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We introduce a category $\mathcal{O}$ of modules over the elliptic quantum group of $\mathfrak{sl}_N$ with well-behaved q-character theory. We construct asymptotic modules as analytic continuation of a family of finite-dimensional modules, the Kirillov–Reshetikhin modules. In the Grothendieck ring of this category we prove two types of identities: Generalized Baxter relations in the spirit of Frenkel–Hernandez between finite-dimensional modules and asymptotic modules. Three-term Baxter TQ relations of infinite-dimensional modules.

Introduction

Fix $\mathfrak{sl}_N$ a special linear Lie algebra, $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ an elliptic curve, and $\hbar$ a complex number. Associated to this triple is the *elliptic quantum group* $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ introduced by G. Felder [1995]. It is a Hopf algebroid (neither commutative nor cocommutative) in the sense of Etingof and Varchenko [1998], so that the tensor product of two $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$-modules is naturally endowed with a module structure. In this paper we study (finite- and infinite-dimensional) representations of the elliptic quantum group.

Suppose $\hbar$ is a formal variable. $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_2)$ is an $\hbar$-deformation [Enriquez and Felder 1998] of the universal enveloping algebra of a Lie algebra $\mathfrak{sl}_2\otimes R_\tau$, where R_τ is an algebra of meromorphic functions of $z\in\mathbb{C}$ built from the Jacobi theta function of the elliptic curve. For $\mathfrak{g}$ an arbitrary finite-dimensional simple Lie algebra, $\mathcal{E}_{\tau,\hbar}(\mathfrak{g})$ is defined [Jimbo et al. 1999] to be a quasi-Hopf algebra twist of the *affine quantum group* $U_\hbar(L\mathfrak{g})$, an $\hbar$-deformation of the loop Lie algebra $\mathfrak{g}\otimes\mathbb{C}[z,z^{-1}]$. It admits a universal

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dynamical R-matrix in a completed tensor square, which provides solutions $R(z;\lambda) \in \mathrm{End}(V \otimes V)$, for V a suitable $\mathcal{E}_{\tau,\hbar}(\mathfrak{g})$-module, to the *quantum dynamical Yang–Baxter equation*:

$$R_{12}(z-w;\lambda+\hbar h^{(3)})R_{13}(z;\lambda)R_{23}(w;\lambda+\hbar h^{(1)}) = R_{23}(w;\lambda)R_{13}(z;\lambda+\hbar h^{(2)})R_{12}(z-w;\lambda) \in \mathrm{End}(V^{\otimes 3}).$$

Here z, w are complex spectral parameters, λ is the dynamical parameter lying in a Cartan subalgebra of $\mathfrak{g}$, the subindices of R indicate the tensor factors of $V^{\otimes 3}$ to be acted on, and the $h^{(i)}$ are grading operators arising from the weight grading on V by the Cartan subalgebra. See the comments following (1.1).

Such R-matrices $R(z;\lambda)$ appeared previously in face-type integrable models [Felder and Varchenko 1996a; Hou et al. 2003]; for instance, the R-matrix of the Andrews–Baxter–Forrester model comes from two-dimensional irreducible modules of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_2)$, as does the 6-vertex model from the affine quantum group $U_\hbar(L\mathfrak{sl}_2)$. The definition of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ in [Felder 1995], by RLL exchange relations, is in the spirit of Faddeev, Reshetikhin and Takhatajan, originated from *quantum inverse scattering method*. We mention that elliptic R-matrices describe the monodromy of the quantized Knizhnik–Zamolodchikov equation associated with representations of affine quantum groups, e.g., [Frenkel and Reshetikhin 1992; Galleas and Stokman 2015; Konno 2006; Tarasov and Varchenko 1997].

Recently Aganagic and Okounkov [2016] proposed the elliptic stable envelope in equivariant elliptic cohomology, as a geometric framework to obtain elliptic R-matrices. This was made explicit [Felder et al. 2017] for cotangent bundles of Grassmannians, resulting in tensor products of two-dimensional irreducible representations of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_2)$. The higher rank case of $\mathfrak{sl}_N$ was studied later by H. Konno [2017].

Meanwhile, Nekrasov, Pestun and Shatashvili [2018] from the 6d quiver gauge theory predicted the elliptic quantum group associated to an arbitrary Kac–Moody algebra, the precise definition of which (as an associative algebra) was proposed by Gautam and Toledano Laredo [2017a]. See also [Yang and Zhao 2017] in the context of quiver geometry.

We are interested in the representation theory of $\mathcal{E}_{\tau,\hbar}(\mathfrak{g})$ with $\hbar \in \mathbb{C}$ generic. The formal twist constructions [Enriquez and Felder 1998; Jimbo et al. 1999] from $U_\hbar(L\mathfrak{g})$ might reduce the problem to the representation theory of affine quantum groups, which is a subject developed intensively in the last three decades from algebraic, geometric and combinatorial aspects. However their work involves formal power series of $\hbar$ and infinite products in the comultiplication of $\mathcal{E}_{\tau,\hbar}(\mathfrak{g})$. Some of these divergence issues were addressed by Etingof and Moura [2002], who defined a fully faithful tenor functor between representation categories of BGG type for $U_\hbar(L\mathfrak{sl}_N)$ and $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$. Towards this functor not much is known: its image, the induced homomorphism of Grothendieck rings, etc.

In this paper we study representations of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ via the RLL presentation [Felder 1995] so as to bypass affine quantum groups, yet along the way we borrow ideas from the affine case. Compared to other works [Cavalli 2001; Etingof and Moura 2002; Felder and Varchenko 1996b; Gautam and Toledano Laredo 2017a; Konno 2009; 2016; Tarasov and Varchenko 2001; Yang and Zhao 2017], our approach emphasizes more on the Grothendieck ring structure of representation category. It is a higher rank extension of a recent joint work with G. Felder [2017].

The presence of the dynamical parameter λ is one of the technical difficulties of elliptic quantum groups. To resolve this, we need a commuting family of elliptic Cartan currents $\phi_j(z) \in \mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ for $j \in J := \{1, 2, \dots, N-1\}$. They act as difference operators on an $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$-module V, and their matrix entries are meromorphic functions of $(z, \lambda) \in \mathbb{C} \times \mathfrak{h}$ where $\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{sl}_N$. As in [Felder and Zhang 2017], we impose the following triangularity condition:[1]

(i) There exists a basis of V, with respect to which the matrices $\phi_j(z)$ are upper triangular and their diagonal entries are independent of λ.

Our category $\mathcal{O}$ is the full subcategory of category BGG [Etingof and Moura 2002] of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$-modules subject to condition (i), see Definition 1.7. It is abelian and monoidal. It contains most of the modules in [Cavalli 2001; Etingof and Moura 2002; Konno 2009; 2016; Tarasov and Varchenko 2001], although the proof is rather indirect. (We believe category $\mathcal{O}$ to be the image of the functor [Etingof and Moura 2002].)

We extend the *q-character* of H. Knight [1995] and Frenkel and Reshetikhin [1999] to the elliptic case. The q-character of a module V encodes the spectra of the $\phi_j(z)$, which are meromorphic functions of z thanks to condition (i). It distinguishes the isomorphism class $[V]$ in the Grothendieck ring $K_0(\mathcal{O})$, and embeds $K_0(\mathcal{O})$ in a commutative ring. Our main results are summarized as follows:

(A) Proposition 4.10 on limit construction of infinite-dimensional *asymptotic* modules $\mathcal{W}_{r,x}$, for $r \in J$ and $x \in \mathbb{C}$, from a distinguished family of finite-dimensional modules, the *Kirillov–Reshetikhin* modules.

(B) Theorem 4.15 on generalized Baxter relations à la Frenkel and Hernandez [2015], the isomorphism class of any finite-dimensional module is a polynomial of the $[\mathcal{W}_{r,x}]/[\mathcal{W}_{r,y}]$ for $r \in J$ and $x, y \in \mathbb{C}$.

(C) Corollary 5.2 relating an asymptotic module $\mathcal{W}$ to a module D and tensor products S' and S'' of asymptotic modules such that $[D][\mathcal{W}] = [S'] + [S'']$.

The above results are known in category HJ of Hernandez and Jimbo [2012] for representations over a Borel subalgebra of an affine quantum group $U_\hbar(L\mathfrak{g})$. Category HJ contains the modules $L^{\pm}_{r,a}$ for $a \in \mathbb{C}$ and r a Dynkin node of $\mathfrak{g}$. The $L^{\pm}_{r,a}$ are "prefundamental" in that their tensor products realize all irreducible objects of HJ as subquotients, and they are not modules over $U_\hbar(L\mathfrak{g})$, which makes Borel subalgebras indispensable. The Grothendieck ring of HJ is commutative.

Result (A) is the asymptotic limit construction [Hernandez and Jimbo 2012] of the $L^{-}_{r,a}$. Result (B) is the relation [Frenkel and Hernandez 2015] between finite-dimensional modules and the $L^{+}_{r,a}$. Result (C) is either QQ^*-system [Feigin et al. 2017a; Hernandez and Leclerc 2016] or $Q\tilde{Q}$-system [Frenkel and Hernandez 2016], as there are two choices of the modules D for $\mathcal{W} = L^{+}_{r,a}$.

Hernandez and Leclerc [2016] interpreted the QQ^*-system as cluster mutations of Fomin–Zelevinsky. They provided conjectural monoidal categorifications of infinite rank cluster algebras by certain subcategories of HJ.

[1]In terms of the $K_i(z)$ from (1.8), we have $\phi_j(z) = K_j(z + \ell_j\hbar)K_{j+1}(z + \ell_j\hbar)^{-1}$ where $\ell_j = (N - j - 1)/2$. These are elliptic deformations of diagonal matrices in $\mathfrak{sl}_N$.

In a quantum integrable system associated to $U_\hbar(L\mathfrak{g})$, the transfer-matrix construction defines an action of the Grothendieck ring $K_0(\mathrm{HJ})$ on the quantum space; to an isomorphism class $[V]$ is attached a transfer matrix $t_V(z)$.

Result (B) is one key step [Frenkel and Hernandez 2015] in solving the conjecture of Frenkel and Reshetikhin [1999] on the spectra of the quantum integrable system, which connects the eigenvalues of the $t_V(z)$ to the q-character of V by the so-called Baxter polynomials [Baxter 1972]. These polynomials are eigenvalues of the $t_{L_{r,a}^+}(z)$ up to an overall factor [Frenkel and Hernandez 2015]. In this sense the $L_{r,a}^+$ have simpler structures than finite-dimensional modules, and the $t_{L_{r,a}^+}(z)$ are defined as *Baxter Q operators*, as an extension of earlier works of V. Bazhanov et al. [1997; 1999; Bazhanov and Tsuboi 2008] for $\mathfrak{g}$ a special linear Lie (super)algebra. Result (C) has as consequence the Bethe Ansatz equations for the roots of Baxter polynomials [Feigin et al. 2017a; Frenkel and Hernandez 2016].

Recently category HJ was studied for quantum toroidal algebras [Feigin et al. 2017b].

For elliptic quantum groups there are no obvious Borel subalgebras. Our idea is to replace the $L_{r,a}^\pm$ over Borel subalgebras by the asymptotic modules $\mathscr{W}_{d,a}^{(r)}$ (with a new parameter $d \in \mathbb{C}$) over the *entire* quantum group, which we now explain.

Let $\theta(z) := \theta(z \mid \tau)$ be the Jacobi theta function. For $r \in J$ a Dynkin node, $a \in \mathbb{C}$ a spectral parameter, and k a positive integer, by [Cavalli 2001; Tarasov and Varchenko 2001] there exists a unique finite-dimensional irreducible module $W_{k,a}^{(r)}$ which contains a nonzero vector ω (highest weight with respect to a triangular decomposition) such that:

$$\phi_j(z)\omega = \omega, \quad \text{if } j \neq r \quad \text{and} \quad \phi_r(z)\omega = \frac{\theta(z + a\hbar + k\hbar)}{\theta(z + a\hbar)}\omega.$$

This is a Kirillov–Reshetikhin (KR) module, a standard terminology for affine quantum groups and Yangians once the θ symbol is removed.

The core of this paper (Section 4) is analytic continuation with respect to k. We modify the asymptotic limits $L_{r,a}^-$ of Hernandez and Jimbo [2012], as in [Felder and Zhang 2017; Zhang 2017].

Firstly the existence of the inductive system $(W_{k,a}^{(r)})_{k>0}$ in [Hernandez and Jimbo 2012] relied on a cyclicity property of M. Kashiwara, Varagnolo–Vasserot and V. Chari, which is unavailable in the elliptic case. We reduce the problem to $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_2)$ by counting "dominant weights" in q-characters (Theorem 3.4), as in the proofs of T-system of KR modules over affine quantum groups by H. Nakajima [2003] and D. Hernandez [2006].

Secondly we express the matrix coefficients of any element of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ acting on the $W_{k,a}^{(r)}$, viewed as functions of $k \in \mathbb{Z}_{>0}$, in products of the $\theta(k\hbar + c)$ where $c \in \mathbb{C}$ is independent of k; see Lemma 4.8. In [Hernandez and Jimbo 2012] these are polynomials in k by induction. Our proof relies on the RLL comultiplication and is explicit.

Since $\theta(k\hbar + c)$ is an entire function of k, we take k in the matrix coefficients to be a fixed complex number d. This results in the asymptotic module $\mathscr{W}_{d,a}^{(r)}$ on the inductive limit $\varinjlim W_{k,a}^{(r)}$. The module $\mathscr{W}_{r,x}$ in (A) is $\mathscr{W}_{x,0}^{(r)}$. All irreducible modules of category $\mathcal{O}$ are subquotients of tensor products of asymptotic modules.

For $\mathfrak{g}$ of general type (A)–(C) and their proofs can be adapted to affine quantum groups, whose asymptotic modules appeared in the appendix of [Zhang 2017], as well as Yangians [Gautam and Toledano Laredo 2016; 2017b]. Borel subalgebras or double Yangians are not needed.

Results (A)–(C) were established for affine quantum general linear Lie superalgebras [Zhang 2018]; their proofs require more than q-characters as counting dominant weights is inefficient [Zhang 2016]. It is interesting to consider elliptic quantum supergroups [Galleas and Stokman 2015].

For elliptic quantum groups associated with other simple Lie algebras, one possible first step would be to derive the RLL presentation; see [Guay et al. 2017; Jing et al. 2017] for Yangians.

The R-matrix of Baxter and Belavin is governed by the vertex-type elliptic quantum group [Jimbo et al. 1999]. The equivalence of representation categories between this elliptic algebra and $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ [Etingof and Schiffmann 1998], a Vertex-IRF correspondence, might give a representation theory meaning to the original Baxter Q operator of the 8-vertex model [Baxter 1972].

The paper is structured as follows. In Section 1 we review the theory of the elliptic quantum group associated to $\mathfrak{sl}_N$ and define category $\mathcal{O}$ of representations. We show that the q-character map is an injective ring homomorphism from the Grothendieck ring $K_0(\mathcal{O})$ to a commutative ring $\mathcal{M}_{\mathrm{t}}$ of meromorphic functions. Then we present the q-character formula of finite-dimensional evaluation modules.

Section 2 is devoted to the proof of the q-character formula.

We derive in Section 3 basic facts on tensor products of KR modules (T-system, fusion) from the q-character formula. They are needed in Section 4 to construct the inductive system of KR modules and the asymptotic modules. We obtain a highest weight classification of irreducible modules in category $\mathcal{O}$. As a consequence, all standard irreducible evaluation modules of [Tarasov and Varchenko 2001] are in category $\mathcal{O}$.

In Section 5 we establish the three-term Baxter TQ relations in $K_0(\mathcal{O})$, which are infinite-dimensional analogs of the T-system. These relations are interpreted as functional relations of transfer matrices in Section 6.

1. Elliptic quantum groups and their representations

Let $N \in \mathbb{Z}_{>0}$. We introduce a category $\mathcal{O}$ (abelian and monoidal) of representations of the elliptic quantum group attached to the Lie algebra $\mathfrak{sl}_N$, and prove that its Grothendieck ring is commutative, based on q-characters.

Fix a complex number $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. Define the Jacobi theta function

$$\theta(z) = \theta(z \mid \tau) := -\sum_{j=-\infty}^{\infty} \exp\left(i\pi\left(j+\tfrac{1}{2}\right)^2\tau + 2i\pi\left(j+\tfrac{1}{2}\right)\left(z+\tfrac{1}{2}\right)\right), \quad i = \sqrt{-1}.$$

It is an entire function of $z \in \mathbb{C}$ with zeros lying on the lattice $\Gamma := \mathbb{Z} + \mathbb{Z}\tau$ and

$$\theta(z+1) = -\theta(z), \quad \theta(z+\tau) = -e^{-i\pi\tau - 2i\pi z}\theta(z), \quad \theta(-z) = -\theta(z).$$

Fix a complex number $\hbar \in \mathbb{C} \setminus (\mathbb{Q} + \mathbb{Q}\tau)$, which is the deformation parameter.

Let $\mathfrak{h}$ be standard Cartan subalgebra of $\mathfrak{sl}_N$; it is a complex vector space generated by the ϵ_i for $1 \le i \le N$ subject to the relation $\sum_{i=1}^N \epsilon_i = 0$. Let $\mathbb{C}^N := \oplus_{i=1}^N \mathbb{C} v_i$ and $E_{ij} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$ be the elementary matrices: $v_k \mapsto \delta_{jk} v_i$ for $1 \le i, j, k \le N$. Define the $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^N \otimes \mathbb{C}^N)$-valued meromorphic functions of $(z; \lambda) \in \mathbb{C} \times \mathfrak{h}$ by

$$\boldsymbol{R}(z;\lambda) = \sum_i E_{ii}^{\otimes 2} + \sum_{i \neq j} \left(\frac{\theta(z)\theta(\lambda_{ij} - \hbar)}{\theta(z+\hbar)\theta(\lambda_{ij})} E_{ii} \otimes E_{jj} + \frac{\theta(z+\lambda_{ij})\theta(\hbar)}{\theta(z+\hbar)\theta(\lambda_{ij})} E_{ij} \otimes E_{ji} \right).$$

In the summations $1 \le i, j \le N$, and $\lambda_{ij} \in \mathfrak{h}^*$ sends $\sum_{i=1}^N c_i \epsilon_i \in \mathfrak{h}$ to $c_i - c_j \in \mathbb{C}$. By [Felder 1995], $\boldsymbol{R}(z;\lambda)$ satisfies the *quantum dynamical Yang–Baxter equation*:

$$\begin{aligned} &\boldsymbol{R}^{12}(z-w; \lambda + \hbar h^{(3)}) \boldsymbol{R}^{13}(z;\lambda) \boldsymbol{R}^{23}(w; \lambda + \hbar h^{(1)}) \\ &\qquad = \boldsymbol{R}^{23}(w;\lambda) \boldsymbol{R}^{13}(z; \lambda + \hbar h^{(2)}) \boldsymbol{R}^{12}(z-w;\lambda) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)^{\otimes 3}. \end{aligned} \tag{1.1}$$

If $\boldsymbol{R}(z;\lambda) = \sum_p c_\lambda^p x_p \otimes y_p$ with $x_p, y_p \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^N)$, then

$$\boldsymbol{R}^{13}(z; \lambda + \hbar h^{(2)}) : u \otimes v_j \otimes w \mapsto \sum_p c^p_{\lambda + \hbar \epsilon_j} x_p(u) \otimes v_j \otimes y_p(w),$$

for $u, w \in \mathbb{C}^N$ and $1 \le j \le N$. The other symbols have a similar meaning.

Let $\mathbb{M} := \mathbb{M}_{\mathfrak{h}}$ be the field of meromorphic functions of $\lambda \in \mathfrak{h}$. It contains the subfield $\mathbb{C}$ of constant functions. A $\mathbb{C}$-linear map Φ of two $\mathbb{M}$-vector spaces will sometimes be denoted by $\Phi(\lambda)$ to emphasize the dependence on λ.

1A. ***Algebraic notions.*** Since the elliptic quantum groups will act on $\mathbb{M}$-vector spaces via difference operators, which are in general not $\mathbb{M}$-linear, we need to recall some basis constructions about difference operators. Our exposition follows largely [Etingof and Varchenko 1998], with minor modifications as in [Felder and Zhang 2017].

Define the category $\mathcal{V}$ as follows. An object is $X = \oplus_{\alpha \in \mathfrak{h}} X[\alpha]$ where each $X[\alpha]$ is an $\mathbb{M}$-vector space and, if nonzero, is called a weight space of weight (or $\mathfrak{h}$-weight) α. Let $\mathrm{wt}(X) \subseteq \mathfrak{h}$ be the set of weights of X. Write $\mathrm{wt}(v) = \alpha$ if $v \in X[\alpha]$.

A morphism $f : X \to Y$ in $\mathcal{V}$ is an $\mathbb{M}$-linear map which respects the weight gradings. Let $\mathcal{V}_{\mathrm{ft}}$ be the full subcategory of $\mathcal{V}$ consisting of X whose weight spaces are finite-dimensional $\mathbb{M}$-vector spaces ("ft" means finite type in [Felder and Varchenko 1996b]).

Viewed as subcategories of the category of $\mathbb{M}$-vector spaces, $\mathcal{V}$ and $\mathcal{V}_{\mathrm{ft}}$ are abelian.

Let X and Y be objects of $\mathcal{V}$. Their *dynamical tensor product* $X \widetilde{\otimes} Y$ is constructed as follows. For $\alpha, \beta \in \mathfrak{h}$, let $X[\alpha] \widetilde{\otimes} Y[\beta]$ be the quotient of the usual tensor product of $\mathbb{C}$-vector spaces $X[\alpha] \otimes_{\mathbb{C}} Y[\beta]$ by the relation

$$g(\lambda) v \otimes_{\mathbb{C}} w = v \otimes_{\mathbb{C}} g(\lambda + \hbar\beta) w, \qquad \text{for } v \in X[\alpha],\ w \in Y[\beta],\ g(\lambda) \in \mathbb{M}.$$

Let $\overline{\otimes}$ denote the image of $\otimes_{\mathbb{C}}$ under the quotient. $X[\alpha]\overline{\otimes}Y[\beta]$ becomes an $\mathbb{M}$-vector space by setting $g(\lambda)(v\overline{\otimes}w)=v\overline{\otimes}g(\lambda)w$. For $\gamma\in\mathfrak{h}$, the weight space $(X\overline{\otimes}Y)[\gamma]$ is then the direct sum of the $X[\alpha]\overline{\otimes}Y[\beta]$ with $\alpha+\beta=\gamma$.

For $\alpha,\beta\in\mathfrak{h}$, a $\mathbb{C}$-linear map $\Phi:X\to Y$ is called a *difference map* of bidegree (α,β) [Etingof and Varchenko 1998, §4.2] if it sends every weight space $X[\gamma]$ to $Y[\gamma+\beta-\alpha]$, and if

$$\Phi(g(\lambda)v)=g(\lambda+\beta\hbar)\Phi(v)\quad\text{for } g(\lambda)\in\mathbb{M} \text{ and } v\in X.$$

Such a map can be recovered from its matrix as in the case of $\mathbb{M}$-linear maps. Choose $\mathbb{M}$-bases $\mathcal{B}$ and $\mathcal{B}'$ for X and Y respectively. Define the $\mathcal{B}'\times\mathcal{B}$ matrix $[\Phi]$ by taking its (b',b)-entry $[\Phi]_{b'b}(\lambda)\in\mathbb{M}$, for $b\in\mathcal{B}$ and $b'\in\mathcal{B}'$, to be the coefficient of b' in $\Phi(b)$. Then for any vector $v=\sum_{b\in\mathcal{B}}g_b(\lambda)b$ of X where $g_b(\lambda)\in\mathbb{M}$, we have[2]

$$\Phi(v)=\sum_{b'\in\mathcal{B}'}b'\sum_{b\in\mathcal{B}}[\Phi]_{b'b}(\lambda)\times g_b(\lambda+\hbar\beta).$$

When $X=Y$, a difference map is also called a difference operator. To define its matrix, we always assume $\mathcal{B}'=\mathcal{B}$.

By an algebra we mean a unital associative algebra over $\mathbb{C}$.

As in [Etingof and Varchenko 1998, Definition 4.1], an $\mathfrak{h}$-*algebra* is an algebra A, endowed with $\mathfrak{h}$-bigrading $A=\oplus_{\alpha,\beta\in\mathfrak{h}}A_{\alpha,\beta}$ which respects the algebra structure and is called the *weight decomposition*, and two algebra embeddings $\mu_{\boldsymbol{l}},\mu_{\boldsymbol{r}}:\mathbb{M}\to A_{0,0}$ called the *left and right moment maps*, such that for $a\in A_{\alpha,\beta}$ and $g(\lambda)\in\mathbb{M}$, we have

$$\mu_{\boldsymbol{l}}(g(\lambda))a=a\mu_{\boldsymbol{l}}(g(\lambda-\hbar\alpha))\quad\text{and}\quad \mu_{\boldsymbol{r}}(g(\lambda))a=a\mu_{\boldsymbol{r}}(g(\lambda-\hbar\beta)).$$

Call (α,β) the bidegree of elements in $A_{\alpha,\beta}$. A morphism of $\mathfrak{h}$-algebras is an algebra morphism preserving the moment maps and the weight decompositions.

From two $\mathfrak{h}$-algebras A and B we construct their tensor product $A\tilde{\otimes}B$ as follows. For $\alpha,\beta,\gamma\in\mathfrak{h}$, let $A_{\alpha,\beta}\tilde{\otimes}B_{\beta,\gamma}$ be $A_{\alpha,\beta}\otimes_{\mathbb{C}}B_{\beta,\gamma}$ modulo the relation

$$\mu_{\boldsymbol{r}}^A(g(\lambda))a\otimes_{\mathbb{C}}b=a\otimes_{\mathbb{C}}\mu_{\boldsymbol{l}}^B(g(\lambda))b\quad\text{for } a\in A_{\alpha,\beta},\ b\in B_{\beta,\gamma},\ g(\lambda)\in\mathbb{M}.$$

Let $(A\tilde{\otimes}B)_{\alpha,\gamma}$ be the direct sum of the $A_{\alpha,\beta}\tilde{\otimes}B_{\beta,\gamma}$ over $\beta\in\mathfrak{h}$ ($\tilde{\otimes}$ denotes the image of $\otimes_{\mathbb{C}}$ under the quotient $\otimes_{\mathbb{C}}\to\tilde{\otimes}$). Multiplication in $A\tilde{\otimes}B$ is induced by $(a\tilde{\otimes}b)(a'\tilde{\otimes}b')=aa'\tilde{\otimes}bb'$. The moment maps are given by

$$\mu_{\boldsymbol{l}}^{A\tilde{\otimes}B}:g(\lambda)\mapsto\mu_{\boldsymbol{l}}^A(g(\lambda))\tilde{\otimes}1\quad\text{and}\quad\mu_{\boldsymbol{r}}^{A\tilde{\otimes}B}:g(\lambda)\mapsto 1\tilde{\otimes}\mu_{\boldsymbol{r}}^B(g(\lambda))\quad\text{for } g(\lambda)\in\mathbb{M}.$$

To an $\mathfrak{h}$-graded vector space one can attach naturally an $\mathfrak{h}$-algebra. Let X be an object of $\mathcal{V}$. Let $\mathbb{D}_{\alpha,\beta}^X$ denote the $\mathbb{C}$-vector space of difference operators $X\to X$ of bidegree (α,β). Then the direct sum

[2] Note that difference maps of bidegree (α,α) make sense for arbitrary $\mathbb{M}$-vector spaces.

