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Martin Orr

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Height bounds and the Siegel property

Martin Orr

Let $\boldsymbol{G}$ be a reductive group defined over $\mathbb{Q}$ and let $\mathfrak{S}$ be a Siegel set in $\boldsymbol{G}(\mathbb{R})$. The Siegel property tells us that there are only finitely many $\gamma \in \boldsymbol{G}(\mathbb{Q})$ of bounded determinant and denominator for which the translate $\gamma.\mathfrak{S}$ intersects $\mathfrak{S}$. We prove a bound for the height of these γ which is polynomial with respect to the determinant and denominator. The bound generalises a result of Habegger and Pila dealing with the case of GL_2, and has applications to the Zilber–Pink conjecture on unlikely intersections in Shimura varieties.

In addition we prove that if $\boldsymbol{H}$ is a subset of $\boldsymbol{G}$, then every Siegel set for $\boldsymbol{H}$ is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set for $\boldsymbol{G}$.

1. Introduction

A Siegel set is a subset of the real points $\boldsymbol{G}(\mathbb{R})$ of a reductive $\mathbb{Q}$-algebraic group of a certain nice form. The notion of a Siegel set was introduced by Borel and Harish-Chandra [1962], in order to prove the finiteness of the covolume of arithmetic subgroups of $\boldsymbol{G}(\mathbb{R})$. In this paper we use a variant of the notion due to Borel [1969] which takes into account the $\mathbb{Q}$-structure of the group $\boldsymbol{G}$, and gives an intrinsic construction of fundamental sets for arithmetic subgroups in $\boldsymbol{G}(\mathbb{R})$.

Let $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ be a Siegel set (see Section 2 for the precise definition). The primary theorem of this paper is a bound for the height of elements of

$$\mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q}) = \{\gamma \in \boldsymbol{G}(\mathbb{Q}) : \gamma.\mathfrak{S} \cap \mathfrak{S} \neq \varnothing\}$$

in terms of their determinant and denominators. This gives a quantitative version of [Borel 1969, Corollaire 15.3], which asserts that $\mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q})$ has only finitely many elements with given determinant and denominators. This in turn implies a quantitative version of the Siegel property, one of the key properties of Siegel sets.

Theorem 1.1. *Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group and let $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ be a Siegel set. Let $\rho : \boldsymbol{G} \to \mathrm{GL}_n$ be a faithful $\mathbb{Q}$-algebraic group representation.*

There exists a constant C_1 (depending on $\boldsymbol{G}$, $\mathfrak{S}$ and ρ) such that, for all

$$\gamma \in \mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q}),$$

if $N = |\det \rho(\gamma)|$ and D is the maximum of the denominators of entries of $\rho(\gamma)$, then

$$\mathrm{H}(\rho(\gamma)) \leq \max(C_1 N D^n, D).$$

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This theorem was inspired by a result of Habegger and Pila [2012, Lemma 5.2]. They dealt with the case $\boldsymbol{G} = \mathrm{GL}_2$, as a step in proving some cases of the Zilber–Pink conjecture on unlikely intersections in $Y(1)^n$. We are motivated by applications of Theorem 1.1 to the Zilber–Pink conjecture in higher-dimensional Shimura varieties, which is the subject of work in progress by the author. The key point for these applications is that the bound is polynomial in the determinant N.

The second main theorem of this paper compares Siegel sets for the group $\boldsymbol{G}$ with Siegel sets for a subgroup $\boldsymbol{H} \subset \boldsymbol{G}$, which can be seen as a result on the functoriality of Siegel sets with respect to injections of $\mathbb{Q}$-algebraic groups. This theorem is used in the proof of Theorem 1.1 to reduce to the case $\boldsymbol{G} = \mathrm{GL}_n$. It also has its own applications to the Zilber–Pink conjecture.

Theorem 1.2. *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be reductive $\mathbb{Q}$-algebraic groups, with $\boldsymbol{H} \subset \boldsymbol{G}$. Let $\mathfrak{S}_{\boldsymbol{H}}$ be a Siegel set in $\boldsymbol{H}(\mathbb{R})$. Then there exist a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ and a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ such that*

$$\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}.$$

Theorem 4.1 gives some additional information about how the Siegel sets $\mathfrak{S}_{\boldsymbol{G}}$ and $\mathfrak{S}_{\boldsymbol{H}}$ are related to each other (in terms of the associated Siegel triples).

1A. ***Previous results: height bounds.*** The primary inspiration for Theorem 1.1 is the following result of Habegger and Pila.

Proposition 1.3 [Habegger and Pila 2012, Lemma 5.2]. *Let $\mathcal{F}$ denote the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane.*

There exists a constant C_2 such that: for all points $x, y \in \mathcal{F}$, if the associated elliptic curves are related by an isogeny of degree N, then there exists $\gamma \in \mathrm{M}_2(\mathbb{Z})$ such that

$$\gamma x = y, \quad \det \gamma = N \quad \textit{and} \quad \mathrm{H}(\gamma) \leq C_2 N^{10}.$$

In order to relate Proposition 1.3 to Theorem 1.1, recall that the upper half-plane $\mathcal{H}$ can be identified with the symmetric space $\mathrm{GL}_2(\mathbb{R})^+/\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R})$, with $\mathrm{GL}_2(\mathbb{R})^+$ acting on $\mathcal{H}$ by Möbius transformations. Under this identification, the standard fundamental domain

$$\mathcal{F} = \left\{ z \in \mathcal{H} : -\tfrac{1}{2} \leq \operatorname{Re} z \leq \tfrac{1}{2},\ |z| \geq 1 \right\}$$

is contained in the image of the standard Siegel set

$$\mathfrak{S} = \Omega_{1/2} A_{\sqrt{3}/2} K \subset \mathrm{GL}_2(\mathbb{R}),$$

as defined in Section 2A.

We further identify the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ with the moduli space $Y(1)$ of elliptic curves over $\mathbb{C}$. It is easy to prove that the elliptic curves associated with points $x, y \in \mathcal{H}$ are related by an isogeny of degree N if and only if there exists $\gamma \in \mathrm{M}_2(\mathbb{Z})$ such that

$$\gamma x = y \quad \text{and} \quad \det \gamma = N. \tag{1}$$

Theorem 1.1 tells us that any γ satisfying (1) has height at most $C_1 N$, improving on the exponent 10 which appears in Proposition 1.3.

Theorem 1.1 also implies a uniform version of the following previous result of the author (which is a combination of [Orr 2015, Lemma 3.3] with [Orr 2017, Theorem 1.3]).

Proposition 1.4. *Let $\mathcal{F}_g$ denote the standard fundamental domain for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on the Siegel upper half-space of rank g. Fix a point $x \in \mathcal{F}_g$.*

There exist constants C_3 and C_4 such that for all points $y \in \mathcal{F}_g$, if the principally polarised abelian varieties associated with x and y are related by a polarised isogeny of degree N, then there exists a matrix $\gamma \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ such that

$$\gamma x = y \quad \text{and} \quad \mathrm{H}(\gamma) \leq C_3 N^{C_4}.$$

In Proposition 1.4, the constant C_3 depends on the fixed point $x \in \mathcal{F}_g$ and only the other point y is allowed to vary. On the other hand, we can apply Theorem 1.1 to the symmetric space $\mathcal{H}_g$ in a similar way to that sketched above for $\mathcal{H}$. This gives a much stronger result in which the constant is uniform in both x and y. Hence Theorem 1.1 can be used to prove results on unlikely intersections in $\mathcal{A}_g \times \mathcal{A}_g$ for which Proposition 1.4 is not sufficient.

Note that [Orr 2015, Lemma 3.3] gives a height bound for unpolarised as well as polarised isogenies. It is not possible to directly deduce a uniform version of this bound for unpolarised isogenies from Theorem 1.1 because [Orr 2015, Lemma 3.3] concerns the homogeneous space $\mathrm{GL}_{2g}(\mathbb{R})/\mathrm{GL}_g(\mathbb{C})$ while Theorem 1.1 applies to the symmetric space $\mathrm{GL}_{2g}(\mathbb{R})/\mathbb{R}^\times \mathrm{O}_{2g}(\mathbb{R})$.

1B. *Previous results: Siegel sets and subgroups.* Let $\boldsymbol{H}$ be a reductive $\mathbb{Q}$-algebraic subgroup of $\boldsymbol{G} = \mathrm{GL}_n$. Borel and Harish-Chandra [1962, Theorem 6.5] gave a recipe for constructing a fundamental set for $\boldsymbol{H}(\mathbb{R})$ which is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set for $\boldsymbol{G}$. However it is not obvious how the resulting fundamental set is related to a Siegel set for $\boldsymbol{H}$. Theorem 1.2 resolves this by directly relating Siegel sets for $\boldsymbol{G}$ and $\boldsymbol{H}$.

Theorem 1.2 can also be interpreted as a result about functoriality of Siegel sets. According to a remark on [Borel 1969, p. 86], if $f : \boldsymbol{H} \to \boldsymbol{G}$ is a *surjective* morphism of reductive $\mathbb{Q}$-algebraic groups and $\mathfrak{S}_{\boldsymbol{H}}$ is a Siegel set in $\boldsymbol{H}(\mathbb{R})$, then $f(\mathfrak{S}_{\boldsymbol{H}})$ is contained in a Siegel set in $\boldsymbol{G}(\mathbb{R})$. Theorem 1.2 gives a similar result for injective morphisms of reductive $\mathbb{Q}$-algebraic groups, where the conclusion must be weakened to saying that the image of a Siegel set is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set. We can of course combine these to conclude that for an arbitrary morphism $f : \boldsymbol{H} \to \boldsymbol{G}$, the image of a Siegel set $\mathfrak{S}_{\boldsymbol{H}} \subset \boldsymbol{H}(\mathbb{R})$ is contained in a finite union of $\boldsymbol{G}(\mathbb{Q})$-translates of a Siegel set in $\boldsymbol{G}(\mathbb{R})$.

The proof of Theorem 1.2 gives an explicit bound for the size of the set $C \subset \boldsymbol{G}(\mathbb{Q})$, namely $\#C$ is at most the size of the $\mathbb{Q}$-Weyl group of $\boldsymbol{G}$. The uniform nature of this bound is less powerful than it might at first appear because the Siegel set $\mathfrak{S}_{\boldsymbol{G}}$ depends on $\mathfrak{S}_{\boldsymbol{H}}$.

1C. *Application to unlikely intersections.* The author's motivation for studying Theorem 1.1 is due to its applications to the Zilber–Pink conjecture on unlikely intersections in Shimura varieties [Pink 2005, Conjecture 1.2]. To illustrate these applications, consider the following special case of the Zilber–Pink conjecture.

Conjecture 1.5. *Let $g \geq 2$ and let $\mathcal{A}_g$ denote the moduli space of principally polarised abelian varieties of dimension g over $\mathbb{C}$. For each point $s \in \mathcal{A}_g$, let (A_s, λ_s) denote the associated principally polarised abelian variety. Let*

$$\Sigma = \{(s_1, s_2) \in \mathcal{A}_g \times \mathcal{A}_g : \text{there exists an isogeny } A_{s_1} \to A_{s_2}\}.$$

Let $V \subset \mathcal{A}_g \times \mathcal{A}_g$ be an irreducible algebraic curve. If $V \cap \Sigma$ is infinite, then V is contained in a proper special subvariety of $\mathcal{A}_g \times \mathcal{A}_g$.

