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Sparsity of p-divisible unramified liftings for subvarieties of abelian varieties with trivial stabilizer

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By means of the theory of strongly semistable sheaves and the theory of the Greenberg transform, we generalize to higher dimensions a result on the sparsity of p-divisible unramified liftings which played a crucial role in Raynaud's proof of the Manin–Mumford conjecture for curves. We also give a bound for the number of irreducible components of the first critical scheme of subvarieties of an abelian variety which are complete intersections.

1. Introduction

The Manin–Mumford conjecture is a significant question concerning the intersection of a subvariety X of an abelian variety A with the group of torsion points of A. Raised independently by Manin and Mumford, the conjecture was originally formulated in the case of curves. Suppose that A is an abelian variety over a number field K and that C is a smooth subcurve of A of genus at least two. Then only finitely many torsion points of $A(\bar{K})$ lie in C. In 1983, Raynaud proved this conjecture and generalized it to higher dimensions: if A/K is as above and X/K is a smooth subvariety of A which does not contain any translate of a nontrivial abelian subvariety, then the set of torsion points of $A(\bar{K})$ lying in X is finite [Raynaud 1983b; 1983c].

Let us fix K, X and A as above. Let U be a nonempty open subscheme of Spec $\mathcal{O}_K$ not containing any ramified primes and such that A/K extends to an abelian scheme $\mathcal{A}/U$ and X extends to a smooth closed integral subscheme $\mathcal{X}$ of $\mathcal{A}$. For any $\mathfrak{p} \in U$, let R and R_n be the ring of Witt vectors and Witt vectors of length $n+1$, respectively, with coordinates in the algebraic closure $\overline{k(\mathfrak{p})}$ of the residue field of $\mathfrak{p}$. Recall that R is a DVR with maximal ideal generated by p such that $R_0 = R/p = \overline{k(\mathfrak{p})}$. Denote by $X_{\mathfrak{p}^n}$ and $A_{\mathfrak{p}^n}$ the R_n-schemes $\mathcal{X} \times_U \operatorname{Spec} R_n$ and $\mathcal{A} \times_U \operatorname{Spec} R_n$, respectively, and consider the reduction map

$$pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \to X_{\mathfrak{p}^0}(R_0). \tag{1}$$

In [Raynaud 1983b] it was shown that, if X is a curve, the image of (1) is not Zariski dense in $X_{\mathfrak{p}^0}$, i.e., it is a finite set. This local result is crucial in Raynaud's proof of the Manin–Mumford conjecture for curves, since it easily implies that only finitely many prime-to-p torsion points of $A(\bar{K})$ lie on X [Raynaud 1983b, Théorème II].

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It is quite natural to expect that a similar result also holds in higher dimensions. More explicitly, one can ask: is it true that, if a smooth subvariety X of A does not contain any translate of a nontrivial abelian subvariety, the image of (1) is not Zariski dense? In this paper we give a positive answer to this question (see Theorem 5.3).

Theorem 1.1 (sparsity of p-divisible unramified liftings). *Suppose that X has trivial stabilizer. For all $\mathfrak{p} \in U$ above a prime $p > (\dim X)^2 \deg(\Omega_X)$ such that $X_{\mathfrak{p}^0}$ has trivial stabilizer, the image of*

$$pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1) \to X_{\mathfrak{p}^0}(R_0)$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

Here $\deg(\Omega_X)$ refers to the degree of the cotangent bundle Ω_X computed with respect to any fixed very ample line bundle on X.

Notice that if X does not contain any translate of a nontrivial abelian subvariety, then it has finite stabilizer. Therefore, replacing A and X with their quotients by the stabilizer of X, one can assume the stabilizer is trivial (see the beginning of the next section for the definition of stabilizer).

A different generalization of Raynaud's local result was given by Rössler [2013] who proved that, if the torsion points of $\mathcal{A}(\overline{\mathrm{Frac}(R)})$ are not dense in $\mathcal{X}(\overline{\mathrm{Frac}(R)})$, then for m big enough the image of

$$p^m A_{\mathfrak{p}^m}(R_m) \cap X_{\mathfrak{p}^m}(R_m) \to X_{\mathfrak{p}^0}(R_0) \tag{2}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$ [Rössler 2013, Theorem 4.1]. Theorem 1.1 makes Rössler's result effective, showing that if the stabilizer of X is trivial, then it is sufficient to consider the map (2) for $m = 1$.

The proof of Theorem 1.1 strongly relies on Rössler's paper [2016] and is done by contradiction. First we use some basic properties of the Greenberg transform to show that, if the image of (1) is Zariski dense in $X_{\mathfrak{p}^0}$, the absolute Frobenius $F_{X_{\mathfrak{p}^0}} : X_{\mathfrak{p}^0} \to X_{\mathfrak{p}^0}$ lifts to an endomorphism of $X_{\mathfrak{p}^1}$. A well-known consequence of this liftability is the existence of a map of sheaves of differentials $F^*_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}} \to \Omega_{X_{\mathfrak{p}^0}}$ which is nonzero. If X is a curve, such a map cannot exist, since $\deg(F^*_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}})$ is strictly bigger than $\deg(\Omega_{X_{\mathfrak{p}^0}})$. This simple observation was in fact used by Raynaud to prove Lemma I.5.4 in [Raynaud 1983a]. By means of the theory of strongly semistable sheaves developed by Rössler [2016], we show that when X has dimension higher than one, there are no nontrivial maps from $F^*_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}}$ to $\Omega_{X_{\mathfrak{p}^0}}$. This gives us the wanted contradiction.

In the last section of this paper, we consider subvarieties of abelian varieties which are complete intersections. If Gr_1 denotes the Greenberg transform of level 1 (see Section 3), then we know that the first critical scheme

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) := [p]_* \,\mathrm{Gr}_1(A_{\mathfrak{p}^1}) \cap \mathrm{Gr}_1(X_{\mathfrak{p}^1})$$

is a scheme over R_0 such that

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0) = pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1).$$

Using exactly the same technique that allowed Buium [1996] to give an effective form of the Manin–Mumford conjecture in the case of curves, we get a bound for the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ when X is a complete intersection (not necessarily with trivial stabilizer).

Theorem 1.2. *Let K be a number field, A/K be an abelian variety of dimension n and let L be a very ample line bundle on A. Let $c \in \mathbb{N}$ be positive and let $H_1, H_2, \ldots, H_c \in |L|$ be general. Suppose that $X := H_1 \cap H_2 \cap \cdots \cap H_c$ is smooth. There exists a nonempty open subscheme $V \subseteq \operatorname{Spec} \mathcal{O}_K$ (see the beginning of Section 6 for its definition) such that if $\mathfrak{p} \in V$, the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ is bounded by*

$$p^{2n}\left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\right)(L^n)^2.$$

Here (L^n) denotes the intersection number of L.

We conclude the introduction with the following remark. Since the field of definition of points in the prime-to-p torsion $\mathrm{Tor}^p(A(\bar{K}))$ is unramified at $\mathfrak{p}$ and the specialization map $\mathcal{A}(R) \to A_{\mathfrak{p}^1}(R_1)$ is injective on the prime-to-p torsion, we have an injection

$$\mathrm{Tor}^p(A(\bar{K})) \cap X(\bar{K}) \subseteq pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1).$$

This implies that, if X is a complete intersection such that $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0)$ is finite, then the bound in Theorem 1.2 is a bound for the cardinality of $\mathrm{Tor}^p(A(\bar{K})) \cap X(\bar{K})$, i.e., an effective form of the Manin–Mumford conjecture for the prime-to-p torsion.