$\mathbb{D}^X := \oplus_{\alpha,\beta\in\mathfrak{h}}\mathbb{D}^X_{\alpha,\beta}$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(X)$. It is an $\mathfrak{h}$-algebra structure with the moment maps

$$\mu_{\boldsymbol{r}}(g(\lambda))v = g(\lambda)v \quad \text{and} \quad \mu_{\boldsymbol{l}}(g(\lambda))v = g(\lambda+\hbar\alpha)v \quad \text{for } v\in X[\alpha],\ g(\lambda)\in\mathbb{M}.$$

Tensor products of difference operators are also difference operators. To be precise, let X and Y be two objects of $\mathcal{V}$. Let $\Phi : X \to X$ and $\Psi : Y \to Y$ be difference operators of bidegree (α,β) and (β,γ), respectively. The $\mathbb{C}$-linear map

$$X\otimes_{\mathbb{C}} Y \to X\overline{\otimes}Y, \quad v\otimes_{\mathbb{C}} w \mapsto \Phi(v)\overline{\otimes}\Psi(w)$$

is easily seen to factor through $X\otimes_{\mathbb{C}} Y \to X\overline{\otimes}Y$ and induces the $\mathbb{C}$-linear map $\Phi\overline{\otimes}\Psi : X\overline{\otimes}Y \to X\overline{\otimes}Y$, which is shown to be a difference operator of bidegree (α,γ). As in [Etingof and Varchenko 1998, Lemma 4.3], the following defines a morphism of $\mathfrak{h}$-algebras

$$\mathbb{D}^X\tilde{\otimes}\mathbb{D}^Y \to \mathbb{D}^{X\overline{\otimes}Y}, \quad \Phi\tilde{\otimes}\Psi \mapsto \Phi\overline{\otimes}\Psi.$$

1B. ***Elliptic quantum groups.*** For $1\le i,j,p,q\le N$ let $R^{ij}_{pq}(z;\lambda)$ be the coefficient of $v_p\otimes v_q$ in $\boldsymbol{R}(z;\lambda)(v_i\otimes v_j)$; it can be viewed as an element of $\mathbb{M}$ after fixing $z\in\mathbb{C}$. The *elliptic quantum group* $\mathcal{E} := \mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$ is an $\mathfrak{h}$-algebra generated by[3]

$$L_{ij}(z)\in\mathcal{E}_{\epsilon_i,\epsilon_j} \quad \text{for } 1\le i,j\le N$$

subject to the dynamical RLL relation [Etingof and Varchenko 1998, §4.4]: for $1\le i,j,m,n\le N$,

$$\sum_{p,q=1}^N \mu_{\boldsymbol{l}}(R^{pq}_{mn}(z-w;\lambda))L_{pi}(z)L_{qj}(w) = \sum_{p,q=1}^N \mu_{\boldsymbol{r}}(R^{ij}_{pq}(z-w;\lambda))L_{nq}(w)L_{mp}(z). \tag{1.2}$$

There is an $\mathfrak{h}$-algebra morphism [Etingof and Varchenko 1998; Felder and Varchenko 1996b]

$$\Delta : \mathcal{E}\to\mathcal{E}\tilde{\otimes}\mathcal{E}, \quad L_{ij}(z)\mapsto \sum_{k=1}^N L_{ik}(z)\tilde{\otimes}L_{kj}(z), \quad \text{for } 1\le i,j\le N \tag{1.3}$$

which is coassociative $(1\tilde{\otimes}\Delta)\Delta = (\Delta\tilde{\otimes}1)\Delta$ and is called the coproduct. For $u\in\mathbb{C}$,

$$\Phi_u : \mathcal{E}\to\mathcal{E}, \quad L_{ij}(z)\mapsto L_{ij}(z+u\hbar) \quad \text{for } 1\le i,j\le N \tag{1.4}$$

extends uniquely to an $\mathfrak{h}$-algebra automorphism (spectral parameter shift).

Strictly speaking, $\mathcal{E}$ is not well-defined as an $\mathfrak{h}$-algebra because of the additional parameter z; this is resolved in [Konno 2016] by viewing $z,\hbar$ as formal variables. In this paper we are mainly concerned with representations in which (1.2)–(1.4) make sense as identities of difference operators depending analytically on z.

[3]We use $\mathfrak{sl}_N$, as in [Felder 1995; Felder and Varchenko 1996b], to emphasize that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}_N$. Other works [Cavalli 2001; Konno 2016] use $\mathfrak{gl}_N$ for the reason that the elliptic quantum determinant is not fixed to be 1.

Let $\mathfrak{S}_N$ be the group of permutations of $\{1, 2, \dots, N\}$. For $1 \le k \le N$, let $\mathfrak{S}^k$ be the subgroup of permutations which fix the last k letters. The k-th *fundamental weight* ϖ_k and *elliptic quantum minor* $\mathcal{D}_k(z)$ are defined by [Tarasov and Varchenko 2001, (2.5)]:

$$\varpi_k := \sum_{i=1}^{k} \epsilon_i \in \mathfrak{h}, \tag{1.5}$$

$$\Theta_k(\lambda) := \prod_{N-k+1 \le i < j \le N} \theta(\lambda_{ij}) \in \mathbb{M}^\times,$$

$$\mathcal{D}_k(z) := \frac{\mu_{\boldsymbol{r}}(\Theta_k(\lambda))}{\mu_{\boldsymbol{l}}(\Theta_k(\lambda))} \sum_{\sigma \in \mathfrak{S}^k} \operatorname{sign}(\sigma) \prod_{i=N}^{N-k+1} L_{\sigma(i),i}(z + (N-i)\hbar) \in \mathcal{E}. \tag{1.6}$$

Here $\operatorname{sign}(\sigma) \in \{\pm 1\}$ denotes the signature of the permutation σ. We take the descending product over $N \ge i \ge N-k+1$ in (1.6). Set $\varpi_0 := 0$.

We shall need the following elements $\hat{L}_k(z)$ of $\mathcal{E}_{\epsilon_k,\epsilon_k}$ as in [Tarasov and Varchenko 2001, (4.1)]:

$$\hat{L}_N(z) := L_{NN}(z) \quad \text{and} \quad \hat{L}_k(z) = L_{kk}(z) \prod_{j=k+1}^{N} \frac{\mu_{\boldsymbol{r}}(\theta(\lambda_{kj}))}{\mu_{\boldsymbol{l}}(\theta(\lambda_{kj}))}. \tag{1.7}$$

Theorem 1.1 [Tarasov and Varchenko 2001, Proposition 2.1] and [Konno 2016, (E.18)]. *$\mathcal{D}_N(z)$ is central in $\mathcal{E}$ and grouplike*: $\Delta(\mathcal{D}_N(z)) = \mathcal{D}_N(z)\tilde{\otimes}\mathcal{D}_N(z)$.

The simple roots $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for $1 \le i < N$ generate a free abelian subgroup $\boldsymbol{Q}$ of $\mathfrak{h}$, called the root lattice. Let $\boldsymbol{Q}_+$ and $\boldsymbol{Q}_-$ be submonoids of $\boldsymbol{Q}$ generated by the α_i and $-\alpha_i$ respectively. Define the lexicographic partial ordering $\prec$ on $\mathfrak{h}$ as follows:

$$\alpha \prec \beta \quad \text{if } \beta - \alpha = n_l \alpha_l + \sum_{i=l+1}^{N-1} n_i \alpha_i \in \boldsymbol{Q} \text{ with } n_l \in \mathbb{Z}_{>0}.$$

This is weaker than the standard ordering $\alpha \le \beta$ if $\beta - \alpha \in \boldsymbol{Q}_+$.

Corollary 1.2. *$\mathcal{D}_k(z)$ commutes with the $L_{ij}(w)$ for $N-k < i, j \le N$ and $\Delta(\mathcal{D}_k(z)) - \mathcal{D}_k(z)\tilde{\otimes}\mathcal{D}_k(z)$ is a finite sum $\sum_\alpha x_\alpha \tilde{\otimes} y_\alpha$ over $\{\alpha \in \mathfrak{h} \mid -\varpi_{N-k} \prec \alpha\}$ where x_α and y_α are of bidegree $(-\varpi_{N-k}, \alpha)$ and $(\alpha, -\varpi_{N-k})$ respectively.*

The proof of the corollary is postponed to Section 2A.

1C. *Categories.* *From now on unless otherwise stated vector spaces, linear maps and bases are defined over* $\mathbb{M}$. Let X be an object of $\mathcal{V}_{\mathrm{ft}}$. A representation of $\mathcal{E}$ on X consists of difference operators $L_{ij}^X(z) : X \to X$ of bidegree (ϵ_i, ϵ_j) for $1 \le i, j \le N$ depending on $z \in \mathbb{C}$ with the following properties:[4]

[4]This is called a representation of *finite type* in [Felder and Varchenko 1996b]. From condition (M1) it follows that the coefficients of the $L_{ij}^X(z)$ are meromorphic functions with respect to *any* basis of X.

(M1) There exists a basis of X with respect to which all the matrix entries of the difference operators $L^X_{ij}(z)$ are meromorphic functions of $(z,\lambda)\in\mathbb{C}\times\mathfrak{h}$.

(M2) Equation (1.2) holds in $\mathbb{D}^X$ with μ_l and μ_r being moment maps in $\mathbb{D}^X$.

Call X an $\mathcal{E}$-module. Property (M2) can be interpreted as an $\mathfrak{h}$-algebra morphism $\mathcal{E}\to\mathbb{D}^X$ sending $L_{ij}(z)\in\mathcal{E}$ to the difference operator $L^X_{ij}(z)$ on X. Applying ρ to the elements of (1.6)–(1.7), one gets difference operators $\mathcal{D}^X_k(z)$ and $\hat{L}^X_k(z)$ acting on X with bidegree $(-\varpi_{N-k},-\varpi_{N-k})$ and (ϵ_k,ϵ_k), respectively. When no confusion arises, we shall drop the superscript X from L^X, $\mathcal{D}^X$ and $\hat{L}^X$ to simplify notations.

A morphism $\Phi: X\to Y$ of $\mathcal{E}$-modules is a linear map which respects the $\mathfrak{h}$-gradings (so that Φ is a morphism in category $\mathcal{V}$) and satisfies $\Phi L^X_{ij}(z)=L^Y_{ij}(z)\Phi$ for $1\le i,j\le N$. The category of $\mathcal{E}$-modules is denoted by Rep. It is a subcategory of $\mathcal{V}_{\mathrm{ft}}$ and is abelian since the kernel and cokernel of a morphism of $\mathcal{E}$-modules, as $\mathfrak{h}$-graded $\mathbb{M}$-vector spaces, are naturally $\mathcal{E}$-modules.[5]

Definition 1.3 [Etingof and Moura 2002, §4]. $\tilde{\mathcal{O}}$ is the full subcategory of Rep whose objects X are such that $\mathrm{wt}(X)$ is contained in a *finite* union of cones $\mu+\boldsymbol{Q}_-$ with $\mu\in\mathfrak{h}$.

For X and Y objects in category $\tilde{\mathcal{O}}$, the $L^{X\tilde{\otimes}Y}_{ij}(z):=\sum_{k=1}^N L^X_{ik}(z)\tilde{\otimes}L^Y_{kj}(z)$ define a representation of $\mathcal{E}$ on $X\tilde{\otimes}Y$ which is easily seen to be in category $\tilde{\mathcal{O}}$. So $\tilde{\mathcal{O}}$ is a monoidal subcategory of $\mathcal{V}$. Similarly, $\tilde{\mathcal{O}}$ is an abelian subcategory of Rep.

Definition 1.4 [Felder and Zhang 2017, §2]. An object in $\mathcal{F}_{\mathrm{mer}}$ consists of a finite-dimensional vector space V equipped with difference operators $D_l(z): V\to V$ of bidegree $(-\varpi_{N-l},-\varpi_{N-l})$ (see footnote 2) for $1\le l\le N$ depending on $z\in\mathbb{C}$ such that:

(M3) There exists an ordered basis of V with respect to which the matrices of the difference operators $D_l(z)$ are upper triangular, the diagonal entries are nonzero meromorphic functions of $z\in\mathbb{C}$, and the off-diagonal entries are meromorphic functions of $(z,\lambda)\in\mathbb{C}\times\mathfrak{h}$.

A morphism $\Phi: V\to W$ in $\mathcal{F}_{\mathrm{mer}}$ is a linear map commuting with the $D_l(z)$. (Namely, $\Phi D^V_l(z)=D^W_l(z)\Phi: V\to W$ for $1\le l\le N$. Here we add the superscripts V and W in the $D_l(z)$ to indicate the space on which they act.)

For V an object of $\mathcal{F}_{\mathrm{mer}}$, the operators $D_l(z)$ being invertible because of the triangularity, one has a unique factorization of operators for $1\le l\le N$:

$$D_l(z)=K_N(z)K_{N-1}(z+\hbar)K_{N-2}(z+2\hbar)\cdots K_{N-l+1}(z+(l-1)\hbar). \tag{1.8}$$

Notably $K_l(z): X\to X$ is a difference operator of bidegree (ϵ_l,ϵ_l). Property (M3) still holds if the $D_l(z)$ are replaced by the $K_l(z)$.

[5] In other works [Cavalli 2001; Etingof and Moura 2002; Felder and Varchenko 1996b; Gautam and Toledano Laredo 2017a; Konno 2009; 2016; Tarasov and Varchenko 2001; Yang and Zhao 2017] a module V is an $\mathfrak{h}$-graded $\mathbb{C}$-vector space; morphisms of modules depend on the dynamical parameter λ, so do their kernel and cokernel; the abelian category structure is nontrivial. The scalar extension gives a module $V\otimes_{\mathbb{C}}\mathbb{M}$ in the present situation. Since our modules and morphisms are $\mathbb{M}$-linear, the dependence of kernels and images on the dynamical parameter does not matter.

The forgetful functor from $\mathcal{F}_{\mathrm{mer}}$ to the category of finite-dimensional vector spaces equips $\mathcal{F}_{\mathrm{mer}}$ with an abelian category structure. (For a proof, we refer to [Felder and Zhang 2017, §2.1] where another characterization of category $\mathcal{F}_{\mathrm{mer}}$ in terms of Jordan–Hölder series is given.) Let us describe its Grothendieck group $K_0(\mathcal{F}_{\mathrm{mer}})$.

The multiplicative group $\mathbb{M}_{\mathbb{C}}^{\times}$ of nonzero meromorphic functions of $z \in \mathbb{C}$ contains a subgroup $\mathbb{C}^{\times}$ of nonzero constant functions. Let $\mathcal{M}$ be the quotient group of $(\mathbb{M}_{\mathbb{C}}^{\times})^N$ by its subgroup formed of $(c_1, c_2, \ldots, c_N) \in (\mathbb{C}^{\times})^N$ such that $c_1 c_2 \cdots c_N = 1$. We show that $K_0(\mathcal{F}_{\mathrm{mer}})$ has a $\mathbb{Z}$-basis indexed by $\mathcal{M}$.

For $\boldsymbol{f} = (f_1(z), f_2(z), \ldots, f_N(z)) \in (\mathbb{M}_{\mathbb{C}}^{\times})^N$, the vector space $\mathbb{M}$ with the following difference operators $D_l(z)$ is an object in category $\mathcal{F}_{\mathrm{mer}}$ denoted by $\mathbb{M}_{\boldsymbol{f}}$:

$$g(\lambda) \mapsto g(\lambda - \hbar\varpi_{N-l}) f_N(z) f_{N-1}(z+\hbar) f_{N-2}(z+2\hbar) \cdots f_{N-l+1}(z+(l-1)\hbar).$$

We have $K_l(z)g(\lambda) = g(\lambda + \hbar\epsilon_l) f_l(z)$. As a consequence of (M3) in Definition 1.4, all irreducible objects of category $\mathcal{F}_{\mathrm{mer}}$ are of this form.

Lemma 1.5. *Let $\boldsymbol{e}, \boldsymbol{f} \in (\mathbb{M}_{\mathbb{C}}^{\times})^N$. The objects $\mathbb{M}_{\boldsymbol{e}}$ and $\mathbb{M}_{\boldsymbol{f}}$ are isomorphic in category $\mathcal{F}_{\mathrm{mer}}$ if and only if $\boldsymbol{e}, \boldsymbol{f}$ have the same image under the quotient $(\mathbb{M}_{\mathbb{C}}^{\times})^N \twoheadrightarrow \mathcal{M}$.*

Proof. Write $\boldsymbol{e} = (e_1(z), e_2(z), \ldots, e_N(z))$ and $\boldsymbol{f} = (f_1(z), f_2(z), \ldots, f_N(z))$.

Sufficiency: assume $e_l(z) = f_l(z)c_l$ with $c_l \in \mathbb{C}^{\times}$ and $c_1 c_2 \cdots c_N = 1$. For $1 \le l < N$, choose b_l such that $c_l = e^{b_l \hbar}$. Set $b_N := -b_1 - b_2 - \cdots - b_{N-1}$. Then $e^{b_N \hbar} = c_1^{-1} c_2^{-1} \cdots c_{N-1}^{-1} = c_N$ and the following is a well-defined element of $\mathbb{M}^{\times}$:

$$\varphi(x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_N\epsilon_N) = e^{b_1x_1 + b_2x_2 + \cdots + b_Nx_N} \quad \text{for } x_1, x_2, \ldots, x_N \in \mathbb{C}.$$

Indeed $\varphi(\alpha+\beta) = \varphi(\alpha)\varphi(\beta)$ and $\varphi(x\epsilon_1 + x\epsilon_2 + \cdots + x\epsilon_N) = 1$ for $x \in \mathbb{C}$. Notably,

$$\varphi(\lambda + \hbar\epsilon_l) = \varphi(\lambda)\varphi(\hbar\epsilon_l) = e^{\hbar b_l}\varphi(\lambda) = c_l\varphi(\lambda).$$

So $\mathbb{M}_{\boldsymbol{e}} \to \mathbb{M}_{\boldsymbol{f}}$, $g(\lambda) \mapsto g(\lambda)\varphi(\lambda)$ is an isomorphism in category $\mathcal{F}_{\mathrm{mer}}$.

Let $\Phi : \mathbb{M}_{\boldsymbol{e}} \to \mathbb{M}_{\boldsymbol{f}}$ be an isomorphism in category $\mathcal{F}_{\mathrm{mer}}$. Set $\varphi(\lambda) := \Phi(1)$. Then $\varphi(\lambda) \in \mathbb{M}^{\times}$. Applying $\Phi K_l(z) = K_l(z)\Phi$ to 1 we get

$$\varphi(\lambda + \hbar\epsilon_l) f_l(z) = \varphi(\lambda) e_l(z).$$

So $e_l(z)/f_l(z) = \varphi(\lambda + \hbar\epsilon_l)/\varphi(\lambda)$, being independent of z, is a constant function $c_l \in \mathbb{C}^{\times}$. We have $e_l(z) = f_l(z)c_l$ and $\varphi(\lambda + \hbar\epsilon_l) = c_l\varphi(\lambda)$. It follows that

$$\varphi(\lambda) = \varphi(\lambda + \hbar\epsilon_1 + \hbar\epsilon_2 + \cdots + \hbar\epsilon_N) = c_1 c_2 \cdots c_N \varphi(\lambda),$$

which implies $c_1 c_2 \cdots c_N = 1$. So $\boldsymbol{e}$ and $\boldsymbol{f}$ have the same image in $\mathcal{M}$. □

For each $\boldsymbol{f} \in \mathcal{M}$, let us fix a preimage $\boldsymbol{f}'$ in $(\mathbb{M}_{\mathbb{C}}^{\times})^N$ and set $\mathbb{M}(\boldsymbol{f}) := \mathbb{M}_{\boldsymbol{f}'}$. Then the isomorphism classes $[\mathbb{M}(\boldsymbol{f})]$ for $\boldsymbol{f} \in \mathcal{M}$ form a $\mathbb{Z}$-basis of $K_0(\mathcal{F}_{\mathrm{mer}})$. When no confusion arises, we identify an element of $(\mathbb{M}_{\mathbb{C}}^{\times})^N$ with its image in $\mathcal{M}$.

Lemma 1.6. *Let V be in category $\mathcal{F}_{\mathrm{mer}}$. Assume B is an ordered basis of V with respect to which the matrices of the difference operators $K_l(z)$ are upper triangular. Then for $b \in B$ and $1 \le l \le N$ there exist $\varphi_b(\lambda) \in \mathbb{M}^\times$ and $f_{b,l}(z) \in \mathbb{M}_{\mathbb{C}}^\times$ such that*

$$[K_l]_{bb}(z;\lambda) = f_{b,l}(z)\frac{\varphi_b(\lambda)}{\varphi_b(\lambda + \hbar\epsilon_l)}.$$

Recall that $[K_l]_{bb}(z;\lambda)$ is the coefficient of b in $K_l(z)b$. This lemma says that if the matrices of the $K_l(z)$ are upper triangular, then their diagonal entries must be of the form $f(z)h(\lambda)$, and the $h(\lambda)$ can be gauged away uniformly.

More precisely, the new basis $\{\varphi_b(\lambda)b \mid b \in B\}$ with the ordering induced from B satisfies (M3) in Definition 1.4; the diagonal entry of $K_l(z)$ associated to $\varphi_b(\lambda)b$ is $f_{b,l}(z)$. This yields the following identity in the Grothendieck group $K_0(\mathcal{F}_{\mathrm{mer}})$:

$$[V] = \sum_{b\in B}[\mathbb{M}(f_{b,1}(z), f_{b,2}(z), \dots, f_{b,N}(z))].$$

Proof. Write $B = \{b_1 < b_2 < \cdots < b_m\}$. We proceed by induction on the dimension $m = \dim(V)$. If $m = 1$, then there exist $\boldsymbol{f} = (f_1(z), f_2(z), \dots, f_N(z)) \in (\mathbb{M}_{\mathbb{C}}^\times)^N$ and an isomorphism $\Phi : \mathbb{M}_{\boldsymbol{f}} \to V$ in category $\mathcal{F}_{\mathrm{mer}}$. Let $\Phi(1) = \varphi(\lambda)b_1$. Then applying $\Phi K_l(z) = K_l(z)\Phi$ to 1 we obtain the desired identity

$$f_l(z)\varphi(\lambda) = [K_l]_{b_1b_1}(z;\lambda)\varphi(\lambda + \hbar\epsilon_l).$$

If $m > 1$, then the subspace V' of V spanned by $(b_1, b_2, \dots, b_{m-1})$ is stable by the $K_l(z)$ and $D_l(z)$ by the triangularity assumption. So V' is an object of category $\mathcal{F}_{\mathrm{mer}}$ and we obtain a short exact sequence $0 \to V' \to V \to V/V' \to 0$. The rest is clear by applying the induction hypothesis to V' and V/V', which have ordered bases $\{b_1 < b_2 < \cdots < b_{m-1}\}$ and $\{b_m + V'\}$ respectively. □

Definition 1.7. $\mathcal{O}$ is the full subcategory of $\tilde{\mathcal{O}}$ consisting of $\mathcal{E}$-modules X such that $X[\mu]$ endowed with the action of the $\mathcal{D}_l(z)$ belongs to $\mathcal{F}_{\mathrm{mer}}$ for all $\mu \in \mathrm{wt}(X)$.

The definition of $\tilde{\mathcal{O}}$ is standard as in the cases of Kac–Moody algebras [Kac 1990] and quantum affinizations [Hernandez 2005]. Definition 1.7 is a special feature of elliptic quantum groups. It is meant to loosen the dependence on the dynamical parameter λ.[6]

$\mathcal{O}$ is an abelian subcategory of $\tilde{\mathcal{O}}$. For X in category $\mathcal{O}$, (1.8) defines difference operators $K_l(z) : X \to X$ of bidegree (ϵ_l, ϵ_l) for $1 \le l \le N$.[7]

Following [Cavalli 2001, Definition 2.1], a nonzero weight vector of a module X in category $\tilde{\mathcal{O}}$ is called *singular* if it is annihilated by the $L_{ij}(z)$ for $1 \le j < i \le N$.

Lemma 1.8. *Let X be in category $\mathcal{O}$. If $v \in X$ is singular, then $K_i(z)v = \hat{L}_i(z)v$ for all $1 \le i \le N$.*

[6]For the elliptic quantum group associated to an arbitrary finite-dimensional simple Lie algebra, Gautam and Toledano Laredo [2017a, §2.3] defined a category of *integrable* modules on which the action of the elliptic Cartan currents, analogs of $\mathcal{D}_k(z)$, is independent of λ. The asymptotic modules that we will construct in Section 4 are not integrable.

[7]The $K_l(z)$ do not come from the elliptic quantum group, yet formally they are elliptic Cartan currents $K_l^+(z)$ in [Konno 2016, Corollary E.24], arising from a Gauss decomposition of an $\hat{L}$-matrix [Ding and Frenkel 1993].

Proof. Descending induction on i: for $i = N$ we have $K_N(z) = L_{NN}(z) = \hat{L}_N(z)$. Assume the statement for $i > N - t$ where $1 \leq t < N$. We need to prove the case $i = N - t$. Let α be the weight of v and let Y be the submodule of X generated by v. By [Cavalli 2001, Lemma 2.3], Y is linearly spanned by vectors of the form

$$L_{p_1q_1}(z_1)L_{p_2q_2}(z_2)\cdots L_{p_nq_n}(z_n)v,$$

where $1 \leq p_l \leq q_l \leq N$ and $z_l \in \mathbb{C}$ for $1 \leq l \leq n$. So $\alpha + \epsilon_p - \epsilon_q \notin \mathrm{wt}(Y)$ for $1 \leq p < q \leq N$, and any nonzero vector $\omega \in Y[\alpha]$ is singular. Apply $\mathcal{D}_k(z)$ to ω. At the right-hand side of (1.6) only the term $\sigma = \mathrm{Id}$ is nonzero and equal to $\hat{L}_N(z)\hat{L}_{N-1}(z+\hbar)\cdots\hat{L}_{N-k+1}(z+(k-1)\hbar)\omega$ by (1.7). It follows that

$$\begin{aligned}\mathcal{D}_{t+1}(z)v &= \underline{\hat{L}_N(z)}\hat{L}_{N-1}(z+\hbar)\cdots\hat{L}_{N-t+1}(z+(t-1)\hbar)\hat{L}_{N-t}(z+t\hbar)v\\ &= K_N(z)\underline{\hat{L}_{N-1}(z+\hbar)}\cdots\hat{L}_{N-t+1}(z+(t-1)\hbar)\hat{L}_{N-t}(z+t\hbar)v\\ &\;\;\vdots\\ &= K_N(z)K_{N-1}(z+\hbar)\cdots K_{N-t+1}(z+(t-1)\hbar)\hat{L}_{N-t}(z+t\hbar)v.\end{aligned}$$

Here we applied the induction hypothesis to $N, N-1, \ldots, N-t+1$ successively to singular vectors to the right of the underlines. Since the $K_l(z)$ are invertible, in view of (1.8) we must have $\hat{L}_{N-t}(z+t\hbar)v = K_{N-t}(z+t\hbar)v$. □

We extend the q-character theory of H. Knight and Frenkel and Reshetikhin to category $\mathcal{O}$, as in [Felder and Zhang 2017, §3]. Take the product group $\mathcal{M}_{\mathrm{w}} := \mathcal{M} \times \mathfrak{h}$, by viewing $\mathfrak{h}$ as an additive group. Let $\varpi : \mathcal{M}_{\mathrm{w}} \twoheadrightarrow \mathfrak{h}$ be the projection to the second component.

As in [Hernandez and Leclerc 2016, §3.2], let $\mathcal{M}_{\mathrm{t}}$ be the set of formal sums $\sum_{\boldsymbol{f} \in \mathcal{M}_{\mathrm{w}}} c_{\boldsymbol{f}} \boldsymbol{f}$ with integer coefficients $c_{\boldsymbol{f}} \in \mathbb{Z}$ such that for $\mu \in \mathfrak{h}$, all but finitely many $c_{\boldsymbol{f}}$ with $\varpi(\boldsymbol{f}) = \mu$ is zero; the set $\{\varpi(\boldsymbol{f}) : c_{\boldsymbol{f}} \neq 0\}$ is contained in a finite union of cones $\nu + \boldsymbol{Q}_-$ with $\nu \in \mathfrak{h}$. Make $\mathcal{M}_{\mathrm{t}}$ into a ring; addition is the usual one of formal sums and multiplication is induced from that of $\mathcal{M}_{\mathrm{w}}$.

