Note that there is some tension, but no inconsistency, between isotriviality and δ-rationality; for example, $(W, 0)$ is far from being δ-rational, instead of there being no new δ-constants in the rational function field we have that δ is trivial on all of $k(W)$. The reason these notions are not inconsistent is that the isotrivial (V, s) is only of the form $(W, 0)$ after base change — that is, over additional parameters — and that makes all the difference.

Fact 2.9. *A k-irreducible D-variety that is at once both δ-rational and isotrivial must be δ-locally closed.*

Proof. Suppose (V, s) is a k-irreducible isotrivial D-variety with $k(V)^\delta \subseteq k^{\mathrm{alg}}$. We want to show that V has a maximum proper D-subvariety over k. The proof we give makes essential use of model theory. We show how the statement translates to the well-known fact that a type internal to the constants but weakly orthogonal to the constants is isolated.

Let p be the Kolchin generic type of $(V,s)^\sharp(K)$ over k. By Lemma 2.6, it suffices to show that p is isolated. That in turn reduces to showing that every extension of p to k^{alg} is isolated. Fix q an extension of p to k^{alg}. So q is the Kolchin generic type of $(\widehat{V},s)^\sharp(K)$ over k^{alg}, for some irreducible component $\widehat{V}$ of V. Isotriviality of (V,s) implies isotriviality of $(\widehat{V},s)$, and this means that q is internal to the constants K^δ; see, for example, [Kowalski and Pillay 2006, Fact 2.6]. By stability, this implies that the binding group $\mathcal{G} = \mathrm{Aut}(q/k^{\mathrm{alg}}(K^\delta))$ is type-definable over k^{alg}; see, for instance, [Hrushovski 2002, Appendix B]. In fact, $\mathcal{G}$ is definable: $\mathcal{G}$ lives in the constants and by ω-stability of the induced structure on the constants, every type-definable group in K^δ is a definable group. So we have a definable group acting definably on the set of realisations of q.

On the other hand, for all $a \models q$ we have that

$$k^{\mathrm{alg}}(a)^\delta \subseteq (k(a)^{\mathrm{alg}})^\delta = (k(a)^\delta)^{\mathrm{alg}} \subseteq k^{\mathrm{alg}},$$

where the last containment uses our assumption on $k(V) = k(a)$. This shows that q is weakly orthogonal to K^δ. So the action of $\mathcal{G}$ on the set of realisations of q is transitive. As $\mathcal{G}$ is definable, the set of realisations of q must be definable — that is, q is isolated. □

Using Lemma 2.7 we can extend Fact 2.9 to the case of D-varieties that are built up by a finite sequence of fibrations by isotrivial D-subvarieties.

Definition 2.10. An irreducible D-variety (V,s) over k is said to be *compound isotrivial* if there exists a sequence of irreducible D-varieties (V_i,s_i) over k, for $i = 0,\dots,\ell$, with dominant D-rational maps over k

$$V = V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \longrightarrow V_{\ell-1} \xrightarrow{f_{\ell-1}} V_\ell = 0,$$

where 0 denotes an irreducible zero-dimensional D-variety, and such that the generic fibres of each f_i are isotrivial. That is, for each $i = 0,\dots,\ell-1$, if η is a k-generic D-point in V_{i+1}, then $f_i^{-1}(\eta)^{\mathrm{Zar}}$, which is a $k(\eta)$-irreducible D-subvariety of (V_i,s_i), is isotrivial. We say (V,s) is compound isotrivial in ℓ *steps*.

While isotriviality is equivalent to the Kolchin generic type being internal to the constants, compound isotriviality corresponds to that type being *analysable* in the constants. As this is a less familiar notion, even among model theorists, we spell out the equivalence here.

Lemma 2.11. *Suppose (V,s) is an irreducible D-variety over k, and $a \in (V,s)^\sharp(K)$ is a k-generic D-point. Then (V,s) is compound isotrivial if and only if the type of a over k in* DCF_0 *is analysable in K^δ.*

Proof. Analysability in K^δ means that there are tuples $a = a_0, a_1, \dots, a_\ell$ such that

(i) a_i is in the δ-field generated by a_{i-1} over k, for $i = 1,\dots,\ell-1$, $a_\ell \in k$, and

(ii) the type of a_i over the algebraic closure of the δ-field generated by $k(a_{i+1})$ is internal to K^δ.

If (V,s) is compound isotrivial one simply takes $a_i = f_{i-1}(a_{i-1})$ for $i = 1,\dots,\ell$. Condition (i) is clear — in fact with "δ-field generated by" replaced by "field generated by" — and condition (ii) follows from the fact that a_i is a $k(a_{i+1})^{\mathrm{alg}}$-generic D-point of one of the irreducible components of $f_i^{-1}(a_{i+1})^{\mathrm{Zar}}$, all of which are isotrivial. For the converse, given $a = a_0, a_1, \dots, a_\ell$ satisfying condition (i) and (ii), one first replaces a_i with $(a_i, \delta(a_i), \dots, \delta^n(a_i))$ for some sufficiently large n, so that a_i is a k-generic D-point of an irreducible D-variety (V_i, s_i) over k. This sequence of D-varieties witnesses the compound isotriviality, using the fact that the irreducible components of $f_i^{-1}(a_{i+1})^{\mathrm{Zar}}$ are all conjugate over $k(a_{i+1})$ and hence the isotriviality of one implies the isotriviality of them all. □

Remark 2.12 (stability under base change). The definition of compound isotriviality seems to be sensitive to parameters; the D-varieties V_i and the D-rational maps f_i need also be defined over k. In fact, the notion is stable under base change: if an irreducible D-variety (V, s) over k is compound isotrivial when viewed as a D-variety over some δ-field extension $F \supseteq k$ then it was already compound isotrivial over k. A model-theoretic restatement of this is the well-known fact that a stationary type with a nonforking extension that is analysable in the constants is already analysable in the constants. We leave it to the reader to formulate a geometric argument.

Note also that (compound) isotriviality is preserved by D-birational maps.

Proposition 2.13. *For an irreducible compound isotrivial D-variety over k, δ-rationality implies δ-local-closedness.*

Proof. Suppose (V, s) is an irreducible compound isotrivial D-variety over k with $k(V)^\delta \subseteq k^{\mathrm{alg}}$. We need to show that V has a maximum proper D-subvariety over k. We proceed by induction on the number of steps witnessing the compound isotriviality. The case $\ell = 0$ is vacuous. Suppose we have a compound isotrivial (V, s) witnessed by

$$V = V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \longrightarrow V_{\ell-1} \xrightarrow{f_{\ell-1}} V_\ell = 0$$

with $\ell \geq 1$. Then V_1 is compound isotrivial in $\ell - 1$ steps, and as $k(V_1)$ is a δ-subfield of $k(V)$ by dominance of f_0, the induction hypothesis applies to give us a maximum proper D-subvariety of V_1 over k.

On the other hand, the generic fibre $V_\eta := f_0^{-1}(\eta)^{\mathrm{Zar}}$ is an isotrivial $k(\eta)$-irreducible D-subvariety of V, where η is a k-generic D-point of V_1. Moreover, as $k(\eta)(V_\eta) = k(V)$, V_η is δ-rational, and therefore Fact 2.9 applies to V_η and we obtain a maximum proper D-subvariety over $k(\eta)$. Now Lemma 2.7 implies that V has a maximum proper D-subvariety over k. □

Corollary 2.14. *Suppose $k \subseteq K^\delta$ and (V, s) is a D-variety over k with the property that every irreducible D-subvariety of V over k^{alg} is compound isotrivial. Then (V, s) satisfies δ-*DME.

Proof. By Proposition 2.3, it suffices to show that every δ-rational k-irreducible D-subvariety (W, s) is δ-locally closed. Note that if $k = k^{\mathrm{alg}}$ then (W, s) is absolutely irreducible, and compound isotrivial by assumption, so that δ-local-closedness follows by Proposition 2.13. In general, let (W_0, s) be an absolutely irreducible component of (W, s). It is over k^{alg}. The δ-rationality of (W, s) over k implies the δ-rationality of (W_0, s) over k^{alg} — see, for example, the last paragraph of the proof of Fact 2.9. By assumption, (W_0, s) is compound isotrivial, and so by Proposition 2.13 it is δ-locally closed over k^{alg}. We have shown that every irreducible component of (W, s) is δ-locally closed over k^{alg}, and it is not hard to see, by taking the union of the maximum proper D-subvarieties of these components, for example, that this implies that (W, s) is δ-locally closed, as desired. □

2F. *D-groups over the constants.* A *D-group* is a D-variety (G, s) over k whose underlying variety G is an algebraic group, and such that the section $s : G \to \tau G$ is a morphism of algebraic groups. (Note that there is a unique algebraic group structure on $\tau(G)$ which makes the embedding $\nabla : G(K) \to \tau G(K)$ a homomorphism.) The notions of *D-subgroup* and *homomorphism of D-groups* are the natural ones, with the caveat that, unless stated otherwise, parameters may come from a larger δ-field. The quotient of a D-group by a normal D-subgroup admits a natural D-group structure. The terms *connected* and *connected component of identity* when applied to D-groups refer just to the underlying algebraic group, though note that the connected component of identity of a D-group over k is a D-subgroup.

In the context of D-groups, isotriviality is better behaved. A connected D-group (G, s) is isotrivial if and only if it is isomorphic as a D-group to one of the form $(H, 0)$, where H is an algebraic group over the constants and 0 is the zero section. So one remains in the category of D-groups, and D-birational equivalence is replaced by D-isomorphism. See the discussion around Fact 2.6 of [Kowalski and Pillay 2006] for a proof of this. In particular, every D-subvariety of an isotrivial D-group is itself isotrivial. Quotients of isotrivial D-groups are also isotrivial. Moreover, by [Pillay 2006, Corollary 3.10], if a D-group (G, s) has a finite normal D-subgroup H such that G/H is isotrivial, then (G, s) must have been isotrivial to start with. We also note that, as (compound) isotriviality is preserved under D-birational maps, when working inside a D-group (compound) isotriviality is preserved under translation by D-points of G (as these translations are in fact D-automorphisms of G).

The following fact is mostly a matter of putting together various results in the literature on D-groups. As we will see, it will imply that every D-subvariety of a D-group over the constants is compound isotrivial in at most 3 steps. At this point it is worth noting that the set of D-points of a D-group is a subgroup definable in DCF_0 of finite Morley rank. Moreover, the $\sharp$ functor is an equivalence between the categories of D-groups over k and finite Morley rank groups in DCF_0 definable over k; see [Kowalski and Pillay 2006, Fact 2.6].

Fact 2.15. *Suppose (G, s) is a connected D-group over the constants.*

(a) *The centre $Z(G)$ is a normal D-subgroup of G over the constants, and the quotient $G/Z(G)$ is an isotrivial D-group.*

(b) *Let H be the algebraic subgroup of points in $Z(G)$, where s agrees with the zero section. Then $Z(G)/H$ is an isotrivial D-group.*

Proof. For a proof that $Z(G)$ is a D-subgroup, see [Kowalski and Pillay 2006, Fact 2.7(iii)]. That $G/Z(G)$ is isotrivial was originally proved by Buium [1992] in the centreless case, and then generalised by Kowalski and Pillay [2006, Theorem 2.10].

For part (b), note first of all that H is a D-subgroup of $Z(G)$ by definition; the zero section does map to the tangent bundle of H. Now, it suffices to show that Z°/H° is isotrivial, where Z° is the connected component of identity of $Z(G)$ and $H^\circ := Z^\circ \cap H$. Let $\mathcal{Z} := (Z^\circ, s)^\sharp(K)$, the subgroup of D-points of Z. Then $(H^\circ, s)^\sharp(K) = \mathcal{Z}(K^\delta)$, the δ-constant points of $\mathcal{Z}$. These are now commutative δ-algebraic groups. As the $\sharp$ functor is an equivalence of categories, isotriviality of Z°/H° follows once we show that $\mathcal{Z}/\mathcal{Z}(K^\delta)$ is definably isomorphic (over some parameters) to $(K^\delta)^n$ for some n. Because Z° is a connected commutative algebraic group over the constants, there exists a δ-algebraic group homomorphism $\ell d : Z^\circ \to L(Z^\circ)$ over k_0, where $L(Z^\circ)$ is the Lie algebra of Z°, the tangent space at the identity. This homomorphism is called the *logarithmic derivative* and is defined as

$$\ell d(X) = \nabla(X) \cdot (s(X))^{-1},$$

with the operations in TZ°. One can check that ℓd is surjective with kernel $Z^\circ(K^\delta)$ (a proof appears in [Marker 2000, §3]; see also [Kolchin 1973, §V.22]). So $\mathcal{Z}/\mathcal{Z}(K^\delta)$ is definably isomorphic to a δ-algebraic subgroup $\mathcal{F}$ of $L(Z^\circ)$. Since $L(Z^\circ)$ is a vector group, $\mathcal{F}$ is a finite-dimensional K^δ-vector subspace (see, for example, [Pillay 1996, Fact 1.3]), and hence definably isomorphic over a basis to some $(K^\delta)^n$. □

Suppose (G, s) is a D-group over a δ-field k and (H, s) is a D-subgroup over k. Even when H is not normal, it makes sense to consider the quotient space G/H as an algebraic variety, and s induces on G/H the natural structure of a D-variety $(G/H, \bar{s})$ over k, in such a way that the quotient map $\pi : G \to G/H$ is a D-morphism. See [Kowalski and Pillay 2006, Fact 2.7(ii)] for details. Now if α is a D-point of $(G/H, \bar{s})$, then the fibre $\pi^{-1}(\alpha)$ is a D-subvariety over $k(\alpha)$; and for β a D-point of this fibre we have $\pi^{-1}(\alpha) = \beta + H$. So each fibre $(\pi^{-1}(\alpha), s)$ is isomorphic to (H, s) over $k(\beta)$. One could develop in this context the notion of "D-homogeneous spaces".

Using Fact 2.15 we obtain the following highly restrictive property on the structure of D-subvarieties of D-groups over the constants.

Proposition 2.16. *Suppose (G, s) is a connected D-group over $k_0 \subseteq K^\delta$. If k is any δ-field extension of k_0 and W is any irreducible D-subvariety of G over k, then W is compound isotrivial in at most 3 steps. In particular, if W is δ-rational then it is δ-locally closed.*

Proof. Consider the normal sequence of D-subgroups

$$G \rhd Z(G) \rhd H \rhd 0,$$

where $Z(G)$ is the centre of G and H is the algebraic subgroup of points in $Z(G)$ where s agrees with the zero section. Consider the corresponding sequence of irreducible D-varieties and D-morphisms over k_0:

$$G \xrightarrow{\pi_0} G/H \xrightarrow{\pi_1} G/Z(G) \xrightarrow{\pi_2} 0.$$

Since $G/Z(G)$, $Z(G)/H$, and H are isotrivial — the first two by Fact 2.15 and the last as $s \restriction_H$ is the zero section — this exhibits G as compound isotrivial in three steps. We can then obtain the same result for any irreducible D-subvariety of G by using the fact that any element of $(G, s)^\sharp(K)$ is a product of two generic elements. Alternatively, we can argue as follows, keeping in mind that every D-subvariety of an isotrivial D-group is itself isotrivial.

If $W \subseteq G$ is an irreducible D-variety over k, then we get a sequence of dominant D-morphisms

$$W \xrightarrow{f_0} W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} 0,$$

where $W_1 \subseteq G/H$ is the Zariski closure of $\pi_0(W)$, $W_2 \subseteq G/Z(G)$ is the Zariski closure of $\pi_1(W_1)$, and the f_i are the appropriate restrictions of the π_i. Then W_2 is isotrivial, as it is a D-subvariety of $G/Z(G)$. If α is a D-point of W_2 then $f_1^{-1}(\alpha)$ is a D-subvariety of $\pi_1^{-1}(\alpha)$ which is isomorphic as a D-variety to $Z(G)/H$. So the fibres of f_1 over D-points are all isotrivial D-subvarieties of W_1. If β is a D-point of W_1 then $f_0^{-1}(\beta)$ is a D-subvariety of $\pi_0^{-1}(\beta)$ which is isomorphic as a D-variety to H. So the fibres of f_0 over D-points are all isotrivial. Hence, W is compound isotrivial in 3 steps.

The "in particular" clause is by Proposition 2.13. □

We have now proved Theorem A of the introduction:

Corollary 2.17. *If $k \subseteq K^\delta$ then every D-group over k satisfies* δ-DME.

Proof. Suppose (G, s) is a D-group over k and W is an irreducible D-subvariety of G over k^{alg}. Then, over k^{alg}, it is isomorphic to an irreducible D-subvariety of the connected component of identity, G_0. Applying Proposition 2.16 to G_0, we have that W is compound isotrivial. So every irreducible D-subvariety of G over k^{alg} is compound isotrivial. The δ-DME now follows from Corollary 2.14. □

2G. *Differential Hopf algebras.* In this section we give equivalent algebraic formulations of the δ-DME and our results so far. This will help us make the connection to the classical DME, which is about noncommutative associative algebras and as such does not have a direct geometric formulation.

We restrict our attention to the case when δ is trivial on the base field k.

As explained in Section 2A, the standard geometry-algebra duality, which assigns to a variety its coordinate ring, induces an equivalence between the category of k-irreducible affine D-varieties (V, s) and that of differential rings (R, δ), where R is an affine k-algebra and δ is a k-linear derivation. This equivalence associates to a k-irreducible D-subvariety of V a prime δ-ideal of R. Using this dictionary, we can easily translate the geometric Definition 2.1, in the case when $k \subseteq K^\delta$, into the following algebraic counterpart.

Definition 2.18 (δ-DME for affine differential algebras). Suppose R is an affine k-algebra equipped with a k-linear derivation δ. We say that (R, δ) *satisfies the* δ-DME if for every prime δ-ideal P of R, the following conditions are equivalent:

(i) P is *δ-primitive*: There exists a maximal ideal $\mathfrak{m}$ of R such that P is maximal among the prime δ-ideals contained in $\mathfrak{m}$.

(ii) P is *δ-locally-closed*: The intersection of all the prime δ-ideals of R that properly contain P is a proper extension of P.

(iii) P is *δ-rational*: $\operatorname{Frac}(R/P)^\delta$ is contained in k^{alg}.

The algebraic counterpart of an affine algebraic group G over k is the commutative Hopf k-algebra $R = k[G]$, where the group law $G \times G \to G$ induces a coproduct $\Delta : R \to R \otimes_k R$. So what is the algebraic counterpart of a D-group (G, s) over k? The following lemma says that it is a *differential Hopf k-algebra*, a commutative Hopf k-algebra R equipped with a k-linear derivation δ that commutes with the coproduct, where δ acts on $R \otimes_k R$ by $\delta(r_1 \otimes r_2) = \delta r_1 \otimes r_2 + r_1 \otimes \delta r_2$.

Lemma 2.19. *Suppose $k \subseteq K^\delta$ and let (G, s) be a D-variety defined over k such that G is a connected affine algebraic group. Let δ on $R = k[G]$ be the corresponding k-linear derivation. Then $s : G \to TG$ is a group homomorphism if and only if δ commutes with the coproduct.*

Proof. Unravelling the facts that s induces the derivation δ on $k[G]$ and that the group operation $m : G \times G \to G$ induces the coproduct Δ on $k[G]$, we have that for all $f \in k[G]$,

$$\Delta(\delta f) = \delta\Delta(f) \iff df(s(m(y, z))) = d(f \circ m)(s(y), s(z)), \tag{2-1}$$

where (y, z) are coordinates for $G \times G$. But

$$d(f \circ m)(s(y), s(z)) = df \circ dm(s(y), s(z)) = df(s(y) * s(z)),$$

where $*$ denotes the group operation $dm : TG \times TG \to TG$. And so the right-hand side of (2-1) is equivalent to

$$df(s(m(y, z))) = df(s(y) * s(z)).$$

But this, asserted for all $f \in k[G]$, is equivalent to $s(m(y, z)) = s(y) * s(z)$, i.e., that s is a group homomorphism. □

In other words, the study of connected affine D-groups over the constants is the same thing as the study of affine differential Hopf k-algebras. So our Theorem A becomes the following.

Theorem 2.20. *Every commutative affine differential Hopf algebra over a field of characteristic zero satisfies* δ-DME.

Proof. By Lemma 2.19 our differential Hopf algebra is of the form $k[G]$ for some connected affine D-group (G, s) with $k \subseteq K^\delta$. By Corollary 2.17, (G, s) satisfies the δ-DME. So $(k[G], \delta)$ satisfies δ-DME. □

3. Twisting by a group-like element

As it turns out, the application to the classical Dixmier–Moeglin problem that we have in mind, and that will be treated in Section 4, requires a generalisation of Theorem 2.20. Instead of working with differential Hopf algebras, we need to consider Hopf algebras equipped with derivations that do not quite commute with the coproduct. Suppose R is a commutative affine Hopf k-algebra. We use Sweedler notation[2] and write $\Delta(r) = \sum r_1 \otimes r_2$ for any $r \in R$. Now, for a k-linear derivation δ to commute with

[2] Recall that in Sweedler notation $\sum r_1 \otimes r_2$ is used to denote an expression of the form $\sum_{j=1}^{d} r_{j,1} \otimes r_{j,2}$. We use Sweedler notation throughout, hopefully without confusion.

Δ on R means that for all $r \in R$,

$$\Delta(\delta r) = \sum \delta r_1 \otimes r_2 + r_1 \otimes \delta r_2.$$

We wish to weaken this condition by asking instead simply that there exists some $a \in R$ satisfying $\Delta(a) = a \otimes a$ — that is, a is a *group-like* element of R — such that for all $r \in R$,

$$\Delta(\delta r) = \sum \delta r_1 \otimes r_2 + a r_1 \otimes \delta r_2. \tag{3-1}$$

That is, we ask δ to be what Panov [2003] calls an *a-coderivation*. We wish to prove the following.

Theorem 3.1. *Suppose k is a field of characteristic zero, R is a commutative affine Hopf k-algebra, and δ is a k-linear derivation on R that is an a-coderivation for some group-like $a \in R$. Then (R, δ) satisfies the δ-DME.*

When $a = 1$ this is just the case of affine differential Hopf k-algebras, and hence is dealt with by Theorem 2.20. The general case requires some work. Throughout this section k is a fixed field of characteristic zero.

Let us begin with a geometric explanation of what this twisting by a group-like element means. First of all, we have $R = k[G]$ for some connected affine algebraic group G over k, with the coproduct Δ on R induced by the group operation on G, and the derivation δ on R induced by a D-variety structure $s : G \to TG$. Note that δ being a k-derivation implies that $k \subseteq K^{\delta}$ and so $\tau G = TG$. Now, as G is an affine algebraic group, we may assume it is an algebraic subgroup of GL_n, so that $TG \subseteq G \times \mathrm{Mat}_n$. Writing $s = (\mathrm{id}, \bar{s})$, where $\bar{s} : G \to \mathrm{Mat}_n$, we want to express as a property of $\bar{s}$ what it means for δ to be an a-coderivation. That $a \in R$ is group-like means that $a : G \to \mathbb{G}_m$ is a homomorphism of algebraic groups.

Lemma 3.2. *Suppose $G \subseteq \mathrm{GL}_n$ is a connected affine algebraic group over k, $a : G \to \mathbb{G}_m$ is a homomorphism, and $s = (\mathrm{id}, \bar{s}) : G \to TG \subseteq G \times \mathrm{Mat}_n$ is a D-variety structure on G over k. Then the corresponding k-linear derivation δ on $k[G]$ is an a-coderivation if and only if*

$$\bar{s}(gh) = \bar{s}(g)h + a(g)g\bar{s}(h) \tag{3-2}$$

for all $g, h \in G$, where all addition and multiplication is in the sense of matrices.

Proof. Note that for $r \in k[G]$, $\Delta(\delta r) \in k[G \times G]$ is given by

$$\Delta(\delta r)(g, h) = d_{gh} r(\bar{s}(gh))$$

for all $g, h \in G$, where $dr : TG \to \mathbb{A}^2$ is the differential of $r : G \to \mathbb{A}^1$. On the other hand, writing $\Delta(r) = \sum r_1 \otimes r_2$ we have

$$\begin{aligned}
\sum(\delta r_1 \otimes r_2 + a r_1 \otimes \delta r_2)(g, h) &= \sum d_g r_1(\bar{s}(g))\, r_2(h) + a(g) r_1(g)\, d_h r_2(\bar{s}(h)) \\
&= d_{(g,h)}\Big(\sum r_1 \otimes r_2\Big)(\bar{s}(g), a(g)\bar{s}(h)) \\
&= d_{(g,h)}(\Delta r)(\bar{s}(g), a(g)\bar{s}(h)),
\end{aligned}$$

where the second equality uses the fact that $a(g)$ is a scalar. Now, as an element of $k[G\times G]$, $\Delta(r)=r\circ m$, where $m: G\times G\to G$ is the restriction of matrix multiplication on GL_n. Note that when we differentiate matrix multiplication we get $d_{(g,h)}m(A,B)=Ah+gB$, for all $g,h\in\mathrm{GL}_n$ and $A,B\in\mathrm{Mat}_n$. Hence,

$$\begin{aligned}\sum(\delta r_1\otimes r_2+ar_1\otimes\delta r_2)(g,h)&=d_{(g,h)}(\Delta r)(\bar{s}(g),a(g)\bar{s}(h))\\&=d_{gh}r\circ d_{(g,h)}m(\bar{s}(g),a(g)\bar{s}(h))\\&=d_{gh}r(\bar{s}(g)h+ga(g)\bar{s}(h))\\&=d_{gh}r(\bar{s}(g)h+a(g)g\bar{s}(h)).\end{aligned}$$

Hence, δ being an a-coderivation, as in (3-1), is equivalent to

$$d_{gh}r(\bar{s}(gh))=d_{gh}r(\bar{s}(g)h+a(g)g\bar{s}(h))$$

for all $r\in k[G]$. But this implies

$$\bar{s}(gh)=\bar{s}(g)h+a(g)g\bar{s}(h),$$

as desired. □

Definition 3.3. When G is an affine algebraic group and (G,s) is a D-variety structure such that (3-2) holds, we say that (G,s) is an *a-twisted D-group*.

The following family of examples of 2-dimensional twisted D-groups will play an important role in the proof.

Example 3.4. Let $c\in k$ be a parameter. Let $R=k\big[x,\frac{1}{x},y\big]$ with δ the k-linear derivation induced by $\delta(x)=xy$ and $\delta(y)=\frac{1}{2}y^2+c(1-x^2)$. Note that R is the coordinate ring of the algebraic subgroup $E\le\mathrm{GL}_2$ made up of matrices of the form

$$\begin{pmatrix}x&y\\0&1\end{pmatrix},$$

and hence is a commutative affine Hopf k-algebra. We denote by (E,t_c) the D-variety structure on E induced by δ. Writing $t_c=(\mathrm{id},\bar{t}_c)$, we have

$$\bar{t}_c\begin{pmatrix}a&b\\0&1\end{pmatrix}=\begin{pmatrix}ab&\frac{1}{2}b^2+c(1-a^2)\\0&0\end{pmatrix}.$$

Now a straightforward computation shows that

$$\begin{aligned}\bar{t}_c\left(\begin{pmatrix}a&b\\0&1\end{pmatrix}\begin{pmatrix}a'&b'\\0&1\end{pmatrix}\right)&=\begin{pmatrix}a^2a'b'+aa'b&\frac{1}{2}a^2(b')^2+abb'+\frac{1}{2}b^2+c(1-(aa')^2)\\0&0\end{pmatrix}\\&=\bar{t}_c\begin{pmatrix}a&b\\0&1\end{pmatrix}\begin{pmatrix}a'&b'\\0&1\end{pmatrix}+a\begin{pmatrix}a&b\\0&1\end{pmatrix}\bar{t}_c\begin{pmatrix}a'&b'\\0&1\end{pmatrix}.\end{aligned}$$

That is, (E,t_c) is an x-twisted D-group. (Note that $x\in R$ is group-like.) Note that since (E,t_c) is not a D-group, we cannot use Theorem 2.20 to deduce the δ-DME. However, since the Krull dimension is two, (E,t_c) does satisfy the δ-DME (see Proposition 2.5).

Our strategy for proving Theorem 3.1 is to show that every a-twisted D-group over the constants admits the example described above as an image, with each fibre having the property that every D-subvariety is compound isotrivial. From the δ-DME for (E, t_c), together with our earlier work around compound isotriviality and maximum proper D-subvarieties, we will then be able to conclude that every a-twisted D-group satisfies the δ-DME.

To relate an arbitrary a-twisted D-group to one of those considered in Example 3.4, we will require the following proposition, whose proof is rather technical, and for which it would be nice to give a conceptual explanation.

Proposition 3.5. *Suppose R is a commutative affine Hopf k-algebra, and δ is a k-linear derivation on R that is an a-coderivation for some group-like $a \in R$. Then for some $c \in k$ we have*

$$a\delta^2 a = \frac{3}{2}(\delta a)^2 + c(a^2 - a^4).$$

We delay the proof of this proposition until we have established the preliminary Lemmas 3.6 and 3.7 below, for which we fix a commutative affine Hopf k-algebra R, equipped with a k-linear derivation δ, such that δ is also an a-coderivation for some group-like $a \in R$. As a is group-like it is invertible in R. We fix the following sequence of elements in R:

$$u_0 := a, \qquad u_1 := \frac{\delta a}{a}, \qquad u_2 := \delta u_1 - \frac{1}{2}u_1^2, \qquad u_m := \delta(u_{m-1}) \quad \text{for } m \geq 3.$$

Note that the desired identity $a\delta^2 a = \frac{3}{2}(\delta a)^2 + c(a^2 - a^4)$ is equivalent to $u_2 = c(1 - a^2)$; this is our eventual aim.

Lemma 3.6. *For all $m \geq 1$, we have*

$$\Delta(u_m) = u_m \otimes 1 + a^m \otimes u_m + \sum_{j=2}^{m-1} c_{j,m} a^j u_{m-j} \otimes u_j + \sum f_i \otimes g_i,$$

where the $c_{j,m}$ are positive (nonzero) integers, the $f_i \in (u_1, \ldots, u_{m-1})^2 k[u_0, \ldots, u_{m-1}]$, and the $g_i \in k[u_0, \ldots, u_{m-1}]$.

Proof. We can compute the coproducts of the elements $u_0, u_1, \ldots$ using the fact that $a = u_0$ is group-like and δ is an a-coderivation:

$$\Delta(u_0) = a \otimes a,$$
$$\Delta(u_1) = \Big(\frac{1}{a} \otimes \frac{1}{a}\Big)(\delta a \otimes a + a^2 \otimes \delta a) = u_1 \otimes 1 + a \otimes u_1,$$
$$\Delta(u_2) = \delta u_1 \otimes 1 + \delta a \otimes u_1 + a^2 \otimes \delta u_1 - \frac{1}{2}u_1^2 \otimes 1 - au_1 \otimes u_1 - a^2 \otimes \frac{1}{2}u_1^2 = u_2 \otimes 1 + a^2 \otimes u_2.$$

Then for $m = 1, 2$, the conclusion of the statement of the lemma follows from the above computations with $f_i = g_i = 0$ and the middle sum being empty. Now one computes $\Delta(u_{m+1}) = \Delta(\delta u_m)$ for $m \geq 2$, using the inductively given expression for $\Delta(u_m)$ and the fact that δ is an a-coderivation. The rest is a straightforward brute force computation that we leave to the reader. □

Lemma 3.7. *There exist $n \geq 1$, a polynomial $P \in k[u_0, \dots, u_{n-1}]$, and some $r \geq 0$ such that*

$$u_n = \frac{P(u_0, \dots, u_{n-1})}{u_0^r}.$$

Proof. Since R is finitely generated as a k-algebra, this sequence (u_m) cannot be algebraically independent over k. Choose n minimal such that $(u_0, \dots, u_n)$ is algebraically dependent over k. Note that if $n = 0$ then $a = u_0$ is a constant and so $u_1 = u_2 = 0$ by definition. So we assume that $n > 0$.

So there is some $d \geq 1$ such that

$$u_n^d + \sum_{i<d} A_i(u_0, \dots, u_{n-1}) u_n^i = 0, \tag{3-3}$$

with $A_0, \dots, A_{d-1}$ rational functions over k. We may assume that d is minimal. Our first step is to show that $d = 1$.

Since $R = k[G]$ for some connected affine algebraic group G over k, we have that $R \otimes R$ is a domain. Indeed, $R \otimes R = k[G \times G]$ and $G \times G$ is a connected affine algebraic group. We can thus work inside the fraction field of $R \otimes R$. Let F be the subfield which is the fraction field of $k[u_0, \dots, u_{n-1}] \otimes_k k[u_0, \dots, u_{n-1}]$. Note that by the minimality of d, $\{1, u_n, \dots, u_n^{d-1}\}$ is linearly independent over $k(u_0, \dots, u_{n-1})$, from which it follows that $\{u_n^i \otimes u_n^j : 0 \leq i, j < d\}$ is linearly independent over F. Applying Δ to both sides of (3-3), Lemma 3.6 gives us that

$$(u_n \otimes 1 + a^n \otimes u_n)^d \in \sum_{i<d} F \cdot (u_n \otimes 1 + a^n \otimes u_n)^i \subseteq \sum_{i+j<d} F \cdot (u_n^i \otimes u_n^j).$$

On the other hand, $u_n^d \otimes 1$ and $a^{nd} \otimes u_n^d$ are also in $\sum_{i+j<d} F \cdot (u_n^i \otimes u_n^j)$ by (3-3). It follows that

$$\sum_{i=1}^{d-1} \binom{d}{i} a^{n(d-i)} u_n^i \otimes u_n^{d-i} \in \sum_{i+j<d} F \cdot (u_n^i \otimes u_n^j).$$

If $d > 1$ then $u_n^{d-1} \otimes u_n$ appears with a nonzero coefficient on the left-hand side but with zero coefficient on the right-hand side. This contradicts the F-linear independence of $(u_n^i \otimes u_n^j : 0 \leq i, j < d)$.

So $d = 1$, and we have that

$$u_n = \frac{P(u_0, \dots, u_{n-1})}{Q(u_0, \dots, u_{n-1})}, \tag{3-4}$$

for some relatively prime polynomials P and Q over k. We aim to show that Q is a monomial in u_0.

First we argue that $Q \otimes Q$ divides $\Delta(Q)$ in $S := k[u_0, \dots, u_{n-1}] \otimes_k k[u_0, \dots, u_{n-1}]$, which we note is the polynomial ring over k in the variables $u_i \otimes 1, 1 \otimes u_j$, and hence is a UFD. Indeed,

$$\begin{aligned}
\Delta(P) &= \Delta(Q)\Delta(u_n) && \text{by applying } \Delta \text{ to both sides of (3-4)} \\
&= \Delta(Q)(u_n \otimes 1 + a^n \otimes u_n + y) && \text{by Lemma 3.6, for some } y \in S \\
&= \Delta(Q)((P/Q) \otimes 1 + a^n \otimes (P/Q) + y).
\end{aligned}$$

We can then multiply both sides by $1 \otimes Q$ to see that $\Delta(Q)((P/Q) \otimes Q) \in S$. Hence, multiplying by $Q \otimes 1$, we see that $Q \otimes 1$ divides $\Delta(Q)(P \otimes Q) = \Delta(Q)(P \otimes 1)(1 \otimes Q)$. Since P and Q are relatively prime, $Q \otimes 1$ divides $\Delta(Q)$. A similar argument shows that $1 \otimes Q$ divides $\Delta(Q)$. Since we are working in a UFD and $1 \otimes Q$ and $Q \otimes 1$ are relatively prime, we see that $Q \otimes Q$ divides $\Delta(Q)$, as desired.

Let $i \le n-1$ be the largest index for which u_i appears in Q. Then we can write

$$Q = \sum_{j=0}^{M} u_i^j Q_j(u_0, u_1, \dots, u_{i-1})$$

with $M > 0$ and Q_M nonzero. So

$$Q \otimes Q = (u_i^M \otimes u_i^M)(Q_M \otimes Q_M) + \sum_{j,k<M} (u_i^j \otimes u_i^k)(Q_j \otimes Q_k)$$

while, if $i \ge 1$, then

$$\Delta(Q) = \sum_{j=0}^{M} \Delta(u_i)^j Q_j(\Delta(u_0), \dots, \Delta(u_{i-1})) = \sum_{\ell+m\le M} f_{\ell,m}(u_i^\ell \otimes u_i^m),$$

where $f_{\ell,m} \in k[u_0, \dots, u_{i-1}] \otimes_k k[u_0, \dots, u_{i-1}]$, by Lemma 3.6. (Note that Lemma 3.6 fails for u_0, so we are using that $i \ge 1$ in the above calculation.) But this contradicts $Q \otimes Q$ dividing $\Delta(Q)$, since $u_i^M \otimes u_i^M$ appears in the former while in the latter no $u_i^\ell \otimes u_i^m$ appears with $\ell, m \ge M$. So it must be that $i = 0$ and we have shown that Q is a polynomial in u_0.

Multiplying by a nonzero scalar if necessary, we may assume that Q is in fact a polynomial in u_0 with leading coefficient 1. Let M denote the degree of Q. Then $Q(u_0) \otimes Q(u_0)$ divides $\Delta(Q) = Q(u_0 \otimes u_0)$ in S, recalling that $u_0 = a$ is group-like. Since both $Q(u_0) \otimes Q(u_0)$ and $Q(u_0 \otimes u_0)$ are polynomials of total degree $2M$ in the variables $u_0 \otimes 1$ and $1 \otimes u_0$ and since they both have leading coefficient 1, we see that they must be the same. In particular, $Q(u_0) \otimes Q(u_0)$ is a polynomial in $u_0 \otimes u_0$ with leading coefficient 1, which implies that $Q(u_0)$ is of the form u_0^r. □

Proof of Proposition 3.5. Let (R, δ), a, and the u_i be as above. We need to show that $u_2 = c(1-a^2)$ for some $c \in k$. By Lemma 3.7 we have that there is some $n \ge 1$, some $r \ge 0$ and some polynomial $P \in k[u_0, \dots, u_{n-1}]$ such that

$$u_n = \frac{P(u_0, \dots, u_{n-1})}{u_0^r}, \tag{3-5}$$

for some polynomial P over k. Our first step is to show that $n \le 2$.

Let $S := k[u_0, \dots, u_{n-1}] \otimes_k k[u_0, \dots, u_{n-1}]$. Let

$$I := (u_1, \dots, u_{n-1})^2 k[u_0, \dots, u_{n-1}]$$

and consider the ideal of S given by

$$J := I \otimes k[u_0, \dots, u_{n-1}] + k[u_0, \dots, u_{n-1}] \otimes I.$$

So these are the elements of S in which each monomial has degree at least 2 in either the variables $u_1 \otimes 1, \dots, u_{n-1} \otimes 1$, or in the variables $1 \otimes u_1, \dots, 1 \otimes u_{n-1}$. Using Lemma 3.6 we can compute that for $1 \le i, j, \ell \le n-1$,

$$\Delta(u_i u_j u_\ell) \in J. \tag{3-6}$$

Moreover, for $1 \le i, j \le n-1$, Lemma 3.6 gives

$$\Delta(u_i u_j) = u_0^j u_i \otimes u_j + u_0^i u_j \otimes u_i \mod J. \tag{3-7}$$

Now, write the polynomial P of (3-5) as

$$P = P_0(u_0) + \sum_{i=1}^{n-1} P_i(u_0) u_i + \sum_{1 \le i \le j \le n-1} P_{i,j}(u_0) u_i u_j + H,$$

where H is of degree at least three in $u_1, \dots, u_{n-1}$. Applying Δ to both sides of (3-5) we get $(u_0 \otimes u_0)^r \Delta(u_n) = \Delta(P)$. We therefore have

$$(u_0 \otimes u_0)^r \Delta(u_n) = P_0(u_0 \otimes u_0) + \sum_{i=1}^{n-1} P_i(u_0 \otimes u_0) \Delta(u_i) + \sum_{i \le j} P_{i,j}(u_0 \otimes u_0) \Delta(u_i u_j) + \Delta(H). \tag{3-8}$$

We claim that this forces $P_i = 0$ for all $i = 1, \dots, n-1$. To prove this, note that by Lemma 3.6 and (3-5), both sides of (3-8) are elements of the polynomial ring

$$k(u_0 \otimes 1, 1 \otimes u_0)[u_1 \otimes 1, \dots, u_{n-1} \otimes 1, 1 \otimes u_1, \dots, 1 \otimes u_{n-1}].$$

We first compute, for both sides of (3-8), the coefficient of $u_i \otimes 1$. On the right-hand side, using equations (3-6) and (3-7), the only term that contributes is $P_i(u_0 \otimes u_0)\Delta(u_i)$. By Lemma 3.6, that contribution is $P_i(u_0 \otimes u_0)$. On the left-hand side, using Lemma 3.6 and (3-5), the coefficient of $u_i \otimes 1$ is $(u_0 \otimes u_0)^r ((P_i(u_0)/u_0^{-r}) \otimes 1)$. So $P_i(u_0) \otimes u_0^r = P_i(u_0 \otimes u_0)$. This forces $P_i = d u_0^r$ for some $d \in k$. On the other hand, comparing the coefficient of $1 \otimes u_i$ on both sides of (3-8) we have that $u_0^{r+n} \otimes P_i(u_0) = P_i(u_0 \otimes u_0)(u_0^i \otimes 1)$. Plugging in $P_i = d u_0^r$ we get that $d(u_0^{r+n} \otimes u_0^r) = d(u_0^{r+i} \otimes u_0^r)$. As $i < n$, this forces $d = 0$ and hence $P_i = 0$.

Equation (3-8) therefore becomes

$$(u_0 \otimes u_0)^r \Delta(u_n) = P_0(u_0 \otimes u_0) + \sum_{1 \le i \le j \le n-1} P_{i,j}(u_0 \otimes u_0)(u_0^j u_i \otimes u_j + u_0^i u_j \otimes u_i) \mod J.$$

Assume towards a contradiction that $n \ge 3$. Then by Lemma 3.6 we must have $u_1 \otimes u_{n-1}$ appearing in $\Delta(u_n)$ on the left with a nonzero coefficient. So $P_{1,n-1} \ne 0$. But then $P_{1,n-1}(u_0 \otimes u_0)(u_0 u_{n-1} \otimes u_1)$ appears on the right, while it does not appear on the left since $u_{n-1} \otimes u_1$ does not appear in $\Delta(u_n)$ by Lemma 3.6.

This contradiction proves that $n \le 2$. Suppose $n = 1$. Then (3-5) says $u_1 = P(u_0)/u_0^r$. Applying Δ to both sides yields

$$P(u_0) \otimes u_0^r + u_0^{r+1} \otimes P(u_0) = P(u_0 \otimes u_0),$$

which is only possible if $P_0 = 0$. Hence $u_1 = u_2 = 0$, as desired.

So we are left to consider the case when $n = 2$. Equation (3-5) becomes

$$u_2 = \frac{1}{u_0^r} \sum_{j=0}^{M} P_j(u_0) u_1^j$$

with $M \geq 1$, the P_j are polynomials over k, and P_M is nonzero. Multiplying by a^r (recall that $u_0 = a$) and applying Δ gives

$$(a^r \otimes a^r)(u_2 \otimes 1 + a^2 \otimes u_2) = \sum_{j=0}^{M} P_j(a \otimes a)(u_1 \otimes 1 + a \otimes u_1)^j,$$

which we can write as

$$\left(\sum_{j=0}^{M} P_j(a) u_1^j \otimes a^r + \sum_{j=0}^{M} a^{r+2} \otimes P_j(a) u_1^j \right) = \sum_{j=0}^{M} P_j(a \otimes a)(u_1 \otimes 1 + a \otimes u_1)^j.$$

Notice that if $M > 1$ then the right-hand side involves terms with $u_1^i \otimes u_1^j$ with $i, j \geq 1$, while the left-hand side does not, and so we cannot have equality. Thus $M = 1$. Writing out the above equation with this in mind we get that

$$P_0(a) \otimes a^r + P_1(a) u_1 \otimes a^r + a^{r+2} \otimes P_0(a) + a^{r+2} \otimes P_1(a) u_1 = P_0(a \otimes a) + P_1(a \otimes a)(u_1 \otimes 1 + a \otimes u_1).$$

We look at this as an equation in $k[a \otimes 1, 1 \otimes a][u_1 \otimes 1, 1 \otimes u_1]$. Then taking the coefficient of $u_1 \otimes 1$ gives that

$$P_1(a) \otimes a^r = P_1(a \otimes a),$$

which can only occur if $P_1 = da^r$ for some $d \in k$. Then computing the coefficient of $1 \otimes u_1$ gives $a^{r+2} \otimes da^r = d(a^{r+1} \otimes a^r)$, and so $d = 0$. Hence $P_1 = 0$. Now taking the constant coefficient (regarding constants as being in $k[a \otimes 1, 1 \otimes a]$) gives that

$$P_0(a) \otimes a^r + a^{r+2} \otimes P_0(a) = P_0(a \otimes a).$$

Now write $P_0(t) = \sum_{j=0}^{L} p_j t^j$. Then we have

$$\sum_{j=0}^{L} p_j (a^j \otimes a^r + a^{r+2} \otimes a^j - a^j \otimes a^j) = 0.$$

Notice that if $j \notin \{r, r+2\}$ then we have that the coefficient of $a^j \otimes a^j$ on the left-hand side is equal to p_j whereas the right-hand side is zero and so $p_j = 0$. It follows that $P_0(t) = p_r t^r + p_{r+2} t^{r+2}$. Then

$$0 = \sum_{j=0}^{L} p_j (a^j \otimes a^r + a^{r+2} \otimes a^j - a^j \otimes a^j) = p_r a^{r+2} \otimes a^r + p_{r+2} a^{r+2} \otimes a^r.$$

This forces $p_r = p_{r+2}$ and so we see that $P_0(t) = c(t^r - t^{r+2})$ for some constant $c \in k$. Thus, $u_2 = \frac{1}{u_0^r}(P_0(u_0) + P_1(u_0)u_1) = c(1 - u_0^2) = c(1 - a^2)$, as desired. $\square$

The following is a geometric interpretation of the proposition.

Proposition 3.8. *Suppose $k \subseteq K^\delta$ and (G,s) is an affine connected a-twisted D-group over k, where $a \in k[G]$ is group-like. Then*

$$g \mapsto \begin{pmatrix} a(g) & \frac{\delta a(g)}{a(g)} \\ 0 & 1 \end{pmatrix}$$

defines a homomorphism $\pi : G \to E$, where $E \le \mathrm{GL}_2$ is the algebraic subgroup made up of matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, there exists some $c \in k$ such that if (E, t_c) is the a-twisted D-group from Example 3.4, then $\pi : (G,s) \to (E,t_c)$ is a D-morphism.

Proof. Recall that as $a \in k[G]$ is group-like, $a : G \to \mathbb{G}_m$ is a homomorphism of algebraic groups. It follows immediately that π is well-defined and does indeed map G to E. We check that it is a group homomorphism. Given $g, h \in G$, note first of all that as $\Delta(a) = a \otimes a$ and δ is an a-coderivation, we have $\Delta(\delta a) = \delta a \otimes a + a^2 \otimes \delta a$ and so

$$\delta a(gh) = \Delta(\delta a)(g,h) = \delta a(g)a(h) + a(g)^2 \delta a(h). \tag{3-9}$$

We can therefore compute

$$\begin{aligned} \pi(g)\pi(h) &= \begin{pmatrix} a(g) & \frac{\delta a(g)}{a(g)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a(h) & \frac{\delta a(h)}{a(h)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a(gh) & a(g)\frac{\delta a(h)}{a(h)} + \frac{\delta a(g)}{a(g)} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a(gh) & \frac{a(g)^2\delta a(h) + \delta a(g) a(h)}{a(gh)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a(gh) & \frac{\delta a(gh)}{a(gh)} \\ 0 & 1 \end{pmatrix} = \pi(gh), \end{aligned}$$

where we have used $a(gh) = a(g)a(h)$ repeatedly and (3-9) in the penultimate equality. We note that we have not up until this point used the parameter $c \in k$; the reason for this is that the groups E_c are isomorphic as algebraic groups.

It remains to show that π is a D-morphism from (G,s) to some (E,t_c). Let c be as given by Proposition 3.5. It suffices to show that π takes D-points to D-points. That is, if $g \in (G,s)^\sharp(K)$ then

$$\begin{pmatrix} a(g) & \frac{\delta a(g)}{a(g)} \\ 0 & 1 \end{pmatrix}$$

should be a D-point of (E,t_c). Writing $t_c = (\mathrm{id}, \bar{t_c})$ we have that

$$\begin{aligned} \bar{t_c}\begin{pmatrix} a(g) & \frac{\delta a(g)}{a(g)} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} \delta a(g) & \frac{\delta a(g)^2}{2a(g)^2} + c(1 - a(g)^2) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \delta a(g) & \frac{a(g)\delta^2 a(g) - \delta a(g)^2 - c(a(g)^2 - a(g)^4)}{a(g)^2} + c(1 - a(g)^2) \\ 0 & 0 \end{pmatrix} \\ &= \delta\begin{pmatrix} a(g) & \frac{\delta a(g)}{a(g)} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where the first step comes from Example 3.4, the second step follows from Proposition 3.5 telling us that $a\delta^2 a = \frac{3}{2}(\delta a)^2 + c(a^2 - a^4)$, and in the final equality we are using the fact that as g is a D-point of G, $\delta(r(g)) = (\delta r)(g)$ for all $r \in k[G]$. This shows that $\pi(g) \in (E, t_c)^\sharp(K)$, as desired. □

We can now complete the proof of the theorem.

Proof of Theorem 3.1. We have already established that (R, δ) is the coordinate ring of an affine connected a-twisted D-group (G, s), where $a \in k[G]$ is group-like. Here recall that $k \subseteq K^\delta$. By Proposition 2.3, it suffices to show that every irreducible δ-rational D-subvariety of G over k is δ-locally-closed.

Let $\pi : (G, s) \to (E, t_c)$ be the D-morphism from Proposition 3.8. We first show that every fibre of this map has the property that all its D-subvarieties, over arbitrary δ-field extensions, are compound isotrivial.

Let us start with the fibre above the identity, that is, $H = \ker(\pi)$. Since π is a D-morphism, (H, s) is a D-subvariety of (G, s). Here, by abuse of notation, we write (H, s) instead of $(H, s\restriction_H)$. Since π is an algebraic group homomorphism H is an algebraic subgroup of G. It follows that (H, s) is an $a\restriction_H$-twisted D-group also. On the other hand, $a\restriction_H = 1$ by the definition of π. So (H, s) is an actual D-group. By Proposition 2.16, every irreducible D-subvariety of H, over any δ-field extension of k, is compound isotrivial.

What about other fibres of π over D-points of (E, t_c)? Any such fibre is a D-subvariety of (G, s) of the form Hg, for some $g \in (G, s)^\sharp(K)$. Since (G, s) is not necessarily a D-group, the multiplication-by-g-on-the-right map, $\rho_g : G \to G$, is not necessarily a D-automorphism. Nevertheless, when we restrict this map to H we do get a D-isomorphism between H and Hg. To see this we need only check that ρ_g takes D-points of H to D-points of Hg. Letting $h \in (H, s)^\sharp(K)$ we compute

$$\begin{aligned}
\bar{s}(hg) &= \bar{s}(h)g + a(h)h\bar{s}(g) && \text{by (3-2)}\\
&= \bar{s}(h)g + h\bar{s}(g) && \text{as } a\restriction_H = 1\\
&= \delta(h)g + h\delta(g) && \text{as } h \text{ and } g \text{ are } D\text{-points}\\
&= \delta(hg) && \text{as } \nabla : G \to TG \text{ is a group homomorphism}
\end{aligned}$$

as desired. So H and Hg are D-isomorphic over $k(g)$. It follows that every fibre of π above a D-point has the property that all its D-subvarieties, over arbitrary δ-field extensions, are compound isotrivial.

Now suppose that $V \subseteq G$ is an irreducible δ-rational D-subvariety over k. We need to prove that it has a maximum proper D-subvariety over k. Let $W \subseteq E$ be the D-subvariety obtained by taking the Zariski closure of the image of V under π, and consider the dominant D-morphism $\pi\restriction_V : (V, s) \to (W, t_c)$. Since $k(W) \subseteq k(V)$, W is also δ-rational. Since (E, t_c) is of dimension two, it satisfies the δ-DME by Proposition 2.5. Hence W has a maximum proper D-subvariety over k. Next, let η be a k-generic D-point of W and consider the fibre V_η. Note that V_η is δ-rational since $k(\eta)(V_\eta) = k(V)$. But V_η is a D-subvariety of the fibre of $\pi : (G, s) \to (E, t_c)$ above the D-point η, and hence, as we have argued above, is compound isotrivial. So, by Proposition 2.13, V_η has a maximum proper D-subvariety over $k(\eta)$. We have shown that both the image and the generic fibre have maximum proper D-subvarieties, and so by Lemma 2.6, (V, s) has a maximum proper D-subvariety over k, as desired. □

Remark 3.9. In the end of the above proof we could also have used the fact that (E, t_c), while not in general isotrivial, is compound isotrivial in two steps. This was observed by Ruizhang Jin, in whose Ph.D. thesis this example will be worked out. In any case, using the compound isotriviality of (E, t_c) the above arguments actually give that every D-subvariety of (G, s) is compound isotrivial (in at most five steps) from which it follows by Corollary 2.14 that (G, s) satisfies the δ-DME.

4. The DME for Ore extensions of commutative Hopf algebras

We now apply the results of the previous sections to the classical study of certain (noncommutative) Hopf algebras. Recall that if A is a noetherian associative algebra over a field k of characteristic zero, then we say that the *Dixmier–Moeglin equivalence* (DME) *holds for A* if for every (two-sided) prime ideal P of A, the following are equivalent:

(i) P is *primitive*: it is the annihilator of a simple left A-module.

(ii) P is *locally closed*: the intersection of all the prime ideals of A that properly contain P is a proper extension of P.

(iii) P is *rational*: the centre of the Goldie quotient ring[3] $\mathrm{Frac}(A/P)$ is an algebraic field extension of k.

Of course, for commutative algebras the DME always holds as the notions of primitive, locally closed, and rational all coincide with maximal.

It is known that in any algebra that satisfies the Nullstellensatz locally closed implies primitive and primitive implies rational; see [Brown and Goodearl 2002, II.7.16]. Thus, the central question is when does rational imply locally closed? Certainly this is not always the case; even in finite Gelfand–Kirillov dimension a counterexample was found in [Bell et al. 2017a]. In [Bell and Leung 2014] the DME was conjectured specifically about all Hopf algebras of finite Gelfand–Kirillov dimension.

We show here that the DME holds for Hopf algebras that arise as certain twisted polynomial rings over commutative Hopf algebras. Recall that if R is a k-algebra equipped with an automorphism σ then a k-linear *σ-derivation* is a k-linear map δ satisfying the twisted Leibniz rule:

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s.$$

Given σ and δ, the *Ore extension* of R, denoted by $R[x; \sigma, \delta]$ is the ring extension of R with the property that it is a free left R-module with basis $\{x^n : n \geq 0\}$ and such that $xr = \sigma(r)x + \delta(r)$ for all $r \in R$. We aim to prove the DME for Hopf algebras that arise as the Ore extensions of commutative Hopf algebras. The next theorem makes this precise.

Theorem 4.1. *Suppose k is a field of characteristic zero and R is a commutative affine Hopf k-algebra equipped with a k-algebra automorphism σ and a k-linear σ-derivation δ. Assume that the Ore extension $A := R[x; \sigma, \delta]$ admits a Hopf algebra structure extending that of R. Then A satisfies the* DME.

[3] The Goldie quotient is an artinian ring of quotients for any prime noetherian ring that imitates the field of fractions construction for integral domains in the commutative case. See [McConnell and Robson 2001, Chapter 2] for details.

This is Theorem B2 of the introduction. Its proof is preceded by a number of preliminaries.

4A. ***Hopf Ore extensions.*** In this section we prove a result (Corollary 4.4, below) that severely restricts what $(R, \Delta, \sigma, \delta)$ can be if $A = R[x; \sigma, \delta]$ is to admit a Hopf algebra structure extending that on R. Actually this was already done in [Brown et al. 2015], answering a question of Panov [2003], in a more general context where R is not necessarily commutative, but under the additional assumption on A that

$$\Delta(x) = a \otimes x + x \otimes b + v(x \otimes x) + w \tag{4-1}$$

for some $a, b \in R$ and $v, w \in R \otimes_k R$. When (4-1) holds, possibly after a change of the variable x, Brown et al. call $R[x; \sigma, \delta]$ a *Hopf Ore* extension. They ask if every Ore extension admitting a Hopf algebra structure extending that on R is a Hopf Ore extension. We prove that this is the case, in a strong way, when R is commutative and affine (this is Theorem C of the introduction):

Theorem 4.2. *Suppose k is an algebraically closed field of characteristic zero and R is a commutative affine Hopf k-algebra equipped with a k-algebra automorphism σ and a k-linear σ-derivation δ. If $R[x; \sigma, \delta]$ admits a Hopf algebra structure extending that of R then, after a linear change of the variable x,*

$$\Delta(x) = a \otimes x + x \otimes b + w$$

for some $a, b \in R$, each of which is either 0 or group-like, and some $w \in R \otimes_k R$.