2. Notations

We fix the following notations

- K a number field,
- $\bar{K}$ an algebraic closure of K,
- A/K an abelian variety,
- $X \subseteq A$ a closed integral subscheme, smooth over K,
- $\mathrm{Stab}_A(X)$ the translation stabilizer of X in A, i.e., the closed subgroup scheme of A characterized uniquely by the fact that for any K-scheme S and any morphism $b : S \to A$, translation by b on the product $A \times_K S$ maps the subscheme $X \times_K S$ to itself if and only if b factors through $\mathrm{Stab}_A(X)$ (for its existence we refer the reader to [SGA 3_{II} 1970, Exemple 6.5(e), Expose VIII]),
- U an open subscheme of $\operatorname{Spec} \mathcal{O}_K$ not containing any ramified prime and such that A/K extends to an abelian scheme $\mathcal{A}/U$ and X extends to a smooth closed integral subscheme $\mathcal{X}$ of $\mathcal{A}$.

For any prime number p, any unramified prime $\mathfrak{p}$ of K above p and any $n \geq 0$, we denote by

- $k(\mathfrak{p})$ the residue field $\mathcal{O}_K/\mathfrak{p}$ for $\mathfrak{p}$,

- $K_{\mathfrak{p}}$ the completion of K with respect to $\mathfrak{p}$,
- $\widehat{K_{\mathfrak{p}}^{\text{unr}}}$ the completion of the maximal unramified extension of $K_{\mathfrak{p}}$,
- $R := W(\overline{k(\mathfrak{p})})$ and $R_n := W_n(\overline{k(\mathfrak{p})})$ the ring of Witt vectors and the ring of Witt vectors of length $n+1$, respectively, with coordinates in $\overline{k(\mathfrak{p})}$. We recall that R can be identified with the ring of integers of $\widehat{K_{\mathfrak{p}}^{\text{unr}}}$ and R_0 with $\overline{k(\mathfrak{p})}$,
- $X_{\mathfrak{p}^n}$ the R_n-scheme $\mathcal{X} \times_U \operatorname{Spec} R_n$ $A_{\mathfrak{p}^n}$ the R_n-scheme $\mathcal{A} \times_U \operatorname{Spec} R_n$.

3. The Greenberg transform and the critical schemes

Now we recall some basic facts about the Greenberg transform (for more details, see [Greenberg 1961; 1963; Bosch et al. 1990, pp. 276–277]).

Fix a prime number p and an unramified prime $\mathfrak{p}$ of K above p.

For any $n \geq 0$, the Greenberg transform of level n is a covariant functor Gr_n from the category of R_n-schemes locally of finite type, to the category of R_0-schemes locally of finite type. If Y_n is an R_n-scheme locally of finite type, $\mathrm{Gr}_n(Y_n)$ is a R_0-scheme with the property

$$Y_n(R_n) = \mathrm{Gr}_n(Y_n)(R_0).$$

More precisely, we can interpret R_n as the set of $\overline{k(\mathfrak{p})}$-valued points of a ring scheme $\mathcal{R}_n$ over $\overline{k(\mathfrak{p})}$. For any R_0-scheme T, we define $\mathbb{W}_n(T)$ as the ringed space over R_n consisting of T as a topological space and of $\mathrm{Hom}_{R_0}(T, \mathcal{R}_n)$ as a structure sheaf. By definition $\mathrm{Gr}_n(Y_n)$ represents the functor from the category of schemes over R_0 to the category of sets given by

$$T \mapsto \mathrm{Hom}_{R_n}(\mathbb{W}_n(T), Y_n)$$

where Hom stands for homomorphisms of ringed spaces. In other words, the functor Gr_n is right adjoint to the functor $\mathbb{W}_n$.

The functor Gr_n respects closed immersions, open immersions, fiber products, smooth, étale morphisms and is the identity for $n = 0$. Furthermore it sends group schemes over R_n to group schemes over R_0. The canonical morphism $R_{n+1} \to R_n$ gives rise to a functorial transition morphism $\pi_{n+1} : \mathrm{Gr}_{n+1} \to \mathrm{Gr}_n$.

Let Y_n be a scheme over R_n locally of finite type. Then for any $m < n$ we define

$$Y_m := Y_n \times_{R_n} R_m.$$

Let us call $F_{Y_0} : Y_0 \to Y_0$ the absolute Frobenius endomorphism of Y_0 and Ω_{Y_0/R_0} the sheaf of relative differentials.

For any finite rank locally free sheaf $\mathcal{F}$ over Y_0 we will write

$$V(\mathcal{F}) := \underline{\operatorname{Spec}}(\operatorname{Sym}(\mathcal{F}^\vee))$$

for the vector bundle over Y_0 associated to $\mathcal{F}$.

Suppose now that Y_n is smooth over R_n, so that Ω_{Y_0/R_0} is locally free. A key result about the Greenberg transform is the following fact [Greenberg 1963, Section 2]:

$$\pi_1 : \mathrm{Gr}_1(Y_1) \to \mathrm{Gr}_0(Y_0) = Y_0$$

is a torsor under the Frobenius tangent bundle

$$V(F_{Y_0}^* \Omega_{Y_0/R_0}^\vee).$$

Let X, A, $\mathcal{X}$, $\mathcal{A}$ and U be as fixed in the previous section and suppose that $\mathfrak{p} \in U$. We refer the reader to Section II.1 in [Raynaud 1983a] for more details on what we will recall from now till the end of the section. For any $n \geq 0$, the kernel of

$$\mathrm{Gr}_n(A_{\mathfrak{p}^n}) \to \mathrm{Gr}_0(A_{\mathfrak{p}^0}) = A_{\mathfrak{p}^0}$$

is unipotent, killed by p^n. Thus, the scheme-theoretic image $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ of multiplication by p^n in $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$ is the greatest abelian subvariety of $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$ and, since R_0 is algebraically closed, $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})(R_0) = p^n \mathrm{Gr}_n(A_{\mathfrak{p}^n})(R_0)$.

We define the n-critical scheme as

$$\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) := [p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n}) \cap \mathrm{Gr}_n(X_{\mathfrak{p}^n}).$$

Notice that $\mathrm{Crit}^n(\mathcal{X}, \mathcal{A})$ is a scheme over R_0 and that $\mathrm{Crit}^0(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0}$.

The transition morphisms $\pi_{n+1} : \mathrm{Gr}_{n+1}(A_{\mathfrak{p}^{n+1}}) \to \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ lead to a projective system of R_0-schemes

$$\cdots \to \mathrm{Crit}^2(\mathcal{X}, \mathcal{A}) \to \mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) \to \mathrm{Crit}^0(\mathcal{X}, \mathcal{A}) = X_{\mathfrak{p}^0},$$

whose connecting morphisms are both affine and proper, hence finite. In fact, transition morphisms are affine and the subscheme $[p^n]_* \mathrm{Gr}_n(A_{\mathfrak{p}^n})$ is proper, being the greatest abelian subvariety of $\mathrm{Gr}_n(A_{\mathfrak{p}^n})$.