Definition 1.9. Let X be in category $\mathcal{O}$. For $\mu \in \mathrm{wt}(X)$, since $X[\mu]$ equipped with the difference operators $\mathcal{D}_k(z)$ is in category $\mathcal{F}_{\mathrm{mer}}$, in the Grothendieck group of which we have $[X[\mu]] = \sum_{i=1}^{\dim X[\mu]} [\mathbb{M}(\boldsymbol{f}^{\mu,i})]$ where $\boldsymbol{f}^{\mu,i} \in \mathcal{M}$ for $1 \leq i \leq \dim X[\mu]$. Each of the $(\boldsymbol{f}^{\mu,i}; \mu) \in \mathcal{M}_{\mathrm{w}}$ is called an *e-weigth* of X. Let $\mathrm{wt}_e(X)$ be the set of e-weigths of X. The *q-character* of X is defined to be

$$\chi_q(X) := \sum_{\mu \in \mathrm{wt}(X)} \sum_{i=1}^{\dim X[\mu]} (\boldsymbol{f}^{\mu,i}; \mu) \in \mathcal{M}_{\mathrm{t}}.$$

Proposition 1.10. *Let X and Y be in category $\mathcal{O}$. The $\mathcal{E}$-module $X \overline{\otimes} Y$ is also in category $\mathcal{O}$ and $\chi_q(X \overline{\otimes} Y) = \chi_q(X)\chi_q(Y)$.*

Proof. Clearly $X \overline{\otimes} Y$ is in category $\tilde{\mathcal{O}}$. Let us verify property (M3) of Definition 1.4. The idea is almost the same as that of [Felder and Zhang 2017, Proposition 3.9], which in turn followed [Frenkel and Reshetikhin 1999, §2.4]. For $\alpha, \beta \in \mathfrak{h}$, let us choose ordered bases $(v_i^\alpha)_{1 \leq i \leq p_\alpha}$ and $(w_j^\beta)_{1 \leq j \leq q_\beta}$ for $X[\alpha]$

and $Y[\beta]$, respectively, satisfying (M3). Note that $(v_i^\alpha \overline{\otimes} w_j^\beta)_{\alpha,\beta,i,j}$ forms a basis $\mathcal{B}$ of $X\overline{\otimes}Y$. Choose a partial order $\trianglelefteq$ on $\mathcal{B}$ with the properties

$$\begin{aligned} v_i^\alpha \overline{\otimes} w_j^\beta &\trianglelefteq v_r^\alpha \overline{\otimes} w_s^\beta \quad \text{if } i \le r \text{ and } j \le s, \\ v_i^\alpha \overline{\otimes} w_j^\beta &\triangleleft v_r^\gamma \overline{\otimes} w_s^\delta \quad \text{if } \gamma \prec \alpha \text{ and } \beta \prec \delta. \end{aligned}$$

For $1 \le k \le N$, by Corollary 1.2, $\mathcal{D}_k^{X\overline{\otimes}Y}(z)(v_r^\gamma \overline{\otimes} w_s^\delta) = \mathcal{D}_k^X(z) v_r^\gamma \overline{\otimes} \mathcal{D}_k^Y(z) w_s^\delta + Z$ where Z is a finite sum of vectors in $X[\gamma + \varpi_{N-k} + \eta]\overline{\otimes} Y[\delta - \varpi_{N-k} - \eta]$ for $\eta \in \mathfrak{h}$ such that $-\varpi_{N-k} \prec \eta$. So the ordered basis $\mathcal{B}$ induces an upper triangular matrix for $\mathcal{D}_k^{X\overline{\otimes}Y}(z)$ whose diagonal entry associated to $v_r^\gamma \overline{\otimes} w_s^\delta$ is the product of those associated to v_r^γ and w_s^δ. This implies (M3) for the weight spaces $(X\overline{\otimes}Y)[\alpha]$ with bases $\mathcal{B} \cap (X\overline{\otimes}Y)[\alpha]$ and the multiplicative formula of q-characters as well. $\square$

For $f(z) \in \mathbb{M}_{\mathbb{C}}^\times$ and $\alpha \in \mathfrak{h}$ we make the simplifications

$$f(z) := (f(z), \ldots, f(z); 0) \quad \text{and} \quad e^\alpha := (1, \ldots, 1; \alpha) \in \mathcal{M}_{\mathrm{w}}.$$

Definition 1.11. Let $1 \le i, k \le N$ such that $i \ne N$. Set $\ell_k := (N-k-1)/2$. For $a \in \mathbb{C}$, define the following elements of $\mathcal{M}_{\mathrm{w}}$:

$$A_{i,a} := \Big(\underbrace{1, \ldots, 1}_{i-1}, \frac{\theta(z+(a-\ell_i)\hbar)}{\theta(z+(a-\ell_i-1)\hbar)}, \frac{\theta(z+(a-\ell_i)\hbar)}{\theta(z+(a-\ell_i+1)\hbar)}, \underbrace{1, \ldots, 1}_{N-i-1}; \alpha_i\Big).$$

$$\Psi_{k,a} := \big(\underbrace{\theta(z+(a-\ell_k)\hbar), \ldots, \theta(z+(a-\ell_k)\hbar)}_{k}, \underbrace{1, \ldots, 1}_{N-k}; a\varpi_k\big).$$

$$Y_{k,a} := \Bigg(\underbrace{\frac{\theta\big(z+\big(a-\ell_k+\frac{1}{2}\big)\hbar\big)}{\theta\big(z+\big(a-\ell_k-\frac{1}{2}\big)\hbar\big)}, \ldots, \frac{\theta\big(z+\big(a-\ell_k+\frac{1}{2}\big)\hbar\big)}{\theta\big(z+\big(a-\ell_k-\frac{1}{2}\big)\hbar\big)}}_{k}, \underbrace{1, \ldots, 1}_{N-k}; \varpi_k\Bigg).$$

$$\boxed{k}_a := \Bigg(\underbrace{\frac{\theta(u+\hbar)\theta(u-\hbar)}{\theta(u)^2}, \ldots, \frac{\theta(u+\hbar)\theta(u-\hbar)}{\theta(u)^2}}_{k-1}, \frac{\theta(u+\hbar)}{\theta(u)}, \underbrace{1, \ldots, 1}_{N-k}; \epsilon_k\Bigg)\Bigg|_{u=z+a\hbar}.$$

$A_{i,a}$, $Y_{k,a}$ and $\Psi_{k,a}$ are elliptic analogs of generalized simple roots, fundamental ℓ-weight [Frenkel and Reshetikhin 1999] and prefundamental weight [Hernandez and Jimbo 2012]. Set $c_{ij} := 2\delta_{ij} - \delta_{i,j\pm 1}$ and $Y_{0,a} = \Psi_{0,a} := 1$. Then (in the products $1 \le j \le N$)

$$Y_{k,a} = \frac{\Psi_{k,a+\frac{1}{2}}}{\Psi_{k,a-\frac{1}{2}}} \quad \text{and} \quad A_{i,a} = \prod_j \frac{\Psi_{j,a+\frac{c_{ij}}{2}}}{\Psi_{j,a-\frac{c_{ij}}{2}}} = Y_{i,a-\frac{1}{2}} Y_{i,a+\frac{1}{2}} \prod_{j=i\pm 1} Y_{j,a}^{-1}. \tag{1.9}$$

The interplay of A, Ψ is the source of the three-term Baxter's Relation (5.32) in category $\mathcal{O}$. Note that A and Y can also be written in terms of $\boxed{k}$:

$$A_{i,a} = \boxed{i}_{a-\ell_i} \boxed{i+1}_{a-\ell_i}^{-1}, \qquad Y_{k,a} = \prod_{j=1}^{k} \boxed{j}_{a-\ell_k-\frac{1}{2}+j-k}, \tag{1.10}$$

$$\Psi_{N,a} = \theta\big(z + \big(a + \tfrac{1}{2}\big)\hbar\big), \qquad Y_{N,a} = \frac{\theta(z+(a+1)\hbar)}{\theta(z+a\hbar)}. \tag{1.11}$$

1D. *Vector representations.* Let $\boldsymbol{V} := \oplus_{i=1}^{N} \mathbb{M} v_i$ with $\mathfrak{h}$-grading $\boldsymbol{V}[\epsilon_i] = \mathbb{M} v_i$. Rewriting (1.1) in the form of (1.2), we obtain an $\mathcal{E}$-module structure on $\boldsymbol{V}$:

$$L_{ij}(z) v_k = \sum_{l=1}^{N} \frac{\theta(z+\hbar)}{\theta(z)} R_{il}^{jk}(z;\lambda) v_l.$$

The factor $\theta(z+\hbar)/\theta(z)$ is used to simplify the q-character, see (1.12).

If $i \le N-k+1$, since $L_{pq}(z)v_i = 0$ for all $N \ge p > q > N-k$, only the term $\sigma = \mathrm{Id}$ in (1.6) survives and

$$\mathcal{D}_k(z) v_i = \frac{\mu_{\boldsymbol{r}}(\Theta_k(\lambda))}{\mu_{\boldsymbol{l}}(\Theta_k(\lambda))} \prod_{j=N}^{N-k+1} L_{jj}(z+(N-j)\hbar) v_i = g_k^i(z;\lambda) v_i,$$

$$g_k^i(z;\lambda) = \prod_{j>N-k} \frac{\theta(\lambda_{ij}+\hbar)}{\theta(\lambda_{ij})} \quad \text{for } i \le N-k,$$

$$g_k^{N-k+1}(z;\lambda) = \frac{\theta(z+k\hbar)}{\theta(z+(k-1)\hbar)}.$$

If $i > N-k+1$, then $L_{N-k+1,i}(z) v_{N-k+1} = \theta(\hbar)\theta(z+\lambda_{N-k+1,i}) v_i / (\theta(z)\theta(\lambda_{N-k+1,i}))$. By Corollary 1.2, $\mathcal{D}_k(z) v_i = g_k^{N-k+1}(z;\lambda) v_i$. Let us perform a change of basis (see [Konno 2016, (E.2)])

$$\tilde{v}_i := v_i \prod_{l>i} \theta(\lambda_{il}+\hbar) \in \boldsymbol{V}[\epsilon_i].$$

After a direct computation, we obtain

$$\mathcal{D}_k(z)\tilde{v}_i = \tilde{v}_i \times \begin{cases} 1 & \text{for } i \le N-k, \\ \frac{\theta(z+k\hbar)}{\theta(z+(k-1)\hbar)} & \text{for } i > N-k. \end{cases} \tag{1.12}$$

The basis $\{\tilde{v}_1 < \tilde{v}_2 < \cdots < \tilde{v}_N\}$ of $\boldsymbol{V}$ satisfies property (M3) of Definition 1.4, so $\boldsymbol{V}$ is in category $\mathcal{O}$. For $a \in \mathbb{C}$, let $\boldsymbol{V}(a)$ be the pullback of $\boldsymbol{V}$ by the spectral parameter shift Φ_a in (1.4). Naturally $\boldsymbol{V}(a)$ is in category $\mathcal{O}$; it is called a *vector representation*. Combining with (1.8) we have

$$\chi_q(\boldsymbol{V}(a)) = \boxed{1}_a + \boxed{2}_a + \cdots + \boxed{N}_a.$$

1E. *Highest weight modules.* Let X be in category $\tilde{\mathcal{O}}$. A nonzero weight vector $v \in X[\alpha]$ is called a *highest weight vector* if it is singular and $\hat{L}_k(z)v = f_k(z)v$ for $1 \le k \le N$; here the $f_k(z) \in \mathbb{M}_{\mathbb{C}}^{\times}$. Call $(f_1(z), f_2(z), \ldots, f_N(z); \alpha) \in \mathcal{M}_{\mathrm{w}}$ the highest weight of v; by Lemma 1.8 it belongs to $\mathrm{wt}_e(X)$ if X is in category $\mathcal{O}$.

If there is a highest weight vector $v \in X[\alpha]$ of X which also generates the whole module, then X is called a *highest weight module*, see [Cavalli 2001, Definition 2.1]. In this case, by [Cavalli 2001, Lemma 2.3], $X[\alpha] = \mathbb{M}v$ and $\mathrm{wt}(X) \subseteq \alpha + \boldsymbol{Q}_-$, so the highest weight vector is unique up to scalar product. This implies that X admits a unique irreducible quotient. The highest weight of v is also called the highest weight of X; it is of multiplicity one in $\chi_q(X)$ if X is in category $\mathcal{O}$.

All irreducible modules in category $\mathcal{O}$ are of highest weight.

By [Cavalli 2001, Theorem 2.8] two irreducible highest weight modules in category $\tilde{\mathcal{O}}$ are isomorphic if and only if their highest weights are identical in $\mathcal{M}_{\mathrm{w}}$; all singular vectors of an irreducible highest weight module in category $\tilde{\mathcal{O}}$ are proportional. It follows that the q-characters distinguish irreducible modules in category $\mathcal{O}$.

Let $\mathcal{R}$ be the set of $\boldsymbol{d} \in \mathcal{M}_{\mathrm{w}}$ which appears as the highest weight of an irreducible module in category $\mathcal{O}$. For $\boldsymbol{d} \in \mathcal{R}$, let us fix an irreducible module $S(\boldsymbol{d})$ in category $\mathcal{O}$ of highest weight $\boldsymbol{d}$. Let $\mathcal{R}_0$ and $\mathcal{R}_{\mathrm{fd}}$ be the set of $\boldsymbol{d} \in \mathcal{R}$ such that $S(\boldsymbol{d})$ is one-dimensional and finite-dimensional, respectively.

We shall need the *completed* Grothendieck group $K_0(\mathcal{O})$. Its definition is the same as that in [Hernandez and Leclerc 2016, §3.2]; elements are formal sums $\sum_{\boldsymbol{d}\in\mathcal{R}} c_{\boldsymbol{d}}[S(\boldsymbol{d})]$ with integer coefficients $c_{\boldsymbol{d}} \in \mathbb{Z}$ such that $\oplus_{\boldsymbol{d}} S(\boldsymbol{d})^{\oplus|c_{\boldsymbol{d}}|}$ is in category $\mathcal{O}$ and addition is the usual one of formal sums. As in the case of Kac–Moody algebras [Kac 1990, §9.6], for $\boldsymbol{d} \in \mathcal{R}$ the multiplicity $m_{\boldsymbol{d},X}$ of $S(\boldsymbol{d})$ in any object X of category $\mathcal{O}$ is well-defined due to Definition 1.7, and $[X] := \sum_{\boldsymbol{d}} m_{\boldsymbol{d},X}[S(\boldsymbol{d})]$ belongs to $K_0(\mathcal{O})$. In the case $X = S(\boldsymbol{d})$ the right-hand side is simply $[S(\boldsymbol{d})]$ as $m_{\boldsymbol{e},S(\boldsymbol{d})} = \delta_{\boldsymbol{d},\boldsymbol{e}}$ for $\boldsymbol{e} \in \mathcal{R}$.

By Proposition 1.10, $K_0(\mathcal{O})$ is endowed with a ring structure with multiplication $[X][Y] = [X \overline{\otimes} Y]$ for X and Y in category $\mathcal{O}$. Together with Definition 1.9, we obtain

Corollary 1.12. *The assignment $[X] \mapsto \chi_q(X)$ defines an injective morphism of rings $\chi_q : K_0(\mathcal{O}) \to \mathcal{M}_{\mathrm{t}}$. In particular, $K_0(\mathcal{O})$ is commutative.*

Let $\mathcal{O}_{\mathrm{fd}}$ be the full subcategory of $\mathcal{O}$ consisting of finite-dimensional modules. It is abelian and monoidal. Its Grothendieck ring $K_0(\mathcal{O}_{\mathrm{fd}})$ admits a $\mathbb{Z}$-basis $[S(\boldsymbol{d})]$ for $\boldsymbol{d} \in \mathcal{R}_{\mathrm{fd}}$, and is commutative as a subring of $K_0(\mathcal{O})$.

By Proposition 1.10, $S(\boldsymbol{d})\overline{\otimes}S(\boldsymbol{e})$ admits an irreducible subquotient $S(\boldsymbol{de})$, so the three sets $\mathcal{R} \supset \mathcal{R}_{\mathrm{fd}} \supset \mathcal{R}_0$ are submonoids of $\mathcal{M}_{\mathrm{w}}$.

Lemma 1.13. *Let $\boldsymbol{d} = ((f_k(z))_{1\le k\le N}; \mu) \in \mathcal{M}_{\mathrm{w}}$.*

(i) *Suppose $\boldsymbol{d} \in \mathcal{R}$. Then for $1 \le k < N$ we have*

$$\frac{f_k(z)}{f_{k+1}(z)} = c \prod_{l=1}^{n} \frac{\theta(z + a_l\hbar)}{\theta(z + b_l\hbar)} \quad \textit{and} \quad \mu_{k,k+1} = \sum_{l=1}^{n} (a_l - b_l)$$

for certain $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{C}$ and $c \in \mathbb{C}^\times$.

(ii) *If $\boldsymbol{d} \in \mathcal{R}_{\mathrm{fd}}$, then (i) holds and after a rearrangement of the a_l, b_l we have $a_l - b_l \in \mathbb{Z}_{\ge 0} + \hbar^{-1}\Gamma$ for all l.*

(iii) $\boldsymbol{d} \in \mathcal{R}_0$ *if and only if (ii) holds with* $a_l - b_l \in \hbar^{-1}\Gamma$ *for all* l.

Proof. Results (i) and (iii) are essentially [Felder and Varchenko 1996b, Theorems 6 and 9], which can be proved as in [Felder and Zhang 2017, Theorem 4.1] by replacing L_{+-}, L_{-+} therein with $L_{k,k+1}$, $L_{k+1,k}$. Result (ii) comes from either [Cavalli 2001, Theorem 5.1] or [Felder and Zhang 2017, Corollary 4.6]. □

As examples $Y_{N,a}, \Psi_{N,a} \in \mathcal{R}_0$. Call an e-weigth $\boldsymbol{e} \in \mathcal{M}_{\mathrm{w}}$ *dominant* or *rational* if $\boldsymbol{e} = \boldsymbol{d}\boldsymbol{m}$ where $\boldsymbol{d} \in \mathcal{R}_0$ and $\boldsymbol{m}$ is a product of the $Y_{i,a}$ or the $\Psi_{i,a}\Psi_{i,b}^{-1}$, respectively with $a, b \in \mathbb{C}$ and $1 \le i \le N$. Lemma 1.13 implies that all elements of $\mathcal{R}_{\mathrm{fd}}$ or $\mathcal{R}$ are dominant or rational, respectively.

Theorem 1.14 [Cavalli 2001]. *$\mathcal{R}_{\mathrm{fd}}$ is the set of dominant e-weigths.*

Proof. It suffices to prove $Y_{n,a} \in \mathcal{R}_{\mathrm{fd}}$ for $1 \le n < N$. Note that $V(w)$ and γ from [Cavalli 2001, (1.19)] correspond to our $\boldsymbol{V}(-w/\hbar)\bar{\otimes}S(\theta(z-w)/\theta(z-w-\hbar))$ and $-\hbar$. Let us rephrase [Cavalli 2001, Theorem 4.4] in terms of the $\boldsymbol{V}$ by replacing z and w in [loc. cit.] with $-a\hbar$ and z.

The $\mathcal{E}$-module $\boldsymbol{V}(a)\bar{\otimes}\boldsymbol{V}(a+1)\bar{\otimes}\cdots\bar{\otimes}\boldsymbol{V}(a+n-1)$ admits an irreducible quotient S which contains a singular vector ω of weight ϖ_n such that $\hat{L}_k(z)\omega = \Lambda_k(z)g_k(\lambda)\omega$ where for $1 \le k \le N$ (set $\delta_{k\le n} = 1$ if $1 \le k \le n$ and $\delta_{k \le n} = 0$ if $n < k \le N$)

$$\Lambda_k(z) = \frac{\theta(z+(a+1)\hbar)}{\theta(z+a\hbar)}\,\frac{\theta(z+(a+n)\hbar)}{\theta(z+(a+n-\delta_{k\le n})\hbar)} \quad \text{for } g_k(\lambda) \in \mathbb{M}^{\times}.$$

As a subquotient of tensor products of vector representations, S belongs to category $\mathcal{O}$. By Lemma 1.6, the $g_k(\lambda)$ can be gauged away, and the highest weight of S is $\Lambda_N(z)Y_{n,a-1+(N+n)/2} \in \mathcal{R}_{\mathrm{fd}}$. This implies $Y_{n,a-1+(N+n)/2} \in \mathcal{R}_{\mathrm{fd}}$. □

A sharp difference from the affine case [Hernandez and Jimbo 2012, Theorem 3.11] is that category $\mathcal{O}$ does not admit prefundamental modules, i.e., $\Psi_{r,a} \notin \mathcal{R}$ if $r < N$. One might want to introduce a larger category with well-behaved q-character theory, so that modules of highest weight $\Psi_{r,a}$ exist. For this purpose, the finite-dimensionality of weight spaces should be dropped because of [Felder and Varchenko 1996b, Theorem 9]. The recent work [Bittmann 2017] on representations of affine quantum groups is in this direction.

1F. *Young tableaux and q-character formula.* Let $\mathcal{P}$ be the set partitions with at most N parts, i.e., N-tuples of nonnegative integers $(\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N)$. To such a partition we associate a Young diagram

$$Y_\mu := \{(i, j) \in \mathbb{Z}^2 \mid 1 \le i \le N, 1 \le j \le \mu_i\},$$

and the set $\mathcal{B}_\mu$ of Young tableaux of shape Y_μ. We put the Young diagram at the northwest position so that $(i, j) \in Y_\mu$ corresponds to the box at the i-th row (from bottom to top) and j-th column (from right to left). By a tableau we mean a function $T : Y_\mu \to \{1 < 2 < \cdots < N\}$ weakly increasing at each row (from left to right) and strictly increasing at each column (from top to bottom).

For $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N) \in \mathcal{P}$ and $a \in \mathbb{C}$, we have the dominant e-weigth

$$\theta_{\mu,a} := \left(\frac{\theta(z+(a+\mu_1)\hbar)}{\theta(z+a\hbar)}, \frac{\theta(z+(a+\mu_2)\hbar)}{\theta(z+a\hbar)}, \ldots, \frac{\theta(z+(a+\mu_N)\hbar)}{\theta(z+a\hbar)}; \sum_{j=1}^{N}\mu_j\epsilon_j\right).$$

The associated irreducible module in category $\mathcal{O}_{\mathrm{fd}}$ is denoted by $S_{\mu,a}$.

Theorem 1.15. *Let $\mu \in \mathcal{P}$ and $a \in \mathbb{C}$. For the $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$-module $S_{\mu,a}$ we have*

$$\chi_q(S_{\mu,a}) = \sum_{T \in \mathcal{B}_\mu} \prod_{(i,j)\in Y_\mu} \boxed{T(i,\, j)}_{a+j-i} \in \mathcal{M}_{\mathrm{t}}. \tag{1.13}$$

For $\nu = (1 \geq 0 \geq 0 \geq \cdots \geq 0)$, we have $S_{\nu,a} \cong \boldsymbol{V}(a)$, and (1.13) specializes to the q-character formula in Section 1D. As an illustration of the theorem, let $N = 3$ and $\mu = (2 \geq 1 \geq 0)$. Pictorially $\mathcal{B}_\mu$ consists of

$$\begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array},\ \begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{3} \end{array},\ \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{2} \end{array},\ \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{3} \end{array},\ \begin{array}{cc} & \boxed{1} \\ \boxed{3} & \boxed{3} \end{array},\ \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \boxed{3} \end{array},\ \begin{array}{cc} & \boxed{2} \\ \boxed{2} & \boxed{3} \end{array},\ \begin{array}{cc} & \boxed{2} \\ \boxed{3} & \boxed{3} \end{array}.$$

The fourth tableau gives rise to the term $\boxed{2}_{a+1}\boxed{3}_{a}\boxed{1}_{a-1}$ in $\chi_q(S_{\mu,a})$.

Remark 1.16. Theorem 1.15 is an elliptic analog of the q-character formula for affine quantum groups [Frenkel and Mukhin 2002, Lemma 4.7]. In principle it can be deduced from the functor of Gautam and Toledano Laredo [2017a, § 6]. This is a functor from finite-dimensional representations of affine quantum groups to those of elliptic quantum groups (including our $S_{\mu,a}$), and it respects affine and elliptic q-characters.

The proof of Theorem 1.15 will be given in Section 2D. It is in the spirit of [Frenkel and Mukhin 2002], based on small elliptic quantum groups of Tarasov and Varchenko [2001].

2. Small elliptic quantum group and evaluation modules

The aim of this section is to prove Corollary 1.2 and Theorem 1.15.

Recall that $\mathfrak{h}$ is the $\mathbb{C}$-vector space generated by the ϵ_i for $1 \leq i \leq N$ subject to the relation

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_N = 0.$$

For $1 \leq k \leq N$, define the $\mathbb{C}$-vector space $\mathfrak{h}_k$ to be the quotient of $\mathfrak{h}$ by $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_{N-k} = 0$. (By convention $\mathfrak{h}_N = \mathfrak{h}$.) The quotient $\mathfrak{h} \twoheadrightarrow \mathfrak{h}_k$ induces an embedding $\mathbb{M}_{\mathfrak{h}_k} \hookrightarrow \mathbb{M}$.

Let $\mathcal{E}_k^{\mathfrak{h}}$ and $\mathcal{E}_k$ be the $\mathfrak{h}$-algebra and $\mathfrak{h}_k$-algebra, respectively, generated by the $L_{ij}(z)$ for $N-k < i,\, j \leq N$ subject to relation (1.2) with summations $N-k < p, q \leq N$. (This makes sense because the $R_{ij}^{pq}(z;\lambda)$ for $N-k < i,\, j,\, p,\, q \leq N$ belong to $\mathbb{M}_{\mathfrak{h}_k}$.) The following defines an $\mathfrak{h}_k$-algebra morphism

$$\Delta_k : \mathcal{E}_k \to \mathcal{E}_k \tilde{\otimes} \mathcal{E}_k \quad \text{and} \quad L_{ij}(z) \mapsto \sum_{p=N-k+1}^{N} L_{ip}(z) \tilde{\otimes} L_{pj}(z).$$

One has natural algebra morphisms $\mathcal{E}_k \to \mathcal{E}_k^{\mathfrak{h}} \to \mathcal{E}$ sending $L_{ij}(z)$ to itself; the second is an $\mathfrak{h}$-algebra morphism. $\mathcal{D}_1(z), \mathcal{D}_2(z), \ldots, \mathcal{D}_k(z)$ from (1.6) are well-defined in $\mathcal{E}_k^{\mathfrak{h}}$ and $\mathcal{E}_k$. Their images in $\mathcal{E}$ are the first k elliptic quantum minors.

2A. ***Proof of Corollary 1.2.*** The $\mathfrak{h}_k$-algebra with coproduct $(\mathcal{E}_k, \Delta_k)$ is isomorphic to the usual elliptic quantum group $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_k)$; here we view $\mathfrak{h}_k$ as a Cartan subalgebra of $\mathfrak{sl}_k$ so that $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_k)$ is an $\mathfrak{h}_k$-algebra. Under this isomorphism, by (1.6), $\mathcal{D}_k(z) \in \mathcal{E}_k$ corresponds to the k-th elliptic quantum minor of $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_k)$. So Theorem 1.1 can be applied to $(\mathcal{E}_k, \mathcal{D}_k(z), \Delta_k)$ and then to the algebra morphism $\mathcal{E}_k \to \mathcal{E}$. The first statement of the corollary is obvious, and the second is based on the fact that for $i, j > N-k$ the difference $\Delta - \Delta_k$ at $L_{ij}(z)$ is a finite sum over $\alpha \in \mathfrak{h}$ of elements in $\mathcal{E}_{\epsilon_i,\alpha}\tilde{\otimes}\mathcal{E}_{\alpha,\epsilon_j}$ with $\epsilon_{N-k+1} \prec \alpha$ and so $\epsilon_i, \epsilon_j \prec \alpha$. $\square$

We believe $0 \neq \alpha + \varpi_{N-k} \in \boldsymbol{Q}_+$ in Corollary 1.2, as in [Damiani 1998, §7] and [Zhang 2016, §3].

2B. ***Small elliptic quantum group of Tarasov–Varchenko.*** Let us define the linear form $\lambda_i \in \mathfrak{h}^*$ by taking i-th component for $1 \leq i \leq N$,

$$x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_N\epsilon_N \mapsto x_i - \frac{1}{N}(x_1 + x_2 + \cdots + x_N).$$

The linear form λ_{ij} of Section 1 is $\lambda_i - \lambda_j$. For $\gamma \in \mathfrak{h}$ and $1 \leq i, j \leq N$, set $\gamma_i := \lambda_i(\gamma)$ and $\gamma_{ij} := \gamma_i - \gamma_j$ as *complex numbers*. We hope this is not to be confused with the previously defined *vectors* $\lambda_i \in \mathfrak{h}^*$ and $\epsilon_i, \alpha_i, \varpi_i \in \mathfrak{h}$.

Following [Tarasov and Varchenko 2001, §3], let $\mathbb{M}_2$ be the ring of meromorphic functions $f(\lambda^{\{1\}}, \lambda^{\{2\}})$ of $(\lambda^{\{1\}}, \lambda^{\{2\}}) \in \mathfrak{h} \oplus \mathfrak{h}$ whose location of singularities in $\lambda^{\{1\}}$ does not depend on $\lambda^{\{2\}}$ and vice versa. For brevity, we write $f(\lambda^{\{1\}})$ or $f(\lambda^{\{2\}})$ instead of $f(\lambda^{\{1\}}, \lambda^{\{2\}})$ if the function does not depend on the other variable.