Proof. Our starting point is [Brown et al. 2015, §2.2, Lemma 1] which says that if $R \otimes_k R$ is a domain (which is true here as R is a commutative domain and k is algebraically closed) then

$$\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + v(x \otimes x) + w, \tag{4-2}$$

where $s, t, v, w \in R \otimes_k R$. We let $A = R[x; \sigma, \delta]$ and let S denote the antipode of A. By making a substitution $x \mapsto x - \lambda$ for some $\lambda \in k$, we may assume that $\epsilon(x) = 0$, where $\epsilon : A \to k$ is the counit. This substitution does not change the form of $\Delta(x)$ given in (4-2).

Our first goal is to show that $v = 0$. Recall that since R is commutative it is of the form $R = k[G]$ for a connected affine algebraic group G. We can therefore view $v \in R \otimes_k R$ as a regular function on $G \times G$. We show first that $v(g^{-1}, g) = 0$ for all $g \in G$, and then that in fact $v = 0$.

We now consider the antipode S. By [Skryabin 2006, Corollary 1], S is bijective on A and its restriction to R is bijective on R. Thus we can write

$$S(x) = a_0 + a_1 x + \cdots + a_d x^d$$

for some $d \geq 1$ and $a_0, \ldots, a_d \in R$ with $a_d \neq 0$. Writing $m : A \otimes_k A \to A$ for the homomorphism induced by multiplication, we have the identity

$$m \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \epsilon(x).$$

So, as $\epsilon(x) = 0$, we may let $\mu = m \circ (S \otimes \mathrm{id})$ and use (4-2) to write

$$0 = \mu(s)x + S(x)\mu(t) + S(x)\mu(v)x + \mu(w).$$

Notice that $m \circ (S \otimes \mathrm{id})(R \otimes R) \subseteq R$ and so if we look at the coefficient of x^{d+1} on the right-hand side, we see that it is $a_d \sigma^d(\mu(v))$. Since $R \otimes R$ is a domain and a_d is nonzero and σ is an automorphism, we see that $\mu(v) = m \circ (S \otimes \mathrm{id})(v) = 0$. Geometrically, this means precisely that $v(g^{-1}, g) = 0$ for all $g \in G$.

Next we apply coassociativity, which tells us that $(\Delta \otimes \mathrm{id})(\Delta(x)) = (\mathrm{id} \otimes \Delta)(\Delta(x))$ in $R \otimes_k R \otimes_k R = k[G \times G \times G]$. Writing this out using (4-2), and equating the coefficients of $x \otimes x \otimes x$, yields

$$(\Delta \otimes \mathrm{id})(v) \cdot (v \otimes 1) = (\mathrm{id} \otimes \Delta)(v) \cdot (1 \otimes v).$$

Evaluating at (g, h^{-1}, h) for any fixed $g, h \in G$ we get

$$v(gh^{-1}, h) v(g, h^{-1}) = v(g, 1_G) v(h^{-1}, h) = 0,$$

where the final equality uses what we proved in the previous paragraph. Now, if $v \neq 0$ then for a Zariski dense set of $(g, h) \in G \times G$, $v(g, h^{-1}) \neq 0$. But then for each such (g, h) the above equation implies that $v(gh^{-1}, h) = 0$. Hence, in fact, $v(gh^{-1}, h) = 0$ for all $(g, h) \in G \times G$. As every element of $G \times G$ can be written in the form (gh^{-1}, h), we have shown that $v = 0$.

We have thus proven that

$$\Delta(x) = s(1 \otimes x) + t(x \otimes 1) + w \tag{4-3}$$

for some $s, t, w \in R \otimes_k R$.

We claim now that either $t = 0$ or $t = 1 \otimes b$ for some group-like $b \in R$. We again apply coassociativity to x, this time using (4-3) and equating the coefficients of $x \otimes 1 \otimes 1$, to get

$$(\Delta \otimes \mathrm{id})(t) \cdot (t \otimes 1) = (\mathrm{id} \otimes \Delta)(t)$$

The geometric interpretation is that

$$t(fg, h) t(f, g) = t(f, gh) \tag{4-4}$$

for all $f, g, h \in G$.

Suppose $t(1_G, g_0) = 0$ for some $g_0 \in G$. We show in this case that $t = 0$. Indeed, for all $h \in G$ we have $0 = t(g_0, h) t(1_G, g_0) = t(1_G, g_0 h)$, by (4-4) with $f = 1_G$. Hence, $t(1_G, h) = 0$ for all $h \in G$. But then, by (4-4) with $f = g^{-1}$, we get $0 = t(1_G, h) = t(g^{-1}g, h) t(g^{-1}, g) = t(g^{-1}, gh)$ for all $g, h \in G$. As every element of $G \times G$ is of the form (g^{-1}, gh) for some $g, h \in G$, we have $t = 0$, as desired.

Suppose on the contrary that $t(1_G, g) \neq 0$ for every $g \in G$. Then $t(g, h) = t(1_G, gh)/t(1_G, g)$ is a never vanishing regular function on $G \times G$, and hence $t = \lambda t'$, where $\lambda \in k^*$ and $t' : G \times G \to \mathbb{G}_m$ is an algebraic group homomorphism (see [Rosenlicht 1961, Theorem 3]). So $t' = b' \otimes b$, where $b', b \in R$ are group-like. But then we have

$$\lambda b'(g) b(h) = t(g, h) = \frac{t(1_G, gh)}{t(1_G, g)} = \frac{\lambda b(gh)}{\lambda b(g)} = \frac{\lambda b(g) b(h)}{\lambda b(g)}$$

for all $g, h \in G$. It follows that $b' = \lambda = 1$ and $t = 1 \otimes b$, as desired.

A similar argument shows that in (4-3) either $s = 0$ or $s = a \otimes 1$ for some group-like $a \in R$. This proves the theorem. $\square$

Remark 4.3. It may be worth pointing out that our proof of Theorem 4.2 made no use of δ. We used only the properties of a Hopf algebra extension and the fact that σ is injective, as well as the fact that every element of A can be written as a left polynomial in x over R.

Corollary 4.4. *Suppose k is an algebraically closed field of characteristic zero and R is a commutative affine Hopf k-algebra equipped with a k-algebra automorphism σ and a k-linear σ-derivation δ. If $R[x;\sigma,\delta]$ admits a Hopf algebra structure extending that of R then (R,Δ,σ,δ) must satisfy the following two conditions*:

(1) *There exists $w \in R \otimes_k R$ and a group-like $a \in R$ such that, for all $r \in R$,*

$$\Delta(\delta(r)) = \sum(\delta(r_1) \otimes r_2 + ar_1 \otimes \delta(r_2)) + w(\Delta(r) - \Delta(\sigma(r))).$$

(2) *There is a character $\chi : R \to k$ such that for all $r \in R$,*

$$\sigma(r) = \sum \chi(r_1)r_2 = \sum r_1\chi(r_2).$$

In the above we are using Sweedler notation, writing $\Delta(r) = \sum r_1 \otimes r_2$ for all $r \in R$.

Proof. Statements (1) and (2) are proven for Hopf Ore extensions in [Brown et al. 2015]. Indeed, remembering that in our case R is commutative, statement (2) is just part (i)(c) of the main theorem of [Brown et al. 2015] (see also their Theorem 2.4(d)), and statement (1) is the identity labelled (21) in [Brown et al. 2015] which is asserted in part (i)(d) of their main theorem. So to prove the corollary it suffices to show that $A = R[x;\sigma,\delta]$ is a Hopf Ore extension, that is, after a change of variable $\Delta(x)$ has the form (4-1) discussed above. But Theorem 4.2 gives us an even stronger form for $\Delta(x)$. ☐

Remark 4.5. The main theorem of [Brown et al. 2015] also includes a converse; namely, assuming that (R,Δ,σ,δ) satisfies (1) and (2), with $w \in R \otimes_k R$ satisfying two other identities, one can always extend in a natural way the Hopf algebra structure from R to $R[x;\sigma,\delta]$. This gives many examples to which our Theorem 4.1 will apply.

4B. *The case when σ is the identity.* When $\sigma = \mathrm{id}$ note that a σ-derivation is just a derivation. In this case we write the Ore extension as $R[x;\delta]$; it is the skew polynomial ring in x over R where $xr = rx + \delta(r)$ for all $r \in R$. Statement (1) of Corollary 4.4 now says that if $R[x;\delta]$ admits a Hopf algebra structure extending that on R, then δ must have been an a-coderivation on R. So Theorem 3.1 applies and we have that (R,δ) satisfies the δ-DME. The following proposition relates the δ-DME for (R,δ) to the DME for $R[x;\delta]$.

Proposition 4.6. *Suppose k is a field of characteristic zero and R is a commutative affine k-algebra equipped with a k-linear derivation δ. If the δ-rational prime δ-ideals of (R,δ) are δ-locally-closed, then the rational prime ideals of $R[x;\delta]$ are locally closed.*

*In particular, if the δ-*DME *holds for (R,δ) then the* DME *holds for $R[x;\delta]$.*

Proof. Suppose P is a rational prime ideal of $R[x;\delta]$. Let $I := P \cap R$. Then I is a prime ideal of R (see [Fisher 1975, Corollary to Lemma 2]). Moreover, I is a δ-ideal since if $a \in P \cap R$ then $\delta(a) = [x, a] \in P \cap R$. It follows easily that $J := IR[x;\delta]$ is an ideal of $R[x;\delta]$ that is contained in P and that $R[x;\delta]/J \cong (R/I)[x;\delta]$, where we use δ to denote the induced derivation on $S := R/I$. Let $F = \mathrm{Frac}(S)$ be the field of fractions of S and extend δ to F. We claim that the δ-constants of F are all algebraic over k. Indeed, note that if $f \in F$ with $\delta(f) = 0$ then f is a central element of $F[x;\delta]$. We let $\tilde{P}$ denote the prime ideal in $S[x;\delta]$ corresponding to P under the isomorphism $R[x;\delta]/J \cong S[x;\delta]$. As $P \cap R = I$, we have that $\tilde{P} \cap S = 0$, so that $\tilde{P}$ lifts to a prime ideal P_0 of $F[x;\delta]$. The image of f in $B := F[x;\delta]/P_0$ is again a central element of B. By construction B is a localization of $S[x;\delta]/\widetilde{P} \cong R[x;\delta]/P$ and thus passing to the full localization gives that f is a central element of $\mathrm{Frac}(R[x;\delta]/P)$. As P is rational f must be algebraic over k.

We have shown that the prime δ-ideal I is δ-rational. By assumption it is therefore δ-locally-closed. Consequently, there is some $g \in R \setminus I$ such that every prime δ-ideal of R properly containing I must contain g.

In order to prove that P is locally closed it now suffices to show that whenever $Q \supsetneq P$ is prime then $Q \cap R \supsetneq P \cap R = I$. Indeed, if this is the case, then we have that

$$g \in \bigcap\{Q \cap R : Q \supsetneq P \text{ prime}\}.$$

Since $g \notin P$, we have in particular that

$$\bigcap\{Q : Q \supsetneq P \text{ prime}\} \neq P.$$

That is, P is locally closed.

Towards a contradiction, therefore, let us assume that there exists a prime ideal $Q \supsetneq P$ such that $Q \cap R = P \cap R = I$. It follows that $F[x;\delta]$ is not simple: under the isomorphism $R[x;\delta]/J \cong S[x;\delta]$, Q corresponds to a nonzero prime ideal $\widetilde{Q}$ in $S[x;\delta]$ whose intersection with $S = R/I$ is trivial, so that $\widetilde{Q}$ lifts to a nonzero prime ideal Q_0 in $F[x;\delta]$. On the other hand, it is well-known that, as F is a field of characteristic zero, if δ is nontrivial on F then $F[x;\delta]$ is a simple ring (indeed this is a consequence of the fact that $F[x;\delta]$ is a left and right PID; see [van der Put and Singer 2003, §2.1]). Thus, δ is trivial on F and so $F[x;\delta] = F[x]$ is a PID. So P_0, the lift of $\widetilde{P}$ from $S[x;\delta]$ to $F[x;\delta]$, must be 0, as it is properly contained in Q_0. That is, $F[x;\delta]$ is a localisation of $R[x;\delta]/P$. Hence, $\mathrm{Frac}(R[x;\delta]/P) = F(x)$, contradicting the rationality of P.

For the "in particular" clause, note $R[x;\delta]$ satisfies the Nullstellensatz — by [Irving 1979, Theorem 2] for example — and hence we already know that local-closedness implies primitivity and primitivity implies rationality. □

Corollary 4.7. *Suppose k is an algebraically closed field of characteristic zero and R is a commutative affine Hopf k-algebra equipped with a k-linear derivation δ that is also an a-coderivation for some group-like $a \in R$. Then $R[x;\delta]$ satisfies the* DME.

Proof. Theorem 3.1 together with Proposition 4.6. □

A special case of Corollary 4.7 is when R is a differential Hopf k-algebra — this yields Theorem B1 of the introduction. But the DME for $R[x;\delta]$ in that case is easier: one uses only Theorem 2.20 and the material in Section 3 is not necessary.

4C. *The case when δ is inner.* If σ is an automorphism of R, and $a \in R$, then the map $r \mapsto a(r-\sigma(r))$ is a σ-derivation on R. Such σ-derivations are called *inner*. Here is a sufficient criterion for a σ-derivation δ being inner.[4]

Lemma 4.8. *Suppose R is a commutative ring with an automorphism σ and a σ-derivation δ. Suppose there exists an element $f \in R$ such that $f-\sigma(f)$ is a unit. Then δ is inner.*

Proof. It is easy to see, using the commutativity of R, that $a := \delta(f)/(f-\sigma(f))$ witnesses the innerness of (R,σ,δ). □

When δ is inner the Dixmier–Moeglin equivalence for $R[x;\sigma,\delta]$ follows easily from known results. It makes use however of one more notion:

Definition 4.9. Let A be a finitely generated algebra over a field k. We say that a k-vector subspace V of A is a *frame* for A if V is finite-dimensional, contains 1_A, and generates A as a k-algebra.

Lemma 4.10. *Suppose R is a commutative affine Hopf algebra over a field k of characteristic zero and σ is a k-algebra automorphism of R satisfying statement (2) of Corollary 4.4. Then there is a frame for R such that $\sigma(V)=V$.*

Proof. Suppose $R=k[G]$, where G is an affine algebraic group. Then G is linear and hence we may embed G into GL_n. This gives us a frame V of R spanned by the restriction to G of 1, the coordinate functions $x_{i,j}$, and $\frac{1}{\det}$. Now, $\Delta(x_{i,j})=\sum x_{i,k}\otimes x_{k,j}$ and $\Delta\big(\frac{1}{\det}\big)=\frac{1}{\det}\otimes\frac{1}{\det}$. So $\Delta(V)\subseteq V\otimes V$. Statement (2) of Corollary 4.4 then implies that $\sigma(V)\subseteq V$, and hence by finite-dimensionality $\sigma(V)=V$.[5] □

Proposition 4.11. *Suppose k is an uncountable algebraically closed field of characteristic zero, R is a finitely generated commutative k-algebra, σ is a k-algebra automorphism of R that preserves a frame, and δ is an inner σ-derivation on R. Then $R[x;\sigma,\delta]$ satisfies the* DME.

Proof. When $\delta=0$ this is [Bell et al. 2017b, Theorem 1.6]; while the theorem there is stated for $k=\mathbb{C}$ it holds for any uncountable algebraically closed field. But if $a\in R$ is such that $\delta(r)=a(r-\sigma(r))$ for all $r\in R$, then $R[x;\sigma,\delta]=R[t;\sigma,0]$, where $t:=x-a$. Indeed,

$$tr=(x-a)r=\sigma(r)x+\delta(r)-ar=\sigma(r)x-a\sigma(r)=\sigma(r)t$$

for all $r\in R$, and $\{t^n : n\geq 0\}$ can be seen to be another left R-basis for $R[x;\sigma,\delta]$ using the fact that, for any polynomial P, $P(t)$ is equal to $P(x)$ plus terms of strictly lower degree. So the inner case reduces to the case when $\delta=0$. □

[4] For a more general statement in the noncommutative case, see [Goodearl 1992, Lemma 2.4(b)].

[5] As a referee pointed out to us, the existence of a frame V with $\Delta(V)\subseteq V\otimes V$ can be deduced for arbitrary finitely generated Hopf algebras by starting with any frame W and extending it to a finite-dimensional subcoalgebra V by the finiteness theorem for coalgebras [Montgomery 1993, Theorem 5.1.1].

4D. ***The general case.*** We fix from now on a field k of characteristic zero. Our proof of Theorem 4.1 will go via reducing to the case either when $\sigma = \mathrm{id}$ or when δ is inner. It will require some preparatory lemmas. First, let us point out that statement (2) of Corollary 4.4 forces (R, σ) to be of a very restricted form.

Lemma 4.12. *Let G be a connected affine algebraic group over k and $\tau : G \to G$ an automorphism of G over k. Let $R = k[G]$, and $\sigma = \tau^*$ the corresponding k-algebra automorphism of R. If (R, σ) satisfies statement* (2) *of Corollary 4.4 then $\tau : G \to G$ is translation by some central element of $G(k)$.*

Proof. Since $\chi : R \to k$ is a homomorphism, there is some $c \in G(k)$ such that $\chi(f) = f(c)$ for all $f \in R$. If we write $\Delta(f) = \sum f_1 \otimes f_2$, then by the definition of the coproduct on R we have $f(ab) = \sum f_1(a) f_2(b)$ for all $a, b \in G$. Now, property (2) gives us that for all $a \in G$,

$$\sigma(f)(a) = \sum \chi(f_1) f_2(a) = \sum f_1(c) f_2(a) = f(ca).$$

The other half of the equality in (2) gives $\sigma(f)(a) = f(ac)$. So $f(ca) = f(ac)$ for all $f \in R$, and hence c is central in G. On the other hand, $f(ca) = \sigma(f)(a) = f(\tau a)$ for all $f \in R$, so τ is translation by c. $\square$

We will make use of the following notion.

Definition 4.13. Suppose σ is an automorphism of a commutative ring R. A *σ-prime ideal* is a σ-ideal I such that whenever J and K are σ-ideals with $JK \subseteq I$ then either $J \subseteq I$ or $K \subseteq I$.

Note that a σ-prime ideal need not be prime. But, at least in the case when R is a commutative noetherian ring, a σ-prime ideal is radical; this follows from the fact that the nilpotent radical of I is a σ-ideal and some power of it is contained in I. We will sometimes need to quotient out by σ-prime ideals that we do not know are prime, which means we will have to work with reduced difference rings that are not necessarily integral domains. The following lemma about such difference rings will be very useful.

Lemma 4.14. *Suppose R is a commutative ring endowed with an automorphism σ such that* (0) *is σ-prime. If $0 \neq f \in R$ satisfies $\sigma(f) \in Rf$ then f is not a zero divisor in R.*

Proof. Let $J = Rf$. Then J is a σ-ideal of R. It follows that $K := \{r \in R : rf = 0\}$ is also a σ-ideal of R. Then by construction, $JK = (0)$. Since (0) is σ-prime and J is nonzero, we see that $K = (0)$ and so we obtain the desired result. $\square$

Finally, we will make use of the following fundamental result on Ore extensions of commutative noetherian rings.

Fact 4.15 [Goodearl 1992]. *Suppose R is a commutative noetherian ring, σ is an automorphism of R, and δ is a σ-derivation. Suppose P is a prime ideal of the Ore extension $R[x; \sigma, \delta]$, and let $I = P \cap R$. Then one of the following three statements must hold*:

(I) *$R[x; \sigma, \delta]/P$ is commutative.*

(II) *I is a (σ, δ)*-ideal *of R — that is, I is preserved under σ and δ — and there is a prime ideal I' of R containing I such that $\sigma(r) - r \in I'$ for all $r \in R$.*

(III) *I is a σ-prime (σ, δ)-ideal of R and $IR[x; \sigma, \delta]$ is a prime ideal of $R[x; \sigma, \delta]$.*

Proof of Theorem 4.1. We have that R is a commutative affine Hopf k-algebra equipped with a k-algebra automorphism σ and a k-linear σ-derivation δ, and that the Ore extension $A := R[x;\sigma,\delta]$ admits a Hopf algebra structure extending that of R. We wish to show that A satisfies the DME. By [Irving 1979, Theorem 2] we have that A satisfies the Nullstellensatz, and so it suffices to prove that if P is a rational prime ideal of A then P is locally closed.

We first reduce to the case when k is algebraically closed. Since $R = k[G]$, where G is an affine algebraic group, and hence smooth, R is integrally closed. Let F denote the field of fractions of R and let $F_0 := k^{\mathrm{alg}} \cap F$. Since R is integrally closed, $F_0 \subseteq R$. Since F is a finitely generated extension of k, F_0 is a finite extension of k. Let $R' := R \otimes_{F_0} k^{\mathrm{alg}}$. Since F_0 is relatively algebraically closed in F, we see that R' is again an integral domain. Thus R' is a commutative affine Hopf k^{alg}-algebra to which we extend σ and δ by k^{alg}-linearity. Suppose we have proven the DME for $R'[x;\sigma,\delta]$. Then Irving–Small reduction techniques (see [Irving and Small 1980] and also [Rowen 1988, Theorem 8.4.27]) give that $A = R[x;\sigma,\delta]$ satisfies the DME over F_0. But since F_0 is a finite extension of k, we get the DME over k also.

Next we reduce to the case when k is uncountable (in order to be able to use Proposition 4.11). Let L be an uncountable algebraically closed extension of k. Then since k is algebraically closed we see that $R \otimes_k L$ is a commutative affine Hopf L-algebra, to which we extend σ and δ by L-linearity, and $B := A \otimes_k L \cong (R \otimes_k L)[x;\sigma,\delta]$. Assume the DME holds for B. Let P be a rational prime ideal of A and let $Q = P \otimes_k L$. Since k is algebraically closed, Q is a prime ideal of B. Since P is rational and $B/Q = (A/P) \otimes_k L$, we see that Q is rational. Hence Q is locally closed. Since the primes in A containing P lift to primes in B containing Q, it follows that P is locally closed in A. So A satisfies the DME.

We may therefore assume that k is uncountable and algebraically closed.

If $\sigma = \mathrm{id}$ then δ is a k-linear derivation on R and statement (1) of Corollary 4.4 tells us that it is also an a-coderivation for some group-like $a \in R$. It follows by Corollary 4.7 that $A = R[x;\delta]$ satisfies the DME. So we may assume $\sigma \neq \mathrm{id}$.

We may also assume that A/P is not commutative. Indeed, if it were, as P is rational, we would have that $\mathrm{Frac}(A/P) \subseteq k$, so that P is a maximal ideal and hence locally closed.

Write $R = k[G]$, where G is a connected affine algebraic group over k. By Lemma 4.12 we know that $\sigma = \tau^*$ where $\tau : G \to G$ is translation by a central (nonidentity) element $c \in G(k)$.

Our next goal is to reduce to the case that $P \cap R = (0)$, though in order to obtain this we will have to give up on R being an integral domain. Let $I = R \cap P$. We have already ruled out case (I) of Fact 4.15. On the other hand, case (II) cannot hold: σ would induce the identity map on R/I', implying that τ is the identity on $V(I')$, which contradicts the fact that it is translation on G by a nonidentity element. Hence case (III) holds; I is a σ-prime (σ,δ)-ideal of R and $J := IA$ is a prime ideal of A. Consider now the reduced quotient ring $\overline{R} := R/I$ with the induced automorphism, which we continue to denote by σ, and the induced σ-derivation, which we continue to denote by δ. Let $\overline{A} = A/J \cong \overline{R}[x;\sigma,\delta]$ and $\overline{P}$ the image of P in $\overline{A}$. Since J is contained in P, $\overline{P}$ is rational in $\overline{A}$ and it suffices to show that $\overline{P}$ is locally closed in $\overline{A}$. Note that we have achieved $\overline{P} \cap \overline{R} = (0)$.

Next, we claim that there is some non-zero-divisor $f \in \overline{R}$ such that (σ, δ) extends to $\widetilde{R} := \overline{R}[1/f]$ and δ is inner on $\widetilde{R}$. To see this, consider the frame V for R, given by Lemma 4.10, that is preserved by σ. The image $\overline{V}$ of V in $\overline{R}$ is then a frame for $\overline{R}$ that is also preserved by σ. Using $k = k^{\mathrm{alg}}$, let f be an eigenvector for the action of σ on $\overline{V}$, say $\sigma(f) = \lambda f$ for some $\lambda \in k^*$. By Lemma 4.14, f is not a zero divisor. Moreover, the multiplicatively closed subset $\{1, f, f^2, \dots\}$ of R is preserved by σ, and hence by [Goodearl 1992, Lemma 1.3], (σ, δ) extends uniquely to the localisation at this set, namely to $\widetilde{R} := \overline{R}[1/f]$. It remains to show that f can be chosen so that δ is inner on $\widetilde{R}$. If the eigenvalue λ is not equal to 1, then $f - \sigma(f) = (1-\lambda)f$ is a unit in $\widetilde{R}$, and we get δ inner by Lemma 4.8. So suppose that 1 is the only eigenvalue for σ on $\overline{V}$. Note that σ is not the identity operator on $\overline{R}$ because τ is not the identity on $V(I)$. Since $\overline{V}$ generates $\overline{R}$ as a k-algebra, σ is not the identity on $\overline{V}$ either. Hence there must be some Jordan block that is of size greater than one, but with eigenvalue 1. So we can choose the eigenvector f in such a way that there exists nonzero $g \in \overline{V}$ with $\sigma(g) = g + f$. Hence $g - \sigma(g)$ is a unit in $\widetilde{R} = \overline{R}[1/f]$, and so by Lemma 4.8 again, δ is inner on $\widetilde{R}$.

To prove that $\overline{P}$ is locally closed let us consider the following partition of the set of prime ideals of $\overline{A}$ that properly extend $\overline{P}$:

$$S_1 := \{Q \supsetneq P : Q \text{ prime, and no power of } f \text{ is in } Q\},$$

$$S_2 := \{Q \supsetneq P : Q \text{ prime, not in } S_1 \text{, and } Q \cap \overline{R} \text{ is a } \sigma\text{-prime } (\sigma, \delta)\text{-ideal}\},$$

$$S_3 := \{Q \supsetneq P : Q \text{ prime, and not in } S_1 \text{ or } S_2\}.$$

It suffices to show that for each of $i = 1, 2, 3$, $\bigcap S_i \neq \overline{P}$.

For $i = 2$, note that as σ-prime implies radical, we have that $f \in Q$ for all $Q \in S_2$, but $f \notin \overline{P}$ as $\overline{P} \cap \overline{R} = (0)$.

For $i = 3$, applying Fact 4.15 to $Q \in S_3$, we have that either $\overline{A}/Q$ is commutative or there is in $\overline{R} = R/I$ a prime ideal $\overline{I} := I'/I$ extending $Q \cap \overline{R}$, and such that σ is the identity on $\overline{R}/\overline{I} = R/I'$. The latter case is impossible using again that $\sigma = \tau^*$ and τ is translation on G by a nonidentity element. So $\overline{A}/Q$ is commutative for all $Q \in S_3$. As $\overline{A}/\overline{P} = A/P$ is not commutative there exist $a, b \in \overline{A}$ such that $g := [a, b] \notin \overline{P}$. But $g \in Q$ for all $Q \in S_3$.

It remains therefore to consider S_1. Let

$$\widetilde{A} = \widetilde{R}[x; \sigma, \delta] = R\left[\frac{1}{f}\right][x; \sigma, \delta].$$

As δ is inner on $\widetilde{R}$, Proposition 4.11 tells us that $\widetilde{A}$ satisfies the Dixmier–Moeglin equivalence. As $\overline{P} \cap \overline{R} = (0)$, we know that no power of f is in $\overline{P}$, and hence $\widetilde{P} := \overline{P}\widetilde{A}$ is a prime ideal. As $\widetilde{A}$ is a localisation of $\overline{A}$ we have that $\widetilde{P}$ is rational, and hence locally closed. If $Q \in S_1$ then $Q\widetilde{A}$ is a prime ideal properly extending $\widetilde{P}$. So there is $\alpha \in \widetilde{A} \setminus \widetilde{P}$ such that $\alpha \in Q\widetilde{A}$ for all $Q \in S_1$. For some $n \geq 0$, we have $f^n\alpha \in \overline{A}$. So

$$f^n\alpha \in Q\widetilde{A} \cap \overline{A} = Q.$$

But $f^n\alpha \notin \overline{P}$. So $\bigcap S_1 \neq \overline{P}$, as desired. □

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Closures in varieties of representations and irreducible components

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Dedicated to the memory of Peter Gabriel

For any truncated path algebra Λ of a quiver, we classify, by way of representation-theoretic invariants, the irreducible components of the parametrizing varieties $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ of the Λ-modules with fixed dimension vector $\boldsymbol{d}$. In this situation, the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ are always among the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$, where $\mathbb{S}$ traces the semisimple sequences with dimension vector $\boldsymbol{d}$, and hence the key to the classification problem lies in a characterization of these closures.

Our first result concerning closures actually addresses arbitrary basic finite-dimensional algebras over an algebraically closed field. In the general case, it corners the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ by means of module filtrations "governed by $\mathbb{S}$"; when Λ is truncated, it pins down the $\overline{\mathbf{Rep}\,\mathbb{S}}$ completely.

The analysis of the varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$ leads to a novel upper semicontinuous module invariant which provides an effective tool towards the detection of components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ in general. It detects all components when Λ is truncated.

1. Introduction

By strong consensus, a classification of all indecomposable finite-dimensional representations of a finite-dimensional algebra Λ is an unattainable goal in general. A far more promising alternative to this impossibly comprehensive problem is that of *generically* classifying the finite-dimensional Λ-modules. This amounts to understanding the generic structure of the modules in the irreducible components of the varieties $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ which parametrize the Λ-modules with dimension vector $\boldsymbol{d}$. By its very nature, this quest comes paired with the task of pinning down the irreducible components of the $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ in representation-theoretic terms.

In the present article, the component problem is solved for arbitrary truncated path algebras Λ over an algebraically closed field K. In tandem, significant headway is made towards determining the generic features of the modules in the components.

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The classification of the components, in turn, relies on a characterization of the modules in the closures of certain representation-theoretically defined locally closed subvarieties of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Our initial round of results regarding such closures, including the description of an associated upper semicontinuous module invariant which serves to test for inclusions, holds for arbitrary basic finite-dimensional K-algebras. The findings lead to partial lists of components in this broad scenario. The results become tight on specialization to the truncated case.

Throughout, we assume K to be an algebraically closed field and Λ a basic finite-dimensional K-algebra. This means that, up to isomorphism, $\Lambda = KQ/I$ for a quiver Q and an admissible ideal I in the path algebra. The maximal length of a path in $KQ \setminus I$ will be denoted by L; in other words, L is minimal with respect to $J^{L+1} = 0$, where J is the Jacobson radical of Λ. Consequently, the *radical layering* $\mathbb{S}(M)$ of a Λ-module M has no more than $L+1$ nonzero entries: $\mathbb{S}(M) = (J^l M / J^{l+1} M)_{0 \le l \le L}$. By $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, we denote the standard affine variety parametrizing the Λ-modules with dimension vector $\boldsymbol{d}$. This variety is partitioned into finitely many locally closed subvarieties $\mathbf{Rep}\,\mathbb{S}$ corresponding to the *semisimple sequences* $\mathbb{S}$ *with dimension vector* $\boldsymbol{d}$; these are the sequences $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ of (isomorphism classes of) semisimple Λ-modules with $\underline{\dim}\,\mathbb{S} := \sum_{0 \le l \le L} \underline{\dim}\,\mathbb{S}_l = \boldsymbol{d}$; here $\mathbf{Rep}\,\mathbb{S}$ consists of those points x in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ which represent modules M_x with $\mathbb{S}(M_x) = \mathbb{S}$.

The closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ are relevant to the problem of describing the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$: indeed, it is readily seen that the components of the ambient variety are always among those of the $\overline{\mathbf{Rep}\,\mathbb{S}}$, where $\mathbb{S}$ traces the $\boldsymbol{d}$-dimensional semisimple sequences. Less obviously, the components of the subvarieties $\mathbf{Rep}\,\mathbb{S}$, and hence those of their closures, may be obtained from Q and I by way of a straightforward algorithm, each component tagged by a "generic minimal projective presentation" of the modules it encodes (see [Babson et al. 2009] and [Huisgen-Zimmermann 2009]). Identifying the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ thus amounts to a sorting problem: for which components $\mathcal{C}$ of $\mathbf{Rep}\,\mathbb{S}$ is the closure $\overline{\mathcal{C}}$ *maximal* among the irreducible subsets of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$? This is an extremely taxing question in general, calling for a thorough understanding of the boundaries of the varieties $\mathbf{Rep}\,\mathbb{S}$.

Our strategy consists of moving back and forth between the varieties $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ and $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$; the latter is a closed subvariety of a vector space Grassmannian which parametrizes the modules with dimension vector $\boldsymbol{d}$ by suitable submodules of a projective cover of the semisimple module with this dimension vector (see Section 2 and [Huisgen-Zimmermann 2009; Huisgen-Zimmermann and Goodearl 2012]). The irreducible components of the projective variety $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ may be studied by "spreading them out" within a suitable flag variety (Theorem 3.9), and the subsequent transfer of information $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda) \longleftrightarrow \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is modeled on the influential work [Gabriel 1975]. In a first step, we show:

Theorem A (cf. Theorem 3.8 and Theorem 4.3; see also Remark 3.7(4)). *Let* $\Lambda = KQ/I$ *be a path algebra modulo relations*, $L+1$ *its Loewy length*, *and* $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ *a* $\boldsymbol{d}$*-dimensional semisimple sequence in* Λ-mod. *Then every module in the closure* $\overline{\mathbf{Rep}\,\mathbb{S}}$ *has a filtration by submodules*,

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{L+1} = 0,$$

which is "governed by $\mathbb{S}$*" in the sense that each quotient* M_l/M_{l+1} *is isomorphic to* $\mathbb{S}_l$ $(0 \le l \le L)$. *In fact*, *the set* $\mathbf{Filt}\,\mathbb{S}$ *consisting of those points in* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ *that correspond to modules with at least one filtration governed by* $\mathbb{S}$ *is always closed.*

If Λ *is a truncated path algebra*, *i.e.*, $\Lambda = KQ/\langle$*all paths of length* $L+1\rangle$, *and* $\mathbf{Rep}\,\mathbb{S}$ *is nonempty*, *then*

$$\overline{\mathbf{Rep}\,\mathbb{S}} = \mathbf{Filt}\,\mathbb{S}.$$

For general Λ, the inclusion $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \mathbf{Filt}\,\mathbb{S}$ may be proper. The question of whether a point in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ belongs to $\mathbf{Filt}\,\mathbb{S}$ may be answered by testing for similarity of certain matrices. By contrast, to date, there is no algorithm for deciding whether a module belongs to $\overline{\mathbf{Rep}\,\mathbb{S}}$.

A semisimple sequence $\mathbb{S}$ is called *realizable* if $\mathbf{Rep}\,\mathbb{S} \neq \varnothing$. (When Λ is a truncated path algebra, realizability is checked via mere inspection of the quiver; see [Huisgen-Zimmermann 2016, Criterion 3.2] and Realizability Criterion 4.1 below.)

Corollary B (cf. Corollary 3.11). *For* $M \in \Lambda$-mod, *let* $\Gamma(M)$ *be the number of those realizable semisimple sequences that govern at least one filtration of* M. *Then*

$$\Gamma_\bullet : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbb{N}, \quad x \mapsto \Gamma(M_x),$$

is an upper semicontinuous function.

In particular, *whenever* $\mathcal{C}$ *is an irreducible component of some* $\mathbf{Rep}\,\mathbb{S}$ *such that* $1 \in \Gamma_\bullet(\mathcal{C})$, *the closure* $\overline{\mathcal{C}}$ *is an irreducible component of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

In the second part of the paper, we derive consequences for truncated path algebras. As is suggested by Theorem A, the component problem simplifies considerably in this situation. Notably, the subvarieties $\overline{\mathbf{Rep}\,\mathbb{S}}$ are all irreducible, and generic minimal projective presentations of the modules in $\mathbf{Rep}\,\mathbb{S}$ are immediate from quiver and Loewy length (see [Babson et al. 2009, Section 5] and Section 5A below). In some prominent special cases, particularly manageable solutions to the problem of sifting out the inclusion-maximal ones among the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ are already available (see [Huisgen-Zimmermann 2016; Huisgen-Zimmermann and Shipman 2017]): for instance, if Λ is either local or based on an acyclic quiver Q, the semisimple sequences singled out by the minimal values of the following upper semicontinuous map furnish a complete, nonrepetitive parametrization of the components $\overline{\mathbf{Rep}\,\mathbb{S}}$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$:

$$\Theta = (\mathbb{S}_{\bullet}, \mathbb{S}^*_{\bullet}) : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbf{Seq}(\boldsymbol{d}) \times \mathbf{Seq}(\boldsymbol{d}), \quad x \mapsto (\mathbb{S}(M_x), \mathbb{S}^*(M_x)); \tag{1-1}$$

here the codomain of Θ is partially ordered by the componentwise dominance order on the set $\mathbf{Seq}(\boldsymbol{d})$ of all $\boldsymbol{d}$-dimensional semisimple sequences (see Section 2), and $\mathbb{S}^*(M_x)$ stands for the socle layering of the module M_x (the dual of the radical layering). The unique minimal sequence $\mathbb{S}^*(M_x)$ attained on $\mathbf{Rep}\,\mathbb{S}$, that is, the generic socle layering of the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$, is supplied by a closed formula based on $\mathbb{S}$, Q and L [Huisgen-Zimmermann and Shipman 2017, Theorem 3.8], which makes the Θ-test very user-friendly. But for general truncated Λ, the map Θ fails to detect all components, even when supplemented by further standard semicontinuous module invariants, such as path ranks or assortments of annihilator dimensions. The map $\Gamma_{\bullet}$, on the other hand, compensates for the blind spots of Θ:

Theorem C (cf. Theorem 4.5). *If Λ is any truncated path algebra, the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ are precisely those closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ on which $\Gamma_{\bullet}$ attains the value* 1.

In other words, $\overline{\mathbf{Rep}\,\mathbb{S}}$ is maximal among the irreducible subsets of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ if and only if there exists a module N in $\mathbf{Rep}\,\mathbb{S}$ such that $N \supseteq JN \supseteq \cdots \supseteq J^{L+1}N$ is the only filtration of N which is governed by a realizable semisimple sequence.

In deciding which semisimple sequences $\mathbb{S}$ are the generic radical layerings of the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, Theorem C thus permits exclusive reliance on $\Gamma_{\bullet}$. However, in practice, combining $\Gamma_{\bullet}$ with the test map Θ is considerably more efficient.

In the pursuit of a generic approach to the structure of Λ-modules, the hereditary case, pioneered in [Kac 1980; 1982] and [Schofield 1992], serves as a model. We further point to a selection of existing contributions to the component problem over nonhereditary algebras: General tools were developed in [Crawley-Boevey and Schröer 2002] and [Babson et al. 2009]. Solutions to the problem over specific classes of tame algebras were given in [Barot and Schröer 2001; Carroll and Weyman 2013; Donald and Flanigan 1977; Geiss and Schröer 2003; 2005; Morrison 1980; Riedtmann et al. 2011; Schröer 2004] for instance; solutions for certain classes of wild nonhereditary algebras can be found in [Bleher et al. 2015; Huisgen-Zimmermann 2016; Huisgen-Zimmermann and Shipman 2017]. As is to be expected, meaningful classifications of the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ in the quoted instances are throughout obtained via partial lists of generic properties of the modules in the components. For a more detailed discussion of prior work on the topic we refer to the introduction of [Huisgen-Zimmermann 2016].

We add a few comments on the foundational nature of truncated path algebras with respect to the component problem. Clearly, given an arbitrary basic K-algebra

$\Lambda = KQ/I$, there is a unique truncated path algebra Λ_{trunc} having the same quiver and Loewy length as Λ. In the general situation, the varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$ typically break up into multiple components. Given that all of them are contained in irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$, it is advantageous to first determine the latter, say

$$\overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(1)}} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(1)}), \quad \dots, \quad \overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(m)}} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(m)}),$$

before aiming at the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Indeed, this confines the need for size comparisons among the closures of components of the varieties $\mathbf{Rep}_\Lambda\,\mathbb{S}$ to the subvarieties $\mathbf{Filt}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(j)} \cap \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$; see Section 6B.

Overview. In Section 2, we provide background for the proofs of the main results and introduce a recurring example. Section 3 addresses the general case, where Λ is basic but otherwise unrestricted. In Sections 4 and 5, we apply the findings to truncated path algebras. Section 4 contains the announced classification of the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, while in Section 5, we discuss generic modules and apply the results of Section 4 to exhibit interconnections among the components. Section 6, finally, illustrates the theory and addresses the interplay $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \longleftrightarrow \mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$.

2. Conventions and prerequisites

To repeat: throughout, we assume $\Lambda = KQ/I$ to be a basic finite-dimensional algebra over $K = \overline{K}$ with Jacobson radical J and Loewy length $L+1$. The composition pq of paths stands for "p after q" when $\text{start}(p) = \text{end}(q)$, while $pq = 0$ in KQ otherwise. By Λ_{trunc} we denote the truncated path algebra associated to Λ, namely,

$$\Lambda_{\text{trunc}} = KQ/\langle\text{the paths of length } L+1\rangle;$$

we make no notational distinction between the Λ- and Λ_{trunc}-structures of the objects in Λ-mod. The vertices $e_1, \dots, e_n$ of Q will be identified with the paths of length zero in KQ, as well as with the corresponding primitive idempotents in Λ. An element x of a Λ-module M is said to be *normed by* e_i if $x = e_i x$, and a normed element in $M \setminus JM$ is called a *top element* of M. A *full sequence of top elements* of M is a generating set of M consisting of top elements which are K-linearly independent modulo JM. The simple module $\Lambda e_i / J e_i$ corresponding to the vertex e_i will be denoted by S_i, and isomorphic semisimple modules will be identified.

The *dominance order* on the set $\mathbf{Seq}(\boldsymbol{d})$ of all semisimple sequences with dimension vector $\boldsymbol{d}$ is defined as follows:

$$(\mathbb{S}_0, \dots, \mathbb{S}_L) \le (\mathbb{S}'_0, \dots, \mathbb{S}'_L) \quad \iff \quad \bigoplus_{0 \le j \le l} \mathbb{S}_j \subseteq \bigoplus_{0 \le j \le l} \mathbb{S}'_j \quad \text{for } 0 \le l \le L.$$

Recall that the radical and socle layerings of a Λ-module M are denoted by $\mathbb{S}(M)$ and $\mathbb{S}^*(M)$. For basic properties of these semisimple sequences, we refer to [Huisgen-Zimmermann 2016, Section 2.B].

We fix our notation for the parametrizing varieties of the $\boldsymbol{d}$-dimensional Λ-modules. The affine variety $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is

$$\left\{(x_\alpha)_{\alpha\in Q_1} \in \prod_{\alpha\in Q_1} \operatorname{Hom}_K(K^{d_{\mathrm{start}(\alpha)}}, K^{d_{\mathrm{end}(\alpha)}}) \;\middle|\; \text{the } x_\alpha \text{ satisfy all relations in } I\right\},$$

where Q_1 is the set of arrows of Q. The orbits of the obvious conjugation action on $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ by the group $\mathrm{GL}(\boldsymbol{d}) := \prod_{1\le i\le n} \mathrm{GL}_{d_i}(K)$ are in natural bijection with the isomorphism classes of the $\boldsymbol{d}$-dimensional Λ-modules. Given $\mathbb{S}\in\mathbf{Seq}(\boldsymbol{d})$, we denote by $\mathbf{Rep}\,\mathbb{S}$ the locally closed subvariety of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ which consists of the points x for which the corresponding module M_x has radical layering $\mathbb{S}$. Clearly, the varieties $\mathbf{Rep}\,\mathbb{S}$, where $\mathbb{S}$ traces the semisimple sequences with $\underline{\dim}\,\mathbb{S}=\boldsymbol{d}$, partition $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. However, in general, this (finite) partition falls short of being a stratification of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ in the strict sense, in that closures of strata need not be unions of strata.

To introduce the projective parametrizing variety $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$, we fix a projective Λ-module $\boldsymbol{P}$ whose top $\boldsymbol{P}/J\boldsymbol{P}$ has dimension vector $\boldsymbol{d}$, and set $d = |\boldsymbol{d}|$. The variety $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ is the closed subvariety of the vector space Grassmannian $\mathrm{Gr}((\dim \boldsymbol{P} - d), \boldsymbol{P})$ consisting of those points $C \in \mathrm{Gr}((\dim \boldsymbol{P} - d), \boldsymbol{P})$ which are Λ-submodules of $\boldsymbol{P}$ with the property that $\underline{\dim}(\boldsymbol{P}/C) = \boldsymbol{d}$. This time, the group action whose orbits determine the isomorphism classes of the quotients $\boldsymbol{P}/C$ in Λ-mod is the canonical action of $\mathrm{Aut}_\Lambda(\boldsymbol{P})$ on $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$. The role played by $\mathbf{Rep}\,\mathbb{S}$ in the affine setting is taken over by $\mathrm{GRASS}(\mathbb{S})$, the locally closed subvariety consisting of those $C \in \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ for which $\mathbb{S}(\boldsymbol{P}/C) = \mathbb{S}$.

The following connection between the affine and projective parametrizing varieties was proved in [Bongartz and Huisgen-Zimmermann 2001, Proposition C]; it was inspired by [Gabriel 1975], as is explained in some detail in Remark 3 of [Bongartz and Huisgen-Zimmermann 2001, Section 2]. We restate the result for convenient reference.

Proposition 2.1. *Consider the natural isomorphism from the lattice of* $\mathrm{GL}(\boldsymbol{d})$*-stable subsets of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ *on one hand to the lattice of* $\mathrm{Aut}_\Lambda(\boldsymbol{P})$*-stable subsets of* $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ *on the other, which pairs orbits encoding isomorphic modules. This correspondence preserves and reflects openness, closures, irreducibility, and smoothness.*

In describing generic projective resolutions of the modules in an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, a key invariant of a $\boldsymbol{d}$-dimensional Λ-module M is its set of skeleta. These skeleta live in a projective cover of M in Λ_{trunc}-mod. In the following definitions, we fix a semisimple sequence $\mathbb{S}$ with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$.

Definitions 2.2 (coordinatized projective modules and skeleta).

(1) Let P_{trunc} be a projective cover of $\mathbb{S}_0$ in Λ_{trunc}-mod. This cover is referred to as a *coordinatized* projective module when it comes equipped with a fixed full sequence of top elements $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$, where $t = \dim \mathbb{S}_0$. In particular, we obtain a decomposition $P_{\text{trunc}} = \bigoplus_{1 \le r \le t} \Lambda_{\text{trunc}}\, \boldsymbol{z}_r$. A *path of length* l in the coordinatized projective module P_{trunc} is any nonzero element $\boldsymbol{p} = p\boldsymbol{z}_r$ where p is a path of length l in Q; thus each $\boldsymbol{z}_r$ is now viewed as a path of length zero. Note that we have a well-defined concept of path length in Λ_{trunc}, and hence also in P_{trunc}. Clearly, each path $\boldsymbol{p} = p\boldsymbol{z}_r \in P_{\text{trunc}}$ is normed by a primitive idempotent, namely by $\text{end}(p)$, and the primitive idempotent norming $\boldsymbol{z}_r$ is $\text{start}(p)$.

(2) An (abstract) *skeleton with layering* $\mathbb{S}$ is a set σ consisting of paths in P_{trunc} which satisfies the following two conditions:

- It is closed under initial subpaths, i.e., whenever $p\boldsymbol{z}_r \in \sigma$, and q is an initial subpath of p (meaning $p = q'q$ for some path q'), the path $q\boldsymbol{z}_r$ again belongs to σ.
- For $0 \le l \le L$, the number of those paths of length l in σ which end in a given vertex e_i coincides with the multiplicity of S_i in the semisimple module $\mathbb{S}_l$.

Note that any skeleton σ with layering $\mathbb{S}$ includes the paths $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$ of length zero.

(3) Let $M \in \Lambda$-mod. An abstract skeleton σ is a *skeleton of* M if M has a full sequence $z_1, \dots, z_t$ of top elements, each z_r normed by the same vertex as $\boldsymbol{z}_r$, such that

- $\{p z_r \mid p\boldsymbol{z}_r \in \sigma\}$ is a K-basis for M, and
- the layering of σ coincides with the radical layering $\mathbb{S}(M)$ of M.

In this situation, we also say that σ is a skeleton of M *relative to* $z_1, \dots, z_t$.

Clearly, the set of skeleta of any finite-dimensional Λ-module M is nonempty, and the set of all skeleta of modules with fixed dimension vector $\boldsymbol{d}$ is finite. The relevance of skeleta towards a generic understanding of the modules in the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is underlined by the following fact:

Observation 2.3. Let $\mathcal{P}$ be the power set of the set of all skeleta with dimension vector $\boldsymbol{d}$. Then the map

$$\mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathcal{P}, \quad x \mapsto \{\text{skeleta of } M_x\},$$

is generically constant on each irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

To see this, let $\mathcal{C} \subseteq \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ be an irreducible component, and $\mathbb{S}$ the generic radical layering of its modules. Then $\mathcal{C} \cap \mathbf{Rep}\,\mathbb{S}$ is open in $\mathcal{C}$, and for any skeleton σ

with layering $\mathbb{S}$, the set

$$\mathbf{Rep}(\sigma) := \{x \in \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \mid \sigma \text{ is a skeleton of } M_x\}$$

is an open subvariety of $\mathbf{Rep}\,\mathbb{S}$; see [Huisgen-Zimmermann 2007, Lemma 3.8]. Hence, a skeleton σ with layering $\mathbb{S}$ arises as a skeleton of the modules in a dense open subset of $\mathcal{C}$ precisely when $\mathcal{C} \cap \mathbf{Rep}(\sigma)$ is nonempty. Given that there are only finitely many eligible skeleta, this proves the claim.

Next, we recall more discerning graphical invariants associated to a finite-dimensional Λ-module, namely its hypergraphs; see Definition 3.9 of [Babson et al. 2009].

Definitions 2.4 (σ-critical paths and hypergraphs). Again, we let P_{trunc} be a coordinatized projective Λ_{trunc}-module with top $\mathbb{S}_0$ and assume $\sigma \subseteq P_{\text{trunc}}$ to be an abstract skeleton with layering $\mathbb{S}$. Recall that the distinguished top elements $\boldsymbol{z}_r$ of P_{trunc} coincide with the paths of length zero in σ.

(1) A *σ-critical path* is a path $\boldsymbol{q} \in P_{\text{trunc}} \setminus \sigma$ such that every proper initial subpath of $\boldsymbol{q}$ belongs to σ. Thus, $\boldsymbol{q} = \alpha \boldsymbol{q}'$, where $\boldsymbol{q}' \in \sigma$ and α is an arrow; in particular, $\text{length}(\boldsymbol{q}) > 0$. Given a σ-critical path $\boldsymbol{q}$, we define a subset $\sigma_{\boldsymbol{q}} \subseteq \sigma$ as follows:

$$\sigma_{\boldsymbol{q}} := \{\text{paths } \boldsymbol{p} \in \sigma \mid \text{length}(\boldsymbol{p}) \geq \text{length}(\boldsymbol{q}) \text{ and } \text{end}(\boldsymbol{p}) = \text{end}(\boldsymbol{q})\}.$$

The final condition in the definition of $\sigma_{\boldsymbol{q}}$ means that all paths in $\sigma_{\boldsymbol{q}}$ are normed (on the left) by the same vertex as $\boldsymbol{q}$.

(2) Suppose $M \in \Lambda\text{-mod}$ has skeleton σ relative to a full sequence $z_1, \dots, z_t$ of top elements. The Λ-structure of M is then determined by the family of expansion coefficients corresponding to the σ-critical paths $\boldsymbol{q} = q\, \boldsymbol{z}_r \in P_{\text{trunc}}$, namely

$$q\, z_r = \sum_{\boldsymbol{p} = p \boldsymbol{z}_s \in \sigma_{\boldsymbol{q}}} c_{\boldsymbol{q},\boldsymbol{p}}\, p\, z_s \tag{2-1}$$

for unique scalars $c_{\boldsymbol{p},\boldsymbol{q}} \in K$.

(3) We refer to any pair

$$\mathcal{G} = (\sigma, (\tau_{\boldsymbol{q}})_{\boldsymbol{q}\ \sigma\text{-critical}}) \quad \text{with } \tau_{\boldsymbol{q}} \subseteq \sigma_{\boldsymbol{q}} \text{ for all } \sigma\text{-critical paths } \boldsymbol{q}$$

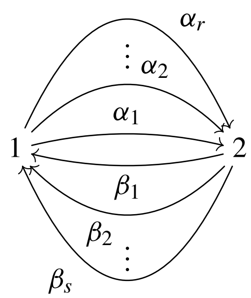
as an (undirected) *hypergraph in P_{trunc}*. The set $\tau_{\boldsymbol{q}}$ is called the *support set* of $\boldsymbol{q}$. Empty support sets are allowed.

In informal terms: the vertices of these hypergraphs are the elements of σ, and a typical (hyper)edge, labeled by an arrow $\gamma \in Q_1$, connects a vertex $\boldsymbol{p} \in \sigma$ to the vertex $\gamma \boldsymbol{p}$ if $\gamma \boldsymbol{p} \in \sigma$ and to the support set $\tau_{\gamma \boldsymbol{p}}$ of vertices if $\gamma \boldsymbol{p}$ is σ-critical.

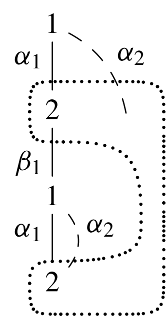
(4) A hypergraph $\mathcal{G}$ as above is called a *hypergraph of* a Λ-module M (relative to a full sequence $z_1, \ldots, z_t$ of top elements of M) if σ is a skeleton of M and, in the expansion (2-1) above, $c_{\boldsymbol{q},\boldsymbol{p}} \neq 0$ precisely when $\boldsymbol{p} \in \tau_{\boldsymbol{q}}$.

While hypergraphs pin down *families* of modules, as opposed to individual isomorphism classes, they provide a useful tool for communicating, in a visually suggestive format, the generic structure of the modules in the components. For our diagrammatic representations of hypergraphs, we refer to [Babson et al. 2009], [Derksen et al. 2014], and to the example below. This example will serve as a staple in the sequel.

Example 2.5. Let $\Lambda = KQ/\langle\text{the paths of length 4}\rangle = \Lambda_{\text{trunc}}$, where Q is the quiver



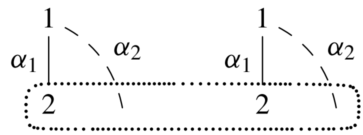
(a) First suppose that $r = 2$ and $s = 1$. Choose $\mathbb{S} := (S_1, S_2, S_1, S_2)$, and let $P_{\text{trunc}} = \Lambda_{\text{trunc}} z$ be the corresponding Λ_{trunc}-projective cover of $\mathbb{S}_0 = S_1$, coordinatized by a fixed top element z. Generically, the modules in $\mathbf{Rep}\,\mathbb{S}$ then have a hypergraph of the form



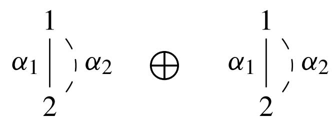
This diagram is to be read as follows: the radical layering of any module G having the above hypergraph (relative to a top element $z \in G$, say) is $\mathbb{S}$, and the skeleton chosen to represent G is $\sigma := \{z, \alpha_1 z, \beta_1\alpha_1 z, \alpha_1\beta_1\alpha_1 z\}$; the edges corresponding to paths in the skeleton σ are drawn as solid edges, while the dashed edges stand for the terminal arrows of σ-critical paths. Moreover, the diagram contains the information that the support sets $\tau_{\boldsymbol{q}}$ for the two σ-critical paths $\boldsymbol{q} = \alpha_2 z$ and $\boldsymbol{q} = \alpha_2\beta_1\alpha_1 z$ in P_{trunc} (in the sense of Definitions 2.4), are $\tau_{\alpha_2 z} = \{\alpha_1 z,\ \alpha_1\beta_1\alpha_1 z\}$ and $\tau_{\alpha_2\beta_1\alpha_1 z} = \{\alpha_1\beta_1\alpha_1 z\}$. Indeed, the "dotted pool" indicates that the element $\alpha_2 z$

of G is a K-linear combination of $\alpha_1 z$ and $\alpha_1\beta_1\alpha_1 z$ with coefficients in K^*; on the other hand, given that the set $\tau_{\alpha_2\beta_1\alpha_1 z}$ is a singleton, no extra pooling device is required to communicate the condition that $\alpha_2\beta_1\alpha_1 z \in G$ be a nonzero scalar multiple of $\alpha_1\beta_1\alpha_1 z$.