We shall write $\mathrm{Exc}^n(\mathcal{X}, \mathcal{A})$ for the scheme theoretic image of the morphism $\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) \to X_{\mathfrak{p}^0}$.

4. The geometry of vector bundles in positive characteristic

In this section we recall some results on the geometry of vector bundles in positive characteristic by Langer [2004] and Rössler [2016]. These results will play a crucial role in the proof of Lemma 5.1 and Theorem 5.3.

Let us start with some basic definitions and facts regarding semistable sheaves in positive characteristic.

Let Y be a smooth projective variety over an algebraically closed field l_0 of positive characteristic. We write as before Ω_{Y/l_0} for the sheaf of differentials of Y over l_0 and $F_Y : Y \to Y$ for the absolute Frobenius endomorphism of Y. Now let L be a very ample line bundle on Y. If V is a torsion free coherent sheaf on Y, we shall write

$$\mu(V) = \mu_L(V) = \deg_L(V) / \mathrm{rk}(V)$$

for the slope of V (with respect to L). Here $\mathrm{rk}(V)$ is the rank of V, i.e., the dimension of the stalk of V at the generic point of Y. Furthermore,

$$\deg_L(V) := \int_Y c_1(V) \cdot c_1(L)^{\dim(Y)-1}$$

where $c_1(\cdot)$ refers to the first Chern class with values in an arbitrary Weil cohomology theory and the integral $\int_Y$ stands for the push-forward morphism to $\operatorname{Spec} l_0$ in that theory. Recall that V is called semistable (with respect to L) if for every coherent subsheaf W of V, we have $\mu(W) \leq \mu(V)$ and it is called strongly semistable if $F_Y^{n,*}V$ is semistable for all $n \geq 0$.

In general, there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

of V by subsheaves, such that the quotients V_i / V_{i-1} are all semistable and such that the slopes $\mu(V_i / V_{i-1})$ are strictly decreasing for $i \geq 1$. This filtration is unique and is called the Harder–Narasimhan (HN) filtration of V. We will say that V has a strongly semistable HN filtration if all the quotients V_i / V_{i-1} are strongly semistable. We shall write

$$\mu_{\min}(V) := \mu(V_r / V_{r-1}) \quad \text{and} \quad \mu_{\max}(V) := \mu(V_1).$$

By the very definition of HN filtration, we have

$$V \text{ is semistable } \Leftrightarrow \mu_{\min}(V) = \mu_{\max}(V).$$

An important consequence of the definitions is the following fact; if V and W are two torsion free sheaves on Y and $\mu_{\min}(V) > \mu_{\max}(W)$, then $\mathrm{Hom}_Y(V, W) = 0$.

For more on the theory of semistable sheaves, see the monograph [Huybrechts and Lehn 2010].

The following two theorems are key results from Langer.

Theorem 4.1 [Langer 2004, Theorem 2.7]. *If V is a torsion free coherent sheaf on Y, then there exists $n_0 \geq 0$ such that $F_Y^{n,*}V$ has a strongly semistable HN filtration for all $n \geq n_0$.*

If V is a torsion free coherent sheaf on Y, we now define

$$\bar{\mu}_{\min}(V) := \lim_{r\to\infty} \mu_{\min}(F_Y^{r,*}V)\,\mathrm{char}(l_0)^r \quad \text{and} \quad \bar{\mu}_{\max}(V) := \lim_{r\to\infty} \mu_{\max}(F_Y^{r,*}V) / \mathrm{char}(l_0)^r.$$

Note that Theorem 4.1 implies that the two sequences $\mu_{\min}(F_Y^{r,*}V) / \mathrm{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V) / \mathrm{char}(l_0)^r$ become constant when r is sufficiently large, so the above definitions of $\bar{\mu}_{\min}$ and $\bar{\mu}_{\max}$ make sense. Furthermore the sequences $\mu_{\min}(F_Y^{r,*}V)\,\mathrm{char}(l_0)^r$ and $\mu_{\max}(F_Y^{r,*}V)\,\mathrm{char}(l_0)^r$ are respectively weakly decreasing and weakly increasing, therefore we have

$$\mu_{\min}(V) \geq \bar{\mu}_{\min}(V) \quad \text{and} \quad \bar{\mu}_{\max}(V) \geq \mu_{\max}(V).$$

Let us define

$$\alpha(V) := \max\{\mu_{\min}(V) - \bar{\mu}_{\min}(V), \bar{\mu}_{\max}(V) - \mu_{\max}(V)\}.$$

Theorem 4.2 [Langer 2004, Cororollary 6.2]. *If V is of rank r, then*

$$\alpha(V) \leq \frac{r-1}{\operatorname{char}(l_0)} \max\{\bar{\mu}_{\max}(\Omega_{Y/l_0}), 0\}.$$

In particular, if $\bar{\mu}_{\max}(\Omega_{Y/l_0}) \geq 0$ and $\operatorname{char}(l_0) \geq d = \dim Y$,

$$\bar{\mu}_{\max}(\Omega_{Y/l_0}) \leq \frac{\operatorname{char}(l_0)}{\operatorname{char}(l_0)+1-d} \mu_{\max}(\Omega_{Y/l_0}).$$

We conclude this section with the following two lemmas from Rössler.

Lemma 4.3 [Rössler 2016, Lemma 3.8]. *Suppose that there is a closed l_0-immersion $i : Y \hookrightarrow B$, where B is an abelian variety over l_0. Suppose that $\operatorname{Stab}_B(Y) = 0$. Then $\Omega_Y^\vee$ is globally generated and for any dominant proper morphism $\phi : Y_0 \to Y$, where Y_0 is integral, we have $H^0(Y_0, \phi^*\Omega_Y^\vee) = 0$. Furthermore, we have $\bar{\mu}_{\min}(\Omega_Y) > 0$.*

Lemma 4.4 [Rössler 2016, Cororollary 3.11]. *Let V be a finite rank, locally free sheaf over Y. Suppose that*

- *for any surjective finite map $\phi : Y' \to Y$ with Y' integral, we have $H^0(Y', \phi^*V) = 0$,*
- *$V^\vee$ is globally generated.*

Then $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for n sufficiently big.

Furthermore, let $T \to Y$ be a torsor under $V(F_Y^{n_0,*}V)$, where n_0 satisfies $H^0(Y, F_Y^{n,*}V \otimes \Omega_{Y/l_0}) = 0$ for all $n > n_0$. Let $\phi : Y' \to Y$ be a finite surjective morphism and suppose that Y' is integral. Then we have the implication

$$\phi^*T \text{ is a trivial } V(\phi^*(F_Y^{n_0,*}V))\text{-torsor} \Longrightarrow T \text{ is a trivial } V(F_Y^{n_0,*}V)\text{-torsor.}$$

The main ingredient of the proof of Lemma 4.4 is a result by Szpiro and Lewin-Ménégaux which we will need later.

Proposition 4.5 [Szpiro 1981, Expose 2, Proposition 1]. *If V is a vector bundle over Y such that $H^0(Y, F_Y^*V \otimes \Omega_{Y/l_0}) = 0$, then the map*

$$H^1(Y, V) \to H^1(Y, F_Y^*V)$$

is injective.