Definition 2.1 [Tarasov and Varchenko 2001]. The *small elliptic quantum group* $\mathfrak{e} := \mathfrak{e}_{\tau,\hbar}(\mathfrak{sl}_N)$ is the algebra with generators $\mathbb{M}_2$ and t_{ij} for $1 \leq i, j \leq N$ and subject to relations: $\mathbb{M}_2$ is a subalgebra. For $f(\lambda^{\{1\}}, \lambda^{\{2\}}) \in \mathbb{M}_2$ and $1 \leq i, j, k, l \leq N$,

$$\begin{aligned} t_{ij} f(\lambda^{\{1\}}, \lambda^{\{2\}}) &= f(\lambda^{\{1\}} + \hbar\epsilon_i, \lambda^{\{2\}} + \hbar\epsilon_j) t_{ij}, \\ t_{ij} t_{ik} &= t_{ik} t_{ij}, \\ t_{ik} t_{jk} &= \frac{\theta(\lambda^{\{1\}}_{ij} - \hbar)}{\theta(\lambda^{\{1\}}_{ij} + \hbar)} t_{jk} t_{ik} \qquad \text{for } i \neq j, \\ \frac{\theta(\lambda^{\{2\}}_{jl} - \hbar)}{\theta(\lambda^{\{2\}}_{jl})} t_{ij} t_{kl} - \frac{\theta(\lambda^{\{1\}}_{ik} - \hbar)}{\theta(\lambda^{\{1\}}_{ik})} t_{kl} t_{ij} &= \frac{\theta(\lambda^{\{1\}}_{ik} + \lambda^{\{2\}}_{jl})\theta(-\hbar)}{\theta(\lambda^{\{1\}}_{ik})\theta(\lambda^{\{2\}}_{jl})} t_{il} t_{kj} \qquad \text{for } i \neq k \text{ and } j \neq l. \end{aligned}$$

Here $\lambda^{\{1\}}_{ij} = \lambda^{\{1\}}_i - \lambda^{\{1\}}_j$ and $\lambda^{\{2\}}_{ij} = \lambda^{\{2\}}_i - \lambda^{\{2\}}_j$.

The small elliptic quantum group $\mathfrak{e}$ is equipped with an $\mathfrak{h}$-algebra structure: Elements of $\mathbb{M}_2$ are of bidegree $(0, 0)$. t_{ij} is of bidegree (ϵ_j, ϵ_i). The moment maps are given by

$$\mu_l(g(\lambda)) = g(\lambda^{\{2\}}) \quad \text{and} \quad \mu_r(g(\lambda)) = g(\lambda^{\{1\}}).$$

Let X be an object of $\mathcal{V}_{\mathrm{ft}}$. A representation ρ of $\mathfrak{e}$ on X is a morphism of $\mathfrak{h}$-algebras $\rho : \mathfrak{e} \to \mathbb{D}^X$ such that for $f(\lambda^{\{1\}}, \lambda^{\{2\}}) \in \mathbb{M}_2$ and $v \in X[\gamma]$,

$$\rho(f(\lambda^{\{1\}}, \lambda^{\{2\}})) : v \mapsto f(\lambda, \lambda + \hbar\gamma)v.$$

A morphism of two representations (ρ, X) and (σ, Y) is a morphism $\Phi : X \to Y$ in $\mathcal{V}_{\mathrm{ft}}$ such that $\Phi\rho(t_{ij}) = \sigma(t_{ij})\Phi$ for $1 \le i, j \le N$. Let rep be the category of $\mathfrak{e}$-modules. The following result is [Tarasov and Varchenko 2001, Corollary 3.4].

Corollary 2.2. *Let (ρ, X) be a representation of $\mathfrak{e}$ on X. Then, for $a \in \mathbb{C}$,*

$$L_{ij}(z) \mapsto \frac{\theta(z + a\hbar + \lambda_i^{\{2\}} - \lambda_j^{\{1\}})}{\theta(z + a\hbar)} \rho(t_{ji})$$

defines a representation of $\mathcal{E}$ on X, called the evaluation module $X(a)$.

There is a flip of the subscripts i and j because the bidegrees of L_{ij} and t_{ij} are flips of each other. See also [Tarasov and Varchenko 2001, (3.6)] where $\mathcal{T}_{ij}(u)$ comes from t_{ji}.

$X \mapsto X(a)$ defines a functor $\mathrm{ev}_a : \mathrm{rep} \to \mathrm{Rep}$. Let $\mathcal{F}$ be the full subcategory of rep whose objects are finite-dimensional $\mathfrak{e}$-modules X with $X(x)$ being in category $\mathcal{O}$. Then ev_a restricts to a functor of abelian categories $\mathcal{F} \to \mathcal{O}_{\mathrm{fd}}$, and induces an injective morphism of Grothendieck groups $K_0(\mathcal{F}) \hookrightarrow K_0(\mathcal{O}_{\mathrm{fd}})$.

For $1 \le k \le N$, define $\hat{t}_k \in \mathfrak{e}$ in the same way as (1.7):[8]

$$\hat{t}_N(z) := t_{NN} \quad \text{and} \quad \hat{t}_k(z) = t_{kk} \prod_{j=k+1}^{N} \frac{\mu_{\boldsymbol{r}}(\theta(\lambda_{kj}))}{\mu_{\boldsymbol{l}}(\theta(\lambda_{kj}))}.$$

Let $\mu \in \mathfrak{h}$. There exists a unique (up to isomorphism) irreducible $\mathfrak{e}$-module V_μ with the property V_μ admits a nonzero vector v of weight μ such that $\hat{t}_k v = v$, $t_{ij}v = 0$ for $1 \le i, j, k \le N$ and $j < k$; it is called *standard* in [Tarasov and Varchenko 2001, §4]. Let L_μ denote the complex irreducible module over the simple Lie algebra $\mathfrak{sl}_N$ of highest weight μ. For $\nu \in \mathfrak{h}$, let $d_\mu[\nu] = \dim_{\mathbb{C}} L_\mu[\nu]$ where $L_\mu[\nu]$ is the weight space of weight ν.

Theorem 2.3 [Tarasov and Varchenko 2001, Theorem 5.9]. *The $\mathfrak{e}$-module V_μ is finite-dimensional if and only if $\mu_{ij} \in \mathbb{Z}_{\ge 0} + \hbar^{-1}\Gamma$ for $1 \le i < j \le N$. If $\tilde{\mu} \in \mathfrak{h}$ is such that $\mu_{ij} - \tilde{\mu}_{ij} \in \hbar^{-1}\Gamma$ and $\tilde{\mu}_{ij} \in \mathbb{Z}_{\ge 0}$ for $i < j$, then* $\dim V_\mu[\mu + \gamma] = d_{\tilde{\mu}}[\tilde{\mu} + \gamma]$ *for $\gamma \in \boldsymbol{Q}_-$.*

In the theorem $\tilde{\mu}$ is uniquely determined by μ since $\mathbb{Z} \cap h^{-1}\Gamma = \{0\}$. Such an $\mathfrak{e}$-module V_μ is in category $\mathcal{F}$. Indeed, the evaluation module $V_\mu(a)$ is irreducible in category $\tilde{\mathcal{O}}$ of highest weight

$$\left(\frac{\theta(z + (\mu_1 + a)\hbar)}{\theta(z + a\hbar)}, \frac{\theta(z + (\mu_2 + a)\hbar)}{\theta(z + a\hbar)}, \ldots, \frac{\theta(z + (\mu_N + a)\hbar)}{\theta(z + a\hbar)}; \mu \right).$$

One checks that such an e-weigth is dominant. So $V_\mu(a)$ is in category $\mathcal{O}_{\mathrm{fd}}$ by Theorem 1.14. The *character* $\chi(V_\mu)$ of V_μ is $\sum_\gamma d_{\tilde{\mu}}[\tilde{\mu} + \gamma] e^{\mu+\gamma} \in \mathcal{M}_{\mathrm{t}}$.

[8] The $\hat{t}_a$ are slightly different from the $\hat{t}_{aa}$ in [Tarasov and Varchenko 2001, (4.1)]. Yet they play the same role.

The isomorphism classes $[V_\mu]$ where $\mu \in \mathfrak{h}$ and $\mu_{ij} \in \mathbb{Z}_{\geq 0} + \hbar^{-1}\Gamma$ for $i < j$ form a $\mathbb{Z}$-basis of $K_0(\mathcal{F})$, and $[V_\mu] \mapsto \chi(V_\mu)$ extends uniquely to a morphism of abelian groups $\chi : K_0(\mathcal{F}) \to \mathcal{M}_{\mathrm{t}}$, which is injective thanks to the linear independence of characters of irreducible representations of the simple Lie algebra $\mathfrak{sl}_N$.

2C. *Category $\mathcal{O}'_{\mathrm{fd}}$.* We are going to prove Theorem 1.15 by induction on N. The idea is to view the irreducible $\mathcal{E}$-module $S_{\mu,a}$ as an $\mathcal{E}^{\mathfrak{h}}_{N-1}$-module and to apply the induction hypothesis. For this purpose, we need to adapt carefully the definitions of finite-dimensional module category $\mathcal{O}_{\mathrm{fd}}$ and its q-characters in Section 1C to $\mathcal{E}^{\mathfrak{h}}_{N-1}$. To distinguish with $\mathcal{E}$ and to simplify notations, we shall add a prime (instead of the index $N-1$) to objects related to $\mathcal{E}^{\mathfrak{h}}_{N-1}$. Notably $\mathfrak{h}' := \mathfrak{h}_{N-1}$.

We define category $\mathcal{O}'_{\mathrm{fd}}$. An object is a *finite-dimensional* $\mathfrak{h}$-graded vector space X (viewed as an object of category $\mathcal{V}_{\mathrm{ft}}$) endowed with difference operators $L^X_{ij}(z) : X \to X$ of bidegree (ϵ_i, ϵ_j) for $2 \leq i, j \leq N$ depending on $z \in \mathbb{C}$ such that:

(M1') There exists a basis of X with respect to which the matrix entries of the difference operators $L^X_{ij}(z)$ are meromorphic functions of $(z, \lambda) \in \mathbb{C} \times \mathfrak{h}$.

(M2') $L_{ij}(z) \mapsto L^X_{ij}(z)$ defines an $\mathfrak{h}$-algebra morphism $\mathcal{E}^{\mathfrak{h}}_{N-1} \to \mathbb{D}^X$.

(M3') X admits an ordered weight basis with respect to which the matrices of the difference operators $\mathcal{D}^X_l(z)$ for $1 \leq l < N$ are upper triangular and their diagonal entries are nonzero meromorphic functions of $z \in \mathbb{C}$.

A morphism in category $\mathcal{O}'_{\mathrm{fd}}$ a linear map $\Phi : X \to Y$ such that $\Phi L^X_{ij}(z) = L^Y_{ij}(z)\Phi$ for $2 \leq i, j \leq N$. Category $\mathcal{O}'_{\mathrm{fd}}$ is an abelian subcategory of $\mathcal{V}_{\mathrm{ft}}$.

The $\mathfrak{h}$-algebra morphism $\mathcal{E}^{\mathfrak{h}}_{N-1} \to \mathcal{E}$ induces restriction functor $\mathcal{O}_{\mathrm{fd}} \to \mathcal{O}'_{\mathrm{fd}}$.

Let X be in category $\mathcal{O}'_{\mathrm{fd}}$. Equation (1.8) defines difference operators $K^X_l(z) : X \to X$ of bidegree (ϵ_l, ϵ_l) for $2 \leq l \leq N$. Condition (M3') implies that for each weight α, the weight space $X[\alpha]$ admits an ordered basis B_α with respect to which the matrix of $K^X_l(z)$ is upper triangular and has as diagonal entries $f_{b,l}(z) \in \mathbb{M}^\times_{\mathbb{C}}$ for $b \in B_\alpha$. Following Definition 1.9, we define the q-character of X to be

$$\chi'_q(X) = \sum_{\alpha \in \mathrm{wt}(X)} \sum_{b \in B_\alpha} (1, f_{b,2}(z), f_{b,3}(z), \ldots, f_{b,N}(z); \alpha) \in \mathcal{M}_{\mathrm{t}}.$$

It is independent of the choice of the bases B_α, as one can use category $\mathcal{F}_{\mathrm{mer}}$ to characterize the $f_{b,l}(z)$, see the comments after Lemma 1.6.

Remark 2.4. Let X be in category $\mathcal{O}_{\mathrm{fd}}$, viewed as an object of $\mathcal{O}'_{\mathrm{fd}}$. Then $\chi'_q(X)$ is obtained from $\chi_q(X)$ by replacing each e-weigth $\boldsymbol{g}$ of the $\mathcal{E}$-module X with $\boldsymbol{g}'$; here for $\boldsymbol{g} = (g_1(z), g_2(z), \ldots, g_N(z); \alpha) \in \mathcal{M}_{\mathrm{t}}$ we define

$$\boldsymbol{g}' := (1, g_2(z), g_3(z), \ldots, g_N(z); \alpha) \in \mathcal{M}_{\mathrm{t}}.$$

Reciprocally, if X is an irreducible $\mathcal{E}$-module in $\mathcal{O}_{\mathrm{fd}}$ of highest weight $(e_1(z), e_2(z), \ldots, e_N(z); \alpha) \in \mathcal{R}_{\mathrm{fd}}$, then $\chi_q(X)$ can be recovered from $\chi'_q(X)$. Indeed, since the N-th elliptic quantum minor is central, by

Schur's lemma, it acts on X as a scalar. Each e-weigth $(f_1(z), f_2(z), \dots, f_N(z); \beta)$ of the $\mathcal{E}$-module X is determined by the its last N components in $\chi'_q(X)$ as follows:

$$e_1(z+(N-1)\hbar)e_2(z+(N-2)\hbar)\cdots e_N(z) = f_1(z+(N-1)\hbar)f_2(z+(N-2)\hbar)\cdots f_N(z).$$

The highest weight theory in Section 1E carries over to category $\mathcal{O}'_{\mathrm{fd}}$ since $\hat{L}_k(z) \in \mathcal{E}^{\mathfrak{h}}_{N-1}$ for $2 \le k \le N$. Irreducible objects in $\mathcal{O}'_{\mathrm{fd}}$ are classified by their highest weight, and the q-character map is an injective morphism from the Grothendieck group $K_0(\mathcal{O}'_{\mathrm{fd}})$ to the additive group $\mathcal{M}_{\mathrm{t}}$. Let $\mathcal{P}'$ be the set of partitions with at most $N-1$ parts $(\nu_2 \ge \nu_3 \ge \cdots \ge \nu_N)$. For such a partition and for $c, a \in \mathbb{C}$,

$$\left(1, \frac{\theta(z+(a+\nu_2)\hbar)}{\theta(z+a\hbar)}, \frac{\theta(z+(a+\nu_3)\hbar)}{\theta(z+a\hbar)}, \dots, \frac{\theta(z+(a+\nu_N)\hbar)}{\theta(z+a\hbar)}; c\epsilon_1 + \sum_{j=2}^{N} \nu_j \epsilon_j\right)$$

is the highest weight of an irreducible $\mathcal{E}^{\mathfrak{h}}_{N-1}$-module in category $\mathcal{O}'_{\mathrm{fd}}$, which is denoted by $S'_{\nu,c,a}$. As in Section 1F, ν is identified with its Young diagram Y_ν. Let $\mathcal{B}'_\nu$ be the set of Young tableaux $Y_\nu \to \{2 < 3 < \cdots < N\}$ of shape ν.

Lemma 2.5. *Assume that Theorem 1.15 is true for $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_{N-1})$-modules. Then for $\nu \in \mathcal{P}'$ and $c, a \in \mathbb{C}$, the q-character of the $\mathcal{E}^{\mathfrak{h}}_{N-1}$-module $S'_{\nu,c,a}$ is*

$$\chi'_q(S'_{\nu,c,a}) = e^{c\epsilon_1} \sum_{T \in \mathcal{B}'_\nu} \prod_{(i,j)\in Y_\nu} \boxed{T(i,j)}'_{a+j-i} \in \mathcal{M}_{\mathrm{t}}.$$

Proof. We shall need $\mathcal{E}_{N-1}$-modules which are $\mathfrak{h}'$-graded $\mathbb{M}_{\mathfrak{h}'}$-vector spaces; similar category of finite-dimensional modules and q-characters are defined, based on the $\mathfrak{h}'$-algebra isomorphism $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_{N-1}) \cong \mathcal{E}_{N-1}$ in Section 2A.

For $\nu := (\nu_2 \ge \nu_3 \ge \cdots \ge \nu_N) \in \mathcal{P}'$ and $a \in \mathbb{C}$ there exists a unique (up to isomorphism) irreducible $\mathcal{E}_{N-1}$-module, denoted by $S'_{\nu,a}$, which contains a nonzero vector ω of $\mathfrak{h}'$-weight $\nu_2\epsilon_2 + \nu_3\epsilon_3 + \cdots + \nu_N\epsilon_N$ such that

$$L_{ij}(z)\omega = 0 \quad \text{and} \quad \hat{L}_k(z)\omega = \frac{\theta(z+(a+\nu_k)\hbar)}{\theta(z+a\hbar)}\omega$$

for $2 \le i, j, k \le N$ with $j < i$. We endow the $\mathbb{M}$-vector space $X := \mathbb{M} \otimes_{\mathbb{M}_{\mathfrak{h}'}} S'_{\nu,a}$ with an $\mathcal{E}^{\mathfrak{h}}_{N-1}$-module structure in category $\mathcal{O}'_{\mathrm{fd}}$.

Let w be a nonzero weight vector in $S'_{\nu,a}$. Its $\mathfrak{h}'$-weight is written uniquely in the form

$$(\nu_2\epsilon_2 + \nu_3\epsilon_3 + \cdots + \nu_N\epsilon_N) + (x_2\alpha_2 + x_3\alpha_3 + \cdots + x_{N-1}\alpha_{N-1}) \in \mathfrak{h}',$$

where $x_j \in \mathbb{Z}_{\le 0}$. Define the $\mathfrak{h}$-weight of $g(\lambda) \otimes_{\mathbb{M}_{\mathfrak{h}'}} w$, for $g(\lambda) \in \mathbb{M}^\times$, to be

$$(c\epsilon_1 + \nu_2\epsilon_2 + \nu_3\epsilon_3 + \cdots + \nu_N\epsilon_N) + (x_2\alpha_2 + x_3\alpha_3 + \cdots + x_{N-1}\alpha_{N-1}) \in \mathfrak{h},$$

and define the action of $L_{ij}(z)$ for $2 \le i, j \le N$ by the formula

$$L_{ij}(z)(g(\lambda) \otimes_{\mathbb{M}_{\mathfrak{h}'}} w) = g(\lambda + \hbar\epsilon_j) \otimes_{\mathbb{M}_{\mathfrak{h}'}} L_{ij}(z)w.$$

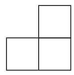
(M1')–(M2') are clear from the $\mathcal{E}_{N-1}$-module structure on $S'_{\nu,a}$. Choose an ordered weight basis B of $S'_{\nu,a}$ over $\mathbb{M}_{\mathfrak{h}'}$ such that the matrices of $\mathcal{D}_k(z)$ for $1 \le k < N$ are upper triangular and their diagonal entries belong to $\mathbb{M}_{\mathbb{C}}^{\times}$. Then the ordered basis $\{1 \otimes_{\mathbb{M}_{\mathfrak{h}'}} b \mid b \in B\} =: B'$ of X satisfies (M3'). So X is in category $\mathcal{O}'_{\mathrm{fd}}$.

The matrices of $\mathcal{D}_k(z)$ with respect to the basis B' of X and the basis B of $S'_{\nu,a}$ are the same. So $\chi'_q(X)$ up to a normalization factor $e^{c\epsilon_1}$, is equal to the q-character of the $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_{N-1})$-module $S_{\nu,a}$. The latter is given by (1.13).

X has a unique (up to scalar) singular vector and is of highest weight, so it is irreducible. A comparison of highest weights shows that $X \cong S'_{\nu,c,a}$. $\square$

Fix $\mu \in \mathcal{P}$ a partition with at most N parts. Given a tableau $T \in \mathcal{B}_\mu$, by deleting the boxes $\boxed{1}$ in T, we obtain a Young diagram $T^{-1}(\{2, 3, \ldots, N\})$ with at most $N-1$ rows, which corresponds to a partition in $\mathcal{P}'$, denoted by ν_T. Let $\mathcal{W}_\mu$ be the set of all such ν_T with $T \in \mathcal{B}_\mu$. For $\nu \in \mathcal{W}_\mu$, define c_ν to be the cardinal of the finite subset $Y_\mu \setminus Y_\nu$ of $\mathbb{Z}^2$.

Again take the example $N = 3$ and $\mu = (2 \ge 1)$ after Theorem 1.15. The eight tableaux in $\mathcal{B}_\mu$ with $\boxed{1}$ deleted give four Young diagrams and partitions

 $= (1)$,  $= (2)$,  $= (1 \ge 1)$,  $= (2 \ge 1)$.

The corresponding integers c_ν are 2, 1, 1, 0.

Lemma 2.6. *Let $\mu \in \mathcal{P}$ and $a \in \mathbb{C}$. In the Grothendieck group $K_0(\mathcal{O}'_{\mathrm{fd}})$*

$$[S_{\mu,a}] = \sum_{\nu \in \mathcal{W}_\mu} [S'_{\nu,c_\nu,a}].$$

Proof. Let $\mathfrak{e}'$ be the subalgebra of $\mathfrak{e}$ generated by $\mathbb{M}_2$ and the t_{ij} for $2 \le i, j \le N$. One can define similar abelian category $\mathcal{F}'$ of $\mathfrak{e}'$-modules (which are $\mathfrak{h}$-graded $\mathbb{M}$-vector spaces) equipped with:

(a) The evaluation functor $\mathrm{ev}'_a : \mathcal{F}' \to \mathcal{O}'_{\mathrm{fd}}$ from $\mathfrak{e}'$-modules to $\mathcal{E}^{\mathfrak{h}}_{N-1}$-modules.

(b) The injective character map $\chi : K_0(\mathcal{F}') \to \mathcal{M}_{\mathfrak{t}}$ from the $\mathfrak{h}$-grading.

Theorem 2.3 applied to the $\mathfrak{h}'$-algebra $\mathfrak{e}_{\tau,\hbar}(\mathfrak{sl}_{N-1})$, from the scalar extension in the proof of Lemma 2.5, one obtains an irreducible object $V'_{\nu,c}$ in category $\mathcal{F}'$ for $\nu = (\nu_2 \ge \nu_3 \ge \cdots \ge \nu_N) \in \mathcal{P}'$ and $c \in \mathbb{C}$ with the following properties:

(c) $V'_{\nu,c}$ admits a nonzero vector v of weight $c\epsilon_1 + \nu_2\epsilon_2 + \nu_3\epsilon_3 + \cdots + \nu_N\epsilon_N$ and $\hat{t}_k v = v$, $t_{ij} v = 0$ for $2 \le i, j, k \le N$ and $i < j$.

(d) $\chi(V'_{\nu,c})$ is equal to the character of the irreducible $\mathfrak{sl}'_{N-1}$-module of highest weight $c\epsilon_1 + \nu_2\epsilon_2 + \nu_3\epsilon_3 + \cdots + \nu_N\epsilon_N$; here $\mathfrak{sl}'_{N-1}$ is the parabolic Lie subalgebra of $\mathfrak{sl}_N$ (with the same Cartan algebra $\mathfrak{h}$) associated to the simple roots $\alpha_2, \alpha_3, \ldots, \alpha_{N-1}$.

By comparing highest weight we observe that $\mathrm{ev}'_a(V'_{\nu,c}) \cong S'_{\nu,c,a}$ in category $\mathcal{O}'_{\mathrm{fd}}$.

Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N) \in \mathscr{P}$. Set $\bar{\mu} := \mu_1\epsilon_1 + \mu_2\epsilon_2 + \cdots + \mu_N\epsilon_N$. Then $S_{\mu,a} \cong \mathrm{ev}_a(V_{\bar{\mu}})$ in category $\mathcal{O}_{\mathrm{fd}}$. By diagram chasing

$$\begin{tikzcd}
K_0(\mathcal{O}_{\mathrm{fd}}) \arrow[d, "\mathrm{Res}"'] & K_0(\mathcal{F}) \arrow[l, "\mathrm{ev}_a"'] \arrow[r, "\chi"] \arrow[d, "\mathrm{Res}"'] & \mathcal{M}_{\mathrm{t}} \\
K_0(\mathcal{O}'_{\mathrm{fd}}) & K_0(\mathcal{F}') \arrow[l, "\mathrm{ev}'_a"'] \arrow[ur, "\chi"'] &
\end{tikzcd}$$

Lemma 2.6 is equivalent to the character identity $\chi(V_{\bar{\mu}}) = \sum_{\nu \in \mathscr{W}_\mu} \chi(V'_{\nu,c_\nu})$. Since the left- and right-hand sides are the character of a representations of $\mathfrak{sl}_N$ by Theorem 2.3 and $\mathfrak{sl}'_{N-1}$ by (d), respectively, this identity is a consequence of the branching rule for representations of the reductive Lie algebras $\mathfrak{sl}_N \supset \mathfrak{sl}'_{N-1}$. □

2D. *Proof of Theorem 1.15.* We proceed by induction on N. For $N = 1$ and $\mu = (n)$, since $S_{\mu,a}$ is one-dimensional, its q-character is equal to its highest weight

$$\left(\frac{\theta(z+(a+n)\hbar)}{\theta(z+a\hbar)}; n\epsilon_1\right) = \prod_{j=1}^{n} \boxed{1}_{a+j-1}.$$

Suppose $N > 1$. By Lemma 2.5, the induction hypothesis in the case of $N-1$ gives the q-character formula for all the $\mathcal{E}^{\mathfrak{h}}_{N-1}$-modules $S'_{\nu,c,a}$ where $\nu \in \mathscr{P}'$ and $c \in \mathbb{C}$. So the q-character $\chi'_q(S_{\mu,a})$ of the $\mathcal{E}^{\mathfrak{h}}_{N-1}$-module $S_{\mu,a}$ is known by Lemma 2.6.

Since $S_{\mu,a}$ is an irreducible $\mathcal{E}$-module in category $\mathcal{O}_{\mathrm{fd}}$, by Remark 2.4, $\chi_q(S_{\mu,a})$ can be recovered from $\chi'_q(S_{\mu,a})$. Since $\mathscr{B}_\mu$ is the disjoint union of the $\mathscr{B}'_\nu$ for $\nu \in \mathscr{W}_\mu$, it suffices to check that for each e-weigth $(m_1^T(z), m_2^T(z), \ldots, m_N^T(z); \alpha)$ at the right-hand side of (1.13), where $T \in \mathscr{B}_\mu$, the following product

$$m^T(z) := m_1^T(z+(N-1)\hbar)m_2^T(z+(N-2)\hbar)\cdots m_N^T(z)$$

is the eigenvalue of scalar action of $\mathcal{D}_N(z)$ on $S_{\mu,a}$. Notice first that

$$\prod_{p=1}^{N} \frac{\theta(z+(a+N-p+\mu_p)\hbar)}{\theta(z+(a+N-p)\hbar)} = \prod_{(i,j)\in Y_\mu} \frac{\theta(z+(a+j-i+N)\hbar)}{\theta(z+(a+j-i+N-1)\hbar)}.$$

By (1.12), each box $\boxed{i}_x$ contributes to $\theta(z+(x+N)\hbar)/\theta(z+(x+N-1)\hbar)$, so the right-hand side of the identity is exactly $m^T(z)$. By Remark 2.4, the left-hand side is the scalar of $\mathcal{D}_N(z)$ acting on $S_{\mu,a}$. This completes the proof of Theorem 1.15. □

3. Kirillov–Reshetikhin modules

We study certain irreducible $\mathcal{E}$-modules via q-characters.

Fix $a \in \mathbb{C}$. For $k \in \mathbb{C}$ and $1 \leq r \leq N$, define the *asymptotic e-weigth*

$$\boldsymbol{w}_{k,a}^{(r)} := \Psi_{r,a+k}\Psi_{r,a}^{-1} \in \mathcal{M}_{\mathrm{w}}. \tag{3.14}$$

Assume $k \in \mathbb{Z}_{\geq 0}$. We identify $k\varpi_r$ with the partition $(k \geq k \geq \cdots \geq k)$ where k appears r times. Then $\boldsymbol{w}_{k,a}^{(r)} = Y_{r,a+\frac{1}{2}} Y_{r,a+\frac{3}{2}} \cdots Y_{r,a+k-\frac{1}{2}} = \theta_{k\varpi_r,a}$ by (1.9)–(1.10), and the finite-dimensional irreducible $\mathcal{E}$-module $S(\boldsymbol{w}_{k,a}^{(r)})$ in category $\mathcal{O}_{\mathrm{fd}}$ is denoted by $W_{k,a}^{(r)}$ and called *Kirillov–Reshetikhin module* (KR module).