Habegger and Pila [2012] used Proposition 1.3 to prove a result similar to Conjecture 1.5 but for the Shimura variety $\mathcal{A}_1^n$ ($n \geq 3$) instead of $\mathcal{A}_g \times \mathcal{A}_g$ ($g \geq 2$) (for reasons of dimension, Conjecture 1.5 is false for $\mathcal{A}_1 \times \mathcal{A}_1$).

In work currently in progress, the author of this paper proves Conjecture 1.5 subject to certain technical conditions and a restricted definition of the set Σ. This work requires the uniform version of Proposition 1.4 which is implied by the GSp_{2g} case of Theorem 1.1. Because Theorem 1.1 applies to all reductive groups, not just GSp_{2g}, it should also be useful for proving statements similar to Conjecture 1.5 where $\mathcal{A}_g$ is replaced by an arbitrary Shimura variety. However, at present it is not known how to prove the Galois bounds which would be required for such a statement.

1D. *Outline of paper.* Section 2 contains the definition of Siegel sets and the associated notation used throughout the paper. In Section 3 we prove Theorem 1.1 for standard Siegel sets in GL_n, and combine this with Theorem 1.2 to deduce the general statement of Theorem 1.1. The proof of the GL_n case is entirely self-contained. Finally Section 4 contains the proof of Theorem 1.2, relying on results on parabolic subgroups and roots from [Borel and Tits 1965].

1E. *Notation.* If $\boldsymbol{G}$ is a real algebraic group, then we write $\boldsymbol{G}(\mathbb{R})^+$ for the identity component of $\boldsymbol{G}(\mathbb{R})$ in the Euclidean topology.

We use a naive definition for the height of a matrix with rational entries, as in [Pila and Wilkie 2006]: if $\gamma \in \mathrm{M}_n(\mathbb{Q})$, then its *height* is

$$\mathrm{H}(\gamma) = \max_{1 \leq i, j \leq n} \mathrm{H}(\gamma_{ij})$$

where the height of a rational number a/b (written in lowest terms) is $\max(|a|, |b|)$. For an algebraic group $\boldsymbol{G}$ other than GL_n, we define the heights of elements of $\boldsymbol{G}(\mathbb{Q})$ via a choice of faithful representation $\boldsymbol{G} \to \mathrm{GL}_n$.

In order to avoid writing uncalculated constant factors in every inequality in the proof of Theorem 1.1, we use the notation

$$X \ll Y$$

to mean that there exists a constant C, depending only on the group $\boldsymbol{G}$, the representation ρ and the Siegel set $\mathfrak{S}$, such that

$$|X| \leq C|Y|.$$

2. Definition of Siegel sets

The definitions of Siegel sets used by different authors (for example, [Borel 1969; Ash et al. 2010]) vary in minor ways, so we state here the precise definition used in this paper. At the same time, we define the notation which we shall use in Sections 3 and 4 for the various ingredients in the construction of Siegel sets.

2A. ***Standard Siegel sets in*** $\mathbf{GL}_n$**.** Before defining Siegel sets in general, we begin with the simpler special case of "standard Siegel sets" in GL_n. Our definition of standard Siegel sets follows [Borel 1969, Définition 1.2]. However, we use the reverse order of multiplication for elements of GL_n and therefore reverse the inequalities in the definition of A_t.

Make the following definitions (all of these are special cases of the corresponding notations for general Siegel sets):

(1) $\boldsymbol{P} \subset \mathrm{GL}_n$ is the Borel subgroup consisting of upper triangular matrices.

(2) $K = \mathrm{O}_n(\mathbb{R})$ is the maximal compact subgroup consisting of orthogonal matrices.

(3) $\boldsymbol{S} \subset \boldsymbol{P}$ is the maximal $\mathbb{Q}$-split torus consisting of diagonal matrices.

(4) A_t is the set $\{\alpha \in \boldsymbol{S}(\mathbb{R})^+ : \alpha_j/\alpha_{j+1} \geq t \text{ for all } j\}$ for any real number $t > 0$.

(5) Ω_u is the compact set

$$\{\nu \in \boldsymbol{P}(\mathbb{R}) : \nu_{ii} = 1 \text{ for all } i \text{ and } |\nu_{ij}| \leq u \text{ for } 1 \leq i < j \leq n\}$$

for any real number $u > 0$.

A *standard Siegel set* in GL_n is a set of the form

$$\mathfrak{S} = \Omega_u A_t K \subset \mathrm{GL}_n(\mathbb{R})$$

for some positive real numbers u and t.

According to [Borel 1969, Théorèmes 1.4 and 4.6], if $t \leq \frac{\sqrt{3}}{2}$ and $u \geq \frac{1}{2}$, then $\mathfrak{S}$ is a fundamental set for $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{GL}_n(\mathbb{R})$.

2B. ***Definition of Siegel sets in general.*** Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group. In order to define a Siegel set in $\boldsymbol{G}(\mathbb{R})$, we begin by making choices of the following subgroups of $\boldsymbol{G}$

(1) $\boldsymbol{P}$ a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$,

(2) K a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$.

Lemma 2.1. *For any* $\boldsymbol{P}$ *and* K, *there exists a unique* $\mathbb{R}$*-torus* $\boldsymbol{S} \subset \boldsymbol{P}$ *satisfying the conditions*

(i) $\boldsymbol{S}$ *is* $\boldsymbol{P}(\mathbb{R})$*-conjugate to a maximal* $\mathbb{Q}$*-split torus in* $\boldsymbol{P}$,

(ii) *$\boldsymbol{S}$ is stabilised by the Cartan involution associated with K.*

Proof. This follows from the lemma in [Ash et al. 2010, Chapter II, section 3.7]. □

We define a *Siegel triple* for $\boldsymbol{G}$ to be a triple $(\boldsymbol{P}, \boldsymbol{S}, K)$ satisfying the conditions of Lemma 2.1. We remark that these conditions could equivalently be stated as:

(i) $\boldsymbol{S}$ is a lift of the unique maximal $\mathbb{Q}$-split torus in $\boldsymbol{P}/R_u(\boldsymbol{P})$.

(ii) Lie $\boldsymbol{S}(\mathbb{R})$ is orthogonal to Lie K with respect to the Killing form of $\boldsymbol{G}$.

Define the following further pieces of notation:

(1) $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{P}$.

(2) $\boldsymbol{M}$ is the preimage in $Z_{\boldsymbol{G}}(\boldsymbol{S})$ of the maximal $\mathbb{Q}$-anisotropic subgroup of $\boldsymbol{P}/\boldsymbol{U}$. (Note that by [Borel and Tits 1965, Corollaire 4.16], $Z_{\boldsymbol{G}}(\boldsymbol{S})$ is a Levi subgroup of $\boldsymbol{P}$ and hence maps isomorphically onto $\boldsymbol{P}/\boldsymbol{U}$.)

(3) Δ is the set of simple roots of $\boldsymbol{G}$ with respect to $\boldsymbol{S}$, using the ordering induced by $\boldsymbol{P}$. (The roots of $\boldsymbol{G}$ with respect to $\boldsymbol{S}$ form a root system because $\boldsymbol{S}$ is conjugate to a maximal $\mathbb{Q}$-split torus in $\boldsymbol{G}$.)

(4) $A_t = \{\alpha \in \boldsymbol{S}(\mathbb{R})^+ : \chi(\alpha) \geq t \text{ for all } \chi \in \Delta\}$ for any real number $t > 0$.

A *Siegel set* in $\boldsymbol{G}(\mathbb{R})$ (with respect to $(\boldsymbol{P}, \boldsymbol{S}, K)$) is a set of the form

$$\mathfrak{S} = \Omega A_t K$$

where Ω is a compact subset of $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})^+$ and t is a positive real number.

2C. *Comparison with other definitions.* In order to reduce confusion caused by definitions of Siegel sets which vary from one author to another, we explain how our definition compares with the definitions used in [Borel and Harish-Chandra 1962; Borel 1969; Ash et al. 2010].

First we compare with [Ash et al. 2010, Chapter II, Section 4.1].

(1) In [Ash et al. 2010], Siegel sets are subsets of the symmetric space $\boldsymbol{G}(\mathbb{R})/K$, while for us they are K-right-invariant subsets of $\boldsymbol{G}(\mathbb{R})$. These two perspectives are related by the quotient map $\boldsymbol{G}(\mathbb{R}) \to \boldsymbol{G}(\mathbb{R})/K$.

(2) In [Ash et al. 2010], Ω is any compact subset of $\boldsymbol{P}(\mathbb{R})$, while we require Ω to be contained in $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})^+$. Every Siegel set in the sense of [Ash et al. 2010] is contained in a Siegel set in our sense and vice versa, so this difference does not matter in applications. We impose the stricter condition on Ω because it ensures that Siegel sets are related to the horospherical decomposition in $\boldsymbol{G}(\mathbb{R})/K$ (as explained in [Borel and Ji 2006, Section I.1.9]).

Now we compare with [Borel 1969, Définition 12.3]. Note that differences (3) and (4) are significant.

(1) We multiply together Ω, A_t and K in the opposite order from [Borel 1969]. This change forces us to reverse the inequalities in the definition of A_t.

(2) In [Borel 1969], Ω is required to be a compact neighbourhood of the identity in $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})^+$ while we allow any compact subset.

(3) Instead of our condition (i) for $\boldsymbol{S}$, [Borel 1969] imposes the condition that $\boldsymbol{S}$ must be a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P}$. This stronger condition is inconvenient when we also impose condition (ii), because there does not exist a maximal $\mathbb{Q}$-split torus satisfying condition (ii) for every choice of $\boldsymbol{P}$ and K. In particular, Theorem 1.2 does not hold if $\boldsymbol{S_G}$ is required to be $\mathbb{Q}$-split.

(4) Our condition (ii) for $\boldsymbol{S}$ is not part of the definition of Siegel set in [Borel 1969]. In [Borel 1969], a Siegel set is called *normal* if condition (ii) is satisfied. We include condition (ii) in the definition of a Siegel set because without it the Siegel property does not necessarily hold. Indeed most of the theorems in [Borel 1969, Chapter 15] apply only to Siegel sets satisfying condition (ii), even though the word "normal" is omitted from their statements. Similarly this paper's Theorem 1.1 does not hold without condition (ii) on $\boldsymbol{S}$.

The definition of "Siegel domain" in [Borel and Harish-Chandra 1962, Section 4] is less fine than the definition used in this paper, or the one in [Borel 1969], because it takes into account only the structure of $\boldsymbol{G}$ as a real algebraic group and not its structure as a $\mathbb{Q}$-algebraic group. Consequently [Borel and Harish-Chandra 1962] could not use their Siegel domains directly to construct fundamental sets for arithmetic subgroups in $\boldsymbol{G}(\mathbb{R})$; instead they constructed such fundamental sets using an embedding of $\boldsymbol{G}$ into GL_n and standard Siegel sets in $\mathrm{GL}_n(\mathbb{R})$.

2D. *Siegel sets and fundamental sets.* The importance of Siegel sets is due to their use in constructing fundamental sets for an arithmetic subgroup Γ in $\boldsymbol{G}(\mathbb{R})$. We say that a set $\Omega \subset \boldsymbol{G}(\mathbb{R})$ is a *fundamental set* for Γ if the following conditions are satisfied:

(F0) $\Omega.K = \Omega$ for a suitable maximal compact subgroup $K \subset \boldsymbol{G}(\mathbb{R})$.

