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A generic slice of the moduli space of line arrangements

Kenneth Ascher and Patricio Gallardo

We study the compactification of the locus parametrizing lines having a fixed intersection with a given line, inside the moduli space of line arrangements in the projective plane constructed for weight one by Hacking, Keel and Tevelev and for general weights by Alexeev. We show that this space is smooth, with normal crossing boundary, and that it has a morphism to the moduli space of marked rational curves which can be understood as a natural continuation of the blow up construction of Kapranov. In addition, we prove that our space is isomorphic to a closed subvariety inside a nonreductive Chow quotient.

1. Introduction

The compact moduli space of weighted hyperplane arrangements in $\mathbb{P}^2$ is a higher dimensional generalization of $\overline{M}_{0,n}$, and has a main component parametrizing equivalence classes of n weighted lines in $\mathbb{P}^2$ and their log canonical degenerations. The moduli space $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n)$ was constructed for lines of weight one by Hacking, Keel and Tevelev [Hacking et al. 2006], and for more general weights $\vec{\beta}$ as a generalization of the weighted Hassett spaces by Alexeev [2015]. The space is expected to satisfy Murphy's law — it can be arbitrarily singular and can contain many irreducible components. The goal of this paper is to describe a naturally appearing locus inside this moduli space which has perhaps unexpected properties — it is smooth with normal crossings boundary.

Given an arrangement of $(n+1)$ labeled lines in $\mathbb{P}^2$, there is a natural restriction morphism; label the line l_{n+1} as l_A and obtain an arrangement of n labeled points on $l_A \cong \mathbb{P}^1$, by intersecting the other n lines with l_A. The restriction morphism induces a morphism $M_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}}$ that has rational fibers of dimension $n-3$ (see Lemma 3.3). Given a generic point $q \in M_{0,\vec{w}}$, we study the closure, which we denote by $R_{\vec{w}}(q)$, in $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ of the fiber of $M_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}}$ over q (see Definition 3.1).

In other words, $R_{\vec{w}}(q)$ compactifies the locus parametrizing equivalence classes of $n+1$ labeled lines having a fixed intersection with the line l_A. Our first theorem characterizes $R_{\vec{w}}(q)$.

Theorem 1.1 (see Theorem 5.14 and Theorem 5.16). *For weights $\vec{w}$ in the set of admissible weights $\mathcal{D}_n^R$ (see Definition 4.1) and generic choice of $q \in M_{0,\vec{w}}$, the locus $R_{\vec{w}}(q)$ is smooth with normal crossings boundary and there are birational morphisms*

$$R_{\vec{w}}(q) \xrightarrow{\pi_2} \overline{M}_{0,\vec{w}} \xrightarrow{\pi_1} \mathbb{P}^{n-3}.$$

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Keywords: moduli spaces, hyperplane arrangements, chow quotient, wonderful compactifications, stable pairs, birational geometry, minimal model program.

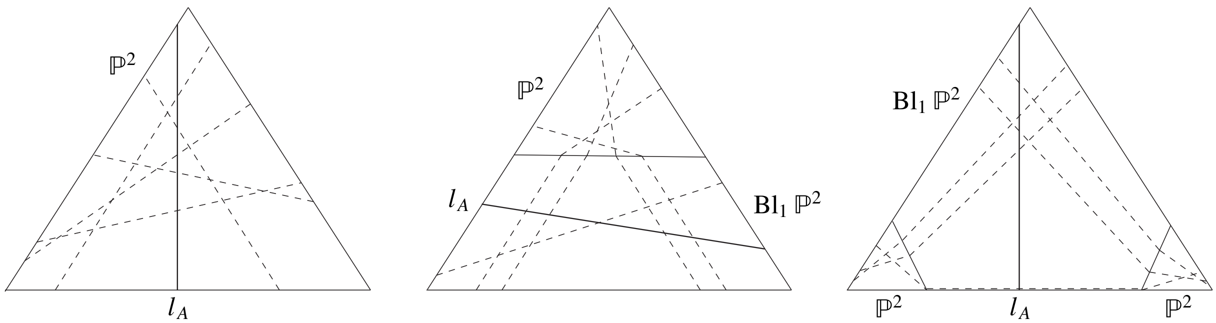

Figure 1. Examples of generic and nongeneric shas parametrized by $R_{1^5}(q)$.

By results of Kapranov [1993b] the morphism $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ factors into the sequence of following morphisms: The blow up of $(n-1)$ points $q_i \in \mathbb{P}^{n-3}$ which are in general position. The blow up of the strict transforms of the $\mathbb{P}^1$s spanned by pairs of the points q_i, and so forth. For $\vec{w} = (1, \dots, 1)$, the morphism π_2 factors in a similar fashion.

Corollary 1.2 (see Corollary 5.18). *The morphism $R_{1^n}(q) \to \overline{M}_{0,n}$ factors into the sequence of following morphisms: The blow up of a point q_n in the interior of $\overline{M}_{0,n}$. The blow up of the strict transforms of the $\mathbb{P}^1$ spanned by pairs $\{q_i, q_n\}$. The blow up of the strict transforms of the $\mathbb{P}^2$ spanned by triples $\{q_i, q_j, q_n\}$ and so forth.*

In contrast to $\overline{M}_{0,n}$, the centers used to construct $R_{1^n}(q)$ are not projectively equivalent to each other. As a result, $R_{1^n}(q)$ depends on the choice of q_n and in general different q_n yields nonisomorphic spaces. Moreover we show the following.

Theorem 1.3. *For a generic choice of q and $n \geq 5$, there do not exist weights $\vec{w}$ such that $R_{\vec{w}}(q) \cong \overline{M}_{0,n}$.*

The objects parametrized by $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$ are called stable hyperplane arrangements, or *shas* (see [Alexeev 2015, Definition 5.3.1]), and they are stable pairs in the sense of the minimal model program, MMP, (see [Alexeev 2015, Theorem 5.3.2]). The shas parametrized by $R_{\vec{w}}(q)$ are described in Section 2. In particular, our setting restricts the possible singularities that appear in our shas (see Remark 3.4, Proposition 4.8 and Figure 1).

Our next main result is that the locus $R_{1^n}(q)$ is the normalization of a nonreductive Chow quotient. In particular, our result fits into a library of examples (see [Gallardo and Giansiracusa 2018; Giansiracusa 2013; Hu 2005; Kapranov et al. 1991; Thaddeus 1999]) where Chow quotients are used to study the geometry of moduli spaces. The following outline generalizes the construction of Kapranov [1993a] in the setting of $R_{1^n}(q)$ (see Remark 6.1); given the collection of n points p_i in the dual projective space $\hat{\mathbb{P}}^2$ such that the point p_i is dual to the line l_i, we consider the locus, in an appropriate Chow variety, that parametrizes the cycles associated to the orbits $\overline{G \cdot (p_1, \dots, p_n)}$ where $G \subset SL(3, \mathbb{C})$ is the group that *fixes the intersection* of the associated lines l_i with l_A. By normalizing the closure of this locus in the Chow variety, we recover $R_{1^n}(q)$ (see Section 6).

Theorem 1.4 (see Theorem 6.12). *For a generic choice of q, the space $R_{1^n}(q)$ is isomorphic to the normalization of a closed subvariety of the Chow quotient $(\hat{\mathbb{P}}^2)^n /\!\!/_{\mathrm{Ch}} G$ where $G \subset \mathrm{SL}(3, \mathbb{C})$ is the group fixing the line l_A pointwise.*

1.5. ***Method of proof of Theorem 1.1.*** We give an outline of our proof that $R_{\vec{w}}(q)$ is smooth with normal crossings boundary. The overall strategy is to prove that $R_{\vec{w}}(q)$ is isomorphic to a wonderful compactification, which is smooth with normal crossings boundary by definition (see Theorem 5.6).

We first construct our space with smallest admissible weights $\vec{w}_0$, show that $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ (see Lemma 4.4), and construct a family over $R_{\vec{w}_0}$ (see Lemma 4.6). In Section 5 we construct the wonderful compactification $\mathrm{Bl}_{\vec{w}}\, R_{\vec{w}_0}$, and in Lemma 5.10 we construct a family of shas over the wonderful compactification. Using this family, we obtain a finite birational (i.e., normalization) morphism from the wonderful compactification to our space: $\mathrm{Bl}_{\vec{w}}\, R_{\vec{w}_0} \to R_{\vec{w}}$. We prove normality of $R_{\vec{w}}$ in Theorem 5.14, which implies that $R_{\vec{w}} \cong \mathrm{Bl}_{\vec{w}}\, R_{\vec{w}_0}$ by Zariski's main theorem. Finally, we note that the key lemma required to prove normality of $R_{\vec{w}}$ is Lemma 4.9.

2. Definition and basic properties

We work only over $\mathbb{C}$ for convenience. We begin with the necessary background on the moduli space $\overline{M}_{\vec{w}}(\mathbb{P}^2, n+1)$, see [Hacking et al. 2006; Alexeev 2015] for a full exposition.

Configurations of $(n+1)$ labeled lines $(l_1, \ldots, l_{n+1})$ in $\mathbb{P}^2$ up to projective equivalence are parametrized by the open moduli space $M(\mathbb{P}^2, n+1)$, which has a family of geometric compactifications $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$ depending on a weight vector $\vec{\beta} := (\beta_1, \ldots, \beta_{n+1})$ (see [Alexeev 2015, Theorem 5.4.2]).

The weight domain of possible weights $\vec{\beta}$ is

$$\mathcal{D}(3, n+1) = \left\{ \vec{\beta} \in \mathbb{Q}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} \beta_i > 3,\ 0 < \beta_i \le 1 \right\}. \tag{2.0.1}$$

In general these compactifications are *not* irreducible. However, they do contain a main irreducible component parametrizing stable pairs in the sense of MMP $\big(X, \sum_{k=1}^{n+1} \beta_k l_k\big)$ appearing as degenerations of the $(n+1)$ lines in $\mathbb{P}^2$.

Definition 2.1. The stable pairs $(X, D) := \big(X, \sum_{i=1}^{n+1} \beta_k l_k\big)$ parametrized by $\overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1)$ are called *shas* of weight $\vec{\beta}$ or just shas if the weight $\vec{\beta}$ is clear from the context.

Notation 2.2. Let $I \subset \{1, 2, \ldots, n\}$ be an index set. A sha (X, D) has a multiple point $p(I)$ if there exists a component X_i of X and divisors $\{l_i = D|_{X_i} \mid i \in I\}$ such that the divisors l_i are concurrent at a point $p(I) \in X_i$.

Remark 2.3. The admissible singularities of the divisors D in the sha (X, D) depend completely on the weights $\vec{\beta}$. Indeed, we cannot have coincident lines $\{l_i \mid i \in I\}$ with weight $\sum_{i \in I} \beta_i > 1$ or multiple points $p(I)$ defined by the concurrent lines $\{l_i \mid i \in I\}$ with total weight $\sum_{i \in I} \beta_i > 2$.

Definition 2.4. Let $\vec{\beta}$ and $\vec{\alpha}$ be two weight vectors in $\mathcal{D}(3, n+1)$. We say that $\vec{\beta} \ge \vec{\alpha}$ if $\beta_i \ge \alpha_i$ for all i.

As in the Hassett spaces $\overline{M}_{0,\vec{w}}$, the shas parametrized by $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ depend solely on the weights $\vec{w}$, and the weight domain admits a wall and chamber decomposition.

Theorem 2.5 [Alexeev 2015, Theorem 5.5.2]. *The domain $\mathcal{D}(3, n+1)$ is divided into finitely many walls and chambers. There are two types of walls*

$$W(I) := \Big(\sum_{i\in I} \beta_i - 2 = 0\Big) \quad \text{and} \quad \widetilde{W}(I) := \Big(\sum_{i\in I} \beta_i - 1 = 0\Big), \tag{2.5.1}$$

for all $I \subset \{1, \dots, n+1\}$, $2 \le |I| \le (n-1)$. Moreover:

(1) *If $\vec{\beta}$ and $\vec{\alpha}$ lie in the same chamber, then the weighted moduli spaces and their families of shas are the same.*

(2) *If $\vec{\beta}$ is in the closure of the chamber containing $\vec{\alpha}$, then there exists a contraction*

$$\overline{M}_{\vec{\alpha}}(\mathbb{P}^2, n+1) \to \overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1).$$

(3) *Further, if $\vec{\beta}$ is in the closure of the chamber containing $\vec{\alpha}$ and $\vec{\alpha} \le \vec{\beta}$ then*

$$\overline{M}_{\vec{\alpha}}(\mathbb{P}^2, n+1) = \overline{M}_{\vec{\beta}}(\mathbb{P}^2, n+1).$$

Remark 2.6. Recall from Remark 2.3 that there are two types of singularities appearing in shas. In this setting, the walls $W(I)$ correspond to multiple points $p(I)$, and the walls $\widetilde{W}(I)$ correspond to coincident lines.

3. Definition of $R_{\vec{w}}(q)$

To construct $R_{\vec{w}}(q)$, we consider arrangements of $n+1$ labeled lines in $\mathbb{P}^2$ and we label the $(n+1)$-st line as l_A to distinguish it. We will always assume l_A has weight 1 and thus will denote our weight set $\beta \in \mathcal{D}(3, n+1)$ as $(\vec{w}, 1)$. In this section, there is no need to restrict the set of weights $\vec{w}$. However in the following sections, we will consider an additional restriction on the weights (see Definition 4.1).

We have a *restriction* morphism

$$\varphi_A : M_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}},$$

induced naturally by considering the intersection of l_A with the lines l_i where $i \in \{1, \dots, n\}$. Next, we take the fiber of this restriction over a generic point $q \in M_{0,\vec{w}}$ and then take closure of this fiber in the compact moduli space of weighted hyperplane arrangements.

Definition 3.1. Let $q \in M_{0,\vec{w}} \subset \overline{M}_{0,\vec{w}}$ be a generic point. We define $R_{\vec{w}}(q)$ as the closure in $\overline{M}_{\vec{w},1}(\mathbb{P}^2, n+1)$ of the fiber product of the following diagram:

$$\begin{array}{ccc} R_{\vec{w}}(q) & \longrightarrow & \overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1) \\ \big\downarrow & & \big\downarrow \varphi_A \\ q & \longrightarrow & \overline{M}_{0,\vec{w}} \end{array}$$

Remark 3.2. B. Hassett gave an example of families $\left(\mathcal{X}, \frac{1}{2}\mathcal{D}\right) \to \mathrm{Spec}(\mathbb{C}[\![t]\!])$ where $\mathcal{D}|_{t=0}$ has embedded points. In general for pairs, the components of the boundary with fractional coefficients $\leq \frac{1}{2}$ need not be Cohen–Macaulay. By [Alexeev 2015, Lemma 1.5.1], the mentioned difficulty will *not* occur for very generic coefficients of the form $\vec{w}$ for which one entry satisfies $w_i = 1$.

Lemma 3.3. *The dimension* $\dim(R_{\vec{w}}(q)) = n - 3$.

Proof. By the fiber product construction we see that

$$\dim(R_{\vec{w}}(q)) = \dim(\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)) - \dim(\overline{M}_{0,n}).$$

The result follows since $\dim(\overline{M}_w(\mathbb{P}^2, n+1)) = 2(n-3)$ (see [Alexeev 2015, p. 84]). □

Remark 3.4. We will show in Proposition 4.8 that:

(1) The only singularities in the shas parametrized by $R_{\vec{w}}(q)$ are multiple points (no overlapping lines), as each line l_i with $1 \leq i \leq n$ intersects the fixed line l_A in a *distinct* point.

(2) The dual graph of X is a rooted tree (see Proposition 4.8 [II]). This allows us to fully describe the shas parametrized by $R_{1^n}(q)$ (see Figure 1).

(3) Each *broken line* l_i can be seen as a chain of lines that starts in the rooted component. The l_i may have several branches, and can be contained in several components.

Definition 3.5. We say that the weight $\vec{\beta}$ *destabilizes* the multiple point $p(K)$ if the sum $\sum_{k\in K}\beta_k$ exceeds 2. We also say $\vec{\beta}$ destabilizes the sha (X, D) if the pair has a singularity destabilized by $\vec{\beta}$.

In what follows, we discuss the stable replacement of shas with multiple points which will be relevant for us (see [Alexeev 2015, Chapter 5] for a complete discussion).

3.6. ***Stable replacement.*** Let $I \subset \{1, 2, \ldots, n\}$ be an index set. We consider two chambers in $\mathcal{D}(3, n+1)$ separated by the wall $W(I)$ as defined in Theorem 2.5. Let $\vec{w} \leq \vec{v}$ be weights in those chambers such that $\sum_{i\in I} w_i < 2$ and $\sum_{i\in I} v_i > 2$. Let $\vec{u}$ be a weight in the wall that separates those chambers, so in particular $\sum_{i\in I} u_i = 2$.

Let (X, D) be a sha parametrized by $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ and suppose that the sha has only a multiple point $p(I)$; notice that the point $p(I)$ will never be supported on l_A (Remark 3.4(1)). By Theorem 2.5(3), changing the weights from $\vec{w}$ to $\vec{u}$ will not modify the moduli spaces, so

$$\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1) \cong \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1).$$

The singularity $p(I)$ is still log canonical with respect to the weights $(\vec{u}, 1)$. Therefore, (X, D) is in the universal family associated to weights $\vec{u}$.

Next, we change the weights from $\vec{u}$ to $\vec{v}$. By Theorem 2.5(2), there is a contraction

$$\pi_{\vec{v},\vec{u}} : \overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1) \to \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1).$$

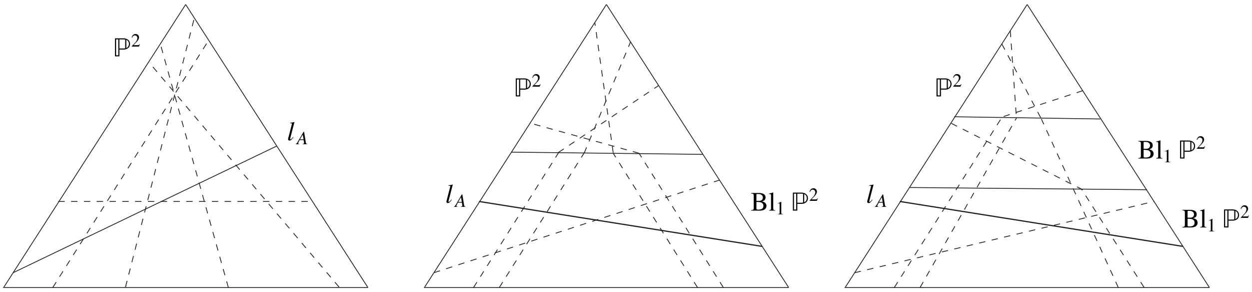

Figure 2. Quadruple point and its generic and nongeneric stable replacement.

By moduli theory, we know that the center of this morphism is the locus parametrizing shas with singularities that are destabilized respect to the new weights $(\vec{v}, 1)$. In particular, the sha (X, D) is no longer parametrized by $\overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1)$ because $\sum_{i\in I} v_i > 2$.

Let $z \in \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1)$ be the point parametrizing the sha (X, D). Next, we describe the sha $(\tilde{X}, \tilde{D})$ parametrized by a generic point in $\pi^{-1}_{\vec{v},\vec{u}}(z)$. We first blow up X at $p(I)$, and we attach a $\mathbb{P}^2$ along the exceptional divisor $E_{p(I)}$ to obtain a new surface

$$\tilde{X} = \mathrm{Bl}_{p(I)} X \cup_{E_{p(I)}} \mathbb{P}^2$$

with the lines $(l_i, i \in I)$ crossing into the new $\mathbb{P}^2$ and defining a new divisor $\tilde{D}$ (see Figure 2). The multiple lines defining $p(I)$ are separated in $\mathrm{Bl}_{p(I)} X$, and they are generically separated in the new component $\mathbb{P}^2$. They may acquire a multiple point, but they cannot overlap with each other, because they are already separated in the double locus.

Example 3.7. Consider a quadruple point in an arrangement of 6 lines — then there are two possible stable replacements. The starting configuration is stable if the total weight of the intersection point of the four lines $l_1, \dots, l_4$ is ≤ 2. Increasing the weights of all the lines to one causes any singularity with multiplicity larger than two to become unstable. Generically, the stable replacement has a new component where the 4 lines are separated. The four lines plus the double locus in $\mathbb{P}^2$ have two dimensional moduli, so that we can further degenerate the configuration to a triple point. In this case, we must blow up the new component, obtaining a surface with three components. Here, the additional surface is a $\mathbb{P}^2$ with three lines. Since a configuration of three lines and the double locus in $\mathbb{P}^2$ has no moduli, we cannot degenerate the configuration any further. These two cases are all of the possible stable replacements.

4. $R_{\vec{w}_0}$ as a GIT quotient and some properties of $R_{\vec{w}}$

The starting point of this section is Lemma 4.4, where we show that there are weights $\vec{w}_0$ such that $R_{\vec{w}_0}(q) \cong \mathbb{P}^{n-3}$. Afterwards, we study some geometric properties of $R_{\vec{w}}$ in general, such as the surfaces parametrized and the singularities that appear (Proposition 4.8), as well as the outcome of wall-crossing on our moduli spaces (Lemma 4.9).

The results of this section do not depend on the q used in the definition of $R_{\vec{w}}(q)$, so we simplify our notation and we just write $R_{\vec{w}}$. First, we define our admissible weights.

Definition 4.1. Let $\vec{w}_0 = (w_{0_1}, \dots, w_{0_n})$ be a set of rational numbers such that for every subset $I \subsetneq \{1, \dots, n\}$ the inequality $\sum_{i \in I} w_{0_i} \leq 2$ holds. The *set of admissible weights* is

$$\mathcal{D}_n^R = \left\{ (w_1, \dots, w_n) \in \mathbb{Q}^n \;\middle|\; 1 \geq w_i > 0, \sum_{i=1}^{n} w_i \geq 2, w_i \geq w_{0_i} \right\}$$

The chamber decomposition of $\mathcal{D}(3, n+1)$ induces a chamber decomposition on $\mathcal{D}_n^R$ where the chambers are separated by the walls $W(I)$ (see Theorem 2.5).

Definition 4.2. We say that two weights $\vec{v}$ and $\vec{u}$ are *adjacent* if each of them belongs to a chamber in $\mathcal{D}_n^R$ and those chambers are separated by a single wall $W(I)$. Sometimes, we say that the weights $\vec{u}$ and $\vec{v}$ are *separated* by $W(I)$.

Moreover, by Remark 3.2, to avoid any subtle technicalities, we will assume all our weighs are very generic.

Before showing that $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ (Lemma 4.4), we prove a key lemma.

Lemma 4.3. *The subgroup of* $\mathrm{SL}(3, \mathbb{C})$ *that fixes*

- *three lines l_n, l_{n-1} and l_A in general position, and*
- *n distinct points $\{l_1 \cap l_A, \dots, l_n \cap l_A\}$ in l_A*

is equal to $\mathbb{C}^*$.

Proof. We can suppose without lost of generality that the lines are

$$l_A := (x_0 = 0), \quad l_{n-1} := (x_1 = 0), \quad l_n := (x_2 = 0).$$

The subgroup that fixes those lines in $\mathbb{P}^2$ is $(\mathbb{C}^*)^2$, and it is given by matrices of the form $g = \mathrm{diag}((g_2 g_1)^{-1}, g_1, g_2)$ which acts on any point in the line l_A by $g \cdot [0 : q_1 : q_2] \to [0 : g_1 q_1 : g_2 q_2]$. By hypothesis, the points $\{l_1 \cap l_A, \dots, l_n \cap l_A\}$ on l_A are fixed, implying that $g_1 = g_2$. $\square$

Lemma 4.4. *Let $\vec{w}_0$ be as in Definition 4.1. Then*

$$R_{\vec{w}_0} \cong \mathbb{P}^{n-3} \subset \overline{M}_{(\vec{w}_0, 1)}(\mathbb{P}^2, n+1)$$

and each fiber of the universal family over $R_{\vec{w}_0}$ is a pair $(\mathbb{P}^2, \sum_{k=1}^{n} w_{0_k} l_k + l_A)$ such that:

1. *The n lines l_i cannot all meet at an n-tuple point.*
2. *Any multiple point of multiplicity strictly smaller than n is allowed.*
3. *None of the lines l_i can overlap with l_A.*

Proof. Let l_A be the line with weight $w_A = 1$ that induces the restriction morphism

$$M_{(\vec{w}_0,1)}(\mathbb{P}^2, n+1) \to M_{0,\vec{w}_0}.$$

To prove (1), recall that an n-tuple point is unstable if and only if the sum of the weights $\sum_{i=1}^{n} w_{0_i}$ exceeds 2, which is true by assumption.

Following the proof of (1), we note that (2) holds because of the assumption that for every subset $I \subsetneq \{1, \dots, n\}$ the sum of the weights is ≤ 2.

To prove (3), we recall that a multiple line is unstable if the sum of the weights is greater than 1. Since the weight w_A of the line l_A is already 1, no other line can overlap with it.

Let (X, D) be any configuration parametrized by $R_{\vec{w}_0}$. By (1) and (3), we can suppose that the lines l_{n-1}, l_n and l_A are fixed and in general position. By definition, the points $\{l_1 \cap l_A, \dots, l_n \cap l_A\} \subset l_A$ induce the equivalence class $q \in M_{0,n}$, and thus we can fix these points.

We can now demonstrate that $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$. First note that the parameter space of each line l_i with $1 \leq i \leq n-2$ is $\mathbb{A}^1$, because the intersection $l_i \cap l_A$ is fixed. We can choose coordinates on each $\mathbb{A}^1$ so that the point $0 \in \mathbb{A}^1$ parametrizes whenever the line l_i coincides with the fixed intersection $l_n \cap l_{n-1}$. Then the parameter space of the $(n-2)$ lines $l_1, \dots, l_{n-2}$ is $(\mathbb{A}^1)^{n-2} \setminus (0, \dots, 0)$, since we cannot have an n-tuple point by (1). Therefore, by Lemma 4.3, we conclude that

$$R_{\vec{w}_0} \cong (\mathbb{A}^{n-2} \setminus (0, \dots, 0)) /\!\!/ \mathbb{C}^* \cong \mathbb{P}^{n-3}. \qquad \square$$

Next, we construct a family of shas over $R_{\vec{w}_0}$. Before doing that, we set up some notation.

Notation 4.5. We choose a coordinate system $[t_0 : t_1 : t_2] \in \mathbb{P}^2$ such that

$$l_A := (t_0 = 0), \quad l_{n-2} \cap l_A := [0:0:1], \quad l_{n-1} := (t_2 = 0), \quad l_n := (t_1 - t_2 = 0).$$

and we select the point $q \in M_{0,\vec{w}_0}$ induced by the following configuration of points in l_A:

$$\{[0 : a_1 : 1], \dots, [0 : a_{n-3} : 1], [0:0:1], [0:1:0], [0:1:1]\}.$$

Under this choice of coordinates, $[s_1 : \cdots : s_{n-2}] \in R_{\vec{w}_0}(q)$ parametrizes the following configuration of lines with $1 \leq i \leq (n-3)$:

$$l_i := (t_1 - a_i t_2 + s_i t_0 = 0), \quad l_{n-2} := (s_{n-2} t_0 + t_1 = 0), \quad l_{n-1} := (t_2 = 0), \quad l_n := (t_1 - t_2 = 0).$$

In the following lemma, we consider $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ with coordinates $[s_1, \dots, s_{n-2}]$ as above and the projective space $\mathbb{P}^{n-1}$ with coordinates $[z_1, \dots, z_n]$. We exclude the $n = 4$ case for convenience of notation (see Example 5.13).

Lemma 4.6. *For $n \geq 5$, let $\mathcal{U}_{\vec{w}_0}$ be the blow up of $\mathbb{P}^{n-1}$ at the line defined by*

$$Z := \{z_k - z_{k+2} = 0 \mid 1 \leq k \leq n-2\},$$

and let σ_i be the strict transform of the following n hyperplanes in $\mathbb{P}^{n-1}$:

$$\begin{aligned} H_i &:= (a_2 z_3 - a_1 z_4) - a_i(z_3 - z_4) + (a_2 - a_1)(z_i - z_{i+2}) = 0, \quad 1 \le i \le n-3, \\ H_{n-2} &:= (a_2 - a_1)(z_{n-2} - z_n) + a_2 z_3 - a_1 z_4 = 0, \\ H_{n-1} &:= z_3 - z_4 = 0, \\ H_n &:= (a_2 - 1) z_3 - (a_1 - 1) z_4 = 0. \end{aligned}$$

Then there exists a flat, proper morphism $\phi_{\vec{w}_0} : \mathcal{U}_{\vec{w}_0} \to R_{\vec{w}_0}$ such that for every $\vec{s} \in R_{\vec{w}_0}$ the fiber $\phi_{\vec{w}_0}^{-1}(\vec{s})$ is isomorphic to $\mathbb{P}^2$. Moreover, if $E_{\vec{w}_0} \subset \mathcal{U}_{\vec{w}_0}$ is the exceptional divisor, then the configuration of lines

$$l_i := \phi_{\vec{w}_0}^{-1}(\vec{s}) \cap \hat{\sigma}_i \quad \textit{and} \quad l_A := \phi_{\vec{w}_0}^{-1}(\vec{s}) \cap E_{\vec{w}_0}$$

define the stable sha of weight $\vec{w}_0$ parametrized by $\vec{s}$.

Proof. Let $\pi_Z : \mathbb{P}^{n-1} \to R_{\vec{w}_0}$ be the projection defined by $\{s_k = z_k - z_{k+2} \mid 1 \le k \le n-2\}$. Note that Z is the indeterminacy loci of π_Z and that, given a point $\vec{s} \in R_{\vec{w}_0}$, we have $\pi_Z^{-1}(\vec{s}) \cong \mathbb{P}^2$. Therefore, the map $\mathcal{U}_{\vec{w}_0} \to R_{\vec{w}_0}$ is a $\mathbb{P}^2$-fibration obtained by the composition $\mathcal{U}_{\vec{w}_0} \to \mathbb{P}^{n-1} \to R_{\vec{w}_0}$.

The following functions, with $2 \le m \le n/2$ if n is even and $2 \le m \le (n+1)/2$ if n is odd,

$$\zeta_1 = t_1 - a_1 t_2 + s_1 t_0, \quad \zeta_2 = t_1 - a_2 t_2 + s_2 t_0, \quad \zeta_{2m-1} = \zeta_1 - t_0 \sum_{k=0}^{m-2} s_{2k+1}, \quad \zeta_{2m} = \zeta_2 - t_0 \sum_{k=1}^{m-1} s_{2k}$$

define, for a fixed $\pi_Z^{-1}(\vec{s})$, a map $\zeta_{\vec{s}} : \mathbb{P}^2 \to \pi_Z^{-1}(\vec{s})$ given by

$$\zeta_{\vec{s}} : [t_0, t_1, t_2] \to [\zeta_1, \zeta_2, \ldots, \zeta_n].$$

Indeed, we can verify the image of the map $\zeta_{\vec{s}}$ is $\pi_Z^{-1}(\vec{s})$ since

$$\pi_Z(\zeta_{\vec{s}}[t_0, t_1, t_2]) = [\zeta_1 - \zeta_3, \zeta_2 - \zeta_4, \ldots, \zeta_{n-2} - \zeta_n] = [s_1 t_0, s_2 t_0, \ldots, s_n t_0].$$

We also note that the map is not defined for $(t_0 = 0)$ because $\zeta_{\vec{s}}^{-1}(Z) = (t_0 = 0)$. Moreover, by the definition of the H_i above and the equations of the lines given in Notation 4.5 it holds that

$$\zeta_{\vec{s}}(l_i) = \pi_Z^{-1}(\vec{X}) \cap H_i \quad \text{and} \quad \zeta_{\vec{s}}(l_A) = Z.$$

These equalities follow at once by observing that $\zeta_3 = \zeta_1 - t_0 s_1$ and $\zeta_4 = \zeta_2 - t_0 s_2$ as well as

$$a_2 \zeta_3 - a_1 \zeta_4 = (a_2 - a_1) t_1, \quad \zeta_3 - \zeta_4 = (a_2 - a_1) t_2, \quad \zeta_i - \zeta_{i+2} = s_i t_0.$$

Finally, we assign the weights given by $\vec{w}_0$ to the n hyperplanes and weight 1 to the exceptional divisor, we get a family of shas with respect to the weights $\vec{w}_0$. □

4.7. ***Generalities on $R_{\vec{w}}$.*** We start with a explicit description of the surfaces parametrized by $R_{\vec{w}}$.

Proposition 4.8. *Let (X, D) be a sha parametrized by $R_{\vec{w}}$.*

(I) *The only singularities in (X, D) are of the form $p(J)$* (see Notation 2.2). *In particular, the shas never have overlapping lines.*

(II) *The dual graph* Graph(X) *of X is a rooted tree where the rooted vertex is the unique surface containing the line l_A.*

(III) *All the components of X are a blow up of $\mathbb{P}^2$ at $k \geq 0$ points. In particular, the stable replacement of any sha parametrized by $R_{\vec{w}}$ is obtained by blowing up isolated points. That is, we never have to blow down a (-1)-curve.*

Proof. Let $\vec{w} \in \mathcal{D}_n^R$ be an admissible weight and consider a sequence of weights $\vec{\gamma}_1, \dots, \vec{\gamma}_m$ such that $\vec{\gamma}_1 := \vec{w}_0$, $\vec{\gamma}_m := \vec{w}$, the weights $\vec{\gamma}_i \leq \vec{\gamma}_{i+1}$ are adjacent to each other (see Definition 4.2), and m is the minimal length of such sequences. We prove our proposition by induction on m. The case $m = 1$ follows from Lemma 4.4. In that case, the dual graph for every pair is a point.

We suppose the statement holds for $m - 1$. Let $\vec{\gamma}_m := \vec{w}$ and let $\vec{\gamma}_{m-1} := \vec{v}$ be two adjacent weights separated by the wall $W(I)$. We highlight that walls of type $\widetilde{W}(K)$ in $\mathcal{D}(3, n+1)$ do not modify neither $R_{\vec{v}}$ nor the shas parametrized by it because the space $R_{\vec{v}}$ only parametrizes shas with isolated multiple points by our inductive hypothesis. By Theorem 2.5(2), there is a contraction

$$\pi_m : \overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1) \to \overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1).$$

Let (X', D') be an arbitrary sha with at least one $p(I)$ singularity and parametrized by a point $z \in R_{\vec{v}}$. We will show that any shas (X', D') parametrized by $\pi_m^{-1}(z)$ have only multiple point singularities.

By Section 3.6, the fibers of π_m parametrize a new sha (X, D) containing a new $\mathbb{P}^2$ component with the lines $\{l_{i_1} \mid i_k \in I\}$. Therefore, the fiber of π_m over the point parametrizing (X', D') is the moduli associated to the pairs $(\mathbb{P}^2, l_{i_1} + \cdots + l_{i_k})$ that satisfy the following conditions:

(1) The lines cannot all overlap in an $|I|$-tuple point, because this is precisely the singularity we destabilized.

(2) The pair can have any singularity of the form $p(J) := \bigcap_{i_k \in J} l_{i_k}$ with J properly contained in I, because we are only destabilizing one type of singularity. We must cross more walls to destabilize $p(J)$.

(3) Let H_0 be the hyperplane obtained by intersecting the new $\mathbb{P}^2$ with the other components of $\tilde{X}$. Then the lines l_{i_s} cannot overlap with H_0.

(4) The equivalence class induced by the intersection of the lines l_{i_s} with the gluing locus is fixed because the sha (X', D') is fixed.

These are precisely the same conditions used in the proof of Lemma 4.4 with the gluing locus playing the role of l_A. Therefore, every positive dimensional fiber of π_m is isomorphic to $\mathbb{P}^{(|I|-3)}$. The new shas (X, D) have at worst multiple point singularities, because the lines $\{l_{i_1} \mid i_k \in I\}$ cannot overlap in the new component $\mathbb{P}^2 \subset X$ by the fourth condition above. The singularities of (X, D) away from this $\mathbb{P}^2$ are also multiple points by our hypothesis on the singularities of (X', D').

Part (II) follows from the previous argument because the wall crossing between two adjacent weights $\vec{v}$ and $\vec{u}$ adds a new vertex to $\mathrm{Graph}(X')$ corresponding to the new $\mathbb{P}^2$. The multiple points never occur in l_A, so l_A is always contained in a single surface which will be our root.

Finally, we prove part (III). In the absence of overlapping lines, as in our case, [Alexeev 2015, Theoarem 5.7.2(ii)] states that a $\mathbb{P}^1 \times \mathbb{P}^1$ component is only obtained from a configuration of points with the following characteristics:

(1) Given a $\mathbb{P}^2$-component with lines $\{l_i\}$, there are exactly two non-log-canonical points in the configuration of those lines.

(2) The line l_k between the two non-log-canonical points have weight 1.

(3) There is not an additional line l_s or a component of the double locus intersecting l_k transversally.

Under the above conditions, one must blow up the two points and contract the strict transform of the line between them (see [Alexeev 2015, Figure 5.8]).

To clarify this last condition, the reader should compare the following shas from [Alexeev 2015, Figure 5.12]. In sha #3, line l_3 intersects l_4 and prevents a line from being contracted in the $\mathrm{Bl}_2\,\mathbb{P}^2$ component, so that we do *not* obtain a $\mathbb{P}^1 \times \mathbb{P}^1$. In contrast, in sha #8, there does not exist a similar line intersecting l_1, in which case the sha has a $\mathbb{P}^1 \times \mathbb{P}^1$ as the corresponding component.

In particular, condition (3) will never happen in our case, as we always have either the double locus or the line l_A intersecting the line l_k transversally. □

The following result will be important for proving that $R_{\vec{w}}$ is smooth.

Lemma 4.9. *Let $\vec{v} \geq \vec{u}$ be adjacent weights in $\mathcal{D}_n^R$ separated by the wall $W(I)$. Let*

$$\pi_{\vec{v},\vec{u}} : \overline{M}_{(\vec{v},1)}(\mathbb{P}^2, n+1) \to \overline{M}_{(\vec{u},1)}(\mathbb{P}^2, n+1)$$

be the associated wall crossing morphism. Then its restriction $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ has (scheme-theoretic) fibers equal to $\mathbb{P}^{(|I|-3)}$.

The morphism $\phi_{\vec{v},\vec{u}}$ has positive dimensional fibers over the loci parametrizing shas that become unstable with respect to the weights $\vec{v}$. In our case, those are the shas with an isolated multiple point $p(I)$ and its fibers are described in the proof of Proposition 4.8. We now prove this scheme-theoretically.

Proof of Lemma 4.9. Let $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ be the wall crossing morphism where $\vec{v} \geq \vec{u}$, let A be the spectrum of an Artinian ring, and let $\psi : A \to R_{\vec{v}}$ be a deformation of $R_{\vec{v}}$. Furthermore, suppose that the total space of the composition $\phi_{\vec{v},\vec{u}} \circ \psi : A \to R_{\vec{u}}$ is constant. We wish to show, by contradiction, that this forces the total space of $\psi : A \to R_{\vec{v}}$ to be the trivial deformation as well.

We may assume that the total space of $\phi_{\vec{v},\vec{u}} \circ \psi : A \to R_{\vec{u}}$ is the trivial deformation of a pair (X, D) where (X, D) is stable with respect to the weights $\vec{u}$ but unstable with respect to $\vec{v}$. Indeed, if (X, D) was stable with respect to both weights, then the morphism $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ is an isomorphism on this locus, and there is nothing to prove.

In particular, there exists $D' \subset D$ such that $D' = \bigcup_{i\in I} L_i$ with $\sum_{i\in I} u_i \leq 2$ and $\sum_{i\in I} v_i > 2$. Then the definition of $\phi_{\vec{v},\vec{u}} : R_{\vec{v}} \to R_{\vec{u}}$ implies that the preimage of the sha (X, D) is $(Y, D_Y + Z)$, where $Y = X' \cup \mathbb{P}^2$ with $X' = \mathrm{Bl}_{p(I)} X$. Recall that $p(I)$ denotes the point we are required to blowup, as there are too many weighted lines passing through that point with respect to $\vec{v}$.

If we denote the gluing locus by $Z_1 \subset X'$ and $Z_2 \subset \mathbb{P}^2$, then it suffices to show that the deformation restricted to the three pairs (X', Z_1), $(\mathbb{P}^2, Z_2)$, and $Z = Z_1 \cong Z_2$ (the gluing locus $X' \cap \mathbb{P}^2$) is trivial. Indeed, we first note that $(\mathbb{P}^2, Z_2)$ is rigid. Furthermore, the deformation restricted to (X', Z_1) is trivial, as the pair (X', Z_1) is uniquely determined by (X, D), which is assumed to be fixed. In particular, (X', Z_1) is obtained as the blowup of a fixed variety at a fixed point. Therefore, it suffices to show that the deformation is trivial on the gluing locus, Z. To do so, we recall how our construction yields this line Z.

Recall that we are blowing up a point $p(I)$ inside a surface X living inside a total space $\overline{X} := X \times A$. In particular, there is an inclusion of normal bundles

$$N_X := N_{p(I)/X} \subset N_{A/\overline{X}} := N_{\overline{X}},$$

where N_X is also the restriction of $N_{\overline{X}}$ on X. Indeed, we obtain $N_{A/\overline{X}}$ as we are blowing up a $p(I)$ inside each fiber, and an entire family of them, thus blowing up a section $A \subset \overline{X}$. The exceptional divisor of the blowup of $p(I)$ inside $X \subset \overline{X}$, is defined by the projectivization of these normal bundles — indeed, the $\mathbb{P}^2$ arises from the projectivization of $N_{\overline{X}}$, and the gluing locus $Z \cong \mathbb{P}^1$ arises from the projectivization of N_X.

As $\phi_{\vec{v},\vec{u}} \circ \psi$ is assumed to be the trivial deformation, the normal bundles N_X and $N_{\overline{X}}$ as well as the inclusion $N_X \to N_{\overline{X}}$ never change. Now it suffices to note that any nontrivial deformation of Z, when composed with the wall crossing $\phi_{\vec{v},\vec{u}}$, would change the inclusion $N_X \to N_{\overline{X}}$, thus contradicting the fact that $\phi_{\vec{v},\vec{u}} \circ \psi$ is a trivial deformation.

Therefore, the moduli is determined by the moduli of the lines $\sum_{i\in I} L_i + Z$ inside $\mathbb{P}^2$, such that $\sum_{i\in I} L_i = 2 + \epsilon$ and $L_I \cap Z$ is a fixed point of $M_{0,n}$, which is $\mathbb{P}^{|I|-3}$ by Lemma 4.4. □

5. Construction of $R_{\vec{w}}$ via wonderful compactifications

As in the previous section, the results of this section do not depend on the q used in the definition of $R_{\vec{w}}(q)$, as long as it is a generic point of $M_{0,\vec{w}_0}$. We simplify our notation and just write $R_{\vec{w}}$.

In Notation 4.5 we showed that the equivalence class of the n lines parametrized by $[s_1 : \cdots : s_{n-2}] \in R_{\vec{w}_0}$ is induced by the lines

$$\begin{aligned} &l_i := (x_1 - a_i x_2 + s_i x_0 = 0), \quad l_{n-2} := (s_{n-2}x_0 + x_1 = 0), \quad l_{n-1} := (x_2 = 0), \\ &l_A := (x_0 = 0), \qquad\qquad\quad l_n := (x_1 - x_2 = 0). \end{aligned}$$

Therefore, the point $[1:0:\cdots:0] \in R_{\vec{w}_0}$ parametrizes a pair with an $(n-1)$-tuple point at $[1:0:0] \in \mathbb{P}^2$ induced by the intersection of the lines $l_2, \ldots, l_n$. Similarly, the hyperplane $(s_1 = 0) \subset R_{\vec{w}_0}$ parametrizes a pair with a triple point at $[1:0:0]$.

We now show that this behavior holds in general.

Lemma 5.1. *For every $I \subset \{1, \ldots, n\}$, there is a linear subspace $\mathbb{P}^{(n-|I|-1)} \cong H(I) \subset R_{\vec{w}_0}$ that generically parametrizes a configuration with an $|I|$-tuple point $p(I)$ given by the intersection of the lines $\{l_i \mid i \in I\}$.*

Proof. A set of lines $\{l_i \mid i \in I\}$ has an $|I|$-multiple point if and only their dual points $\{y_i \mid i \in I\}$ are collinear. Taking any subset of three of these points, the associated matrix $[y_j, y_k, y_l]$ has determinant equal to zero. In particular, these equations are linear on the variables s_i and define $H(I)$. Finally, the dimension count is $(n-3) - (|I|-2) = n - |I| - 1$. □

Example 5.2. We use the equation of the lines as given in Notation 4.5. For example associated to the points $y_1 = [s_1, 1, -a_1]$, $y_2 = [s_1, 1, -a_2]$, and $y_3 = [s_1, 1, -a_3]$, we have the equation

$$s_1(a_2 - a_3) - s_2(a_1 - a_3) + s_3(a_1 - a_2) = 0.$$

The sets $H(I)$ will generate the centers of the morphism $R_{\vec{1}^n} \to R_{\vec{w}_0}$. These morphisms are induced by changing the weights, and the description of these linear subspaces will be crucial for the next subsection.

5.3. *Wonderful compactifications.* In what follows, we review the pertinent definitions of *wonderful compactifications* following [Li 2009]. We note that the theory of wonderful compactifications originated in [De Concini and Procesi 1995].

Definition 5.4. An *arrangement* of subvarieties of a nonsingular variety Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed subvarieties $S_i \subset Y$ closed under scheme-theoretic intersection. Given $\dim(Y) = (n-3)$, we say that a finite collection of k nonsingular subvarieties $S_1, \ldots, S_k$ intersect *transversely*, if either $k = 1$ or for any $y \in Y$ the following conditions holds (see [Li 2009, §5.1.2]):

(a) There exist a system of local parameters $x_1, \ldots, x_{(n-3)}$ on Y at y that are regular on an affine neighborhood U of y such that y is defined by the maximal ideal $(x_1, \ldots, x_{(n-3)})$.

(b) There are integers $0 = r_0 \leq r_1 \leq \cdots \leq r_k \leq (n-3)$ such that the subvariety S_i is defined by the ideal

$$(x_{r_{i-1}+1}, x_{r_{i-1}+2}, \ldots, x_{r_i}),$$

for all $1 \leq i \leq k$.

If $r_{i-1} = r_i$ then the ideal is assumed to be the ideal containing units, which means geometrically that the restriction of S_i to U is empty.

Definition 5.5. A subset $\mathcal{G} \subset \mathcal{S}$ is called a *building set* of $\mathcal{S}$ if for all $S_k \in \mathcal{S}$, the minimal elements of $\mathcal{G}$ containing S_k intersect transversally and their intersection is equal to S_k (by convention, the condition is satisfied if $S_k \in \mathcal{G}$). These minimal elements are called the *$\mathcal{G}$-factors* of S_k. Let $\mathcal{G}$ be a building set and set $Y^o :=$

$Y \setminus \bigcup_{S_k \in \mathcal{G}} S_k$. The closure of the image of the natural locally closed embedding [Li 2009, Definition 1.1]

$$Y^o \hookrightarrow \prod_{S_k \in \mathcal{G}} \mathrm{Bl}_{S_k} Y$$

is called the *wonderful compactification* of Y with respect to $\mathcal{G}$.

Theorem 5.6 [Li 2009, Theorem 1.3]. *Let $\mathcal{G}$ be a building set and let $\mathrm{Bl}_{\mathcal{G}} Y$ be the wonderful compactification of Y with respect to $\mathcal{G}$. Then $\mathrm{Bl}_{\mathcal{G}} Y$ is smooth with normal crossing boundary and for each $S_k \in \mathcal{G}$ there is a nonsingular divisor $D_{S_k} \subset Y_{\mathcal{G}}$. Moreover, the union of the divisors is $Y_{\mathcal{G}} \setminus Y^o$ and any set of these divisors, with nonempty intersection, meet transversally.*

Example 5.7. A building set $\mathcal{H}$ in $R_{\vec{w}_0}$ is given by 5 points $H(J)$ with $|J| = 4$ and 10 lines $H(I)$ with $|I| = 3$ parametrizing configurations with either a quadruple or a triple point respectively. The arrangement $\mathcal{S}$ is the set of all possible intersections among them. The 10 lines, which are not in general position, intersect along 20 points given by:

(1) The point $H(I) \cap H(J)$ with $|I \cap J| = 2$ parametrizes the quadruple point $p(I \cup J)$.

(2) The point $H(I) \cap H(J)$ with $|I \cap J| = 1$ parametrizes a configuration with two triple points associated to I and J. There are 15 of these points.

The above example illustrates the general behavior.

Lemma 5.8. *Let $\mathcal{S}_{\vec{w}}$ be the set of all possible intersections of collections of subvarieties from*

$$\mathcal{H}_{\vec{w}} = \left\{ H(J) \,\Big|\, \sum_{i \in J} w_i > 2,\ |J| \subset \{1, \dots, n\} \right\}.$$

Then, $\mathcal{S}_{\vec{w}}$ is an arrangement and $\mathcal{H}_{\vec{w}}$ is a building set.

Proof. $\mathcal{S}_{\vec{w}}$ is an arrangement by Definition 5.4. For the last statement, let S_k be an arbitrary element of $\mathcal{S}_{\vec{w}}$. By definition, S_k is an arbitrary nonempty intersection $S_k := H(I_1) \cap \cdots \cap H(I_m)$. We need to prove two conditions: (I) The minimal elements of $\mathcal{H}_{\vec{w}}$ that contain S_k intersect transversally. (II) Their intersection is equal to S_k.

For (I), we observe that any S_k can be written uniquely as an intersection of the form $H(J_1) \cap \cdots \cap H(J_s)$, where $|J_i \cap J_k| \le 1$ and each of the J_i is a union of I_j. Indeed, if $|I_1 \cap I_2| \ge 2$ and $I_1 \cap I_2 \ne \{1, \dots, n\}$, then their intersection must parametrize an $(|I_1| + |I_2|)$-tuple point. This implies that $H(I_1) \cap H(I_2)$ is either the empty set or $H(I_1 \cup I_2) \in \mathcal{H}_{\vec{w}}$. In the latter case, we can dismiss $H(I_1)$ and $H(I_2)$ while keeping $H(I_1) \cap H(I_2)$. Iterating this process, we can find all the minimal elements $J_i \in \mathcal{H}_{\vec{w}}$ containing S_k.

Part (I) now reduces to showing that the intersection of the linear subspaces $\mathbb{P}^{(n-|J_i|-1)}$, $1 \le i \le s$, along S_k is transversal. By Definition 5.4, it is enough to exhibit numbers $0 = r_0 \le r_1 \le \cdots \le r_s \le (n-3)$ that satisfy the conditions of the aforementioned definition. We can take

$$r_0 := 0 \quad \text{and} \quad r_m := \sum_{i=1}^{m} (|J_1| - 2) \ \text{ with } 1 \le m \le s.$$

Indeed, $r_s \leq (n-3)$ because

$$0 \leq \dim(S_k) = (n-3) - \sum_{i=1}^{s}(|J_1|-2),$$

since S_k is nonempty. We can take the linear subspace $H(J_m) = \mathbb{P}^{n-|J_m|-1}$ to be defined by the ideal

$$(x_{(r_{m-1}+1)}, \ldots, x_{r_m}),$$

because counting its number of generators, we obtain

$$r_m - (r_{m-1}+1) + 1 = \left(\sum_{i=1}^{i=m}(|J_1|-2)\right) - \left(\sum_{i=1}^{i=m-1}(|J_1|-2)\right) = (|J_m|-2),$$

which is the codimension of $H(J_m)$.

Finally, as we are intersecting linear subspaces in projective space condition (II) follows by the definition of the $H(J_i)$. □

Definition 5.9. Let $\vec{w} \in \mathcal{D}_n^R$ be an admissible weight and let $\mathcal{H}_{\vec{w}}$ be as in Lemma 5.8. Then the *wonderful compactification* of $R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ with respect to $\mathcal{H}_{\vec{w}}$ is denoted by $\mathrm{Bl}_{\vec{w}}\ R_{\vec{w}_0}$.

Lemma 5.10. *Let $\vec{w}$ be an admissible weight vector in $\mathcal{D}_n^R$. There exists a smooth variety $\mathcal{U}_{\vec{w}}$, a flat proper morphism $\phi_{\vec{w}}$,*

$$\begin{array}{ccc} \mathcal{U}_{\vec{w}} & \xrightarrow{\ \tau\ } & \mathcal{U}_{\vec{w}_0} \\ {\scriptstyle \phi_{\vec{w}}}\downarrow & & \downarrow{\scriptstyle \phi_{\vec{w}_0}} \\ \mathrm{Bl}_{\vec{w}}\ R_{\vec{w}_0} & \longrightarrow & R_{\vec{w}_0} \end{array}$$

and n hypersurfaces $\sigma_i(\vec{w}) \subset \mathcal{U}_{\vec{w}}$ such that for every $\vec{s} \in \mathrm{Bl}_{\vec{w}}\ R_{\vec{w}_0}$ the fiber $\phi_{\vec{w}}^{-1}(\vec{s})$ and the divisors

$$\phi_{\vec{w}}^{-1}(\vec{s}) \cap \sigma_i(\vec{w}) \quad \text{and} \quad l_A := \phi_{\vec{w}}^{-1}(\vec{s}) \cap \tau^{-1}(E_{\vec{w}_0})$$

define a stable sha of weight $\vec{w}$ ($E_{\vec{w}_0}$ is defined in Lemma 4.6).

Proof. Let $\vec{w} \in \mathcal{D}_n^R$ be an admissible weight and consider a sequence of weights $\vec{\gamma}_1, \ldots, \vec{\gamma}_{m+1}$ such that $\vec{\gamma}_1 := \vec{w}_0$, $\vec{\gamma}_{m+1} := \vec{w}$, the weights $\vec{\gamma}_i \leq \vec{\gamma}_{i+1}$ are adjacent to each other (see Definition 4.2) and $m+1$ is the minimal length of such sequences. We prove our Lemma by induction. The base case is proven in Lemma 4.6.

Next, we describe the inductive step. We suppose that the statement holds for γ_m. In particular, there exists a smooth variety $\mathcal{U}_{\vec{\gamma}_m}$ with a flat proper morphism $\phi_{\vec{\gamma}_m} : \mathcal{U}_{\vec{\gamma}_m} \to \mathrm{Bl}_{\vec{\gamma}_m}\ R_{\vec{w}_0}$, and n hypersurfaces $\sigma_i(\vec{\gamma}_m) \subset \mathcal{U}_{\vec{\gamma}_m}$ such that for every $\vec{s} \in \mathrm{Bl}_{\vec{\gamma}_m}\ R_{\vec{w}_0}$ the fiber $\phi_{\vec{\gamma}_m}^{-1}(\vec{s})$ and the divisors $\phi_{\vec{\gamma}_m}^{-1}(\vec{s}) \cap \sigma_i(\vec{\gamma}_m)$ and $l_A := \phi_{\vec{\gamma}_m}^{-1}(\vec{s}) \cap \tau^{-1}(E_{\vec{w}_0})$ define a stable sha of weight γ_m.

Let $W(I)$ be the wall separating $\vec{\gamma}_m$ and $\vec{\gamma}_{m+1} = \vec{w}$, we denote the singularity destabilized by this wall crossing by $p(I)$.

Let $\overline{H}(I)$ be the closure of the locus in $\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0}$ parametrizing all shas (X, D) with a multiple point $p(I)$, and let $S(I) \subset \mathcal{U}_{\gamma_m}$ be the locus supporting $p(I)$. We will show that the diagram

$$\begin{array}{ccccc} \mathcal{U}_{\vec{w}} := \mathrm{Bl}_{\eta^{-1}(S(I))}(\mathrm{Bl}_{\vec{w}} R_{\vec{w}_0} \times_{(\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0})} \mathcal{U}_{\vec{\gamma}_m}) & \longrightarrow & \mathrm{Bl}_{\vec{w}} R_{\vec{w}_0} \times_{(\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0})} \mathcal{U}_{\vec{\gamma}_m} & \xrightarrow{\eta} & \mathcal{U}_{\vec{\gamma}_m} \\ & \searrow^{\phi_{\vec{w}}} & \downarrow \tilde{\pi} & & \downarrow \gamma_{\vec{\gamma}_m} \\ & & \mathrm{Bl}_{\vec{w}} R_{\vec{w}_0} & \xrightarrow{\rho} & \mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0} \end{array}$$

yields our family $\phi_{\vec{w}} : \mathcal{U}_{\vec{w}} \to \mathrm{Bl}_{\vec{w}} R_{\vec{w}_0}$.

Notice that $S(I) \cong \overline{H}(I)$ because the projection $S(I) \to \overline{H}(I)$ is finite, generically one-to-one, and $\overline{H}(I)$ is the smooth strict transform of $H(I) \subset R_{\vec{w}_0}$ in $\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0}$. Therefore, the isomorphism $S(I) \cong \overline{H}(I)$ follows by Zariski's main theorem.

By definition of the wonderful blow up, we have that

$$\mathrm{Bl}_{\vec{w}} R_{\vec{w}_0} = \mathrm{Bl}_{\overline{H}(I)}(\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0}).$$

On another hand, by the inductive hypothesis, $\phi_{\vec{\gamma}_m} : \mathcal{U}_{\vec{\gamma}_m} \to \mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0}$ is flat. Since blowing up commutes with flat base change, we obtain

$$\mathrm{Bl}_{\vec{w}} R_{\vec{w}_0} \times_{(\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0})} \mathcal{U}_{\vec{\gamma}_m} \cong \mathrm{Bl}_{\phi_{\vec{\gamma}_m}^{-1}(\overline{H}(I))} \mathcal{U}_{\vec{\gamma}_m}, \tag{5.10.1}$$

which implies

$$\mathrm{Bl}_{\eta^{-1}(S(I))}(\mathrm{Bl}_{\vec{w}} R_{\vec{w}_0} \times_{(\mathrm{Bl}_{\vec{\gamma}_m} R_{\vec{w}_0})} \mathcal{U}_{\vec{\gamma}_m}) = \mathrm{Bl}_{\eta^{-1}(S(I))}(\mathrm{Bl}_{\phi_{\vec{\gamma}_m}^{-1}(\overline{H}(I))} \mathcal{U}_{\vec{\gamma}_m}).$$

Let E_ρ and E_η be the exceptional divisors of ρ and η respectively. Next, we describe the fiber $\tilde{\pi}^{-1}(z)$ for $z \in E_\rho$. Given $y \in \overline{H}(I)$, the fiber $\phi_{\vec{\gamma}_m}^{-1}(y)$ is a surface X.

We find, by dimension counting, that $\rho^{-1}(y) \cong \mathbb{P}^{(|I|-3)}$ and $\eta^{-1}(\phi_{\vec{\gamma}_m}^{-1}(y))$ is a $\mathbb{P}^{(|I|-3)}$-fibration over X. Due to the fiber product construction, there is a morphism $\eta^{-1}(\phi_{\vec{\gamma}_m}^{-1}(z)) \to \rho^{-1}(z)$. So $\eta^{-1}(\phi_{\vec{\gamma}_m}^{-1}(z))$ is a fibration over $\mathbb{P}^{(|I|-3)}$ with fibers isomorphic to X.

Therefore, for all $z \in E_\rho$ it holds that $\tilde{\pi}^{-1}(z) \cong X$ and the strict transform

$$\{\eta_*^{-1}(\sigma_i(\vec{\gamma}_m)) \mid i \in I\}$$

of the sections $\{\sigma_i(\vec{\gamma}_m) \mid i \in I\}$ induces a divisor in $\tilde{\pi}^{-1}(z)$ with an $(n-1)$ multiple point. Blowing up $\eta^{-1}(S(I))$ generically separates those sections in $\mathcal{U}_{\vec{w}}$, because the intersection of the hypersurfaces $\{\eta_*^{-1}(\sigma_i(\vec{\gamma}_m)) \mid i \in I\}$ is locally an intersection of $|I|$ hyperplanes in affine space. Indeed, recall our sections are the strict transforms of $\sigma_i \subset R_{\vec{w}_0}$ and that $\mathcal{U}_{\vec{w}_0} \cong \mathrm{Bl}_Z \mathbb{P}^{n-1}$ with $Z \cong \mathbb{P}^1$ and $Z \cap \sigma_i = \varnothing$.

Finally, we describe the fibers of $\phi_{\vec{w}}$. The locus $\eta^{-1}(q_I) \cong \mathbb{P}^{|I|-3}$ intersects $\tilde{\pi}(z) \cong X$ at the point x supporting the multiple point $q(I)$. The locus $S(I) \subset \mathcal{U}_{\vec{\gamma}_m}$ has dimension $(n - |I| - 1)$. Therefore, $\dim(\eta^{-1}(S(I)) = (n-4)$ which implies the divisor of the blow up

$$\mathcal{U}_{\vec{w}} \to \mathrm{Bl}_{\phi_{\vec{\gamma}_m}^{-1}(\overline{H}(I))}(\mathcal{U}_{\vec{\gamma}_m})$$

is a $\mathbb{P}^2$-fibration over $\eta^{-1}(S(I))$. So, $\phi_{\vec{w}}^{-1}(z)$ is equal to

$$\mathbb{P}^2 \bigcup_{L=E} \mathrm{Bl}_x(\tilde{\pi}^{-1}(z)) \cong \mathbb{P}^2 \bigcup_{L=E} \mathrm{Bl}_x X, \tag{5.10.2}$$

where $E \subset \mathrm{Bl}_x X$ is the exceptional divisor obtained by blowing up x and L is a line in $\mathbb{P}^2$.

The $\mathbb{P}^2$ component is a fiber of $\mathcal{U}_{\vec{w}} \to \mathrm{Bl}_{\phi_{\vec{\gamma}_m}^{-1}(p_I)} \mathcal{U}_{\vec{\gamma}_m}$, so the strict transforms of the sections $\{\sigma_i(\vec{\gamma}_m) \mid i \in I\}$ define a configuration of lines on it. Those lines do not overlap in a $|I|$-tuple point, because that is the multiple point we just separated. Therefore, the resultant pair defined by the surface in (5.10.2) and its intersection with the strict transform $\sigma_i(\vec{w})$ of the hypersurfaces $\sigma_i(\vec{\gamma}_m)$ in $\mathcal{U}_{\vec{w}}$ defines a stable sha with respect to $\vec{w}$. □

In the following lemma, we recall that n points in $\mathbb{P}^{n-3}$ are in general position if there are no two of them supported in a point, no three of them contained on a line, no four of them contained in a plane, and so forth.

Lemma 5.11. *For $n \geq 5$, there are n points $q_1, \dots, q_n$ in $R_{\vec{w}_0}$ in general position, a sequence of weights $\vec{w}_k$ with $1 \leq k \leq (n-3)$, and morphisms of smooth varieties*

$$\mathrm{Bl}_{\vec{w}_{(n-3)}} R_{\vec{w}_0} \to \cdots \to \mathrm{Bl}_{\vec{w}_k} R_{\vec{w}_0} \to \cdots \to R_{\vec{w}_0},$$

where:

- $\mathrm{Bl}_{\vec{w}_1} R_{\vec{w}_0}$ *is the blow up of* $R_{\vec{w}_0}$ *along* $q_1, \dots, q_n$ *in any order.*
- $\mathrm{Bl}_{\vec{w}_2} R_{\vec{w}_0}$ *is the blow up of* $\mathrm{Bl}_{\vec{w}_1} R_{\vec{w}_0}$ *along the strict transform of lines spanned by all pairs of points* $\{q_i, q_j\}$, *in any order.*

⋮

- $\mathrm{Bl}_{\vec{w}_{(n-3)}} R_{\vec{w}_0}$ *is the blow up of* $\mathrm{Bl}_{\vec{w}_{(n-4)}} R_{\vec{w}_0}$ *along the strict transforms of the* $(n-4)$*-planes spanned by all* $(n-3)$*-tuples of the* q_i, $i = 1, \dots, n$ *in any order.*

Proof. The wonderful blowup is by definition a sequence of iterative blow ups along the strict transforms of the elements in the building set $\mathcal{H}_{1^n}$. The points q_i correspond to $H(I)$ with $|I| = (n-1)$, the lines spanned by the points q_i correspond to $H(J)$ with $|J| = (n-2)$, and so on. The order of the blow-ups can be taken to be any order of increasing dimension by [Li 2009, Theorem 1.3]. □

5.12. *$R_{\vec{w}}$ is isomorphic to a wonderful compactification.* Our aim is to show that $R_{\vec{w}}$ is isomorphic to the wonderful compactification $\mathrm{Bl}_{\vec{w}} \mathbb{P}^{n-3}$. First we review $R_{\vec{w}}$ for small values of n.

Example 5.13. (1) If $\vec{w} \in \mathcal{D}_3^R$, then $R_{\vec{w}}$ is a point.

(2) If $\vec{w} \in \mathcal{D}_4^R$, then $R_{\vec{w}} \cong \mathbb{P}^1$, as $\overline{M}_{1^6}(\mathbb{P}^2, 5) \cong \overline{M}_{0,5}$.

(3) If $\vec{w} \in \mathcal{D}_5^R$, then $R_{\vec{w}} \cong \mathrm{Bl}_{\vec{w}} \mathbb{P}^2$. In particular, the morphism $R_{1^5} \to R_{\vec{w}_0} \cong \mathbb{P}^2$ is the blow up of $\mathbb{P}^2$ at five points and the morphisms induced by wall crossings inside $\mathcal{D}_5^R$ are either smooth blow ups or isomorphisms. Indeed, it is known that $\overline{M}_{1^6}(\mathbb{P}^2, 6)$ has isolated singularities (see [Luxton 2008,

Theorem 4.2.4]). Therefore, by the construction of $R_{\vec{w}}$ as in Definition 3.1 it follows that R_{1^5} is smooth. We note that the building set $\mathcal{H}_{1^5}$ is described in Example 5.7 and that the smoothness of $R_{\vec{w}}$ follows from the smoothness of R_{1^5} and Theorem 5.16.

Theorem 5.14. *For any choice of n and $\vec{w} \in \mathcal{D}_n^R$, it holds that $R_{\vec{w}} \cong \mathrm{Bl}_{\vec{w}} R_{\vec{w}_0}$ and thus $R_{\vec{w}}$ is smooth with normal crossings boundary.*

Proof. Our proof is by induction on the weight vector. The base case is $R_{\vec{w}_0}$ which is discussed in Lemmas 4.4 and 4.6. Let $\vec{v} \geq \vec{u}$ be two adjacent weights separated by the wall $W(I)$ which destabilizes the multiple point $p(I)$. Now consider the following diagram:

$$\begin{array}{ccc} \mathrm{Bl}_{\vec{v}} R_{\vec{w}_0} & \xrightarrow{f_{\vec{v}}} & R_{\vec{v}} \\ \downarrow{\scriptstyle \psi_{\vec{v},\vec{u}}} & & \downarrow{\scriptstyle \phi_{\vec{v},\vec{u}}} \\ \mathrm{Bl}_{\vec{u}} R_{\vec{w}_0} & \xrightarrow{\sim} & R_{\vec{u}} \end{array}$$

where the morphism $\psi_{\vec{v},\vec{u}}$ is the blowup

$$\mathrm{Bl}_{\vec{v}} R_{\vec{w}_0} \cong \mathrm{Bl}_{\overline{H}(I)}(\mathrm{Bl}_{\vec{u}} R_{\vec{w}_0}) \to \mathrm{Bl}_{\vec{u}} R_{\vec{w}_0}$$

induced by the wonderful compactification, and $\phi_{\vec{v},\vec{u}}$ is the wall crossing morphism induced by changing the weights. By induction, we assume that $\mathrm{Bl}_{\vec{u}} R_{\vec{w}_0} \cong R_{\vec{u}}$ and thus $R_{\vec{u}}$ is smooth. We must now show that $R_{\vec{v}}$ is also smooth.

By Lemma 5.10, there is a flat family $\mathcal{U}_{\vec{v}} \to \mathrm{Bl}_{\vec{v}} R_{\vec{w}_0}$ whose fibers are stable shas with respect to $\vec{v}$. On the other hand, $\overline{M}_{(\vec{w},1)}(\mathbb{P}^2, n+1)$ is a fine moduli space by [Alexeev 2008, Lemma 7.7]. Therefore, there is a map $f_{\vec{v}} : \mathrm{Bl}_{\vec{v}} R_{\vec{w}_0} \to R_{\vec{v}}$. Let $E_{\vec{v}} \subset \mathrm{Bl}_{\vec{v} R_{\vec{w}_0}}$ and $F_{\vec{v}} \subset R_{\vec{v}}$ be the exceptional divisors of $\phi_{\vec{v},\vec{u}}$ and $\varphi_{\vec{v},\vec{u}}$ respectively. By construction $f_{\vec{v}}$ is an isomorphism when restricted to the open sets

$$(\mathrm{Bl}_{\vec{v}} R_{\vec{w}_0}) \setminus E_{\vec{v}} \to R_{\vec{v}} \setminus F_{\vec{v}}$$

and the restriction $f_{\vec{v}} : E_{\vec{v}} \to F_{\vec{v}}$ is a finite morphism because both exceptional divisors are $\mathbb{P}^{|I|-3}$ fibrations over $\overline{H}(I)$.

In particular, the above argument implies the morphism $f_{\vec{v}}$ is the normalization. Therefore, since $\mathrm{Bl}_{\vec{v}} R_{\vec{w}_0}$ is smooth, by Zariski's main theorem, it suffices to show that $R_{\vec{v}}$ is normal. To do so, we consider the exact sequence arising from normalization

$$0 \to \mathcal{O}_{R_{\vec{v}}} \to f_* \mathcal{O}_{\mathrm{Bl}_{\vec{v}} R_{\vec{w}_0}} \to \delta \to 0.$$

Our goal is to prove that $\delta = 0$. If $p \in R_{\vec{u}}$ is a point parametrizing a configuration which is stable with respect to both weights $\vec{v}$ and $\vec{u}$, then the morphisms $\psi_{\vec{v},\vec{u}}$ and $\phi_{\vec{v},\vec{u}}$ are both isomorphisms and there is nothing to prove. Therefore, we may assume that p is a point which induces a blowup.

To look at the fiber over the point p we tensor by $\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}$ to obtain

$$\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}} \to (f_*\mathcal{O}_{\mathrm{Bl}_{\vec{v}} \mathbb{P}^{n-3}}) \otimes (\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}) \to \delta \otimes (\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}) \to 0.$$

The wonderful compactification is a sequence of iterative smooth blows, so by dimension counting the fiber of $\psi_{\vec{v},\vec{u}}$ over p is a $\mathbb{P}^{|I|-3}$. Furthermore, by Lemma 4.9, the fiber of $\phi_{\vec{v},\vec{u}}$ over p is also a $\mathbb{P}^{|I|-3}$. As $f_{\vec{v}}$ is the normalization, and both $\phi^{-1}_{\vec{v},\vec{u}}(p)$ and $\psi^{-1}_{\vec{v},\vec{u}}(p)$ are scheme theoretically $\mathbb{P}^{|I|-3}$, the projective spaces must be isomorphic. As the first arrow above is an isomorphism, we see that

$$\delta \otimes (\mathcal{O}_{R_{\vec{v}}}/I_p\mathcal{O}_{R_{\vec{v}}}) = 0.$$

As this is true for all $p \in R_{\vec{v}}$, we see that $\delta = 0$, and thus $R_{\vec{v}}$ is normal. □

5.15. ***Consequences of the blow up construction.***

Theorem 5.16. *There is a birational projective morphism $R_{\vec{w}} \to R_{\vec{w}_0} \cong \mathbb{P}^{n-3}$ that can be understood as a sequence of smooth blowups. In particular, the morphism $R_{1^n} \to \mathbb{P}^{n-3}$ can be understood as completing the steps descried in Lemma 5.11.*

Proof. The theorem follows from Theorem 5.14. □

Lemma 5.17 [Hassett 2003]. *Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a set of weights where*

$$\alpha_1 = 1, \quad \alpha_2 = 1 - \frac{(n-2)}{n-1} + \frac{1}{2(n-1)}, \quad \alpha_3 = \cdots = \alpha_n = \frac{1}{n-1}.$$

Then $\overline{M}_{0,\vec{\alpha}} = \mathbb{P}^{n-3}$. Let $\delta_I \subset \mathbb{P}^{n-3}$ be the locus parametrizing configurations of n points in $\mathbb{P}^1$ such that $\{p_{i_1} = \cdots = p_{i_s} \mid i_k \in I\}$. Then, for every $\vec{w} > \vec{\alpha}$, it follows that $\overline{M}_{0,\vec{w}}$ is the wonderful compactification of $\mathbb{P}^{n-3}$ with respect to the building set

$$\mathcal{S}_{\vec{w}} := \left\{ \mathbb{P}^{(n-|I|)-2} \cong \delta_I \subset \mathbb{P}^{n-3} \;\middle|\; \sum_{i \in I} w_i > 1, I \subset \{2, \dots, n\}, 2 \le |I| \le (n-2) \right\}.$$

Proof. The existence of a set of weights $\vec{\alpha}$ such that $\overline{M}_{0,\vec{\alpha}} \cong \mathbb{P}^{n-3}$ is well known (see [Hassett 2003, §6.2]). The condition $\vec{w} > \vec{\alpha}$ guarantees the existence of a morphism $\overline{M}_{0,\vec{w}} \to \overline{M}_{0,\vec{\alpha}}$ (see [Hassett 2003, Theorem 4.1]). The set $\mathcal{S}_{\vec{w}}$ is the locus in $\mathbb{P}^{n-3}$ that becomes unstable with respect to the weights $\vec{w}$. In particular, the condition $1 \notin I$ is necessary for $\delta_I \subset \mathbb{P}^{n-3}$, otherwise δ_I is unstable with respect to $\vec{\alpha}$. □

Corollary 5.18. *Given a set of weights $\vec{w} = (1, w_2, \dots, w_n)$, there is a morphism $R_{\vec{w}} \to \overline{M}_{0,\vec{w}}$ which can be interpreted as a continuation of a blow up construction $\overline{M}_{0,\vec{w}} \to \mathbb{P}^{n-3}$.*

Proof. The weights of l_A and l_1 are one by construction, then we can define the morphism $\psi : R_{\vec{w}} \to \overline{M}_{0,\vec{w}}$ by intersecting the broken lines $\{l_A, l_2, \dots, l_n\}$ with l_1. That is

$$\left(X, l_A + \sum_{k=1}^{n} w_k l_k\right) \to \left(l_1, (l_A + \sum_{k=2}^{n} w_k l_k)\Big|_{l_1}\right).$$

The morphism is well defined by adjunction. Notice that the set $\{H(I) \in \mathcal{H}_{\vec{w}} \mid 1 \in I\}$ is isomorphic to $\mathcal{S}_{\vec{w}}$ as in Lemma 5.17 above. Indeed, for an index set $I \subset \{1, \dots n\}$ such that $1 \in I$, it holds that

$$\sum_{i \in I} w_i > 2 \iff \sum_{i \in I \setminus 1} w_i > 1.$$

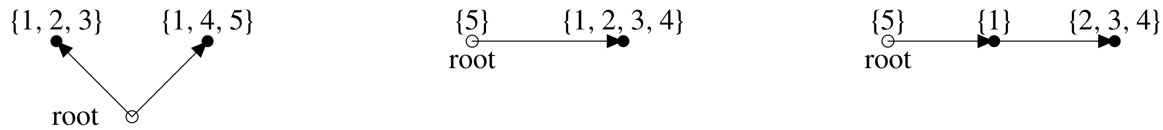

Figure 3. From left to right, the dual graphs associated to the last sha of Figure 1 and the last 2 of Figure 2.

Moreover, $\mathbb{P}^{(n-|I-1|)-2} \cong \delta_{I\setminus 1} \cong H(I) \cong \mathbb{P}^{(n-|I|)-1}$ by Lemma 5.1, and if I and K are indices containing 1, then $\delta_{I\setminus 1} \cap \delta_{K\setminus 1} \neq \varnothing$ if and only if $H(I) \cap H(K) \neq \varnothing$. Finally, we use that $\mathbb{P}^{n-3} \cong R_{\vec{\alpha}} \cong \overline{M}_{0,\vec{\alpha}}$ to identify these sets.

By [Li 2009, Theorem 1.3.ii], the wonderful blowup does not change if we rearrange the elements of $\mathcal{H}_{\vec{w}}$ so that the first k terms form a building set for any $1 \leq k \leq n$. Therefore, by Theorem 5.14, we have

$$R_{\vec{w}} = \mathrm{Bl}_{\mathcal{H}_{\vec{w}}}(\mathbb{P}^{n-3}) = \mathrm{Bl}_{\mathcal{H}_{\vec{w}}\setminus\mathcal{S}_{\vec{w}}}(\mathrm{Bl}_{\mathcal{S}_{\vec{w}}}\,\mathbb{P}^{n-3}) = \mathrm{Bl}_{\mathcal{H}_{\vec{w}}\setminus\mathcal{S}_{\vec{w}}}(\overline{M}_{0,\vec{w}}),$$

where $\mathrm{Bl}_{\mathcal{H}_\beta\setminus S_\beta}$ denotes the blow up along the strict transform of the elements in the set $\mathcal{H}_\beta \setminus S_\beta$.

The description in the statement of our result follows by comparing Lemma 5.11 with the blow up construction of $\overline{M}_{0,n}$ outlined in the introduction. □

We now show that there do not exist weights $\vec{w}$ so that $R_{\vec{w}} \cong \overline{M}_{0,n}$.

Proof of Theorem 1.3. Let $\vec{w} \in \mathcal{D}_n^R$, we will show that $\mathcal{H}_{\vec{w}}$ cannot be equal to the locus $S_{\vec{w}}$ required to construct $\overline{M}_{0,n}$ as described in [Hassett 2003, §6.2]. If we suppose otherwise, then $\vec{w}$ destabilizes $(n-1)$ points and all the linear subspaces spanned by them while the n-th point is stable with respect to $\vec{w}$. In other words, let $H(I_k) \in \mathcal{H}_{\vec{w}}$, where $|I_k| = (n-1)$ for $k = 1, \ldots, n-1$ and $H(I_n) \notin \mathcal{H}_w$ where $|I_n| = (n-1)$. The existence of $\vec{w}$ is equivalent to the existence of a solution for the following system of inequalities.

$$w_{i_1} + w_{i_2} + w_{i_3} > 2 \quad \forall \{i_1, i_2, i_3\} \subset I_k, \tag{5.18.1}$$

$$\sum_{i\in I_n} w_i \leq 2 \quad 0 < w_i \leq 1. \tag{5.18.2}$$

The inequality (5.18.1) is associated to destabilizing the $(n-2)$-planes generated by $H(I_k)$ with $1 \leq i \leq n$. The inequality (5.18.2) follows because $H(I_n)$ is stable with respect to $\vec{w}$. Without loss of generality, we set $I_n = \{2, \ldots, n\}$. Since $|I_k| = (n-1)$, there is at least one I_k such that $I_k \cap I_n$ has at least three distinct elements i_1, i_2, i_3 and so the inequality (5.18.1) for these three elements contradicts (5.18.2). □

6. R_{1^n} as a nonreductive Chow quotient

In this section, we discuss the proof of Theorem 1.4. An important step of the proof is based on the fact that the dual graphs of the pairs parametrized by $R_{\vec{w}}$ are always rooted trees, with the root vertex corresponding to the component containing the line l_A. To keep track of the lines l_i, we mark the vertices corresponding to the last component containing the broken line l_i.

We highlight that there is a configuration space known as $T_{d,n}$ which generalizes $\overline{M}_{0,n}$ (see [Chen et al. 2009]), and there is a nonreductive Chow quotient under the same group [Gallardo and Giansiracusa 2018]. The objects parametrized by $T_{d,n}$ are known as stable rooted trees and are the union of surfaces $X \cong \mathrm{Bl}_m \mathbb{P}^2$, as in our space, but with markings given by points rather than lines.