5. Sparsity of p-divisible unramified liftings

In this section we prove our result on the sparsity of p-divisible unramified liftings (see Theorem 5.3).

Let K, A, X and U be as fixed in Section 2 and let $\operatorname{Stab}_A(X)$ be trivial. The construction of the stabilizer commutes with the base change, so we have

$$\operatorname{Stab}_A(X) = \operatorname{Stab}_{\mathcal{A}}(\mathcal{X}) \times_U \operatorname{Spec} K.$$

Since $\mathrm{Stab}_A(X)$ is trivial, by generic flatness and finiteness, we can restrict the map $\pi : \mathrm{Stab}_{\mathcal{A}}(\mathcal{X}) \to U$ to the inverse image of a nonempty open subscheme $U' \subset U$ to obtain a finite flat commutative group scheme of degree one

$$\pi_{|\pi^{-1}(U')} : \pi^{-1}(U') \to U'.$$

This implies that $\pi_{|\pi^{-1}(U')}$ is an isomorphism and for any $\mathfrak{q} \in U'$ we have that $\mathrm{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0})$ is trivial. We will denote by $\tilde{U} \subseteq U$ the nonempty open subscheme

$$\tilde{U} := \{\mathfrak{q} \in U \mid \mathrm{Stab}_{A_{\mathfrak{q}^0}}(X_{\mathfrak{q}^0}) \text{ is trivial}\}.$$

For any $\mathfrak{p} \in U$ we denote by $F_{\overline{k(\mathfrak{p})}}$ the Frobenius endomorphism on $\overline{k(\mathfrak{p})}$ and by F_{R_1} the endomorphism of R_1 induced by $F_{\overline{k(\mathfrak{p})}}$ by functoriality. We define

$$X'_{\mathfrak{p}^0} := X_{\mathfrak{p}^0} \times_{F_{\overline{k(\mathfrak{p})}}} \overline{k(\mathfrak{p})} \quad \text{and} \quad X'_{\mathfrak{p}^1} := X_{\mathfrak{p}^1} \times_{F_{R_1}} R_1$$

and we write

$$F_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}} : X_{\mathfrak{p}^0} \to X'_{\mathfrak{p}^0}$$

for the relative Frobenius on $X_{\mathfrak{p}^0}$. For brevity's sake, from now on we will write

$$\Omega_{X_{\mathfrak{p}^0}}, \Omega_{X'_{\mathfrak{p}^0}}, \Omega_{X_{\mathfrak{p}^1}}, \Omega_{X'_{\mathfrak{p}^1}} \text{ and } \Omega_X$$

instead of

$$\Omega_{X_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}, \Omega_{X'_{\mathfrak{p}^0}/\overline{k(\mathfrak{p})}}, \Omega_{X_{\mathfrak{p}^1}/R_1}, \Omega_{X'_{\mathfrak{p}^1}/R_1} \text{ and } \Omega_{X/K}.$$

Observe that since U is normal, $\mathcal{A}$ is projective over U [Raynaud 1970, Theorem XI 1.4]. Therefore there exists a U-very ample line bundle L on $\mathcal{X}$. For any $\mathfrak{p} \in U$ different from the generic point ξ, let us denote by $L_{\mathfrak{p}}$ the inverse image of L on $X_{\mathfrak{p}^0}$. Similarly we denote by L_ξ the inverse image of L on X. From now on, for any vector bundle $G_{\mathfrak{p}}$ over $X_{\mathfrak{p}^0}$, we will write $\deg(G_{\mathfrak{p}})$ for the degree of $G_{\mathfrak{p}}$ with respect to $L_{\mathfrak{p}}$. Analogously, if G_ξ is a vector bundle over X, we will write $\deg(G_\xi)$ for the degree of G_ξ with respect to L_ξ. Now consider the vector bundle $\Omega_{\mathcal{X}/U}$ over $\mathcal{X}$. For any natural number m, the map from U to $\mathbb{Z}$ defined by

$$\mathfrak{p} \mapsto \chi((\Omega_{\mathcal{X}/U} \otimes L^m)_{\mathfrak{p}}) = \chi(\Omega_{X_{\mathfrak{p}^0}} \otimes L^m_{\mathfrak{p}}) \quad \text{and} \quad \xi \mapsto \chi((\Omega_{\mathcal{X}/U} \otimes L^m)_\xi) = \chi(\Omega_X \otimes L^m_\xi)$$

(here χ refers to the Euler characteristic) is constant on U [Mumford 1970, Chapter II, Section 5]. Therefore we have the equality

$$\chi(\Omega_{X_{\mathfrak{p}^0}} \otimes L^m_{\mathfrak{p}}) = \chi(\Omega_X \otimes L^m_\xi)$$

for all $m \in \mathbb{N}$ and for all $\mathfrak{p} \in U$. In other words, the Hilbert polynomial of $\Omega_{X_{\mathfrak{p}^0}}$ with respect to $L_{\mathfrak{p}}$ coincides with the Hilbert polynomial of Ω_X with respect to L_ξ. Since the degree of a vector bundle we defined at the beginning of this section can be described in terms of its Hilbert polynomial [Huybrechts and Lehn 2010, Definition 1.2.11], we obtain that for every $\mathfrak{p} \in U$ we have $\deg(\Omega_{X_{\mathfrak{p}^0}}) = \deg(\Omega_X)$.

The following lemma is a fundamental step to prove our sparsity Theorem 5.3.

Lemma 5.1. *Let K, A, X and U be as fixed in Section 2, let $\mathrm{Stab}_A(X)$ be trivial and let n be the dimension of X over K. Then*

$$\mathrm{Hom}_{X_{\mathfrak{p}0}}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}, \Omega_{X_{\mathfrak{p}0}}) = 0$$

for any $k \geq 1$ and any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Let us notice first that, if $n = 1$, then X is a curve of genus g at least 2 and

$$\mathrm{Hom}_{X_{\mathfrak{p}0}}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}, \Omega_{X_{\mathfrak{p}0}}) = 0$$

is a simple consequence of the fact

$$\deg(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) = p^k(2g-2) > 2g-2 = \deg \Omega_{X_{\mathfrak{p}0}}.$$

To treat the general case, let us fix $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$. We know that if

$$\mu_{\min}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) > \mu_{\max}(\Omega_{X_{\mathfrak{p}0}})$$

then $\mathrm{Hom}_{X_{\mathfrak{p}0}}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}, \Omega_{X_{\mathfrak{p}0}}) = 0$. Since $\mu_{\min} \geq \bar{\mu}_{\min}$ and $\bar{\mu}_{\max} \geq \mu_{\max}$, it is sufficient to show that, for every $k \geq 1$

$$\bar{\mu}_{\min}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) > \bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{3}$$

Since $\mathrm{Stab}_{A_{\mathfrak{p}0}}(X_{\mathfrak{p}0})$ is trivial, we can apply Lemma 4.3 to obtain $\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}}) > 0$. In particular $\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) > 0$ and $\deg(\Omega_{X_{\mathfrak{p}0}}) > 0$. Using this and the equality $\bar{\mu}_{\min}(F^{k,*}_{X_{\mathfrak{p}0}}\Omega_{X_{\mathfrak{p}0}}) = p^k\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}})$, we see that (3) is implied by

$$p\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}}) > \bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{4}$$