(F1) $\Gamma.\Omega = \boldsymbol{G}(\mathbb{R})$.

(F2) For every $\theta \in \boldsymbol{G}(\mathbb{R})$, the set

$$\{\gamma \in \Gamma : \gamma.\Omega \cap \theta.\Omega \neq \varnothing\}$$

is finite (the Siegel property).

The following two theorems show that, if we make suitable choices of Siegel set $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ and finite set $C \subset \boldsymbol{G}(\mathbb{Q})$, then $C.\mathfrak{S}$ is a fundamental set for Γ in $\boldsymbol{G}(\mathbb{R})$.

Theorem 2.2 [Borel 1969, Théorème 13.1]. *Let Γ be an arithmetic subgroup of $\boldsymbol{G}(\mathbb{Q})$. Let $(\boldsymbol{P}, \boldsymbol{S}, K)$ be a Siegel triple for $\boldsymbol{G}(\mathbb{R})$.*

There exist a Siegel set $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ with respect to $(\boldsymbol{P}, \boldsymbol{S}, K)$ and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that

$$\boldsymbol{G}(\mathbb{R}) = \Gamma.C.\mathfrak{S}.$$

Theorem 2.3 [Borel 1969, Théorème 15.4]. *Let Γ be an arithmetic subgroup of $\boldsymbol{G}(\mathbb{Q})$. Let $\mathfrak{S} \subset \boldsymbol{G}(\mathbb{R})$ be a Siegel set.*

For any finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ and any element $\theta \in \boldsymbol{G}(\mathbb{Q})$, the set

$$\{\gamma \in \Gamma : \gamma.C.\mathfrak{S} \cap \theta.C.\mathfrak{S} \neq \varnothing\}$$

is finite.

As remarked in Section 2C, Theorem 2.3 requires the torus $\boldsymbol{S}$ used in the definition of a Siegel set to satisfy condition (ii) from Section 2B, even though this condition is erroneously omitted from the statement in [Borel 1969].

This paper's Theorem 1.1 implies [loc. cit., Corollaire 15.3] and therefore it implies Theorem 2.3, by the same argument as in the proof of [loc. cit., Théorème 15.4]. Since our proof of Theorem 1.1 is independent of Borel's proof of [loc. cit., Corollaire 15.3], this gives a new proof of Theorem 2.3.

3. Proof of main height bound

In this section we prove Theorem 1.1. Most of the section deals with the case of standard Siegel sets in GL_n. At the end we show how to deduce the general statement of Theorem 1.1 from this case, using Theorem 1.2.

Thus let $\boldsymbol{G} = \mathrm{GL}_n$ and let $\mathfrak{S}$ be a standard Siegel set in $\boldsymbol{G}$. As in the statement of Theorem 1.1, we are given an element

$$\gamma \in \mathfrak{S}.\mathfrak{S}^{-1} \cap \boldsymbol{G}(\mathbb{Q}),$$

with $N = |\det \gamma|$ and with D denoting the maximum of the denominators of entries of γ. Since $\gamma \in \mathfrak{S}.\mathfrak{S}^{-1}$, using the notation from Section 2A, we can write

$$\gamma = \nu\beta\kappa\alpha^{-1}\mu^{-1} \tag{2}$$

with $\alpha, \beta \in A_t$, $\mu, \nu \in \Omega_u$ and $\kappa \in K$. Rearranging this equation, we obtain

$$\gamma\mu\alpha = \nu\beta\kappa. \tag{3}$$

Our aim is to bound the height of γ by a polynomial in N and D. The proof has three stages. First we compare entries of the diagonal matrices α and β, showing that $\alpha_j \ll D\beta_i$ for certain pairs of indices (i, j). Secondly, we prove that

$$\beta_j \ll N D^{n-1} \alpha_i \tag{4}$$

whenever i and j lie in the same segment of a certain partition of $\{1, \dots, n\}$. Finally we expand out (2) and use inequality (4).

3A. ***Partitioning the indices.*** An important device in the proof of Theorem 1.1 for standard Siegel sets is a partition of the set of indices $\{1, \dots, n\}$ into subintervals which we call "segments" (depending on γ). The *segments* are defined to be the subintervals of $\{1, \dots, n\}$ such that

(i) γ is block upper triangular with respect to the chosen partition,

(ii) γ is not block upper triangular with respect to any finer partition of $\{1, \dots, n\}$ into subintervals.

We define a *leading entry* to be a pair of indices $(i, j) \in \{1, \dots, n\}^2$ such that γ_{ij} is the left-most nonzero entry in the i-th row of γ.

The following lemma describes segments in terms of leading entries. This lemma also has a converse, which we will not need: if $i > j$ and there exists a sequence satisfying condition $(*)$, then i and j are in the same segment.

Lemma 3.1. *If $i > j$ and i and j are in the same segment, then there exists a sequence of leading entries $(i_1, j_1), \dots, (i_s, j_s)$ such that*

$$i \le i_1, \qquad j_p \le i_{p+1} \quad \textit{for every} \quad p \in \{1, \dots, s-1\}, \qquad \textit{and} \qquad j_s \le j. \tag{$*$}$$

Proof. First, for each k such that $j < k \le i$, we show that there exists a leading entry (i', j') such that $j' < k \le i'$. Because segments give the finest partition according to which γ is block upper triangular, γ cannot be block upper triangular with respect to the partition

$$\{1, \dots, k-1\}, \{k, \dots, n\}.$$

So there exists some $i' \ge k$ such that the i'-th row of γ has a nonzero entry in the first $k-1$ columns. Choosing j' to be the index of the left-most nonzero entry in the i'-th row, we get the desired leading entry with $j' < k \le i'$.

Let $s = i - j$. For each p such that $1 \le p \le s$ we apply the above argument to $k = i - p + 1$ and get a leading entry (i_p, j_p) such that $j_p < i - p + 1 \le i_p$. The resulting sequence $(i_1, j_1), \dots, (i_s, j_s)$ satisfies condition $(*)$. □

We define $\boldsymbol{Q}$ to be the subgroup of GL_n consisting of block upper triangular matrices according to the segments defined above (thus $\boldsymbol{Q}$ depends on γ). Observe that $\boldsymbol{Q}$ could equivalently be defined as the smallest standard parabolic subgroup of GL_n which contains γ.

We define $\boldsymbol{L}$ to be the subgroup of GL_n consisting of block diagonal matrices according to the same partition into segments. Thus $\boldsymbol{L}$ could equivalently be defined as the Levi subgroup of $\boldsymbol{Q}$ containing the torus of diagonal matrices.

3B. *Example partitions for* $\mathbf{GL_3}$. To illustrate the definition of segments and Lemma 3.1, we show the various cases which occur for GL_3. Table 1 shows classes of matrix in GL_3, depending on the region of zeros adjacent to the bottom left corner of the matrix, and gives the associated partitions of $\{1, 2, 3\}$ into segments. Every matrix in GL_3 falls into exactly one of the classes in Table 1.

In Table 1, $*$ represents an entry which must be nonzero, while $\cdot$ represents an entry which may be either zero or nonzero. Every entry to the left of a $*$ is zero, so each $*$ is a leading entry. For rows which do not contain a $*$, there is not enough information to determine the leading entry; these rows' leading entries are not important for Lemma 3.1.

Comparing the two classes of matrices in the right-hand column of Table 1, we see that it is possible for matrices to have different patterns of zeros adjacent to the bottom left corner, yet still be associated with the same partition of $\{1, 2, 3\}$. This is related to the fact that matrices in the lower class of this column do

γ	segments	γ	segments
$\begin{pmatrix} * & \cdot & \cdot \\ 0 & * & \cdot \\ 0 & 0 & * \end{pmatrix}$	{1}, {2}, {3}	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ * & \cdot & \cdot \end{pmatrix}$	{1, 2, 3}
$\begin{pmatrix} * & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & * & \cdot \end{pmatrix}$	{1}, {2, 3}	$\begin{pmatrix} \cdot & \cdot & \cdot \\ * & \cdot & \cdot \\ 0 & * & \cdot \end{pmatrix}$	{1, 2, 3}
$\begin{pmatrix} \cdot & \cdot & \cdot \\ * & \cdot & \cdot \\ 0 & 0 & * \end{pmatrix}$	{1, 2}, {3}		

Table 1. Partitions into segments for $\gamma \in \mathrm{GL}_3$.

not form a subgroup of GL_3: the smallest standard parabolic subgroup containing such a matrix is the full group GL_3, the same as for the upper class.

On the other hand, the difference between the two classes in the right-hand column of Table 1 is important for finding sequences of leading entries as in Lemma 3.1. In the upper class of this column, the sequence consisting just of the leading entry $(3, 1)$ satisfies condition $(*)$ for every pair (i, j). In the lower class, in order to construct a sequence satisfying condition $(*)$ which goes from $i = 3$ to $j = 1$, we need both the leading entries $(3, 2)$ and $(2, 1)$.

3C. ***Ratios between diagonal matrices (leading entries).*** In the first stage of the proof, we compare α_j with β_i when (i, j) is a leading entry. This is based on comparing the lengths of the i-th rows on either side of (3).

Lemma 3.2. *If (i, j) is a leading entry for γ, then*

$$\alpha_j \ll D\beta_i.$$

Proof. Recall (3):

$$\gamma\mu\alpha = \nu\beta\kappa.$$

Because $\kappa \in \mathrm{O}_n(\mathbb{R})$, multiplying by κ on the right does not change the length of a row vector. Hence expanding out the lengths of the i-th rows on either side of (3) gives

$$\sum_{p=1}^{n} \left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qp} \right)^2 \alpha_p^2 = \sum_{p=1}^{n} \nu_{ip}^2 \beta_p^2. \tag{5}$$

Look first at the right-hand side of (5), comparing it to β_i^2. Because ν is upper triangular, nonzero terms on the right-hand side of (5) must have $p \geq i$ and hence (by the definition of A_t) $\beta_p \ll \beta_i$. Since ν

is in the fixed compact set Ω_u, there is a uniform bound for the entries ν_{ip}. Thus we get

$$\sum_{p=1}^{n} \nu_{ip}^2 \beta_p^2 \ll \beta_i^2. \tag{6}$$

Now look at the left-hand side of (5), comparing it to α_j^2. We pull out the $p = j$ term. Because squares are nonnegative we have

$$\left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qj}\right)^2 \alpha_j^2 \le \sum_{p=1}^{n}\left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qp}\right)^2 \alpha_p^2. \tag{7}$$

Because (i, j) is a leading entry, if $\gamma_{iq} \neq 0$ then $q \geq j$. Because μ is upper triangular, if $\mu_{qj} \neq 0$ then $q \leq j$. Combining these facts, the only nonzero term on the left-hand side of (7) is the term with $q = j$. In other words,

$$\gamma_{ij}^2 \mu_{jj}^2 \alpha_j^2 = \left(\sum_{q=1}^{n} \gamma_{iq}\mu_{qj}\right)^2 \alpha_j^2. \tag{8}$$

Because $\mu \in \Omega_u$, we have $\mu_{jj} = 1$. Because (i, j) is a leading entry, $\gamma_{ij} \neq 0$. Because entries of γ are rational numbers with denominator at most D, this implies that $|\gamma_{ij}| \geq D^{-1}$. Combining these facts, we get

$$D^{-2} \le \gamma_{ij}^2 \mu_{jj}^2. \tag{9}$$

Using successively the inequalities and equations (9), (8), (7), (5) and (6) gives

$$D^{-2}\alpha_j^2 \ll \beta_i^2. \qquad \square$$

3D. ***Ratios between diagonal matrices (in each segment).*** In the second stage of the proof of Theorem 1.1, we prove a series of inequalities comparing entries of α and β. This concludes with an inequality between α_i and β_j valid whenever i and j are in the same segment. (Note that the final inequality, Lemma 3.5, is in the opposite direction to the starting point of Lemma 3.2.)