Remark 6.1. We recall Kapranov's construction of $\overline{M}_{0,n}$ as a Chow quotient (see [Kapranov 1993a]). Given a collection of n generic points p_i in $\mathbb{P}^1$, we consider the cycle associated to the closure of the orbit: $\overline{\mathrm{SL}_3 \cdot (p_1, \ldots, p_n)} \subset (\mathbb{P}^1)^n$. Varying the points, we obtain cycles parametrized by an open locus in the appropriate Chow variety. Taking the closure of this open set, we obtain the Chow quotient $(\mathbb{P}^1)^n /\!\!/_{\mathrm{Ch}} \mathrm{SL}_2$ which is isomorphic to $\overline{M}_{0,n}$.

We fix our line l_A once and for all, and denote by $\hat{\mathbb{P}}^2$ the dual projective space. The lines $\{l_1, \ldots l_n\}$ are parametrized by points $p_1, \ldots, p_n \in (\hat{\mathbb{P}}^2)^n$. Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be the group acting on $\mathbb{P}^2$ that fixes the line $l_A \subset \mathbb{P}^2$ pointwise. Then $G \cong \mathbb{G}_m \rtimes \mathbb{G}_a^2$, $\dim(G) = 3$, and if $l_A := (x_0 = 0)$, the group consists of elements of the form

$$G = \begin{pmatrix} t^{-2} & 0 & 0 \\ s_0 & t & 0 \\ s_1 & 0 & t \end{pmatrix}.$$

Given a point $p_i = [a_0 : a_1 : a_2] \in \hat{\mathbb{P}}^2$, the line associated to it by projective duality can be written as $l(\vec{x}) := (p_i \cdot x = 0)$. Then we have $l(g \cdot x) = (p_i \cdot g)(x)$ from which we obtain the following action of G on $\hat{\mathbb{P}}^2$.

Definition 6.2. Let $g \in G$ be as above, then we define the action on $\hat{\mathbb{P}}^2$ as

$$g \cdot [a_0 : a_1 : a_2] := \left[t^{-3} a_0 + \frac{s_0}{t} a_1 + \frac{s_1}{t} a_2 : a_2 : a_3 \right].$$

After acting with the group, the line $l(x) = (a_0 x_0 + a_1 x_1 + a_2 x_2 = 0)$ becomes

$$\left(t^{-3} a_0 + \frac{s_0}{t} a_1 + \frac{s_1}{t} a_2 \right) x_0 + a_1 x_1 + a_2 x_2.$$

In particular, the intersection point $l(x) \cap (x_0 = 0)$ is invariant under the action of G.

Inside $(\hat{\mathbb{P}}^2)^n$, we define the loci

$$U(q) := \{(p_1, \ldots, p_n) \in (\hat{\mathbb{P}}^2)^n \mid l_i \cap l_A \text{ are fixed with equivalence class } q \in M_{0,n}\}.$$

Notice that $\dim(U_n(q)) = n$. We select once and for all a connected component of the closure of $U(q_n)$ and we denote it, by abuse of notation, as $\overline{U}(q_n)$. In particular, we fix an intersection $\{l_i \cap l_A\}$ once and for all for the rest of this chapter, so we omit it after here and just write $\overline{U}$.

Proposition 6.3. *The Chow quotient $\overline{U} /\!\!/_{\mathrm{Ch}} G$ is birational to R_{1^n}.*

Proof. By shrinking if necessary, we can find an open subset $U' \subset U$ contained in a G-invariant open locus in $(\hat{\mathbb{P}}^2)^n$, so that there is a natural map $\psi : U' \to R_{1^n}$. Furthermore, the G-action fixes the line l_A

pointwise, and thus fixes $l_i \cap l_A$. As a result, all configurations in the orbit $G \cdot l_i$ are isomorphic as line arrangements in $\mathbb{P}^2$, and thus are equivalent in R_{1^n}. Therefore, ψ is G-invariant and induces a morphism $\overline{\psi} : U'/G \to R_{1^n}$. This morphism is injective on an open set in R_{1^n}, because if generic $p, p' \in U'$ satisfy $\overline{\psi}(p) = \overline{\psi}(p')$, then there is a $g \in \mathrm{SL}(3, \mathbb{C})$ such that $g \cdot p = p'$. This last equality implies g fixes the line l_A as well as all of the intersections $l_i \cap l_A$ and so $g \in G$ and p and p' are in the same G-orbit. The map $\overline{\psi}$ is dominant, because for a generic isomorphic class of lines parametrized by R_{1^n}, we can choose a representative where l_A and $l_i \cap l_A$ are as in the beginning of this section, and that representative is parametrized by U'. □

Next, we show that the birational map $\rho : R_{1^n} \dashrightarrow \overline{U} /\!\!/_{\mathrm{Ch}} G$ is a regular morphism. This is done by associating a cycle to each sha X parametrized by $\overline{R}_{1^n}$. We recall that each component X_v of X is either $\mathbb{P}^2$, the blow up of $\mathbb{P}^2$ at finite number of points, or $\mathbb{P}^1 \times \mathbb{P}^1$ (see Proposition 4.8), and that there is a *contraction morphism* $\varphi_v : X \to \mathbb{P}^2$ that contracts X_v to $\mathbb{P}^2$ while also contracting all other components. For each $v \in I$, the contraction morphism induces a line arrangement $\varphi_v(X)$ defined up to choice of coordinates. We always select a representative which, by an abuse of notation, we denote by $\varphi_v(X)$, so $l_A := (x_0 = 0)$ and the points $l_A \cap l_i$ are the same as the ones used to define U.

Definition 6.4. Fix a closed point of R_{1^n} parametrizing the sha $X = \bigcup_{v \in I} X_v$. The *configuration cycle* $Z(X)$ is

$$Z(X) := \sum_{v \in I} \overline{G \cdot \varphi_v(X)} \subsetneq (\hat{\mathbb{P}}^2)^n.$$

We must show that these configuration cycles all have the same dimension and homology class. Let $\vec{m} := \{m_1, \dots, m_n\}$ be a set of integers such that $\sum_{i=1}^n m_i = 3$ and $0 \le m_i \le 2$. By the Künneth formula, a basis for the homology in $(\hat{\mathbb{P}}^2)^n$ is $[\hat{\mathbb{P}}^{m_1}] \otimes \cdots \otimes [\hat{\mathbb{P}}^{m_n}]$. Let $\mathbb{L}_{\vec{m}} := L_1 \times \cdots \times L_n$ be a collection of generic linear subspaces $L_i \subseteq \hat{\mathbb{P}}^2$ of codimension m_i. The homology class of the orbit $\overline{G \cdot p}$ is

$$[\bar{G} \cdot p] = \sum_{\vec{m}} c_{\vec{m}} ([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]),$$

where $(\overline{G \cdot p}) \cdot \mathbb{L}_{\vec{m}}$ is the intersection of the orbit $\overline{G \cdot p}$ with the generic linear subspaces $\mathbb{L}_{\vec{m}}$.

Proposition 6.5. *Let $\vec{m}$ be as above and $X = \bigcup_{v \in I} X_v$, then the homology class $[Z(X)]$ of the cycle $Z(X)$ is*

$$[Z(X)] := \sum_{\vec{m} = m_1, \dots, m_n} \left(\sum_{v \in I} \overline{G \cdot \varphi_v(X)} \cdot \mathbb{L}_{\vec{m}} \right) ([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]). \tag{6.5.1}$$

In particular, if X is a generic point of R_{1^n} (i.e., X is supported on a single $\mathbb{P}^2$). Then

$$[Z(X)] = \sum_{\vec{m}} c_{\vec{m}} ([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]),$$

where $c_{\vec{m}}$ is either 0 *or* 1.

Proof. The result follows verbatim from the analogous [Kapranov 1993a, Proposition 2.1.7]. The main idea is as follows: Let $p_i \in \hat{\mathbb{P}}^2$ be the points parametrizing the lines l_i in $\varphi_v(X)$. Then, $c_{\vec{m}} = 1$ if and only if there is a unique $g \in G \subset SL(3, \mathbb{C})$ such that $g \cdot p_i \in L_i$ for all $1 \leq i \leq n$ and $c_{\vec{m}}$ is zero if there is no such as $g \in G$. For generic X those are the only cases, so we only have those coefficients. □

It will turn out that we only need to calculate the homology of the cycles associated to the maximal degenerations parametrized by R_{1^n}.

Lemma 6.6. *A closed point $X = \bigcup_{v \in I} X_v$ in R_{1^n} is* maximally degenerate, *that is it lies on a minimal* (*i.e.*, *deepest*) *stratum of the boundary stratification*, *if and only if the configuration of lines $\varphi_v(X_v)$ has exactly three lines l_i with $1 \leq i \leq n$ in general position for every $v \in I$, not including l_A or its image.*

Proof. Recall that the group G is three dimensional. If $\varphi_v(X_v)$ has more than three lines, not including l_A or its image, in general position, then X_v has moduli larger than zero, and it can be degenerated further. □

Proposition 6.7. *If the sha $X \in R_{1^n}$ is maximally degenerated*, *then the homology class of $Z(X)$ has all coefficients $c_{\vec{m}}$ equal to* 1 *if and only if for all $m_i \in \vec{m}$ we have that $m_i \neq 2$.*

Proof. ($\Rightarrow$) We show this by proving the contrapositive. Suppose that there is an $m_i \in \vec{m}$ such that $m_i = 2$. Then we claim that for each component X_v of X, we have that $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}'} = 0$. Indeed, $m_i = 2$ implies that there is a generic linear subspace $L_i \in \mathbb{L}_{\vec{m}'}$ such that $L_i \cong \mathbb{P}^0_i \subset \hat{\mathbb{P}}^2$ is a point. By projective duality, we obtain a line $\mathbb{P}^1_i$ in $\mathbb{P}^2$ that has generic intersection with l_A. However, there does not exist a $g \in G$ such that $g \cdot l_i = \mathbb{P}^1_i$, because this would imply that both l_i and the $\mathbb{P}^1_i$ would intersect l_A at the same point. This is impossible given our action of G, because G restricts to the identity in l_A.

($\Leftarrow$) We divide the set of lines in $\varphi_v(X) \cong \mathbb{P}^2$ into sets $I_i(v)$ and $I_A(v)$, where $I_i(v)$ denotes the set of lines associated to the multiple points $p(I_i(v)) \in \varphi_v(X)$ (i.e., points of multiplicity ≥ 3), and the set $I_A(v)$ denotes the lines overlapping with l_A. By construction, $I_i(v) \cap I_A(v) = \varnothing$. However, the sets $I_i(v)$ are not necessarily disjoint, as lines can support more than one multiple point. Of course, if the configuration only has double points, then $I_i(v) = \varnothing$. We define the numbers $m_i(v) := \sum_{k \in I_i(v)} m_k$ and $m_A(v) := \sum_{k \in I_A(v)} m_k$. If $I_i(v) = \varnothing$, then we take $m_i(v) := 0$, and similarly for $I_A(v)$. We make the following claim.

Claim 6.8. $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} > 0 \iff m_A(v) = 0$, $m_i(v) \leq 2$, *and* $m_k \leq 1$ *for all i and $m_k \in \vec{m}$.*

Proof of Claim 6.8.

($\Rightarrow$) If $m_A(v) > 0$, then we have a generic line $L_i \subset \hat{\mathbb{P}}^2$ with $i \in I_A(v)$, and thus a generic point $\mathbb{P}^0_i \subset \mathbb{P}^2$ in the dual space. We must find a $g \in G$ such that $\mathbb{P}^0 \in g(l_i)$ for a line l_i that overlaps with l_A. This is impossible, because G does not move l_A, and so $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

Next, suppose that $m_i(v) = 3$. By the previous argument, we know that if $m_i = 2$, then $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$. Then up to relabeling, we can assume that $m_1 = m_2 = m_3 = 1$ and that $\{1, 2, 3\} \subset I_1(v)$. The generic lines L_1, L_2 and L_3 in $\hat{\mathbb{P}}^2$ induce three generic points $\mathbb{P}^0_s$ in $\mathbb{P}^2$. We need to find a $g \in G$ such that the points $\mathbb{P}^0_s \in g \cdot l_s$ for $s \in 1, 2, 3$. Again, this is impossible by the geometry of the problem. Indeed, recall that the intersection points of the lines $l_s \cap l_A$ are fixed. We can find two lines passing through $\mathbb{P}^0_1$ and $\mathbb{P}^0_2$, but those two lines will intersect at $p(I_1)$, and thus determine the position of all the other lines in $I_1(v)$. Therefore, a generic $\mathbb{P}^0_3$ will not be contained in $g \cdot l_3$, and therefore $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

($\Leftarrow$) There are three L_s of codimension one, and we can suppose that $s \in \{1, 2, 3\}$. By duality, they induce three points in general position in $\mathbb{P}^2$. The statement follows because we can find three lines that pass through these three points as along as the lines are in general position. This holds, because $m_i(v) \leq 2$ implies that $\{1, 2, 3\}$ is not a subset of $I_i(v)$ for any i. □

By (6.5.1) in Proposition 6.5, our statement follows if we prove that for a given $\vec{m}$, and any sha $X = \bigcup_v X_v$ parametrized by R_{1^n}, there exists a unique component X_v satisfying the criteria of Claim 6.8. The following argument uses the description of the dual graph of the X, which is a rooted tree by Proposition 4.8. We start with the root component X_0. There is no line coinciding with l_A in $\varphi_0(X)$, and so $m_A(0) = 0$. Thus there are two options,

(1) either $m_i(0) \leq 2$ for all i, or

(2) there exists an i such that $m_i(0) = 3$.

Case 1: If $m_i(0) \leq 2$ for all i, then $\varphi_0(X) \cdot \mathbb{L} > 0$. To show uniqueness, recall that $\sum_{i=1,2,3} m_i = 3$, and that $m_i(v) \leq 2$ for all i. Therefore, the root has at least two branches, and each of those branches has at least one index i_0 such that $m_{i_0} = 1$. Then $I_A(v)$ contains at least one of these indices for every other component $v \neq 0$, because at least one those branches is contracted with its line i_0 that overlaps with l_A. Therefore, $m_A(v) > 0$, and thus $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

Case 2: If there exists an i such that $m_i(0) = 3$, then $\varphi_0(X) \cdot \mathbb{L}_{\vec{m}} = 0$. Thus we may suppose after relabeling, that $m_1(0) = 3$, and that $m_1 = m_2 = m_3 = 1$ with $\{1, 2, 3\} \subset I_1(0)$. This means that there is a branch starting from the root which contains the lines $\{1, 2, 3\}$. Let $X_{v'}$ be the component in that branch that intersects with the rooted component. We claim that $m_A(v') = 0$, because $I_A(v')$ denotes the set of lines in the other branches which are not in I_1. Those indices do not include $\{1, 2, 3\}$, and these indices are the only ones of weight one. Thus, we have two options:

(1) If $m_i(v') \leq 2$ for every i, then we have that $\varphi_{v'}(X) \cdot \mathbb{L}_{\vec{m}} > 0$. Uniqueness follows by same argument used above. There are at least two branches starting from v' with an index j such that $m_j = 1$. Any other $\varphi_v(X)$ will contain that index in $I_A(v)$, and so $\varphi_v(X) \cdot \mathbb{L}_{\vec{m}} = 0$.

(2) If $m_i(v') = 3$ for some i, then there is a branch starting from the vertex v' that contains the lines $\{1, 2, 3\}$.

In the last case, we repeat the above argument with the surface $X_{v''}$ that intersects v' and belongs to the branch containing the lines with indices $\{1, 2, 3\}$. Since for any sha the tree is finite, one of the next two things must happen:

(1) We find a component $\hat{v}$ such that $\varphi_{\hat{v}}(X) \cdot \mathbb{L}_{\vec{m}} > 0$. It is unique by above arguments.

(2) We arrive to the last vertex of a branch that we call v_f.

In the last case, we have at most three lines in general position on X_{v_f}, because by assumption X is maximally degenerated and there are no multiple points. Following our labeling, those lines are precisely $\{1, 2, 3\}$ and so $m_A(v_f) = m_i(v_f) = 0$ and $\varphi_{v_f}(X) \cdot \mathbb{L}_{\vec{m}} > 0$. □

Next, we extend the birational map $\rho : R_{1^n} \dashrightarrow \overline{U} /\!\!/_{\mathrm{Ch}} G$ to a regular morphism. Note that there exists at most one extension, since the image is dense and the Chow variety is separated. Furthermore, the image of an extension as above is contained in $\overline{U} /\!\!/_{\mathrm{Ch}} G$, since this Chow quotient is closed in the Chow variety. We begin with a crucial lemma.

Definition 6.9 [Giansiracusa and Gillam 2014, Definition 7.2]. Let $(A, \mathfrak{m})$ be a DVR with residue field k and fraction field K and let Y be a proper scheme. By the valuative criterion, any map $g : \operatorname{Spec} K \to Y$ extends to a map $g : \operatorname{Spec} A \to Y$. We write $\lim g$ for the point $g(\mathfrak{m}) \in Y$.

Lemma 6.10 [Giansiracusa and Gillam 2014, Theorem 7.3]. *Suppose X_1 and X_2 are proper schemes over a noetherian scheme S with X_1 normal. Let $U \subset X_1$ be an open dense set and $f : U \to X_2$ an S-morphism. Then f extends to an S-morphism $\hat{f} : X_1 \to X_2$ if and only if for any DVR $(K, \mathfrak{m})$ and any morphism $g : \operatorname{Spec}(K) \to U$, the point* $\lim fg$ *of X_2 is uniquely determined by the point* $\lim g$ *of X_1.*

Our argument for the following result follows the same structure as the one used for the proof of $\overline{M}_{0,n}$ [Giansiracusa 2013, Theorem 1.1], and $T_{d,n}$ [Gallardo and Giansiracusa 2018, §4.3].

Proposition 6.11. *There is a morphism $\rho : R_{1^n} \to (\hat{\mathbb{P}}^2)^n /\!\!/_{\mathrm{Ch}} G$ that associates to each closed point $X = \bigcup_{v \in I} X_v$ of R_{1^n} a cycle with homology class*

$$\sum_{\vec{m}:=(m_1,\ldots,m_n)} ([\mathbb{P}^{m_1}] \otimes \cdots \otimes [\mathbb{P}^{m_n}]) \quad \textit{for } 0 \le m_i \le 1,\ \sum_{i=1}^{n} m_i = 3.$$

Proof. Consider a flat proper 1-parameter family $X_\Delta \to \Delta$ where the generic fiber X_t is a sha parametrized by the interior $R^\circ_{1^n}$. Then X_t is supported in $\mathbb{P}^2$ without any multiple point of multiplicity larger than two, and the central fiber $X_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is an arbitrary closed point of R_{1^n}. The cycle $[Z(X_t)]$ associated to a generic fiber in X_t is three dimensional, and its homology class is δ (see Proposition 6.5). Therefore, we have a 1-parameter family of cycles whose limit in the Chow variety we denote as $\lim_{t\to 0}[Z(X_t)]$. By Proposition 6.3 and Lemma 6.10, the existence of the morphism then follows if we show that $\lim_{t\to 0}[Z(X_t)]$ is uniquely determined by $X_{\mathbb{C}}$. It suffices to show that

$$\lim_{t\to 0}[Z(X_t)] = [Z(X_{\mathbb{C}})], \tag{6.11.1}$$

where $[Z(X_{\mathbb{C}})]$ is equal to the cycle defined in Proposition 6.5.

First we show that $Z(X_{\mathbb{C}}) \subseteq \lim_{t\to 0} Z(X_t)$ as subvarieties of $(\hat{\mathbb{P}}^2)^n$. Since $X_{\mathbb{C}} = \bigcup_v X_v$, by definition of $Z(X_{\mathbb{C}})$, our claim follows if for every component X_v of $X_{\mathbb{C}}$, we have that

$$\varphi_v(X_{\mathbb{C}}) \subset \lim_{t\to 0} Z(X_t) \subset (\hat{\mathbb{P}}^2)^n.$$

By construction $\lim_{t\to 0} Z(X_t)$ is closed and G-invariant. Therefore, our claim follows if φ_v maps the points $(p_{1_0}, \dots p_{n_0}) \in (\hat{\mathbb{P}}^2)^n$ associated to the lines in $\varphi_v(X_{\mathbb{C}})$, into

$$\lim_{t\to 0} Z(X_t) \subset (\hat{\mathbb{P}}^2)^n.$$

We recall that in general for shas, the contraction morphism $\varphi_v : X_{\mathbb{C}} \to \mathbb{P}^2$ is induced by a line bundle L_v that satisfies $h^i(X, L_v) = 0$ for all $1 \geq i$, since φ_v is degree 1 on the X_v component and degree 0 elsewhere. Then, by Grauert's theorem [Hartshorne 1977, Corollary III.12.9], the morphism φ_v lifts to a morphism from the central fiber to our 1-parameter family X_Δ. Let $\varphi_v : X_\Delta \to (\hat{\mathbb{P}}^2)^n$ be that lift. For $t \neq 0$, the map φ_v sends the points $p_{i_t} \in (\hat{\mathbb{P}}^2)^n$ associated to the lines in $\varphi_v(X_t)$ to $Z(X_t)$, and the morphism φ_v is continuous. Then, $\varphi_v(X_{\mathbb{C}}) \subset \lim_{t\to 0} Z(X_t)$ and we have

$$[Z(X_{\mathbb{C}})] \leq \lim_{t\to 0}[Z(X_t)]. \tag{6.11.2}$$

Next, we show the equality. By Proposition 6.5, we know that the homology class of the generic orbit has coefficients equal to either 0 or 1. By the argument in the proof of Proposition 6.7, we conclude that the homology class of the generic orbit has coefficient $c_{\vec{m}} = 0$ if there is an $m_i \in \vec{m}$ such that $m_i = 2$. Indeed, it will induce a generic line $\mathbb{P}^1_i \subset \mathbb{P}^2$ and we cannot move any lines l_i to such a line because the intersections $l_i \cap l_A$ are fixed. On the other hand, for $t_0 \neq 0$ we see that

$$\lim_{t\to 0}[Z(X_t)] = [Z(X_{t_0})], \tag{6.11.3}$$

because we are taking the limit inside a Chow variety. Consequently, the homology class of the limit is the same as the homology class of the generic fiber.

Expressions (6.11.2) and (6.11.3) imply that the coefficients $c^{\text{gen}}_{\vec{m}}$ in the homology class of the generic element $Z(X_{t_0})$ are necessarily larger than or equal to the coefficients $c^0_{\vec{m}}$ associated to the central fiber $Z(X_{\mathbb{C}})$. Therefore we have the following inequality:

$$1 \leq c^0_{\vec{m}} \leq c^{\text{gen}}_{\vec{m}} \leq 1. \tag{6.11.4}$$

The left inequality follows by Proposition 6.7 and because the homology class only decreases whenever degenerating, as seen in (6.11.2). The right inequality follows from Proposition 6.5. We conclude that there is a morphism $\rho : R_{1^n} \to (\hat{\mathbb{P}}^2)^n /\!\!/_{\text{Ch}} G$. □

Finally, we prove that R_{1^n} is isomorphic to the normalization of our Chow quotient.

Theorem 6.12. *Let $\overline{U}^n /\!\!/_{\text{Ch}} G$ be the normalization of the Chow quotient, and let ρ^n be the morphism obtained from the Stein factorization of ρ. Then the morphism*

$$\rho^n : R_{1^n} \to \overline{U}^n /\!\!/_{\text{Ch}} G$$

is an isomorphism.

Proof. We use the Zariski's main theorem which asserts that a quasifinite birational morphism to a normal, Noetherian scheme is an open immersion. R_{1^n} is normal and our morphism ρ factors through the normalization of the Chow quotient. Then, ρ^n is surjective and birational and the crux of the result is to prove that ρ is quasifinite. By Proposition 6.3, we already know the map ρ is injective on the interior $R^{\circ}_{1^n}$ and we observe that no point of the boundary divisor in R_{1^n} can be sent to the same cycle as a point of the open stratum, since the image of the latter is an irreducible cycle whereas the image of the former is not. Therefore, we only need to show that the restriction of ρ to the boundary in R_{1^n} is quasifinite. The boundary is the union of a finite number of divisors, and so it will be enough to show our claim for a single component D_I of the boundary. The general point of the divisor D_I parametrizes a sha $X = \mathbb{P}^2 \cup \mathrm{Bl}_x(\mathbb{P}^2)$, where $\mathrm{Bl}_x(\mathbb{P}^2)$ contains the line l_A. For example, the second sha in Figure 1 is parametrized by D_{2345}. The morphism ρ sends X to the union of the two cycles

$$\overline{G \cdot \varphi_0(X)} \cup \overline{G \cdot \varphi_1(X)}.$$

If another sha $\tilde{X}$ parametrized by the interior of D_I has the same image as X, that is $\rho(X) = \rho(\tilde{X})$, then their cycles coincide. This means that the image of their reduction morphisms satisfy $\varphi_i(\tilde{X}) \in G \cdot \varphi_i(X)$. However, $G \subset SL(3, \mathbb{C})$, which implies that $X \cong \tilde{X}$. Therefore, ρ is injective on the interior of D_I. A straightforward iteration of this argument, using the fact that our dual graphs are always trees, applies to the deeper strata and shows that ρ is injective on D_I itself. □

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Parabolic induction and extensions

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Let G be a p-adic reductive group. We determine the extensions between admissible smooth mod p representations of G parabolically induced from supersingular representations of Levi subgroups of G, in terms of extensions between representations of Levi subgroups of G and parabolic induction. This proves for the most part a conjecture formulated by the author in a previous article and gives some strong evidence for the remaining part. In order to do so, we use the derived functors of the left and right adjoints of the parabolic induction functor, both related to Emerton's δ-functor of derived ordinary parts. We compute the latter on parabolically induced representations of G by pushing to their limits the methods initiated and expanded by the author in previous articles.

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1. Introduction

The study of representations of a p-adic reductive group G over a field of characteristic p has a strong motivation in the search for a possible mod p Langlands correspondence for G. Recently, Abe, Henniart, Herzig and Vignéras [Abe et al. 2017a] gave a complete classification of the irreducible admissible smooth representations of G over an algebraically closed field of characteristic p in terms of supersingular representations of the Levi subgroups of G and parabolic induction, generalising the results of Barthel and Livné [1994] for GL_2, Herzig [2011] for GL_n and Abe [2013] for a split G.

Two major difficulties come into play when trying to extend the mod p Langlands correspondence beyond $\mathrm{GL}_2(\mathbb{Q}_p)$. First, the supersingular representations of G remain completely unknown, except for some reductive groups of relative semisimple rank 1 over $\mathbb{Q}_p$ [Abdellatif 2014; Cheng 2013; Kozioł 2016] using the classification of Breuil [2003] for $\mathrm{GL}_2(\mathbb{Q}_p)$. Second, it is expected that such a correspondence would involve representations of G with many irreducible constituents; see, e.g., [Breuil and Herzig 2015]. This phenomenon already appears for $\mathrm{GL}_2(\mathbb{Q}_p)$ when the Galois representation is an extension between two characters, in which case the corresponding representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is an extension between two principal series [Colmez 2010]. This raises the question of the extensions between representations of G.

In this article, we intend to compute the extensions between admissible smooth mod p representations of G parabolically induced from supersingular representations of Levi subgroups of G, in terms of extensions between representations of Levi subgroups of G and parabolic induction.

In order to do so, we use the derived functors of the left and right adjoints of the parabolic induction functor, namely the Jacquet functor and the ordinary parts functor [Emerton 2010a], both related to Emerton's δ-functor of derived ordinary parts [Emerton 2010b]. We compute the latter on parabolically induced representations of G by pushing to their limits the methods initiated in [Hauseux 2016a] and expanded in [Hauseux 2016b].

These computations have also been used to study the deformations of parabolically induced admissible smooth mod p representations of G in a joint work with T. Schmidt and C. Sorensen [Hauseux et al. 2016].

Presentation of the main results. We let $F/\mathbb{Q}_p$ and $k/\mathbb{F}_p$ be finite extensions. We fix a connected reductive algebraic F-group $\boldsymbol{G}$, a minimal parabolic subgroup $\boldsymbol{B} \subseteq \boldsymbol{G}$ and a maximal split torus $\boldsymbol{S} \subseteq \boldsymbol{B}$. We write the corresponding groups of F-points G, B, S, etc. We let Δ denote the set of simple roots of $\boldsymbol{S}$ in $\boldsymbol{B}$. To each $\alpha \in \Delta$ there corresponds a simple reflection s_α and a root subgroup $\boldsymbol{U}_\alpha \subset \boldsymbol{B}$. We put $\Delta^1 := \{\alpha \in \Delta \mid \dim_F \boldsymbol{U}_\alpha = 1\}$.

Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup. We write $\Delta_{\boldsymbol{L}} \subseteq \Delta$ for the corresponding subset and we put $\Delta_{\boldsymbol{L}}^{\perp} := \{\alpha \in \Delta \mid \langle \alpha, \beta^\vee \rangle = 0 \ \forall \beta \in \Delta_{\boldsymbol{L}}\}$ and $\Delta_{\boldsymbol{L}}^{\perp,1} := \Delta_{\boldsymbol{L}}^{\perp} \cap \Delta^1$. For $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$, conjugation by (any representative of) s_α stabilises $\boldsymbol{L}$ and α extends to an algebraic character of $\boldsymbol{L}$; see the proof of Lemma 5.1.4.

We let $\boldsymbol{P}^-$ denote the opposite parabolic subgroup. Recall the parabolic induction functor $\mathrm{Ind}_{P^-}^G$ from the category of admissible smooth representations of L over k to the category of admissible smooth representations of G over k, which is k-linear, fully faithful and exact [Emerton 2010a]. In particular, it induces a k-linear injection $\mathrm{Ext}^1_L \hookrightarrow \mathrm{Ext}^1_G$.

Let σ be an admissible smooth representation of L over k. For $\alpha \in \Delta_L^{\perp,1}$, we consider the admissible smooth representation $\sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ of L over k where σ^α is the s_α-conjugate of σ and $\omega : F^\times \to \mathbb{F}_p^\times \subseteq k^\times$ is the mod p cyclotomic character. We say that σ is *supersingular* if it is absolutely irreducible and $\overline{\mathbb{F}}_p \otimes_k \sigma$ is supersingular [Abe et al. 2017a].

In cases (iii) and (iv) of the following conjecture, "otherwise" means that the conditions of case (ii) are not all satisfied.

Conjecture 1.1 [Hauseux 2016b, Conjecture 3.17]. *Assume $\boldsymbol{G}$ split with connected centre and simply connected derived subgroup. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}' = \boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups and σ, σ' be supersingular representations of L, L' over k, respectively. Assume $\mathrm{Ind}_{P^-}^G \sigma$ and $\mathrm{Ind}_{P'^-}^G \sigma'$ irreducible or $p \neq 2$.*

(i) *If $\boldsymbol{P}' \not\subseteq \boldsymbol{P}$ and $\boldsymbol{P} \not\subseteq \boldsymbol{P}'$, then $\mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big) = 0$.*

(ii) *If $F = \mathbb{Q}_p$, $\boldsymbol{P}' = \boldsymbol{P}$ and $\sigma' \cong \sigma^\alpha \otimes \big(\omega^{-1} \circ \alpha\big) \ncong \sigma$ for some $\alpha \in \Delta_L^\perp$, then*

$$\dim_k \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big) = 1.$$

(iii) *Otherwise if $\boldsymbol{P}' \subseteq \boldsymbol{P}$, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}^1_L\big(\mathrm{Ind}_{L\cap P'^-}^L \sigma', \sigma\big) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big).$$

(iv) *Otherwise if $\boldsymbol{P} \subseteq \boldsymbol{P}'$, then the functor $\mathrm{Ind}_{P'^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}^1_{L'}\big(\sigma', \mathrm{Ind}_{L'\cap P^-}^{L'} \sigma\big) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big).$$

We prove cases (ii), (iii), and (iv) of this conjecture and give some strong evidence for case (i). We actually work without any assumption on $\boldsymbol{G}$ and our results hold true for broader classes of representations; see Section 5.2 for more precise statements. We also prove similar results for unitary continuous p-adic representations; see Section 5.3.

We treat the cases $F = \mathbb{Q}_p$ and $F \neq \mathbb{Q}_p$ separately. They are in fact the degree 1 case of a more general (but conditional to a conjecture of Emerton) result on the k-linear morphism $\mathrm{Ext}^n_L \to \mathrm{Ext}^n_G$ induced by $\mathrm{Ind}_{P^-}^G$ in all degrees $n \leq [F : \mathbb{Q}_p]$; see Remark 5.2.6.

Theorem 1.2 (Theorem 5.2.2). *Assume $F = \mathbb{Q}_p$. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}' = \boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups and σ, σ' be supersingular representations of L, L' over k, respectively.*

(i) *If $\boldsymbol{P}' = \boldsymbol{P}$ and $\sigma' \ncong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ for all $\alpha \in \Delta_L^{\perp,1}$, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}^1_L\big(\sigma', \sigma\big) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big).$$

(ii) *If $\boldsymbol{P}' \subsetneq \boldsymbol{P}$, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}^1_L\big(\mathrm{Ind}_{L\cap P'^-}^L \sigma', \sigma\big) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big).$$

(iii) *If $\boldsymbol{P} \subsetneqq \boldsymbol{P}'$, then the functor* $\operatorname{Ind}_{P'^-}^G$ *induces a k-linear isomorphism*

$$\operatorname{Ext}^1_{L'}\big(\sigma', \operatorname{Ind}_{L'\cap P^-}^{L'} \sigma\big) \xrightarrow{\sim} \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P'^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big).$$

If $\boldsymbol{P}' = \boldsymbol{P}$, we do not know the exact dimension of the cokernel of the k-linear injection $\operatorname{Ext}^1_L(\sigma', \sigma) \hookrightarrow \operatorname{Ext}^1_G(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma)$ induced by $\operatorname{Ind}_{P^-}^G$ in general, but we prove that it is at most $\operatorname{card}\{\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1} \mid \sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)\}$; see Remark 5.2.3 for more details. Further, we compute it when $\boldsymbol{G}$ is split with connected centre; see Theorem 1.4 below. Note that in cases (ii) and (iii), the source of the isomorphism can be nonzero [Hu 2017].

Theorem 1.3 (Theorem 5.2.4). *Assume $F \neq \mathbb{Q}_p$. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup. The functor* $\operatorname{Ind}_{P^-}^G$ *induces a k-linear isomorphism*

$$\operatorname{Ext}^1_L\big(\sigma', \sigma\big) \xrightarrow{\sim} \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big)$$

for all admissible smooth representations σ, σ' of L over k.

In particular, Theorem 1.2(ii) and (iii) hold true for any admissible smooth representations σ, σ' of L, L' respectively over k when $F \neq \mathbb{Q}_p$; see Corollary 5.2.5.

We complete Theorem 1.2(i) when $\boldsymbol{G}$ is split with connected centre (see also Remark 5.2.8 for a more general, but conditional to a conjecture of Emerton, result on the k-linear morphism $\operatorname{Ext}_L^{[F:\mathbb{Q}_p]} \to \operatorname{Ext}_G^{[F:\mathbb{Q}_p]}$ induced by $\operatorname{Ind}_{P^-}^G$).

Theorem 1.4 (Theorem 5.2.7). *Assume $F = \mathbb{Q}_p$ and $\boldsymbol{G}$ split with connected centre. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup and σ, σ' be supersingular representations of L over k.*

(i) *If $\sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha) \not\cong \sigma$ for some $\alpha \in \Delta_{\boldsymbol{L}}^{\perp}$, then $\operatorname{Ext}^1_L(\sigma', \sigma) = 0$ and*

$$\dim_k \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) = 1.$$

(ii) *If either $\sigma' \cong \sigma$ and $p \neq 2$, or $\sigma' \not\cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ for any $\alpha \in \Delta_{\boldsymbol{L}}^{\perp}$, then the functor* $\operatorname{Ind}_{P^-}^G$ *induces a k-linear isomorphism*

$$\operatorname{Ext}^1_L\big(\sigma', \sigma\big) \xrightarrow{\sim} \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big).$$

(iii) *If $p = 2$, then the functor* $\operatorname{Ind}_{P^-}^G$ *induces a k-linear injection*

$$\operatorname{Ext}^1_L\big(\sigma', \sigma\big) \hookrightarrow \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big)$$

whose cokernel is of dimension $\operatorname{card}\{\alpha \in \Delta_{\boldsymbol{L}}^{\perp} \mid \sigma' \cong \sigma^\alpha\}$.

Finally, we treat the case where there is no inclusion between the two parabolic subgroups, assuming a special case of Conjecture 1.7 below; see also Remark 3.3.6.

Proposition 1.5 (Proposition 5.2.1). *Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}' = \boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups and σ, σ' be supersingular representations of L, L' over k, respectively. Assume Conjecture 1.7 is true for $A = k$, $n = 1$ and ${}^I w^J = 1$. If $\boldsymbol{P}' \not\subseteq \boldsymbol{P}$ and $\boldsymbol{P} \not\subseteq \boldsymbol{P}'$, then*

$$\operatorname{Ext}^1_G\big(\operatorname{Ind}_{P'^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) = 0.$$

As a consequence, Conjecture 1.1 is true under the same assumption when $\boldsymbol{G}$ is split with connected centre (without assuming the derived subgroup of $\boldsymbol{G}$ simply connected).

Corollary 1.6 (Corollary 5.2.9). *Assume $\boldsymbol{G}$ split with connected centre. If Conjecture 1.7 is true for $A = k$, $n = 1$ and ${}^{I}w^{J} = 1$, then Conjecture 1.1 is true.*

Strategy of proof and methods used. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field k. We work more generally with smooth representations over an Artinian local $\mathcal{O}$-algebra A with residue field k.

The main tools to compute extensions between parabolically induced representations are two exact sequences related to Emerton's δ-functor of derived ordinary parts (see below (1) which is due to Emerton and (2) which is a new feature of this article).

Using these, most of the previous results reduce to computing the derived ordinary parts of parabolically induced representations. We formulate a conjecture on these computations (see Conjecture 1.7 below). We prove it in low degree (see Theorem 1.8 below) and give some strong evidence for it in general.

We proceed in two steps: first we construct filtrations of parabolically induced representations related to the Bruhat decomposition; second we partially compute the derived ordinary parts of the associated graded representations using some dévissages.

Derived ordinary parts and extensions. Let $\boldsymbol{P} \subseteq \boldsymbol{G}$ be a parabolic subgroup and $\boldsymbol{L} \subseteq \boldsymbol{P}$ be a Levi factor. We let $\boldsymbol{P}^{-} \subseteq \boldsymbol{G}$ denote the parabolic subgroup opposed to $\boldsymbol{P}$ with respect to $\boldsymbol{L}$. Emerton [2010a; 2010b] constructed a cohomological δ-functor $\mathrm{H}^{\bullet}\mathrm{Ord}_P$ from the category of admissible smooth representations of G over A to the category of admissible smooth representations of L over A, which is the right adjoint functor Ord_P of $\mathrm{Ind}_{P^-}^{G}$ in degree 0. From this, he derived a natural exact sequence of A-modules

$$0 \to \mathrm{Ext}^1_L\left(\sigma, \mathrm{Ord}_P\,\pi\right) \to \mathrm{Ext}^1_G\left(\mathrm{Ind}_{P^-}^{G}\,\sigma, \pi\right) \to \mathrm{Hom}_L\left(\sigma, \mathrm{H}^1\mathrm{Ord}_P\,\pi\right) \tag{1}$$

for all admissible smooth representations σ and π of L and G respectively over A.

In Section 4.2, we construct a second exact sequence in which parabolic induction is on the right. The construction is somewhat dual to that of (1) but not exactly (see Remark 4.2.1(ii)). We let d denote the integer $\dim_F \boldsymbol{N}$ and δ denote the algebraic character of the adjoint representation of $\boldsymbol{L}$ on $\det_F(\mathrm{Lie}\,\boldsymbol{N})$. The key fact is that the A-linear functors

$$\mathrm{H}_{\bullet}\left(N, -\right) := \mathrm{H}^{[F:\mathbb{Q}_p]d-\bullet}\mathrm{Ord}_P \otimes (\omega \circ \delta)\,.$$

form a homological δ-functor from the category of admissible smooth representations of G over A to the category of admissible smooth representations of L over A, which is isomorphic to the left adjoint functor $(-)_N$ of Ind_P^G in degree 0 (hence the notation). From this and using a result of Oort [1964] to compute extensions using pro-categories (see Section 4.1), we derive a natural exact sequence of A-modules

$$0 \to \mathrm{Ext}^1_L\left(\pi_N, \sigma\right) \to \mathrm{Ext}^1_G\left(\pi, \mathrm{Ind}_P^G\,\sigma\right) \to \mathrm{Hom}_L\left(\mathrm{H}_1\left(N, \pi\right), \sigma\right) \tag{2}$$

for all admissible smooth representations π and σ of G and L respectively over A.

Computation of derived ordinary parts. We let W be the Weyl group of $(\boldsymbol{G}, \boldsymbol{S})$. For $I \subseteq \Delta$, we write $\boldsymbol{P}_I = \boldsymbol{L}_I \boldsymbol{N}_I$ for the corresponding standard parabolic subgroup, $\boldsymbol{B}_I \subseteq \boldsymbol{L}_I$ for the minimal parabolic subgroup $\boldsymbol{B} \cap \boldsymbol{L}_I$ and $W_I \subseteq W$ for the subgroup generated by $(s_\alpha)_{\alpha \in I}$.

Let $I, J \subseteq \Delta$, σ be a locally admissible smooth representation of L_I over A and $n \in \mathbb{N}$. We intend to compute the smooth representation of L_J over A

$$\mathrm{H}^n \mathrm{Ord}_{P_J}\big(\mathrm{Ind}_{P_I^-}^G \sigma\big).$$

In Section 2.2, we use the generalised Bruhat decomposition

$$G = \bigsqcup_{{}^I w^J \in {}^I W^J} P_I^- \, {}^I w^J P_J$$

where ${}^I W^J$ is the system of representatives of minimal length of the double cosets $W_I \backslash W / W_J$ (see Section 2.1) to define a natural filtration $\mathrm{Fil}_{P_J}^\bullet(\mathrm{Ind}_{P_I^-}^G \sigma)$ of $\mathrm{Ind}_{P_I^-}^G \sigma$ by $A[P_J]$-submodules indexed by ${}^I W^J$ (with the Bruhat order). We also adapt the notion of graded representation associated with such a filtration (in particular, the grading has values in ${}^I W^J$) and we prove that for all ${}^I w^J \in {}^I W^J$, there is a natural $A[P_J]$-linear isomorphism

$$\mathrm{Gr}_{P_J}^{{}^I w^J}\big(\mathrm{Ind}_{P_I^-}^G \sigma\big) \cong \text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J} \sigma.$$

We prove that $\mathrm{Fil}_{P_J}^\bullet(\mathrm{Ind}_{P_I^-}^G \sigma)$ induces a filtration of $\mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)$ by $A[L_J]$-submodules indexed by ${}^I W^J$ (see Proposition 3.3.1).

Finally, we attach to each ${}^I w^J \in {}^I W^J$ an integer $d_{{}^I w^J}$ and an algebraic character $\delta_{{}^I w^J}$ of $\boldsymbol{L}_{J \cap {}^I w^{J-1}(I)}$ (see Notation 2.3.3 and Remark 2.3.4), and we formulate the following conjecture.

Conjecture 1.7 (Conjecture 3.3.4). *Let σ be a locally admissible smooth representation of L_I over A, ${}^I w^J \in {}^I W^J$ and $n \in \mathbb{N}$. There is a natural $A[L_J]$-linear isomorphism*

$$\mathrm{H}^n \mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J} \sigma\big) \cong \mathrm{Ind}_{L_J \cap P_{J \cap {}^I w^{J-1}(I)}^-}^{L_J} \big((\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma)^{{}^I w^J} \otimes (\omega^{-1} \circ \delta_{{}^I w^J}) \big).$$

We give some strong evidence for this conjecture (see Theorem 3.3.3): we prove that these two representations have natural filtrations by $A[B_J]$-submodules indexed by ${}^{J \cap {}^I w^{J-1}(I)} W_J$ (the system of representatives of minimal length of the right cosets $W_{J \cap {}^I w^{J-1}(I)} \backslash W_J$) such that the associated graded representations are naturally isomorphic; see the subsection below.

We prove this conjecture in several cases (see Proposition 3.3.5): whenever the right-hand side is either zero or a trivially induced representation, in which cases the aforementioned filtrations of both sides are trivial; when $n = 0$, in which case we deduce the result from the computation of $\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)$ in [Abe et al. 2017b]. This allows us to compute $\mathrm{H}^\bullet \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)$ in low degree when there is an inclusion between I and J; see Proposition 3.3.7. In particular, we obtain the following result in the case $I = J$.

Theorem 1.8 (Corollary 3.3.8). *Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup and σ be a locally admissible smooth representation of L over A.*

(i) *For all $n \in \mathbb{N}$ such that $0 < n < [F : \mathbb{Q}_p]$, we have $\mathrm{H}^n \mathrm{Ord}_P(\mathrm{Ind}_{P^-}^G \sigma) = 0$.*

(ii) *If* $\mathrm{Ord}_{L\cap s_\alpha P s_\alpha^{-1}}\,\sigma = 0$ *for all* $\alpha \in \Delta^1 \setminus (\Delta_{\boldsymbol{L}} \cup \Delta_{\boldsymbol{L}}^{\perp})$, *then there is a natural $A[L]$-linear isomorphism*

$$\mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_P\big(\mathrm{Ind}_{P^-}^G\,\sigma\big) \cong \bigoplus_{\alpha\in\Delta_L^{\perp,1}} \sigma^\alpha \otimes \big(\omega^{-1}\circ\alpha\big).$$

Note that for all $\alpha \in \Delta \setminus \Delta_{\boldsymbol{L}}$, $\boldsymbol{L}\cap s_\alpha \boldsymbol{P} s_\alpha^{-1}$ is the standard parabolic subgroup of $\boldsymbol{L}$ corresponding to $\Delta_{\boldsymbol{L}} \cap s_\alpha(\Delta_{\boldsymbol{L}})$ and it is proper if and only if $\alpha \notin \Delta_{\boldsymbol{L}}^{\perp}$. In particular, the condition in (ii) is satisfied when σ is supersingular.

In Section 4.3, we adapt the previous results in order to partially compute $\mathrm{H}_n(N_J, \mathrm{Ind}_{P_I}^G\,\sigma)$. We obtain an analogue of Theorem 1.8; see Corollary 4.3.4.

Filtrations and dévissages. Let ${}^Iw^J \in {}^IW^J$. We explain the partial computation of the smooth representation of L_J over A

$$\mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw^J P_J}\sigma\big).$$

In Section 2.2, we use again the Bruhat decomposition to construct a natural filtration $\mathrm{Fil}_B^\bullet\big(\text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw^J P_J}\sigma\big)$ by $A[B]$-submodules indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$, and we prove that for all $w_J \in {}^{J\cap {}^Iw^{J-1}(I)}W_J$ there is a natural $A[B]$-linear isomorphism

$$\mathrm{Gr}_B^{w_J}\big(\text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw^J P_J}\sigma\big) \cong \text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw^J w_J B}\sigma.$$

We prove that $\mathrm{Fil}_B^\bullet\big(\text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw^J P_J}\sigma\big)$ induces a filtration of $\mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw^J P_J}\sigma\big)$ by $A[B_J]$-submodules indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$ (see Proposition 3.3.2). Likewise, we construct a natural filtration

$$\mathrm{Fil}_{B_J}^\bullet\big(\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J}\tilde{\sigma}\big)$$

by $A[B_J]$-submodules indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$ for any smooth representation $\tilde{\sigma}$ of $L_{J\cap {}^Iw^{J-1}(I)}$ over A.

Let $w_J \in {}^{J\cap {}^Iw^{J-1}(I)}W_J$ and set ${}^Iw := {}^Iw^J w_J$ and $\pi_{{}^Iw} := \text{c-ind}_{P_I^-}^{P_I^-\,{}^Iw B}\sigma$. We want to compute the A-module $\mathrm{H}^n(N_{J,0}, \pi_{{}^Iw})$ endowed with the Hecke action of B_J^+ (see Section 3.1), where $N_{J,0} \subseteq N_J$ is a compact open subgroup and $B_J^+ \subseteq B_J$ is the open submonoid stabilising N_0 by conjugation (we use similar notation for subgroups of $\boldsymbol{N}_J$ and $\boldsymbol{B}_J$ by taking intersections with $N_{J,0}$ and B_J^+ respectively).

In Section 2.3, we define closed subgroups $\boldsymbol{N}_{J,{}^Iw} \subseteq \boldsymbol{N}_J$ and $\boldsymbol{B}_{J,w_J} \subseteq \boldsymbol{B}_J$ such that there is a semidirect product $\boldsymbol{B}_{J,w_J} \ltimes \boldsymbol{N}_{J,{}^Iw}$ and we give an explicit description of the actions of $N_{J,{}^Iw}$ and B_{J,w_J} on $\pi_{{}^Iw}$ for all $w_J \in {}^{J\cap {}^Iw^{J-1}(I)}W_J$. Then, we compute the A-module $\mathrm{H}^n(N_{J,{}^Iw,0}, \pi_{{}^Iw})$ with the Hecke action of B^+_{J,w_J} (see Proposition 3.2.6).

The idea is to use a semidirect product $\boldsymbol{N}_{J,{}^Iw} = \boldsymbol{N}''_{J,{}^Iw} \ltimes \boldsymbol{N}'_{J,{}^Iw}$ (also defined in Section 2.3) where $\boldsymbol{N}'_{J,{}^Iw} \subseteq \boldsymbol{N}_{J,{}^Iw}$ is a closed subgroup stable under conjugation by $\boldsymbol{B}_{J,w_J}$ such that $\pi_{{}^Iw}$ is $N'_{J,{}^Iw,0}$-acyclic and there is an $A[B^+_{J,w_J}]$-linear surjection with a locally nilpotent kernel from $\pi_{{}^Iw}^{N'_{J,{}^Iw,0}}$ onto

$$\mathrm{Gr}_{B_J}^{w_J}\big(\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J}(\sigma_{|L_{I\cap {}^Iw^J(J)}})^{{}^Iw^J}\big).$$

Then, taking the $N''_{J,{}^Iw,0}$-cohomology changes $\sigma_{|L_{I\cap {}^Iw^J(J)}}$ into $\mathrm{H}^n\mathrm{Ord}_{L_I\cap P_{I\cap {}^Iw^J(J)}}\,\sigma$ in the target and the inflation map is an $A[B^+_{J,w_J}]$-linear isomorphism between the source and $\mathrm{H}^n(N_{J,{}^Iw,0}, \pi_{{}^Iw})$.

Finally, by a technical result on dévissages (see Proposition 3.1.2) and a finiteness property of the A-modules $\mathrm{H}^\bullet(N_{J,{}^Iw,0}, \pi_{{}^Iw})$, we can compute the A-module $\mathrm{H}^n(N_{J,0}, \pi_{{}^Iw})$ with the Hecke action of

B^+_{J,w_J} from $\mathrm{H}^n(N_{J,{}^I w,0}, \pi_{{}^I w})$. It is this dévissage that introduces the degree shift and the twist (i.e., $d_{{}^I w^J}$ and $\delta_{{}^I w^J}$) in the formulas.

Notation and terminology. Let $F/\mathbb{Q}_p$ be a finite extension. A linear algebraic F-group will be denoted by a boldface letter like $\boldsymbol{H}$ and the group of its F-points $\boldsymbol{H}(F)$ will be denoted by the corresponding ordinary letter H. We will also write $\boldsymbol{H}^{\mathrm{der}}$ for its derived subgroup and $\boldsymbol{H}^\circ$ for its identity component. The group of algebraic characters of $\boldsymbol{H}$ will be denoted by $\mathrm{X}^*(\boldsymbol{H})$, the group of algebraic cocharacters of $\boldsymbol{H}$ will be denoted by $\mathrm{X}_*(\boldsymbol{H})$, and we will write $\langle -, - \rangle : \mathrm{X}^*(\boldsymbol{H}) \times \mathrm{X}_*(\boldsymbol{H}) \to \mathbb{Z}$ for the natural pairing. We now turn to reductive groups. The main reference for these is [Borel and Tits 1965].

Let $\boldsymbol{G}$ be a connected reductive algebraic F-group. We write $\boldsymbol{Z}$ for the centre of $\boldsymbol{G}$. Let $\boldsymbol{S} \subseteq \boldsymbol{G}$ be a maximal split torus. We write $\boldsymbol{\mathcal{Z}}$ (resp. $\boldsymbol{\mathcal{N}}$) for the centraliser (resp. normaliser) of $\boldsymbol{S}$ in $\boldsymbol{G}$ and W for the Weyl group $\boldsymbol{\mathcal{N}}/\boldsymbol{\mathcal{Z}} = \mathcal{N}/\mathcal{Z}$. We write $\Phi \subseteq \mathrm{X}^*(\boldsymbol{S})$ for the set of roots of $\boldsymbol{S}$ in $\boldsymbol{G}$ and $\Phi_0 \subseteq \Phi$ for the subset of reduced roots. To each $\alpha \in \Phi$ correspond a coroot $\alpha^\vee \in \mathrm{X}_*(\boldsymbol{S})$, a reflection $s_\alpha \in W$ and a root subgroup $\boldsymbol{U}_\alpha \subset \boldsymbol{G}$ (which is denoted by $\boldsymbol{U}_{(\alpha)}$ in [loc. cit.]). For $\alpha, \beta \in \Phi$, we write $\alpha \perp \beta$ if and only if $\langle \alpha, \beta^\vee \rangle = 0$. For $I \subseteq \Delta$, we put

$$I^\perp := \{\alpha \in \Delta \mid \alpha \perp \beta \ \forall \beta \in I\}.$$

Let $\boldsymbol{B} \subseteq \boldsymbol{G}$ be a minimal parabolic subgroup containing $\boldsymbol{S}$. We write $\boldsymbol{U}$ for the unipotent radical of $\boldsymbol{B}$ (so that $\boldsymbol{B} = \boldsymbol{\mathcal{Z}}\boldsymbol{U}$), $\Phi^+ \subseteq \Phi$ for the subset of roots of $\boldsymbol{S}$ in $\boldsymbol{U}$ and $\Delta \subseteq \Phi^+$ for the subset of simple roots. We set $\Phi_0^+ := \Phi_0 \cap \Phi^+$. A simple reflection is a reflection $s_\alpha \in W$ with $\alpha \in \Delta$. A reduced decomposition of $w \in W$ is any decomposition into simple reflections $w = s_1 \dots s_n$ with $n \in \mathbb{N}$ minimal, which is called the length of w and denoted by $\ell(w)$. We write w_0 for the element of maximal length in W.

We say that $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ is a standard parabolic subgroup if $\boldsymbol{P} \subseteq \boldsymbol{G}$ is a parabolic subgroup containing $\boldsymbol{B}$ with unipotent radical $\boldsymbol{N}$ and $\boldsymbol{L} \subseteq \boldsymbol{P}$ is the Levi factor containing $\boldsymbol{S}$ (we say that $\boldsymbol{L}$ is a standard Levi subgroup). In this case, we write $\boldsymbol{P}^-$ for the parabolic subgroup of $\boldsymbol{G}$ opposed to $\boldsymbol{P}$ with respect to $\boldsymbol{L}$ (i.e., $\boldsymbol{P} \cap \boldsymbol{P}^- = \boldsymbol{L}$) and $\boldsymbol{N}^-$ for the unipotent radical of $\boldsymbol{P}^-$. We write $\boldsymbol{Z_L}$ for the centre of $\boldsymbol{L}$, $\boldsymbol{B_L} \subseteq \boldsymbol{L}$ for the minimal parabolic subgroup $\boldsymbol{B} \cap \boldsymbol{L}$, $\boldsymbol{U_L} \subseteq \boldsymbol{B_L}$ for the unipotent radical $\boldsymbol{U} \cap \boldsymbol{L}$ (so that $\boldsymbol{B_L} = \boldsymbol{\mathcal{Z}}\boldsymbol{U_L}$ and $\boldsymbol{U} = \boldsymbol{U_L} \ltimes \boldsymbol{N}$) and $\Delta_{\boldsymbol{L}} \subseteq \Delta$ for the subset of simple roots of $\boldsymbol{S}$ in $\boldsymbol{U_L}$.

Each parabolic subgroup of $\boldsymbol{G}$ is conjugate to exactly one standard parabolic subgroup and the map $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N} \mapsto \Delta_{\boldsymbol{L}}$ yields a bijection between standard parabolic subgroups of $\boldsymbol{G}$ and subsets of Δ. For $I \subseteq \Delta$, we write $\boldsymbol{P}_I = \boldsymbol{L}_I\boldsymbol{N}_I$ for the corresponding standard parabolic subgroup (i.e., $\Delta_{\boldsymbol{L}_I} = I$), $\boldsymbol{Z}_I$, $\boldsymbol{B}_I$, $\boldsymbol{U}_I$ instead of $\boldsymbol{Z}_{\boldsymbol{L}_I}$, $\boldsymbol{B}_{\boldsymbol{L}_I}$, $\boldsymbol{U}_{\boldsymbol{L}_I}$ respectively, $W_I \subseteq W$ for the subgroup generated by $(s_\alpha)_{\alpha \in I}$ (so that $\boldsymbol{P}_I = \boldsymbol{B} W_I \boldsymbol{B}$), $w_{I,0}$ for the element of maximal length in W_I, $\Phi_I \subseteq \Phi$ for the subset of roots of $\boldsymbol{S}$ in $\boldsymbol{L}_I$ and $\Phi_I^+ \subseteq \Phi^+$ for the subset of roots of $\boldsymbol{S}$ in $\boldsymbol{U}_I$.

Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field k. We let A be an Artinian local $\mathcal{O}$-algebra with residue field k. We write $\varepsilon : F^\times \to \mathbb{Z}_p^\times \subseteq \mathcal{O}^\times$ for the p-adic cyclotomic character (defined by

$$\varepsilon(x) = \mathrm{Nrm}_{F/\mathbb{Q}_p}(x) |\mathrm{Nrm}_{F/\mathbb{Q}_p}(x)|_p$$

for all $x \in F^\times$) and $\omega : F^\times \to A^\times$ for its image in $A^\times$.

We use the terminology and notation of [Emerton 2010a, §2] for representations of a p-adic Lie group H over A. An H-representation is a smooth representation of H over A and a morphism between H-representations is A-linear. We write $\mathrm{Mod}_H^{\mathrm{sm}}(A)$ for the category of H-representations and H-equivariant morphisms, and $\mathrm{Mod}_H^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_H^{\mathrm{l.adm}}(A)$, $\mathrm{Mod}_H^{\mathrm{sm}}(A)^{Z_H-\mathrm{l.fin}}$) for the full subcategory of admissible (resp. locally admissible, locally Z_H-finite) H-representations (here Z_H denotes the centre of H).

Assume $H \subseteq G$ is closed and π is an H-representation. For $g \in G$, we write π^g for the $g^{-1}Hg$-representation with the same underlying A-module as π on which $g^{-1}hg$ acts as h for all $h \in H$. If $g \in H$, then $g^{-1}Hg = H$ and the action of g on π induces a natural H-equivariant isomorphism $\pi \xrightarrow{\sim} \pi^g$.

Assume furthermore $\mathcal{Z} \subseteq H$. For $w \in W$, we write π^w for the $n^{-1}Hn$-representation π^n where $n \in \mathcal{N}$ is any representative of w (neither $n^{-1}Hn$ nor π^n depend on the choice of n up to isomorphism). For $\alpha \in \Delta$, we simply write π^α instead of π^{s_α}.

For a topological space X and an A-module V, we write $\mathcal{C}^{\mathrm{sm}}(X, V)$ for the A-module of locally constant functions $f : X \to V$ and $\mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}(X, V)$ for the A-submodule consisting of those functions with compact support (the support of f is the open and closed subset $\operatorname{supp} f := f^{-1}(V \backslash \{0\}) \subseteq X$).

2. Generalised Bruhat filtrations

The aim of this section is to define filtrations of parabolically induced representations and describe the associated graded representations. In Section 2.1, we review some properties of the representatives of minimal length of certain double cosets in W and some variants of the Bruhat decomposition. In Section 2.2, we define the notion of filtration indexed by a poset and we construct filtrations of induced representations indexed by subsets of W with the Bruhat order using the previous decompositions. In Section 2.3, we define several subgroups of $\boldsymbol{U}$ that we use to describe the graded representations associated with the previous filtrations as spaces of locally constant functions with compact support.

2.1. ***Double cosets.*** We recall some facts about certain right cosets in W; see [Borel and Tits 1972, Proposition 3.9]. For any $I \subseteq \Delta$, we define a system of representatives of the right cosets $W_I \backslash W$ by setting

$$^{I}W := \left\{ w \in W \;\middle|\; w \text{ is of minimal length in } W_I w \right\}.$$

For all $w \in W$, there exists a unique decomposition $w = w_I {}^{I}w$ with $w_I \in W_I$ and ${}^{I}w \in {}^{I}W$. This decomposition is characterised by the equality

$$\Phi_I^+ \cap w(\Phi^+) = \Phi_I^+ \cap w_I(\Phi_I^+).$$

In particular, we have ${}^{I}w^{-1}(\Phi_I^+) \subseteq \Phi^+$. Furthermore, we have $\ell(w) = \ell(w_I) + \ell({}^{I}w)$.

We now recall some properties of certain double cosets in W; see, for example, [Digne and Michel 1991, Lemma 5.4]. For any $I, J \subseteq \Delta$, we define a system of representatives of the double cosets $W_I \backslash W / W_J$ by setting

$$^{I}W^{J} := {}^{I}W \cap ({}^{J}W)^{-1}.$$

For all ${}^I w \in {}^I W$, there exists a unique decomposition ${}^I w = {}^I w^J w_J$ with ${}^I w^J \in {}^I W^J$ and $w_J \in W_J$. In fact $w_J \in {}^{J \cap {}^I w^{J-1}(I)} W_J$. This decomposition is characterised by the equality

$$\Phi_J^+ \cap {}^I w^{-1}(\Phi^+) = \Phi_J^+ \cap w_J^{-1}(\Phi_J^+). \tag{3}$$

In particular, we have ${}^I w^J(\Phi_J^+) \subseteq \Phi^+$. Furthermore, we have $\ell({}^I w) = \ell({}^I w^J) + \ell(w_J)$. Conversely, for all ${}^I w^J \in {}^I W^J$ and $w_J \in W_J$, we have ${}^I w^J w_J \in {}^I W$ if and only if $w_J \in {}^{J \cap {}^I w^{J-1}(I)} W_J$. Note that the projections $W \twoheadrightarrow {}^I W$ and ${}^I W \twoheadrightarrow {}^I W^J$ respect the Bruhat order;[1] see [Björner and Brenti 2005, Proposition 2.5.1].

Lemma 2.1.1. *We have the following equalities in* $\boldsymbol{G}$.

(i) $\boldsymbol{L}_I \cap {}^I w^J \boldsymbol{U}_J {}^I w^{J-1} = \boldsymbol{U}_{I \cap {}^I w^J(J)}$

(ii) $\boldsymbol{L}_I \cap {}^I w^J \boldsymbol{L}_J {}^I w^{J-1} = \boldsymbol{L}_{I \cap {}^I w^J(J)}$

(iii) $\boldsymbol{L}_I \cap {}^I w^J \boldsymbol{N}_J {}^I w^{J-1} = \boldsymbol{L}_I \cap \boldsymbol{N}_{I \cap {}^I w^J(J)}$

(iv) $\boldsymbol{L}_I \cap {}^I w^J \boldsymbol{P}_J {}^I w^{J-1} = \boldsymbol{L}_I \cap \boldsymbol{P}_{I \cap {}^I w^J(J)}$

Proof. First, we prove the following equalities in Φ:

$$\Phi_I \cap {}^I w^J(J) = I \cap {}^I w^J(J), \tag{4}$$

$$\Phi_I \cap {}^I w^J\left(\Phi_J^+\right) = \Phi^+_{I \cap {}^I w^J(J)}. \tag{5}$$

We prove the nontrivial inclusion of (4). Assume $\Phi_I \cap {}^I w^J(J) \neq \varnothing$ and let $\alpha \in \Phi_I \cap {}^I w^J(J)$. Since ${}^I w^J(J) \subseteq \Phi^+$, $\alpha \in \Phi_I^+$ so that there exists $(r_\beta)_{\beta \in I} \in \mathbb{N}^I$ such that $\alpha = \sum_{\beta \in I} r_\beta \beta$. Then ${}^I w^{J-1}(\alpha) = \sum_{\beta \in I} r_\beta {}^I w^{J-1}(\beta) \in \Delta$. Since ${}^I w^{J-1}(\beta) \in \Phi^+$ for all $\beta \in I$, $r_\beta = 0$ for all $\beta \in I \backslash \{\alpha\}$ and $r_\alpha = 1$. Thus $\alpha \in I$. We prove the nontrivial inclusion of (5). Assume $\Phi_I \cap {}^I w^J(\Phi_J^+) \neq \varnothing$ and let $\alpha \in \Phi_I \cap {}^I w^J(\Phi_J^+)$. There exists $(r_\beta)_{\beta \in J} \in \mathbb{N}^J$ such that $\alpha = \sum_{\beta \in J} r_\beta {}^I w^J(\beta)$. Since ${}^I w^J(\beta) \in \Phi^+$ for all $\beta \in J$, ${}^I w^J(\beta) \in \Phi_I^+$ so that ${}^I w^J(\beta) \in I$ by (4) for all $\beta \in J$ such that $r_\beta > 0$. Thus $\alpha \in \Phi^+_{I \cap {}^I w^J(J)}$.

Now, by considering the Lie algebras, (5) yields (i), (5) and its opposite yield (ii), the equality $\Phi_I \cap {}^I w^J(\Phi^+ \backslash \Phi_J^+) = \Phi_I^+ \backslash \Phi^+_{I \cap {}^I w^J(J)}$ (which follows from (5) and the fact that $\Phi_I \cap {}^I w^J(\Phi^+) = \Phi_I^+$ since ${}^I w^J \in {}^I W$) yields (iii), and we deduce (iv) from (ii) and (iii). □

Finally, we give certain decompositions in double cosets (for the notion of "lower set", see foonote 2 on p. 789).

Lemma 2.1.2. (i) *We have* $G = \bigsqcup_{{}^I w^J \in {}^I W^J} P_I^- {}^I w^J P_J$ *and for any lower set* ${}^I W_1^J \subseteq {}^I W^J$, *the subset* $P_I^- {}^I W_1^J P_J \subseteq G$ *is open.*

(ii) *We have* $P_I^- {}^I w^J P_J = \bigsqcup_{w_J \in {}^{J \cap {}^I w^{J-1}(I)} W_J} P_I^- {}^I w^J w_J B$ *and for any lower set* $W_J' \subseteq {}^{J \cap {}^I w^{J-1}(I)} W_J$, *the subset* $P_I^- {}^I w^J W_J' B \subseteq P_I^- {}^I w^J P_J$ *is open.*

(iii) *We have* $L_J = \bigsqcup_{w_J \in {}^{J \cap {}^I w^{J-1}(I)} W_J} L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} w_J B_J$ *and for any lower set* $W_J' \subseteq {}^{J \cap {}^I w^{J-1}(I)} W_J$, *the subset* $L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} W_J' B_J \subseteq L_J$ *is open.*

[1] The Bruhat order on W is defined by $w \leq w'$ if and only if there exist a reduced decomposition $w' = s_1 \dots s_{\ell(w')}$ and integers $1 \leq i_1 < \cdots < i_{\ell(w)} \leq \ell(w')$ such that $w = s_{i_1} \dots s_{i_{\ell(w)}}$.

Proof. We have $G = \bigsqcup_{{}^I w \in {}^I W} P_I^- {}^I w B$ and for any ${}^I w \in {}^I W$, the closure of $P_I^- {}^I w B$ in G is $\bigsqcup_{{}^I w' \geq {}^I w} P_I^- {}^I w' B$. (This can be deduced from the Bruhat decomposition, see, e.g., [Hauseux 2016a, §2.3].) Furthermore, for any ${}^I w^J \in {}^I W^J$, we have

$$P_I^- {}^I w^J P_J = \bigcup_{w_J \in W_J} P_I^- w_0 B w_0 {}^I w^J w_{J,0} B w_J B = \bigcup_{w_J \in W_J} P_I^- w_0 B w_0 {}^I w^J w_{J,0} w_J B = \bigcup_{w_J \in W_J} P_I^- {}^I w^J w_J B.$$

(The first and third equalities follow from the inclusion $w_0 B w_0 = B^- \subseteq P_I^-$ and the decomposition $P_J = B W_J B$, and the second equality follows from [Borel and Tits 1972, Lemme 3.4(iv); Björner and Brenti 2005, Proposition 2.5.4].) From this we deduce (ii), and also (i) using the fact that the projection ${}^I W \twoheadrightarrow {}^I W^J$ is order-preserving. Finally, (iii) is (i) for the double cosets $L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} \backslash L_J / B_J$ instead of $P_I^- \backslash G / P_J$. □

Remark 2.1.3 (case $w_J = 1$). Note that $P_I^- {}^I w^J B$ is $P_{J \cap {}^I w^{J-1}(I)}$-invariant by right translation. In general, the stabiliser of $P_I^- {}^I w B$ in G for the action by right translation is the (nonstandard) parabolic subgroup $B {}^I w^{-1} W_I {}^I w B$. Likewise, $L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} B_J$ is $L_J \cap P_{J \cap {}^I w^{J-1}(I)}$-invariant by right translation.

2.2. *Definition of filtrations.*

Filtration indexed by a poset. Let H be a p-adic Lie group, π be an H-representation and $(\widetilde{W}, \leq)$ be a poset. A filtration of π indexed by $\widetilde{W}$ is a morphism of complete lattices $\mathrm{Fil}_H^\bullet \pi$ from the complete lattice of lower sets[2] of $\widetilde{W}$ to the complete lattice of H-subrepresentations of π, i.e., an H-subrepresentation $\mathrm{Fil}_H^{\widetilde{W}'} \pi \subseteq \pi$ for each lower set $\widetilde{W}' \subseteq \widetilde{W}$ such that for any family $(\widetilde{W}_i)_{i \in \mathcal{I}}$ of lower sets of $\widetilde{W}$, we have the following equalities in π:

$$\mathrm{Fil}_H^{\bigcap_{i \in \mathcal{I}} \widetilde{W}_i} \pi = \bigcap_{i \in \mathcal{I}} \mathrm{Fil}_H^{\widetilde{W}_i} \pi, \quad \mathrm{Fil}_H^{\bigcup_{i \in \mathcal{I}} \widetilde{W}_i} \pi = \sum_{i \in \mathcal{I}} \mathrm{Fil}_H^{\widetilde{W}_i} \pi.$$

When $\widetilde{W}$ is finite, these two equalities are equivalent (by induction) to the following conditions: $\mathrm{Fil}_H^\bullet \pi$ is inclusion-preserving with $\mathrm{Fil}_H^{\varnothing} \pi = 0$ and $\mathrm{Fil}_H^{\widetilde{W}} \pi = \pi$ (i.e., the empty family case), and for any lower sets $\widetilde{W}_1, \widetilde{W}_2 \subseteq \widetilde{W}$ the short sequence of H-representations

$$0 \to \mathrm{Fil}_H^{\widetilde{W}_1 \cap \widetilde{W}_2} \pi \to \mathrm{Fil}_H^{\widetilde{W}_1} \pi \oplus \mathrm{Fil}_H^{\widetilde{W}_2} \pi \to \mathrm{Fil}_H^{\widetilde{W}_1 \cup \widetilde{W}_2} \pi \to 0,$$

defined by $v \mapsto (v, -v)$ and $(v_1, v_2) \mapsto v_1 + v_2$, is exact.

Each $\tilde{w} \in \widetilde{W}$ defines a principal lower set $\{\tilde{w}' \in \widetilde{W} \mid \tilde{w}' \leq \tilde{w}\}$ and we write $\mathrm{Fil}_H^{\tilde{w}} \pi$ for the corresponding H-subrepresentation of π. Note that for any lower set $\widetilde{W}' \subseteq \widetilde{W}$, we have the following equality in π:

$$\mathrm{Fil}_H^{\widetilde{W}'} \pi = \sum_{\tilde{w}' \in \widetilde{W}'} \mathrm{Fil}_H^{\tilde{w}'} \pi.$$

Thus, we can recover the whole filtration from the H-subrepresentations of π corresponding to the elements of $\widetilde{W}$; hence the terminology. We define the graded representation $\mathrm{Gr}_H^\bullet \pi$ associated with the filtration $\mathrm{Fil}_H^\bullet \pi$ by setting

$$\mathrm{Gr}_H^{\tilde{w}} \pi := \mathrm{Fil}_H^{\tilde{w}} \pi \Big/ \sum_{\tilde{w}' < \tilde{w}} \mathrm{Fil}_H^{\tilde{w}'} \pi \quad \text{for each } \tilde{w} \in \widetilde{W}.$$

[2] A lower set of $\widetilde{W}$ is a subset $\widetilde{W}'$ such that $\tilde{w} \leq \tilde{w}' \Rightarrow \tilde{w} \in \widetilde{W}'$ for any $\tilde{w} \in \widetilde{W}$ and $\tilde{w}' \in \widetilde{W}'$.

Let $\tilde{\ell}:\widetilde{W}\to\mathbb{Z}$ be a monotonic map (i.e., $\tilde{w}\le\tilde{w}'\Rightarrow\tilde{\ell}(\tilde{w})\le\tilde{\ell}(\tilde{w}')$ for any $\tilde{w},\tilde{w}'\in\widetilde{W}$). For each $n\in\mathbb{Z}$, we set

$$\mathrm{Fil}_H^{\tilde{\ell},n}\pi := \sum_{\tilde{\ell}(\tilde{w})\le n}\mathrm{Fil}_H^{\tilde{w}}\pi.$$

We obtain a filtration of π indexed by $\mathbb{Z}$ (in the usual sense).

Lemma 2.2.1. *Assume $\tilde{\ell}:\widetilde{W}\to\mathbb{Z}$ is strictly monotonic (i.e., $\tilde{w}<\tilde{w}'\Rightarrow\tilde{\ell}(\tilde{w})<\tilde{\ell}(\tilde{w}')$ for any $\tilde{w},\tilde{w}'\in\widetilde{W}$). For all $n\in\mathbb{Z}$, there is a natural H-equivariant isomorphism*

$$\mathrm{Gr}_H^{\tilde{\ell},n}\pi\cong\bigoplus_{\tilde{\ell}(\tilde{w})=n}\mathrm{Gr}_H^{\tilde{w}}\pi.$$

Proof. Let $n\in\mathbb{Z}$. By definition of $\mathrm{Fil}_H^{\tilde{\ell},n}\pi$ and $\mathrm{Gr}_H^{\tilde{\ell},n}\pi$, there are natural H-equivariant surjections

$$\bigoplus_{\tilde{\ell}(\tilde{w})\le n}\mathrm{Fil}_H^{\tilde{w}}\pi\twoheadrightarrow\mathrm{Fil}_H^{\tilde{\ell},n}\pi\twoheadrightarrow\mathrm{Gr}_H^{\tilde{\ell},n}\pi. \tag{6}$$

The kernel of (6) contains $\bigoplus_{\tilde{\ell}(\tilde{w})\le n}\mathrm{Fil}_H^{\tilde{w}}\pi\cap\mathrm{Fil}_H^{\tilde{\ell},n-1}\pi$, and $\mathrm{Fil}_H^{\tilde{w}}\pi\cap\mathrm{Fil}_H^{\tilde{\ell},n-1}\pi=\mathrm{Fil}_H^{\tilde{w}}\pi$ for all $\tilde{w}\in\widetilde{W}$ such that $\tilde{\ell}(\tilde{w})<n$. Now, for any $\tilde{w}_0\in\widetilde{W}$ such that $\tilde{\ell}(\tilde{w}_0)=n$, we have the following equality in π:

$$\mathrm{Fil}_H^{\tilde{w}_0}\pi\cap\sum_{\substack{\tilde{\ell}(\tilde{w})\le n\\ \tilde{w}\ne\tilde{w}_0}}\mathrm{Fil}_H^{\tilde{w}}\pi=\sum_{\tilde{w}<\tilde{w}_0}\mathrm{Fil}_H^{\tilde{w}}\pi,$$

which results from the following equality in $\widetilde{W}$:

$$\left\{\tilde{w}'\in\widetilde{W}\;\middle|\;\tilde{w}'\le\tilde{w}_0\right\}\cap\bigcup_{\substack{\tilde{\ell}(\tilde{w})\le n\\ \tilde{w}\ne\tilde{w}_0}}\left\{\tilde{w}'\in\widetilde{W}\;\middle|\;\tilde{w}'\le\tilde{w}\right\}=\bigcup_{\tilde{w}<\tilde{w}_0}\left\{\tilde{w}'\in\widetilde{W}\;\middle|\;\tilde{w}'\le\tilde{w}\right\},$$

which in turn follows from the fact that $\tilde{w}_0\not\le\tilde{w}$ for all $\tilde{w}\in\widetilde{W}\backslash\{\tilde{w}_0\}$ such that $\tilde{\ell}(\tilde{w})\le n$ by strict monotonicity of $\tilde{\ell}$. We deduce that the kernel of (6) is $\bigoplus_{\tilde{\ell}(\tilde{w})\le n}\mathrm{Fil}_H^{\tilde{w}}\pi\cap\mathrm{Fil}_H^{\tilde{\ell},n-1}\pi$, and that $\mathrm{Fil}_H^{\tilde{w}}\pi\cap\mathrm{Fil}_H^{\tilde{\ell},n-1}\pi=\sum_{\tilde{w}'<\tilde{w}}\mathrm{Fil}_H^{\tilde{w}'}\pi$ for all $\tilde{w}\in\widetilde{W}$ such that $\tilde{\ell}(\tilde{w})=n$. We conclude that (6) induces an isomorphism as in the statement. □

Filtrations of induced representations. Let $I,J\subseteq\Delta$ and σ be an L_I-representation. Recall that for any locally closed subset $X\subseteq G$ and for any open subset $Y\subseteq X$, both P_I^--invariant by left translation, there is a natural short exact sequence of A-modules

$$0\to\text{c-ind}_{P_I^-}^{Y}\sigma\to\text{c-ind}_{P_I^-}^{X}\sigma\to\text{c-ind}_{P_I^-}^{X\backslash Y}\sigma\to 0.$$

(See [Bernšteĭn and Zelevinskiĭ 1976, Proposition 1.8]; see also the proof of [Hauseux 2016a, Proposition 2.1.3].) Note that there is a natural A-linear isomorphism $\text{c-ind}_{P_I^-}^{G}\sigma\xrightarrow{\sim}\mathrm{Ind}_{P_I^-}^{G}\sigma$ since $P_I^-\backslash G$ is compact.

For each lower set ${}^IW_1^J\subseteq{}^IW^J$, we define a P_J-subrepresentation of $\mathrm{Ind}_{P_I^-}^G\sigma$ by setting

$$\mathrm{Fil}_{P_J}^{{}^IW_1^J}\left(\mathrm{Ind}_{P_I^-}^G\sigma\right):=\text{c-ind}_{P_I^-}^{P_I^-\,{}^IW_1^J P_J}\sigma.$$

Using Lemma 2.1.2(i), we obtain a filtration of $\mathrm{Ind}_{P_I^-}^G \sigma$ indexed by ${}^IW^J$ such that for all ${}^Iw^J \in {}^IW^J$, there is a natural P_J-equivariant isomorphism

$$\mathrm{Gr}_{P_J}^{{}^Iw^J}\big(\mathrm{Ind}_{P_I^-}^G \sigma\big) \cong \text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma. \tag{7}$$

Let ${}^Iw^J \in {}^IW^J$. For each lower set $W'_J \subseteq {}^{J\cap {}^Iw^{J-1}(I)}W_J$, we define a B-subrepresentation of $\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma$ by setting

$$\mathrm{Fil}_B^{W'_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big) := \text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J W'_J B} \sigma.$$

Using Lemma 2.1.2(ii), we obtain a filtration of $\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma$ indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$ such that for all $w_J \in {}^{J\cap {}^Iw^{J-1}(I)}W_J$, there is a natural B-equivariant isomorphism

$$\mathrm{Gr}_B^{w_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big) \cong \text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J w_J B} \sigma. \tag{8}$$

Likewise, for any $L_{J\cap {}^Iw^{J-1}(I)}$-representation $\tilde\sigma$ and using Lemma 2.1.2(iii), we define for each lower set $W'_J \subseteq {}^{J\cap {}^Iw^{J-1}(I)}W_J$ a B_J-subrepresentation of $\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J} \tilde\sigma$ by setting

$$\mathrm{Fil}_{B_J}^{W'_J}\big(\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J} \tilde\sigma\big) := \text{c-ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)} W'_J B_J} \tilde\sigma$$

and we obtain a filtration of $\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J} \tilde\sigma$ indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$ such that for all $w_J \in {}^{J\cap {}^Iw^{J-1}(I)}W_J$, there is a natural B_J-equivariant isomorphism

$$\mathrm{Gr}_{B_J}^{w_J}\big(\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J} \tilde\sigma\big) \cong \text{c-ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)} w_J B_J} \tilde\sigma. \tag{9}$$

Remark 2.2.2 (case $w_J = 1$). Note that

$$\mathrm{Gr}_B^1(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma) \cong \text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J B} \sigma$$

is a $P_{J\cap {}^Iw^{J-1}(I)}$-subrepresentation of $\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma$ and likewise

$$\mathrm{Gr}_{B_J}^1(\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J} \tilde\sigma) \cong \text{c-ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)} B_J} \tilde\sigma$$

is an $L_J \cap P_{J\cap {}^Iw^{J-1}(I)}$-subrepresentation of $\mathrm{Ind}_{L_J\cap P^-_{J\cap {}^Iw^{J-1}(I)}}^{L_J} \tilde\sigma$ (see Remark 2.1.3).