Theorem 4.2 gives us the following inequality

$$p\bar{\mu}_{\min}(\Omega_{X_{\mathfrak{p}0}}) \geq p\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) + (1-n)\bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}),$$

so that (4) is satisfied if

$$p\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) > n\bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{5}$$

Since $p > n^2 \deg(\Omega_X) \geq n$, we can apply the second part of Theorem 4.2

$$\bar{\mu}_{\max}(\Omega_{X_{\mathfrak{p}0}}) \leq \frac{p}{p+1-n}\mu_{\max}(\Omega_{X_{\mathfrak{p}0}}),$$

so that inequality (5) is implied by

$$(p+1-n)\mu_{\min}(\Omega_{X_{\mathfrak{p}0}}) > n\mu_{\max}(\Omega_{X_{\mathfrak{p}0}}). \tag{6}$$

If $\Omega_{X_{\mathfrak{p}0}}$ is semistable, (6) gives $p > 2n-1$. Otherwise, we can estimate $\mu_{\max}(\Omega_{X_{\mathfrak{p}0}})$ and $\mu_{\min}(\Omega_{X_{\mathfrak{p}0}})$ in the following way. We know that

$$\mu_{\max}(\Omega_{X_{\mathfrak{p}0}}) = \frac{\deg(M)}{\mathrm{rk}(M)}$$

for some subsheaf $0 \neq M \subsetneq \Omega_{X_{\mathfrak{p}^0}}$. Therefore we have $\mu_{\max}(\Omega_{X_{\mathfrak{p}^0}}) \leq \deg(M)$. Furthermore, since $\overline{\mu}_{\min}(\Omega_{X_{\mathfrak{p}^0}}) > 0$, we have that $\deg(\Omega_{X_{\mathfrak{p}^0}}/M) > 0$. This and the additivity of the degree on short exact sequences gives us

$$\mu_{\max}(\Omega_{X_{\mathfrak{p}^0}}) \leq \deg(M) \leq \deg(\Omega_{X_{\mathfrak{p}^0}}) - 1.$$

Similarly,

$$\mu_{\min}(\Omega_{X_{\mathfrak{p}^0}}) = \frac{\deg(Q)}{\mathrm{rk}(Q)}$$

for some Q quotient of $\Omega_{X_{\mathfrak{p}^0}}$, so $\mu_{\min}(\Omega_{X_{\mathfrak{p}^0}}) \geq 1/n$. Inequality (6) is then implied by

$$p > n^2 \deg(\Omega_{X_{\mathfrak{p}^0}}) + (n - 1 - n^2).$$

Since $n - 1 - n^2$ is always negative, we are reduced to $p > n^2 \deg(\Omega_{X_{\mathfrak{p}^0}})$. Now $\deg(\Omega_{X_{\mathfrak{p}^0}})$ is greater or equal to one, so $n^2 \deg(\Omega_{X_{\mathfrak{p}^0}}) \geq 2n - 1$ for any n. This ensures us that the condition

$$p > n^2 \deg(\Omega_{X_{\mathfrak{p}^0}})$$

is sufficient to have $\mu_{\min}(F^{k,*}_{X_{\mathfrak{p}^0}} \Omega_{X_{\mathfrak{p}^0}}) > \mu_{\max}(\Omega_{X_{\mathfrak{p}^0}})$ for every $k \geq 1$ whether $\Omega_{X_{\mathfrak{p}^0}}$ is semistable or not. To conclude it is enough to remember that $\deg(\Omega_{X_{\mathfrak{p}^0}})$ coincides with $\deg(\Omega_X)$. □

Corollary 5.2. *The map*

$$H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \to H^1(X_{\mathfrak{p}^0}, F^{k,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}})$$

is injective for every $k \geq 1$ and every $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$.

Proof. Lemma 5.1 and Proposition 4.5 imply that

$$H^1(X_{\mathfrak{p}^0}, F^{h,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \to H^1(X_{\mathfrak{p}^0}, F^{h+1,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}})$$

is injective for every $h \geq 0$. Therefore the composition

$$H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \hookrightarrow H^1(X_{\mathfrak{p}^0}, F^{2,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{p}^0}}) \hookrightarrow \cdots \hookrightarrow H^1(X_{\mathfrak{p}^0}, F^{k,*}_{X_{\mathfrak{p}^0}} \Omega^\vee_{X_{\mathfrak{q}^0}})$$

is an injective map. □

We are now ready to prove our sparsity result.

Theorem 5.3. *With the same hypotheses as in Lemma 5.1, for any $\mathfrak{p} \in \tilde{U}$ above a prime $p > n^2 \deg(\Omega_X)$, the set*

$$\{P \in X_{\mathfrak{p}^0}(R_0) \mid P \textit{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)\}$$

is not Zariski dense in $X_{\mathfrak{p}^0}$.

Proof. Let us fix $\mathfrak{p}$ as in the hypotheses. Since

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0) = pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1),$$

we have that

$$\{P \in X_{\mathfrak{p}^0}(R_0) \mid P \text{ lifts to an element of } pA_{\mathfrak{p}^1}(R_1) \cap X_{\mathfrak{p}^1}(R_1)\}$$

coincides with the image of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})(R_0) \to X_{\mathrm{p}^0}(R_0)$.

Let us assume by contradiction that this image is dense in $X_{\mathrm{p}^0}(R_0)$. This implies that $\pi_1 : \mathrm{Gr}_1(X_{\mathrm{p}^1}) \to X_{\mathrm{p}^0}$ is a trivial torsor; the argument we use to show this is taken from Rössler (see the beginning of the proof of Theorem 2.2 in [Rössler 2016]). First of all the closed map $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A}) \to X_{\mathrm{p}^0}$ is surjective and so we can choose an irreducible component

$$\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})_0 \hookrightarrow \mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$$

which dominates X_{p^0}. Lemmas 4.3 and 5.1 allow us to apply the second part of Lemma 4.4 with $V = \Omega^\vee_{X_{\mathrm{p}^0}}$, $Y = X_{\mathrm{p}^0}$, $n_0 = 1$, $T = \mathrm{Gr}_1(X_{\mathrm{p}^1})$ and ϕ equal to $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})_0 \to X_{\mathrm{p}^0}$. We have that $\phi^* \mathrm{Gr}_1(X_{\mathrm{p}^1})$ is trivial as a $V(\phi^* F^*_{X_{\mathrm{p}^0}} \Omega^\vee_{X_{\mathrm{p}^0}})$-torsor, since $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})_0$ is contained in $\mathrm{Gr}_1(X_{\mathrm{p}^1})$. Hence $\pi_1 : \mathrm{Gr}_1(X_{\mathrm{p}^1}) \to X_{\mathrm{p}^0}$ is trivial as a $V(F^*_{X_{\mathrm{p}^0}} \Omega^\vee_{X_{\mathrm{p}^0}})$-torsor. Let us take a section $\sigma : X_{\mathrm{p}^0} \to \mathrm{Gr}_1(X_{\mathrm{p}^1})$. By the definition of the Greenberg transform, the map σ over R_0 corresponds to a map $\bar{\sigma} : \mathbb{W}_1(X_{\mathrm{p}^0}) \to X_{\mathrm{p}^1}$ over R_1. We can precompose $\bar{\sigma}$ with the morphism $t : X_{\mathrm{p}^1} \to \mathbb{W}_1(X_{\mathrm{p}^0})$ corresponding to