Lemma 3.3. *For all $k \in \{1, \dots, n\}$,*

$$\alpha_k \ll D\beta_k.$$

Proof. The key point is that there exists a leading entry (i, j) such that

$$j \le k \le i.$$

To prove this, observe that since γ is invertible there must be some $i \geq k$ such that the i-th row of γ contains a nonzero entry in or to the left of the k-th column. Choosing j to be the index of the left-most nonzero entry in the i-th row of γ gives the required leading entry.

Taking such a leading entry (i, j), we can use Lemma 3.2 (for the middle inequality) and the definition of A_t (for the outer inequalities) to prove that

$$\alpha_k \ll \alpha_j \ll D\beta_i \ll D\beta_k. \qquad \square$$

Lemma 3.4. *For every set $J \subset \{1, \dots, n\}$,*

$$\prod_{j \in J} \beta_j \ll N D^{n-\#J} \prod_{j \in J} \alpha_j.$$

Proof. Because α and β are diagonal matrices with positive diagonal entries,

$$\prod_{j \in J} \beta_j \cdot \det \alpha = \prod_{j \in J} \beta_j \cdot \prod_{k=1}^{n} \alpha_k \ll D^{n-\#J} \prod_{j \in J} \alpha_j \cdot \prod_{k=1}^{n} \beta_k = D^{n-\#J} \prod_{j \in J} \alpha_j \cdot \det \beta \tag{10}$$

where the middle inequality uses Lemma 3.3 for all indices $k \in \{1, \dots, n\} \setminus J$.

All of μ, ν and κ have determinant ± 1. Hence (3) implies that

$$\det \beta = N \det \alpha.$$

Combining this with inequality (10) proves the lemma. □

Lemma 3.5. *If i and j are in the same segment, then*

$$\beta_j \ll N D^{n-1} \alpha_i.$$

Proof. If $i \leq j$, then we apply Lemma 3.4 to the singleton $\{j\}$ to obtain

$$\beta_j \ll N D^{n-1} \alpha_j.$$

Combining this with $\alpha_j \ll \alpha_i$ proves the lemma in the case $i \leq j$.

Otherwise, $i > j$ so we can use Lemma 3.1 to find a sequence of leading entries $(i_1, j_1), \dots, (i_s, j_s)$ satisfying condition (∗). We may assume that $i_1, \dots, i_s$ are distinct — otherwise we could simply delete the subsequence between two occurrences of the same i_p. Similarly, we may assume that none of $i_1, \dots, i_s$ are equal to j.

Therefore we can apply Lemma 3.4 to the set $\{i_1, \dots, i_s, j\}$ to get

$$\beta_j \prod_{p=1}^{s} \beta_{i_p} \ll N D^{n-(s+1)} \alpha_j \prod_{p=1}^{s} \alpha_{i_p}. \tag{11}$$

For each $p \in \{1, \dots, s-1\}$, the fact that $j_p \leq i_{p+1}$ and Lemma 3.2 tell us that

$$\alpha_{i_{p+1}} \ll \alpha_{j_p} \ll D \beta_{i_p}.$$

Similarly because $j_s \leq j$ we have

$$\alpha_j \ll \alpha_{j_s} \ll D \beta_{i_s}.$$

Multiplying these inequalities together and also multiplying by β_j gives the first inequality below, while (11) gives the second:

$$\beta_j \alpha_j \prod_{p=2}^{s} \alpha_{i_p} \ll D^s \beta_j \prod_{p=1}^{s} \beta_{i_p} \ll N D^{n-1} \alpha_j \prod_{p=1}^{s} \alpha_{i_p}.$$

Canceling $\alpha_j \prod_{p=2}^{s} \alpha_{i_p}$ shows that

$$\beta_j \ll N D^{n-1} \alpha_{i_1}.$$

Since $i \leq i_1$, we have $\alpha_{i_1} \ll \alpha_i$. This completes the proof of the lemma. □

3E. *Conclusion of proof for standard Siegel sets.* In the final stage of the proof, we expand out (2). When we do this, we get terms of the form $\beta_p \kappa_{pq} \alpha_q^{-1}$. In order to bound this using Lemma 3.5, we need to know that κ_{pq} is zero if p and q are not in the same segment. In other words we have to begin by proving that κ is in the group $\boldsymbol{L}(\mathbb{R})$ of block diagonal matrices.

Lemma 3.6. $\kappa \in \boldsymbol{L}(\mathbb{R})$.

Proof. By construction, γ, μ, α, ν, β are all in the group $\boldsymbol{Q}(\mathbb{R})$ of block upper triangular matrices. Hence (2) tells us that also $\kappa \in \boldsymbol{Q}(\mathbb{R})$.

If a matrix is both block upper triangular and orthogonal, then it is block diagonal according to the same blocks (because the inverse-transpose of a block upper triangular matrix is block lower triangular). In other words,

$$\boldsymbol{Q}(\mathbb{R}) \cap K \subset \boldsymbol{L}(\mathbb{R}).$$

This proves the lemma. □

Lemma 3.7. *For all $i, j \in \{1, \ldots, n\}$, we have*

$$|\gamma_{ij}| \ll N D^{n-1}.$$

Proof. We expand out the matrix product in (2), which we recall:

$$\gamma = \nu \beta \kappa \alpha^{-1} \mu^{-1}.$$

Because α and β are diagonal, the pq-th entry of $\beta \kappa \alpha^{-1}$ is equal to

$$\beta_p \kappa_{pq} \alpha_q^{-1}.$$

If p and q are not in the same segment, then Lemma 3.6 tells us that $\kappa_{pq} = 0$. On the other hand if p and q are in the same segment, then we can apply Lemma 3.5 to bound $\beta_p \alpha_q^{-1}$. Furthermore, because κ is in the compact subgroup K, there is a uniform upper bound for entries of κ. We conclude that

$$\beta_p \kappa_{pq} \alpha_q^{-1} \ll N D^{n-1}. \tag{12}$$

Because μ and ν are in the fixed compact set Ω_u and because all elements of Ω_u are invertible, there is a uniform upper bound for entries of ν and of μ^{-1}. Thus inequality (12) together with (2) implies the lemma. □

To complete the proof of Theorem 1.1 for standard Siegel sets in GL_n, we just have to note that the definition of $\mathrm{H}(\gamma)$ implies that

$$\mathrm{H}(\gamma) \leq D \max(1, |\gamma_{ij}|)$$

where the maximum is over all indices $(i, j) \in \{1, \dots, n\}^2$. Hence Lemma 3.7 implies that

$$\mathrm{H}(\gamma) \leq \max(D, C_5 N D^n),$$

where C_5 denotes the implied constant from Lemma 3.7.

3F. ***Deducing general case from standard Siegel sets.*** To complete the proof of Theorem 1.1, we deduce the general statement from the case of standard Siegel sets in GL_n. This has two steps. Lemma 3.8 allows us to generalise from standard Siegel sets to arbitrary Siegel sets in GL_n. Theorem 1.2 (proved in Section 4) allows us to generalise from GL_n to arbitrary reductive groups $\boldsymbol{G}$.

Lemma 3.8. *Let $\mathfrak{S}$ be a Siegel set in $\mathrm{GL}_n(\mathbb{R})$. Then there exist $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ and $\sigma \in \mathrm{GL}_n(\mathbb{R})$ such that $\gamma^{-1}.\mathfrak{S}.\gamma\sigma$ is contained in a standard Siegel set.*

Proof. Let $(\boldsymbol{P}, \boldsymbol{S}, K)$ be the Siegel triple associated with the Siegel set $\mathfrak{S}$, and write $\mathfrak{S} = \Omega.A_t.K$ using the notation of Section 2B.

Let $(\boldsymbol{P}_0, \boldsymbol{S}_0, K_0)$ be the standard Siegel triple in GL_n. Write $A_{0,t}$ and $\Omega_{0,u}$ for the sets called A_t and Ω_u in the definition of standard Siegel sets.

Since $\boldsymbol{P}$ and $\boldsymbol{P}_0$ are minimal $\mathbb{Q}$-parabolic subgroups of GL_n, there exists $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ such that $\boldsymbol{P}_0 = \gamma^{-1}\boldsymbol{P}\gamma$.

Since K_0 and $\gamma^{-1}K\gamma$ are maximal compact subgroups of $\mathrm{GL}_n(\mathbb{R})$, there exists $\sigma \in \mathrm{GL}_n(\mathbb{R})$ such that $\gamma^{-1}K\gamma = \sigma K_0 \sigma^{-1}$. Applying the Iwasawa decomposition

$$\mathrm{GL}_n(\mathbb{R}) = \boldsymbol{U}_0(\mathbb{R}).\boldsymbol{S}_0(\mathbb{R})^+.K_0,$$

we may assume that $\sigma = \tau\beta$ where $\beta \in \boldsymbol{S}_0(\mathbb{R})^+$ and $\tau \in \boldsymbol{U}_0(\mathbb{R})$.

Under this assumption, $\sigma \in \boldsymbol{P}_0(\mathbb{R})$. Hence $\sigma^{-1}\gamma^{-1}.\boldsymbol{P}.\gamma\sigma = \boldsymbol{P}_0$. By Lemma 2.1, $\sigma^{-1}\gamma^{-1}.\boldsymbol{S}.\gamma\sigma = \boldsymbol{S}_0$. Thus $\sigma^{-1}\gamma^{-1}.A_t.\gamma\sigma = A_{0,t}$.

Now

$$\gamma^{-1}\mathfrak{S}\gamma\sigma = \gamma^{-1}\Omega\gamma.\sigma.\sigma^{-1}\gamma^{-1}A_t\gamma\sigma.\sigma^{-1}\gamma^{-1}K\gamma\sigma = \gamma^{-1}\Omega\gamma.\tau\beta.A_{0,t}.K_0.$$

Here $\gamma^{-1}\Omega\gamma\tau$ is a compact subset of $\boldsymbol{U}_0(\mathbb{R})$ so it is contained in $\Omega_{0,u}$ for a suitable $u > 0$. Meanwhile $\beta.A_{0,t}$ is contained in $A_{0,s}$ for a suitable $s > 0$. Thus $\gamma^{-1}\mathfrak{S}\gamma\sigma$ is contained in the standard Siegel set $\Omega_{0,u}.A_{0,s}.K_0$, as required. □

4. Siegel sets and subgroups

In this section we prove Theorem 1.2. The proof gives additional information on the relationship between the Siegel triples for $\boldsymbol{G}$ and $\boldsymbol{H}$, as follows.

Theorem 4.1. *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be reductive $\mathbb{Q}$-algebraic groups, with $\boldsymbol{H} \subset \boldsymbol{G}$. Let $\mathfrak{S}_{\boldsymbol{H}}$ be a Siegel set in $\boldsymbol{H}(\mathbb{R})$ with respect to the Siegel triple $(\boldsymbol{P}_{\boldsymbol{H}}, \boldsymbol{S}_{\boldsymbol{H}}, K_{\boldsymbol{H}})$. Then there exist a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that*

$$\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}.$$

Furthermore if $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ *denotes the Siegel triple associated with* $\mathfrak{S}_{\boldsymbol{G}}$*, then* $R_u(\boldsymbol{P_H}) \subset R_u(\boldsymbol{P_G})$, $\boldsymbol{S_H} = \boldsymbol{S_G} \cap \boldsymbol{H}$ *and* $K_{\boldsymbol{H}} = K_{\boldsymbol{G}} \cap \boldsymbol{H}(\mathbb{R})$.