2.3. ***Computation of the associated graded representations.*** For each $w \in W$, we define a closed subgroup of $\boldsymbol{U}$ stable under conjugation by $\boldsymbol{\mathcal{Z}}$ by setting

$$\boldsymbol{U}_w := \boldsymbol{U} \cap w^{-1}\boldsymbol{U} w$$

and we let $\boldsymbol{B}_w \subseteq \boldsymbol{B}$ be the closed subgroup $\boldsymbol{\mathcal{Z}}\boldsymbol{U}_w$. For any order on $\Phi^+ \cap w^{-1}(\Phi_0^+)$, the product induces an isomorphism of F-varieties

$$\prod_{\alpha\in\Phi^+\cap w^{-1}(\Phi_0^+)} \boldsymbol{U}_\alpha \xrightarrow{\sim} \boldsymbol{U}_w. \tag{10}$$

Let $I \subseteq \Delta$ and ${}^I w \in {}^I W$. We define closed subgroups of $\boldsymbol{U}_{{}^I w}$ stable under conjugation by $\boldsymbol{\mathcal{Z}}$ by setting

$$\boldsymbol{U}'_{{}^I w} := \boldsymbol{U} \cap {}^I w^{-1} \boldsymbol{N}_I {}^I w, \quad \boldsymbol{U}''_{{}^I w} := \boldsymbol{U} \cap {}^I w^{-1} \boldsymbol{U}_I {}^I w,$$

and we let $\boldsymbol{B}''_{{}^I w} \subseteq \boldsymbol{B}_{{}^I w}$ be the closed subgroup $\boldsymbol{\mathcal{Z}} \boldsymbol{U}''_{{}^I w}$. We have semidirect products $\boldsymbol{U}_{{}^I w} = \boldsymbol{U}''_{{}^I w} \ltimes \boldsymbol{U}'_{{}^I w}$ and $\boldsymbol{B}_{{}^I w} = \boldsymbol{B}''_{{}^I w} \ltimes \boldsymbol{U}'_{{}^I w}$.

Let σ be an L_I-representation. The product induces an isomorphism of F-varieties

$$P_I^- \times \left\{ {}^I w \right\} \times U'_{{}^I w} \xrightarrow{\sim} P_I^- {}^I w B,$$

hence an A-linear isomorphism

$$\text{c-ind}_{P_I^-}^{P_I^- {}^I w B} \sigma \cong \mathcal{C}_c^{\text{sm}} \left(U'_{{}^I w}, \sigma^{{}^I w} \right) \tag{11}$$

via which $U'_{{}^I w}$ acts on $\mathcal{C}_c^{\text{sm}}(U'_{{}^I w}, \sigma^{{}^I w})$ by right translation and the action of $b'' \in B''_{{}^I w}$ on $f \in \mathcal{C}_c^{\text{sm}}(U'_{{}^I w}, \sigma^{{}^I w})$ is given by

$$(b'' \cdot f)(u') = b'' \cdot f(b''^{-1} u' b'')$$

for all $u' \in U'_{{}^I w}$.

Let $J \subseteq \Delta$. We write ${}^I w = {}^I w^J w_J$ with ${}^I w^J \in {}^I W^J$ and $w_J \in W_J$. We define closed subgroups of $\boldsymbol{N}_J$ and $\boldsymbol{U}_J$ stable under conjugation by $\boldsymbol{\mathcal{Z}}$ by setting

$$\boldsymbol{N}_{J, {}^I w} := \boldsymbol{N}_J \cap \boldsymbol{U}_{{}^I w} = \boldsymbol{N}_J \cap {}^I w^{-1} \boldsymbol{U}^I w, \quad \boldsymbol{U}_{J, w_J} := \boldsymbol{U}_J \cap \boldsymbol{U}_{{}^I w} = \boldsymbol{U}_J \cap {}^I w^{-1} \boldsymbol{U}^I w = \boldsymbol{U}_J \cap w_J^{-1} \boldsymbol{U}_J w_J,$$

the last equality resulting from (3), and we let $\boldsymbol{B}_{J, w_J} \subseteq \boldsymbol{B}_J$ be the closed subgroup $\boldsymbol{\mathcal{Z}} \boldsymbol{U}_{J, w_J}$. We have semidirect products $\boldsymbol{U}_{{}^I w} = \boldsymbol{U}_{J, w_J} \ltimes \boldsymbol{N}_{J, {}^I w}$ and $\boldsymbol{B}_{{}^I w} = \boldsymbol{B}_{J, w_J} \ltimes \boldsymbol{N}_{J, {}^I w}$. We define closed subgroups of $\boldsymbol{N}_{J, {}^I w}$ and $\boldsymbol{U}_{J, w_J}$ stable under conjugation by $\boldsymbol{\mathcal{Z}}$ by setting

$$\begin{aligned} \boldsymbol{N}'_{J, {}^I w} &:= \boldsymbol{N}_J \cap \boldsymbol{U}'_{{}^I w} = \boldsymbol{N}_J \cap {}^I w^{-1} \boldsymbol{N}_I {}^I w, \\ \boldsymbol{N}''_{J, {}^I w} &:= \boldsymbol{N}_J \cap \boldsymbol{U}''_{{}^I w} = \boldsymbol{N}_J \cap {}^I w^{-1} \boldsymbol{U}_I {}^I w, \\ \boldsymbol{U}'_{J, w_J} &:= \boldsymbol{U}_J \cap \boldsymbol{U}'_{{}^I w} = \boldsymbol{U}_J \cap {}^I w^{-1} \boldsymbol{N}_I {}^I w, \\ \boldsymbol{U}''_{J, w_J} &:= \boldsymbol{U}_J \cap \boldsymbol{U}''_{{}^I w} = \boldsymbol{U}_J \cap {}^I w^{-1} \boldsymbol{U}_I {}^I w, \end{aligned}$$

and we let $\boldsymbol{B}''_{J, w_J} \subseteq \boldsymbol{B}_J$ be the closed subgroup $\boldsymbol{\mathcal{Z}} \boldsymbol{U}''_{J, w_J}$. We have semidirect products $\boldsymbol{N}_{J, {}^I w} = \boldsymbol{N}''_{J, {}^I w} \ltimes \boldsymbol{N}'_{J, {}^I w}$, $\boldsymbol{U}_{J, w_J} = \boldsymbol{U}''_{J, w_J} \ltimes \boldsymbol{U}'_{J, w_J}$ and $\boldsymbol{B}_{J, w_J} = \boldsymbol{B}''_{J, w_J} \ltimes \boldsymbol{U}'_{J, w_J}$. Note that $\boldsymbol{U}'_{J, w_J}$ and $\boldsymbol{U}''_{J, w_J}$ actually depend on ${}^I w$ (not only on w_J).

Likewise, for any $L_{J \cap {}^I w^{J^{-1}}(I)}$-representation $\tilde{\sigma}$ and using Lemma 2.1.1 with I and J swapped and ${}^I w^J$ inverted, the product induces an isomorphism of F-varieties

$$L_J \cap P^-_{J \cap {}^I w^{J^{-1}}(I)} \times \{w_J\} \times U'_{J, w_J} \xrightarrow{\sim} L_J \cap P^-_{J \cap {}^I w^{J^{-1}}(I)} w_J B_J,$$

hence an A-linear isomorphism

$$\text{c-ind}_{L_J \cap P^-_{J \cap {}^I w^{J^{-1}}(I)}}^{L_J \cap P^-_{J \cap {}^I w^{J^{-1}}(I)} w_J B_J} \tilde{\sigma} \cong \mathcal{C}_c^{\text{sm}} \left(U'_{J, w_J}, \tilde{\sigma}^{w_J} \right) \tag{12}$$

via which U'_{J,w_J} acts on $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \tilde{\sigma}^{w_J})$ by right translation and the action of $b'' \in B''_{J,w_J}$ on $f \in \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \tilde{\sigma}^{w_J})$ is given by

$$(b'' \cdot f)(u') = b'' \cdot f(b''^{-1} u' b'')$$

for all $u' \in U'_{J,w_J}$. In particular with $\tilde{\sigma} = \sigma^{{}^I w^J}$, we have defined a natural smooth A-linear action of B_{J,w_J} on $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w})$.

We have a semidirect product $\boldsymbol{U}'_{{}^I w} = \boldsymbol{U}'_{J,w_J} \ltimes \boldsymbol{N}'_{J,{}^I w}$, so that (11) composed with the A-linear morphism defined by $f \mapsto (n' \mapsto (u' \mapsto f(u'n')))$ is an A-linear isomorphism

$$\operatorname{c-ind}_{P_I^-}^{P_I^- {}^I w B} \sigma \cong \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(N'_{J,{}^I w}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \sigma^{{}^I w}\big)\big) \tag{13}$$

via which $N'_{J,{}^I w}$ acts on $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(N'_{J,{}^I w}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w}))$ by right translation, the action of $b \in B_{J,w_J}$ on $f \in \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(N'_{J,{}^I w}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w}))$ is given by

$$(b \cdot f)(n') = b \cdot f(b^{-1} n' b)$$

for all $n' \in N'_{J,{}^I w}$ and the action of $N''_{J,{}^I w}$ on $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(N'_{J,{}^I w}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w}))$ is given by the following result.

Lemma 2.3.1. *Let* $f \in \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(N'_{J,{}^I w}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w}))$ *and* $n'' \in N''_{J,{}^I w}$. *Via* (13), *the action of* n'' *on* f *is given by*

$$(n'' \cdot f)(n')(u') = n'' \cdot f(u'^{-1} n''^{-1} u' n' n'')(u')$$

for all $n' \in N'_{J,{}^I w}$ *and* $u' \in U'_{J,w_J}$.

Proof. Let $n' \in N'_{J,{}^I w}$ and $u' \in U'_{J,w_J}$. We have

$${}^I w u' n' n'' = ({}^I w n'' {}^I w^{-1}) {}^I w u' (u'^{-1} n''^{-1} u' n' n'').$$

Thus, it is enough to check that $u'^{-1} n''^{-1} u' n' n'' \in N'_{J,{}^I w}$. Since $u' \in U_J$ and $n', n'' \in N_J$, we have $(u'^{-1} n''^{-1} u') n' n'' \in N_J$. Since $n'' \in {}^I w^{-1} U_I {}^I w$ and $n', u' \in {}^I w^{-1} N_I {}^I w$, we have $u'^{-1}(n''^{-1}(u'n')n'') \in {}^I w^{-1} N_I {}^I w$. Hence the result. ☐

Remark 2.3.2 (case $w_J = 1$). We can also give the action of $L_{J \cap {}^I w^{J-1}(I)}$ (which normalises $U'_{J,1}$, $N'_{J,{}^I w^J}$, $N''_{J,{}^I w^J}$, and thus $U'_{{}^I w^J}$, $N_{J,{}^I w^J}$) on $\operatorname{c-ind}_{P_I^-}^{P_I^- {}^I w^J B} \sigma$ and $\operatorname{c-ind}_{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)}}^{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} B_J} \tilde{\sigma}$ (see Remark 2.2.2) via (13) and (12) respectively, by replacing $B_{J,w_J} = B''_{J,w_J} \ltimes U'_{J,w_J}$ by $L_J \cap P_{J \cap {}^I w^{J-1}(I)} = L_{J \cap {}^I w^{J-1}(I)} \ltimes U'_{J,1}$.

We end this subsection with some more notation.

Notation 2.3.3. For each $w \in W$, we let d_w be the integer $\dim_F(\boldsymbol{U}/\boldsymbol{U}_w)$ and $\delta_w \in \mathrm{X}^*(\boldsymbol{S})$ be the algebraic character of the adjoint representation of $\boldsymbol{S}$ on $\det_F((\operatorname{Lie} \boldsymbol{U})/(\operatorname{Lie} \boldsymbol{U}_w))$. Note that $d_w \geq \ell(w)$ and δ_w extends to an algebraic character of $\boldsymbol{\mathcal{Z}}$. For $\alpha \in \Delta$, we have $d_{s_\alpha} = \dim_F \boldsymbol{U}_\alpha$ and $\delta_{s_\alpha} = d_{s_\alpha} \alpha$. We define a subset of Δ by setting

$$\Delta^1 := \left\{ \alpha \in \Delta \,\middle|\, \dim_F \boldsymbol{U}_\alpha = 1 \right\}.$$

For $I \subseteq \Delta$, we put $I^1 := I \cap \Delta^1$.

Remark 2.3.4. For ${}^Iw^J \in {}^IW^J$, we have $\boldsymbol{U}_J \subseteq \boldsymbol{U}_{{}^Iw^J}$ and $\boldsymbol{L}_{J \cap {}^Iw^{J-1}(I)}$ normalises $\boldsymbol{N}_{J,{}^Iw^J}$. Thus, the inclusion $\boldsymbol{N}_J \hookrightarrow \boldsymbol{U}$ induces an isomorphism of F-varieties

$$\boldsymbol{N}_J/\boldsymbol{N}_{J,{}^Iw^J} \xrightarrow{\sim} \boldsymbol{U}/\boldsymbol{U}_{{}^Iw^J}$$

and there is an adjoint action of $\boldsymbol{L}_{J \cap {}^Iw^{J-1}(I)}$ on $(\mathrm{Lie}\,\boldsymbol{N}_J)/(\mathrm{Lie}\,\boldsymbol{N}_{J,{}^Iw^J})$. Therefore, we have $d_{{}^Iw^J} = \dim_F(\boldsymbol{N}_J/\boldsymbol{N}_{J,{}^Iw^J})$ and $\delta_{{}^Iw^J}$ extends to an algebraic character of $\boldsymbol{L}_{J \cap {}^Iw^{J-1}(I)}$.

3. Derived ordinary parts

The aim of this section is to compute the derived ordinary parts of a parabolically induced representation. In Section 3.1, we show how to compute the cohomology of certain groups with a Hecke action from the cohomology of certain subgroups with the induced Hecke action, provided the latter satisfy some finiteness condition. In Section 3.2, we make a computation of cohomology and Hecke action on a compactly induced representation. In Section 3.3, we use the previous results to partially compute the derived ordinary parts of the graded representations associated with the Bruhat filtrations, we formulate a conjecture on the complete result and we prove it in many cases in low degree.

3.1. ***Cohomology, Hecke action and dévissage.*** Let $\widetilde{\boldsymbol{L}}$ be a linear algebraic F-group and $\widetilde{\boldsymbol{N}}$ be a unipotent algebraic F-group endowed with an action of $\widetilde{\boldsymbol{L}}$ that we identify with the conjugation in $\widetilde{\boldsymbol{L}} \ltimes \widetilde{\boldsymbol{N}}$. We let $\tilde{d}$ denote the integer $\dim_F \widetilde{\boldsymbol{N}}$ and $\tilde{\delta} \in \mathrm{X}^*(\widetilde{\boldsymbol{L}})$ denote the algebraic character of the adjoint representation of $\widetilde{\boldsymbol{L}}$ on $\det_F(\mathrm{Lie}\,\widetilde{\boldsymbol{N}})$.

Let $\widetilde{L}^+ \subseteq \widetilde{L}$ be an open submonoid and $\widetilde{N}_0 \subseteq \widetilde{N}$ be a standard[3] compact open subgroup stable under conjugation by $\widetilde{L}^+$. If π is an $\widetilde{L}^+ \ltimes \widetilde{N}_0$-representation,[4] then the A-modules of $\widetilde{N}_0$-cohomology $\mathrm{H}^\bullet(\widetilde{N}_0, \pi)$ computed using locally constant cochains (or equivalently an N_0-injective resolution of π; see [Emerton 2010b, Proposition 2.2.6]) are naturally endowed with the Hecke action of $\widetilde{L}^+$ (denoted $\overset{\mathrm{H}}{\cdot}$), defined for every $\tilde{l} \in \widetilde{L}^+$ as the composite

$$\mathrm{H}^\bullet(\widetilde{N}_0, \pi) \to \mathrm{H}^\bullet(\tilde{l}\widetilde{N}_0\tilde{l}^{-1}, \pi) \to \mathrm{H}^\bullet(\widetilde{N}_0, \pi)$$

where the first morphism is induced by the action of $\tilde{l}$ on π and the second morphism is the corestriction from $\tilde{l}\widetilde{N}_0\tilde{l}^{-1}$ to $\widetilde{N}_0$ (this defines a natural smooth A-linear action of $\widetilde{L}^+$ in degree 0 [Emerton 2010a, Lemma 3.1.4] that extends in higher degrees by universality of $\mathrm{H}^\bullet(\widetilde{N}_0, -)$). We obtain a universal δ-functor

$$\mathrm{H}^\bullet(\widetilde{N}_0, -) : \mathrm{Mod}^{\mathrm{sm}}_{\widetilde{L}^+ \ltimes \widetilde{N}_0}(A) \to \mathrm{Mod}^{\mathrm{sm}}_{\widetilde{L}^+}(A),$$

since an injective $\widetilde{L}^+ \ltimes \widetilde{N}_0$-representation is $\widetilde{N}_0$-acyclic [Emerton 2010b, Proposition 2.1.11; Hauseux 2016a, Lemme 3.1.1].

[3]The exponential map $\exp : \mathrm{Lie}\,\widetilde{\boldsymbol{N}} \to \widetilde{\boldsymbol{N}}$ is an isomorphism of F-varieties [Demazure and Gabriel 1970, Chapitre IV, §2, Proposition 4.1] and we say that $\widetilde{N}_0$ is *standard* if $\mathrm{Lie}\,\widetilde{N}_0 := \exp^{-1}(\widetilde{N}_0) \subseteq \mathrm{Lie}\,\widetilde{N}$ is a $\mathbb{Z}_p$-Lie subalgebra. The identity of $\widetilde{N}$ admits a basis of neighbourhoods consisting of standard compact open subgroups [Emerton 2010b, Lemma 3.5.2]

[4]Given a p-adic Lie group H and an open submonoid $H^+ \subseteq H$, a representation of H^+ over A is *smooth* if its restriction to an open subgroup of H contained in H^+ is smooth.

Let $\widetilde{\boldsymbol{Z}} \subseteq \widetilde{\boldsymbol{L}}$ be a central split torus and $\widetilde{Z}^+ \subseteq \widetilde{Z}$ be the open submonoid $\widetilde{Z} \cap \widetilde{L}^+$. Since $\widetilde{\boldsymbol{Z}}$ is split, its adjoint representation on Lie $\widetilde{\boldsymbol{N}}$ is a direct sum of weights. We assume that there exists $\tilde{\lambda} \in \mathrm{X}_*(\widetilde{\boldsymbol{Z}})$ such that $\langle \tilde{\mu}, \tilde{\lambda} \rangle > 0$ for any weight $\tilde{\mu}$ of $\widetilde{\boldsymbol{Z}}$ in Lie $\widetilde{\boldsymbol{N}}$. We fix an element $\tilde{z} := \tilde{\lambda}(p^j) \in \widetilde{Z}$ with $j \in \mathbb{N}$ large enough so that $\tilde{z}$ is *strictly contracting* $\widetilde{N}_0$, i.e., $(\tilde{z}^i \widetilde{N}_0 \tilde{z}^{-i})_{i \in \mathbb{N}}$ is a basis of neighbourhoods of the identity in $\widetilde{N}_0$; see [Emerton 2010b, Lemma 3.1.3] using the fact that $\mathrm{ord}_p(\tilde{\mu}(\tilde{z})) = \langle \tilde{\mu}, \tilde{\lambda} \rangle j$ for any weight $\tilde{\mu}$ of $\widetilde{\boldsymbol{Z}}$ in Lie $\widetilde{\boldsymbol{N}}$). In particular $\tilde{z} \in \widetilde{Z}^+$.

If π is a $\widetilde{Z}^+$-representation, we say that π is *locally $\tilde{z}$-finite* if for every $v \in \pi$, the A-submodule $A[\tilde{z}] \cdot v$ is of finite type, and we say that the action of $\tilde{z}$ on π is *locally nilpotent* if for every $v \in \pi$, there exists $i \in \mathbb{N}$ such that $\tilde{z}^i \cdot v = 0$.

Lemma 3.1.1. *Let π be a locally $\tilde{z}$-finite $\widetilde{L}^+ \ltimes \widetilde{N}_0$-representation and $n \in \mathbb{N}$.*

(i) *If $n = [F : \mathbb{Q}_p]\tilde{d}$, then the action of $\tilde{z}$ on the kernel of the natural $\widetilde{L}^+$-equivariant surjection $\pi \otimes (\omega^{-1} \circ \tilde{\delta}) \twoheadrightarrow \mathrm{H}^n(\widetilde{N}_0, \pi)$ is locally nilpotent.*

(ii) *If $n < [F : \mathbb{Q}_p]\tilde{d}$, then the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0, \pi)$ is locally nilpotent.*

Proof. We prove (i). The natural $\widetilde{L}^+$-equivariant surjection in the statement is the composite

$$\pi \otimes (\omega^{-1} \circ \tilde{\delta}) \twoheadrightarrow \pi_{\widetilde{N}_0} \otimes (\omega^{-1} \circ \tilde{\delta}) \cong \mathrm{H}^{[F:\mathbb{Q}_p]\tilde{d}}(\widetilde{N}_0, \pi)$$

where the first morphism is the natural projection onto the $\widetilde{N}_0$-coinvariants of π and the second morphism is the natural isomorphism [Hauseux 2016b, (2.2)] which is due to Emerton (in loc. cit. $\tilde{\alpha} \in \mathrm{X}^*(\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{L}})$ is the algebraic character of the adjoint representation of $\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{L}}$ on $\det_{\mathbb{Q}_p}(\mathrm{Lie}(\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{N}}))$ so that $\tilde{\alpha} = \mathrm{Nrm}_{F/\mathbb{Q}_p} \circ \tilde{\delta}$ as $\mathbb{Q}_p^\times$-valued characters of $\widetilde{L}$, hence $\tilde{\alpha}^{-1} |\tilde{\alpha}|_p^{-1} = \omega^{-1} \circ \tilde{\delta}$ as $\mathbb{Q}_p^\times$-valued characters of $\widetilde{L}$). For every $v \in \pi$, there exists $i \in \mathbb{N}$ such that $\tilde{z}^i \widetilde{N}_0 \tilde{z}^{-i}$ fixes $A[\tilde{z}] \cdot v$ (since π is locally $\tilde{z}$-finite and $\tilde{z}$ is strictly contracting $\widetilde{N}_0$), so that for all $\tilde{n} \in \widetilde{N}_0$ we have

$$\tilde{z}^i \cdot (\tilde{n} \cdot v - v) = (\tilde{z}^i \tilde{n} \tilde{z}^{-i}) \cdot (\tilde{z}^i \cdot v) - (\tilde{z}^i \cdot v) = 0.$$

Thus the action of $\tilde{z}$ on the kernel of the above surjection is locally nilpotent.

We prove (ii). Let $(\tilde{\mu}_r)_{r \in [\![0, m-1]\!]}$ be an enumeration of the weights of $\widetilde{\boldsymbol{Z}}$ in Lie $\widetilde{\boldsymbol{N}}$ such that the sequence $(\langle \tilde{\mu}_r, \tilde{\lambda} \rangle)_{r \in [\![0, m-1]\!]}$ is increasing. If $\tilde{\mu}_i + \tilde{\mu}_j = \tilde{\mu}_r$ with $i, j, r \in [\![0, m-1]\!]$, then $r > \max\{i, j\}$ (since $\langle \tilde{\mu}_r, \tilde{\lambda} \rangle > \max\{\langle \tilde{\mu}_i, \tilde{\lambda} \rangle, \langle \tilde{\mu}_j, \tilde{\lambda} \rangle\}$). Thus for all $r \in [\![0, m]\!]$, the direct sum of the weight spaces corresponding to $\tilde{\mu}_r, \dots, \tilde{\mu}_{m-1}$ is an ideal of Lie $\widetilde{\boldsymbol{N}}$ stable under the adjoint action of $\widetilde{\boldsymbol{Z}}$ and we let $\widetilde{\boldsymbol{N}}^{(r)} \subseteq \widetilde{\boldsymbol{N}}$ be the corresponding closed normal subgroup stable under conjugation by $\widetilde{\boldsymbol{Z}}$, $\tilde{d}_r$ denote the integer $\dim_F \widetilde{\boldsymbol{N}}^{(r)}$, $\tilde{\delta}_r \in \mathrm{X}^*(\widetilde{\boldsymbol{L}})$ denote the algebraic character of the adjoint representation of $\widetilde{\boldsymbol{L}}$ on $\det_F(\mathrm{Lie}\, \widetilde{\boldsymbol{N}}^{(r)})$ and $\widetilde{N}_0^{(r)} \subseteq \widetilde{N}^{(r)}$ be the standard compact open subgroup $\widetilde{N}^{(r)} \cap \widetilde{N}_0$ stable under conjugation by $\widetilde{Z}^+$.

Let $r \in [\![0, m]\!]$. We assume that $n < [F : \mathbb{Q}_p]\tilde{d}_r$ and we prove that the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r)}, \pi)$ is locally nilpotent by induction on r. The result is trivial for $r = m$. We assume $r < m$ and the result true for $r + 1$. We have a short exact sequence of topological groups

$$1 \to \widetilde{N}_0^{(r+1)} \to \widetilde{N}_0^{(r)} \to \widetilde{N}_0^{(r)} / \widetilde{N}_0^{(r+1)} \to 1.$$

The Lyndon–Hochschild–Serre spectral sequence corresponding to this dévissage is naturally a spectral sequence of $\widetilde{L}^+$-representations (see [Hauseux 2016b, (2.3)])

$$\mathrm{H}^i\big(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^j\big(\widetilde{N}_0^{(r+1)}, \pi\big)\big) \Rightarrow \mathrm{H}^{i+j}\big(\widetilde{N}_0^{(r)}, \pi\big). \tag{14}$$

Let $i, j \in \mathbb{N}$ be such that $i+j=n$. If $j < [F:\mathbb{Q}_p]\tilde{d}_{r+1}$, the Hecke action of $\tilde{z}$ on $\mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)$ is locally nilpotent by the induction hypothesis; thus the Hecke action of $\tilde{z}$ on $\mathrm{H}^i(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi))$ is also locally nilpotent (since the image of a locally constant cochain is finite by compactness). If $j = [F:\mathbb{Q}_p]\tilde{d}_{r+1}$, then $i < [F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$ and we deduce from (i) with $\widetilde{\boldsymbol{N}}^{(r+1)}$ and j instead of $\widetilde{\boldsymbol{N}}$ and n respectively that $\mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)$ is locally $\tilde{z}$-finite; thus the Hecke action of $\tilde{z}$ on $\mathrm{H}^i(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi))$ is locally nilpotent by the sublemma below with $\tilde{\mu} = \tilde{\mu}_r$ and $\widetilde{\boldsymbol{N}}^{(r)}/\widetilde{\boldsymbol{N}}^{(r+1)}$, $\mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)$, i instead of $\widetilde{\boldsymbol{N}}, \pi, n$ respectively. If $j > [F:\mathbb{Q}_p]\tilde{d}_{r+1}$, then $\mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi) = 0$ by [Emerton 2010b, Lemma 3.5.4]; thus $\mathrm{H}^i(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)) = 0$. Using (14), we conclude that the action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r)}, \pi)$ is locally nilpotent. □

Sublemma. *Let π be a locally $\tilde{z}$-finite $\widetilde{Z}^+ \ltimes \widetilde{N}_0$-representation, $\tilde{\mu} \in \mathrm{X}^*(\widetilde{\boldsymbol{Z}})$ and $n \in \mathbb{N}$. Assume that the adjoint action of $\widetilde{\boldsymbol{Z}}$ on* Lie $\widetilde{\boldsymbol{N}}$ *factors through $\tilde{\mu}$. If $n < [F:\mathbb{Q}_p]\tilde{d}$, then the Hecke action of $\tilde{z}$ on* $\mathrm{H}^n(\widetilde{N}_0, \pi)$ *is locally nilpotent.*

Proof. Let $\widetilde{\boldsymbol{S}} \subseteq \mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{Z}}$ be the maximal split subtorus, $\widetilde{S} \subseteq \widetilde{Z}$ be the closed subgroup $\widetilde{\boldsymbol{S}}(\mathbb{Q}_p)$ and $\widetilde{S}^+ \subseteq \widetilde{S}$ be the open submonoid $\widetilde{S} \cap \widetilde{Z}^+$. Every algebraic (co)character of $\widetilde{\boldsymbol{Z}}$ induces by restriction of scalars a (co)character of $\widetilde{\boldsymbol{S}}$ (since the image of a split torus by a morphism of algebraic groups is a split torus [Borel and Tits 1965, §1.4]). In particular, the restriction of $\tilde{\lambda} : F^\times \to \widetilde{Z}$ to $\mathbb{Q}_p^\times$ takes values in $\widetilde{S}$ and the restriction of $\tilde{\mu} : \widetilde{Z} \to F^\times$ to $\widetilde{S}$ takes values in $\mathbb{Q}_p^\times$.

We deduce on the one hand that $\tilde{z} \in \widetilde{S}^+$, and on the other hand that the adjoint action of $\widetilde{\boldsymbol{S}}$ on $\mathrm{Lie}(\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{N}})$ factors through an algebraic character so that any closed subgroup of $\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{N}}$ is stable under conjugation by $\widetilde{\boldsymbol{S}}$. Since $\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{N}}$ is unipotent, there exists a composition series

$$\mathrm{Res}_{F/\mathbb{Q}_p} \widetilde{\boldsymbol{N}} = \widetilde{\boldsymbol{N}}^{(0)} \rhd \widetilde{\boldsymbol{N}}^{(1)} \rhd \cdots \rhd \widetilde{\boldsymbol{N}}^{([F:\mathbb{Q}_p]\tilde{d})} = 1$$

whose successive quotients are isomorphic to the additive group over $\mathbb{Q}_p$ and for all $r \in [\![0, [F:\mathbb{Q}_p]\tilde{d}]\!]$, we let $\widetilde{N}^{(r)} \subseteq \widetilde{N}$ be the closed subgroup $\widetilde{\boldsymbol{N}}^{(r)}(\mathbb{Q}_p)$ and $\widetilde{N}_0^{(r)} \subseteq \widetilde{N}^{(r)}$ be the standard compact open subgroup $\widetilde{N}^{(r)} \cap \widetilde{N}_0$ stable under conjugation by $\widetilde{S}^+$.

Let $r \in [\![0, [F:\mathbb{Q}_p]\tilde{d}]\!]$. We assume that $n < [F:\mathbb{Q}_p]\tilde{d} - r$ and we prove by induction on r that the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r)}, \pi)$ is locally nilpotent. The result is trivial for $r = [F:\mathbb{Q}_p]\tilde{d}$. We assume $r < [F:\mathbb{Q}_p]\tilde{d}$ and the result true for $r+1$. Since $\dim_{\mathbb{Q}_p}(\widetilde{\boldsymbol{N}}^{(r)}/\widetilde{\boldsymbol{N}}^{(r+1)}) = 1$, we have a short exact sequence of $\widetilde{S}^+$-representations (see [Hauseux 2016b, (2.4)])

$$0 \to \mathrm{H}^1\big(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^{n-1}\big(\widetilde{N}_0^{(r+1)}, \pi\big)\big) \to \mathrm{H}^n\big(\widetilde{N}_0^{(r)}, \pi\big) \to \mathrm{H}^n\big(\widetilde{N}_0^{(r+1)}, \pi\big)^{\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}} \to 0. \tag{15}$$

The Hecke action of $\tilde{z}$ on $\mathrm{H}^{n-1}(\widetilde{N}_0^{(r+1)}, \pi)$ is locally nilpotent by the induction hypothesis; thus the Hecke action of $\tilde{z}$ on $\mathrm{H}^1(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^{n-1}(\widetilde{N}_0^{(r+1)}, \pi))$ is also locally nilpotent. If $n < [F:\mathbb{Q}_p]\tilde{d} - (r+1)$, then the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r+1)}, \pi)$ is locally nilpotent by induction; thus the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r+1)}, \pi)^{\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}}$ is also locally nilpotent. If $n = [F:\mathbb{Q}_p]\tilde{d} - (r+1)$, then we have a natural $\widetilde{S}^+$-equivariant surjection $\pi \otimes \tilde{\mu}^{-n}|\tilde{\mu}|_p^{-n} \twoheadrightarrow \mathrm{H}^n(\widetilde{N}_0^{(r+1)}, \pi)$ [op. cit., (2.2)] and we deduce that $\mathrm{H}^n(\widetilde{N}_0^{(r+1)}, \pi)$ is locally $\tilde{z}$-finite. In this case, we put $\widetilde{N}_0'' := \widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}$. For every $v \in \mathrm{H}^n(\widetilde{N}_0^{(r+1)}, \pi)$, there exists $i \in \mathbb{N}$ such that $\tilde{z}^i \widetilde{N}_0'' \tilde{z}^{-i}$ fixes $A[\tilde{z}] \cdot v$, so that for all $j \in \mathbb{N}$ we have

$$\begin{aligned}\tilde{z}^{i+j} \overset{\mathrm{H}}{\cdot} v &= \sum_{\tilde{n}'' \in \widetilde{N}_0''/\tilde{z}^{i+j}\widetilde{N}_0''\tilde{z}^{-(i+j)}} \tilde{n}'' \cdot (\tilde{z}^{i+j} \cdot v) \\ &= (\tilde{z}^i \widetilde{N}_0'' \tilde{z}^{-i} : \tilde{z}^{i+j} \widetilde{N}_0'' \tilde{z}^{-(i+j)}) \sum_{\tilde{n}'' \in \widetilde{N}_0''/\tilde{z}^{i}\widetilde{N}_0''\tilde{z}^{-i}} \tilde{n}'' \cdot (\tilde{z}^{i+j} \cdot v) \\ &= (\widetilde{N}_0'' : \tilde{z}^j \widetilde{N}_0'' \tilde{z}^{-j}) \sum_{\tilde{n}'' \in \widetilde{N}_0''/\tilde{z}^{i}\widetilde{N}_0''\tilde{z}^{-i}} \tilde{n}'' \cdot (\tilde{z}^{i+j} \cdot v).\end{aligned}$$

Now $\widetilde{N}_0''$ is an infinite pro-p group, $\tilde{z}$ is strictly contracting $\widetilde{N}_0''$ and A is Artinian. Thus $(\widetilde{N}_0'' : \tilde{z}^j \widetilde{N}_0'' \tilde{z}^{-j})$ is zero in A for $j \in \mathbb{N}$ large enough. Therefore, the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r+1)}, \pi)^{\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}}$ is locally nilpotent. Using (15), we conclude that the Hecke action of $\tilde{z}$ on $\mathrm{H}^n(\widetilde{N}_0^{(r)}, \pi)$ is locally nilpotent. □

Let $\widetilde{\boldsymbol{N}}' \subseteq \widetilde{\boldsymbol{N}}$ be a closed subgroup such that $\mathrm{Lie}\, \widetilde{\boldsymbol{N}}' \subseteq \mathrm{Lie}\, \widetilde{\boldsymbol{N}}$ is a direct sum of weight spaces of $\widetilde{\boldsymbol{Z}}$. We stress that $\widetilde{\boldsymbol{N}}'$ need not be normal. Since $\widetilde{\boldsymbol{Z}}$ is central in $\widetilde{\boldsymbol{L}}$, $\mathrm{Lie}\, \widetilde{\boldsymbol{N}}'$ is stable under the adjoint action of $\widetilde{\boldsymbol{L}}$; thus $\widetilde{\boldsymbol{N}}'$ is stable under conjugation by $\widetilde{\boldsymbol{L}}$. We let $\tilde{d}'$ denote the integer $\dim_F \widetilde{\boldsymbol{N}}'$ and $\tilde{\delta}' \in \mathrm{X}^*(\widetilde{\boldsymbol{L}})$ denote the algebraic character of the adjoint representation of $\widetilde{\boldsymbol{L}}$ on $\det_F(\mathrm{Lie}\, \widetilde{\boldsymbol{N}}')$. We let $\widetilde{N}_0' \subseteq \widetilde{N}'$ be the standard compact open subgroup $\widetilde{N}' \cap \widetilde{N}_0$ stable under conjugation by $\widetilde{L}^+$.

Proposition 3.1.2. *Let π be an $\widetilde{L}^+ \ltimes \widetilde{N}_0$-representation. For all $n \in \mathbb{N}$, there is a natural $\widetilde{L}^+$-equivariant morphism*

$$\mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}-\tilde{d}')}(\widetilde{N}_0', \pi) \otimes (\omega^{-1} \circ (\tilde{\delta} - \tilde{\delta}')) \to \mathrm{H}^n(\widetilde{N}_0, \pi).$$

Furthermore, the Hecke action of $\tilde{z}$ on its kernel and cokernel is locally nilpotent if the $\widetilde{L}^+$-representations $\mathrm{H}^\bullet(\widetilde{N}_0', \pi)$ are locally $\tilde{z}$-finite.

Proof. Let $(\tilde{\mu}_r)_{r \in [\![0, m-m'-1]\!]}$ be an enumeration of the weights of $\widetilde{\boldsymbol{Z}}$ in $(\mathrm{Lie}\, \widetilde{\boldsymbol{N}})/(\mathrm{Lie}\, \widetilde{\boldsymbol{N}}')$ such that the sequence $(\langle \tilde{\mu}_r, \tilde{\lambda} \rangle)_{r \in [\![0, m-m'-1]\!]}$ is increasing and $(\tilde{\mu}_r)_{r \in [\![m-m', m-1]\!]}$ be an enumeration of the weights of $\widetilde{\boldsymbol{Z}}$ in $\mathrm{Lie}\, \widetilde{\boldsymbol{N}}'$ such that the sequence $(\langle \tilde{\mu}_r, \tilde{\lambda} \rangle)_{r \in [\![m-m', m-1]\!]}$ is increasing. If $\tilde{\mu}_i + \tilde{\mu}_j = \tilde{\mu}_r$ with $i, j, r \in [\![0, m-1]\!]$, then $r > \min\{i, j\}$ (since $\langle \tilde{\mu}_r, \tilde{\lambda} \rangle > \max\{\langle \tilde{\mu}_i, \tilde{\lambda} \rangle, \langle \tilde{\mu}_j, \tilde{\lambda} \rangle\}$). Thus for all $r \in [\![0, m-m']\!]$, the direct sum of the weight spaces corresponding to $\tilde{\mu}_r, \ldots, \tilde{\mu}_{m-1}$ is a Lie subalgebra of $\mathrm{Lie}\, \widetilde{\boldsymbol{N}}$ stable under the adjoint action of $\widetilde{\boldsymbol{Z}}$ and we use the notations $\widetilde{\boldsymbol{N}}^{(r)}$, $\tilde{d}_r$, $\tilde{\delta}_r$ and $\widetilde{N}_0^{(r)}$ as in the proof of Lemma 3.1.1(ii). Moreover for all $r \in [\![0, m-m'-1]\!]$, $\mathrm{Lie}\, \widetilde{\boldsymbol{N}}^{(r+1)}$ is an ideal of $\mathrm{Lie}\, \widetilde{\boldsymbol{N}}^{(r)}$ so that $\widetilde{\boldsymbol{N}}^{(r+1)}$ is a normal subgroup of $\widetilde{\boldsymbol{N}}^{(r)}$.

Let $r \in [\![0, m-m']\!]$. We prove by induction on r that for all $n \in \mathbb{N}$, there is a natural $\widetilde{L}^+$-equivariant morphism

$$\mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}')}(\widetilde{N}_0', \pi) \otimes (\omega^{-1} \circ (\tilde{\delta}_r - \tilde{\delta}')) \to \mathrm{H}^n(\widetilde{N}_0^{(r)}, \pi). \tag{16}$$

The result is trivial for $r = m - m'$. We assume $r < m - m'$ and the result true for $r + 1$. Let $n \in \mathbb{N}$. Since $\dim_F(\widetilde{N}^{(r)}/\widetilde{N}^{(r+1)}) = \tilde{d}_r - \tilde{d}_{r+1}$, we deduce from [Emerton 2010b, Lemma 3.5.4] that (14) yields a natural $\widetilde{L}^+$-equivariant morphism

$$\mathrm{H}^{[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}\big(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}(\widetilde{N}_0^{(r+1)}, \pi)\big) \to \mathrm{H}^n(\widetilde{N}_0^{(r)}, \pi) \tag{17}$$

whose kernel and cokernel are built out of subquotients of $\mathrm{H}^i\big(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)\big)$ with $i, j \in \mathbb{N}$ such that $i < [F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$. Furthermore, Lemma 3.1.1(i) with $\widetilde{\boldsymbol{N}}$, π, n and $\tilde{\delta}$ replaced by $\widetilde{\boldsymbol{N}}^{(r)}/\widetilde{\boldsymbol{N}}^{(r+1)}$, $\mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}(\widetilde{N}_0^{(r+1)}, \pi)$, $[F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$ and $\tilde{\delta}_r - \tilde{\delta}_{r+1}$ respectively yields a natural $\widetilde{L}^+$-equivariant surjection

$$\begin{aligned}\mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}&(\widetilde{N}_0^{(r+1)}, \pi) \otimes (\omega^{-1} \circ (\tilde{\delta}_r - \tilde{\delta}_{r+1}))\\ &\twoheadrightarrow \mathrm{H}^{[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}\big(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}(\widetilde{N}_0^{(r+1)}, \pi)\big).\end{aligned} \tag{18}$$

Finally, by the induction hypothesis with $n - [F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$ instead of n, there is a natural $\widetilde{L}^+$-equivariant morphism

$$\mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}')}(\widetilde{N}'_0, \pi) \otimes (\omega^{-1} \circ (\tilde{\delta}_r - \tilde{\delta}')) \to \mathrm{H}^{n-[F:\mathbb{Q}_p](\tilde{d}_r-\tilde{d}_{r+1})}(\widetilde{N}_0^{(r+1)}, \pi) \otimes (\omega^{-1} \circ (\tilde{\delta}_r - \tilde{\delta}_{r+1})). \tag{19}$$

The composition of (17), (18) and (19) yields the natural $\widetilde{L}^+$-equivariant morphism (16).

Now, we assume that the $\widetilde{L}^+$-representations $\mathrm{H}^\bullet(\widetilde{N}'_0, \pi)$ are locally $\tilde{z}$-finite and we prove by induction on r that for all $n \in \mathbb{N}$, the Hecke action of $\tilde{z}$ on the kernel and cokernel of (16) is locally nilpotent, or equivalently that the localisation of (16) with respect to $\tilde{z}^{\mathbb{N}}$ is an isomorphism. The result is trivial for $r = m - m'$. We assume $r < m - m'$ and the result true for $r+1$. Let $n \in \mathbb{N}$. By composition, it is enough to prove that the Hecke action of $\tilde{z}$ on the kernels and cokernels of (17), (18) and (19) is locally nilpotent. By the induction hypothesis with j instead of n, the Hecke action of $\tilde{z}$ on the kernel and cokernel of the natural $\widetilde{L}^+$-equivariant morphism

$$\mathrm{H}^{j-[F:\mathbb{Q}_p](\tilde{d}_{r+1}-\tilde{d}')}(\widetilde{N}'_0, \pi) \otimes (\omega^{-1} \circ (\tilde{\delta}_{r+1} - \tilde{\delta}')) \to \mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)$$

is locally nilpotent for all $j \in \mathbb{N}$. With $j = n - [F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$, we deduce that the Hecke action of $\tilde{z}$ on the kernel and cokernel of (19) is locally nilpotent. Furthermore, we deduce that $\mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)$ is locally $\tilde{z}$-finite for all $j \in \mathbb{N}$ and we use Lemma 3.1.1 with $\widetilde{\boldsymbol{N}}^{(r)}/\widetilde{\boldsymbol{N}}^{(r+1)}$, $\mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi)$ and i instead of $\widetilde{\boldsymbol{N}}$, π and n respectively: we deduce from (i) with $i = [F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$ that the Hecke action of $\tilde{z}$ on the kernel of (18) is locally nilpotent, and we deduce from (ii) that the Hecke action of $\tilde{z}$ on the kernel and cokernel of (17) is locally nilpotent (since the Hecke action of $\tilde{z}$ on $\mathrm{H}^i(\widetilde{N}_0^{(r)}/\widetilde{N}_0^{(r+1)}, \mathrm{H}^j(\widetilde{N}_0^{(r+1)}, \pi))$ is locally nilpotent for all $i, j \in \mathbb{N}$ such that $i < [F:\mathbb{Q}_p](\tilde{d}_r - \tilde{d}_{r+1})$). □

We end this subsection by reviewing and generalising the construction of Emerton's δ-functor of derived ordinary parts [Emerton 2010b, §3.3]. Let $\widetilde{\boldsymbol{Z}}_{\widetilde{\boldsymbol{L}}}$ denote the centre of $\widetilde{\boldsymbol{L}}$. Assume that $\widetilde{\boldsymbol{Z}}_{\widetilde{\boldsymbol{L}}}^\circ$ is a torus, that $\widetilde{Z}_{\widetilde{L}}$ is generated by $\widetilde{Z}_{\widetilde{L}}^+ := \widetilde{Z}_{\widetilde{L}} \cap \widetilde{L}^+$ as a group, and that $\widetilde{L}$ is generated by $\widetilde{L}^+$ and $\widetilde{Z}_{\widetilde{L}}$ as a monoid. Then, the product induces a group isomorphism $\widetilde{L}^+ \times_{\widetilde{Z}_{\widetilde{L}}^+} \widetilde{Z}_{\widetilde{L}} \xrightarrow{\sim} \widetilde{L}$ [Emerton 2006, Proposition 3.3.6]. Thus, for any $\widetilde{L}^+$-representation π, the A-module $\mathrm{Hom}_{A[\widetilde{Z}_{\widetilde{L}}^+]}(A[\widetilde{Z}_{\widetilde{L}}], \pi)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}}$ is naturally an $\widetilde{L}$-representation [Emerton 2010a, Lemma 3.1.7]. Therefore, we obtain an A-linear left-exact functor $\mathrm{Mod}_{\widetilde{L}^+}^{\mathrm{sm}}(A) \to \mathrm{Mod}_{\widetilde{L}}^{\mathrm{sm}}(A)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}}$ which commutes with inductive limits [op. cit., Lemma 3.2.2].

Remark 3.1.3. Let $\tilde{z} \in \widetilde{Z}_{\widetilde{L}}^+$. Assume moreover that $\widetilde{Z}_{\widetilde{L}}$ is generated by $\widetilde{Z}_{\widetilde{L}}^+$ and $\tilde{z}^{-1}$ as a monoid. Then, for any locally finite $\widetilde{Z}_{\widetilde{L}}^+$-representation π, there is a natural $\widetilde{Z}_{\widetilde{L}}$-equivariant isomorphism

$$\mathrm{Hom}_{A[\widetilde{Z}_{\widetilde{L}}^+]}(A[\widetilde{Z}_{\widetilde{L}}], \pi)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}} \xrightarrow{\sim} A[\tilde{z}^{\pm 1}] \otimes_{A[\tilde{z}]} \pi$$

[Emerton 2010b, Lemma 3.2.1]. Thus, the functor $\mathrm{Hom}_{A[\widetilde{Z}_{\widetilde{L}}^+]}(A[\widetilde{Z}_{\widetilde{L}}], -)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}}$ restricted to the category $\mathrm{Mod}_{\widetilde{L}^+}^{\mathrm{sm}}(A)^{\widetilde{Z}_{\widetilde{L}}^+-\mathrm{l.fin}}$ is isomorphic to the localisation with respect to $\tilde{z}^{\mathbb{N}}$. In particular, it is exact.

Definition 3.1.4. For a connected linear algebraic F-group $\widetilde{\boldsymbol{P}}$ with unipotent radical $\widetilde{\boldsymbol{N}}$ such that $\widetilde{\boldsymbol{P}} \cong \widetilde{\boldsymbol{L}} \ltimes \widetilde{\boldsymbol{N}}$, we define A-linear functors $\mathrm{Mod}_{\widetilde{P}}^{\mathrm{sm}}(A) \to \mathrm{Mod}_{\widetilde{L}}^{\mathrm{sm}}(A)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}}$ which commute with inductive limits by setting

$$\mathrm{H}^\bullet\mathrm{Ord}_{\widetilde{P}} := \mathrm{Hom}_{A[\widetilde{Z}_{\widetilde{L}}^+]}\left(A[\widetilde{Z}_{\widetilde{L}}], \mathrm{H}^\bullet(\widetilde{N}_0, -)\right)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}}.$$

If $\widetilde{\boldsymbol{B}} \subseteq \widetilde{\boldsymbol{P}}$ is a connected closed subgroup containing $\widetilde{\boldsymbol{N}}$ and $\widetilde{\boldsymbol{Z}}_{\widetilde{\boldsymbol{L}}}$, then $\widetilde{B}_{\widetilde{L}} := \widetilde{B} \cap \widetilde{L}$ is generated by $\widetilde{B}_{\widetilde{L}}^+ := \widetilde{B}_{\widetilde{L}} \cap \widetilde{L}^+$ and $\widetilde{Z}_{\widetilde{L}}$ as a monoid, so that $\mathrm{H}^\bullet\mathrm{Ord}_{\widetilde{P}}$ naturally extend to A-linear functors $\mathrm{Mod}_{\widetilde{B}}^{\mathrm{sm}}(A) \to \mathrm{Mod}_{\widetilde{B}_{\widetilde{L}}}^{\mathrm{sm}}(A)^{\widetilde{Z}_{\widetilde{L}}-\mathrm{l.fin}}$ which commute with inductive limits.

3.2. ***Computations on the associated graded representations.*** Let $J \subseteq \Delta$. We fix a totally decomposed[5] standard compact open subgroup $N_{J,0} \subseteq N_J$ and we define an open submonoid of L_J by setting

$$L_J^+ := \left\{ l \in L_J \mid l N_{J,0} l^{-1} \subseteq N_{J,0} \right\}.$$

We let $Z_J^+ \subseteq Z_J$ be the open submonoid $Z_J \cap L_J^+$. Note that Z_J is generated by Z_J^+ as a group and L_J is generated by L_J^+ and Z_J as a monoid [Emerton 2006, Proposition 3.3.2]. Moreover, any $\lambda \in \mathrm{X}_*(\boldsymbol{S})$ corresponding to $\boldsymbol{P}_J$ has its image contained in the maximal split subtorus $\boldsymbol{S}_J$ of $\boldsymbol{Z}_J^\circ$ and satisfies $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \Phi^+ \backslash \Phi_J^+$; thus the assumption of Section 3.1 with $\widetilde{\boldsymbol{N}} = \boldsymbol{N}_J$ and $\widetilde{\boldsymbol{Z}} = \boldsymbol{S}_J$ is satisfied. We fix $z \in Z_J^+$ strictly contracting $N_{J,0}$ (equivalently Z_J is generated by Z_J^+ and z^{-1} as a monoid).

Let $I \subseteq \Delta$ and ${}^I w \in {}^I W$. We write ${}^I w = {}^I w^J w_J$ with ${}^I w^J \in {}^I W^J$ and $w_J \in W_J$. Let σ be an L_I-representation. We set[6]

$$\pi_{{}^I w} := \text{c-ind}_{P_I^-}^{P_I^- {}^I w B} \sigma.$$

We use the notation of Section 2.3. The subgroup $\boldsymbol{N}_{J,{}^I w} \subseteq \boldsymbol{N}_J$ is stable under conjugation by $\boldsymbol{B}_{J,w_J}$, and we have a semidirect product $\boldsymbol{N}_{J,{}^I w} = \boldsymbol{N}''_{J,{}^I w} \ltimes \boldsymbol{N}'_{J,{}^I w}$. The subgroup $\boldsymbol{N}'_{J,{}^I w}$ is stable under conjugation by $\boldsymbol{B}_{J,w_J}$, and we endow $\boldsymbol{N}''_{J,{}^I w}$ (which may not be stable under conjugation by $\boldsymbol{B}_{J,w_J}$) with the quotient action of $\boldsymbol{B}_{J,w_J}$ via the isomorphism $\boldsymbol{N}''_{J,{}^I w} \cong \boldsymbol{N}_{J,{}^I w}/\boldsymbol{N}'_{J,{}^I w}$. We let $N_{J,{}^I w,0} \subseteq N_{J,{}^I w}$ (resp. $N'_{J,{}^I w,0} \subseteq N'_{J,{}^I w}$, $N''_{J,{}^I w,0} \subseteq N''_{J,{}^I w}$) be the totally decomposed standard compact open subgroup $N_{J,{}^I w} \cap N_{J,0}$ (resp. $N'_{J,{}^I w} \cap N_{J,0}$, $N''_{J,{}^I w} \cap N_{J,0}$) and $B_{J,w_J}^+ \subseteq B_{J,w_J}$ be the open submonoid $B_{J,w_J} \cap L_J^+$. Since $N_{J,{}^I w,0}$ is totally decomposed, we have a short exact sequence of topological groups

$$1 \to N'_{J,{}^I w,0} \to N_{J,{}^I w,0} \to N''_{J,{}^I w,0} \to 1. \tag{20}$$

[5]Given a closed subgroup $\widetilde{\boldsymbol{U}} \subseteq \boldsymbol{U}$ stable under conjugation by $\boldsymbol{S}$, we say that a compact open subgroup $\widetilde{U}_0 \subseteq \widetilde{U}$ is *totally decomposed* if the product induces a homeomorphism $\prod_{\alpha \in \Phi_0^+} (U_\alpha \cap \widetilde{U}_0) \xrightarrow{\sim} \widetilde{U}_0$ for any order on Φ_0^+ (e.g., $\widetilde{U}_0 = \widetilde{U} \cap K$ where $K \subseteq G$ is a maximal compact subgroup which is special with respect to $\mathcal{Z}$ [Henniart and Vignéras 2015, §6.6, Remark 2]).

[6]The naturality of a morphism involving $\pi_{{}^I w}$ will mean its functoriality with respect to σ.

In particular, $N''_{J,{}^I w,0}$ is stable under the quotient action of B^+_{J,w_J} on $N''_{J,{}^I w}$.

Lemma 3.2.1. *For all $n \in \mathbb{N}$, the inflation map is a natural B^+_{J,w_J}-equivariant isomorphism*

$$\mathrm{H}^n\big(N''_{J,{}^I w,0}, \pi_{{}^I w}^{N'_{J,{}^I w,0}}\big) \xrightarrow{\sim} \mathrm{H}^n(N_{J,{}^I w,0}, \pi_{{}^I w}).$$

Proof. The Lyndon–Hochschild–Serre spectral sequence corresponding to (20) is naturally a spectral sequence of B^+_{J,w_J}-representations [Hauseux 2016b, (2.3)]

$$\mathrm{H}^i\big(N''_{J,{}^I w,0}, \mathrm{H}^j(N'_{J,{}^I w,0}, \pi_{{}^I w})\big) \Rightarrow \mathrm{H}^{i+j}(N_{J,{}^I w,0}, \pi_{{}^I w}). \tag{21}$$

The inflation maps are the edge maps of (21) for $j = 0$; thus they are B^+_{J,w_J}-equivariant and in order to prove that they are bijective, it is enough to show that (21) degenerates, i.e., that $\mathrm{H}^j(N'_{J,{}^I w,0}, \pi_{{}^I w}) = 0$ for all integers $j > 0$.

Since the left cosets $N'_{J,{}^I w}/N'_{J,{}^I w,0}$ form an open partition of $N'_{J,{}^I w}$, we deduce from (13) a natural $N'_{J,{}^I w,0}$-equivariant isomorphism

$$\pi_{{}^I w} \cong \bigoplus_{n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}} \mathcal{C}^{\mathrm{sm}}\big(n' N'_{J,{}^I w,0}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \sigma^{{}^I w}\big)\big)$$

where $N'_{J,{}^I w,0}$ acts by right translation on the terms of the direct sum. The latter are $N'_{J,{}^I w,0}$-acyclic by Shapiro's lemma (since they are induced discrete $A[N'_{J,{}^I w,0}]$-modules) and the $N'_{J,{}^I w,0}$-cohomology commutes with direct sums (since the image of a locally constant cochain is finite by compactness); thus $\pi_{{}^I w}$ is $N'_{J,{}^I w,0}$-acyclic. □

There is a natural smooth A-linear action of $B''_{J,w_J} \ltimes (U'_{J,w_J} \times N''_{J,{}^I w})$ on $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w})$: we already defined the action of $B_{J,w_J} = B''_{J,w_J} \ltimes U'_{J,w_J}$ in Section 2.3 and we define the action of $n'' \in N''_{J,{}^I w}$ on $f \in \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^I w})$ by setting

$$(n'' \cdot f)(u') := n'' \cdot f(u')$$

for all $u' \in U'_{J,w_J}$.

Lemma 3.2.2. *For all $n \in \mathbb{N}$, there is a natural B^+_{J,w_J}-equivariant morphism*

$$\mathrm{H}^n\big(N''_{J,{}^I w,0}, \pi_{{}^I w}^{N'_{J,{}^I w,0}}\big) \to \mathrm{H}^n\big(N''_{J,{}^I w,0}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \sigma^{{}^I w}\big)\big)$$

such that the Hecke action of z on its kernel and cokernel is locally nilpotent.

Proof. We will implicitly make use of the isomorphism (13). For each $n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}$, evaluation at n' induces a natural A-linear surjection

$$\mathrm{ev}_{n'} : \pi_{{}^I w}^{N'_{J,{}^I w,0}} \twoheadrightarrow \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \sigma^{{}^I w}\big).$$

We define a natural A-linear surjection

$$\mathrm{Ev} := \sum_{n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}} \mathrm{ev}_{n'} : \pi_{{}^I w}^{N'_{J,{}^I w,0}} \twoheadrightarrow \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \sigma^{{}^I w}\big).$$

We prove that Ev is B^+_{J,w_J}-equivariant: for any $f \in \pi_{{}^I w}^{N'_{J,{}^I w,0}}$ and $b \in B^+_{J,w_J}$, we have

$$\begin{aligned}
\mathrm{Ev}(b \overset{\mathrm{H}}{\cdot} f) &= \sum_{n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}} \sum_{n'_0 \in N'_{J,{}^I w,0}/bN'_{J,{}^I w,0}b^{-1}} (n'_0 b \cdot f)(n') \\
&= \sum_{n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}} \sum_{n'_0 \in N'_{J,{}^I w,0}/bN'_{J,{}^I w,0}b^{-1}} b \cdot f(b^{-1}n'n'_0 b) \\
&= \sum_{n' \in N'_{J,{}^I w}/bN'_{J,{}^I w,0}b^{-1}} b \cdot f(b^{-1}n'b) \\
&= b \cdot \mathrm{Ev}(f)
\end{aligned}$$

where the last equality results from the change of variable $n' \mapsto bn'b^{-1}$. We prove that Ev is also $N''_{J,{}^I w,0}$-equivariant: for any $f \in \pi_{{}^I w}^{N'_{J,{}^I w,0}}$, $n'' \in N''_{J,{}^I w,0}$ and $u' \in U'_{J,w_J}$, we have

$$\begin{aligned}
\mathrm{Ev}(n'' \cdot f)(u') &= \sum_{n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}} (n'' \cdot f)(n')(u') \\
&= \sum_{n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}} n'' \cdot f(u'^{-1}n''^{-1}u'n'n'')(u') \\
&= n'' \cdot \mathrm{Ev}(f)(u'),
\end{aligned}$$

where the last equality results from the fact that when n' runs among $N'_{J,{}^I w}/N'_{J,{}^I w,0}$ we have

$$u'^{-1}n''^{-1}u'n'n'' = (u'^{-1}n''^{-1}u'n'')(n''^{-1}n'n'')$$

with on the one hand $n''^{-1}n'n''$ running among $N'_{J,{}^I w}/N'_{J,{}^I w,0}$ and on the other hand $u'^{-1}n''^{-1}u'n'' \in N'_{J,{}^I w}$ being constant. We deduce that Ev induces natural B^+_{J,w_J}-equivariant morphisms in $N''_{J,{}^I w,0}$-cohomology.

We prove that the Hecke action of z on the kernel of Ev is locally nilpotent: for any $f \in \pi_{{}^I w}^{N'_{J,{}^I w,0}}$ there exists $i \in \mathbb{N}$ such that $\mathrm{supp}(f) \subseteq z^{-i}N'_{J,{}^I w,0}z^i$ (since z is strictly contracting $N'_{J,{}^I w,0}$ which is open in $N'_{J,{}^I w}$), thus for any $n' \in N'_{J,{}^I w}/N'_{J,{}^I w,0}$, we have

$$\begin{aligned}
(z^i \overset{\mathrm{H}}{\cdot} f)(n') &= \sum_{n'_0 \in N'_{J,{}^I w,0}/z^i N'_{J,{}^I w,0}z^{-i}} (n'_0 z^i \cdot f)(n') \\
&= \sum_{n'_0 \in N'_{J,{}^I w,0}/z^i N'_{J,{}^I w,0}z^{-i}} z^i \cdot f\left((z^{-i}n'z^i)(z^{-i}n'_0 z^i)\right) \\
&= \begin{cases} z^i \cdot \mathrm{Ev}(f) & \text{if } n' \in N'_{J,{}^I w,0}, \\ 0 & \text{if } n' \notin N'_{J,{}^I w,0}. \end{cases}
\end{aligned}$$

Using the long exact sequence of $N''_{J,{}^I w,0}$-cohomology, we deduce that the Hecke action of z on the kernels and cokernels of the morphisms induced by Ev in $N''_{J,{}^I w,0}$-cohomology is locally nilpotent. □

The subgroup $\boldsymbol{B}''_{J,w_J} \subseteq \boldsymbol{B}_{J,w_J}$ normalises $\boldsymbol{N}''_{J,{}^Iw}$ and the conjugation action coincides with the action induced by the quotient action of $\boldsymbol{B}_{J,w_J}$ on $\boldsymbol{N}''_{J,{}^Iw}$. We define an open submonoid $B''^+_{J,w_J} \subseteq B''_{J,w_J}$ by setting

$$B''^+_{J,w_J} := \left\{b'' \in B''_{J,w_J} \;\middle|\; b'' N''_{J,{}^Iw,0} b''^{-1} \subseteq N''_{J,{}^Iw,0}\right\}.$$

Lemma 3.2.3. *We have $B^+_{J,w_J} \subseteq B''^+_{J,w_J} \ltimes U'_{J,w_J}$.*

Proof. We have a semidirect product $\boldsymbol{B}_{J,w_J} = \boldsymbol{B}''_{J,w_J} \ltimes \boldsymbol{U}'_{J,w_J}$. Let $b \in B^+_{J,w_J}$. We write $b = b''u'$ with $b'' \in B''_{J,w_J}$ and $u' \in U'_{J,w_J}$. We prove that $b'' \in B''^+_{J,w_J}$. Let $n'' \in N''_{J,{}^Iw,0}$. Proceeding as in the proof of Lemma 2.3.1, we see that $u'n''u'^{-1}n''^{-1} \in N'_{J,{}^Iw}$ so that $u'n''u'^{-1} = n'n''$ with $n' \in N'_{J,{}^Iw}$. Thus $bn''b^{-1} = (b''n'b''^{-1})(b''n''b''^{-1}) \in N_{J,{}^Iw,0}$, and since $N_{J,{}^Iw,0}$ is totally decomposed, we deduce that $b''n''b''^{-1} \in N''_{J,{}^Iw,0}$. ☐

Lemma 3.2.4. *For all $n \in \mathbb{N}$, there is a natural $B''^+_{J,w_J} \ltimes U'_{J,w_J}$-equivariant isomorphism*

$$\mathrm{H}^n\left(N''_{J,{}^Iw,0}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \sigma^{{}^Iw})\right) \cong \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, \mathrm{H}^n(N''_{J,{}^Iw,0}, \sigma^{{}^Iw})\right).$$

Proof. Let $\tilde{\sigma}$ be a $B''^+_{J,w_J} \ltimes N''_{J,{}^Iw,0}$-representation. The A-modules $\mathrm{H}^\bullet\left(N''_{J,{}^Iw,0}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \tilde{\sigma})\right)$ and $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, \mathrm{H}^\bullet(N''_{J,{}^Iw,0}, \tilde{\sigma})\right)$ are naturally $B''^+_{J,w_J} \ltimes U'_{J,w_J}$-representations. The identity of $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \tilde{\sigma})$ induces a natural U'_{J,w_J}-equivariant isomorphism

$$\iota : \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, \tilde{\sigma}\right)^{N''_{J,{}^Iw,0}} \xrightarrow{\sim} \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, \tilde{\sigma}^{N''_{J,{}^Iw,0}}\right).$$

We prove that ι is also B''^+_{J,w_J}-equivariant: for any $f \in \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, \tilde{\sigma})^{N''_{J,{}^Iw,0}}$, $b'' \in B''^+_{J,w_J}$ and $u' \in U'_{J,w_J}$, we have

$$\begin{aligned}
\iota(b'' \overset{\mathrm{H}}{\cdot} f)(u') &= \sum_{n'' \in N''_{J,{}^Iw,0}/b''N''_{J,{}^Iw,0}b''^{-1}} \iota(n''b'' \cdot f)(u') \\
&= \sum_{n'' \in N''_{J,{}^Iw,0}/b''N''_{J,{}^Iw,0}b''^{-1}} n''b'' \cdot \iota(f)(b''^{-1}u'b'') \\
&= b'' \overset{\mathrm{H}}{\cdot} \iota(f)(b''^{-1}u'b'') \\
&= (b'' \cdot \iota(f))(u').
\end{aligned}$$

We will prove that deriving ι yields the desired isomorphisms with $\tilde{\sigma} = \sigma^{{}^Iw}$.

The functor $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, -)$ is A-linear and exact and the δ-functor $\mathrm{H}^\bullet(N''_{J,{}^Iw,0}, -)$ is universal; thus, denoting by $\mathrm{R}^\bullet$ the right derived functors on the category $\mathrm{Mod}^{\mathrm{sm}}_{B''^+_{J,w_J} \ltimes N''_{J,{}^Iw,0}}(A)$, we have morphisms of functors

$$\begin{aligned}
\mathrm{R}^\bullet\left(\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, (-)^{N''_{J,{}^Iw,0}}\right)\right) &\cong \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, \mathrm{H}^\bullet(N''_{J,{}^Iw,0}, -)\right), \\
\mathrm{R}^\bullet\left(\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, -\right)^{N''_{J,{}^Iw,0}}\right) &\to \mathrm{H}^\bullet\left(N''_{J,{}^Iw,0}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\left(U'_{J,w_J}, -\right)\right).
\end{aligned}$$

In order to show that the second one is also an isomorphism, it is enough to prove that $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, -)$ takes injective objects of $\mathrm{Mod}^{\mathrm{sm}}_{B''^+_{J,w_J} \ltimes N''_{J,{}^Iw,0}}(A)$ to $N''_{J,{}^Iw,0}$-acyclic objects. If $\tilde{\sigma}$ is an A-module, then we have a natural $N''_{J,{}^Iw,0}$-equivariant isomorphism

$$\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \mathcal{C}^{\mathrm{sm}}\big(N''_{J,{}^Iw,0}, \tilde{\sigma}\big)\big) \cong \mathcal{C}^{\mathrm{sm}}\big(N''_{J,{}^Iw,0}, \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \tilde{\sigma}\big)\big),$$

so $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \mathcal{C}^{\mathrm{sm}}(N''_{J,{}^Iw,0}, \tilde{\sigma})\big)$ is $N''_{J,{}^Iw,0}$-acyclic. Now if $\tilde{\sigma}$ is an injective object of $\mathrm{Mod}^{\mathrm{sm}}_{B''^{+}_{J,w_J} \ltimes N''_{J,{}^Iw,0}}(A)$, it is also an injective object of $\mathrm{Mod}^{\mathrm{sm}}_{N''_{J,{}^Iw,0}}(A)$ [Emerton 2010b, Proposition 2.1.11; Hauseux 2016a, Lemme 3.1.1]; thus the natural $N''_{J,{}^Iw,0}$-equivariant injection $\tilde{\sigma} \hookrightarrow \mathcal{C}^{\mathrm{sm}}(N''_{J,{}^Iw,0}, \tilde{\sigma})$ defined by $v \mapsto (n'' \mapsto n'' \cdot v)$ admits an $N''_{J,{}^Iw,0}$-equivariant retraction, so that $\tilde{\sigma}$ is a direct factor of $\mathcal{C}^{\mathrm{sm}}(N''_{J,{}^Iw,0}, \tilde{\sigma})$, and therefore $\tilde{\sigma}$ is $N''_{J,{}^Iw,0}$-acyclic. □

We now assume that σ is locally admissible.

Lemma 3.2.5. *For all $n \in \mathbb{N}$, there is a natural $B''^{+}_{J,w_J} \ltimes U'_{J,w_J}$-equivariant morphism*

$$\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \mathrm{H}^n\big(N''_{J,{}^Iw,0}, \sigma^{{}^Iw}\big)\big) \to \text{c-ind}_{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)}}^{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)} w_J B_J} \big(\mathrm{H}^n \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw^J}$$

such that the action of z on its kernel and cokernel is locally nilpotent.

Proof. We have natural B''^{+}_{J,w_J}-equivariant isomorphisms

$$\mathrm{H}^\bullet\big(N''_{J,{}^Iw,0}, \sigma^{{}^Iw}\big) \cong \mathrm{H}^\bullet\big({}^IwN''_{J,{}^Iw,0}{}^Iw^{-1}, \sigma\big)^{{}^Iw}. \tag{22}$$

Since ${}^Iw\boldsymbol{N}''_{J,{}^Iw}{}^Iw^{-1} = \boldsymbol{U}_I \cap {}^Iw^J \boldsymbol{N}_J {}^Iw^{J-1}$ is the unipotent radical of $\boldsymbol{L}_I \cap \boldsymbol{P}_{I \cap {}^Iw^J(J)}$ (see Lemma 2.1.1(iii)), we define an open submonoid of $L_{I \cap {}^Iw^J(J)}$ by setting

$$L^+_{I \cap {}^Iw^J(J)} := \big\{ l \in L_{I \cap {}^Iw^J(J)} \;\big|\; l\,{}^IwN''_{J,{}^Iw,0}{}^Iw^{-1}l^{-1} \subseteq {}^IwN''_{J,{}^Iw,0}{}^Iw^{-1} \big\}.$$

We have ${}^IwB''^{+}_{J,w_J}{}^Iw^{-1} = {}^IwB''_{J,w_J}{}^Iw^{-1} \cap L^+_{I \cap {}^Iw^J(J)}$. We let $Z^+_{I \cap {}^Iw^J(J)} \subseteq Z_{I \cap {}^Iw^J(J)}$ be the open submonoid $Z_{I \cap {}^Iw^J(J)} \cap L^+_{I \cap {}^Iw^J(J)}$. Since σ is locally admissible, $\mathrm{H}^\bullet({}^IwN''_{J,{}^Iw,0}{}^Iw^{-1}, \sigma)$ is locally $Z^+_{I \cap {}^Iw^J(J)}$-finite [Emerton 2010b, Theorem 3.4.7(1)], and thus locally ${}^Iwz{}^Iw^{-1}$-finite. Note that ${}^Iwz{}^Iw^{-1} \in Z^+_{I \cap {}^Iw^J(J)}$ is strictly contracting ${}^IwN''_{J,{}^Iw,0}{}^Iw^{-1}$. Therefore, localising with respect to $({}^Iwz{}^Iw^{-1})^{\mathbb{N}}$ gives rise to $L^+_{I \cap {}^Iw^J(J)}$-equivariant morphisms

$$\mathrm{H}^\bullet\big({}^IwN''_{J,{}^Iw,0}{}^Iw^{-1}, \sigma\big) \to \mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma$$

such that the action of ${}^Iwz{}^Iw^{-1}$ on their kernels and cokernel is locally nilpotent (see Remark 3.1.3). Using (22), we deduce B''^{+}_{J,w_J}-equivariant morphisms

$$\mathrm{H}^\bullet\big(N''_{J,{}^Iw,0}, \sigma^{{}^Iw}\big) \to \big(\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw}$$

such that the action of z on their kernels and cokernels is locally nilpotent. Applying the functor $\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}(U'_{J,w_J}, -)$, we obtain $B''^{+}_{J,w_J} \ltimes U'_{J,w_J}$-equivariant morphisms

$$\mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \mathrm{H}^\bullet\big(N''_{J,{}^Iw,0}, \sigma^{{}^Iw}\big)\big) \to \mathcal{C}^{\mathrm{sm}}_{\mathrm{c}}\big(U'_{J,w_J}, \big(\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw}\big)$$

such that the action of z on their kernel and cokernel is still locally nilpotent (because the functions in their sources and targets have finite images). We conclude using the inverse of the B_{J,w_J}-equivariant isomorphism (12) with $\tilde{\sigma} = (\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma)^{{}^Iw^J}$. □

We combine the previous results into the following one.

Proposition 3.2.6. *Let σ be a locally admissible L_I-representation and ${}^Iw \in {}^IW$. We write ${}^Iw = {}^Iw^J w_J$ with ${}^Iw^J \in {}^IW^J$ and $w_J \in W_J$. For all $n \in \mathbb{N}$, there is a natural B^+_{J,w_J}-equivariant morphism*

$$\mathrm{H}^n\big(N_{J,{}^Iw,0}, \operatorname{c-ind}_{P_I^-}^{P_I^- {}^Iw B} \sigma\big) \to \operatorname{c-ind}_{L_J\cap P^-_{J\cap {}^Iw^{J^{-1}}(I)}}^{L_J\cap P^-_{J\cap {}^Iw^{J^{-1}}(I)} w_J B_J} \big(\mathrm{H}^n \mathrm{Ord}_{L_I\cap P_{I\cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw^J}$$

such that the action of z on its kernel and cokernel is locally nilpotent. Furthermore, this morphism is even $L_J^+ \cap P_{J\cap {}^Iw^{J^{-1}}(I)}$-equivariant when $w_J = 1$ (see Remark 2.2.2).

Proof. Combining Lemmas 3.2.1, 3.2.2, 3.2.4, 3.2.5, and using Lemma 3.2.3, we obtain the desired morphism. If $w_J = 1$, then the previous lemmas and their proofs are true verbatim with $\boldsymbol{L}_J \cap \boldsymbol{P}_{J\cap {}^Iw^{J^{-1}}(I)}$ and $\boldsymbol{L}_{J\cap {}^Iw^{J^{-1}}(I)}$ instead of $\boldsymbol{B}_{J,w_J}$ and $\boldsymbol{B}''_{J,w_J}$ respectively (see Remark 2.3.2); thus the morphism is $L_J^+ \cap P_{J\cap {}^Iw^{J^{-1}}(I)}$-equivariant. □

3.3. ***Computations on parabolically induced representations.*** Let $I, J \subseteq \Delta$, σ be a locally admissible L_I-representation and $n \in \mathbb{N}$. For any lower set ${}^IW_1^J \subseteq {}^IW^J$, the natural P_J-equivariant injection $\mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \hookrightarrow \mathrm{Ind}_{P_I^-}^G \sigma$ induces an L_J-equivariant morphism

$$\mathrm{H}^n \mathrm{Ord}_{P_J}\big(\mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to \mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma), \tag{23}$$

and by taking its image we define an L_J-subrepresentation

$$\mathrm{Fil}_{P_J}^{{}^IW_1^J}\big(\mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \subseteq \mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma).$$

Proposition 3.3.1. *The L_J-subrepresentations $\mathrm{Fil}_{P_J}^{\bullet}\big(\mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big)$ form a natural filtration of $\mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)$ indexed by ${}^IW^J$. Furthermore, for all ${}^Iw^J \in {}^IW^J$ there is a natural L_J-equivariant isomorphism*

$$\mathrm{Gr}_{P_J}^{{}^Iw^J}\big(\mathrm{H}^n \mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \cong \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\operatorname{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big).$$

Proof. First, we prove for any lower sets ${}^IW_2^J \subseteq {}^IW_1^J \subseteq {}^IW^J$, the short exact sequence of P_J-representations

$$0 \to \mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \to \mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \to \mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma)/\mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \to 0 \tag{24}$$

induces a short exact sequence of L_J-representations

$$0 \to \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \\ \to \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma)/\mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to 0. \tag{25}$$

In particular, (23) is injective and (7) induces the isomorphism in the statement.

Let $N_{J,0} \subseteq N_J$, $L_J^+ \subseteq L_J$, $Z_J^+ \subseteq Z_J$ and $z \in Z_J^+$ be as in Section 3.2. Proceeding as in the proof of [Hauseux 2016a, Proposition 2.2.3], we see that the first nontrivial morphism of (24) induces an injection in $N_{J,0}$-cohomology. Using the long exact sequence of $N_{J,0}$-cohomology, we deduce that (24) induces a short exact sequence of L_J^+-representations

$$0 \to \mathrm{H}^n\big(N_{J,0}, \mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to \mathrm{H}^n\big(N_{J,0}, \mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to \mathrm{H}^n\big(N_{J,0}, \mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma)/\mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to 0. \quad (26)$$

Since σ is locally admissible, $\mathrm{Ind}_{P_I^-}^G \sigma$ is locally admissible [Emerton 2010a, Proposition 4.1.7]; thus $\mathrm{H}^\bullet(N_{J,0}, \mathrm{Ind}_{P_I^-}^G \sigma)$ is locally Z_J^+-finite [Emerton 2010b, Theorem 3.4.7(1)]. We deduce that each term of (26) is locally Z_J^+-finite (as a subquotient). We conclude that localising (26) with respect to $z^{\mathbb{N}}$ yields (25) (see Remark 3.1.3).

We now prove that $\mathrm{Fil}_{P_J}^\bullet(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma))$ is a filtration of $\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)$ indexed by ${}^IW^J$. Since ${}^IW^J$ is finite and $\mathrm{Fil}_{P_J}^\bullet(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma))$ is inclusion-preserving with $\mathrm{Fil}_{P_J}^{\varnothing}(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)) = 0$ and $\mathrm{Fil}_{P_J}^{{}^IW^J}(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)) = \mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)$ by construction, it remains to prove that for any lower sets ${}^IW_1^J, {}^IW_2^J \subseteq {}^IW^J$, the natural short exact sequence of P_J-representations

$$0 \to \mathrm{Fil}_{P_J}^{{}^IW_1^J \cap {}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \to \mathrm{Fil}_{P_J}^{{}^IW_1^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \oplus \mathrm{Fil}_{P_J}^{{}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \to \mathrm{Fil}_{P_J}^{{}^IW_1^J \cup {}^IW_2^J}(\mathrm{Ind}_{P_I^-}^G \sigma) \to 0$$

induces a short exact sequence of L_J-representations

$$0 \to \mathrm{Fil}_{P_J}^{{}^IW_1^J \cap {}^IW_2^J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to \mathrm{Fil}_{P_J}^{{}^IW_1^J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \oplus \mathrm{Fil}_{P_J}^{{}^IW_2^J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to \mathrm{Fil}_{P_J}^{{}^IW_1^J \cup {}^IW_2^J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma)\big) \to 0.$$

This follows from the same arguments as above. □

Let ${}^Iw^J \in {}^IW^J$. For any lower set $W_J' \subseteq {}^{J\cap {}^Iw^{J-1}(I)}W_J$, the natural B-equivariant (resp. $P_{J\cap {}^Iw^{J-1}(I)}$-equivariant when $W_J' = \{1\}$; see Remark 2.2.2) injection $\mathrm{Fil}_B^{W_J'}(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma) \hookrightarrow \text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma$ induces a B_J-equivariant (resp. $L_J \cap P_{J\cap {}^Iw^{J-1}(I)}$-equivariant when $W_J' = \{1\}$) morphism

$$\mathrm{H}^n\mathrm{Ord}_{P_J}\big(\mathrm{Fil}_B^{W_J'}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big)\big) \to \mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big), \quad (27)$$

and by taking its image we define a B_J-subrepresentation (resp. $L_J \cap P_{J\cap {}^Iw^{J-1}(I)}$-subrepresentation when $W_J' = \{1\}$)

$$\mathrm{Fil}_B^{W_J'}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big)\big) \subseteq \mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big).$$

Proceeding as in the proof of Proposition 3.3.1 and using (8), we prove that (27) is injective and the following result.