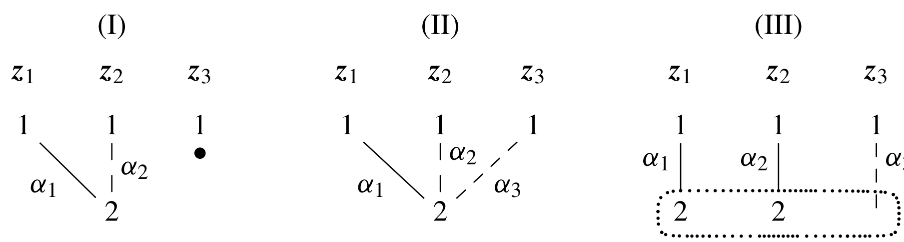
Next, we consider the semisimple sequence $\mathbb{S}' := (S_1^2, S_2^2, 0, 0)$. The modules in **Rep** $\mathbb{S}'$ generically look as follows, relative to top elements z_1, z_2, say:



Here, the dotted pool serves double duty in indicating that both $\alpha_2 z_1$ and $\alpha_2 z_2$ are linear combinations of $\alpha_1 z_1$ and $\alpha_1 z_2$ with (unspecified) nonzero coefficients. In the sequel, we will use the fact that, generically, the modules in **Rep** $\mathbb{S}'$ decompose in the form

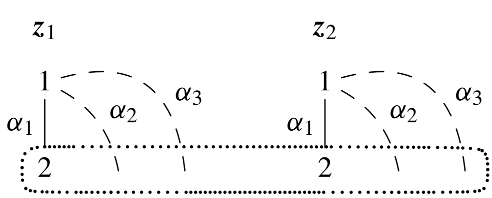


(b) Now let $r = 3$. The hypergraphs



are hypergraphs of modules $M_i = \big(\bigoplus_{1\le j\le 3} \Lambda z_j\big)/U_i$, where $z_j = e_1$ for $j = 1, 2, 3$. Here the submodule U_1 is generated by $\alpha_2 z_2 - \alpha_1 z_1$, $\alpha_3 z_3$ and $\alpha_j z_k$ for $j \neq k$, while U_2 is generated by $\alpha_2 z_2 - \alpha_1 z_1$, $\alpha_3 z_3 - \alpha_1 z_1$ and $\alpha_j z_k$ for $j \neq k$; finally, U_3 is generated by $\alpha_3 z_3 - (\alpha_1 z_1 + \alpha_2 z_2)$ and $\alpha_j z_k$ for $j \neq k$. The chosen reference skeleton of M_1 and M_2 is $\sigma := \{z_1, z_2, z_3, \alpha_1 z_1\}$, and that of M_3 is $\sigma \cup \{\alpha_2 z_2\}$. Note that the dimension of JM_3 is 2, the number of displayed vertices in the second row of the hypergraph.

Generically, the modules with radical layering $\mathbb{S}' := (S_1^2, S_2^2, 0, 0)$ are indecomposable and have hypergraphs of the form



The modules in $\mathbf{Rep}\,\mathbb{S}$, where $\mathbb{S} := (S_1, S_2, S_1, S_2)$, generically have a hypergraph akin to the first one shown in part (a). $\square$

3. The main results for general Λ

3A. ***Pared-down parametrizing varieties.*** Towards a description of $\overline{\mathbf{Rep}\,\mathbb{S}}$, we present lower-dimensional, more manageable varieties parametrizing the modules with radical layering $\mathbb{S}$.

Definition 3.1 (decompositions of $K^{|\boldsymbol{d}|}$ induced by semisimple sequences). Let $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ be a realizable semisimple sequence in Λ-mod with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$, and write $d = |\boldsymbol{d}|$. Consider a vector space decomposition of K^d which is *induced by* $\mathbb{S}$ in the following sense: namely,

$$K^d = \bigoplus_{\substack{0 \le l \le L \\ 1 \le i \le n}} \mathcal{K}_{(l,i)}$$

with the property that $\dim \mathcal{K}_{(l,i)} = \dim e_i \mathbb{S}_l$ for all eligible indices l and i. Set $\mathcal{K}_l = \bigoplus_{1 \le i \le n} \mathcal{K}_{(l,i)}$ for $l \le L$, and $\mathcal{K}_{L+1} = \mathcal{K}_{(L+1,i)} = 0$. Given a family $(f_\alpha)_{\alpha \in Q_1}$ of K-endomorphisms of K^d, the following notation will be convenient: whenever $p = \alpha_l \cdots \alpha_1$ is a path of positive length l in Q, we set $f_p = f_{\alpha_l} \circ \cdots \circ f_{\alpha_1}$; if p is a path of length 0, say $p = e_i$, then f_p is defined to be the canonical projection $K^d \to \bigoplus_{0 \le l \le L} \mathcal{K}_{(l,i)} \subseteq K^d$ relative to the above decomposition. Thus, we obtain a K-algebra homomorphism $KQ \to \operatorname{End}_K(K^d)$ such that $p \mapsto f_p$ for all paths p in Q.

By $Q_{\ge l}$ we denote the set of paths of length at least l in Q. The following lemma is an upgraded version of [Huisgen-Zimmermann 2016, Lemma 5.1] and is proved analogously.

Lemma 3.2 (triangular points in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$). *We refer to the above notation. Suppose that $f = (f_\alpha)_{\alpha \in Q_1}$ is a family of K-linear maps $K^d \to K^d$ satisfying the following three conditions: For any arrow α from e_i to e_j and any index $l \in \{0, \dots, L\}$,*

(i) $f_\alpha(\mathcal{K}_{(l,r)}) = 0$ *for all* $r \ne i$;

(ii) $f_\alpha(\mathcal{K}_{(l,i)}) \subseteq \bigoplus_{l+1 \le m \le L} \mathcal{K}_{(m,j)}$;

(iii) *whenever $c_1, \dots, c_m \in K$ and $p_1, \dots, p_m$ are paths of length $\le L$ in Q, which have a common starting vertex and a common terminal vertex,*

$$\sum_{1 \le j \le m} c_j p_j \in I \quad \Longrightarrow \quad \sum_{1 \le j \le m} c_j f_{p_j} = 0.$$

Then the following statements (I)–(III) *hold*:

(I) *The tuple f is a point in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, and the radical layering of the corresponding Λ-module M_f satisfies $\mathbb{S}(M_f) \geq \mathbb{S}$. Moreover, all Λ-modules with radical layering $\mathbb{S}$ are represented by suitable points $f \in \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ satisfying* (i)–(iii).

(II) $J^l M_f = \sum_{p \in Q_{\geq l}} \operatorname{Im}(f_p)$ *for all* $l \in \{0, \dots, L\}$.

(III) $\mathbb{S}(M_f) = \mathbb{S}$ *precisely when, for each* $h \in \{0, \dots, L\}$, *the linear map*

$$(\mathcal{K}_0)^{Q_{\geq h}} \to \bigoplus_{l \geq h} \mathcal{K}_l, \quad (x_q)_{q \in Q_{\geq h}} \mapsto \sum_q f_q(x_q),$$

has maximal rank, namely $\sum_{l \geq h} \dim \mathcal{K}_l$. □

The lemma prompts an analysis of the following two subvarieties of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

3.3 (the varieties Δ-$\mathbf{Rep}(\geq\mathbb{S})$ and Δ-$\mathbf{Rep}\,\mathbb{S}$). Keep $\mathbb{S}$ and a decomposition of K^d induced by $\mathbb{S}$ fixed. The collection of all $f = (f_\alpha)$ satisfying conditions (i)–(iii) of Lemma 3.2 is a closed subvariety of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ which we denote by Δ-$\mathbf{Rep}(\geq\mathbb{S})$. Indeed, the inclusion map

$$\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S}) \hookrightarrow \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$$

provided by part (I) of Lemma 3.2 is a closed immersion.

To see this, take $B_{(l,\mu)} = (b^1_{(l,\mu)}, \dots, b^{d_{l,\mu}}_{(l,\mu)})$ to be an ordered basis for $\mathcal{K}_{(l,\mu)}$ and B to be the lexicographically ordered union of the $B_{(l,\mu)}$. Relative to this basis for K^d, the image of the above embedding consists of all those families (F_α) of matrices in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ such that each F_α has a strictly lower triangular form of the following ilk:

- the only nonzero entries in any column labeled $(l, \mu)^{(j)}$ are confined to positions with lower label $(l+1, \nu), \dots, (L, \nu)$, provided α is an arrow $e_\mu \to e_\nu$, and
- condition (iii) of Lemma 3.2 is satisfied.

The latter requirement translates into polynomial equations for the entries of the F_α. This shows that the considered embedding is indeed a closed immersion.

Moreover, observe that, up to isomorphism, the variety Δ-$\mathbf{Rep}(\geq\mathbb{S})$ is determined by $\mathbb{S}$, irrespective of the choice of a decomposition $K^d = \bigoplus_{l,i} \mathcal{K}_{(l,i)}$ induced by $\mathbb{S}$. Lemma and Definition 3.6 below will show that the $\mathrm{GL}(\boldsymbol{d})$-stable hull $\mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})) \subseteq \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is, in fact, unique in the strict sense.

We will identify Δ-$\mathbf{Rep}(\geq\mathbb{S})$ with its image under the above immersion whenever convenient. The subset of Δ-$\mathbf{Rep}(\geq\mathbb{S})$ consisting of the points which correspond to modules with radical layering $\mathbb{S}$ will be denoted by Δ-$\mathbf{Rep}\,\mathbb{S}$. In view of part (III) of Lemma 3.2, Δ-$\mathbf{Rep}\,\mathbb{S}$ is an open subvariety of Δ-$\mathbf{Rep}(\geq\mathbb{S})$.

Next, we consider the effect of conjugation by $\mathrm{GL}(\boldsymbol{d})$ on the varieties Δ-$\mathbf{Rep}(\geq\mathbb{S})$ and Δ-$\mathbf{Rep}(\mathbb{S})$.

3.4 (Δ-$\mathbf{Rep}(\geq\mathbb{S})$ under the $\mathrm{GL}(\boldsymbol{d})$-action). Viewed as subvarieties of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, the varieties Δ-$\mathbf{Rep}(\geq\mathbb{S})$ and Δ-$\mathbf{Rep}(\mathbb{S})$ fail to be stable under the $\mathrm{GL}(\boldsymbol{d})$-action in all nontrivial cases. However, each of these varieties carries a conjugation action by the subgroup $\mathrm{GL}(\mathbb{S})$ of $\mathrm{GL}(\boldsymbol{d})$ which consists of the sequences $(g_1, \dots, g_n)$ with the property that each g_i leaves the subspaces $\bigoplus_{j\geq l} \mathcal{K}_{(j,i)}$ invariant for all l. Caveat: the $\mathrm{GL}(\mathbb{S})$-action does not separate the isomorphism classes of the pertinent modules in general.

By part (I) of Lemma 3.2, the closure of Δ-$\mathbf{Rep}(\geq\mathbb{S})$ under the $\mathrm{GL}(\boldsymbol{d})$-action on $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is contained in the closed subvariety $\bigcup_{\mathbb{S}'\geq\mathbb{S}} \mathbf{Rep}\,\mathbb{S}'$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. In fact, in view of the lemma,

$$\mathbf{Rep}\,\mathbb{S} = \mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\mathbb{S})) \subseteq \mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})) \subseteq \bigcup_{\mathbb{S}'\geq\mathbb{S}} \mathbf{Rep}\,\mathbb{S}'.$$

Either inclusion may be proper. This is obvious for the first. Regarding the second, let $\Lambda = KQ/\langle\beta^2\rangle$, for instance, where

$$Q := \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd}.$$

Moreover, take $\mathbb{S} := (S_1^2, S_2^2)$ and $\widetilde{\mathbb{S}} := (S_1^2 \oplus S_2, S_2)$. Then $\widetilde{\mathbb{S}} \geq \mathbb{S}$, but the module $N := S_1^2 \oplus \Lambda e_2$ in $\mathbf{Rep}(\widetilde{\mathbb{S}})$ is not isomorphic to a module in Δ-$\mathbf{Rep}(\geq\mathbb{S})$. Indeed, since $\mathcal{K}_{(0,2)} = 0$ and $\dim \mathcal{K}_{(1,2)} = 2$ in the decomposition of K^4 induced by $\mathbb{S}$, we have $S_2^2 \subseteq \operatorname{soc} M$ for all M in Δ-$\mathbf{Rep}(\geq\mathbb{S})$, while this is not the case for N.

3B. ***The closure of*** $\mathbf{Rep}\,\mathbb{S}$ ***in*** $\mathbf{Rep}_{\boldsymbol{d}}(\boldsymbol{\Lambda})$**.** We start with an elementary lemma characterizing the modules corresponding to the points in Δ-$\mathbf{Rep}(\geq\mathbb{S})$. For a given realizable semisimple sequence $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$, we fix a decomposition of $K^{|\boldsymbol{d}|}$ induced by $\mathbb{S}$ as in Definition 3.1. As we already pointed out, modulo isomorphism of varieties, this choice has no bearing on Δ-$\mathbf{Rep}(\geq\mathbb{S})$.

Definition 3.5 (filtrations governed by $\mathbb{S}$). Let M be a Λ-module. A *filtration of M governed by* $\mathbb{S}$ is any chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{L+1} = 0$$

such that each factor M_l/M_{l+1} is isomorphic to $\mathbb{S}_l$; in other words, $JM_l \subseteq M_{l+1}$ and $\underline{\dim}\, M_l/M_{l+1} = \underline{\dim}\,\mathbb{S}_l$ for $0 \leq l \leq L$. Filtrations with these properties will also be referred to more briefly as $\mathbb{S}$-*filtrations*.

Lemma and Definition 3.6 (the variety $\mathbf{Filt}\,\mathbb{S}$). *Let $\Lambda = KQ/I$ be an arbitrary basic finite-dimensional K-algebra. Moreover, let $\mathbb{S}$ be a semisimple sequence with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$. Then the following conditions are equivalent for a Λ-module M*:

(1) *M belongs to* $\mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S}))$, *the* $\mathrm{GL}(\boldsymbol{d})$*-stable hull of* Δ-$\mathbf{Rep}(\geq\mathbb{S})$.

(2) *M has a filtration governed by* $\mathbb{S}$.

In particular, $\mathrm{GL}(\boldsymbol{d}).(\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S}))$ *is independent of the choice of a decomposition of* $K^{|\boldsymbol{d}|}$ *induced by* $\mathbb{S}$. *Motivated by the above equivalence*, *we will denote this subvariety of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ *by* $\mathbf{Filt}\,\mathbb{S}$.

Proof. (1) $\implies$ (2): Suppose that M is represented by some point $f = (f_\alpha) \in \Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})$. This means that, up to isomorphism, M equals K^d, equipped with the Λ-module structure of Lemma 3.2. In particular, we obtain a filtration of M governed by $\mathbb{S}$ by setting $M_l = \bigoplus_{j\geq l,\,1\leq i\leq n} \mathcal{K}_{(j,i)}$.

(2) $\implies$ (1): Given an $\mathbb{S}$-filtration $(M_l)_{0\leq l\leq L+1}$ of M, we take $M_{(l,i)}$ to be a vector space complement of $e_i M_{l+1}$ in $e_i M_l$ for $0 \leq l \leq L$. Moreover, we set $f = (f_\alpha)_{\alpha\in Q_1}$, where $f_\alpha(x) = \alpha x$ for $x \in M$. Then the decomposition $M = \bigoplus_{0\leq l\leq L,\,1\leq i\leq n} M_{(l,i)}$ satisfies conditions (i)–(iii) of Lemma 3.2, and thus can be shifted to a decomposition $\bigoplus_{0\leq l\leq L,\,1\leq i\leq n} \mathcal{K}_{(l,i)}$ of K^d induced by $\mathbb{S}$ via a suitable family $h = (h_{(l,i)})$ of isomorphisms $h_{(l,i)} : M_{(l,i)} \to \mathcal{K}_{(l,i)}$. We conclude that $hfh^{-1} \in \Delta\text{-}\mathbf{Rep}(\geq\mathbb{S})$ and that $M_{hfh^{-1}} \cong M$. □

The upcoming remarks (1)–(3) will be tacitly used throughout the sequel.

Remarks 3.7. (1) $\mathbf{Filt}\,\mathbb{S}$ is always nonempty, irrespective of whether $\mathbb{S}$ is realizable. Indeed, the semisimple module $\bigoplus_{0\leq l\leq L} \mathbb{S}_l$ has a filtration governed by $\mathbb{S}$.

(2) For any $M \in \Lambda$-mod, the chain $M \supseteq JM \supseteq \cdots \supseteq J^{L+1}M = 0$ is the only filtration of M governed by $\mathbb{S}(M)$; moreover, if $\mathbb{S}'$ is any semisimple sequence governing a filtration of M, then $\mathbb{S}' \leq \mathbb{S}(M)$.

(3) The socle layering $\mathbb{S}^*(M)$ of M governs the socle filtration, provided the traditional indexing of the latter is reversed; i.e., if $\mathbb{S}^*(M) = (\mathbb{S}^*_0, \dots, \mathbb{S}^*_m, 0, \dots, 0)$ with $\mathbb{S}^*_m \neq 0$, then the filtration

$$\mathrm{soc}_m M = M \supseteq \mathrm{soc}_{m-1} M \supseteq \cdots \supseteq \mathrm{soc}_0 M = \mathrm{soc}\, M \supseteq 0$$

is governed by the semisimple sequence $(\mathbb{S}^*_m, \dots, \mathbb{S}^*_0, 0, \dots, 0)$ (which is not necessarily realizable). In particular, $(\mathbb{S}^*_m, \dots, \mathbb{S}^*_0, 0, \dots, 0) \leq \mathbb{S}(M)$.

(4) K. Bongartz pointed out to us that the upcoming Theorem 3.8 may alternatively be derived from a useful result of Steinberg. We state it below, but omit detail. We do fully anchor our own steppingstone to Theorem 3.8 (namely Theorem 3.9), though. The embedding of $\mathrm{GRASS}(\mathbb{S})$ into a flag variety, as specified there, is instrumental in a further analysis of the closure of $\mathrm{GRASS}(\mathbb{S})$ in $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$.

Lemma [Steinberg 1974, Lemma 2, p. 68]. Let V be a quasiprojective variety carrying a morphic action by a connected linear algebraic group G. Moreover, let U be a closed subvariety of V which is stable under the action of some parabolic subgroup of G. Then the G-stable hull $G.U$ of U in V is in turn closed.

Theorem 3.8. *Let Λ be an arbitrary basic finite-dimensional algebra, and let $\mathbb{S}$ be a semisimple sequence in* Λ-mod *with $\underline{\dim}\,\mathbb{S} = \boldsymbol{d}$. Then the* $\mathrm{GL}(\boldsymbol{d})$*-stable set* $\mathbf{Filt}\,\mathbb{S}$, *which consists of the points in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ encoding modules with $\mathbb{S}$-filtrations, is a closed subvariety of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.*

In particular, $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \mathbf{Filt}\,\mathbb{S}$, *meaning that every module in* $\overline{\mathbf{Rep}\,\mathbb{S}}$ *has an $\mathbb{S}$-filtration.*

To prove Theorem 3.8, we switch back and forth between the affine and projective settings, $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ and $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$, using Proposition 2.1 to transfer information from one to the other. Again, we denote by $\boldsymbol{P}$ the Λ-projective cover of $\bigoplus_{1\le i\le n} S_i^{d_i}$ in whose submodule lattice the points of $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ are located. We start by establishing a natural embedding of $\mathrm{GRASS}(\mathbb{S})$ into a projective variety consisting of submodule flags $D_{L+1} \subseteq D_L \subseteq \cdots \subseteq D_0 = \boldsymbol{P}$ of $\boldsymbol{P}$ which are governed by $\mathbb{S}$. It is this embedding which makes information about the closure of $\mathrm{GRASS}(\mathbb{S})$ in $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ more accessible.

Theorem 3.9. *Consider the subset $\mathfrak{U}$ of the partial flag variety $\mathfrak{Flag}(\partial_0, \dots, \partial_{L+1}, \boldsymbol{P})$ of $\boldsymbol{P}$, where $\partial_i := (\dim \boldsymbol{P} - |\boldsymbol{d}|) + \sum_{l=L+1-i}^{L} |\underline{\dim}\,\mathbb{S}_l|$, consisting of the Λ-submodule flags*

$$0 \subseteq D_{L+1} \subseteq D_L \subseteq \cdots \subseteq D_0 = \boldsymbol{P} \quad \textit{with } D_l/D_{l+1} \cong \mathbb{S}_l \textit{ for } 0 \le l \le L.$$

Then $\mathfrak{U}$ is closed, and there is a natural embedding of varieties

$$\Phi : \mathrm{GRASS}(\mathbb{S}) \to \mathfrak{U},$$

which induces an isomorphism onto its image.

Proof of Theorem 3.9. Recall that a module N belongs to $\mathrm{GRASS}(\mathbb{S})$, meaning that $N \cong \boldsymbol{P}/C$ with $C \in \mathrm{GRASS}(\mathbb{S})$, precisely when

$$\underline{\dim}\,\mathbb{S}_l = \underline{\dim}\, J^l N/J^{l+1}N = \underline{\dim}(C + J^l\boldsymbol{P})/(C + J^{l+1}\boldsymbol{P})$$

for all eligible l. Set $\boldsymbol{d}^{(L+1)} = \boldsymbol{d}$ and $\boldsymbol{d}^{(l)} = \boldsymbol{d} - \sum_{l\le r\le L} \underline{\dim}\,\mathbb{S}_r$ for $0 \le l \le L$. In particular, we obtain $\mathrm{GRASS}_{\boldsymbol{d}^{(L+1)}}(\Lambda) = \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$, and $\mathrm{GRASS}_{\boldsymbol{d}^{(0)}}(\Lambda) = \{\boldsymbol{P}\}$.

Clearly, $\mathfrak{U}$ is a subset of the projective variety

$$\mathrm{GRASS}_{\boldsymbol{d}^{(L+1)}}(\Lambda) \times \mathrm{GRASS}_{\boldsymbol{d}^{(L)}}(\Lambda) \times \cdots \times \mathrm{GRASS}_{\boldsymbol{d}^{(0)}}(\Lambda);$$

namely, $\mathfrak{U}$ consists of those points $(D_{L+1}, \dots, D_0)$ in the direct product that correspond to flags $D_{L+1} \subseteq D_L \subseteq \cdots \subseteq D_0 = \boldsymbol{P}$ of Λ-submodules of $\boldsymbol{P}$ satisfying

$$JD_l \subseteq D_{l+1} \quad \text{and} \quad \underline{\dim}\,D_l/D_{l+1} = \underline{\dim}\,\mathbb{S}_l \qquad \text{for } 0 \le l \le L. \qquad (\ddagger)$$

To verify that the set $\mathfrak{U}$ is closed in the given direct product of module Grassmannians, note that the equalities under ($\ddagger$), specifying the dimension vectors of the consecutive quotients D_l/D_{l+1}, are actually automatic; this is due to the placement of the D_l in

$\mathrm{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda)$, respectively. As for the inclusions under (‡): it is well-known that, given any $f \in \mathrm{End}_K(\boldsymbol{P})$, the requirement "$f(D_l) \subseteq D_{l+1}$ for all l" cuts a closed subset out of the variety

$$\left\{ (D_l) \in \prod_{0 \le l \le L+1} \mathrm{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda) \;\middle|\; D_{l+1} \subseteq D_l \text{ for } 0 \le l \le L \right\}$$

of partial submodule flags. Applying this to the linear maps $\boldsymbol{P} \to \boldsymbol{P}$ given by $x \mapsto \alpha x$ for $\alpha \in Q_1$, and investing the fact that the displayed partial flag variety is closed in the given product of Grassmannians, one finds that $\mathfrak{U}$ is indeed closed. In particular, $\mathfrak{U}$ is a projective variety.

We have a natural embedding of $\mathrm{GRASS}(\mathbb{S})$ into $\mathfrak{U}$, namely

$$\Phi : \mathrm{GRASS}(\mathbb{S}) \to \mathfrak{U}, \quad C \mapsto (C + J^{L+1}\boldsymbol{P}, C + J^L\boldsymbol{P}, \dots, C + J\boldsymbol{P}, C + J^0\boldsymbol{P}),$$

where the leftmost entry $C + J^{L+1}\boldsymbol{P}$ of the sequence equals C, and the rightmost entry equals $\boldsymbol{P}$.

To see that Φ is a morphism, we use the open affine cover $(\mathrm{GRASS}(\sigma))_\sigma$ of $\mathrm{GRASS}(\mathbb{S})$, where σ traces the skeleta with layering $\mathbb{S}$ and $\mathrm{GRASS}(\sigma) \neq \varnothing$. For that purpose, recall the following description of $\mathrm{GRASS}(\sigma)$ from [Huisgen-Zimmermann 2009]. We view the Λ-projective cover P of $\mathbb{S}_0$ as a direct summand of the projective cover $\boldsymbol{P} = \bigoplus_{1 \le r \le |\boldsymbol{d}|} \Lambda \boldsymbol{z}_r$ of $\bigoplus_{0 \le l \le L} \mathbb{S}_l$, say $P = \bigoplus_{1 \le r \le t} \Lambda \boldsymbol{z}_r$. On identifying the top elements $\boldsymbol{z}_r$ of P with those of P_{trunc} (see Definitions 2.2), we retrieve each of the subsets σ of P_{trunc} as a subset of P; as such, σ consists of $|\boldsymbol{d}|$ linearly independent elements of $\boldsymbol{P}$. Define $s := \dim \boldsymbol{P} - |\boldsymbol{d}|$, and let $\mathbf{Schu}(\sigma)$ be the big open Schubert cell of $\mathrm{Gr}(s, \boldsymbol{P})$ consisting of the vector space complements of the subspace $\bigoplus_{\boldsymbol{p} \in \sigma} K\boldsymbol{p}$ in $\boldsymbol{P}$. Then $\mathrm{GRASS}(\sigma) := \mathrm{GRASS}(\mathbb{S}) \cap \mathbf{Schu}(\sigma)$ is open in $\mathrm{GRASS}(\mathbb{S})$, and the union of the $\mathrm{GRASS}(\sigma)$, with σ as specified, equals $\mathrm{GRASS}(\mathbb{S})$; see [Huisgen-Zimmermann 2009, Observation 3.6]. By [ibid., Theorem 3.17], the $\mathrm{GRASS}(\sigma)$ are affine; in fact, they can readily be realized as closed subsets of the K-space $\bigwedge^s \boldsymbol{P}$ relative to the Plücker coordinates $[c_1 \wedge \cdots \wedge c_s]$ of $\mathbf{Schu}(\sigma)$.

Hence it suffices to show that, for each such skeleton σ, the restriction Φ_σ of Φ to $\mathrm{GRASS}(\sigma)$ is a morphism. For $0 \le j \le L$, let σ_j be the set of all paths of length j in σ. Enumerate the elements of σ so that increasing indices correspond to weakly decreasing lengths. If $t_l := |\sigma_l| + \cdots + |\sigma_L|$, we thus obtain $\bigsqcup_{l \le j \le L} \sigma_j$ in the form

$$\bigsqcup_{l \le j \le L} \sigma_j = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_{t_l}\} \quad \text{for } 0 \le l \le L.$$

We deduce that, given any K-basis $c_1, \dots, c_s$ for a point $C \in \mathrm{GRASS}(\sigma)$, the elements $c_1, \dots, c_s, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{t_l}$ form a K-basis for $C + J^l\boldsymbol{P}$: indeed, $J^l\boldsymbol{P}$ is generated by the paths in $\boldsymbol{P}$ of the form $q\boldsymbol{z}_r$, where q is a path of length $\ge l$ in KQ

and $r \le |\boldsymbol{d}|$. Moreover, by the definition of $\mathrm{GRASS}(\sigma)$, $\boldsymbol{p}_1, \dots, \boldsymbol{p}_{t_l}$ induce a basis for $J^l(\boldsymbol{P}/C) = (J^l\boldsymbol{P} + C)/C$. This shows that the restriction Φ_σ sends any point $C \in \mathrm{GRASS}(\sigma)$ to

$$\big([c_1 \wedge \cdots \wedge c_s], [c_1 \wedge \cdots \wedge c_s \wedge \boldsymbol{p}_1 \wedge \cdots \wedge \boldsymbol{p}_{t_L}], \dots, [c_1 \wedge \cdots \wedge c_s \wedge \boldsymbol{p}_1 \wedge \cdots \wedge \boldsymbol{p}_{t_0}]\big),$$

whence Φ_σ is indeed a morphism.

Finally, we observe that Φ induces an isomorphism onto its image. Indeed, the inverse is the restriction to $\mathrm{Im}(\Phi)$ of the projection onto the leftmost component of the direct product of the $\mathrm{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda)$, namely the restriction of

$$\Psi : \prod_{0 \le l \le L+1} \mathrm{GRASS}_{\boldsymbol{d}^{(l)}}(\Lambda) \to \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda), \quad (D_{L+1}, \dots, D_0) \mapsto D_{L+1},$$

to $\mathrm{Im}(\Phi)$. Therefore $\Phi^{-1} : \mathrm{Im}(\Phi) \to \mathrm{GRASS}(\mathbb{S})$ is a morphism. □

Proof of Theorem 3.8. We refer to the notation in the proof of Theorem 3.9. Since $\mathfrak{U}$ is a projective variety, so is $\Psi(\mathfrak{U})$. In particular, $\Psi(\mathfrak{U})$ is closed in $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$.

By condition (‡) spelled out in the proof of Theorem 3.9, the image $\Psi(\mathfrak{U}) \subseteq \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ consists precisely of those points $C \in \mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$ which have the property that $\boldsymbol{P}/C$ has a filtration governed by $\mathbb{S}$; in particular $\Psi(\mathfrak{U})$ is stable under the $\mathrm{Aut}_\Lambda(\boldsymbol{P})$-action of $\mathrm{GRASS}_{\boldsymbol{d}}(\Lambda)$. In light of Lemma and Definition 3.6, Proposition 2.1 thus matches up $\Psi(\mathfrak{U})$ with the $\mathrm{GL}(\boldsymbol{d})$-stable subset $\mathbf{Filt}\,\mathbb{S}$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ and tells us that $\mathbf{Filt}\,\mathbb{S}$ is in turn closed.

For the final claim, it suffices to observe that $\mathbf{Rep}\,\mathbb{S} \subseteq \mathbf{Filt}\,\mathbb{S}$. □

Theorem 3.8 prompts us to introduce a new module invariant, which will turn out to be highly informative in gauging the overlaps among the closed varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$.

Definition 3.10 (the module invariant Γ). For $M \in \Lambda$-mod, let $\Gamma(M)$ denote the number of realizable semisimple sequences which govern some filtration of M.

Corollary 3.11. *The map $\Gamma_\bullet : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbb{N}$ sending x to $\Gamma(M_x)$ is upper semicontinuous.*

In particular, whenever $\mathcal{C}$ is an irreducible component of some $\mathbf{Rep}\,\mathbb{S}$ *such that $1 \in \Gamma_\bullet(\mathcal{C})$, the closure $\overline{\mathcal{C}}$ is an irreducible component of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

Proof. Let $\mathcal{R}$ be the set of all realizable semisimple sequences with dimension vector $\boldsymbol{d}$. Moreover, for $a \in \mathbb{N}$, let $\mathcal{R}(a)$ be the collection of all those intersections $\bigcap_i \mathbf{Filt}(\mathbb{S}^{(i)})$ which involve at least a distinct sequences $\mathbb{S}^{(i)} \in \mathcal{R}$. Then the preimage $\Gamma_\bullet^{-1}([a, \infty))$ is the union of the sets in $\mathcal{R}(a)$. Since each $\mathbf{Filt}(\mathbb{S}^{(i)})$ is closed in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ by Theorem 3.8 and $\mathcal{R}(a)$ is finite, the union $\Gamma_\bullet^{-1}([a, \infty))$ is closed. This proves the claim regarding upper semicontinuity.

To justify the final assertion, suppose that $\overline{\mathcal{C}}$ is properly contained in some irreducible component $\mathcal{C}'$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Then $\mathcal{C}'$ is an irreducible component of some $\overline{\mathbf{Rep}\,\mathbb{S}'}$ with $\mathbb{S}' < \mathbb{S}$. Since $\overline{\mathbf{Rep}\,\mathbb{S}'} \subseteq \mathbf{Filt}(\mathbb{S}')$ by Theorem 3.8, all modules in $\bar{\mathcal{C}}$ have a filtration governed by $\mathbb{S}'$ in this situation, whence $\Gamma(M) > 1$ for all $M \in \mathbf{Rep}\,\mathbb{S}$. □

Now let $D = \mathrm{Hom}_K(-, K) : \Lambda\text{-mod} \to \text{mod-}\Lambda$ be the standard duality. Clearly, $M \in \Lambda$-mod contains a descending submodule chain governed by $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ if and only if $D(M)$ contains an ascending chain $M'_{-1} = 0 \subseteq M'_0 \subseteq \cdots \subseteq M'_L = D(M)$ which is *cogoverned* by $D(\mathbb{S}) = (D(\mathbb{S}_0), \dots, D(\mathbb{S}_L))$, in the sense that each of the consecutive quotients M'_l/M'_{l-1} is isomorphic to $D(\mathbb{S}_l)$. We define $\mathbf{Cofilt}\,\mathbb{S}'$ to be the subset of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ whose points correspond to the modules which are cogoverned by a semisimple sequence $\mathbb{S}'$. The duality $\widehat{D} : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda\text{-mod}) \to \mathbf{Rep}_{\boldsymbol{d}}(\text{mod-}\Lambda)$ of [Huisgen-Zimmermann and Shipman 2017, Section 2.C] thus yields the following dual of Theorem 3.8; we spell it out since, in size comparisons of $\overline{\mathcal{C}^{(i)}}$ versus $\overline{\mathcal{C}^{(j)}}$, for irreducible components $\mathcal{C}^{(k)}$ of $\mathbf{Rep}\,\mathbb{S}$, one gains mileage in combining Theorem 3.8 with its dual. (Recall that the process of filtering the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ out of

$$\{\bar{\mathcal{C}} \mid \mathcal{C} \text{ is a component of some } \mathbf{Rep}\,\mathbb{S} \text{ with } \underline{\dim}\,\mathbb{S} = \boldsymbol{d}\}$$

rests on comparisons of this ilk.)

Theorem 3.12 (dual of Theorem 3.8). *If $\mathbb{S}^* = (\mathbb{S}^*_0, \dots, \mathbb{S}^*_L)$ is a semisimple sequence in* Λ-mod *with dimension vector $\boldsymbol{d}$, let* $\mathbf{Corep}\,\mathbb{S}^*$ (*resp.* $\mathbf{Cofilt}\,\mathbb{S}^*$) *be the set of all points in* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ *which correspond to modules with socle series* $\mathbb{S}^*$ (*resp. to modules with filtrations cogoverned by* $\mathbb{S}^*$).

Then $\mathbf{Cofilt}(\mathbb{S}^*)$ *is a closed subvariety of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, *and hence* $\overline{\mathbf{Corep}\,\mathbb{S}^*} \subseteq \mathbf{Cofilt}\,\mathbb{S}^*$. *In particular, if $\mathcal{C}$ is an irreducible component of* $\mathbf{Rep}\,\mathbb{S}$ *such that, generically, the modules in $\mathcal{C}$ have socle layering $\mathbb{S}^*$, then* $\bar{\mathcal{C}} \subseteq \mathbf{Filt}\,\mathbb{S} \cap \mathbf{Cofilt}\,\mathbb{S}^*$. □

We close the section with an example to the effect that, in general, the inclusion $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \mathbf{Filt}\,\mathbb{S}$ may be proper and the final implication of Corollary 3.11 need not be reversible. This contrasts with the situation where $\Lambda = \Lambda_{\text{trunc}}$, as we will see in Section 4.

Example 3.13. Consider the quiver Q of Example 2.5 with $r = 2$ and $s = 1$, and set

$$\Lambda = KQ/\langle \beta_1\alpha_2,\ \alpha_2\beta_1, \text{ all paths of length } 4\rangle.$$

Let $\boldsymbol{d} := (2, 2)$, $\mathbb{S} := (S_1, S_2, S_1, S_2)$, and $\mathbb{S}' := (S_1^2, S_2^2, 0, 0)$. Then the varieties $\mathbf{Rep}\,\mathbb{S}$ and $\mathbf{Rep}\,\mathbb{S}'$ are irreducible, and generically their modules have hypergraphs as shown in Figure 1, whence both are contained in $\mathbf{Filt}\,\mathbb{S}$. Clearly, $\overline{\mathbf{Rep}\,\mathbb{S}} \not\subseteq \overline{\mathbf{Rep}\,\mathbb{S}'}$, due to the generic Loewy lengths of the modules in $\mathbf{Rep}\,\mathbb{S}$ and $\mathbf{Rep}\,\mathbb{S}'$. By comparing generic α_2-ranks, one finds, moreover, that $\overline{\mathbf{Rep}\,\mathbb{S}'} \not\subseteq \overline{\mathbf{Rep}\,\mathbb{S}}$. In conclusion, both $\overline{\mathbf{Rep}\,\mathbb{S}}$ and $\overline{\mathbf{Rep}\,\mathbb{S}'}$ are components of $\mathbf{Filt}\,\mathbb{S}$. In fact, both of these closures are

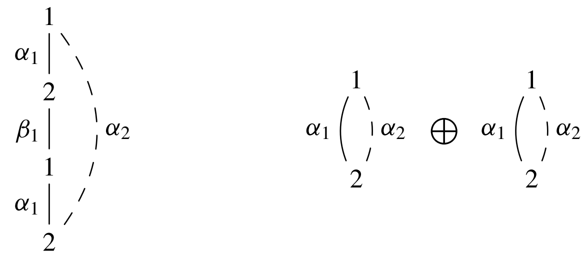


Figure 1. Hypergraphs of generic modules for $\mathbf{Rep}\,\mathbb{S}$ (left) and $\mathbf{Rep}\,\mathbb{S}'$ (right) for Example 3.13.

even irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, the latter failing to satisfy the sufficient condition of Corollary 3.11. Indeed, $\Gamma(M) = 2$ for all M in $\overline{\mathbf{Rep}\,\mathbb{S}'}$.

It is readily verified that the total number of components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is three, the remaining component being $\overline{\mathbf{Rep}\,\mathbb{S}''} = \mathbf{Filt}(\mathbb{S}'')$ for $\mathbb{S}'' = (S_2, S_1, S_2, S_1)$. By contrast, on replacing Λ by the associated truncated path algebra Λ_{trunc}, two of the three components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ fuse into a single component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$; see Example 6.1(b) below. □

4. The main results for truncated Λ

Throughout this section, Λ *stands for a truncated path algebra of Loewy length* $L+1$*, i.e.,* $\Lambda = \Lambda_{\text{trunc}}$. In particular, the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ are among the $\overline{\mathbf{Rep}\,\mathbb{S}}$, where $\mathbb{S}$ traces the $\boldsymbol{d}$-dimensional realizable semisimple sequences. The upcoming theory characterizes these components in terms of their generic radical layerings $\mathbb{S}$ (or, equivalently, in terms of their *generic modules* in the sense of Section 5 below). As in the special cases already mastered — the local case and that of an acyclic quiver Q — the classification may be implemented on a computer; see Section 5B. However, the general algorithm is considerably more labor-intensive than the Θ-test which applies to the local and acyclic cases.

As we will recall in Section 5, the generic properties of the modules in any component $\overline{\mathbf{Rep}\,\mathbb{S}}$ may be accessed via a single generic module $G(\mathbb{S})$. A key asset of the truncated situation lies in the fact that such a module $G(\mathbb{S})$ is available on sight from $\mathbb{S}$; detail will follow in Section 5A below.

Moreover, it is particularly easy to recognize realizability of semisimple sequences over truncated path algebras. We recall the following from [Huisgen-Zimmermann 2016, Criterion 3.2]:

Realizability Criterion 4.1. Let $\boldsymbol{B} = (\boldsymbol{B}_{ij})$ be the adjacency matrix of Q, i.e., $\boldsymbol{B}_{ij}$ is the number of arrows from e_i to e_j. Then $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ is realizable if and only if $\underline{\dim}\,\mathbb{S}_l \le (\underline{\dim}\,\mathbb{S}_{l-1}) \cdot \boldsymbol{B}$ for all $1 \le l \le L$; the latter, in turn, is equivalent to realizability of the two-term sequences $(\mathbb{S}_l, \mathbb{S}_{l+1})$ in (Λ/J^2)-mod for $l < L$. □

In more intuitive terms: $\mathbb{S}$ is realizable if and only if there exists an abstract skeleton with layering $\mathbb{S}$. Moreover, note that in the positive case, any such skeleton belongs to the *generic* set of skeleta of the modules in $\overline{\mathbf{Rep}}\,\mathbb{S}$.

Next, we find that the description of Δ-$\mathbf{Rep}(\geq\mathbb{S})$ may be simplified in the truncated situation, in that requirement (iii) of Lemma 3.2 is now void.

Observation 4.2 (Δ-$\mathbf{Rep}(\geq\mathbb{S})$ is an affine space). Referring to the decomposition of K^d induced by $\mathbb{S}$ in Definition 3.1, we obtain that Δ-$\mathbf{Rep}(\geq\mathbb{S})$ consists of those points $f = (f_\alpha)_\alpha \in (\mathrm{End}_\Lambda(K^d))^{Q_1}$ which satisfy the following conditions: for any arrow α from e_i to e_j,

- $f_\alpha(\mathcal{K}_{(l,r)}) = 0$ for all $r \neq i$, and
- $f_\alpha(\mathcal{K}_{(l,i)}) \subseteq \bigoplus_{l+1\leq m\leq L} \mathcal{K}_{(m,j)}$.

In particular, Δ-$\mathbf{Rep}(\geq\mathbb{S})$ is a full affine space in this situation. Indeed, the image of the closed immersion Δ-$\mathbf{Rep}(\geq\mathbb{S}) \hookrightarrow \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, which we presented in 3.3, consists of *all* sequences of $d_i \times d_i$ matrices of the described lower triangular format. Consequently, $\mathbf{Filt}\,\mathbb{S}$, being a morphic image of $\mathrm{GL}(\boldsymbol{d}) \times \Delta$-$\mathbf{Rep}(\geq\mathbb{S})$, is irreducible as well.

This observation, in turn, allows us to derive a full characterization of the modules in $\overline{\mathbf{Rep}}\,\mathbb{S}$ from Theorem 3.8.

Theorem 4.3. *Suppose Λ is a truncated path algebra and $\mathbb{S}$ a realizable semisimple sequence. Then*

$$\overline{\mathbf{Rep}\,\mathbb{S}} = \mathbf{Filt}\,\mathbb{S}.$$

In other words, a module M belongs to $\overline{\mathbf{Rep}\,\mathbb{S}}$ precisely when M has a filtration governed by $\mathbb{S}$.

Dually, $\overline{\mathbf{Corep}\,\mathbb{S}^*} = \mathbf{Cofilt}\,\mathbb{S}^*$, where $\mathbb{S}^*$ is the generic socle layering of the modules in $\mathbf{Rep}\,\mathbb{S}$. If $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, then

$$\overline{\mathbf{Corep}\,\mathbb{S}^*} = \mathbf{Cofilt}\,\mathbb{S}^* = \mathbf{Filt}\,\mathbb{S} = \overline{\mathbf{Rep}\,\mathbb{S}}.$$

Proof. Concerning the first equality: In light of Observation 4.2, the variety Δ-$\mathbf{Rep}(\geq\mathbb{S})$ is irreducible. Therefore the open subset Δ-$\mathbf{Rep}\,\mathbb{S}$ is dense in it, meaning that the closure $\overline{\Delta\text{-}\mathbf{Rep}\,\mathbb{S}}$ in $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ contains Δ-$\mathbf{Rep}(\geq \mathbb{S})$. Moreover, Δ-$\mathbf{Rep}\,\mathbb{S} \subseteq \Delta$-$\mathbf{Rep}(\geq\mathbb{S})$ by construction, whence we obtain

$$\Delta\text{-}\mathbf{Rep}(\geq\mathbb{S}) \subseteq \overline{\Delta\text{-}\mathbf{Rep}\,\mathbb{S}} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}}.$$

Given that $\overline{\mathbf{Rep}\,\mathbb{S}}$ is $\mathrm{GL}(\boldsymbol{d})$-stable, it follows that $\mathbf{Filt}\,\mathbb{S} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}}$ due to Lemma and Definition 3.6. The reverse inclusion was established in Theorem 3.8. The second assertion follows by duality (see Theorem 3.12 and [Huisgen-Zimmermann and Shipman 2017, Corollary 3.4.b]).

In particular, duality guarantees that the varieties $\overline{\mathbf{Corep}\,\mathbb{S}^*}$ are again irreducible. For arbitrary $\mathbb{S}$ we find, moreover, that $\mathbf{Rep}\,\mathbb{S} \subseteq \overline{\mathbf{Corep}\,\mathbb{S}^*}$, since the modules in a dense open subset of $\mathbf{Rep}\,\mathbb{S}$ have socle layering $\mathbb{S}^*$. Therefore, $\overline{\mathbf{Rep}\,\mathbb{S}} = \overline{\mathbf{Corep}\,\mathbb{S}^*}$ whenever $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. $\square$

The following consequence, addressing the relative sizes of the closures $\overline{\mathbf{Rep}\,\mathbb{S}}$, is now immediate. It was independently obtained by I. Shipman with different methods; he also developed an algorithm for checking the considered inclusion via matrices of dimension vectors (Shipman, personal communication, 2016). Algorithmic counterparts to the upcoming Corollary 4.4 and Theorem 4.5 will be addressed in Section 5B.

Corollary 4.4 (comparing the varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$). *Let Λ be a truncated path algebra. Moreover, suppose that $\mathbb{S}$ and $\mathbb{S}'$ are realizable semisimple sequences with the same dimension vector. Then $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}'}$ if and only if (generically) the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ have filtrations governed by $\mathbb{S}'$.* $\square$

The upper semicontinuous map $\Gamma_{\bullet} : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbb{N}$ of Corollary 3.11 detects all irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Indeed, $\mathbb{S}$ is the generic radical layering of an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ if and only if $\Gamma_{\bullet}$ attains the value 1 on $\mathbf{Rep}\,\mathbb{S}$. We record this as follows.

Theorem 4.5. *Let Λ be a truncated path algebra. If $\mathbb{S}^{(1)}, \ldots, \mathbb{S}^{(m)}$ are the distinct $\boldsymbol{d}$-dimensional semisimple sequences $\mathbb{S}$ with $1 \in \Gamma_{\bullet}(\mathbf{Rep}\,\mathbb{S})$, then*

$$\mathbf{Filt}(\mathbb{S}^{(1)}) = \overline{\mathbf{Rep}\,\mathbb{S}^{(1)}}, \quad \ldots, \quad \mathbf{Filt}(\mathbb{S}^{(m)}) = \overline{\mathbf{Rep}\,\mathbb{S}^{(m)}}$$

are the distinct irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

Proof. Suppose $\mathbb{S}$ is a realizable $\boldsymbol{d}$-dimensional semisimple sequence. If $1 \in \Gamma_{\bullet}(\mathbf{Rep}\,\mathbb{S})$, then $\mathbf{Rep}\,\mathbb{S} \not\subseteq \mathbf{Filt}(\mathbb{S}') = \overline{\mathbf{Rep}\,\mathbb{S}'}$ for any semisimple sequence $\mathbb{S}' \neq \mathbb{S}$, whence $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

If, on the other hand, $1 \notin \Gamma_{\bullet}(\mathbf{Rep}\,\mathbb{S})$, then every module in $\mathbf{Rep}\,\mathbb{S}$ is contained in some variety $\mathbf{Filt}\,\mathbb{S}'$, where $\mathbb{S}'$ is a realizable semisimple sequence different from $\mathbb{S}$. Therefore,

$$\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \bigcup_{\substack{\mathbb{S}' \text{ realizable} \\ \mathbb{S}' \neq \mathbb{S}}} \mathbf{Filt}\,\mathbb{S}' = \bigcup_{\substack{\mathbb{S}' \text{realizable} \\ \mathbb{S}' \neq \mathbb{S}}} \overline{\mathbf{Rep}\,\mathbb{S}'},$$

the final equality being part of Theorem 4.3. Irreducibility of $\overline{\mathbf{Rep}\,\mathbb{S}}$ thus implies $\overline{\mathbf{Rep}\,\mathbb{S}} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}'}$ for some $\mathbb{S}' \neq \mathbb{S}$, which shows that $\overline{\mathbf{Rep}\,\mathbb{S}}$ fails to be maximal irreducible. $\square$

5. Applications of Section 4: Generic modules for the components over truncated path algebras

Barring Example 5.2(b), Λ *will, throughout this section, stand for a truncated path algebra of Loewy length* $L+1$. *Moreover,* $\boldsymbol{d}$ *will be a dimension vector of* Λ.

If one extends the base field K of Λ to an algebraically closed field of infinite transcendence degree over its prime field K_0, neither the description of the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ nor the generic properties of their modules will be affected; see [Huisgen-Zimmermann and Shipman 2017, Section 2.B]. This means that, in developing a generic representation theory for the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, one does not lose generality in assuming that $\operatorname{trdeg}(K : K_0) = \infty$.

5A. ***Generic modules.*** Assume that K has infinite transcendence degree over K_0, and let $\mathbb{S}$ be a realizable $\boldsymbol{d}$-dimensional semisimple sequence. Given that $\Lambda = \Lambda_{\text{trunc}}$, we will denote the coordinatized projective Λ_{trunc}-projective cover $P_{\text{trunc}} = \bigoplus_{1\le r\le t} \Lambda z_r$ of $\mathbb{S}_0$ (see Section 2) more simply by P.

Let σ be any skeleton with layering $\mathbb{S}$. Then the following module $G = G(\mathbb{S})$ is generic for $\overline{\mathbf{Rep}\,\mathbb{S}}$ in the strict sense of [Babson et al. 2009, Definition 4.2]:

$$G = P/C, \quad \text{where } C = \sum_{\boldsymbol{q}\ \sigma\text{-critical}} \Lambda\Bigl(\boldsymbol{q} - \sum_{\boldsymbol{p}\in\sigma_{\boldsymbol{q}}} c_{\boldsymbol{q},\boldsymbol{p}}\,\boldsymbol{p}\Bigr)$$

for some family $(c_{\boldsymbol{q},\boldsymbol{p}})_{\boldsymbol{q}\ \sigma\text{-critical},\ \boldsymbol{p}\in\sigma_{\boldsymbol{q}}}$ of scalars which is algebraically independent over K_0. That G is *generic* means that G has all those generic properties of the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ which are invariant under Morita self-equivalences Λ-mod $\to$ Λ-mod induced by automorphisms of K over $\overline{K_0}$. Moreover, G is unique relative to this property, up to such a Morita self-equivalence. We refer to [ibid., Theorem 5.12], and to [ibid., Section 4] for a more general statement addressing arbitrary path algebras modulo relations.

Filtrations of generic modules. In particular, the preceding comments ensure that tests for semisimple sequences which generically govern filtrations of the modules in $\overline{\mathbf{Rep}\,\mathbb{S}}$ may be confined to "the" generic module $G = G(\mathbb{S})$.

Caveat: Suppose G is a generic module for an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. While the combination of Corollary 3.11 and Theorem 4.5 guarantees that the radical layering $\mathbb{S}(G)$ is the only *realizable* semisimple sequence to govern a filtration of G, there will in general be further, nonrealizable, sequences governing suitable filtrations. For instance, let Q be the quiver $4 \longleftarrow 1 \xrightarrow{\alpha} 2 \longrightarrow 3$ and Λ any truncated path algebra based on Q. If $\boldsymbol{d} = (0, 1, 1, 1)$, then $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is irreducible with generic module $G = \Lambda\alpha \oplus S_4$ for any truncation Λ of KQ. In particular, $\mathbb{S}(G) = (S_2 \oplus S_4,\ S_3)$ is the only realizable semisimple sequence governing all modules with dimension vector $\boldsymbol{d}$. If Λ has Loewy length 2, the sequence $(S_2,\ S_3 \oplus S_4)$ also governs a filtration of G; if the Loewy length of Λ is 3, then (S_2, S_3, S_4) and (S_4, S_2, S_3) are additional (nonrealizable) semisimple sequences governing filtrations of G.

5B. ***Algorithmic aspect of Corollary 4.4 and Theorem 4.5.*** Section 5A tells us that, for any two realizable $\boldsymbol{d}$-dimensional semisimple sequences $\mathbb{S}$ and $\mathbb{S}'$, we have

$$\mathbf{Rep}\,\mathbb{S} \subseteq \overline{\mathbf{Rep}\,\mathbb{S}'} \quad \iff \quad G(\mathbb{S}) \in \mathbf{Filt}\,\mathbb{S}'.$$

From Lemma and Definition 3.6 we know, moreover, that $\mathbf{Filt}\,\mathbb{S}'$ is the $\mathrm{GL}(\boldsymbol{d})$-stable hull of Δ-$\mathbf{Rep}(\geq\mathbb{S}')$. Hence, if the point $(G_\alpha)_{\alpha\in Q_1} \in \mathbf{Rep}\,\mathbb{S}$ represents the isomorphism class of $G(\mathbb{S})$, the question of whether $G(\mathbb{S})$ lies in $\mathbf{Filt}\,\mathbb{S}'$ boils down to the question of whether the matrices G_α are "simultaneously" similar (i.e., similar by way of a single element of $\mathrm{GL}(\boldsymbol{d})$) to matrices having the lower triangular format F_α characterizing the points in Δ-$\mathbf{Rep}(\geq \mathbb{S}')$. This format is spelled out in 3.3.

Given that there are only finitely many $\boldsymbol{d}$-dimensional semisimple sequences to be compared, this means in particular that the decision of whether or not $\overline{\mathbf{Rep}\,\mathbb{S}}$ is a component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ is algorithmic.

5C. ***Interconnections among the components.*** The following statement rephrases a result of Crawley-Boevey and Schröer [2002, Theorem 1.1] in terms of generic modules: if G is a generic module for an irreducible component $\mathcal{C}$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ and $G = \bigoplus_{1\leq j\leq s} G_j$ is a decomposition into direct summands, then each G_j is generic for an irreducible component of $\mathbf{Rep}_{\underline{\dim}\, G_j}(\Lambda)$. Over a truncated path algebra, this result may be sharpened as follows.

Call a submodule M of N *layer-stably embedded* in N if $J^l M = M \cap J^l N$ for all $l \leq L$. As a consequence of Theorem 4.5, we obtain:

Theorem 5.1. *Suppose that Λ is a truncated path algebra and $\overline{\mathbf{Rep}\,\mathbb{S}}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ with generic module G. If $G' \subseteq G$ is a layer-stably embedded submodule of G with $\mathbb{S}(G') = \mathbb{S}'$ and $\underline{\dim}\, G' = \boldsymbol{d}'$, then $\overline{\mathbf{Rep}\,\mathbb{S}'}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$ with generic module G'.*

Proof. Let $H := G'$ be layer-stably embedded in G. From [Huisgen-Zimmermann and Shipman 2017, Corollary 3.2] we know that H is generic for $\mathbf{Rep}\,\mathbb{S}' = \mathbf{Rep}\,\mathbb{S}(H)$. Thus only the status of $\overline{\mathbf{Rep}\,\mathbb{S}'}$ as a potential component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$ needs to be addressed.

Assume that $\mathbf{Rep}\,\mathbb{S}'$ fails to be an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$. In view of Theorem 4.5, this means that H has a filtration governed by some realizable semisimple sequence $\mathbb{S}''$ which is strictly smaller than $\mathbb{S}'$, say $H = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_L \supseteq H_{L+1} = 0$; by definition, $\mathbb{S}''_l = H_l/H_{l+1}$. We aim at constructing a submodule filtration $G = G_0 \supseteq \cdots \supseteq G_L \supseteq 0$ which, in turn, is governed by a realizable semisimple sequence $\widehat{\mathbb{S}}$ strictly smaller than $\mathbb{S}$. Another application of Theorem 4.5

will then show that $\overline{\mathbf{Rep}\,\mathbb{S}}$ is not an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, contrary to our hypothesis.

For $l \le L$, let $\pi_l : G \to G/J^{l+1}G$ denote the quotient map. We recursively choose submodules U_l of J^lG such that

$$J^{l+1}G \subseteq U_l \subseteq J^lG, \quad U_l \subseteq JU_{l-1}, \quad J^lG/J^{l+1}G = \pi_l(J^lH) \oplus \pi_l(U_l). \tag{5-1}$$

First, semisimplicity of G/JG implies that $G/JG = \pi_0(H) \oplus \pi_0(U_0)$ for some $U_0 \subseteq G$, and since U_0 may be replaced by $U_0 + JG$, there is no loss of generality in assuming that $JG \subseteq U_0$. If $U_0, \dots, U_k$ for some $k < L$ have been chosen so as to satisfy (5-1), we have $J^kG = J^kH + U_k$ by Nakayama's lemma, whence $J^{k+1}G = J^{k+1}H + JU_k$. Consequently, $J^{k+1}G/J^{k+2}G = \pi_{k+1}(J^{k+1}H) \oplus \pi_{k+1}(U_{k+1})$ for some $U_{k+1} \subseteq JU_k$. On replacing U_{k+1} by $U_{k+1} + J^{k+2}G$, we obtain (5-1) for $l = k+1$. Finally, set $U_{L+1} := 0$.

Now define $G_l := H_l + U_l$ for $l \le L+1$. That the consecutive factors of the sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_L \supseteq G_{L+1} = 0 \tag{5-2}$$

are semisimple, i.e., $JG_l \subseteq G_{l+1}$ for $l \le L$, is straightforward from our construction. Indeed, $JH_l \subseteq H_{l+1}$ and

$$JU_l \subseteq J^{l+1}G = J^{l+1}H + U_{l+1} \subseteq H_{l+1} + U_{l+1}.$$

Let $\widehat{\mathbb{S}}$ be the semisimple sequence governing the filtration (5-2). Remark 3.7(2) tells us that $\widehat{\mathbb{S}} \le \mathbb{S}$.