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathrm{p}^0}}) &\to \mathcal{O}_{X_{\mathrm{p}^1}} \\ (a_0, a_1) &\mapsto \tilde{a}_0^p + \tilde{a}_1 p, \end{aligned}$$

where $\tilde{a}_i$ lifts a_i. Consider the following diagram

$$\begin{tikzcd} X_{\mathrm{p}^1} \arrow[r, "t"] & \mathbb{W}_1(X_{\mathrm{p}^0}) \arrow[r, "\bar{\sigma}"] & X_{\mathrm{p}^1} \\ X_{\mathrm{p}^0} \arrow[u] \arrow[r, "F_{X_{\mathrm{p}^0}}"] & X_{\mathrm{p}^0} \arrow[u] \arrow[r, "\mathrm{Id}"] & X_{\mathrm{p}^0} \arrow[u] \end{tikzcd}$$

Its left square is commutative, since the composition

$$X_{\mathrm{p}^0} \longrightarrow X_{\mathrm{p}^1} \xrightarrow{t} \mathbb{W}_1(X_{\mathrm{p}^0})$$

simply corresponds to the map

$$\begin{aligned} W_1(\mathcal{O}_{X_{\mathrm{p}^0}}) &\to \mathcal{O}_{X_{\mathrm{p}^0}} \\ (a_0, a_1) &\mapsto a_0^p. \end{aligned}$$

For the commutativity of the right square, notice that by the very definition of the transition morphism $\pi_1 : \mathrm{Gr}_1(X_{\mathrm{p}^1}) \to X_{\mathrm{p}^0}$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{R_1}(\mathbb{W}_1(X_{\mathrm{p}^0}), X_{\mathrm{p}^1}) \arrow[r] \arrow[d, "\text{reduction mod } p"'] & \mathrm{Hom}_{R_0}(X_{\mathrm{p}^0}, \mathrm{Gr}_1(X_{\mathrm{p}^1})) \arrow[dl, "(\pi_1 \circ -)"] \\ \mathrm{Hom}_{R_0}(X_{\mathrm{p}^0}, X_{\mathrm{p}^0}) & \end{tikzcd}$$

In particular, $\mathrm{Id}_{X_{\mathrm{p}0}} = \pi_1 \circ \sigma = (\text{reduction mod } p)(\bar{\sigma})$, which is exactly what we wanted to verify. We obtain therefore that $\bar{\sigma} \circ t : X_{\mathrm{p}^1} \to X_{\mathrm{p}^1}$ is a lift of the Frobenius $F_{X_{\mathrm{p}0}}$.

The diagram below is also commutative

$$\begin{tikzcd}
X_{\mathrm{p}^1} \arrow[r, "t"] \arrow[d] & \mathbb{W}_1(X_{\mathrm{p}^0}) \arrow[r, "\bar{\sigma}"] \arrow[d] & X_{\mathrm{p}^1} \arrow[d] \\
\mathrm{Spec}(R_1) \arrow[r, "F_{R_1}"] & \mathrm{Spec}(R_1) \arrow[r, "\mathrm{Id}"] & \mathrm{Spec}(R_1)
\end{tikzcd}$$

In fact, by definition, $\bar{\sigma}$ is a morphism over R_1, so the right square is commutative. The commutativity of the left square is easy to check, since we know explicitly t and F_{R_1}. Therefore $\bar{\sigma} \circ t$ is a lift of the Frobenius $F_{X_{\mathrm{p}0}}$ compatible with F_{R_1}; this implies the existence of a morphism of R_1-schemes

$$\tilde{F} : X_{\mathrm{p}^1} \to X'_{\mathrm{p}^1}$$

lifting the relative Frobenius $F_{X_{\mathrm{p}0}/R_0}$.

As shown in part (b) of the proof of Théorème 2.1 in [Deligne and Illusie 1987], since the image of $\tilde{F}^* : \Omega_{X'_{\mathrm{p}1}} \to \tilde{F}_* \Omega_{X_{\mathrm{p}^1}}$ is contained in $p\tilde{F}_*\Omega_{X_{\mathrm{p}^1}}$ and the multiplication by p induces an isomorphism $p : F_{X_{\mathrm{p}0}/R_0,*}\Omega_{X_{\mathrm{p}0}} \xrightarrow{\sim} p\tilde{F}_*\Omega_{X_{\mathrm{p}^1}}$, there exists a unique map

$$f := p^{-1}\tilde{F}^* : \Omega_{X'_{\mathrm{p}0}} \to F_{X_{\mathrm{p}0}/R_0,*}\Omega_{X_{\mathrm{p}0}},$$

making the diagram below commutative.

$$\begin{tikzcd}
\Omega_{X'_{\mathrm{p}1}} \arrow[r, "\tilde{F}^*"] \arrow[d, two heads] & p\tilde{F}_*\Omega_{X_{\mathrm{p}1}} \\
\Omega_{X'_{\mathrm{p}0}} \arrow[r, "f"] & F_{X_{\mathrm{p}0}/R_0,*}\Omega_{X_{\mathrm{p}0}} \arrow[u, "p"']
\end{tikzcd}$$

Proposition 3 in [Xin 2016] states that the adjoint of f,

$$\bar{f} : F^*_{X_{\mathrm{p}0}}\Omega_{X_{\mathrm{p}0}} = F^*_{X_{\mathrm{p}0}/R_0}\Omega_{X'_{\mathrm{p}0}} \to \Omega_{X_{\mathrm{p}0}},$$

is generically bijective. This clearly contradicts Lemma 5.1. □

6. The number of irreducible components of the critical scheme of complete intersections

In this last section we provide an upper bound for the number of irreducible components of the critical scheme $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ in the case in which X is a smooth complete intersection.

Let A/K be an abelian variety of dimension n and let L be a very ample line bundle on A. Let $c \in \mathbb{N}$ be positive and let $H_1, H_2, \dots, H_c \in |L|$ be general. We define $X := H_1 \cap H_2 \cap \cdots \cap H_c$. Suppose that X is smooth.

Let us take a sufficiently small open $V \subseteq \mathrm{Spec}(\mathcal{O}_K)$ such that A extends over V to an abelian scheme $\mathcal{A}$, L extends to a V-very ample line bundle $\mathcal{L}$, H_i extends to $\mathcal{H}_i$ for every i and $\mathcal{X} := \mathcal{H}_1 \cap \mathcal{H}_2 \cap \cdots \cap \mathcal{H}_c$ is smooth. We can restrict V if necessary and suppose $K/\mathbb{Q}$ is unramified at $\mathfrak{p}$.

Theorem 6.1. *Let K be a number field, A/K be an abelian variety of dimension n and let L be a very ample line bundle on A. Let $c \in \mathbb{N}$ be positive and let $H_1, H_2, \dots, H_c \in |L|$ be general. Suppose that $X := H_1 \cap H_2 \cap \cdots \cap H_c$ is smooth. If $\mathfrak{p}$ is in the open subscheme V defined above, then the number of irreducible components of* $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ *is bounded by*

$$p^{2n}\left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\right)(L^n)^2.$$

Here (L^n) denotes the intersection number of L.