The $Y_{i,a+m}$ and $A_{i,a+m}$ for $1 \leq i \leq N$ and $m \in \frac{1}{2}\mathbb{Z}$ are linearly independent in the abelian group $\mathcal{M}_{\mathrm{w}}$, and generate the subgroups $\mathcal{P}_a$ and $\mathcal{Q}_a$ and the submonoids $\mathcal{P}_a^+$ and $\mathcal{Q}_a^+$, respectively. The inverses of these submonoids are denoted by $\mathcal{P}_a^-$ and $\mathcal{Q}_a^-$ respectively. By (1.13) and (1.10),

$$\mathrm{wt}_e(S_{\mu,a}) \subset \theta_{\mu,a}\mathcal{Q}_a^- \subset \mathcal{P}_a \quad \text{for } \mu \in \mathscr{P}.$$

Indeed, let $T_\mu \in \mathscr{B}_\mu$ be such that the associated monomial in (1.13) is $\theta_{\mu,a}$. Then for $S \in \mathscr{B}_\mu$, we must have $S(i,j) \geq T_\mu(i,j)$ for all $(i,j) \in Y_\mu$.

Following [Frenkel and Mukhin 2001, § 6], we call $\boldsymbol{f} \in \mathcal{P}_a$ *right-negative* if the factors $Y_{i,a+m}$ with $1 \leq i < N$ appearing in $\boldsymbol{f}$, for which $m \in \frac{1}{2}\mathbb{Z}$ is minimal, have negative powers.

Lemma 3.1 [Frenkel and Mukhin 2001]. *Let $\boldsymbol{e}, \boldsymbol{f} \in \mathcal{P}_a$. If $\boldsymbol{e}$ and $\boldsymbol{f}$ are right-negative, then so is $\boldsymbol{ef}$.*

All elements in $\mathcal{Q}_a^-$ different from 1 are right-negative by (1.9).

Lemma 3.2. *Let $k \in \mathbb{Z}_{>0}$ and $1 \leq r < N$.*

1. *For $1 \leq l \leq k$, $\boldsymbol{w}_{k,a}^{(r)} A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r,a+l-1}^{-1}$ is an e-weigth of $W_{k,a}^{(r)}$ of multiplicity one in $\chi_q(W_{k,a}^{(r)})$.*
2. *An e-weigth of $W_{k,a}^{(r)}$ different from those in (1) and from $\boldsymbol{w}_{k,a}^{(r)}$ must belong to $\boldsymbol{w}_{k,a}^{(r)} A_{r,a}^{-1} A_{s,a-\frac{1}{2}}^{-1} \mathcal{Q}_a^-$ for certain $1 \leq s < N$ with $s = r \pm 1$.*
3. *Any e-weigth of $W_{k,a}^{(r)}$ is either $\boldsymbol{w}_{k,a}^{(r)}$ or right-negative.*

Proof. The Young diagram $Y_{k\varpi_r}$ is a rectangle of r rows and k columns. For (1)–(2) the proof of [Zhang 2018, Lemma 3.4] works by applying Theorem 1.15 to $W_{k,a}^{(r)} \cong S_{k\varpi_r, a-\ell_r}$. For (3), $\boldsymbol{w}_{k,a}^{(r)} A_{r,a}^{-1}$ is right-negative, and so is any element of $\boldsymbol{w}_{k,a}^{(r)} A_{r,a}^{-1} \mathcal{Q}_a^-$. ☐

For $1 \leq r < N$ and $k, t, a \in \mathbb{C}$, define as in [Frenkel and Hernandez 2016, §4.3] and [Zhang 2018, Remark 3.2]

$$\boldsymbol{d}_{k,a}^{(r,t)} := \frac{\Psi_{r,a+t}}{\Psi_{r,a}} \prod_{s=r\pm 1} \frac{\Psi_{s,a-\frac{1}{2}}}{\Psi_{s,a-\frac{1}{2}-k}} = \boldsymbol{w}_{t,a}^{(r)} \prod_{s=r\pm 1} \boldsymbol{w}_{k,a-\frac{1}{2}-k}^{(s)} \in \mathcal{M}_{\mathrm{w}}. \tag{3.15}$$

If $k, t \in \mathbb{Z}_{\geq 0}$, then $\boldsymbol{d}_{k,a}^{(r,t)} \in \mathcal{R}_{\mathrm{fd}}$ and set $D_{k,a}^{(r,t)} := S(\boldsymbol{d}_{k,a}^{(r,t)})$.

Lemma 3.3. *Let $1 \leq r < N$ and $m, k \in \mathbb{Z}_{>0}$.*

1. *The dominant e-weigths of $W_{k+m-1,1}^{(r)} \overline{\otimes} W_{k,0}^{(r)}$ and $W_{k-1,1}^{(r)} \overline{\otimes} W_{k+m,0}^{(r)}$ are*
$$\begin{aligned} &\boldsymbol{w}_{k+m-1,1}^{(r)} \boldsymbol{w}_{k,0}^{(r)} \quad \textit{and} \quad \boldsymbol{w}_{k+m-1,1}^{(r)} \boldsymbol{w}_{k,0}^{(r)} A_{r,1}^{-1} A_{r,2}^{-1} \cdots A_{r,l}^{-1} \quad \textit{for } 1 \leq l \leq k, \\ &\boldsymbol{w}_{k-1,1}^{(r)} \boldsymbol{w}_{k+m,0}^{(r)} \quad \textit{and} \quad \boldsymbol{w}_{k-1,1}^{(r)} \boldsymbol{w}_{k+m,0}^{(r)} A_{r,1}^{-1} A_{r,2}^{-1} \cdots A_{r,l}^{-1} \quad \textit{for } 1 \leq l < k, \end{aligned}$$
respectively. All such e-weigths are of multiplicity one.
2. *The module $W_{k-1,1}^{(r)} \overline{\otimes} W_{k+m,0}^{(r)}$ is irreducible.*

Proof. For (1), one can copy the last two paragraphs of the proof of [Fourier and Hernandez 2014, Theorem 4.1], since the right-negativity property of KR modules in the elliptic case (Lemma 3.2) is the same as in the affine case. Let T be the tensor product module of (2). Suppose T is not irreducible. Then there exists $1 \le l \le k-1$ such that T admits an irreducible subquotient $S \cong S(\boldsymbol{d}_l)$ where by (1.9)

$$\boldsymbol{d}_l := \boldsymbol{w}^{(r)}_{k-1,1}\boldsymbol{w}^{(r)}_{k+m,0}\prod_{j=1}^{l} A_{r,j}^{-1} = \frac{\Psi_{r,k}\Psi_{r,k+m}}{\Psi_{r,l+1}\Psi_{r,l}}\prod_{s=r\pm 1}\frac{\Psi_{s,l+\frac{1}{2}}}{\Psi_{s,\frac{1}{2}}}.$$

Set $\mu := \varpi(\boldsymbol{d}_l)$. The weight space $S[\mu-\alpha_r]$ is nonzero since the Ψ_r do not cancel in $\boldsymbol{d}_l$, and its possible e-weigths are $\boldsymbol{d}_l A_{r,l}^{-1}$, $\boldsymbol{d}_l A_{r,l+1}^{-1}$ since S is a subquotient of $W^{(r)}_{k-l-1,l+1}\overline{\otimes} W^{(r)}_{k-l+m,l}\overline{\otimes}(\overline{\otimes}_{s=r\pm 1} W^{(s)}_{l,\frac{1}{2}})$. If $\boldsymbol{d}_l A_{r,l}^{-1}$ is an e-weigth of S, then

$$\boldsymbol{w}^{(r)}_{k-1,1}\boldsymbol{w}^{(r)}_{k+m,0}A_{r,1}^{-1}A_{r,2}^{-1}\cdots A_{r,l-1}^{-1}A_{r,l}^{-2} \in \mathrm{wt}_e(T) = \mathrm{wt}_e(W^{(r)}_{k-1,1})\,\mathrm{wt}_e(W^{(r)}_{k+m,0})$$

which contradicts with the q-characters of KR modules in Lemma 3.2. So $k > l+1$ and $S[\mu-\alpha_r] = \mathbb{M}v \neq 0$. Let ω be a highest weight vector of S. Then

$$p := \mu_{r,r+1} = 2k-2l+m-1, \quad L_{r,r+1}(z)\omega = A(z;\lambda)v, \quad L_{r+1,r}(z)v = B(z;\lambda)\omega$$

for some meromorphic functions A and B of $(z,\lambda) \in \mathbb{C}\times\mathfrak{h}$. For $1 \le i \le N$, let $g_i(z) \in \mathbb{M}_{\mathbb{C}}^{\times}$ be the i-th component of $\boldsymbol{d}_l \in \mathcal{M}_{\mathrm{w}}$. Then $L_{ii}(z)\omega = g_i(z)\varphi_i(\lambda)\omega$ for certain $\varphi_i(\lambda) \in \mathbb{M}^{\times}$ by (1.7). Set $h(z) := g_r(z)/g_{r+1}(z)$. We have

$$h(z) = \frac{\theta(z+(k-\ell_r)\hbar)\theta(z+(k+m-\ell_r)\hbar)}{\theta(z+(l+1-\ell_r)\hbar)\theta(z+(l-\ell_r)\hbar)} = \frac{\theta(z-w_1)\theta(z-w_2)}{\theta(z-w_3)\theta(z-w_4)},$$

where $w_1 := (\ell_r-k)\hbar$, $w_2 := (\ell_r-k-m)\hbar$ and so on. Applying (1.2) with $(i,j) = (r+1,r) = (n,m)$ to ω, as in the proof of [Felder and Zhang 2017, Theorem 4.1], we obtain

$$\left(\frac{\theta(z-w+\lambda_{r,r+1}+p\hbar)\theta(\hbar)}{\theta(z-w+\hbar)\theta(\lambda_{r,r+1}+p\hbar)}g_{r+1}(z)g_r(w) - \frac{\theta(z-w+\lambda_{r,r+1})\theta(\hbar)}{\theta(z-w+\hbar)\theta(\lambda_{r,r+1})}g_{r+1}(w)g_r(z)\right)\varphi_r(\lambda+\hbar\epsilon_{r+1})\varphi_{r+1}(\lambda)$$
$$= \frac{\theta(z-w)\theta(\lambda_{r,r+1}+\hbar)}{\theta(z-w+\hbar)\theta(\lambda_{r,r+1})}B(w;\lambda)A(z;\lambda+\hbar\epsilon_r).$$

Multiplying both sides by $\theta(z-w+\hbar)/(g_{r+1}(z)g_{r+1}(w))$ and noticing $g_{r+1}(z) = \theta(z-w_3)/\theta(z-w_3-l\hbar)$, one can evaluate w at w_1 and w_2 to obtain identities of meromorphic functions of (z,λ):

$$\tilde{A}(z;\lambda)x_i(\lambda) = \frac{\theta(z-w_i+\lambda_{r,r+1})}{\theta(z-w_i)}f(\lambda)h(z) \quad \text{for } i = 1,2.$$

Here we set $\varphi(\lambda) := \varphi_r(\lambda+\hbar\epsilon_{r+1})\varphi_{r+1}(\lambda)$ and

$$\tilde{A}(z;\lambda) := \frac{A(z;\lambda+\hbar\epsilon_r)}{g_{r+1}(z)}, \quad x_i(\lambda) := \frac{B(w_i;\lambda)}{g_{r+1}(w_i)}, \quad f(\lambda) := -\frac{\theta(\hbar)\varphi(\lambda)}{\theta(\lambda_{r,r+1}+\hbar)}.$$

Since $f(\lambda)h(z) \neq 0$, we have $x_i(\lambda) \neq 0$ and so

$$\frac{\theta(z-w_1+\lambda_{r,r+1})\theta(z-w_2)}{x_1(\lambda)\theta(z-w_3)\theta(z-w_4)} = \frac{\theta(z-w_2+\lambda_{r,r+1})\theta(z-w_1)}{x_2(\lambda)\theta(z-w_3)\theta(z-w_4)}$$

as nonzero meromorphic functions of (z, λ). This forces $w_1 - w_2 = m\hbar \in \mathbb{Z} + \mathbb{Z}\tau$, which certainly does not hold. This proves (3). $\square$

Theorem 3.4. *For $1 \leq r < N$, $t \in \mathbb{Z}_{\geq 0}$ and $k > 0$, we have the following identities in the Grothendieck ring of category $\mathcal{O}_{\mathrm{fd}}$:*

$$[D^{(r,t)}_{k,k+1}] + [W^{(r)}_{k-1,1}][W^{(r)}_{k+t+1,0}] = [W^{(r)}_{k+t,1}][W^{(r)}_{k,0}], \tag{3.16}$$

$$[D^{(r,t+1)}_{k,k}][W^{(r)}_{k+t,0}] = [D^{(r,0)}_{k+t+1,k+t+1}][W^{(r)}_{k-1,0}] + [D^{(r,t)}_{k,k}][W^{(r)}_{k+t+1,0}]. \tag{3.17}$$

Proof. Set $T := W^{(r)}_{k+t,1} \bar{\otimes} W^{(r)}_{k,0}$ and $\boldsymbol{d} := \boldsymbol{w}^{(r)}_{k+t,1}\boldsymbol{w}^{(r)}_{k,0}$. Then $S := S(\boldsymbol{d})$ is an irreducible subquotient of T and by (3.14)–(3.15)

$$\boldsymbol{d} = \boldsymbol{w}^{(r)}_{k-1,1}\boldsymbol{w}^{(r)}_{k+t+1,0} \quad \text{and} \quad \boldsymbol{d}^{(r,t)}_{k,k+1} = A^{-1}_{r,1}A^{-1}_{r,2}\cdots A^{-1}_{r,k}\boldsymbol{d}.$$

Set $m = t+1$ in Lemma 3.3. Then $S \cong W^{(r)}_{k-1,1} \bar{\otimes} W^{(r)}_{k+t+1,0}$, and there is exactly one dominant e-weigth (counted with multiplicity) in $\mathrm{wt}_e(T) \setminus \mathrm{wt}_e(S)$, namely $\boldsymbol{d}^{(r,t)}_{k,k+1}$. This proves (3.16), which implies after taking spectral parameter shifts

$$[D^{(r,t+1)}_{k,k}] = [W^{(r)}_{k+t+1,0}][W^{(r)}_{k,-1}] - [W^{(r)}_{k-1,0}][W^{(r)}_{k+t+2,-1}],$$

$$[D^{(r,0)}_{k+t+1,k+t+1}] = [W^{(r)}_{k+t+1,0}][W^{(r)}_{k+t+1,-1}] - [W^{(r)}_{k+t,0}][W^{(r)}_{k+t+2,-1}],$$

$$[D^{(r,t)}_{k,k}] = [W^{(r)}_{k+t,0}][W^{(r)}_{k,-1}] - [W^{(r)}_{k-1,0}][W^{(r)}_{k+t+1,-1}].$$

Equation (3.17) becomes a trivial identity involving only KR modules. $\square$

$D^{(r,t)}_{k,k+1}$ is *special* in the sense of [Nakajima 2003] as it contains only one dominant e-weigth. For $t = 0$, $D^{(r,0)}_{k,k+1} \cong W^{(r-1)}_{k,\frac{1}{2}} \bar{\otimes} W^{(r+1)}_{k,\frac{1}{2}}$ by showing that the tensor product is special as in [Nakajima 2003], and (3.16) is the T-system of KR modules.

Corollary 3.5. *Let $1 \leq r < N$, $a \in \mathbb{C}$ and $k, t \in \mathbb{Z}_{>0}$.*

(1) $\boldsymbol{d}^{(r,t)}_{k,a} A^{-1}_{r,a}A^{-1}_{r,a+1}\cdots A^{-1}_{r,a+l-1} \in \mathrm{wt}_e(D^{(r,t)}_{k,a})$ *for* $1 \leq l \leq t$.

(2) *Any e-weigth of $D^{(r,t)}_{k,a}$ different from those in* (1) *and $\boldsymbol{d}^{(r,d)}_{k,a}$ belongs to $\boldsymbol{d}^{(r,t)}_{k,a}\{A^{-1}_{r,a-k-1}, A^{-1}_{s,a-k-\frac{1}{2}}\}\mathcal{Q}^{-}_{a}$, for certain $1 \leq s < N$ with $s = r \pm 1$.*

Proof. This comes from Lemma 3.2 and Theorem 3.4. $\square$

Lemma 3.6. *Let $1 \leq r < N$ and $t \in \mathbb{Z}_{\geq 0}$. There is a short exact sequence*

$$0 \to D^{(r,t)}_{1,a} \to W^{(r)}_{t+1,a-1} \bar{\otimes} W^{(r)}_{1,a-2} \to W^{(r)}_{t+2,a-2} \to 0$$

of $\mathcal{E}$-modules in category $\mathcal{O}_{\mathrm{fd}}$.

Proof. Let T and S be the second and third terms above (zero excluded). Let ω_1 and ω_2 be highest weight vectors of $W^{(r)}_{t+1,a-1}$ and $W^{(r)}_{1,a-2}$ respectively. Then $\omega_1 \bar{\otimes} \omega_2$ is a highest weight vector of T and generates a submodule T'. Suppose $T' = T$. Then T is a highest weight module whose highest weight is equal to that of the irreducible module S. There is a surjective morphism of modules $T \to S$, the kernel of which is $D_{1,a}^{(r,t)}$ by (3.16) (one applies a spectral parameter shift Φ_{a-2} to the equation with $k=1$). This is the desired short exact sequence.

Suppose $T \neq T'$. Then $[T'] = [S]$ or $[T'] = [D_{k,a}^{(r,t)}]$. By comparing highest weights, we have $[T'] = [S]$. So the weight space $T'[(t+2)\varpi_r - \alpha_r]$ is one-dimensional. Corollary 2.2 applied to $W_{t+1,a-1}^{(r)} \cong S_{(t+1)\varpi_r, a-\ell_r-1}$, one finds $g(\lambda) \in \mathbb{M}^\times$ such that $L_{r+1,r+1}(z)\omega_1 = \omega_1$ and (set $b := a - \ell_r - 1$)

$$L_{r,r+1}(z)\omega_1 = \frac{\theta(z+(b+t)\hbar+\lambda_{r,r+1})}{\theta(z+b\hbar)}\omega_1' \quad \text{and} \quad L_{rr}(z)\omega_1 = \frac{\theta(z+(b+t+1)\hbar)}{\theta(z+b\hbar)}g(\lambda)\omega_1,$$

where $0 \neq \omega_1'$ is of weight $(t+1)\varpi_r - \alpha_r$. Similarly $L_{r+1,r+1}(z)\omega_2 = \omega_2$ and

$$L_{r,r+1}(z)\omega_2 = \frac{\theta(z+(b-1)\hbar+\lambda_{r,r+1})}{\theta(z+(b-1)\hbar)}\omega_2'$$

with $\omega_2' \neq 0$ of weight $\varpi_r - \alpha_r$. Since ω_1, ω_2 are highest weight vectors, we have

$$\begin{aligned}
&L_{r,r+1}(z)(\omega_1 \overline{\otimes} \omega_2) \\
&= L_{r,r+1}(z)\omega_1 \overline{\otimes} L_{r+1,r+1}(z)\omega_2 + L_{rr}(z)\omega_1 \overline{\otimes} L_{r,r+1}(z)\omega_2 \\
&= \left(\frac{\theta(z+(b+t)\hbar+\lambda_{r,r+1})}{\theta(z+b\hbar)}\omega_1'\right)\overline{\otimes}\omega_2 + \frac{\theta(z+(b+t+1)\hbar)}{\theta(z+b\hbar)}g_1(\lambda)\omega_1 \overline{\otimes}\left(\frac{\theta(z+(b-1)\hbar+\lambda_{r,r+1})}{\theta(z+(b-1)\hbar)}\omega_2'\right).
\end{aligned}$$

Setting $z = -(b+t+1)\hbar$ we obtain $\omega_1' \overline{\otimes} \omega_2 \in T'$, and so $\omega_1 \otimes \omega_2' \in T'$. The weight space $T'[(t+2)\varpi_r - \alpha_r]$ is at least two-dimensional, a contradiction. $\square$

Lemma 3.6 is inspired by [Moura and Pereira 2017, § 5.3], to transform identities in the Grothendieck group into exact sequences by restriction to $\mathfrak{sl}_2$, see [Chari 2002]. More generally, we have the short exact sequences in category $\mathcal{O}_{\mathrm{fd}}$ by [Felder and Zhang 2017, Proposition 4.3 and Corollary 4.5]:[9]

$$0 \to D_{k,k+1}^{(r,t)} \to W_{k+t,1}^{(r)} \overline{\otimes} W_{k,0}^{(r)} \to W_{k-1,1}^{(r)} \overline{\otimes} W_{k+t+1,0}^{(r)} \to 0,$$

$$0 \to D_{k+t+1,k+t+1}^{(r,0)} \overline{\otimes} W_{k-1,0}^{(r)} \to D_{k,k}^{(r,t+1)} \overline{\otimes} W_{k+t,0}^{(r)} \to D_{k,k}^{(r,t)} \overline{\otimes} W_{k+t+1,0}^{(r)} \to 0.$$

These exact sequences hold for affine quantum (super)groups [Fourier and Hernandez 2014; Zhang 2018]. In the super case the proof is more delicate since Lemma 3.2(3) fails.

4. Asymptotic representations

We construct infinite-dimensional modules in category $\mathcal{O}$ as inductive limits ($k \to \infty$) of the KR modules $W_{k,a}^{(r)}$ for fixed $1 \leq r < N$ and $a := \ell_r$.

The general strategy follows that of Hernandez and Jimbo [2012]:

(i) Produce an inductive system of vector spaces $W_{0,a}^{(r)} \subseteq W_{1,a}^{(r)} \subseteq W_{2,a}^{(r)} \subseteq \cdots$.

(ii) Prove that the matrix entries of the $L_{ij}(z)$ are *good* functions of $k \in \mathbb{Z}_{\geq 0}$.

[9] The elliptic quantum group of [Felder and Zhang 2017] is slightly different as it is defined by another R-matrix, which is gauge equivalent to the present $\boldsymbol{R}$ by [Enriquez and Felder 1998].

(iii) Define the module structure on the inductive limit of (i).

Step (i) is done in Lemma 4.2, step (ii) in Lemma 4.8, and step (iii) in Proposition 4.10. We shall see that the proofs in each step are different from [Hernandez and Jimbo 2012].

In what follows, by $k > l$ we implicitly assume that $k, l \in \mathbb{Z}_{\geq 0}$. For $k > l$, set $Z_{kl} := W^{(r)}_{k-l,a+l} \cong S_{(k-l)\varpi_r,l}$ and fix a highest weight vector $\omega_{kl} \in Z_{kl}$. By (1.7), we have for $1 \leq i \leq r < j \leq N$,

$$L_{ii}(z)\omega_{kl} = \omega_{kl}\frac{\theta(z+k\hbar)}{\theta(z+l\hbar)}\prod_{q=r+1}^{N}\frac{\theta(\lambda_{iq}+(k-l+1)\hbar)}{\theta(\lambda_{iq}+\hbar)} \quad \text{for } L_{jj}(z)\omega_{kl} = \omega_{kl}.$$

Note that $Z_{k0} = W^{(r)}_{k,a}$, and we simply write $\omega_{k0} =: \omega_k$.

Lemma 4.1. *Let $t > k > l > m$. There exists a unique morphism of $\mathcal{E}$-modules*

$$\mathcal{G}^l_{k,m} : Z_{kl}\overline{\otimes}Z_{lm} \to Z_{km}$$

such that $\mathcal{G}^l_{k,m}(\omega_{kl}\overline{\otimes}\omega_{lm}) = \omega_{km}$. Moreover the following diagram commutes:

$$\begin{tikzcd} Z_{tk}\overline{\otimes}Z_{kl}\overline{\otimes}Z_{lm} \arrow[r, "\mathcal{G}^k_{t,l}\overline{\otimes}\mathrm{Id}"] \arrow[d, "\mathrm{Id}\,\overline{\otimes}\mathcal{G}^l_{k,m}"'] & Z_{tl}\overline{\otimes}Z_{lm} \arrow[d, "\mathcal{G}^l_{t,m}"] \\ Z_{tk}\overline{\otimes}Z_{km} \arrow[r, "\mathcal{G}^k_{t,m}"] & Z_{tm} \end{tikzcd} \tag{4.18}$$

Proof. (Uniqueness) Let F and G be two such morphisms and let X be the image of $F - G$. Then $\omega_{km} \notin X$. If $X \neq 0$, then X has a highest weight vector $v \neq 0$, which is proportional to ω_{km} by the irreducibility of Z_{km}, a contradiction. So $X = 0$ and $F = G$. The commutativity of (4.18) is proved in the same way.

(Existence) Let $b \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$. By Lemma 3.6 there exists a surjective $\mathcal{E}$-linear map

$$W^{(r)}_{n-1,b+1}\overline{\otimes}W^{(r)}_{1,b} \to W^{(r)}_{n,b}.$$

An induction on n shows that the $\mathcal{E}$-module $W^{(r)}_{1,b+n-1}\overline{\otimes}W^{(r)}_{1,b+n-2}\overline{\otimes}\cdots\overline{\otimes}W^{(r)}_{1,b+1}\overline{\otimes}W^{(r)}_{1,b}$ can be projected onto $W^{(r)}_{n,b}$. Setting $(n, b) = (k-m, a+m)$ we obtain a surjective $\mathcal{E}$-linear map

$$g : Z_{k,k-1}\overline{\otimes}Z_{k-1,k-2}\overline{\otimes}\cdots\overline{\otimes}Z_{m+2,m+1}\overline{\otimes}Z_{m+1,m} =: T \twoheadrightarrow Z_{km}.$$

Taking (n, b) to be $(k-l, a+l)$ and $(l-m, a+m)$, we project the first $k-l$ and the last $l-m$ tensor factors of T onto Z_{kl} and Z_{lm} respectively. The tensor product of these projections gives $f : T \twoheadrightarrow Z_{kl}\overline{\otimes}Z_{lm}$. Since $\omega_{kl}\overline{\otimes}\omega_{lm}$, ω_{km} and $\omega := \omega_{k,k-1}\overline{\otimes}\omega_{k-1,k-2}\overline{\otimes}\cdots\overline{\otimes}\omega_{m+2,m+1}\overline{\otimes}\omega_{m+1,m} \in T$ are highest weight vectors of the same e-weigth, by surjectivity one can assume $f(\omega) = \omega_{kl}\overline{\otimes}\omega_{lm}$ and $g(\omega) = \omega_{km}$. It suffices to prove that g factorizes through f, and so $g = \mathcal{G}^l_{k,m}f$. Set $Y := \ker(f)$ and $Z := \ker(g)$. The image of g being irreducible, Z is a maximal submodule of T. Since $\omega \notin Y + Z$, we have $Y + Z = Z$ and $Y \subseteq Z$. $\square$

We need two special cases of the $\mathcal{G}$; for $k > l$ and $t-1 > l$,

$$\mathcal{F}_{k,l} = \mathcal{G}^{l}_{k,0} : Z_{kl}\overline{\otimes} W^{(r)}_{l,a} \to W^{(r)}_{k,a} \quad \text{and} \quad \mathcal{G}_{t,l} = \mathcal{G}^{l+1}_{t,l} : Z_{t,l+1}\overline{\otimes} Z_{l+1,l} \to Z_{tl}.$$

As in [Hernandez and Jimbo 2012, §4.2], for $k > l$ define the restriction map

$$F_{k,l} : W^{(r)}_{l,a} \to W^{(r)}_{k,a}, \quad v \mapsto \mathcal{F}_{k,l}(\omega_{kl}\overline{\otimes} v).$$

It is a difference map of bidegree $((l-k)\varpi_r, 0)$.

Applying (4.18) with $t > k > l > 0$ to $\omega_{tk}\overline{\otimes}\omega_{kl}\overline{\otimes} W^{(r)}_{l,a}$ gives $F_{t,k}F_{k,l} = F_{t,l}$. So $(W^{(r)}_{l,a}, F_{k,l})$ is an inductive system of vector spaces.[10]

Applying (4.18) with $k > l+1 > l > 0$ to $\omega_{k,l+1}\overline{\otimes} Z_{l+1,l}\overline{\otimes} W^{(r)}_{l,a}$, we obtain

$$\mathcal{F}_{k,l}(\mathcal{G}_{k,l}(\omega_{k,l+1}\overline{\otimes} v)\overline{\otimes} w) = F_{k,l}\mathcal{F}_{l+1,l}(v\overline{\otimes} w) \quad \text{for } v\overline{\otimes} w \in Z_{l+1,l}\overline{\otimes} W^{(r)}_{l,a}. \tag{4.19}$$

Lemma 4.2. *The linear maps $F_{k,l}$ are injective.*

Proof. Assume $K := \ker(F_{k,l}) \neq 0$; it is a graded subspace of $W^{(r)}_{l,a}$. Choose $\mu \in \mathrm{wt}(K)$ such that $\mu + \alpha_i \notin \mathrm{wt}(K)$ for all $1 \le i < N$ and fix $0 \neq w \in K[\mu]$. We show that w is a singular vector, so $w \in \mathbb{M}\omega_l$ and $\omega_l \in K$, a contradiction. It suffices to prove that $L_{ji}(z)w \in K$ for all $1 \le i < j \le N$; this implies $L_{ji}(z)w = 0$ because by assumption on μ the weight space $K[\mu + \epsilon_i - \epsilon_j]$ vanishes.