Suppose m is minimal with the property that $J^mH \subsetneqq H_m$. Such an index m exists, since $\mathbb{S}'' < \mathbb{S}'$. Then $m \ge 1$. Using layer-stability of H in G, we derive

$$J^mG = J^mH + U_m \subsetneqq H_m + U_m = G_m.$$

On the other hand, $G_l = J^lG$ for $l < m$, so that the first discrepancy between the downward filtration (5-2) and the radical filtration of G occurs at $l = m$. More specifically,

$$\underline{\dim}\,\widehat{\mathbb{S}}_{m-1} = \underline{\dim}(G_{m-1}/G_m) < \underline{\dim}(G_{m-1}/J^mG) = \underline{\dim}\,J^{m-1}G/J^mG = \underline{\dim}\,\mathbb{S}_{m-1}.$$

This yields $\widehat{\mathbb{S}} < \mathbb{S}$.

It remains to be verified that $\widehat{\mathbb{S}}$ is realizable. To do so, we make repeated use of Realizability Criterion 4.1. Again, $\boldsymbol{B}$ is the adjacency matrix of Q. First we note that realizability of $\mathbb{S}$ and $\mathbb{S}''$ entails

$$\begin{gathered} \underline{\dim}\,J^lG/J^{l+1}G \le (\underline{\dim}\,J^{l-1}G/J^lG)\cdot \boldsymbol{B}, \\ \underline{\dim}\,H_l/H_{l+1} \le (\underline{\dim}\,H_{l-1}/H_l)\cdot \boldsymbol{B} \end{gathered} \tag{5-3}$$

for $1 \le l \le L$. Therefore $\underline{\dim}\,G_l/G_{l+1} \le (\underline{\dim}\,G_{l-1}/G_l)\cdot \boldsymbol{B}$ for $1 \le l \le m-2$.

Invoking (5-1), we find that, for $1 \le l \le L$,

$$G_l/J^{l+1}G = (H_l + J^{l+1}G)/J^{l+1}G \oplus (U_l/J^{l+1}G)$$

and

$$G_{l+1}/J^{l+1}G = (H_{l+1} + J^{l+1}G)/J^{l+1}G,$$

where the sum in the first equation is direct because $H_l \cap U_l \subseteq H \cap J^l G = J^l H$ implies $H_l \cap U_l = J^l H \cap U_l \subseteq J^{l+1}G$. We also have

$$(H_l + J^{l+1}G)/(H_{l+1} + J^{l+1}G) \cong H_l/H_{l+1},$$

since layer-stability of H in G guarantees that $H_l \cap J^{l+1}G \subseteq J^{l+1}H \subseteq H_{l+1}$. Consequently,

$$G_l/G_{l+1} \cong (H_l/H_{l+1}) \oplus (U_l/J^{l+1}G) \quad \text{for } 1 \le l \le L. \tag{5-4}$$

Since $U_l \subseteq JU_{l-1}$ we obtain, moreover, that

$$\underline{\dim}\, U_l/J^{l+1}G \le \underline{\dim}\, JU_{l-1}/J(J^l G) \le (\underline{\dim}\, U_{l-1}/J^l G)\cdot \boldsymbol{B} \quad \text{for } 1 \le l \le L. \tag{5-5}$$

Combining (5-5) with (5-3) and (5-4) yields $\underline{\dim}\, G_l/G_{l+1} \le (\underline{\dim}\, G_{l-1}/G_l)\cdot \boldsymbol{B}$ for $1 \le l \le L$, which shows that $\widehat{\mathbb{S}}$ is realizable as required. $\square$

The following examples demonstrate: (a) that the conclusion of Theorem 5.1 does not extend to arbitrary top-stably embedded submodules G' of G, i.e., to submodules G' satisfying only $JG' = G' \cap JG$, and (b) that Theorem 5.1 has no analogue for nontruncated Λ in general.

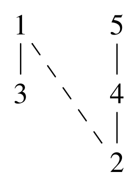
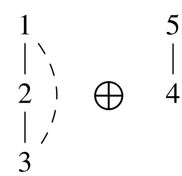
Examples 5.2 (demonstrating the sharpness of Theorem 5.1). Consider the quivers

$$Q_1: \quad \begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3 & 4 \arrow[ll, bend left] & 5 \arrow[l] \end{tikzcd} \qquad Q_2: \quad \begin{tikzcd} 4 \arrow[rr, bend left, "\delta"] & 1 \arrow[r] \arrow[rr, bend right] & 2 \arrow[r, "\beta"] & 3 \end{tikzcd}$$

(a) Let Λ be the truncated path algebra of Loewy length 3 based on the quiver Q_1. For $\boldsymbol{d} = (1, 1, 1, 1, 1)$, the variety $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ has two irreducible components, with generic radical layerings

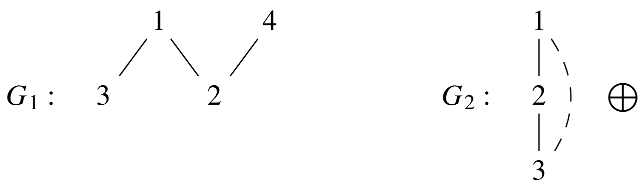
$$\mathbb{S}^{(1)} := (S_1 \oplus S_5,\, S_3 \oplus S_4,\, S_2) \quad \text{and} \quad \mathbb{S}^{(2)} := (S_1 \oplus S_5,\, S_2 \oplus S_4,\, S_3)$$

and generic modules G_1 and G_2 as graphed below:

G_1 :  G_2 : 

Clearly, the top-stably embedded submodule G' of G_1 generated by any element $z = e_1 z \in G_1$ has dimension vector $\boldsymbol{d}' := (1, 1, 1, 0, 0)$. On the other hand, the sequence $\mathbb{S}(G') = (S_1, S_2 \oplus S_3, 0)$ fails to be the generic radical layering of an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$, the latter variety being irreducible with uniserial generic modules.

(b) Now let $\Lambda = KQ_2/\langle \beta\delta \rangle$ and $\boldsymbol{d} := (1, 1, 1, 1)$. Then, again, $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ consists of two irreducible components. Their generic modules are graphed below:



The submodule G' of G_1 generated by any element $z = e_1 z \in G_1$ has dimension vector $\boldsymbol{d}' := (1, 1, 1, 0)$ and is layer-stably embedded in G_1 this time. Nonetheless, $\overline{\mathbf{Rep}\,\mathbb{S}(G')}$ fails to be an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$. Indeed, once again, $\mathbf{Rep}_{\boldsymbol{d}'}(\Lambda)$ is irreducible and its generic modules are uniserial. $\square$

6. Examples illustrating the theory. The interplay $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \longleftrightarrow \mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$

6A. ***Illustrations of the truncated case.*** *In this subsection, Λ denotes a truncated path algebra.*

In sifting the radical layerings of the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ out of the set $\mathbf{Seq}(\boldsymbol{d})$, it is computationally advantageous to supplement $\Gamma_{\bullet}$ by the map Θ of equation (1-1), or by the upgraded map Θ^+ to be introduced next.

Example 4.8 in [Huisgen-Zimmermann 2016] shows that Θ fails to detect all irreducible components in the general truncated case. However, in that instance (as in many others), supplementing Θ by path ranks compensates for the blind spots of Θ. Here the *path rank* of a finite-dimensional Λ-module M is the tuple $(\dim pM)_p \in \mathbb{Z}^\tau$, where τ is the set of paths in $KQ \setminus I$. Set $f(M) = (-\dim pM)_p$, and let $f^*(M)$ be the negative of the path rank of the right Λ-module $D(M)$. Clearly, the map

$$\Theta^+ : \mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \to \mathbf{Seq}(\boldsymbol{d}) \times \mathbf{Seq}(\boldsymbol{d}) \times \mathbb{Z}^\tau \times \mathbb{Z}^\tau,$$
$$x \mapsto (\mathbb{S}(M_x), \mathbb{S}^*(M_x), f(M_x), f^*(M_x)),$$

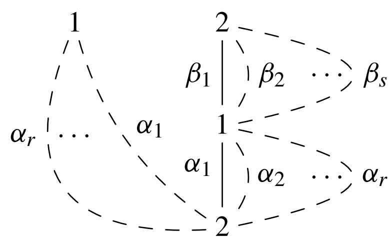
is in turn upper semicontinuous. Therefore, it is generically constant on the varieties $\mathbf{Rep}\,\mathbb{S}$. In particular, those closures $\overline{\mathbf{Rep}\,\mathbb{S}}$ on which Θ^+ attains its minimal values (relative to the componentwise partial order on the codomain) are components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Yet, part (c) of the next example attests to the fact that the augmented upper semicontinuous map Θ^+ still leaves certain components undetected in general. We use $\Gamma_\bullet$ to fill in what Θ^+ fails to pick up.

Example 6.1. Let Λ be the truncated path algebra of Loewy length 4 based on the quiver Q of Example 2.5, and take $\boldsymbol{d} = (2, 2)$. The semisimple sequences which are in the running as potential generic radical layerings of components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ are

$$\begin{aligned}
&\mathbb{S}^{(1)} = (S_1, S_2, S_1, S_2), && \mathbb{S}^{(4)} = (S_2, S_1^2, S_2, 0), && \mathbb{S}^{(7)} = (S_1 \oplus S_2, S_1 \oplus S_2, 0, 0),\\
&\mathbb{S}^{(2)} = (S_2, S_1, S_2, S_1), && \mathbb{S}^{(5)} = (S_1^2, S_2^2, 0, 0), && \mathbb{S}^{(8)} = (S_1 \oplus S_2, S_1, S_2, 0),\\
&\mathbb{S}^{(3)} = (S_1, S_2^2, S_1, 0), && \mathbb{S}^{(6)} = (S_2^2, S_1^2, 0, 0), && \mathbb{S}^{(9)} = (S_1 \oplus S_2, S_2, S_1, 0).
\end{aligned}$$

The list excludes the sequences which are not realizable for any choice of r and s, such as $(S_1, S_1 \oplus S_2, S_2, 0)$ and $(S_1, S_2, S_1 \oplus S_2, 0)$, as well as the radical layering $\mathbb{S}^{(0)}$ of the semisimple module, given that $\mathbf{Rep}\,\mathbb{S}^{(0)}$ is contained in all nonempty varieties $\overline{\mathbf{Rep}\,\mathbb{S}}$. Except for $\mathbb{S}^{(3)}$ and $\mathbb{S}^{(4)}$, all sequences on the list are realizable for arbitrary positive integers r, s.

Theorem 4.5 allows us to discard $\mathbb{S}^{(j)}$ for $j = 7, 8, 9$ from the list of possible generic radical layerings of irreducible components: indeed, the modules in $\mathbf{Rep}\,\mathbb{S}^{(7)}$ are generically decomposable, which makes it evident that they have filtrations governed by both $\mathbb{S}^{(1)}$ and $\mathbb{S}^{(2)}$. Any generic module G_8 for $\mathbf{Rep}\,\mathbb{S}^{(8)}$ has hypergraph



Clearly, G_8 is generated by elements $z_1 = e_1 z_1$ and $z_2 = e_2 z_2$, and the following submodule chain is governed by $\mathbb{S}^{(1)}$:

$$G_8 \supseteq \Lambda z_2 \supseteq \Lambda\beta_1 z_2 \supseteq \Lambda\alpha_1\beta_1 z_2 \supseteq 0.$$

Consequently, $\mathbf{Rep}\,\mathbb{S}^{(8)} \subseteq \mathbf{Filt}\,\mathbb{S}^{(1)}$ by Corollary 4.4. An analogous argument shows $\mathbf{Rep}\,\mathbb{S}^{(9)} \subseteq \mathbf{Filt}\,\mathbb{S}^{(2)}$.

On the other hand, $\mathcal{C}_j := \overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, 2$ are components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ for all choices of $r, s \geq 1$ by Theorem 4.5, since $\Gamma(U) = 1$ for any uniserial module U. Hence only the sequences $\mathbb{S}^{(j)}$ for $3 \leq j \leq 6$ require discussion by cases. We consider only the cases when $r \geq s$, due to the symmetry of the quiver Q.

(a) Let $r = s = 1$. Then $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ has precisely two irreducible components, namely $\mathcal{C}_j = \overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, 2$. We rule out the remaining sequences. First, $\mathbb{S}^{(3)}$ and $\mathbb{S}^{(4)}$ fail to be realizable when $r = s = 1$. Generically, the modules in

$\mathbf{Rep}\,\mathbb{S}^{(5)}$ are direct sums of two uniserials with radical layering $(S_1, S_2, 0, 0)$, and such a module has a filtration governed by $\mathbb{S}^{(1)}$. Thus, $\mathbf{Rep}\,\mathbb{S}^{(5)} \subseteq \mathbf{Filt}\,\mathbb{S}^{(1)} = \mathcal{C}_1$. Similarly, $\mathbf{Rep}\,\mathbb{S}^{(6)} \subseteq \mathcal{C}_2$.

(b) Let $r = 2$, $s = 1$. Then $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ again has precisely two irreducible components, $\mathcal{C}_1$ and $\mathcal{C}_2$. Concerning $\mathbb{S}^{(3)}$: a generic module G_3 for $\mathbf{Rep}\,\mathbb{S}^{(3)}$ has a hypergraph of the form

$$\begin{tikzcd} & 1 \arrow[dl, dash, "\alpha_1"'] \arrow[dr, dash, "\alpha_2"] & \\ 2 \arrow[dr, dash, "\beta_1"'] & & 2 \arrow[dl, dash, dashed, "\beta_1"] \\ & 1 & \end{tikzcd}$$

In particular, the socle of $G_3 = \Lambda z$ contains a copy of S_2, namely $\Lambda(\alpha_1 - k\alpha_2)z$ for a suitable scalar $k \in K^*$. We deduce that the submodule chain

$$G_3 \supseteq JG_3 \supseteq \Lambda(\alpha_1 - k\alpha_2)z + \Lambda\beta_1\alpha_1 z \supseteq \Lambda(\alpha_1 - k\alpha_2)z \supseteq 0$$

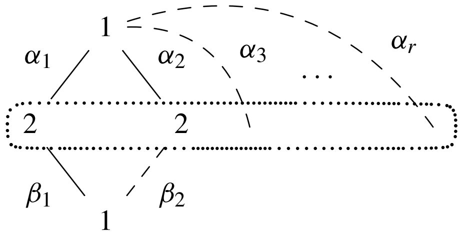
is governed by $\mathbb{S}^{(1)}$, showing $\mathbf{Rep}\,\mathbb{S}^{(3)} \subseteq \mathbf{Filt}\,\mathbb{S}^{(1)} = \mathcal{C}_1$. (On the side, we mention that $\mathbf{Rep}\,\mathbb{S}^{(3)}$ is not contained in $\mathcal{C}_2$ because the sequences $\mathbb{S}^{(2)}$ and $\mathbb{S}^{(3)}$ are not comparable under the dominance order.)

The sequence $\mathbb{S}^{(4)}$ fails to be realizable for $s = 1$. As for $\mathbb{S}^{(5)}$: generically, the modules in $\mathbf{Rep}\,\mathbb{S}^{(5)}$ decompose in the form shown at the end of Example 2.5(a), whence $\mathbf{Rep}\,\mathbb{S}^{(5)} \subseteq \mathcal{C}_1$. (Clearly, $\mathbf{Rep}\,\mathbb{S}^{(5)} \not\subseteq \mathcal{C}_2$, because $\mathbb{S}^{(5)}$ is not comparable to $\mathbb{S}^{(2)}$.) A routine check shows that $\mathbf{Rep}\,\mathbb{S}^{(6)}$ is contained in $\mathcal{C}_2$, but not in $\mathcal{C}_1$.

(c) Let $r \geq 3$, $s = 1$. Then the variety $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ has three irreducible components, namely $\mathcal{C}_j = \overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$, for $j = 1, 2, 5$. The status of $\mathcal{C}_1, \mathcal{C}_2$ being clear, we focus on the variety $\mathbf{Rep}\,\mathbb{S}^{(5)}$ with generic module G_5 as depicted at the end of Example 2.5(b). Again, we prove our claim regarding $\mathcal{C}_5$ via Theorem 4.5: to see that $\mathbb{S}^{(5)} = \mathbb{S}(G_5)$ is the only realizable semisimple sequence governing a filtration of G_5, we note that the only other realizable sequence not ruled out by Θ (i.e., with a Θ-value less than $\Theta(G_5)$) is $\mathbb{S}^{(1)}$. To verify, without computational effort, that $\mathbb{S}^{(1)}$ does not govern any filtration of G_5, it suffices to observe that, for any module N in $\mathbf{Filt}\,\mathbb{S}^{(1)}$, we have $S_1 \subseteq N/\Lambda x$ for some $x \in e_2 N$. On the other hand, it is readily checked that $S_1 \not\subseteq G_5/\Lambda x$ for all elements $x \in e_2 G_5$, which shows $\Gamma(G_5) = 1$ as required. Finally, to link up with the remarks preceding Example 6.1, we point out that $\Theta^+(G_1) < \Theta^+(G_5)$, whence the Θ^+-test fails to detect the status of $\overline{\mathbf{Rep}\,\mathbb{S}^{(5)}}$ as an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

To see that $\mathbb{S}^{(j)}$ for $j = 3, 4, 6$ do not arise as generic radical layerings of irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, one may follow the patterns of part (b).

(d) Moving to $r \geq 3$ and $s = 2$ raises the number of irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ to five. We first show that $\overline{\mathbf{Rep}\,\mathbb{S}^{(3)}}$ is now a component. Generically, the modules in $\mathbf{Rep}\,\mathbb{S}^{(3)}$ have hypergraph

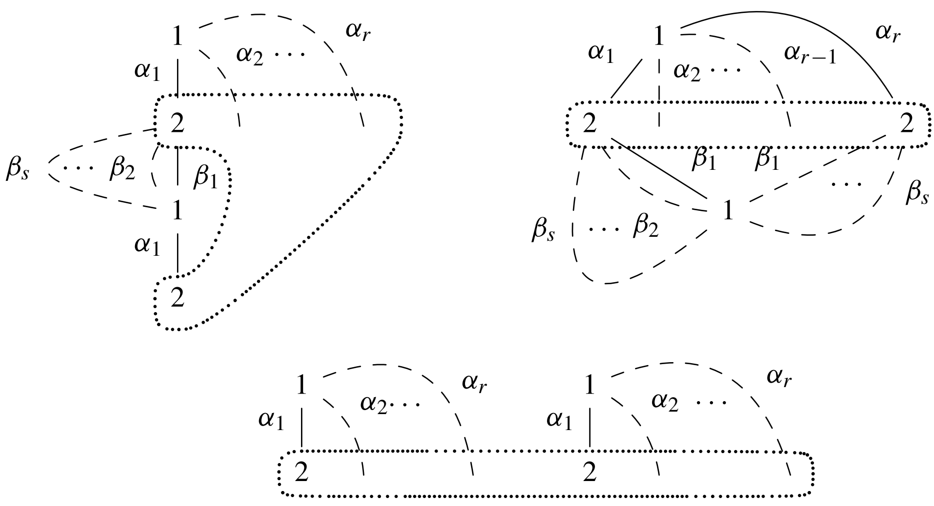


Again, the only $\overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ (for $j \le 6$) potentially containing $\mathbf{Rep}\,\mathbb{S}^{(3)}$ is $\overline{\mathbf{Rep}\,\mathbb{S}^{(1)}} = \mathbf{Filt}\,\mathbb{S}^{(1)}$. Since the modules in $\mathbf{Filt}\,\mathbb{S}^{(1)}$ clearly contain a copy of S_2 in their socle, while G_3 does not, this possibility is ruled out, and our claim is justified.

The discussion of $\overline{\mathbf{Rep}\,\mathbb{S}^{(4)}}$ is analogous, in that the only $\overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ (for $j \le 6$) potentially containing $\mathbf{Rep}\,\mathbb{S}^{(4)}$ is $\overline{\mathbf{Rep}\,\mathbb{S}^{(2)}} = \mathbf{Filt}\,\mathbb{S}^{(2)}$, and the modules in $\mathbf{Filt}\,\mathbb{S}^{(2)}$ contain a copy of S_1 in their socle, while a generic module for $\mathbf{Rep}\,\mathbb{S}^{(4)}$ does not.

As in part (c), one shows that $\overline{\mathbf{Rep}\,\mathbb{S}^{(5)}}$ is a component of $\mathbf{Rep}_d(\Lambda)$. On the other hand, $\overline{\mathbf{Rep}\,\mathbb{S}^{(6)}}$ still fails to be a component; the argument used in part (b) (in that case, to exclude $\overline{\mathbf{Rep}\,\mathbb{S}^{(5)}}$ from the list of components for $r = 2$) may now be applied to $s = 2$.

(e) Finally, let $r \ge 3$ and $s \ge 3$. Then all of the varieties $\overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, \dots, 6$ are irreducible components of $\mathbf{Rep}_d(\Lambda)$. The argument backing the status of $\mathbb{S}^{(6)}$ follows the reasoning we used to confirm $\overline{\mathbf{Rep}\,\mathbb{S}^{(5)}}$ as a component of $\mathbf{Rep}_d(\Lambda)$ in part (c). For $r = s = 3$, hypergraphs of generic modules for the components $\overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ for $j = 1, 3, 5$ are shown below:

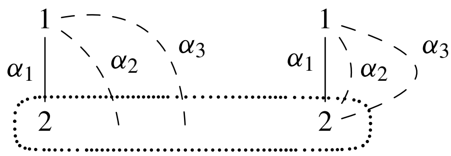


Due to symmetry, the generic structure of the modules in the remaining components is obtained by swapping the roles played by the vertices 1 and 2. $\square$

Consequences of the "truncated" theory, exemplified by Example 6.1.

(1) *Allocation of modules to the components.* Once the irreducible components $\overline{\mathbf{Rep}\,\mathbb{S}^{(j)}}$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ have been pinned down, by way of Theorem 4.5, say, one is in a position to list the components containing any given $\boldsymbol{d}$-dimensional Λ-module M. Indeed, compiling this list amounts to deciding which of the $\mathbb{S}^{(j)}$ govern filtrations of M; as was pointed out in Section 5B, there is an algorithm for carrying out this task.

In Example 6.1 with $r = 3$ and $s \geq 1$, for instance, any module M with hypergraph



belongs to the components $\mathcal{C}_1 = \mathbf{Filt}\,\mathbb{S}^{(1)}$ and $\mathcal{C}_5 = \mathbf{Filt}\,\mathbb{S}^{(5)}$, but does not have a filtration governed by $\mathbb{S}^{(j)}$ for $j \in \{2, 3, 4, 6\}$. Therefore, M belongs to precisely two of the irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$, namely to $\mathcal{C}_1$ and $\mathcal{C}_5$.

(2) *Comparing the generic behavior of the finite-dimensional Λ-modules to that of the finite-dimensional KQ-modules.* Examples 6.1(a)–(e) place a spotlight on the fact that, in the presence of oriented cycles, the generic representation theory of the path algebra KQ may be "disjoint" from that of its truncations in the following sense: for $r, s \geq 1$, we have $J(KQ) = 0$, and for $\boldsymbol{d} = (2, 2)$ the modules in the irreducible variety $\mathbf{Rep}_{\boldsymbol{d}}(KQ)$ are generically simple. Since generically the latter modules are not annihilated by any path in KQ, we find the variety $\mathbf{Rep}_{\boldsymbol{d}}(KQ/\langle\text{the paths of length } 4\rangle)$ to be contained in the boundary of a dense open subset of $\mathbf{Rep}_{\boldsymbol{d}}(KQ)$.

6B. ***Information on the components of $\mathbf{Rep}_{\boldsymbol{d}}(\boldsymbol{\Lambda})$ from those of $\mathbf{Rep}\,\boldsymbol{d}(\boldsymbol{\Lambda}_{\mathbf{trunc}})$.*** We conclude with a first installment of observations on how to pull information about the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ from knowledge of the components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$. Suppose that the distinct irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$ are

$$\overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(1)}} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(1)}), \quad \ldots, \quad \overline{\mathbf{Rep}_{\Lambda_{\text{trunc}}}\,\mathbb{S}^{(m)}} = \mathbf{Filt}_{\Lambda_{\text{trunc}}}(\mathbb{S}^{(m)}).$$

Moreover, suppose that $\mathcal{C}$ is an irreducible component of some $\mathbf{Rep}_{\Lambda}\,\mathbb{S}$ with generic module G (recall that, for any Λ, these components and their generic modules may be algorithmically accessed from quiver and relations of Λ). To compare with $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\text{trunc}})$, one first determines which among the $\mathbb{S}^{(j)}$ govern a filtration of G. Suppose the pertinent sequences are $\mathbb{S}^{(1)}, \ldots, \mathbb{S}^{(r)}$, that is, $\mathcal{C} \subseteq \mathbf{Filt}_{\Lambda}\,\mathbb{S}^{(j)}$ precisely when $j \leq r$.

Observation 6.2. The closure $\overline{\mathcal{C}}$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ if and only if $\overline{\mathcal{C}}$ is maximal irreducible in $\mathbf{Filt}_{\Lambda}\,\mathbb{S}^{(j)}$ for all $j \leq r$.

Proof. The claim is immediate from the fact that every irreducible subvariety $\mathcal{D}$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ which contains $\overline{\mathcal{C}}$ is contained in one of the intersections

$$\mathbf{Rep}_{\boldsymbol{d}}(\Lambda) \cap \mathbf{Filt}_{\Lambda_{\mathrm{trunc}}} \mathbb{S}^{(j)} = \mathbf{Filt}_{\Lambda} \mathbb{S}^{(j)}. \qquad \square$$

This leads to a lower bound for the number of irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$. Computing it in specific instances typically requires a nonnegligible effort, as it is not simply based on the *number* of components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\mathrm{trunc}})$. The bound is sharp in general. Indeed, if Δ denotes the algebra of Example 6.1(e) and $\Lambda = \Delta/\langle \beta_i \alpha_j \beta_k \mid i, j, k \in \{1, 2, 3\}\rangle$, then $\Delta = \Lambda_{\mathrm{trunc}}$ and the number of irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ coincides with the lower bound given below.

Corollary 6.3. *Again, let* $\boldsymbol{d}$ *be a dimension vector of a basic* K*-algebra* Λ, *and adopt the above notation for the irreducible components of* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\mathrm{trunc}})$. *Moreover, set*

$$A_j := \mathbf{Rep}_{\boldsymbol{d}}(\Lambda_{\mathrm{trunc}}) \setminus \bigcup_{\substack{i \le m \\ i \ne j}} \mathbf{Filt}_{\Lambda_{\mathrm{trunc}}} \mathbb{S}^{(i)} \quad \text{for } j \le m.$$

Then the number of irreducible components of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ *is bounded from below by the number of* A_j *which have nonempty intersection with* $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$.

Proof. Suppose $A_1, \dots, A_s$ are the A_j which intersect $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ nontrivially, and let $\mathfrak{U}_j$ be an irreducible subvariety of $A_j \cap \mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ for $j \le s$. Among the $\mathbf{Filt}_{\Lambda_{\mathrm{trunc}}} \mathbb{S}^{(i)}$, the variety $\mathbf{Filt}\, \mathbb{S}^{(j)}$ is then the only one to contain $\mathfrak{U}_j$. Consequently, any maximal irreducible subset $\mathcal{D}_j$ of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ containing $\mathfrak{U}_j$ is an irreducible component of $\mathbf{Rep}_{\boldsymbol{d}}(\Lambda)$ by the preceding observation. By construction, the resulting $\mathcal{D}_j$ are pairwise different. $\square$

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Sparsity of p-divisible unramified liftings for subvarieties of abelian varieties with trivial stabilizer

Danny Scarponi

By means of the theory of strongly semistable sheaves and the theory of the Greenberg transform, we generalize to higher dimensions a result on the sparsity of p-divisible unramified liftings which played a crucial role in Raynaud's proof of the Manin–Mumford conjecture for curves. We also give a bound for the number of irreducible components of the first critical scheme of subvarieties of an abelian variety which are complete intersections.

1. Introduction

The Manin–Mumford conjecture is a significant question concerning the intersection of a subvariety X of an abelian variety A with the group of torsion points of A. Raised independently by Manin and Mumford, the conjecture was originally formulated in the case of curves. Suppose that A is an abelian variety over a number field K and that C is a smooth subcurve of A of genus at least two. Then only finitely many torsion points of $A(\bar{K})$ lie in C. In 1983, Raynaud proved this conjecture and generalized it to higher dimensions: if A/K is as above and X/K is a smooth subvariety of A which does not contain any translate of a nontrivial abelian subvariety, then the set of torsion points of $A(\bar{K})$ lying in X is finite [Raynaud 1983b; 1983c].

Let us fix K, X and A as above. Let U be a nonempty open subscheme of Spec $\mathcal{O}_K$ not containing any ramified primes and such that A/K extends to an abelian scheme $\mathcal{A}/U$ and X extends to a smooth closed integral subscheme $\mathcal{X}$ of $\mathcal{A}$. For any $\mathfrak{p} \in U$, let R and R_n be the ring of Witt vectors and Witt vectors of length $n+1$, respectively, with coordinates in the algebraic closure $\overline{k(\mathfrak{p})}$ of the residue field of $\mathfrak{p}$. Recall that R is a DVR with maximal ideal generated by p such that $R_0 = R/p = \overline{k(\mathfrak{p})}$. Denote by $X_{\mathfrak{p}^n}$ and $A_{\mathfrak{p}^n}$ the R_n-schemes $\mathcal{X} \times_U \operatorname{Spec} R_n$ and $\mathcal{A} \times_U \operatorname{Spec} R_n$, respectively, and consider the reduction map

$$pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \to X_{\mathfrak{p}^0}(R_0). \tag{1}$$

In [Raynaud 1983b] it was shown that, if X is a curve, the image of (1) is not Zariski dense in $X_{\mathfrak{p}^0}$, i.e., it is a finite set. This local result is crucial in Raynaud's proof of the Manin–Mumford conjecture for curves, since it easily implies that only finitely many prime-to-p torsion points of $A(\bar{K})$ lie on X [Raynaud 1983b, Théorème II].

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It is quite natural to expect that a similar result also holds in higher dimensions. More explicitly, one can ask: is it true that, if a smooth subvariety X of A does not contain any translate of a nontrivial abelian subvariety, the image of (1) is not Zariski dense? In this paper we give a positive answer to this question (see Theorem 5.3).

Theorem 1.1 (sparsity of p-divisible unramified liftings). *Suppose that X has trivial stabilizer. For all $\mathfrak{p} \in U$ above a prime $p > (\dim X)^2 \deg(\Omega_X)$ such that $X_{\mathfrak{p}^0}$ has trivial stabilizer, the image of*

$$pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \to X_{\mathfrak{p}^0}(R_0)$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

Here $\deg(\Omega_X)$ refers to the degree of the cotangent bundle Ω_X computed with respect to any fixed very ample line bundle on X.

Notice that if X does not contain any translate of a nontrivial abelian subvariety, then it has finite stabilizer. Therefore, replacing A and X with their quotients by the stabilizer of X, one can assume the stabilizer is trivial (see the beginning of the next section for the definition of stabilizer).

A different generalization of Raynaud's local result was given by Rössler [2013] who proved that, if the torsion points of $\mathcal{A}(\overline{\mathrm{Frac}(R)})$ are not dense in $\mathcal{X}(\overline{\mathrm{Frac}(R)})$, then for m big enough the image of

$$p^m A_{\mathfrak{p}^m}(R_m) \cap X_{\mathfrak{p}^m}(R_m) \to X_{\mathfrak{p}^0}(R_0) \tag{2}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$ [Rössler 2013, Theorem 4.1]. Theorem 1.1 makes Rössler's result effective, showing that if the stabilizer of X is trivial, then it is sufficient to consider the map (2) for $m = 1$.

The proof of Theorem 1.1 strongly relies on Rössler's paper [2016] and is done by contradiction. First we use some basic properties of the Greenberg transform to show that, if the image of (1) is Zariski dense in $X_{\mathfrak{p}^0}$, the absolute Frobenius $F_{X_{\mathfrak{p}^0}} : X_{\mathfrak{p}^0} \to X_{\mathfrak{p}^0}$ lifts to an endomorphism of $X_{\mathfrak{p}^1}$. A well-known consequence of this liftability is the existence of a map of sheaves of differentials $F^*_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}} \to \Omega_{X_{\mathfrak{p}^0}}$ which is nonzero. If X is a curve, such a map cannot exist, since $\deg(F^*_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}})$ is strictly bigger than $\deg(\Omega_{X_{\mathfrak{p}^0}})$. This simple observation was in fact used by Raynaud to prove Lemma I.5.4 in [Raynaud 1983a]. By means of the theory of strongly semistable sheaves developed by Rössler [2016], we show that when X has dimension higher than one, there are no nontrivial maps from $F^*_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}}$ to $\Omega_{X_{\mathfrak{p}^0}}$. This gives us the wanted contradiction.

In the last section of this paper, we consider subvarieties of abelian varieties which are complete intersections. If Gr_1 denotes the Greenberg transform of level 1 (see Section 3), then we know that the first critical scheme

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) := [p]_* \, \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \cap \mathrm{Gr}_1(X_{\mathfrak{p}^1})$$

is a scheme over R_0 such that

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0) = pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1).$$

Using exactly the same technique that allowed Buium [1996] to give an effective form of the Manin–Mumford conjecture in the case of curves, we get a bound for the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ when X is a complete intersection (not necessarily with trivial stabilizer).

Theorem 1.2. *Let K be a number field, A/K be an abelian variety of dimension n and let L be a very ample line bundle on A. Let $c \in \mathbb{N}$ be positive and let $H_1, H_2, \dots, H_c \in |L|$ be general. Suppose that $X := H_1 \cap H_2 \cap \cdots \cap H_c$ is smooth. There exists a nonempty open subscheme $V \subseteq \operatorname{Spec} \mathcal{O}_K$ (see the beginning of Section 6 for its definition) such that if $\mathfrak{p} \in V$, the number of irreducible components of* $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ *is bounded by*

$$p^{2n}\left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\right)(L^n)^2.$$

Here (L^n) denotes the intersection number of L.

We conclude the introduction with the following remark. Since the field of definition of points in the prime-to-p torsion $\mathrm{Tor}^p(A(\bar{K}))$ is unramified at $\mathfrak{p}$ and the specialization map $\mathcal{A}(R) \to A_{\mathfrak{p}^1}(R_1)$ is injective on the prime-to-p torsion, we have an injection

$$\mathrm{Tor}^p(A(\bar{K})) \cap X(\bar{K}) \subseteq pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1).$$

This implies that, if X is a complete intersection such that $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0)$ is finite, then the bound in Theorem 1.2 is a bound for the cardinality of $\mathrm{Tor}^p(A(\bar{K})) \cap X(\bar{K})$, i.e., an effective form of the Manin–Mumford conjecture for the prime-to-p torsion.

2. Notations

We fix the following notations

- K a number field,
- $\bar{K}$ an algebraic closure of K,
- A/K an abelian variety,
- $X \subseteq A$ a closed integral subscheme, smooth over K,
- $\mathrm{Stab}_A(X)$ the translation stabilizer of X in A, i.e., the closed subgroup scheme of A characterized uniquely by the fact that for any K-scheme S and any morphism $b : S \to A$, translation by b on the product $A \times_K S$ maps the subscheme $X \times_K S$ to itself if and only if b factors through $\mathrm{Stab}_A(X)$ (for its existence we refer the reader to [SGA 3_{II} 1970, Exemple 6.5(e), Expose VIII]),
- U an open subscheme of $\operatorname{Spec} \mathcal{O}_K$ not containing any ramified prime and such that A/K extends to an abelian scheme $\mathcal{A}/U$ and X extends to a smooth closed integral subscheme $\mathcal{X}$ of $\mathcal{A}$.

For any prime number p, any unramified prime $\mathfrak{p}$ of K above p and any $n \geq 0$, we denote by

- $k(\mathfrak{p})$ the residue field $\mathcal{O}_K/\mathfrak{p}$ for $\mathfrak{p}$,

- $K_{\mathfrak{p}}$ the completion of K with respect to $\mathfrak{p}$,
- $\widehat{K_{\mathfrak{p}}^{\mathrm{unr}}}$ the completion of the maximal unramified extension of $K_{\mathfrak{p}}$,
- $R := W(\overline{k(\mathfrak{p})})$ and $R_n := W_n(\overline{k(\mathfrak{p})})$ the ring of Witt vectors and the ring of Witt vectors of length $n+1$, respectively, with coordinates in $\overline{k(\mathfrak{p})}$. We recall that R can be identified with the ring of integers of $\widehat{K_{\mathfrak{p}}^{\mathrm{unr}}}$ and R_0 with $\overline{k(\mathfrak{p})}$,
- $X_{\mathfrak{p}^n}$ the R_n-scheme $\mathcal{X} \times_U \operatorname{Spec} R_n$ $A_{\mathfrak{p}^n}$ the R_n-scheme $\mathcal{A} \times_U \operatorname{Spec} R_n$.

3. The Greenberg transform and the critical schemes

Now we recall some basic facts about the Greenberg transform (for more details, see [Greenberg 1961; 1963; Bosch et al. 1990, pp. 276–277]).

Fix a prime number p and an unramified prime $\mathfrak{p}$ of K above p.

For any $n \geq 0$, the Greenberg transform of level n is a covariant functor Gr_n from the category of R_n-schemes locally of finite type, to the category of R_0-schemes locally of finite type. If Y_n is an R_n-scheme locally of finite type, $\mathrm{Gr}_n(Y_n)$ is a R_0-scheme with the property

$$Y_n(R_n) = \mathrm{Gr}_n(Y_n)(R_0).$$

More precisely, we can interpret R_n as the set of $\overline{k(\mathfrak{p})}$-valued points of a ring scheme $\mathcal{R}_n$ over $\overline{k(\mathfrak{p})}$. For any R_0-scheme T, we define $\mathbb{W}_n(T)$ as the ringed space over R_n consisting of T as a topological space and of $\mathrm{Hom}_{R_0}(T, \mathcal{R}_n)$ as a structure sheaf. By definition $\mathrm{Gr}_n(Y_n)$ represents the functor from the category of schemes over R_0 to the category of sets given by

$$T \mapsto \mathrm{Hom}_{R_n}(\mathbb{W}_n(T), Y_n)$$

where Hom stands for homomorphisms of ringed spaces. In other words, the functor Gr_n is right adjoint to the functor $\mathbb{W}_n$.

The functor Gr_n respects closed immersions, open immersions, fiber products, smooth, étale morphisms and is the identity for $n = 0$. Furthermore it sends group schemes over R_n to group schemes over R_0. The canonical morphism $R_{n+1} \to R_n$ gives rise to a functorial transition morphism $\pi_{n+1} : \mathrm{Gr}_{n+1} \to \mathrm{Gr}_n$.

Let Y_n be a scheme over R_n locally of finite type. Then for any $m < n$ we define

$$Y_m := Y_n \times_{R_n} R_m.$$

Let us call $F_{Y_0} : Y_0 \to Y_0$ the absolute Frobenius endomorphism of Y_0 and Ω_{Y_0/R_0} the sheaf of relative differentials.

For any finite rank locally free sheaf $\mathcal{F}$ over Y_0 we will write

$$V(\mathcal{F}) := \underline{\mathrm{Spec}}(\mathrm{Sym}(\mathcal{F}^{\vee}))$$

for the vector bundle over Y_0 associated to $\mathcal{F}$.

Suppose now that Y_n is smooth over R_n, so that Ω_{Y_0/R_0} is locally free. A key result about the Greenberg transform is the following fact [Greenberg 1963, Section 2]:

$$\pi_1 : \mathrm{Gr}_1(Y_1) \to \mathrm{Gr}_0(Y_0) = Y_0$$

is a torsor under the Frobenius tangent bundle

$$V(F_{Y_0}^* \Omega_{Y_0/R_0}^\vee).$$

Let X, A, $\mathcal{X}$, $\mathcal{A}$ and U be as fixed in the previous section and suppose that $\mathfrak{p} \in U$. We refer the reader to Section II.1 in [Raynaud 1983a] for more details on what we will recall from now till the end of the section. For any $n \geq 0$, the kernel of

$$\mathrm{Gr}_n(A_{\mathfrak{p}^n}) \to \mathrm{Gr}_0(A_{\mathfrak{p}^0}) = A_{\mathfrak{p}^0}$$

is unipotent, killed by p^n. Thus, the scheme-theoretic image $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ of multiplication by p^n in $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$ is the greatest abelian subvariety of $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$ and, since R_0 is algebraically closed, $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})(R_0) = p^n \mathrm{Gr}_n(A_{\mathfrak{p}^n})(R_0)$.

We define the n-critical scheme as

$$\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) := [p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n}) \cap \mathrm{Gr}_n(X_{\mathfrak{p}^n}).$$

Notice that $\mathrm{Crit}^n(\mathcal{X}, \mathcal{A})$ is a scheme over R_0 and that $\mathrm{Crit}^0(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$.

The transition morphisms $\pi_{n+1} : \mathrm{Gr}_{n+1}(A_{\mathfrak{p}^{n+1}}) \to \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ lead to a projective system of R_0-schemes

$$\cdots \to \mathrm{Crit}^2(\mathcal{X}, \mathcal{A}) \to \mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) \to \mathrm{Crit}^0(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0},$$

whose connecting morphisms are both affine and proper, hence finite. In fact, transition morphisms are affine and the subscheme $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ is proper, being the greatest abelian subvariety of $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$.

We shall write $\mathrm{Exc}^n(\mathcal{X}, \mathcal{A})$ for the scheme theoretic image of the morphism $\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) \to X_{\mathfrak{p}^0}$.

4. The geometry of vector bundles in positive characteristic

In this section we recall some results on the geometry of vector bundles in positive characteristic by Langer [2004] and Rössler [2016]. These results will play a crucial role in the proof of Lemma 5.1 and Theorem 5.3.

Let us start with some basic definitions and facts regarding semistable sheaves in positive characteristic.

Let Y be a smooth projective variety over an algebraically closed field l_0 of positive characteristic. We write as before Ω_{Y/l_0} for the sheaf of differentials of Y over l_0 and $F_Y : Y \to Y$ for the absolute Frobenius endomorphism of Y. Now let L be a very ample line bundle on Y. If V is a torsion free coherent sheaf on Y, we shall write

$$\mu(V) = \mu_L(V) = \deg_L(V) / \mathrm{rk}(V)$$

for the slope of V (with respect to L). Here $\mathrm{rk}(V)$ is the rank of V, i.e., the dimension of the stalk of V at the generic point of Y. Furthermore,

$$\deg_L(V) := \int_Y c_1(V)\cdot c_1(L)^{\dim(Y)-1}$$

where $c_1(\cdot)$ refers to the first Chern class with values in an arbitrary Weil cohomology theory and the integral $\int_Y$ stands for the push-forward morphism to $\operatorname{Spec} l_0$ in that theory. Recall that V is called semistable (with respect to L) if for every coherent subsheaf W of V, we have $\mu(W) \le \mu(V)$ and it is called strongly semistable if $F_Y^{n,*}V$ is semistable for all $n \ge 0$.

In general, there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

of V by subsheaves, such that the quotients V_i/V_{i-1} are all semistable and such that the slopes $\mu(V_i/V_{i-1})$ are strictly decreasing for $i \ge 1$. This filtration is unique and is called the Harder–Narasimhan (HN) filtration of V. We will say that V has a strongly semistable HN filtration if all the quotients V_i/V_{i-1} are strongly semistable. We shall write

$$\mu_{\min}(V) := \mu(V_r/V_{r-1}) \quad \text{and} \quad \mu_{\max}(V) := \mu(V_1).$$

By the very definition of HN filtration, we have

$$V \text{ is semistable } \Leftrightarrow \mu_{\min}(V) = \mu_{\max}(V).$$

An important consequence of the definitions is the following fact; if V and W are two torsion free sheaves on Y and $\mu_{\min}(V) > \mu_{\max}(W)$, then $\mathrm{Hom}_Y(V, W) = 0$.

For more on the theory of semistable sheaves, see the monograph [Huybrechts and Lehn 2010].

The following two theorems are key results from Langer.

Theorem 4.1 [Langer 2004, Theorem 2.7]. *If V is a torsion free coherent sheaf on Y, then there exists $n_0 \ge 0$ such that $F_Y^{n,*}V$ has a strongly semistable HN filtration for all $n \ge n_0$.*

If V is a torsion free coherent sheaf on Y, we now define

$$\bar{\mu}_{\min}(V) := \lim_{r\to\infty} \mu_{\min}(F_Y^{r,*}V)\,\mathrm{char}(l_0)^r \quad \text{and} \quad \bar{\mu}_{\max}(V) := \lim_{r\to\infty} \mu_{\max}(F_Y^{r,*}V)/\,\mathrm{char}(l_0)^r.$$

Note that Theorem 4.1 implies that the two sequences $\mu_{\min}(F_Y^{r,*}V)/\,\mathrm{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)/\,\mathrm{char}(l_0)^r$ become constant when r is sufficiently large, so the above definitions of $\bar{\mu}_{\min}$ and $\bar{\mu}_{\max}$ make sense. Furthermore the sequences $\mu_{\min}(F_Y^{r,*}V)\,\mathrm{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)\,\mathrm{char}(l_0)^r$ are respectively weakly decreasing and weakly increasing, therefore we have

$$\mu_{\min}(V) \ge \bar{\mu}_{\min}(V) \quad \text{and} \quad \bar{\mu}_{\max}(V) \ge \mu_{\max}(V).$$

Let us define

$$\alpha(V) := \max\{\mu_{\min}(V) - \bar{\mu}_{\min}(V), \bar{\mu}_{\max}(V) - \mu_{\max}(V)\}.$$

Theorem 4.2 [Langer 2004, Cororollary 6.2]. *If V is of rank r, then*

$$\alpha(V) \leq \frac{r-1}{\mathrm{char}(l_0)} \max\{\bar{\mu}_{\max}(\Omega_{Y/l_0}), 0\}.$$

In particular, if $\bar{\mu}_{\max}(\Omega_{Y/l_0}) \geq 0$ and $\mathrm{char}(l_0) \geq d = \dim Y$,

$$\bar{\mu}_{\max}(\Omega_{Y/l_0}) \leq \frac{\mathrm{char}(l_0)}{\mathrm{char}(l_0)+1-d} \mu_{\max}(\Omega_{Y/l_0}).$$

We conclude this section with the following two lemmas from Rössler.

Lemma 4.3 [Rössler 2016, Lemma 3.8]. *Suppose that there is a closed l_0-immersion $i : Y \hookrightarrow B$, where B is an abelian variety over l_0. Suppose that $\mathrm{Stab}_B(Y) = 0$. Then $\Omega_Y^\vee$ is globally generated and for any dominant proper morphism $\phi : Y_0 \to Y$, where Y_0 is integral, we have $H^0(Y_0, \phi^*\Omega_Y^\vee) = 0$. Furthermore, we have $\bar{\mu}_{\min}(\Omega_Y) > 0$.*

Lemma 4.4 [Rössler 2016, Cororollary 3.11]. *Let V be a finite rank, locally free sheaf over Y. Suppose that*

- *for any surjective finite map $\phi : Y' \to Y$ with Y' integral, we have $H^0(Y', \phi^*V) = 0$,*
- *$V^\vee$ is globally generated.*

Then $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for n sufficiently big.

Furthermore, let $T \to Y$ be a torsor under $V(F_Y^{n_0,*}V)$, where n_0 satisfies $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for all $n > n_0$. Let $\phi : Y' \to Y$ be a finite surjective morphism and suppose that Y' is integral. Then we have the implication

$$\phi^*T \text{ is a trivial } V(\phi^*(F_Y^{n_0,*}V))\text{-torsor} \Longrightarrow T \text{ is a trivial } V(F_Y^{n_0,*}V)\text{-torsor.}$$

The main ingredient of the proof of Lemma 4.4 is a result by Szpiro and Lewin-Ménégaux which we will need later.

Proposition 4.5 [Szpiro 1981, Expose 2, Proposition 1]. *If V is a vector bundle over Y such that $H^0(Y, F_Y^*V \otimes \Omega_{Y/l_0}) = 0$, then the map*

$$H^1(Y, V) \to H^1(Y, F_Y^*V)$$

is injective.

5. Sparsity of p-divisible unramified liftings

In this section we prove our result on the sparsity of p-divisible unramified liftings (see Theorem 5.3).

Let K, A, X and U be as fixed in Section 2 and let $\mathrm{Stab}_A(X)$ be trivial. The construction of the stabilizer commutes with the base change, so we have

$$\mathrm{Stab}_A(X) = \mathrm{Stab}_{\mathcal{A}}(\mathcal{X}) \times_U \mathrm{Spec}\, K.$$

Since $\mathrm{Stab}_A(X)$ is trivial, by generic flatness and finiteness, we can restrict the map $\pi : \mathrm{Stab}_{\mathcal{A}}(\mathcal{X}) \to U$ to the inverse image of a nonempty open subscheme $U' \subset U$ to obtain a finite flat commutative group scheme of degree one

$$\pi_{|\pi^{-1}(U')} : \pi^{-1}(U') \to U'.$$

This implies that $\pi_{|\pi^{-1}(U')}$ is an isomorphism and for any $\mathfrak{q} \in U'$ we have that $\mathrm{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0})$ is trivial. We will denote by $\tilde{U} \subseteq U$ the nonempty open subscheme

$$\tilde{U} := \{\mathfrak{q} \in U \mid \mathrm{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0}) \text{ is trivial}\}.$$

For any $\mathfrak{p} \in U$ we denote by $F_{\overline{k(\mathfrak{p})}}$ the Frobenius endomorphism on $\overline{k(\mathfrak{p})}$ and by F_{R_1} the endomorphism of R_1 induced by $F_{\overline{k(\mathfrak{p})}}$ by functoriality. We define

$$X'_{\mathfrak{p}^0} := X_{\mathfrak{p}^0} \times_{F_{\overline{k(\mathfrak{p})}}} \overline{k(\mathfrak{p})} \quad \text{and} \quad X'_{\mathfrak{p}^1} := X_{\mathfrak{p}^1} \times_{F_{R_1}} R_1$$

and we write

$$F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}} : X_{\mathfrak{p}^0} \to X'_{\mathfrak{p}^0}$$

for the relative Frobenius on $X_{\mathfrak{p}^0}$. For brevity's sake, from now on we will write

$$\Omega_{X_{\mathfrak{p}^0}}, \Omega_{X'_{\mathfrak{p}^0}}, \Omega_{X_{\mathfrak{p}^1}}, \Omega_{X'_{\mathfrak{p}^1}} \text{ and } \Omega_X$$

instead of

$$\Omega_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}, \Omega_{X'_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}, \Omega_{X_{\mathfrak{p}^1}/R_1}, \Omega_{X'_{\mathfrak{p}^1}/R_1} \text{ and } \Omega_{X/K}.$$

Observe that since U is normal, $\mathcal{A}$ is projective over U [Raynaud 1970, Theorem XI 1.4]. Therefore there exists a U-very ample line bundle L on $\mathcal{X}$. For any $\mathfrak{p} \in U$ different from the generic point ξ, let us denote by $L_{\mathfrak{p}}$ the inverse image of L on $X_{\mathfrak{p}^0}$. Similarly we denote by L_ξ the inverse image of L on X. From now on, for any vector bundle $G_{\mathfrak{p}}$ over $X_{\mathfrak{p}^0}$, we will write $\deg(G_{\mathfrak{p}})$ for the degree of $G_{\mathfrak{p}}$ with respect to $L_{\mathfrak{p}}$. Analogously, if G_ξ is a vector bundle over X, we will write $\deg(G_\xi)$ for the degree of G_ξ with respect to L_ξ. Now consider the vector bundle $\Omega_{\mathcal{X}/U}$ over $\mathcal{X}$. For any natural number m, the map from U to $\mathbb{Z}$ defined by

$$\mathfrak{p} \mapsto \chi((\Omega_{\mathcal{X}/U} \otimes L^m)_{\mathfrak{p}}) = \chi(\Omega_{X_{\mathfrak{p}^0}} \otimes L^m_{\mathfrak{p}}) \quad \text{and} \quad \xi \mapsto \chi((\Omega_{\mathcal{X}/U} \otimes L^m)_\xi) = \chi(\Omega_X \otimes L^m_\xi)$$

(here χ refers to the Euler characteristic) is constant on U [Mumford 1970, Chapter II, Section 5]. Therefore we have the equality

$$\chi(\Omega_{X_{\mathfrak{p}^0}} \otimes L^m_{\mathfrak{p}}) = \chi(\Omega_X \otimes L^m_\xi)$$

for all $m \in \mathbb{N}$ and for all $\mathfrak{p} \in U$. In other words, the Hilbert polynomial of $\Omega_{X_{\mathfrak{p}^0}}$ with respect to $L_{\mathfrak{p}}$ coincides with the Hilbert polynomial of Ω_X with respect to L_ξ. Since the degree of a vector bundle we defined at the beginning of this section can be described in terms of its Hilbert polynomial [Huybrechts and Lehn 2010, Definition 1.2.11], we obtain that for every $\mathfrak{p} \in U$ we have $\deg(\Omega_{X_{\mathfrak{p}^0}}) = \deg(\Omega_X)$.

The following lemma is a fundamental step to prove our sparsity Theorem 5.3.

Lemma 5.1. *Let K, A, X and U be as fixed in Section 2, let $\mathrm{Stab}_A(X)$ be trivial and let n be the dimension of X over K. Then*

$$\mathrm{Hom}_{X_{\mathfrak{p}0}}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}, \Omega_{X_{\mathfrak{p}0}}) = 0$$

for any $k \geq 1$ and any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Let us notice first that, if $n = 1$, then X is a curve of genus g at least 2 and

$$\mathrm{Hom}_{X_{\mathfrak{p}0}}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}, \Omega_{X_{\mathfrak{p}0}}) = 0$$

is a simple consequence of the fact

$$\deg(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) = p^k(2g-2) > 2g-2 = \deg \Omega_{X_{\mathfrak{p}0}}.$$

To treat the general case, let us fix $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$. We know that if

$$\mu_{\min}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) > \mu_{\max}(\Omega_{X_{\mathfrak{p}0}})$$

then $\mathrm{Hom}_{X_{\mathfrak{p}0}}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}, \Omega_{X_{\mathfrak{p}0}}) = 0$. Since $\mu_{\min} \geq \bar{\mu}_{\min}$ and $\bar{\mu}_{\max} \geq \mu_{\max}$, it is sufficient to show that, for every $k \geq 1$

$$\bar{\mu}_{\min}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) > \bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{3}$$

Since $\mathrm{Stab}_{A_{\mathfrak{p}0}}(X_{\mathfrak{p}0})$ is trivial, we can apply Lemma 4.3 to obtain $\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}}) > 0$. In particular $\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) > 0$ and $\deg(\Omega_{X_{\mathfrak{p}0}}) > 0$. Using this and the equality $\bar{\mu}_{\min}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) = p^k\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}})$, we see that (3) is implied by

$$p\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}}) > \bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{4}$$

Theorem 4.2 gives us the following inequality

$$p\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}}) \geq p\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) + (1-n)\bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}),$$

so that (4) is satisfied if

$$p\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) > n\bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{5}$$

Since $p > n^2 \deg(\Omega_X) \geq n$, we can apply the second part of Theorem 4.2

$$\bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}) \leq \frac{p}{p+1-n}\mu_{\max}(\Omega_{X_{\mathfrak{p}0}}),$$

so that inequality (5) is implied by

$$(p+1-n)\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) > n\mu_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{6}$$

If $\Omega_{X_{\mathfrak{p}0}}$ is semistable, (6) gives $p > 2n-1$. Otherwise, we can estimate $\mu_{\max}(\Omega_{X_{\mathfrak{p}0}})$ and $\mu_{\min}(\Omega_{X_{\mathfrak{p}0}})$ in the following way. We know that

$$\mu_{\max}(\Omega_{X_{\mathfrak{p}0}}) = \frac{\deg(M)}{\mathrm{rk}(M)}$$

for some subsheaf $0 \neq M \subsetneq \Omega_{X_{\mathfrak{p}^0}}$. Therefore we have $\mu_{\max}(\Omega_{X_{\mathfrak{p}^0}}) \leq \deg(M)$. Furthermore, since $\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}^0}}) > 0$, we have that $\deg(\Omega_{X_{\mathfrak{p}^0}}/M) > 0$. This and the additivity of the degree on short exact sequences gives us

$$\mu_{\max}(\Omega_{X_{\mathfrak{p}^0}}) \leq \deg(M) \leq \deg(\Omega_{X_{\mathfrak{p}^0}}) - 1.$$

Similarly,

$$\mu_{\min}(\Omega_{X_{\mathfrak{p}^0}}) = \frac{\deg(Q)}{\mathrm{rk}(Q)}$$

for some Q quotient of $\Omega_{X_{\mathfrak{p}^0}}$, so $\mu_{\min}(\Omega_{X_{\mathfrak{p}^0}}) \geq 1/n$. Inequality (6) is then implied by

$$p > n^2 \deg(\Omega_{X_{\mathfrak{p}^0}}) + (n - 1 - n^2).$$

Since $n - 1 - n^2$ is always negative, we are reduced to $p > n^2 \deg(\Omega_{X_{\mathfrak{p}^0}})$. Now $\deg(\Omega_{X_{\mathfrak{p}^0}})$ is greater or equal to one, so $n^2 \deg(\Omega_{X_{\mathfrak{p}^0}}) \geq 2n - 1$ for any n. This ensures us that the condition

$$p > n^2 \deg(\Omega_{X_{\mathfrak{p}^0}})$$

is sufficient to have $\mu_{\min}(F^{k,*}_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}}) > \mu_{\max}(\Omega_{X_{\mathfrak{p}^0}})$ for every $k \geq 1$ whether $\Omega_{X_{\mathfrak{p}^0}}$ is semistable or not. To conclude it is enough to remember that $\deg(\Omega_{X_{\mathfrak{p}^0}})$ coincides with $\deg(\Omega_X)$. $\square$

Corollary 5.2. *The map*

$$H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \to H^1(X_{\mathfrak{p}^0}, F^{k,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}})$$

is injective for every $k \geq 1$ and every $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Lemma 5.1 and Proposition 4.5 imply that

$$H^1(X_{\mathfrak{p}^0}, F^{h,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \to H^1(X_{\mathfrak{p}^0}, F^{h+1,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}})$$

is injective for every $h \geq 0$. Therefore the composition

$$H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \hookrightarrow H^1(X_{\mathfrak{p}^0}, F^{2,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \hookrightarrow \cdots \hookrightarrow H^1(X_{\mathfrak{p}^0}, F^{k,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{q}^0}})$$

is an injective map. $\square$

We are now ready to prove our sparsity result.