Proposition 3.3.2. *The B_J-subrepresentations $\mathrm{Fil}_B^\bullet\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma)\big)$ form a natural filtration of $\mathrm{H}^n\mathrm{Ord}_{P_J}(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma)$ indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$. Furthermore, for all $w_J \in {}^{J\cap {}^Iw^{J-1}(I)}W_J$ there is a natural B_J-equivariant isomorphism*

$$\mathrm{Gr}_B^{w_J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J} \sigma\big)\big) \cong \mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J w_J B} \sigma\big)$$

which is even $L_J \cap P_{J\cap {}^Iw^{J-1}(I)}$-equivariant when $w_J = 1$ (see Remark 2.2.2).

We now state the main result of this section using Notation 2.3.3 and Remark 2.3.4.

Theorem 3.3.3. *Let σ be a locally admissible L_I-representation, ${}^I w^J \in {}^I W^J$ and $n \in \mathbb{N}$. For all $w_J \in {}^{J \cap {}^I w^{J-1}(I)} W_J$, there is a natural B_{J,w_J}-equivariant isomorphism*

$$\mathrm{Gr}_B^{w_J}\left(\mathrm{H}^n \mathrm{Ord}_{P_J}\left(\text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J} \sigma\right)\right) \cong \mathrm{Gr}_{B_J}^{w_J}\left(\mathrm{Ind}_{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)}}^{L_J}\left(\left(\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma\right)^{{}^I w^J} \otimes (\omega^{-1} \circ \delta_{{}^I w^J})\right)\right)$$

which is even $L_J \cap P_{J \cap {}^I w^{J-1}(I)}$-equivariant when $w_J = 1$ (see Remark 2.2.2).

Proof. We use the notation of Section 2.3. We let $w_J \in {}^{J \cap {}^I w^{J-1}(I)} W_J$ and we put ${}^I w := {}^I w^J w_J$. We let $N_{J,0} \subseteq N_J$, $L_J^+ \subseteq L_J$, $Z_J^+ \subseteq Z_J$, $z \in Z_J^+$ and $\pi_{{}^I w}$ be as in Section 3.2. In the course of the proof of Proposition 3.3.2, we see that $\mathrm{H}^n(N_{J,0}, \pi_{{}^I w})$ is locally Z_J^+-finite (as we saw it for $\mathrm{H}^n(N_{J,0}, \text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J} \sigma)$ in the course of the proof of Proposition 3.3.1).

Since σ is locally admissible, the $L_{I \cap {}^I w^J(J)}$-representations $\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma$ are locally admissible by [Emerton 2010b, Theorem 3.4.7(2)]; thus locally $Z_{I \cap {}^I w^J(J)}$-finite by [Emerton 2010a, Lemma 2.3.4]. Therefore, the B_J-representations

$$\text{c-ind}_{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)}}^{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} w_J B_J}\left(\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma\right)^{{}^I w^J}$$

are locally Z_J-finite; thus locally z-finite. We deduce from Proposition 3.2.6 that the B_{J,w_J}^+-representations $\mathrm{H}^\bullet(N_{J,{}^I w,0}, \pi_{{}^I w})$ are locally z-finite and that there is a natural B_{J,w_J}^+-equivariant (resp. $L_J^+ \cap P_{J \cap {}^I w^{J-1}(I)}$-equivariant when $w_J = 1$) morphism

$$\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}}(N_{J,{}^I w,0}, \pi_{{}^I w}) \otimes (\omega^{-1} \circ \delta_{{}^I w^J})^{w_J} \to \left(\text{c-ind}_{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)}}^{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} w_J B_J}\left(\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma\right)^{{}^I w^J}\right) \otimes (\omega^{-1} \circ \delta_{{}^I w^J})^{w_J} \quad (28)$$

such that the action of z on its kernel and cokernel is locally nilpotent.

Using Proposition 3.1.2 with $\widetilde{\boldsymbol{L}} = \boldsymbol{B}_{J,w_J}$ (resp. $\widetilde{\boldsymbol{L}} = \boldsymbol{L}_J \cap \boldsymbol{P}_{J \cap {}^I w^{J-1}(I)}$ when $w_J = 1$), $\widetilde{\boldsymbol{N}} = \boldsymbol{N}_J$, $\widetilde{\boldsymbol{N}}' = \boldsymbol{N}_{J,{}^I w}$ (so that $\tilde{d} - \tilde{d}' = d_{{}^I w^J}$ and $\tilde{\delta} - \tilde{\delta}' = w_J^{-1}(\delta_{{}^I w^J})$ since conjugation by w_J induces an isomorphism of F-varieties $\boldsymbol{N}_J / \boldsymbol{N}_{J,{}^I w} \xrightarrow{\sim} \boldsymbol{N}_J / \boldsymbol{N}_{J,{}^I w^J}$), $\tilde{z} = z$ and $\pi = \pi_{{}^I w}$, we deduce a natural B_{J,w_J}^+-equivariant (resp. $L_J^+ \cap P_{J \cap {}^I w^{J-1}(I)}$-equivariant when $w_J = 1$) morphism

$$\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}}(N_{J,{}^I w,0}, \pi_{{}^I w}) \otimes (\omega^{-1} \circ \delta_{{}^I w^J})^{w_J} \to \mathrm{H}^n(N_{J,0}, \pi_{{}^I w}) \quad (29)$$

and the Hecke action of z on its kernel and cokernel is locally nilpotent.

Using Proposition 3.3.2, (9) with $\tilde{\sigma} = (\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma)^{{}^I w^J} \otimes (\omega^{-1} \circ \delta_{{}^I w^J})$ and the natural B_J-equivariant (resp. $L_J \cap P_{J \cap {}^I w^{J-1}(I)}$-equivariant when $w_J = 1$) isomorphism

$$\left(\text{c-ind}_{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)}}^{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} w_J B_J}\left(\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma\right)^{{}^I w^J}\right) \otimes (\omega^{-1} \circ \delta_{{}^I w^J})^{w_J} \xrightarrow{\sim} \text{c-ind}_{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)}}^{L_J \cap P^-_{J \cap {}^I w^{J-1}(I)} w_J B_J}\left(\left(\mathrm{H}^{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^I w^J(J)}} \sigma\right)^{{}^I w^J} \otimes (\omega^{-1} \circ \delta_{{}^I w^J})\right),$$

the localisation of (28) with respect to $z^{\mathbb{N}}$ and the inverse of the localisation of (29) with respect to $z^{\mathbb{N}}$ yield the desired isomorphism (see Remark 3.1.3). $\square$

In particular with $w_J = 1$ and $\tilde{\sigma} := \big(\mathrm{H}^{n-[F:\mathbb{Q}_p]d_{{}^Iw^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw^J} \otimes (\omega^{-1} \circ \delta_{{}^Iw^J})$, there is a natural $L_J \cap P_{J \cap {}^Iw^{J-1}(I)}$-equivariant injection

$$\mathrm{Gr}^1_{B_J}\big(\mathrm{Ind}^{L_J}_{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)}} \tilde{\sigma}\big) \hookrightarrow \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma\big),$$

hence a natural L_J-equivariant morphism

$$A[L_J] \otimes_{A[L_J \cap P_{J \cap {}^Iw^{J-1}(I)}]} \mathrm{Gr}^1_{B_J}\big(\mathrm{Ind}^{L_J}_{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)}} \tilde{\sigma}\big) \to \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma\big).$$

In the proof of [Emerton 2010a, Theorem 4.4.6], it is shown that such a morphism factors uniquely through the natural L_J-equivariant surjection

$$A[L_J] \otimes_{A[L_J \cap P_{J \cap {}^Iw^{J-1}(I)}]} \mathrm{Gr}^1_{B_J}\big(\mathrm{Ind}^{L_J}_{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)}} \tilde{\sigma}\big) \twoheadrightarrow \mathrm{Ind}^{L_J}_{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)}} \tilde{\sigma}.$$

Thus, the previous injection naturally extends to an L_J-equivariant morphism

$$\mathrm{Ind}^{L_J}_{L_J \cap P^-_{J \cap {}^Iw^{J-1}(I)}}\Big(\big(\mathrm{H}^{n-[F:\mathbb{Q}_p]d_{{}^Iw^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw^J} \otimes (\omega^{-1} \circ \delta_{{}^Iw^J})\Big) \to \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma\big). \quad (30)$$

Conjecture 3.3.4. *The natural morphism* (30) *is an isomorphism.*

We prove Conjecture 3.3.4 in some special cases.

Proposition 3.3.5. (i) *If* $\mathrm{H}^{n-[F:\mathbb{Q}_p]d_{{}^Iw^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma = 0$, *then*

$$\mathrm{H}^n \mathrm{Ord}_{P_J}\big(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma\big) = 0.$$

(ii) *If* ${}^Iw^J(J) \subseteq I$, *then* (30) *is a natural* L_J*-equivariant isomorphism*

$$\big(\mathrm{H}^{n-[F:\mathbb{Q}_p]d_{{}^Iw^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma\big)^{{}^Iw^J} \otimes (\omega^{-1} \circ \delta_{{}^Iw^J}) \xrightarrow{\sim} \mathrm{H}^n \mathrm{Ord}_{P_J}\big(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma\big).$$

(iii) *If* $n = 0$ *and* ${}^Iw^J = 1$, *then* (30) *is a natural* L_J*-equivariant isomorphism*

$$\mathrm{Ind}^{L_J}_{L_J \cap P_I^-}\big(\mathrm{Ord}_{L_I \cap P_J} \sigma\big) \xrightarrow{\sim} \mathrm{Ord}_{P_J}\big(\text{c-ind}^{P_I^- P_J}_{P_I^-} \sigma\big).$$

Proof. We first use Theorem 3.3.3.

If $\mathrm{H}^{n-[F:\mathbb{Q}_p]d_{{}^Iw^J}} \mathrm{Ord}_{L_I \cap P_{I \cap {}^Iw^J(J)}} \sigma = 0$, we deduce that $\mathrm{Gr}^\bullet_B\big(\mathrm{H}^n \mathrm{Ord}_{P_J}(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma)\big) = 0$, hence (i).

If ${}^Iw^J(J) \subseteq I$, we deduce from [Emerton 2010b, Proposition 3.6.1] that $\mathrm{Gr}^\bullet_B\big(\mathrm{H}^n \mathrm{Ord}_{P_J}(\text{c-ind}^{P_I^- {}^Iw^J P_J}_{P_I^-} \sigma)\big)$ is concentrated in degree 1; thus (30) is an isomorphism, hence (ii).

We now prove (iii). Since all the functors involved commute with inductive limits, we reduce to the case where σ is admissible. By [Abe et al. 2017b, Corollaries 4.13 and 5.9], there is a natural L_J-equivariant isomorphism

$$\mathrm{Ind}^{L_J}_{L_J \cap P_I^-}\big(\mathrm{Ord}_{L_I \cap P_J} \sigma\big) \xrightarrow{\sim} \mathrm{Ord}_{P_J}\big(\mathrm{Ind}^G_{P_I^-} \sigma\big). \quad (31)$$

Using (i), we deduce from Proposition 3.3.1 with $n=0$ that $\mathrm{Gr}^\bullet_{P_J}(\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G \sigma))$ is concentrated in degree 1, hence a natural L_J-equivariant isomorphism

$$\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- P_J} \sigma\big) \xrightarrow{\sim} \mathrm{Ord}_{P_J}\big(\mathrm{Ind}_{P_I^-}^G \sigma\big). \tag{32}$$

The composition of (30) with $n=0$ and ${}^I w^J = 1$, (32) and the inverse of (31) yields an L_J-equivariant endomorphism φ of $\mathrm{Ind}_{L_J\cap P_I^-}^{L_J}(\mathrm{Ord}_{L_I\cap P_J}\sigma)$ which is injective in restriction to $\mathrm{Fil}^1_{B_J}\big(\mathrm{Ind}_{L_J\cap P_I^-}^{L_J}(\mathrm{Ord}_{L_I\cap P_J}\sigma)\big)$. From [Emerton 2010a, Lemma 4.3.1 and Proposition 4.3.4] and the left-exactness of $\mathrm{Ord}_{L_I\cap P_J}$, we deduce that $\mathrm{Ord}_{L_J\cap P_I}\varphi$ is an injective $L_{I\cap J}$-equivariant endomorphism of $\mathrm{Ord}_{L_I\cap P_J}\sigma$. Since the latter is admissible by [op. cit., Theorem 3.3.3], it is Artinian (see Section 4.1 below), and thus co-Hopfian so that $\mathrm{Ord}_{L_I\cap P_J}\varphi$ is an isomorphism. We deduce that φ is an isomorphism using [op. cit., Proposition 4.3.4 and Theorem 4.4.6]. We conclude that (30) with $n=0$ and ${}^I w^J = 1$ is an isomorphism as in the statement. □

Remark 3.3.6. Let $\mathrm{R}^\bullet\mathrm{Ord}_{L_I\cap P_J}$ denote the derived functors of $\mathrm{Ord}_{L_I\cap P_J}$ on $\mathrm{Mod}_{L_I}^{\mathrm{l,adm}}(A)$. By universality of derived functors, the isomorphism in (iii) extends uniquely to a morphism of δ-functors

$$\mathrm{Ind}_{L_J\cap P_I^-}^{L_J} \circ \mathrm{R}^\bullet\mathrm{Ord}_{L_I\cap P_J} \to \mathrm{H}^\bullet\mathrm{Ord}_{P_J} \circ \text{c-ind}_{P_I^-}^{P_I^- P_J} \tag{33}$$

(the left-hand side is the derived functor of $\mathrm{Ind}_{L_J\cap P_I^-}^{L_J} \circ \mathrm{Ord}_{L_I\cap P_J}$ by exactness of $\mathrm{Ind}_{L_J\cap P_I^-}^{L_J}$, and the right-hand side is a δ-functor by the same arguments as in the proof of Proposition 3.3.1). Now, assume that [Emerton 2010b, Conjecture 3.7.2] is true for $L_I\cap P_J$, i.e., $\mathrm{R}^\bullet\mathrm{Ord}_{L_I\cap P_J} \xrightarrow{\sim} \mathrm{H}^\bullet\mathrm{Ord}_{L_I\cap P_J}$. Then Conjecture 3.3.4 for ${}^I w^J = 1$ is equivalent to (33) being an isomorphism. We could prove this if we knew that the isomorphism of Theorem 3.3.3 with ${}^I w^J = 1$ were B_J-equivariant for all $w_J \in {}^{J\cap I}W_J$.

Finally, we compute the derived ordinary parts of a parabolically induced representation in low degree when there is an inclusion between I and J.

Proposition 3.3.7. (i) *If $I \subseteq J$ and $0 < n < [F:\mathbb{Q}_p]$, then* $\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma) = 0$.

(ii) *If $J \subseteq I$ and $n < [F:\mathbb{Q}_p]$, then there is a natural L_J-equivariant isomorphism*

$$\mathrm{H}^n\mathrm{Ord}_{L_I\cap P_J}\sigma \xrightarrow{\sim} \mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma).$$

(iii) *If $J \subseteq I$ and $\mathrm{Ord}_{L_I\cap P_{I\cap s_\alpha(J)}}\sigma = 0$ for all $\alpha \in \Delta^1 \setminus (I\cup J^\perp)$, then there is a natural short exact sequence of L_J-representations*

$$0 \to \mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{L_I\cap P_J}\sigma \to \mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma) \to \bigoplus_{\alpha\in J^{\perp,1}\setminus I} (\mathrm{Ord}_{L_I\cap P_J}\sigma)^\alpha \otimes (\omega^{-1}\circ\alpha) \to 0.$$

Proof. We use Proposition 3.3.1 and Lemma 2.2.1 with $\ell : {}^I W^J \to \mathbb{N}$ to obtain a filtration

$$\mathrm{Fil}^{\ell,\bullet}_{P_J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma)\big)$$

indexed by $\mathbb{N}$ such that for all $i\in\mathbb{N}$, there is a natural L_J-equivariant isomorphism

$$\mathrm{Gr}^{\ell,i}_{P_J}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma)\big) \cong \bigoplus_{\ell({}^I w^J)=i} \mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J}\sigma\big). \tag{34}$$

Assume $n < [F:\mathbb{Q}_p]$. If ${}^I w^J \neq 1$ (i.e., $d_{{}^I w^J} > 0$), then $\mathrm{H}^n\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J}\sigma\big) = 0$ by Proposition 3.3.5(i) since $n - [F:\mathbb{Q}_p]d_{{}^I w^J} < 0$; thus we deduce from (34) that $\mathrm{Gr}_{P_J}^{\ell,\bullet}\big(\mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma)\big)$ is concentrated in degree 0, so that assuming Conjecture 3.3.4 for ${}^I w^J = 1$, we obtain a natural L_J-equivariant isomorphism

$$\mathrm{Ind}_{L_J\cap P_I^-}^{L_J}\big(\mathrm{H}^n\mathrm{Ord}_{L_I\cap P_J}\sigma\big) \xrightarrow{\sim} \mathrm{H}^n\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma). \tag{35}$$

Now, Conjecture 3.3.4 is true for ${}^I w^J = 1$ in the following cases: $n > 0$ and $I \subseteq J$ by Proposition 3.3.5(i) since $\mathrm{H}^n\mathrm{Ord}_{L_I\cap P_J} = \mathrm{H}^n\mathrm{Ord}_{L_I} = 0$ [Emerton 2010b, Proposition 3.6.1], in which case the source of (35) is zero, hence (i); $J \subseteq I$ by Proposition 3.3.5(ii), in which case the source of (35) is $\mathrm{H}^n\mathrm{Ord}_{L_I\cap P_J}\sigma$, hence (ii).

Likewise, if ${}^I w^J \neq 1$ and ${}^I w^J \neq s_\alpha$ for all $\alpha \in \Delta^1 \setminus (I \cup J)$ (i.e., $d_{{}^I w^J} > 1$), then

$$\mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{P_J}\big(\text{c-ind}_{P_I^-}^{P_I^- {}^I w^J P_J}\sigma\big) = 0$$

by Proposition 3.3.5(i) since $[F:\mathbb{Q}_p] - [F:\mathbb{Q}_p]d_{{}^I w^J} < 0$; thus we deduce from (34) that

$$\mathrm{Gr}_{P_J}^{\ell,\bullet}\big(\mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma)\big)$$

is concentrated in degrees 0 and 1, so that assuming Conjecture 3.3.4 for $n = [F:\mathbb{Q}_p]$ and ${}^I w^J = 1$ or ${}^I w^J = s_\alpha$ for all $\alpha \in \Delta^1 \setminus (I \cup J)$, we obtain a short exact sequence of L_J-representations

$$\begin{aligned} 0 \to \mathrm{Ind}_{L_J\cap P_I^-}^{L_J}\big(\mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{L_I\cap P_J}\sigma\big) &\to \mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{P_J}(\mathrm{Ind}_{P_I^-}^G\sigma) \\ &\to \bigoplus_{\alpha\in\Delta^1\setminus(I\cup J)} \mathrm{Ind}_{L_J\cap P_{J\cap s_\alpha(I)}^-}^{L_J}\big(\big(\mathrm{Ord}_{L_I\cap P_{I\cap s_\alpha(J)}}\sigma\big)^\alpha \otimes (\omega^{-1}\circ\alpha)\big) \to 0. \end{aligned} \tag{36}$$

Assume $J \subseteq I$ and $\mathrm{Ord}_{L_I\cap P_{I\cap s_\alpha(J)}}\sigma = 0$ for all $\alpha \in \Delta^1 \setminus (I \cup J^\perp)$. Then Conjecture 3.3.4 is indeed true for $n = [F:\mathbb{Q}_p]$ in the following cases: ${}^I w^J = 1$ by Proposition 3.3.5(ii), and the first nontrivial term of (36) is $\mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_{L_I\cap P_J}\sigma$; ${}^I w^J = s_\alpha$ with $\alpha \in \Delta^1 \setminus (I \cup J^\perp)$ by Proposition 3.3.5(i) and the hypothesis on σ, and the corresponding summand of the last nontrivial term of (36) is zero; ${}^I w^J = s_\alpha$ with $\alpha \in J^{\perp,1}\setminus I$ by Proposition 3.3.5(ii) since $s_\alpha(J) = J \subseteq I$, and the corresponding summand of the last nontrivial term of (36) is $(\mathrm{Ord}_{L_I\cap P_J}\sigma)^\alpha \otimes (\omega^{-1}\circ\alpha)$. Hence (iii). ☐

We reformulate Proposition 3.3.7 in the case $I = J$, using the fact that in this case $\mathrm{H}^n\mathrm{Ord}_{L_I\cap P_J} = 0$ if $n > 0$ [Emerton 2010b, Proposition 3.6.1]. Note that if $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ is a standard parabolic subgroup, then for all $\alpha \in \Delta \setminus \Delta_{\boldsymbol{L}}$ the standard parabolic subgroup of $\boldsymbol{L}$ corresponding to $\Delta_{\boldsymbol{L}} \cap s_\alpha(\Delta_{\boldsymbol{L}})$ is $\boldsymbol{L} \cap s_\alpha \boldsymbol{P} s_\alpha^{-1}$ and it is proper if and only if $\alpha \notin \Delta_{\boldsymbol{L}}^\perp$.

Corollary 3.3.8. *Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup and σ be a locally admissible L-representation.*

(i) *For all $n \in \mathbb{N}$ such that $0 < n < [F:\mathbb{Q}_p]$, we have $\mathrm{H}^n\mathrm{Ord}_P(\mathrm{Ind}_{P^-}^G\sigma) = 0$.*

(ii) *If $\mathrm{Ord}_{L\cap s_\alpha P s_\alpha^{-1}}\sigma = 0$ for all $\alpha \in \Delta^1 \setminus (\Delta_{\boldsymbol{L}} \cup \Delta_{\boldsymbol{L}}^\perp)$, then there is a natural L-equivariant isomorphism*

$$\mathrm{H}^{[F:\mathbb{Q}_p]}\mathrm{Ord}_P(\mathrm{Ind}_{P^-}^G\sigma) \cong \bigoplus_{\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}} \sigma^\alpha \otimes (\omega^{-1}\circ\alpha).$$

4. Derived Jacquet functors

The aim of this section is to study the derived functors of the Jacquet functor. In Section 4.1, we review some results on pro-categories. In Section 4.2, we relate the left derived functors of the Jacquet functor in a pro-category with the derived ordinary parts functors and we construct a new exact sequence to compute extensions by a parabolically induced representation. In Section 4.3, we adapt the results of Section 3.3 in order to partially compute the derived Jacquet functors on a parabolically induced representation.

4.1. ***Pro-categories.*** Let H be a p-adic Lie group. Let $\mathfrak{C}$ be the category whose objects are the $A[H]$-modules such that for some (equivalently any) compact open subgroup $H_0 \subseteq H$, the $A[H_0]$-action extends to a structure of $A[\![H_0]\!]$-module of finite type, and whose morphisms are the $A[H]$-linear maps. Since the completed group rings are Noetherian [Emerton 2010a, Theorem 2.1.2], the category $\mathfrak{C}$ is A-abelian and Noetherian, i.e., it is essentially small[7] and its objects are Noetherian. Let $\mathfrak{C}^\wedge$ be the category of contravariant functors $\mathfrak{C} \to \mathrm{Set}$ and Ind-$\mathfrak{C}$ be the full subcategory of $\mathfrak{C}^\wedge$ whose objects are the functors isomorphic to a small inductive limit in $\mathfrak{C}^\wedge$ of objects of $\mathfrak{C}$ (using the Yoneda embedding $\mathfrak{C} \to \mathfrak{C}^\wedge$). By [Kashiwara and Schapira 2006, Theorem 8.6.5], the category Ind-$\mathfrak{C}$ is a Grothendieck category[8] (in particular it has enough injectives; see [Kashiwara and Schapira 2006, Theorem 9.6.2]) and the natural A-linear functor $\mathfrak{C} \to$ Ind-$\mathfrak{C}$ is fully faithful and exact.

Now, Pontryagin duality induces an equivalence of categories [Emerton 2010a, (2.2.12)]

$$\mathrm{Mod}_H^{\mathrm{adm}}(A) \cong \mathfrak{C}^{\mathrm{op}}.$$

Thus, the category $\mathrm{Mod}_H^{\mathrm{adm}}(A)$ is Artinian, the pro-category

$$\text{Pro-}\mathrm{Mod}_H^{\mathrm{adm}}(A) := (\text{Ind-}\mathfrak{C})^{\mathrm{op}}$$

has enough projectives, and the natural A-linear functor

$$\mathrm{Mod}_H^{\mathrm{adm}}(A) \to \text{Pro-}\mathrm{Mod}_H^{\mathrm{adm}}(A) \tag{37}$$

is fully faithful and exact. We let $\mathrm{Ext}_H^\bullet$ and $\mathrm{Ext}_{\text{Pro-}H}^\bullet$ denote the bifunctors of Yoneda extensions in the categories $\mathrm{Mod}_H^{\mathrm{adm}}(A)$ and $\text{Pro-}\mathrm{Mod}_H^{\mathrm{adm}}(A)$ respectively. By [Oort 1964, Theorem 3.5], (37) induces A-linear isomorphisms

$$\mathrm{Ext}_H^\bullet(\pi', \pi) \xrightarrow{\sim} \mathrm{Ext}_{\text{Pro-}H}^\bullet(\pi', \pi) \tag{38}$$

for all objects π, π' of $\mathrm{Mod}_H^{\mathrm{adm}}(A)$.

4.2. ***A second exact sequence.*** Let $\boldsymbol{P} \subseteq \boldsymbol{G}$ be a parabolic subgroup and $\boldsymbol{L} \subseteq \boldsymbol{P}$ be a Levi factor. We let $\boldsymbol{P}^- \subseteq \boldsymbol{G}$ denote the parabolic subgroup opposed to $\boldsymbol{P}$ with respect to $\boldsymbol{L}$. There is a natural exact sequence of A-modules [Emerton 2010b, (3.7.6)]

$$0 \to \mathrm{Ext}_L^1(\sigma, \mathrm{Ord}_P\, \pi) \to \mathrm{Ext}_G^1(\mathrm{Ind}_{P^-}^G\, \sigma, \pi) \to \mathrm{Hom}_L(\sigma, \mathrm{H}^1\mathrm{Ord}_P\, \pi) \tag{39}$$

[7] A category is essentially small if it is equivalent to a small category, i.e., if the isomorphism classes of its objects form a set.

[8] A Grothendieck category is an abelian category that admits a generator and small direct sums, and in which inductive limits are exact.

for all objects σ and π of $\mathrm{Mod}_L^{\mathrm{adm}}(A)$ and $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ respectively. We construct a second exact sequence, in which parabolic induction is on the right.

By [Emerton 2010a, Proposition 4.1.5 and Proposition 4.1.7], parabolic induction induces an A-linear exact functor

$$\mathrm{Ind}_P^G : \mathrm{Mod}_L^{\mathrm{adm}}(A) \to \mathrm{Mod}_G^{\mathrm{adm}}(A).$$

By [Emerton 2010b, Corollary 3.6.7], taking N-coinvariants induces an A-linear right-exact functor (the so-called Jacquet functor)

$$(-)_N : \mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_L^{\mathrm{adm}}(A).$$

By Frobenius reciprocity and the universal property of coinvariants, there is a natural A-linear isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_L(\pi_N, \sigma). \tag{40}$$

for all objects π and σ of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ and $\mathrm{Mod}_L^{\mathrm{adm}}(A)$ respectively.

We deduce from [Kashiwara and Schapira 2006, Proposition 6.1.9] that these functors and the adjunction relation extend to the corresponding pro-categories. By [op. cit., Corollary 8.6.8], Ind_P^G is still exact so that $(-)_N$ still preserves projectives. Thus, denoting by $\mathrm{L}_\bullet(N,-)$ the left derived functors of $(-)_N$ in $\text{Pro-}\mathrm{Mod}_G^{\mathrm{adm}}(A)$, there is a Grothendieck spectral sequence of A-modules

$$\mathrm{Ext}^i_{\text{Pro-}L}(\mathrm{L}_j(N,\pi),\sigma) \Rightarrow \mathrm{Ext}^{i+j}_{\text{Pro-}G}(\pi, \mathrm{Ind}_P^G \sigma)$$

whose low degree terms form a natural exact sequence of A-modules

$$0 \to \mathrm{Ext}^1_{\text{Pro-}L}(\pi_N,\sigma) \to \mathrm{Ext}^1_{\text{Pro-}G}(\pi, \mathrm{Ind}_P^G \sigma) \to \mathrm{Hom}_{\text{Pro-}L}(\mathrm{L}_1(N,\pi),\sigma) \to \mathrm{Ext}^2_{\text{Pro-}L}(\pi_N,\sigma) \to \mathrm{Ext}^2_{\text{Pro-}G}(\pi, \mathrm{Ind}_P^G \sigma) \tag{41}$$

for all objects π and σ of $\text{Pro-}\mathrm{Mod}_G^{\mathrm{adm}}(A)$ and $\text{Pro-}\mathrm{Mod}_L^{\mathrm{adm}}(A)$ respectively.

We let d denote the integer $\dim_F \boldsymbol{N}$ and $\delta \in \mathrm{X}^*(\boldsymbol{L})$ denote the algebraic character of the adjoint representation of $\boldsymbol{L}$ on $\det_F(\mathrm{Lie}\,\boldsymbol{N})$. We define A-linear functors by setting

$$\mathrm{H}_\bullet(N,-) := \mathrm{H}^{[F:\mathbb{Q}_p]d-\bullet}\mathrm{Ord}_P \otimes (\omega \circ \delta).$$

We deduce from [Emerton 2010b, Corollary 3.4.8 and Proposition 3.6.1] that we obtain a homological δ-functor

$$\mathrm{H}_\bullet(N,-) : \mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_L^{\mathrm{adm}}(A)$$

and proceeding as in the proof of [Kashiwara and Schapira 2006, Corollary 8.6.8], we see that it extends to a homological δ-functor between the corresponding pro-categories.

By [Emerton 2010b, Proposition 3.6.2], there is an isomorphism of functors (hence the notation)

$$\mathrm{H}_0(N,-) \cong (-)_N$$

which, by universality of derived functors, extends uniquely to a morphism of δ-functors

$$\mathrm{H}_\bullet(N,-) \to \mathrm{L}_\bullet(N,-) \tag{42}$$

which is bijective in degree 0, and thus surjective in degree 1 (by a dimension-shifting argument). Using (38), we deduce from (41) a natural exact sequence of A-modules

$$0 \to \mathrm{Ext}^1_L(\pi_N, \sigma) \to \mathrm{Ext}^1_G(\pi, \mathrm{Ind}_P^G \sigma) \to \mathrm{Hom}_L(\mathrm{H}_1(N, \pi), \sigma) \tag{43}$$

for all objects π and σ of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ and $\mathrm{Mod}_L^{\mathrm{adm}}(A)$ respectively.

Remark 4.2.1. (i) Nothing is known (to the author at least) regarding the nature of the morphism (42) in degree greater than 1.

(ii) Let H be a p-adic Lie group. Taking inductive limits induces an A-linear exact functor

$$\varinjlim : \text{Ind-}\mathrm{Mod}_H^{\mathrm{adm}}(A) \to \mathrm{Mod}_H^{\mathrm{l.adm}}(A)$$

which is essentially surjective, but not faithful nor full in general. Thus the situation here (i.e., deriving in $\text{Pro-}\mathrm{Mod}_H^{\mathrm{adm}}(A)$) is not exactly dual to that of [Emerton 2010b, §3.7] (i.e., deriving in $\mathrm{Mod}_H^{\mathrm{l.adm}}(A)$).

4.3. ***Adaptation of the computations.*** Let $I, J \subseteq \Delta$, σ be an L_I-representation and $n \in \mathbb{N}$. We let ${}^Iw_0 = w_{I,0}w_0$ (resp. ${}^{J\cap {}^Iw^{J-1}(I)}w_{J,0} = w_{J\cap {}^Iw^{J-1}(I),0}w_{J,0}$) denote the image of w_0 (resp. $w_{J,0}$) in ${}^IW$ (resp. ${}^{J\cap {}^Iw^{J-1}(I)}W_J$) and we define an auxiliary subset of Δ by setting $I' := {}^Iw_0^{-1}(I)$. We have $\boldsymbol{L}_I = {}^Iw_0\boldsymbol{L}_{I'}{}^Iw_0^{-1}$ and $\boldsymbol{P}_I = {}^Iw_0\boldsymbol{P}_{I'}^{-}{}^Iw_0^{-1}$, hence a natural G-equivariant isomorphism

$$\mathrm{Ind}_{P_I}^G \sigma \cong \mathrm{Ind}_{P_{I'}^-}^G \sigma^{{}^Iw_0} \tag{44}$$

defined by $f \mapsto (g \mapsto f({}^Iw_0 g))$.

Lemma 4.3.1. *The map ${}^IW^J \to {}^{I'}W^J$ defined by ${}^Iw^J \mapsto {}^Iw_0^{-1}{}^Iw^J\,{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}$ is an order-reversing bijection.*

Proof. First, note that $W_I = w_0W_{I'}w_0$, so that left translation by w_0 induces a bijection

$$W_I\backslash W/W_J \xrightarrow{\sim} W_{I'}\backslash W/W_J.$$

In particular, $\mathrm{card}\,{}^IW^J = \mathrm{card}\,{}^{I'}W^J$. Thus, it is enough to prove that the order-reversing composite

$${}^IW^J \hookrightarrow W \xrightarrow{\sim} W$$

where the first arrow is defined by ${}^Iw^J \mapsto w_{I,0}{}^Iw^J\,{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}$ (it is injective since ${}^IW^J$ is a system of representatives of the double cosets $W_I\backslash W/W_J$, and order-preserving since the projection $W \twoheadrightarrow {}^IW^J$ is order-preserving) and the second arrow is the left multiplication by w_0 (it is an order-reversing bijection; see [Björner and Brenti 2005, Proposition 2.3.4(i)]), takes values in ${}^{I'}W^J$.

Now, let ${}^Iw^J \in {}^IW^J$. For all ${}^Iw \in {}^IW$ and $w_{I'} \in W_{I'}$ (using [op. cit., Proposition 2.3.2(ii)]),

$$\begin{aligned}\ell\big(w_{I'}{}^Iw_0^{-1}{}^Iw\big) &= \ell\big(w_0(w_0w_{I'}w_0)w_{I,0}{}^Iw\big)\\ &= \ell(w_0) - \big(\big(\ell(w_{I,0}) - \ell(w_0w_{I'}w_0)\big) + \ell({}^Iw)\big)\\ &= \ell(w_{I'}) + \ell(w_0w_{I,0}{}^Iw).\end{aligned}$$

Since ${}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0} \in {}^I W$, we deduce that ${}^I w_0^{-1}\, {}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0} \in {}^{I'} W$. Likewise, for all $w^J \in W^J$, we have $w_0 w^J w_{J,0} \in W^J$. Since

$$\begin{aligned} {}^I w_0^{-1}\, {}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0} &= w_0 w_{I,0}\, {}^I w^J\, w_{J\cap {}^I w^{J-1}(I),0} w_{J,0} \\ &= w_0 w_{I,0} w_{I\cap {}^I w^J(J),0}\, {}^I w^J w_{J,0} \\ &= w_0 w_{I,0}^{I\cap {}^I w^J(J)}\, {}^I w^J w_{J,0} \end{aligned}$$

and $w_{I,0}^{I\cap {}^I w^J(J)}\, {}^I w^J \in W^J$, we deduce that ${}^I w_0^{-1}\, {}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0} \in W^J$. We conclude that

$$ {}^I w_0^{-1}\, {}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0} \in {}^{I'} W^J. \qquad \square$$

We see from Lemma 4.3.1 that the left translate by ${}^I w_0$ of the decomposition $G = \bigsqcup_{{}^{I'} w^J \in {}^{I'} W^J} P_{I'}^{-}\, {}^{I'} w^J P_J$ is the decomposition $G = \bigsqcup_{{}^I w^J \in {}^I W^J} P_I\, {}^I w^J P_J$ with the opposite closure relations. Proceeding as in Section 2.2, we can construct a natural filtration $\mathrm{Fil}^\bullet_{P_J}(\mathrm{Ind}_{P_I}^G \sigma)$ by P_J-subrepresentations indexed by ${}^I W^J$ with the opposite Bruhat order, and there is a natural P_J-equivariant isomorphism

$$\mathrm{Gr}_{P_J}^{{}^I w^J}(\mathrm{Ind}_{P_I}^G \sigma) \cong \text{c-ind}_{P_I}^{P_I {}^I w^J P_J} \sigma$$

for all ${}^I w^J \in {}^I W^J$. Furthermore, (44) identifies this filtration with $\mathrm{Fil}^\bullet_{P_J}(\mathrm{Ind}_{P_{I'}^-}^G \sigma^{{}^I w_0})$, using Lemma 4.3.1 to identifiy the indexing posets, and induces a natural P_J-equivariant isomorphism

$$\text{c-ind}_{P_I}^{P_I {}^I w^J P_J} \sigma \cong \text{c-ind}_{P_{I'}^-}^{P_{I'}^- {}^{I'} w^J P_J} \sigma^{{}^I w_0} \tag{45}$$

for all ${}^I w^J \in {}^I W^J$ with ${}^{I'} w^J = {}^I w_0^{-1}\, {}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0}$. Therefore, by Proposition 3.3.1, $\mathrm{Fil}^\bullet_{P_J}(\mathrm{Ind}_{P_I}^G \sigma)$ induces a filtration $\mathrm{Fil}^\bullet_{P_J}(\mathrm{H}_n(N_J, \mathrm{Ind}_{P_I}^G \sigma))$ by L_J-subrepresentations indexed by ${}^I W^J$ with the opposite Bruhat order and there is a natural L_J-equivariant isomorphism

$$\mathrm{Gr}_{P_J}^{{}^I w^J}\big(\mathrm{H}_n(N_J, \mathrm{Ind}_{P_I}^G \sigma)\big) \cong \mathrm{H}_n\big(N_J, \text{c-ind}_{P_I}^{P_I {}^I w^J P_J} \sigma\big)$$

for all ${}^I w^J \in {}^I W^J$.

Let ${}^I w^J \in {}^I W^J$ and set ${}^{I'} w^J := {}^I w_0^{-1}\, {}^I w^{J\, J\cap {}^I w^{J-1}(I)} w_{J,0}$. We let $\tilde\sigma$ be an $L_{J\cap {}^I w^{J-1}(I)}$-representation. Note that $J \cap {}^I w^{J-1}(I) = {}^{J\cap {}^I w^{J-1}(I)} w_{J,0}(J \cap {}^{I'} w^{J-1}(I'))$. We have

$$\boldsymbol{L}_{J\cap {}^I w^{J-1}(I)} = {}^{J\cap {}^I w^{J-1}(I)} w_{J,0} \boldsymbol{L}_{J\cap {}^{I'} w^{J-1}(I')}\, {}^{J\cap {}^I w^{J-1}(I)} w_{J,0}^{-1}$$

and

$$\boldsymbol{L}_J \cap \boldsymbol{P}_{J\cap {}^I w^{J-1}(I)} = {}^{J\cap {}^I w^{J-1}(I)} w_{J,0} \boldsymbol{L}_J \cap \boldsymbol{P}^-_{J\cap {}^{I'} w^{J-1}(I')}\, {}^{J\cap {}^I w^{J-1}(I)} w_{J,0}^{-1},$$

hence a natural L_J-equivariant isomorphism

$$\mathrm{Ind}_{L_J\cap P_{J\cap {}^I w^{J-1}(I)}}^{L_J} \tilde\sigma \cong \mathrm{Ind}_{L_J\cap P^-_{J\cap {}^{I'} w^{J-1}(I')}}^{L_J} \tilde\sigma^{{}^{J\cap {}^I w^{J-1}(I)} w_{J,0}} \tag{46}$$

defined by $f \mapsto (l \mapsto f({}^{J\cap {}^I w^{J-1}(I)} w_{J,0} l))$. Proceeding as in the proof of Lemma 4.3.1, we obtain the following result.

Lemma 4.3.2. *The map ${}^{J\cap {}^I w^{J-1}(I)} W_J \to {}^{J\cap {}^{I'} w^{J-1}(I')} W_J$ defined by $w_J \mapsto {}^{J\cap {}^I w^{J-1}(I)} w_{J,0}^{-1} w_J$ is an order-reversing bijection.*

We deduce from Lemma 4.3.2 that the left translate by ${}^Iw_0$ (resp. ${}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}$) of the decomposition

$$P_{I'}^{-}\,{}^{I'}w^J P_J = \bigsqcup_{w'_J\in {}^{J\cap {}^{I'}w^{J-1}(I')}W_J} P_{I'}^{-}\,{}^{I'}w^J w'_J B \quad \left(\text{resp. } L_J = \bigsqcup_{w'_J\in {}^{J\cap {}^{I'}w^{J-1}(I')}W_J} L_J\cap P^{-}_{J\cap {}^{I'}w^{J-1}(I')} w'_J B_J\right)$$

is the decomposition

$$P_I\,{}^Iw^J P_J = \bigsqcup_{w_J\in {}^{J\cap {}^Iw^{J-1}(I)}W_J} P_I\,{}^Iw^J w_J B \quad \left(\text{resp. } L_J = \bigsqcup_{w_J\in {}^{J\cap {}^Iw^{J-1}(I)}W_J} L_J\cap P_{J\cap {}^Iw^{J-1}(I)} w_J B_J\right)$$

with the opposite closure relations. Proceeding as in Section 2.2, we can construct a natural filtration $\mathrm{Fil}^\bullet_B(\text{c-ind}_{P_I}^{P_I\,{}^Iw^J P_J}\sigma)$ (resp. $\mathrm{Fil}^\bullet_{B_J}(\mathrm{Ind}_{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}}^{L_J}\tilde\sigma)$) by B- (resp. B_J-)subrepresentations indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$ with the opposite Bruhat order, and there is a natural B- (resp. B_J-)equivariant isomorphism

$$\mathrm{Gr}_B^{w_J}\big(\text{c-ind}_{P_I}^{P_I\,{}^Iw^J P_J}\sigma\big)\cong \text{c-ind}_{P_I}^{P_I\,{}^Iw^J w_J B}\sigma \quad \left(\text{resp. } \mathrm{Gr}_{B_J}^{w_J}\big(\mathrm{Ind}_{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}}^{L_J}\tilde\sigma\big)\cong \text{c-ind}_{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}}^{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}w_J B_J}\tilde\sigma\right)$$

for all $w_J\in {}^{J\cap {}^Iw^{J-1}(I)}W_J$. Further, (45) (resp. (46)) identifies this filtration with

$$\mathrm{Fil}^\bullet_B\big(\text{c-ind}_{P_{I'}^{-}}^{P_{I'}^{-}\,{}^{I'}w^J P_J}\sigma^{{}^Iw_0}\big) \quad \left(\text{resp. } \mathrm{Fil}^\bullet_{B_J}\big(\mathrm{Ind}_{L_J\cap P^{-}_{J\cap {}^{I'}w^{J-1}(I')}}^{L_J}\tilde\sigma^{{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}}\big)\right),$$

using Lemma 4.3.2 to identifiy the indexing posets, and induces a natural B- (resp. B_J-)equivariant isomorphism

$$\text{c-ind}_{P_I}^{P_I\,{}^Iw^J w_J B}\sigma\cong \text{c-ind}_{P_{I'}^{-}}^{P_{I'}^{-}\,{}^{I'}w^J w'_J B}\sigma^{{}^Iw_0}$$

$$\left(\text{resp. } \text{c-ind}_{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}}^{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}w_J B_J}\tilde\sigma\cong \text{c-ind}_{L_J\cap P^{-}_{J\cap {}^{I'}w^{J-1}(I')}}^{L_J\cap P^{-}_{J\cap {}^{I'}w^{J-1}(I')}w'_J B_J}\tilde\sigma^{{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}}\right)$$

for all $w_J\in {}^{J\cap {}^Iw^{J-1}(I)}W_J$ with $w'_J = {}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}^{-1}w_J$. Therefore, by Proposition 3.3.2, $\mathrm{Fil}^\bullet_B(\text{c-ind}_{P_I}^{P_I\,{}^Iw^J P_J}\sigma)$ induces a filtration $\mathrm{Fil}^\bullet_B(\mathrm{H}_n(N_J,\text{c-ind}_{P_I}^{P_I\,{}^Iw^J P_J}\sigma))$ by B_J-subrepresentations indexed by ${}^{J\cap {}^Iw^{J-1}(I)}W_J$ with the opposite Bruhat order and there is a natural B_J-equivariant isomorphism

$$\mathrm{Gr}_B^{w_J}\big(\mathrm{H}_n\big(N_J,\text{c-ind}_{P_I}^{P_I\,{}^Iw^J P_J}\sigma\big)\big)\cong \mathrm{H}_n\big(N_J,\text{c-ind}_{P_I}^{P_I\,{}^Iw^J w_J B}\sigma\big)$$

for all $w_J\in {}^{J\cap {}^Iw^{J-1}(I)}W_J$.

Theorem 4.3.3. *Let σ be a locally admissible L_I-representation, ${}^Iw^J\in {}^IW^J$ and $n\in\mathbb{N}$. For all $w_J\in {}^{J\cap {}^Iw^{J-1}(I)}W_J$, there is a natural $B_{J,{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}^{-1}w_J}$-equivariant isomorphism*

$$\mathrm{Gr}_B^{w_J}\big(\mathrm{H}_n\big(N_J,\text{c-ind}_{P_I}^{P_I\,{}^Iw^J P_J}\sigma\big)\big)$$
$$\cong \mathrm{Gr}_{B_J}^{w_J}\Big(\mathrm{Ind}_{L_J\cap P_{J\cap {}^Iw^{J-1}(I)}}^{L_J}\Big(\mathrm{H}_{n-[F:\mathbb{Q}_p]d_{{}^Iw^J}}\big(L_I\cap N_{I\cap {}^Iw^J(J)},\sigma\big)^{{}^Iw^J}\otimes(\omega\circ\delta_{{}^Iw^J})\Big)\Big)$$

which is even $L_J\cap P_{J\cap {}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}^{-1}\,{}^Iw^{J-1}(I)}$-equivariant when $w_J = {}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}$.

Proof. We set ${}^{I'}w^J := {}^Iw_0^{-1}\,{}^Iw^J\,{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}$ and we define an $L_{J\cap {}^Iw^{J-1}(I)}$-representation by setting

$$\tilde\sigma := \Big(\big(\mathrm{H}^{[F:\mathbb{Q}_p](d_J-d_{{}^{I'}w^J})-n}\mathrm{Ord}_{L_{I'}\cap P_{I'\cap {}^{I'}w^J(J)}}\sigma^{{}^Iw_0}\big)^{{}^{I'}w^J}\otimes(\omega\circ(\delta_J-\delta_{{}^{I'}w^J}))\Big)^{{}^{J\cap {}^Iw^{J-1}(I)}w_{J,0}^{-1}},$$

where d_J denotes the integer $\dim_F \boldsymbol{N}_J$ and $\delta_J \in \mathrm{X}^*(\boldsymbol{L}_J)$ denotes the algebraic character of the adjoint representation of $\boldsymbol{L}_J$ on $\det_F(\mathrm{Lie}\, \boldsymbol{N}_J)$. We prove that there is a natural $L_{J\cap {}^I w^{J-1}(I)}$-equivariant isomorphism

$$\tilde{\sigma} \cong \mathrm{H}_{n-[F:\mathbb{Q}_p]d_{{}^I w^J}} \left(L_I \cap N_{I\cap {}^I w^J(J)}, \sigma\right)^{{}^I w^J} \otimes (\omega \circ \delta_{{}^I w^J}) \,. \tag{47}$$

We have $\boldsymbol{L}_{I\cap {}^I w^J(J)} = {}^I w_0 \boldsymbol{L}_{I'\cap {}^{I'} w^J(J)} {}^I w_0^{-1}$ and $\boldsymbol{L}_I \cap \boldsymbol{P}_{I\cap {}^I w^J(J)} = {}^I w_0 \boldsymbol{L}_{I'} \cap \boldsymbol{P}_{I'\cap {}^{I'} w^J(J)} {}^I w_0^{-1}$, hence natural $L_{I'\cap {}^{I'} w^J(J)}$-equivariant isomorphisms

$$\mathrm{H}^\bullet \mathrm{Ord}_{L_{I'}\cap P_{I'\cap {}^{I'} w^J(J)}} \sigma^{{}^I w_0} \cong \left(\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I\cap {}^I w^J(J)}} \sigma\right)^{{}^I w_0}.$$

Using Lemma 2.1.1(iii), we have (with notations analogous to d_J and δ_J)

$$\mathrm{H}^\bullet \mathrm{Ord}_{L_I \cap P_{I\cap {}^I w^J(J)}} = \mathrm{H}_{[F:\mathbb{Q}_p](d_{I\cap {}^I w^J(J)} - d_I) - \bullet}(L_I \cap N_{I\cap {}^I w^J(J)}, -) \otimes (\omega^{-1} \circ (\delta_{I\cap {}^I w^J(J)} - \delta_I)).$$

Thus in order to prove (47), it remains to check that

$$d_J = (d_{I\cap {}^I w^J(J)} - d_I) + d_{{}^I w^J} + d_{{}^{I'} w^J}, \quad \delta_J = {}^I w^{J-1}(\delta_{I\cap {}^I w^J(J)} - \delta_I) + \delta_{{}^I w^J} + {}^{J\cap {}^I w^{J-1}(I)} w_{J,0}(\delta_{{}^{I'} w^J}).$$

We do these computations on the corresponding Lie algebras: d_J and δ_J correspond to $\Phi^+ \backslash \Phi_J^+$, $(d_{I\cap {}^I w^J(J)} - d_I)$ and ${}^I w^{J-1}(\delta_{I\cap {}^I w^J(J)} - \delta_I)$ correspond to $(\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(\Phi_I^+)$, $d_{{}^I w^J}$ and $\delta_{{}^I w^J}$ correspond to $(\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(-\Phi^+)$, and $d_{{}^{I'} w^J}$ and ${}^{J\cap {}^I w^{J-1}(I)} w_{J,0}(\delta_{{}^{I'} w^J})$ correspond to $(\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(\Phi^+ \backslash \Phi_I^+)$ (noting that ${}^I w_0(-\Phi^+) = (-\Phi_I^+) \sqcup (\Phi^+ \backslash \Phi_I^+)$ and $(\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(-\Phi_I^+) = \varnothing$). Thus, the two equalities above follow from the partition

$$\Phi^+ \backslash \Phi_J^+ = \left((\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(\Phi_I^+)\right) \sqcup \left((\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(-\Phi^+)\right) \sqcup \left((\Phi^+ \backslash \Phi_J^+) \cap {}^I w^{J-1}(\Phi^+ \backslash \Phi_I^+)\right),$$

which is obtained from the partition $\Phi = \Phi_I^+ \sqcup (-\Phi^+) \sqcup (\Phi^+ \backslash \Phi_I^+)$ by applying ${}^I w^{J-1}$ and taking the intersection with $\Phi^+ \backslash \Phi_J^+$.

Let $w_J \in {}^{J\cap {}^I w^{J-1}(I)} W_J$ and set $w'_J := {}^{J\cap {}^I w^{J-1}(I)} w_{J,0}^{-1} w_J$. By construction, (45) induces a natural B_J-equivariant isomorphism

$$\mathrm{Gr}_B^{w_J}\left(\mathrm{H}_n\left(N_J, \text{c-ind}_{P_I}^{P_I {}^I w^J P_J} \sigma\right)\right) \cong \mathrm{Gr}_B^{w'_J}\left(\mathrm{H}^{[F:\mathbb{Q}_p]d_J - n} \mathrm{Ord}_{P_J}\left(\text{c-ind}_{P_{I'}^-}^{P_{I'}^- {}^{I'} w^J P_J} \sigma^{{}^I w_0}\right) \otimes (\omega \circ \delta_J)\right).$$

By Theorem 3.3.3, there is a natural B_{J,w'_J}-equivariant isomorphism

$$\mathrm{Gr}_B^{w'_J}\left(\mathrm{H}^{[F:\mathbb{Q}_p]d_J - n} \mathrm{Ord}_{P_J}\left(\text{c-ind}_{P_{I'}^-}^{P_{I'}^- {}^{I'} w^J P_J} \sigma^{{}^I w_0}\right) \otimes (\omega \circ \delta_J)\right) \cong \mathrm{Gr}_{B_J}^{w'_J}\left(\mathrm{Ind}_{L_J \cap P^-_{J\cap {}^{I'} w^{J-1}(I')}}^{L_J} \tilde{\sigma}^{{}^{J\cap {}^I w^{J-1}(I)} w_{J,0}}\right)$$

which is even $L_J \cap P_{J\cap {}^{I'} w^{J-1}(I')}$-equivariant when $w'_J = 1$. By construction, (46) and (47) induce a natural B_J-equivariant isomorphism

$$\begin{aligned}\mathrm{Gr}_{B_J}^{w'_J}&\left(\mathrm{Ind}_{L_J \cap P^-_{J\cap {}^{I'} w^{J-1}(I')}}^{L_J} \tilde{\sigma}^{{}^{J\cap {}^I w^{J-1}(I)} w_{J,0}}\right)\\ &\cong \mathrm{Gr}_{B_J}^{w_J}\left(\mathrm{Ind}_{L_J \cap P_{J\cap {}^I w^{J-1}(I)}}^{L_J}\left(\mathrm{H}_{n-[F:\mathbb{Q}_p]d_{{}^I w^J}}(L_I \cap N_{I\cap {}^I w^J(J)}, \sigma)^{{}^I w^J} \otimes (\omega \circ \delta_{{}^I w^J})\right)\right).\end{aligned}$$

Composing these three isomorphisms yields the result. □

Theorem 4.3.3 with $w_J = {}^{J \cap {}^I w^{J-1}(I)} w_{J,0}$ yields a natural L_J-equivariant morphism analogous to (30),

$$\mathrm{Ind}_{L_J \cap P_{J \cap {}^I w^{J-1}(I)}}^{L_J} \left(\mathrm{H}_{n-[F:\mathbb{Q}_p] d_{{}^I w^J}} \left(L_I \cap N_{I \cap {}^I w^J(J)}, \sigma \right)^{{}^I w^J} \otimes (\omega \circ \delta_{{}^I w^J}) \right) \to \mathrm{H}_n \left(N_J, \text{c-ind}_{P_I}^{P_I {}^I w^J P_J} \sigma \right), \quad (48)$$

and Conjecture 3.3.4 is equivalent to (48) being an isomorphism. We also have analogues of Propositions 3.3.5 and 3.3.7. In the case $I = J$, we obtain the following analogue of Corollary 3.3.8.

Corollary 4.3.4. *Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup and σ be a locally admissible L-representation.*

(i) *For all $n \in \mathbb{N}$ such that $0 < n < [F : \mathbb{Q}_p]$, we have* $\mathrm{H}_n(N, \mathrm{Ind}_P^G \pi) = 0$.

(ii) *If $\sigma_{L \cap s_\alpha N s_\alpha^{-1}} = 0$ for all $\alpha \in \Delta^1 \setminus (\Delta_{\boldsymbol{L}}^1 \cup \Delta_{\boldsymbol{L}}^{\perp,1})$, then there is a natural L-equivariant isomorphism*

$$\mathrm{H}_{[F:\mathbb{Q}_p]} \left(N, \mathrm{Ind}_P^G \sigma \right) \cong \bigoplus_{\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}} \sigma^\alpha \otimes (\omega \circ \alpha) .$$

Remark 4.3.5. The results hold true with P^-, N^- and ω^{-1} instead of P, N and ω respectively.

5. Application to extensions

The aim of this section is to compute the extensions between parabolically induced representations of G. In Section 5.1, we review some cuspidality and genericity properties and we prove some preliminary results on extensions which will be used in the case where $\boldsymbol{G}$ is split and $\boldsymbol{Z}$ is connected. Then, the main results are proved in Section 5.2. Finally, some of these results are lifted to characteristic 0 in Section 5.3.

5.1. *Preliminaries.* We fix a standard parabolic subgroup $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$. We first define some cuspidality properties and discuss the relations between them.

Definition 5.1.1. We say that an admissible smooth representation σ of L over k is:

- *supersingular* if $\overline{\mathbb{F}}_p \otimes_k \sigma$ is supersingular (in the sense of [Abe et al. 2017a]),
- *supercuspidal* if it is irreducible and not a subquotient of $\mathrm{Ind}_Q^L \tau$ for any proper parabolic subgroup $\boldsymbol{Q} \subset \boldsymbol{L}$ with Levi quotient $\boldsymbol{L_Q}$ and any irreducible admissible smooth representation τ of L_Q over k,
- *right* (resp. *left*) *cuspidal* if $\mathrm{Ord}_Q \, \sigma = 0$ (resp. $\sigma_{N_Q} = 0$) for any proper parabolic subgroup $\boldsymbol{Q} \subset \boldsymbol{L}$ with unipotent radical $\boldsymbol{N_Q}$.

Remark 5.1.2. In [Abe et al. 2017b, Definition 6.3], left and right cuspidality are defined for smooth representations using the left and right adjoint functors of Ind_Q^G, namely L_Q^L and R_Q^L. Since $\mathrm{L}_Q^L = (-)_{N_Q}$ and the restriction of R_Q^L to admissible representations is Ord_{Q^-} [op. cit., Corollary 4.13]), these definitions coincide for admissible representations.

Lemma 5.1.3. *Let σ be an irreducible admissible smooth representation of L over k. The following conditions are equivalent.*

(i) *σ is supercuspidal.*

(ii) *σ is left and right cuspidal.*

(iii) $\overline{\mathbb{F}}_p \otimes_k \sigma$ *is a (finite) direct sum of supersingular representations.*

In particular, σ is supersingular if and only if it is absolutely irreducible and supercuspidal.

Proof. Over $\overline{\mathbb{F}}_p$, the equivalence between (i) and (ii) is [op. cit., Corollary 6.9], and the equivalence between "supercuspidal" and "supersingular" is [Abe et al. 2017a, Theorem 5]. By [Emerton 2010b, Lemma 4.1.2], $\overline{\mathbb{F}}_p \otimes_k \sigma$ is a finite direct sum of irreducible admissible smooth representations of L over $\overline{\mathbb{F}}_p$. Since Ind_Q^L, $(-)_{N_Q}$ and Ord_Q commute with $\overline{\mathbb{F}}_p \otimes_k -$, we deduce the equivalences over k. □

We now study some genericity property for smooth representations of L over k with central character. We assume that $\Delta_{\boldsymbol{L}}^{\perp,1} \neq \varnothing$.

Lemma 5.1.4. *Let σ be a smooth representation of L over k with central character $\zeta : Z_L \to k^\times$ and $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$. If $\zeta \circ \alpha^\vee = \omega^{-1}$, then $\sigma^\alpha \otimes (\omega^{-1} \circ \alpha) \cong \sigma$.*

Proof. For convenience, we recall the construction of the representation $\sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$. Let $\boldsymbol{G}_\alpha \subseteq \boldsymbol{G}$ be the standard Levi subgroup corresponding to α. We fix a representative $n_\alpha \in \mathcal{N}$ of s_α. For every $\beta \in \Delta_{\boldsymbol{L}}$ and for all integers $i, j > 0$, $i\alpha + j\beta \notin \Phi$ (since $\alpha \perp \beta$), thus $\boldsymbol{U}_\alpha$ and $\boldsymbol{U}_\beta$ commute for every $\beta \in \Delta_{\boldsymbol{L}}$ [Borel and Tits 1965, Proposition 2.5], or more directly using the Baker–Campbell–Hausdorff formula. We deduce that $\boldsymbol{G}_\alpha$ and $\boldsymbol{L}$ normalise each other (since $\boldsymbol{G}_\alpha$ and $\boldsymbol{L}$ are generated by $\boldsymbol{\mathcal{Z}}$ and respectively $\boldsymbol{U}_{\pm\alpha}$ and $(\boldsymbol{U}_\beta)_{\beta \in \pm\Delta_{\boldsymbol{L}}}$). In particular, n_α normalises L (since $n_\alpha \in G_\alpha$) and the n_α-conjugate σ^α does not depend on the choice of n_α in $n_\alpha \mathcal{Z}$ up to isomorphism (since $\mathcal{Z} \subseteq L$). Furthermore, $\boldsymbol{L}$ normalises $\boldsymbol{U}_\alpha$ and α extends (uniquely) to an algebraic character of $\boldsymbol{L}$ (since $\alpha \in \Delta^1$).

We let $I_\alpha \subseteq L$ be the kernel of $\alpha : L \to F^\times$. Note that $L = SI_\alpha$. We may and will assume that n_α lies in the subgroup of G_α generated by $U_{\pm\alpha}$ [Abe et al. 2017a, §II.4] so that n_α commutes with I_α. Thus, the action of I_α on $\sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ and σ is the same.

Now, assume $\zeta \circ \alpha^\vee = \omega^{-1}$. For any $\lambda \in \mathrm{X}_*(\boldsymbol{S})$ we have $\lambda - s_\alpha(\lambda) = \langle \alpha, \lambda \rangle \alpha^\vee$, so that

$$\lambda - s_\alpha(\lambda) \in \mathrm{X}_*(\boldsymbol{S} \cap \boldsymbol{Z_L}) \quad \text{and} \quad \zeta \circ (\lambda - s_\alpha(\lambda)) = (\zeta \circ \alpha^\vee)^{\langle \alpha, \lambda \rangle} = \omega^{-\langle \alpha, \lambda \rangle} = (\omega^{-1} \circ \alpha) \circ \lambda.$$

We deduce that for any $s \in S$, $s(n_\alpha s n_\alpha^{-1})^{-1} \in S \cap Z_L$ and $\zeta(s(n_\alpha s n_\alpha^{-1})^{-1}) = (\omega^{-1} \circ \alpha)(s)$. Thus, the action of S on $\sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ and σ is the same. □

The following result yields a converse to Lemma 5.1.4 when $\boldsymbol{G}$ is split and $\boldsymbol{Z}$ is connected [Breuil and Herzig 2015, Proposition 2.1.1].

Lemma 5.1.5. *Let σ be a smooth representation of L over k with central character $\zeta : Z_L \to k^\times$ and $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$. Assume that there exists $\lambda \in \mathrm{X}_*(\boldsymbol{Z_L})$ such that $\langle \alpha, \lambda \rangle = 1$ and $\langle \beta, \lambda \rangle = 0$ for all $\beta \in \Delta_{\boldsymbol{L}}^{\perp,1} \backslash \{\alpha\}$. If $\zeta \circ \alpha^\vee \neq \omega^{-1}$, then $s_\alpha(\zeta)(\omega^{-1} \circ \alpha) \neq \zeta$ and $s_\alpha(\zeta)(\omega^{-1} \circ \alpha) \neq s_\beta(\zeta)(\omega^{-1} \circ \beta)$ for all $\beta \in \Delta_{\boldsymbol{L}}^{\perp,1} \backslash \{\alpha\}$.*

Proof. We have $(s_\alpha(\zeta)(\omega^{-1} \circ \alpha)) \circ \lambda = (\zeta \circ \lambda)((\zeta \circ \alpha^\vee)\omega)^{-1}$ and $(s_\beta(\zeta)(\omega^{-1} \circ \beta)) \circ \lambda = (\zeta \circ \lambda)$ for all $\beta \in \Delta_{\boldsymbol{L}}^{\perp,1} \backslash \{\alpha\}$. Thus, if $s_\alpha(\zeta)(\omega^{-1} \circ \alpha) = \zeta$ or $s_\alpha(\zeta)(\omega^{-1} \circ \alpha) = s_\beta(\zeta)(\omega^{-1} \circ \beta)$ for some $\beta \in \Delta_{\boldsymbol{L}}^{\perp,1} \backslash \{\alpha\}$, then precomposing each side of the equality with λ yields the equality $(\zeta \circ \alpha^\vee)\omega = 1$. □

We now give some preliminary results on extensions. Let H be a p-adic Lie group. For locally admissible smooth representations π, π' of L over A, we let $\mathrm{Ext}^\bullet_H(\pi', \pi)$ denote the A-modules of extensions computed in $\mathrm{Mod}^{\mathrm{l.adm}}_H(A)$ à la Yoneda or using an injective resolution of π. If π, π' are admissible, then in degree 1 it is equivalent to compute $\mathrm{Ext}^\bullet_H(\pi', \pi)$ in $\mathrm{Mod}^{\mathrm{adm}}_H(A)$ à la Yoneda, but this is not known in higher degree [Emerton 2010b, Remark 3.7.8], except when $H = \mathrm{GL}_2(\mathbb{Q}_p)$ [Paškūnas 2013, Corollary 5.17]. Let $\widetilde{Z} \subseteq H$ be a central closed subgroup and $\zeta : \widetilde{Z} \to A^\times$ be a smooth character. We write $\mathrm{Mod}^{\mathrm{l.adm}}_{H,\zeta}(A)$ for the full subcategory of $\mathrm{Mod}^{\mathrm{l.adm}}_H(A)$ whose objects are the representations on which $\widetilde{Z}$ acts via ζ. If $\widetilde{Z}$ acts on π, π' via ζ, then we let $\mathrm{Ext}^\bullet_{H,\zeta}(\pi', \pi)$ denote the A-modules of extensions computed in $\mathrm{Mod}^{\mathrm{l.adm}}_{H,\zeta}(A)$ à la Yoneda, or equivalently using an injective resolution of π.

We now assume that $\boldsymbol{G}$ is split and we write $\boldsymbol{T}$ for the maximal split torus $\boldsymbol{S} = \boldsymbol{\mathcal{Z}}$. Using Notation 2.3.3, we have $d_w = \ell(w)$ for all $w \in W$, so that in particular $\Delta^1 = \Delta$. Let $\boldsymbol{L}' \subseteq \boldsymbol{G}$ be a standard Levi subgroup such that $\Delta_{\boldsymbol{L}} \perp \Delta_{\boldsymbol{L}'}$. Note that $\boldsymbol{L}\boldsymbol{L}'$ is the standard Levi subgroup corresponding to $\Delta_{\boldsymbol{L}} \sqcup \Delta_{\boldsymbol{L}'}$. Let σ be a locally admissible smooth representation of L over k with central character $\zeta : Z_L \to k^\times$. The following construction was communicated to me by N. Abe.

First, we assume that $\boldsymbol{G}^{\mathrm{der}}$ is simply connected and we let $\widetilde{\boldsymbol{Z}} \subseteq \boldsymbol{Z}$ be a closed subgroup. Recall that this is equivalent to the existence of fundamental weights $(\mu_\alpha)_{\alpha \in \Delta}$ [Breuil and Herzig 2015, Proposition 2.1.1]. We set $\chi := \zeta \circ \sum_{\alpha \in \Delta_{L'}} (\alpha^\vee \circ \mu_\alpha)$. Thus $\chi \circ \alpha^\vee = 1$ for all $\alpha \in \Delta_{\boldsymbol{L}}$ and $\chi \circ \alpha^\vee = \zeta \circ \alpha^\vee$ for all $\alpha \in \Delta_{\boldsymbol{L}'}$, so that χ extends uniquely to L and $\sigma_0 := \sigma \otimes \chi^{-1}$ extends uniquely to a locally admissible smooth representation of LL' over k [Abe 2013, Lemma 3.2]. We let $\chi' : T \to k^\times$ be a smooth character such that $\chi'_{|Z_{L'}} = \chi_{|Z_{L'}}$, so that χ' extends uniquely to L, and we set $\sigma' := \sigma_0 \otimes \chi'$. There is a commutative diagram of k-vector spaces

$$\begin{array}{ccc}
\mathrm{Ext}^\bullet_{T,\chi_{|Z_{L'}}}(\chi', \chi) & \longrightarrow & \mathrm{Ext}^\bullet_{L',\chi_{|Z_{L'}}}\big(\mathrm{Ind}^{L'}_{B^-_{L'}} \chi', \mathrm{Ind}^{L'}_{B^-_{L'}} \chi\big) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet_{L,\chi_{|\widetilde{Z}}}(\chi', \chi) & \longrightarrow & \mathrm{Ext}^\bullet_{LL',\chi_{|\widetilde{Z}}}\big(\mathrm{Ind}^{LL'}_{LB^-_{L'}} \chi', \mathrm{Ind}^{LL'}_{LB^-_{L'}} \chi\big) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet_{L,\zeta_{|\widetilde{Z}}}(\sigma', \sigma) & \longrightarrow & \mathrm{Ext}^\bullet_{LL',\zeta_{|\widetilde{Z}}}\big(\mathrm{Ind}^{LL'}_{LB^-_{L'}} \sigma', \mathrm{Ind}^{LL'}_{LB^-_{L'}} \sigma\big)
\end{array} \tag{49}$$

where the horizontal arrows are induced by the functors $\mathrm{Ind}^{L'}_{B^-_{L'}}$ and $\mathrm{Ind}^{LL'}_{LB^-_{L'}}$, the upper vertical arrows are induced by extending representations to L and LL', and the lower vertical arrows are induced by tensoring representations with σ_0. Furthermore, the lower horizontal arrow of (49) composed with the k-linear morphism induced by the functor $\mathrm{Ind}^G_{P^-L'}$:

$$\mathrm{Ext}^\bullet_{LL',\zeta_{|\widetilde{Z}}}\big(\mathrm{Ind}^{LL'}_{LB^-_{L'}} \sigma', \mathrm{Ind}^{LL'}_{LB^-_{L'}} \sigma\big) \to \mathrm{Ext}^\bullet_{G,\zeta_{|\widetilde{Z}}}(\mathrm{Ind}^G_{P^-} \sigma', \mathrm{Ind}^G_{P^-} \sigma), \tag{50}$$

is the k-linear morphism induced by the functor $\mathrm{Ind}^G_{P^-}$:

$$\mathrm{Ext}^\bullet_{L,\zeta_{|\widetilde{Z}}}(\sigma', \sigma) \to \mathrm{Ext}^\bullet_{G,\zeta_{|\widetilde{Z}}}(\mathrm{Ind}^G_{P^-} \sigma', \mathrm{Ind}^G_{P^-} \sigma). \tag{51}$$

Lemma 5.1.6. (i) *In degree* 1, *there is a k-linear injection from the cokernel of the upper horizontal arrow of* (49) *into the cokernel of* (51).

(ii) *Assume $\boldsymbol{Z}$ connected. In all degrees, there is a k-linear injection from the kernel of the upper horizontal arrow of* (49) *into the kernel of* (51).

Proof. We prove (i). The map in question is induced by the composite right-hand side vertical arrow of (49) composed with (50). Let $\mathcal{E}$ be an extension of $\mathrm{Ind}_{B_{L'}^-}^{L'} \chi'$ by $\mathrm{Ind}_{B_{L'}^-}^{L'} \chi$ with central character $\chi_{|Z_{L'}}$ (so that $\mathcal{E}$ extends to LL'). Then $\mathrm{Ind}_{P^- L'}^G(\sigma_0 \otimes \mathcal{E})$ is an extension of $\mathrm{Ind}_{P^-}^G \sigma'$ by $\mathrm{Ind}_{P^-}^G \sigma$ on which $\widetilde{Z}$ acts via ζ. There are L-equivariant isomorphisms

$$\mathrm{Ord}_P\left(\mathrm{Ind}_{P^- L'}^G(\sigma_0 \otimes \mathcal{E})\right) \cong \mathrm{Ord}_{LB_{L'}}(\sigma_0 \otimes \mathcal{E}) \cong \sigma_0 \otimes \mathrm{Ord}_{B_{L'}} \mathcal{E}.$$

The first one results from [Emerton 2010a, Proposition 4.3.4] and the second one from the fact that $U_{L'}$ acts trivially on σ_0 (note that $\mathrm{Ord}_{B_{L'}} \mathcal{E}$ extends to L). If the class of $\mathcal{E}$ is not in the image of the upper horizontal arrow of (49), then there is a T-equivariant isomorphism $\mathrm{Ord}_{B_{L'}} \mathcal{E} \cong \chi$, hence an L-equivariant isomorphism $\mathrm{Ord}_P(\mathrm{Ind}_{P^- L'}^G(\sigma_0 \otimes \mathcal{E})) \cong \sigma$; thus the class of $\mathrm{Ind}_{P^- L'}^G(\sigma_0 \otimes \mathcal{E})$ is not in the image of (51).

We prove (ii). The map in question is induced by the left-hand side composite vertical arrow of (49). Thus, it is enough to prove that the latter is injective. We assume $\boldsymbol{Z}$ connected. Recall that this is equivalent to the existence of fundamental coweights $(\lambda_\alpha)_{\alpha \in \Delta}$ [Breuil and Herzig 2015, Proposition 2.1.1]. We let $\boldsymbol{T}' \subseteq \boldsymbol{T}$ be the closed subgroup generated by the images of $(\lambda_\alpha)_{\alpha \in \Delta_{L'}}$, so that $\boldsymbol{T}' \subseteq \boldsymbol{Z_L}$ and the product induces an isomorphism $\boldsymbol{T}' \times \boldsymbol{Z_{L'}} \xrightarrow{\sim} \boldsymbol{T}$. There is a commutative diagram of k-vector spaces

$$\begin{array}{ccc}
\mathrm{Ext}^\bullet_{T,\chi_{|Z_{L'}}}(\chi', \chi) & \xrightarrow{\sim} & \mathrm{Ext}^\bullet_{T'}\big(\chi'_{|T'}, \chi_{|T'}\big) \\
\downarrow & & \Vert \\
\mathrm{Ext}^\bullet_{L,\chi_{|\widetilde{Z}}}(\chi', \chi) & \longrightarrow & \mathrm{Ext}^\bullet_{T'}\big(\chi'_{|T'}, \chi_{|T'}\big) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet_{L,\zeta_{|\widetilde{Z}}}(\sigma', \sigma) & \longrightarrow & \mathrm{Ext}^\bullet_{T'}\big(\sigma'_{|T'}, \sigma_{|T'}\big)
\end{array}$$

where the horizontal arrows are induced by restricting representations to T' (the upper one is bijective with inverse induced by tensoring representations with $\chi_{|Z_{L'}}$, and the middle and lower ones are well-defined since a locally admissible smooth representation of L over k is locally Z_L-finite [Emerton 2010a, Lemma 2.3.4], the left-hand side vertical arrows are the same as in (49) and the lower right-hand side vertical arrow is induced by tensoring representations with $\sigma_{0|T'}$ (it is injective since T' acts on σ_0 via $\zeta\chi^{-1}$). □

Now, we do not assume $\boldsymbol{G}^{\mathrm{der}}$ simply connected. Instead we take a z-extension of $\boldsymbol{G}$, i.e., an exact sequence of linear algebraic F-groups

$$1 \to \widetilde{\boldsymbol{Z}} \to \widetilde{\boldsymbol{G}} \to \boldsymbol{G} \to 1$$

such that $\widetilde{\boldsymbol{G}}$ is reductive with simply connected derived subgroup and $\widetilde{\boldsymbol{Z}}$ is a central torus [Colliot-Thélène 2008, §3.1]. The projection $\widetilde{\boldsymbol{G}} \twoheadrightarrow \boldsymbol{G}$ identifies the corresponding root systems. We let $\widetilde{\boldsymbol{P}} \subseteq \widetilde{\boldsymbol{G}}$ be the standard parabolic subgroups corresponding to $\boldsymbol{P}$ and $\widetilde{\boldsymbol{L}} \subseteq \widetilde{\boldsymbol{P}}$ be the standard Levi subgroup corresponding to $\boldsymbol{L}$. Note that $\widetilde{\boldsymbol{L}}$ is a z-extensions of $\boldsymbol{L}$. We let σ' be a locally admissible smooth representation of L over k with central character ζ. By inflation, we obtain locally admissible smooth representations $\tilde{\sigma}$ and $\tilde{\sigma}'$ of $\widetilde{L}$ over k. There is a commutative diagram of k-vector spaces

$$\begin{array}{ccc} \mathrm{Ext}^\bullet_L(\sigma',\sigma) & \longrightarrow & \mathrm{Ext}^\bullet_G\big(\mathrm{Ind}_{P^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Ext}^\bullet_{\widetilde{L},\zeta_{|\widetilde{Z}}}(\tilde{\sigma}',\tilde{\sigma}) & \longrightarrow & \mathrm{Ext}^\bullet_{\widetilde{G},\zeta_{|\widetilde{Z}}}\big(\mathrm{Ind}_{\widetilde{P}^-}^{\widetilde{G}} \tilde{\sigma}', \mathrm{Ind}_{\widetilde{P}^-}^{\widetilde{G}} \tilde{\sigma}\big) \end{array} \tag{52}$$

where the horizontal arrows are induced by the functors $\mathrm{Ind}_{P^-}^G$ and $\mathrm{Ind}_{\widetilde{P}^-}^{\widetilde{G}}$ and the vertical arrows are induced by inflating representations to $\widetilde{L}$ and $\widetilde{G}$ (they are well defined and bijective since $\zeta_{|\widetilde{Z}}$ is trivial).

Proposition 5.1.7. *Assume $F = \mathbb{Q}_p$ and $\boldsymbol{G}$ split. Let σ be a locally admissible smooth representation of L over k with central character $\zeta : Z_L \to k^\times$.*

(i) *Assume $\Delta_{\boldsymbol{L}}^\perp \neq \varnothing$ and let $\alpha \in \Delta_{\boldsymbol{L}}^\perp$. If $\zeta \circ \alpha^\vee \neq \omega^{-1}$, then the k-linear injection*

$$\mathrm{Ext}^1_L\left(\sigma^\alpha \otimes (\omega^{-1}\circ\alpha), \sigma\right) \hookrightarrow \mathrm{Ext}^1_G\left(\mathrm{Ind}_{P^-}^G \sigma^\alpha \otimes (\omega^{-1}\circ\alpha), \mathrm{Ind}_{P^-}^G \sigma\right)$$

induced by the functor $\mathrm{Ind}_{P^-}^G$ is not surjective.

(ii) *If $p=2$, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear injection*

$$\mathrm{Ext}^1_L(\sigma,\sigma) \hookrightarrow \mathrm{Ext}^1_G\left(\mathrm{Ind}_{P^-}^G \sigma, \mathrm{Ind}_{P^-}^G \sigma\right)$$

whose cokernel is of dimension at least $\mathrm{card}\{\alpha \in \Delta_{\boldsymbol{L}}^\perp \mid \zeta\circ\alpha^\vee = 1\}$.

Remark 5.1.8. We expect the results to hold true for a nonsplit reductive group with $\Delta_{\boldsymbol{L}}^{\perp,1}$ instead of $\Delta_{\boldsymbol{L}}^\perp$.

Proof. By taking a z-extension of $\boldsymbol{G}$ and using (52), we can and do assume that $\boldsymbol{G}^{\mathrm{der}}$ is simply connected and prove analogous results for the morphism (51).