Proof. To obtain Theorem 6.1, we follow the approach of [Buium 1996, Theorem 1.11], proving the Manin–Mumford conjecture for curves; we first show that $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ can be realized as the intersection of two projective varieties (see $\mathbb{P}(E_X)$ and $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})$ below) and then use the product of their degrees to bound the number of its irreducible components. Since X is not necessarily of dimension one, the computation of the degree of $\mathbb{P}(E_X)$ is slightly more demanding here than the corresponding one in Buium's work.

Let us fix $\mathfrak{p} \in V$. The torsors $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \to X_{\mathfrak{p}^0}$ and $\mathrm{Gr}_1(A_{\mathfrak{p}^1}) \to A_{\mathfrak{p}^0}$ correspond to elements $\eta_X \in H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}0}} \Omega^\vee_{X_{\mathfrak{p}0}})$ and $\eta_A \in H^1(A_{\mathfrak{p}^0}, F^*_{A_{\mathfrak{p}0}} \Omega^\vee_{A_{\mathfrak{p}0}})$, respectively. Under the natural isomorphisms

$$H^1(X_{\mathfrak{p}^0}, F^*_{X_{\mathfrak{p}0}} \Omega^\vee_{X_{\mathfrak{p}0}}) \simeq \mathrm{Ext}^1(F^*_{X_{\mathfrak{p}0}} \Omega_{X_{\mathfrak{p}^0}}, \mathcal{O}_{X_{\mathfrak{p}0}}) \quad \text{and} \quad H^1(A_{\mathfrak{p}^0}, F^*_{A_{\mathfrak{p}0}} \Omega^\vee_{A_{\mathfrak{p}0}}) \simeq \mathrm{Ext}^1(F^*_{A_{\mathfrak{p}0}} \Omega_{A_{\mathfrak{p}^0}}, \mathcal{O}_{A_{\mathfrak{p}0}}),$$

η_X and η_A correspond to extensions of vector bundles

$$0 \to \mathcal{O}_{X_{\mathfrak{p}0}} \to E_X \to F^*_{X_{\mathfrak{p}0}} \Omega_{X_{\mathfrak{p}0}} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{A_{\mathfrak{p}0}} \to E_A \to F^*_{A_{\mathfrak{p}0}} \Omega_{A_{\mathfrak{p}0}} \to 0.$$

For any locally free sheaf W over a base S of finite type over a field, we shall write $\mathbb{P}(W)$ for the projective bundle associated to W, i.e., the S-scheme representing the functor on S-schemes

$T \mapsto \{$isomorphism classes of surjective morphisms of $\mathcal{O}_T$-modules $W_T \to Q$,
where Q is locally free of rank $1\}$.

As shown in paragraph 1 of [Martin-Deschamps 1984], the two extensions above give us two divisors

$$D_X := \mathbb{P}(F^*_{X_{\mathfrak{p}0}} \Omega_{X_{\mathfrak{p}0}}) \subseteq \mathbb{P}(E_X) \quad \text{and} \quad D_A := \mathbb{P}(F^*_{A_{\mathfrak{p}0}} \Omega_{A_{\mathfrak{p}0}}) \subseteq \mathbb{P}(E_A),$$

belonging respectively to the linear systems $|\mathcal{O}_{\mathbb{P}(E_X)}(1)|$ and $|\mathcal{O}_{\mathbb{P}(E_A)}(1)|$, and

$$\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_X) \setminus D_X \quad \text{and} \quad \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_A) \setminus D_A.$$

If i denotes the closed immersion $i : X_{\mathfrak{p}^0} \to A_{\mathfrak{p}^0}$, then it is not difficult to show that there is a natural restriction homomorphism $i^* E_A \to E_X$ prolonging the homomorphism $i^*\Omega_{A_{\mathfrak{p}^0}} \to \Omega_{X_{\mathfrak{p}^0}}$. The homomorphism $i^* E_A \to E_X$ is clearly surjective, so it induces a closed immersion $j : \mathbb{P}(E_X) \hookrightarrow \mathbb{P}(E_A)$ prolonging $\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \hookrightarrow \mathrm{Gr}_1(A_{\mathfrak{p}^1})$. Therefore we have a commutative diagram

$$
\begin{tikzcd}
 & {[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})} \arrow[d, hook] \arrow[ddd, bend left=60, "T"] \\
\mathrm{Gr}_1(X_{\mathfrak{p}^1}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \arrow[d, hook] \\
\mathbb{P}(E_X) \arrow[r, hook, "j"] \arrow[d, "\pi_X"] & \mathbb{P}(E_A) \arrow[d, "\pi_A"] \\
X_{\mathfrak{p}^0} \arrow[r, hook, "i"] & A_{\mathfrak{p}^0}
\end{tikzcd}
$$

Let us denote by $\mathcal{L}_{\mathfrak{p}}$ the base change of $\mathcal{L}$ to $A_{\mathfrak{p}^0}$. It is standard to prove that

$$\mathcal{H} := \pi_A^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_A)}(1)$$

is very ample on $\mathbb{P}(E_A)$ [Buium and Voloch 1996, p. 4]. We have

$$\mathcal{H}|_{\mathbb{P}(E_X)} = \pi_X^* i^* \mathcal{L}_{\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_X)}(1) \quad \text{and} \quad \mathcal{H}|_{[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})} = T^* \mathcal{L}_{\mathfrak{p}},$$

since $D_A \in |\mathcal{O}_{\mathbb{P}(E_A)}(1)|$ and $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \subseteq \mathrm{Gr}_1(A_{\mathfrak{p}^1}) \simeq \mathbb{P}(E_A) \setminus D_A$. We know that $[p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})$ is the maximal abelian subvariety of $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ and we know that the multiplication by p map on $\mathrm{Gr}_1(A_{\mathfrak{p}^1})$ factors through the isogeny T. This implies that T has degree at most p^{2n}, so we have the following estimate

$$\deg_{\mathcal{H}}([p]_* \mathrm{Gr}_1(A_{\mathfrak{p}^1})) \leq p^{2n} (\mathcal{L}_{\mathfrak{p}}^n).$$

Let us now consider $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$. It coincides with

$$\int_{\mathbb{P}(E_X)} c_1(\mathcal{H}|_{\mathbb{P}(E_X)})^{2n-2c} \tag{7}$$

where c_1 stands for the first Chern class in the Chow ring and $\int_{\mathbb{P}(E_X)}$ stands for the push-forward morphism to $\mathrm{Spec}(R_0)$ in the Chow theory. Since

$$c_1(\mathcal{H}|_{\mathbb{P}(E_X)}) = c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}}) + c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))$$

we can rewrite (7) as

$$\int_{\mathbb{P}(E_X)} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(\pi_X^* i^* \mathcal{L}_{\mathfrak{p}})^h \cdot c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h}.$$

Equivalently

$$\int_{X_{\mathfrak{p}0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot \pi_{X,*}(c_1(\mathcal{O}_{\mathbb{P}(E_X)}(1))^{2n-2c-h})$$

and by definition of Segre class this is

$$\int_{X_{\mathfrak{p}0}} \sum_{h=0}^{2n-2c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X^\vee).$$