Suppose $j > r$. If $1 \le p \le N$ and $p \neq j$, then $(k-l)\varpi_r + \epsilon_p - \epsilon_j \notin \mathrm{wt}(Z_{kl})$ by Theorem 1.15. It follows that for $v \in W^{(r)}_{l,a}$ we have in $Z_{kl}\overline{\otimes} W^{(r)}_{l,a}$,

$$L_{ji}(z)(\omega_{kl}\overline{\otimes} v) = L_{jj}(z)\omega_{kl}\overline{\otimes} L_{ji}(z)v = \omega_{kl}\overline{\otimes} L_{ji}(z)v.$$

It follows that $L_{ji}(z)K \subseteq K$ because of the commutativity:

$$L_{ji}(z)F_{k,l} = F_{k,l}L_{ji}(z) \quad \text{for } 1 \le i, j \le N \text{with } j > r. \tag{4.20}$$

Suppose $j \le r$. For $p > r$ since $r \ge j > r$ we have $L_{pi}(z)w \in K$ and so $L_{pi}(z)w = 0$. For $p \le r$, by Theorem 1.15, $L_{jp}(z)\omega_{kl} = 0$ if $p \neq j$. This implies

$$L_{ji}(z)(\omega_{kl}\overline{\otimes} w) = L_{jj}(z)\omega_{kl}\overline{\otimes} L_{ji}(z)w = \frac{\theta(z+k\hbar)}{\theta(z+l\hbar)} g(\lambda)(\omega_{kl}\overline{\otimes} L_{ji}(z)w)$$

for certain $g(\lambda) \in \mathbb{M}^\times$. Applying $\mathcal{F}_{k,l}$ we obtain $F_{k,l}L_{ji}(z)w = 0$, as desired. □

In what follows k and l denote positive integers, while $i, j, m, n, p, q, s, t, u$ and v denote the integers between 1 and N related to the Lie algebra $\mathfrak{sl}_N$.

Lemma 4.3. *For $k > l$ and $1 \le i \le N$ we have*

$$K_i(z)F_{k,l} = \left(\frac{\theta(z+k\hbar)}{\theta(z+l\hbar)}\right)^{\delta_{i\le r}} F_{k,l}K_i(z). \tag{4.21}$$

[10] In the affine case [Hernandez and Jimbo 2012, (4.26)] the structure map comes from the stronger fact that $Z_{kl}\overline{\otimes} Z_{lm}$ is of highest weight with Z_{km} being the irreducible quotient.

Proof. We compute $\mathcal{D}_i(z)(\omega_{kl}\overline{\otimes}v)$ for $v \in W_{l,a}^{(r)}$ based on the coproduct of Corollary 1.2. If $-\varpi_{N-k} \prec \alpha$ then $\alpha + \varpi_{N-k} \notin \boldsymbol{Q}_-$ and $(k-l)\omega_{kl} + \alpha + \varpi_{N-k} \notin \mathrm{wt}(Z_{kl})$. The extra terms $x_\alpha \tilde{\otimes} y_\alpha$ in the coproduct do not contribute, and so $\mathcal{D}_i(z)(\omega_{kl}\overline{\otimes}v) = \mathcal{D}_i(z)\omega_{kl}\overline{\otimes}\mathcal{D}_i(z)v$. By (1.8) a similar identity holds when $\mathcal{D}_i(z)$ is replaced by $K_i(z)$, because $K_i(z)\omega_{kl} = (\theta(z+k\hbar)/\theta(z+l\hbar))^{\delta_{i\le r}}\omega_{kl}$ is independent of λ. By applying $\mathcal{F}_{k,l}$ to the new identity involving $K_i(z)$, we obtain (4.21). □

From now until Corollary 4.7, we shall fix integers j and p with the condition $1 \le j \le r < p \le N$. By Corollary 2.2, there are elements $\omega_{kl}^{jp} \in Z_{kl}$ for $k > l$ such that

$$L_{jp}(z)\omega_{kl} = \frac{\theta(z+(k-1)\hbar+\lambda_{jp})}{\theta(z+l\hbar)}\omega_{kl}^{jp}.$$

Indeed $\omega_{kl}^{jp} = t_{pj}\omega_{kl}$ in the evaluation module $Z_{kl} \cong V_{(k-l)\varpi_r}(l)$. Since $Y_{(k-l)\varpi_r}$ is a rectangle, $\mathbb{M}\omega_{kl}^{jp}$ is the weight space of weight $(k-l)\varpi_r + \epsilon_p - \epsilon_j$.

Lemma 4.4. *In the $\mathcal{E}$-module Z_{kl} we have $\omega_{kl}^{jp} \ne 0$ and*

$$L_{pj}(z)\omega_{kl}^{jp} = -\omega_{kl}\frac{\theta(z+l\hbar-\lambda_{jp})\theta((k-l)\hbar)\theta(\hbar)}{\theta(z+l\hbar)\theta(\lambda_{jp})\theta(\lambda_{jp}+\hbar)}\prod_{r<q\ne p}\frac{\theta(\lambda_{jq}+(k-l+1)\hbar)}{\theta(\lambda_{jq}+\hbar)}.$$

The product is taken over integers q such that $r+1 \le q \le N$ and $q \ne p$.

Proof. The weight grading on $Z_{kl} = S_{(k-l)\varpi_r,l}$ indicates $t_{jp}\omega_{kl}^{jp} = g(\lambda)\omega_{kl}$ for certain $g(\lambda) \in \mathbb{M}$. The last relation of Definition 2.1 with $a = d = j$ and $c = b = p$ is applied to the highest weight vector ω_{kl}, the second term vanishes and

$$\frac{\theta(\lambda_{jp}+(k-l+1)\hbar)}{\theta(\lambda_{jp}+(k-l)\hbar)}g(\lambda) = \frac{\theta((k-l)\hbar)\theta(-\hbar)}{\theta(\lambda_{jp})\theta(\lambda_{jp}+(k-l)\hbar)}\prod_{q>r}\frac{\theta(\lambda_{jq}+(k-l+1)\hbar)}{\theta(\lambda_{jq}+\hbar)}.$$

This implies $\omega_{kl}^{jp} \ne 0$. We conclude that $L_{pj}(z)\omega_{kl}^{jp} = \theta(z+l\hbar-\lambda_{jp})g(\lambda)\omega_{kl}/\theta(z+l\hbar)$. □

Lemma 4.5. *Let $k-1 > l$. In the $\mathcal{E}$-module $Z_{k,l+1}\overline{\otimes}Z_{l+1,l}$ we have*

$$L_{pj}(z)(\boldsymbol{a}_{jp}^{(l)}(k;\lambda)(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l}^{jp}) - \omega_{k,l+1}^{jp}\overline{\otimes}\omega_{l+1,l}) = 0,$$

where

$$\boldsymbol{a}_{jp}^{(l)}(k;\lambda) := \frac{\theta((k-l-1)\hbar)\theta(\lambda_{jp}-\hbar)}{\theta(\hbar)\theta(\lambda_{jp})}\prod_{r<q\ne p}\frac{\theta(\lambda_{jq}+(k-l)\hbar)}{\theta(\lambda_{jq}+\hbar)}.$$

Furthermore $\mathcal{G}_{k,l}(\boldsymbol{a}_{jp}^{(l)}(k;\lambda)(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l}^{jp}) - \omega_{k,l+1}^{jp}\overline{\otimes}\omega_{l+1,l}) = 0$.

Proof. We compute $L_{pj}(z)(\omega_{k,l+1}^{jp}\overline{\otimes}\omega_{l+1,l}) = \sum_{q=1}^{N} L_{pq}(z)\omega_{k,l+1}^{jp}\overline{\otimes}L_{qj}(z)\omega_{l+1,l}$. Since $\omega_{l+1,l}$ is a highest weight vector, the terms with $q > j$ vanish. The weight of $L_{qj}(z)\omega_{k,l+1}^{jp}$ is $(k-l-1)\varpi_r + \epsilon_q - \epsilon_j$, which does not belong to $\mathrm{wt}(Z_{k,l+1})$ for $q < j$. So only the term $q = j$ survives. By Lemma 4.4

$$
\begin{aligned}
&L_{pj}(z)(\omega^{jp}_{k,l+1}\overline{\otimes}\omega_{l+1,l})\\
&\qquad = L_{pj}(z)\omega^{jp}_{k,l+1}\overline{\otimes}L_{jj}(z)\omega_{l+1,l}\\
&\qquad = -\frac{\theta(z+(l+1)\hbar-\lambda_{jp})\theta((k-l-1)\hbar)\theta(\hbar)}{\theta(z+(l+1)\hbar)\theta(\lambda_{jp})\theta(\lambda_{jp}+\hbar)}\prod_{r<q\neq p}\frac{\theta(\lambda_{jq}+(k-l)\hbar)}{\theta(\lambda_{jq}+\hbar)}\omega_{k,l+1}\\
&\qquad\qquad\qquad\overline{\otimes}\frac{\theta(z+(l+1)\hbar)}{\theta(z+l\hbar)}\prod_{q>r}\frac{\theta(\lambda_{jq}+2\hbar)}{\theta(\lambda_{jq}+\hbar)}\omega_{l+1,l}\\
&\qquad = -\frac{\theta(z+l\hbar-\lambda_{jp})\theta((k-l-1)\hbar)\theta(\hbar)}{\theta(z+l\hbar)\theta(\lambda_{jp}+\hbar)^2}\prod_{r<q\neq p}\frac{\theta(\lambda_{jq}+(k-l+1)\hbar)}{\theta(\lambda_{jq}+\hbar)}(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l}).
\end{aligned}
$$

Similar arguments lead to

$$
\begin{aligned}
L_{pj}(z)(\omega_{k,l+1}\overline{\otimes}\omega^{jp}_{l+1,l}) &= L_{pp}(z)\omega_{k,l+1}\overline{\otimes}L_{pj}(z)\omega^{jp}_{l+1,l}\\
&= -\frac{\theta(z+l\hbar-\lambda_{jp})\theta(\hbar)^2}{\theta(z+l\hbar)\theta(\lambda_{jp})\theta(\lambda_{jp}+\hbar)}\prod_{r<q\neq p}\frac{\theta(\lambda_{jq}+2\hbar)}{\theta(\lambda_{jq}+\hbar)}(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l}).
\end{aligned}
$$

The ratio of the two coefficients of $\omega_{kl}\overline{\otimes}\omega_{l+1,l}$ above is $\boldsymbol{a}^{(l)}_{jp}(k;\lambda+\hbar\epsilon_j)$, which is easily seen to be independent of z. For the last identity, let x be the vector in the argument of $\mathcal{G}_{k,l}$. Then both $\mathcal{G}_{k,l}(x)$ and ω^{jp}_{kl} belong to the one-dimensional weight space of weight $(k-l)\varpi_r+\epsilon_j-\epsilon_p$. These two vectors are proportional, the first is annihilated by $L_{pj}(z)$, while the second is not. So $\mathcal{G}_{k,l}(x)=0$. □

Corollary 4.6. *Let $k-1>l$. In the $\mathcal{E}$-module Z_{kl} we have*

$$L_{jp}(z)\omega_{kl}=\mathcal{G}_{k,l}(\omega_{k,l+1}\overline{\otimes}\omega^{jp}_{l+1,l})\times\boldsymbol{b}^{(l)}_{jp}(k,z;\lambda),$$

where

$$\boldsymbol{b}^{(l)}_{jp}(k,z;\lambda):=\frac{\theta(z+(k-1)\hbar+\lambda_{jp})\theta((k-l)\hbar)}{\theta(z+l\hbar)\theta(\hbar)}\prod_{r<q\neq p}\frac{\theta(\lambda_{jq}+(k-l)\hbar)}{\theta(\lambda_{jq}+\hbar)}.$$

Proof. The idea is similar to [Zhang 2018, Lemma 7.6]. We compute $L_{jp}(z)(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l})$. As in the proof of Lemma 4.5, only two terms survive:

$$
\begin{aligned}
L_{jp}(z)(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l}) &= L_{jj}(z)\omega_{k,l+1}\overline{\otimes}L_{jp}(z)\omega_{l+1,l}+L_{jp}(z)\omega_{k,l+1}\overline{\otimes}L_{pp}(z)\omega_{l+1,l}\\
&= \frac{\theta(z+k\hbar)}{\theta(z+(l+1)\hbar)}\prod_{q>r}\frac{\theta(\lambda_{jq}+(k-l)\hbar)}{\theta(\lambda_{jq}+\hbar)}\omega_{k,l+1}\overline{\otimes}\frac{\theta(z+l\hbar+\lambda_{jp})}{\theta(z+l\hbar)}\omega^{jp}_{l+1,l}\\
&\qquad+\frac{\theta(z+(k-1)\hbar+\lambda_{jp})}{\theta(z+(l+1)\hbar)}\omega^{jp}_{k,l+1}\overline{\otimes}\omega_{l+1,l}\\
&= \boldsymbol{e}^{(l)}_{jp}(k,z;\lambda)(\omega_{k,l+1}\overline{\otimes}\omega^{jp}_{l+1,l})+\frac{\theta(z+k\hbar+\lambda_{jp})}{\theta(z+(l+1)\hbar)}(\omega^{jp}_{k,l+1}\overline{\otimes}\omega_{l+1,l}).
\end{aligned}
$$

Here $\boldsymbol{e}^{(l)}_{jp}(k,z;\lambda)$ is the following meromorphic function of $(k,z,\lambda)\in\mathbb{C}\times\mathbb{C}\times\mathfrak{h}$:

$$\frac{\theta(z+k\hbar)\theta(z+l\hbar+\lambda_{jp})}{\theta(z+(l+1)\hbar)\theta(z+l\hbar)}\frac{\theta(\lambda_{jp}+(k-l-1)\hbar)}{\theta(\lambda_{jp})}\prod_{r<q\neq p}\frac{\theta(\lambda_{jq}+(k-l)\hbar)}{\theta(\lambda_{jq}+\hbar)}.$$

Set $x:=\boldsymbol{a}^{(l)}_{jp}(k;\lambda)(\omega_{k,l+1}\overline{\otimes}\omega^{jp}_{l+1,l})-\omega^{jp}_{k,l+1}\overline{\otimes}\omega_{l+1,l}$, which is in the kernel of $\mathcal{G}_{k,l}$ by Lemma 4.5. It follows that for any $g(\lambda)\in\mathbb{M}$ we have

$$L_{jp}(z)\omega_{kl}=L_{jp}(z)\mathcal{G}_{k,l}(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l})=\mathcal{G}_{k,l}(L_{jp}(z)(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l})+g(\lambda)x).$$

Let us fix $g(z;\lambda):=\theta(z+k\hbar+\lambda_{jp})/\theta(z+(l+1)\hbar)$. Then $L_{jp}(z)(\omega_{k,l+1}\overline{\otimes}\omega_{l+1,l})+g(z;\lambda)x$ is proportional to $\omega_{k,l+1}\overline{\otimes}\omega^{jp}_{l+1,l}$ and $L_{jp}(z)\omega_{kl}=\mathcal{G}_{k,l}(\omega_{k,l+1}\overline{\otimes}\omega^{jp}_{l+1,l})\times\boldsymbol{b}^{(l)}_{jp}(k,z;\lambda)$ where

$$\begin{aligned}\boldsymbol{b}^{(l)}_{jp}(k,z;\lambda)&=\boldsymbol{e}^{(l)}_{jp}(k,z;\lambda)+g(z;\lambda)\boldsymbol{a}^{(l)}_{jp}(k;\lambda)\\&=\boldsymbol{b}(k,z;\lambda)\times\prod_{r<q\neq p}\frac{\theta(\lambda_{jq}+(k-l)\hbar)}{\theta(\lambda_{jq}+\hbar)},\\ \boldsymbol{b}(k,z;\lambda)&:=\frac{\theta(z+k\hbar)\theta(z+l\hbar+\lambda_{jp})}{\theta(z+(l+1)\hbar)\theta(z+l\hbar)}\frac{\theta(\lambda_{jp}+(k-l-1)\hbar)}{\theta(\lambda_{jp})}+\frac{\theta(z+k\hbar+\lambda_{jp})\theta((k-l-1)\hbar)\theta(\lambda_{jp}-\hbar)}{\theta(z+(l+1)\hbar)\theta(\lambda_{jp})\theta(\hbar)}.\end{aligned}$$

The function $\boldsymbol{b}(k,z;\lambda)$, viewed as an entire function of k, satisfies the same double periodicity as $\theta(k\hbar)\theta(k\hbar+z+\lambda_{jp}-(l+1)\hbar)$. One checks that $\boldsymbol{b}(l,z;\lambda)=0$. This implies

$$\boldsymbol{b}(k,z;\lambda)=\theta(k\hbar+z-\hbar+\lambda_{jp})\theta(k\hbar-l\hbar)f(z;\lambda),$$

where $f(z;\lambda)$ is a meromorphic function of $(z;\lambda)\in\mathbb{C}\times\mathfrak{h}$ independent of k. Now setting $k\hbar=-z$, we obtain $f(z;\lambda)=1/(\theta(z+l\hbar)\theta(\hbar))$. □

Corollary 4.7. *Let $1\le i,j\le N$ with $j\le r$. For $k-1>l$ and $x\in W^{(r)}_{l,a}$*

$$\begin{aligned}L_{ji}(z)F_{k,l}(x)=F_{k,l}\frac{\theta(z+k\hbar)}{\theta(z+l\hbar)}\mu_l\Bigg(\prod_{q=r+1}^{N}\frac{\theta(\lambda_{jq}+(k-l+1)\hbar)}{\theta(\lambda_{jq}+\hbar)}\Bigg)L_{ji}(z)x\\+F_{k,l+1}\mathcal{F}_{l+1,l}\Bigg(\sum_{p=r+1}^{N}\omega^{jp}_{l+1,l}\overline{\otimes}\mu_l(\boldsymbol{b}^{(l)}_{jp}(k,z;\lambda))L_{pi}(z)x\Bigg).\end{aligned}\tag{4.22}$$

Proof. Consider $L_{ji}(z)F_{k,l}(x)=\mathcal{F}_{k,l}\big(\sum_{p=1}^{N}L_{jp}(z)\omega_{kl}\overline{\otimes}L_{pi}(z)x\big)$. As in the proof of Lemma 4.5, $L_{jp}(z)\omega_{kl}=0$ if $p\notin\{j,r+1,r+2,\dots,N\}$. For $p=j$, we obtain the first row of (4.22), while for $r<p\le N$, Corollary 4.6 and (4.19) with $v=\omega^{jp}_{l+1,l}$ give the second row. □

Fix weight bases $\mathcal{B}_l$ of $W^{(r)}_{l,a}$ for $l>0$ uniformly so that $F_{k,l}(\mathcal{B}_l)\subseteq\mathcal{B}_k$.

We view $\boldsymbol{b}_{jp}^{(l)}(c,z;\lambda)$ in Corollary 4.6 as a meromorphic function of $(c,z,\lambda)\in\mathbb{C}^2\times\mathfrak{h}$. For $1\le i,j\le N$, $l>0$ and $c,z\in\mathbb{C}$, define $\mathscr{L}_{ji}^{(l)}(c,z):W_{l,a}^{(r)}\to W_{l+1,a}^{(r)}$ by

$$\mathscr{L}_{ji}^{(l)}(c,z)x=F_{l+1,l}L_{ji}(z)x=\frac{\theta(z+(\gamma_j+\delta_{ij}-1)\hbar+\lambda_{ji})}{\theta(z)}F_{l+1,l}t_{ij}x \qquad \text{for } j>r,$$

$$\mathscr{L}_{ji}^{(l)}(c,z)x=\frac{\theta(z+c\hbar)}{\theta(z+l\hbar)}\prod_{q=r+1}^{N}\frac{\theta(\lambda_{jq}+(c+\gamma_{jq}+\delta_{ij}-\delta_{iq})\hbar)}{\theta(\lambda_{jq}+(l+\gamma_{jq}+\delta_{ij}-\delta_{iq})\hbar)}F_{l+1,l}L_{ji}(z)x$$
$$+\sum_{p=r+1}^{N}\boldsymbol{b}_{jp}^{(l)}(c,z;\lambda+(\gamma+l\varpi_r+\epsilon_i-\epsilon_p)\hbar)\mathscr{F}_{l+1,l}(\omega_{l+1,l}^{jp}\bar{\otimes}L_{pi}(z)x) \qquad \text{for } j\le r.$$

Here $x\in W_{l,a}^{(r)}[\gamma+l\varpi_r]$ and δ_{ij} is the usual Kronecker symbol. Corollary 2.2 applied to the evaluation module $W_{l,a}^{(r)}\cong V_{l\varpi_r}(0)$ indicates that for $b'\in\mathcal{B}_{l+1}$ and $b\in\mathcal{B}_l$

> $\mathscr{L}_{ji}^{(l)}(c,z)$ is a difference map of bidegree $(\epsilon_j-\varpi_r,\epsilon_i)$. Its matrix entry $[\mathscr{L}_{ji}^{(l)}]_{b'b}(c,z;\lambda)$ is a meromorphic function of $(c,z,\lambda)\in\mathbb{C}^2\times\mathfrak{h}$. Moreover, $\theta(z)\theta(z+l\hbar)[\mathscr{L}_{ji}^{(l)}]_{b'b}(c,z;\lambda)$ is entire on (c,z) for generic λ.

As a unification of (4.20) and (4.22), we have

$$L_{ji}(z)F_{k,l}=F_{k,l+1}\mathscr{L}_{ji}^{(l)}(k,z) \quad \text{for } k>l+1. \tag{4.23}$$

For $k\in\mathbb{Z}_{>0}$ and $z\in\mathbb{C}$ let $\Xi(c;k,z)$ be the set of entire functions $F(c)$ of $c\in\mathbb{C}$ with the following double periodicity:

$$F(c+\hbar^{-1})=(-1)^kF(c) \quad \text{and} \quad F(c+\tau\hbar^{-1})=(-1)^ke^{-ki\pi\tau-2ki\pi c\hbar-2i\pi z}F(c).$$

A typical example is $\theta(c\hbar)^{k-1}\theta(c\hbar+z)$. Such a function is called *homogeneous*. If $f(c),g(c)\in\Xi(c;k,z)$, then we write $f(c)\approx g(c)$.

Note that $\Xi(c;k,z)\Xi(c;k',z')\subseteq\Xi(c;k+k',z+z')$.

Lemma 4.8. *Let $b\in\mathcal{B}_l$ be of weight $\gamma+l\varpi_r$ and $b'\in\mathcal{B}_{l+1}$. For $j>r$ the matrix entry $[\mathscr{L}_{ji}^{(l)}]_{b'b}(c,z;\lambda)$ is independent of c. For $j\le r$ as entire functions of c*

$$[\mathscr{L}_{ji}^{(l)}]_{b'b}(c,z;\lambda)\approx\theta(z+c\hbar)\prod_{q=r+1}^{N}\theta(\lambda_{jq}+(c+\gamma_{jq}+\delta_{ij}-\delta_{iq})\hbar).$$

Moreover, $\theta(z)[\mathscr{L}_{ji}^{(l)}]_{b'b}(c,z;\lambda)$ is an entire function of (c,z) for generic λ.

Proof. In the case $j>r$, Corollary 2.2 is applied to $W_{l,a}^{(r)}\cong S_{l\varpi_r,0}$, the matrix entry is of the form $\theta(z+(\gamma_j+\delta_{ij}-1)\hbar+\lambda_{ji})g_1(\lambda)/\theta(z)$ for $g_1(\lambda)\in\mathbb{M}$. Assume $j\le r$. By Corollary 4.6 the matrix entry is of the form $E(c,z;\lambda)g_2(\lambda)/(\theta(z)\theta(z+l\hbar))$, where $g_2(\lambda)\in\mathbb{M}$ and $E(c,z;\lambda)$ is an entire function of

$(c, z, \lambda) \in \mathbb{C} \times \mathbb{C} \times \mathfrak{h}$. As functions of z and c, respectively, we have

$$E(c, z\hbar; \lambda) \in \Xi(z; 2, (c + l + \gamma_j + \delta_{ij} - 1)\hbar + \lambda_{ji}),$$
$$E(c, z; \lambda) \in \Xi(c; N - r + 1, \sum_{q=r+1}^{N} (\lambda_{jq} + (\gamma_{jq} + \delta_{ij} - \delta_{iq})\hbar) + z).$$

On the other hand, for $k > l + 1$ we have by Corollary 2.2 and (4.23),

$$F_{k,l+1}\mathscr{L}^{(l)}_{ji}(k, z)b = L_{ji}(z)F_{k,l}v = \frac{\theta(z + \lambda_{ji} + (k + \gamma_j + \delta_{ij} - 1)\hbar)}{\theta(z)} t_{ij} F_{k,l} b.$$

The right-hand side as a function of z is regular at $z = -l\hbar$, so are any of the coefficients of the left-hand side $E(k, z; \lambda)g_2(\lambda)/(\theta(z)\theta(z + l\hbar))$. This forces $E(k, -l\hbar; \lambda) = 0$ and

$$E(c, z; \lambda) = \theta(z + l\hbar)\theta(z + (c + \gamma_j + \delta_{ij} - 1)\hbar + \lambda_{ji})D(c; \lambda)g_3(\lambda),$$

where $g_3(\lambda) \in \mathbb{M}$ and $D(c; \lambda)$ is an entire function of (c, λ). Applying the double periodicity with respect to c once more, we obtain the desired result. □

Lemma 4.9. *Let $f(c)$ be a homogeneous entire function. If $f(k) = 0$ for infinitely many integers k, then $f(c)$ is identically zero.*

Proof. By definition the homogeneous entire function $f(c)$, if nonzero, can be written as a product of theta functions $\theta(c\hbar + z)$. Since $\hbar \notin \mathbb{Q} + \mathbb{Q}\tau$, each of these theta functions of c can not have zeroes at infinitely many integers. □

Let W_∞ be the inductive limit of the inductive system $(W^{(r)}_{l,a}, F_{k,l})$ of vector spaces (over $\mathbb{M}$), with the $F_l : W^{(r)}_{l,a} \to W_\infty$ for $l > 0$ being the structural maps.

From now on fix $d \in \mathbb{C}$. A vector $0 \neq w \in W_\infty$ is of weight $d\varpi_r + \gamma$ if there exist $l > 0$ and $w' \in W^{(r)}_{l,a}[l\varpi_r + \gamma]$ such that $w = F_l(w')$. The weight grading is independent of the choice of l because $F_{k,l}$ sends $W^{(r)}_{l,a}[l\varpi_r + \gamma]$ to $W^{(r)}_{k,a}[k\varpi_r + \gamma]$. Let W^d_∞ denote the resulting object of $\mathcal{V}$. By construction $\mathrm{wt}(W^d_\infty) \subseteq d\varpi_r + \boldsymbol{Q}_-$, and $F_l : W^{(r)}_{l,a} \to W^d_\infty$ is a difference map of bidegree $((l - d)\varpi_r, 0)$.

Let $\gamma \in \boldsymbol{Q}_-$. The injective maps $F_{k,l}$ together with Theorems 1.15 and 2.3 imply that

$$\dim(W^{(r)}_{k,a}[k\varpi_r + \gamma]) = d_{k\varpi_r}[k\varpi_r + \gamma],$$

as $k \to \infty$, converges to an integer which is exactly $\dim(W^d_\infty[d\varpi_r + \gamma])$. So W^d_∞ is an object of $\mathcal{V}_{\mathrm{ft}}$. Our goal is to make W^d_∞ into an $\mathcal{E}$-module in category $\mathcal{O}$ with favorable q-character.[11]

[11] In the affine case, the matrix entries of analogs of $\mathscr{L}^{(l)}_{ji}(k, z)$ are Laurent polynomials of $e^{k\hbar}$. Hernandez and Jimbo [2012] proved this by using elimination theorems of q-characters, and then took the limit $e^{k\hbar} \to 0$ as $k \to \infty$ to obtain modules over Borel subalgebras of affine quantum groups. Later in [Zhang 2017; 2018] an elementary proof of polynomiality was given based on $\mathfrak{sl}_2$-representation theory, which by taking limit $e^{k\hbar} \to e^{d\hbar}$ as $k \to \infty$ (with $d \in \mathbb{C}$ a new parameter) resulted in modules over affine quantum groups. Here we adapt the second approach to the elliptic case.