Assume $\Delta_{\boldsymbol{L}}^\perp \neq \varnothing$ and let $\alpha \in \Delta_{\boldsymbol{L}}^\perp$. We use Lemma 5.1.6(i) with $\boldsymbol{L}'$ defined by $\Delta_{\boldsymbol{L}'} = \{\alpha\}$, $\chi = \zeta\circ\alpha^\vee\circ\mu_\alpha$, and $\chi' = s_\alpha(\chi)(\omega^{-1}\circ\alpha)$, so that $\sigma' = \sigma^\alpha \otimes (\omega^{-1}\circ\alpha)$ (since $\sigma_0^\alpha = \sigma_0$ by Lemma 5.1.4 with 1 instead of ω). If $\zeta\circ\alpha^\vee \neq \omega^{-1}$, then the upper horizontal arrow of (49) in degree 1 is not surjective by the mod p analogue of [Hauseux 2017, Lemme 3.1.4] (since $\chi\circ\alpha^\vee = \zeta\circ\alpha^\vee = 1$); thus (51) in degree 1 is not surjective, hence (i).

We use Lemma 5.1.6(i) with $\boldsymbol{L}'$ defined by $\Delta_{\boldsymbol{L}'} = \{\alpha \in \Delta_{\boldsymbol{L}}^\perp \mid \zeta\circ\alpha^\vee = 1\}$, $\chi = \zeta \circ \sum_{\alpha\in\Delta_{\boldsymbol{L}'}} (\alpha^\vee\circ\mu_\alpha)$, and $\chi' = \chi$, so that $\sigma' = \sigma$. If $p=2$, then the cokernel of the upper horizontal arrow of (49) in degree 1 is of dimension at least $\mathrm{card}\,\Delta_{\boldsymbol{L}'}$ [Hauseux 2017, Théorème 3.2.4(ii) and Remarque 3.2.5(ii)] (since $\chi\circ\alpha^\vee = \zeta\circ\alpha^\vee = 1$ so that $s_\alpha(\chi) = \chi$ by Lemma 5.1.4 for all $\alpha \in \Delta_{\boldsymbol{L}'}$), noting that all the extensions constructed there have a central character [op. cit., Lemme 3.1.5]; thus the cokernel of (51) in degree 1 is of dimension at least $\mathrm{card}\,\Delta_{\boldsymbol{L}'}$, hence (ii). ☐

Proposition 5.1.9. *Assume $F=\mathbb{Q}_p$, $\boldsymbol{G}$ split and $\boldsymbol{Z}$ connected. Let σ be a locally admissible smooth representation of L over k with central character $\zeta : Z_L \to k^\times$. If $p \neq 2$, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear morphism*

$$\mathrm{Ext}_L^2(\sigma,\sigma) \to \mathrm{Ext}_G^2\left(\mathrm{Ind}_{P^-}^G \sigma, \mathrm{Ind}_{P^-}^G \sigma\right)$$

whose kernel is of dimension at least $\mathrm{card}\{\alpha \in \Delta_{\boldsymbol{L}}^{\perp} \mid \zeta\circ\alpha^\vee = \omega^{-1}\}$.

Proof. By taking a z-extension of $\boldsymbol{G}$ (noting that the centre of $\widetilde{\boldsymbol{G}}$ is also connected because $\widetilde{\boldsymbol{Z}}$ is connected) and using (52), we can and do assume that $\boldsymbol{G}^{\mathrm{der}}$ is simply connected and prove an analogous result for the morphism (51).

We use Lemma 5.1.6(ii) with $\boldsymbol{L}'$ defined by $\Delta_{\boldsymbol{L}'} = \{\alpha \in \Delta_{\boldsymbol{L}}^{\perp} \mid \zeta\circ\alpha^\vee = \omega^{-1}\}$ and $\chi'=\chi$, so that $\sigma'=\sigma$. If $p\neq 2$, then we see in the proof of [Hauseux 2017, Théorème 3.2.4(i)] that the kernel of the upper horizontal arrow of (49) in degree 2 is of dimension at least $\mathrm{card}\,\Delta_{\boldsymbol{L}'}$ (since $\chi\circ\alpha^\vee = \zeta\circ\alpha^\vee = \omega^{-1}$ for all $\alpha\in\Delta_{\boldsymbol{L}'}$); thus the kernel of (49) in degree 2 is also of dimension at least $\mathrm{card}\,\Delta_{\boldsymbol{L}'}$, hence the result. □

5.2. ***Extensions between parabolically induced representations.*** We begin with a result when there is no inclusion between the two parabolic subgroups, assuming a special case of Conjecture 3.3.4 (see also Remark 3.3.6).

Proposition 5.2.1. *Let $\boldsymbol{P}=\boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}'=\boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups and σ, σ' be admissible smooth representations of L, L' respectively over k. Assume Conjecture 3.3.4 is true for $A=k$, $n=1$ and ${}^Iw^J=1$. If $\boldsymbol{P}'\not\subseteq\boldsymbol{P}$, $\boldsymbol{P}\not\subseteq\boldsymbol{P}'$, and σ, σ' are right, left cuspidal respectively, then*

$$\mathrm{Ext}_G^1\left(\mathrm{Ind}_{P'^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\right)=0.$$

Proof. We put $I:=\Delta_{\boldsymbol{L}}$ and $J:=\Delta_{\boldsymbol{L}'}$. Using (31), (39) with $\pi=\mathrm{Ind}_{P_I^-}^G\sigma$ and $\boldsymbol{P}_J$, $\boldsymbol{L}_J$, σ' instead of $\boldsymbol{P}$, $\boldsymbol{L}$, σ respectively yields an exact sequence of k-vector spaces

$$0\to \mathrm{Ext}_{L_J}^1\left(\sigma', \mathrm{Ind}_{L_J\cap P_I^-}^{L_J}\left(\mathrm{Ord}_{L_I\cap P_J}\sigma\right)\right)\to \mathrm{Ext}_G^1\left(\mathrm{Ind}_{P_J^-}^G\sigma', \mathrm{Ind}_{P_I^-}^G\sigma\right)\to \mathrm{Hom}_{L_J}\left(\sigma', \mathrm{H}^1\mathrm{Ord}_{P_J}\left(\mathrm{Ind}_{P_I^-}^G\sigma\right)\right). \tag{53}$$

Assume $I\not\subseteq J$ and σ right cuspidal. Then $\mathrm{Ord}_{L_I\cap P_J}\sigma=0$ (since $\boldsymbol{L}_I\cap\boldsymbol{P}_J$ is a proper parabolic subgroup of $\boldsymbol{L}_I$) and there is a natural L_J-equivariant isomorphism

$$\mathrm{Ind}_{L_J\cap P_I^-}^{L_J}\left(\mathrm{H}^1\mathrm{Ord}_{L_I\cap P_J}\sigma\right) \xrightarrow{\sim} \mathrm{H}^1\mathrm{Ord}_{P_J}\left(\mathrm{Ind}_{P_I^-}^G\sigma\right). \tag{54}$$

Indeed, by assumption (30) is a natural L_J-equivariant isomorphism

$$\mathrm{Ind}_{L_J\cap P_I^-}^{L_J}\left(\mathrm{H}^1\mathrm{Ord}_{L_I\cap P_J}\sigma\right) \xrightarrow{\sim} \mathrm{H}^1\mathrm{Ord}_{P_J}\left(\text{c-ind}_{P_I^-}^{P_I^- P_J}\sigma\right)$$

and by Proposition 3.3.5(i) with $n=1$, we have $\mathrm{H}^1\mathrm{Ord}_{P_J}(\text{c-ind}_{P_I^-}^{P_I^- {}^Iw^J P_J}\sigma)=0$ for all ${}^Iw^J\in{}^IW^J$ such that ${}^Iw^J\neq1$ (since either $d_{{}^Iw^J}=1$ and $\mathrm{Ord}_{L_I\cap P_{I\cap {}^Iw^J(J)}}\sigma=0$, or $d_{{}^Iw^J}>1$ and $1-[F:\mathbb{Q}_p]d_{{}^Iw^J}<0$). Thus, we deduce (54) from Proposition 3.3.1.

Assume $J\not\subseteq I$ and σ' left cuspidal. Then $\sigma'_{L_J\cap N_I^-}=0$ (since $\boldsymbol{L}_J\cap\boldsymbol{P}_I^-$ is a proper parabolic subgroup of $\boldsymbol{L}_J$) and using (40) with $\pi=\sigma'$ and $\boldsymbol{L}_J$, $\boldsymbol{L}_J\cap\boldsymbol{P}_I^-$, $\boldsymbol{L}_{J\cap I}$, $\boldsymbol{L}_J\cap\boldsymbol{N}_I^-$, $\mathrm{H}^1\mathrm{Ord}_{L_I\cap P_J}\sigma$ instead of $\boldsymbol{G}$, $\boldsymbol{P}$, $\boldsymbol{L}$,

$\boldsymbol{N}$, σ respectively, we obtain

$$\mathrm{Hom}_{L_J}\big(\sigma', \mathrm{Ind}_{L_J\cap P_I^-}^{L_J}\big(\mathrm{H}^1\mathrm{Ord}_{L_I\cap P_J}\,\sigma\big)\big)=0.$$

We conclude using (53). □

Now, we prove unconditional results whenever there is an inclusion between the two parabolic subgroups. We treat the cases $F=\mathbb{Q}_p$ and $F\neq\mathbb{Q}_p$ separately.

Theorem 5.2.2. *Assume $F=\mathbb{Q}_p$. Let $\boldsymbol{P}=\boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}'=\boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups and σ,σ' be admissible smooth representations of L, L' respectively over k.*

(i) *If $\boldsymbol{P}'=\boldsymbol{P}$, σ,σ' are supercuspidal and $\sigma'\not\cong\sigma^\alpha\otimes(\omega^{-1}\circ\alpha)$ for all $\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}$, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}_L^1(\sigma',\sigma)\xrightarrow{\sim}\mathrm{Ext}_G^1\big(\mathrm{Ind}_{P^-}^G\,\sigma',\mathrm{Ind}_{P^-}^G\,\sigma\big).$$

(ii) *If $\boldsymbol{P}'\subsetneq\boldsymbol{P}$ and σ is right cuspidal, then the functor $\mathrm{Ind}_{P^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}_L^1\big(\mathrm{Ind}_{L\cap P'^-}^L\,\sigma',\sigma\big)\xrightarrow{\sim}\mathrm{Ext}_G^1\big(\mathrm{Ind}_{P'^-}^G\,\sigma',\mathrm{Ind}_{P^-}^G\,\sigma\big).$$

(iii) *If $\boldsymbol{P}\subsetneq\boldsymbol{P}'$ and σ' is left cuspidal, then the functor $\mathrm{Ind}_{P'^-}^G$ induces a k-linear isomorphism*

$$\mathrm{Ext}_{L'}^1\big(\sigma',\mathrm{Ind}_{L'\cap P^-}^{L'}\,\sigma\big)\xrightarrow{\sim}\mathrm{Ext}_G^1\big(\mathrm{Ind}_{P'^-}^G\,\sigma',\mathrm{Ind}_{P^-}^G\,\sigma\big).$$

Remark 5.2.3. Assume $\boldsymbol{P}'=\boldsymbol{P}$ and σ,σ' irreducible. In general, we do not know the dimension of the cokernel of the k-linear injection $\mathrm{Ext}_L^1(\sigma',\sigma)\hookrightarrow\mathrm{Ext}_G^1(\mathrm{Ind}_{P^-}^G\,\sigma',\mathrm{Ind}_{P^-}^G\,\sigma)$ induced by $\mathrm{Ind}_{P^-}^G$, but we prove that it is at most $\mathrm{card}\{\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}\mid\sigma'\cong\sigma^\alpha\otimes(\omega^{-1}\circ\alpha)\}$ whenever σ is right cuspidal or σ' is left cuspidal (see the proof). If σ,σ' are supersingular, then letting $\zeta:Z_L\to k^\times$ denote the central character of σ [Emerton 2010b, Lemma 4.1.7], we expect this dimension to be equal to

$$\mathrm{card}\big\{\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}\;\big|\;\sigma'\cong\sigma^\alpha\otimes\big(\omega^{-1}\circ\alpha\big)\text{ and }\zeta\circ\alpha^\vee\neq\omega^{-1}\big\}$$

except when $p=2$ and in some exceptional cases [Hauseux 2017, Remarque 3.2.5] when $\boldsymbol{G}$ is split and $\boldsymbol{P}=\boldsymbol{B}$. We prove this when $\boldsymbol{G}$ is split and $\boldsymbol{Z}$ is connected (see Theorem 5.2.7 below), in which case the cardinal above is equal to 1 if $\sigma'\cong\sigma^\alpha\otimes(\omega^{-1}\circ\alpha)\not\cong\sigma$ for some $\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}$ and 0 otherwise by Lemma 5.1.5. When $\boldsymbol{G}$ is split but $\boldsymbol{Z}$ is not connected, one could prove that the cardinal above is a lower bound using Proposition 5.1.7(i) and some generalisation of [Hauseux 2017, §2.2] for $\boldsymbol{P}\neq\boldsymbol{B}$.

Proof. We prove slightly more general results.

Assume $\boldsymbol{P}'\subseteq\boldsymbol{P}$ and σ satisfies the condition in Corollary 3.3.8(ii), e.g., σ is right cuspidal. Using [Emerton 2010a, Proposition 4.3.4] and Corollary 3.3.8(ii), (39) with $\pi=\mathrm{Ind}_{P^-}^G\,\sigma$ and $\mathrm{Ind}_{L\cap P'^-}^L\,\sigma'$ instead of σ yields an exact sequence of k-vector spaces

$$0\to\mathrm{Ext}_L^1\big(\mathrm{Ind}_{L\cap P'^-}^L\,\sigma',\sigma\big)\to\mathrm{Ext}_G^1\big(\mathrm{Ind}_{P'^-}^G\,\sigma',\mathrm{Ind}_{P^-}^G\,\sigma\big)\to\bigoplus_{\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}}\mathrm{Hom}_L\big(\mathrm{Ind}_{L\cap P'^-}^L\,\sigma',\sigma^\alpha\otimes(\omega^{-1}\circ\alpha)\big).\tag{55}$$

If $\boldsymbol{P}' = \boldsymbol{P}$ and σ, σ' are irreducible, then $\sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ is also irreducible for all $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$, and thus the last term of (55) has dimension equal to $\operatorname{card}\{\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1} \mid \sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)\}$, hence (i). If $\boldsymbol{P}' \subsetneqq \boldsymbol{P}$ and σ is right cuspidal, then $\boldsymbol{L} \cap \boldsymbol{P}'$ is a proper parabolic subgroup of $\boldsymbol{L}$ and

$$\operatorname{Ord}_{L\cap P'}\big(\sigma^\alpha \otimes \big(\omega^{-1} \circ \alpha\big)\big) \cong (\operatorname{Ord}_{L\cap P'} \sigma)^\alpha \otimes \big(\omega^{-1} \circ \alpha\big) = 0$$

for all $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$; thus the last term of (55) is zero by [Emerton 2010a, Theorem 4.4.6], hence (ii).

Assume $\boldsymbol{P} \subseteq \boldsymbol{P}'$ and σ' satisfies the condition in Corollary 4.3.4(ii) for $\boldsymbol{P}'^- = \boldsymbol{L}'\boldsymbol{N}'^-$, e.g., σ' is left cuspidal. Using [Vignéras 2016, Theorem 5.3, 3] and Corollary 4.3.4(ii) for $\boldsymbol{P}'^- = \boldsymbol{L}'\boldsymbol{N}'^-$ (see Remark 4.3.5), (43) with $\pi = \operatorname{Ind}_{P'^-}^G \sigma'$ and $\boldsymbol{P}'^-, \boldsymbol{L}', \boldsymbol{N}'^-, \operatorname{Ind}_{L'\cap P^-}^{L'} \sigma$ instead of $\boldsymbol{P}, \boldsymbol{L}, \boldsymbol{N}, \sigma$ respectively yields an exact sequence of k-vector spaces

$$0 \to \operatorname{Ext}^1_{L'}\big(\sigma', \operatorname{Ind}_{L'\cap P^-}^{L'} \sigma\big) \to \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P'^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) \to \bigoplus_{\alpha \in \Delta_{L'}^{\perp,1}} \operatorname{Hom}_{L'}\big(\sigma'^\alpha \otimes (\omega^{-1} \circ \alpha), \operatorname{Ind}_{L'\cap P^-}^{L'} \sigma\big). \tag{56}$$

If $\boldsymbol{P}' = \boldsymbol{P}$ and σ, σ' are irreducible, then $\sigma'^\alpha \otimes (\omega^{-1} \circ \alpha)$ is also irreducible for all $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$, and thus the last term of (56) has dimension equal to $\operatorname{card}\{\alpha \in \Delta_{\boldsymbol{L}'}^{\perp,1} \mid \sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)\}$, hence (i). If $\boldsymbol{P} \subsetneqq \boldsymbol{P}'$ and σ' is left cuspidal, then $\boldsymbol{L}' \cap \boldsymbol{P}^-$ is a proper parabolic subgroup of $\boldsymbol{L}'$ and

$$(\sigma'^\alpha \otimes (\omega^{-1} \circ \alpha))_{L'\cap N^-} \cong (\sigma'_{L'\cap N^-})^\alpha \otimes (\omega^{-1} \circ \alpha) = 0$$

for all $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$; thus the last term of (56) is zero using (40) with $\pi = \sigma'^\alpha \otimes (\omega^{-1} \circ \alpha)$ and $\boldsymbol{L}', \boldsymbol{L}' \cap \boldsymbol{P}^-$, $\boldsymbol{L}' \cap \boldsymbol{L}$, $\boldsymbol{L}' \cap \boldsymbol{N}^-$ instead of $\boldsymbol{G}, \boldsymbol{P}, \boldsymbol{L}, \boldsymbol{N}$ respectively, hence (iii). □

Theorem 5.2.4. *Assume $F \neq \mathbb{Q}_p$. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup. The functor* $\operatorname{Ind}_{P^-}^G$ *induces an A-linear isomorphism*

$$\operatorname{Ext}^1_L(\sigma', \sigma) \xrightarrow{\sim} \operatorname{Ext}^1_G\left(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\right)$$

for all locally admissible smooth representations σ, σ' of L over A.

Proof. Let σ, σ' be locally admissible smooth representations of L over A. Using [Emerton 2010a, Proposition 4.3.4] and Corollary 3.3.8(i), (39) with $\pi = \operatorname{Ind}_{P^-}^G \sigma$ and σ' instead of σ yields the isomorphism in the statement. □

Corollary 5.2.5. *Assume $F \neq \mathbb{Q}_p$. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}' = \boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups and σ, σ' be admissible smooth representations of L, L' respectively over k.*

(i) *If $\boldsymbol{P}' \subseteq \boldsymbol{P}$, then the functor* $\operatorname{Ind}_{P^-}^G$ *induces a k-linear isomorphism*

$$\operatorname{Ext}^1_L\big(\operatorname{Ind}_{L\cap P'^-}^L \sigma', \sigma\big) \xrightarrow{\sim} \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P'^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big).$$

(ii) *If $\boldsymbol{P} \subseteq \boldsymbol{P}'$, then the functor* $\operatorname{Ind}_{P'^-}^G$ *induces a k-linear isomorphism*

$$\operatorname{Ext}^1_{L'}\big(\sigma', \operatorname{Ind}_{L'\cap P^-}^{L'} \sigma\big) \xrightarrow{\sim} \operatorname{Ext}^1_G\big(\operatorname{Ind}_{P'^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big).$$

Remark 5.2.6. Theorem 5.2.2(i) and Theorem 5.2.4 are encompassed in a more general (but conditional to a conjecture of Emerton) result. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup, σ, σ' be locally admissible smooth representations of L over A and $n \in \mathbb{N}$. The functor $\operatorname{Ind}_{P^-}^G$ induces an A-linear morphism

$$\operatorname{Ext}_L^n(\sigma', \sigma) \to \operatorname{Ext}_G^n\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) \tag{57}$$

and there is a spectral sequence of A-modules [Emerton 2010b, (3.7.4)]

$$\operatorname{Ext}_L^i\big(\sigma', \mathrm{R}^j\mathrm{Ord}_P\big(\operatorname{Ind}_{P^-}^G \sigma\big)\big) \Rightarrow \operatorname{Ext}_G^{i+j}\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) \tag{58}$$

where $\mathrm{R}^\bullet\mathrm{Ord}_P$ denotes the right derived functors of $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{l.adm}}(A) \to \mathrm{Mod}_L^{\mathrm{l.adm}}(A)$. Now assume that [op. cit., Conjecture 3.7.2] is true, i.e., $\mathrm{R}^\bullet\mathrm{Ord}_P \xrightarrow{\sim} \mathrm{H}^\bullet\mathrm{Ord}_P$. Using Corollary 3.3.8, one can deduce from (58) that:

- if $n < [F : \mathbb{Q}_p]$, then (57) is an isomorphism;
- if $n = [F : \mathbb{Q}_p]$, then (57) is injective and if furthermore σ, σ' are supercuspidal, then the dimension of its cokernel is at most $\operatorname{card}\big\{\alpha \in \Delta_L^{\perp,1} \mid \sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)\big\}$.

One can also generalise Proposition 5.2.1 and Theorem 5.2.2 (ii) and (iii) in all degrees $n \leq [F : \mathbb{Q}_p]$.

Finally, we complete Theorem 5.2.2(i) when $\boldsymbol{G}$ is split and $\boldsymbol{Z}$ is connected.

Theorem 5.2.7. *Assume $F = \mathbb{Q}_p$, $\boldsymbol{G}$ split and $\boldsymbol{Z}$ connected. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup and σ, σ' be supersingular representations of L over k.*

(i) *If $\sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha) \not\cong \sigma$ for some $\alpha \in \Delta_L^\perp$, then $\operatorname{Ext}_L^1(\sigma', \sigma) = 0$ and*

$$\dim_k \operatorname{Ext}_G^1\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) = 1.$$

(ii) *If either $\sigma' \cong \sigma$ and $p \neq 2$, or $\sigma' \not\cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ for any $\alpha \in \Delta_L^\perp$, then the functor $\operatorname{Ind}_{P^-}^G$ induces a k-linear isomorphism*

$$\operatorname{Ext}_L^1(\sigma', \sigma) \xrightarrow{\sim} \operatorname{Ext}_G^1\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big).$$

(iii) *If $p = 2$, then the functor $\operatorname{Ind}_{P^-}^G$ induces a k-linear injection*

$$\operatorname{Ext}_L^1(\sigma', \sigma) \hookrightarrow \operatorname{Ext}_G^1\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big)$$

whose cokernel is of dimension $\operatorname{card}\big\{\alpha \in \Delta_L^\perp \mid \sigma' \cong \sigma^\alpha\big\}$.

Proof. Since σ is absolutely irreducible, it has a central character $\zeta : Z_L \to k^\times$ [Emerton 2010b, Lemma 4.1.7].

We first assume that $\sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha) \not\cong \sigma$ for some $\alpha \in \Delta_L^\perp$. We have $\zeta \circ \alpha^\vee \neq \omega^{-1}$ by Lemma 5.1.4, so that σ and σ' have distinct central characters by Lemma 5.1.5; thus $\operatorname{Ext}_L^1(\sigma', \sigma) = 0$ [Paškūnas 2010, Proposition 8.1]. Furthermore, $\sigma' \not\cong \sigma^\beta \otimes (\omega^{-1} \circ \beta)$ for any $\beta \in \Delta_L^\perp \setminus \{\alpha\}$ (since the central characters are distinct by Lemma 5.1.5). Using (55) with $\boldsymbol{P}' = \boldsymbol{P}$ and $\boldsymbol{L}' = \boldsymbol{L}$, we deduce that

$$\dim_k \operatorname{Ext}_G^1\big(\operatorname{Ind}_{P^-}^G \sigma', \operatorname{Ind}_{P^-}^G \sigma\big) \leq 1.$$

Since the left-hand side is nonzero by Proposition 5.1.7(i), this proves (i).

We now prove (ii). If $\sigma' \ncong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha)$ for any $\alpha \in \Delta_L^\perp$, then the result follows from Theorem 5.2.2(i). Assume that $\sigma' \cong \sigma$ and $p \neq 2$. The terms of low degree of (58) form an exact sequence of k-vector spaces

$$0 \to \mathrm{Ext}^1_L(\sigma, \sigma) \to \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P^-}^G \sigma, \mathrm{Ind}_{P^-}^G \sigma\big) \to \mathrm{Hom}_L\big(\sigma, \mathrm{R}^1\mathrm{Ord}_P\big(\mathrm{Ind}_{P^-}^G \sigma\big)\big) \\ \to \mathrm{Ext}^2_L\big(\sigma, \sigma\big) \to \mathrm{Ext}^2_G\big(\mathrm{Ind}_{P^-}^G \sigma, \mathrm{Ind}_{P^-}^G \sigma\big). \quad (59)$$

Since there is an injection of functors $\mathrm{R}^1\mathrm{Ord}_P \hookrightarrow \mathrm{H}^1\mathrm{Ord}_P$ [Emerton 2010b, Remark 3.7.3], we deduce from Corollary 3.3.8(ii) and Lemma 5.1.5 that

$$\dim_k \mathrm{Hom}_L\big(\sigma, \mathrm{R}^1\mathrm{Ord}_P\big(\mathrm{Ind}_{P^-}^G \sigma\big)\big) \leq \mathrm{card}\big\{\alpha \in \Delta_L^\perp \mid \zeta \circ \alpha^\vee = \omega^{-1}\big\}.$$

Thus, we deduce from Proposition 5.1.9 that the third arrow of (59) is zero, hence the result.

Finally, we assume $p = 2$ and we prove (iii). By Proposition 5.1.7(ii) we have a lower bound and using (55) with $\boldsymbol{P}' = \boldsymbol{P}$ and $\boldsymbol{L}' = \boldsymbol{L}$ we obtain an upper bound. Using Lemmas 5.1.4 and 5.1.5 together with the fact that $\omega = 1$, we see that both are equal to $\mathrm{card}\{\alpha \in \Delta_L^\perp \mid \sigma' \cong \sigma^\alpha\}$. □

Remark 5.2.8. Theorem 5.2.7(i) can also be generalised in the context of Remark 5.2.6. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup and σ, σ' be supersingular representations of L over k such that $\sigma' \cong \sigma^\alpha \otimes (\omega^{-1} \circ \alpha) \ncong \sigma$ for some $\alpha \in \Delta_L^\perp$. Assume $\boldsymbol{G}$ split, $\boldsymbol{Z}$ connected and [Emerton 2010b, Conjecture 3.7.2] is true. Using Corollary 3.3.8 and Lemma 5.1.5, one can deduce from (58) that $\mathrm{Ext}_L^{[F:\mathbb{Q}_p]}(\sigma', \sigma) = 0$ and

$$\dim_k \mathrm{Ext}_G^{[F:\mathbb{Q}_p]}\big(\mathrm{Ind}_{P^-}^G \sigma', \mathrm{Ind}_{P^-}^G \sigma\big) = 1.$$

Corollary 5.2.9. *Assume $\boldsymbol{G}$ split and $\boldsymbol{Z}$ connected. If* Conjecture 3.3.4 *is true for $A = k$, $n = 1$ and ${}^Iw^J = 1$, then* [Hauseux 2016b, Conjecture 3.17] *is true.*

Proof. Even though [Hauseux 2016b, Conjecture 3.17] is formulated under the hypotheses $\boldsymbol{G}$ split, $\boldsymbol{Z}$ connected and $\boldsymbol{G}^{\mathrm{der}}$ simply connected, we do not need the last one to prove it: (i) is Proposition 5.2.1, which is conditional to Conjecture 3.3.4 for $A = k$, $n = 1$ and ${}^Iw^J = 1$; (ii) is Theorem 5.2.7(i); (iii) and (iv) are Corollary 5.2.5(i) and (ii) respectively when $F \neq \mathbb{Q}_p$, Theorem 5.2.2(ii) and (iii) respectively when $F = \mathbb{Q}_p$ and $\boldsymbol{P}' \neq \boldsymbol{P}$, Theorem 5.2.7(ii) when $F = \mathbb{Q}_p$, $\boldsymbol{P}' = \boldsymbol{P}$ and $p \neq 2$, and Theorem 5.2.7(iii) when $F = \mathbb{Q}_p$, $\boldsymbol{P}' = \boldsymbol{P}$ and $p = 2$ (noting that if $p = 2$, then $\omega = 1$ and $\mathrm{Ind}_{P^-}^G \sigma$ is irreducible if and only if $\sigma^\alpha \ncong \sigma$ for all $\alpha \in \Delta_L^\perp$; see [Abe 2013, Lemma 5.8] and Lemma 5.1.5). □

5.3. ***Results for unitary continuous p-adic representations.*** Let H be a p-adic Lie group. A *continuous representation* of H over E is an E-Banach space Π endowed with an E-linear action of H such that the map $H \times \Pi \to \Pi$ is continuous. It is *admissible* if the continuous dual $\Pi^* := \mathrm{Hom}_H^{\mathrm{cont}}(\Pi, E)$ is of finite type over the Iwasawa algebra $E \otimes_{\mathcal{O}} \mathcal{O}[\![H_0]\!]$ for some (equivalently any) compact open subgroup $H_0 \subseteq H$ [Schneider and Teitelbaum 2002]. It is *unitary* if there exists an H-stable bounded open $\mathcal{O}$-lattice $\Pi^0 \subseteq \Pi$. We write $\mathrm{Ban}_H^{\mathrm{adm,u}}(E)$ for the category of admissible unitary continuous representations of H over E and H-equivariant E-linear continuous morphisms. It is an E-abelian category.

We fix a uniformiser ϖ of $\mathcal{O}$. Following [Emerton 2010a, §2.4], we let $\mathrm{Mod}_H^{\varpi-\mathrm{adm}}(\mathcal{O})^{\mathrm{fl}}$ be the category of ϖ-torsion-free ϖ-adically complete and separated $\mathcal{O}$-modules Π^0 such that $\Pi^0/\varpi\Pi^0$ is admissible as a smooth representation of H over k and H-equivariant $\mathcal{O}$-linear morphisms. It is an $\mathcal{O}$-abelian category and the localised category $E \otimes_{\mathcal{O}} \mathrm{Mod}_H^{\varpi-\mathrm{adm}}(\mathcal{O})^{\mathrm{fl}}$ is equivalent to $\mathrm{Ban}_H^{\mathrm{adm,u}}(E)$.

The E-vector spaces $\mathrm{Ext}^1_H(\Pi', \Pi)$ of Yoneda extensions between admissible unitary continuous representations Π, Π' of H over E are computed in $\mathrm{Ban}_H^{\mathrm{adm,u}}(E)$. For all $n \geq 1$, the $\mathcal{O}/\varpi^n\mathcal{O}$-modules $\mathrm{Ext}^1_H(\pi', \pi)$ of Yoneda extensions between admissible smooth representations π, π' of H over $\mathcal{O}/\varpi^n\mathcal{O}$ are computed in $\mathrm{Mod}_H^{\mathrm{adm}}(\mathcal{O}/\varpi^n\mathcal{O})$. The following result is a slight generalisation of [Hauseux 2016a, Proposition B.2].

Proposition 5.3.1. *Let H be a p-adic Lie group*, Π, Π' *be admissible unitary continuous representations of H and π, π' be the reductions mod ϖ of H-stable bounded open $\mathcal{O}$-lattices* Π^0, Π'^0 *of* Π, Π' *respectively. Assume that* $\dim_k \mathrm{Hom}_H(\pi', \pi) < \infty$. *There is an E-linear isomorphism*

$$\mathrm{Ext}^1_H(\Pi', \Pi) \cong E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1} \big(\mathrm{Ext}^1_H(\Pi'^0/\varpi^n\Pi'^0, \Pi^0/\varpi^n\Pi^0)\big).$$

Furthermore, $\dim_E \mathrm{Ext}^1_H(\Pi', \Pi) \leq \dim_k \mathrm{Ext}^1_H(\pi', \pi)$.

Proof. In the proof of [Hauseux 2016a, Proposition B.2], the hypothesis that π' is of finite length is only used to prove that $\mathrm{Hom}_H(\Pi'^0/\varpi^n\Pi'^0, \Pi^0/\varpi^n\Pi^0)$ is of finite type over $\mathcal{O}/\varpi^n\mathcal{O}$ for all $n \geq 1$. But this can be proved by induction using that $\dim_k \mathrm{Hom}_H(\pi', \pi) < \infty$. □

Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a parabolic subgroup. We recall that the continuous parabolic induction functor is defined for any continuous representation Σ of L over E by

$$\mathrm{Ind}_{P^-}^G \Sigma := \left\{ f : G \to \Sigma \text{ continuous} \;\middle|\; f(pg) = p \cdot f(g)\ \forall p \in P^-, \forall g \in G \right\}.$$

We obtain an E-linear exact functor $\mathrm{Ind}_{P^-}^G : \mathrm{Ban}_L^{\mathrm{adm,u}}(E) \to \mathrm{Ban}_G^{\mathrm{adm,u}}(E)$ [Emerton 2010a, §4.1]. Furthermore, there is a natural G-equivariant E-linear continuous isomorphism [op. cit., Lemma 4.1.3]

$$\mathrm{Ind}_{P^-}^G \Sigma \cong E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1} \big(\mathrm{Ind}_{P^-}^G(\Sigma^0/\varpi^n\Sigma^0)\big).$$

We extend the definition of the ordinary parts functor to any admissible unitary continuous representation Π of G over E by setting

$$\mathrm{Ord}_P \Pi := E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1} \big(\mathrm{Ord}_P(\Pi^0/\varpi^n\Pi^0)\big)$$

for some (equivalently any) G-stable bounded open $\mathcal{O}$-lattice $\Pi^0 \subseteq \Pi$. We obtain an E-linear left-exact functor $\mathrm{Ord}_P : \mathrm{Ban}_G^{\mathrm{adm,u}}(E) \to \mathrm{Ban}_L^{\mathrm{adm,u}}(E)$ which is a left quasi-inverse and the right adjoint of $\mathrm{Ind}_{P^-}^G$ [Emerton 2010a, Theorem 3.4.8, Corollary 4.3.5 and Theorem 4.4.6].

Definition 5.3.2. We say that an admissible unitary continuous representation Σ of L over E is *right cuspidal* if $\mathrm{Ord}_Q \Sigma = 0$ for any proper parabolic subgroup $\boldsymbol{Q} \subset \boldsymbol{L}$.

Remark 5.3.3. We also extend the Jacquet functor to continuous representations of G over E by taking the Hausdorff completion of the N-coinvariants. We obtain the left adjoint of Ind_P^G by Frobenius reciprocity and the universal property of coinvariants. However, we do not know whether it preserves admissibility. For unitary representations, it does not behave well with respect to reduction mod ϖ^n $(n \geq 1)$. Nevertheless, we say that an admissible unitary continuous representation Σ of L over E is *left cuspidal* if $\Sigma_{N_Q} = 0$ for any proper parabolic subgroup $\boldsymbol{Q} \subset \boldsymbol{L}$ with unipotent radical $\boldsymbol{N_Q}$.

We now turn to extensions computations. Our main tool is the following result, which gives a weak p-adic analogue of the exact sequence (39).

Proposition 5.3.4. *Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup, Σ, Σ' be admissible unitary continuous representations of L respectively over E and σ, σ' be the reductions mod ϖ of L-stable bounded open $\mathcal{O}$-lattices Σ^0, Σ'^0 of Σ, Σ' respectively. Assume that* $\dim_k \operatorname{Hom}_L(\sigma', \sigma) < \infty$. *There is a natural exact sequence of E-vector spaces*

$$0 \to \operatorname{Ext}^1_L(\Sigma', \Sigma) \to \operatorname{Ext}^1_G\big(\operatorname{Ind}^G_{P^-} \Sigma', \operatorname{Ind}^G_{P^-} \Sigma\big) \to \operatorname{Hom}_L\Big(\Sigma', E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1}\big(\mathrm{H}^1\mathrm{Ord}_P\big(\operatorname{Ind}^G_{P^-}(\Sigma^0/\varpi^n\Sigma^0)\big)\big)\Big).$$

Proof. For all $n \geq 1$, (39) with $A = \mathcal{O}/\varpi^n\mathcal{O}$, $\pi = \operatorname{Ind}^G_{P^-}(\Sigma^0/\varpi^n\Sigma^0)$ and using [Emerton 2010a, Proposition 4.3.4] $\sigma = \Sigma'^0/\varpi^n\Sigma'^0$ yields an exact sequence of $\mathcal{O}/\varpi^n\mathcal{O}$-modules

$$0 \to \operatorname{Ext}^1_L(\Sigma'^0/\varpi^n\Sigma'^0, \Sigma^0/\varpi^n\Sigma^0) \to \operatorname{Ext}^1_G\left(\operatorname{Ind}^G_{P^-}(\Sigma'^0/\varpi^n\Sigma'^0), \operatorname{Ind}^G_{P^-}(\Sigma^0/\varpi^n\Sigma^0)\right) \to \operatorname{Hom}_L\left(\Sigma'^0/\varpi^n\Sigma'^0, \mathrm{H}^1\mathrm{Ord}_P\left(\operatorname{Ind}^G_{P^-}(\Sigma^0/\varpi^n\Sigma^0)\right)\right). \quad (60)$$

The composite $\mathrm{H}^1\mathrm{Ord}_P \circ \operatorname{Ind}^G_{P^-} : \operatorname{Mod}^{\mathrm{adm}}_L(\mathcal{O}/\varpi^n\mathcal{O}) \to \operatorname{Mod}^{\mathrm{adm}}_L(\mathcal{O}/\varpi^n\mathcal{O})$ is left-exact for all $n \geq 1$ by [Emerton 2010a, Proposition 4.3.4; Emerton 2010b, Corollary 3.4.8]. Thus

$$\varprojlim_{n\geq 1}\big(\mathrm{H}^1\mathrm{Ord}_P(\operatorname{Ind}^G_{P^-}(\Sigma^0/\varpi^n\Sigma^0))\big)$$

is a ϖ-adically admissible representation of L over $\mathcal{O}$ by [Emerton 2010a, Corollary 3.4.5]. Furthermore, it is ϖ-torsion-free and the projective limit topology coincide with the ϖ-adic topology [op. cit., Proposition 3.4.3(1) and (3)]. Thus

$$E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1}\big(\mathrm{H}^1\mathrm{Ord}_P\big(\operatorname{Ind}^G_{P^-}(\Sigma^0/\varpi^n\Sigma^0)\big)\big)$$

is an admissible unitary continuous representation of L over E. Taking the projective limit over $n \geq 1$ of (60) and inverting ϖ and using Proposition 5.3.1 yields the desired exact sequence. $\square$

Remark 5.3.5. In order to obtain an analogue of (39) for any admissible unitary continuous representations Σ, Π of L, G respectively over E, one has to prove that the ϖ-torsion of $\varprojlim_{n\geq 1}(\mathrm{H}^1\mathrm{Ord}_P(\Pi^0/\varpi^n\Pi^0))$ is of bounded exponent (i.e., annihilated by a power of ϖ) for some (equivalently any) G-stable bounded open $\mathcal{O}$-lattice $\Pi^0 \subseteq \Pi$.

We now use Proposition 5.3.4 to compute extensions between parabolically induced representations.

Theorem 5.3.6. *Assume $F = \mathbb{Q}_p$. Let $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}' = \boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups, Σ, Σ' be admissible unitary continuous representations of L, L' over E and σ, σ' be the reductions mod ϖ of L, L'-stable bounded open $\mathcal{O}$-lattices of Σ, Σ'. Assume that* $\dim_k \operatorname{Hom}_G(\operatorname{Ind}^G_{P'^-} \sigma', \operatorname{Ind}^G_{P^-} \sigma) < \infty$ *and Σ is right cuspidal.*

(i) *If $\boldsymbol{P}' = \boldsymbol{P}$, Σ, Σ' are topologically irreducible and $\Sigma' \ncong \Sigma^\alpha \otimes (\varepsilon^{-1} \circ \alpha)$ for all $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$, then the functor $\mathrm{Ind}_{P^-}^G$ induces an E-linear isomorphism*

$$\mathrm{Ext}_L^1(\Sigma', \Sigma) \xrightarrow{\sim} \mathrm{Ext}_G^1\big(\mathrm{Ind}_{P^-}^G \Sigma', \mathrm{Ind}_{P^-}^G \Sigma\big).$$

(ii) *If $\boldsymbol{P}' \subsetneq \boldsymbol{P}$, then the functor $\mathrm{Ind}_{P^-}^G$ induces an E-linear isomorphism*

$$\mathrm{Ext}_L^1\big(\mathrm{Ind}_{L\cap P'^-}^L \Sigma', \Sigma\big) \xrightarrow{\sim} \mathrm{Ext}_G^1\big(\mathrm{Ind}_{P'^-}^G \Sigma', \mathrm{Ind}_{P^-}^G \Sigma\big).$$

Remark 5.3.7. Assume $\boldsymbol{P}' = \boldsymbol{P}$ and Σ, Σ' topologically irreducible. We do not know the dimension of the cokernel of the E-linear injection $\mathrm{Ext}_L^1(\Sigma', \Sigma) \hookrightarrow \mathrm{Ext}_G^1(\mathrm{Ind}_{P^-}^G \Sigma', \mathrm{Ind}_{P^-}^G \Sigma)$ induced by $\mathrm{Ind}_{P^-}^G$, but we prove that it is at most $\mathrm{card}\{\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1} \mid \Sigma' \cong \Sigma^\alpha \otimes (\varepsilon^{-1} \circ \alpha)\}$ (see the proof). If Σ, Σ' are absolutely topologically irreducible and supercuspidal, then letting $\zeta : Z_L \to \mathcal{O}^\times \subset E^\times$ be the central character of Σ [Dospinescu and Schraen 2013, Theorem 1.1(2)], we expect this dimension to be equal to

$$\mathrm{card}\left\{\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1} \;\middle|\; \Sigma' \cong \Sigma^\alpha \otimes \left(\varepsilon^{-1} \circ \alpha\right) \text{ and } \zeta \circ \alpha^\vee \neq \varepsilon^{-1}\right\}.$$

Proof. Let $\Sigma^0 \subseteq \Sigma$ be an L-stable bounded open $\mathcal{O}$-lattice. For all $n \geq 1$, we deduce from Propositions 3.3.1 and 3.3.5(i) that there is a natural L-equivariant $\mathcal{O}/\varpi^n\mathcal{O}$-linear isomorphism

$$\mathrm{H}^1\mathrm{Ord}_P\big(\mathrm{Ind}_{P^-}^G(\Sigma^0/\varpi^n\Sigma^0)\big) \cong \bigoplus_{\alpha\in\Delta^1\setminus\Delta_L} \mathrm{H}^1\mathrm{Ord}_P\big(\text{c-ind}_{P^-}^{P^-s_\alpha P}(\Sigma^0/\varpi^n\Sigma^0)\big). \tag{61}$$

Furthermore, if $\alpha \in \Delta_{\boldsymbol{L}}^{\perp,1}$, then there is a natural L-equivariant $\mathcal{O}/\varpi^n\mathcal{O}$-linear isomorphism

$$\mathrm{H}^1\mathrm{Ord}_P\big(\text{c-ind}_{P^-}^{P^-s_\alpha P}(\Sigma^0/\varpi^n\Sigma^0)\big) \cong (\Sigma^0/\varpi^n\Sigma^0)^\alpha \otimes (\omega^{-1} \circ \alpha)$$

hence a natural L-equivariant E-linear continuous isomorphism

$$E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1}\big(\mathrm{H}^1\mathrm{Ord}_P\big(\text{c-ind}_{P^-}^{P^-s_\alpha P}(\Sigma^0/\varpi^n\Sigma^0)\big)\big) \cong \Sigma^\alpha \otimes (\varepsilon^{-1} \circ \alpha),$$

whereas if $\alpha \notin \Delta_{\boldsymbol{L}}^{\perp}$, there is natural filtration of $\mathrm{H}^1\mathrm{Ord}_P(\text{c-ind}_{P^-}^{P^-s_\alpha P}(\Sigma^0/\varpi^n\Sigma^0))$ by B_L-subrepresentations such that each term of the associated graded representation is isomorphic as an $\mathcal{O}/\varpi^n\mathcal{O}$-modules to

$$\mathcal{C}_{\mathrm{c}}^{\mathrm{sm}}\big(U'_L, \mathrm{Ord}_{L\cap s_\alpha P s_\alpha^{-1}}(\Sigma^0/\varpi^n\Sigma^0)\big)$$

for some closed subgroup $\boldsymbol{U}'_{\boldsymbol{L}} \subseteq \boldsymbol{U}_{\boldsymbol{L}}$, and since $\mathrm{Ord}_{L\cap s_\alpha P s_\alpha^{-1}} \Sigma = 0$ we deduce that

$$E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1}\big(\mathrm{H}^1\mathrm{Ord}_P\big(\text{c-ind}_{P^-}^{P^-s_\alpha P}(\Sigma^0/\varpi^n\Sigma^0)\big)\big) = 0.$$

Thus, taking the projective limit of (61) over $n \geq 1$ and inverting ϖ yields a natural L-equivariant E-linear continuous isomorphism

$$E \otimes_{\mathcal{O}} \varprojlim_{n\geq 1}\left(\mathrm{H}^1\mathrm{Ord}_P\left(\mathrm{Ind}_{P^-}^G(\Sigma^0/\varpi^n\Sigma^0)\right)\right) \cong \bigoplus_{\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}} \Sigma^\alpha \otimes (\varepsilon^{-1} \circ \alpha). \tag{62}$$

Now Proposition 5.3.4 with $\mathrm{Ind}_{L\cap P'^-}^{L}\,\Sigma'$ instead of Σ' yields, using (62), an exact sequence of E-vector spaces

$$0 \to \mathrm{Ext}^1_L\big(\mathrm{Ind}_{L\cap P'^-}^{L}\,\Sigma', \Sigma\big) \to \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^{G}\,\Sigma', \mathrm{Ind}_{P^-}^{G}\,\Sigma\big) \to \bigoplus_{\alpha\in\Delta_L^{\perp,1}} \mathrm{Hom}_L\big(\mathrm{Ind}_{L\cap P'^-}^{L}\,\Sigma', \Sigma^\alpha\otimes(\varepsilon^{-1}\circ\alpha)\big). \quad (63)$$

If $\boldsymbol{P}'=\boldsymbol{P}$ and Σ, Σ' are topologically irreducible, then $\Sigma^\alpha\otimes(\varepsilon^{-1}\circ\alpha)$ is also topologically irreducible for all $\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}$, and thus the last term of (63) has dimension equal to $\mathrm{card}\{\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1} \mid \Sigma'\cong\Sigma^\alpha\otimes(\varepsilon^{-1}\circ\alpha)\}$, hence (i). If $\boldsymbol{P}'\subsetneq\boldsymbol{P}$, then $\boldsymbol{L}\cap\boldsymbol{P}'$ is a proper parabolic subgroup of $\boldsymbol{L}$ so that

$$\mathrm{Ord}_{L\cap P'}(\Sigma^\alpha\otimes(\varepsilon^{-1}\circ\alpha)) = (\mathrm{Ord}_{L\cap P'}\,\Sigma)^\alpha\otimes(\varepsilon^{-1}\circ\alpha)=0$$

for all $\alpha\in\Delta_{\boldsymbol{L}}^{\perp,1}$; thus the last term of (63) is zero, hence (ii). ☐

Remark 5.3.8. Theorem 5.2.2(iii) cannot be directly lifted to characteristic 0 because we do not have a weak p-adic analogue of the exact sequence (43) (since it uses the Jacquet functor, see Remark 5.3.3). However, assuming Conjecture 3.3.4 true for $A=\mathcal{O}/\varpi^r\mathcal{O}$ $(r\geq1)$, $n=1$, $I\subsetneq J$ and ${}^I w^J=s_\alpha$ $(\alpha\in\Delta^1\setminus J)$, one can recover this case: with notation and assumptions as in Theorem 5.3.6, if $\boldsymbol{P}\subsetneq\boldsymbol{P}'$ and Σ' is left cuspidal, then the functor $\mathrm{Ind}_{P'^-}^{G}$ induces an E-linear isomorphism

$$\mathrm{Ext}^1_{L'}\big(\Sigma', \mathrm{Ind}_{L'\cap P^-}^{L}\,\Sigma\big) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^{G}\,\Sigma', \mathrm{Ind}_{P^-}^{G}\,\Sigma\big).$$

Theorem 5.3.9. *Assume $F\neq\mathbb{Q}_p$. Let $\boldsymbol{P}=\boldsymbol{L}\boldsymbol{N}$ be a standard parabolic subgroup, Σ, Σ' be admissible unitary continuous representations of L over E and σ,σ' be the reductions mod ϖ of L-stable bounded open $\mathcal{O}$-lattices of Σ, Σ' respectively. Assume that $\dim_k\mathrm{Hom}_L(\sigma',\sigma)<\infty$. Then, the functor $\mathrm{Ind}_{P^-}^{G}$ induces an E-linear isomorphism*

$$\mathrm{Ext}^1_L(\Sigma',\Sigma) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{P^-}^{G}\,\Sigma', \mathrm{Ind}_{P^-}^{G}\,\Sigma\big).$$

Proof. Let $\Sigma^0\subseteq\Sigma$ be an L-stable bounded open $\mathcal{O}$-lattice. Then $\mathrm{H}^1\mathrm{Ord}_P\big(\mathrm{Ind}_{P^-}^{G}(\Sigma^0/\varpi^n\Sigma^0)\big)=0$ for all $n\geq1$, by Corollary 3.3.8(i). Thus, the result follows from Proposition 5.3.4. ☐

We end with a remark on the case where there is no inclusion between the two parabolic subgroups.

Remark 5.3.10. Let $\boldsymbol{P}=\boldsymbol{L}\boldsymbol{N}$, $\boldsymbol{P}'=\boldsymbol{L}'\boldsymbol{N}'$ be standard parabolic subgroups, Σ, Σ' be admissible unitary continuous representations of L, L' respectively over E and σ,σ' be the reductions mod ϖ of L, L'-stable bounded open $\mathcal{O}$-lattices of Σ, Σ' respectively. Assume Conjecture 3.3.4 is true for $A=\mathcal{O}/\varpi^r\mathcal{O}$ $(r\geq1)$, $n=1$ and ${}^I w^J=1$. Assume further $\dim_k\mathrm{Hom}_G(\mathrm{Ind}_{P'^-}^{G}\,\sigma', \mathrm{Ind}_{P^-}^{G}\,\sigma)<\infty$ and the ϖ-torsion of $\varprojlim_{n\geq1}(\mathrm{H}^1\mathrm{Ord}_{L\cap P}(\Sigma^0/\varpi^n\Sigma^0))$ is of bounded exponent (see Remark 5.3.5). Then, one can prove the following p-adic analogue of Proposition 5.2.1: if $\boldsymbol{P}'\not\subseteq\boldsymbol{P}$, $\boldsymbol{P}\not\subseteq\boldsymbol{P}'$, and Σ, Σ' are right,left cuspidal respectively, then

$$\mathrm{Ext}^1_G\big(\mathrm{Ind}_{P'^-}^{G}\,\Sigma', \mathrm{Ind}_{P^-}^{G}\,\Sigma\big)=0.$$

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Akizuki–Witt maps and Kaletha's global rigid inner forms

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We give an explicit construction of global Galois gerbes constructed more abstractly by Kaletha to define global rigid inner forms. This notion is crucial to formulate Arthur's multiplicity formula for inner forms of quasisplit reductive groups. As a corollary, we show that any global rigid inner form is almost everywhere unramified, and we give an algorithm to compute the resulting local rigid inner forms at all places in a given finite set. This makes global rigid inner forms as explicit as global pure inner forms, up to computations in local and global class field theory.

1. Introduction

Let F be a number field, and G a connected reductive group over F. Following seminal contributions in [Labesse and Langlands 1979; Langlands 1983; Langlands and Shelstad 1987], Kottwitz [1984] and Arthur [1989] conjectured a multiplicity formula for discrete automorphic representations for G, in terms of Arthur–Langlands parameters $\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$. The formulation of this conjecture on automorphic multiplicities requires a precise version of the local Arthur–Langlands correspondence for $G_{F_v} := G \times_F F_v$ at all places v of F, describing individual elements of local packets using the theory of endoscopy. For this it is necessary to endow each G_{F_v} with a *rigidifying datum*. For places v such that G_{F_v} is quasisplit, that is for all but finitely many places of F, this can take the form of a Whittaker datum $\mathfrak{w}_v$. If G is quasisplit, then one can choose a global Whittaker datum $\mathfrak{w}$, and it is expected that taking localizations $\mathfrak{w}_v$ of $\mathfrak{w}$ yields a coherent family of precise versions of the local Arthur–Langlands correspondence. This coherence is crucial for the automorphic multiplicity formula to hold. For example

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this is the setting used in [Arthur 2013] and [Mok 2015]. Note that even though a choice of global Whittaker datum is necessary to express the formula for automorphic multiplicities, these multiplicities are canonical, as one can easily deduce from [Kaletha 2013, Theorem 4.3].

In general the connected reductive group G might not be quasisplit, and G is only an inner form of a unique quasisplit group. Recall (see [Borel 1979]) that two connected reductive groups have isomorphic Langlands dual groups if and only if they are inner forms of each other. Vogan [1993] and Kottwitz conjectured a formulation of the local Langlands correspondence in the case where G_{F_v} is a *pure* inner form of a quasisplit group. In this case a rigidifying datum is a quadruple $(G_v^*, \Xi_v, z_v, \mathfrak{w}_v)$ where G_v^* is a connected reductive quasisplit group over F_v, $\Xi_v : (G_v^*)_{\overline{F_v}} \to G_{\overline{F_v}}$ is an isomorphism, and $z_v \in Z^1(F_v, G_v^*)$ is such that for any $\sigma \in \mathrm{Gal}(\overline{F_v}/F_v)$ we have $\Xi_v^{-1}\sigma(\Xi_v) = \mathrm{Ad}(z_v(\sigma))$. If globally G is a pure inner form of a quasisplit group, one can choose a similar global quadruple $(G^*, \Xi, z, \mathfrak{w})$, and localizing at all places of F seems to yield a coherent family of rigidifying data. Away from a finite set S of places of F, the restriction z_v of z to a decomposition group $\mathrm{Gal}(\overline{F_v}/F_v)$ is cohomologically trivial, and writing it as a coboundary yields an isomorphism $\Xi'_v : G^*_{F_v} \simeq G_{F_v}$ well defined up to conjugation by $G(F_v)$, which endows G_{F_v} with a Whittaker datum $(\Xi'_v)_*(\mathfrak{w}_v)$ in a canonical way. Furthermore, up to enlarging S this can be done integrally, that is over a finite étale extension of $\mathcal{O}(F_v)$, so that Ξ'_v is an isomorphism between the canonical models of G^* and G over $\mathcal{O}(F_v)$.

Unfortunately not all connected reductive groups can be realized as pure inner forms of quasisplit groups, due to the fact that $H^1(F, G^*) \to H^1(F, G^*_{\mathrm{ad}})$ can fail to be surjective. The simplest example is certainly the group of elements having reduced norm equal to 1 in a nonsplit quaternion algebra, an inner form of SL_2, considered in [Labesse and Langlands 1979]. To circumvent this problem, Kaletha defined larger Galois cohomology groups in [Kaletha 2016] for the local case and in [Kaletha 2018] for the global case. More precisely, he constructed central extensions (Galois gerbes bound by commutative groups in the terminology of [Langlands and Rapoport 1987])

$$1 \to P_v \to \mathcal{E}_v \to \mathrm{Gal}(\overline{F_v}/F_v) \to 1$$

in the local case, v any place of F, and

$$1 \to P \to \mathcal{E} \to \mathrm{Gal}(\bar{F}/F) \to 1$$

in the global case. Here P_v and P are inverse limits of finite commutative algebraic groups defined over F_v or F, and we have denoted by $P_v \to \mathcal{E}_v$ the extension denoted by $u \to W$ in [Kaletha 2016], to emphasize the analogy between the local and global cases. The central extensions are obtained from certain classes $\xi_v \in H^2(F_v, P_v)$, $\xi \in H^2(F, P)$. Using these central extensions Kaletha defined, for Z a finite central algebraic subgroup of G^*, certain sets of 1-cocycles

$$Z^1(P_v \to \mathcal{E}_v, Z(\overline{F_v}) \to G^*(\overline{F_v})) \supset Z^1(F_v, G^*_{F_v}), \quad \text{resp.} \quad Z^1(P \to \mathcal{E}, Z(\bar{F}) \to G^*(\bar{F})) \supset Z^1(F, G^*),$$

which naturally map to $Z^1(F_v, G^*_{\mathrm{ad}, F_v})$ (resp. $Z^1(F, G^*_{\mathrm{ad}})$), so that such cocycles give rise to inner forms of G^*. Kaletha also proposed precise formulations of the local Langlands conjecture and Arthur multiplicity formula, using rigidifying data $(G_v^*, \Xi_v, z_v, \mathfrak{w}_v)$ (resp. $(G^*, \Xi, z, \mathfrak{w})$) where now z_v (resp. z) belongs to this larger group of 1-cocycles. For Z large enough, for example if Z contains the center of the derived subgroup of G^*, the map between the resulting cohomology sets

$$H^1(P \to \mathcal{E}, Z(\bar{F}) \to G^*(\bar{F})) \to H^1(F, G^*_{\mathrm{ad}})$$

is surjective, and so any G can be endowed with such a rigidifying datum $(G^*, \Xi, z, \mathfrak{w})$. From such a global rigidifying datum, one obtains local rigidifying data by localization. Each localization $z_v = \mathrm{loc}_v(z)$ of z is defined by pulling back via a morphism of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_v & \longrightarrow & \mathcal{E}_v & \longrightarrow & \mathrm{Gal}(\overline{F_v}/F_v) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P & \longrightarrow & \mathcal{E} & \longrightarrow & \mathrm{Gal}(\bar{F}/F) & \longrightarrow & 1 \end{array} \tag{1.0.1}$$

and extending coefficients from $G^*(\bar{F})$ to $G^*(\overline{F_v})$.

In this paper we give an explicit, bottom-up realization of the central extension

$$1 \to P \to \mathcal{E} \to \mathrm{Gal}(\bar{F}/F) \to 1$$

constructed in [Kaletha 2018]. Here "bottom-up" means that our construction is naturally an inverse limit over $k \geq 0$ of central extensions

$$1 \to P_k \to \mathcal{E}_k \to \mathrm{Gal}(E'_k/F) \to 1,$$

where E'_k/F is finite Galois extension, P_k is a finite commutative algebraic group over F such that $P_k(E'_k) = P_k(\bar{F})$, and $P = \varprojlim_{k\geq 0} P_k$. We also give bottom-up realizations of localization morphisms (1.0.1) and generalized Tate–Nakayama morphisms for tori ([Kaletha 2018, Theorem 3.7.3], which generalizes [Tate 1966]), as well as compatibilities between them. We also show (Proposition 5.5.2) that our construction recovers the "canonical class" defined abstractly in [Kaletha 2018, §3.5]. Apart from giving alternative proofs of some results in that work, a benefit of our construction is that it allows one to compute with global rigid inner forms "at finite level", that is using a *finite* Galois extension of the base field F. In particular, we deduce that global rigid inner forms are almost everywhere unramified (Proposition 6.1.1), a fact which is obvious for pure inner forms, but surprisingly not for rigid inner forms. In the future our construction could be used to prove further properties of Kaletha's canonical class.

Our direct construction is also useful for explicit applications using Arthur's formula for automorphic multiplicities. Computing spaces of automorphic forms, along with action of a Hecke algebra, is possible for definite reductive groups thanks to reduction theory. Unfortunately noncommutative definite reductive groups are not quasisplit. Once such spaces are computed, one would like to interpret Hecke eigenforms as being related to (ersatz) motives, and Arthur's multiplicity formula makes this relation precise (see [Taïbi 2015] for some cases for which rigid inner forms are needed). For this it is necessary to compute localizations of rigidifying data, more precisely to solve the following problem.

Problem. Given a connected reductive group G over a number field F, find

- a global rigidifying datum $\mathcal{D} = (G^*, \Xi, z, \mathfrak{w})$,
- a finite set S of places of F containing all archimedean places and all nonarchimedean places v such that G_{F_v} is ramified,
- a reductive model of $\underline{G}$ over the ring $\mathcal{O}_{F,S}$ of S-integers in F such that for any $v \notin S$, the localization $\mathcal{D}_v$ of $\mathcal{D}$ at v is unramified with respect to the integral model $\underline{G}_{\mathcal{O}_{F_v}}$ of G_{F_v},
- for each $v \in S$, an explicit description of the localization $\mathcal{D}_v$ of $\mathcal{D}$ at v.

Above, "unramified" means that $\mathrm{loc}_v(z) \in B^1(F_v, G)$, and that the resulting isomorphism $\Xi'_v : G^*_{F_v} \simeq G_{F_v}$, which is well-defined up to composing with conjugation by an element of $G(F_v)$, identifies the conjugacy class of $\mathfrak{w}_v$ with a Whittaker datum for G_{F_v} compatible with the integral model $\underline{G}_{\mathcal{O}(F_v)}$, in the sense of [Casselman and Shalika 1980]. At almost all places this is implied by the fact that $\mathfrak{w}_v$ is compatible with the canonical model of G^* and the fact that $\mathrm{loc}_v(z) \in Z^1(F_v^{\mathrm{unr}}/F_v, G^*)$, but for applications it is desirable to keep S as small as possible. For $v \in S$, the meaning of "explicit description of $\mathcal{D}_v$" is somewhat vague. In the case where $\mathrm{loc}_v(z)$ is cohomologically trivial this simply means a Whittaker datum for G_{F_v}. In general it means describing the localization $\mathcal{D}_v$ in a purely local fashion, so that it could be compared to a reference rigidifying datum. We give detailed steps to solve this problem in Section 7, reducing the computation of localizations at places in S to computations in local and global class field theory. We give an example in Section 7.2 in a case where G is a definite inner form of SL_2 over $F = \mathbb{Q}(\sqrt{3})$ which is split at all finite places, and for S the set of archimedean places, that is in "level one". It can be generalized effortlessly, and without additional computations, to the analogous inner forms of Sp_{2n} over F, for arbitrary $n \geq 2$.

Let us explain why this problem does not appear to be directly solvable using constructions in [Kaletha 2018], which might be surprising when one considers the case of pure inner forms, as it is straightforward to restrict a 1-cocycle to a decomposition group. For explicit computations one can only work with finite extensions of F, and finite Galois modules. Although the localization maps (1.0.1) are canonical, unfortunately they do not arise from *canonical* morphism of central extensions of Galois groups by *finite* Galois modules, because of the possible nonvanishing of $H^1(F_v, P_k)$, where $P = \varprojlim_k P_k$. Similarly, the possible nonvanishing of $H^1(F, P_k)$ means that inflation morphisms

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_{k+1} & \longrightarrow & \mathcal{E}_{k+1} & \longrightarrow & \mathrm{Gal}(\overline{F}/F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P_k & \longrightarrow & \mathcal{E}_k & \longrightarrow & \mathrm{Gal}(\overline{F}/F) & \longrightarrow & 1 \end{array} \tag{1.0.2}$$

are not defined canonically, where $\mathcal{E}_k$ is the central extension obtained using a 2-cocycle in the cohomology class of the image of ξ in $H^2(F, P_k)$. For applications to generalized Tate–Nakayama isomorphisms, Kaletha shows that these ambiguities are innocuous using a clever indirect argument (Lemma 3.7.10 in

[Kaletha 2018]) in cohomology (but only in cohomology). Note that in the local case, Kaletha gave an explicit construction of the inflation maps analogous to (1.0.2): see [Kaletha 2016, §4.5].

Our construction is a global analogue. The main difficulty lies in formulating and proving the analogue of [Kaletha 2016, Lemma 4.4] (which draws on [Langlands 1983, §VI.1]) in the global case. First we reinterpret [Kaletha 2016, Lemma 4.4] using a modification AW^2 of the Akizuki–Witt map on 2-cocycles [Artin and Tate 1968, Chapter XV] occurring in the construction of Weil groups attached to class formations. We study this modification systematically in Section 3.1, in particular we observe that it is more flexible while retaining the interpretation in terms of central extensions. It is not difficult to establish the analogue of [Kaletha 2016, Lemma 4.4] where local fundamental cocycles are replaced by global fundamental cocycles. However, in Tate–Nakayama isomorphisms these global fundamental cocycles control Galois cohomology groups such as $H^1(E/F, T(\mathbb{A}_E)/T(E))$, where T is a torus over F split by the finite Galois extension E/F, whereas we are interested in cohomology groups such as $H^1(E/F, T(E))$. These are controlled by *Tate cocycles* defined by Tate [1966], essentially as a consequence of the compatibility between local and global fundamental 2-cocycles. Unfortunately these do not seem to have an interpretation using the Akizuki–Witt map, and this makes the global case more challenging. We give an ad hoc definition of a certain map AWES^2 in Definition 4.2.1, which is compatible with the corestriction map in Eckmann–Shapiro's lemma for modules which are *twice* induced. This definition is crucial for the main technical result of this article, Theorem 4.4.2, constructing a family of Tate cocycles compatible under AWES^2, as well as local-global compatibility with local fundamental cocycles. We give a second proof as preparation for the algorithm in Section 7. Once this is proved, we construct Kaletha's generalized Tate–Nakayama morphisms at the level of cocycles in Section 5, and prove compatibilities with respect to inflation and localization. In particular we obtain an explicit version of Kaletha's localization maps at finite level and for cocycles. Although these explicit localization maps are not canonical, as they depend on a number of choices detailed in the paper to form cocycles, they are compatible with inflation and so yield a localization map between towers of central extensions (see Proposition 5.4.5).

As mentioned above, a consequence is that global rigid inner forms are unramified away from a finite set (Proposition 6.1.1), which is not obvious from the definition using cohomology classes. After the first version of this paper was written, we found a short proof of this ramification property using only Kaletha's characterization of the canonical class in [Kaletha 2018, §3.5]. This proof is included in Section 6, along with an example of a "noncanonical" class, which does not satisfy this ramification property.

2. Notation

Let F be a number field. We denote by $\mathbb{A}$ the ring of adeles for F. Let $\overline{F}$ be an algebraic closure of F. All algebraic extensions of F considered will be subextensions of $\overline{F}$. If E is an algebraic extension of F, let $\mathcal{O}(E)$ be its ring of integers, $\mathbb{A}_E = E \otimes_F \mathbb{A}$, $I(E) = \mathbb{A}_E^\times$ the group of ideles and $C(E) = I(E)/E^\times$ the group of idele classes. Let $\bar{\mathbb{A}} = \mathbb{A}_{\overline{F}}$. Let V be the set of all places of F. If $S \subset V$ and E is an algebraic extension of F, denote by S_E the set of places of E above S. If S is a set of places of F or E containing all archimedean places, let $I(E, S)$ be the subgroup of $I(E)$ consisting of ideles which are integral units away from S, and $\mathcal{O}(E, S)$ the ring of S-integral elements of E. For $S \subset V$ let $\overline{F}_S$ be the maximal subextension of $\overline{F}/F$ unramified outside S, and $\mathcal{O}_S = \mathcal{O}(\overline{F}_S, S)$. For E an algebraic extension of F and $u \in V_E$, we will denote by pr_u the projection $\mathbb{A}_E \to E_u$. For $v \in V$ we will denote by pr_v the projection $\mathbb{A}_{\overline{F}} \to \overline{F} \otimes_F F_v$.

As in [Kaletha 2018] we fix a tower $(E_k)_{k\geq 0}$ of increasing finite Galois extensions of F, with $E_0 = F$ and $\bigcup_k E_k = \overline{F}$. Choose an increasing sequence $(S_k)_{k\geq 0}$ of finite subsets of V such that S_0 contains all archimedean places of F, S_k contains all nonarchimedean places of F ramifying in E_k, and $I(E_k, S_k)$ maps onto $C(E_k)$. We also fix a set $\dot{V} \subset V_{\overline{F}}$ of representatives for the action of $\mathrm{Gal}(\overline{F}/F)$, that is $\dot{V}$ contains a place of $\overline{F}$ above every place of F. For E a Galois extension of F and $S' \subset V$ let $\dot{S}'_E$ be the set of places of E below $\dot{V}$ and above S', so that $\dot{S}'_E$ is a set of representatives for the action of $\mathrm{Gal}(E/F)$ on S'_E. We can assume that $\dot{V}$ is chosen so that for any finite Galois extension E/F and $\sigma \in \mathrm{Gal}(E/F)$, there exists $\dot{v} \in \dot{V}_E$ such that $\sigma \cdot \dot{v} = \dot{v}$. This follows from Chebotarev's density theorem by an inductive process as in [Kaletha 2018, (3.8)]. For $v \in V$ and $k \geq 0$ we will denote by $\dot{v}_k$ the unique place in $\dot{V}_{E_k}$ above v. To avoid double subscripts we let $E_{k,\dot{v}} = E_{k,\dot{v}_k}$. For $v \in S$ let $\overline{F_v} = \varinjlim_k E_{k,\dot{v}_k}$, an algebraic closure of F_v, so that we have a well-defined inclusion $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$.

Remark 2.0.1. The above hypotheses on $(S_k)_{k\geq 0}$ are weaker than Conditions 3.3.1 in [Kaletha 2018]. For effective computations (see Section 7) it is useful to have S_k as small as possible, and so we have only imposed conditions on $(S_k)_{k\geq 0}$ that are necessary for constructions in the present article.

The condition on the choice of $\dot{V}$ (corresponding to Condition 3.3.1.4 in [Kaletha 2018]) will not be used for the main constructions in this article. However, the extension $P \to \mathcal{E} \to \mathrm{Gal}(\overline{F}/F)$ and the morphism ι in Corollary 5.2.4 depend on the choice of $(E_k)_{k\geq 0}$ and $\dot{V}$, and so the above condition on $\dot{V}$ is necessary to obtain objects isomorphic to those in that work. Note that Condition 3.3.1.4 in [Kaletha 2018] is first used in Lemma 3.3.2, 3 there, and so it is also used in Lemma 3.6.1 there to obtain surjectivity of

$$H^1(P \to \mathcal{E}, Z \to G) \to H^1(F, G/Z)$$

for any connected reductive group G over F and finite central subgroup Z. This is crucial for applications to automorphic forms (see §4.3 there).

Condition 3.3.1.3 there, which we have not imposed, is used to prove that certain inflation maps are injective (Lemma 3.1.10, Lemma 3.2.7, Proposition 3.7.12).

If A is a commutative group, $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. If A is commutative group and $N \geq 1$ is an integer, $A[N]$ denotes the N-torsion subgroup of A. If A is a finite commutative group, $\exp(A)$ is the exponent of A, i.e., the smallest $N \geq 1$ such that $A[N] = A$. We will denote the group law of most abelian groups multiplicatively, except notably for groups of characters or cocharacters of tori. If G is a group and A a G-module, $A^G \subset A$ is the subgroup of G-invariants. If in addition $G = \mathrm{Gal}(E/F)$, we will write $N_{E/F}$ for the norm map, and $A^{N_{E/F}}$ for the subgroup of elements killed by $N_{E/F}$.

3. Preliminaries

3.1. ***A modification of the Akizuki–Witt map.*** Consider G a finite group, N a normal subgroup. If $s : G/N \to G$ is a section such that $s(1) = 1$ and A is a G-module, with group law written multiplicatively, then for $\alpha \in Z^2(G, A)$,

$$\widetilde{\mathrm{AW}}(\alpha) : (\sigma, \tau) \mapsto \prod_{n \in N} \frac{n(\alpha(s(\sigma), s(\tau))) \times \alpha(n, s(\sigma)s(\tau))}{\alpha(n, s(\sigma\tau))} \tag{3.1.1}$$

defines an element of $Z^2(G/N, A^N)$, the cohomology class of which only depends on that of α [Artin and Tate 1968, Chapter XIII, §3], so that $\widetilde{\mathrm{AW}}$ descends to a map $H^2(G, A) \to H^2(G/N, A^N)$. We refer to [Artin and Tate 1968, Chapter XIII, §3] for the natural interpretation of $\widetilde{\mathrm{AW}}$ in terms of central group extensions. Using the 2-cocycle relation for α at $(n, s(\sigma), s(\tau))$ we can express (3.1.1) as

$$\prod_{n \in N} \frac{\alpha(n, s(\sigma)) \times \alpha(ns(\sigma), s(\tau))}{\alpha(n, s(\sigma\tau))} = \prod_{n \in N} \frac{\alpha(n, s(\sigma)) \times \alpha(\tilde{\sigma}n, s(\tau))}{\alpha(n, s(\sigma\tau))},$$

where $\tilde{\sigma} \in G$ is any lift of σ, not necessarily equal to $s(\sigma)$. Using the 2-cocycle relation for α at $(\tilde{\sigma}, n, s(\tau))$ we can also rewrite this as

$$\widetilde{\mathrm{AW}}(\alpha)(\sigma, \tau) = \prod_{n \in N} \Big(\frac{\alpha(n, s(\sigma)) \times \tilde{\sigma}(\alpha(n, s(\tau)))}{\alpha(n, s(\sigma\tau))} \times \frac{\alpha(\tilde{\sigma}, ns(\tau))}{\alpha(\tilde{\sigma}, n)} \Big). \tag{3.1.2}$$

The following shows that with an appropriate choice of α in its cohomology class, this expression simplifies.

Lemma 3.1.1. *In any cohomology class in $H^2(G, A)$, there is a 2-cocycle α such that for all $n \in N$ and $\sigma \in G/N$, $\alpha(n, s(\sigma)) = 1$.*

Proof. It is well known that any cohomology class contains a 2-cocycle α such that for all $\sigma \in G$, $\alpha(\sigma, 1) = 1 = \alpha(1, \sigma)$. We choose such an α, and we will construct $\beta : G \to A$ such that $\alpha\, \mathrm{d}(\beta)$ satisfies the required property. Let $\beta(1) = 1$, and choose the values of β on $N \smallsetminus \{1\}$ and $s(G/N \smallsetminus \{1\})$ arbitrarily. For $n \in N$ and $\sigma \in G/N$,

$$\mathrm{d}\,\beta(n, s(\sigma)) = \frac{\beta(n) \times n(\beta(s(\sigma)))}{\beta(ns(\sigma))},$$

and we are led to define $\beta(ns(\sigma)) = \alpha(n, s(\sigma)) \times \beta(n) \times n(\beta(s(\sigma)))$ for $n \in N \smallsetminus \{1\}$ and $\sigma \in G/N \smallsetminus \{1\}$. Note that this equality also holds when $n = 1$ or $\sigma = 1$. □

This motivates to the following modification AW^2 of the Akizuki–Witt map $\widetilde{\mathrm{AW}}$.

Definition 3.1.2. Let Γ be an extension of G, i.e., Γ is a group endowed with a surjective morphism $\Gamma \to G$. Let A be a commutative group, with group law written multiplicatively. For $\alpha : \Gamma \times G \to A$, define $\mathrm{AW}^2(\alpha) : \Gamma \times G/N \to A$ by

$$\mathrm{AW}^2(\alpha)(\sigma, \tau) = \prod_{n \in N} \frac{\alpha(\sigma, n\tilde{\tau})}{\alpha(\sigma, n)},$$

where $\sigma \in \Gamma$, $\tau \in G/N$ and $\tilde{\tau} \in G$ is any lift of τ.

Although this coincides with the original Akizuki–Witt map a priori only for classes α as in Lemma 3.1.1 (for A a G-module and $\Gamma = G$), this definition has the advantage that it does not require a choice of section s, and will be more convenient for taking cup products. Moreover it is defined in a slightly more general setting, since it does not involve an action of G on A. This property will make "extracting N-th roots" in Section 5 almost harmless. The definition has the disadvantage that, even when A is a G-module, $\Gamma = G$ and $\alpha \in Z^2(G, A)$, it is not automatic that $\mathrm{AW}^2(\alpha)$ factors through $G/N \times G/N$ or takes values in A^N.

For Γ an extension of G and A a commutative group recall [Kaletha 2016, §4.3] for $i \geq j \geq 0$ the commutative group $C^{i,j}(\Gamma, G, A)$ of functions $\Gamma^{i-j} \times G^j \to A$, which is naturally a subgroup of $C^i(\Gamma, A)$. If A is a Γ-module, the differential d maps $C^{i,j}(\Gamma, G, A)$ to $C^{i+1,j}(\Gamma, G, A)$. Let $Z^{i,j}(\Gamma, G, A)$ be its kernel.

The following proposition is the first evidence that AW^2 behaves nicely under weaker conditions than the one imposed in Lemma 3.1.1, retaining the interpretation in terms of central extensions.

Proposition 3.1.3. *Let Γ be an extension of G.*

(1) *For $\alpha \in Z^{2,1}(\Gamma, G, A)$, we have $\mathrm{AW}^2(\alpha) \in Z^{2,1}(\Gamma, G/N, A)$.*

(2) *If $\Gamma = G$ then $\sigma \mapsto \prod_{n \in N} \alpha(n, \sigma)$ descends to a map $G/N \to A/A^N$ mapping 1 to 1.*

(3) *If $\Gamma = G$, the following are equivalent:*

  (a) *$\mathrm{AW}^2(\alpha)$ factors through $G/N \times G/N$,*
  (b) *for all $\sigma \in N$ and $\tau \in G/N$, $\mathrm{AW}^2(\alpha)(\sigma, \tau) = 1$,*
  (c) *for all $\sigma \in G$, $\prod_{n \in N} \alpha(n, \sigma) \in A^N$.*

(4) *If $\Gamma = G$ and the above conditions are satisfied, then $\mathrm{AW}^2(\alpha) \in Z^2(G/N, A^N)$ belongs to the same cohomology class as $\widetilde{\mathrm{AW}}(\alpha)$ and we have a morphism of central extensions*

$$A \underset{\alpha}{\boxtimes} G \to A^N \underset{\mathrm{AW}^2(\alpha)}{\boxtimes} G/N, \quad x \boxtimes \sigma \mapsto \left(\prod_{n \in N} n(x) \alpha(n, \sigma) \right) \boxtimes \bar{\sigma}. \tag{3.1.3}$$

We only sketch the proof, since this proposition is not logically necessary for the rest of the paper.

Proof. **(1)** This is an easy computation.

(2) Suppose that $\Gamma = G$. Using the cocycle relation for α, for every $\tau, \gamma \in N$,

$$\tau\left(\prod_{n \in N} \alpha(n, \gamma) \right) = \prod_{n \in N} \alpha(\tau n, \gamma) \alpha(\tau, n) / \alpha(\tau, n\gamma) = \prod_{n \in N} \alpha(n, \gamma)$$

and so $\prod_{n \in N} \alpha(n, \gamma) \in A^N$ for any $\gamma \in N$. Now for $\gamma \in N$ and $\sigma \in G$, using the cocycle relation again,

$$\prod_{n \in N} \alpha(n, \gamma\sigma) = \prod_{n \in N} \alpha(n\gamma, \sigma) \alpha(n, \gamma) n(\alpha(\gamma, \sigma)) \equiv \prod_{n \in N} \alpha(n, \sigma) \mod A^N.$$

(3) Using the cocycle relation we can write

$$\mathrm{AW}^2(\alpha)(\sigma,\tau)=\prod_{n\in N}\frac{\alpha(\sigma n,\tilde{\tau})}{\sigma(\alpha(n,\tilde{\tau}))}.$$

The numerator only depends on $\alpha \bmod N$, and the equivalence between (a) and (c) follows easily. The equivalence between (b) and (c) is obtained by taking $\sigma\in N$.

(4) The fact that $\mathrm{AW}^2(\alpha)$ is cohomologous to $\widetilde{\mathrm{AW}}^2(\alpha)$ follows from the expression (3.1.2) for $\widetilde{\mathrm{AW}}$ and condition (c). This give an isomorphism $A^N\boxtimes_{\mathrm{AW}^2(\alpha)} G/N\simeq A^N\boxtimes_{\widetilde{\mathrm{AW}}^2(\alpha)} G/N$. Since we have an explicit map $A\boxtimes_\alpha G\to A^N\boxtimes_{\widetilde{\mathrm{AW}}^2(\alpha)} G/N$ by construction in [Artin and Tate 1968, Chapter XIII, §3], finding formula (3.1.3) is a simple computation. Alternatively, one can directly check that (3.1.3) is a morphism. □

In order to investigate the effect on $\mathrm{AW}^2(\alpha)$ of the choice of α in its cohomology class, let us define a second map AW^1 on 1-cochains.

Definition 3.1.4. Let A be a commutative group. For $\beta:G\to A$, define $\mathrm{AW}^1(\beta):G/N\to A$ by the formula $\mathrm{AW}^1(\beta)(\sigma)=\prod_{n\in N}\beta(n\tilde{\sigma})/\beta(n)$, where $\tilde{\sigma}\in G$ is any lift of $\sigma\in G/N$.