Theorem 5.3. *With the same hypotheses as in Lemma 5.1, for any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$, the set*

$$\{P \in X_{\mathfrak{p}^0}(R_0) \mid P \textit{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

Proof. Let us fix $\mathfrak{p}$ as in the hypotheses. Since

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0) = pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1),$$

we have that

$$\{P \in X_{\mathfrak{p}^0}(R_0) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)\}$$

coincides with the image of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0) \to X_{\mathrm{p}^0}(R_0)$.

Let us assume by contradiction that this image is dense in $X_{\mathrm{p}^0}(R_0)$. This implies that $\pi_1 : \mathrm{Gr}_1(X_{\mathrm{p}^1}) \to X_{\mathrm{p}^0}$ is a trivial torsor; the argument we use to show this is taken from Rössler (see the beginning of the proof of Theorem 2.2 in [Rössler 2016]). First of all the closed map $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) \to X_{\mathrm{p}^0}$ is surjective and so we can choose an irreducible component

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})_0 \hookrightarrow \mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$$

which dominates X_{p^0}. Lemmas 4.3 and 5.1 allow us to apply the second part of Lemma 4.4 with $V = \Omega^\vee_{X_{\mathrm{p}^0}}$, $Y = X_{\mathrm{p}^0}$, $n_0 = 1$, $T = \mathrm{Gr}_1(X_{\mathrm{p}^1})$ and ϕ equal to $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})_0 \to X_{\mathrm{p}^0}$. We have that $\phi^* \mathrm{Gr}_1(X_{\mathrm{p}^1})$ is trivial as a $V(\phi^* F^*_{X_{\mathrm{p}^0}} \Omega^\vee_{X_{\mathrm{p}^0}})$-torsor, since $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})_0$ is contained in $\mathrm{Gr}_1(X_{\mathrm{p}^1})$. Hence $\pi_1 : \mathrm{Gr}_1(X_{\mathrm{p}^1}) \to X_{\mathrm{p}^0}$ is trivial as a $V(F^*_{X_{\mathrm{p}^0}} \Omega^\vee_{X_{\mathrm{p}^0}})$-torsor. Let us take a section $\sigma : X_{\mathrm{p}^0} \to \mathrm{Gr}_1(X_{\mathrm{p}^1})$. By the definition of the Greenberg transform, the map σ over R_0 corresponds to a map $\bar{\sigma} : \mathbb{W}_1(X_{\mathrm{p}^0}) \to X_{\mathrm{p}^1}$ over R_1. We can precompose $\bar{\sigma}$ with the morphism $t : X_{\mathrm{p}^1} \to \mathbb{W}_1(X_{\mathrm{p}^0})$ corresponding to

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathrm{p}^0}}) &\to \mathcal{O}_{X_{\mathrm{p}^1}} \\ (a_0, a_1) &\mapsto \tilde{a}_0^p + \tilde{a}_1 p, \end{aligned}$$

where $\tilde{a}_i$ lifts a_i. Consider the following diagram

$$\begin{tikzcd} X_{\mathrm{p}^1} \arrow[r, "t"] & \mathbb{W}_1(X_{\mathrm{p}^0}) \arrow[r, "\bar{\sigma}"] & X_{\mathrm{p}^1} \\ X_{\mathrm{p}^0} \arrow[u] \arrow[r, "F_{X_{\mathrm{p}^0}}"] & X_{\mathrm{p}^0} \arrow[u] \arrow[r, "\mathrm{Id}"] & X_{\mathrm{p}^0} \arrow[u] \end{tikzcd}$$

Its left square is commutative, since the composition

$$X_{\mathrm{p}^0} \longrightarrow X_{\mathrm{p}^1} \xrightarrow{t} \mathbb{W}_1(X_{\mathrm{p}^0})$$

simply corresponds to the map

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathrm{p}^0}}) &\to \mathcal{O}_{X_{\mathrm{p}^0}} \\ (a_0, a_1) &\mapsto a_0^p. \end{aligned}$$

For the commutativity of the right square, notice that by the very definition of the transition morphism $\pi_1 : \mathrm{Gr}_1(X_{\mathrm{p}^1}) \to X_{\mathrm{p}^0}$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{R_1}(\mathbb{W}_1(X_{\mathrm{p}^0}), X_{\mathrm{p}^1}) \arrow[r] \arrow[d, "\text{reduction mod } p"'] & \mathrm{Hom}_{R_0}(X_{\mathrm{p}^0}, \mathrm{Gr}_1(X_{\mathrm{p}^1})) \arrow[dl, "(\pi_1 \circ -)"] \\ \mathrm{Hom}_{R_0}(X_{\mathrm{p}^0}, X_{\mathrm{p}^0}) & \end{tikzcd}$$

In particular, $\mathrm{Id}_{X_{\mathfrak{p}0}} = \pi_1 \circ \sigma =$ (reduction mod p)$(\bar{\sigma})$, which is exactly what we wanted to verify. We obtain therefore that $\bar{\sigma} \circ t : X_{\mathfrak{p}^1} \to X_{\mathfrak{p}^1}$ is a lift of the Frobenius $F_{X_{\mathfrak{p}0}}$.

The diagram below is also commutative

$$\begin{tikzcd} X_{\mathfrak{p}^1} \arrow[r, "t"] \arrow[d] & \mathbb{W}_1(X_{\mathfrak{p}^0}) \arrow[r, "\bar{\sigma}"] \arrow[d] & X_{\mathfrak{p}^1} \arrow[d] \\ \mathrm{Spec}(R_1) \arrow[r, "F_{R_1}"] & \mathrm{Spec}(R_1) \arrow[r, "\mathrm{Id}"] & \mathrm{Spec}(R_1) \end{tikzcd}$$

In fact, by definition, $\bar{\sigma}$ is a morphism over R_1, so the right square is commutative. The commutativity of the left square is easy to check, since we know explicitly t and F_{R_1}. Therefore $\bar{\sigma} \circ t$ is a lift of the Frobenius $F_{X_{\mathfrak{p}0}}$ compatible with F_{R_1}; this implies the existence of a morphism of R_1-schemes

$$\tilde{F} : X_{\mathfrak{p}^1} \to X'_{\mathfrak{p}^1}$$

lifting the relative Frobenius $F_{X_{\mathfrak{p}0}/R_0}$.

As shown in part (b) of the proof of Théorème 2.1 in [Deligne and Illusie 1987], since the image of $\tilde{F}^* : \Omega_{X'_{\mathfrak{p}1}} \to \tilde{F}_* \Omega_{X_{\mathfrak{p}^1}}$ is contained in $p\tilde{F}_*\Omega_{X_{\mathfrak{p}^1}}$ and the multiplication by p induces an isomorphism $p : F_{X_{\mathfrak{p}0}/R_0,*}\Omega_{X_{\mathfrak{p}0}} \xrightarrow{\sim} p\tilde{F}_*\Omega_{X_{\mathfrak{p}^1}}$, there exists a unique map

$$f := p^{-1}\tilde{F}^* : \Omega_{X'_{\mathfrak{p}0}} \to F_{X_{\mathfrak{p}0}/R_0,*}\Omega_{X_{\mathfrak{p}0}},$$

making the diagram below commutative.

$$\begin{tikzcd} \Omega_{X'_{\mathfrak{p}1}} \arrow[r, "\tilde{F}^*"] \arrow[d, two heads] & p\tilde{F}_*\Omega_{X_{\mathfrak{p}^1}} \\ \Omega_{X'_{\mathfrak{p}0}} \arrow[r, "f"] & F_{X_{\mathfrak{p}0}/R_0,*}\Omega_{X_{\mathfrak{p}0}} \arrow[u, "p"'] \end{tikzcd}$$

Proposition 3 in [Xin 2016] states that the adjoint of f,

$$\bar{f} : F^*_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}} = F^*_{X_{\mathfrak{p}0}/R_0}\Omega_{X'_{\mathfrak{p}0}} \to \Omega_{X_{\mathfrak{p}0}},$$

is generically bijective. This clearly contradicts Lemma 5.1. □

6. The number of irreducible components of the critical scheme of complete intersections

In this last section we provide an upper bound for the number of irreducible components of the critical scheme $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ in the case in which X is a smooth complete intersection.

Let A/K be an abelian variety of dimension n and let L be a very ample line bundle on A. Let $c \in \mathbb{N}$ be positive and let $H_1, H_2, \ldots, H_c \in |L|$ be general. We define $X := H_1 \cap H_2 \cap \cdots \cap H_c$. Suppose that X is smooth.

Let us take a sufficiently small open $V \subseteq \mathrm{Spec}(\mathcal{O}_K)$ such that A extends over V to an abelian scheme $\mathcal{A}$, L extends to a V-very ample line bundle $\mathcal{L}$, H_i extends to $\mathcal{H}_i$ for every i and $\mathcal{X} := \mathcal{H}_1 \cap \mathcal{H}_2 \cap \cdots \cap \mathcal{H}_c$ is smooth. We can restrict V if necessary and suppose $K/\mathbb{Q}$ is unramified at $\mathfrak{p}$.

Theorem 6.1. *Let K be a number field, A/K be an abelian variety of dimension n and let L be a very ample line bundle on A. Let $c \in \mathbb{N}$ be positive and let $H_1, H_2, \ldots, H_c \in |L|$ be general. Suppose that $X := H_1 \cap H_2 \cap \cdots \cap H_c$ is smooth. If $\mathfrak{p}$ is in the open subscheme V defined above, then the number of irreducible components of* $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ *is bounded by*

$$p^{2n}\biggl(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\biggr)(L^n)^2.$$

Here (L^n) denotes the intersection number of L.

Proof. To obtain Theorem 6.1, we follow the approach of [Buium 1996, Theorem 1.11], proving the Manin–Mumford conjecture for curves; we first show that $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ can be realized as the intersection of two projective varieties (see $\mathbb{P}(E_X)$ and $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})$ below) and then use the product of their degrees to bound the number of its irreducible components. Since X is not necessarily of dimension one, the computation of the degree of $\mathbb{P}(E_X)$ is slightly more demanding here than the corresponding one in Buium's work.

Let us fix $\mathfrak{p} \in V$. The torsors $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \to X_{\mathfrak{p}^0}$ and $\mathrm{Gr}_1(A_{\mathfrak{p}^1}) \to A_{\mathfrak{p}^0}$ correspond to elements $\eta_X \in H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}0}} \Omega^\vee_{X_{\mathfrak{p}0}})$ and $\eta_A \in H^1(A_{\mathfrak{p}^0}, F^*_{A_{\mathfrak{p}0}} \Omega^\vee_{A_{\mathfrak{p}0}})$, respectively. Under the natural isomorphisms

$$H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}0}} \Omega^\vee_{X_{\mathfrak{p}0}}) \simeq \mathrm{Ext}^1(F^*_{X_{\mathfrak{p}0}} \Omega_{X_{\mathfrak{p}^0}}, \mathcal{O}_{X_{\mathfrak{p}0}}) \quad \text{and} \quad H^1(A_{\mathfrak{p}^0}, F^*_{A_{\mathfrak{p}0}} \Omega^\vee_{A_{\mathfrak{p}0}}) \simeq \mathrm{Ext}^1(F^*_{A_{\mathfrak{p}0}} \Omega_{A_{\mathfrak{p}^0}}, \mathcal{O}_{A_{\mathfrak{p}0}}),$$

η_X and η_A correspond to extensions of vector bundles

$$0 \to \mathcal{O}_{X_{\mathfrak{p}0}} \to E_X \to F^*_{X_{\mathfrak{p}0}} \Omega_{X_{\mathfrak{p}0}} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{A_{\mathfrak{p}0}} \to E_A \to F^*_{A_{\mathfrak{p}0}} \Omega_{A_{\mathfrak{p}0}} \to 0.$$

For any locally free sheaf W over a base S of finite type over a field, we shall write $\mathbb{P}(W)$ for the projective bundle associated to W, i.e., the S-scheme representing the functor on S-schemes

$T \mapsto$ {isomorphism classes of surjective morphisms of $\mathcal{O}_T$-modules $W_T \to Q$,
where Q is locally free of rank 1}.

As shown in paragraph 1 of [Martin-Deschamps 1984], the two extensions above give us two divisors

$$D_X := \mathbb{P}(F^*_{X_{\mathfrak{p}0}} \Omega_{X_{\mathfrak{p}0}}) \subseteq \mathbb{P}(E_X) \quad \text{and} \quad D_A := \mathbb{P}(F^*_{A_{\mathfrak{p}0}} \Omega_{A_{\mathfrak{p}0}}) \subseteq \mathbb{P}(E_A),$$

belonging respectively to the linear systems $|\mathcal{O}_{\mathbb{P}(E_X)}(1)|$ and $|\mathcal{O}_{\mathbb{P}(E_A)}(1)|$, and

$$\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_X) \setminus D_X \quad \text{and} \quad \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_A) \setminus D_A.$$

If i denotes the closed immersion $i : X_{\mathfrak{p}^0} \to A_{\mathfrak{p}^0}$, then it is not difficult to show that there is a natural restriction homomorphism $i^*E_A \to E_X$ prolonging the homomorphism $i^*\Omega_{A_{\mathfrak{p}^0}} \to \Omega_{X_{\mathfrak{p}^0}}$. The homomorphism $i^*E_A \to E_X$ is clearly surjective, so it induces a closed immersion $j : \mathbb{P}(E_X) \hookrightarrow \mathbb{P}(E_A)$ prolonging $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \hookrightarrow \mathrm{Gr}_1(A_{\mathfrak{p}^1})$. Therefore we have a commutative diagram

$$
\begin{tikzcd}
 & {[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})} \arrow[d, hook] \arrow[ddd, bend left, "T"] \\
\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \arrow[d, hook] \\
\mathbb{P}(E_X) \arrow[r, hook, "j"] \arrow[d, "\pi_X"] & \mathbb{P}(E_A) \arrow[d, "\pi_A"] \\
X_{\mathfrak{p}^0} \arrow[r, hook, "i"] & A_{\mathfrak{p}^0}
\end{tikzcd}
$$

Let us denote by $\mathcal{L}_{\mathfrak{p}}$ the base change of $\mathcal{L}$ to $A_{\mathfrak{p}^0}$. It is standard to prove that

$$\mathcal{H} := \pi_A^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_A)}(1)$$

is very ample on $\mathbb{P}(E_A)$ [Buium and Voloch 1996, p. 4]. We have

$$\mathcal{H}|_{\mathbb{P}(E_X)} = \pi_X^* i^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_X)}(1) \quad \text{and} \quad \mathcal{H}|_{[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})} = T^* \mathcal{L}_{\mathfrak{p}},$$

since $D_A \in |\mathcal{O}_{\mathbb{P}(E_A)}(1)|$ and $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \subseteq \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_A) \setminus D_A$. We know that $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})$ is the maximal abelian subvariety of $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ and we know that the multiplication by p map on $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ factors through the isogeny T. This implies that T has degree at most p^{2n}, so we have the following estimate

$$\deg_{\mathcal{H}}([p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})) \le p^{2n} (\mathcal{L}_{\mathfrak{p}}^n).$$

Let us now consider $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$. It coincides with

$$\int_{\mathbb{P}(E_X)} c_1(\mathcal{H}|_{\mathbb{P}(E_X)})^{2n-2c} \tag{7}$$

where c_1 stands for the first Chern class in the Chow ring and $\int_{\mathbb{P}(E_X)}$ stands for the push-forward morphism to $\mathrm{Spec}(R_0)$ in the Chow theory. Since

$$c_1(\mathcal{H}|_{\mathbb{P}(E_X)}) = c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}}) + c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))$$

we can rewrite (7) as

$$\int_{\mathbb{P}(E_X)} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}})^h \cdot c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h}.$$

Equivalently

$$\int_{X_{\mathfrak{p}0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot \pi_{X,*}(c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h})$$

and by definition of Segre class this is

$$\int_{X_{\mathfrak{p}0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X^\vee).$$

Notice that the Segre classes of the dual of E_X appear in our formula; this is due to the fact that we are not using Fulton's geometric notation for the projective bundle associated to a vector bundle (see the note at the end of B.5.5 in [Fulton 1998]). Since $s_k = 0$ if $k < 0$, we end up with

$$\int_{X_{\mathfrak{p}0}} \sum_{h=0}^{n-c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X^\vee).$$

Now the exact sequence

$$0 \to \mathcal{O}_{X_{\mathfrak{p}0}} \to E_X \to F_{X_{\mathfrak{p}0}}^* \Omega_{X_{\mathfrak{p}0}} \to 0$$

implies

$$s_{n-c-h}(E_X^\vee) = s_{n-c-h}(F_{X_{\mathfrak{p}0}}^* \Omega_{X_{\mathfrak{p}0}}^\vee)$$

and so

$$s_{n-c-h}(E_X^\vee) = p^{n-c-h} s_{n-c-h}(\Omega_{X_{\mathfrak{p}0}}^\vee)$$

(here we have used the following fact: the pullback of a cycle η of codimension j through the Frobenius map coincides with $p^j \eta$). Therefore we have to study the following sum

$$\sum_{h=0}^{n-c} \binom{2n-2c}{h} p^{n-c-h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(\Omega_{X_{\mathfrak{p}0}}^\vee). \tag{8}$$

The short exact sequence

$$0 \to \Omega_{X_{\mathfrak{p}0}}^\vee \to i^* \Omega_{A_{\mathfrak{p}0}}^\vee \to N \to 0$$

(where N is the normal bundle for i) gives

$$c_t(\Omega_{X_{\mathfrak{p}0}}^\vee) c_t(N) = c_t(i^* \Omega_{A_{\mathfrak{p}0}}^\vee) = 1,$$

so that $c_t(N) = s_t(\Omega_{X_{\mathfrak{p}0}}^\vee)$. Recalling that

$$c_t(N) = (1 + c_1(i^* \mathcal{L}_{\mathfrak{p}}) t)^c$$

we obtain

$$s_{n-c-h}(\Omega^\vee_{X_{\mathfrak{p}^0}}) = c_{n-c-h}(N) = \binom{c}{n-c-h} c_1(i^*\mathcal{L}_\mathfrak{p})^{n-c-h}.$$

Substituting in (8), we obtain

$$\biggl(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\biggr)c_1(i^*\mathcal{L}_\mathfrak{p})^{n-c}.$$

Therefore $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\biggl(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\biggr)\int_{X_{\mathfrak{p}^0}} c_1(i^*\mathcal{L}_\mathfrak{p})^{n-c}.$$

Since $X_{\mathfrak{p}^0} = H_{1,\mathfrak{p}} \cap \cdots \cap H_{c,\mathfrak{p}}$ where $H_{1,\mathfrak{p}}, \ldots, H_{c,\mathfrak{p}}$ belong to $|\mathcal{L}_\mathfrak{p}|$, we have

$$\int_{X_{\mathfrak{p}^0}} c_1(i^*\mathcal{L}_\mathfrak{p})^{n-c} = \int_{A_{\mathfrak{p}^0}} c_1(\mathcal{L}_\mathfrak{p})^n = (\mathcal{L}_\mathfrak{p}^n)$$

and

$$\deg_{\mathcal{H}}(\mathbb{P}(E_X)) = \biggl(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\biggr)(\mathcal{L}_\mathfrak{p}^n).$$

Now Bézout's theorem in Fulton's form [1998, p. 148] says that the number of irreducible components in the intersection of two projective varieties of degrees d_1 and d_2 cannot exceed $d_1 d_2$. In particular, the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ is less than or equal to

$$p^{2n}\biggl(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\cdot\biggr)(\mathcal{L}_\mathfrak{p}^n)^2.$$

Notice that $(\mathcal{L}_\mathfrak{p}^n) = (L^n)$, by the same reasoning as before Lemma 5.1. □

Remark 6.2. One can consider any intersection $X := H_1 \cap H_2 \cap \cdots \cap H_c$ where $H_i \in |L_i|$ for some very ample line bundles L_i. In this more general case, the computations in our proof become a bit more complex, but it is still possible to give an explicit bound for the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$. We have

$$c_j(N) = \sum_{1\le i_1<\cdots<i_j\le c}\prod_{k=i_1}^{i_j} c_1(i^*\mathcal{L}_{k,\mathfrak{p}})$$

which implies

$$s_{n-c-h}(\Omega^\vee_{X_{\mathfrak{p}^0}}) = \sum_{1\le i_1<\cdots<i_{n-c-h}\le c}\prod_{k=i_1}^{i_{n-c-h}} c_1(i^*\mathcal{L}_{k,\mathfrak{p}}).$$

Therefore, defining $\mathcal{H} := \pi_A^*\mathcal{L}_{1,\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_A)}(1)$, then $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\sum_{h=0}^{n-c}\binom{2n-2c}{h}p^{n-c-h}\sum_{1\le i_1<\cdots<i_{n-c-h}\le c}\biggl(\int_{X_{\mathfrak{p}^0}} c_1(i^*\mathcal{L}_{1,\mathfrak{p}})^h \prod_{k=i_1}^{i_{n-c-h}} c_1(i^*\mathcal{L}_{k,\mathfrak{p}})\biggr).$$

We have

$$\int_{X_{\mathfrak{p}0}} c_1(i^*\mathcal{L}_{1,\mathfrak{p}})^h \prod_{k=i_1}^{i_{n-c-h}} c_1(i^*\mathcal{L}_{k,\mathfrak{p}}) = I_{i_1,\dots,i_{n-c-h}}$$

where $I_{i_1,\dots,i_{n-c-h}}$ is the following intersection number

$$I_{i_1,\dots,i_{n-c-h}} := (\overbrace{L_1\cdots L_1}^{h+1 \text{ times}} \cdot L_2\cdot L_3\cdots L_c\cdot L_{i_1}\cdot L_{i_2}\cdots L_{i_{n-c-h}}).$$

We obtain that $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\sum_{h=0}^{n-c}\binom{2n-2c}{h}p^{n-c-h}\sum_{1\le i_1<\dots<i_{n-c-h}\le c} I_{i_1,\dots,i_{n-c-h}},$$

and therefore the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X},\mathcal{A})$ is bounded by

$$p^{2n}(L_1^n)\sum_{h=0}^{n-c}\binom{2n-2c}{h}p^{n-c-h}\sum_{1\le i_1<\dots<i_{n-c-h}\le c} I_{i_1,\dots,i_{n-c-h}}.$$

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On a conjecture of Kato and Kuzumaki

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In 1986, Kato and Kuzumaki stated several conjectures in order to give a diophantine characterization of cohomological dimension of fields in terms of projective hypersurfaces of small degree and Milnor K-theory. We establish these conjectures for finite extensions of $\mathbb{C}(x_1, \dots, x_n)$ and $\mathbb{C}(x_1, \dots, x_n)((t))$, and we prove new local-global principles over number fields and global fields of positive characteristic in the context of Kato and Kuzumaki's conjectures.

Introduction

In 1986, in the article [Kato and Kuzumaki 1986], Kato and Kuzumaki stated a set of conjectures which aimed at giving a diophantine characterization of cohomological dimension of fields. For this purpose, they introduced variants of the C_i-properties of fields involving Milnor K-theory and projective hypersurfaces of small degree, and they hoped that these variants would characterize fields of small cohomological dimension.

More precisely, fix a field L and two nonnegative integers q and i. Let $K_q^M(L)$ be the q-th Milnor K-group of L. For each finite extension L' of L, one can define a norm morphism $N_{L'/L} : K_q^M(L') \to K_q^M(L)$; see [Kato 1980, section 1.7]. Thus, if Z is a scheme of finite type over L, one can introduce the subgroup $N_q(Z/L)$ of $K_q^M(L)$ generated by the images of the norm morphisms $N_{L'/L}$ when L' describes the finite extensions of L such that $Z(L') \neq \varnothing$. One then says that the field L is C_i^q if, for each $n \geq 1$, for each finite extension L' of L and for each hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$, one has $N_q(Z/L') = K_q^M(L')$. For example, the field L is C_i^0 if, for each finite extension L' of L, every hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$ has a 0-cycle of degree 1. The field L is C_0^q if, for each tower of finite extensions $L''/L'/L$, the norm morphism $N_{L''/L'} : K_q^M(L'') \to K_q^M(L')$ is surjective.

Kato and Kuzumaki conjectured that, for $i \geq 0$ and $q \geq 0$, a perfect field is C_i^q if and only if it is of cohomological dimension at most $i + q$. This conjecture

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generalizes a question raised by Serre [1965] asking whether the cohomological dimension of a C_i-field is at most i. In an unpublished paper, Acquista [2005] proved Kato and Kuzumaki's conjecture for $i=0$: in other words, a perfect field is C_0^q if and only if it is of cohomological dimension at most q. As it was already pointed out at the end of Kato and Kuzumaki's original paper [1986], such a result also follows from the Bloch–Kato conjecture, which has been established by Rost and Voevodsky. However, it turns out that the conjectures of Kato and Kuzumaki are wrong in general. For example, Merkurjev [1991] constructed a field of characteristic 0 and of cohomological dimension 2 which did not satisfy property C_2^0. Similarly, Colliot-Thélène and Madore [2004] produced in a field of characteristic 0 and of cohomological dimension 1 which did not satisfy property C_1^0. These counterexamples were all constructed by a method using transfinite induction due to Merkurjev and Suslin. The conjecture of Kato and Kuzumaki is therefore still completely open for fields that usually appear in number theory or in algebraic geometry.

Very recently Wittenberg [2015] made an important step forward: he proved that p-adic fields, the field $\mathbb{C}((t_1))((t_2))$ and totally imaginary number fields all satisfy property C_1^1. His method consists of introducing and proving a property which is stronger than property C_1^1: more precisely, he says that a field L is strongly C_1^1 if, for each finite extension L' of L, each proper scheme Z over L' and each coherent sheaf E on Z, the Euler–Poincaré characteristic $\chi(Z, E)$ kills the abelian group $K_q^M(L')/N_q(Z/L')$. It turns out that this notion behaves much better with respect to dévissage than the C_1^1-property of Kato and Kuzumaki: this allows Wittenberg to use methods that had been previously developed in [Esnault et al. 2015].

Wittenberg's article leaves open the question of the C_1^1-property for the following fields: the field of rational functions $\mathbb{C}(x, y)$, the field of Laurent series in two variables $\mathbb{C}((x, y))$, and the fields $\mathbb{C}(x)((y))$ and $\mathbb{C}((x))(y)$. That the property is satisfied by $\mathbb{C}(x, y)$ and $\mathbb{C}(x)((y))$ is a particular case of the general theorems that are established in the present paper (see theorems C and D below).

The article is divided into three parts that can be read almost independently and that deal with Kato and Kuzumaki's conjectures for different fields. In the first section, we focus on the cases of number fields and of function fields of curves over finite fields. In the case of number fields, we establish a local-global principle in the context of the conjecture of Kato and Kuzumaki for varieties containing a geometrically integral closed subscheme. Such a result was previously only known for smooth, projective, geometrically irreducible varieties (see theorem 4 of [Kato and Saito 1983]) or for proper varieties of Euler–Poincaré characteristic equal to 1 [Wittenberg 2015, Proposition 6.2]:

Theorem A (Theorem 1.4, number field case). *Let K be a number field and let Ω_K be the set of places of K. Let Z be a K-variety containing a geometrically integral* *closed subscheme. For each $v \in \Omega_K$, let K_v be the completion of K with respect to v and Z_v be the K_v-scheme $Z \times_K K_v$. Then*

$$\operatorname{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0.$$

This theorem, which is established by using Hilbertianity properties of number fields as well as results due to Demarche and Wei [2014] concerning the local-global principle for torsors under normic tori, then allows us to deduce a simplified and more effective proof of Wittenberg's result concerning the C_1^1-property for totally imaginary number fields (see Corollary 1.9 and page 438). The explicitness of our proof allows us to give new and more precise results in some situations (see Proposition 1.14).

In the case of global fields of positive characteristic, we prove a local-global principle similar to the one in Theorem A but which involves a variant of the group $N_1(Z/K)$:

Theorem B (Theorem 1.4, function field case). *Let K be the function field of a curve over a finite field of characteristic $p > 0$ and let Ω_K be the set of places of K. Let Z be a proper K-scheme containing a geometrically irreducible closed subscheme. For $v \in \Omega_K$, let K_v be the completion of K with respect to v and Z_v be the K_v-scheme $Z \times_K K_v$. Let $N_1^s(Z/K)$ be the subgroup of $K^\times$ spanned by the images of the norm homomorphisms $N_{L_s/K} : L_s^\times \to K^\times$ where L describes finite extensions of K such that $Z(L) \neq \varnothing$ and L_s stands for the separable closure of K in L. Then*

$$\operatorname{Ker}\left(K^\times/N_1^s(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^s(Z_v/K_v)\right) = 0.$$

This theorem then allows us to prove that global fields of positive characteristic have the C_1^1-property "away from p" (Theorem 1.18).

In the second part, by means of a surprisingly simple argument, we prove Kato and Kuzumaki's conjectures for function fields over $\mathbb{C}$ of arbitrary dimension:

Theorem C (Theorem 2.2). *Let k be an algebraically closed field of characteristic* 0. *Then the function field of an n-dimensional integral k-variety satisfies the C_i^q-property for all $i \geq 0$ and $q \geq 0$ such that $i + q = n$.*

In particular, this shows that the field $\mathbb{C}(x, y)$ satisfies the C_1^1-property, and hence answers question (3) in paragraph 7.3 of [Wittenberg 2015] positively.

In the third and last part, we prove Kato and Kuzumaki's properties for complete discrete valuation fields whose residue field is the function field of a variety over an algebraically closed field of characteristic zero:

Theorem D (Theorem 3.9). *Let k be an algebraically closed field of characteristic zero. Let K be the function field of an n-dimensional integral k-variety. Then the complete field $K((t))$ satisfies the C_i^q-property for all $i, q \geq 0$ such that $i+q=n+1$.*

This theorem, whose proof relies on subtle refinements of Artin's approximation theorem, implies in particular that $\mathbb{C}(x)((t))$ is C_1^1.

Remark 0.1. The C_1^1-property for the fields $\mathbb{C}(x, y)$ and $\mathbb{C}(x)((t))$, which is a special case of theorems C and D, cannot be obtained by the methods developed in [Wittenberg 2015] because those fields are not strongly C_1^1 (see remark 7.6 there).

Preliminaries. Let L be any field and let q be a nonnegative integer. The q-th Milnor K-group of L is by definition the group $K_0^M(K)=\mathbb{Z}$ if $q=0$ and

$$K_q^M(L) := \underbrace{L^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L^\times}_{q \text{ times}} / \langle x_1 \otimes \cdots \otimes x_q \mid \text{there exist } i, j,\ i \neq j,\ x_i + x_j = 1 \rangle$$

if $q>0$. For $x_1,\dots,x_q \in L^\times$, the symbol $\{x_1,\dots,x_q\}$ denotes the class of $x_1 \otimes \cdots \otimes x_q$ in $K_q^M(L)$. More generally, for r and s nonnegative integers such that $r+s=q$, there is a natural pairing

$$K_r^M(L) \times K_s^M(L) \to K_q^M(L),$$

which we will denote $\{\cdot\,,\cdot\}$.

When L' is a finite extension of L, one can construct a norm homomorphism $N_{L'/L} : K_q^M(L') \to K_q^M(L)$ (see section 1.7 of [Kato 1980]) satisfying the following properties:

- for $q=0$, the map $N_{L'/L} : K_0^M(L') \to K_0^M(L)$ is given by multiplication by $[L' : L]$;
- for $q=1$, the map $N_{L'/L} : K_1^M(L') \to K_1^M(L)$ coincides with the usual norm $L'^\times \to L^\times$;
- if r and s are nonnegative integers such that $r+s=q$, we have $N_{L'/L}(\{x, y\}) = \{x, N_{L'/L}(y)\}$ for $x \in K_r^M(L)$ and $y \in K_s^M(L')$;
- if L'' is a finite extension of L', we have $N_{L''/L} = N_{L'/L} \circ N_{L''/L'}$.

For each L-scheme of finite type, we denote by $N_q(Z/L)$ the subgroup of $K_q^M(L)$ generated by the images of the maps $N_{L'/L} : K_q^M(L') \to K_q^M(L)$ when L' describes the finite extensions of L such that $Z(L') \neq \varnothing$. In particular, $N_0(Z/L)$ is the subgroup of $\mathbb{Z}$ generated by the index of Z (i.e., the gcd of the degrees $[L' : L]$ when L' describes the finite extensions of L such that $Z(L') \neq \varnothing$). For $i \geq 0$, we say that L satisfies the C_i^q-property if, for every finite extension L' of L and for every hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$, we have $N_q(Z/L') = K_q^M(L')$. In particular, L is C_i^0 if, for each finite extension L' of L, every hypersurface Z in $\mathbb{P}^n_{L'}$ of degree d with $d^i \leq n$ has index 1.

The field L is C_0^q if, for each tower of finite extensions $L''/L'/L$, the norm $N_{L''/L'} : K_q^M(L'') \to K_q^M(L')$ is surjective. As was already pointed out by Kato and Kuzumaki at the end of [Kato and Kuzumaki 1986], by using the Bloch–Kato conjecture which identifies the groups $K_q^M(L)/n$ and $H^q(L, \mu_n^{\otimes q})$ for n prime to the characteristic of L and which has been proved by Rost and Voevodsky, one can show that a field of characteristic zero is C_0^q if and only if it is of cohomological dimension at most q.

1. Global fields

Proof of theorems A and B. This section is devoted to number fields and function fields of curves over finite fields. The main goal consists of establishing theorems A and B. Whenever K is a global field, Ω_K stands for the set of places of K, and for $v \in \Omega_K$, we denote by K_v the completion of K with respect to v and by $\mathcal{O}_v$ the ring of integers in K_v.

We start with a preliminary lemma concerning Hilbertian fields. For a definition of Hilbertian fields, the reader may refer to section 12.1 of [Fried and Jarden 2008].

Lemma 1.1. *Let K be a Hilbertian field and fix an algebraic closure $\bar{K}$ of K. Let F be a finite Galois extension of K and let Y be a geometrically integral K-variety. Then there exists a finite extension F_0 of K such that $Y(F_0) \neq \varnothing$ and $F_0 \cap F = K$.*

Proof. Of course, we can assume that $\dim Y > 0$. By applying Bertini's theorem to an open dense quasiprojective subset of Y, one shows that Y contains a quasiprojective geometrically integral curve C over K. Since Y is geometrically reduced, one can find a curve C' in $\mathbb{P}^2_K$ birationally equivalent to C. Let $g \in K[X, Y, Z]$ be a homogeneous polynomial which is irreducible over $\bar{K}$ and such that C' is the curve defined by the equation $g = 0$. Let U' be a nonempty subset of C' which is isomorphic to an open subset of C. We now distinguish two cases:

- if K has characteristic $p > 0$, we know that $g \in K[X, Y, Z] \setminus K[X^p, Y^p, Z^p]$, and we can thus assume without loss of generality that $g \in K[X, Y, Z] \setminus K[X^p, Y, Z]$. Hence we may consider an integer $m \geq 1$ and a polynomial $h \in K[Y, Z] \setminus \{0\}$ such that p does not divide m and the coefficient of X^m in g is h. We also consider the set
$$H := \{(y, z) \in F^2 \mid g(X, y, z) \in F[X] \text{ is irreducible}, h(y, z) \neq 0\}.$$
- if K has characteristic 0, we can assume without loss of generality that $g \notin K[Y, Z]$ and we consider the set
$$H := \{(y, z) \in F^2 \mid g(X, y, z) \in F[X] \text{ is irreducible}\}.$$

  To unify notation with the positive characteristic case, we also set $h(Y, Z) := 1 \in K[Y, Z]$.

As g is irreducible over F and separable in the variable X, the set H is by definition a separable Hilbert subset of F^2. According to [Fried and Jarden 2008, Corollary 12.2.3], H contains a separable Hilbert subset H' of K^2. Since K is a Hilbertian field, the second paragraph of Section 12.1 of the same work implies that H' is Zariski dense in K^2. In particular, the set H' is infinite, and there exists an infinite number of pairs $(y, z) \in K^2$ such that $g(X, y, z)$ is irreducible over F and $h(y, z) \neq 0$. Each of these pairs corresponds to a point $w \in (C')^{(1)}$ such that $K(w) \cap F = K$ and the extension $K(w)/K$ is separable. Since $C' \setminus U'$ is finite, we conclude that there exists $w \in (U')^{(1)}$ such that $K(w)$ is a finite separable extension of K satisfying $K(w) \cap F = K$. By setting $F_0 = K(w)$, we get $Y(F_0) \neq \varnothing$ and $F_0 \cap F = K$. □

Corollary 1.2. *Let K be a Hilbertian field and fix an algebraic closure $\bar{K}$ of K. Let F be a finite Galois extension of K and let Y be a geometrically irreducible K-variety. Then there exists a finite extension F_0 of K such that $Y(F_0) \neq \varnothing$ and $F_0 \cap F = K$.*

Proof. If K has characteristic 0, the corollary immediately follows from Lemma 1.1. Assume that K has positive characteristic. Let K' be a purely inseparable finite extension of K such that $(Y_{K'})^{\mathrm{red}}$ is geometrically integral. By Lemma 1.1, there exists a finite extension F_1 of K' such that $Y(F_1) \neq \varnothing$ and $F_1 \cap (K' \cdot F) = K'$, where $K' \cdot F$ denotes the subfield of $\bar{K}$ generated by K' and F. Then we also have $F_1 \cap F = K$. □

We now introduce a variant of the group $N_1(Z/K)$ which will allow us to treat in a unified way number fields and function fields of curves over finite fields:

Definition 1.3. Let K be a field and let Z be a K-scheme of finite type. We denote by $N_1^s(Z/K)$ the subgroup of $K^\times$ spanned by the images of the norm morphisms $N_{L_s/K} : L_s^\times \to K^\times$, where L describes finite extensions of K such that $Z(L) \neq \varnothing$ and L_s stands for the separable closure of K in L.

Note that, if K is a field of characteristic 0 and Z is a K-scheme of finite type, then $N_1^s(Z/K) = N_1(Z/K)$. We are now ready to prove the main theorem of this section:

Theorem 1.4. *Let K be a number field or the function field of a curve over a finite field. Let Z be a K-variety containing a geometrically irreducible closed subscheme. For $v \in \Omega_K$, we denote by Z_v the K_v-scheme $Z \times_K K_v$. Then*

$$\mathrm{Ker}\left(K^\times / N_1^s(Z/K) \to \prod_{v \in \Omega_K} K_v^\times / N_1^s(Z_v/K_v) \right) = 0.$$

Notation 1.5. Whenever M denotes a Galois module over K, we define the first Tate–Shafarevich group of M by

$$Ш^1(K, M) := \mathrm{Ker}\left(H^1(K, M) \to \prod_{v \in \Omega_K} H^1(K_v, M) \right).$$

Proof. In the sequel, we fix an algebraic closure $\bar{K}$ of K: all finite extensions of K will therefore be considered as subfields of $\bar{K}$.

Now fix $x \in K^\times$ whose class modulo $N_1^s(Z/K)$ lies in

$$\operatorname{Ker}\left(K^\times/N_1^s(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1^s(Z_v/K_v)\right).$$

We want to prove that $x \in N_1^s(Z/K)$. To do so, we consider a finite normal extension L of K such that $Z(L) \neq \varnothing$. Let L_s be the separable closure of K in L: it is a finite Galois extension of K. Let $S \subseteq \Omega_K$ be the set of places v of K satisfying one of the following properties:

(i) v is finite and the extension L_s/K is ramified at v;

(ii) v is finite and x is not a unit in $\mathcal{O}_v$;

(iii) v is infinite.

Of course, S is a finite subset of Ω_K.

Now fix $v \in \Omega_K$. Two main cases arise:

- Assume in the first place that $v \in \Omega_K \setminus S$. In this case, v is a finite place, and as the extension $L_{s\,v}/K_v$ is unramified, we know that $N_{L_{s\,v}/K_v}(L_{s\,v}^\times)$ contains $\mathcal{O}_v^\times$. Since $x \in \mathcal{O}_v^\times$, we conclude that $x \in N_{L_{s\,v}/K_v}(L_{s\,v}^\times)$.
- Assume now that $v \in S$ and fix an algebraic closure $\overline{K_v}$ of K_v. By assumption, $x \in N_1^s(Z_v/K_v)$. Let then $M_1^{(v)}, \dots, M_{n_v}^{(v)}$ be finite extensions of K_v contained in $\overline{K_v}$ such that, if $M_{i,s}^{(v)}$ denotes the separable closure of K_v in $M_i^{(v)}$, we have $x \in \langle N_{M_{i,s}^{(v)}/K_v}(M_{i,s}^{(v)\times}) \mid 1 \leq i \leq n_v\rangle \subseteq K_v^\times$ and $Z(M_i^{(v)}) \neq \varnothing$ for each i. According to Greenberg's approximation theorem [1966, Theorem 1] if v is finite and Tarski–Seidenberg principle if v is real [Pirutka $\geq$ 2018, Corollary 4.1.6], we have $Z(M_i^{(v)} \cap \bar{K}) \neq \varnothing$. We can therefore consider a finite extension $L_i^{(v)}$ of K contained in $M_i^{(v)} \cap \bar{K}$ such that $Z(L_i^{(v)}) \neq \varnothing$. Let $L_{i,s}^{(v)}$ be the separable closure of K in $L_i^{(v)}$. The valuation on $M_i^{(v)}$ induces by restriction a place w of $L_i^{(v)}$ which divides v and such that the completion of $L_{i,s}^{(v)}$ with respect to w is a subextension of $M_{i,s}^{(v)}/K_v$. Hence

$$N_{M_{i,s}^{(v)}/K_v}(M_{i,s}^{(v)\times}) \subseteq N_{L_{i,s}^{(v)}\otimes_K K_v/K_v}((L_{i,s}^{(v)} \otimes_K K_v)^\times) \subseteq K_v^\times.$$

Since $x \in \langle N_{M_{i,s}^{(v)}/K_v}(M_{i,s}^{(v)\times}) \mid 1 \leq i \leq n_v\rangle \subseteq K_v^\times$, we deduce that

$$x \in \langle N_{L_{i,s}^{(v)}\otimes_K K_v/K_v}((L_{i,s}^{(v)} \otimes_K K_v)^\times) \mid 1 \leq i \leq n_v\rangle \subseteq K_v^\times.$$

To summarize, we have just proved that, if T is the normic torus $R^1_{E/K}(\mathbb{G}_m)$ with $E = L_s \times \prod_{v\in S}\prod_{i=1}^{n_v} L_{i,s}^{(v)}$ and if $[x]$ stands for the image of x in

$$H^1(K, T) = K^\times / N_{E/K}(E^\times),$$

then

$$[x] \in Ш^1(K, T). \tag{1}$$

Now let F be the smallest finite Galois extension of K containing L_s and all the $L_{i,s}^{(v)}$. Since Z contains a geometrically irreducible closed subscheme, Corollary 1.2 shows that Z has a point in a finite extension F_0 of K such that $F_0 \cap F = K$. Denote by $F_{0,s}$ the separable closure of K in F_0.

According to Theorem 1 of [Demarche and Wei 2014], since $F_{0,s} \cap F = K$ and the extension F/K is Galois, we have

$$Ш^1(K, Q) = 0,$$

where Q denotes the normic torus $R^1_{E'/K}(\mathbb{G}_m)$, with $E' = L_s \times F_{0,s} \times \prod_{v\in S}\prod_{i=1}^{n_v} L_{i,s}^{(v)}$. Noting that the torus T naturally embeds in Q and using (1), we conclude that the class of x in $H^1(K, Q)$ is trivial. Since $Z(L) \neq \varnothing$, $Z(F_0) \neq \varnothing$ and $Z(L_i^{(v)}) \neq \varnothing$ for each v and each i, this shows that $x \in N_1^s(Z/K)$ as desired. □

Remark 1.6. Let K be a number field and keep the notation and the assumptions of Theorem 1.4. The proof implies that, if $L_1, \dots, L_r$ are finite extensions of K such that

$$\langle N_{L_i\otimes_K K_v/K_v}((L_i \otimes_K K_v)^\times) \mid 1 \le i \le r\rangle = K_v^\times$$

for each $v \in \Omega_K$, then there exists a finite extension L_{r+1} of K such that

$$\langle N_{L_i/K}(L_i^\times) \mid 1 \le i \le r+1\rangle = K^\times.$$

Moreover, if L is a finite Galois extension of K containing all the L_i, the field L_{r+1} can be chosen to be any finite extension of K which is linearly disjoint from L.

Number fields. In this paragraph, we focus on the case when K is a number field. We give a new proof of the C_1^1-property for totally imaginary number fields, and we see how this proof allows one to study some concrete examples.

Property C_1^1 for totally imaginary number fields. In Theorem 1.4, the assumption that Z contains a geometrically integral closed subscheme cannot be removed. Indeed, one can for example choose $K = \mathbb{Q}$, take for Z a variety over $L = \mathbb{Q}(\sqrt{13}, \sqrt{17})$ having a rational point in L and see Z as a K-variety. In this case, Theorem 1.4 fails since the affine $\mathbb{Q}$-variety defined by the equation

$$N_{L/\mathbb{Q}}(x + y\sqrt{13} + z\sqrt{17} + t\sqrt{221}) = -1$$

does not satisfy the local-global principle.

Nevertheless, the assumption that Z contains a geometrically integral closed subscheme can be slightly weakened:

Corollary 1.7. *Let K be a number field and let Z be K-variety. For $v \in \Omega_K$, we denote by Z_v the K_v-scheme $Z \times_K K_v$. Assume that there exist finite extensions $K_1, \dots, K_r$ of K such that Z_{K_i} contains a geometrically integral closed subscheme for each i and the gcd of the degrees $[K_i : K]$ is* 1. *Then*

$$\operatorname{Ker}\Bigl(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\Bigr) = 0.$$

Remark 1.8. This corollary was previously only known for smooth, projective, geometrically irreducible K-varieties [Kato and Saito 1983, Theorem 4] and for proper varieties with Euler–Poincaré characteristic equal to 1 [Wittenberg 2015, Proposition 6.2]. It generalizes those results according to Proposition 3.3 there.

Proof. According to Theorem 1.4, for each i, we have

$$\operatorname{Ker}\Bigl(K_i^\times/N_1(Z_{K_i}/K_i) \to \prod_{w\in\Omega_{K_i}} K_{i,w}^\times/N_1(Z_{K_{i,w}}/K_{i,w})\Bigr) = 0.$$

Therefore a restriction-corestriction argument shows that the group

$$\operatorname{Ker}\Bigl(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\Bigr)$$

is of $[K_i : K]$-torsion for each i, hence trivial. □

Wittenberg [2015, Theorem 6.1] has recently proved property C_1^1 for totally imaginary number fields. Theorem 1.4 allows us to obtain this result by a different method. The passage from local results to global results is simpler and more explicit than in Section 6 there.

Corollary 1.9. *Let K be a number field and let Z be a hypersurface of degree d in $\mathbb{P}^n_K$ such that $d \le n$ and $N_1(Z_v/K_v) = K_v^\times$ for each real place v of K. Then $N_1(Z/K) = K^\times$.*

Proof. By Exercise I.7.2(c) of [Hartshorne 1977], we know that the Euler–Poincaré characteristic $\chi(Z, \mathcal{O}_Z) := \sum_{i\ge 0} \dim_K H^i_{\mathrm{Zar}}(Z, \mathcal{O}_Z)$ is equal to 1. Hence, [Wittenberg 2015, Proposition 3.3] establishes the existence of finite extensions $K_1, \dots, K_r$ of K satisfying the assumptions of Corollary 1.7. We deduce that

$$\operatorname{Ker}\Bigl(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\Bigr) = 0.$$

But by Corollary 5.5 there, we have $N_1(Z_v/K_v) = K_v^\times$ for each finite place v of K. By assumption, we also know that $N_1(Z_v/K_v) = K_v^\times$ for each infinite place v of K. We conclude that $N_1(Z/K) = K^\times$. □

Remark 1.10. Instead of using [Wittenberg 2015, Proposition 3.3] and Corollary 1.7 to prove that

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v\in\Omega_K} K_v^\times/N_1(Z_v/K_v)\right) = 0,$$

we could have combined Theorem 2 of [Kollár 2007] (which asserts that a projective hypersurface in $\mathbb{P}^n_K$ of degree d with $d \le n$ always contains a geometrically integral closed subscheme) with Theorem 1.4. The proof of Proposition 3.3 of [Wittenberg 2015] is nevertheless more elementary than the one of Theorem 2 of [Kollár 2007].

Some concrete examples over number fields. It is interesting to notice that the proof we have given of the C_1^1-property for totally imaginary number fields is quite explicit: by this, we mean that in many numerical examples, it allows us to establish more precise results than just the C_1^1-property. To see this, we first establish the following lemma:

Lemma 1.11. *Let $n \ge 1$ be an integer. Let M be a field of characteristic prime to n. Fix an algebraic closure $\overline{M}$ of M. Assume that M contains all n-th roots of unity and that $M^\times/M^{\times n}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$. Let $a_0, \dots, a_n$ be $n+1$ elements of $M^\times$. For $0 \le i, j \le n$ with $i \ne j$, set $M_{ij} = M\big(\sqrt[n]{a_i a_j^{-1}}\big)$. Then*

$$M^\times = \langle N_{M_{ij}/M}(M_{ij}^\times) \mid 1 \le i,\ j \le n,\ i \ne j\rangle.$$

Proof. Write $n = p_1^{r_1} \cdots p_s^{r_s}$ with $p_1, \dots, p_s$ pairwise distinct prime numbers and $r_1, \dots, r_s$ positive integers. Since $\langle N_{M_{ij}/M}(M_{ij}^\times) \mid 1 \le i,\ j \le n,\ i \ne j\rangle$ contains $M^{\times n}$ and

$$M^\times/M^{\times n} \cong \prod_{t=1}^{s} M^\times/M^{\times p_t^{r_t}},$$

it is enough to show that for each $t \in \{1, \dots, s\}$, the group $M^\times$ is spanned by the subgroups $M^{\times p_t^{r_t}}$ and $N_{M_{ij}/M}(M_{ij}^\times)$ for $1 \le i, j \le n, i \ne j$.

We henceforth fix $t \in \{1, \dots, s\}$. If there exist integers i and j with $0 \le i, j \le n$ and $i \ne j$ such that $a_i a_j^{-1} \in M^{\times p_t^{r_t}}$, there is nothing to prove. We can therefore assume that $a_i a_j^{-1} \notin M^{\times p_t^{r_t}}$ for all $0 \le i, j \le n$ with $i \ne j$.

For $0 \le i, j \le n$ with $i \ne j$, let e_{ij} be the largest divisor of $p_t^{r_t}$ such that there exists $y_{ij} \in M^\times$ satisfying $y_{ij}^{e_{ij}} = a_i a_j^{-1}$. The following properties are satisfied for $0 \le i, j \le n$ with $i \ne j$:

(i) the integer $p_t^{r_t}$ does not divide e_{ij}, because $a_i a_j^{-1} \notin M^{\times p_t^{r_t}}$,

(ii) the order of y_{ij} in $M^\times/M^{\times p_t^{r_t}}$ is $p_t^{r_t}$ because $M^\times/M^{\times p_t^{r_t}}$ is isomorphic to $(\mathbb{Z}/p_t^{r_t}\mathbb{Z})^2$,

(iii) one has $y_{ij}^{e_{ij}} = y_{i0}^{e_{i0}} \cdot y_{j0}^{-e_{j0}}$,

and we want to prove that the group $M^\times/M^{\times p_t^{r_t}}$ is spanned by the $N_{M_{ij}/M}(M_{ij}^\times)$ for $0 \leq i,\ j \leq n,\ i \neq j$. Since the group $N_{M_{ij}/M}(M_{ij}^\times)$ contains y_{ij} for each i and j and $M^\times/M^{\times p_t^{r_t}}$ is isomorphic to $(\mathbb{Z}/p_t^{r_t}\mathbb{Z})^2$, it is enough to prove the following abstract sublemma provided that one chooses $\Lambda = M^\times/M^{\times p_t^{r_t}}$, $x_{ij} = y_{ij}$ for $1 \leq i, j \leq p_t^{r_t}$ with $i \neq j$ and $x_{ii} = y_{i0}$ for $i \in \{1, \dots, p_t^{r_t}\}$. □

Sublemma 1.12. *Let p be a prime number and $r \geq 1$ an integer. Set $n = p^r$ and let $\Lambda = (\mathbb{Z}/n\mathbb{Z})^2$. For each $i \in \{1, \dots, n\}$ and each $j \in \{1, \dots, n\}$, let x_{ij} be an element of Λ and let e_{ij} be a positive integer. Assume that*

(i) *for $1 \leq i,\ j \leq n$, the integer p^r does not divide e_{ij}.*

(ii) *for each i and each j, the order of x_{ij} in Λ is n.*

(iii) *for each i and each j such that $i \neq j$, one has $e_{ij}x_{ij} = e_{ii}x_{ii} - e_{jj}x_{jj}$.*

Then Λ is spanned by all the x_{ij}.

Proof. Consider an automorphism ϕ of Λ such that $\phi(x_{1,1}) = (1, 0)$. By assumptions (i) and (ii), we have $e_{ij}x_{ij} \neq 0$ for all i and j. Hence $\phi(e_{11}x_{11}), \dots, \phi(e_{nn}x_{nn})$ are pairwise distinct and nonzero. According to the pigeonhole principle, we deduce that we are in one of the following situations:

Case 1: there exists $i_0 \in \{1, \dots, n\}$ such that $\phi(e_{i_0i_0}x_{i_0i_0}) \in \{0\} \times (\mathbb{Z}/n\mathbb{Z} \setminus \{0\})$. We then conclude that x_{11} and $x_{i_0i_0}$ span Λ.

Case 2: there exist $i_0, j_0 \in \{1, \dots, n\}$ such that $\phi(e_{i_0i_0}x_{i_0i_0}) - \phi(e_{j_0j_0}x_{j_0j_0}) \in \{0\} \times (\mathbb{Z}/n\mathbb{Z} \setminus \{0\})$. We conclude that x_{11} and $x_{i_0j_0}$ span the group Λ. □

In the sequel, we will also need the following easy lemma:

Lemma 1.13. *Let n be a positive integer and let $q(n)$ be the number of prime divisors of n. Let X be a generating set of $\Delta := \mathbb{Z}/n\mathbb{Z}$. Then X contains a subset X' which has at most $q(n)$ elements and which spans Δ.*

Proof. We proceed by induction on $q(n)$.

If $q(n) = 1$, then $n = p^a$ for some prime number p and some integer a. The set X contains an element x which is not divisible by p, and one can simply choose $X' = \{x\}$.

Now let q be a positive integer and assume that the lemma is known when $q(n) \leq q$. Take $n \geq 1$ such that $q(n) = q + 1$ and write $n = p_1^{a_1} \cdots p_{q+1}^{a_{q+1}}$ for $p_1, \dots, p_{q+1}$ pairwise distinct prime numbers and $a_1, \dots, a_{q+1}$ positive integers. The set X contains an element x which is not divisible by p_{q+1}. The quotient group $\Delta/\langle x\rangle$ is spanned by the image $\overline{X}$ of X in $\Delta/\langle x\rangle$. Since $\Delta/\langle x\rangle$ is a cyclic group whose order m satisfies $q(m) \leq q$, the induction hypothesis shows that one can find a subset $\overline{X_0}$ of $\overline{X}$ which has at most q elements and which spans $\Delta/\langle x\rangle$. By choosing any lifting $X_0 \subseteq X$ of $\overline{X_0} \subseteq \overline{X}$ having at most q elements, one sees that $\{x\} \cup X_0$ is a subset of X which has at most $q(n)$ elements and which spans Δ. $\square$

Lemma 1.11 applies to p-adic fields containing n-th roots of unity and such that p does not divide n. From this, our proof of Kato and Kuzumaki's conjecture yields the following proposition:

Proposition 1.14. *Let $n \geq 1$ be an integer. Let K be a totally imaginary number field containing n-th roots of unity. Let $f \in K[X_0, \dots, X_n]$ be a homogeneous polynomial of degree n of the form*

$$f = a_0 X_0^n + \cdots + a_n X_n^n + g(X_0, \dots, X_n),$$

where each monomial appearing in g contains at least three different variables. Set

$$N = \tfrac{1}{2} n(n+1) + 1 + [K : \mathbb{Q}] q(n)(q(n)+1),$$

where $q(n)$ denotes the number of prime divisors of n. Then there exist N finite extensions $K_1, \dots, K_N$ of K such that

(i) *the equation $f = 0$ has nontrivial solutions in K_i for each i,*

(ii) *$K^\times$ is spanned by the subgroups $N_{K_i/K}(K_i^\times)$ for $1 \leq i \leq N$.*

Proof. For $0 \leq i < j \leq n$, consider the field $K_{ij} = K\big(\sqrt[n]{a_i a_j^{-1}}\big)$. Fix v a place of K not dividing n and denote by $k(v)$ the residue field of K_v. Since K contains n-th roots of unity, n divides the order of $k(v)^\times$. Hence Proposition II.5.7 of [Neukirch 1999] implies that $K_v^\times/K_v^{\times n} \cong (\mathbb{Z}/n\mathbb{Z})^2$. Lemma 1.11 then shows that $K_v^\times$ is spanned by the subgroups $N_{K_{ij}\otimes_K K_v/K_v}((K_{ij} \otimes_K K_v)^\times)$.