Notice that the Segre classes of the dual of E_X appear in our formula; this is due to the fact that we are not using Fulton's geometric notation for the projective bundle associated to a vector bundle (see the note at the end of B.5.5 in [Fulton 1998]). Since $s_k = 0$ if $k < 0$, we end up with

$$\int_{X_{\mathfrak{p}0}} \sum_{h=0}^{n-c} \binom{2n-2c}{h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(E_X^\vee).$$

Now the exact sequence

$$0 \to \mathcal{O}_{X_{\mathfrak{p}0}} \to E_X \to F_{X_{\mathfrak{p}0}}^* \Omega_{X_{\mathfrak{p}0}} \to 0$$

implies

$$s_{n-c-h}(E_X^\vee) = s_{n-c-h}(F_{X_{\mathfrak{p}0}}^* \Omega_{X_{\mathfrak{p}0}}^\vee)$$

and so

$$s_{n-c-h}(E_X^\vee) = p^{n-c-h} s_{n-c-h}(\Omega_{X_{\mathfrak{p}0}}^\vee)$$

(here we have used the following fact: the pullback of a cycle η of codimension j through the Frobenius map coincides with $p^j \eta$). Therefore we have to study the following sum

$$\sum_{h=0}^{n-c} \binom{2n-2c}{h} p^{n-c-h} c_1(i^* \mathcal{L}_{\mathfrak{p}})^h \cdot s_{n-c-h}(\Omega_{X_{\mathfrak{p}0}}^\vee). \tag{8}$$

The short exact sequence

$$0 \to \Omega_{X_{\mathfrak{p}0}}^\vee \to i^* \Omega_{A_{\mathfrak{p}0}}^\vee \to N \to 0$$

(where N is the normal bundle for i) gives

$$c_t(\Omega_{X_{\mathfrak{p}0}}^\vee) c_t(N) = c_t(i^* \Omega_{A_{\mathfrak{p}0}}^\vee) = 1,$$

so that $c_t(N) = s_t(\Omega_{X_{\mathfrak{p}0}}^\vee)$. Recalling that

$$c_t(N) = (1 + c_1(i^* \mathcal{L}_{\mathfrak{p}}) t)^c$$

we obtain

$$s_{n-c-h}(\Omega^{\vee}_{X_{\mathfrak{p}^0}}) = c_{n-c-h}(N) = \binom{c}{n-c-h} c_1(i^*\mathcal{L}_{\mathfrak{p}})^{n-c-h}.$$

Substituting in (8), we obtain

$$\left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\right)c_1(i^*\mathcal{L}_{\mathfrak{p}})^{n-c}.$$

Therefore $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\right)\int_{X_{\mathfrak{p}^0}} c_1(i^*\mathcal{L}_{\mathfrak{p}})^{n-c}.$$

Since $X_{\mathfrak{p}^0} = H_{1,\mathfrak{p}} \cap \cdots \cap H_{c,\mathfrak{p}}$ where $H_{1,\mathfrak{p}}, \ldots, H_{c,\mathfrak{p}}$ belong to $|\mathcal{L}_{\mathfrak{p}}|$, we have

$$\int_{X_{\mathfrak{p}^0}} c_1(i^*\mathcal{L}_{\mathfrak{p}})^{n-c} = \int_{A_{\mathfrak{p}^0}} c_1(\mathcal{L}_{\mathfrak{p}})^n = (\mathcal{L}_{\mathfrak{p}}^n)$$

and

$$\deg_{\mathcal{H}}(\mathbb{P}(E_X)) = \left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\right)(\mathcal{L}_{\mathfrak{p}}^n).$$

Now Bézout's theorem in Fulton's form [1998, p. 148] says that the number of irreducible components in the intersection of two projective varieties of degrees d_1 and d_2 cannot exceed $d_1 d_2$. In particular, the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$ is less than or equal to

$$p^{2n}\left(\sum_{h=0}^{n-c}\binom{2n-2c}{h}\binom{c}{n-c-h}p^{n-c-h}\cdot\right)(\mathcal{L}_{\mathfrak{p}}^n)^2.$$

Notice that $(\mathcal{L}_{\mathfrak{p}}^n) = (L^n)$, by the same reasoning as before Lemma 5.1. □

Remark 6.2. One can consider any intersection $X := H_1 \cap H_2 \cap \cdots \cap H_c$ where $H_i \in |L_i|$ for some very ample line bundles L_i. In this more general case, the computations in our proof become a bit more complex, but it is still possible to give an explicit bound for the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X}, \mathcal{A})$. We have

$$c_j(N) = \sum_{1\le i_1<\cdots<i_j\le c}\prod_{k=i_1}^{i_j} c_1(i^*\mathcal{L}_{k,\mathfrak{p}})$$

which implies

$$s_{n-c-h}(\Omega^{\vee}_{X_{\mathfrak{p}^0}}) = \sum_{1\le i_1<\cdots<i_{n-c-h}\le c}\prod_{k=i_1}^{i_{n-c-h}} c_1(i^*\mathcal{L}_{k,\mathfrak{p}}).$$

Therefore, defining $\mathcal{H} := \pi_A^*\mathcal{L}_{1,\mathfrak{p}} \otimes \mathcal{O}_{\mathbb{P}(E_A)}(1)$, then $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\sum_{h=0}^{n-c}\binom{2n-2c}{h}p^{n-c-h}\sum_{1\le i_1<\cdots<i_{n-c-h}\le c}\left(\int_{X_{\mathfrak{p}^0}} c_1(i^*\mathcal{L}_{1,\mathfrak{p}})^h\prod_{k=i_1}^{i_{n-c-h}} c_1(i^*\mathcal{L}_{k,\mathfrak{p}})\right).$$

We have

$$\int_{X_{\mathfrak{p}0}} c_1(i^*\mathcal{L}_{1,\mathfrak{p}})^h \prod_{k=i_1}^{i_{n-c-h}} c_1(i^*\mathcal{L}_{k,\mathfrak{p}}) = I_{i_1,\dots,i_{n-c-h}}$$

where $I_{i_1,\dots,i_{n-c-h}}$ is the following intersection number

$$I_{i_1,\dots,i_{n-c-h}} := (\overbrace{L_1 \cdots L_1}^{h+1 \text{ times}} \cdot L_2 \cdot L_3 \cdots L_c \cdot L_{i_1} \cdot L_{i_2} \cdots L_{i_{n-c-h}}).$$

We obtain that $\deg_{\mathcal{H}}(\mathbb{P}(E_X))$ is

$$\sum_{h=0}^{n-c} \binom{2n-2c}{h} p^{n-c-h} \sum_{1\le i_1<\cdots<i_{n-c-h}\le c} I_{i_1,\dots,i_{n-c-h}},$$

and therefore the number of irreducible components of $\mathrm{Crit}^1(\mathcal{X},\mathcal{A})$ is bounded by

$$p^{2n}(L_1^n)\sum_{h=0}^{n-c} \binom{2n-2c}{h} p^{n-c-h} \sum_{1\le i_1<\cdots<i_{n-c-h}\le c} I_{i_1,\dots,i_{n-c-h}}.$$

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