We denote sets used in the construction of the Siegel sets $\mathfrak{S}_{\boldsymbol{G}}$ and $\mathfrak{S}_{\boldsymbol{H}}$ by the notation from Section 2B with the subscript $\boldsymbol{G}$ or $\boldsymbol{H}$ added as appropriate. Thus we write

$$\mathfrak{S}_{\boldsymbol{H}} = \Omega_{\boldsymbol{H}}.A_{\boldsymbol{H},t}.K_{\boldsymbol{H}},$$

where $\Omega_{\boldsymbol{H}}$ is a compact subset of $\boldsymbol{U_H}(\mathbb{R})\boldsymbol{M_H}(\mathbb{R})^+$, $K_{\boldsymbol{H}}$ is a maximal compact subgroup of $\boldsymbol{H}(\mathbb{R})$ and

$$A_{\boldsymbol{H},t} = \{\alpha \in \boldsymbol{S_H}(\mathbb{R})^+ : \chi(\alpha) \geq t \text{ for all } \chi \in \Delta_{\boldsymbol{H}}\}.$$

4A. ***Reduction to a split torus*** $\boldsymbol{S_H}$***.*** We begin by reducing the proof of Theorem 4.1 to the case in which the torus $\boldsymbol{S_H}$ is $\mathbb{Q}$-split. Note that, even when $\boldsymbol{S_H}$ is $\mathbb{Q}$-split, it is not always possible to choose a $\mathbb{Q}$-split torus for $\boldsymbol{S_G}$.

According to the definition of a Siegel set, we can choose $u \in \boldsymbol{P_H}(\mathbb{R})$ such that $u\boldsymbol{S_H}u^{-1}$ is a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_H}$. Using the Levi decomposition $\boldsymbol{P_H} = Z_{\boldsymbol{H}}(\boldsymbol{S_H}) \ltimes \boldsymbol{U_H}$, we may assume that $u \in \boldsymbol{U_H}(\mathbb{R})$.

Now $\Omega_{\boldsymbol{H}}u^{-1}$ is a compact subset of $\boldsymbol{U_H}(\mathbb{R}).u\boldsymbol{M_H}(\mathbb{R})^+u^{-1}$ so

$$\mathfrak{S}_{\boldsymbol{H}}.u^{-1} = \Omega_{\boldsymbol{H}}u^{-1}.uA_{\boldsymbol{H},t}u^{-1}.uK_{\boldsymbol{H}}u^{-1}.$$

is a Siegel set with respect to the Siegel triple $(\boldsymbol{P_H}, u\boldsymbol{S_H}u^{-1}, uK_{\boldsymbol{H}}u^{-1})$.

We prove below that Theorem 4.1 holds when $\boldsymbol{S_H}$ is $\mathbb{Q}$-split. Hence there exist a Siegel set $\mathfrak{S}'_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that

$$\mathfrak{S}_{\boldsymbol{H}}.u^{-1} \subset C.\mathfrak{S}'_{\boldsymbol{G}}.$$

Let $(\boldsymbol{P_G}, \boldsymbol{S}'_{\boldsymbol{G}}, K'_{\boldsymbol{G}})$ denote the Siegel triple associated with $\mathfrak{S}'_{\boldsymbol{G}}$. According to Theorem 4.1, $\boldsymbol{U_H} \subset R_u(\boldsymbol{P_G})$ and so $u \in R_u(\boldsymbol{P_G})(\mathbb{R})$. Therefore

$$\mathfrak{S}_{\boldsymbol{G}} = \mathfrak{S}'_{\boldsymbol{G}}.u$$

is a Siegel set for $\boldsymbol{G}(\mathbb{R})$ with respect to the Siegel triple $(\boldsymbol{P_G}, u^{-1}\boldsymbol{S}'_{\boldsymbol{G}}u, u^{-1}K'_{\boldsymbol{G}}u)$. We clearly have $\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}$ and the Siegel triple associated with $\mathfrak{S}_{\boldsymbol{G}}$ satisfies the conditions of Theorem 4.1 relative to $(\boldsymbol{P_H}, \boldsymbol{S_H}, K_{\boldsymbol{H}})$.

4B. ***Choosing the Siegel triple.*** We henceforth assume that $\boldsymbol{S_H}$ is $\mathbb{Q}$-split. As the first step in proving Theorem 4.1 for this case, we choose a Siegel triple $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ for $\boldsymbol{G}$.

The main difficulty lies in choosing $\boldsymbol{P_G}$. The obvious idea is to choose a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ which contains $\boldsymbol{P_H}$, but such a subgroup does not always exist (for example, if $\boldsymbol{G}$ is $\mathbb{Q}$-split and $\boldsymbol{H}$ is $\mathbb{Q}$-anisotropic). Instead we construct a larger parabolic $\mathbb{Q}$-subgroup $\boldsymbol{Q} \subset \boldsymbol{G}$ which contains $\boldsymbol{P_H}$, and then define $\boldsymbol{P_G}$ to be a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{Q}$.

Let us write

$$\boldsymbol{Z} = Z_{\boldsymbol{G}}(\boldsymbol{S_H}).$$

Lemma 4.2. *There exists a parabolic $\mathbb{Q}$-subgroup $\boldsymbol{Q} \subset \boldsymbol{G}$ such that*

(i) *$\boldsymbol{Z}$ is a Levi subgroup of $\boldsymbol{Q}$, and*

(ii) *$\boldsymbol{U_H} \subset R_u(\boldsymbol{Q})$.*

Proof. Let $\Phi_{\boldsymbol{H}}^+$ denote the set of roots $\Phi(\boldsymbol{S_H}, \boldsymbol{P_H})$. By [Borel and Tits 1965, Proposition 3.1] there exists an order $>_{\boldsymbol{Q}}$ on $X^*(\boldsymbol{S_H})$ with respect to which all elements of $\Phi_{\boldsymbol{H}}^+$ are positive.

Let

$$\Phi_{\boldsymbol{Q}} = \{\chi \in \Phi(\boldsymbol{S_H}, \boldsymbol{G}) : \chi >_{\boldsymbol{Q}} 0\}$$

and let $\boldsymbol{Q}$ denote the group $\boldsymbol{G}_{\Phi_{\boldsymbol{Q}}}$ (using the notation of [loc. cit., Paragraph 3.8] with respect to the torus $\boldsymbol{S_H}$). By [loc. cit., Théorème 4.15], $\boldsymbol{Q}$ is a parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ and $\boldsymbol{Z}$ is a Levi subgroup of $\boldsymbol{Q}$.

Since all weights of $\boldsymbol{S_H}$ on $\boldsymbol{U_H}$ are contained in $\Phi_{\boldsymbol{H}}^+$, which is a subset of $\Phi_{\boldsymbol{Q}}$, [loc. cit., Proposition 3.12] tells us that $\boldsymbol{U_H} \subset \boldsymbol{G}^*_{\Phi_{\boldsymbol{Q}}}$, again using the notation of [loc. cit., Paragraph 3.8]. By [loc. cit., Théorème 3.13], $\boldsymbol{G}^*_{\Phi_{\boldsymbol{Q}}} = R_u(\boldsymbol{Q})$. This completes the proof that $\boldsymbol{U_H} \subset R_u(\boldsymbol{Q})$. □

We will make no use of the following lemma, but it sheds some light on the significance of the group $\boldsymbol{Q}$.

Lemma 4.3. $\boldsymbol{P_H} = \boldsymbol{Q} \cap \boldsymbol{H}$.

Proof. We use the notation from the proof of Lemma 4.2. By construction, we have that $\Phi(\boldsymbol{S_H}, \boldsymbol{P_H}) = \Phi_{\boldsymbol{H}}^+ \subset \Phi_{\boldsymbol{Q}}$. Hence by [Borel and Tits 1965, Proposition 3.12], $\boldsymbol{P_H} \subset \boldsymbol{G}_{\Phi_{\boldsymbol{Q}}} = \boldsymbol{Q}$.

For the reverse inclusion, observe that $\Phi(\boldsymbol{S_H}, \boldsymbol{Q} \cap \boldsymbol{H}) \subset \Phi_{\boldsymbol{H}}^+$. Hence applying [loc. cit., Proposition 3.12], this time inside $\boldsymbol{H}$, we get

$$\boldsymbol{Q} \cap \boldsymbol{H} \subset \boldsymbol{H}_{\Phi_{\boldsymbol{H}}^+} = \boldsymbol{P_H}.$$ □

Choose the following subgroups of $\boldsymbol{G}$:

(1) $\boldsymbol{P_G}$, a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{Q}$.

(2) $K_{\boldsymbol{G}}$, a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$ containing $K_{\boldsymbol{H}}$.

Define the following notation for subgroups of $\boldsymbol{G}$ which are uniquely determined by $\boldsymbol{P_G}$ and $K_{\boldsymbol{G}}$:

(1) $\boldsymbol{S_G}$ is the unique torus such that $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ is a Siegel triple for $\boldsymbol{G}$.

(2) $\boldsymbol{U_G} = R_u(\boldsymbol{P_G})$.

(3) $\boldsymbol{P_Z} = \boldsymbol{P_G} \cap \boldsymbol{Z}$ and $\boldsymbol{U_Z} = R_u(\boldsymbol{P_Z})$.

(4) $K_{\boldsymbol{Z}} = K_{\boldsymbol{G}} \cap \boldsymbol{Z}(\mathbb{R})$.

Lemma 4.4. *$K_{\boldsymbol{Z}}$ is a maximal compact subgroup of $\boldsymbol{Z}(\mathbb{R})$.*

Proof. Let Θ be the Cartan involution of $\boldsymbol{G}$ associated with the maximal compact subgroup $K_{\boldsymbol{G}}$. Because $K_{\boldsymbol{H}} = K_{\boldsymbol{G}} \cap \boldsymbol{H}(\mathbb{R})$, Θ restricts to the Cartan involution of $\boldsymbol{H}$ associated with $K_{\boldsymbol{H}}$.

From the definition of Siegel triple applied to $(\boldsymbol{P_H}, \boldsymbol{S_H}, K_{\boldsymbol{H}})$, Θ stabilizes $\boldsymbol{S_H}$. Hence Θ also stabilizes $\boldsymbol{Z}$. Therefore the fixed points of Θ in $\boldsymbol{Z}(\mathbb{R})$, namely $K_{\boldsymbol{Z}}$, form a maximal compact subgroup of $\boldsymbol{Z}(\mathbb{R})$. □

Lemma 4.5. $\boldsymbol{S_H} \subset \boldsymbol{S_G}$.

Proof. Note that $\boldsymbol{Z}$ is a reductive group defined over $\mathbb{Q}$, because $\boldsymbol{S_H}$ is defined over $\mathbb{Q}$. Thus it makes sense to talk about Siegel triples in $\boldsymbol{Z}$. By [Borel and Tits 1965, Proposition 4.4], $\boldsymbol{P_Z}$ is a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{Z}$.