Fix now v a place of K dividing n. Since the maximal unramified extension of K_v is a C_1-field [Lang 1952, Theorem 12], there exists a finite unramified extension $L_{v,0}$ of K_v such that the equation $f = 0$ has a nontrivial solution in $L_{v,0}$. As $L_{v,0}/K_v$ is unramified, the group $N_{L_{v,0}/K_v}(L_{v,0}{}^\times)$ contains $\mathcal{O}_v^\times$. Moreover, by Corollary 5.5 of [Wittenberg 2015], the group $K_v^\times$ is spanned by the images of the norm morphisms N_{M/K_v} when M describes finite extensions of K_v such that the equation $f = 0$ has nontrivial solutions in M; hence, by applying Lemma 1.13 to the group $\Delta = (K_v^\times/K_v^{\times n})\big/(\mathcal{O}_v^\times/\mathcal{O}_v^{\times n})$ (which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$), one can find $q(n)$ finite extensions $L_{v,1}, \dots, L_{v,q(n)}$ of K_v such that the equation $f = 0$ has nontrivial solutions in $L_{v,i}$ for each i and the subgroup of $K_v^\times$ spanned by the subgroups $N_{L_{v,i}/K_v}(L_{v,i}^\times)$ contains a uniformizer. Hence the group $K_v^\times$ is spanned by the subgroups $N_{L_{v,i}\otimes_K K_v/K_v}(L_{v,i}^\times)$ for $0 \leq i \leq q(n)$. By Greenberg's approximation theorem, we deduce that there exist finite extensions $M_{v,0}, M_{v,1}, \dots, M_{v,q(n)}$ of K

such that the equation $f=0$ has nontrivial solutions in $M_{v,i}$ for $0 \le i \le q(n)$ and $K_v^\times$ is spanned by the subgroups $N_{M_{v,i}\otimes_K K_v/K_v}((M_{v,i}\otimes_K K_v)^\times)$.

Let M be a Galois extension of K containing all the K_{ij} and all the $M_{v,i}$. Let L be a finite field extension of K which is linearly disjoint from M and such that the equation $f=0$ has a nontrivial zero in L. Such an extension L exists by Theorem 2 of [Kollár 2007] and by Lemma 1.1. Then, by Remark 1.6, the group $K^\times$ is spanned by the subgroups $N_{K_{ij}/K}(K_{ij}^\times)$ (for $0 \le i < j \le n$), $N_{M_{v,i}/K}(M_{v,i}^\times)$ (for $0 \le i \le q(n)$) and $N_{L/K}(L^\times)$. The corollary follows since the number of finite extensions of K that enter the game here is at most N. □

Here is a concrete example:

Example 1.15. Consider the case where $K=\mathbb{Q}(i)$ and

$$f = X_0^2 + 2X_1^2 + aX_2^2 \in K[X_0, X_1, X_2]$$

for some $a \in \mathbb{Z}$ such that a is congruent to 1, 3, 9, 11, 17, 19, 25 or 27 modulo 32. Let v_2 be the unique place of K above 2 and note that we have

$$1+i = N_{K_{v_2}(\sqrt{2})/K_{v_2}}\big(1+\tfrac{1}{2}(1-i)\sqrt{2}\big),$$

hence

$$1+i \in N_{K_{v_2}(\sqrt{2})/K_{v_2}}\big(K_{v_2}(\sqrt{2})^\times\big). \tag{2}$$

Moreover, one easily checks that the assumptions on a imply that the extension $K_{v_2}(\sqrt{a})/K_{v_2}$ is unramified. Hence

$$\mathcal{O}_{v_2}^\times \subseteq N_{K_{v_2}(\sqrt{a})/K_{v_2}}\big(K_{v_2}(\sqrt{a})^\times\big). \tag{3}$$

From the inclusions (2) and (3), we get that the group $K_{v_2}^\times$ is spanned by the subgroups $N_{K_{v_2}(\sqrt{2})/K_{v_2}}(K_{v_2}(\sqrt{2})^\times)$ and $N_{K_{v_2}(\sqrt{a})/K_{v_2}}(K_{v_2}(\sqrt{a})^\times)$. Using Lemma 1.11, we deduce that for each place v of K, the group $K_v^\times$ is spanned by the subgroups $N_{K_v(\sqrt{b})/K_v}(K_v(\sqrt{b})^\times)$ for $b \in \{2, a, 2a\}$. One can then easily check that the extensions $K(\sqrt{2}, \sqrt{a})$ and $K(\sqrt{a+2})$ are always linearly disjoint over K. Therefore $K^\times$ is spanned by the subgroups $N_{K(\sqrt{b})/K}(K(\sqrt{b})^\times)$ for $b \in \{2, a, 2a, a+2\}$. Of course, for such b, the equation $f=0$ has nontrivial solutions in $K(\sqrt{b})$.

Global fields of positive characteristic. In this paragraph, we focus on the case when K is a global field of positive characteristic. We prove of the C_1^1-property « away from p », and, as in the case of number fields, we see how the proof allows one to study some concrete examples.

We start by introducing a variant of the group $N_1(Z/K)$ which will allow us to study the C_1^1-property away from p for global fields of positive characteristic:

Definition 1.16. Let K be a field of characteristic $p > 0$. Let Z be a K-scheme of finite type. We denote by $N_1^p(Z/K)$ the set of $x \in K^\times$ such that there exists an integer $r \geq 1$ satisfying $x^{p^r} \in N_1(Z/K)$.

The following proposition is a consequence of Theorem 1.4:

Proposition 1.17. *Let K be the function field of a curve over a finite field of characteristic $p > 0$. Let Z be a K-variety containing a geometrically irreducible closed subscheme. For $v \in \Omega_K$, we denote by Z_v the K_v-scheme $Z \times_K K_v$. Then the abelian group*

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v \in \Omega_K} K_v^\times/N_1^p(Z_v/K_v)\right)$$

is a p-primary group.

Proof. Consider an element $x \in K^\times$ whose class modulo $N_1(Z/K)$ lies in

$$\mathrm{Ker}\left(K^\times/N_1(Z/K) \to \prod_{v \in \Omega_K} K_v^\times/N_1^p(Z_v/K_v)\right).$$

By assumption, for each $v \in \Omega_K$, there exists $r_v \geq 0$ such that $x^{p^{r_v}} \in N_1(Z_v/K_v) \subseteq N_1^s(Z_v/K_v)$. Furthermore, there exists an integer $m \geq 0$ such that $x^m \in N_1^s(Z_v/K_v)$ for each $v \in \Omega_K$. We conclude that there exists $r \geq 0$ such that $x^{p^r} \in N_1^s(Z_v/K_v)$ for each $v \in \Omega_K$. According to Theorem 1.4, this shows that $x^{p^r} \in N_1^s(Z/K)$. We can therefore consider finite extensions $K_1, \dots, K_n$ of K such that, if $K_{i,s}$ denotes the separable closure of K in K_i, we have $x^{p^r} \in \langle N_{K_{i,s}/K}(K_{i,s}^\times) \mid 1 \leq i \leq n\rangle$ and $Z(K_i) \neq \varnothing$ for each i. Since all the degrees $[K_i : K_{i,s}]$ are powers of p, this implies that there exists an integer $r' \geq 0$ such that $(x^{p^r})^{p^{r'}} \in \langle N_{K_i/K}(K_i^\times) \mid 1 \leq i \leq n\rangle$. We conclude that $x^{p^{r+r'}} \in N_1(Z/K)$, which finishes the proof of the corollary. $\square$

We are now ready to prove the C_1^1-property away from p for global fields of characteristic p.

Theorem 1.18. *Let K be the function field of a curve over a finite field of characteristic $p > 0$ and let Z be a hypersurface of degree d in $\mathbb{P}^n_K$ such that $d \leq n$. Then the exponent of the group $K^\times/N_1(Z/K)$ is a power of p.*

For the proof, it is useful to recall from [Wittenberg 2015] that a field L is said to be strongly C_1^1 away from p if, for each finite extension L' of L, each proper scheme Z over L' and each coherent sheaf E on Z, the Euler–Poincaré characteristic $\chi(Z, E)$ kills every element of $K_q^M(L')/N_q(Z/L')$ whose order is not divisible by p.

Proof. If A is a torsion abelian group, we denote by $A\{p'\}$ the subgroup of A constituted by elements of A whose order is not divisible by p. For each proper

K-scheme Z, we define

$$H_1(Z/K) = K^\times / N_1(Z/K)$$

and we denote by n_Z the exponent of the abelian group $H_1(Z/K)\{p'\}$ if Z is nonempty or 0 otherwise. We say that Z satisfies property P if Z is normal. We are now going to check the three assumptions that appear in Proposition 2.1 of [Wittenberg 2015].

(1) This is obvious, because a morphism of proper K-schemes $Y \to Z$ induces a surjective morphism $H_1(Y/K) \to H_1(Z/K)$.

(2) Let Z be a proper normal K-scheme. Let K' be the algebraic closure of K in $K(Z)$. Then Z is naturally endowed with a structure of proper geometrically irreducible K'-scheme. According to Proposition 1.17,

$$\operatorname{Ker}\left(H_1(Z/K') \to \prod_{v \in \Omega_K} H_1(Z_v/K'_v) \right)\{p'\} = 0.$$

Moreover, since K'_v is strongly C_1^1 away from p for each $v \in \Omega_K$ according to Corollary 4.7 of [Wittenberg 2015], the group $H_1(Z_v/K'_v)\{p'\}$ is killed by $\chi_{K'}(Z, \mathcal{O}_Z)$. We deduce that the group $H_1(Z/K')\{p'\}$ is also killed by $\chi_{K'}(Z, \mathcal{O}_Z)$. But $\chi_K(Z, \mathcal{O}_Z) = [K' : K]\chi_{K'}(Z, \mathcal{O}_Z)$. Hence a restriction-corestriction argument shows that $\chi_K(Z, \mathcal{O}_Z)$ kills $H_1(Z/K)\{p'\}$. The integer n_Z has therefore to divide $\chi_K(Z, \mathcal{O}_Z)$.

(3) It suffices to choose the normalization morphism.

We can therefore apply Proposition 2.1 of [Wittenberg 2015] and deduce that the field K is strongly C_1^1 away from p. The corollary then follows from the fact an $(n-1)$-dimensional projective hypersurface of degree d with $d \leq n$ has Euler–Poincaré characteristic 1. □

Remark 1.19. While Corollary 1.9 was already proved in [Wittenberg 2015], Theorem 1.18 is new.

In the same way as in the case of number fields, one can get more precise results. For example, one can prove the following proposition similarly to Proposition 1.14:

Proposition 1.20. *Let $n \geq 1$ be an integer. Let K be the function field of a curve over a finite field and assume that K contains n-th roots of unity. Let $f \in K[X_0, \dots, X_n]$ be a homogeneous polynomial of degree n of the form*

$$f = a_0 X_0^n + \cdots + a_n X_n^n + g(X_0, \dots, X_n),$$

where each monomial appearing in g contains at least three different variables. Assume that the projective hypersurface defined by $f = 0$ is geometrically irreducible.

Set

$$N = \tfrac{1}{2}n(n+1) + 1.$$

Then there exist N finite extensions $K_1, \dots, K_N$ of K such that

(i) *the equation $f = 0$ has nontrivial solutions in K_i for each i,*

(ii) *$K^\times$ is spanned by the subgroups $N_{K_i/K}(K^\times)$ for $1 \le i \le N$.*

2. Function fields of varieties over an algebraically closed field

In this section, we are going to establish Kato and Kuzumaki's conjectures for function fields of varieties over an algebraically closed field of characteristic 0. We have already recalled that the Bloch–Kato conjecture implies that a field of characteristic 0 is C_0^q if and only if it is of cohomological dimension at most q. The proposition that follows is a particular case of this result. Anyway, we give an elementary proof, because its ideas will be useful in the sequel in order to establish Theorems 2.2 and 3.9:

Proposition 2.1. *Let k be an algebraically closed field of characteristic* 0*. Then the field $K = k(t_1, \dots, t_q)$ satisfies property C_0^q.*

Proof. We proceed by induction on q. The result is obvious for $q = 0$. Assume now that we have proved the proposition for some $q \ge 0$ and consider the field $K = k(t_1, \dots, t_{q+1})$. Let L_1 be a finite extension of K and L_2 be a finite extension of L_1. Let $u_1, \dots, u_{q+1}$ be elements of $L_1^\times$. We are going to prove that $\{u_1, \dots, u_{q+1}\} \in N_{L_2/L_1}(K_{q+1}^M(L_2))$.

To do so, we construct a family $(w_1, \dots, w_s)$ of elements in $L_1^\times$ in the following way:

- if $u_1, \dots, u_{q+1}$ are not algebraically independent over k, we consider a transcendence basis $(v_1, \dots, v_r)$ of the extension $L_1/k(u_1, \dots, u_{q+1})$ and we set $(w_1, \dots, w_s) = (u_1, \dots, u_{q+1}, v_1, \dots, v_{r-1})$;
- if $u_1, \dots, u_{q+1}$ are algebraically independent over k, we set $(w_1, \dots, w_s) = (u_1, \dots, u_q)$.

Let M_1 (resp. M_2) be the algebraic closure of $k(w_1, \dots, w_s)$ in L_1 (resp. L_2). Let C_1 (resp. C_2) be a geometrically integral curve over M_1 (resp. M_2) such that $M_1(C_1) = L_1$ (resp. $M_2(C_2) = L_2$). Since $\overline{M}_1(C_1)$ is a C_1 field and $\overline{M}_2(C_2)/\overline{M}_1(C_1)$ is a finite extension, Propositions 10 and 11 of Section X.7 of [Serre 1979] imply that $u_{q+1} \in N_{\overline{M}_2(C_2)/\overline{M}_1(C_1)}(\overline{M}_2(C_2)^\times)$. And so there exist a finite extension F of M_2 and $y \in F(C_2)^\times$ such that $u_{q+1} = N_{F(C_2)/F(C_1)}(y)$. Moreover, by the inductive assumption, M_1 satisfies property C_0^q, and hence there exists $x \in K_q^M(F)$ such that

$\{u_1, \ldots, u_q\} = N_{F/M_1}(x)$. We deduce that

$$\begin{aligned} N_{L_2/L_1}(N_{F(C_2)/L_2}(\{x, y\})) &= N_{F(C_2)/M_1(C_1)}(\{x, y\}) \\ &= N_{F(C_1)/M_1(C_1)}(N_{F(C_2)/F(C_1)}(\{x, y\})) \\ &= N_{F(C_1)/M_1(C_1)}(\{x, u_{q+1}\}) \\ &= \{u_1, \ldots, u_{q+1}\}. \end{aligned}$$

We have therefore proved that $\{u_1, \ldots, u_{q+1}\} \in N_{L_2/L_1}(K^M_{q+1}(L_2))$. As a consequence, the field K satisfies the C^q_0-property. □

We are now ready to establish Kato and Kuzumaki's conjectures for the function field of a variety over an algebraically closed field of characteristic 0:

Theorem 2.2. *Let k be an algebraically closed field of characteristic* 0. *Then the function field of a q-dimensional integral k-variety satisfies the C^j_i-property for all $i \geq 0$ and $j \geq 0$ such that $i + j = q$.*

Proof. Let K be the function field of a q-dimensional integral k-variety. Let $i \geq 0$ and $j \geq 0$ be integers such that $i + j = q$. If $j = 0$, there is nothing to prove because the field K is C_q. If $i = 0$, the result follows from the previous proposition. Hence we can now assume that $i \neq 0$ and $j \neq 0$.

Fix a finite extension L of K. Let Z be a hypersurface of degree d in $\mathbb{P}^n_L$ with $d^i \leq n$ and let $u_1, \ldots, u_j$ be elements of $L^\times$. We will show that the symbol $\{u_1, \ldots, u_j\}$ is in $N_j(Z/K)$. Let $(v_1, \ldots, v_r)$ be a transcendence basis of the extension $L/k(u_1, \ldots, u_j)$ (with $r \geq 0$). Let M be the algebraic closure of $k(u_1, \ldots, u_j, v_{q-j+1}, \ldots, v_r)$ in L (so that the transcendence degree of M/k is j) and let X be a geometrically integral M-variety of dimension i such that $M(X) = L$. Since the field $\overline{M}(X)$ is C_i, the variety Z has points in $\overline{M}(X)$. Therefore, there exists a finite extension F of M such that $Z(F(X)) \neq \varnothing$. Moreover, since the norm $N_{F/M} : K^M_j(F) \to K^M_j(M)$ is surjective according to Proposition 2.1 and $\{u_1, \ldots, u_j\} \in K^M_j(M)$, we get $\{u_1, \ldots, u_j\} \in N_{F/M}(K^M_j(F))$. As a consequence, $\{u_1, \ldots, u_j\} \in N_{F(X)/M(X)}(K^M_j(F(X)))$, and K has the C^j_i-property. □

Remark 2.3. In the previous theorem, we have in fact proved that, if L is a finite extension of $k(t_1, \ldots, t_q)$ and Z is a hypersurface of degree d in $\mathbb{P}^n_L$ with $d^i \leq n$, then for each j-symbol $x \in K^M_j(L)$, there exists a finite extension M of L such that $Z(M) \neq \varnothing$ and $x \in N_{M/L}(K^M_j(M))$. In particular, if $i = q - 1$ and $j = 1$, for each element x in $L^\times$, there exists a finite extension M of L such that $Z(M) \neq \varnothing$ and $x \in N_{M/L}(M^\times)$.

3. Local fields with a function field as residue field

Problem and strategy. The goal of this section is to prove the conjectures of Kato and Kuzumaki for complete discrete valuation fields whose residue field is the function field of a variety over an algebraically closed field of characteristic 0. In particular, this applies to the field $\mathbb{C}(x)((t))$, for which properties C_0^2 and C_2^0 are already known and for which we are going to establish property C_1^1.

The main difficulty we face in order to establish the C_1^1-property for the field $K = \mathbb{C}(x)((t))$ lies in proving that, if Z is a hypersurface in $\mathbb{P}^n_K$ of degree $d \leq n$, then $t \in N_1(Z/K)$. To do so, we are going to show that if we adjoin all the roots of t to K, then the field K_∞ we obtain is C_1: this will imply that $Z(K_\infty) \neq \varnothing$. In order to establish the C_1-property for K_∞, we will have to establish a modular criterion allowing us to determine whether an affine variety over K_∞ has a rational point (Corollary 3.8). For this purpose we will heavily use the constructions of the article [Greenberg 1966].

Greenberg's approximation theorem revisited. We start by recalling the theorem:

Theorem 3.1 [Greenberg 1966, Theorem 1]. *Let R be a henselian discrete valuation ring with field of fractions K. Let t be a uniformizer of R. Let $\underline{F} = (F_1, \dots, F_r)$ be a system of r polynomials in n variables with coefficients in R. We assume that K has characteristic* 0. *Then there exist integers $N \geq 1$, $c \geq 1$ and $s \geq 0$ (depending exclusively on the ideal $\underline{F}R[\underline{X}]$ of $R[\underline{X}]$ generated by $F_1, \dots, F_r$) such that, for each $\nu \geq N$ and each $\underline{x} \in R^n$ satisfying*

$$\underline{F}(\underline{x}) \equiv 0 \mod t^\nu,$$

there exists $\underline{y} \in R^n$ such that

$$\underline{y} \equiv \underline{x} \mod t^{[\nu/c]-s} \quad \textit{and} \quad \underline{F}(\underline{y}) = 0.$$

In particular, if the system $\underline{F} = 0$ has solutions modulo t^m for each $m \geq 1$, then it has a solution in R.

From now on, fix a henselian discrete valuation ring R with field of fractions K. Assume that K has characteristic 0 and fix an algebraic closure $\bar{K}$ of K. Let t be a uniformizer of R and choose a compatible system $\{t^{1/q}\}_{q\geq 1}$ of roots of t in $\bar{K}$: by this, we mean that the elements $t^{1/q}$ of $\bar{K}$ satisfy the relation $(t^{1/(qq')})^{q'} = t^{1/q}$ for each $q, q' \geq 1$. For $q \geq 1$, we set $K_q = K(t^{1/q})$ and $R_q = \mathcal{O}_{K_q}$. We also set $K_\infty = \bigcup_{q\geq 1} K_q$ and $R_\infty = \bigcup_{q\geq 1} R_q$. We want to establish a similar result to Theorem 3.1 for the field K_∞. In that respect, we start by proving a simple lemma in commutative algebra.

Definition 3.2. We say that an ideal I of $R[\underline{X}]$ is t-saturated if, for each $f \in R[\underline{X}]$ such that $tf \in I$, we have $f \in I$.

Remark 3.3. Of course, the previous definition is independent of the choice of the uniformizer t. But since in the sequel we will have to replace R by R_q, it will be useful to systematically track a uniformizer of the ring we will be working on.

Lemma 3.4. *Let I be an ideal of $R[\underline{X}]$.*

(i) *If I is t-saturated, then $IR_q[\underline{X}]$ is $t^{1/q}$-saturated for each $q \geq 1$.*

(ii) *If I is radical and t-saturated, then $IR_q[\underline{X}]$ is radical for each $q \geq 1$.*

Proof. (i) Assume that I is t-saturated. Fix an integer $q \geq 1$ and let $f \in R_q[\underline{X}]$ such that $t^{1/q} f \in IR_q[\underline{X}]$. Write $t^{1/q} f = \sum_{i=1}^r f_i g_i$, with $f_i \in I$ and $g_i \in R_q[\underline{X}]$. For each $i \in \{1, \ldots, n\}$, let h_i be a polynomial in $R[\underline{X}]$ such that $t^{1/q}$ divides $g_i - h_i$ in R_q (i.e., the valuation of $g_i - h_i$ is strictly positive); this can be achieved because R and R_q have the same residue field. We can now write

$$t^{1/q} f = \sum_{i=1}^{r} f_i (g_i - h_i) + \sum_{i=1}^{r} f_i h_i.$$

Thus, t divides $\sum_{i=1}^r f_i h_i$ in $R[\underline{X}]$. Since I is t-saturated and $\sum_{i=1}^r f_i h_i \in I$, we deduce that $\sum_{i=1}^r f_i h_i / t \in I$. The equality

$$f = \sum_{i=1}^{r} f_i \frac{g_i - h_i}{t^{1/q}} + t^{(q-1)/q} \cdot \frac{\sum_{i=1}^r f_i h_i}{t}$$

then implies that $f \in IR_q[\underline{X}]$ and hence the ideal $IR_q[\underline{X}]$ is $t^{1/q}$-saturated.

(ii) Assume that I is radical and t-saturated. Fix $q \geq 1$ and let f be a polynomial in $R_q[\underline{X}]$ such that $f^n \in IR_q[\underline{X}]$ for some $n > 0$. Since I is radical, one immediately checks that $IK[\underline{X}]$ is also radical. This implies that $IK_q[\underline{X}]$ is also radical, because the extension K_q/K is separable. Hence $f \in IK_q[\underline{X}]$. This means that there exists $r \geq 1$ such that $t^{r/q} f \in IR_q[\underline{X}]$. Since $IR_q[\underline{X}]$ is $t^{1/q}$-saturated (by part (i)), we deduce that $f \in IR_q[\underline{X}]$. Hence the ideal $IR_q[\underline{X}]$ is indeed radical. □

In order to prove a similar result to Theorem 3.1 for K_∞, we need to work simultaneously with all the fields K_q, which all satisfy Theorem 3.1. More precisely, if we fix a system of polynomial equations over R, we can see it as a system of equations with coefficients in R_q for each $q \geq 1$; Theorem 3.1 then gives us integers N_q, c_q and s_q, and our goal in the sequel is to control these integers when q varies. It is therefore quite natural to introduce the following technical definition:

Definition 3.5. Let $\underline{F} = (F_1, \ldots, F_r)$ be a system of r polynomials in n variables with coefficients in R.

(i) Fix $q \in \mathbb{N}$. We say that a triplet $(N, c, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0$ is associated to $(R, t, q, \underline{F})$ if it satisfies the following property: for each $\nu \geq N$ and each

$\underline{x} \in R_q^n$ such that

$$\underline{F}(\underline{x}) \equiv 0 \mod t^{\nu/q},$$

there exists $\underline{y} \in R_q^n$ such that

$$\underline{y} \equiv \underline{x} \mod t^{([\nu/c]-s)/q} \quad \text{and} \quad \underline{F}(\underline{y}) = 0.$$

(ii) We say that a 4-tuple $(q_0, N, c, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0$ is $(R, t, \underline{F})$-admissible if, for each $q \geq 1$, the triplet (qN, c, qs) is associated to $(R, t, qq_0, \underline{F})$.

We are now ready to state the following theorem:

Theorem 3.6. *Let R be a henselian discrete valuation ring with field of fractions K. Assume that K has characteristic 0 and fix a uniformizer t of R. For $q \geq 1$, we set $K_q = K(t^{1/q})$ and $R_q = \mathcal{O}_{K_q}$. Let $\underline{F} = (F_1, \dots, F_r)$ be a system of r polynomials in n variables with coefficients in R. Then there exists a 4-tuple (q_0, N, c, s) which is $(R, t, \underline{F})$-admissible.*

In order to establish this theorem, we are going to use considerably the constructions developed in the proof of Theorem 3.1 (see [Greenberg 1966]).

Proof. Denote by V the affine K-variety defined by $\underline{F} = 0$ and let m be its dimension. We are going to prove by induction on m that there exists a $(R, t, \underline{F})$-admissible 4-tuple of integers.

- If $m = -1$ (i.e., $V = \varnothing$), there exists a nonzero $u \in R \cap \underline{F}R[\underline{X}]$. Then the 4-tuple $(1, \mathrm{val}_R(u) + 1, 1, 0)$ is $(R, t, \underline{F})$-admissible, since for these values of q_0, N, c, s, the assumption appearing in Definition 3.5(i) fails.

- Now assume that $m \geq 0$.

∘ Assume in the first place that $\underline{F}R[\underline{X}]$ is radical and t-saturated, and that V_{K_∞} is irreducible. In this case, Lemma 3.4 shows that the ideal $\underline{F}R_q[\underline{X}]$ of $R_q[\underline{X}]$ is radical for each $q \geq 1$. Let J be the jacobian matrix of $\underline{F}$ and let $\underline{D}$ be the system of minors of size $n-m$ in J. By the inductive assumption, there exists a 4-tuple (q_0', N', c', s') which is $(R, t, \underline{F}, \underline{D})$-admissible. For $I \subseteq \{1, \dots, r\}$ with $|I| = n-m$, denote by $\underline{F}_I$ the system constituted by the polynomials F_i for $i \in I$. Let V_I be the K-variety defined by the system $\underline{F}_I = 0$. Let V_I^+ be the union of the irreducible components of V_I which are m-dimensional and different from V. Let $\underline{G}_I$ be a system of generators of the ideal of V_I^+ in $R[\underline{X}]$. By the inductive assumption, there exists a 4-tuple $(q_{0,I}, N_I, c_I, s_I)$ which is $(R, t, \underline{G}_I, \underline{F})$-admissible. Set

$$q_0 = q_0' \prod_{\substack{|I|=n-m \\ I \subseteq \{1,\dots,n\}}} q_{0,I}.$$

According to the proof of Theorem 1 of [Greenberg 1966], for each $q \geq 1$, the triplet $(N^{(q)}, c^{(q)}, s^{(q)})$ defined by

$$\begin{aligned} N^{(q)} &= 2 + 2qq_0 \max\Big\{\frac{N'}{q_0'}, \max\Big\{\frac{N_I}{q_{0,I}} \;\Big|\; I \subseteq \{1, \dots, n\},\ |I| = n-m\Big\}\Big\} \\ c^{(q)} &= 2\max\{c', \max\{c_I \mid I \subseteq \{1, \dots, n\},\ |I| = n-m\}\} \\ s^{(q)} &= 1 + qq_0 \max\Big\{\frac{s'}{q_0'}, \max\Big\{\frac{s_I}{q_{0,I}} \;\Big|\; I \subseteq \{1, \dots, n\},\ |I| = n-m\Big\}\Big\} \end{aligned}$$

is associated to $(R, t, qq_0, \underline{F})$. We deduce that the 4-tuple (q_0, N, c, s) defined by

$$\begin{aligned} N &= 2 + 2q_0 \max\Big\{\frac{N'}{q_0'}, \max\Big\{\frac{N_I}{q_{0,I}} \;\Big|\; I \subseteq \{1, \dots, n\}, |I| = n-m\Big\}\Big\} \\ c &= 2\max\{c', \max\{c_I \mid I \subseteq \{1, \dots, n\}, |I| = n-m\}\} \\ s &= 1 + q_0 \max\Big\{\frac{s'}{q_0'}, \max\Big\{\frac{s_I}{q_{0,I}} \;\Big|\; I \subseteq \{1, \dots, n\}, |I| = n-m\Big\}\Big\} \end{aligned}$$

is $(R, t, \underline{F})$-admissible.

∘ We no longer make any assumptions on the ideal $\underline{F}R[\underline{X}]$ or the variety V_{K_∞}. Let $q_0' \geq 1$ be an integer such that the irreducible components $W_1, \dots, W_u$ of $V_{K_{q_0'}}$ remain irreducible over K_∞. For each $j \in \{1, \dots, u\}$, let I_j' be the prime ideal of $K_{q_0'}[\underline{X}]$ defining W_j. Consider the ideal

$$I_j := I_j' \cap R_{q_0'}[\underline{X}].$$

Let $\underline{G}_j$ be a system of generators of I_j. The ideal I_j is radical and $t^{1/q_0'}$-saturated. Moreover, the $K_{q_0'}$-variety defined by I_j is W_j; it is a variety of dimension at most m and $(W_j)_{K_\infty}$ is irreducible. We deduce that there exists a 4-tuple $(q_{0,j}, N_j, c_j, s_j)$ which is $(R_{q_0'}, t^{1/q_0'}, \underline{G}_j)$-admissible. Note now that there exists an integer $w \in \mathbb{N}^*$ such that $(I_1' \cdots I_u')^w \subseteq \underline{F}K_{q_0'}[\underline{X}]$. Hence there exists $v \in \mathbb{N}^*$ such that

$$t^{uvw/q_0'}(I_1 \cdots I_u)^w \subseteq \underline{F}R_{q_0'}[\underline{X}]. \tag{4}$$

Set $q_0 = q_0' \prod_j q_{0,j}$, and consider an integer $q \geq 1$. Denote by $\mathrm{val} : R_{qq_0} \to \mathbb{Z} \cup \{\infty\}$ the valuation on R_{qq_0}, and introduce the integers

$$\begin{aligned} N^{(q)} &= uw\frac{qq_0}{q_0'}\Big(\max\Big\{\frac{N_j}{q_{0,j}}\Big\} + v\Big), \\ c^{(q)} &= uw\max\{c_j\}, \\ s^{(q)} &= 1 + \frac{qq_0}{q_0'}\Big(v + \max\Big\{\frac{s_j}{q_{0,j}}\Big\}\Big). \end{aligned}$$

Fix $\nu \geq N^{(q)}$ and $\underline{x} \in R^n_{qq_0}$ such that $\underline{F}(\underline{x}) \equiv 0 \mod t^{\nu/(qq_0)}$. If we are given polynomials $g_1 \in I_1 R_{qq_0}[\underline{X}], \dots, g_u \in I_u R_{qq_0}[\underline{X}]$, the inclusion (4) implies that

$$\nu \leq \operatorname{val}\left[t^{uvw/q'_0}\left(\prod_{j=1}^{u} g_j(\underline{x})\right)^w\right] = \frac{qq_0}{q'_0}uvw + w\sum_j \operatorname{val}(g_j(\underline{x})).$$

Hence there exists an integer $j_0 \in \{1, \dots, u\}$ such that

$$\operatorname{val}(g_{j_0}(\underline{x})) \geq \frac{\nu}{uw} - \frac{qq_0}{q'_0}v.$$

Since this is true whatever the chosen polynomials $g_1 \in I_1 R_{qq_0}[\underline{X}], \dots, g_u \in I_u R_{qq_0}[\underline{X}]$ are, we conclude that there exists $j_1 \in \{1, \dots, u\}$ such that

$$\text{for all } g \in I_{j_1} R_{qq_0}[\underline{X}], \quad \operatorname{val}(g(\underline{x})) \geq \frac{\nu}{uw} - \frac{qq_0}{q'_0}v. \tag{5}$$

As $\nu \geq N^{(q)}$, we also have

$$\frac{\nu}{uw} - \frac{qq_0}{q'_0}v \geq \frac{qq_0}{q'_0 q_{0,j_0}} N_{j_0}. \tag{6}$$

Since the 4-tuple $(q_{0,j_0}, N_{j_0}, c_{j_0}, s_{j_0})$ is $(R_{q'_0}, t^{1/q'_0}, \underline{G}_{j_0})$-admissible, the triplet $((qq_0/q'_0 q_{0,j})N_{j_0}, c_{j_0}, (qq_0/q'_0 q_{0,j_0})s_{j_0})$ is associated to $(R_{q'_0}, t^{1/q'_0}, (qq_0/q'_0), \underline{G}_{j_0})$. We then deduce from (5) and (6) that there exists $\underline{y} \in R^n_{qq_0}$ such that

$$\underline{y} \equiv \underline{x} \mod t^{\mu/(qq_0)} \quad \text{and} \quad \underline{G}_{j_0}(\underline{y}) = 0,$$

where $\mu = [\nu/c_{j_0}uw] - (qq_0/q'_0 c_{j_0})v - 1 - (qq_0/q'_0 q_{0,j_0})s_{j_0}$. This implies that

$$\underline{y} \equiv \underline{x} \mod t^{([\nu/c^{(q)}] - s^{(q)})/(qq_0)} \quad \text{and} \quad \underline{F}(\underline{y}) = 0,$$

and hence the triplet $(N^{(q)}, c^{(q)}, s^{(q)})$ is associated to $(R, t, qq_0, \underline{F})$. Therefore the 4-tuple (q_0, N, c, s) defined by

$$N = uw\frac{q_0}{q'_0}\left(\max\left\{\frac{N_j}{q_{0,j}}\right\} + v\right), \quad c = uw\max\{c_j\}, \quad s = 1 + \frac{q_0}{q'_0}\left(v + \max\left\{\frac{s_j}{q_{0,j}}\right\}\right)$$

is $(R, t, \underline{F})$-admissible. □

Corollary 3.7. *Let R be a henselian discrete valuation ring with field of fractions K. Assume that K has characteristic 0 and fix a uniformizer t of R. For $q \geq 1$, we set $K_q = K(t^{1/q})$ and $R_q = \mathcal{O}_{K_q}$. We also set $K_\infty = \bigcup_{q \geq 1} K_q$ and $R_\infty = \bigcup_{q\geq 1} R_q$. Let $\underline{F} = (F_1, \dots, F_r)$ be a system of r polynomials in n variables with coefficients in R_∞. There exists $M \in \mathbb{Q}_{>0}$, $\gamma \in \mathbb{N}$ and $\sigma \in \mathbb{Q}_{>0}$ satisfying the following property: for each rational number $\mu \geq M$ and each $\underline{x} \in R^n_\infty$ such that*

$$\underline{F}(\underline{x}) \equiv 0 \mod t^\mu,$$

there exists $\underline{y} \in R_\infty^n$ *such that*

$$\underline{y} \equiv \underline{x} \mod t^{\mu/\gamma-\sigma} \quad \textit{and} \quad \underline{F}(\underline{y}) = 0.$$

Proof. By replacing R by R_q for some sufficiently large q, we can assume that the system $\underline{F}$ has coefficients in R. According to Theorem 3.6, there exists a $(R, t, \underline{F})$-admissible 4-tuple (q_0, N, c, s). Set $M = N/q_0$, $\gamma = c$ and $\sigma = (s+1)/q_0$. Consider $\mu \in \mathbb{Q}$ such that $\mu \geq M$ and write $\mu = a/b$ with $a, b \in \mathbb{N}$. Assume that there exists $\underline{x} \in R_\infty^n$ such that $\underline{F}(\underline{x}) \equiv 0 \mod t^\mu$. Let $q_1 \geq 1$ be such that $\underline{x} \in R_{q_1}^n$. We know that, for each $q \geq 1$, the triplet (qN, c, qs) is associated to $(R, t, qq_0, \underline{F})$. In particular, the triplet (bq_1N, c, bq_1s) is associated to $(R, t, bq_1q_0, \underline{F})$. Since $\underline{F}(\underline{x}) \equiv 0 \mod t^\mu$ and $\mu \geq N/q_0$, we deduce that there exists $\underline{y} \in R_{bq_0q_1}^n$ such that $\underline{F}(\underline{y}) = 0$ and

$$\underline{y} \equiv \underline{x} \mod t^\lambda \quad \text{with } \lambda = \frac{1}{bq_1q_0}\left(\left\lceil \frac{aq_1q_0}{c} \right\rceil - bq_1s\right).$$

This finishes the proof because $\lambda \geq \mu/c - \sigma$. □

Corollary 3.8. *Under the assumptions of Corollary 3.7, if the congruence* $\underline{F}(\underline{x}) \equiv 0$ mod t^ν *has solutions in* R_∞ *for each integer* $\nu \geq 1$, *then the equation* $\underline{F}(\underline{x}) = 0$ *has solutions in* R_∞.

Statement for the field $\mathbb{C}(x_1, \ldots, x_m)((t))$**.** We are finally ready to establish Kato and Kuzumaki's conjectures for complete discrete valuation fields whose residue field is the function field of a variety over an algebraically closed field of characteristic 0:

Theorem 3.9. *Let k be an algebraically closed field of characteristic zero and fix $m \geq 1$. Let Y be an m-dimensional integral k-variety and set $K = k(Y)((t))$. Then the complete field K satisfies the C_i^j-property for all $i \geq 0$ and $j \geq 0$ such that $i + j = m + 1$.*

Proof. Since the field K is C_{m+1} and has cohomological dimension $m+1$, we can assume that $j \neq 0$ and $i \neq 0$. In the sequel, we fix an algebraic closure $\bar{K}$ of K. All fields will be understood as subfields of $\bar{K}$.

Let Z be a hypersurface of $\mathbb{P}_K^n$ of degree d, with $d^i \leq n$. We want to prove that $N_j(Z/K) = K_j^M(K)$.

Fix first a j-tuple $(u_1, \ldots, u_j) \in k(Y)^{\times j}$. We are going to prove that $\{u_1, \ldots, u_j\} \in N_j(Z/K)$. For this purpose, let $(v_1, \ldots, v_r)$ be a transcendence basis of the extension $k(Y)/k(u_1, \ldots, u_j)$ and denote by M the algebraic closure of the field $k(u_1, \ldots, u_j, v_{m-j+1}, \ldots, v_r)$ in K; it is a field of transcendence degree j over k. Let Y' be a geometrically integral M-variety of dimension $i-1$ such that $k(Y) = M(Y')$.

The field $\overline{M}(Y')$ is C_{i-1}, and therefore the field

$$K_M := \bigcup_{F/M \text{ finite}} F(Y')((t))$$

is C_i. We deduce that $Z(K_M) \neq \varnothing$, and hence there exists a finite extension F of M such that $Z(F(Y')((t))) \neq \varnothing$. Since M is C_0^j, we have $\{u_1, \ldots, u_j\} \in N_{F/M}(K_j^M(F))$, and hence $\{u_1, \ldots, u_j\} \in N_j(Z/K)$ as desired.

Fix now a $(j-1)$-tuple $(u_1, \ldots, u_{j-1}) \in k(Y)^{\times j-1}$. We are going to prove that $\{u_1, \ldots, u_{j-1}, t\} \in N_j(Z/K)$. For this purpose, consider a homogeneous polynomial $f \in k(Y)[[t]][X_0, \ldots, X_n]$ defining Z. Let $(v_1, \ldots, v_r)$ be a transcendence basis of the extension $k(Y)/k(u_1, \ldots, u_{j-1})$ and denote by M the algebraic closure of $k(u_1, \ldots, u_{j-1}, v_{m-j+2}, \ldots, v_r)$ in K: it is a field of transcendence degree $j-1$ over k. Let Y' be a geometrically integral M-variety of dimension i such that $k(Y) = M(Y')$. We set

$$K_M := \bigcup_{F/M \text{ finite}} F(Y')((t)), \quad R_M := \bigcup_{F/M \text{ finite}} F(Y')[[t]].$$

The ring R_M is a henselian discrete valuation ring with fraction field K_M, uniformizer t and residue field $\overline{M}(Y')$. We also set

$$K_\infty := \bigcup_{q \geq 1} K_M(t^{1/q}), \quad R_\infty := \bigcup_{q \geq 1} R_M[t^{1/q}].$$

Let $\mathfrak{m}_\infty$ be the maximal ideal of R_∞ and fix an integer $\nu \geq 1$. Let f_ν and g_ν be homogeneous polynomials of degree d, $f_\nu \in M((t))(Y')[X_0, \ldots, X_n]$ and $g_\nu \in R_\infty[X_0, \ldots, X_n]$ such that

$$f = f_\nu + t^\nu g_\nu.$$

Since $\overline{M((t))}(Y')$ is C_i and is contained in K_∞, there exists $(x_0, \ldots, x_n) \in R_\infty^{n+1} \setminus \mathfrak{m}_\infty^{n+1}$ such that $f_\nu(x_0, \ldots, x_n) = 0$. We therefore have

$$f(x_0, \ldots, x_n) \equiv 0 \mod t^\nu.$$

Since this is satisfied for each $\nu \geq 1$, we deduce from Corollary 3.8 that $Z(K_\infty) \neq \varnothing$. We can then consider a finite extension F/M and an integer $q \geq 1$ such that $Z(F(Y')((t^{1/q}))) \neq \varnothing$. As M has the C_0^{j-1}-property, there exists $x \in K_{j-1}^M(F)$ such that $N_{F/K}(x) = \{u_1, \ldots, u_{j-1}\}$. Hence:

$$\begin{aligned} N_{F(Y')((t^{1/q}))/K}(\{x, t^{1/q}\}) &= N_{F(Y')((t))/M(Y')((t))}\big(N_{F(Y')((t^{1/q}))/F(Y')((t))}(\{x, t^{1/q}\})\big) \\ &= N_{F(Y')((t))/M(Y')((t))}(\{x, \pm t\}) \\ &= \{u_1, \ldots, u_{j-1}, \pm t\}. \end{aligned}$$

We conclude that $\{u_1, \ldots, u_{j-1}, t\} \in N_j(Z/K)$.

Since the group $K_j^M(K)/d$ is spanned by symbols of the form $\{u_1, \dots, u_j\}$ and $\{u_1, \dots, u_{j-1}, t\}$ with $(u_1, \dots, u_j) \in k(Y)^{\times j}$, we get $N_j(Z/K) = K_j^M(K)$. □

Remark 3.10. Let k be an algebraically closed field of characteristic 0 and let Y be an integral k-variety of dimension m. The previous proof shows in fact that, if M is an extension of transcendence degree $j-1$ over k contained in $k(Y)$ and if Y' is an integral M-variety such that $M(Y') = k(Y)$, then the field

$$K_\infty = \bigcup_{q\geq 1} \bigcup_{F/M \text{ finite}} F(Y')((t))(t^{1/q})$$

is C_{m+1-j}. In particular, the field $\bigcup_{q\geq 1} \mathbb{C}(x)((t))(t^{1/q})$ is C_1.

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Height bounds and the Siegel property

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Let $\boldsymbol{G}$ be a reductive group defined over $\mathbb{Q}$ and let $\mathfrak{S}$ be a Siegel set in $\boldsymbol{G}(\mathbb{R})$. The Siegel property tells us that there are only finitely many $\gamma \in \boldsymbol{G}(\mathbb{Q})$ of bounded determinant and denominator for which the translate $\gamma.\mathfrak{S}$ intersects $\mathfrak{S}$. We prove a bound for the height of these γ which is polynomial with respect to the determinant and denominator. The bound generalises a result of Habegger and Pila dealing with the case of GL_2, and has applications to the Zilber–Pink conjecture on unlikely intersections in Shimura varieties.

In addition we prove that if $\boldsymbol{H}$ is a subset of $\boldsymbol{G}$, then every Siegel set for $\boldsymbol{H}$ is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set for $\boldsymbol{G}$.

1. Introduction

A Siegel set is a subset of the real points $\boldsymbol{G}(\mathbb{R})$ of a reductive $\mathbb{Q}$-algebraic group of a certain nice form. The notion of a Siegel set was introduced by Borel and Harish-Chandra [1962], in order to prove the finiteness of the covolume of arithmetic subgroups of $\boldsymbol{G}(\mathbb{R})$. In this paper we use a variant of the notion due to Borel [1969] which takes into account the $\mathbb{Q}$-structure of the group $\boldsymbol{G}$, and gives an intrinsic construction of fundamental sets for arithmetic subgroups in $\boldsymbol{G}(\mathbb{R})$.

Let $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ be a Siegel set (see Section 2 for the precise definition). The primary theorem of this paper is a bound for the height of elements of

$$\mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q}) = \{\gamma \in \boldsymbol{G}(\mathbb{Q}) : \gamma.\mathfrak{S} \cap \mathfrak{S} \neq \varnothing\}$$

in terms of their determinant and denominators. This gives a quantitative version of [Borel 1969, Corollaire 15.3], which asserts that $\mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q})$ has only finitely many elements with given determinant and denominators. This in turn implies a quantitative version of the Siegel property, one of the key properties of Siegel sets.

Theorem 1.1. *Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group and let $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ be a Siegel set. Let $\rho : \boldsymbol{G} \to \mathrm{GL}_n$ be a faithful $\mathbb{Q}$-algebraic group representation.*

There exists a constant C_1 (depending on $\boldsymbol{G}$, $\mathfrak{S}$ and ρ) such that, for all

$$\gamma \in \mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q}),$$

if $N = |\det \rho(\gamma)|$ and D is the maximum of the denominators of entries of $\rho(\gamma)$, then

$$\mathrm{H}(\rho(\gamma)) \leq \max(C_1 N D^n, D).$$

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This theorem was inspired by a result of Habegger and Pila [2012, Lemma 5.2]. They dealt with the case $\boldsymbol{G} = \mathrm{GL}_2$, as a step in proving some cases of the Zilber–Pink conjecture on unlikely intersections in $Y(1)^n$. We are motivated by applications of Theorem 1.1 to the Zilber–Pink conjecture in higher-dimensional Shimura varieties, which is the subject of work in progress by the author. The key point for these applications is that the bound is polynomial in the determinant N.

The second main theorem of this paper compares Siegel sets for the group $\boldsymbol{G}$ with Siegel sets for a subgroup $\boldsymbol{H} \subset \boldsymbol{G}$, which can be seen as a result on the functoriality of Siegel sets with respect to injections of $\mathbb{Q}$-algebraic groups. This theorem is used in the proof of Theorem 1.1 to reduce to the case $\boldsymbol{G} = \mathrm{GL}_n$. It also has its own applications to the Zilber–Pink conjecture.

Theorem 1.2. *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be reductive $\mathbb{Q}$-algebraic groups, with $\boldsymbol{H} \subset \boldsymbol{G}$. Let $\mathfrak{S}_{\boldsymbol{H}}$ be a Siegel set in $\boldsymbol{H}(\mathbb{R})$. Then there exist a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ and a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ such that*

$$\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}.$$

Theorem 4.1 gives some additional information about how the Siegel sets $\mathfrak{S}_{\boldsymbol{G}}$ and $\mathfrak{S}_{\boldsymbol{H}}$ are related to each other (in terms of the associated Siegel triples).

1A. ***Previous results: height bounds.*** The primary inspiration for Theorem 1.1 is the following result of Habegger and Pila.

Proposition 1.3 [Habegger and Pila 2012, Lemma 5.2]. *Let $\mathcal{F}$ denote the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane.*

There exists a constant C_2 such that: for all points x, $y \in \mathcal{F}$, if the associated elliptic curves are related by an isogeny of degree N, then there exists $\gamma \in \mathrm{M}_2(\mathbb{Z})$ such that

$$\gamma x = y, \quad \det \gamma = N \quad \textit{and} \quad \mathrm{H}(\gamma) \leq C_2 N^{10}.$$

In order to relate Proposition 1.3 to Theorem 1.1, recall that the upper half-plane $\mathcal{H}$ can be identified with the symmetric space $\mathrm{GL}_2(\mathbb{R})^+/\mathbb{R}^\times \, \mathrm{SO}_2(\mathbb{R})$, with $\mathrm{GL}_2(\mathbb{R})^+$ acting on $\mathcal{H}$ by Möbius transformations. Under this identification, the standard fundamental domain

$$\mathcal{F} = \left\{ z \in \mathcal{H} : -\tfrac{1}{2} \leq \operatorname{Re} z \leq \tfrac{1}{2},\ |z| \geq 1 \right\}$$

is contained in the image of the standard Siegel set

$$\mathfrak{S} = \Omega_{1/2} A_{\sqrt{3}/2} K \subset \mathrm{GL}_2(\mathbb{R}),$$

as defined in Section 2A.

We further identify the quotient $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ with the moduli space $Y(1)$ of elliptic curves over $\mathbb{C}$. It is easy to prove that the elliptic curves associated with points x, $y \in \mathcal{H}$ are related by an isogeny of degree N if and only if there exists $\gamma \in \mathrm{M}_2(\mathbb{Z})$ such that

$$\gamma x = y \quad \text{and} \quad \det \gamma = N. \tag{1}$$

Theorem 1.1 tells us that any γ satisfying (1) has height at most $C_1 N$, improving on the exponent 10 which appears in Proposition 1.3.

Theorem 1.1 also implies a uniform version of the following previous result of the author (which is a combination of [Orr 2015, Lemma 3.3] with [Orr 2017, Theorem 1.3]).

Proposition 1.4. *Let $\mathcal{F}_g$ denote the standard fundamental domain for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on the Siegel upper half-space of rank g. Fix a point $x \in \mathcal{F}_g$.*

There exist constants C_3 and C_4 such that for all points $y \in \mathcal{F}_g$, if the principally polarised abelian varieties associated with x and y are related by a polarised isogeny of degree N, then there exists a matrix $\gamma \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ such that

$$\gamma x = y \quad \textit{and} \quad \mathrm{H}(\gamma) \leq C_3 N^{C_4}.$$

In Proposition 1.4, the constant C_3 depends on the fixed point $x \in \mathcal{F}_g$ and only the other point y is allowed to vary. On the other hand, we can apply Theorem 1.1 to the symmetric space $\mathcal{H}_g$ in a similar way to that sketched above for $\mathcal{H}$. This gives a much stronger result in which the constant is uniform in both x and y. Hence Theorem 1.1 can be used to prove results on unlikely intersections in $\mathcal{A}_g \times \mathcal{A}_g$ for which Proposition 1.4 is not sufficient.

Note that [Orr 2015, Lemma 3.3] gives a height bound for unpolarised as well as polarised isogenies. It is not possible to directly deduce a uniform version of this bound for unpolarised isogenies from Theorem 1.1 because [Orr 2015, Lemma 3.3] concerns the homogeneous space $\mathrm{GL}_{2g}(\mathbb{R})/\mathrm{GL}_g(\mathbb{C})$ while Theorem 1.1 applies to the symmetric space $\mathrm{GL}_{2g}(\mathbb{R})/\mathbb{R}^\times \mathrm{O}_{2g}(\mathbb{R})$.

1B. ***Previous results: Siegel sets and subgroups.*** Let $\boldsymbol{H}$ be a reductive $\mathbb{Q}$-algebraic subgroup of $\boldsymbol{G} = \mathrm{GL}_n$. Borel and Harish-Chandra [1962, Theorem 6.5] gave a recipe for constructing a fundamental set for $\boldsymbol{H}(\mathbb{R})$ which is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set for $\boldsymbol{G}$. However it is not obvious how the resulting fundamental set is related to a Siegel set for $\boldsymbol{H}$. Theorem 1.2 resolves this by directly relating Siegel sets for $\boldsymbol{G}$ and $\boldsymbol{H}$.

Theorem 1.2 can also be interpreted as a result about functoriality of Siegel sets. According to a remark on [Borel 1969, p. 86], if $f : \boldsymbol{H} \to \boldsymbol{G}$ is a *surjective* morphism of reductive $\mathbb{Q}$-algebraic groups and $\mathfrak{S}_{\boldsymbol{H}}$ is a Siegel set in $\boldsymbol{H}(\mathbb{R})$, then $f(\mathfrak{S}_{\boldsymbol{H}})$ is contained in a Siegel set in $\boldsymbol{G}(\mathbb{R})$. Theorem 1.2 gives a similar result for injective morphisms of reductive $\mathbb{Q}$-algebraic groups, where the conclusion must be weakened to saying that the image of a Siegel set is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set. We can of course combine these to conclude that for an arbitrary morphism $f : \boldsymbol{H} \to \boldsymbol{G}$, the image of a Siegel set $\mathfrak{S}_{\boldsymbol{H}} \subset \boldsymbol{H}(\mathbb{R})$ is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set in $\boldsymbol{G}(\mathbb{R})$.

The proof of Theorem 1.2 gives an explicit bound for the size of the set $C \subset \boldsymbol{G}(\mathbb{Q})$, namely $\#C$ is at most the size of the $\mathbb{Q}$-Weyl group of $\boldsymbol{G}$. The uniform nature of this bound is less powerful than it might at first appear because the Siegel set $\mathfrak{S}_{\boldsymbol{G}}$ depends on $\mathfrak{S}_{\boldsymbol{H}}$.

1C. ***Application to unlikely intersections.*** The author's motivation for studying Theorem 1.1 is due to its applications to the Zilber–Pink conjecture on unlikely intersections in Shimura varieties [Pink 2005, Conjecture 1.2]. To illustrate these applications, consider the following special case of the Zilber–Pink conjecture.

Conjecture 1.5. *Let $g \geq 2$ and let $\mathcal{A}_g$ denote the moduli space of principally polarised abelian varieties of dimension g over $\mathbb{C}$. For each point $s \in \mathcal{A}_g$, let (A_s, λ_s) denote the associated principally polarised abelian variety. Let*

$$\Sigma = \{(s_1, s_2) \in \mathcal{A}_g \times \mathcal{A}_g : \text{there exists an isogeny } A_{s_1} \to A_{s_2}\}.$$

Let $V \subset \mathcal{A}_g \times \mathcal{A}_g$ be an irreducible algebraic curve. If $V \cap \Sigma$ is infinite, then V is contained in a proper special subvariety of $\mathcal{A}_g \times \mathcal{A}_g$.

Habegger and Pila [2012] used Proposition 1.3 to prove a result similar to Conjecture 1.5 but for the Shimura variety $\mathcal{A}_1^n$ ($n \geq 3$) instead of $\mathcal{A}_g \times \mathcal{A}_g$ ($g \geq 2$) (for reasons of dimension, Conjecture 1.5 is false for $\mathcal{A}_1 \times \mathcal{A}_1$).

In work currently in progress, the author of this paper proves Conjecture 1.5 subject to certain technical conditions and a restricted definition of the set Σ. This work requires the uniform version of Proposition 1.4 which is implied by the GSp_{2g} case of Theorem 1.1. Because Theorem 1.1 applies to all reductive groups, not just GSp_{2g}, it should also be useful for proving statements similar to Conjecture 1.5 where $\mathcal{A}_g$ is replaced by an arbitrary Shimura variety. However, at present it is not known how to prove the Galois bounds which would be required for such a statement.

1D. ***Outline of paper.*** Section 2 contains the definition of Siegel sets and the associated notation used throughout the paper. In Section 3 we prove Theorem 1.1 for standard Siegel sets in GL_n, and combine this with Theorem 1.2 to deduce the general statement of Theorem 1.1. The proof of the GL_n case is entirely self-contained. Finally Section 4 contains the proof of Theorem 1.2, relying on results on parabolic subgroups and roots from [Borel and Tits 1965].

1E. ***Notation.*** If $\boldsymbol{G}$ is a real algebraic group, then we write $\boldsymbol{G}(\mathbb{R})^+$ for the identity component of $\boldsymbol{G}(\mathbb{R})$ in the Euclidean topology.

We use a naive definition for the height of a matrix with rational entries, as in [Pila and Wilkie 2006]: if $\gamma \in \mathrm{M}_n(\mathbb{Q})$, then its *height* is

$$\mathrm{H}(\gamma) = \max_{1 \leq i, j \leq n} \mathrm{H}(\gamma_{ij})$$

where the height of a rational number a/b (written in lowest terms) is $\max(|a|, |b|)$. For an algebraic group $\boldsymbol{G}$ other than GL_n, we define the heights of elements of $\boldsymbol{G}(\mathbb{Q})$ via a choice of faithful representation $\boldsymbol{G} \to \mathrm{GL}_n$.