Proposition 3.1.5. *Suppose Γ is an extension of G, and A is a Γ-module. For any $\beta:G\to A$, we have* $\mathrm{d}(\mathrm{AW}^1(\beta))=\mathrm{AW}^2(\mathrm{d}(\beta))$ *in* $Z^{2,1}(\Gamma,G/N,A)$.

Proof. For $\sigma\in\Gamma$ and $\tau\in G/N$, denoting $\bar{\sigma}$ the image of σ in G, we have

$$\mathrm{d}(\mathrm{AW}^1(\beta))(\sigma,\tau)=\prod_{n\in N}\frac{\beta(n\bar{\sigma})}{\beta(n)}\frac{\sigma(\beta(n\tilde{\tau}))}{\sigma(\beta(n))}\frac{\beta(n)}{\beta(n\bar{\sigma}\tilde{\tau})}=\prod_{n\in N}\frac{\beta(n\bar{\sigma})\sigma(\beta(n\tilde{\tau}))}{\beta(n\bar{\sigma}\tilde{\tau})\sigma(\beta(n))}$$

and

$$\mathrm{AW}^2(\mathrm{d}(\beta))(\sigma,\tau)=\prod_{n\in N}\frac{\beta(\bar{\sigma})\sigma(\beta(n\tilde{\tau}))}{\beta(\bar{\sigma}n\tilde{\tau})}\frac{\beta(\bar{\sigma}n)}{\beta(\bar{\sigma})\sigma(\beta(n))}=\prod_{n\in N}\frac{\sigma(\beta(n\tilde{\tau}))}{\beta(\bar{\sigma}n\tilde{\tau})}\frac{\beta(\bar{\sigma}n)}{\sigma(\beta(n))}.$$ □

Lemma 3.1.6. *The maps*

$$\{\beta:G\to A\mid\beta(1)=1\}\to\{\beta:G/N\to A\mid\beta(1)=1\}$$

induced by AW^1 *and*

$$\{\alpha:\Gamma\times G\to A\mid\alpha(\sigma,1)=1\text{ for all }\sigma\in\Gamma\}\to\{\alpha:\Gamma\times G/N\to A\mid\alpha(\sigma,1)=1\text{ for all }\sigma\in\Gamma\}$$

induced by AW^2 *are both surjective.*

Proof. Let $s:G/N\to G$ be a section such that $s(1)=1$. Restricting AW^1 to the set of $\beta:G\to A$ such that $\beta|_N=1$ and $\beta(ns(\sigma))=1$ for $\sigma\in G/N\smallsetminus\{1\}$ and $n\in N\smallsetminus\{1\}$ yields a bijective map onto $\{\beta:G/N\to A\mid\beta(1)=1\}$.

Similarly, restricting AW^2 to the set of $\alpha:\Gamma\times G\to A$ such that

- for all $\sigma\in\Gamma$ and $n\in N$, $\alpha(\sigma,n)=1$,
- for all $\sigma\in\Gamma$, $n\in N\smallsetminus\{1\}$ and $\tau\in G/N\smallsetminus\{1\}$, $\alpha(\sigma,ns(\tau))=1$,

yields a bijective map onto $\{\alpha:\Gamma\times G/N\to A\mid\alpha(\sigma,1)=1\text{ for all }\sigma\in\Gamma\}$. □

The following corollary is readily deduced from Lemmas 3.1.1 and 3.1.6 and Proposition 3.1.5.

Corollary 3.1.7. *Suppose that A is a G-module. Consider $c \in H^2(G, A)$, and let $\alpha_N \in Z^2(G/N, A^N)$ be in the cohomology class of the image of c under $\widetilde{\mathrm{AW}}$. Assume that $\alpha_N(1, 1) = 1$. Then there exists $\alpha \in c$ such that $\alpha(1, 1) = 1$ and $\mathrm{AW}^2(\alpha) = \alpha_N$.*

Note that we have not imposed that α should satisfy the property in Lemma 3.1.1, and indeed it can happen that no such α maps to α_N under AW^2. A simple computation shows that if we fix a section $s : G/N \to G$ as above, then for $\alpha, \alpha' \in c$ as in Lemma 3.1.1, $\mathrm{AW}^2(\alpha/\alpha') \in B^2(G/N, N_N(A))$ where

$$N_N(A) = \left\{ \prod_{n \in N} n(x) \,\middle|\, x \in A \right\}.$$

3.2. *Explicit Eckmann–Shapiro.* Let G be a finite group acting transitively on the left on a set X. Choose $x_0 \in X$ and let H be the stabilizer of x_0, so that we have an identification of G-sets $X \simeq G/H$ mapping x_0 to the trivial coset.

Let A be a left H-module. Define

$$\mathrm{Ind}_H^G(A) = \{ f : G \to A \mid \text{for all } h \in H, g \in G,\ f(hg) = h \cdot f(g) \}.$$

It is naturally a left G-module by defining $(g_1 \cdot f)(g_2) = f(g_2 g_1)$. Evaluation at 1 defines a surjective morphism of H-modules $\pi : \mathrm{Ind}_H^G(A) \to A$, which admits a natural splitting: we can identify A with the H-submodule of $\mathrm{Ind}_H^G(A)$ consisting of all functions whose support is contained in H. Choose R a set of representatives for G/H. Then $\mathrm{Ind}_H^G(A) = \bigoplus_{r \in R} r \cdot A$. For simplicity we assume that $1 \in R$.

If A happens to be a G-module, then

$$f \mapsto (gH \mapsto g \cdot f(g^{-1})) \tag{3.2.1}$$

defines an isomorphism of G-modules ϕ_H^G between $\mathrm{Ind}_H^G(A)$ and $\mathrm{Maps}(X, A)$. The H-submodule A of $\mathrm{Ind}_H^G(A)$ corresponds to functions supported on x_0 under this isomorphism.

The Eckmann–Shapiro lemma states that for any $i \geq 0$, the composite

$$H^i(G, \mathrm{Ind}_H^G(A)) \to H^i(H, \mathrm{Ind}_H^G(A)) \to H^i(H, A)$$

is an isomorphism, where the first map is restriction and the second map is induced by π. See, e.g., [Serre 1994, Chapter I, §2.5]. It is well known (for example [Tate 1966, p.713]) that the inverse is obtained as the composite

$$H^i(H, A) \to H^i(H, \mathrm{Ind}_H^G(A)) \to H^i(G, \mathrm{Ind}_H^G(A))$$

where the first map is induced by the embedding of H-modules $A \to \mathrm{Ind}_H^G(A)$ mentioned above and the second map is corestriction. In this paper we will use explicit formulas for this inverse map, especially in degree 2.

Proposition-Definition 3.2.1. *As above, G is a finite group, H is a subgroup of G, R is a set of representatives for G/H containing 1, and A is a G-module.*

(1) *For $i \geq 0$ and $c \in C^i(H, A)$, define $\mathrm{ES}^i_R(c) \in C^i(G, \mathrm{Ind}^G_H(A))$ by*

$$\mathrm{ES}^i_R(c)(r_1h_1r_2^{-1}, r_2h_2r_3^{-1}, \ldots, r_ih_ir_{i+1}^{-1})(h_{i+1}r_1^{-1}) = h_{i+1}(c(h_1, h_2, \ldots, h_i)),$$

where $r_1, \ldots, r_{i+1} \in R$ and $h_1, \ldots, h_{i+1} \in H$. If A happens to be a G-module, then using the identification (3.2.1) *we can write*

$$\phi^G_H(\mathrm{ES}^i_R(c)(r_1h_1r_2^{-1}, r_2h_2r_3^{-1}, \ldots, r_ih_ir_{i+1}^{-1}))(r_1 \cdot x_0) = r_1(c(h_1, h_2, \ldots, h_i)). \tag{3.2.2}$$

(2) *For $i \geq 0$ and $c \in C^i(H, A)$, $\mathrm{d}(\mathrm{ES}^i_R(c)) = \mathrm{ES}^{i+1}_R(\mathrm{d}(c))$. Thus ES^i_R induces a map $H^i(H, A) \to H^i(G, \mathrm{Ind}^G_H)$, which is an isomorphism that we still denote by ES^i_R.*

Proof. The formula for $\mathrm{ES}^i_R(c)$ follows from the explicit formula for corestriction for homogeneous cochains found in [Neukirch et al. 2008, Chapter I, §5.4. p. 48] specialized to the case at hand where c takes values in $A \subset \mathrm{Ind}^G_H(A)$. □

4. Construction of Tate cocycles in a tower

Let us recall from [Tate 1966] the construction of the Tate–Nakayama isomorphism, which gives a relatively simple description of Galois cohomology groups of tori over F. Consider E a finite Galois extension of F, and S a not necessarily finite set of places of F containing all archimedean places and all nonarchimedean places that ramify in E, and such that $I(E, S)$ surjects to $C(E)$. Tate introduced the $\mathrm{Gal}(E/F)$-module $\mathrm{Ta}(E, S)$ which consists of all morphisms from the short exact sequence

$$\mathbb{Z}[S_E]_0 \to \mathbb{Z}[S_E] \to \mathbb{Z}$$

to the short exact sequence

$$\mathcal{O}(E, S)^\times \to I(E, S) \to C(E).$$

Equivalently,

$$\mathrm{Ta}(E, S) = \mathrm{Hom}(\mathbb{Z}[S_E], I(E, S)) \underset{\mathrm{Hom}(\mathbb{Z}[S_E], C(E))}{\times} C(E) \subset \mathrm{Maps}(S_E, I(E, S)).$$

Tate constructed, using local and global fundamental classes and compatibility between them, a *Tate class* $\alpha \in H^2(E/F, \mathrm{Ta}(E, S))$. Consider a torus T over F which is split by E, let $Y = X_*(T)$ be the associated $\mathrm{Gal}(E/F)$-module of cocharacters. The main result of [Tate 1966] is that taking cup product with α gives isomorphisms in every degree $i \in \mathbb{Z}$

$$\widehat{H}^i(E/F, Y[S_E]_0) \to \widehat{H}^{i+2}(E/F, T(\mathcal{O}(E, S))) \tag{4.0.1a}$$

$$\widehat{H}^i(E/F, Y[S_E]) \to \widehat{H}^{i+2}(E/F, T(\mathbb{A}_E, S)) \tag{4.0.1b}$$

$$\widehat{H}^i(E/F, Y) \to \widehat{H}^{i+2}(E/F, T(\mathbb{A}_E)/T(E)) \tag{4.0.1c}$$

where

$$T(\mathbb{A}_E, S) = Y \otimes_{\mathbb{Z}} I(E, S) = \prod_{w \in S_E} T(E_w) \times \prod_{w \notin S_E} T(\mathcal{O}_{E_w}).$$

We shall see that varying S among the sets of places containing a fixed finite set S_0 satisfying the above conditions does not result in any difficulty. Varying E (for example in the tower E_k that is fixed in this paper), however, leads to the surprising phenomenon that it is not completely obvious that all three isomorphisms (4.0.1) are compatible with inflation of cohomology classes on the right hand side. See [Kaletha 2018, Lemma 3.1.4] for a precise statement and a proof in cohomology.

Our first goal is to construct a *compatible* family of Tate *cocycles*

$$\alpha_k \in Z^2(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$$

for the Galois extensions E_k/F. We will give a precise meaning to technical notion of "compatibility" in Theorem 4.4.2. For now we simply mention that this compatibility is a global analogue of [Kaletha 2016, Lemma 4.4].

Unwinding the definition, one can see that for a fixed k, a Tate cocycle α_k for E_k/F is obtained as follows.

(1) Choose a representative $\bar{\alpha}_k \in Z^2(E_k/F, C(E_k))$ of the fundamental class for E_k/F.

(2) For each place v of F, choose a representative $\alpha_{k,v} \in Z^2(E_{k,\dot{v}}/F_v, E_{k,\dot{v}}^\times)$ of the fundamental class for $E_{k,\dot{v}}/F_v$. Let $\alpha'_k \in Z^2(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$ be such that for any $v \in V$, the 2-cocycle

$$\mathrm{Gal}(E_{k,\dot{v}}/F_v)^2 \to I(E_k), \quad (\sigma, \tau) \mapsto \alpha'_k(\sigma, \tau)(\dot{v}_k)$$

is cohomologous to $\alpha_{k,v}$ composed with $j_{k,v} : E_{k,\dot{v}}^\times \hookrightarrow I(E_k)$. Explicitly, α'_k can be obtained from $(\alpha_{k,v})_{v\in V}$ using (3.2.2).

(3) Choose $\bar{\beta}_k \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, C(E_k)))$ such that $\bar{\alpha}_k/\overline{\alpha'_k} = \mathrm{d}(\bar{\beta}_k)$, where $\bar{\alpha}_k$ is seen as taking values in the set of constant maps $V_{E_k} \to C(E_k)$ and $\overline{\alpha'_k}$ is the composition of α'_k with the natural map $\mathrm{Maps}(V_{E_k}, I(E_k)) \to \mathrm{Maps}(V_{E_k}, C(E_k))$.

(4) Lift $\bar{\beta}_k$ to $\beta_k \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$ arbitrarily, and define $\alpha_k = \alpha'_k \times \mathrm{d}(\beta_k)$.

In this section we will show that each step can be done compatibly with Akizuki–Witt-like maps. For cocycles $\alpha_{k,v}$ this was done in [Kaletha 2016, Lemma 4.4], we will however give a slightly different construction, using Corollary 3.1.7. The case of $\bar{\alpha}_k$ is very similar. A key point of the construction will be the definition (see 4.2.1) of an "Akizuki–Witt–Eckmann–Shapiro" map relating the maps AW for local and global Galois groups, and formula (3.2.2) (see Lemma 4.2.2).

4.1. ***Global fundamental cocycles.*** For any $k \geq 0$, the image in $H^2(E_{k+1}/F, C(E_{k+1}))$ of the fundamental class under the Akizuki–Witt map (3.1.1) (for the normal subgroup $\mathrm{Gal}(E_{k+1}/E_k)$, and any choice of section) is the fundamental class in $H^2(E_k/F, C(E_k))$. This is a direct consequence of [Artin and Tate 1968, Chapter XIII, Theorem 6]. For $i \in \{1, 2\}$ write AW_k^i for the maps AW^i defined in Section 3.1, for the normal subgroup $\mathrm{Gal}(E_{k+1}/E_k)$ of $\mathrm{Gal}(E_{k+1}/F)$. Using Corollary 3.1.7 we see that there exists a family $(\bar{\alpha}_k)_{k\geq 0}$ where each $\bar{\alpha}_k \in Z^2(E_k/F, C(E_k))$ represents the fundamental class, and such that for all $k \geq 0$ we have $\bar{\alpha}_k = \mathrm{AW}_k^2(\bar{\alpha}_{k+1})$.

Remark 4.1.1. Alternatively, one could construct such a family using a method similar to [Kaletha 2016, §4.4] (and so [Langlands 1983, §VI.1]), that is by choosing sections $\mathrm{Gal}(E_{k+1}/E_k) \to W_{E_k}$, where W_{E_k} is the Weil group of E_k, and multiplying them to produce sections $\mathrm{Gal}(E_k/F) \to W_{E_k/F}$, yielding fundamental cocycles compatible with AW^2_k.

A third way would be to use a compactness argument and Lemma 3.1.1, as in the proof of Theorem 4.4.2 (using 2-cochains instead of 1-cochains). The details for this last alternative are left to the reader.

4.2. ***Local and adelic fundamental classes.*** Fix $v \in V$. For $i \in \{1, 2\}$ write $\mathrm{AW}^i_{k,v}$ for the maps AW^i defined in Section 3.1, for the normal subgroup $\mathrm{Gal}(E_{k+1,\dot v}/E_{k,\dot v})$ of $\mathrm{Gal}(E_{k+1,\dot v}/F_v)$. As in the global case we can use Corollary 3.1.7 inductively to produce a family $(\alpha_{k,v})_{k\geq 0}$ where $\alpha_{k,v} \in Z^2(E_{k,\dot v}/F_v, E^\times_{k,\dot v})$ represents the fundamental class and for all $k \geq 0$, we have $\alpha_{k,v} = \mathrm{AW}^2_{k,v}(\alpha_{k+1,v})$. Alternatively we could simply use [Kaletha 2016, Lemma 4.4]: see Remark 4.1.1.

For each $k \geq 1$, choose representatives for $\mathrm{Gal}(E_k/E_{k-1})/\mathrm{Gal}(E_{k,\dot v}/E_{k-1,\dot v})$, and choose lifts of these representatives in $\mathrm{Gal}(\bar F/E_{k-1})$ to obtain a finite set $R_{k,v} \subset \mathrm{Gal}(\bar F/E_{k-1})$. We can and do assume that $1 \in R_{k,v}$. For convenience we also define $R_{0,v} = \{1\} \subset \mathrm{Gal}(\bar F/F)$. For any $k \geq 0$, $R'_{k,v} := R_{0,v}R_{1,v}\cdots R_{k,v} \subset \mathrm{Gal}(\bar F/F)$ projects to a set of representatives for $\mathrm{Gal}(E_k/F)/\mathrm{Gal}(E_{k,\dot v}/F_v)$. For $v \in V$ and $k \geq 0$ let $\zeta_{k,v} : \{v\}_{E_k} \to \{v\}_{E_{k+1}}$ be the section such that for all $r \in R'_{k,v}$, $\zeta_{k,v}(r \cdot \dot v_k) = r \cdot \dot v_{k+1}$. Define $j_{k,v} : E^\times_{k,\dot v} \hookrightarrow I(E_k)$ by $(j_{k,v}(x))_{\dot v_k} = x$ and $(j_{k,v}(x))_w = 1$ for $w \neq \dot v_k$. We have natural inclusions $E^\times_{k,\dot v} \subset E^\times_{k+1,\dot v}$ and for $x \in E^\times_{k,\dot v}$ we have

$$j_{k,v}(x) = \prod_{r \in R_{k+1,v}} r(j_{k+1,v}(x)). \tag{4.2.1}$$

Following Proposition-Definition 3.2.1 define, for all $k \geq 0$, $\alpha'_k \in Z^2(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$ by

$$\alpha'_k(r_1\sigma r_2^{-1}, r_2\tau r_3^{-1})(r_1 \cdot \dot v_k) = r_1(j_{k,v}(\alpha_{k,v}(\sigma, \tau))) \tag{4.2.2}$$

for $v \in V$, $\sigma, \tau \in \mathrm{Gal}(E_{k,\dot v}/F_v)$ and $r_1, r_2, r_3 \in R'_{k,v}$. That is, α'_k is obtained by aggregating

$$\phi^{\mathrm{Gal}(E_k/F)}_{\mathrm{Gal}(E_{k,\dot v}/F_v)}\big(\mathrm{ES}^2_{R'_{k,v}}(j_{k,v}(\alpha_{k,v}))\big) \in Z^2\big(E_k/F, \mathrm{Maps}(\{v\}_{E_k}, I(E_k))\big) \quad \text{for } v \in V.$$

Definition 4.2.1. Suppose that A is a commutative group. For $k \geq 0$ and $\alpha : \mathrm{Gal}(\bar F/F) \times \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$, define

$$\mathrm{AWES}^2_k(\alpha) : \mathrm{Gal}(\bar F/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$$

by

$$\mathrm{AWES}^2_k(\alpha)(\sigma, \tau)(\sigma_k\tau \cdot w) := \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\alpha(\sigma, n\tilde\tau)(\sigma_{k+1}n\tilde\tau \cdot \zeta_{k,v}(w))}{\alpha(\sigma, n)(\sigma_{k+1}n \cdot \zeta_{k,v}(\tau \cdot w))}.$$

In this formula $\sigma \in \mathrm{Gal}(\bar F/F)$ has image σ_{k+1} in $\mathrm{Gal}(E_{k+1}/F)$ and σ_k in $\mathrm{Gal}(E_k/F)$, $\tau \in \mathrm{Gal}(E_k/F)$ and $\tilde\tau \in \mathrm{Gal}(E_{k+1}/F)$ is any lift of τ, $v \in V$ and $w \in \{v\}_{E_k}$.

Note that AWES^2_k depends on the choice of representatives $R'_{k,v}$ only via $\zeta_{k,v}$.

Lemma 4.2.2. *For all $k \geq 0$ we have* $\mathrm{AWES}^2_k(\alpha'_{k+1}) = \alpha'_k$.

Note that a priori the left hand side is only a map $\mathrm{Gal}(E_{k+1}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, I(E_{k+1}))$. The lemma implies that it is inflated from a map $\mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(V_{E_k}, I(E_k))$.

Proof. Fix $\sigma \in \mathrm{Gal}(E_{k+1}/F)$, $\tau \in \mathrm{Gal}(E_k/F)$ and $\gamma \in R'_{k,v}$. In $\mathrm{Gal}(E_k/F)$ write $\tau\gamma = r_2 g_2$, where $r_2 \in R'_{k,v}$ and $g_2 \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. Let $\tilde{\tau} \in \mathrm{Gal}(E_{k+1}/F)$ be any lift of τ and let $\widetilde{g_2} \in \mathrm{Gal}(E_{k+1,\dot{v}}/F_v)$ be any lift of g_2. Note that

$$\{n\tilde{\tau} \mid n \in \mathrm{Gal}(E_{k+1}/E_k)\} = \{r_2 u n_v \widetilde{g_2} \gamma^{-1} \mid u \in R_{k+1,v},\ n_v \in \mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})\},$$
$$\mathrm{Gal}(E_{k+1}/E_k) = \{r_2 u n_v r_2^{-1} \mid u \in R_{k+1,v},\ n_v \in \mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})\}.$$

In $\mathrm{Gal}(E_k/F)$ write $\sigma_k r_2 = r_1 g_1$ where $r_1 \in R'_{k,v}$ and $g_1 \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. Choose $\widetilde{g_1} \in \mathrm{Gal}(E_{k+1,\dot{v}}/F_v)$ lifting g_1. For every $u \in R_{k+1,v}$ we can decompose $\sigma r_2 u \in \mathrm{Gal}(E_{k+1}/F)$ as follows: $\sigma r_2 u = r_1 u' \widetilde{g_1} x_v$ where $u' \in R_{k+1,v}$ and $x_v \in \mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})$ depend on u. Moreover $u \mapsto u'$ realizes a bijection from $R_{k+1,v}$ to itself: $r_1^{-1}\sigma r_2 \widetilde{g_1}^{-1} \in \mathrm{Gal}(E_{k+1}/E_k)$ induces a permutation of the set of places of E_{k+1} lying above $\dot{v}_k$.

$$\begin{aligned}\mathrm{AWES}^2_k(\alpha'_{k+1})(\sigma,\tau)(\sigma_k\tau\gamma\cdot\dot{v}_k) &= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\alpha'_{k+1}(\sigma, n\tilde{\tau})(\sigma n\tilde{\tau}\gamma\cdot\dot{v}_{k+1})}{\alpha'_{k+1}(\sigma, n)(\sigma n r_2\cdot\dot{v}_{k+1})} \\ &= \prod_{u,n_v} \frac{\alpha'_{k+1}(r_1u'\widetilde{g_1}x_v(r_2u)^{-1}, r_2un_v\widetilde{g_2}\gamma^{-1})(r_1u'\cdot\dot{v}_{k+1})}{\alpha'_{k+1}(r_1u'\widetilde{g_1}x_v(r_2u)^{-1}, r_2un_vr_2^{-1})(r_1u'\cdot\dot{v}_{k+1})}\end{aligned}$$

using the above bijections. Now apply definition (4.2.2) of α'_{k+1} to the numerator (resp. denominator), with (r_1, r_2, r_3) replaced by (r_1u', r_2u, γ) (resp. (r_1u', r_2u, r_2)):

$$\begin{aligned}\mathrm{AWES}^2_k(\alpha'_{k+1})(\sigma,\tau)(\sigma_k\tau\gamma\cdot\dot{v}_k) &= \prod_u r_1u'\left(\prod_{n_v} \frac{j_{k+1,v}(\alpha_{k+1,v}(\widetilde{g_1}x_v, n_v\widetilde{g_2}))}{j_{k+1,v}(\alpha_{k+1,v}(\widetilde{g_1}x_v, n_v))}\right) \\ &= \prod_u r_1u'(j_{k+1,v}(\alpha_{k,v}(g_1,g_2))) \\ &= r_1(j_{k,v}(\alpha_{k,v}(g_1,g_2))) \\ &= \alpha'_k(r_1g_1r_2^{-1}, r_2g_2\gamma^{-1})(r_1\cdot\dot{v}_k) \\ &= \alpha'_k(\sigma,\tau)(\sigma\tau\gamma\cdot\dot{v}_k).\end{aligned}$$

The second equality follows from $\alpha_{k,v} = \mathrm{AW}^2_{k,v}(\alpha_{k+1,v})$. The third is a consequence of (4.2.1). The fourth follows from the definition (4.2.2) of α'_k, and the last from the definition of r_1, r_2, g_1, g_2. □

Remark 4.2.3. One could define AWES^2 axiomatically, as we did for AW^2 in Section 3.1, for general quadruples $(G, N, H, R_{G/N}, R_N)$ where G is a finite group, N a normal subgroup of G, H a subgroup of G, $R_{G/N} \subset G$ a set of representatives for $G/HN = (G/N)/(HN/N)$ such that $1 \in R_{G/N}$, and $R_N \subset N$ a set of representatives for $N/(N \cap H)$ such that $1 \in R_N$. One could also state the generalization of Lemma 4.2.2 in this context, with an identical proof. Note that it would apply to 2-cocycles α' taking values in a *twice* induced module, that is $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} \mathrm{Ind}_H^G(A)$ for some H-module A. Indeed Definition 4.2.1 is essentially used with $A = (E_k \otimes_F F_v)^\times = \prod_{w|v} E_w^\times$, which is already induced with respect to the subgroup $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$ of $\mathrm{Gal}(E_k/F)$. We will not need this generality, however.

4.3. ***Properties of*** $\mathbf{AWES}_k^2$**.** To establish the analogue of Proposition 3.1.5, we introduce variants of AWES_k^2 in degrees 0 and 1.

Definition 4.3.1. Fix $k \geq 0$.

(1) Suppose that A is a commutative group. For $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$, define $\mathrm{AWES}_k^1(\beta) : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$ by

$$\mathrm{AWES}_k^1(\beta)(\sigma)(\sigma \cdot w) = \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\beta(n\tilde{\sigma})(n\tilde{\sigma} \cdot \zeta_{k,v}(w))}{\beta(n)(n \cdot \zeta_{k,v}(\sigma \cdot w))}$$

for $\sigma \in \mathrm{Gal}(E_k/F)$ and $w \in \{v\}_{E_k}$. In this formula $\tilde{\sigma} \in \mathrm{Gal}(E_{k+1}/F)$ is any lift of σ.

(2) Suppose that A is a $\mathrm{Gal}(E_{k+1}/E_k)$-module. For $\beta \in \mathrm{Maps}(V_{E_{k+1}}, A)$ define

$$\mathrm{AWES}_k^0(\beta) \in \mathrm{Maps}(V_{E_k}, A^{\mathrm{Gal}(E_{k+1}/E_k)}) \quad \text{by} \quad \mathrm{AWES}_k^0(\beta)(w) = N_{E_{k+1}/E_k}\big(\beta(\zeta_{k,v}(w))\big) \text{ for } w \in \{v\}_{E_k}.$$

Lemma 4.3.2. *Fix* $k \geq 0$.

(1) *Suppose that* A *is a* $\mathrm{Gal}(\bar{F}/F)$*-module. For* $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$, *we have the equality of maps* $\mathrm{Gal}(\bar{F}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$

$$\mathrm{AWES}_k^2(\mathrm{d}(\beta)) = \mathrm{d}(\mathrm{AWES}_k^1(\beta)).$$

(2) *Suppose that* A *is a* $\mathrm{Gal}(E_{k+1}/F)$*-module. For* $\beta \in \mathrm{Maps}(V_{E_{k+1}}, A)$, *we have the equality of maps* $\mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_k}, A)$

$$\mathrm{AWES}_k^1(\mathrm{d}(\beta)) = \mathrm{d}(\mathrm{AWES}_k^0(\beta)).$$

The right hand side is a map $\mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, N_{E_{k+1}/E_k}(A))$.

Proof. **(1)** Let $v \in S$, $w \in \{w\}_k$, $\sigma \in \mathrm{Gal}(E_{k+1}/F)$ and $\tau \in \mathrm{Gal}(E_k/F)$. Let $\bar{\sigma}$ be the image of σ in $\mathrm{Gal}(E_k/F)$, and fix $\tilde{\tau} \in \mathrm{Gal}(E_{k+1}/F)$ lifting τ. We have

$$\begin{aligned}
&\mathrm{d}(\mathrm{AWES}_k^1(\beta))(\sigma, \tau)(\bar{\sigma}\tau \cdot w) \\
&= \frac{\mathrm{AWES}_k^1(\beta)(\bar{\sigma})(\bar{\sigma}\tau \cdot w)\sigma(\mathrm{AWES}_k^1(\beta)(\tau))(\bar{\sigma}\tau \cdot w)}{\mathrm{AWES}_k^1(\beta)(\bar{\sigma}\tau)(\bar{\sigma}\tau \cdot w)} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\beta(n\sigma)(n\sigma \cdot \zeta_{k,v}(\tau \cdot w))}{\beta(n)(n \cdot \zeta_{k,v}(\bar{\sigma}\tau \cdot w))} \times \sigma\Big(\frac{\beta(n\tilde{\tau})(n\tilde{\tau} \cdot \zeta_{k,v}(w))}{\beta(n)(n \cdot \zeta_{k,v}(\tau \cdot w))}\Big) \times \frac{\beta(n)(n \cdot \zeta_{k,v}(\bar{\sigma}\tau \cdot w))}{\beta(n\sigma\tilde{\tau})(n\sigma\tilde{\tau} \cdot \zeta_{k,v}(w))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\sigma(\beta(n\tilde{\tau})(n\tilde{\tau} \cdot \zeta_{k,v}(w)))}{\beta(n\sigma\tilde{\tau})(n\sigma\tilde{\tau} \cdot \zeta_{k,v}(w))} \times \frac{\beta(n\sigma)(n\sigma \cdot \zeta_{k,v}(\tau \cdot w))}{\sigma(\beta(n)(n \cdot \zeta_{k,v}(\tau \cdot w)))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\sigma(\beta(n\tilde{\tau})(n\tilde{\tau} \cdot \zeta_{k,v}(w)))}{\beta(\sigma n\tilde{\tau})(\sigma n\tilde{\tau} \cdot \zeta_{k,v}(w))} \times \frac{\beta(\sigma n)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w))}{\sigma(\beta(n)(n \cdot \zeta_{k,v}(\tau \cdot w)))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(\sigma, n\tilde{\tau})(\sigma n\tilde{\tau} \cdot \zeta_{k,v}(w))}{\beta(\sigma)(\sigma n\tilde{\tau} \cdot \zeta_{k,v}(w))} \times \frac{\beta(\sigma)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w)))}{\mathrm{d}\beta(\sigma, n)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(\sigma, n\tilde{\tau})(\sigma n\tilde{\tau} \cdot \zeta_{k,v}(w))}{\mathrm{d}\beta(\sigma, n)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w))} = \mathrm{AWES}_k^2(d\beta)(\sigma, \tau)(\bar{\sigma}\tau \cdot w).
\end{aligned}$$

We have used the fact that for any $u \in \{v\}_{E_{k+1}}$,

$$\mathrm{card}\{n \in \mathrm{Gal}(E_{k+1}/E_k) \mid n\tilde{\tau} \cdot \zeta_{k,v}(w) = u\} = \mathrm{card}\{n \in \mathrm{Gal}(E_{k+1}/E_k) \mid n \cdot \zeta_{k,v}(\tau \cdot w)) = u\}$$

that implies

$$\prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \beta(\sigma)(\sigma n \tilde{\tau} \cdot \zeta_{k,v}(w)) = \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \beta(\sigma)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w)).$$

(2) Let $v \in S$ and $w \in \{v\}_{E_k}$. Let $\sigma \in \mathrm{Gal}(E_k/F)$ and fix $\tilde{\sigma} \in \mathrm{Gal}(E_{k+1}/F)$ lifting σ.

$$\begin{aligned}
\mathrm{d}(\mathrm{AWES}_k^0(\beta))(\sigma)(\sigma \cdot w) &= \frac{\sigma(\mathrm{AWES}_k^0(\beta))(\sigma \cdot w)}{\mathrm{AWES}_k^0(\beta)(\sigma \cdot w)} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\tilde{\sigma} n(\beta(\zeta_{k,v}(w)))}{n(\beta(\zeta_{k,v}(\sigma \cdot w)))} = \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{n\tilde{\sigma}(\beta(\zeta_{k,v}(w)))}{n(\beta(\zeta_{k,v}(\sigma \cdot w)))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(n\tilde{\sigma})(n\tilde{\sigma} \cdot \zeta_{k,v}(w)) \times \beta(n\tilde{\sigma} \cdot \zeta_{k,v}(w))}{\mathrm{d}\beta(n)(n \cdot \zeta_{k,v}(\sigma \cdot w)) \times \beta(n \cdot \zeta_{k,v}(\sigma \cdot w))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(n\tilde{\sigma})(n\tilde{\sigma} \cdot \zeta_{k,v}(w))}{\mathrm{d}\beta(n)(n \cdot \zeta_{k,v}(\sigma \cdot w))} = \mathrm{AWES}_k^1(\mathrm{d}\beta)(\sigma)(\sigma \cdot w).
\end{aligned}$$

Again we have used the fact that for any $u \in \{v\}_{E_{k+1}}$,

$$\mathrm{card}\{n \in \mathrm{Gal}(E_{k+1}/E_k) \mid n\tilde{\sigma} \cdot \zeta_{k,v}(w) = u\} = \mathrm{card}\{n \in \mathrm{Gal}(E_{k+1}/E_k) \mid n \cdot \zeta_{k,v}(\sigma \cdot w)) = u\}$$

that implies

$$\prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \beta(n\tilde{\sigma} \cdot \zeta_{k,v}(w)) = \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \beta(n \cdot \zeta_{k,v}(\sigma \cdot w))). \qquad \square$$

Corollary 4.3.3. *Fix $k \geq 0$, and suppose that A is a $\mathrm{Gal}(E_{k+1}/F)$-module.*

(1) *Let $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$ be such that $\mathrm{AWES}_k^2(\mathrm{d}(\beta))$ factors through $\mathrm{Gal}(E_k/F)^2$. Then $\mathrm{AWES}_k^1(\beta)$ takes values in $\mathrm{Maps}(V_{E_k}, A^{\mathrm{Gal}(E_{k+1}/E_k)})$.*

(2) *If $\beta \in Z^1(\mathrm{Gal}(E_{k+1}/F), \mathrm{Maps}(V_{E_{k+1}}, A))$ then*

$$\mathrm{AWES}_k^1(\beta) \in Z^1(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, A^{\mathrm{Gal}(E_{k+1}/E_k)})).$$

Proof. **(1)** Recall that a priori $\mathrm{AWES}_k^1(\beta) : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$. By the previous lemma, for all $w \in V_{E_k}$, $\sigma \in \mathrm{Gal}(E_{k+1}/F)$ and $\tau \in \mathrm{Gal}(E_k/F)$, the quotient

$$\frac{\mathrm{AWES}_k^1(\beta)(\bar{\sigma})(\bar{\sigma}\tau \cdot w) \times \sigma(\mathrm{AWES}_k^1(\beta)(\tau)(\tau \cdot w))}{\mathrm{AWES}_k^1(\beta)(\bar{\sigma}\tau)(\bar{\sigma}\tau \cdot w)}$$

depends on σ only via its image $\bar{\sigma} \in \mathrm{Gal}(E_k/F)$. Taking $\sigma \in \mathrm{Gal}(E_{k+1}/E_k)$ shows $\mathrm{AWES}_k^1(\beta)(\tau)(\tau \cdot w)$ is invariant under $\mathrm{Gal}(E_{k+1}/E_k)$.

(2) This follows directly from the first point and a second application of the previous lemma. $\square$

We now establish the analogue of Lemma 3.1.6 for AWES_k^1 and AWES_k^2.

Lemma 4.3.4. *Let $k \geq 0$. Suppose that A is a commutative group.*

(1) *The map*

$$\{\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A) \mid \beta(1) = 1\} \to \{\beta : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A) \mid \beta(1) = 1\}$$

induced by AWES_k^1 *is surjective.*

(2) *Let $K \subset \bar{F}$ be a Galois extension of F containing E_{k+1}. The map*

$$\{\alpha : \mathrm{Gal}(K/F) \times \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A) \mid \textit{for all } \sigma \in \mathrm{Gal}(K/F),\ \alpha(\sigma, 1) = 1\}$$
$$\to \{\alpha : \mathrm{Gal}(K/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A) \mid \textit{for all } \sigma \in \mathrm{Gal}(K/F),\ \alpha(\sigma, 1) = 1\}$$

induced by AWES_k^2 *is surjective.*

Proof. As in the proof of Lemma 3.1.6, in each case we exhibit a subset of the source such that restricting to this subset yields a bijection. Choose a section $s : \mathrm{Gal}(E_k/F) \to \mathrm{Gal}(E_{k+1}/F)$ such that $s(1) = 1$.

(1) Restrict to the set of $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$ such that for $n \in \mathrm{Gal}(E_{k+1}/E_k)$, $\sigma \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $u \in \{v\}_{E_{k+1}}$, $\beta(ns(\sigma))(ns(\sigma) \cdot u) = 1$ unless $n = 1$, $\sigma \neq 1$ and u belongs to the image of $\zeta_{k,v} : \{v\}_{E_k} \to \{v\}_{E_{k+1}}$.

(2) Restrict to the set of $\alpha : \mathrm{Gal}(K/F) \times \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$ such that for $\sigma \in \mathrm{Gal}(K/F)$, $n \in \mathrm{Gal}(E_{k+1}/E_k)$, $\tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $u \in \{v\}_{E_{k+1}}$, $\alpha(\sigma, ns(\tau))(\sigma ns(\tau) \cdot u) = 1$ unless $n = 1$, $\tau \neq 1$ and u belongs to the image of $\zeta_{k,v} : \{v\}_{E_k} \to \{v\}_{E_{k+1}}$. □

4.4. ***Tate cocycles.*** Recall that for every $k \geq 0$ the kernel $C(E_k)^1$ of the surjective norm map $\|\cdot\|_k : C(E_k) \to \mathbb{R}_{>0}$ is compact, and that these norm maps commute with the norm maps for the Galois action $N_{E_{k+1}/E_k} : C(E_{k+1}) \to C(E_k)$, that is $\|x\|_{k+1} = \|N_{E_{k+1}/E_k}(x)\|_k$ for all $x \in C(E_{k+1})$. In this section we will see the fundamental cocycles $\bar{\alpha}_k \in \mathbb{Z}^2(E_k/F, C(E_k))$ defined in Section 4.1 as taking values in $\mathrm{Maps}(V_{E_k}, C(E_k))$, by seeing elements of $C(E_k)$ as constant functions $V_{E_k} \to C(E_k)$.

Lemma 4.4.1. *There exists a family* $(\bar{\beta}_k^{(0)})_{k \geq 0}$, *where* $\bar{\beta}_k^{(0)} : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, C(E_k))$, *such that*:

(1) *For any $k \geq 0$ we have* $\bar{\alpha}_k/\overline{\alpha'_k} = \mathrm{d}(\bar{\beta}_k^{(0)})$, *where* $\overline{\alpha'_k} := \alpha'_k \bmod E_k^\times$.

(2) *For any $k \geq 0$ we have*

$$\mathrm{AWES}_k^1(\bar{\beta}_{k+1}^{(0)}) \in \mathrm{Maps}(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k))).$$

(3) *For any $k \geq 0$ we have* $\|\mathrm{AWES}_k^1(\bar{\beta}_{k+1}^{(0)})\|_k = \|\bar{\beta}_k^{(0)}\|_k$, *as functions* $\mathrm{Gal}(E_k/F) \times V_{E_k} \to \mathbb{R}_{>0}$.

Proof. For a given k, the existence of $\bar{\beta}_k^{(0)}$ satisfying the first condition is a consequence of compatibility between local and global fundamental classes; see [Tate 1966]. Note that if $\bar{\beta}_{k+1}^{(0)}$ is such that $\bar{\alpha}_{k+1}/\overline{\alpha'_{k+1}} = \mathrm{d}(\bar{\beta}_{k+1}^{(0)})$, then by Lemma 4.3.2

$$\mathrm{d}(\mathrm{AWES}_k^1(\bar{\beta}_{k+1}^{(0)})) = \mathrm{AWES}_k^2(\mathrm{d}(\bar{\beta}_{k+1}^{(0)})) = \mathrm{AWES}_k^2(\bar{\alpha}_{k+1})/\overline{\mathrm{AWES}_k^2(\alpha'_{k+1})} = \bar{\alpha}_k/\overline{\alpha'_k} \tag{4.4.1}$$

factors through $\mathrm{Gal}(E_k/F)^2$, and by Corollary 4.3.3 $\mathrm{AWES}^1_k(\bar\beta^{(0)}_{k+1})$ takes values in $\mathrm{Maps}(V_{E_k}, C(E_k))$. So the second condition in the lemma is a consequence of the first one.

Let us start with a family $(\bar\beta^{(0)}_k)_{k\geq 0}$ satisfying the first condition, and show that we can inductively multiply $\bar\beta^{(0)}_k$, $k \geq 1$, by a 1-coboundary so that the third condition is also satisfied. By (4.4.1) we know that

$$\mathrm{AWES}^1_k(\bar\beta^{(0)}_{k+1})/\bar\beta^{(0)}_k \in Z^1(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k)))$$

and by vanishing of $H^1(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k)))$ there exists $b_k : V_{E_k} \to C(E_k)$ such that $\mathrm{AWES}^1_k(\bar\beta^{(0)}_{k+1})/\bar\beta^{(0)}_k = \mathrm{d}(b_k)$. Choose $\tilde b_k : V_{E_{k+1}} \to C(E_{k+1})$ such that for any $w \in \{v\}_{E_k}$, we have $\|\tilde b_k(\zeta_{k,v}(w))\|_{k+1} = \|b_k(\tau \cdot \dot v_k)\|_k$. Equivalently, $\|\mathrm{AWES}^0_k(\tilde b_k)\|_k = \|b_k\|_k$. Substituting $\bar\beta^{(0)}_{k+1}/\,\mathrm{d}(\tilde b_k)$ for $\bar\beta^{(0)}_{k+1}$, the third condition becomes satisfied. □

Theorem 4.4.2. *There exists a family $(\beta_k)_{k\geq 0}$ with $\beta_k \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k, S_k)))$ such that*

1. *For any $k \geq 0$ we have $\bar\alpha_k/\overline{\alpha'_k} = \mathrm{d}(\overline{\beta_k})$, where $\overline{\beta_k} \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, C(E_k)))$ is the projection of β_k.*
2. *For any $k \geq 0$ we have $\mathrm{AWES}^1_k(\beta_{k+1}) = \beta_k$.*

Therefore, the family $(\alpha_k)_{k\geq 0}$ defined by $\alpha_k = \alpha'_k \times \mathrm{d}(\beta_k)$ is a family of Tate cocycles, compatible in the sense that $\mathrm{AWES}^2_k(\alpha_{k+1}) = \alpha_k$ for all $k \geq 0$.

Proof. Let $(\bar\beta^{(0)}_k)_{k\geq 0}$ be a family as in the previous Lemma. The space

$$X_k := \left\{\bar\beta_k : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, C(E_k)) \mid \|\bar\beta_k\|_k = \|\bar\beta^{(0)}_k\|_k \text{ and } \bar\alpha_k/\overline{\alpha'_k} = \mathrm{d}(\bar\beta_k)\right\}$$

is compact for the topology induced by the product topology on

$$\mathrm{Maps}(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k))) = \prod_{\mathrm{Gal}(E_k/F)\times V_{E_k}} C(E_k).$$

Moreover $\bar\beta^{(0)}_k \in X_k$. The inverse system $((X_k)_{k\geq 0}, (\mathrm{AWES}^1_k : X_{k+1} \to X_k)_{k\geq 0})$ consists of nonempty compact topological spaces and continuous maps between them; therefore $\varprojlim_{k\geq 0} X_k \neq \varnothing$. Choose $(\bar\beta_k)_k \in \varprojlim X_k$. Such a family satisfies the two conditions in the proposition, but note that $\bar\beta_k$ takes values in $C(E_k)$.

Let us inductively choose lifts β_k of $\bar\beta_k$ such that $\mathrm{AWES}^1_k(\beta_{k+1}) = \beta_k$. Note that this imposes $\beta_k(1) = 1$ for all k. Choose any β_0 lifting $\bar\beta_0$ such that $\beta_0(1) = 1$. Suppose that β_k is given. If β is any lift of $\bar\beta_{k+1}$ such that $\beta(1) = 1$, then $\beta_k/\,\mathrm{AWES}^1_k(\beta)$ is a mapping $\mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, \mathcal{O}(E_{k+1}, S_{k+1}))$. By Lemma 4.3.4, there exists $\nu : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, \mathcal{O}(E_{k+1}, S_{k+1}))$ such that $\nu(1) = 1$ and $\beta_k/\,\mathrm{AWES}^1_k(\beta) = \mathrm{AWES}^1_k(\nu)$, and we let $\beta_{k+1} = \beta \times \nu$. □

Remark 4.4.3. This result solves two problems at once:

1. Constructing a family of Tate cocycles $(\alpha_k)_{k\geq 0}$ compatible with respect to AWES^2_k, which will be useful to compare (generalized) Tate–Nakayama isomorphisms in the tower $(E_k)_{k\geq 0}$, by taking cup products (Lemma 5.2.1 and Proposition 5.2.3).

(2) Constructing a family $(\beta_k)_{k\geq 0}$ compatible with respect to AWES^1_k and realizing local-global compatibility, which will be useful to compare local and global (generalized) Tate–Nakayama isomorphisms (Lemmas 5.4.1 and 5.4.4 and Propositions 5.4.3 and 5.4.5).

The proof suggests that it is not possible to solve the first problem separately from the second. One can show that if families $(\alpha_{k,v})_{k\geq 0, v\in V}$, $(R_{k,v})_{k\geq 0, v\in V}$ and $(\bar{\alpha}_k)_{k\geq 0}$ as above are fixed, then $(\overline{\beta_k})_{k\geq 0}$ is determined up to

$$B^1(\mathrm{Gal}(\bar{F}/F), \varprojlim_{k\geq 0} C(E_k)^0)$$

where $C(E_k)^0$ is the connected component of 1 in $C(E_k)$, i.e., the closure of $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}$ in $C(E_k)$, where $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}$ is the connected component of 1 in $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times}$.

Note that while $\alpha_{k,v}$, α_k and $R_{k,v}$ can simply be chosen sequentially as k grows, the existence of a family $(\beta_k)_{k\geq 0}$ in Theorem 4.4.2 follows from a compactness argument. Let us give an alternative, constructive but more intricate argument for the existence of $(\beta_k)_{k\geq 0}$. For simplicity we assume that for any $k\geq 0$, E_{k+1} contains the narrow Hilbert class field of E_k, i.e., $N_{E_{k+1}/E_k}(C(E_{k+1}))$ is contained in the image of $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}\times\widehat{\mathcal{O}(E_k)}^{\times}$ in $C(E_k)$. This can be achieved by discarding some of the E_k. Choose $\bar{\beta}_1^{(0)}$ such that $\mathrm{d}(\bar{\beta}_1^{(0)})=\bar{\alpha}_1/\overline{\alpha_1'}$. Note that $\bar{\beta}_0^{(1)}:=\mathrm{AWES}^1_0(\bar{\beta}_1^{(0)})=1$. For good measure let $\beta_0^{(1)}=1$ and $\alpha_0=1$. We now proceed to inductively construct $\bar{\beta}_{k+1}^{(0)}$, $\beta_k^{(1)}$ and ϵ_{k-1} for $k\geq 1$, satisfying the following properties.

(1) $\bar{\beta}_{k+1}^{(0)}:\mathrm{Gal}(E_{k+1}/F)\to\mathrm{Maps}(V_{E_{k+1}}, C(E_{k+1}))$ is such that $\overline{\alpha_{k+1}'}\times\mathrm{d}(\bar{\beta}_{k+1}^{(0)})=\bar{\alpha}_{k+1}$.

(2) $\beta_k^{(1)}:\mathrm{Gal}(E_k/F)\to\mathrm{Maps}(V_{E_k}, I(E_k,S_k))$ is a lift of $\mathrm{AWES}^1_k(\bar{\beta}_{k+1}^{(0)})$ such that $\beta_k^{(1)}(1)=1$.

(3) $\epsilon_k\in\mathrm{Maps}(V_{E_k},\widehat{\mathcal{O}(E_k)}^{\times})$ is such that $\mathrm{AWES}^1_{k-1}(\beta_k^{(1)})=\beta_{k-1}^{(1)}\,\mathrm{d}(\epsilon_{k-1})$.

Let $k\geq 0$, assume that $\bar{\beta}_{k+1}^{(0)}$ and $\beta_k^{(1)}$ are constructed. First choose any $\bar{\beta}_{k+2}^{(0)}:\mathrm{Gal}(E_{k+2}/F)\to\mathrm{Maps}(V_{E_{k+2}}, C(E_{k+2}))$ such that $\overline{\alpha_{k+2}'}\times\mathrm{d}(\bar{\beta}_{k+2}^{(0)})=\bar{\alpha}_{k+2}$. As we saw in the proof of Lemma 4.4.1, there exists $\bar{z}_{k+1}\in\mathrm{Maps}(V_{E_{k+1}}, C(E_{k+1}))$ such that $\mathrm{AWES}^1_{k+1}(\bar{\beta}_{k+2}^{(0)})=\bar{\beta}_{k+1}^{(0)}\times\mathrm{d}(\bar{z}_{i+1})$. Applying AWES^1_k, we get

$$\mathrm{AWES}^1_k\circ\mathrm{AWES}^1_{k+1}(\bar{\beta}_{k+2}^{(0)})=\mathrm{AWES}^1_k(\bar{\beta}_{k+1}^{(0)})\times\mathrm{d}(\mathrm{AWES}^0_k(\bar{z}_{k+1}))$$

and we would like to let $\epsilon_k\in\mathrm{Maps}(V_{E_k},(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}\times\widehat{\mathcal{O}(E_k)}^{\times})$ be a lift of $\mathrm{AWES}^0_k(\bar{z}_{k+1})$, which exists thanks to the hypothesis that E_{k+1} contains the narrow Hilbert class field of E_k. This is not quite right, since we want $\epsilon_k\in\mathrm{Maps}(V_{E_k},\widehat{\mathcal{O}(E_k)}^{\times})$. By surjectivity of

$$\mathrm{AWES}^0_k\circ\mathrm{AWES}^0_{k+1}:\mathrm{Maps}(V_{E_{k+2}},(\mathbb{R}\otimes_{\mathbb{Q}} E_{k+2})^{\times,0})\to\mathrm{Maps}(V_{E_k},(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}),$$

we see that up to dividing $\bar{\beta}_{k+2}^{(0)}$ by an element of $B^1(\mathrm{Gal}(E_{k+2}/F),\mathrm{Maps}(V_{E_{k+2}},(\mathbb{R}\otimes_{\mathbb{Q}} E_{k+2})^{\times,0}))$, we can find $\epsilon_k\in\mathrm{Maps}(V_{E_k},\widehat{\mathcal{O}(E_k)}^{\times})$. Now let $\beta_k^{(2)}=\beta_k^{(1)}\times\mathrm{d}(\epsilon_k)$, and as we saw in the proof of Theorem 4.4.2, there exists $\beta_{k+1}^{(1)}:\mathrm{Gal}(E_{k+1}/F)\to\mathrm{Maps}(V_{E_{k+1}}, I(E_{k+1},S_{k+1}))$ a lift of $\mathrm{AWES}^1_{k+1}(\bar{\beta}_{k+2}^{(0)})$ such that $\beta_{k+1}^{(1)}(1)=1$ and $\mathrm{AWES}^1_k(\beta_{k+1}^{(1)})=\beta_k^{(2)}$. This concludes the construction of $(\bar{\beta}_{k+2}^{(0)},\beta_{k+1}^{(1)},\epsilon_k)$.

Define inductively $\beta_k^{(i+1)} = \mathrm{AWES}_k^1(\beta_{k+1}^{(i)})$ for $i \geq 0$. Then for all $i > k \geq 0$, we have

$$\beta_k^{(i+2-k)} = \beta_k^{(i+1-k)} \times \mathrm{d}(\mathrm{AWES}_k^0 \circ \cdots \circ \mathrm{AWES}_{i-1}^0(\epsilon_i))$$

and since $\mathrm{AWES}_k^0 \circ \cdots \circ \mathrm{AWES}_{i-1}^0(\epsilon_i) \in \mathrm{Maps}(V_{E_k}, N_{E_i/E_k}(\widehat{\mathcal{O}(E_i)}^{\times}))$, by the existence theorem in local class field theory and Krasner's lemma the sequences $(\beta_k^{(i)})_{i>0}$ converge and we can define $\beta_k = \lim_{i \to +\infty} \beta_k^{(i)}$.

5. Generalized Tate–Nakayama morphisms

In this section we will construct N-th roots of the cochains $(\alpha_{k,v})_{v \in V}$, α'_k, β_k and α_k for all $N \geq 1$ and $k \geq 0$. This is necessary to establish the global analogue of [Kaletha 2016, §4.5], i.e., to make explicit the morphism $\iota_{\dot{V}}$ of [Kaletha 2018, Theorem 3.7.3] for the tower $(E_k)_{k \geq 0}$, and to study the localization map, (3.19) there.

5.1. *Choice of N-th roots.*

Proposition 5.1.1. *For any $v \in V$, there exists a family $(\sqrt[N]{\alpha_{k,v}})_{N \geq 1, k \geq 0}$ where $\sqrt[N]{\alpha_{k,v}} : \mathrm{Gal}(E_{k,\dot{v}}/F_v)^2 \to \overline{F_v}^{\times}$ such that*

(1) *for all $k \geq 0$, $\sqrt[1]{\alpha_{k,v}} = \alpha_{k,v}$,*

(2) *for all $k \geq 0$ and $N, N' \geq 1$ such that N divides N', $\sqrt[N']{\alpha_{k,v}}^{N'/N} = \sqrt[N]{\alpha_{k,v}}$,*

(3) *for all $k \geq 0$ and $N \geq 1$, $\mathrm{AW}_{k,v}^2(\sqrt[N]{\alpha_{k+1,v}}) = \sqrt[N]{\alpha_{k,v}}$.*

Proof. Using Bézout identities, we see that it is enough to construct families $(\sqrt[\ell^m]{\alpha_{k,v}})_{m \geq 0, k \geq 0}$ for all primes ℓ. So fix a prime number ℓ. For a fixed $k \geq 0$, there exists a family $(\sqrt[\ell^m]{\alpha_{k,v}})_{m \geq 0}$ satisfying the first two conditions in the proposition, and such that for all $m \geq 0$ and $\sigma \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$, $\sqrt[\ell^m]{\alpha_{k,v}}(\sigma, 1) = 1$. If we choose two such families for k and $k+1$, the last condition might not be satisfied, i.e., for some $m \geq 1$ the obstruction

$$\frac{\mathrm{AW}_{k,v}^2(\sqrt[\ell^m]{\alpha_{k+1,v}})}{\sqrt[\ell^m]{\alpha_{k,v}}} : \mathrm{Gal}(E_{k+1,\dot{v}}/F_v) \times \mathrm{Gal}(E_{k,\dot{v}}/F_v) \to \mu_{\ell^m}$$

could be nontrivial. Note that the target is contained in μ_{ℓ^m} because $\mathrm{AW}_{k,v}^2(\alpha_{k+1,v}) = \alpha_{k,v}$. Recall that $\mathbb{Z}_\ell(1)$ is defined as $\varprojlim_{m \geq 0} \mu_{\ell^m}$. By the second condition these obstructions, as m varies, glue to give a mapping

$$\mathrm{Gal}(E_{k+1,\dot{v}}/F_v) \times \mathrm{Gal}(E_{k,\dot{v}}/F_v) \to \mathbb{Z}_\ell(1)$$

which maps any element of $\mathrm{Gal}(E_{k+1,\dot{v}}/F_v) \times \{1\}$ to 1. Applying Lemma 3.1.6 with $A = \mathbb{Z}_\ell(1)$, we obtain that $(\sqrt[\ell^m]{\alpha_{k+1,v}})_{m \geq 0}$ can be chosen so that $\mathrm{AW}_{k,v}^2(\sqrt[\ell^m]{\alpha_{k+1,v}}) = \sqrt[\ell^m]{\alpha_{k,v}}$ for all $m \geq 0$. □

Fix such a family for each $v \in V$. Recall from Section 4.2 the embedding $j_{k,v} : E_{k,\dot{v}}^\times \hookrightarrow I(E_k)$. We now want to extend to $j_{k,v} : \overline{F_v}^\times \hookrightarrow I(\overline{F})$. For $x \in \overline{F_v}^\times$, there exists $i \geq 0$ such that $x \in E_{k+i,\dot{v}}^\times$. Define

$$j_{k,v}(x) = \prod_{r \in R_{k+1,v} \cdots R_{k+i,v}} r(j_{k+i,v}(x)),$$

which does not depend on the choice of a big enough i. These extended embeddings $j_{k,v}$ also satisfy a compatibility formula similar to (4.2.1): for any $x \in \overline{F_v}^\times$ we have

$$j_{k,v}(x) = \prod_{r \in R_{k+1,v}} r(j_{k+1,v}(x)). \tag{5.1.1}$$

For $N \geq 1$ define $\sqrt[N]{\alpha'_k} : \mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(V_{E_k}, I(\overline{F}))$ by

$$\sqrt[N]{\alpha'_k}(r_1 \sigma r_2^{-1}, r_2 \tau r_3^{-1})(r_1 \cdot \dot{v}_k) = r_1(j_{k,v}(\sqrt[N]{\alpha_{k,v}}(\sigma, \tau)))$$

for $r_1, r_2, r_3 \in R'_{k,v}$ and $\sigma, \tau \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. Obviously $\sqrt[1]{\alpha'_k} = \alpha'_k$ and whenever N divides N', $\sqrt[N']{\alpha'_k}^{N'/N} = \sqrt[N]{\alpha'_k}$. By the same proof as Lemma 4.2.2, thanks to (5.1.1), we have

$$\mathrm{AWES}_k^2\big(\sqrt[N]{\alpha'_{k+1}}\big) = \sqrt[N]{\alpha'_k}.$$

Note that for any $k \geq 0$ and $v \in V$, there exists $i \geq 0$ such that $\sqrt[N]{\alpha_{k,v}}$ takes values in $E_{k+i,\dot{v}}^\times$ and so for any $w \in \{v\}_{E_k}$, $\sqrt[N]{\alpha'_k}(-,-)(w)$ takes values in $\mathbb{A}_{E_{k+i}}^\times$.

We now want to construct N-th roots $\sqrt[N]{\alpha_k}$ of the Tate classes α_k constructed in Section 4.4. For this it is necessary to take N-th roots of ideles, which may not be ideles. For S' a finite subset of V, let $\mathcal{I}(F, S') \subset \prod_{v \in V} (\overline{F} \otimes_F F_v)^\times$ be the set of families $(x_v)_v$ such that for any $v \notin S'$, there exists a finite Galois extension K/F unramified above v such that $x_v \in (\mathcal{O}_K \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v})^\times = \prod_{w|v} \mathcal{O}_{K_w}^\times$. Let $\mathcal{I}(F) = \varinjlim_{S'} \mathcal{I}(F, S')$. Recall (Theorem 4.4.2) that $\alpha_k : \mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(V_{E_k}, I(E_k))$ has the following properties:

- for all $\sigma, \tau \in \mathrm{Gal}(E_k/F)$ and $w_1, w_2 \in V_{E_k}$, $\alpha_k(\sigma, \tau)(w_1)/\alpha_k(\sigma, \tau)(w_2) \in E_k^\times$;
- for all $\sigma, \tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $w \in \{v\}_{E_k}$, $\alpha_k(\sigma, \tau)(w) \in I(E_k)$ is a unit away from $S_{k,E_k} \cup \{v\}_{E_k}$.

It is crucial for $\sqrt[N]{\alpha_k}$ to enjoy similar properties.

Proposition 5.1.2. *There exists a family* $(\sqrt[N]{\alpha_k})_{N \geq 1, k \geq 0}$ *where* $\sqrt[N]{\alpha_k} : \mathrm{Gal}(E_k/F)^2 \to \mathcal{I}(F)$ *such that*

(1) *for all* $k \geq 0$, $\sqrt[1]{\alpha_k} = \alpha_k$,

(2) *for all* $k \geq 0$ *and* $N, N' \geq 1$ *such that* N *divides* N', $\sqrt[N']{\alpha_k}^{N'/N} = \sqrt[N]{\alpha_k}$,

(3) *for all* $k \geq 0$ *and* $N \geq 1$, $\mathrm{AWES}_k^2(\sqrt[N]{\alpha_{k+1}}) = \sqrt[N]{\alpha_k}$,

(4) *for all* $k \geq 0$, $N \geq 1$, $\sigma, \tau \in \mathrm{Gal}(E_k/F)$ *and* $w_1, w_2 \in V_{E_k}$,

$$\sqrt[N]{\alpha_k}(\sigma, \tau)(w_1) / \sqrt[N]{\alpha_k}(\sigma, \tau)(w_2) \in \overline{F}^\times,$$

(5) *for all* $k \geq 0$, $N \geq 1$, $\sigma, \tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ *and* $w \in \{v\}_{E_k}$, $\sqrt[N]{\alpha_k}(\sigma, \tau)(w) \in \mathcal{I}(F, S_k \cup \{v\} \cup N)$.

Proof. It will be convenient to fix an archimedean place u of F, so that in particular $\dot{u}_k \in S_{k,E_k}$ for all $k \geq 0$. As in the proof of Proposition 5.1.1 it is enough to restrict to powers of a fixed prime ℓ.

First we show how to construct a family $(\sqrt[\ell^m]{\alpha_k})_{m\geq 0}$ for a fixed $k \geq 0$. For $m \geq 0$ and $\sigma, \tau \in \mathrm{Gal}(E_k/F)$ choose roots $\sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(\sigma\tau\cdot\dot{u}_k) \in \mathcal{I}(F, S_k \cup \ell)$ such that $\sqrt[\ell^{m+1}]{\alpha_k}(\sigma,\tau)(\sigma\tau\cdot\dot{u}_k)^{\ell} = \sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(\sigma\tau\cdot\dot{u}_k)$. We can further impose that $\sqrt[\ell^m]{\alpha_k}(\sigma,1)(\sigma\cdot\dot{u}_k) = 1$ for all $\sigma \in \mathrm{Gal}(E_k/F)$. Then choose, for $\sigma, \tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $w \in \{v\}_{E_k} \smallsetminus \{\sigma\tau\cdot\dot{u}_k\}$, ℓ^m-th roots of $\alpha_k(\sigma,\tau)(w)/\alpha_k(\sigma,\tau)(\sigma\tau\cdot\dot{u}_k)$ in $(\bar{F}_{S_k\cup\{v\}\cup\ell})^{\times}$, and define $\sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(w)$ as the products of these ℓ^m-th roots with $\sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(\sigma\tau\cdot\dot{u}_k)$. This can be done compatibly as m varies. Again we can impose $\sqrt[\ell^m]{\alpha_k}(\sigma,1)(w) = 1$ for all $\sigma \in \mathrm{Gal}(E_k/F)$. We obtain a family $(\sqrt[\ell^m]{\alpha_k})_{m\geq 0}$ satisfying all conditions in the proposition except for the third one.

The fact that these choices can be made compatibly as k varies, i.e., in such a way that the third condition is also satisfied, can be proved as in Proposition 5.1.1, using the fact that $\mathrm{AWES}^2_k(\alpha_{k+1}) = \alpha_k$ and Lemma 4.3.4 instead of Lemma 3.1.6. □

Fix a family $(\sqrt[N]{\alpha_k})_{N\geq 1,k\geq 0}$ as in the proposition. We want to compare $\sqrt[N]{\alpha'_k}$ and $\sqrt[N]{\alpha_k}$. Recall (Theorem 4.4.2) that $\alpha_k = \alpha'_k\, \mathrm{d}(\beta_k)$, where $\beta_k : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, I(E_k, S_k))$.

Proposition 5.1.3. *There exists a family* $(\sqrt[N]{\beta_k})_{N\geq 1,k\geq 0}$, *where*

$$\sqrt[N]{\beta_k} : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, \mathcal{I}(F, S_k \cup N))$$

such that

(1) *for all* $k \geq 0$, $\sqrt[1]{\beta_k} = \beta_k$,

(2) *for all* $k \geq 0$ *and* $N, N' \geq 1$ *such that* N *divides* N', $\sqrt[N']{\beta_k}^{N'/N} = \sqrt[N]{\beta_k}$,

(3) *for all* $k \geq 0$ *and* $N \geq 1$, $\mathrm{AWES}^1_k(\sqrt[N]{\beta_{k+1}}) = \sqrt[N]{\beta_k}$.

Proof. Only the third condition is nontrivial, and the proof proceeds as in Propositions 5.1.1 and 5.1.2. □

Fix a family $(\sqrt[N]{\beta_k})_{N\geq 1,k\geq 0}$ as in the proposition. Note that $\mathrm{d}(\sqrt[N]{\beta_k}) : \mathrm{Gal}(\bar{F}_{S_k\cup N}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, \mathcal{I}(F, S_k \cup N))$.

Definition 5.1.4. For $k \geq 0$ and $N \geq 1$, let

$$\delta_k(N) = \frac{\sqrt[N]{\alpha_k}}{\sqrt[N]{\alpha'_k}\ \mathrm{d}(\sqrt[N]{\beta_k})} : \mathrm{Gal}(\bar{F}_{S_k\cup N}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, \mathcal{I}(F)[N]),$$

where $\mathcal{I}(F)[N]$ is the subgroup of N-torsion elements in $\mathcal{I}(F)$.

By construction, we have:

- For all $k \geq 0$, $N \geq 1$ and $w \in V_{E_k}$, there exists a finite Galois extension K of F containing E_k such that $\delta_k(N)(w)$ factors through $\mathrm{Gal}(K/F) \times \mathrm{Gal}(E_k/F)$.
- For all $k \geq 0$, $N \geq 1$, $\sigma \in \mathrm{Gal}(\bar{F}_{S_k\cup N}/F)$, $\tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $w \in \{v\}_{E_k}$,
$$\delta_k(N)(\sigma,\tau)(w) \in \mathcal{I}(F, S_k \cup \{v\} \cup N)[N].$$
- For all $k \geq 0$ and $N, N' \geq 1$ such that N divides N', we have $\delta_k(N')^{N'/N} = \delta_k(N)$.
- For all $k \geq 0$ and $N \geq 1$, $\mathrm{AWES}^2_k(\delta_{k+1}(N)) = \delta_k(N)$.

5.2. *Generalized Tate–Nakayama morphism for the global tower.* Using the compatible families of cochains constructed in the previous section, we now want to recast several of Kaletha's constructions in cohomology, but for actual cochains. First we describe the extension $P_{\dot{V}} \to \mathcal{E}_{\dot{V}} \to \mathrm{Gal}(\overline{F}/F)$ explicitly as a projective limit of extensions $P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N) \to \mathrm{Gal}(\overline{F}_{S' \cup N}/F)$ constructed using $\sqrt[N]{\alpha_k}$, for varying k, S', N. This is the global analogue of [Kaletha 2016, §4.5]. Then we make explicit the morphism $\iota_{\dot{V}}$ of [Kaletha 2018, Theorem 3.7.3] using this projective limit. To avoid repeating similar calculations we deduce these two constructions from Lemma 5.2.1 below.

Let us recall notation from [Kaletha 2018, Lemma 3.1.7]. Suppose that $S' \subset V$. If M is an abelian group, define $!_k : M[S'_{E_k}] \to M[S'_{E_{k+1}}]$ by $!_k(\Lambda)(\zeta_{k,v}(w)) = \Lambda(w)$ for $v \in S'$ and $w \in \{v\}_{E_k}$, and $!_k(\Lambda)(u) = 0$ if $u \notin \{\zeta_{k,v}(w) \mid v \in S',\ w \in \{v\}_{E_k}\}$. Here $\zeta_{k,v}$ is the section of the natural projection $\{v\}_{E_{k+1}} \to \{v\}_{E_k}$ defined in Section 4.2.

Recall also the notion of unbalanced cup product $\sqcup$ from [Kaletha 2016, §4.3].

Lemma 5.2.1. *Let T be a torus defined over F. Denote $Y = X_*(T)$. Let k be big enough so that E_k splits T. Let $N \geq 1$ be an integer. Let S' be a finite subset of V containing S_{k+1}. Let $\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}} = \widehat{Z}^{-1}(\mathrm{Gal}(E_k/F), Y[S'_{E_k}]_0)$. Then we have an equality of maps $\mathrm{Gal}(\overline{F}_{S' \cup N}/F) \to T(\mathcal{O}_{S' \cup N})$:*

$$\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda = \sqrt[N]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda).$$

Note that if $S_k \subset S'' \subset S'$ and the support of Λ is contained in S''_{E_k}, then the left hand side is inflated from a map $\mathrm{Gal}(\overline{F}_{S'' \cup N}/F) \to T(\mathcal{O}_{S'' \cup N})$.

Proof. For $\sigma \in \mathrm{Gal}(\overline{F}_{S' \cup N}/F)$ we have

$$\Big(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda\Big)(\sigma) = \prod_{\tau \in \mathrm{Gal}(E_k/F)} \sqrt[N]{\alpha_k}(\sigma, \tau) \otimes \sigma\tau(\Lambda) = \prod_{\tau \in \mathrm{Gal}(E_k/F)} \prod_{w \in S'_{E_k}} \sqrt[N]{\alpha_k}(\sigma, \tau)(w) \otimes \sigma\tau(\Lambda)(w).$$

Note that in this last expression, the tensor products land in $\mathcal{I}(F, S' \cup N) \otimes_{\mathbb{Z}} Y$, but the product over S'_{E_k} belongs to $\mathcal{O}^{\times}_{S' \cup N} \otimes_{\mathbb{Z}} Y = T(\mathcal{O}_{S' \cup N})$ because $\sum_{w \in S'_{E_k}} \Lambda(w) = 0$, using the third condition in Proposition 5.1.2. Compare with the pairing [Kaletha 2018, (3.24)]. Recall that $\sqrt[N]{\alpha_k} = \mathrm{AWES}^2_k(\sqrt[N]{\alpha_{k+1}})$ by construction in Theorem 4.4.2, so that

$$\Big(\sqrt[N]{\alpha_k} \underset{E_{k+1}/F}{\sqcup} \Lambda\Big)(\sigma) = \prod_{\tau \in \mathrm{Gal}(E_k/F)} \prod_{v \in S'} \prod_{w \in \{v\}_{E_k}} \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\alpha(\sigma, n\tilde{\tau})(\sigma_{k+1} n\tilde{\tau} \cdot \zeta_{k,v}(w))}{\alpha(\sigma, n)(\sigma_{k+1} n \cdot \zeta_{k,v}(\tau \cdot w))} \otimes \sigma\tau(\Lambda(w)),$$

where σ_{k+1} is the image of σ in $\mathrm{Gal}(E_{k+1}/F)$. We recognize $(\sqrt[N]{\alpha_{k+1}} \sqcup_{E_{k+1}/F} !_k(\Lambda))$ at the numerator, by writing the product over $\tau \in \mathrm{Gal}(E_k/F)$ and $n \in \mathrm{Gal}(E_{k+1}/E_k)$ as a product over $\tau' \in \mathrm{Gal}(E_{k+1}/F)$ with $\tau' = n\tilde{\tau}$. We obtain

$$\Big(\sqrt[N]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda)\Big)(\sigma) / \Big(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda\Big)(\sigma)$$
$$= \prod_{\tau \in \mathrm{Gal}(E_k/F)} \prod_{v \in S'} \prod_{w \in \{v\}_{E_k}} \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \sqrt[N]{\alpha_{k+1}}(\sigma, n)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w)) \otimes \sigma\tau(\Lambda(w)).$$

To simplify this expression we use the change of variable $u = \tau \cdot w$ to get

$$\prod_{\substack{v \in S' \\ n \in \mathrm{Gal}(E_{k+1}/E_k)}} \prod_{u \in \{v\}_{E_k}} \sqrt[N]{\alpha_{k+1}}(\sigma, n)(\sigma n \cdot \zeta_{k,v}(u)) \otimes \sigma\Bigg(\sum_{\tau \in \mathrm{Gal}(E_k/F)} \tau(\Lambda(\tau^{-1} \cdot u))\Bigg)$$

and the sum over τ vanishes since $N_{E_k/F}(\Lambda) = 0$ by assumption. □

Let $k \geq 0$ and $N \geq 1$, and let S' be a finite subset of V containing S_k. Recall the finite $\mathrm{Gal}(E_k/F)$-submodule $M(E_k, \dot{S}'_{E_k}, N)$ of $\mathrm{Maps}\big(\mathrm{Gal}(E_k/F) \times S'_{E_k}, \frac{1}{N}\mathbb{Z}/\mathbb{Z}\big)$ defined in [Kaletha 2018, §3.3], and the finite commutative algebraic group $P(E_k, \dot{S}'_{E_k}, N)$ such that $X^*(P(E_k, \dot{S}'_{E_k}, N)) = M(E_k, \dot{S}'_{E_k}, N)$. For any finite commutative algebraic group Z over F such that $\exp(Z) \mid N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$, we have an identification $\Psi(E_k, S', N) : \mathrm{Hom}(P(E_k, \dot{S}'_{E_k}, N), Z) \simeq A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$ (see Lemma 3.3.2 there). Recall also the 2-cocycle $\xi_k \in Z^2(\mathrm{Gal}(\bar{F}_{S' \cup N}/F), P(E_k, \dot{S}'_{E_k}, N))$ from (3.5) of the same work, defined using an unbalanced cup product:

$$\xi_k(S', N) = \mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} c_{\mathrm{univ}}(E_k, S', N) \tag{5.2.1}$$

where $c_{\mathrm{univ}}(E_k, S', N) \in M(E_k, \dot{S}'_{E_k}, N)^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$ is the image of $\mathrm{Id}_{P(E_k, \dot{S}'_{E_k}, N)}$ under $\Psi(E_k, S', N)$. Explicitly, for $w \in S'_{E_k}$ and $f \in M(E_k, \dot{S}'_{E_k}, N)$, $c_{\mathrm{univ}}(E_k, S', N)(w)(f) = f(1, w)$. The restriction of $\mathrm{d}(\sqrt[N]{\alpha_k})$ to S'_{E_k} is a 3-cocycle

$$\mathrm{Gal}(\bar{F}_{S' \cup N}/F) \times \mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(S'_{E_k}, \mathcal{I}(F, S' \cup N)[N])$$

such that

$$\frac{\mathrm{d}(\sqrt[N]{\alpha_k})(\sigma_1, \sigma_2, \sigma_3)(w_1)}{\mathrm{d}(\sqrt[N]{\alpha_k})(\sigma_1, \sigma_2, \sigma_3)(w_2)} \in \mu_N(\bar{F}) \subset \mathcal{I}(F, S' \cup N)[N].$$

This property allows us to pair $\mathrm{d}(\sqrt[N]{\alpha_k})(\sigma_1, \sigma_2, \sigma_3)$ with an element of $M(E_k, \dot{S}'_{E_k}, N)^\vee[\dot{S}'_{E_k}]_0$ to get an element of $P(E_k, \dot{S}'_{E_k}, N)$, as in [Kaletha 2018, Fact 3.2.3]. This is the pairing used in the definition of $\xi_k(S', N)$ (5.2.1). The 2-cocycle $\xi_k(S', N)$ is universal in the sense that for any morphism of algebraic groups $f : P(E_k, \dot{S}'_{E_k}, N) \to Z$ over F we have

$$f_*(\xi_k(S', N)) = \mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} \Psi(E_k, S', N)(f). \tag{5.2.2}$$

Definition 5.2.2. Let $k \geq 0$ and $N \geq 1$, and let S' be a finite subset of V containing S_k. Define $\mathcal{E}_k(S', N)$ as the central extension $P(E_k, \dot{S}'_{E_k}, N) \boxtimes_{\xi_k(S', N)} \mathrm{Gal}(\bar{F}_{S' \cup N}/F)$.

Recall that set-theoretically this is $P(E_k, \dot{S}'_{E_k}, N) \times \mathrm{Gal}(\bar{F}_{S' \cup N}/F)$, with group law

$$(x \boxtimes \sigma)(y \boxtimes \tau) = x\sigma(y)\xi_k(S', N)(\sigma, \tau) \boxtimes \sigma\tau.$$

Suppose $Z \hookrightarrow T$ is an injective morphism of commutative algebraic groups over F with Z finite, $\exp(Z) \mid N$ and T a torus split by E_k. Denote $A = X^*(Z)$, $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$, so that we have a short exact sequence $0 \to Y \to \bar{Y} \to A^\vee \to 0$. Recall from [Kaletha 2018, §3.7] the subgroup $\bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]$ of $\bar{Y}[S'_{E_k}]$ consisting of all elements whose image in $A^\vee[S'_{E_k}]$ is supported on $\dot{S}'_{E_k}$. Also let

$\bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0 = \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}] \cap \bar{Y}[S'_{E_k}]_0$ and $\bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}} = \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}] \cap \bar{Y}[S'_{E_k}]_0^{N_{E_k/F}}$. As shown in [Kaletha 2018, Proposition 3.7.8], we have a morphism

$$\begin{aligned}\iota_k(S', N) : \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}} &\to Z^1(P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N), Z \to T(\mathcal{O}_{S' \cup N}))\\ \Lambda &\mapsto (x \boxtimes \sigma \mapsto \Psi(E_k, S', N)^{-1}([\Lambda])(x) \times (\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} N\Lambda)(\sigma))\end{aligned}$$

where $[\Lambda]$ is the image of Λ in $A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$. As explained in the proof of [Kaletha 2018, Proposition 3.7.8], the fact that $\iota_k(S', N)(\Lambda)$ is a 1-cocycle is essentially equivalent to

$$\mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} N\Lambda = \mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} [\Lambda]. \tag{5.2.3}$$

Note that different pairings are used to form cup products in this equality: [Kaletha 2018, (3.24)] on the left, [Kaletha 2018, (3.3)] on the right. To be rigorous we should point out that Proposition 3.7.8 there is stated with additional assumptions on S', but it is easy to check that the first point in this proposition does not use these assumptions.

As N and S' vary, there are natural morphisms between the extensions $\mathcal{E}_k(S', N)$, compatible with $\iota_k(S', N)$. Verifying this is purely formal, so we omit this verification.

The more challenging and interesting compatibility is when k varies. This is the main goal of this paper, and we can finally harvest the fruit of our labor. Assume that S' also contains S_{k+1}. Recall [Kaletha 2018, (3.7)] the natural injection $M(E_k, \dot{S}'_{E_k}, N) \hookrightarrow M(E_{k+1}, \dot{S}'_{E_{k+1}}, N)$ mapping f to

$$(\sigma, w) \mapsto \begin{cases} f(\bar{\sigma}, \bar{w}) & \text{if } \sigma^{-1} \cdot w \in \dot{V}_{E_{k+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\bar{\sigma}$ (resp. $\bar{w}$) is the image of σ in $\mathrm{Gal}(E_k/F)$ (resp. V_{E_k}), and the dual surjective morphism $\rho_k(S', N) : P(E_{k+1}, \dot{S}'_{E_{k+1}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$.