By Lemma 2.1, there exists a unique torus $\boldsymbol{S_Z} \subset \boldsymbol{Z}$ such that $(\boldsymbol{P_Z}, \boldsymbol{S_Z}, K_{\boldsymbol{Z}})$ is a Siegel triple for $\boldsymbol{Z}$. This means that:

(i) $\boldsymbol{S_Z}$ is $\boldsymbol{P_Z}(\mathbb{R})$-conjugate to a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_Z}$. Note that a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_Z}$ is also a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_G}$.

(ii) The Cartan involution of $\boldsymbol{Z}$ associated with $K_{\boldsymbol{Z}}$ normalises $\boldsymbol{S_Z}$. This involution is the restriction of the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$.

Thus $\boldsymbol{S_Z}$ satisfies the conditions of Lemma 2.1 with respect to $(\boldsymbol{P_G}, K_{\boldsymbol{G}})$. By the uniqueness in Lemma 2.1, we conclude that $\boldsymbol{S_Z} = \boldsymbol{S_G}$.

Because $\boldsymbol{S_Z}$ is $\boldsymbol{Z}(\mathbb{R})$-conjugate to a maximal $\mathbb{Q}$-split torus in $\boldsymbol{Z}$, it contains every $\mathbb{Q}$-split subtorus of the centre of $\boldsymbol{Z}$. In particular $\boldsymbol{S_H} \subset \boldsymbol{S_Z}$. □

Let $\boldsymbol{S}'_{\boldsymbol{G}}$ be a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_Z}$. Because $(\boldsymbol{P_Z}, \boldsymbol{S_Z}, K_{\boldsymbol{Z}})$ is a Siegel triple, there exists $u \in \boldsymbol{P_Z}(\mathbb{R})$ such that $\boldsymbol{S}'_{\boldsymbol{G}} = u\boldsymbol{S}'_{\boldsymbol{Z}}u^{-1}$. Because of the Levi decomposition $\boldsymbol{P_Z} = Z_{\boldsymbol{G}}(\boldsymbol{S_G}) \ltimes \boldsymbol{U_Z}$, we may assume that $u \in \boldsymbol{U_Z}(\mathbb{R})$.

The following lemma is not needed in our proof of Theorem 1.2, but it contains extra information about $\boldsymbol{S_G}$ which is included in the statement of Theorem 4.1.

Lemma 4.6. $\boldsymbol{S_H} = \boldsymbol{S_G} \cap \boldsymbol{H}$.

Proof. Let q denote the quotient map $\boldsymbol{P_G} \to \boldsymbol{P_G}/\boldsymbol{U_G}$. Observe that $\boldsymbol{U_G} \cap \boldsymbol{P_H}$ is a normal unipotent subgroup of $\boldsymbol{P_H}$, so it is contained in $\boldsymbol{U_H}$. On the other hand,

$$\boldsymbol{U_H} \subset R_u(\boldsymbol{Q}) \cap \boldsymbol{P_H} \subset \boldsymbol{U_G} \cap \boldsymbol{P_H}.$$

Hence $\boldsymbol{U_G} \cap \boldsymbol{P_H} = \boldsymbol{U_H}$, so q restricts to the quotient map $\boldsymbol{P_H} \to \boldsymbol{P_H}/\boldsymbol{U_H}$.

According to the definition of a Siegel triple, $q(\boldsymbol{S_G})$ is a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_G}/\boldsymbol{U_G}$. Furthermore, $\boldsymbol{S_G} \cap \boldsymbol{H} \subset \boldsymbol{Q} \cap \boldsymbol{H} = \boldsymbol{P_H}$. Hence $q(\boldsymbol{S_G} \cap \boldsymbol{H})$ is a $\mathbb{Q}$-split torus in $\boldsymbol{P_H}/\boldsymbol{U_H}$.

Since $\boldsymbol{S_H} \subset \boldsymbol{S_G} \cap \boldsymbol{H}$ and $q(\boldsymbol{S_H})$ is a maximal $\mathbb{Q}$-split torus in $\boldsymbol{P_H}/\boldsymbol{U_H}$, we conclude that $q(\boldsymbol{S_H}) = q(\boldsymbol{S_G} \cap \boldsymbol{H})$. Because $\boldsymbol{S_G} \cap \boldsymbol{U_G} = \{1\}$, $q_{|\boldsymbol{S_G}}$ is injective. Thus $\boldsymbol{S_H} = \boldsymbol{S_G} \cap \boldsymbol{H}$. □

4C. ***Comparing $A_{\boldsymbol{H},t}$ with $A_{\boldsymbol{G},t'}$.*** We now compare the sets $A_{\boldsymbol{H},t} \subset \boldsymbol{S_H}(\mathbb{R})$ and $A_{\boldsymbol{G},t'} \subset \boldsymbol{S_G}(\mathbb{R})$. We would like to have $A_{\boldsymbol{H},t} \subset A_{\boldsymbol{G},t'}$, but it is not always possible to choose $t' \in \mathbb{R}_{>0}$ such that this holds. This is because there may be simple roots in $\Phi(\boldsymbol{S_G}, \boldsymbol{G})$ whose restrictions to $\boldsymbol{S_H}$ are not positive combinations of simple roots in $\Phi(\boldsymbol{S_H}, \boldsymbol{H})$. The values of such a root are bounded below by a positive constant on $A_{\boldsymbol{G},t'}$ but can be arbitrarily close to zero on $A_{\boldsymbol{H},t}$.

Instead we show that for a suitable value of t', every $\alpha \in A_{\boldsymbol{H},t'}$ can be conjugated into $A_{\boldsymbol{G},t'}$ by an element of the Weyl group $N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$. This element of the Weyl group must also satisfy certain other conditions which will be used later in the proof of Theorem 4.1.

Write

$$W = N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G}) \quad \text{and} \quad W' = N_{\boldsymbol{G}}(\boldsymbol{S'_G})/Z_{\boldsymbol{G}}(\boldsymbol{S'_G}).$$

Since $\boldsymbol{S'_G} = u\boldsymbol{S_G}u^{-1}$, conjugation by u induces an isomorphism $W \to W'$.

Proposition 4.7. *There exists $t' > 0$ (depending only on $\boldsymbol{G}$, $\boldsymbol{H}$, and t) such that for every $\alpha \in A_{\boldsymbol{H},t}$, there exists $w \in W$ such that*:

(i) $\boldsymbol{U_Z} \subset w\boldsymbol{U_G}w^{-1}$.

(ii) $\boldsymbol{U_H} \subset w\boldsymbol{U_G}w^{-1}$.

(iii) $\alpha \in wA_{\boldsymbol{G},t}w^{-1}$.

Note that the statement of the proposition makes sense because $w\boldsymbol{U_G}w^{-1}$ and $wA_{\boldsymbol{G},t'}w^{-1}$ do not depend on the choice of representative of w in $N_{\boldsymbol{G}}(\boldsymbol{S_G})$.

Construction of $\boldsymbol{Q}_\alpha$. Suppose that we are given $\alpha \in A_{\boldsymbol{H},t}$. In order to find $w \in W$ as in Proposition 4.7, we construct a parabolic subgroup $\boldsymbol{P}_{\boldsymbol{G},\alpha} = w\boldsymbol{P_G}w^{-1}$ by a refinement of the construction of $\boldsymbol{P_G}$ from Section 4B. First we construct a larger parabolic subgroup $\boldsymbol{Q}_\alpha$ which satisfies conditions (i) and (ii) from Lemma 4.2, as well as the following additional condition:

(iii) There exists $t' > 0$ (independent of α) such that, for every $\alpha \in A_{\boldsymbol{H},t}$ and every $\chi \in \Phi(\boldsymbol{S_H}, \boldsymbol{Q}_\alpha)$, $\chi(\alpha) \geq t'$.

Similar to the proof of Lemma 4.2, we construct $\boldsymbol{Q}_\alpha$ by choosing a suitable order $>_\alpha$ on $X^*(\boldsymbol{S_H})$.

Given $\alpha \in \boldsymbol{S_H}(\mathbb{R})^+$, choose a set $\Psi_\alpha \subset \Phi(\boldsymbol{S_H}, \boldsymbol{G})$ which is maximal with respect to the following conditions:

(a) The set $\Phi_{\boldsymbol{H}}^+ \cup \Psi_\alpha$ is $\mathbb{R}_{>0}$-independent. (Recall that $\Phi_{\boldsymbol{H}}^+ = \Phi(\boldsymbol{S_H}, \boldsymbol{P_H})$.)

(b) For all $\chi \in \Psi_\alpha$, $\chi(\alpha) \geq 1$.

There always exists at least one set satisfying conditions (a) and (b), namely the empty set. Since $\Phi(\boldsymbol{S_H}, \boldsymbol{G})$ is finite, we deduce that there is a maximal set Ψ_α satisfying the conditions.

By (a) there exists an order $>_\alpha$ on $X^*(\boldsymbol{S_H})$ with respect to which all elements of $\Phi_{\boldsymbol{H}}^+ \cup \Psi_\alpha$ are positive. Let

$$\Phi_\alpha = \{\chi \in \Phi(\boldsymbol{S_H}, \boldsymbol{G}) : \chi >_\alpha 0\}$$

and let $\boldsymbol{Q}_\alpha = \boldsymbol{G}_{\Phi_\alpha}$ (in the notation of [Borel and Tits 1965, Paragraph 3.8] with respect to $\boldsymbol{S_H}$).

The only condition on the order $>_{\boldsymbol{Q}}$ in the proof of Lemma 4.2 was that all elements of $\Phi_{\boldsymbol{H}}^+$ are positive with respect to $>_{\boldsymbol{Q}}$. By definition, $>_\alpha$ satisfies this condition. Hence the proof of Lemma 4.2

also applies to $\boldsymbol{Q}_\alpha$. We conclude that $\boldsymbol{Q}_\alpha$ is a parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ satisfying conclusions (i) and (ii) of Lemma 4.2.

Lemma 4.8. *Every root $\chi \in \Phi_\alpha$ is a $\mathbb{R}_{>0}$-combination of $\Delta_{\boldsymbol{H}} \cup \Psi_\alpha$.*

Proof. If $\chi \in \Psi_\alpha$, the result is trivial. So we may assume that $\chi \notin \Psi_\alpha$.

Since $\chi >_\alpha 0$, $\Psi_\alpha \cup \{\chi\}$ satisfies (a). Since $\chi \notin \Psi_\alpha$, the maximality of Ψ_α tells us that $\Psi_\alpha \cup \{\chi\}$ does not satisfy (b). Thus $\chi(\alpha) < 1$.

Hence $\Psi_\alpha \cup \{-\chi\}$ satisfies (b). But $-\chi <_\alpha 0$, so $-\chi \notin \Psi_\alpha$. Again by the maximality of Ψ_α, we conclude that $\Psi_\alpha \cup \{-\chi\}$ does not satisfy (a). Thus there exist $m_i, n_j, x \in \mathbb{R}_{>0}$, $\chi_i \in \Phi_{\boldsymbol{H}}^+$ and $\psi_j \in \Psi_\alpha$ such that

$$\sum_i m_i \chi_i + \sum_j n_j \psi_j + x(-\chi) = 0.$$

(The coefficient of $-\chi$ in this equation must be nonzero because $\Phi_{\boldsymbol{H}}^+ \cup \Psi_\alpha$ is $\mathbb{R}_{>0}$-independent.)