In order to avoid writing uncalculated constant factors in every inequality in the proof of Theorem 1.1, we use the notation

$$X \ll Y$$

to mean that there exists a constant C, depending only on the group $\boldsymbol{G}$, the representation ρ and the Siegel set $\mathfrak{S}$, such that

$$|X| \leq C|Y|.$$

2. Definition of Siegel sets

The definitions of Siegel sets used by different authors (for example, [Borel 1969; Ash et al. 2010]) vary in minor ways, so we state here the precise definition used in this paper. At the same time, we define the notation which we shall use in Sections 3 and 4 for the various ingredients in the construction of Siegel sets.

2A. ***Standard Siegel sets in*** $\mathbf{GL}_n$**.** Before defining Siegel sets in general, we begin with the simpler special case of "standard Siegel sets" in GL_n. Our definition of standard Siegel sets follows [Borel 1969, Définition 1.2]. However, we use the reverse order of multiplication for elements of GL_n and therefore reverse the inequalities in the definition of A_t.

Make the following definitions (all of these are special cases of the corresponding notations for general Siegel sets):

(1) $\boldsymbol{P} \subset \mathrm{GL}_n$ is the Borel subgroup consisting of upper triangular matrices.

(2) $K = \mathrm{O}_n(\mathbb{R})$ is the maximal compact subgroup consisting of orthogonal matrices.

(3) $\boldsymbol{S} \subset \boldsymbol{P}$ is the maximal $\mathbb{Q}$-split torus consisting of diagonal matrices.

(4) A_t is the set $\{\alpha \in \boldsymbol{S}(\mathbb{R})^+ : \alpha_j/\alpha_{j+1} \geq t \text{ for all } j\}$ for any real number $t > 0$.

(5) Ω_u is the compact set

$$\{\nu \in \boldsymbol{P}(\mathbb{R}) : \nu_{ii} = 1 \text{ for all } i \text{ and } |\nu_{ij}| \leq u \text{ for } 1 \leq i < j \leq n\}$$

for any real number $u > 0$.

A *standard Siegel set* in GL_n is a set of the form

$$\mathfrak{S} = \Omega_u A_t K \subset \mathrm{GL}_n(\mathbb{R})$$

for some positive real numbers u and t.

According to [Borel 1969, Théorèmes 1.4 and 4.6], if $t \leq \frac{\sqrt{3}}{2}$ and $u \geq \frac{1}{2}$, then $\mathfrak{S}$ is a fundamental set for $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{GL}_n(\mathbb{R})$.

2B. ***Definition of Siegel sets in general.*** Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group. In order to define a Siegel set in $\boldsymbol{G}(\mathbb{R})$, we begin by making choices of the following subgroups of $\boldsymbol{G}$

(1) $\boldsymbol{P}$ a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$,

(2) K a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$.

Lemma 2.1. *For any* $\boldsymbol{P}$ *and* K, *there exists a unique* $\mathbb{R}$-*torus* $\boldsymbol{S} \subset \boldsymbol{P}$ *satisfying the conditions*

(i) $\boldsymbol{S}$ *is* $\boldsymbol{P}(\mathbb{R})$-*conjugate to a maximal* $\mathbb{Q}$-*split torus in* $\boldsymbol{P}$,

(ii) *$\boldsymbol{S}$ is stabilised by the Cartan involution associated with K.*

Proof. This follows from the lemma in [Ash et al. 2010, Chapter II, section 3.7]. □

We define a *Siegel triple* for $\boldsymbol{G}$ to be a triple $(\boldsymbol{P}, \boldsymbol{S}, K)$ satisfying the conditions of Lemma 2.1. We remark that these conditions could equivalently be stated as:

(i) $\boldsymbol{S}$ is a lift of the unique maximal $\mathbb{Q}$-split torus in $\boldsymbol{P}/R_u(\boldsymbol{P})$.

(ii) Lie $\boldsymbol{S}(\mathbb{R})$ is orthogonal to Lie K with respect to the Killing form of $\boldsymbol{G}$.

Define the following further pieces of notation:

(1) $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{P}$.

(2) $\boldsymbol{M}$ is the preimage in $Z_{\boldsymbol{G}}(\boldsymbol{S})$ of the maximal $\mathbb{Q}$-anisotropic subgroup of $\boldsymbol{P}/\boldsymbol{U}$. (Note that by [Borel and Tits 1965, Corollaire 4.16], $Z_{\boldsymbol{G}}(\boldsymbol{S})$ is a Levi subgroup of $\boldsymbol{P}$ and hence maps isomorphically onto $\boldsymbol{P}/\boldsymbol{U}$.)

(3) Δ is the set of simple roots of $\boldsymbol{G}$ with respect to $\boldsymbol{S}$, using the ordering induced by $\boldsymbol{P}$. (The roots of $\boldsymbol{G}$ with respect to $\boldsymbol{S}$ form a root system because $\boldsymbol{S}$ is conjugate to a maximal $\mathbb{Q}$-split torus in $\boldsymbol{G}$.)

(4) $A_t = \{\alpha \in \boldsymbol{S}(\mathbb{R})^+ : \chi(\alpha) \geq t \text{ for all } \chi \in \Delta\}$ for any real number $t > 0$.

A *Siegel set* in $\boldsymbol{G}(\mathbb{R})$ (with respect to $(\boldsymbol{P}, \boldsymbol{S}, K)$) is a set of the form

$$\mathfrak{S} = \Omega A_t K$$

where Ω is a compact subset of $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})^+$ and t is a positive real number.

2C. ***Comparison with other definitions.*** In order to reduce confusion caused by definitions of Siegel sets which vary from one author to another, we explain how our definition compares with the definitions used in [Borel and Harish-Chandra 1962; Borel 1969; Ash et al. 2010].

First we compare with [Ash et al. 2010, Chapter II, Section 4.1].

(1) In [Ash et al. 2010], Siegel sets are subsets of the symmetric space $\boldsymbol{G}(\mathbb{R})/K$, while for us they are K-right-invariant subsets of $\boldsymbol{G}(\mathbb{R})$. These two perspectives are related by the quotient map $\boldsymbol{G}(\mathbb{R}) \to \boldsymbol{G}(\mathbb{R})/K$.

(2) In [Ash et al. 2010], Ω is any compact subset of $\boldsymbol{P}(\mathbb{R})$, while we require Ω to be contained in $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})^+$. Every Siegel set in the sense of [Ash et al. 2010] is contained in a Siegel set in our sense and vice versa, so this difference does not matter in applications. We impose the stricter condition on Ω because it ensures that Siegel sets are related to the horospherical decomposition in $\boldsymbol{G}(\mathbb{R})/K$ (as explained in [Borel and Ji 2006, Section I.1.9]).

Now we compare with [Borel 1969, Définition 12.3]. Note that differences (3) and (4) are significant.

(1) We multiply together Ω, A_t and K in the opposite order from [Borel 1969]. This change forces us to reverse the inequalities in the definition of A_t.

(2) In [Borel 1969], Ω is required to be a compact neighbourhood of the identity in $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})^+$ while we allow any compact subset.

(3) Instead of our condition (i) for $\boldsymbol{S}$, [Borel 1969] imposes the condition that $\boldsymbol{S}$ must be a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P}$. This stronger condition is inconvenient when we also impose condition (ii), because there does not exist a maximal $\mathbb{Q}$-split torus satisfying condition (ii) for every choice of $\boldsymbol{P}$ and K. In particular, Theorem 1.2 does not hold if $\boldsymbol{S_G}$ is required to be $\mathbb{Q}$-split.

(4) Our condition (ii) for $\boldsymbol{S}$ is not part of the definition of Siegel set in [Borel 1969]. In [Borel 1969], a Siegel set is called *normal* if condition (ii) is satisfied. We include condition (ii) in the definition of a Siegel set because without it the Siegel property does not necessarily hold. Indeed most of the theorems in [Borel 1969, Chapter 15] apply only to Siegel sets satisfying condition (ii), even though the word "normal" is omitted from their statements. Similarly this paper's Theorem 1.1 does not hold without condition (ii) on $\boldsymbol{S}$.

The definition of "Siegel domain" in [Borel and Harish-Chandra 1962, Section 4] is less fine than the definition used in this paper, or the one in [Borel 1969], because it takes into account only the structure of $\boldsymbol{G}$ as a real algebraic group and not its structure as a $\mathbb{Q}$-algebraic group. Consequently [Borel and Harish-Chandra 1962] could not use their Siegel domains directly to construct fundamental sets for arithmetic subgroups in $\boldsymbol{G}(\mathbb{R})$; instead they constructed such fundamental sets using an embedding of $\boldsymbol{G}$ into GL_n and standard Siegel sets in $\mathrm{GL}_n(\mathbb{R})$.

2D. ***Siegel sets and fundamental sets.*** The importance of Siegel sets is due to their use in constructing fundamental sets for an arithmetic subgroup Γ in $\boldsymbol{G}(\mathbb{R})$. We say that a set $\Omega \subset \boldsymbol{G}(\mathbb{R})$ is a *fundamental set* for Γ if the following conditions are satisfied:

(F0) $\Omega.K = \Omega$ for a suitable maximal compact subgroup $K \subset \boldsymbol{G}(\mathbb{R})$.

(F1) $\Gamma.\Omega = \boldsymbol{G}(\mathbb{R})$.

(F2) For every $\theta \in \boldsymbol{G}(\mathbb{R})$, the set
$$\{\gamma \in \Gamma : \gamma.\Omega \cap \theta.\Omega \neq \varnothing\}$$
is finite (the Siegel property).

The following two theorems show that, if we make suitable choices of Siegel set $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ and finite set $C \subset \boldsymbol{G}(\mathbb{Q})$, then $C.\mathfrak{S}$ is a fundamental set for Γ in $\boldsymbol{G}(\mathbb{R})$.

Theorem 2.2 [Borel 1969, Théorème 13.1]. *Let Γ be an arithmetic subgroup of $\boldsymbol{G}(\mathbb{Q})$. Let $(\boldsymbol{P}, \boldsymbol{S}, K)$ be a Siegel triple for $\boldsymbol{G}(\mathbb{R})$.*

There exist a Siegel set $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ with respect to $(\boldsymbol{P}, \boldsymbol{S}, K)$ and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that
$$\boldsymbol{G}(\mathbb{R}) = \Gamma.C.\mathfrak{S}.$$

Theorem 2.3 [Borel 1969, Théorème 15.4]. *Let Γ be an arithmetic subgroup of $\boldsymbol{G}(\mathbb{Q})$. Let $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ be a Siegel set.*

For any finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ and any element $\theta \in \boldsymbol{G}(\mathbb{Q})$, the set

$$\{\gamma \in \Gamma : \gamma.C.\mathfrak{S} \cap \theta.C.\mathfrak{S} \neq \varnothing\}$$

is finite.

As remarked in Section 2C, Theorem 2.3 requires the torus $\boldsymbol{S}$ used in the definition of a Siegel set to satisfy condition (ii) from Section 2B, even though this condition is erroneously omitted from the statement in [Borel 1969].

This paper's Theorem 1.1 implies [loc. cit., Corollaire 15.3] and therefore it implies Theorem 2.3, by the same argument as in the proof of [loc. cit., Théorème 15.4]. Since our proof of Theorem 1.1 is independent of Borel's proof of [loc. cit., Corollaire 15.3], this gives a new proof of Theorem 2.3.

3. Proof of main height bound

In this section we prove Theorem 1.1. Most of the section deals with the case of standard Siegel sets in GL_n. At the end we show how to deduce the general statement of Theorem 1.1 from this case, using Theorem 1.2.

Thus let $\boldsymbol{G} = \mathrm{GL}_n$ and let $\mathfrak{S}$ be a standard Siegel set in $\boldsymbol{G}$. As in the statement of Theorem 1.1, we are given an element

$$\gamma \in \mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q}),$$

with $N = |\det \gamma|$ and with D denoting the maximum of the denominators of entries of γ. Since $\gamma \in \mathfrak{S}.\mathfrak{S}^{-1}$, using the notation from Section 2A, we can write

$$\gamma = \nu\beta\kappa\alpha^{-1}\mu^{-1} \tag{2}$$

with $\alpha, \beta \in A_t$, $\mu, \nu \in \Omega_u$ and $\kappa \in K$. Rearranging this equation, we obtain

$$\gamma\mu\alpha = \nu\beta\kappa. \tag{3}$$

Our aim is to bound the height of γ by a polynomial in N and D. The proof has three stages. First we compare entries of the diagonal matrices α and β, showing that $\alpha_j \ll D\beta_i$ for certain pairs of indices (i, j). Secondly, we prove that

$$\beta_j \ll ND^{n-1}\alpha_i \tag{4}$$

whenever i and j lie in the same segment of a certain partition of $\{1, \dots, n\}$. Finally we expand out (2) and use inequality (4).

3A. ***Partitioning the indices.*** An important device in the proof of Theorem 1.1 for standard Siegel sets is a partition of the set of indices $\{1, \dots, n\}$ into subintervals which we call "segments" (depending on γ). The *segments* are defined to be the subintervals of $\{1, \dots, n\}$ such that

(i) γ is block upper triangular with respect to the chosen partition,

(ii) γ is not block upper triangular with respect to any finer partition of $\{1, \dots, n\}$ into subintervals.

We define a *leading entry* to be a pair of indices $(i, j) \in \{1, \dots, n\}^2$ such that γ_{ij} is the left-most nonzero entry in the i-th row of γ.

The following lemma describes segments in terms of leading entries. This lemma also has a converse, which we will not need: if $i > j$ and there exists a sequence satisfying condition $(*)$, then i and j are in the same segment.

Lemma 3.1. *If $i > j$ and i and j are in the same segment, then there exists a sequence of leading entries $(i_1, j_1), \dots, (i_s, j_s)$ such that*

$$i \leq i_1, \qquad j_p \leq i_{p+1} \quad \text{for every} \quad p \in \{1, \dots, s-1\}, \qquad \text{and} \qquad j_s \leq j. \tag{$*$}$$

Proof. First, for each k such that $j < k \leq i$, we show that there exists a leading entry (i', j') such that $j' < k \leq i'$. Because segments give the finest partition according to which γ is block upper triangular, γ cannot be block upper triangular with respect to the partition

$$\{1, \dots, k-1\}, \{k, \dots, n\}.$$

So there exists some $i' \geq k$ such that the i'-th row of γ has a nonzero entry in the first $k-1$ columns. Choosing j' to be the index of the left-most nonzero entry in the i'-th row, we get the desired leading entry with $j' < k \leq i'$.

Let $s = i - j$. For each p such that $1 \leq p \leq s$ we apply the above argument to $k = i - p + 1$ and get a leading entry (i_p, j_p) such that $j_p < i - p + 1 \leq i_p$. The resulting sequence $(i_1, j_1), \dots, (i_s, j_s)$ satisfies condition $(*)$. $\square$

We define $\boldsymbol{Q}$ to be the subgroup of GL_n consisting of block upper triangular matrices according to the segments defined above (thus $\boldsymbol{Q}$ depends on γ). Observe that $\boldsymbol{Q}$ could equivalently be defined as the smallest standard parabolic subgroup of GL_n which contains γ.

We define $\boldsymbol{L}$ to be the subgroup of GL_n consisting of block diagonal matrices according to the same partition into segments. Thus $\boldsymbol{L}$ could equivalently be defined as the Levi subgroup of $\boldsymbol{Q}$ containing the torus of diagonal matrices.

3B. ***Example partitions for* $\mathbf{GL_3}$.** To illustrate the definition of segments and Lemma 3.1, we show the various cases which occur for GL_3. Table 1 shows classes of matrix in GL_3, depending on the region of zeros adjacent to the bottom left corner of the matrix, and gives the associated partitions of $\{1, 2, 3\}$ into segments. Every matrix in GL_3 falls into exactly one of the classes in Table 1.

In Table 1, $*$ represents an entry which must be nonzero, while $\cdot$ represents an entry which may be either zero or nonzero. Every entry to the left of a $*$ is zero, so each $*$ is a leading entry. For rows which do not contain a $*$, there is not enough information to determine the leading entry; these rows' leading entries are not important for Lemma 3.1.

Comparing the two classes of matrices in the right-hand column of Table 1, we see that it is possible for matrices to have different patterns of zeros adjacent to the bottom left corner, yet still be associated with the same partition of $\{1, 2, 3\}$. This is related to the fact that matrices in the lower class of this column do

γ	segments	γ	segments
$\begin{pmatrix} * & \cdot & \cdot \\ 0 & * & \cdot \\ 0 & 0 & * \end{pmatrix}$	$\{1\}, \{2\}, \{3\}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ * & \cdot & \cdot \end{pmatrix}$	$\{1, 2, 3\}$
$\begin{pmatrix} * & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & * & \cdot \end{pmatrix}$	$\{1\}, \{2, 3\}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ * & \cdot & \cdot \\ 0 & * & \cdot \end{pmatrix}$	$\{1, 2, 3\}$
$\begin{pmatrix} \cdot & \cdot & \cdot \\ * & \cdot & \cdot \\ 0 & 0 & * \end{pmatrix}$	$\{1, 2\}, \{3\}$		

Table 1. Partitions into segments for $\gamma \in \mathrm{GL}_3$.

not form a subgroup of GL_3: the smallest standard parabolic subgroup containing such a matrix is the full group GL_3, the same as for the upper class.

On the other hand, the difference between the two classes in the right-hand column of Table 1 is important for finding sequences of leading entries as in Lemma 3.1. In the upper class of this column, the sequence consisting just of the leading entry $(3, 1)$ satisfies condition $(*)$ for every pair (i, j). In the lower class, in order to construct a sequence satisfying condition $(*)$ which goes from $i = 3$ to $j = 1$, we need both the leading entries $(3, 2)$ and $(2, 1)$.

3C. ***Ratios between diagonal matrices (leading entries).*** In the first stage of the proof, we compare α_j with β_i when (i, j) is a leading entry. This is based on comparing the lengths of the i-th rows on either side of (3).

Lemma 3.2. *If* (i, j) *is a leading entry for* γ, *then*

$$\alpha_j \ll D\beta_i.$$

Proof. Recall (3):

$$\gamma\mu\alpha = \nu\beta\kappa.$$

Because $\kappa \in \mathrm{O}_n(\mathbb{R})$, multiplying by κ on the right does not change the length of a row vector. Hence expanding out the lengths of the i-th rows on either side of (3) gives

$$\sum_{p=1}^{n} \biggl(\sum_{q=1}^{n} \gamma_{iq}\mu_{qp} \biggr)^2 \alpha_p^2 = \sum_{p=1}^{n} \nu_{ip}^2 \beta_p^2. \tag{5}$$

Look first at the right-hand side of (5), comparing it to β_i^2. Because ν is upper triangular, nonzero terms on the right-hand side of (5) must have $p \geq i$ and hence (by the definition of A_t) $\beta_p \ll \beta_i$. Since ν

is in the fixed compact set Ω_u, there is a uniform bound for the entries ν_{ip}. Thus we get

$$\sum_{p=1}^{n} \nu_{ip}^2 \beta_p^2 \ll \beta_i^2. \tag{6}$$

Now look at the left-hand side of (5), comparing it to α_j^2. We pull out the $p = j$ term. Because squares are nonnegative we have

$$\left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qj}\right)^2 \alpha_j^2 \le \sum_{p=1}^{n}\left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qp}\right)^2 \alpha_p^2. \tag{7}$$

Because (i, j) is a leading entry, if $\gamma_{iq} \neq 0$ then $q \geq j$. Because μ is upper triangular, if $\mu_{qj} \neq 0$ then $q \leq j$. Combining these facts, the only nonzero term on the left-hand side of (7) is the term with $q = j$. In other words,

$$\gamma_{ij}^2 \mu_{jj}^2 \alpha_j^2 = \left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qj}\right)^2 \alpha_j^2. \tag{8}$$

Because $\mu \in \Omega_u$, we have $\mu_{jj} = 1$. Because (i, j) is a leading entry, $\gamma_{ij} \neq 0$. Because entries of γ are rational numbers with denominator at most D, this implies that $|\gamma_{ij}| \geq D^{-1}$. Combining these facts, we get

$$D^{-2} \leq \gamma_{ij}^2 \mu_{jj}^2. \tag{9}$$

Using successively the inequalities and equations (9), (8), (7), (5) and (6) gives

$$D^{-2}\alpha_j^2 \ll \beta_i^2. \qquad \square$$

3D. ***Ratios between diagonal matrices (in each segment).*** In the second stage of the proof of Theorem 1.1, we prove a series of inequalities comparing entries of α and β. This concludes with an inequality between α_i and β_j valid whenever i and j are in the same segment. (Note that the final inequality, Lemma 3.5, is in the opposite direction to the starting point of Lemma 3.2.)

Lemma 3.3. *For all $k \in \{1, \dots, n\}$,*

$$\alpha_k \ll D\beta_k.$$

Proof. The key point is that there exists a leading entry (i, j) such that

$$j \leq k \leq i.$$

To prove this, observe that since γ is invertible there must be some $i \geq k$ such that the i-th row of γ contains a nonzero entry in or to the left of the k-th column. Choosing j to be the index of the left-most nonzero entry in the i-th row of γ gives the required leading entry.

Taking such a leading entry (i, j), we can use Lemma 3.2 (for the middle inequality) and the definition of A_t (for the outer inequalities) to prove that

$$\alpha_k \ll \alpha_j \ll D\beta_i \ll D\beta_k. \qquad \square$$

Lemma 3.4. *For every set $J \subset \{1, \dots, n\}$,*

$$\prod_{j \in J} \beta_j \ll N D^{n-\#J} \prod_{j \in J} \alpha_j.$$

Proof. Because α and β are diagonal matrices with positive diagonal entries,

$$\prod_{j \in J} \beta_j \cdot \det \alpha = \prod_{j \in J} \beta_j \cdot \prod_{k=1}^{n} \alpha_k \ll D^{n-\#J} \prod_{j \in J} \alpha_j \cdot \prod_{k=1}^{n} \beta_k = D^{n-\#J} \prod_{j \in J} \alpha_j \cdot \det \beta \tag{10}$$

where the middle inequality uses Lemma 3.3 for all indices $k \in \{1, \dots, n\} \setminus J$.

All of μ, ν and κ have determinant ± 1. Hence (3) implies that

$$\det \beta = N \det \alpha.$$

Combining this with inequality (10) proves the lemma. □

Lemma 3.5. *If i and j are in the same segment, then*

$$\beta_j \ll N D^{n-1} \alpha_i.$$

Proof. If $i \leq j$, then we apply Lemma 3.4 to the singleton $\{j\}$ to obtain

$$\beta_j \ll N D^{n-1} \alpha_j.$$

Combining this with $\alpha_j \ll \alpha_i$ proves the lemma in the case $i \leq j$.

Otherwise, $i > j$ so we can use Lemma 3.1 to find a sequence of leading entries $(i_1, j_1), \dots, (i_s, j_s)$ satisfying condition $(*)$. We may assume that $i_1, \dots, i_s$ are distinct — otherwise we could simply delete the subsequence between two occurrences of the same i_p. Similarly, we may assume that none of $i_1, \dots, i_s$ are equal to j.

Therefore we can apply Lemma 3.4 to the set $\{i_1, \dots, i_s, j\}$ to get

$$\beta_j \prod_{p=1}^{s} \beta_{i_p} \ll N D^{n-(s+1)} \alpha_j \prod_{p=1}^{s} \alpha_{i_p}. \tag{11}$$

For each $p \in \{1, \dots, s-1\}$, the fact that $j_p \leq i_{p+1}$ and Lemma 3.2 tell us that

$$\alpha_{i_{p+1}} \ll \alpha_{j_p} \ll D \beta_{i_p}.$$

Similarly because $j_s \leq j$ we have

$$\alpha_j \ll \alpha_{j_s} \ll D \beta_{i_s}.$$

Multiplying these inequalities together and also multiplying by β_j gives the first inequality below, while (11) gives the second:

$$\beta_j \alpha_j \prod_{p=2}^{s} \alpha_{i_p} \ll D^s \beta_j \prod_{p=1}^{s} \beta_{i_p} \ll N D^{n-1} \alpha_j \prod_{p=1}^{s} \alpha_{i_p}.$$

Canceling $\alpha_j \prod_{p=2}^{s} \alpha_{i_p}$ shows that

$$\beta_j \ll N D^{n-1} \alpha_{i_1}.$$

Since $i \leq i_1$, we have $\alpha_{i_1} \ll \alpha_i$. This completes the proof of the lemma. □

3E. ***Conclusion of proof for standard Siegel sets.*** In the final stage of the proof, we expand out (2). When we do this, we get terms of the form $\beta_p \kappa_{pq} \alpha_q^{-1}$. In order to bound this using Lemma 3.5, we need to know that κ_{pq} is zero if p and q are not in the same segment. In other words we have to begin by proving that κ is in the group $\boldsymbol{L}(\mathbb{R})$ of block diagonal matrices.

Lemma 3.6. $\kappa \in \boldsymbol{L}(\mathbb{R})$.

Proof. By construction, γ, μ, α, ν, β are all in the group $\boldsymbol{Q}(\mathbb{R})$ of block upper triangular matrices. Hence (2) tells us that also $\kappa \in \boldsymbol{Q}(\mathbb{R})$.

If a matrix is both block upper triangular and orthogonal, then it is block diagonal according to the same blocks (because the inverse-transpose of a block upper triangular matrix is block lower triangular). In other words,

$$\boldsymbol{Q}(\mathbb{R}) \cap K \subset \boldsymbol{L}(\mathbb{R}).$$

This proves the lemma. □

Lemma 3.7. *For all* $i, j \in \{1, \ldots, n\}$, *we have*

$$|\gamma_{ij}| \ll N D^{n-1}.$$

Proof. We expand out the matrix product in (2), which we recall:

$$\gamma = \nu \beta \kappa \alpha^{-1} \mu^{-1}.$$

Because α and β are diagonal, the pq-th entry of $\beta \kappa \alpha^{-1}$ is equal to

$$\beta_p \kappa_{pq} \alpha_q^{-1}.$$

If p and q are not in the same segment, then Lemma 3.6 tells us that $\kappa_{pq} = 0$. On the other hand if p and q are in the same segment, then we can apply Lemma 3.5 to bound $\beta_p \alpha_q^{-1}$. Furthermore, because κ is in the compact subgroup K, there is a uniform upper bound for entries of κ. We conclude that

$$\beta_p \kappa_{pq} \alpha_q^{-1} \ll N D^{n-1}. \tag{12}$$

Because μ and ν are in the fixed compact set Ω_u and because all elements of Ω_u are invertible, there is a uniform upper bound for entries of ν and of μ^{-1}. Thus inequality (12) together with (2) implies the lemma. □

To complete the proof of Theorem 1.1 for standard Siegel sets in GL_n, we just have to note that the definition of $\mathrm{H}(\gamma)$ implies that

$$\mathrm{H}(\gamma) \leq D \max(1, |\gamma_{ij}|)$$

where the maximum is over all indices $(i, j) \in \{1, \dots, n\}^2$. Hence Lemma 3.7 implies that

$$\mathrm{H}(\gamma) \leq \max(D, C_5 N D^n),$$

where C_5 denotes the implied constant from Lemma 3.7.

3F. *Deducing general case from standard Siegel sets.* To complete the proof of Theorem 1.1, we deduce the general statement from the case of standard Siegel sets in GL_n. This has two steps. Lemma 3.8 allows us to generalise from standard Siegel sets to arbitrary Siegel sets in GL_n. Theorem 1.2 (proved in Section 4) allows us to generalise from GL_n to arbitrary reductive groups $\boldsymbol{G}$.

Lemma 3.8. *Let $\mathfrak{S}$ be a Siegel set in $\mathrm{GL}_n(\mathbb{R})$. Then there exist $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ and $\sigma \in \mathrm{GL}_n(\mathbb{R})$ such that $\gamma^{-1}.\mathfrak{S}.\gamma\sigma$ is contained in a standard Siegel set.*

Proof. Let $(\boldsymbol{P}, \boldsymbol{S}, K)$ be the Siegel triple associated with the Siegel set $\mathfrak{S}$, and write $\mathfrak{S} = \Omega.A_t.K$ using the notation of Section 2B.

Let $(\boldsymbol{P}_0, \boldsymbol{S}_0, K_0)$ be the standard Siegel triple in GL_n. Write $A_{0,t}$ and $\Omega_{0,u}$ for the sets called A_t and Ω_u in the definition of standard Siegel sets.

Since $\boldsymbol{P}$ and $\boldsymbol{P}_0$ are minimal $\mathbb{Q}$-parabolic subgroups of GL_n, there exists $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ such that $\boldsymbol{P}_0 = \gamma^{-1}\boldsymbol{P}\gamma$.

Since K_0 and $\gamma^{-1}K\gamma$ are maximal compact subgroups of $\mathrm{GL}_n(\mathbb{R})$, there exists $\sigma \in \mathrm{GL}_n(\mathbb{R})$ such that $\gamma^{-1}K\gamma = \sigma K_0 \sigma^{-1}$. Applying the Iwasawa decomposition

$$\mathrm{GL}_n(\mathbb{R}) = \boldsymbol{U}_0(\mathbb{R}).\boldsymbol{S}_0(\mathbb{R})^+.K_0,$$

we may assume that $\sigma = \tau\beta$ where $\beta \in \boldsymbol{S}_0(\mathbb{R})^+$ and $\tau \in \boldsymbol{U}_0(\mathbb{R})$.

Under this assumption, $\sigma \in \boldsymbol{P}_0(\mathbb{R})$. Hence $\sigma^{-1}\gamma^{-1}.\boldsymbol{P}.\gamma\sigma = \boldsymbol{P}_0$. By Lemma 2.1, $\sigma^{-1}\gamma^{-1}.\boldsymbol{S}.\gamma\sigma = \boldsymbol{S}_0$. Thus $\sigma^{-1}\gamma^{-1}.A_t.\gamma\sigma = A_{0,t}$.

Now

$$\gamma^{-1}\mathfrak{S}\gamma\sigma = \gamma^{-1}\Omega\gamma.\sigma.\sigma^{-1}\gamma^{-1}A_t\gamma\sigma.\sigma^{-1}\gamma^{-1}K\gamma\sigma = \gamma^{-1}\Omega\gamma.\tau\beta.A_{0,t}.K_0.$$

Here $\gamma^{-1}\Omega\gamma\tau$ is a compact subset of $\boldsymbol{U}_0(\mathbb{R})$ so it is contained in $\Omega_{0,u}$ for a suitable $u > 0$. Meanwhile $\beta.A_{0,t}$ is contained in $A_{0,s}$ for a suitable $s > 0$. Thus $\gamma^{-1}\mathfrak{S}\gamma\sigma$ is contained in the standard Siegel set $\Omega_{0,u}.A_{0,s}.K_0$, as required. □

4. Siegel sets and subgroups

In this section we prove Theorem 1.2. The proof gives additional information on the relationship between the Siegel triples for $\boldsymbol{G}$ and $\boldsymbol{H}$, as follows.

Theorem 4.1. *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be reductive $\mathbb{Q}$-algebraic groups, with $\boldsymbol{H} \subset \boldsymbol{G}$. Let $\mathfrak{S}_{\boldsymbol{H}}$ be a Siegel set in $\boldsymbol{H}(\mathbb{R})$ with respect to the Siegel triple $(\boldsymbol{P}_{\boldsymbol{H}}, \boldsymbol{S}_{\boldsymbol{H}}, K_{\boldsymbol{H}})$. Then there exist a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that*

$$\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}.$$

Furthermore if $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ *denotes the Siegel triple associated with* $\mathfrak{S}_{\boldsymbol{G}}$, *then* $R_u(\boldsymbol{P_H}) \subset R_u(\boldsymbol{P_G})$, $\boldsymbol{S_H} = \boldsymbol{S_G} \cap \boldsymbol{H}$ *and* $K_{\boldsymbol{H}} = K_{\boldsymbol{G}} \cap \boldsymbol{H}(\mathbb{R})$.

We denote sets used in the construction of the Siegel sets $\mathfrak{S}_{\boldsymbol{G}}$ and $\mathfrak{S}_{\boldsymbol{H}}$ by the notation from Section 2B with the subscript $\boldsymbol{G}$ or $\boldsymbol{H}$ added as appropriate. Thus we write

$$\mathfrak{S}_{\boldsymbol{H}} = \Omega_{\boldsymbol{H}}.A_{\boldsymbol{H},t}.K_{\boldsymbol{H}},$$

where $\Omega_{\boldsymbol{H}}$ is a compact subset of $\boldsymbol{U_H}(\mathbb{R})\boldsymbol{M_H}(\mathbb{R})^+$, $K_{\boldsymbol{H}}$ is a maximal compact subgroup of $\boldsymbol{H}(\mathbb{R})$ and

$$A_{\boldsymbol{H},t} = \{\alpha \in \boldsymbol{S_H}(\mathbb{R})^+ : \chi(\alpha) \geq t \text{ for all } \chi \in \Delta_{\boldsymbol{H}}\}.$$

4A. ***Reduction to a split torus*** $\boldsymbol{S_H}$**.** We begin by reducing the proof of Theorem 4.1 to the case in which the torus $\boldsymbol{S_H}$ is $\mathbb{Q}$-split. Note that, even when $\boldsymbol{S_H}$ is $\mathbb{Q}$-split, it is not always possible to choose a $\mathbb{Q}$-split torus for $\boldsymbol{S_G}$.

According to the definition of a Siegel set, we can choose $u \in \boldsymbol{P_H}(\mathbb{R})$ such that $u\boldsymbol{S_H}u^{-1}$ is a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_H}$. Using the Levi decomposition $\boldsymbol{P_H} = Z_{\boldsymbol{H}}(\boldsymbol{S_H}) \ltimes \boldsymbol{U_H}$, we may assume that $u \in \boldsymbol{U_H}(\mathbb{R})$.

Now $\Omega_{\boldsymbol{H}}u^{-1}$ is a compact subset of $\boldsymbol{U_H}(\mathbb{R}).u\boldsymbol{M_H}(\mathbb{R})^+u^{-1}$ so

$$\mathfrak{S}_{\boldsymbol{H}}.u^{-1} = \Omega_{\boldsymbol{H}}u^{-1}.uA_{\boldsymbol{H},t}u^{-1}.uK_{\boldsymbol{H}}u^{-1}.$$

is a Siegel set with respect to the Siegel triple $(\boldsymbol{P_H}, u\boldsymbol{S_H}u^{-1}, uK_{\boldsymbol{H}}u^{-1})$.

We prove below that Theorem 4.1 holds when $\boldsymbol{S_H}$ is $\mathbb{Q}$-split. Hence there exist a Siegel set $\mathfrak{S}'_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that

$$\mathfrak{S}_{\boldsymbol{H}}.u^{-1} \subset C.\mathfrak{S}'_{\boldsymbol{G}}.$$

Let $(\boldsymbol{P_G}, \boldsymbol{S}'_{\boldsymbol{G}}, K'_{\boldsymbol{G}})$ denote the Siegel triple associated with $\mathfrak{S}'_{\boldsymbol{G}}$. According to Theorem 4.1, $\boldsymbol{U_H} \subset R_u(\boldsymbol{P_G})$ and so $u \in R_u(\boldsymbol{P_G})(\mathbb{R})$. Therefore

$$\mathfrak{S}_{\boldsymbol{G}} = \mathfrak{S}'_{\boldsymbol{G}}.u$$

is a Siegel set for $\boldsymbol{G}(\mathbb{R})$ with respect to the Siegel triple $(\boldsymbol{P_G}, u^{-1}\boldsymbol{S}'_{\boldsymbol{G}}u, u^{-1}K'_{\boldsymbol{G}}u)$. We clearly have $\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}$ and the Siegel triple associated with $\mathfrak{S}_{\boldsymbol{G}}$ satisfies the conditions of Theorem 4.1 relative to $(\boldsymbol{P_H}, \boldsymbol{S_H}, K_{\boldsymbol{H}})$.

4B. ***Choosing the Siegel triple.*** We henceforth assume that $\boldsymbol{S_H}$ is $\mathbb{Q}$-split. As the first step in proving Theorem 4.1 for this case, we choose a Siegel triple $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ for $\boldsymbol{G}$.

The main difficulty lies in choosing $\boldsymbol{P_G}$. The obvious idea is to choose a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ which contains $\boldsymbol{P_H}$, but such a subgroup does not always exist (for example, if $\boldsymbol{G}$ is $\mathbb{Q}$-split and $\boldsymbol{H}$ is $\mathbb{Q}$-anisotropic). Instead we construct a larger parabolic $\mathbb{Q}$-subgroup $\boldsymbol{Q} \subset \boldsymbol{G}$ which contains $\boldsymbol{P_H}$, and then define $\boldsymbol{P_G}$ to be a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{Q}$.

Let us write

$$\boldsymbol{Z} = Z_{\boldsymbol{G}}(\boldsymbol{S_H}).$$

Lemma 4.2. *There exists a parabolic $\mathbb{Q}$-subgroup $\boldsymbol{Q} \subset \boldsymbol{G}$ such that*

(i) *$\boldsymbol{Z}$ is a Levi subgroup of $\boldsymbol{Q}$, and*

(ii) *$\boldsymbol{U_H} \subset R_u(\boldsymbol{Q})$.*

Proof. Let $\Phi_{\boldsymbol{H}}^+$ denote the set of roots $\Phi(\boldsymbol{S_H}, \boldsymbol{P_H})$. By [Borel and Tits 1965, Proposition 3.1] there exists an order $>_{\boldsymbol{Q}}$ on $X^*(\boldsymbol{S_H})$ with respect to which all elements of $\Phi_{\boldsymbol{H}}^+$ are positive.

Let

$$\Phi_{\boldsymbol{Q}} = \{\chi \in \Phi(\boldsymbol{S_H}, \boldsymbol{G}) : \chi >_{\boldsymbol{Q}} 0\}$$

and let $\boldsymbol{Q}$ denote the group $\boldsymbol{G}_{\Phi_{\boldsymbol{Q}}}$ (using the notation of [loc. cit., Paragraph 3.8] with respect to the torus $\boldsymbol{S_H}$). By [loc. cit., Théorème 4.15], $\boldsymbol{Q}$ is a parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ and $\boldsymbol{Z}$ is a Levi subgroup of $\boldsymbol{Q}$.

Since all weights of $\boldsymbol{S_H}$ on $\boldsymbol{U_H}$ are contained in $\Phi_{\boldsymbol{H}}^+$, which is a subset of $\Phi_{\boldsymbol{Q}}$, [loc. cit., Proposition 3.12] tells us that $\boldsymbol{U_H} \subset \boldsymbol{G}^*_{\Phi_{\boldsymbol{Q}}}$, again using the notation of [loc. cit., Paragraph 3.8]. By [loc. cit., Théorème 3.13], $\boldsymbol{G}^*_{\Phi_{\boldsymbol{Q}}} = R_u(\boldsymbol{Q})$. This completes the proof that $\boldsymbol{U_H} \subset R_u(\boldsymbol{Q})$. □

We will make no use of the following lemma, but it sheds some light on the significance of the group $\boldsymbol{Q}$.

Lemma 4.3. *$\boldsymbol{P_H} = \boldsymbol{Q} \cap \boldsymbol{H}$.*

Proof. We use the notation from the proof of Lemma 4.2. By construction, we have that $\Phi(\boldsymbol{S_H}, \boldsymbol{P_H}) = \Phi_{\boldsymbol{H}}^+ \subset \Phi_{\boldsymbol{Q}}$. Hence by [Borel and Tits 1965, Proposition 3.12], $\boldsymbol{P_H} \subset \boldsymbol{G}_{\Phi_{\boldsymbol{Q}}} = \boldsymbol{Q}$.

For the reverse inclusion, observe that $\Phi(\boldsymbol{S_H}, \boldsymbol{Q} \cap \boldsymbol{H}) \subset \Phi_{\boldsymbol{H}}^+$. Hence applying [loc. cit., Proposition 3.12], this time inside $\boldsymbol{H}$, we get

$$\boldsymbol{Q} \cap \boldsymbol{H} \subset \boldsymbol{H}_{\Phi_{\boldsymbol{H}}^+} = \boldsymbol{P_H}. \qquad \square$$

Choose the following subgroups of $\boldsymbol{G}$:

(1) $\boldsymbol{P_G}$, a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{Q}$.

(2) $K_{\boldsymbol{G}}$, a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$ containing $K_{\boldsymbol{H}}$.

Define the following notation for subgroups of $\boldsymbol{G}$ which are uniquely determined by $\boldsymbol{P_G}$ and $K_{\boldsymbol{G}}$:

(1) $\boldsymbol{S_G}$ is the unique torus such that $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ is a Siegel triple for $\boldsymbol{G}$.

(2) $\boldsymbol{U_G} = R_u(\boldsymbol{P_G})$.

(3) $\boldsymbol{P_Z} = \boldsymbol{P_G} \cap \boldsymbol{Z}$ and $\boldsymbol{U_Z} = R_u(\boldsymbol{P_Z})$.

(4) $K_{\boldsymbol{Z}} = K_{\boldsymbol{G}} \cap \boldsymbol{Z}(\mathbb{R})$.

Lemma 4.4. *$K_{\boldsymbol{Z}}$ is a maximal compact subgroup of $\boldsymbol{Z}(\mathbb{R})$.*

Proof. Let Θ be the Cartan involution of $\boldsymbol{G}$ associated with the maximal compact subgroup $K_{\boldsymbol{G}}$. Because $K_{\boldsymbol{H}} = K_{\boldsymbol{G}} \cap \boldsymbol{H}(\mathbb{R})$, Θ restricts to the Cartan involution of $\boldsymbol{H}$ associated with $K_{\boldsymbol{H}}$.

From the definition of Siegel triple applied to $(\mathbf{P_H}, \mathbf{S_H}, K_{\mathbf{H}})$, Θ stabilizes $\mathbf{S_H}$. Hence Θ also stabilizes $\mathbf{Z}$. Therefore the fixed points of Θ in $\mathbf{Z}(\mathbb{R})$, namely $K_{\mathbf{Z}}$, form a maximal compact subgroup of $\mathbf{Z}(\mathbb{R})$. □

Lemma 4.5. $\mathbf{S_H} \subset \mathbf{S_G}$.

Proof. Note that $\mathbf{Z}$ is a reductive group defined over $\mathbb{Q}$, because $\mathbf{S_H}$ is defined over $\mathbb{Q}$. Thus it makes sense to talk about Siegel triples in $\mathbf{Z}$. By [Borel and Tits 1965, Proposition 4.4], $\mathbf{P_Z}$ is a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{Z}$.

By Lemma 2.1, there exists a unique torus $\mathbf{S_Z} \subset \mathbf{Z}$ such that $(\mathbf{P_Z}, \mathbf{S_Z}, K_{\mathbf{Z}})$ is a Siegel triple for $\mathbf{Z}$. This means that:

(i) $\mathbf{S_Z}$ is $\mathbf{P_Z}(\mathbb{R})$-conjugate to a maximal $\mathbb{Q}$-split torus in $\mathbf{P_Z}$. Note that a maximal $\mathbb{Q}$-split torus in $\mathbf{P_Z}$ is also a maximal $\mathbb{Q}$-split torus in $\mathbf{P_G}$.

(ii) The Cartan involution of $\mathbf{Z}$ associated with $K_{\mathbf{Z}}$ normalises $\mathbf{S_Z}$. This involution is the restriction of the Cartan involution of $\mathbf{G}$ associated with $K_{\mathbf{G}}$.

Thus $\mathbf{S_Z}$ satisfies the conditions of Lemma 2.1 with respect to $(\mathbf{P_G}, K_{\mathbf{G}})$. By the uniqueness in Lemma 2.1, we conclude that $\mathbf{S_Z} = \mathbf{S_G}$.

Because $\mathbf{S_Z}$ is $\mathbf{Z}(\mathbb{R})$-conjugate to a maximal $\mathbb{Q}$-split torus in $\mathbf{Z}$, it contains every $\mathbb{Q}$-split subtorus of the centre of $\mathbf{Z}$. In particular $\mathbf{S_H} \subset \mathbf{S_Z}$. □

Let $\mathbf{S}'_{\mathbf{G}}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{P_Z}$. Because $(\mathbf{P_Z}, \mathbf{S_Z}, K_{\mathbf{Z}})$ is a Siegel triple, there exists $u \in \mathbf{P_Z}(\mathbb{R})$ such that $\mathbf{S}'_{\mathbf{G}} = u\mathbf{S}'_{\mathbf{Z}}u^{-1}$. Because of the Levi decomposition $\mathbf{P_Z} = Z_{\mathbf{G}}(\mathbf{S_G}) \ltimes \mathbf{U_Z}$, we may assume that $u \in \mathbf{U_Z}(\mathbb{R})$.

The following lemma is not needed in our proof of Theorem 1.2, but it contains extra information about $\mathbf{S_G}$ which is included in the statement of Theorem 4.1.

Lemma 4.6. $\mathbf{S_H} = \mathbf{S_G} \cap \mathbf{H}$.

Proof. Let q denote the quotient map $\mathbf{P_G} \to \mathbf{P_G}/\mathbf{U_G}$. Observe that $\mathbf{U_G} \cap \mathbf{P_H}$ is a normal unipotent subgroup of $\mathbf{P_H}$, so it is contained in $\mathbf{U_H}$. On the other hand,

$$\mathbf{U_H} \subset R_u(\mathbf{Q}) \cap \mathbf{P_H} \subset \mathbf{U_G} \cap \mathbf{P_H}.$$

Hence $\mathbf{U_G} \cap \mathbf{P_H} = \mathbf{U_H}$, so q restricts to the quotient map $\mathbf{P_H} \to \mathbf{P_H}/\mathbf{U_H}$.

According to the definition of a Siegel triple, $q(\mathbf{S_G})$ is a maximal $\mathbb{Q}$-split torus in $\mathbf{P_G}/\mathbf{U_G}$. Furthermore, $\mathbf{S_G} \cap \mathbf{H} \subset \mathbf{Q} \cap \mathbf{H} = \mathbf{P_H}$. Hence $q(\mathbf{S_G} \cap \mathbf{H})$ is a $\mathbb{Q}$-split torus in $\mathbf{P_H}/\mathbf{U_H}$.

Since $\mathbf{S_H} \subset \mathbf{S_G} \cap \mathbf{H}$ and $q(\mathbf{S_H})$ is a maximal $\mathbb{Q}$-split torus in $\mathbf{P_H}/\mathbf{U_H}$, we conclude that $q(\mathbf{S_H}) = q(\mathbf{S_G} \cap \mathbf{H})$. Because $\mathbf{S_G} \cap \mathbf{U_G} = \{1\}$, $q_{|\mathbf{S_G}}$ is injective. Thus $\mathbf{S_H} = \mathbf{S_G} \cap \mathbf{H}$. □

4C. *Comparing $A_{\mathbf{H},t}$ with $A_{\mathbf{G},t'}$.* We now compare the sets $A_{\mathbf{H},t} \subset \mathbf{S_H}(\mathbb{R})$ and $A_{\mathbf{G},t'} \subset \mathbf{S_G}(\mathbb{R})$. We would like to have $A_{\mathbf{H},t} \subset A_{\mathbf{G},t'}$, but it is not always possible to choose $t' \in \mathbb{R}_{>0}$ such that this holds. This is because there may be simple roots in $\Phi(\mathbf{S_G}, \mathbf{G})$ whose restrictions to $\mathbf{S_H}$ are not positive combinations of simple roots in $\Phi(\boldsymbol{S_H}, \boldsymbol{H})$. The values of such a root are bounded below by a positive constant on $A_{\boldsymbol{G},t'}$ but can be arbitrarily close to zero on $A_{\boldsymbol{H},t}$.

Instead we show that for a suitable value of t', every $\alpha \in A_{\boldsymbol{H},t'}$ can be conjugated into $A_{\boldsymbol{G},t'}$ by an element of the Weyl group $N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$. This element of the Weyl group must also satisfy certain other conditions which will be used later in the proof of Theorem 4.1.

Write

$$W = N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G}) \quad \text{and} \quad W' = N_{\boldsymbol{G}}(\boldsymbol{S'_G})/Z_{\boldsymbol{G}}(\boldsymbol{S'_G}).$$

Since $\boldsymbol{S'_G} = u\boldsymbol{S_G}u^{-1}$, conjugation by u induces an isomorphism $W \to W'$.

Proposition 4.7. *There exists $t' > 0$ (depending only on $\boldsymbol{G}$, $\boldsymbol{H}$, and t) such that for every $\alpha \in A_{\boldsymbol{H},t}$, there exists $w \in W$ such that*:

(i) $\boldsymbol{U_Z} \subset w\boldsymbol{U_G}w^{-1}$.

(ii) $\boldsymbol{U_H} \subset w\boldsymbol{U_G}w^{-1}$.

(iii) $\alpha \in wA_{\boldsymbol{G},t}w^{-1}$.

Note that the statement of the proposition makes sense because $w\boldsymbol{U_G}w^{-1}$ and $wA_{\boldsymbol{G},t'}w^{-1}$ do not depend on the choice of representative of w in $N_{\boldsymbol{G}}(\boldsymbol{S_G})$.

Construction of $\boldsymbol{Q}_\alpha$. Suppose that we are given $\alpha \in A_{\boldsymbol{H},t}$. In order to find $w \in W$ as in Proposition 4.7, we construct a parabolic subgroup $\boldsymbol{P}_{\boldsymbol{G},\alpha} = w\boldsymbol{P_G}w^{-1}$ by a refinement of the construction of $\boldsymbol{P_G}$ from Section 4B. First we construct a larger parabolic subgroup $\boldsymbol{Q}_\alpha$ which satisfies conditions (i) and (ii) from Lemma 4.2, as well as the following additional condition:

(iii) There exists $t' > 0$ (independent of α) such that, for every $\alpha \in A_{\boldsymbol{H},t}$ and every $\chi \in \Phi(\boldsymbol{S_H}, \boldsymbol{Q}_\alpha)$, $\chi(\alpha) \geq t'$.

Similar to the proof of Lemma 4.2, we construct $\boldsymbol{Q}_\alpha$ by choosing a suitable order $>_\alpha$ on $X^*(\boldsymbol{S_H})$.

Given $\alpha \in \boldsymbol{S_H}(\mathbb{R})^+$, choose a set $\Psi_\alpha \subset \Phi(\boldsymbol{S_H}, \boldsymbol{G})$ which is maximal with respect to the following conditions:

(a) The set $\Phi^+_{\boldsymbol{H}} \cup \Psi_\alpha$ is $\mathbb{R}_{>0}$-independent. (Recall that $\Phi^+_{\boldsymbol{H}} = \Phi(\boldsymbol{S_H}, \boldsymbol{P_H})$.)

(b) For all $\chi \in \Psi_\alpha$, $\chi(\alpha) \geq 1$.

There always exists at least one set satisfying conditions (a) and (b), namely the empty set. Since $\Phi(\boldsymbol{S_H}, \boldsymbol{G})$ is finite, we deduce that there is a maximal set Ψ_α satisfying the conditions.

By (a) there exists an order $>_\alpha$ on $X^*(\boldsymbol{S_H})$ with respect to which all elements of $\Phi^+_{\boldsymbol{H}} \cup \Psi_\alpha$ are positive. Let

$$\Phi_\alpha = \{\chi \in \Phi(\boldsymbol{S_H}, \boldsymbol{G}) : \chi >_\alpha 0\}$$

and let $\boldsymbol{Q}_\alpha = \boldsymbol{G}_{\Phi_\alpha}$ (in the notation of [Borel and Tits 1965, Paragraph 3.8] with respect to $\boldsymbol{S_H}$).

The only condition on the order $>_{\boldsymbol{Q}}$ in the proof of Lemma 4.2 was that all elements of $\Phi^+_{\boldsymbol{H}}$ are positive with respect to $>_{\boldsymbol{Q}}$. By definition, $>_\alpha$ satisfies this condition. Hence the proof of Lemma 4.2

also applies to $\boldsymbol{Q}_\alpha$. We conclude that $\boldsymbol{Q}_\alpha$ is a parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ satisfying conclusions (i) and (ii) of Lemma 4.2.

Lemma 4.8. *Every root $\chi \in \Phi_\alpha$ is a $\mathbb{R}_{>0}$-combination of $\Delta_{\boldsymbol{H}} \cup \Psi_\alpha$.*

Proof. If $\chi \in \Psi_\alpha$, the result is trivial. So we may assume that $\chi \notin \Psi_\alpha$.

Since $\chi >_\alpha 0$, $\Psi_\alpha \cup \{\chi\}$ satisfies (a). Since $\chi \notin \Psi_\alpha$, the maximality of Ψ_α tells us that $\Psi_\alpha \cup \{\chi\}$ does not satisfy (b). Thus $\chi(\alpha) < 1$.

Hence $\Psi_\alpha \cup \{-\chi\}$ satisfies (b). But $-\chi <_\alpha 0$, so $-\chi \notin \Psi_\alpha$. Again by the maximality of Ψ_α, we conclude that $\Psi_\alpha \cup \{-\chi\}$ does not satisfy (a). Thus there exist $m_i, n_j, x \in \mathbb{R}_{>0}$, $\chi_i \in \Phi_{\boldsymbol{H}}^+$ and $\psi_j \in \Psi_\alpha$ such that

$$\sum_i m_i \chi_i + \sum_j n_j \psi_j + x(-\chi) = 0.$$

(The coefficient of $-\chi$ in this equation must be nonzero because $\Phi_{\boldsymbol{H}}^+ \cup \Psi_\alpha$ is $\mathbb{R}_{>0}$-independent.)

We can rearrange this equation to write χ as a $\mathbb{R}_{>0}$-combination of $\Phi_{\boldsymbol{H}}^+ \cup \Psi_\alpha$. Since every element of $\Phi_{\boldsymbol{H}}^+$ is a $\mathbb{R}_{>0}$-combination of elements of $\Delta_{\boldsymbol{H}}$, we deduce that χ is a $\mathbb{R}_{>0}$-combination of $\Delta_{\boldsymbol{H}} \cup \Psi_\alpha$. $\square$

Lemma 4.9. *There exists $t' > 0$ (depending on $\boldsymbol{G}$, $\boldsymbol{H}$ and t but not on α) such that for every $\alpha \in A_{\boldsymbol{H},t}$ and every $\chi \in \Phi_\alpha$, $\chi(\alpha) \geq t'$.*

Proof. Consider all pairs (χ, Ξ) where $\chi \in \Phi_{\boldsymbol{G}}$ and Ξ is a subset of $\Phi_{\boldsymbol{G}}$ such that χ can be written as a $\mathbb{R}_{>0}$-combination of elements of Ξ. There are only finitely many such pairs, so we can find M (depending only on the root system $\Phi_{\boldsymbol{G}}$) such that, for every such pair, there exist $m_i \in \mathbb{R}_{>0}$ and $\xi_i \in \Xi$ satisfying

$$\chi = \sum_i m_i \xi_i \quad \text{and} \quad \sum_i m_i \leq M.$$

Suppose that $\chi \in \Phi_\alpha$. Using Lemma 4.8, we can write χ as a combination

$$\chi = \sum_i m_i \chi_i + \sum_j n_j \psi_j$$

where $\chi_i \in \Delta_{\boldsymbol{H}}$, $\psi_j \in \Psi_\alpha$, $m_i, n_i \in \mathbb{R}_{>0}$. By the definition of M, we may assume that $\sum_i m_i + \sum_j n_j \leq M$.

By the definition of $A_{\boldsymbol{H},t}$, we have $\chi_i(\alpha) \geq t$ for all i. By condition (b) on Ψ_α, we have $\psi_j(\alpha) \geq 1$ for all j. Therefore $\chi(\alpha) \geq \min(1, t)^M$. $\square$

Proof of Proposition 4.7. Because $\boldsymbol{Q}_\alpha$ satisfies conclusion (i) of Lemma 4.2, $\boldsymbol{Z}$ is a Levi subgroup of $\boldsymbol{Q}_\alpha$. Let $\boldsymbol{P}_{\boldsymbol{G},\alpha} = \boldsymbol{P}_{\boldsymbol{Z}} \ltimes R_u(\boldsymbol{Q}_\alpha)$. By [Borel and Tits 1965, Proposition 4.4], $\boldsymbol{P}_{\boldsymbol{G},\alpha}$ is a minimal $\mathbb{Q}$-parabolic subgroup of $\boldsymbol{G}$.

By [Borel and Tits 1965, Corollaire 5.9], the Weyl group W' acts transitively on the minimal parabolic $\mathbb{Q}$-subgroups of $\boldsymbol{G}$ containing the maximal $\mathbb{Q}$-split torus $\boldsymbol{S}'_{\boldsymbol{G}}$. Since $\boldsymbol{S}'_{\boldsymbol{G}} \subset \boldsymbol{P}_{\boldsymbol{Z}} \subset \boldsymbol{P}_{\boldsymbol{G},\alpha}$, we conclude that there exists $w' \in W'$ (depending on α) such that $\boldsymbol{P}_{\boldsymbol{G},\alpha} = w' \boldsymbol{P}_{\boldsymbol{G}} w'^{-1}$.