It is formal to check that for any finite commutative algebraic group Z over F such that $\exp(Z) \mid N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$ and any finite $s' \subset V$, the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}(P(E_k, \dot{S}'_{E_k}, N), Z) & \xrightarrow{\Psi(E_k, S', N)} & A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}} \\ \downarrow{\scriptstyle \rho_k(S',N)^*} & & \downarrow{\scriptstyle !_k} \\ \mathrm{Hom}(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N), Z) & \xrightarrow{\Psi(E_{k+1}, S', N)} & A^\vee[\dot{S}'_{E_{k+1}}]_0^{N_{E_{k+1}/F}} \end{array} \tag{5.2.4}$$

Proposition 5.2.3. *Let $k \geq 0$ and $N \geq 1$, and let S' be a finite subset of V containing S_{k+1}.*

(1) *Composition with $\rho_k(S', N)$ maps $\xi_{k+1}(S', N)$ to $\xi_k(S', N)$. In particular, we have a natural surjective morphism of extensions*

$$\mathcal{E}_{k+1}(S', N) \to \mathcal{E}_k(S', N), \quad x \boxtimes \sigma \mapsto \rho_k(S', N)(x) \boxtimes \sigma.$$

(2) *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus split by E_k. Assume that* $\exp(Z) \mid N$. *Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. Then the following diagram commutes*

$$\begin{array}{ccc} \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}} & \xrightarrow{\iota_k(S',N)} & Z^1(P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N), Z \to T(\mathcal{O}_{S' \cup N})) \\ \downarrow !_k & & \downarrow \\ \bar{Y}[S'_{E_{k+1}}, \dot{S}'_{E_{k+1}}]_0^{N_{E_{k+1}/F}} & \xrightarrow{\iota_{k+1}(S',N)} & Z^1(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N) \to \mathcal{E}_{k+1}(S', N), Z \to T(\mathcal{O}_{S' \cup N})) \end{array}$$

where the right vertical map is the inflation map induced by the morphism of extensions defined above.

Proof. **(1)** We use an argument similar to the proof of [Kaletha 2018, Lemma 3.2.8]. We will apply Lemma 5.2.1. This way we avoid explicit computations with 3-cocycles $\mathrm{d}(\sqrt[N]{\alpha_k})$. Denote $Z = P(E_k, \dot{S}'_{E_k}, N)$ and $A = X^*(Z)$. Fix a surjective morphism $X \to A$ where X is a free $\mathbb{Z}[\mathrm{Gal}(E_k/F)]$-module, and let $\overline{X}$ be the kernel. Associated to $X, \overline{X}$ are tori $T, \overline{T}$ and a short exact sequence $1 \to Z \to T \to \overline{T} \to 1$. Let $Y = X_*(T) = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and $\bar{Y} = X_*(\overline{T}) = \mathrm{Hom}_{\mathbb{Z}}(\overline{X}, \mathbb{Z})$. We have a short exact sequence $0 \to Y[S'_{E_k}]_0 \to \bar{Y}[S'_{E_k}]_0 \to A^\vee[S'_{E_k}]_0 \to 0$, where $A = \mathrm{Hom}(X^*(Z), \mathbb{Q}/\mathbb{Z})$. The $\mathrm{Gal}(E_k/F)$-modules Y and $Y[S'_{E_k}]$ are cohomologically trivial (for Tate cohomology) and we have a short exact sequence $0 \to Y[S'_{E_k}]_0 \to Y[S'_{E_k}] \to Y \to 0$, therefore $Y[S'_{E_k}]_0$ is also cohomologically trivial. This implies in particular that there exists $\Lambda \in \bar{Y}[S'_{E_k}]_0^{N_{E_k/F}}$ mapping to the class of $c_{\mathrm{univ}}(E_k, S', N)$ in $A^\vee[S'_{E_k}]_0^{N_{E_k/F}} / I_{E_k/F}(A^\vee[S'_{E_k}]_0)$. Since $I_{E_k/F}(\bar{Y}[S'_{E_k}]_0)$ surjects to $I_{E_k/F}(A^\vee[S'_{E_k}]_0)$, we can even assume that the image $[\Lambda]$ of Λ in $A^\vee[S'_{E_k}]_0^{N_{E_k/F}}$ equals $c_{\mathrm{univ}}(E_k, S', N)$. Then $\Lambda \in \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, and applying Lemma 5.2.1 to $N\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}}$ and taking the coboundary, we obtain the identity between 2-cocycles taking values in Z

$$\mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} N\Lambda = \mathrm{d}(\sqrt[N]{\alpha_{k+1}}) \underset{E_{k+1}/F}{\sqcup} !_k(N\Lambda).$$

Using identity (5.2.3) on both sides, we obtain

$$\xi_k(S', N) = \mathrm{d}(\sqrt[N]{\alpha_{k+1}}) \underset{E_{k+1}/F}{\sqcup} [!_k(\Lambda)].$$

Moreover

$$[!_k(\Lambda)] = !_k([\Lambda]) = !_k(c_{\mathrm{univ}}(E_k, S', N)) = !_k(\Psi(E_k, S', N)(\mathrm{Id}_{P(E_k, \dot{S}'_{E_k}, N)}))$$

equals $\Psi(E_{k+1}, S', N)(\rho_k(S', N))$ by commutativity of diagram (5.2.4). Therefore

$$\xi_k(S', N) = \mathrm{d}(\sqrt[N]{\alpha_{k+1}}) \underset{E_{k+1}/F}{\sqcup} \Psi(E_{k+1}, S', N)(\rho_k(S', N)) = \rho_k(S', N)_*(\xi_{k+1}(S', N)).$$

(2) Let $\Lambda \in \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$. The inflation of $\iota_k(S', N)(\Lambda)$ is the element of

$$Z^1(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N) \to \mathcal{E}_{k+1}(S', N), Z \to T(\mathcal{O}_{S' \cup N}))$$

mapping $x \boxtimes \sigma \in \mathcal{E}_{k+1}(S', N)$ to

$$\Psi(E_k, S', N)^{-1}([\Lambda])(\rho_k(S', N)(x)) \times (\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} N\Lambda)(\sigma).$$

By (5.2.4) we have $\Psi(E_k, S', N)^{-1}([\Lambda]) \circ \rho_k(S', N) = \Psi(E_{k+1}, S', N)(!_k([\Lambda]))$ and moreover $!_k([\Lambda]) = [!_k(\Lambda)]$. The conclusion then follows from Lemma 5.2.1 applied to $N\Lambda$. ☐

Thanks to the first part of Proposition 5.2.3 and obvious compatibilities with respect to enlarging S' and replacing N by a multiple, we can now define the extension $P \to \mathcal{E}$ of $\mathrm{Gal}(\overline{F}/F)$ as the projective limit of the extensions $P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N)$ over triples (k, N, S') such that $S' \supset S_k$.

Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus. Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$, and denote

$$\bar{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F} = \varinjlim_{k,S'} \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}},$$

where the limit is over pairs k, S' such that E_k splits T and $S' \supset S_k$.

Corollary 5.2.4. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus. Let $\overline{T} = T/Z$ and let $Y = X_*(T)$, $\bar{Y} = X_*(\overline{T})$. Then the morphisms $(\iota_k(S', N))_{k,S',N}$, for k, S', N such that E_k splits T, $\exp(Z) \mid N$ and $S' \supset S_k$, splice into a morphism*

$$\iota : \bar{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F} \to Z^1(P \to \mathcal{E}, Z \to T(\overline{F})). \tag{5.2.5}$$

In Section 5.5 we will check that the class of the extension $P \to \mathcal{E}$ coincides with Kaletha's "canonical class" from [Kaletha 2018]. Granting this, it is clear that ι in (5.2.5) lifts the cohomological isomorphism $\iota_{\dot{V}}$ of Theorem 3.7.3 there.

5.3. ***Generalized Tate–Nakayama morphism for the local towers.*** In this section we fix $v \in V$. We want to study the relation of the map ι defined in Corollary 5.2.4 with the localization map loc_v defined in [Kaletha 2018, §3.6]. This will necessitate defining loc_v (for varying k, S', N) for cochains rather than in cohomology. The first step is to recall several constructions from [Kaletha 2016]. We choose notation similar to the global case instead of notation used there. For $k \geq 0$ and $N \geq 1$, we have a central extension

$$P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N) \to \mathrm{Gal}(\overline{F_v}/F_v),$$

where $P(E_{k,\dot{v}}, N) := \mathrm{Res}_{E_{k,\dot{v}}/F_v}(\mu_N)/\mu_N$. In particular, $M(E_{k,\dot{v}}, N) := X^*(P(E_{k,\dot{v}}, N))$ can be identified with $\mathbb{Z}/N\mathbb{Z}[\mathrm{Gal}(E_{k,\dot{v}}/F_v)]_0$. The central extension

$$\mathcal{E}_{k,v}(N) := P(E_{k,\dot{v}}, N) \underset{\xi_{k,v}(N)}{\boxtimes} \mathrm{Gal}(\overline{F_v}/F_v)$$

is defined using the 2-cocycle

$$\xi_{k,v}(N) := \mathrm{d}(\sqrt[N]{\alpha_{k,v}}) \underset{E_{k,\dot{v}}/F_v}{\sqcup} c_{\mathrm{univ}}(E_{k,\dot{v}}, N),$$

where $c_{\mathrm{univ}}(E_{k,\dot{v}}, N) \in X^*(P(E_{k,\dot{v}}, N))^\vee$ is killed by $N_{E_{k,\dot{v}}/F_v}$, and is defined as $f \mapsto f(1)$.

Suppose $Z \hookrightarrow T$ is an injective morphism of commutative algebraic groups over F_v with Z finite, $\exp(Z) \mid N$ and T a torus split by $E_{k,\dot{v}}$. Denote $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. We have a morphism

$$\begin{aligned}\iota_{k,v}(N) : \bar{Y}^{N_{E_{k,\dot{v}}/F_v}} &\to Z^1(P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N), Z \to T(\overline{F_v})) \\ \Lambda &\mapsto \big(x \boxtimes \sigma \mapsto \Psi(E_{k,\dot{v}}, N)^{-1}([\Lambda])(x) \times (\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} N\Lambda)(\sigma)\big)\end{aligned}$$

The following lemma and proposition, using formulations analogous to those of Lemma 5.2.1 and Proposition 5.2.3, are essentially proved in [Kaletha 2016, Lemmas 4.5 and 4.7]. Note that we have arranged for the 1-cochain denoted α_k in Lemma 4.5 there to be trivial. This slightly simplifies formulae. Then Kaletha's proof becomes a simpler analogue of that of Lemma 5.2.1, using $\mathrm{AW}_k^2(\sqrt[N]{\alpha_{k+1,v}}) = \sqrt[N]{\alpha_{k,v}}$ instead of $\mathrm{AWES}_k^2(\sqrt[N]{\alpha_{k+1}}) = \sqrt[N]{\alpha_k}$.

Lemma 5.3.1. *Let T be a torus defined over F_v. Denote $Y = X_*(T)$. Let k be big enough so that $E_{k,\dot{v}}$ splits T. Let $N \geq 1$ be an integer. Let $\Lambda \in Y^{N_{E_{k,\dot{v}}/F_v}}$. Then we have an equality of maps* $\mathrm{Gal}(\overline{F_v}/F_v) \to T(\overline{F_v})$:

$$\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} \Lambda = \sqrt[N]{\alpha_{k+1,v}} \underset{E_{k+1,\dot{v}}/F_v}{\sqcup} \Lambda.$$

As in the global case, there are natural morphisms $\rho_{k,v}(N) : P(E_{k+1,\dot{v}}, N) \to P(E_{k,\dot{v}}, N)$, denoted p in [Kaletha 2016, (3.2)]. There are also natural morphisms as N varies, which we do not bother to name. As in the global case (5.2.4), for any finite commutative algebraic group Z over F_v such that $\exp(Z) \mid N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(P(E_{k,\dot{v}}, N), Z) & \xrightarrow{\Psi(E_{k,\dot{v}},N)} & (A^\vee)^{N_{E_{k,\dot{v}}/F_v}} \\ \big\downarrow{\scriptstyle \rho_{k,v}(N)^*} & & \big\downarrow \\ \mathrm{Hom}(P(E_{k+1,\dot{v}}, N), Z) & \xrightarrow{\Psi(E_{k+1,\dot{v}},N)} & (A^\vee)^{N_{E_{k+1,\dot{v}}/F_v}} \end{array} \tag{5.3.1}$$

Proposition 5.3.2. *Let $k \geq 0$ and $N \geq 1$.*

(1) *Composition with $\rho_{k,v}(N)$ maps $\xi_{k+1,v}(N)$ to $\xi_{k,v}(N)$. In particular, we have a natural morphism of extensions*

$$\mathcal{E}_{k+1,v}(N) \to \mathcal{E}_{k,v}(N), \quad x \boxtimes \sigma \mapsto \rho_{k,v}(N)(x) \boxtimes \sigma.$$

(2) *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F_v with Z finite and T a torus split by $E_{k,\dot{v}}$. Assume that* $\exp(Z) \mid N$. *Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. Then the following diagram commutes*

$$\begin{array}{ccc} \bar{Y}^{N_{E_{k,\dot{v}}/F_v}} & \xrightarrow{\iota_{k,v}(N)} & Z^1(P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N), Z \to T(\overline{F_v})) \\ \Big\| & & \big\downarrow \\ \bar{Y}^{N_{E_{k+1,\dot{v}}/F_v}} & \xrightarrow{\iota_{k+1,v}(N)} & Z^1(P(E_{k+1,v}, N) \to \mathcal{E}_{k+1,v}(N), Z \to T(\overline{F_v})) \end{array}$$

where the right vertical map is inflation for the morphism of extensions defined above.

Proof. The proof is similar to that of Proposition 5.2.3, in fact slightly easier, so we omit it. □

Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F_v with Z finite and T a torus. Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. Denote $\bar{Y}^{N/F_v} = \bar{Y}^{N_{E_{k,\dot{v}}/F_v}}$ for any k such that $E_{k,\dot{v}}$ splits T.

Corollary 5.3.3. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F_v with Z finite and T a torus. Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. Then the morphisms $(\iota_{k,v}(N))_{k,N}$, for k, N such that $E_{k,\dot{v}}$ splits T and* $\exp(Z) \mid N$, *splice into a morphism*

$$\iota_v : \bar{Y}^{N/F_v} \to Z^1(P_v \to \mathcal{E}_v, Z \to T(\overline{F_v}))$$

lifting the morphism in cohomology of [Kaletha 2016, Theorem 4.8].

5.4. ***Localization.*** In this section fix $v \in V$. We want to study the relationship between ι (Corollary 5.2.4), ι_v (Corollary 5.3.3) and loc_v [Kaletha 2018, §3.6]. We study it for fixed $k \geq 0$ first.

Recall [Kaletha 2018, (3.11)] the morphisms $\mathrm{loc}_{k,v}(S', N) : P(E_{k,\dot{v}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$. If $v \in S'$ it is dual to $f \mapsto (\sigma \mapsto f(\sigma, \dot{v}))$. We define it to be trivial if $v \notin S'$. It is $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$-equivariant, and there are obvious commuting diagrams as S' and N vary.

For M a $\mathrm{Gal}(E_k/F)$-module, recall the morphism $l_{k,v} : M[S'_{E_k}]^{N_{E_k/F}} \to M^{N_{E_{k,\dot{v}}/F_v}}$ (denoted l^k_v in Lemma 3.7.2 there) defined by

$$l_{k,v}(\Lambda) = \sum_{r \in R'_{k,v}} r^{-1}(\Lambda(r \cdot \dot{v}_k))$$

if $v \in S'$, and zero otherwise.

Lemma 5.4.1. *Let T be a torus defined over F. Denote $Y = X_*(T)$. Let k be big enough so that E_k splits T. Let $N \geq 1$ be an integer. Let S' be a finite subset of V containing S_k. Let $\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}}$.*

Let $i \geq 0$ be big enough so that $\sqrt[N]{\alpha_{k,v}}$ takes values in $E^\times_{k+i,\dot{v}}$. Then we have an equality of maps $\mathrm{Gal}(\bar{F}/F) \to T(\bar{F} \otimes_F F_v)$:

$$\mathrm{pr}_v(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda) = \mathrm{ES}^1_{R'_{k+i,v}}(\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}(\Lambda)) \times \mathrm{d}(\mathrm{pr}_v(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda) \times (\mathrm{pr}_v(\delta_k(N)) \underset{E_k/F}{\sqcup} \Lambda).$$

In particular, upon restriction to $\mathrm{Gal}(\overline{F_v}/F_v)$ *and projection to* $T(\overline{F_v})$:

$$\mathrm{pr}_{\dot{v}}(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda) = (\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}(\Lambda)) \times \mathrm{d}(\mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda) \times (\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} \Lambda).$$

Note that the first equality implicitly uses the identification

$$\mathrm{Ind}_{\mathrm{Gal}(E_{k+i,\dot{v}}/F_v)}^{\mathrm{Gal}(E_{k+i}/F)}(E^\times_{k+i,\dot{v}}) \xrightarrow{\sim} (E_{k+i} \otimes_F F_v)^\times, \quad f \mapsto \prod_{g \in \mathrm{Gal}(E_{k+i,\dot{v}}/F_v)) \backslash \mathrm{Gal}(E_{k+i}/F)} g^{-1}(f(g))$$

to see $\mathrm{ES}^1_{R'_{k+i,v}}(\sqrt[N]{\alpha_{k,v}} \sqcup_{E_{k,\dot{v}}/F_v} l_{k,v}(\Lambda))$ as a map $\mathrm{Gal}(E_{k+i}/F) \to T(E_{k+i} \otimes_F F_v)$.

Proof. Recall that by definition of $\delta_k(N)$, we have $\sqrt[N]{\alpha_k} = \sqrt[N]{\alpha'_k}\, \mathrm{d}(\sqrt[N]{\beta_k})\delta_k(N)$, and we compute unbalanced cup products with these three terms separately. In the case of $\delta_k(N)$ there is nothing to prove, so we first consider $\mathrm{d}(\sqrt[N]{\beta_k})$. By [Kaletha 2016, Fact 4.3] we have

$$\mathrm{d}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda = \mathrm{d}(\sqrt[N]{\beta_k} \underset{E_k/F}{\sqcup} \Lambda)$$

and thus upon restriction to $\mathrm{Gal}(\overline{F_v}/F_v)$,

$$\mathrm{pr}_{\dot{v}}(\mathrm{d}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda) = \mathrm{d}(\mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k} \underset{E_k/F}{\sqcup} \Lambda)).$$

Let us now consider $\sqrt[N]{\alpha'_k}$. For $\sigma \in \mathrm{Gal}(E_k/F)$ we have

$$\mathrm{pr}_v\Big(\Big(\sqrt[N]{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda\Big)(\sigma)\Big) = \prod_{\gamma \in R'_{k,v}} \prod_{\tau \in \mathrm{Gal}(E_k/F)} \sqrt[N]{\alpha'_k}(\sigma, \tau)(\sigma\tau\gamma \cdot \dot{v}_k) \otimes \sigma\tau(\Lambda(\gamma \cdot \dot{v}_k)).$$

Write $\tau\gamma = r\tau'$ and $\sigma r = r'\sigma'$, where $r, r' \in R'_{k,v}$ and $\tau', \sigma' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$ are functions of (σ, γ, τ). For σ and γ fixed the map $\tau \mapsto (r, \tau')$ is bijective onto $R'_{k,v} \times \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. We obtain

$$\mathrm{pr}_v\Big(\Big(\sqrt[N]{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda\Big)(\sigma)\Big) = \prod_{\gamma \in R'_{k,v}} \prod_{r \in R'_{k,v}} \prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} \sqrt[N]{\alpha'_k}(r'\sigma' r^{-1}, r\tau'\gamma^{-1})(r' \cdot \dot{v}_k) \otimes r'\sigma'\tau'\gamma^{-1}(\Lambda(\gamma \cdot \dot{v}_k)),$$

where $r'\sigma' = \sigma r$, $r' \in R'_{k,v}$ and $\sigma' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$ being functions of r. Recall that by definition,

$$\sqrt[N]{\alpha'_k}(r'\sigma' r^{-1}, r\tau'\gamma^{-1})(r' \cdot \dot{v}_k) = r'(j_{k,v}(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau'))).$$

Therefore

$$\begin{aligned}\mathrm{pr}_v\Big(\Big(\sqrt[N]{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda\Big)(\sigma)\Big) &= \prod_{\gamma \in R'_{k,v}} \prod_{r \in R'_{k,v}} \prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} r'\big(j_{k,v}(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau')) \otimes \sigma'\tau'\gamma^{-1}(\Lambda(\gamma \cdot \dot{v}_k))\big) \\ &= \prod_{r \in R'_{k,v}} r'\Big(\prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} j_{k,v}(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau')) \otimes \sigma'\tau'(l_{k,v}(\Lambda))\Big).\end{aligned}$$

The map $r \mapsto r'$ from $R'_{k,v}$ to itself is bijective, so we can write this as

$$\prod_{r' \in R'_{k,v}} r'\Big(\prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} j_{k,v}(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau')) \otimes \sigma'\tau'(l_{k,v}(\Lambda))\Big),$$

where σ' depends on r' and is the unique element of $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$ such that $\sigma^{-1}r'\sigma' \in R'_{k,v}$. Choose $i \geq 0$ such that for any $\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$, $\sqrt[N]{\alpha_{k,v}}(\sigma', \tau') \in E^{\times}_{k+i,\dot{v}}$. Using (5.1.1) we obtain

$$\mathrm{pr}_v\Big(\Big(\sqrt[N]{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda\Big)(\sigma)\Big) = \prod_{r' \in R'_{k+i,v}} r'\Big(\prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} j_{k+i,v}(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau')) \otimes \sigma'\tau'(l_{k,v}(\Lambda))\Big)$$

and it is easy to check that this is equal to $\mathrm{ES}^1_{R'_{k+i,v}}(\sqrt[N]{\alpha_{k,v}} \sqcup_{E_k/F} l_{k,v}(\Lambda))(\sigma)$. ☐

It is formal to check that for any finite commutative algebraic group Z over F such that $\exp(Z) \mid N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$, and any finite set of places S' of F such that $S' \supset S_k$, the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}(P(E_k, \dot{S}'_{E_k}, N), Z) & \xrightarrow{\Psi(E_k,S',N)} & A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}} \\ \big\downarrow (\mathrm{loc}_{k,v}(S',N))^* & & \big\downarrow l_{k,v} \\ \mathrm{Hom}(P(E_{k,\dot{v}}, N), Z) & \xrightarrow{\Psi(E_{k,\dot{v}},S',N)} & (A^\vee)^{N_{E_{k,\dot{v}}/F_v}} \end{array} \tag{5.4.1}$$

Definition 5.4.2. For $k \geq 0$, $N \geq 1$ and S' a finite subset of V containing S_k, let $\eta_{k,v}(S', N) : \mathrm{Gal}(\overline{F_v}/F_v) \to P(E_k, \dot{S}'_{E_k}, N)$ be the restriction of $\mathrm{pr}_{\dot{v}}(\delta_k(N)) \sqcup_{E_k/F} c_{\mathrm{univ}}(E_k, S', N)$ to $\mathrm{Gal}(\overline{F_v}/F_v)$.

Proposition 5.4.3. *Let $k \geq 0$, $N \geq 1$ and S' a finite subset of V containing S_k.*

(1) *The restriction of the 2-cocycle $\xi_k(S', N)$ to $\mathrm{Gal}(\overline{F_v}/F_v)$ equals*

$$(\mathrm{loc}_{k,v}(S', N))_*(\xi_{k,v}(N)) \times \mathrm{d}(\eta_{k,v}(S', N))$$

and so the morphism $\mathrm{loc}_{k,v}(S', N) : P(E_{k,\dot{v}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$ can be extended to a morphism of extensions

$$\mathrm{loc}_{k,v}(S', N) : \mathcal{E}_{k,v}(N) \to \mathcal{E}_k(S', N), \quad x \boxtimes \sigma \mapsto \frac{\mathrm{loc}_{k,v}(S', N)(x)}{\eta_{k,v}(S', N)(\sigma)} \boxtimes \sigma.$$

(2) *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus split by E_k. Assume that $\exp(Z) \mid N$. Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. Then for any $\Lambda \in \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, the following identity holds in $Z^1(P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N), Z \to T(\overline{F_v}))$:*

$$\mathrm{pr}_{\dot{v}}(\iota_k(S', N)(\Lambda) \circ \mathrm{loc}_{k,v}(S', N)) = \iota_{k,v}(N)(l_{k,v}(\Lambda)) \times \mathrm{d}(\mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} N\Lambda). \tag{5.4.2}$$

Proof. The proof is similar to that of Proposition 5.2.3, and we will be more concise.

(1) Let $Z = P(E_k, \dot{S}'_{E_k}, N)$ and $A = X^*(Z)$. As in the proof of Proposition 5.2.3 we can find an embedding $Z \hookrightarrow T$ where T is a torus over F, split over E_k and such that $Y := X_*(T)$ is a free $\mathbb{Z}[\mathrm{Gal}(E_k/F)]$-module. Let $\bar{Y} = X_*(T/Z)$. There exists $\Lambda \in \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$ such that its image $[\Lambda]$ in $A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$ equals $c_{\mathrm{univ}}(E_k, S', N)$. Applying Lemma 5.4.1 to $N\Lambda \in Y$ and taking the coboundary, we obtain the identity between 2-cocycles $\mathrm{Gal}(\overline{F_v}/F_v)^2 \to T(\overline{F_v})$

$$\mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} N\Lambda = (\mathrm{d}(\sqrt[N]{\alpha_{k,v}}) \underset{E_{k,\dot{v}}/F_v}{\sqcup} N l_{k,v}(\Lambda)) \times \mathrm{d}(\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} N\Lambda).$$

Since $\mathrm{d}(\sqrt[N]{\alpha_k})^N = 1$, $\mathrm{d}(\sqrt[N]{\alpha_{k,v}})^N = 1$ and $\delta_k(N)^N = 1$ all three terms take values in $Z \subset T(\overline{F_v})$ and the equality can be written

$$\mathrm{d}(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} [\Lambda] = \big(\mathrm{d}(\sqrt[N]{\alpha_{k,v}}) \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}([\Lambda])\big) \times \mathrm{d}(\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} [\Lambda])$$

using the pairing $\mu_N \times A^\vee \to Z$. Using the fact that

$$l_{k,v}(c_{\mathrm{univ}(E_k,S',N)}) = \Psi(E_{k,\dot{v},S',N})(\mathrm{loc}_{k,v}(S',N))$$

thanks to (5.4.1), we obtain the desired equality.

(2) This is a direct consequence of Lemma 5.4.1 applied to $N\Lambda$, using also the commutative diagram (5.4.1) with $[\Lambda]$ in the top right corner. □

Lemma 5.4.4. *Let T be a torus defined over F. Denote $Y = X_*(T)$. Let k be big enough so that E_k splits T. Let $N \geq 1$ be an integer. Let S' be a finite subset of V containing S_{k+1}. Let $\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}}$. Then we have an equality of maps* $\mathrm{Gal}(\bar{F}_{S'\cup N}/F) \to Y \otimes_{\mathbb{Z}} \mathcal{I}(F, S' \cup N)[N]$:

$$\delta_k(N) \underset{E_k/F}{\sqcup} \Lambda = \delta_{k+1}(N) \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda) \tag{5.4.3}$$

and an equality in $Y \otimes_{\mathbb{Z}} \mathcal{I}(F, S' \cup N)$:

$$\sqrt[N]{\beta_k} \underset{E_k/F}{\sqcup} \Lambda = \sqrt[N]{\beta_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda). \tag{5.4.4}$$

Note that in (5.4.4) the left hand side belongs to $Y \otimes_{\mathbb{Z}} \mathcal{I}(F, S_k \cup N)$.

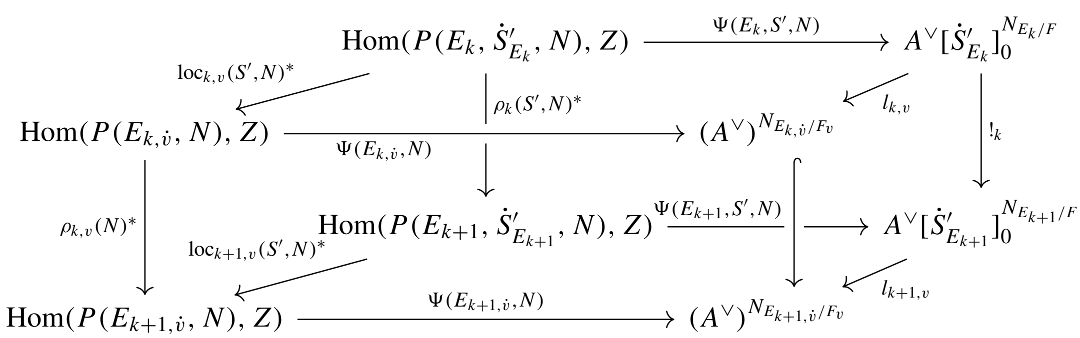
Proof. For (5.4.3) the proof is identical to that of Lemma 5.2.1. For (5.4.4) the proof is similar and easier, so we omit it. □

The localization maps $l_{k,v}$ are compatible with increasing k, i.e., $l_{k+1,v} \circ !_k = l_{k,v}$. This is proved in [Kaletha 2018, Lemma 3.7.2]. Thus for any embedding $Z \hookrightarrow T$ of commutative algebraic groups over F with Z finite and T a torus, they splice into

$$l_v : \bar{Y}[V_{\bar{F}}, \dot{V}]_0^{N/F} \to \bar{Y}^{N/F_v},$$

where $\bar{Y} = X_*(T/Z)$.

The localization morphisms $\mathrm{loc}_{k,v}(S', N) : P(E_{k,\dot{v}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$ are also compatible with varying k. We formulate this compatibility, together with (5.2.4), (5.3.1) and (5.4.1), using a commutative cubic diagram below. For any finite commutative algebraic group Z over F such that $\exp(Z) \mid N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$, and any finite set of places S' of F such that $S' \supset S_{k+1}$, the following cubic diagram is commutative.



(5.4.5)

In fact the commutativity of the left face follows from the commutativity of the other faces and the fact that the morphisms Ψ are isomorphisms.

Proposition 5.4.5. (1) *For any $k \geq 0$, $N \geq 1$ and S' a finite subset of V containing S_{k+1} we have $\eta_{k,v}(S', N) = \rho_k(S', N)_*(\eta_{k+1,v}(S', N))$, and a commutative diagram of central extensions*

$$\begin{array}{ccc} \mathcal{E}_{k+1,v}(N) & \xrightarrow{\mathrm{loc}_{k+1,v}(S',N)} & \mathcal{E}_{k+1}(S', N) \\ \downarrow & & \downarrow \\ \mathcal{E}_{k,v}(N) & \xrightarrow{\mathrm{loc}_{k,v}(S',N)} & \mathcal{E}_k(S', N) \end{array} \tag{5.4.6}$$

Therefore as k, S', N vary, the morphisms $\mathrm{loc}_{k,v}(S', N)$ yield $\mathrm{loc}_v : \mathcal{E}_v \to \mathcal{E}$.

(2) *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus. Let $Y = X_*(T)$ and $\bar{Y} = X_*(T/Z)$. Let $\Lambda \in \bar{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F}$. For k, S', N such that E_k splits T, $N \geq 1$ is divisible by $\exp(Z)$, S' contains S_k and Λ comes from an element $\Lambda_k \in \bar{Y}[S'_k, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, let $\kappa_v(\Lambda) = \mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k}) \sqcup_{E_k/F} N\Lambda_k \in T(\overline{F_v})$. As the notation suggests, it does not depend on the choice of k, S', N. Then the following identity holds in $Z^1(P_v \to \mathcal{E}_v, Z \to T(\overline{F_v}))$:*

$$\mathrm{pr}_{\dot{v}}(\iota(\Lambda) \circ \mathrm{loc}_v) = \iota_v(l_v(\Lambda)) \times \mathrm{d}(\kappa_v(\Lambda)). \tag{5.4.7}$$

Proof. **(1)** The equality $\eta_{k,v}(S', N) = \rho_k(S', N)_*(\eta_{k+1,v}(S', N))$ follows from (5.4.3) in Lemma 5.4.4, using the same argument as in the proof of Proposition 5.2.3. Commutativity of diagram (5.4.6) follows from this equality and the equality $\mathrm{loc}_{k,v}(S', N) \circ \rho_{k,v}(N) = \rho_k(S', N) \circ \mathrm{loc}_{k+1,v}(S', N)$, which is equivalent to commutativity of the left face of (5.4.5) for $Z = P(E_k, \dot{S}'_{E_k}, N)$.

(2) The fact that $\kappa_v(\Lambda)$ does not depend on the choice of k, S', N follows from (5.4.4) in Lemma 5.4.4, and (5.4.7) is (5.4.2) in Proposition 5.4.3. □

5.5. ***Comparison with Kaletha's canonical class.*** We follow the convention in [Kaletha 2018] and define, for a projective system $(Q_k)_{k\geq 0}$, $(Q_{k+1} \to Q_k)_{k\geq 0}$ of commutative algebraic groups over F and R a F-algebra, $(\varprojlim_k Q_k)(R) = \varprojlim_k Q_k(R)$. In particular

$$\varinjlim_{E/F \text{ finite}} ((\varprojlim_k Q_k)(E)) \to (\varprojlim_k Q_k)(\overline{F})$$

is not surjective in general. For $\mathrm{Gal}(\overline{F}/F)$- or $\mathrm{Gal}(\overline{F_v}/F_v)$-modules which arise naturally as projective limits (such as $Q(\overline{F})$, $Q(\overline{F_v})$ or $Q(\mathbb{A})$ for $Q = \varprojlim_k Q_k$ as above), we will only consider *continuous* cochains, for the topology on projective limits induced by the discrete topology on each term.

As in that work, we let $P = \varprojlim_{k,S',N} P(E_k, \dot{S}'_{E_k}, N)$. Each term $P(E_k, \dot{S}'_{E_k}, N)$ is finite, so that we can also simply consider the profinite $\mathrm{Gal}(\overline{F}/F)$-module $P(\overline{F})$, which equals $P(\overline{F_v})$ for any $v \in V$.

The 2-cocycles $\xi_k(S', N)$ are compatible by Proposition 5.2.3, and so we obtain a 2-cocycle $\xi \in Z^2(F, P)$ which corresponds to the extension $P \to \mathcal{E}$ of $\mathrm{Gal}(\overline{F}/F)$ introduced at the end of Section 5.2.

The goal of this section is to check that ξ represents the canonical class in $H^2(\mathrm{Gal}(\bar{F}/F), P)$ defined in [Kaletha 2018, §3.5], so that our $P \to \mathcal{E}$ is isomorphic to Kaletha's, canonically by Proposition 3.4.6 there.

As in §3.3 there, fix a cofinal sequence $(N_k)_{k\geq 0}$ in $\mathbb{Z}_{>0}$ (for the partial order defined by divisibility) with $N_0 = 1$ and such that for any $k \geq 0$, S_k contains all places dividing N_k (this is possible up to enlarging the finite sets S_k). To simplify notation we write $P_k = P(E_k, \dot{S}_{k,E_k}, N_k)$, $M_k = M(E_k, \dot{S}_{k,E_k}, N_k) = X^*(P_k)$, $\rho_k : P_{k+1} \twoheadrightarrow P_k$ and $c_{\mathrm{univ},k} = c_{\mathrm{univ}}(E_k, S_k, N_k)$.

First we need to go back to the construction of a resolution of P by pro-tori in [Kaletha 2018, Lemma 3.5.1].

Lemma 5.5.1. *There exists a family of resolutions, for $k \geq 0$,*

$$1 \to P_k \to T_k \to \overline{T}_k \to 1$$

of P_k by tori T_k, $\overline{T}_k$ defined over F and split by E_k, and morphisms $r_k : T_{k+1} \to T_k$ and $\bar{r}_k : \overline{T}_{k+1} \to \overline{T}_k$, such that

(1) *For all $k \geq 0$, the diagram*

$$\begin{array}{ccccc} P_{k+1} & \longrightarrow & T_{k+1} & \longrightarrow & \overline{T}_{k+1} \\ \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle r_k} & & \downarrow{\scriptstyle \bar{r}_k} \\ P_k & \longrightarrow & T_k & \longrightarrow & \overline{T}_k \end{array} \tag{5.5.1}$$

is commutative and r_k, $\bar{r}_k$ are surjective with connected kernels.

(2) *Letting $Y_k = X_*(T_k)$ and $\bar{Y}_k = X_*(\overline{T}_k)$, there exists a family $(\Lambda_k)_{k\geq 0}$ where $\Lambda_k \in \bar{Y}_k[S_{k,E_k}, \dot{S}_{k,E_k}]_0^{N_{E_k/F}}$ maps to $c_{\mathrm{univ},k} \in M_k^\vee[\dot{S}_{k,E_k}]_0^{N_{E_k/F}}$ and $!_k(\Lambda_k) = \bar{r}_k(\Lambda_{k+1})$ in $\bar{Y}_k[S_{k+1,E_{k+1}}, \dot{S}_{k+1,E_{k+1}}]_0^{N_{E_{k+1}/F}}$.*

Proof. For $k \geq 0$ let $X'_k = \mathbb{Z}[\mathrm{Gal}(E_k/F)][M_k]$, so that there is a canonical surjective map of $\mathbb{Z}[\mathrm{Gal}(E_k/F)]$-modules $X'_k \to M_k$. Let $X_0 = X'_0$, and for $k \geq 0$ let $X_{k+1} = X_k \oplus X'_{k+1}$. We have a natural surjective morphism $X_k \to M_k$, which for $k > 0$ is obtained as the sum of $X_{k-1} \to M_{k-1} \hookrightarrow M_k$ and $X'_k \to M_k$. Let T_k be the torus over F such that $X^*(T_k) = X_k$, and let $U_k = T_k/P_k$. Compared to the construction in [Kaletha 2018, Lemma 3.5.1], the only difference is that X'_{k+1} is free with basis M_{k+1} instead of $M_{k+1} \smallsetminus M_k$. Let $Y_k = X_*(T_k)$ and $\bar{Y}_k = X_*(U_k)$, so that we have an exact sequence

$$0 \to Y_k \to \bar{Y}_k \to M_k^\vee \to 0.$$

Let $\bar{X}'_k = \ker(X'_k \to M_k)$, $Y'_k = \mathrm{Hom}_{\mathbb{Z}}(X'_k, \mathbb{Z})$ and $\bar{Y}'_k = \mathrm{Hom}_{\mathbb{Z}}(\bar{X}'_k, \mathbb{Z})$ Since X'_k is a free $\mathbb{Z}[\mathrm{Gal}(E_k/F]$-module, using the same argument as in Proposition 5.2.3 we can find $\Upsilon_k \in \bar{Y}'_k[S_{k,E_k}, \dot{S}_{k,E_k}]_0^{N_{E_k/F}}$ mapping to $c_{\mathrm{univ},k}$. For all $k \geq 0$ we can identify $\bar{Y}_{k+1}$ with the group of $f \oplus g \in \bar{Y}_k \oplus \bar{Y}'_{k+1}$ such that $[f] = [g]$ in $M_k^\vee$. We use these identifications to construct Λ_k inductively from Υ_k. Let $\Lambda_0 = \Upsilon_0$, and for $k \geq 0$ let $\Lambda_{k+1} = !_k(\Lambda_k) \oplus \Upsilon_{k+1} \in (\bar{Y}_k \oplus \bar{Y}'_{k+1})[S_{k+1,E_{k+1}}, \dot{S}_{k+1,E_{k+1}}]_0^{N_{E_{k+1}/F}}$. Thanks to the equality $!_k(c_{\mathrm{univ},k}) = \rho_k(c_{\mathrm{univ},k+1})$, we have that $\Lambda_{k+1} \in \bar{Y}_{k+1}[S_{k+1,E_{k+1}}, \dot{S}_{k+1,E_{k+1}}]_0^{N_{E_{k+1}/F}}$. □

Let us now recall how Kaletha pins down the canonical class ξ in [Kaletha 2018, Proposition 3.5.2]. For $v \in V$, let $k_{0,v}$ be the minimal $k \geq 0$ such that $v \in S_k$. For $k \geq k_{0,v}$ let $P_{k,v} = P(E_{k,\dot{v}}, N_k)$. As in the global case $(\xi_{k,v})_{k \geq k_{0,v}}$ induce a continuous 2-cocycle $\xi_v \in Z^2(\mathrm{Gal}(\overline{F_v}/F_v, P_v)$ where $P_v = \varprojlim_k P_{k,v}$. Note that unlike in the global case, the cohomology class of ξ_v is simply characterized by the property that its image in $H^2(\mathrm{Gal}(\overline{F_v}/F_v), P_{k,v})$ is that of $\xi_{k,v}$ for every $k \geq k_{0,v}$. Uniqueness follows from vanishing of $\varinjlim^1_k H^1(\mathrm{Gal}(\overline{F_v}/F_v), P_{k,v})$.

For $v \in V$ denote $R'_v = (R'_{k,v})_{k \geq 0}$. Consider a projective system $(Q_k)_{k \geq 0}, (Q_{k+1} \to Q_k)_{k \geq 0}$ of commutative algebraic groups over F, and let $Q = \varprojlim_k Q_k$. The Eckmann–Shapiro maps, for $k, i, j \geq 0$,

$$\mathrm{ES}^j_{R'_{k+i,v}} : C^j(\mathrm{Gal}(E_{k+i,\dot{v}}/F_v), Q_k(E_{k+i,\dot{v}})) \to C^j(\mathrm{Gal}(E_{k+i}/F), Q_k(E_{k+i} \otimes_F F_v))$$

are compatible (for k fixed and varying i, and then also for varying k) and yield a pro-Eckmann–Shapiro map

$$\mathrm{ES}^j_{R'_v} : C^j(F_v, Q(\overline{F_v})) \to C^j(F, Q(\bar{F} \otimes_F F_v)).$$

This is explained in Appendix B of the same work, although notations differ: our set of *right* coset representatives $R'_{k,v}$ corresponds to the image of the composition in Lemma B.1, 1 there, by mapping $r \in R'_{k,v}$ to r^{-1}.

Define $x_k \in Z^2(\mathrm{Gal}(\bar{F}/F), P_k(\bar{\mathbb{A}}))$ by $x_k = \prod_{v \in S_k} \mathrm{ES}^2_{R'_v}(\mathrm{loc}_{k,v}(\xi_{k,v})) \in Z^2(\mathbb{A}, P_k)$. The family $(x_k)_{k \geq 0}$ is easily seen to be compatible and so it defines a continuous 2-cocycle $x \in Z^2(\mathrm{Gal}(\bar{F}/F), P(\bar{\mathbb{A}}))$. Kaletha checks that the class of x in $H^2(\mathrm{Gal}(\bar{F}/F), P(\bar{\mathbb{A}}))$ does not depend on the choice of sets of representatives $R_{k,v}$, nor does it depend on the choice of ξ_v in its cohomology class.

Kaletha shows [2018, Proposition 3.5.2] that there is a unique class $\mathrm{cl}(\xi_{\mathrm{can}}) \in H^2(\mathrm{Gal}(\bar{F}/F), P(\bar{F}))$ such that

(1) for any $k \geq 0$, the image of $\mathrm{cl}(\xi_{\mathrm{can}})$ in $H^2(F, P_k)$ is $\mathrm{cl}(\xi_k)$;

(2) the image of $\mathrm{cl}(\xi_{\mathrm{can}})$ in $H^2(\mathbb{A}, T \to \overline{T})$ coincides with the image of $\mathrm{cl}(x)$.

Adelic cohomology groups of complexes of tori were defined and studied in [Kottwitz and Shelstad 1999, Appendix C], see [Kaletha 2018, §3.5] for the case of projective systems of complexes of tori satisfying a Mittag-Leffler condition. The class $\mathrm{cl}(\xi_{\mathrm{can}})$ does not depend on the choice of a suitable pro-resolution $T \to \overline{T}$ of P by pro-tori, but for the following proposition it will be convenient to use the pro-resolution introduced in Lemma 5.5.1.

Proposition 5.5.2. *The* 2*-cocycle* ξ *belongs to the canonical class* $\mathrm{cl}(\xi_{\mathrm{can}}) \in H^2(F, P)$ *defined in* [Kaletha 2018, Definition 3.5.4].

Proof. The first property above is obviously satisfied. The second property is equivalent to the existence of a compatible family $(a_k, b_k)_{k \geq 0}$ where $a_k \in C^1(F, T_k)$ and $b_k \in \overline{T}_k(\mathbb{A}_{\bar{F}})$ are such that $\overline{a_k} = \mathrm{d}(b_k)$ in $C^1(\mathbb{A}, \overline{T}_k)$ and

$$\xi_k = \prod_{v \in S_k} \mathrm{ES}^2_{R'_{k+i,v}}(\mathrm{loc}_{k,v}(\xi_{k,v})) \times \mathrm{d}(a_k)$$

in $Z^2(\mathbb{A}, T_k)$, for $i \geq 0$ large enough.

By Lemma 5.4.1 and thanks to the fact that Λ_k has support in the finite set S_{k,E_k}, for $i \geq 0$ big enough we have

$$\sqrt[N_k]{\alpha'_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k = \prod_{v \in S_k} \mathrm{ES}^1_{R'_{k+i,v}}\big(\sqrt[N_k]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} N_k l_{k,v}(\Lambda_k)\big)$$

as maps $\mathrm{Gal}(E_k/F) \to T_k(\mathbb{A}_{E_{k+i}})$. Using an argument similar to the proof of Proposition 5.4.3, we deduce

$$\mathrm{d}\Big(\sqrt[N_k]{\alpha'_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k \Big) = \prod_{v \in S_k} \mathrm{ES}^2_{R'_{k+i,v}}\big(\mathrm{d}(\sqrt[N_k]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} N_k l_{k,v}(\Lambda_k))\big) = \prod_{v \in S_k} \mathrm{ES}^2_{R'_{k+i,v}}(\mathrm{loc}_{k,v}(\xi_{k,v}))$$

in $Z^2(\mathrm{Gal}(\overline{F}/F), \ker(T_k(\mathbb{A}_{\overline{F}}) \to \overline{T}_k(\mathbb{A}_{\overline{F}})))$. This leads us to define

$$a_k = \frac{\sqrt[N_k]{\alpha_k}}{\sqrt[N_k]{\alpha'_k}} \underset{E_k/F}{\sqcup} N_k \Lambda_k \in C^1(\mathrm{Gal}(E_k/F), T_k(\mathbb{A}_{E_{k+i}})).$$

Then

$$\overline{a_k} = \frac{\alpha_k}{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda_k = \mathrm{d}(b_k),$$

where $b_k = \beta_k \sqcup_{E_k/F} \Lambda_k \in \overline{T}(\mathbb{A}_{E_k})$.

The fact that $\bar{r}_k(b_{k+1}) = b_k$ for all $k \geq 0$ follows directly from (5.4.4) in Lemma 5.4.4. Using $\sqrt[N_{k+1}]{\alpha_k}^{N_{k+1}/N_k} = \sqrt[N_k]{\alpha_k}$ and Lemma 5.2.1 we find

$$\sqrt[N_k]{\alpha_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k = \sqrt[N_{k+1}]{\alpha_k} \underset{E_k/F}{\sqcup} N_{k+1} \Lambda_k = \sqrt[N_{k+1}]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} N_{k+1} !_k(\Lambda_k).$$

Lemma 5.2.1 also holds with $\sqrt[N]{\alpha_?}$ replaced by $\sqrt[N]{\alpha'_?}$ because this family also satisfies $\mathrm{AWES}^2_k\big(\sqrt[N]{\alpha'_{k+1}}\big) = \sqrt[N]{\alpha'_k}$, and so we similarly find

$$\sqrt[N_k]{\alpha'_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k = \sqrt[N_{k+1}]{\alpha'_{k+1}} \underset{E_{k+1}/F}{\sqcup} N_{k+1} !_k(\Lambda_k).$$

The fact that $r_k(a_{k+1}) = a_k$ for all $k \geq 0$ follows from these two equalities and $\bar{r}_k(\Lambda_{k+1}) = !_k(\Lambda_k)$ (Lemma 5.5.1). □

6. On ramification

6.1. ***A ramification property.*** We deduce a ramification property for Kaletha's generalized Galois cocycles from our explicit construction. Such a property is important to state Arthur's multiplicity formula in [Kaletha 2018, §4.5], namely to guarantee that the global adelic packets Π_φ are well defined; see Lemma 4.5.1 there.

Proposition 6.1.1. *Let G be a connected reductive group over F, and Z a finite central subgroup defined over F. For any $z \in Z^1(P \to \mathcal{E}, Z \to G)$, there exists a finite subset S' of V containing all archimedean places such that for any $v \in V \smallsetminus S'$, $\mathrm{pr}_{\dot{v}}(z \circ \mathrm{loc}_v)$ is unramified, i.e., inflated from an element of $Z^1(\mathrm{Gal}(K(v)/F_v), G(\mathcal{O}(K(v))))$ for some finite unramified extension $K(v)/F_v$.*

Proof. Let us first check that for $z' \in Z^1(P \to \mathcal{E}, Z \to G)$ in the same class as z, this ramification property holds for z if and only if it holds for z' (in general for distinct finite sets of places). There exists $g \in G(\overline{F})$ such that for any $w \in \mathcal{E}$, $z'(w) = g^{-1} z(w) w(g)$. Note that the action of $\mathcal{E}$ on $G(\overline{F})$ factors through $\mathrm{Gal}(\overline{F}/F)$. There exists a finite set $S'' \subset V$ containing all archimedean places and a finite Galois extension E/F unramified away from S'' such that $g \in G(\mathcal{O}(E, S''))$. Thus if z satisfies the ramification property for S', z' satisfies it for $S' \cup S''$.

Thanks to [Kaletha 2018, Lemma 3.6.2] it is enough to prove the statement in the case where G is a torus T. We remark that this reduction could force us to enlarge S'. As usual let $\bar{Y} = X_*(T/Z)$. Let $N = \exp(Z)$. There exists $k \geq 0$ such that E_k splits T and a finite $S' \subset V$ containing all places dividing N and S_k such that z is inflated from a unique element of $Z^1(P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N), Z \to T(\mathcal{O}_{S'}))$, which we also denote by z. By Proposition 3.7.8, 3 there, up to replacing z with a cohomologous cocycle we can assume that $z = \iota_k(S', N)(\Lambda)$ for some $\Lambda \in \bar{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, up to enlarging S' so that Conditions 3.3.1 there are satisfied.

For $v \in V \smallsetminus S'$, the morphism $\mathrm{loc}_{k,v}(S', N) : \mathcal{E}_{k,v}(N) \to \mathcal{E}_k(S', N)$ is trivial on $P(E_{k,\dot{v}}, N)$ and so it factors through $\mathrm{Gal}(\overline{F_v}/F_v)$. Thanks to ramification properties of $\delta_k(N)$ (see Definition 5.1.4) and by definition of $\eta_{k,v}(S', N)$ (see Definition 5.4.2), $\eta_{k,v}(S', N) : \mathrm{Gal}(\overline{F_v}/F_v) \to P(E_k, \dot{S}'_{E_k}, N)$ factors through $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v)$. By construction in Proposition 5.1.3, $\sqrt[N]{\beta_k}$ takes values in $\mathcal{I}(F, S_k \cup N)$. Thus by definition of $\kappa_v(\Lambda)$ in Proposition 5.4.5, $\kappa_v(\Lambda) \in T(\mathcal{O}(F_v^{\mathrm{nr}}))$. The equality (5.4.7) in Proposition 5.4.5, which is inflated from (5.4.2) in Proposition 5.4.3, shows that $\mathrm{pr}_{\dot{v}}(z \circ \mathrm{loc}_v)$ is unramified. □

Note that it does not seem possible to choose $K(v) = K_v$ for some finite extension K/F.

6.2. ***Alternative proof.*** As announced in the introduction to this paper, we now give an alternative proof of Proposition 6.1.1, which relies solely on Kaletha's definition of the canonical class, and not on constructions in the present paper.

Alternative proof of Proposition 6.1.1. For $v \in V$ temporarily let $\xi_v \in Z^2(\mathrm{Gal}(\overline{F_v}/F_v), P_v)$ be *any* element of $Z^2(\mathrm{Gal}(\overline{F_v}/F_v), P_v)$ representing the class defined in [Kaletha 2016]. Choose a tower of resolutions $(1 \to P_k \to T_k \to U_k \to 1)_{k \geq 0}$ as in [Kaletha 2018, Lemma 3.5.1], and as before write $T(\bar{\mathbb{A}}) = \varprojlim_k T_k(\bar{\mathbb{A}})$ and $U(\bar{\mathbb{A}}) = \varprojlim_k U_k(\bar{\mathbb{A}})$. Temporarily let ξ be *any* element of $Z^2(\mathrm{Gal}(\overline{F}/F), P)$ representing the canonical class defined in §3.5 there. Of course the 2-cocycles constructed in this paper are examples of elements of these cohomology classes, but we want to emphasize that the present proof does not require constructions in previous sections.

By definition of the canonical class there exists $a \in C^1(\mathbb{A}, T)$ and $b \in U(\bar{\mathbb{A}})$ such that

$$\xi = \prod_{v \in V} \mathrm{ES}^2_{R'_v}(\mathrm{loc}_v(\xi_v)) \times \mathrm{d}(a)$$

in $Z^2(\mathbb{A}, T)$ and $\bar{a} = \mathrm{d}(b)$ in $C^1(\mathbb{A}, U)$. In particular for any $v \in V$ we have

$$\mathrm{res}_v(\xi) = \mathrm{loc}_v(\xi_v) \times \mathrm{d}(a_v),$$

where res_v denotes restriction to $\mathrm{Gal}(\overline{F_v}/F_v)$ and $a_v = \mathrm{pr}_{\dot{v}}(\mathrm{res}_v(a))$. This equality holds in $Z^2(F_v, T)$, but ξ and $\mathrm{loc}_v(\xi_v)$ both take values in P. Let $b_v = \mathrm{pr}_{\dot{v}}(b)$, and choose a lift $\tilde{b}_v$ of b_v in $T(\overline{F_v})$. This is possible thanks to the surjectivity of all maps $P_{k+1} \to P_k$, by a simple diagram chasing argument (or more conceptually using vanishing of $\varprojlim_k^1 P_k$). Let $a'_v = a_v / \mathrm{d}(\tilde{b}_v)$. Then $a'_v \in C^1(F_v, P)$, and we have the equality

$$\mathrm{res}_v(\xi) = \mathrm{loc}_v(\xi_v) \times \mathrm{d}(a'_v) \quad \text{in } Z^2(F_v, P).$$

Fix $k \geq 0$. For $v \in V$ denote by $a_{k,v}$ (resp. $b_{k,v}$, $\tilde{b}_{k,v}$, $a'_{k,v}$) the image of a_v (resp. b_v, $\tilde{b}_v$, a'_v) in $C^1(F_v, T_k)$ (resp. $U_k(\overline{F_v})$, $T_k(\overline{F_v})$, $C^1(F_v, P_k)$). Let us check that there is a finite set S' of places of F such that for all $v \notin S'$, $a'_{k,v} \in C^1(F_v, P_k)$ is unramified. There exists a finite set $S' \supset S_k$ and a finite Galois extension K of F containing E_k, splitting T_k and unramified away from S' such that $a_k \in C^1(K/F, T_k(\mathbb{A}_K)_{S'})$ and $b_k \in U_k(\mathbb{A}_K)_{S'}$ where $T_k(\mathbb{A}_K)_{S'}$ is defined as $X_*(T_k) \otimes_{\mathbb{Z}} I(K, S')$. So for $v \notin S'$, $a_{k,v} \in C^1(K_{\dot{v}}/F_v, T_k(\mathcal{O}(K_{\dot{v}})))$ is unramified. The group $P_k = \ker(T_k \to U_k)$ is killed by N_k, and so there is a unique morphism $U_k \to T_k$ such that the composition $U_k \to T_k \to U_k$ is the N_k-power map. Thus for any $v \notin S'$, $\tilde{b}_{k,v} \in T_k(\mathcal{O}(K_{\dot{v}})^{(N_k)})$ where $\mathcal{O}(K_{\dot{v}})^{(N_k)}$ is the finite étale extension of $\mathcal{O}(K_{\dot{v}})$ obtained by adjoining all N_k-th roots of elements in $\mathcal{O}(K_{\dot{v}})^\times$. We conclude that for $v \notin S'$, $a'_{k,v} \in C^1(\mathrm{Gal}(\mathcal{O}(K_{\dot{v}})^{(N_k)}/\mathcal{O}(F_v)), P_k)$ and

$$\mathrm{res}_v(\xi_k) = \mathrm{d}(a'_{k,v}) \quad \text{in } Z^2(F_v, P_k),$$

where ξ_k is ξ composed with the surjection $P \to P_k$. This easily implies Proposition 6.1.1. □

Note that the fact that for a fixed k, $\mathrm{res}_v(\xi_k)$ is the coboundary of an unramified 1-cochain for almost all $v \in V$ is straightforward from the definition. What the proof above shows is that the cochain $a'_{k,v}$ coming from "infinite level", which is unique up multiplication by a 1-coboundary, is unramified for almost all $v \in V$.

6.3. *A noncanonical class failing the ramification property.*

Proposition 6.3.1. *Assume that $N_1 = 2$ and that S_1 is big enough so that P_1 is nontrivial. Then there exists $\xi^{\mathrm{bad}} \in Z^2(F, P)$ which coincides with the canonical class in $\varprojlim_k H^2(F, P_k)$ and such that for infinitely many places v of F, the 1-cochain $a_v \in C^1(F_v, P)$ such that $\mathrm{res}_v(\xi^{\mathrm{bad}}) = \mathrm{loc}_v(\xi_v)\, \mathrm{d}(a_v)$ is such that its image $a_{1,v} \in C^1(F_v, P_1)$ is ramified.*

Note that a_v is unique up to a 1-coboundary by [Kaletha 2018, Proposition 3.4.5], and so the property "$a_{1,v}$ is unramified" is well defined at all places $v \in V \smallsetminus S_1$.

Proof. Fix a tower of resolutions $(T_k \to U_k)_{k \geq 0}$ of P_k by tori as in §3.5 of that work, and denote by π_k the morphism $(T_{k+1} \to U_{k+1}) \to (T_k \to U_k)$. Recall (discussion before Proposition 3.5.2 in that work and [Weibel 1994, Theorem 3.5.8]) that for any $j \geq 0$ the following short sequences are exact:

$$1 \to \varprojlim_k{}^1 H^j(F, P_k) \to H^{j+1}(F, P) \to \varprojlim_k H^{j+1}(F, P_k) \to 1 \tag{6.3.1}$$

$$1 \to \varprojlim_k{}^1 H^j(\mathbb{A}, T_k \to U_k) \to H^{j+1}(\mathbb{A}, T \to U) \to \varprojlim_k H^{j+1}(\mathbb{A}, T_k \to U_k) \to 1.$$

For any $k \geq 0$ and $j \geq 0$ the natural map $H^j(F, P_k) \to H^j(F, T_k \to U_k)$ is an isomorphism because

$$1 \to P_k(\bar{F}) \to T_k(\bar{F}) \to U_k(\bar{F}) \to 1$$

is exact (whereas $T_k(\bar{\mathbb{A}}) \to U_k(\bar{\mathbb{A}})$ is not surjective in general). By the five lemma this implies that the first short exact sequence (6.3.1) is isomorphic to

$$1 \to \varprojlim_k{}^1 H^j(F, T_k \to U_k) \to H^{j+1}(F, T \to U) \to \varprojlim_k H^{j+1}(F, T_k \to U_k) \to 1.$$

One could also check that $H^j(F, P) \to H^j(F, T \to U)$ is an isomorphism more directly by manipulating cocycles.

By [Kaletha 2018, Lemma 3.5.3] the natural morphism

$$\varprojlim_k{}^1 H^1(F, P_k) \to \varprojlim_k{}^1 H^1(\mathbb{A}, T_k \to U_k) \tag{6.3.2}$$

is an isomorphism. So let us first define a nontrivial element of $\varprojlim_k{}^1 H^1(\mathbb{A}, T_k \to U_k)$. Choose, for any $k \geq 1$, a place $v_k \in V \smallsetminus S_1$ such that E_k/F is split above v_k and the v_k are distinct. For any $k \geq 1$, the tori T_k, U_k, T_1 and U_1 are split over F_{v_k}, and the surjective morphism of tori $U_k \to U_1$ splits over F_{v_k} since it has connected kernel. Therefore

$$H^1(F_{v_k}, P_k) = H^1(F_{v_k}, T_k \to U_k) \simeq U_k(F_{v_k})/T_k(F_{v_k})$$

maps onto

$$H^1(F_{v_k}, P_1) = H^1(F_{v_k}, T_1 \to U_1) \simeq U_1(F_{v_k})/T_1(F_{v_k}).$$

Since we have assumed $N_1 = 2$, over F_{v_k} the multiplicative group P_1 is isomorphic to μ_2^r for some $r > 1$. For each $k \geq 1$ let $c_{k,v_k} \in Z^1(F_{v_k}, P_k) \subset Z^1(F_{v_k}, T_k \to U_k)$ be such that its image in $H^1(F_{v_k}, P_1)$ is ramified. Recall that $H^1(\mathbb{A}, T_k \to U_k)$ decomposes as a restricted direct product over places in V [Kottwitz and Shelstad 1999, Lemma C.1.B]. Define $c_k \in Z^1(\mathbb{A}, T_k \to U_k)$ by

$$\mathrm{pr}_v(c_k) = \begin{cases} 1 & \text{if } v \neq v_k, \\ \mathrm{ES}^1_{R'_v}(c_{k,v_k}) & \text{if } v = v_k. \end{cases}$$

If $\tilde{c}_k \in C^1(\mathbb{A}, T \to U)$ lifts c_k, then $\mathrm{d}(\tilde{c}_k) \in Z^2(\mathbb{A}, T \to U)$ has trivial image in $Z^2(\mathbb{A}, T_k \to U_k)$. The family $(c_k)_{k\geq 1}$ defines an element of $\varprojlim^1_k H^1(\mathbb{A}, T_k \to U_k)$, whose image in $H^2(\mathbb{A}, T \to U)$ is the class of the convergent product $\prod_{k\geq 1} \mathrm{d}(\tilde{c}_k)$, for any choice of lifts $(\tilde{c}_k)_{k\geq 1}$. For simplicity we choose a lift $\tilde{c}_{k,v_k} \in C^1(F_v, T \to U)$ of c_{k,v_k} and define $\tilde{c}_k$ by

$$\mathrm{pr}_v(\tilde{c}_k) = \begin{cases} 1 & \text{if } v \neq v_k, \\ \mathrm{ES}^1_{R'_v}(\tilde{c}_{k,v_k}) & \text{if } v = v_k. \end{cases}$$

By surjectivity of (6.3.2), there exists a family $(b_k)_{k\geq 1}$ with $b_k \in Z^1(\mathbb{A}, T_k \to U_k)$ such that for every $k \geq 1$, the class of $c_k b_k/\pi_k(b_{k+1})$ belongs to the image of $H^1(F, T_k \to U_k) \to H^1(\mathbb{A}, T_k \to U_k)$. This means that there exists $e_k \in C^0(\mathbb{A}, T_k \to U_k) = T_k(\bar{\mathbb{A}})$ such that for every $k \geq 0$,

$$f_k := c_k \frac{b_k}{\pi_k(b_{k+1})} \, \mathrm{d}(e_k) \in Z^1(F, T_k \to U_k).$$

Choose lifts $\tilde{b}_k \in C^1(\mathbb{A}, T \to U)$ of b_k, $\tilde{e}_k \in C^0(\mathbb{A}, T \to U) = T(\bar{\mathbb{A}})$ of e_k, and $\tilde{f}_k \in C^1(F, T \to U)$ of f_k. Then

$$g_k := \tilde{c}_k \frac{\tilde{b}_k}{\tilde{b}_{k+1}} \, \mathrm{d}(\tilde{e}_k) \tilde{f}_k^{-1} \in C^1(\mathbb{A}, T \to U)$$

takes values in the complex

$$\ker(T(\bar{\mathbb{A}}) \to T_k(\bar{\mathbb{A}})) \to \ker(U(\bar{\mathbb{A}}) \to U_k(\bar{\mathbb{A}}))$$

and so $\prod_{k \geq 1} g_k$ is convergent in $C^1(\mathbb{A}, T \to U)$. Let $q = \prod_{k \geq 1} \mathrm{d}(\tilde{f}_k) \in Z^2(F, T \to U)$, which converges because f_k is a cocycle. In $Z^2(\mathbb{A}, T \to U)$ we have a factorization

$$q = \mathrm{d}(\tilde{b}_1) \times \left(\prod_{k \geq 1} \mathrm{d}(\tilde{c}_k) \right) \times \mathrm{d}\left(\prod_{k \geq 1} g_k^{-1} \right).$$

Moreover q defines a class in $H^2(F, T \to U) = H^2(F, P)$. Choose $a^{(1)} \in C^1(F, T \to U)$ such that $q \times \mathrm{d}(a^{(1)}) \in Z^2(F, P)$.

Let $\xi_{\mathrm{bad}} = \xi \times q \times \mathrm{d}(a^{(1)})$ in $Z^2(F, P)$, where $\xi \in Z^2(F, P)$ belongs to the canonical class. For any $v \in V$, by vanishing of $\varprojlim_k^1 H^1(F_v, P) = \varprojlim_k^1 H^1(F_v, T \to U)$ we know a priori that $\mathrm{res}_v(q)$ is the trivial class in $H^2(F_v, P)$. The point of the diagonal construction above is that we can write $\mathrm{res}_v(q)$ more explicitly as a coboundary. Let $a^{(2)} = \tilde{b}_1 \prod_{k \geq 1} g_k^{-1} \in C^1(\mathbb{A}, T \to U)$. Then for any place v, letting $a_v^{(2)} = \mathrm{pr}_{\dot{v}}(\mathrm{res}_v(a^{(2)}))$,

$$\mathrm{res}_v(q) = \begin{cases} \mathrm{d}(a_v^{(2)}) & \text{if } v \notin \{v_k \mid k \geq 1\}, \\ \mathrm{d}(a_v^{(2)} \times c_k^{(v)}) & \text{if } v = v_k. \end{cases}$$

Since ξ belongs to the canonical class, as in the alternative proof in Section 6.2 there exists $a^{(3)} \in C^1(\mathbb{A}, T \to U)$ such that for any place v, $\mathrm{res}_v(\xi) = \mathrm{loc}_v(\xi_v) \times \mathrm{d}(a_v^{(3)})$. Let $a = a^{(1)} a^{(2)} a^{(3)} \in C^1(\mathbb{A}, T \to U)$. Then for every place v, letting $a_v = \mathrm{pr}_{\dot{v}}(\mathrm{res}_v(a))$,

$$\mathrm{res}_v(\xi_{\mathrm{bad}}) / \mathrm{loc}_v(\xi_v) = \begin{cases} \mathrm{d}(a_v) & \text{if } v \notin \{v_k \mid k \geq 1\}, \\ \mathrm{d}(a_v \times c_k^{(v)}) & \text{if } v = v_k. \end{cases}$$

By the same argument as in Section 6.2, in this equality we can replace $a_v \in C^1(F_v, T \to U)$ by $a'_v \in C^1(F_v, P)$, and for almost all places v the image $a'_{1,v}$ of a'_v in $C^1(F_v, P_1)$ is unramified. We conclude that for almost all $k \geq 1$, $\mathrm{res}_{v_k}(\xi_{\mathrm{bad}}) / \mathrm{loc}_{v_k}(\xi_{v_k})$ is the coboundary of an element of $C^1(F_{v_k}, P)$ whose image in $C^1(F_{v_k}, P_1)$ is ramified. □

This example shows that for [Kaletha 2018, Lemma 4.5.1], it is important to use the canonical class and not an arbitrary lift in $H^2(F, P)$ of the canonical element of $\varprojlim_k H^2(F, P_k)$. More precisely, suppose that we form an extension $\mathcal{E}^{\mathrm{bad}}$ of $\mathrm{Gal}(\bar{F}/F)$ by P using a noncanonical class ξ^{bad} as above. Suppose that G is a reductive group that is an inner form of a quasisplit reductive group G^* over F. Realize G as a global rigid inner form (Ξ, z) of G^* with $z \in Z^1(P \to \mathcal{E}^{\text{bad}}, Z \to G^*)$ for some finite central subgroup Z of G^*. Let $k \geq 0$ be big enough so that

(1) G^* and G admit reductive models over $\mathcal{O}(F, S_k)$, that we fix,

(2) G^* admits a global Whittaker datum $\mathfrak{w}$ compatible with this model at all $v \notin S_k$ in the sense of [Casselman and Shalika 1980],

(3) the restriction of z to P factors through a morphism $P(E_k, \dot{S}'_{E_k}, N_k) \to Z$, and for any $v \notin S_k$ the localization $z_v \in Z^1(F_v, G^*)$ is cohomologically trivial.

It can happen that the set V^{bad} of finite places $v \notin S_k$ such that the conjugacy classes of hyperspecial maximal compact subgroups $G(\mathcal{O}_{F_v})$ and $G^*(\mathcal{O}_{F_v})$ are *not* conjugate under the trivialization of (Ξ_v, z_v) is infinite. Using Proposition 6.3.1 one can easily give such examples with $G^* = \mathrm{Sp}_{2n}$ for any $n \geq 1$. Suppose for simplicity that G^* is split and that for a finite place v of F there are exactly two conjugacy classes of hyperspecial maximal compact subgroups in $G^*(F_v)$, as is the case for $G^* = \mathrm{Sp}_{2n}$. Suppose that φ is a global discrete Langlands parameter for G and that for every place v of F, φ_v is relevant for G_{F_v}, i.e., that the local L-packet Π_{φ_v} is nonempty. Let V_φ^{bad} be the set of $v \in V^{\text{bad}}$ such that the local parameter φ_v is unramified and endoscopic, i.e., the centralizer of $\varphi(\mathrm{Frob}_v)$ in $\widehat{G}$ is not connected. For every such v, Π_{φ_v} has two elements and the base point of this set for the rigidifying datum $(G^*_{F_v}, \Xi_v, z_v, \mathfrak{w}_v)$ is *not* $G(\mathcal{O}_{F_v})$-spherical. If V_φ^{bad} is infinite, no element of the adelic L-packet considered in [Kaletha 2018, §4.5] is admissible, which is a problem to formulate a multiplicity formula for automorphic representations. In Example 6.3.2 below we point out that by [Elkies 1987] there are infinitely many examples of (unconditional substitutes for) global Langlands parameters φ such that φ_v is endoscopic for infinitely many v. We do not know if there are examples with V_φ^{bad} infinite, but Proposition 6.3.1 and Example 6.3.2 certainly justify caution.

Example 6.3.2. Consider first a prime number p and the group $\mathrm{SL}_2(\mathbb{Q}_p)$. There are two conjugacy classes of hyperspecial maximal compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$, represented by $K_1 = \mathrm{SL}_2(\mathbb{Z}_p)$ and its conjugate K_2 under $\mathrm{diag}(p, 1) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Therefore, for any Satake parameter $c = \mathrm{cl}(\mathrm{diag}(x, 1))$, a semisimple conjugacy class in $\mathrm{PGL}_2(\mathbb{C})$, a priori there are two associated unramified representations of $\mathrm{SL}_2(\mathbb{Q}_p)$, say $\pi_{1,x}, \pi_{2,x}$ such that $\dim_{\mathbb{C}} \pi_{i,x}^{K_i} = 1$. Let $T = \{\mathrm{diag}(t, t^{-1}) \mid t \in \mathbb{Q}_p^\times\}$, a maximal torus in $\mathrm{SL}_2(\mathbb{Q}_p)$, and χ_x the unramified character $\mathrm{diag}(t, t^{-1}) \mapsto x^{v_p(t)}$ of T, where v_p is the p-adic valuation such that $v_p(p) = 1$. Let B be a Borel subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ containing T. Then $\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi_x)$ is irreducible and isomorphic to $\pi_{1,x} \simeq \pi_{2,x}$ if $x \notin \{-1, p, p^{-1}\}$, whereas $\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi_{-1}) \simeq \pi_{1,-1} \oplus \pi_{2,-1}$ with $\pi_{1,-1} \not\simeq \pi_{2,-1}$. This is related to the fact that $\mathrm{diag}(-1, 1)$ is, up to conjugation, the only semisimple element of $\mathrm{PGL}_2(\mathbb{C})$ whose centralizer is not connected (it has two connected components).

Now let E be an elliptic curve over $\mathbb{Q}$. Let $f = \sum_{n \geq 1} a_n q^n$ be the associated [Breuil et al. 2001] newform. By [Elkies 1987] there are infinitely many primes p such that $a_p = 0$. In terms of the cuspidal automorphic representation $\pi = \bigotimes'_v \pi_v$ corresponding to f, this means that for infinitely many primes p, the Satake parameter of the unramified representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ (a semisimple conjugacy class in

$\mathrm{GL}_2(\mathbb{C})$) has trace zero. Equivalently, its image in $\mathrm{PGL}_2(\mathbb{C})$ is $\mathrm{cl}(\mathrm{diag}(-1, 1))$. Consider the *conjectural* associated Langlands parameter $\varphi_E : L_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ of π, where $L_{\mathbb{Q}}$ is the hypothetical Langlands group of $\mathbb{Q}$. Then its projection $\overline{\varphi_E}$ to $\mathrm{PGL}_2(\mathbb{C})$ is such that for infinitely many unramified primes p, $\overline{\varphi_E}(\mathrm{Frob}_p)$ is conjugated to $\mathrm{diag}(-1, 1)$.

This phenomenon has the following *unconditional* consequence. Let $\widetilde{G}$ be an inner form of $\mathrm{GL}_2/\mathbb{Q}$, i.e., the group of invertible elements of a central simple algebra of degree 2 over $\mathbb{Q}$. Assume that E is relevant for $\widetilde{G}$, i.e., that for any prime p such that $\widetilde{G}_{\mathbb{Q}_p}$ is not split, π_p is a twist of the Steinberg representation or a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. By the Jacquet–Langlands correspondence [Jacquet and Langlands 1970], there is a unique automorphic cuspidal representation π' for $\widetilde{G}$ corresponding to π. Let G be the derived subgroup of $\widetilde{G}$, an inner form of $\mathrm{SL}_2/\mathbb{Q}$. By [Labesse and Langlands 1979] and [Ramakrishnan 2000], the restriction of π' to $G(\mathbb{A})$ (at the real place, one should consider $(\mathfrak{g}, K)$-modules) embeds in the space of cuspidal automorphic forms for G. This restriction is admissible but has infinite length: for any prime $p > 3$ such that $G_{\mathbb{Q}_p}$ is split and E has good supersingular reduction, $\pi'_p|_{G(\mathbb{Q}_p)}$ has length 2.

Interestingly, the algorithm in [Elkies 1987] uses primes which do *not* split in certain quadratic extensions of $\mathbb{Q}$, while the counterexample in 6.3.1 is constructed using primes split in arbitrarily large extensions of the base field.

7. Effective localization

We conclude by explaining how the constructive proof of the existence of a family of "local-global compatibility" cochains $(\beta_k)_{k\geq 0}$ at the end of Section 4.4 allows one to explicitly compute all localizations of a global rigidifying datum, as promised in the introduction to this article.

7.1. ***A general procedure.*** Let G^* be a quasisplit connected reductive group over F. Fix a global Whittaker datum $\mathfrak{w}$ of G^*, i.e., choose a Borel subgroup B^* of G^* defined over F, let U be the unipotent radical of B^*, let χ be a generic unitary character of $U(\mathbb{A})/U(F)$, and let $\mathfrak{w}$ be the $G^*(F)$-conjugacy class of (B^*, χ). Let T a maximal torus of G^* defined over F, and E a finite Galois extension of F splitting T. Let S be a finite set of places of F such that

(1) S contains all archimedean places of F and all places of F which ramify in E, and the (always injective) morphism $I(E, S)/\mathcal{O}(E, S)^\times \to C(E)$ is surjective (i.e., $\mathrm{Pic}(\mathcal{O}(E, S)) = 1$).

(2) G^* admits a reductive model $\underline{G}^*$ over $\mathcal{O}(F, S)$ in the sense of [SGA 3_{III} 1970, Exposé XIX, Définition 2.7] such that the schematic closure $\underline{T}$ of T in $\underline{G}^*$, which is a flat group scheme over $\mathcal{O}(F, S)$ since this ring is Dedekind, is a torus in the sense of [SGA 3_{II} 1970, Exposé IX, Définition 1.3].

(3) For any $v \notin S$, the Whittaker datum $\mathfrak{w}$ is compatible with the $G^*(F_v)$-conjugacy class of the hyperspecial maximal compact subgroup $\underline{G}^*(\mathcal{O}(F_v))$, in the sense of [Casselman and Shalika 1980].

Let Z be a finite central subgroup of G, $N = \exp(Z)$ and $\overline{T} = T/Z$. Let $\underline{Z}$ be the schematic closure of Z in $\underline{T}$ (or $\underline{G}$), then $\underline{Z}$ is a group scheme of multiplicative type over $\mathcal{O}(F, S)$. Moreover $\underline{\overline{T}} := \underline{T}/\underline{Z}$ is a maximal torus of the reductive group scheme $\underline{G}^*/\underline{Z}$; see [SGA 3_{III} 1970, Exposé XXII, Corollaire 4.3.2].

Let $\dot{S}_E$ be a set of representatives for the action of $\mathrm{Gal}(E/F)$ on S_E. Finally, choose $\Lambda \in \overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}$. If

$$\alpha_{E/F} \in Z^2(\mathrm{Gal}(E/F), \mathrm{Hom}(\mathbb{Z}[S_E]_0, \mathcal{O}(E, S))^\times)$$

is any Tate cocycle (as in [Tate 1966]), then taking the cup product of $\alpha_{E/F}$ with Λ yields

$$\bar{z} \in Z^1\big(\mathrm{Gal}(\mathcal{O}(E, S)/\mathcal{O}(F, S)), \underline{\overline{T}}(\mathcal{O}(E, S))\big), \tag{7.1.1}$$

i.e., a Čech cocycle for the étale sheaf $\underline{\overline{T}}$ and the covering $\mathrm{Spec}(\mathcal{O}(E, S)) \to \mathrm{Spec}(\mathcal{O}(F, S))$. In particular we obtain a reductive group $\underline{G}$ over $\mathcal{O}(F, S)$ by twisting $\underline{G^*}$ with the image $\bar{\bar{z}}$ of $\bar{z}$ in

$$Z^1\big(\mathrm{Gal}(\mathcal{O}(E, S)/\mathcal{O}(F, S)), \underline{G^*}_{\mathrm{ad}}(\mathcal{O}(E, S))\big).$$

This realizes the generic fiber G of $\underline{G}$ as an inner twist $(\Xi, \bar{\bar{z}})$ of G^*.

Remark 7.1.1. The fact that any connected reductive group G over F arises in this way is a consequence of [Kaletha 2018, Lemmas A.1 and 3.6.1].

More directly, that is without making use of Lemma A.1 there, Steinberg's theorem on rational conjugacy classes in quasisplit semisimple simply connected algebraic groups [Steinberg 1965] implies that if we start with a reductive group G and a maximal torus T of G, then it can be realized as an inner twist $(G^*, \Xi, \bar{\bar{z}})$ with $\bar{\bar{z}}$ taking values in $\Xi^{-1}(T_{\mathrm{ad}}(\overline{F}))$.

We now use the constructive proof of Theorem 4.4.2 at the end of Section 4.4. Let $E_1 = E$ and $S_1 = S$ and choose a finite Galois extension E_2 of F which is totally complex and such that for every $v \in S$ nonarchimedean,

$$N_{E_2/E}\left(\prod_{w|v} \mathcal{O}(E_{2,w})^\times\right)$$

is contained in the subgroup of N-th powers in $\prod_{w|v} \mathcal{O}(E_w)^\times$. Finally, let E_3 be any finite Galois extension of F containing the Hilbert class field of E_2. Choose global fundamental classes $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ such that $\bar{\alpha}_k = \mathrm{AW}_k^2(\bar{\alpha}_{k+1})$ for $k \in \{1, 2\}$ and $\bar{\alpha}_3$ is normalized, i.e., $\bar{\alpha}_3(1, 1) = 1$. Fix finite sets of places $S_3 \supset S_2 \supset S$ as in Section 2. For each $v \in S_3$ fix a place $\dot{v}_3 \in S_{E_3}$. Choose local fundamental classes $\alpha_{k,v}$ for $v \in S$ and $k \in \{1, 2, 3\}$. Choose sets of representatives $(R_{k,v})_{1\le k\le 3, v\in S}$ as in Section 4.2, or rather, choose their image $\overline{R}_{k,v}$ in $\mathrm{Gal}(E_3/F)$. These families $(S_k)_{k\le 3}$, $(\bar{\alpha}_k)_{k\le 3}$, $(\alpha_{k,v})_{k\le 3, v\in S}$, $(\overline{R}_{k,v})_{k\le 3, v\in S}$ can be extended to $k \ge 0$ and $v \in V$, as explained in sections 4.1, 4.2 and 4.4. Moreover $\{\dot{v}_3\}_{v\in S}$ can be lifted and extended to yield $\dot{V}$ as in Section 2.