We can rearrange this equation to write χ as a $\mathbb{R}_{>0}$-combination of $\Phi_{\boldsymbol{H}}^+ \cup \Psi_\alpha$. Since every element of $\Phi_{\boldsymbol{H}}^+$ is a $\mathbb{R}_{>0}$-combination of elements of $\Delta_{\boldsymbol{H}}$, we deduce that χ is a $\mathbb{R}_{>0}$-combination of $\Delta_{\boldsymbol{H}} \cup \Psi_\alpha$. □

Lemma 4.9. *There exists $t' > 0$ (depending on $\boldsymbol{G}$, $\boldsymbol{H}$ and t but not on α) such that for every $\alpha \in A_{\boldsymbol{H},t}$ and every $\chi \in \Phi_\alpha$, $\chi(\alpha) \geq t'$.*

Proof. Consider all pairs (χ, Ξ) where $\chi \in \Phi_{\boldsymbol{G}}$ and Ξ is a subset of $\Phi_{\boldsymbol{G}}$ such that χ can be written as a $\mathbb{R}_{>0}$-combination of elements of Ξ. There are only finitely many such pairs, so we can find M (depending only on the root system $\Phi_{\boldsymbol{G}}$) such that, for every such pair, there exist $m_i \in \mathbb{R}_{>0}$ and $\xi_i \in \Xi$ satisfying

$$\chi = \sum_i m_i \xi_i \quad \text{and} \quad \sum_i m_i \leq M.$$

Suppose that $\chi \in \Phi_\alpha$. Using Lemma 4.8, we can write χ as a combination

$$\chi = \sum_i m_i \chi_i + \sum_j n_j \psi_j$$

where $\chi_i \in \Delta_{\boldsymbol{H}}$, $\psi_j \in \Psi_\alpha$, $m_i, n_i \in \mathbb{R}_{>0}$. By the definition of M, we may assume that $\sum_i m_i + \sum_j n_j \leq M$.

By the definition of $A_{\boldsymbol{H},t}$, we have $\chi_i(\alpha) \geq t$ for all i. By condition (b) on Ψ_α, we have $\psi_j(\alpha) \geq 1$ for all j. Therefore $\chi(\alpha) \geq \min(1, t)^M$. □

Proof of Proposition 4.7. Because $\boldsymbol{Q}_\alpha$ satisfies conclusion (i) of Lemma 4.2, $\boldsymbol{Z}$ is a Levi subgroup of $\boldsymbol{Q}_\alpha$. Let $\boldsymbol{P}_{\boldsymbol{G},\alpha} = \boldsymbol{P}_{\boldsymbol{Z}} \ltimes R_u(\boldsymbol{Q}_\alpha)$. By [Borel and Tits 1965, Proposition 4.4], $\boldsymbol{P}_{\boldsymbol{G},\alpha}$ is a minimal $\mathbb{Q}$-parabolic subgroup of $\boldsymbol{G}$.

By [Borel and Tits 1965, Corollaire 5.9], the Weyl group W' acts transitively on the minimal parabolic $\mathbb{Q}$-subgroups of $\boldsymbol{G}$ containing the maximal $\mathbb{Q}$-split torus $\boldsymbol{S}'_{\boldsymbol{G}}$. Since $\boldsymbol{S}'_{\boldsymbol{G}} \subset \boldsymbol{P}_{\boldsymbol{Z}} \subset \boldsymbol{P}_{\boldsymbol{G},\alpha}$, we conclude that there exists $w' \in W'$ (depending on α) such that $\boldsymbol{P}_{\boldsymbol{G},\alpha} = w' \boldsymbol{P}_{\boldsymbol{G}} w'^{-1}$.

Let w be the element of W which corresponds to $w' \in W'$ via conjugation by u. Since $u \in \boldsymbol{U_Z}(\mathbb{R}) \subset \boldsymbol{P_G}(\mathbb{R}) \cap \boldsymbol{P}_{\boldsymbol{G},\alpha}(\mathbb{R})$, we have

$$\boldsymbol{P}_{\boldsymbol{G},\alpha} = w\boldsymbol{P_G}w^{-1}.$$

Since $\boldsymbol{Q}_\alpha$ satisfies conclusion (ii) of Lemma 4.2, we have

$$\boldsymbol{U_H} \subset R_u(\boldsymbol{Q}_\alpha) \subset R_u(\boldsymbol{P}_{\boldsymbol{G},\alpha}) = w\boldsymbol{U_G}w^{-1}.$$

Furthermore $\boldsymbol{P_Z} \subset \boldsymbol{P}_{\boldsymbol{G},\alpha}$ and so $\boldsymbol{U_Z} \subset R_u(\boldsymbol{P}_{\boldsymbol{G},\alpha})$. This proves conclusions (i) and (ii) of Proposition 4.7.

Since $\boldsymbol{P}_{\boldsymbol{G},\alpha} \subset \boldsymbol{Q}_\alpha$, if $\chi \in \Phi(\boldsymbol{S_G}, \boldsymbol{P}_{\boldsymbol{G},\alpha})$ then $\chi_{|\boldsymbol{S_H}} \in \Phi_\alpha$. Hence by Lemma 4.9,

$$\chi(\alpha) \geq t' \text{ for all } \alpha \in A_{\boldsymbol{H},t} \text{ and } \chi \in \Phi(\boldsymbol{S_G}, \boldsymbol{P}_{\boldsymbol{G},\alpha}).$$

Noting that

$$wA_{\boldsymbol{G},t'}w^{-1} = \{\beta \in \boldsymbol{S_G}(\mathbb{R})^+ : \chi(\beta) \geq t' \text{ for all simple roots of } \boldsymbol{P}_{\boldsymbol{G},\alpha}\}$$

we conclude that $\alpha \in wA_{\boldsymbol{G},t'}w^{-1}$, proving conclusion (iii) of Proposition 4.7. □

4D. *Weyl group representatives.* We need to choose two representatives for each element w in the Weyl group $W = N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$.

Firstly we would like to choose representatives for W in $\boldsymbol{G}(\mathbb{Q})$. However this is not usually possible because the torus $\boldsymbol{S_G}$ is not defined over $\mathbb{Q}$. Instead, recall that conjugation by u induces an isomorphism $W \to W'$. Given $w \in W$, let w' denote the corresponding element of W'. By [Borel and Tits 1965, Théorème 5.3], we can choose $w'_{\mathbb{Q}} \in \boldsymbol{G}(\mathbb{Q})$ which represents w'. We then get a representative for w by setting

$$w_{\mathbb{Q}} = u^{-1}w'_{\mathbb{Q}}u.$$

Secondly we choose representatives for W in $K_{\boldsymbol{G}}$.

Lemma 4.10. *Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group. Let $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$ be a Siegel triple in $\boldsymbol{G}$. Every $w \in N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$ has a representative $w_K \in K_{\boldsymbol{G}}$.*

Proof. Let $\boldsymbol{T_G}$ be a maximal $\mathbb{R}$-split torus in $\boldsymbol{G}$ which contains $\boldsymbol{S_G}$.

Let $\boldsymbol{N} = N_{\boldsymbol{G}}(\boldsymbol{S_G}) \cap N_{\boldsymbol{G}}(\boldsymbol{T_G})$. Because $\boldsymbol{S_G}$ is conjugate to a maximal $\mathbb{Q}$-split torus of $\boldsymbol{G}$, [Borel and Tits 1965, Corollaire 5.5] implies that

$$N_{\boldsymbol{G}}(\boldsymbol{S_G}) = \boldsymbol{N}.Z_{\boldsymbol{G}}(\boldsymbol{S_G}).$$

Therefore we can choose $\sigma \in \boldsymbol{N}(\mathbb{C})$ such that $w = \sigma.Z_{\boldsymbol{G}}(\boldsymbol{S_G})$.

According to the final displayed equation from [Borel and Tits 1965, Section 14], every element of $N_{\boldsymbol{G}}(\boldsymbol{T_G})/Z_{\boldsymbol{G}}(\boldsymbol{T_G})$ has a representative in $K_{\boldsymbol{G}}$. In particular, there exists $w_K \in N_{\boldsymbol{G}}(\boldsymbol{T_G})(\mathbb{R}) \cap K_{\boldsymbol{G}}$ which represents $\sigma.Z_{\boldsymbol{G}}(\boldsymbol{T_G})$. Then

$$w_K\sigma^{-1} \in Z_{\boldsymbol{G}}(\boldsymbol{T_G})(\mathbb{C}) \subset Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{C}).$$

It follows that w_K normalises $\boldsymbol{S_G}$ and represents $w \in N_{\boldsymbol{G}}(\boldsymbol{S_G})/Z_{\boldsymbol{G}}(\boldsymbol{S_G})$. □

Since the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$ stabilizes $\boldsymbol{S_G}$, it also stabilizes $Z_{\boldsymbol{G}}(\boldsymbol{S_G})$. Hence $K_{\boldsymbol{G}} \cap Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$ is a maximal compact subgroup of $Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$. By [Hochschild 1965, Chapter XV, Theorem 3.1], $K_{\boldsymbol{G}} \cap Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$ meets every connected component of $Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})$. When choosing w_K as in Lemma 4.10, we may therefore assume that $w_K \in w_{\mathbb{Q}}.Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})^+$.

We will need the following lemma about $w_{\mathbb{Q}}$ and $w'_{\mathbb{Q}}$. This lemma does not hold for every element of W, so we restrict our attention to elements which satisfy conditions (i) and (ii) of Proposition 4.7, that is, elements of the set

$$W^\dagger = \{w \in W : \boldsymbol{U_Z} \subset w\boldsymbol{U_G}w^{-1} \text{ and } \boldsymbol{U_H} \subset w\boldsymbol{U_G}w^{-1}\}.$$

Lemma 4.11. *If $w \in W^\dagger$, then $w'^{-1}_{\mathbb{Q}} w_{\mathbb{Q}} \in \boldsymbol{U_G}(\mathbb{R})$.*

Proof. By definition,

$$w'^{-1}_{\mathbb{Q}} w_{\mathbb{Q}} = u w_{\mathbb{Q}}^{-1} u^{-1} w_{\mathbb{Q}}.$$

Because $w \in W^\dagger$ and $u \in \boldsymbol{U_Z}(\mathbb{R})$, we have

$$w_{\mathbb{Q}}^{-1} u^{-1} w_{\mathbb{Q}} \in \boldsymbol{U_G}(\mathbb{R}).$$

Multiplying this by $u \in \boldsymbol{U_G}(\mathbb{R})$ proves the lemma. □

4E. ***Construction of the compact set $\Omega_{\boldsymbol{G}}$.*** By the Langlands decomposition in $\boldsymbol{P_H}$, the multiplication map

$$\boldsymbol{U_H}(\mathbb{R}) \times \boldsymbol{M_H}(\mathbb{R})^+ \to \boldsymbol{U_H}(\mathbb{R}).\boldsymbol{M_H}(\mathbb{R})^+$$

is a homeomorphism. Hence there exist compact sets $\Omega_{\boldsymbol{U_H}} \subset \boldsymbol{U_H}(\mathbb{R})$ and $\Omega_{\boldsymbol{M_H}} \subset \boldsymbol{M_H}(\mathbb{R})^+$ such that

$$\Omega_{\boldsymbol{H}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{M_H}}. \tag{13}$$