Let w be the element of W which corresponds to $w' \in W'$ via conjugation by u. Since $u \in \boldsymbol{U_Z}(\mathbb{R}) \subset \boldsymbol{P_G}(\mathbb{R}) \cap \boldsymbol{P}_{\boldsymbol{G},\alpha}(\mathbb{R})$, we have

$$\boldsymbol{P}_{\boldsymbol{G},\alpha} = w \boldsymbol{P_G} w^{-1}.$$

Since $\boldsymbol{Q}_\alpha$ satisfies conclusion (ii) of Lemma 4.2, we have

$$\boldsymbol{U_H} \subset R_u(\boldsymbol{Q}_\alpha) \subset R_u(\boldsymbol{P}_{\boldsymbol{G},\alpha}) = w \boldsymbol{U_G} w^{-1}.$$

Furthermore $\boldsymbol{P_Z} \subset \boldsymbol{P}_{\boldsymbol{G},\alpha}$ and so $\boldsymbol{U_Z} \subset R_u(\boldsymbol{P}_{\boldsymbol{G},\alpha})$. This proves conclusions (i) and (ii) of Proposition 4.7.

Since $\boldsymbol{P}_{\boldsymbol{G},\alpha} \subset \boldsymbol{Q}_\alpha$, if $\chi \in \Phi(\boldsymbol{S_G}, \boldsymbol{P}_{\boldsymbol{G},\alpha})$ then $\chi_{|\boldsymbol{S_H}} \in \Phi_\alpha$. Hence by Lemma 4.9,

$$\chi(\alpha) \geq t' \text{ for all } \alpha \in A_{\boldsymbol{H},t} \text{ and } \chi \in \Phi(\boldsymbol{S_G}, \boldsymbol{P}_{\boldsymbol{G},\alpha}).$$

Noting that

$$w A_{\boldsymbol{G},t'} w^{-1} = \{\beta \in \boldsymbol{S_G}(\mathbb{R})^+ : \chi(\beta) \geq t' \text{ for all simple roots of } \boldsymbol{P}_{\boldsymbol{G},\alpha}\}$$

we conclude that $\alpha \in w A_{\boldsymbol{G},t'} w^{-1}$, proving conclusion (iii) of Proposition 4.7. □

4D. ***Weyl group representatives.*** We need to choose two representatives for each element w in the Weyl group $W = N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$.

Firstly we would like to choose representatives for W in $\boldsymbol{G}(\mathbb{Q})$. However this is not usually possible because the torus $\boldsymbol{S_G}$ is not defined over $\mathbb{Q}$. Instead, recall that conjugation by u induces an isomorphism $W \to W'$. Given $w \in W$, let w' denote the corresponding element of W'. By [Borel and Tits 1965, Théorème 5.3], we can choose $w'_{\mathbb{Q}} \in \boldsymbol{G}(\mathbb{Q})$ which represents w'. We then get a representative for w by setting

$$w_{\mathbb{Q}} = u^{-1} \, w'_{\mathbb{Q}} \, u.$$

Secondly we choose representatives for W in $K_{\boldsymbol{G}}$.

Lemma 4.10. *Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group. Let $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ be a Siegel triple in $\boldsymbol{G}$. Every $w \in N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$ has a representative $w_K \in K_{\boldsymbol{G}}$.*

Proof. Let $\boldsymbol{T_G}$ be a maximal $\mathbb{R}$-split torus in $\boldsymbol{G}$ which contains $\boldsymbol{S_G}$.

Let $\boldsymbol{N} = N_{\boldsymbol{G}}(\boldsymbol{S_G}) \cap N_{\boldsymbol{G}}(\boldsymbol{T_G})$. Because $\boldsymbol{S_G}$ is conjugate to a maximal $\mathbb{Q}$-split torus of $\boldsymbol{G}$, [Borel and Tits 1965, Corollaire 5.5] implies that

$$N_{\boldsymbol{G}}(\boldsymbol{S_G}) = \boldsymbol{N}.Z_{\boldsymbol{G}}(\boldsymbol{S_G}).$$

Therefore we can choose $\sigma \in \boldsymbol{N}(\mathbb{C})$ such that $w = \sigma.Z_{\boldsymbol{G}}(\boldsymbol{S_G})$.

According to the final displayed equation from [Borel and Tits 1965, Section 14], every element of $N_{\boldsymbol{G}}(\boldsymbol{T_G})/Z_{\boldsymbol{G}}(\boldsymbol{T_G})$ has a representative in $K_{\boldsymbol{G}}$. In particular, there exists $w_K \in N_{\boldsymbol{G}}(\boldsymbol{T_G})(\mathbb{R}) \cap K_{\boldsymbol{G}}$ which represents $\sigma.Z_{\boldsymbol{G}}(\boldsymbol{T_G})$. Then

$$w_K \sigma^{-1} \in Z_{\boldsymbol{G}}(\boldsymbol{T_G})(\mathbb{C}) \subset Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{C}).$$

It follows that w_K normalises $\boldsymbol{S_G}$ and represents $w \in N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$. □

Since the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$ stabilizes $\boldsymbol{S_G}$, it also stabilizes $Z_{\boldsymbol{G}}(\boldsymbol{S_G})$. Hence $K_{\boldsymbol{G}} \cap Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$ is a maximal compact subgroup of $Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$. By [Hochschild 1965, Chapter XV, Theorem 3.1], $K_{\boldsymbol{G}} \cap Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$ meets every connected component of $Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$. When choosing w_K as in Lemma 4.10, we may therefore assume that $w_K \in w_{\mathbb{Q}}.Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})^+$.

We will need the following lemma about $w_{\mathbb{Q}}$ and $w'_{\mathbb{Q}}$. This lemma does not hold for every element of W, so we restrict our attention to elements which satisfy conditions (i) and (ii) of Proposition 4.7, that is, elements of the set

$$W^\dagger = \{w \in W : \boldsymbol{U_Z} \subset w\boldsymbol{U_G}w^{-1} \text{ and } \boldsymbol{U_H} \subset w\boldsymbol{U_G}w^{-1}\}.$$

Lemma 4.11. *If $w \in W^\dagger$, then $w'^{-1}_{\mathbb{Q}} w_{\mathbb{Q}} \in \boldsymbol{U_G}(\mathbb{R})$.*

Proof. By definition,

$$w'^{-1}_{\mathbb{Q}} w_{\mathbb{Q}} = u w_{\mathbb{Q}}^{-1} u^{-1} w_{\mathbb{Q}}.$$

Because $w \in W^\dagger$ and $u \in \boldsymbol{U_Z}(\mathbb{R})$, we have

$$w_{\mathbb{Q}}^{-1} u^{-1} w_{\mathbb{Q}} \in \boldsymbol{U_G}(\mathbb{R}).$$

Multiplying this by $u \in \boldsymbol{U_G}(\mathbb{R})$ proves the lemma. ☐

4E. *Construction of the compact set $\Omega_{\boldsymbol{G}}$.* By the Langlands decomposition in $\boldsymbol{P_H}$, the multiplication map

$$\boldsymbol{U_H}(\mathbb{R}) \times \boldsymbol{M_H}(\mathbb{R})^+ \to \boldsymbol{U_H}(\mathbb{R}).\boldsymbol{M_H}(\mathbb{R})^+$$

is a homeomorphism. Hence there exist compact sets $\Omega_{\boldsymbol{U_H}} \subset \boldsymbol{U_H}(\mathbb{R})$ and $\Omega_{\boldsymbol{M_H}} \subset \boldsymbol{M_H}(\mathbb{R})^+$ such that

$$\Omega_{\boldsymbol{H}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{M_H}}. \tag{13}$$

Since $\boldsymbol{M_H}$ need not be contained in $\boldsymbol{M_G}$, we need to further decompose $\Omega_{\boldsymbol{M_H}}$. Let $\boldsymbol{B_Z}$ be a minimal $\mathbb{R}$-parabolic subgroup of $\boldsymbol{Z} = Z_{\boldsymbol{G}}(\boldsymbol{S_H})$ contained in $\boldsymbol{P_Z}$. By the Iwasawa decomposition in $\boldsymbol{Z}$, the multiplication map $\boldsymbol{B_Z}(\mathbb{R})^+ \times K_{\boldsymbol{Z}} \to \boldsymbol{Z}(\mathbb{R})$ is a homeomorphism so there exists a compact set $\Omega_{\boldsymbol{B_Z}} \subset \boldsymbol{B_Z}(\mathbb{R})^+$ such that

$$\Omega_{\boldsymbol{M_H}} \subset \Omega_{\boldsymbol{B_Z}}.K_{\boldsymbol{Z}}. \tag{14}$$

For each $w \in W^\dagger$, choose w_K, $w_{\mathbb{Q}}$ and $w'_{\mathbb{Q}}$ as in Section 4D. We have $w_K w_{\mathbb{Q}}^{-1} \in Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})^+ \subset \boldsymbol{P_Z}(\mathbb{R})^+$ and $\boldsymbol{B_Z}(\mathbb{R})^+ \subset \boldsymbol{P_Z}(\mathbb{R})^+$, so $\Omega_{\boldsymbol{B_Z}}.w_K w_{\mathbb{Q}}^{-1}$ is a compact subset of $\boldsymbol{P_Z}(\mathbb{R})^+$. Noting that $Z_{\boldsymbol{G}}(\boldsymbol{S_G})$ is a Levi subgroup of $\boldsymbol{P_Z}$, the Langlands decomposition in $\boldsymbol{P_Z}$ [Borel and Ji 2006, Equation (I.1.8)] tells us that the multiplication map

$$\boldsymbol{U_Z}(\mathbb{R}) \times \boldsymbol{M_G}(\mathbb{R})^+ \times \boldsymbol{S_G}(\mathbb{R})^+ \to \boldsymbol{P_Z}(\mathbb{R})^+$$

is a homeomorphism. Therefore there exist compact sets $\Omega^{[w]}_{\boldsymbol{U_Z}} \subset \boldsymbol{U_Z}(\mathbb{R})$, $\Omega^{[w]}_{\boldsymbol{M_G}} \subset \boldsymbol{M_G}(\mathbb{R})^+$ and $\Omega^{[w]}_{\boldsymbol{S_G}} \subset \boldsymbol{S_G}(\mathbb{R})^+$ such that

$$\Omega_{\boldsymbol{B_Z}}.w_K w_{\mathbb{Q}}^{-1} \subset \Omega^{[w]}_{\boldsymbol{U_Z}}.\Omega^{[w]}_{\boldsymbol{M_G}}.\Omega^{[w]}_{\boldsymbol{S_G}}. \tag{15}$$

Let

$$\Omega_{\boldsymbol{G}} = \bigcup_{w \in W^\dagger} w_{\mathbb{Q}}'^{-1}.\Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.w_{\mathbb{Q}}.$$

Since $W^\dagger$ is finite, $\Omega_{\boldsymbol{G}}$ is compact.

Lemma 4.12. $\Omega_{\boldsymbol{G}} \subset \boldsymbol{U_G}(\mathbb{R})\boldsymbol{M_G}(\mathbb{R})^+$.

Proof. For each $w \in W^\dagger$, by Lemma 4.11, $w_{\mathbb{Q}}'^{-1} w_{\mathbb{Q}} \in \boldsymbol{U_G}(\mathbb{R})$. Using the definition of $W^\dagger$, we have

$$w_{\mathbb{Q}}^{-1}\Omega_{\boldsymbol{U_H}} w_{\mathbb{Q}} \subset \boldsymbol{U_G}(\mathbb{R}) \quad \text{and} \quad w_{\mathbb{Q}}^{-1}\Omega_{\boldsymbol{U_Z}}^{[w]} w_{\mathbb{Q}} \subset \boldsymbol{U_G}(\mathbb{R}).$$

Multiplying these together, we conclude that

$$w_{\mathbb{Q}}'^{-1}.\Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{U_Z}}^{[w]}.w_{\mathbb{Q}} \subset \boldsymbol{U_G}(\mathbb{R}). \tag{16}$$

Since $\boldsymbol{S_G}$ is $\boldsymbol{G}(\mathbb{R})$-conjugate to a maximal $\mathbb{Q}$-split torus in $\boldsymbol{G}$, we can use [Borel and Tits 1965, Corollaire 5.4] to show that $\boldsymbol{M_G}$ is normal in $N_{\boldsymbol{G}}(\boldsymbol{S_G})$. It follows that $w_{\mathbb{Q}}$ normalises $\boldsymbol{M_G}(\mathbb{R})^+$ and so

$$w_{\mathbb{Q}}^{-1}\Omega_{\boldsymbol{M_G}}^{[w]} w_{\mathbb{Q}} \subset \boldsymbol{M_G}(\mathbb{R})^+. \tag{17}$$

Combining (16) and (17) proves the lemma. ☐

Lemma 4.13. *For each* $w \in W^\dagger$, $w_{\mathbb{Q}}'^{-1}\Omega_{\boldsymbol{H}} \subset \Omega_{\boldsymbol{G}}.w_K^{-1}.\Omega_{\boldsymbol{S_G}}^{[w]}.K_{\boldsymbol{Z}}$.

Proof. Noting that $w_{\mathbb{Q}} w_K^{-1}$ commutes with $\boldsymbol{S_G}$, we can rearrange (15) to obtain

$$\Omega_{\boldsymbol{B_Z}} \subset \Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.w_{\mathbb{Q}} w_K^{-1}.\Omega_{\boldsymbol{S_G}}^{[w]}.$$

Combining this with (13) and (14), we get

$$\Omega_{\boldsymbol{H}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{M_H}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{B_Z}}.K_{\boldsymbol{Z}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.w_{\mathbb{Q}} w_K^{-1}.\Omega_{\boldsymbol{S_G}}^{[w]}.K_{\boldsymbol{Z}}.$$

We can now read off the lemma using the definition of $\Omega_{\boldsymbol{G}}$. ☐

4F. ***The Siegel set for G.*** For each $w \in W^\dagger$, $w_K^{-1}\Omega_{\boldsymbol{S_G}}^{[w]} w_K$ is a compact subset of $\boldsymbol{S_G}(\mathbb{R})^+$. Hence there exists $s > 0$ such that $\chi(\beta) \geq s$ for all $\chi \in \Delta_{\boldsymbol{G}}$ and all $\beta \in w_K^{-1}\Omega_{\boldsymbol{S_G}}^{[w]} w_K$ (since $W^\dagger$ is finite, we can choose a single value of s which works for all $w \in W^\dagger$).

Let $\mathfrak{S}_{\boldsymbol{G}}$ be the Siegel set

$$\mathfrak{S}_{\boldsymbol{G}} = \Omega_{\boldsymbol{G}}.A_{\boldsymbol{G},t's}.K_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R}),$$

using t' from Proposition 4.7 and $\Omega_{\boldsymbol{G}}$ from Section 4E. Let C be the finite set

$$C = \{w_{\mathbb{Q}}' : w \in W^\dagger\} \subset \boldsymbol{G}(\mathbb{Q}).$$

Proposition 4.14. $\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}$.

Proof. Given $\sigma \in \mathfrak{S}_{\boldsymbol{H}}$, we can write

$$\sigma = \mu\alpha\kappa$$

with $\mu \in \Omega_{\boldsymbol{H}}$, $\alpha \in A_{\boldsymbol{H},t}$ and $\kappa \in K_{\boldsymbol{H}}$.

By Proposition 4.7, we can choose $w \in W^\dagger$ such that $\alpha \in wA_{\boldsymbol{G},t'}w^{-1}$. By Lemma 4.13, we can write

$$w_{\mathbb{Q}}'^{-1}\mu = \nu w_K^{-1}\beta\lambda$$

where $\nu \in \Omega_{\boldsymbol{G}}$, $\beta \in \Omega_{\boldsymbol{S_G}}^{[w]}$ and $\lambda \in K_{\boldsymbol{Z}}$. Therefore

$$w_{\mathbb{Q}}'^{-1}\sigma = \nu w_K^{-1}\beta\lambda\,\alpha\kappa.$$

Since $\lambda \in K_{\boldsymbol{Z}} \subset \boldsymbol{Z}(\mathbb{R})$, λ commutes with $\alpha \in \boldsymbol{S_H}(\mathbb{R})$ so we can rewrite this as

$$w_{\mathbb{Q}}'^{-1}\sigma = \nu.w_K^{-1}\,\beta\alpha w_K.w_K^{-1}\lambda\kappa.$$

By definition, $\nu \in \Omega_{\boldsymbol{G}}$. By the definition of s, we have $w_K^{-1}\beta w_K \in A_{\boldsymbol{G},s}$ while $w_K^{-1}\alpha w_K \in A_{\boldsymbol{G},t'}$ by Proposition 4.7. Hence

$$w_K^{-1}\beta\alpha w_K \in A_{\boldsymbol{G},t's}.$$

Finally, w_K^{-1}, λ and κ are all in the group $K_{\boldsymbol{G}}$, so their product is also in $K_{\boldsymbol{G}}$.

Thus we have shown that $w_{\mathbb{Q}}'^{-1}\sigma \in \mathfrak{S}_{\boldsymbol{G}}$, and so $\sigma \in C.\mathfrak{S}_{\boldsymbol{G}}$. □

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Quadric surface bundles over surfaces and stable rationality

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We prove a general specialization theorem which implies stable irrationality for a wide class of quadric surface bundles over rational surfaces. As an application, we solve, with the exception of two cases, the stable rationality problem for any very general complex projective quadric surface bundle over $\mathbb{P}^2$, given by a symmetric matrix of homogeneous polynomials. Both exceptions degenerate over a plane sextic curve, and the corresponding double cover is a K3 surface.

1. Introduction

Recently, Hassett, Pirutka, and Tschinkel [Hassett et al. 2016b; 2016c; 2017] found the first three examples of families of quadric surface bundles over $\mathbb{P}^2$, where the very general member is not stably rational. In each case, the degeneration locus is a plane octic curve. Smooth quadric surface bundles over rational surfaces typically deform to smooth bundles with a section, hence to smooth rational fourfolds. This allowed them to produce the first examples of smooth nonrational varieties that deform to rational ones.

In [Schreieder 2018], we introduced a variant of the method of Voisin [2015] and Colliot-Thélène and Pirutka [2016a], which allowed us to disprove stable rationality via a degeneration argument where a universally CH_0-trivial resolution of the special fiber is not needed. The purpose of this paper is to show that one can use this technique to simplify the arguments in [Hassett et al. 2016b; 2016c; 2017] and to apply them to large classes of quadric surface bundles.

The main result is the following general specialization theorem without resolutions; see Section 1.1 below for what it means that a variety specializes to another variety.

Theorem 1. *Let X and Y be complex projective varieties of dimension four. Suppose that X specializes to Y and that there is a morphism $f : Y \to S$ to a rational surface S, such that*

(1) *the generic fiber of f is a smooth quadric surface Q over $K = \mathbb{C}(S)$,*

(2) *the discriminant $d \in K^*/(K^*)^2$ of Q is nontrivial, and*

(3) $H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2) \neq 0$.

Then X is not stably rational.

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Since $H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2) = H^2_{nr}(K(Y)/\mathbb{C}, \mathbb{Z}/2)$, the assumptions in the above theorem concern only the generic fiber of f. In particular, f need not be flat and there is no assumption on the singularities of Y at points which do not dominate S. A universally CH_0-trivial resolution of Y is not needed. For a more general version which works also if the discriminant of Q is possibly trivial, X and Y have arbitrary dimension, and the generic fiber of f is only stably birational to Q, see Theorem 9 below.

The second unramified cohomology group in item (3) coincides with the 2-torsion subgroup of the Brauer group of any resolution of singularities of Y. Pirutka [2016, Theorem 3.17] computed this group explicitly for any quadric surface over $\mathbb{C}(\mathbb{P}^2)$ which satisfies (2). This gives rise to many examples to which the above theorem applies. In this paper we will apply it only to a single example of Hassett, Pirutka, and Tschinkel [Hassett et al. 2016b, Proposition 11].

The proof of Theorem 1 uses results of Pirutka [2016] on the unramified cohomology of quadric surfaces over $\mathbb{C}(\mathbb{P}^2)$, together with our aforementioned method from [Schreieder 2018], which builds on [Voisin 2015; Colliot-Thélène and Pirutka 2016a].

To give an application of Theorem 1, let us consider a generically nondegenerate line bundle valued quadratic form $q : \mathscr{E} \to \mathscr{O}_{\mathbb{P}^2}(n)$, where $\mathscr{E} = \bigoplus_{i=0}^3 \mathscr{O}_{\mathbb{P}^2}(-r_i)$ is split and such that the quadratic form q_s on the fiber $\mathscr{E}_s$ is nonzero for all $s \in \mathbb{P}^2$. Then, $X = \{q = 0\} \subset \mathbb{P}(\mathscr{E})$ defines a quadric surface bundle over $\mathbb{P}^2$. We may also regard q as a symmetric matrix $A = (a_{ij})$, where a_{ij} is a global section of $\mathscr{O}_{\mathbb{P}^2}(r_i + r_j + n)$. Locally over $\mathbb{P}^2$, X is given by

$$\sum_{i,j=0}^{3} a_{ij} z_i z_j = 0, \tag{1}$$

where z_i denotes a local coordinate which trivializes $\mathscr{O}_{\mathbb{P}^2}(-r_i) \subset \mathscr{E}$.

If X is smooth, its deformation type depends only on the integers $d_i := 2r_i + n$; we call any such quadric surface bundle of type (d_0, d_1, d_2, d_3). The degeneration locus of $X \to \mathbb{P}^2$ is a plane curve of degree $\sum_i d_i$, which is always even. If some d_i is negative, then $a_{ii} = 0$ and so $X \to \mathbb{P}^2$ admits a section; hence, X is rational. We may thus from now on restrict ourselves to the case $d_i \geq 0$ for all i.

Corollary 2. *Let d_0, d_1, d_2, and d_3 be nonnegative integers of the same parity, and let $X \to \mathbb{P}^2$ be a very general complex projective quadric surface bundle of type (d_0, d_1, d_2, d_3). If $\sum_i d_i \neq 6$, then*

(1) *X is rational if $\sum_i d_i \leq 4$ or if $d_i = d_j = 0$ for some $i \neq j$ and*

(2) *X is not stably rational otherwise.*

As we will see in the proof, the bundles in item (1) of the above corollary have a rational section, and so already the generic fiber of X over $\mathbb{P}^2$ is rational.

Up to reordering, the only cases left open by the above corollary are types $(1, 1, 1, 3)$ and $(0, 2, 2, 2)$. The former corresponds to blow-ups of cubic fourfolds containing a plane, see e.g. [Auel et al. 2017b], and the latter are Verra fourfolds [Camere et al. 2017; Iliev et al. 2017], i.e., double covers of $\mathbb{P}^2 \times \mathbb{P}^2$, branched along a hypersurface of bidegree $(2, 2)$. In both exceptions, the degeneration locus of the quadric bundle is a sextic curve in $\mathbb{P}^2$, and so the associated double cover is a K3 surface, see e.g. [Auel et al. 2015].

Specializing to $a_{33} = 0$ in (1) shows that all examples in the above corollary deform to smooth quadric surface bundles with a section, hence to smooth rational fourfolds.

Many quadric surface bundles over $\mathbb{P}^2$ are birational to fourfolds which arise naturally in projective geometry, see e.g. [Schreieder 2018, §3.5]. For instance, Corollary 2 implies that

(I) a very general complex hypersurface of bidegree $(d, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not stably rational if $d \geq 2$,

(II) a very general complex hypersurface $X \subset \mathbb{P}^5$ of degree $d+2$ and with multiplicity d along a 2-plane is not stably rational if $d \geq 2$, and

(III) a double cover $X \xrightarrow{2:1} \mathbb{P}^4$, branched along a very general complex hypersurface $Y \subset \mathbb{P}^4$ of even degree $d+2$ and with multiplicity d along a line, is not stably rational if $d \geq 2$.

The case $d = 2$ in items (I) and (III) corresponds to the aforementioned results in [Hassett et al. 2016b; 2016c]. For stable rationality properties of smooth hypersurfaces and double covers, see [Beauville 2016; Colliot-Thélène and Pirutka 2016a; 2016b; Hassett et al. 2016c; Okada 2016; Totaro 2016; Voisin 2015]; for results on conic bundles, see [Ahmadinezhad and Okada 2018; Artin and Mumford 1972; Auel et al. 2016; Beauville 2016; Böhning and von Bothmer 2018; Hassett et al. 2016a; Voisin 2015].

In [Schreieder 2018], we studied rationality properties of quadric bundles with arbitrary fiber dimensions. Our uniform treatment sufficed to prove (I) and (II) for $d \geq 5$, and (III) for $d \geq 8$. On the other hand, the results in [Schreieder 2018] left open infinitely many cases in Corollary 2. For instance, the types $(1, 1, d_2, d_3)$ and $(0, 2, d_2, d_3)$ with $d_2 \leq 7$ and arbitrary d_3 are not covered by [Schreieder 2018] and there are more cases which were not accessible; see [Schreieder 2018, Remark 36].

Our method applies also to quadric surface bundles over other rational surfaces S. We treat in this paper the case $S = \mathbb{P}^1 \times \mathbb{P}^1$ and obtain similar results as those in Corollary 2 above; see Corollaries 11 and 12 below.

1.1. ***Conventions and notations.*** All schemes are separated. A variety is an integral scheme of finite type over a field. A property is said to hold at a very general point of a scheme, if it holds at all closed points outside a countable union of proper closed subsets.

Let k be an algebraically closed field. We say that a variety X over a field L specializes (or degenerates) to a variety Y over k, if there is a discrete valuation ring R with residue field k and fraction field F with an injection of fields $F \hookrightarrow L$, together with a flat proper morphism $\mathcal{X} \to \operatorname{Spec} R$ of finite type, such that Y is isomorphic to the special fiber $Y \simeq \mathcal{X} \times_R k$ and $X \simeq \mathcal{X} \times_R L$ is isomorphic to a base change of the generic fiber. If $\mathcal{Y} \to B$ is a flat proper morphism of complex varieties with integral fibers, then for any closed points $0, t \in B$ with t very general, the fiber Y_t specializes to Y_0 in the above sense [Schreieder 2018, Lemma 8].

A morphism $f : X \to Y$ of varieties over a field k is universally CH_0-trivial, if $f_* : \mathrm{CH}_0(X \times L) \xrightarrow{\simeq} \mathrm{CH}_0(Y \times L)$ is an isomorphism for all field extensions L of k.

A quadric surface bundle is a flat morphism $f : X \to S$ between projective varieties such that the generic fiber is a smooth quadric surface; the degeneration locus is given by all $s \in S$ such that $f^{-1}(s)$ is singular. If f is not assumed flat, then we call X a weak quadric surface bundle over S. Quadric surface bundles over surfaces have been studied in detail in [Auel et al. 2015].

We denote by $\mu_2 \subset \mathbb{G}_m$ the group of second roots of unity. If X is a proper variety over a field k of characteristic different from 2, the unramified cohomology group $H^i_{nr}(k(X)/k, \mu_2^{\otimes i})$ is the subgroup of all elements of the Galois cohomology group $H^i(k(X), \mu_2^{\otimes i})$ which have trivial residue at all discrete valuations of rank one on $k(X)$ over k [Colliot-Thélène and Ojanguren 1989]. This is a stable birational invariant of X [Colliot-Thélène and Ojanguren 1989, Proposition 1.2]. If X is smooth and proper over k, then $H^i_{nr}(k(X)/k, \mu_2^{\otimes i})$ coincides with the subgroup of elements of $H^i(k(X), \mu_2^{\otimes i})$ that have trivial residue at any codimension-one point of X [Colliot-Thélène 1995, Theorem 4.1.1].

2. Second unramified cohomology of quadric surfaces

Let K be a field of characteristic different from 2. It will be convenient to identify the Galois cohomology group $H^i(K, \mu_2^{\otimes i})$ with the étale cohomology group $H^i_{\text{ét}}(\operatorname{Spec}(K), \mu_2^{\otimes i})$. We also use the identification $H^1(K, \mu_2) \simeq K^*/(K^*)^2$, induced by the Kummer sequence. For $a, b \in K^*$, we denote by $(a, b) \in H^2(K, \mu_2^{\otimes 2})$ the cup product of the classes given by a and b. If S is a normal variety over a field k and with fraction field $k(S) = K$, then for any $\alpha \in H^2(K, \mu_2^{\otimes 2})$, the ramification divisor $\operatorname{ram}(\alpha) \subset S$ is given by (the closure of) all codimension-one points $x \in S^{(1)}$ with $\partial^2_x \alpha \neq 0$. Here, $\partial^2_x : H^2(K, \mu_2^{\otimes 2}) \to H^1(\kappa(x), \mu_2)$ denotes the residue induced by the local ring $\mathcal{O}_{S,x} \subset K$.

To any nondegenerate quadratic form q over K, one associates the discriminant $\operatorname{discr}(q) \in K^*/(K^*)^2$ and the Clifford invariant $\operatorname{cl}(q) \in H^2(K, \mu_2^{\otimes 2})$. If q has even dimension, then the discriminant $\operatorname{discr}(q)$ depends only on the quadric hypersurface $Q = \{q = 0\}$ and the Clifford invariant satisfies $\operatorname{cl}(\lambda \cdot q) = \operatorname{cl}(q) + (\lambda, \operatorname{discr}(q))$ for all $\lambda \in K^*$ [Lam 1973, Chapter 5, (3.16)]. If Q is a surface, then up to similarity, $q \simeq \langle 1, -a, -b, abd \rangle$ for some $a, b, d \in K^*$. In this case, $\operatorname{discr}(q) = d$ and $\operatorname{cl}(q) = (-a, -b) + (ab, d)$. We will need the following [Arason 1975; Kahn et al. 1998, Corollary 8]:

Theorem 3. *Let K be a field with $\operatorname{char}(K) \neq 2$, and let $f : Q \to \operatorname{Spec} K$ be a smooth projective quadric surface over K. Denote by $d \in K^*/(K^*)^2$ the discriminant of Q and by $\beta \in H^2(K, \mu_2^{\otimes 2})$ the Clifford invariant of some quadratic form q with $Q = \{q = 0\}$. Then*

$$f^* : H^2(K, \mu_2^{\otimes 2}) \to H^2_{nr}(K(Q)/K, \mu_2^{\otimes 2})$$

is an isomorphism if d is nontrivial. If $d \in (K^*)^2$, then $\ker(f^*) = \{1, \beta\}$.

Pirutka [2016, Theorem 3.17] computed the unramified cohomology group $H^2_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes 2})$ of a smooth quadric surface Q with nonzero discriminant over the function field of a smooth complex surface. The following reflects one half of her result:

Theorem 4 (Pirutka). *Let $f : Q \to \operatorname{Spec} K$ be a smooth projective quadric surface over the function field K of some smooth surface S over $\mathbb{C}$. Let $d \in K^*/(K^*)^2$ denote the discriminant and $\beta \in H^2(K, \mu_2^{\otimes 2})$ the Clifford invariant of some quadratic form q with $Q = \{q = 0\}$. If for some $\alpha \in H^2(K, \mu_2^{\otimes 2})$ the pullback $f^*(\alpha) \in H^2_{nr}(K(Q)/K, \mu_2^{\otimes 2})$ is unramified over $\mathbb{C}$, then the following holds:*

(∗) *If the residue* $\partial^2_x\alpha$ *at some codimension-one point* $x \in S^{(1)}$ *is nonzero, then*

(a) $\partial^2_x\alpha = \partial^2_x\beta$ *and*

(b) d *becomes a square in the fraction field of the completion* $\widehat{\mathcal{O}_{S,x}}$.

Proof. The condition on d is by Hensel's lemma equivalent to asking that, up to multiplication by a square, d is a unit in $\mathcal{O}_{S,x}$ whose image in $\kappa(x)$ is a square. The theorem follows therefore from [Pirutka 2016, §3.6.2]. (In [Pirutka 2016, Theorem 3.17], the assumption that d is not a square is only used to invoke bijectivity of f^* via Theorem 3; the assumption that $\mathrm{ram}(\beta)$ is a simple normal crossing divisor on S is only used in [Pirutka 2016, §3.6.1].) □

Remark 5. Up to replacing S by some blow-up, one can always assume that $\mathrm{ram}(\beta)$ is a simple normal crossing divisor on S. Under this assumption, the analysis of Pirutka [2016, §3.6.1] shows that the following converse of the above theorem is also true: if $\alpha \in H^2(K, \mu_2^{\otimes 2})$ is such that condition (∗) holds, then $f^*\alpha \in H^2_{nr}(K(Q)/K, \mu_2^{\otimes 2})$ is unramified over $\mathbb{C}$; nontriviality can be checked via Theorem 3.

The result of Pirutka [2016, Theorem 3.17] applies to the following important example, due to Hassett, Pirutka, and Tschinkel [Hassett et al. 2016b, Proposition 11]; for a reinterpretation in terms of conic bundles, see [Auel et al. 2016].

Proposition 6 (Hassett, Pirutka, and Tschinkel). *Let* $K = \mathbb{C}(x, y)$ *be the function field of* $\mathbb{P}^2$, *and consider the quadratic form* $q = \langle y, x, xy, F(x, y, 1)\rangle$ *over* K, *where*

$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz).$$

If $f : Q \to \operatorname{Spec} K$ *denotes the corresponding projective quadric surface over* K, *then*

$$0 \neq f^*((x, y)) \in H^2_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes 2}).$$

3. A vanishing result

The following general vanishing result is the key ingredient of this paper.

Proposition 7. *Let* Y *be a smooth complex projective variety, and let* S *be a smooth complex projective surface. Let* $f : Y \dashrightarrow S$ *be a dominant rational map whose generic fiber* Y_η *is stably birational to a smooth quadric surface* Q *over* $K = \mathbb{C}(S)$. *Suppose that there is some* $\alpha \in H^2(K, \mu_2^{\otimes 2})$, *such that* $\alpha' := f^*\alpha \in H^2_{nr}(K(Y_\eta)/K, \mu_2^{\otimes 2})$ *is unramified over* $\mathbb{C}$. *Then for any prime divisor* $E \subset Y$ *which does not dominate* S, *the restriction of* α' *to* E *vanishes*:

$$\alpha'|_E = 0 \in H^2(\mathbb{C}(E), \mu_2^{\otimes 2}).$$

Proof. Since unramified cohomology is a functorial stable birational invariant [Colliot-Thélène and Ojanguren 1989], we may up to replacing Y by $Y \times \mathbb{P}^m$ assume that Y_η is birational to $Q \times \mathbb{P}^r_K$ for some $r \geq 0$. This birational map induces a dominant rational map $Y_\eta \dashrightarrow Q$.

Since Y is smooth, f is defined at the generic point y of E. By [Merkurjev 2008, Propositions 1.4 and 1.7; Schreieder 2018, §5], we may up to replacing S by a different smooth projective model assume that the image $x := f(y) \in S^{(1)}$ is a codimension-one point on S. Consider the local ring $A := \mathcal{O}_{S,x}$, and let $\hat{A}$ be its completion with field of fractions $\widehat{K} := \mathrm{Frac}(\hat{A})$. The local ring $B := \mathcal{O}_{Y,y}$ contains A. We let $\widehat{B}$ be the completion of B and $\hat{L} := \mathrm{Frac}(\widehat{B})$ be its field of fractions. Since $Y_\eta \dashrightarrow Q$ is dominant, inclusion of fields induces the sequence

$$H^2(K, \mu_2^{\otimes 2}) \xrightarrow{\varphi_1} H^2(\widehat{K}, \mu_2^{\otimes 2}) \xrightarrow{\varphi_2} H^2(\widehat{K}(Q), \mu_2^{\otimes 2}) \xrightarrow{\varphi_3} H^2(\hat{L}, \mu_2^{\otimes 2}). \tag{2}$$

Lemma 8. *If some $\gamma \in H^2(K, \mu_2^{\otimes 2})$ satisfies $\partial_x^2 \gamma = 0$, then $\varphi_1(\gamma) = 0 \in H^2(\widehat{K}, \mu_2^{\otimes 2})$.*

Proof. Since $\partial_x^2 \gamma = 0$, the image of γ in $H^2(\widehat{K}, \mu_2^{\otimes 2})$ is contained in $H^2_{\text{ét}}(\mathrm{Spec}\, \hat{A}, \mu_2^{\otimes 2}) \subset H^2(\widehat{K}, \mu_2^{\otimes 2})$ [Colliot-Thélène 1995, §3.3 and §3.8]. It thus suffices to show that $H^2_{\text{ét}}(\mathrm{Spec}\, \hat{A}, \mu_2^{\otimes 2})$ vanishes. Since $\hat{A}$ is a henselian local ring, restriction to the closed point gives an isomorphism $H^2_{\text{ét}}(\mathrm{Spec}\, \hat{A}, \mu_2^{\otimes 2}) \simeq H^2(\kappa(x), \mu_2^{\otimes 2})$ [Milne 1980, Corollary VI.2.7]. By Tsen's theorem, $H^2(\kappa(x), \mu_2^{\otimes 2}) = 0$. This concludes the lemma. □

Since $f^*\alpha$ is unramified, we know that

$$\varphi_3 \circ \varphi_2 \circ \varphi_1(\alpha) \in H^2_{\text{ét}}(\mathrm{Spec}\, \widehat{B}, \mu_2^{\otimes 2}) \subset H^2(\hat{L}, \mu_2^{\otimes 2}) \tag{3}$$

[Colliot-Thélène 1995, §3.3 and §3.8] and the compatibility of the residue map illustrated in [Colliot-Thélène and Ojanguren 1989, p. 143]. We aim to show that this class vanishes, which is enough to conclude the proposition, because $\alpha'|_E$ is obtained as the restriction of the above class to the closed point $\mathrm{Spec}\, \mathbb{C}(E)$.

In order to show that (3) vanishes, we choose some quadratic form q with $Q = \{q = 0\}$ and denote by $d \in K^*/(K^*)^2$ and $\beta \in H^2(K, \mu_2^{\otimes 2})$ the discriminant and the Clifford invariant of q, respectively. If $\partial_x^2 \alpha = 0$, then (3) vanishes by Lemma 8. If $\partial_x^2 \alpha \neq 0$, then $\partial_x^2(\alpha - \beta) = 0$ by Theorem 4, because Y_η is stably birational to Q and unramified cohomology is a stable birational invariant. By Lemma 8, it then suffices to show that β maps to zero via (2). By Theorem 4, d becomes a square in $\widehat{K}$, and so the latter follows from Theorem 3, applied to φ_2 in (2). This concludes the proof of the proposition. □

4. Proof of Theorem 1

The following is a generalization of Theorem 1, stated in the introduction. For what it exactly means that a variety specializes to another variety, see Section 1.1 above.

Theorem 9. *Let X be a proper variety which specializes to a complex projective variety Y. Suppose that there is a dominant rational map $f : Y \dashrightarrow \mathbb{P}^2$ with the properties that*

(a) *some Zariski open and dense subset $U \subset Y$ admits a universally CH_0-trivial resolution of singularities $\widetilde{U} \to U$ such that the induced rational map $\widetilde{U} \dashrightarrow \mathbb{P}^2$ is a morphism whose generic fiber is proper over $\mathbb{C}(\mathbb{P}^2)$ and*

(b) *the generic fiber Y_η of f is stably birational to a smooth projective quadric surface $g : Q \to \mathrm{Spec}\, K$ over $K = \mathbb{C}(\mathbb{P}^2)$, such that there is a class $\alpha \in H^2(K, \mu_2^{\otimes 2})$ whose pullback $g^*\alpha$ is nontrivial and* *unramified over* $\mathbb{C}$:

$$0 \neq g^*\alpha \in H^2_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes 2}) = H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mu_2^{\otimes 2}).$$

Then no resolution of singularities of X admits an integral decomposition of the diagonal. In particular, X is not stably rational.

Proof. Since $g^*\alpha \neq 0$ is unramified over $\mathbb{C}$ and unramified cohomology is a stable birational invariant, $\alpha' := f^*\alpha \in H^2(\mathbb{C}(Y), \mu_2^{\otimes 2})$ is a nontrivial class which is unramified over $\mathbb{C}$. By Hironaka's theorem, there exists a resolution of singularities $\tau : \widetilde{Y} \to Y$, such that $\tau^{-1}(U)$ identifies with the resolution of singularities $\widetilde{U}$ of U given in (a), and such that $E := \widetilde{Y} \setminus \widetilde{U}$ is a simple normal crossing divisor in $\widetilde{Y}$. Our assumption on $\widetilde{U}$ then implies that $\tau^{-1}(U) \to U$ is universally CH_0-trivial. Moreover, each component E_i of E is smooth and does not dominate $\mathbb{P}^2$. Therefore, Proposition 7 implies that the nontrivial class α' restricts to zero on E_i for all i and so Theorem 9 follows from the new key technique in [Schreieder 2018, §4]. □

Proof of Theorem 1. Condition (1) in Theorem 1 implies condition (a) in Theorem 9 with $\widetilde{U} = U$. By Theorem 3, conditions (1), (2), and (3) in Theorem 1 imply condition (b) in Theorem 9. Theorem 1 follows therefore from Theorem 9. □

5. Applications

5.1. *Quadric surface bundles over* $\mathbb{P}^2$. If the symmetric matrix $A = (a_{ij})$ in (1) is of diagonal form, i.e., $a_{ij} = 0$ for all $i \neq j$, then we say that the corresponding quadric surface bundle X is given by the quadratic form $q = \langle a_{00}, \ldots, a_{33} \rangle$. The condition that X is flat over $\mathbb{P}^2$ means that the a_{ii} have no common zero. If the homogeneous polynomials a_{ii} degenerate and acquire common zeros, then the same formula still defines a weak quadric bundle as long as the a_{ii} are nonzero and have no common factor. We will use such degenerations in the proofs below.

Proof of Corollary 2. In the notation of (1), let $A = (a_{ij})_{0 \leq i, j \leq 3}$ be the symmetric matrix which corresponds to the very general quadric surface bundle X of type (d_0, d_1, d_2, d_3) over $\mathbb{P}^2$. We may without loss of generality assume $0 \leq d_0 \leq d_1 \leq d_2 \leq d_3$. If $d_1 = 0$, then also $d_0 = 0$ and $a_{ij} \in \mathbb{C}$ is constant for $i, j \in \{0, 1\}$. The quadric $\{a_{00}z_0^2 + 2a_{01}z_0z_1 + a_{11}z_1^2 = 0\}$ thus has a point over $\mathbb{C}$ and so $X \to \mathbb{P}^2$ has a section. Hence, X is rational. If $d_i = 1$ for all i, then X is a hypersurface of bidegree $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ and so projection to the second factor shows that X is rational. Since the d_i have all the same parity, this shows that X is rational if $\sum d_i \leq 4$ or $d_1 = 0$.

The case $d_i = 2$ for all i is due to [Hassett et al. 2016b]; a quick proof follows from [Hassett et al. 2016b, Proposition 11] (= Proposition 6 above) and Theorem 1.

It remains to deal with the case where $\sum_i d_i \geq 8$, $d_1 \geq 1$, and $d_3 \geq 3$. Recall that all d_i are either even or odd. Consider the weak quadric surface bundle $Y_i := \{q_i = 0\} \subset \mathbb{P}(\mathscr{E})$ of type (d_0, d_1, d_2, d_3), given by the diagonal forms

$$q_1 := \langle z^{d_0}, x^{d_1}, xyz^{d_2-2}, yz^{d_3-3}F(x,y,z)\rangle,$$
$$q_2 := \langle z^{d_0}, xz^{d_1-1}, x^{d_2-1}y, yz^{d_3-3}F(x,y,z)\rangle,$$
$$q_3 := \langle z^{d_0}, x^{d_1}, yz^{d_2-1}, xyz^{d_3-4}F(x,y,z)\rangle,$$

where F is the quadratic polynomial from Proposition 6.

Note that Y_i is integral, because the entries in the diagonal form are coprime. Consider the natural projection $Y_i \to \mathbb{P}^2$. The generic fiber is a smooth quadric surface Q_i over $K = \mathbb{C}(\mathbb{P}^2)$. Setting $z = 1$ shows that Q_1 is given by the quadratic form $q_1' = \langle 1, x^{d_1}, xy, yF(x,y,1)\rangle$, Q_2 is given by $q_2' = \langle 1, x, x^{d_2-1}y, yF(x,y,1)\rangle$, and Q_3 is given by $q_3' = \langle 1, x^{d_1}, y, xyF(x,y,1)\rangle$.

If d_0 is even, then so is d_2. Multiplying through by y, absorbing squares and reordering the entries thus shows in this case that q_2' is similar to the quadratic form $q = \langle y, x, xy, F(x,y,1)\rangle$ from Proposition 6. If d_0 is odd, then so is d_1 and so q_1' is isomorphic to $\langle 1, x, xy, yF(x,y,1)\rangle$ and q_3' is isomorphic to $\langle 1, x, y, xyF(x,y,1)\rangle$. Again, q_1' and q_3' are both similar to q. Hence, $H^2_{nr}(K(Q_i)/\mathbb{C}, \mu_2^{\otimes 2}) \neq 0$ for $i \equiv d_0 \bmod 2$ by [Hassett et al. 2016b, Proposition 11] (= Proposition 6 above).

Since $d_1, d_2 \geq 1$ and $d_3 \geq 3$, the very general quadric surface bundle $X \subset \mathbb{P}(\mathscr{E})$ as in Corollary 2 degenerates to Y_2. If d_0 is odd, X also degenerates to Y_1 or Y_3, depending on whether $d_2 \geq 3$ or $d_2 = 1$. Depending on the parity of d_0 and the size of d_2, we can choose one of the three degenerations together with Theorem 1 (or 9) to conclude. □

Remark 10. Pirutka informed me that for any total degree $d := \sum_i d_i \geq 8$, one can reprove some cases of Corollary 2 via degenerations to similar quadric surface bundles as in [Hassett et al. 2016b], for which [Pirutka 2016, Theorem 3.17] applies, and for which one can compute universally CH_0-trivial resolutions explicitly [Auel et al. 2017a].

5.2. ***Quadric surface bundles over*** $\mathbb{P}^1 \times \mathbb{P}^1$**.** As a second example where Theorem 1 applies, we consider quadric surface bundles X over $\mathbb{P}^1 \times \mathbb{P}^1$ that are given by a line bundle valued quadratic form $q : \mathscr{E} \to \mathscr{O}(m,n)$, where $\mathscr{E} = \bigoplus_{i=0}^3 \mathscr{O}(-p_i, -q_i)$ is split. Locally, $X := \{q = 0\} \subset \mathbb{P}(\mathscr{E})$ is given by (1) where a_{ij} is a global section of $\mathscr{O}(p_i + p_j + m, q_i + q_j + n)$. If $a_{ij} = 0$ for $i \neq j$, we say that X is given by the quadratic form $q = \langle a_{00}, \dots, a_{33}\rangle$. If the a_{ii} degenerate and acquire common zeros, then the same formulas still define a hypersurface in $\mathbb{P}(\mathscr{E})$ which is a weak quadric surface bundle over $\mathbb{P}^2$ as long as the a_{ii} are nonzero and have no common factor. The deformation type of X depends only on the integers $d_i := m + 2p_i$ and $e_i := n + 2q_i$, and we call $(d_i, e_i)_{0 \leq i \leq 3}$ the type of X. Note that the d_i as well as the e_i have the same parity for all i. We say that the type $(d_i, e_i)_{0 \leq i \leq 3}$ is lexicographically ordered, if $d_i < d_{i+1}$, or $d_i = d_{i+1}$ and $e_i \leq e_{i+1}$.

Corollary 11. *Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a very general quadric surface bundle of lexicographically ordered type $(d_i, e_i)_{0 \leq i \leq 3}$, with $d_i, e_i \geq 0$ and $d_3, e_3 \geq 3$. Then*

(1) *X is rational if $d_2 = 0$, $d_1 = e_1 = e_0 = 0$ or $e_0 = e_1 = e_2 = 0$ and*

(2) *X is not stably rational otherwise.*

All examples in Corollary 11 deform to smooth rational varieties of dimension four; see for instance [Schreieder 2018, §3.5]. The condition $d_3, e_3 \geq 3$ in the above theorem could be replaced by a weaker but more complicated assumption; we collect in Corollary 12 below the remaining cases where our method works.

Proof of Corollary 11. Let $A = (a_{ij})_{0 \leq i,j \leq 3}$ be a symmetric matrix, where a_{ij} is a very general global section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(p_i + p_j + m, q_i + q_j + n)$, and consider the corresponding quadric surface bundle X over $\mathbb{P}^1 \times \mathbb{P}^1$. Here the integers $d_i := 2p_i + m$ and $e_i := 2q_i + n$ are assumed to satisfy the assumptions of Corollary 11; i.e., $(d_i, e_i)_{0 \leq i \leq 3}$ is lexicographically ordered with $d_i, e_i \geq 0$ and $d_3, e_3 \geq 3$.

If $d_1 = e_1 = e_0 = 0$, then $(a_{ij})_{0 \leq i,j \leq 1}$ is a constant matrix and so X has a section. If $d_2 = 0$, then $(a_{ij})_{0 \leq i,j \leq 2}$ is a matrix of polynomials which are constant along the first factor. Since any conic bundle over $\mathbb{P}^1$ has a section, X also admits a section. If $e_0 = e_1 = e_2 = 0$, then $(a_{ij})_{0 \leq i,j \leq 2}$ is a matrix of polynomials, constant along the second factor, and so X has a section as before. Since X is general and $d_i, e_i \geq 0$, the generic fiber of X over $\mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric surface and so X is rational in each of the above cases.

The case where $(e_0, e_1, e_2) \neq (0, 0, 0)$, $(d_1, e_0, e_1) \neq (0, 0, 0)$, and $d_2 \neq 0$ is similar to the proof of Corollary 2. The main point is that we can always degenerate X to weak quadric surface bundle Y over $\mathbb{P}^1 \times \mathbb{P}^1$ whose generic fiber is isomorphic to the example in Proposition 6. To find such a degeneration, we consider coordinates x_0, x_1 and y_0, y_1 on the first and second factors of $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, and consider the bidegree-(2, 2) polynomial

$$h := x_1^2 y_0^2 + x_0^2 y_1^2 + x_0^2 y_0^2 - 2(x_1 y_1 x_0 y_0 + x_1 x_0 y_0^2 + y_1 y_0 x_0^2). \tag{4}$$

We then start with the quadratic form $q = \langle 1, y_1, x_1, x_1 y_1 h\rangle$. Putting $x_0 = y_0 = 1$ shows that the corresponding quadric surface over $K = \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to the one in Proposition 6. The point is that the isomorphism type of this quadric surface does not change if we perform any of the following operations to the quadratic form q:

- multiply some entries with even powers of x_1 and y_1,
- multiply some entries with arbitrary powers of x_0 and y_0, or
- reorder the entries of the quadratic form.

Our aim is to produce a quadratic form of given type $(e_i, d_i)_{0 \leq i \leq 3}$ whose entries are coprime, since the latter guarantees that the associated quadratic form defines a weak quadric surface bundle Y over $\mathbb{P}^1 \times \mathbb{P}^1$. Once this is achieved, Corollary 11 will follow from Proposition 6 and Theorem 1.

By assumption, $d_2 \geq 1$, and if $e_0 = e_1 = 0$, then $d_1 \geq 1$ and $e_2 \geq 1$. This leads to Cases A, B, and C below. We divide into further subcases and provide each time a quadratic form (produced via the above process) with the properties we want. Recall that the d_i, as well as the e_i, have the same parity.

Case A ($e_1 \geq 1$). (1) If d_0 and e_0 are even, then we take

$$\langle x_1^{d_0} y_1^{e_0}, x_0^{d_1} y_0^{e_1 - 1} y_1, x_0^{d_2 - 1} x_1 y_0^{e_2}, x_0^{d_3 - 3} y_0^{e_3 - 3} x_1 y_1 h\rangle.$$

(2) If d_0 is odd and e_0 is even, then we take

$$\langle x_0^{d_0} y_1^{e_0}, x_0^{d_1} y_0^{e_1-1} y_1, x_1^{d_2} y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(3) If d_0 is even and e_0 is odd, then we take

$$\langle x_1^{d_0} y_0^{e_0}, x_0^{d_1} y_1^{e_1}, x_0^{d_2-1} x_1 y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(4) If d_0 and e_0 are odd, then we take

$$\langle x_0^{d_0} y_0^{e_0}, x_0^{d_1} y_1^{e_1}, x_1^{d_2} y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

Case B ($e_0 \geq 1$ and $e_1 = 0$; hence, e_i is even for all i). (1) If d_0 is even, then we take

$$\langle x_1^{d_0} y_0^{e_0-1} y_1, x_0^{d_1}, x_0^{d_2-1} x_1 y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(2) If d_0 is odd, then we take

$$\langle x_0^{d_0} y_0^{e_0-1} y_1, x_0^{d_1}, x_1^{d_2} y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

Case C ($d_1, e_2 \geq 1$ and $e_0 = e_1 = 0$; hence, e_i is even for all i). (1) If d_0 is even, then we take

$$\langle x_1^{d_0}, x_0^{d_1-1} x_1, x_0^{d_2} y_0^{e_2-1} y_1, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(2) If d_0 is odd, then we take

$$\langle x_0^{d_0}, x_1^{d_1}, x_0^{d_2} y_0^{e_2-1} y_1, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

In each of the above cases, putting $x_0 = y_0 = 1$ and reordering the factors if necessary shows that the corresponding weak quadric surface bundle Y over $\mathbb{P}^1 \times \mathbb{P}^1$ has generic fiber which is isomorphic to $\langle 1, y_1, x_1, x_1 y_1 F(x_1, y_1, 1)\rangle$. Corollary 11 therefore follows from [Hassett et al. 2016b, Proposition 11] (see Proposition 6 above) and Theorem 1. □

Corollary 12. *Let $(d_i, e_i)_{0\leq i\leq 3}$ be a lexicographically ordered tuple of pairs of nonnegative integers with $d_i + d_j$ and $e_i + e_j$ even for all i, j. Suppose that one of the following holds*:

(1) $d_1 \geq 1$, $d_3 \geq 2$, $e_1 + e_2 \geq 1$, *and* $e_3 \geq 3$ *or*

(2) $d_1 \geq 1$, $d_3 \geq 2$, $e_0 \geq 1$, $e_1 + e_2 \geq 1$, *and* $e_2 \geq 2$.

Then a very general complex projective quadric surface bundle X over $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(d_i, e_i)_{0\leq i\leq 3}$ is not stably rational.

Proof. We start with the quadratic forms $q_1 := \langle 1, x_1, x_1 y_1, y_1 h\rangle$ and $q_2 := \langle y_1, x_1, x_1 y_1, h\rangle$, where h is as in (4). If condition (1) holds, then we can use q_1 and if (2) holds, then we can use q_2 to obtain, via the procedure explained in the proof of Corollary 11, a quadratic form of type $(d_i, e_i)_{0\leq i\leq 3}$ whose coefficients are coprime. This yields a special fiber to which Theorem 1 applies. The details are similar as in the proof of Corollary 11, and we leave them to the reader. □

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Correction to the article Finite generation of the cohomology of some skew group algebras

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For the class of examples in Section 5 of the article in question, the proof of finite generation of cohomology is incomplete. We give here a proof of existence of a polynomial subalgebra needed there. The rest of the proof of finite generation given by the authors then applies.

Let k be a field of characteristic $p > 2$. Let A be the augmented k-algebra generated by a and b, with relations

$$a^p = 0, \quad b^p = 0, \quad ba = ab + \tfrac{1}{2}a^2,$$

and augmentation $\varepsilon : A \to k$ given by $\varepsilon(a) = \varepsilon(b) = 0$. Let G be a cyclic group of order p with generator g, acting on A by

$$g(a) = a, \quad g(b) = a + b.$$

The corresponding skew group algebra $A\#kG$ is a pointed Hopf algebra described in [Cibils et al. 2009, Corollary 3.14]. We remark that in Section 4 of the article we are correcting, referred to as [NW 2014], we used the left G-module structure with $g(a) = a$ and $g(b) = b - a$, whereas the authors in [Cibils et al. 2009; Nguyen et al. 2017] used the right G-module structure given as above. We will apply the results in [Nguyen et al. 2017] to prove that the cohomology $\mathrm{H}^*(A\#kG, k) := \mathrm{Ext}^*_{A\#kG}(k, k)$ is finitely generated, and this will fill a gap in the proof in [NW 2014, Section 5]. Thus we will now also adopt the choices of group actions in [Cibils et al. 2009; Nguyen et al. 2017] instead of that in [NW 2014]. This change does not affect the results discussed in [NW 2014, Section 4].

Let k be an $A\#kG$-module via the augmentation map ε. To prove finite generation of $\mathrm{H}^*(A\#kG, k)$, we wish to apply [NW 2014, Theorem 3.1]. We use results in [Nguyen et al. 2017], where the notation is slightly different, with x in place of a and y in place of b. There it is shown that there are 2-cocycles ξ_a, ξ_b in $\mathrm{H}^*(A, k)$ generating a polynomial subring $k[\xi_a, \xi_b]$. These 2-cocycles are not both G-invariant, as was claimed in [NW 2014]; specifically, in [Nguyen et al. 2017] it is shown that ξ_a is G-invariant while ξ_b is not. The claimed G-invariance was used in [NW 2014, Section 5] to show that ξ_a and ξ_b are

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in the image $\mathrm{Im}(\mathrm{res}_{A\#kG,A})$ of the restriction map from $\mathrm{H}^*(A\#kG, k)$ to $\mathrm{H}^*(A, k)$. However, results in [Nguyen et al. 2017, Section 5.1] imply directly that ξ_a, ξ_b are in $\mathrm{Im}(\mathrm{res}_{A\#kG,A})$; the needed elements in $\mathrm{H}^*(A\#kG, k)$ are constructed explicitly using a twisted tensor product resolution in [Nguyen et al. 2017, Section 3.3]. Now the rest of the finite generation proof in [NW 2014, Section 5] can proceed as before, since it is shown there that the rest of the hypotheses of [NW 2014, Theorem 3.1] are satisfied. An alternative proof is given in [Nguyen et al. 2017, Section 5.1].

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