For $1 \le i, j \le N$ and $z \in \mathbb{C}$ with $\theta(z) \neq 0$, the $\mathscr{L}_{ji}^{(l)}(d, z)$ constitute a morphism of inductive system of $\mathbb{C}$-vector spaces:

$$\begin{tikzcd} W_{l,a}^{(r)} \arrow[r, "\mathscr{L}_{ji}^{(l)}(d,z)"] \arrow[d, "F_{l',l}"'] & W_{l+1,a}^{(r)} \arrow[d, "F_{l'+1,l+1}"] \\ W_{l',a}^{(r)} \arrow[r, "\mathscr{L}_{ji}^{(l')}(d,z)"] & W_{l'+1,a}^{(r)} \end{tikzcd} \quad \text{for } l' > l.$$

Indeed, the matrix entries of $F_{l'+1,l+1}\mathscr{L}_{ji}^{(l)}(c, z)$ and $\mathscr{L}_{ji}^{(l')}(c, z)F_{l',l}$, as difference maps $W_{l,a}^{(r)} \to W_{l'+1,a}^{(r)}$, are homogeneous entire functions of c with the same double periodicity by Lemma 4.8 and are equal at all integers c larger than $l'+1$ by (4.23). By Lemma 4.9 these two maps coincide for all $c \in \mathbb{C}$. Define

$$\mathscr{L}_{ji}^{d}(z) := \lim_{\to} \mathscr{L}_{ji}^{(l)}(d, z) \in \mathrm{Hom}_{\mathbb{C}}(W_\infty^d, W_\infty^d).$$

For $x \in W_\infty^d[d\varpi_r + \gamma]$ with $x = F_l(x')$ and $x' \in W_{l,a}^{(r)}[l\varpi_r + \gamma]$, we have

$$\mathscr{L}_{ji}^{d}(z)x = F_{l+1}\mathscr{L}_{ji}^{(l)}(d, z)x'. \tag{4.24}$$

The difference maps $\mathscr{L}_{ji}^{(l)}(d, z)$ and F_{l+1} are of bidegree $(\epsilon_j - \varpi_r, \epsilon_i)$ and $((l+1-d)\varpi_r, 0)$ respectively. So $\mathscr{L}_{ji}^{d}(z)$ is a difference operator of bidegree (ϵ_j, ϵ_i).

Proposition 4.10. *$(W_\infty^d, \mathscr{L}_{ji}^d(z))$ is an $\mathcal{E}$-module in category $\mathcal{O}$. Moreover,*

$$\chi_q(W_\infty^d, \mathscr{L}_{ji}^d(z)) = \boldsymbol{w}_{d,a}^{(r)} \times \lim_{k\to\infty} (\boldsymbol{w}_{k,a}^{(r)})^{-1}\chi_q(W_{k,a}^{(r)}). \tag{4.25}$$

Proof. We need to prove conditions (M1)–(M3) of Section 1C. First (M1) follows from (4.24) and from the comments before (4.23). To prove (M2), let $x \in W_\infty^d[d\varpi_r + \gamma]$ and $x' \in W_{l,a}^{(r)}[l\varpi_r + \gamma]$ such that $x = F_l(x')$. We assume l so large that $W_\infty^d[d\varpi_r + \gamma]$ and $W_{l,a}^{(r)}[l\varpi_r + \gamma]$ have the same dimension.

Step I: Proof of (M2) We need to show that for $1 \le i, j, m, n \le N$

$$\sum_{p,q} R_{mn}^{pq}(z-w; \lambda + (\epsilon_i + \epsilon_j + d\varpi_r + \gamma)\hbar)\mathscr{L}_{pi}^d(z)\mathscr{L}_{qj}^d(w)x = \sum_{s,t} R_{st}^{ij}(z-w; \lambda)\mathscr{L}_{nt}^d(w)\mathscr{L}_{ms}^d(z)x \in W_\infty^d.$$

Here at the right-hand side we have used $R_{mn}^{pq}(z; \lambda) = R_{mn}^{pq}(z; \lambda + \hbar\epsilon_p + \hbar\epsilon_q)$ to move R to the left. By (4.24) it is enough to prove the equation

$$\sum_{p,q} R_{mn}^{pq}(z-w; \lambda + (\epsilon_i + \epsilon_j + c\varpi_r + \gamma)\hbar)\mathscr{L}_{pi}^{(l+1)}(c, z)\mathscr{L}_{qj}^{(l)}(c, w)x'$$
$$= \sum_{s,t} R_{st}^{ij}(z-w; \lambda)\mathscr{L}_{nt}^{(l+1)}(c, w)\mathscr{L}_{ms}^{(l)}(c, z)x' \in W_{l+2,a}^{(r)}. \tag{4.26}$$

Let $A_1(c, z, w)$ and $A_2(c, z, w)$ denote the left-hand side and the right-hand side of this equation without x'. These are difference maps $W_{l,a}^{(r)} \to W_{l+2,a}^{(r)}$ of bidegree $(\epsilon_m + \epsilon_n - 2\varpi_r, \epsilon_i + \epsilon_j)$, as $R_{mn}^{pq} \neq 0$ implies $\epsilon_m + \epsilon_n = \epsilon_p + \epsilon_q$.

Claim 1. *For $b \in \mathcal{B}_l$ of weight $l\varpi_r + \gamma$ and $b' \in \mathcal{B}_{l+2}$, as entire functions of c,*

$$[A_1]_{b'b}(c, z, w; \lambda) \approx [A_2]_{b'b}(c, z, w; \lambda).$$

This is divided into four cases. For simplicity let us drop b', b, z, w and λ from A_1 and A_2.

Case 1.1: $m, n > r$. $A_1(c)$ and $A_2(c)$ are independent of c by Lemma 4.8.

Case 1.2: $m, n \leq r$. At the left-hand side of (4.26) we have $\{p, q\} = \{m, n\}$ and so R_{mn}^{pq} is independent of c. At the right-hand side $\{s, t\} = \{i, j\}$. Therefore

$$A_1(c) \approx \theta(c\hbar + z) \prod_{u>r} \theta(\lambda_{pu} + (c + \gamma_{pu} + \delta_{ip} - \delta_{iu} + \delta_{jp} - \delta_{ju} - \delta_{qp} + \delta_{qu})\hbar) \\ \times \theta(c\hbar + w) \prod_{v>r} \theta(\lambda_{qv} + (c + \gamma_{qv} + \delta_{jq} - \delta_{jv} + \delta_{iq} - \delta_{iv})\hbar),$$

$$A_2(c) \approx \theta(c\hbar + w) \prod_{u>r} \theta(\lambda_{nu} + (c + \gamma_{nu} + \delta_{tn} - \delta_{tu} + \delta_{sn} - \delta_{su} - \delta_{mn} + \delta_{mu})\hbar) \\ \times \theta(c\hbar + z) \prod_{v>r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{sm} - \delta_{sv} + \delta_{tm} - \delta_{tv})\hbar).$$

These formulas are deduced from Lemma 4.8. One needs to take into account the shifts of γ and λ. For example at the left-hand side of (4.26), the terms $\mathscr{L}_{qj}$ and $\mathscr{L}_{pi}$ shift γ and λ by $\epsilon_j - \epsilon_q$ and $\hbar\epsilon_i$, respectively. The right-hand sides of these two formulas lie in $\Xi(c; 2+2N-2r, e)$ with $e \in \mathbb{C}$ independent of the choices of p, q, s and t.

Case 1.3: $m \leq r < n$. At the right-hand side $\{s, t\} = \{i, j\}$ and

$$A_2(c) \approx \theta(c\hbar + z) \prod_{v>r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{sm} - \delta_{sv} + \delta_{tm} - \delta_{tv})\hbar) \\ \approx \theta(c\hbar + z) \prod_{v>r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{im} - \delta_{iv} + \delta_{jm} - \delta_{jv})\hbar).$$

The last term is independent of s and t. On the other hand $A_1(c) = E(c) + F(c)$ where E and F correspond to $(p, q) = (m, n)$ and $(p, q) = (n, m)$ respectively and so

$$E(c) \approx \frac{\theta(f - \hbar)}{\theta(f)} \theta(c\hbar + z) \prod_{u>r} \theta(\lambda_{mu} + (c + \gamma_{mu} + \delta_{im} - \delta_{iu} + \delta_{jm} - \delta_{ju} + \delta_{nu})\hbar),$$

$$F(c) \approx \frac{\theta(f + z - w)}{\theta(f)} \theta(c\hbar + w) \prod_{v>r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{jm} - \delta_{jv} + \delta_{im} - \delta_{iv})\hbar).$$

Here $f := c\hbar + \lambda_{mn} + (\gamma_{mn} + \delta_{im} - \delta_{in} + \delta_{jm} - \delta_{jn})\hbar$. We observe easily that $A_2(c) \approx E(c) \approx F(c)$ and so $A_1(c) \approx A_2(c)$ are homogeneous.

Case 1.4: $n \leq r < m$. This is parallel to the third case.

Claim 2. *In Claim 1 equality holds for $c = k \in \mathbb{Z}_{>l+2}$.*

Let us apply $F_{k,l+2}$ to (4.26) with $c = k$ and $x' = b$. By (4.23)

$$F_{k,l+2}\mathscr{L}_{pi}^{(l+1)}(k, z)\mathscr{L}_{qj}^{(l)}(k, w) = L_{pi}(z)F_{k,l+1}\mathscr{L}_{qj}^{(l)}(k, w)x' = L_{pi}(z)L_{qj}(w)F_{k,l},$$

and similarly $F_{k,l+2}\mathscr{L}^{(l+1)}_{nt}(d,w)\mathscr{L}^{(l)}_{ms}(d,z)b = L_{nt}(w)L_{ms}(z)F_{k,l}b$. We obtain the defining relation $RLL = LLR$ of the $\mathcal{E}$-module $W^{(r)}_{k,a}$ applied to the vector $F_{k,l}(b)$. Since $F_{k,l+2}$ is injective, (4.26) holds for $c = k$ and $x' = b$. This proves Claim 2.

Together with Lemma 4.9, we obtain equality in Claim 1 for all $c \in \mathbb{C}$. This proves (4.26).

Step II: Let $1 \le i \le N$. We have by (1.6) and (4.26)

$$\mathcal{D}_i(z)x = \frac{\Theta_i(\lambda)}{\Theta_i(\lambda + (d\varpi_r + \gamma)\hbar)} F_{l+i}\mathscr{D}^{(l)}_i(d,z)x'. \tag{4.27}$$

Here $\mathscr{D}^{(l)}_i(c,z) = \sum_{\sigma\in\mathfrak{S}^i} T_\sigma(c,z)$ and $T_\sigma(c,z) : W^{(i)}_{l,a} \to W^{(i)}_{l+i,a}$, for $\sigma \in \mathfrak{S}^i$, is given by

$$\operatorname{sign}(\sigma)\mathscr{L}^{(l+i-1)}_{\sigma(N),N}(c,z)\mathscr{L}^{(l+i-2)}_{\sigma(N-1),N-1}(c,z+\hbar)\cdots\mathscr{L}^{(l)}_{\sigma(N-i+1),N-i+1}(c,z+(i-1)\hbar).$$

Each $T_\sigma(c,z)$ is a difference map of bidegree $(-\varpi_{N-i} - i\varpi_r, -\varpi_{N-i})$. Define the meromorphic function of $(c,z) \in \mathbb{C}^2$ (note that l is fixed)

$$g(c,z) = 1 \ \text{ if } i < N+1-r \quad \text{and} \quad g(c,z) = \prod_{p=N-i+1}^{r} \frac{\theta(z+(N-p+c)\hbar)}{\theta(z+(N-p+l)\hbar)} \ \text{ otherwise.}$$

Claim 3. *For $b \in \mathcal{B}_l$ of weight $l\varpi_r + \gamma$ and $b' \in \mathcal{B}_{l+i}$, as entire functions of $c \in \mathbb{C}$,*

$$[T_\sigma]_{b'b}(c,z;\lambda) \approx g_i(c,z)\Theta_i(\lambda + (c\varpi_r + \gamma)\hbar).$$

The idea is the same as Claim 1, based on Lemma 4.8. If $N - i + 1 > r$, then $T^b_\sigma(c,z;\lambda)$ and $\Theta_i(\lambda + (c\varpi_r + \gamma)\hbar)$ are independent of c, and we are done.

Assume $N - i + 1 \le r$. By (1.5) and Lemma 4.8,

$$\Theta_i(\lambda + (c\varpi_r + \gamma)\hbar) \approx \prod_{p=N-i+1}^{r}\prod_{u=r+1}^{N} \theta(\lambda_{pu} + (c+\gamma_{pu})\hbar),$$

$$[T_{\mathrm{Id}}]_{b,b'}(c,z;\lambda) \approx \prod_{p=N-i+1}^{r} \theta(z+(c+N-p)\hbar) \prod_{u=r+1}^{N} \theta(\lambda^{(p)}_{pu} + (c+\gamma_{pu}+1)\hbar).$$

Here $\lambda^{(p)} = \lambda + \hbar\sum_{v=p+1}^{N}\epsilon_v$ and so $\lambda^{(p)}_{pu} = \lambda_{pu} - \hbar$ for $p \le r < u$. The case $\sigma = \mathrm{Id}$ in Claim 3 is now obvious. It remains to show $[T_\sigma]_{b'b}(c,z;\lambda) \approx [T_{\sigma'}]_{b'b}(c,z;\lambda)$ for all $\sigma,\sigma' \in \mathfrak{S}^i$. One can assume $\sigma' = \sigma s_j$ where $s_j = (j, j+1)$ is a simple transposition with $N - i + 1 \le j < N - 1$. Let us define

$$p := \sigma(j+1), q := \sigma(j), \quad l' := l + i + j - 1 - N, \quad w := z + (N-j)\hbar.$$

Then we have the decomposition of difference maps

$$T_\sigma(c,z) = \operatorname{sign}(\sigma)A(c,z)U_{pq}(c,w)B(c,z) \quad \text{and} \quad T_{\sigma'}(c,z) = \operatorname{sign}(\sigma)A(c,z)U_{qp}(c,w)B(c,z).$$

The difference maps A, B, U are defined by (descending order in the products)

$$A(c,z)=\prod_{u=N}^{j+2}\mathcal{L}^{(l+i-1-N+u)}_{\sigma(u),u}(c,z+(N-u)\hbar):W^{(r)}_{l'+2,a}\to W^{(r)}_{l+i-1,a},$$

$$B(c,z)=\prod_{u=j-1}^{N-i+1}\mathcal{L}^{(l+i-1-N+u)}_{\sigma(u),u}(c,z+(N-u)\hbar):W^{(r)}_{l,a}\to W^{(r)}_{j-1,a},$$

$$U_{pq}(c,w)=\mathcal{L}^{(l'+1)}_{p,j+1}(c,w-\hbar)\mathcal{L}^{(l')}_{qj}(c,w):W^{(r)}_{l',a}\to W^{(r)}_{l'+2,a}.$$

Flipping p and q one gets U_{qp}. Now $[T_\sigma]_{b'b}(c,z;\lambda)\approx[T_{\sigma'}]_{b'b}(c,z;\lambda)$ is a consequence of the following claim.

Claim 4. *For $y\in\mathcal{B}_{l'}$ of weight $l'\varpi_r+\eta$ and $y'\in\mathcal{B}_{l'+2}$, as entire functions of c,*

$$[U_{pq}]_{y'y}(c,w;\lambda)\approx[U_{qp}]_{y'y}(c,w;\lambda).$$

If $p,q\le r$, then by Lemma 4.8 (setting $\eta'=\eta+\epsilon_j-\epsilon_q$ and $\lambda'=\lambda+\hbar\epsilon_{j+1}$)

$$[U_{pq}]_{y'y}(c,w;\lambda)\approx\theta(w+(c-1)\hbar)\prod_{u=r+1}^{N}\theta(\lambda_{pu}+(c+\eta'_{pu}+\delta_{p,j+1}-\delta_{u,j+1})\hbar)$$
$$\times\,\theta(w+c\hbar)\prod_{v=r+1}^{N}\theta(\lambda'_{qv}+(c+\eta_{qv}+\delta_{jq}-\delta_{jv})\hbar).$$

We have $U^{b'}_{pq}(c,w;\lambda)\in\Xi(c;2N-2r+2,e)$ with $e=e(p,q)$ symmetric on p,q. So $[U_{pq}]_{y'y}(c,w;\lambda)\approx[U_{qp}]_{y'y}(c,w;\lambda)$.

The other cases of p and q are proved in the same way as in Claim 1.

Step III: Proof of (M3) Let $k>l+i$. Notice that $\mathcal{D}_i(z)\omega_{kl}=g_i(k,z)\omega_{kl}$. From the proof of Lemma 4.3 and from (4.23) and (4.27) we get

$$g_i(k,z)F_{k,l}\mathcal{D}_i(z)x'=\mathcal{D}_i(z)F_{k,l}(x')=F_{k,l+i}\frac{\Theta_i(\lambda)}{\Theta_i(\lambda+(k\varpi_r+\gamma)\hbar)}\mathscr{D}^{(l)}_i(k,z)x'.$$

Applying F_k to this identity and multiplying by $\Theta_i(\lambda+(k\varpi_r+\gamma)\hbar)$ we have

$$\Theta_i(\lambda+(k\varpi_r+\gamma)\hbar)g_i(k,z)F_l\mathcal{D}_i(z)x'=\Theta_i(\lambda)F_{l+i}\mathscr{D}^{(l)}_i(k,z)x'\quad\text{for }k>l+i.$$

Both sides after taking coefficients with respect to a basis of $W^d_\infty[d\varpi_r+\gamma]$ can be viewed as entire functions of $k\in\mathbb{C}$, and they satisfy the same double periodicity by Claim 3. By Lemma 4.9, the above identity holds for all $k\in\mathbb{C}$. Taking $k=d$, by (4.27), we obtain $\mathcal{D}_i(z)x=g_i(d,z)F_l\mathcal{D}_i(z)x'$.

Let B be a basis of $W^{(r)}_{l,a}[l\varpi_r+\gamma]$ satisfying the upper triangular property of (M3). Then the basis $F_l(B)$ of $W^d_\infty[d\varpi_r+\gamma]$ satisfies the same property. The $\mathcal{E}$-module W^d_∞ is in category $\mathcal{O}$. The diagonal entry of $\mathcal{D}_i(z)$ associated to $F_l(x')\in W^d_\infty$ for $x'\in B$ is equal to that of $\mathcal{D}_i(z)$ associated to $x'\in W^{(r)}_{l,a}$ multiplied by $g_i(d,z)$. The q-character formula in (4.25) follows from the explicit formula of $g_i(d,z)$. □

Question 4.11. Let $F(c)$ be a finite sum of homogeneous entire functions. If $F(k) = 0$ for infinitely many integers k, then is $F(c)$ identically zero?

If the answer to this question is affirmative, then the proof of Proposition 4.10 can be largely simplified, Claims 1, 3 and 4 are not necessary.[12]

Remark 4.12. By Lemma 4.8, $W_\infty^d \cong \mathcal{W}(0)$ with $\mathcal{W}$ an $\mathfrak{e}$-module of character $\lim_{k\to\infty} e^{(d-k)\varpi_r}\chi(W_{k,a}^{(r)})$, so it is in the image of the functor [Etingof and Moura 2002, Proposition 4.1]. By Lemma 4.2 and its proof, $\mathcal{W}$ contains a unique highest weight vector up to scalar. Let Q be the quotient of standard Verma module $M_{d\varpi_r,1}$ in [Tarasov and Varchenko 2001, Proposition 4.7] by $t_{a+1,a}v_{d\varpi_r,1}$ for $a \neq r$. Then $\mathcal{W}$ is the contragradient module to Q in [Tarasov and Varchenko 2001, §6]. It is interesting to have a direct proof of $\mathcal{W}(0)$ being in category $\mathcal{O}$.

For $x \in \mathbb{C}$ let $\mathcal{W}_{d,x}^{(r)}$ be the pullback of W_∞^d by Φ_{x-a} in (1.4); it is called *an asymptotic module*. Set $\mathcal{W}_{d,x}^{(N)} := S(\boldsymbol{w}_{d,x}^{(N)})$ and $\mathcal{W}_{s,x} := \mathcal{W}_{x,0}^{(s)}$ for $1 \leq s \leq N$.

Corollary 4.13. (i) *$\mathcal{R}$ is the set of rational e-weigths.*

(ii) *For any $\mathcal{E}$-module M in category $\mathcal{O}$, we have $\mathrm{wt}_e(M) \subset \mathcal{R}$.*

(iii) *For $d, x \in \mathbb{C}$ and $1 \leq r \leq N$ we have in $\mathcal{M}_\mathfrak{t}$ and $K_0(\mathcal{O})$ respectively*

$$\chi_q(\mathcal{W}_{d,x}^{(r)}) = \boldsymbol{w}_{d,x}^{(r)} \times \chi_q(\mathcal{W}_{0,x}^{(r)}) \quad \textit{and} \quad [\mathcal{W}_{d,0}^{(r)}][\mathcal{W}_{0,x}^{(r)}] = [\mathcal{W}_{d-x,x}^{(r)}][\mathcal{W}_{x,0}^{(r)}]. \tag{4.28}$$

Proof. Conclusion (iii) comes from (4.25), as in the proof of [Felder and Zhang 2017, Theorem 3.11]. As a highest weight of $\mathcal{W}_{d,x}^{(r)}$, $\boldsymbol{w}_{d,x}^{(r)}$ belongs to $\mathcal{R}$. Together with Lemma 1.13 we obtain (i). In (ii) one may assume M irreducible. Then M is a subquotient of a tensor product of asymptotic modules. Since e-weigths of an asymptotic module are rational, we conclude from the multiplicative structure of q-characters in Proposition 1.10. □

In Section 2B the evaluation module $V_\mu(x)$ is an irreducible highest weight module in category $\tilde{\mathcal{O}}$. Its highest weight is easily shown to be rational.

Corollary 4.14. *$V_\mu(x)$ is in category $\mathcal{O}$ for $\mu \in \mathfrak{h}$ and $x \in \mathbb{C}$.*

Finite-dimensional modules in category $\mathcal{O}$ are related to the asymptotic modules by *generalized Baxter relations* in the sense of Frenkel and Hernandez [2015, Theorem 4.8]; see [Felder and Zhang 2017, Corollary 4.7] and [Zhang 2017, Theorem 5.11] for a closer situation.

Theorem 4.15. *Let V be a finite-dimensional $\mathcal{E}$-module in category $\mathcal{O}$. Then*

$$[V] = \sum_{j=1}^{\dim V} [S(\boldsymbol{d}_j)] \times \boldsymbol{m}_j \tag{4.29}$$

[12] In the affine case, by footnote 11 the situation is much easier; a Laurent polynomial vanishing at infinitely many integers must be zero, see [Zhang 2017, §2].

in a fraction ring of the Grothendieck ring of $\mathcal{O}$. Here $\boldsymbol{d}_j \in \mathcal{R}_0$ and $\boldsymbol{m}_j$ is a product of the $[\mathcal{W}_{r,x}]/[\mathcal{W}_{r,y}]$ with $x, y \in \mathbb{C}$ and $1 \le r < N$.

Proof. The idea is the same as [Frenkel and Hernandez 2015]. Since the q-character map is injective, one can replace isomorphism classes with q-characters. $\chi_q(V)$ is the sum of its e-weigths, the number of which is $\dim V$. By Corollary 4.13, any e-weigth $\boldsymbol{e}$ is of the form $\boldsymbol{d}\prod \Psi_{r,x}/\Psi_{r,y} = \chi_q(S(\boldsymbol{d}))\prod \chi_q(\mathcal{W}_{r,x})/\chi_q(\mathcal{W}_{r,y})$, where $\boldsymbol{d} \in \mathcal{R}_0$ and the product is over $1 \le r < N$ and $x, y \in \mathbb{C}$. This proves (4.29) in terms of q-characters. □

To compare with [Frenkel and Hernandez 2015, Theorem 4.8], one imagines that for $1 \le r < N$ and $x \in \mathbb{C}$ there existed a *positive prefundamental module* $L^+_{r,x}$ in category $\mathcal{O}$ with q-character $\chi_q(L^+_{r,x}) = \Psi_{r,x} \times \chi(L^+_{r,0})$ as in [Frenkel and Hernandez 2015, Theorem 4.1]. Then $[\mathcal{W}_{r,x}]/[\mathcal{W}_{r,y}] = [L^+_{r,x}]/[L^+_{r,y}]$. Note that the q-character of $\mathcal{W}^{(r)}_{0,x}$ in (4.28) is different from its character.

Example 4.16. Let $N = 3$. Consider the vector representation $\boldsymbol{V}$ of Section 1D:

$$\boxed{1}_0 = \frac{\Psi_{1,\frac{3}{2}}}{\Psi_{1,\frac{1}{2}}}, \quad \boxed{2}_0 = \frac{\Psi_{1,-\frac{1}{2}}}{\Psi_{1,\frac{1}{2}}}\frac{\Psi_{2,1}}{\Psi_{2,0}}, \quad \boxed{3}_0 = \frac{\theta(z+\hbar)}{\theta(z)}\frac{\Psi_{2,-1}}{\Psi_{2,0}},$$

$$[\boldsymbol{V}] = \frac{[\mathcal{W}_{1,\frac{3}{2}}]}{[\mathcal{W}_{1,\frac{1}{2}}]} + \frac{[\mathcal{W}_{1,-\frac{1}{2}}]}{[\mathcal{W}_{1,\frac{1}{2}}]}\frac{[\mathcal{W}_{2,1}]}{[\mathcal{W}_{2,0}]} + \frac{[\mathcal{W}_{3,\frac{1}{2}}]}{[\mathcal{W}_{3,-\frac{1}{2}}]}\frac{[\mathcal{W}_{2,-1}]}{[\mathcal{W}_{2,0}]}.$$

Example 4.17. Let us construct the $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_2)$-module $\mathcal{W}_{1,\Lambda}$ from [Felder and Varchenko 1996b, Theorem 3]. As in [loc. cit.] set $\eta = -\frac{1}{2}\hbar$, $\lambda = \lambda_{12}$ and $(a, b, c, d) = (L_{11}, L_{12}, L_{21}, L_{22})$. For $\Lambda \in \mathbb{Z}_{>0}$, consider the evaluation module $L_\Lambda((\Lambda-1)\eta)$ with basis $(e_k)_{0\le k\le\Lambda}$. Note that k indicates the basis vectors, while Λ the integer parameter of a KR module. Let us make a change of basis (the second product is empty if $k = 0$)

$$v_k := e_k \prod_{i=1}^{\Lambda} \frac{\theta(\lambda+(i-k)\hbar)}{\theta(\hbar)} \times \prod_{j=1}^{k} \frac{\theta(\lambda-j\hbar)}{\theta((\Lambda-k+j)\hbar)} \quad \text{for } 0 \le k \le \Lambda.$$

Tensoring $L_\Lambda((\Lambda-1)\eta)$ with the one-dimensional module of highest weight $\theta(w+\Lambda\hbar)/\theta(w)$, we obtain another irreducible module V_Λ with basis $(v_k = v_k \widetilde{\otimes} 1)_{0\le k\le\Lambda}$; here to follow [loc. cit.] w denotes z. We have $\mathrm{wt}(v_k) = (\Lambda-k)\epsilon_1 + k\epsilon_2$ and

$$\begin{aligned}
a(w)v_k &= \frac{\theta(w+(\Lambda-k)\hbar)}{\theta(w)}\frac{\theta(\lambda+(\Lambda-k+1)\hbar)}{\theta(\lambda+(1-k)\hbar)}v_k,\\
b(w)v_k &= \frac{\theta(w+\lambda+(\Lambda-k-1)\hbar)}{\theta(w)}\frac{\theta((\Lambda-k)\hbar)\theta(\hbar)}{\theta(\lambda-\hbar)\theta(\lambda)}v_{k+1},\\
c(w)v_k &= -\frac{\theta(w-\lambda+(k-1)\hbar)}{\theta(w)}\frac{\theta(k\hbar)}{\theta(\hbar)}v_{k-1},\\
d(w)v_k &= \frac{\theta(w+k\hbar)}{\theta(w)}\frac{\theta(\lambda-(k+1)\hbar)\theta(\lambda-k\hbar)}{\theta(\lambda-\hbar)\theta(\lambda)}v_k.
\end{aligned}$$

We have $t_{12}v_k = -v_{k-1}\theta(k\hbar)/\theta(\hbar)$ and v_0 is of highest weight $\boldsymbol{w}^{(1)}_{\Lambda,0}$, so $V_\Lambda \cong W^{(1)}_{\Lambda,0}$. The bases (v_k) trivialize the inductive system (V_Λ) because the inductive maps commute with t_{12} by (4.20). For $\Lambda \in \mathbb{C}$, the above formulas define an $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_2)$-module structure on $\oplus_{k=0}^{\infty}\mathbb{M}v_k$, with $\mathrm{wt}(v_k) = (\Lambda - k)\epsilon_1 + k\epsilon_2$. This is the desired $\mathcal{W}_{1,\Lambda}$.