Since $\boldsymbol{M_H}$ need not be contained in $\boldsymbol{M_G}$, we need to further decompose $\Omega_{\boldsymbol{M_H}}$. Let $\boldsymbol{B_Z}$ be a minimal $\mathbb{R}$-parabolic subgroup of $\boldsymbol{Z} = Z_{\boldsymbol{G}}(\boldsymbol{S_H})$ contained in $\boldsymbol{P_Z}$. By the Iwasawa decomposition in $\boldsymbol{Z}$, the multiplication map $\boldsymbol{B_Z}(\mathbb{R})^+ \times K_{\boldsymbol{Z}} \to \boldsymbol{Z}(\mathbb{R})$ is a homeomorphism so there exists a compact set $\Omega_{\boldsymbol{B_Z}} \subset \boldsymbol{B_Z}(\mathbb{R})^+$ such that

$$\Omega_{\boldsymbol{M_H}} \subset \Omega_{\boldsymbol{B_Z}}.K_{\boldsymbol{Z}}. \tag{14}$$

For each $w \in W^\dagger$, choose w_K, $w_{\mathbb{Q}}$ and $w'_{\mathbb{Q}}$ as in Section 4D. We have $w_K w_{\mathbb{Q}}^{-1} \in Z_{\boldsymbol{G}}(\boldsymbol{S_G})(\mathbb{R})^+ \subset \boldsymbol{P_Z}(\mathbb{R})^+$ and $\boldsymbol{B_Z}(\mathbb{R})^+ \subset \boldsymbol{P_Z}(\mathbb{R})^+$, so $\Omega_{\boldsymbol{B_Z}}.w_K w_{\mathbb{Q}}^{-1}$ is a compact subset of $\boldsymbol{P_Z}(\mathbb{R})^+$. Noting that $Z_{\boldsymbol{G}}(\boldsymbol{S_G})$ is a Levi subgroup of $\boldsymbol{P_Z}$, the Langlands decomposition in $\boldsymbol{P_Z}$ [Borel and Ji 2006, Equation (I.1.8)] tells us that the multiplication map

$$\boldsymbol{U_Z}(\mathbb{R}) \times \boldsymbol{M_G}(\mathbb{R})^+ \times \boldsymbol{S_G}(\mathbb{R})^+ \to \boldsymbol{P_Z}(\mathbb{R})^+$$

is a homeomorphism. Therefore there exist compact sets $\Omega_{\boldsymbol{U_Z}}^{[w]} \subset \boldsymbol{U_Z}(\mathbb{R})$, $\Omega_{\boldsymbol{M_G}}^{[w]} \subset \boldsymbol{M_G}(\mathbb{R})^+$ and $\Omega_{\boldsymbol{S_G}}^{[w]} \subset \boldsymbol{S_G}(\mathbb{R})^+$ such that

$$\Omega_{\boldsymbol{B_Z}}.w_K w_{\mathbb{Q}}^{-1} \subset \Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.\Omega_{\boldsymbol{S_G}}^{[w]}. \tag{15}$$

Let

$$\Omega_{\boldsymbol{G}} = \bigcup_{w \in W^\dagger} w_{\mathbb{Q}}'^{-1}.\Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.w_{\mathbb{Q}}.$$

Since $W^\dagger$ is finite, $\Omega_{\boldsymbol{G}}$ is compact.

Lemma 4.12. $\Omega_{\boldsymbol{G}} \subset \boldsymbol{U_G}(\mathbb{R})\boldsymbol{M_G}(\mathbb{R})^+$.

Proof. For each $w \in W^\dagger$, by Lemma 4.11, $w_{\mathbb{Q}}'^{-1} w_{\mathbb{Q}} \in \boldsymbol{U_G}(\mathbb{R})$. Using the definition of $W^\dagger$, we have

$$w_{\mathbb{Q}}^{-1}\Omega_{\boldsymbol{U_H}} w_{\mathbb{Q}} \subset \boldsymbol{U_G}(\mathbb{R}) \quad \text{and} \quad w_{\mathbb{Q}}^{-1}\Omega_{\boldsymbol{U_Z}}^{[w]} w_{\mathbb{Q}} \subset \boldsymbol{U_G}(\mathbb{R}).$$

Multiplying these together, we conclude that

$$w_{\mathbb{Q}}'^{-1}.\Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{U_Z}}^{[w]}.w_{\mathbb{Q}} \subset \boldsymbol{U_G}(\mathbb{R}). \tag{16}$$

Since $\boldsymbol{S_G}$ is $\boldsymbol{G}(\mathbb{R})$-conjugate to a maximal $\mathbb{Q}$-split torus in $\boldsymbol{G}$, we can use [Borel and Tits 1965, Corollaire 5.4] to show that $\boldsymbol{M_G}$ is normal in $N_{\boldsymbol{G}}(\boldsymbol{S_G})$. It follows that $w_{\mathbb{Q}}$ normalises $\boldsymbol{M_G}(\mathbb{R})^+$ and so

$$w_{\mathbb{Q}}^{-1}\Omega_{\boldsymbol{M_G}}^{[w]} w_{\mathbb{Q}} \subset \boldsymbol{M_G}(\mathbb{R})^+. \tag{17}$$

Combining (16) and (17) proves the lemma. □

Lemma 4.13. *For each $w \in W^\dagger$, $w_{\mathbb{Q}}'^{-1}\Omega_{\boldsymbol{H}} \subset \Omega_{\boldsymbol{G}}.w_K^{-1}.\Omega_{\boldsymbol{S_G}}^{[w]}.K_{\boldsymbol{Z}}$.*

Proof. Noting that $w_{\mathbb{Q}} w_K^{-1}$ commutes with $\boldsymbol{S_G}$, we can rearrange (15) to obtain

$$\Omega_{\boldsymbol{B_Z}} \subset \Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.w_{\mathbb{Q}} w_K^{-1}.\Omega_{\boldsymbol{S_G}}^{[w]}.$$

Combining this with (13) and (14), we get

$$\Omega_{\boldsymbol{H}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{M_H}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{B_Z}}.K_{\boldsymbol{Z}} \subset \Omega_{\boldsymbol{U_H}}.\Omega_{\boldsymbol{U_Z}}^{[w]}.\Omega_{\boldsymbol{M_G}}^{[w]}.w_{\mathbb{Q}} w_K^{-1}.\Omega_{\boldsymbol{S_G}}^{[w]}.K_{\boldsymbol{Z}}.$$

We can now read off the lemma using the definition of $\Omega_{\boldsymbol{G}}$. □

4F. ***The Siegel set for G.*** For each $w \in W^\dagger$, $w_K^{-1}\Omega_{\boldsymbol{S_G}}^{[w]} w_K$ is a compact subset of $\boldsymbol{S_G}(\mathbb{R})^+$. Hence there exists $s > 0$ such that $\chi(\beta) \geq s$ for all $\chi \in \Delta_{\boldsymbol{G}}$ and all $\beta \in w_K^{-1}\Omega_{\boldsymbol{S_G}}^{[w]} w_K$ (since $W^\dagger$ is finite, we can choose a single value of s which works for all $w \in W^\dagger$).

Let $\mathfrak{S}_{\boldsymbol{G}}$ be the Siegel set

$$\mathfrak{S}_{\boldsymbol{G}} = \Omega_{\boldsymbol{G}}.A_{\boldsymbol{G},t's}.K_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R}),$$

using t' from Proposition 4.7 and $\Omega_{\boldsymbol{G}}$ from Section 4E. Let C be the finite set

$$C = \{w_{\mathbb{Q}}' : w \in W^\dagger\} \subset \boldsymbol{G}(\mathbb{Q}).$$

Proposition 4.14. $\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}$.

Proof. Given $\sigma \in \mathfrak{S}_{\boldsymbol{H}}$, we can write

$$\sigma = \mu\alpha\kappa$$

with $\mu \in \Omega_{\boldsymbol{H}}$, $\alpha \in A_{\boldsymbol{H},t}$ and $\kappa \in K_{\boldsymbol{H}}$.

By Proposition 4.7, we can choose $w \in W^{\dagger}$ such that $\alpha \in wA_{\boldsymbol{G},t'}w^{-1}$. By Lemma 4.13, we can write

$$w_{\mathbb{Q}}'^{-1}\mu = \nu w_K^{-1}\beta\lambda$$

where $\nu \in \Omega_{\boldsymbol{G}}$, $\beta \in \Omega_{\boldsymbol{S_G}}^{[w]}$ and $\lambda \in K_{\boldsymbol{Z}}$. Therefore

$$w_{\mathbb{Q}}'^{-1}\sigma = \nu w_K^{-1}\beta\lambda\,\alpha\kappa.$$

Since $\lambda \in K_{\boldsymbol{Z}} \subset \boldsymbol{Z}(\mathbb{R})$, λ commutes with $\alpha \in \boldsymbol{S_H}(\mathbb{R})$ so we can rewrite this as

$$w_{\mathbb{Q}}'^{-1}\sigma = \nu.w_K^{-1}\,\beta\alpha w_K.w_K^{-1}\lambda\kappa.$$

By definition, $\nu \in \Omega_{\boldsymbol{G}}$. By the definition of s, we have $w_K^{-1}\beta w_K \in A_{\boldsymbol{G},s}$ while $w_K^{-1}\alpha w_K \in A_{\boldsymbol{G},t'}$ by Proposition 4.7. Hence

$$w_K^{-1}\beta\alpha w_K \in A_{\boldsymbol{G},t's}.$$

Finally, w_K^{-1}, λ and κ are all in the group $K_{\boldsymbol{G}}$, so their product is also in $K_{\boldsymbol{G}}$.

Thus we have shown that $w_{\mathbb{Q}}'^{-1}\sigma \in \mathfrak{S}_{\boldsymbol{G}}$, and so $\sigma \in C.\mathfrak{S}_{\boldsymbol{G}}$. □

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References

[Ash et al. 2010] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, 2nd ed., Cambridge University Press, 2010. With the collaboration of Peter Scholze. MR Zbl

[Borel 1969] A. Borel, *Introduction aux groupes arithmétiques*, Publ. Inst. Math. Univ. Strasbourg **1341**, Hermann, Paris, 1969. MR Zbl

[Borel and Harish-Chandra 1962] A. Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups", *Ann. of Math.* (2) **75** (1962), 485–535. MR

[Borel and Ji 2006] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications **2**, Birkhäuser, Boston, 2006. MR Zbl

[Borel and Tits 1965] A. Borel and J. Tits, "Groupes réductifs", *Inst. Hautes Études Sci. Publ. Math.* 27 (1965), 55–150. MR Zbl

[Habegger and Pila 2012] P. Habegger and J. Pila, "Some unlikely intersections beyond André–Oort", *Compos. Math.* **148**:1 (2012), 1–27. MR Zbl

[Hochschild 1965] G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965. MR Zbl

[Orr 2015] M. Orr, "Families of abelian varieties with many isogenous fibres", *J. Reine Angew. Math.* **705** (2015), 211–231. MR Zbl

[Orr 2017] M. Orr, "On compatibility between isogenies and polarizations of abelian varieties", *Int. J. Number Theory* **13**:3 (2017), 673–704. MR Zbl

[Pila and Wilkie 2006] J. Pila and A. J. Wilkie, "The rational points of a definable set", *Duke Math. J.* **133**:3 (2006), 591–616. MR Zbl

[Pink 2005] R. Pink, "A common generalization of the conjectures of André–Oort, Manin–Mumford, and Mordell–Lang", preprint, 2005, Available at http://www.math.ethz.ch/~pink/ftp/AOMMML.pdf.

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m.orr@imperial.ac.uk *Department of Mathematics, Imperial College London, South Kensington, London, United Kingdom*

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