General formulas for the $\mathcal{E}_{\tau,\hbar}(\mathfrak{sl}_N)$-module $\mathcal{W}_{1,\Lambda}$ can be found in [Cavalli 2001, §3.4].

5. Baxter TQ relations

We derive three-term relations in the Grothendieck ring $K_0(\mathcal{O})$ for the asymptotic modules. For $1 \le r < N$ and $k, x, t \in \mathbb{C}$, by Corollary 4.13, $\boldsymbol{d}^{(r,t)}_{k,x} \in \mathcal{R}$ and is the highest weight of an irreducible module $D^{(r,t)}_{k,x}$ in category $\mathcal{O}$.

Call a complex number $c \in \mathbb{C}$ *generic* if $c \notin \frac{1}{2}\mathbb{Z} + \frac{1}{\hbar}(\mathbb{Z} + \mathbb{Z}\tau)$. This condition is equivalent to $\mathcal{Q}_a \cap \mathcal{Q}_{a+c} = \{1\}$ for all $a \in \mathbb{C}$.

Theorem 5.1. *Let $1 \le r < N$, $t \in \mathbb{Z}_{>0}$ and $k, a, b \in \mathbb{C}$ with k generic. Then*

$$\chi_q(D^{(r,t)}_{k,a}) = \boldsymbol{d}^{(r,t)}_{k,a}\Big(1 + \sum_{l=1}^{t} A^{-1}_{r,a}A^{-1}_{r,a+1}\cdots A^{-1}_{r,a+l-1}\Big)\prod_{s=r\pm1}\chi_q(\mathcal{W}^{(s)}_{0,a-k-\frac{1}{2}}) \tag{5.30}$$

and $D^{(r,0)}_{k,a} \cong \overline{\otimes}_{s=r\pm1}\mathcal{W}^{(s)}_{k,a-k-\frac{1}{2}}$.

Proof. Set $x := a - k - \frac{1}{2}$. Define $\boldsymbol{d} := \boldsymbol{d}^{(r,t)}_{k,a}$ and for $1 \le l \le t$,

$$\boldsymbol{m}_l := \boldsymbol{m}_0 A^{-1}_{r,a}A^{-1}_{r,a+1}\cdots A^{-1}_{r+a+l-1} \quad \text{and} \quad \boldsymbol{m}_0 := \boldsymbol{d}\prod_{j=1}^{t}\boldsymbol{w}^{(r)}_{k+j-\frac{1}{2},x}.$$

By (1.9) and (3.14)–(3.15), we have for $0 \le l \le t$

$$\boldsymbol{m}_l = \prod_{j=l+1}^{t}\frac{\Psi_{r,a+j}}{\Psi_{r,x}} \times \prod_{j=1}^{l}\frac{\Psi_{r,a+j-2}}{\Psi_{r,x}} \times \prod_{s=r\pm1}\frac{\Psi_{s,a+l-\frac{1}{2}}}{\Psi_{s,x}}.$$

Let us introduce the tensor products for $0 \le l \le t$,

$$S^l := (\overline{\otimes}^t_{j=l+1}\mathcal{W}^{(r)}_{k+j+\frac{1}{2},x})\overline{\otimes}(\overline{\otimes}^l_{j=1}\mathcal{W}^{(r)}_{k+j-\frac{3}{2},x})\overline{\otimes}(\overline{\otimes}_{s=r\pm1}\mathcal{W}^{(s)}_{k+l,x}),$$
$$T := D^{(r,t)}_{k,a}\overline{\otimes}(\overline{\otimes}^t_{j=1}\mathcal{W}^{(r)}_{k+j-\frac{1}{2},x}).$$

Equation (5.30) is equivalent to $\chi_q(T) = \sum_{l=0}^{t}\chi_q(S^l)$ in view of (4.28).

Given two elements $\chi = \sum_f c_f \boldsymbol{f}$ and $\chi' := \sum_f c'_f \boldsymbol{f}$ of $\mathcal{M}_{\mathrm{t}}$, we say that χ is bounded above by χ' if $c_f \le c'_f$ for all $\boldsymbol{f} \in \mathcal{M}_{\mathrm{w}}$. When this is the case, χ' is bounded below by χ. If χ is bounded below and above by χ', then $\chi = \chi'$.

Claim 1. *The S^l are irreducible. In particular, $D^{(r,0)}_{k,a} \cong \overline{\otimes}_{s=r\pm1}\mathcal{W}^{(s)}_{k,x}$.*

Fix $0 \le l \le t$. Let $S' := S(\boldsymbol{m}_l)$. For $n \in \mathbb{Z}_{>0}$, set

$$S'_n := (W^{(r)}_{n,x})^{\overline{\otimes} t} \overline{\otimes} (\overline{\otimes}_{s=r\pm 1} W^{(s)}_{n,x}) \quad \text{and} \quad \boldsymbol{s}'_n := (\boldsymbol{w}^{(r)}_{n,x})^t \prod_{s=r\pm 1} \boldsymbol{w}^{(s)}_{n,x}.$$

By Lemma 3.2, any e-weigth $\boldsymbol{s}'_n \boldsymbol{e} \in \mathcal{P}_x$ of S'_n different from $\boldsymbol{s}'_n$ is right-negative. So S'_n is irreducible. Viewing S'_n as an irreducible subquotient of

$$S' \overline{\otimes} (\overline{\otimes}_{j=l+1}^t \mathscr{W}^{(r)}_{n-k-j-\frac{1}{2},a+j}) \overline{\otimes} (\overline{\otimes}_{j=1}^l \mathscr{W}^{(r)}_{n-k-j+\frac{3}{2},a+j-2}) \overline{\otimes} (\overline{\otimes}_{s=r\pm 1} \mathscr{W}^{(s)}_{n-k-l,a+l-\frac{1}{2}}),$$

we have $\boldsymbol{e} = \boldsymbol{e}' \prod_{j=1}^t \boldsymbol{e}_j \prod_{s=r\pm 1} \boldsymbol{e}^{(s)}$ where $\boldsymbol{m}_l \boldsymbol{e}'$, $\boldsymbol{w}^{(r)}_{n-k-j-\frac{1}{2},a+j} \boldsymbol{e}_j$ for $l < j \le t$, $\boldsymbol{w}^{(r)}_{n-k-j+\frac{3}{2},a+j-2} \boldsymbol{e}_j$ for $1 \le j \le l$, and $\boldsymbol{w}^{(s)}_{n-k-l,a+l-\frac{1}{2}} \boldsymbol{e}^{(s)}$ are e-weigths of the corresponding tensor factors. By Lemma 3.2 and Proposition 4.10,

$$\boldsymbol{e}, \boldsymbol{e}' \in \mathcal{Q}_x^- \quad \text{and} \quad \boldsymbol{e}_j, \boldsymbol{e}^{(s)} \in \mathcal{Q}_a^-.$$

Since $a - x = k + \frac{1}{2}$ is generic, $\mathcal{Q}_a^- \cap \mathcal{Q}_x^- = \{1\}$ and so $\boldsymbol{e} = \boldsymbol{e}'$. The normalized q-character of S' is bounded below by that of S'_n for all $n \in \mathbb{Z}_{>0}$. On the other hand, viewing S' as an irreducible subquotient of S^l and applying (4.25) to S^l, we see that the normalized q-character of S' is bounded above by the limit of that of S'_n as $n \to \infty$. Therefore $S^l \cong S'$ is irreducible.

Claim 2. *For $1 \le l \le t$, we have $\boldsymbol{d} A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r,a+l-1}^{-1} \in \mathrm{wt}_e(D^{(r,t)}_{k,a})$. It follows that $\boldsymbol{m}_l \in \mathrm{wt}_e(T)$.*

Let us view the KR module $W^{(r)}_{t,a}$ as an irreducible subquotient of

$$D^{(r,t)}_{k,a} \overline{\otimes} (\overline{\otimes}_{s=r\pm 1} \mathscr{W}^{(s)}_{-k,a-\frac{1}{2}}).$$

By Lemma 3.2, $\boldsymbol{w}^{(r)}_{t,a} A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r,a+l-1}^{-1} \in \mathrm{wt}_e(W^{(r)}_{t,a})$. The $A_{r,a+j}^{-1}$ must arise from $\mathrm{wt}_e(D^{(r,t)}_{k,a})$ instead of any of the $\mathrm{wt}_e(W^{(s)}_{-k,a-\frac{1}{2}})$ with $s \neq r$.

For $0 \le j, l \le t$, since $\mathrm{wt}_e(S^l) \subset \boldsymbol{m}_l \mathcal{Q}_x^-$ and $\boldsymbol{m}_j \in \boldsymbol{m}_l \mathcal{Q}_a$, we have $\boldsymbol{m}_j \in \mathrm{wt}_e(S^l)$ if and only if $l = j$. Therefore, all the S^l appear as irreducible subquotients of T, and they are mutually nonisomorphic. So $\chi_q(T)$ is bounded below by $\sum_{l=0}^t \chi_q(S^l)$.

Claim 3. *$\chi_q(D^{(r,t)}_{k,a})$ is bounded above by*

$$\boldsymbol{d} \left(1 + \sum_{l=1}^t A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r+a+l-1}^{-1} \right) \prod_{s=r\pm 1} \chi_q(\mathscr{W}^{(s)}_{0,x}).$$

Fix $\boldsymbol{d}\boldsymbol{f} \in \mathrm{wt}_e(D^{(r,t)}_{k,a})$. For $n \in \mathbb{Z}_{>0}$, viewing $D^{(r,t)}_{k,a}$ as a subquotient of

$$D^{(r,t)}_{n,a} \overline{\otimes} (\overline{\otimes}_{s=r\pm 1} \mathscr{W}^{(s)}_{k-n,x})$$

gives $\boldsymbol{f} = \boldsymbol{f}_n \prod_{s=r\pm 1} \boldsymbol{f}^{(s)}$ where by Lemma 3.2 and Corollary 3.5

$$\boldsymbol{f}_n \boldsymbol{d}^{(r,t)}_{n,a} \in \mathrm{wt}_e(D^{(r,t)}_{n,a}) \quad \text{and} \quad \boldsymbol{f}^{(s)} \boldsymbol{w}^{(s)}_{k-n,x} \in \mathrm{wt}_e(\mathscr{W}^{(s)}_{k-n,x}) = \boldsymbol{w}^{(s)}_{k-n,x} \mathrm{wt}_e(\mathscr{W}^{(s)}_{0,x}).$$

It follows that $\boldsymbol{f}_n \in \mathcal{Q}_a^-$, $\boldsymbol{f}^s \in \mathcal{Q}_x^-$ and $\boldsymbol{f} \in \mathcal{Q}_a^- \mathcal{Q}_x^-$.

Let $n \in \mathbb{Z}_{>0}$ be large enough so that $\boldsymbol{f} \in \mathcal{Q}_{a;n}^- \mathcal{Q}_x^-$ where $\mathcal{Q}_{a;n}^-$ is the submonoid of $\mathcal{Q}_a^-$ generated by the $A_{i,a+m}^{-1}$ for $1 \le i < N$ and $m \in \frac{1}{2}\mathbb{Z}$ with $m > -n$. Since $a - x = k + \frac{1}{2}$ is generic, Corollary 3.5 implies that

$$\boldsymbol{f}_n \in \{1, A_{r,a}^{-1}, A_{r,a}^{-1} A_{r,a+1}^{-1}, \dots, A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r+a+t-1}^{-1}\}$$

is uniquely determined by $\boldsymbol{f}$. The coefficient of $\boldsymbol{d}\boldsymbol{f}$ in $\chi_q(M_{k,a}^{(r)})$ is bounded above by that of $\prod_{s=r\pm 1} \boldsymbol{f}^{(s)}$ in $\prod_{s=r\pm 1} \chi_q(\mathscr{W}_{0,x}^{(s)})$. This proves the claim.

It follows from Claim 3 that $\chi_q(T)$ is bounded above by

$$\boldsymbol{m}_0 \left(1 + \sum_{l=1}^{t} A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r+a+l-1}^{-1}\right) \prod_{s=r\pm 1} \chi_q(\mathscr{W}_{0,x}^{(s)}) \times \prod_{j=1}^{t} \chi_q(\mathscr{W}_{0,x}^{(r)}) = \sum_{l=0}^{t} \chi_q(S^l).$$

Since "bounded below" also holds, we obtain the exact formula for $\chi_q(T)$, which implies (5.30). This completes the proof of the theorem. □

Claim 1 is in the spirit of [Frenkel and Hernandez 2015, Theorem 4.11], and Claim 3 [Hernandez and Leclerc 2016, (6.14)], [Frenkel and Hernandez 2016, §4.3] and [Zhang 2018, Theorem 3.3], the main difference being the nonexistence of prefundamental modules. If both k and t are generic, then $\chi_q(D_{k,a}^{(r,t)})$ is obtained from the right-hand side of (5.30) by replacing $\sum_{l=1}^{t}$ therein with $\sum_{l=1}^{\infty}$.

Corollary 5.2. *Let $k \in \mathbb{C}$ be generic and $1 \le r < N$. In $K_0(\mathcal{O})$ holds*

$$[D_{k,k+\frac{1}{2}}^{(r,1)}][\mathscr{W}_{r,k+\frac{1}{2}}] = [\mathscr{W}_{r,k-\frac{1}{2}}] \prod_{s=r\pm 1} [\mathscr{W}_{s,k+1}] + [\mathscr{W}_{r,k+\frac{3}{2}}] \prod_{s=r\pm 1} [\mathscr{W}_{s,k}]. \tag{5.31}$$

Proof. From (5.30) and the injectivity of the q-character map we obtain

$$[D_{k,a}^{(r,t)}][\mathscr{W}_{a-b+t-1,b}^{(r)}] = [D_{k+t,a+t}^{(r,0)}][\mathscr{W}_{a-b-1,b}^{(r)}] + [D_{k,a}^{(r,t-1)}][\mathscr{W}_{a-b+t,b}^{(r)}] \tag{5.32}$$

for $a, b \in \mathbb{C}$ and $t \in \mathbb{Z}_{>0}$. (5.31) is the special case $(t, a, b) = \left(1, k + \frac{1}{2}, 0\right)$ of this identity in view of the tensor product decomposition of $D_{k,a}^{(r,0)}$ in Theorem 5.1. □

Equation (5.32) can be viewed as a generic version of (3.17).

6. Transfer matrices and Baxter operators

We have obtained three types of identities (4.28), (4.29), and (5.31) in the Grothendieck ring $K_0(\mathcal{O})$. These are viewed as universal functional relations [Bazhanov et al. 1997; 1999; Bazhanov and Tsuboi 2008] in the sense that when specialized to quantum integrable systems they imply functional relations of transfer matrices. In this section, we study one such example, with the quantum space being a tensor product of vector representations [Hou et al. 2003].

Fix $\ell := N\kappa$ with $\kappa \in \mathbb{Z}_{>0}$ and $a_1, a_2, \dots, a_\ell \in \mathbb{C} \setminus \Gamma$. Set $I := \{1, 2, \dots, N\}$. Let I_0^ℓ be the subset of I^ℓ formed of $\underline{i}$ such that $\epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_\ell} = 0 \in \mathfrak{h}$. Upon identification $\underline{i} := v_{i_1} \overline{\otimes} v_{i_2} \overline{\otimes} \cdots \overline{\otimes} v_{i_\ell}$, the weight space $V^{\overline{\otimes}\ell}[0]$ has basis I_0^ℓ.

Let $\mathbb{D}_p$ be the set of formal sums $\sum_{\alpha\in\mathfrak{h}} p^\alpha T_\alpha f_\alpha(z;\lambda)$ such that: The $f_\alpha(z;\lambda)$ are meromorphic functions of $(z,\lambda)\in\mathbb{C}\times\mathfrak{h}$. The set $\{\alpha : f_\alpha\neq 0\}$ is contained in a finite union of cones $\nu+\boldsymbol{Q}_-$ with $\nu\in\mathfrak{h}$. Make $\mathbb{D}_p$ into a ring: Addition is the usual one of formal sums. Multiplication is induced from

$$p^\alpha T_\alpha f(z;\lambda)\times p^\beta T_\beta g(z;\lambda)=p^{\alpha+\beta}T_{\alpha+\beta} f(z;\lambda+\hbar\beta)g(z;\lambda). \tag{6.33}$$

As in [Felder and Varchenko 1996a; Felder and Zhang 2017], we construct a ring morphism $[X]\mapsto t_X(z)$ from $K_0(\mathcal{O})$ to the ring $\mathrm{M}(I_0^\ell;\mathbb{D}_p)$ of $I_0^\ell\times I_0^\ell$ matrices with coefficients in $\mathbb{D}_p$. (We think of $\mathrm{M}(I_0^\ell;\mathbb{D}_p)$ as a ring of formal difference operators on $\boldsymbol{V}^{\overline{\otimes}\ell}[0]$.)

Let X be an object of category $\mathcal{O}$. To $\underline{i},\underline{j}\in I_0^\ell$ we associate

$$L^X_{\underline{i}\,\underline{j}}(z):=L^X_{i_1j_1}(z+a_1)L^X_{i_2j_2}(z+a_2)\cdots L^X_{i_\ell j_\ell}(z+a_\ell)\in(D_X)_{0,0}.$$

Since $(D_X)_{0,0}\subseteq \mathrm{End}_{\mathbb{M}}(X)$, one can take trace of $L^X_{\underline{i}\,\underline{j}}(z)$ over weight spaces of X.

Definition 6.1. The *transfer matrix* associated to an object X in category $\mathcal{O}$ is the matrix $t_X(z)\in\mathrm{M}(I_0^\ell;\mathbb{D}_p)$ whose $(\underline{i},\underline{j})$-th entry for $\underline{i},\underline{j}\in I_0^\ell$ is

$$\sum_{\alpha\in\mathrm{wt}(X)} p^\alpha T_\alpha\times \mathrm{Tr}_{X[\alpha]}(L^X_{\underline{i}\,\underline{j}}(z)|_{X[\alpha]})\in\mathbb{D}_p.$$

Almost all of the results and comments in [Felder and Zhang 2017, §5] hold true after slight modification in our present situation. In the following, we focus on the modification of these results, referring to [Felder and Zhang 2017] for their proofs.

We remark that $t_{\boldsymbol{V}}(z)|_{p=1}$ can be identified with the transfer matrix $T(z)$ in [Hou et al. 2003, (2.22)] where the $E_{\tau,\eta}(sl_n)$-module W is $V_{\Lambda_1}(a_1)\otimes V_{\Lambda_1}(a_2)\otimes\cdots\otimes V_{\Lambda_1}(a_\ell)$.

The transfer matrix associated to the one-dimensional module of highest weight $g(z)\in\mathbb{M}_{\mathbb{C}}^\times$ is the scalar matrix $\prod_{i=1}^\ell g(z+a_i)$.

For $1\le r\le N$ and $x\in\mathbb{C}$, consider the $\mathcal{E}$-module $\mathscr{W}'_{r,x}:=\mathscr{W}_{r,x}\overline{\otimes}S(\theta(z-\ell_r\hbar))$ in category $\mathcal{O}$. By Lemma 4.8, the matrix entries of the difference operators $L_{ij}(z)$ for $1\le i,j\le N$, with respect to any basis of $\mathscr{W}'_{r,x}$, are entire functions of $z\in\mathbb{C}$.

Definition 6.2. The r-th *Baxter Q-operator* for $1\le r\le N$ is defined to be

$$Q_r(u):=t_{\mathscr{W}'_{r,u\hbar^{-1}}}(z)|_{z=0}\quad\text{for } u\in\mathbb{C}. \tag{6.34}$$

Since $\mathscr{W}'_{N,x}=S\big(\theta\big(z+\big(x+\tfrac12\big)\hbar\big)\big)$ is one-dimensional, $Q_N(z)=\prod_{i=1}^\ell\theta\big(z+a_i+\tfrac12\hbar\big)$.

Let $1\le r<N$. Then $Q_r(z)=p^{z\hbar^{-1}\varpi_r}T_{z\hbar^{-1}\varpi_r}\tilde{Q}(z)$ and $\tilde{Q}(z)$ is a power series in the $p^{-\alpha_i}T_{-\alpha_i}$ for $1\le i<N$. The leading term $\tilde{Q}_0(z)$ of $\tilde{Q}(z)$ is invertible. Indeed $\tilde{Q}_0(0)$ is the scalar matrix $\prod_{j=1}^\ell\theta(a_j)\in\mathrm{M}(I_\ell;\mathbb{C})$, which is invertible because $\theta(a_j)\neq 0$ by assumption. (One can prove furthermore that with respect to certain order on I_0^ℓ, the matrix $\tilde{Q}_0(z)$ is upper triangular, whose entries are meromorphic functions of $(z;\lambda)\in\mathbb{C}\times\mathfrak{h}$ and entire on z.) Therefore $Q_r(z)\in\mathrm{GL}(I_0^\ell;\mathbb{D}_p)$.

Similarly one can show that $t_{\mathscr{W}'_{r,x}}(z)$ is invertible for $x\in\mathbb{C}$.

Proposition 6.3. *Let X and Y be in category $\mathcal{O}$ and let $x, u \in \mathbb{C}$.*

(i) $t_{\Psi_u^* X}(z) = t_X(z + u\hbar)$.

(ii) $t_X(z) t_Y(z) = t_{X \bar{\otimes} Y}(z)$.

(iii) $t_{\mathscr{W}_{r,x}}(z) t_{\mathscr{W}_{r,0}}(z + u\hbar) = t_{\mathscr{W}_{r,x-u}}(z + u\hbar) t_{\mathscr{W}_{r,u}}(z)$.

(iv) $t_X(z) t_Y(w) = t_Y(w) t_X(z)$.

In (iv), we replace one of the z in (6.33) with w to define the multiplication. It is proved as in [Frenkel and Hernandez 2015, Theorem 5.3]; the commutativity of transfer matrices is a consequence of the commutativity of the Grothendieck ring $K_0(\mathcal{O})$. The standard proof by using the Yang–Baxter equation [Baxter 1972] would require braiding in category $\mathcal{O}$, whose existence is not clear.

Conclusion (ii) and the fact that $t_X(z)$ only depends on the isomorphism class $[X]$ of X imply that $[X] \mapsto t_X(z)$ is a ring homomorphism $\mathrm{tr}_p : K_0(\mathcal{O}) \to \mathrm{M}(I_0^\ell; \mathbb{D}_p)$. Applying tr_p to (4.28) we obtain (iii). Replace $(\mathscr{W}, x, u, z)$ with $(\mathscr{W}', z\hbar^{-1} + x, z\hbar^{-1}, 0)$ in (iii) and take the inverse of $Q_r(z)$ and $t_{\mathscr{W}'_{r,0}}(z)$. We have

$$\frac{t_{\mathscr{W}_{r,x}}(z)}{t_{\mathscr{W}_{r,0}}(z)} = \frac{t_{\mathscr{W}'_{r,x}}(z)}{t_{\mathscr{W}'_{r,0}}(z)} = \frac{Q_r(z + x\hbar)}{Q_r(z)}, \tag{6.35}$$

as in [Felder and Zhang 2017, Theorem 5.6(i)]. Now applying tr_p to (4.29), we obtain

Corollary 6.4. *Let V be a finite-dimensional $\mathcal{E}$-module in category $\mathcal{O}$. Then in (4.29) replacing V, $S(\boldsymbol{d}_j)$ and the $[\mathscr{W}_{r,a}]/[\mathscr{W}_{r,b}]$ with $t_V(z)$, $t_{S(\boldsymbol{d}_j)}(z)$ and $Q_r(z+a\hbar)/Q_r(z+b\hbar)$ respectively, we obtain an identity in $\mathrm{M}(I_0^\ell; \mathbb{D}_p)$.*

This forms the generalized Baxter relations for transfer matrices. If the prefundamental modules $L_{r,a}^+$ before Example 4.16 existed, then we would have defined alternatively the r-th Baxter operator $Q_r^{\mathrm{FH}}(z) = t_{L_{r,0}^+}(z)$ as a *real* transfer matrix [Frenkel and Hernandez 2015, §5.5] and so $Q_r(z+a\hbar)/Q_r(z+b\hbar) = Q_r^{\mathrm{FH}}(z+a\hbar)/Q_r^{\mathrm{FH}}(z+b\hbar)$ based on $[\mathscr{W}_{r,a}]/[\mathscr{W}_{r,b}] = [L_{r,a}^+]/[L_{r,b}^+]$.

As an illustration of the corollary, let us be in the situation of Example 4.16:

$$t_V(z) = \frac{Q_1\big(z+\frac{3}{2}\hbar\big)}{Q_1\big(z+\frac{1}{2}\hbar\big)} + \frac{Q_1\big(z-\frac{1}{2}\hbar\big)}{Q_1\big(z+\frac{1}{2}\hbar\big)} \frac{Q_2(z+\hbar)}{Q_2(z)} + \frac{Q_2(z-\hbar)}{Q_2(z)} \prod_{j=1}^{\ell} \frac{\theta(z+a_j+\hbar)}{\theta(z+a_j)}.$$

Apply tr_p to (5.31), divide both sides by the second term, and then perform the change of variable $z + \big(k+\frac{1}{2}\big)\hbar \mapsto w$. By (6.35) and Proposition 6.3(i)

$$X_k^{(r)}(w) \frac{Q_r(w)}{Q_r(w-\hbar)} = 1 + \frac{Q_r(w+\hbar)}{Q_r(w-\hbar)} \prod_{s=r\pm1} \frac{Q_s\big(w-\frac{1}{2}\hbar\big)}{Q_s\big(w+\frac{1}{2}\hbar\big)}. \tag{6.36}$$

This forms three-term Baxter TQ relations for transfer matrices, where

$$X_k^{(r)}(w) = t_{D_{k,0}^{(r,1)}}(w) \prod_{s=r\pm1} t^{-1}_{\mathscr{W}_{s,k+1}}\big(w - \big(k+\tfrac{1}{2}\big)\hbar\big).$$

By (6.36), $X_k^{(r)}(z) \in \mathrm{M}(I_0^\ell; \mathbb{D}_p)$ is independent of the choice of generic $k \in \mathbb{C}$.

In the homogeneous case $a_1 = a_2 = \cdots = a_\ell = a$, the entries of the matrix $\tilde{Q}_r(z)$, as entire functions of z, in general do not satisfy the uniform double periodicity of [Felder and Zhang 2017, Theorem 5.6(ii)]. By "uniform" we mean the multipliers with respect to $z+1$ and $z+\tau$ only depend on (a, z, ℓ). This is because the transfer matrix construction in [Felder and Zhang 2017] is based on a slightly different elliptic quantum group; see footnote 9.

We follow [Frenkel and Hernandez 2016, §5] to derive the Bethe Ansatz equations from (6.36). Let u be a zero of $Q_r(z)$. Suppose $X_k^{(r)}(z)$, $Q_r(z-\hbar)$ and $Q_s\big(z+\frac{1}{2}\hbar\big)$ for $s \neq r \pm 1$ have no poles at $z = u$. (This is a genericity condition.) Then as in [Frenkel and Hernandez 2016, (5.16)]

$$\frac{Q_r(u+\hbar)}{Q_r(u-\hbar)} \prod_{s=r\pm 1} \frac{Q_s\big(u-\frac{1}{2}\hbar\big)}{Q_s\big(u+\frac{1}{2}\hbar\big)} = -1. \tag{6.37}$$

To compare with [Frenkel and Hernandez 2016], we can assume furthermore that eigenvalues of $Q_r(z)$ are of the form $p^{z\hbar^{-1}\varpi_r} \prod_{i=1}^{d_r} \theta(z-u_{r;i})$ based on [Felder and Zhang 2017, Remark 5.8]. Then

$$p^{\alpha_r} \prod_{i=1}^{d_r} \frac{\theta(u_{r;k}+\hbar-u_{r;i})}{\theta(u_{r;k}-\hbar-u_{r;i})} \prod_{s=r\pm 1} \prod_{j=1}^{d_s} \frac{\theta\big(u_{r;k}+\frac{1}{2}\hbar-u_{s;j}\big)}{\theta\big(u_{r;k}+\frac{1}{2}\hbar-u_{s;j}\big)} = -1 \quad \text{for } 1 \leq k \leq d_r.$$

We remark that similar Bethe Ansatz equations for $\mathcal{E}$ appeared in [Hou et al. 2003, (3.45)].

For affine quantum groups and toroidal $\mathfrak{gl}_1$, the genericity condition of Bethe Ansatz equations has been dropped in [Feigin et al. 2017a; 2017b].

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