Now choose $\bar{\beta}_3^{(0)} : \mathrm{Gal}(E_3/F) \to \mathrm{Maps}(S_{E_3}, C(E_3))$ such that $\mathrm{d}(\bar{\beta}_3^{(0)}) = \bar{\alpha}_3/\overline{\alpha_3'}$. Choose $\beta_2^{(1)} : \mathrm{Gal}(E_2/F) \to \mathrm{Maps}(S_{E_2}, I(E_2, S_2))$ lifting $\mathrm{AWES}_2^1(\bar{\beta}_3^{(0)})$ such that $\beta_2^{(1)}(1) = 1$ and $\beta_1^{(2)} := \mathrm{AWES}_1^1(\beta_2^{(1)})$ takes values in $\mathrm{Maps}(S_{E_1}, I(E, S))$. Let $\alpha_1 = \alpha_1' \times \mathrm{d}(\beta_1^{(2)})$. At the end of Section 4.4 we constructed a family $(\beta_k)_{k\ge 0}$ such that there exists $\epsilon_2' \in \mathrm{Maps}(S_{E_2}, \widehat{\mathcal{O}(E_2)}^\times)$ satisfying $\beta_2|_{S_{E_2}} = \beta_2^{(1)} \times \mathrm{d}(\epsilon_2')$, more precisely ϵ'_2 is the restriction to S_{E_2} of

$$\lim_{n\to+\infty}\prod_{2\le i\le n}\mathrm{AWES}^0_2\circ\cdots\circ\mathrm{AWES}^0_{i-1}(\epsilon_i).$$

Therefore $\beta_1|_{S_E}=\mathrm{AWES}^1_1(\beta_2)=\beta_1^{(2)}\times\mathrm{d}(x)$ where $x=\mathrm{AWES}^0_1(\epsilon'_2)$ is a map

$$S_E\to N_{E_2/E}(\widehat{\mathcal{O}(E_2)}^{\times}).$$

In particular, for every nonarchimedean $v\in S$ there exists a map $y_v:S_E\to\prod_{w|v}\mathcal{O}(E_w)^\times$ such that $y_v^N=\mathrm{pr}_v(x)$. For $v\in S$ archimedean, simply let $y_v=1$. Recall that $N=\exp(Z)$. Going back to the construction of N'-th roots in Propositions 5.1.1, 5.1.2 and 5.1.3, we see that for any choice of N-th root $\sqrt[N]{\beta_1^{(2)}}:\mathrm{Gal}(E/F)\to\mathrm{Maps}(S_E,\mathcal{I}(E,S\cup N))$, we can choose the N-th root $\sqrt[N]{\beta_1}$ so that for all $v\in S$,

$$\mathrm{pr}_v(\sqrt[N]{\beta_1})|_{S_E}=\mathrm{pr}_v\Big(\sqrt[N]{\beta_1^{(2)}}\Big)\times\mathrm{d}(y_v).$$

If α_1 is chosen to form $\bar{z}$ in (7.1.1), the generic fiber G of $\underline{G}$ is endowed with a global rigidifying datum $(G^*,\Xi,z,\mathfrak{w})$ where $z=\iota(\Lambda)$. For $v\in V$, the localization of this rigidifying datum at v is $(G^*_{F_v},\Xi_v,z_v,\mathfrak{w}_v)$ where $\Xi_v=\Xi_{\overline{F_v}}$ and $z_v=\mathrm{pr}_{\dot{v}}(z\circ\mathrm{loc}_v)$.

Let $z'_v=\iota_v(l_v(\Lambda))$ and fix a rigid inner twist (G'_v,Ξ'_v) of $G^*_{F_v}$ by z'_v, which is well defined up to conjugation by $G'_v(F_v)$ (see [Kaletha 2016, Fact 5.1]). We now compare the rigid inner twists (G_{F_v},Ξ_v) and (G'_v,Ξ'_v) of $G^*_{F_v}$. Recall (Proposition 5.4.5) that

$$\mathrm{pr}_{\dot{v}}(z\circ\mathrm{loc}_v)=\iota_v(l_v(\Lambda))\times\mathrm{d}(\kappa_v(\Lambda)),$$

where $\kappa_v(\Lambda)=\mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_1})\underset{E/F}{\sqcup}N\Lambda\in T(\overline{F_v})$. Therefore we have an isomorphism of rigid inner twists of $G^*_{F_v}$

$$(f_v,\kappa_v(\Lambda)):(G_{F_v},\Xi_v,z_v)\xrightarrow{\sim}(G'_v,\Xi'_v,z'_v),$$

where f_v is obtained from $\Xi'_v\circ\mathrm{Ad}(\kappa_v(\Lambda))\circ\Xi_v^{-1}$ by Galois descent. Thus $f_v:G_{F_v}\simeq G^*_{F_v}$ identifies the rigidifying datum $(G^*_{F_v},\Xi_v,z_v,\mathfrak{w}_v)$ for G_{F_v} with the rigidifying datum $(G^*_{F_v},\Xi'_v,z'_v,\mathfrak{w}_v)$ for G'_v.

- For $v\in V\smallsetminus S$, $l_v(\Lambda_v)=0$ and we can simply take $G'_v=G^*_{F_v}$ and $\Xi'_v=\mathrm{Id}$. In particular G_{F_v} is quasisplit and we can simply take as rigidifying datum the pullback $f_v^*(\mathfrak{w}_v)$ of the Whittaker datum $\mathfrak{w}_v$. The image $\bar{\kappa}_v(\Lambda)$ of $\kappa_v(\Lambda)$ in $\overline{T}(\overline{F_v})$ equals

$$\mathrm{pr}_{\dot{v}}(\beta_1)\underset{E/F}{\sqcup}\Lambda\in\underline{\overline{T}}(\mathcal{O}(E_{\dot{v}}))$$

  and so $\mathrm{Ad}(\kappa_v(\Lambda))$ is an automorphism of the reductive group scheme $\underline{G^*}_{\mathcal{O}(E_{\dot{v}})}$. Since Ξ_v is obtained as the generic fiber of an isomorphism $\underline{G^*}_{\mathcal{O}(E_{\dot{v}})}\simeq\underline{G}_{\mathcal{O}(E_{\dot{v}})}$, we see that f_v descends from an isomorphism $\underline{G}_{\mathcal{O}(E_{\dot{v}})}\simeq\underline{G^*}_{\mathcal{O}(E_{\dot{v}})}$ and so f_v can be extended to an isomorphism of reductive models $\underline{G}_{\mathcal{O}(F_v)}\simeq\underline{G^*}_{\mathcal{O}(F_v)}$. This shows that $f_v^*(\mathfrak{w}_v)$ is compatible with the $G(F_v)$-conjugacy class of hyperspecial maximal compact subgroups represented by $\underline{G}(\mathcal{O}(F_v))$. Note that this holds even for $v\notin S$ dividing N.

- For $v \in S$, one can compute the element $\kappa_v(\Lambda)$ up to an element of $T(F_v)$, since

$$\mathrm{d}(y_v) \underset{E/F}{\sqcup} N\Lambda = N_{E/F}(y_v \underset{E/F}{\sqcup} N\Lambda) \in T(F_v),$$

and so $\mathrm{d}(\kappa_v(\Lambda)) = \mathrm{d}(\kappa_v'(\Lambda))$, where

$$\kappa_v'(\Lambda) = \mathrm{pr}_{\dot{v}}\left(\sqrt[N]{\beta_1^{(2)}}\right) \underset{E/F}{\sqcup} N\Lambda$$

is computable. Thus $(f_v, \kappa_v'(\Lambda))$ is also an isomorphism of rigid inner twists of $G^*_{F_v}$. Note that to compute f_v it is enough to compute the image of $\kappa_v'(\Lambda)$ in $\overline{T}(\overline{F_v})$, i.e.,

$$\mathrm{pr}_{\dot{v}}(\beta_1^{(2)}) \underset{E/F}{\sqcup} \Lambda \in \overline{T}(E_{\dot{v}}),$$

and so in practice it is not necessary to compute an N-th root of $\beta_1^{(2)}$.

7.2. ***A simple example.*** Let us illustrate this on a simple example, where almost no computation of cocycles is needed.

Definition of the group G. Let $F = \mathbb{Q}(s)$ with $s^2 = 3$. Let D be a quaternion algebra over F such that D is definite at both real places of F, and split at all nonarchimedean places of F. Let $N_D \in \mathrm{Sym}^2(D^*)$ be the reduced norm, and G the reductive group scheme over F defined by

$$G(R) = \{x \in R \otimes_F D \mid N_D(x) = 1 \text{ in } R\}$$

for any F-algebra R.

A reductive model of G. The class group of F is trivial, and the narrow class group of F is $\mathbb{Z}/2\mathbb{Z}$, corresponding to the totally complex and everywhere unramified extension $E = F(\zeta)$ of F, where $\zeta^2 - s\zeta + 1 = 0$ (ζ is a primitive 12-th root of unity). The class group of E is also trivial. Write σ for the nontrivial $\mathcal{O}(F)$-automorphism of $\mathcal{O}(E)$. Let S be the set of real places of F, so that $S = \{v_+, v_-\}$, where the image of s in $F_{v_+} = \mathbb{R}$ is positive. We still denote by v_+, v_- the unique complex places of E above v_+, v_-. The group $\mathcal{O}(E)^\times$ is generated by ζ and $\zeta - 1$, which has infinite order. The group $\mathcal{O}(F)^\times$ is generated by -1 and $2 - s = N_{E/F}(\zeta - 1)$, which has infinite order.

Let $\underline{G^*} = \mathrm{SL}_2$ over $\mathcal{O}(F)$ and let $\underline{T} \subset \underline{G^*}$ be the torus defined by

$$\underline{T}(R) = \left\{ \begin{pmatrix} x & -y \\ y & x + sy \end{pmatrix} \,\middle|\, x, y \in R,\ x^2 + sxy + y^2 = 1 \right\}$$

for any $\mathcal{O}(F)$-algebra R. Then $\underline{T}$ splits over $\mathcal{O}(E)$. Let $\underline{Z} \simeq \mu_2$ be the center of $\underline{G^*}$ and $\overline{\underline{T}} = \underline{T}/\underline{Z}$. The element $(x = s, y = -2) \in \underline{T}(\mathcal{O}_F)$ maps to the unique element of order 2 in $\overline{T}(F)$, and so we have a 1-cocycle

$$\bar{z} : \sigma \mapsto \overline{(x = s, y = -2)} \in \mathrm{PGL}_2(\mathcal{O}(F)).$$

Since PGL_2 is also the automorphism group of the matrix algebra M_2, we obtain an Azumaya algebra $\mathcal{O}(D)$ over $\mathcal{O}(F)$ by twisting $\mathrm{M}_2(\mathcal{O}(F))$ using $\bar{z}$. Explicitly, it has basis $(1, Z, I, ZI)$ over $\mathcal{O}(F)$, where

$$Z = \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 2\zeta - s \\ 2\zeta - s & 0 \end{pmatrix}.$$

We have $Z^{12} = 1$ and $I^2 = -1$. Let $D = F \otimes_{\mathcal{O}(F)} \mathcal{O}(D)$. Let $\underline{G}$ be the inner twist of $\underline{G^*}$ by $\bar{z}$, so that

$$\underline{G}(R) = \{x \in R \otimes_{\mathcal{O}(F)} \mathcal{O}(D) \mid N_D(x) = 1\}$$

for any $\mathcal{O}(F)$-algebra R.

The group G as a rigid inner twist. If we identify $Y = X_*(T)$ with $\mathbb{Z}$, then $\bar{Y} = X_*(\overline{T})$ is identified with $\frac{1}{2}\mathbb{Z}$. Let $\Lambda \in \bar{Y}[\dot{S}_E]_0^{N_{E/F}}$ be defined by $\Lambda(v_+) = \frac{1}{2}$ and $\Lambda(v_-) = -\frac{1}{2}$. An easy computation shows that one can choose the Tate cocycle α_1 for E/F such that

$$\alpha_1(\sigma, \sigma)(v_+)/\alpha_1(\sigma, \sigma)(v_-) = -1,$$

and so $\bar{z} = \alpha_1 \sqcup_{E/F} \Lambda$. Using $z = \iota(\Lambda)$, we obtain a realization of G as a rigid inner twist (Ξ, z) of G^*.

Choice(s) of Whittaker data. Let ψ be the unitary character of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ such that $\psi_\infty(x) = \exp(2i\pi x)$, so that for every prime p we have $\ker(\psi_p) = \mathbb{Z}_p$. Fortunately the different ideal of $F/\mathbb{Q}$ is principal, generated by $2s$, and so for any choice of sign the global Whittaker datum $\mathfrak{w}$ for G^*

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U(\mathbb{A}_F) \mapsto \psi\big(\pm \mathrm{Tr}_{F/\mathbb{Q}}(x/(2s))\big) \tag{7.2.1}$$

is compatible with the model $\underline{G^*}_{\mathcal{O}(F_v)}$ at *every* finite place v of F. Therefore the global rigidifying datum $\mathcal{D} = (G^*, \Xi, z, \mathfrak{w})$ for G is such that for any finite place v of F, the localization $\mathcal{D}_v$ is unramified and compatible with the $G(F_v)$-conjugacy class of hyperspecial maximal compact subgroups $\underline{G}(\mathcal{O}(F_v))$.

Real places. At any real place v of F, we could compute explicit coboundaries expressing local-global compatibility, but this is not necessary since the parametrization of Arthur–Langlands packets for the compact Lie groups $G(F_v) \simeq \mathrm{SU}(2)$ is simply determined by the Whittaker datum $\mathfrak{w}_v$ and the *cohomology class* of z_v in $H^1(P_v \to \mathcal{E}, Z \to T)$ (see [Kaletha 2016, §5.6] and [Taïbi 2015, §3.2]), which only depends on $l_v(\Lambda)$. This simplification is particular to anisotropic real groups, for which Langlands packets have at most one element.

In order to formulate the local Langlands correspondence at each real place v of F it is necessary to identify an algebraic closure of the base field F_v, occurring in the definition of the Weil group W_{F_v}, with the coefficient field $\mathbb{C}$. We have natural algebraic closures E_{v_+} and E_{v_-} of F_{v_+} and F_{v_-}. Choose $\tau_+ : \zeta \mapsto \exp(2i\pi/12)$ (resp. $\tau_- : \zeta \mapsto \exp(5 \times 2i\pi/12)$) identifying E_{v_+} (resp. E_{v_-}) with $\mathbb{C}$. There is a natural identification θ_+ (resp. θ_-) of $G^*_{F_{v_+}}$ (resp. $G^*_{F_{v_-}}$) with the usual split group SL_2 over $\mathbb{R}$, compatibly with the canonical isomorphisms $F_{v_+} = \mathbb{R}$ and $F_{v_-} = \mathbb{R}$. Let $\sqrt{3}$ be the positive square root of 3 in $\mathbb{R}$, so that $\tau_+(s) = \sqrt{3}$ and $\tau_-(s) = -\sqrt{3}$. In particular for any choice of sign in (7.2.1), the Whittaker data $(\theta_+)_*(\mathfrak{w}_{v_+})$ and $(\theta_-)_*(\mathfrak{w}_{v_-})$ differ. Associated to $\mathfrak{w}_+$ is a Borel subgroup B_+ of $G^*_{F_{v_+}} \times_{F_{v_+}} \mathbb{C}$ containing $T_{F_{v_+}} \times_{F_{v_+}} \mathbb{C}$

(see [Taïbi 2015]), corresponding to the generic discrete series representations of $G^*(F_{v_+})$. Using τ_+ we see B_+ as a Borel subgroup of $G^*_{E_{v_+}}$, and since T is defined over F and split over E we see that B_+ comes from a well-defined Borel subgroup of G^*_E containing T_E, which we still denote by B_+. Similarly, we have a Borel subgroup B_- of G^*_E containing T_E. Up to changing the sign in (7.2.1), we can assume that B_+ is such that the unique root of T_E in B_+ is $\alpha_+ : (x, y) \mapsto (x + \zeta y)^2$. Let us determine B_- using θ_+ and θ_-. For this we need to conjugate $\theta_+(T_{F_{v_+}})$ and $\theta_-(T_{F_{v_-}})$ by an element of $\mathrm{SL}_2(\mathbb{R})$. The matrix

$$g = \begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

conjugates $\theta_-(T_{F_{v_-}})$ into $\theta_+(T_{F_{v_+}})$, mapping $\theta_-(x, y)$ to $\theta_+(x-\sqrt{3}y, y)$. Since $(\theta_+)_*(\mathfrak{w}_+)$ and $(\theta_-)_*(\mathfrak{w}_-)$ differ, the root α_- of T_E in B_- is *not* equal to

$$(\tau_-)^{-1} \circ \tau_+ \circ \alpha_+ \circ \tau_+^{-1} \circ (\theta_+)_{\mathbb{C}}^{-1} \circ \mathrm{Ad}(g) \circ (\theta_-)_{\mathbb{C}} \circ \tau_-,$$

which equals α_+. Therefore $\alpha_- \neq \alpha_+$ and $B_- \neq B_+$. Note that other choices for τ_+, τ_- would lead to other Borel subgroups, and some choices would give equal Borel subgroups.

Let us now consider Arthur–Langlands packets of unitary representations of $G(F_{v_+})$ and $G(F_{v_-})$. We refer to [Taïbi 2015, §3.2.2] for the parametrization of "cohomological" Arthur–Langlands packets for inner forms of symplectic or special orthogonal groups, following Shelstad, Adams–Johnson and Kaletha. The present case is much simpler. Note also that since $G(F_{v_+})$ and $G(F_{v_-})$ are compact, any nonempty Arthur–Langlands packet is "cohomological", i.e., is a packet of Adams–Johnson representations. For $v \in \{v_+, v_-\}$ there is only one Arthur–Langlands parameter

$$W_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

which is nontrivial on $\mathrm{SL}_2(\mathbb{C})$ and yields a nonempty packet, namely the principal representation

$$\mathrm{SL}_2(\mathbb{C}) \to \widehat{G} \simeq \mathrm{PGL}_2(\mathbb{C}),$$

with corresponding packet containing the trivial representation with multiplicity one. Any other Arthur–Langlands parameter yielding a nonempty packet of representations is tempered and discrete, and so up to conjugation by $\widehat{G}$ it is of the form

$$\varphi_{k_+} : W_{F_{v_+}} \to \mathrm{PGL}_2(\mathbb{C}), \quad z \in E_{v_+}^\times \mapsto \begin{pmatrix} \tau_+(z/\bar{z})^{k_++1} & 0 \\ 0 & 1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for some $k_+ \in \mathbb{Z}_{\geq 0}$, and similarly discrete tempered parameters for $G_{F_{v_-}}$ are parametrized by integers $k_- \geq 0$, using τ_-. Above j is any element of $W_{F_{v_+}} \smallsetminus E_{v_+}^\times$ such that $j^2 = -1$. Note that we have put φ_{k_+} in dominant form for the upper-triangular Borel subgroup $\mathcal{B}$ of $\widehat{G}$. Using B_+ we have an identification between the group $\mathcal{T}$ of diagonal matrices in $\mathrm{PGL}_2(\mathbb{C})$ and $\widehat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$. So we can identify $l_{v_+}(\Lambda) = \Lambda(v_+) \in X_*(\overline{T})^{N_{E/F}}$ with an element of $X^*(\overline{\mathcal{T}})$, where $\overline{\mathcal{T}}$ is the preimage of $\mathcal{T}$ in $\widehat{\widehat{G}} = \mathrm{SL}_2(\mathbb{C})$.

The preimage $\mathcal{S}^+_{\varphi_{k_+}}$ of $\mathcal{S}_{\varphi_{k_+}} = \mathrm{Cent}(\varphi_{k_+}, \widehat{G})$ in $\widehat{\widehat{G}}$ has 4 elements and is generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \overline{\mathcal{T}}.$$

The class of $l_{v_+}(\Lambda)$ modulo $(1-\sigma)X_*(T)$ defines a character of $\mathcal{S}^+_{\varphi_{k_+}}$. There is a unique element π_{v_+,k_+} in the Arthur–Langlands packet attached to (the $\widehat{G}$-conjugacy class of) φ_{k_+}, that is the unique irreducible representation of $G(F_{v_+})$ in dimension $k_+ + 1$. The character $\langle \cdot, \pi_{v_+,k_+} \rangle$ of $\mathcal{S}^+_{\varphi_{k_+}}$ is the one defined by $l_{v_+}(\Lambda)$.

Similarly, each discrete series L-packet for $G_{F_{v_-}}$ has a unique element π_{v_-,k_-}, and a character $\langle \cdot, \pi_{v_-,k_-} \rangle$ of $\mathcal{S}^+_{\varphi_{k_-}}$ coming from the character $l_{v_-}(\Lambda) = \Lambda(v_-)$ of $\overline{\mathcal{T}}$. Note that since B_- and B_+ differ and $\Lambda(v_-) = -\Lambda(v_+)$, the characters of $\overline{\mathcal{T}}$ corresponding to $\Lambda(v_+)$ and $\Lambda(v_-)$ are equal.

Automorphic representations. To lighten notation we let $K = \underline{G}(\widehat{\mathcal{O}(F)})$. We can now formulate precisely the endoscopic decomposition of the space of $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$-finite functions on $G(F) \setminus G(\mathbb{A}_F)/K$, with commuting actions of $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$ and of the Hecke algebra in level K. Let V_+ (resp. V_-) be the irreducible representation of $G(F_{v_+})$ (resp. $G(F_{v_-})$) of dimension $k_+ + 1$ (resp. $k_- + 1$). Note that $V_\pm$ is obtained by restricting an irreducible algebraic representation of $G_{E_{v_\pm}}$. Recall [Gross 1999] that we can cut out the $V_+ \otimes V_-$-isotypical subspace inside the space of all automorphic forms for G, and define the space $M_{k_+,k_-}(K)$ of automorphic forms of weight (k_+, k_-) and level K as the space of $G(F)$-equivariant functions

$$G(\mathbb{A}_{F,f})/K \to V_+ \otimes V_-,$$

which is a finite-dimensional vector space over $\mathbb{C}$ endowed with a semisimple action of the commutative Hecke algebra in level K. Moreover it is easy to check that $M_{k_+,k_-}(K)$ has a natural E-structure.

The automorphic multiplicity formula for SL_2 and its inner forms was proved in [Labesse and Langlands 1979], although at the time there was no general definition of transfer factors, let alone Kaletha's normalization of transfer factors for inner forms. Formally we can use the main result of [Taïbi 2015], but of course a careful reading of [Labesse and Langlands 1979] and a comparison of transfer factors with the later definition in [Langlands and Shelstad 1987] and [Kaletha 2016], [Kaletha 2018] should give a more direct proof. In the present case, automorphic representations for G in level K fall into three categories:

- the trivial representation,
- representations corresponding to self-dual automorphic cuspidal representations of PGL_3/F which are algebraic regular at both infinite places and unramified at all finite places,
- representations "automorphically induced" from certain algebraic Hecke characters for E.

The multiplicity formula is nontrivial only in the third case. Making it explicit allows one to enumerate representations in the (most interesting) second case.

Global endoscopic parameters. Let $\chi : C(E) \to \mathbb{C}^\times$ be a continuous unitary character which is trivial on $C(F) = C(E)^{\mathrm{Gal}(E/F)}$. In particular, $\chi^\sigma = \chi^{-1}$. Using χ we can form the parameter

$$\varphi_\chi : W_{E/F} \to \mathrm{PGL}_2(\mathbb{C}), \quad z \in C(E) \mapsto \begin{pmatrix} \chi(z) & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\sigma} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\tilde{\sigma} \in W_{E/F}$ is any lift of $\sigma \in \mathrm{Gal}(E/F)$. The parameters φ_χ and $\varphi_{\chi^{-1}}$ are conjugated by $\mathrm{PGL}_2(\mathbb{C})$. We only consider characters χ such that the restriction of φ_χ to the Weil groups at both real places of F are discrete, i.e., we impose that $\chi_{v_+} = \chi|_{E_{v_+}^\times}$ and $\chi_{v_-} = \chi|_{E_{v_-}^\times}$ are nontrivial. Therefore there are $a_+, a_- \in \mathbb{Z} \smallsetminus \{0\}$ such that

$$\chi_{v_+}(z) = \tau_+(z/\bar{z})^{a_+}, \quad \chi_{v_-}(z) = \tau_-(z/\bar{z})^{a_-}.$$

Moreover we impose that χ is everywhere unramified, i.e., at every finite place w of E, χ_w is trivial on $\mathcal{O}(E_w)^\times$. Since E has class number 1 the map

$$E_{v_+}^\times \times E_{v_-}^\times \times \prod_{w \text{ finite}} \mathcal{O}(E_w)^\times \to C(E)$$

is surjective, and its kernel is $\mathcal{O}(E)^\times$. Thus for $a_+, a_- \in \mathbb{Z} \smallsetminus \{0\}$ there is at most one everywhere unramified χ as above, and there exists one if and only if $\chi_{v_+} \times \chi_{v_-}$ is trivial on $\mathcal{O}(E)^\times$, which is generated by ζ and $\zeta - 1$. A simple computation shows that this is equivalent to

$$a_+ + 5a_- = 0 \mod 12.$$

For such a character χ, at a finite place w of E we have:

- If w is fixed by σ (inert case), then there is a uniformizer $\varpi_w \in \mathcal{O}(F)$, and so χ_w is trivial.
- If w is not fixed by σ (split case), then if $\varpi_w \in \mathcal{O}(E)$ is a uniformizer, we have

$$\chi_w(\varpi_w) = \chi_{v_+}(\varpi_w)^{-1} \chi_{v_-}(\varpi_w)^{-1}.$$

This concludes the description of all endoscopic global parameters for G which are discrete at both real places and unramified at all finite places. They are parametrized by pairs $(a_+, a_-) \in (\mathbb{Z} \smallsetminus \{0\})^2$ such that $a_+ + 5a_- = 0 \bmod 12$, modulo $(a_+, a_-) \sim (-a_+, -a_-)$.

Let χ be a character as above. Then the centralizer $\mathcal{S}_{\varphi_\chi}$ of φ_χ is

$$\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathcal{T}$$

and so it coincides with the local centralizers at v_+, v_-. Up to replacing χ by χ^{-1}, we are in exactly one of the following cases:

- $a_+ > 0$ and $a_- > 0$, i.e., $\chi_{v_+}(z) = \tau_+(z/\bar{z})^{k_+ + 1}$ for $k_+ \geq 0$ and $\chi_{v_-}(z) = \tau_-(z/\bar{z})^{k_- + 1}$ for $k_- \geq 0$. Then $\langle \cdot, \pi_{v_+, k_+} \rangle \times \langle \cdot, \pi_{v_-, k_-} \rangle$ is the nontrivial character of $\mathcal{S}_{\varphi_\chi}$.
- $a_+ > 0$ and $a_- < 0$, i.e., $\chi_{v_+}(z) = \tau_+(z/\bar{z})^{k_+ + 1}$ for $k_+ \geq 0$ and $\chi_{v_-}(z) = \tau_-(z/\bar{z})^{-k_- - 1}$ for $k_- \geq 0$. Then $\langle \cdot, \pi_{v_+} \rangle \times \langle \cdot, \pi_{v_-} \rangle$ is the trivial character of $\mathcal{S}_{\varphi_\chi}$.

By the multiplicity formula, in weight (k_+, k_-) and level $\underline{G}(\widehat{\mathcal{O}(F)})$, there is at most one endoscopic automorphic representation, and there is one if and only if

$$(k_+ + 1) - 5(k_- + 1) = 0 \mod 12. \tag{7.2.2}$$

In low weight, we have computed Hecke operators for small primes and verified this condition.

Comments. The class number

$$\mathrm{card}(G(F) \setminus G(\mathbb{A}_{F,f}) / \underline{G}(\widehat{\mathcal{O}(F)})) = 1$$

as one can check when computing a Hecke operator at any finite place, by strong approximation. Note that $\underline{G}$ is *not* the only reductive model of G, even up to the action of $G_{\mathrm{ad}}(F)$. By splitting the Azumaya algebra $\mathcal{O}(D)$ modulo $(2) = (s-1)^2$, we can compute an $(s-1)$-Kneser neighbor of $\mathcal{O}(D)$, that is another maximal order $\mathcal{O}'(D)$ of D, having basis over $\mathcal{O}(F)$

$$1,\ Z + sI,\ (1-s)(s+ZI),\ (1-s)^{-1}(1+I+sZI).$$

It gives rise to a second model $\underline{G}'$ of G, which is not isomorphic to $\underline{G}$ since one can compute using reduction theory that $\underline{G}(\mathcal{O}_F)$ is a dihedral group of order 24 (generated by Z and I, with $IZI^{-1} = Z^{-1}$), whereas $\underline{G}'(\mathcal{O}(F))$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ (an isomorphism is given by reduction modulo s). One can also check that the class number

$$\mathrm{card}(G_{\mathrm{ad}}(F) \setminus G_{\mathrm{ad}}(\mathbb{A}_{F,f}) / \underline{G}_{\mathrm{ad}}(\widehat{\mathcal{O}(F)})) = 2,$$

and so $\underline{G}$ and $\underline{G}'$ are up to isomorphism the only two reductive models of G over $\mathcal{O}(F)$. So we have two distinct notions of "level one" for automorphic representations for G, and although the relevant Arthur–Langlands parameters are the same in both cases, the automorphic multiplicities differ. More precisely, any algebraic Hecke character χ for E as above contributes an automorphic representation for G either in level $\underline{G}(\widehat{\mathcal{O}(F)})$ or in level $\underline{G}'(\widehat{\mathcal{O}(F)})$.

Higher rank. Alternatively, one could explicitly compute the geometric transfer factors defined in [Labesse and Langlands 1979] for G and the endoscopic group H isomorphic to the unique anisotropic torus over F of dimension 1 which is split by E. Although one would lose the interpretation in terms of characters of centralizers of Langlands parameters, this would probably lead to a proof that the multiplicity formula for G in level $\underline{G}(\widehat{\mathcal{O}(F)})$ reduces to (7.2.2).

Note however that the approach in the present paper generalizes easily to higher rank. For example, using the embedding $(\mathrm{SL}_2)^n \hookrightarrow \mathrm{Sp}_{2n}$, it is easy to generalize the above example to the case where G is the inner form of $G^* = \mathrm{Sp}_{2n}$ over F which is definite (i.e., $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact) and split at all finite places. This does not require additional computation, and so one can make explicit Arthur's multiplicity formula (also known in this case; see [Taïbi 2015]) in "level one". Moreover, using also *pure* inner forms of quasisplit special orthogonal groups, namely definite special orthogonal groups obtained using copies of $(x, y) \mapsto x^2 + sxy + y^2$ and (in odd dimension) $x \mapsto x^2$, it is possible to carry out the same inductive strategy as in [Taïbi 2017], but using definite groups as in [Chenevier and Renard 2015], which makes explicit computations much simpler. Therefore the above example makes it possible to explicitly compute automorphic cuspidal self-dual representations for general linear groups over F which are unramified at all finite places and algebraic regular at both real places.

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(φ,Γ)-modules de de Rham et fonctions L p-adiques

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On développe une variante des méthodes de Coleman et Perrin-Riou permettant, pour une représentation galoisienne de de Rham, construire des fonctions L p-adiques à partir d'un système compatible d'éléments globaux. On obtient de la sorte des fonctions analytiques sur un ouvert de l'espace des poids contenant les caractères localement algébriques de conducteur assez grand. Appliqué au système d'Euler de Kato, cela fournit des fonctions L p-adiques pour les courbes elliptiques à mauvaise réduction additive et, plus généralement, pour les formes modulaires supercuspidales en p. En dimension 2, nous prouvons une équation fonctionnelle pour nos fonctions L p-adiques.

We develop a variant of Coleman and Perrin-Riou's methods giving, for a de Rham p-adic Galois representation, a construction of p-adic L-functions from a compatible system of global elements. As a result, we construct analytic functions on an open set of the p-adic weight space containing all locally algebraic characters of large enough conductor. Applied to Kato's Euler system, this gives p-adic L-functions for elliptic curves with additive bad reduction and, more generally, for modular forms which are supercuspidal at p. In the case of dimension 2, we prove a functional equation for our p-adic L-functions.

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Mots-clefs : p-adic L-functions, modular forms, (φ,Γ)-modules, p-adic Hodge theory.

Introduction

Cette article est consacré à l'étude des fonctions L p-adiques associées aux formes modulaires. En utilisant la théorie des (φ,Γ)-modules et en généralisant certains résultats de Perrin-Riou, on montre comment construire des fonctions L p-adiques associées à une représentation p-adique du groupe de Galois absolu $\mathscr{G}_{\mathbb{Q}_p}$ de $\mathbb{Q}_p$ (possiblement à mauvaise réduction en p) munie d'un système compatible de classes de cohomologie. En particulier, ceci fournit des fonctions L p-adiques pour une forme modulaire supercuspidale en p tordue par des caractères suffisamment ramifiés.

Soit

$$f = \sum_{n=1}^{+\infty} a_n q^n \in S_k(\Gamma_1(N), \omega_f) \otimes \mathbb{C}$$

une forme primitive (i.e., cuspidale, nouvelle, propre pour les opérateurs de Hecke et normalisée) de poids $k \geq 2$, niveau N et caractère $\omega_f : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Pour η un caractère de Dirichlet, notons

$$L(f, \eta, s) = \sum_{n=1}^{+\infty} a_n \eta(n) n^{-s}$$

la fonction L complexe associée à f et η. La série définissant $L(f, \eta, s)$ converge pour $\operatorname{Re} s > 1$, admet un prolongement analytique à tout le plan complexe et elle satisfait une équation fonctionnelle reliant les valeurs $L(f, \eta, s)$ et $L(\check{f}, \eta^{-1}, k-s)$, où $\check{f} = \sum_{n \geq 1} \bar{a}_n q^n \in S_k(\Gamma_1(N), \omega_f^{-1})$ est la forme conjuguée à f. La théorie des symboles modulaires permet de montrer l'existence des périodes complexes Ω_f^+ et Ω_f^- telles que, si η est un caractère de Dirichlet, j est un entier tel que $1 \leq j \leq k-1$ et $\pm$ est tel que $\eta(-1)(-1)^j = \pm 1$, alors

$$\frac{\Gamma(j)}{(2i\pi)^j} \frac{L(f, \eta, j)}{\Omega_f^\pm} \in \overline{\mathbb{Q}}.$$

Ceci permet, en fixant une immersion $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$ et en notant $\Lambda_\infty(f, \eta, s) = (\Gamma(s)/(2i\pi)^s) L(f, \eta, s)$, de voir ces valeurs dans le monde p-adique en posant

$$\iota_p(\Lambda_\infty(f, \eta, j)) = \frac{\Lambda_\infty(f, \eta, j)}{\Omega_f^\pm} \in \overline{\mathbb{Q}}_p.$$

Les fonctions L p-adiques peuvent être vues naturellement comme des fonctions rigides analytiques sur l'espace des poids p-adiques $\mathfrak{X}$.[1] Si $\eta : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ est un caractère d'ordre fini, une telle fonction $L_p \in \mathcal{O}(\mathfrak{X})$ est déterminée par ses valeurs sur les caractères de la forme $x \mapsto \eta(x) x^k$, $k \in \mathbb{Z}$. En particulier, si l'on pose, pour $s \in \mathbb{Z}_p$ et $x \in \mathbb{Z}_p^\times$, $\langle x \rangle^s = \exp(s \log x)$, une fonction analytique sur l'espace des poids donne naissance à une famille de fonctions $L_p(\eta, s)$ d'une variable p-adique $s \in \mathbb{Z}_p$, pour η parcourant les caractères d'ordre fini.

1. L'espace $\mathfrak{X}$ est un espace analytique rigide dont les L-points, pour L une extension finie de $\mathbb{Q}_p$, sont donnés par $\mathfrak{X}(L) = \operatorname{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, L)$. Il est une union de boules ouvertes et donc quasi-Stein. Par un théorème d'Amice, les distributions sur $\mathbb{Z}_p^\times$ correspondent aux fonctions (rigides) analytiques sur $\mathfrak{X}$.

Fixons un isomorphisme $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ et notons

$$X^2 - a_p X + \omega_f(p) p^{k-1}$$

le polynôme de Hecke en p de la forme f et $\alpha, \beta \in \overline{\mathbb{Q}}_p$ ses racines. On dit que f est de pente finie si au moins une de ces racines est non nulle. Remarquons simplement que, si $p \nmid N$, alors f est de pente finie, et que, si p divise N, alors f est de pente finie si et seulement si $a_p \neq 0$. Une des premières constructions de la fonction L p-adique de f dépend du choix d'une racine non nulle, disons α, du polynôme de Hecke en p de la forme f.

Théorème 0.1 [Manin 1973; Amice et Vélu 1975; Višik 1976; Mazur et al. 1986; Pollack et Stevens 2013; Bellaïche 2012; Delbourgo 2002]. *Soient f de pente finie et α comme ci-dessus. Il existe une* (*unique si* $v_p(\alpha) < k-1$) *fonction $L_{p,\alpha}(f) \in \mathcal{O}(\mathfrak{X})$, d'ordre $v_p(\alpha)$, telle que, si $\eta : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ est un caractère de Dirichlet de conducteur p^n et j un entier tel que* $0 \leq j \leq k-2$, *on a*

$$L_{p,\alpha}(f)(\eta\chi^j) = e_{p,\alpha}(f, \eta, j) \frac{p^{n(j+1)}}{G(\eta^{-1})} \cdot \iota_p(\Lambda_\infty(f, \eta^{-1}, j+1)),$$

où $G(\eta^{-1}) = \sum_{a=0}^{p^n-1} \eta^{-1}(a) \exp(2i\pi a/p^n) \in \overline{\mathbb{Q}}_p$ dénote la somme de Gauss du caractère η^{-1} et le facteur $e_{p,\alpha}(f, \eta, j)$ est défini par la formule

$$e_{p,\alpha}(f, \eta, j) = \begin{cases} \alpha^{-n} & \textit{si } n > 0, \\ (1-\alpha^{-1}\omega_f(p)p^{k-2-j})(1-\alpha^{-1}p^j) & \textit{si } n = 0. \end{cases}$$

La construction peut être adaptée [Delbourgo 2002] à la situation où f est de pente infinie mais $f \otimes \xi$ est de pente finie pour un certain caractère ξ. Si $f \otimes \xi$ n'est pas de pente finie pour aucun caractère ξ, on dit que la forme f est *supercuspidale*. Cette condition correspond [Loeffler et Weinstein 2012, Proposition 2.8] à ce que la représentation lisse de $\mathrm{GL}_2(\mathbb{Q}_p)$ associée à f soit supercuspidale.

0A. ***Le système d'Euler de Kato.*** Il existe une construction alternative [Kato 2004; Colmez 2004] de la fonction L p-adique, qui est moins élémentaire mais a pourtant l'avantage de permettre de relier les valeurs en certains points entiers de la fonction L p-adique à des quantités de nature cohomologique, ce qui permet, par exemple, de démontrer des instances de la conjecture de Birch et Swinnerton-Dyer p-adique et la conjecture principale d'Iwasawa pour les représentations galoisiennes attachées aux formes modulaires. Nous rappelons dans ce qui suit la construction de Kato de la fonction L p-adique dans le cas où p est un nombre premier ne divisant pas le niveau N de f.[2]

Soit L une extension finie de $\mathbb{Q}_p$ contenant les coefficients de Fourier de la forme f et notons $\mathscr{O}_L$ l'anneau des entiers de L. Soit $V(f)$ la L-représentation du groupe de Galois absolu $\mathscr{G}_{\mathbb{Q}}$ de $\mathbb{Q}$ associée à f par Shimura–Deligne : elle est de dimension 2, non ramifiée en dehors Np, de Rham en p et caractérisée par le fait que, pour tout $\ell \nmid Np$, alors

$$\det(1 - \mathrm{Frob}_\ell^{-1} X | V(f)^{I_\ell}) = 1 - a_\ell X + \ell^{k-1}\omega_f(\ell) X^2,$$

2. Voir [Colmez 2004] pour les modifications nécessaires dans le cas semi-stable et [Delbourgo 2002] pour le cas où la représentation dévient cristalline sur une extension abélienne de $\mathbb{Q}_p$.

où Frob_ℓ désigne le Frobenius arithmétique en ℓ et $I_\ell \subseteq \mathscr{G}_{\mathbb{Q}_\ell} \subseteq \mathscr{G}_{\mathbb{Q}}$ dénote le groupe d'inertie du groupe de Galois absolu $\mathscr{G}_{\mathbb{Q}_\ell} = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ de $\mathbb{Q}_\ell$ vu dedans $\mathscr{G}_{\mathbb{Q}}$ comme un groupe de décomposition. Notons aussi par $V(f)$ la restriction de $V(f)$ au groupe $\mathscr{G}_{\mathbb{Q}_p}$, qui est une représentation de dimension 2, de Rham (cristalline si $p \nmid N$) à poids de Hodge–Tate 0 et $1-k$.

Soient $F_\infty = \bigcup_n \mathbb{Q}_p(\mu_{p^n})$ l'extension cyclotomique de $\mathbb{Q}_p$,

$$\Gamma_n = \mathrm{Gal}(F_\infty/\mathbb{Q}_p(\mu_{p^n})) \subseteq \Gamma = \mathrm{Gal}(F_\infty/\mathbb{Q}_p)$$

et $\chi : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$ le caractère cyclotomique. La construction de Kato repose sur la construction d'un système d'Euler[3] z_{Kato} attaché à f dans la représentation $V = V(f)^*(1) = V(\check{f})(k)$ (qui est de Rham à poids de Hodge–Tate 1 et k en p) et dont les niveaux en les différentes puissances de p fournissent un élément, aussi noté z_{Kato}, de la cohomologie d'Iwasawa de la représentation V définie par

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) = \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T) \otimes \mathbb{Q}_p \cong H^1(\mathscr{G}_{\mathbb{Q}_p}, \Lambda \otimes V),$$

où T dénote n'importe quel $\mathscr{O}_L$-réseau de V stable par $\mathscr{G}_{\mathbb{Q}_p}$, $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ dénote l'algèbre d'Iwasawa de Γ et où la limite projective est prise par rapport aux applications de corestriction (le dernier isomorphisme étant une conséquence du lemme de Shapiro).

Un théorème fondamental de Kato [2004, Theorem 12.5] montre que l'élément z_{Kato} est intimement lié aux valeurs spéciales de la fonction L complexe de la forme f. Si $p \nmid N$ (i.e., si $V(f)$ est cristalline), ce théorème permet à Kato [2004] d'appliquer la machine à fonctions L p-adiques de Perrin-Riou [1994; 1995]

$$\mathrm{Log}_V : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \otimes_{\Lambda(\Gamma)} \mathscr{D}(\Gamma) \to \mathcal{O}(\mathfrak{X}) \otimes \boldsymbol{D}_{\mathrm{cris}}(V),$$

où $\mathscr{D}(\Gamma)$ dénote l'algèbre de distributions sur Γ,[4] interpolant p-adiquement les applications exponentielles et (d'après un théorème de Colmez, Benois et Kato–Kurihara–Tsuji) exponentielles duales de Bloch–Kato pour des différentes tordues de la représentation en question, pour obtenir une nouvelle construction de la fonction L p-adique de f.

Théorème 0.2 [Kato 2004, Theorem 16.6]. *Soit $e_\alpha \in \boldsymbol{D}_{\mathrm{cris}}(V(f)) = \boldsymbol{D}_{\mathrm{cris}}(V^*(1))$ un vecteur propre du Frobenius cristallin de valeur propre α. On a alors*

$$L_{p,\alpha}(f) = \langle \mathrm{Log}_V(z_{\mathrm{Kato}}), e_\alpha \rangle,$$

où $\langle\ ,\ \rangle : \boldsymbol{D}_{\mathrm{cris}}(V) \times \boldsymbol{D}_{\mathrm{cris}}(V^*(1)) \to L$ dénote l'accouplement de Poincaré.

Remarquons que l'on dispose dans tous les cas d'un système d'Euler z_{Kato} et que l'application de Perrin-Riou a été généralisée pour les représentations de de Rham par des travaux de Colmez [1998] et Cherbonnier et Colmez [1999] et, pour un (φ,Γ)-module sur l'anneau de Robba $\mathscr{R}$, par Nakamura [2014].

3. Les éléments zêta de Kato dépendent d'un certain nombre de choix que l'on ignore dans cette introduction afin de simplifier l'exposition.

4. D'après un théorème d'Amice, $\mathscr{D}(\Gamma)$ et $\mathcal{O}(\mathfrak{X})$ sont isomorphes, or on garde la notation $\mathscr{D}(\Gamma)$, qui est plus souvent utilisée.

Or ces applications ne s'expriment pas naturellement en termes de fonctions analytiques sur l'espace des poids et ne fournissent malheureusement (ou heureusement) pas si simplement des fonctions L p-adiques.

0B. *Le cas supercuspidal.* Décrivons brièvement le résultat principal de cet article. L'isomorphisme $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ que l'on a fixé permet de voir $\eta : \mathbb{Z}_p^\times \to L^\times$ comme un caractère de Dirichlet de conducteur p^n, $n \geq 0$, et on note $f \otimes \eta^{-1}$ la forme tordue de f par η^{-1}, qui est aussi une forme primitive (de niveau Np^{2n} et caractère $\omega_f \eta^{-2}$, au moins si n est assez grand). Rappelons que l'on a posé

$$\Lambda_\infty(f, \eta^{-1}, s) = \frac{\Gamma(s)}{(2i\pi)^s} \cdot L(f, \eta^{-1}, s)$$

et que la forme f donne lieu à une représentation automorphe $\pi(f) = \prod'_v \pi_v(f)$ de $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, où v parcourt l'ensemble des places de $\mathbb{Q}$ et $\pi_v(f)$ est une représentation lisse de $\mathrm{GL}_2(\mathbb{Q}_v)$. Pour η un caractère de Dirichlet, vu comme un caractère des adèles, et v une place de $\mathbb{Q}$, on note $\varepsilon(\pi_v(f) \otimes \eta)$ les facteurs epsilon des composantes locales de la représentation $\pi(f) \otimes \eta$,[5] de sorte que le facteur epsilon global associé à f et η est donné par la formule

$$\varepsilon(\pi(f) \otimes \eta) = \varepsilon(\pi_\infty(f)) \cdot \prod_{\ell \mid N} \varepsilon(\pi_\ell(f) \otimes \eta).$$

La fonction $\Lambda_\infty(f, \eta^{-1}, s)$ satisfait alors l'équation fonctionnelle

$$\Lambda_\infty(f, \eta^{-1}, j) = i^k (-1)^j \varepsilon(\pi(f) \otimes \eta^{-1} \otimes |\cdot|^{j-(k-1)/2}) \cdot \Lambda_\infty(f, \eta, k-j), \quad j \in \mathbb{Z}$$

On obtient le théorème suivant :

Théorème 0.3 (définition 3 + théorème I.1 + lemme II.6 + théorème II.10). *Soit $f \in S_k(\Gamma_1(N), \omega_f)$ une forme primitive et soit $V = V(\check{f})(k-1)$. Il existe des plongements naturels*

$$\Lambda_\infty(f, \eta^{-1}, j) \mapsto \iota_p(\Lambda_\infty(f, \eta^{-1}, j)) \in \boldsymbol{D}_{\mathrm{dR}}(V), \quad j \leq k-1,$$

un ouvert $\mathfrak{U}_f \subseteq \mathfrak{X}$ ne dépendant que de la puissance de p divisant le niveau N de la forme f et contenant tous les caractères d'ordre p^n pour n assez grand, et une unique fonction rigide analytique $\Lambda_p(f) \in \mathcal{O}(\mathfrak{U}_f) \otimes \boldsymbol{D}_{\mathrm{dR}}(V)$ telle que, si $\eta : \mathbb{Z}_p^\times \to L^\times$ est un caractère de conducteur p^n et $j < k-1$ sont tels que $\eta x^j \in \mathfrak{U}_f$, alors

$$\Lambda_p(f)(\eta x^j) = \frac{p^{n(j+1)}}{G(\eta^{-1})} \cdot \iota_p(\Lambda_\infty(f, \eta^{-1}, j+1)).$$

De plus, la fonction $\Lambda_p(f)$ satisfait une équation fonctionnelle de la forme[6]

$$\Lambda_p(f)(\eta x^j) = C(f, \eta, j) \cdot \Lambda_p(\check{f})(\eta^{-1} x^{k-2-j}),$$

5. On a $\varepsilon(\pi_\infty(f)) = i^k$.
6. La formulation de l'équation fonctionnelle est légèrement imprécise, cf. le théorème II.10.

où

$$C(f,\eta,j)=p^n\,\varepsilon\big(\eta\otimes|\cdot|^{-j+(k-1)/2}\big)^2\varepsilon\big(\pi_p(\check f)\otimes\eta\otimes|\cdot|^{-j+k-1}\big)^{-1}\cdot\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check f)\otimes\eta^{-1}\otimes|\cdot|^{-j+(k-1)/2}\big)^{-1},$$

où N' dénote la partie de N première à p et $\varepsilon(\eta\otimes|\cdot|^s)=p^{-ns}\eta(p)^nG(\eta^{-1})$.

Finalement, si $p\nmid N$ alors $\mathfrak{U}_f=\mathfrak{X}$ et, si α est une valeur propre du polynôme de Hecke en p de f et $e_\alpha\in\boldsymbol{D}_{\mathrm{cris}}(V(f))=\boldsymbol{D}_{\mathrm{cris}}(V)^*$ est un vecteur propre du Frobenius cristallin de valeur propre α, on a

$$L_{p,\alpha}(f)=\langle\Lambda_p(f),e_\alpha\rangle.$$

Remarque 0.4. Les méthodes du théorème fournissent, plus généralement, pour toute représentation $V\in\mathrm{Rep}_L\,\mathscr{G}_{\mathbb{Q}_p}$ de Rham de dimension d et tout $z\in H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)$, un ouvert $\mathfrak{U}_V\subseteq\mathfrak{X}$, ne dépendant que de la valuation p-adique du discriminant de la plus petite extension finie galoisienne $K/\mathbb{Q}_p$ sur laquelle V dévient semi-stable, et une unique fonction $\Lambda_{V,z}\in\mathcal{O}(\mathfrak{U}_V)$ interpolant des exponentielles et exponentielles duales de différentes spécialisations de l'élément z. Dans le cas général, on n'a pas, hélas ! une interprétation si satisfaisante des valeurs interpolées.

0C. *Démonstration.* Disons quelques mots sur la démonstration du théorème.

0C1. *Plongements p-adiques des valeurs spéciales.* Soit $M=M(f\otimes\eta^{-1})$ le motif associé à la forme $f\otimes\eta^{-1}$ et considérons, pour $j\geq 0$,

$$M^*(1+j)=M(\check f\otimes\eta)(k+j),$$

dont $V(\eta\chi^{j+1})$ est la réalisation p-adique. En utilisant les symboles d'Eisenstein définis par Beilinson [1986], on peut construire des éléments (cf. [Gealy 2006, §3.2]) $\mathscr{Z}(\check f\otimes\eta,j)\in H^1(M^*(1+j))$ et un résultat de Gealy [2006, Theorem 4.1.1] montre qu'ils satisfont une variante de la conjecture de Bloch–Kato.

Par ailleurs, grâce aux travaux de Gros [1990], Niziol [1997], Besser [2000], et Nekovář et Nizioł [2016], on dispose aussi des régulateurs p-adiques

$$r_p:H^1(M^*(1+j))\to\boldsymbol{D}_{\mathrm{dR}}(V(\eta\chi^{j+1})),$$

qui satisfont la relation de commutativité

$$r_{\text{ét}}=\exp\circ r_p,$$

où exp est l'exponentielle de Bloch–Kato.

Le résultat de Gealy mentionné ci-dessus suggère de considérer les régulateurs p-adiques des éléments $\mathscr{Z}(\check f\otimes\eta,j)$ comme les valeurs p-adiques naturels à interpoler. Notons que les éléments

$$r_p(\mathscr{Z}(\check f\otimes\eta,j))\in\boldsymbol{D}_{\mathrm{dR}}(V(\eta\chi^{j+1}))$$

vivent dans des modules différents. Nous allons expliquer comment les voir tous naturellement comme des éléments dans $\boldsymbol{D}_{\mathrm{dR}}(V)$.

Si $\eta : \mathbb{Z}_p^\times \to L^\times$ est un caractère d'ordre fini, $G(\eta)$ dénote sa somme de Gauss et $a \in \mathbb{Z}_p^\times$, alors $\sigma_a(G(\eta)) = \eta(a)^{-1}G(\eta)$ et, si e_η dénote une base du module $L(\eta)$,[7] le groupe Γ (et donc aussi $\mathscr{G}_{\mathbb{Q}_p}$) agit trivialement sur l'élément $\boldsymbol{e}_\eta^{\mathrm{dR}} = G(\eta)\cdot e_\eta \in \mathbb{B}_{\mathrm{dR}} \otimes L(\eta)$ et est donc une base du L-espace vectoriel $\boldsymbol{D}_{\mathrm{dR}}(L(\eta))$. Si $j \in \mathbb{Z}$, il en est de même de $\boldsymbol{e}_j^{\mathrm{dR}} = t^{-j}e_j \in \mathbb{B}_{\mathrm{dR}} \otimes L(\chi^j)$. Notons

$$\boldsymbol{e}_{\eta,j}^{\mathrm{dR}} = \boldsymbol{e}_\eta^{\mathrm{dR}} \otimes \boldsymbol{e}_j^{\mathrm{dR}} = G(\eta)e_\eta \otimes t^{-j},\, e_j$$

qui est une base du module $\boldsymbol{D}_{\mathrm{dR}}(L(\eta\chi^j))$ et, pour $e_\eta^\vee = e_{\eta^{-1}}$ et e_{-j} les éléments duaux de e_η et e_j, on note

$$\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee} = G(\eta)^{-1}e_\eta^\vee \otimes t^j e_{-j},$$

qui est une base du module $\boldsymbol{D}_{\mathrm{dR}}(L(\eta\chi^j)^*)$.

Si $V \in \mathrm{Rep}_L\, \mathscr{G}_{\mathbb{Q}_p}$ est de Rham, alors $V(\eta\chi^j)$ l'est aussi et on a, par ce qui précède,

$$\boldsymbol{D}_{\mathrm{dR}}(V(\eta\chi^j)) = (\mathbb{B}_{\mathrm{dR}} \otimes V \otimes L(\eta\chi^j))^{\mathscr{G}_{\mathbb{Q}_p}} = (\mathbb{B}_{\mathrm{dR}} \otimes V)^{\mathscr{G}_{\mathbb{Q}_p}} \otimes \boldsymbol{e}_{\eta,j}^{\mathrm{dR}} = \boldsymbol{D}_{\mathrm{dR}}(V) \otimes \boldsymbol{e}_{\eta,j}^{\mathrm{dR}},$$

de sorte que l'application $x \mapsto x \otimes \boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}$ induit un isomorphisme

$$\boldsymbol{D}_{\mathrm{dR}}(V(\eta\chi^j)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(V).$$

Notons

$$\Gamma^*(j+1) = \begin{cases} j! & \text{si } j \geq 0, \\ \dfrac{(-1)^{j-1}}{(-j-1)!} & \text{si } j < 0 \end{cases}$$

le coefficient principal de la série de Laurent de la fonction $\Gamma(s)$ en $s = j+1$. On pose, pour $j \geq 0$,

$$\iota_p(\Lambda_\infty(f, \eta^{-1}, -j)) = \Gamma^*(j)\, G(\eta) \cdot r_p(\mathscr{Z}(\check{f} \otimes \eta, j)) \otimes \boldsymbol{e}_{\eta,j+1}^{\mathrm{dR},\vee} \in \boldsymbol{D}_{\mathrm{dR}}(V).$$

Remarquons que, dans la formule définissant $\iota_p(\Lambda_\infty(f, \eta^{-1}, -j))$, on “multiplie” et “divise” par $G(\eta)$, de sorte que son introduction n'a moralement aucun effet. La valeur $\Gamma^*(j)$ correspond donc à $\Gamma(j)$ et la puissance t^{j+1} correspond à la puissance de $2i\pi$ dans la définition de $\Lambda_\infty(f, \eta^{-1}, -j)$.

0C2. *Interpolation : un premier exemple.* La construction de la fonction L locale repose sur la théorie des (φ,Γ)-modules, et est inspirée de la ressemblance entre celle-ci et l'analyse fonctionnelle p-adique, via le dictionnaire d'analyse fonctionnelle p-adique de Colmez, qui est aussi à la base même de la construction de la correspondance de Langlands p-adique pour $\mathrm{GL}_2(\mathbb{Q}_p)$.

Soit $\mathrm{LA}(\mathbb{Z}_p, L)$ l'espace des fonctions localement analytiques sur $\mathbb{Z}_p$ à valeurs dans L et $\mathscr{D}(\mathbb{Z}_p, L)$, son L-dual topologique, l'espace des distributions sur $\mathbb{Z}_p$. Si $\phi \in \mathrm{LA}(\mathbb{Z}_p, L)$ et $\mu \in \mathscr{D}(\mathbb{Z}_p, L)$, on note $\int_{\mathbb{Z}_p} \phi \cdot \mu$ l'évaluation de μ en ϕ. On a des actions du groupe Γ et des opérateurs φ et ψ sur l'espace de distributions définis par les formules

$$\int_{\mathbb{Z}_p} \phi(x)\cdot\sigma_a(\mu) = \int_{\mathbb{Z}_p} \phi(ax)\cdot\mu, \quad \int_{\mathbb{Z}_p} \phi(x)\cdot\varphi(\mu) = \int_{\mathbb{Z}_p} \phi(px)\cdot\mu, \quad \int_{\mathbb{Z}_p} \phi(x)\cdot\psi(\mu) = \int_{p\mathbb{Z}_p} \phi(x/p)\cdot\mu,$$

7. $L(\eta)$ dénote le L-espace vectoriel de dimension 1 sur lequel $\mathscr{G}_{\mathbb{Q}_p}$ agit à travers η.

et une opération de “multiplication par x” définie par $\int_{\mathbb{Z}_p} \phi(x) \cdot m_x(\mu) = \int_{\mathbb{Z}_p} \phi(x) x \cdot \mu$. Notons que $\psi(\mu) = 0$ si et seulement si la distribution μ est supportée sur $\mathbb{Z}_p^\times$ et que, si $\psi(\mu) = \mu$, alors $(1-\varphi)\mu$ est la restriction à $\mathbb{Z}_p^\times$ de μ.

La transformée d’Amice

$$\mu \mapsto \mathscr{A}_\mu = \int_{\mathbb{Z}_p} (1+T)^x \cdot \mu$$

induit un isomorphisme $\mathscr{D}(\mathbb{Z}_p, L) \xrightarrow{\sim} \mathscr{R}^+$, où $\mathscr{R}^+ = \mathscr{R} \cap L[\![T]\!]$. L’opérateur différentiel $\partial = (1+T)\frac{d}{dT}$ sur $\mathscr{R}$ correspond à la multiplication par x sur les distributions au sens que $\partial \mathscr{A}_\mu = \mathscr{A}_{m_x(\mu)}$ et les actions de φ et de $\sigma_a \in \Gamma$ correspondent à $\varphi(T) = (1+T)^p - 1$ et $\sigma_a(T) = (1+T)^a - 1$. En fixant un système compatible $(\zeta_{p^n})_{n\in\mathbb{N}}$ de racines primitives de l’unité (c’est-à-dire, $\zeta_{p^n} \in \overline{\mathbb{Q}}_p$ est une racine primitive p^n-ième de 1 et $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ pour tout $n \in \mathbb{N}$), on peut définir des applications de “localisation”

$$\varphi^{-n} : \mathscr{E}^{]0,r_n]} \to L_\infty[\![t]\!], \quad T \mapsto \zeta_{p^n} e^{t/p^n} - 1,$$

où $\mathscr{E}^{]0,r_n]}$ dénote les éléments de $\mathscr{R}$ qui convergent sur la couronne $0 < v_p(z) \le r_n = 1/(p^{n-1}(p-1))$ et $L_\infty[\![t]\!] = \varinjlim L_n[\![t]\!]$. L’opérateur ∂ stabilise $\mathscr{E}^{]0,r_n]}$ et on a l’identité $\varphi^{-n} \circ \partial = p^n \frac{d}{dt} \circ \varphi^{-n}$. Si $x = \sum_{l\in\mathbb{N}} a_l t^l \in L_\infty[\![t]\!]$, on note $[x]_0 = a_0$. Le lemme suivant nous permet de réécrire l’intégration p-adique en termes des applications de localisation.

Lemme 0.5. *Pour tout $f \in \mathscr{R}^{\psi=0}$, il existe une unique fonction analytique $\kappa \mapsto \kappa(\partial) f : \mathfrak{X} \to \mathscr{R}^{\psi=0}$ interpolant les valeurs $\partial^j f$, $j \in \mathbb{Z}$, aux caractères x^j. De plus, si $\mu \in \mathscr{D}(\mathbb{Z}_p, L)^{\psi=1}$, $\eta : \mathbb{Z}_p^\times \to L^\times$ est un caractère de conducteur p^n, $n > 0$, et $\kappa \in \mathfrak{X}$, alors*[8]

$$\int_{\mathbb{Z}_p^\times} \eta^{-1} \kappa \cdot \mu = G(\eta)^{-1} \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta(a) \sigma_a [\varphi^{-n} \kappa(\partial)(1-\varphi) \mathscr{A}_\mu]_0.$$

L’avantage du lemme ci-dessus est que le terme de droite a un sens, pour tout (φ,Γ)-module D de de Rham, si l’on remplace $\mathscr{A}_\mu$ par $z \in \mathbb{N}_{\mathrm{rig}}(D)^{\psi=1}$, où $\mathbb{N}_{\mathrm{rig}}(D)$ est l’équation différentielle p-adique associée à D par Berger, dès que n est assez grand, comme on le verra à continuation.

0C3. *Interpolation : le cas général.* Comme suggéré par les résultats de Kato, la construction de $\Lambda_p(f)$ demande à étendre la construction du logarithme de Perrin-Riou aux (φ,Γ)-modules de de Rham.

Soit $V \in \mathrm{Rep}_L \mathscr{G}_{\mathbb{Q}_p}$ et notons $D = \boldsymbol{D}_{\mathrm{rig}}(V)$ le (φ,Γ)-module sur l’anneau de Robba $\mathscr{R}$ qui lui est associé par l’équivalence de catégories de Fontaine [1990], Cherbonnier et Colmez [1998] et Kedlaya [2004]. On a un isomorphisme, dû à Fontaine et Pottharst [Cherbonnier et Colmez 1999; Pottharst 2012],

$$\mathrm{Exp}^* : H^1(\mathbb{Q}_p, \mathscr{D}(\Gamma) \otimes V)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{rig}}(V)^{\psi=1}.$$

Supposons V de Rham et notons $\Delta = \mathbb{N}_{\mathrm{rig}}(D)$ l’équation différentielle p-adique associée à D par Berger [2002; 2003]. Si D est à poids de Hodge–Tate positifs, on a $D \subseteq \Delta$. On dispose d’un opérateur de connexion ∂ sur Δ au-dessus de l’opérateur $\partial = (1+T)\frac{d}{dT}$ de “multiplication par x” sur $\mathscr{R}$.

8. Si $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, on note $\sigma_a \in \Gamma_n$ l’élément lui correspondant par le caractère cyclotomique.

Pour $n \gg 0$, il existe des sous-$\mathscr{E}^{]0,r_n]}$-modules $\Delta^{]0,r_n]} \subseteq \Delta$ tels que $\Delta^{]0,r_n]} \otimes_{\mathscr{E}^{]0,r_n]}} \mathscr{R} = \Delta$. Ces modules sont stables par l'action de Γ et satisfont $\varphi(\Delta^{]0,r_n]}) \subseteq \Delta^{]0,r_{n+1}]}$. On a, pour tout $n \gg 0$, des applications de localisation

$$\varphi^{-n} : \Delta^{]0,r_n]} \to L_n[\![t]\!] \otimes \boldsymbol{D}_{\mathrm{dR}}(V),$$

et on note $[\,\cdot\,]_0 : L_n[\![t]\!] \otimes \boldsymbol{D}_{\mathrm{dR}}(V) \to L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(V)$ l'application $\sum_l a_l t^l \otimes d_l \mapsto a_0 \otimes d_0$. L'opérateur stabilise $\Delta^{]0,r_n]}$ et on a $\varphi^{-n} \circ \partial = p^n \frac{d}{dt} \circ \varphi^{-n}$.

Si $z \in D^{\psi=1} \subseteq \Delta^{\psi=1}$, η est un caractère de conducteur p^n, $n \gg 0$ et $j \geq 0$, on a l'égalité suivante dans $L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$, qui est l'analogue, en termes de théorie d'Iwasawa, de l'égalité du lemme 0.5 ci-dessus :

$$G(\eta)^{-1} \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta(a)\sigma_a[\varphi^{-n}\partial^j(1-\varphi)z]_0 = p^{n(j+1)}\Gamma^*(j+1) \cdot \exp^*\Big(\int_{\mathbb{Z}_p^\times} \eta\chi^{-j}\mu_z\Big) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j}, \tag{1}$$

où $\exp^* : H^1(\mathbb{Q}_p, V(\eta\chi^{-j})) \to \boldsymbol{D}_{\mathrm{dR}}(V(\eta\chi^{-j})) = \boldsymbol{D}_{\mathrm{dR}}(V) \otimes \boldsymbol{e}^{\mathrm{dR}}_{\eta,-j}$ dénote l'exponentielle duale de Bloch–Kato. La proposition suivante permet de prolonger analytiquement le terme de gauche de cette égalité.

Proposition 0.6 (proposition I.13). *Pour tout $z \in \Delta^{\psi=0}$, il existe une unique fonction rigide analytique $\kappa \mapsto \kappa(\partial)z$, $\mathfrak{X} \to \Delta^{\psi=0}$ interpolant les valeurs $\partial^j z$, $j \in \mathbb{Z}$, aux caractères x^j.*

La restriction à $\mathbb{Z}_p^\times$ (i.e., l'application de l'opérateur $1-\varphi$) permet d'interpoler p-adiquement l'opérateur ∂ mais, en revanche, l'application de spécialisation φ^{-n} n'est pas définie sur tout le module $\Delta^{\psi=0}$. Une étude du rayon de convergence de l'élément $\kappa(\partial)$, pour un caractère $\kappa \in \mathfrak{X}(L)$, montre que le terme de gauche de l'équation définit bien une fonction analytique sur une boule ouverte autour η dans l'espace des poids. Le théorème suivant, qui est une généralisation du Logarithme de Perrin-Riou et le résultat principal de cet article, appliqué à $D = \boldsymbol{D}_{\mathrm{rig}}(V(f)(k-1))$, fournit la construction de la fonction $\Lambda_p(f)$. On renvoie au texte pour les notations qui n'ont pas encore été introduites.

Théorème 0.7 (théorème I.1). *Soient D un (φ,Γ)-module qui est de Rham à poids de Hodge–Tate positifs et $z \in D^{\psi=1}$. Notons $\mu_z = \mathrm{Exp}^*(z) \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, D)$. Il existe un ouvert $\mathfrak{U}_D \subseteq \mathfrak{X}$, ne dépendant que du module $\boldsymbol{D}_{\mathrm{pst}}(D)$, et une unique fonction analytique $\Lambda_{D,z} \in \mathcal{O}(\mathfrak{U}_D) \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ telle que, pour tout $\eta\chi^j \in \mathfrak{U}_D$, où η est un caractère de conducteur p^n et $j \in \mathbb{Z}$, on a*

$$\Lambda_{D,z}(\eta\chi^j) = p^{n(j+1)}\Gamma^*(j+1) \cdot \begin{cases} \exp^{-1}\big(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\big) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j} & si\ j \ll 0, \\ \exp^*\big(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\big) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j} & si\ j \geq 0. \end{cases}$$

Ce théorème, accouplé à un deuxième résultat de Gealy reliant les éléments motiviques du proposition II.3 et le système d'Euler de Kato (cf. proposition II.4), permet de démontrer l'existence de l'application $\Lambda_p(f)$ du théorème 0.3. Le fait que dans le cas cristallin l'on récupère la fonction L p-adique classique associée à la forme f suit du fait que notre application est une généralisation du logarithme de Perrin-Riou et de la description de la fonction L p-adique de f en termes du système d'Euler de Kato.

0C4. *L'équation fonctionnelle en dimension* 2 *et valeurs aux entiers positifs.* Il est naturel de se demander si ce que la fonction $\Lambda_p(f)$ interpole aux entiers positifs s'interprète aussi en termes de valeurs spéciales de la fonction L complexe. Rappelons que, dans la bande critique $1 \leq j \leq k-1$, les valeurs $L(f, \eta, j)$ sont naturellement interprétées p-adiquement en les divisant par les périodes complexes de la forme f. Si $j > k-2$, une équation fonctionnelle purement locale démontrée dans [Rodrigues Jacinto 2018], accouplée avec une équation fonctionnelle satisfaite par le système d'Euler de Kato démontrée par Nakamura (proposition II.8), fournit l'équation fonctionnelle du théorème 0.3.

0D. ***Plan de l'article.*** L'organisation de l'article ne reflet pas celle de l'introduction. Les premières sections présentent des résultats purement locaux, reportant l'application aux formes modulaire à la fin.

Le premier chapitre contient le premier résultat principal de l'article, à savoir l'extension analytique (partielle) de l'application logarithme de Perrin-Riou pour les représentations de de Rham, fournissant des fonctions rigides analytiques sur des ouverts de l'espace des poids interpolant les applications exponentielle et exponentielle duale de Bloch et Kato. On y trouvera aussi des rappels et généralités sur les (φ,Γ)-modules, ainsi qu'une estimation précise du rayon de convergence de Δ en termes du module $\boldsymbol{D}_{\mathrm{pst}}(D)$, permettant de décrire l'ouvert $\mathfrak{U}_D$ du théorème 0.7. Dans le cas étale de dimension 2, en utilisant le résultat principal de [Rodrigues Jacinto 2018], on obtient une équation fonctionnelle pour notre fonction L locale.

Finalement, on montre, à l'aide d'un théorème de M. Gealy et des conjectures de Bloch–Kato pour les formes modulaires, comment la construction de la fonction L locale peut être utilisée pour donner une construction de la fonction L p-adique d'une forme modulaire, sans aucune hypothèse sur sa pente, et montrer qu'elle interpole des valeurs spéciales de la forme (dûment interprétées p-adiquement) en tout caractère algébrique de composante finie suffisamment ramifiée.

0E. ***Notations.***

— Soient p un nombre premier, $\mathbb{Q}_p$ le corps des nombres p-adiques, $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ l'anneau des entiers de $\mathbb{Q}_p$, $\mathbb{Z}_p^\times$ le groupe des unités et $\mathbb{C}_p$ la complétion p-adique de la clôture algébrique $\overline{\mathbb{Q}}_p$ de $\mathbb{Q}_p$.

— On fixe un système $(\zeta_{p^n})_{n\in\mathbb{N}}$, où $\zeta_{p^n} \in \overline{\mathbb{Q}}_p$, de racines p^n-ièmes primitives de l'unité satisfaisant la relation $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ dès que $n \geq 0$. Si $n \in \mathbb{N}$, on note $F_n = \mathbb{Q}_p(\zeta_{p^n})$ le n-ième niveau de la tour cyclotomique et $F_\infty = \bigcup_n F_n$.

— Soit $\mathscr{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ le groupe de Galois absolu de $\mathbb{Q}_p$. On définit le caractère cyclotomique $\chi : \mathscr{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ par la formule $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$, pour tout $n \in \mathbb{N}$, et on note $\mathscr{H} = \mathscr{H}_{\mathbb{Q}_p}$ le noyau de χ, qui s'identifie au groupe de Galois absolu de F_∞, et $\Gamma = \Gamma_{\mathbb{Q}_p} = \mathscr{G}_{\mathbb{Q}_p}/\mathscr{H}_{\mathbb{Q}_p}$ qui s'identifie à $\mathrm{Gal}(F_\infty/\mathbb{Q}_p)$. Le caractère cyclotomique induit un isomorphisme $\chi : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$. Si $a \in \mathbb{Z}_p^\times$, on note $\sigma_a \in \Gamma$ son inverse par χ.

— Soit L une extension finie de $\mathbb{Q}_p$, qui sera notre corps de coefficients, et notons $L_n = L \otimes F_n = L(\mu_{p^n})$ et $L_\infty = \bigcup_n L_n$. On note $\mathrm{Rep}_L(\mathscr{G}_{\mathbb{Q}_p})$ la catégorie des L-espaces vectoriels de dimension finie munis d'une action L-linéaire continue du groupe $\mathscr{G}_{\mathbb{Q}_p}$. On omettra souvent L des notations, cependant il faut être conscient que L est le corps sous-jacent et qu'il change au fur et à mesure des besoins. Par exemple, si η est un caractère de $\mathscr{G}_{\mathbb{Q}_p}$ et si on note $\mathbb{Q}_p(\eta)$ la représentation de dimension 1 engendrée par un élément e_η sur lequel l'action de $\mathscr{G}_{\mathbb{Q}_p}$ est définie par la formule $g(e_\eta) = \eta(g)e_\eta$, on sous-entendra que le corps L de coefficients contient les valeurs prises par η.

0E1. *Anneaux des séries de Laurent.* Commençons par quelques définitions classiques.

— On définit le corps $\mathscr{E}$ par

$$\mathscr{E} = \left\{ \sum_{k\in\mathbb{Z}} a_k T^k : \ a_k \in L, \ \liminf_{k\to+\infty} v_p(a_k) > -\infty, \ \lim_{k\to-\infty} v_p(a_k) = +\infty \right\},$$

muni de la valuation donnée par $v_{\mathscr{E}}\big(\sum_{k\in\mathbb{Z}} a_k T^k\big) = \inf_k v_p(a_k)$, ce qui fait de $\mathscr{E}$ un corps valué de dimension 2 dont on note $\mathscr{O}_{\mathscr{E}}$ l'anneau des entiers.

— Si $0 < r < s$, on définit $\mathscr{E}^{[r,s]}$ comme l'anneau des fonctions analytiques à valeurs dans L sur la couronne $C_{[r,s]} = \{z \in \mathbb{C}_p : r \le v_p(z) \le s\}$. On a

$$\mathscr{E}^{[r,s]} = \left\{ \sum_{k\in\mathbb{Z}} a_k T^k : a_k \in L, \lim_{k\to-\infty} v_p(a_k) + ks = +\infty, \text{ et } \lim_{k\to+\infty} v_p(a_k) + kr = +\infty \right\}.$$

L'anneau $\mathscr{E}^{[r,s]}$ est principal, de Banach pour la valuation $v^{[r,s]}$ définie par

$$v^{[r,s]}\left(\sum_{k\in\mathbb{Z}} a_k T^k\right) = \inf_{r\le v_p(z)\le s} v_p(f(z)) = \min\left\{\inf_{k\in\mathbb{Z}}(v_p(a_k)+kr),\ \inf_{k\in\mathbb{Z}}(v_p(a_k)+ks)\right\}.$$

— Pour $0 < r < s$ on note

$$\mathscr{E}^{]r,s]} = \varprojlim_{t>r} \mathscr{E}^{[t,s]}$$

l'anneau de fonctions analytiques sur la couronne $C_{]r,s]} = \{z \in \mathbb{C}_p : r < v_p(z) \le s\}$, qui est un anneau de Fréchet pour la famille de valuations $v^{[t,s]}$ pour $t \in]r, s]$.

— Soit $r_n = 1/(p^{n-1}(p-1)) = v_p(\zeta_{p^n} - 1)$. On note

$$\mathscr{R} = \varinjlim_{s>0} \mathscr{E}^{]0,s]}$$

l'anneau de Robba et $\mathscr{E}^\dagger \subseteq \mathscr{R}$ son sous-anneau d'éléments bornés. C'est l'anneau des séries de Laurent (resp. des séries de Laurent à coefficients bornés) qui convergent sur une couronne $C_{]0,s]}$ pour s assez petit (qui dépend de chaque fonction). On remarquera que l'on obtient $\mathscr{E}$ et $\mathscr{R}$ en complétant $\mathscr{E}^\dagger$, respectivement, par la topologie p-adique et par la topologie de Fréchet définie par la famille de normes $v^{[r,s]}$, $0 < r < s$.

Si $\mathscr{A}$ est un des anneaux définis ci-haut, on note $\mathscr{A}^+$ son intersection avec $L[\![T]\!]$. On a, par exemple, $\mathscr{E}^+ = \mathscr{O}_L[\![T]\!][1/p]$ et $\mathscr{R}^+$ s'identifie à l'anneau des fonctions analytiques sur la boule ouverte unité. On a une action de Γ sur tous les anneaux définis et une action de l'opérateur φ sur les anneaux $\mathscr{E}$, $\mathscr{E}^\dagger$ et $\mathscr{R}$, définies par les formules

$$\sigma_a(T) = (1+T)^a - 1, \quad \varphi(T) = (1+T)^p - 1.$$

Tous ces anneaux portent des topologies naturelles pour lesquelles les actions de Γ et φ sont continues. Posons $\mathscr{A} \in \{\mathscr{E}, \mathscr{R}\}$. L'anneau $\mathscr{A}$ est muni d'une action de l'opérateur ψ : $\mathscr{A}$ est une extension de degré p de $\varphi(\mathscr{A})$ avec une base formée par les éléments $(1+T)^i$, $i = 0, \dots, p-1$, et on pose

$$\psi\left(\sum_{i=0}^{p-1}(1+T)^i\varphi(f_i)\right) = f_0.$$

Ceci peut être écrit comme

$$\psi = p^{-1}\varphi^{-1} \circ \mathrm{Tr}_{\mathscr{A}/\varphi(\mathscr{A})}.$$

L'opérateur ψ ainsi construit est un inverse à gauche de φ.

Pour $\mathscr{A}$ comme ci-dessus, on note $\Phi\Gamma(\mathscr{A})$ la catégorie des (φ,Γ)-modules sur $\mathscr{A}$.

0E2. *Caractères de $\mathbb{Z}_p^\times$.* Soit $\eta : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ un caractère de conducteur p^n. On définit, pour $b \in \mathbb{Z}_p$, $G(\eta, b) = \sum_{a=1}^{p^n-1} \eta(a)\zeta_{p^n}^{ab}$ la somme de Gauss tordue et on note $G(\eta) = G(\eta, 1)$. On note η^{-1} le caractère de Dirichlet modulo p^n, défini par $\eta^{-1}(n) = \eta(n)^{-1}$ pour $n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. Rappelons deux résultats classiques de la théorie des caractères :

Proposition 0.8. *Soit $\eta : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ un caractère de conducteur p^n. Alors*

— $G(\eta, b) = \eta^{-1}(b)G(\eta, 1)$ *pour tout $b \in \mathbb{Z}_p$.*

— $G(\eta)G(\eta^{-1}) = \eta(-1)p^n$.

Si $f \in \mathrm{LC_c}(\mathbb{Q}_p, \mathbb{C}_p)$ est une fonction localement constante modulo p^n et à support compact, on peut définir sa transformée de Fourier discrète par la formule

$$\hat{f}(x) = p^{-m} \sum_{y \bmod p^m} f(y)e^{-2\pi i x y},$$

où m est un entier arbitraire tel que $m \geq \sup(n, -v_p(x))$, et $e^{-2\pi i x y}$ est la racine de l'unité d'ordre une puissance de p définie par l'application $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[1/p]/\mathbb{Z}$ et par le choix du système $(\zeta_{p^n})_{n\in\mathbb{N}}$ de racines de l'unité : si $k \in \mathbb{N}$ est tel que $a = p^k yx \in \mathbb{Z}_p$, alors $e^{-2\pi i x y} = \zeta_{p^k}^{-a}$. Si η est un caractère de Dirichlet de conducteur p^n, on a

$$\hat{\eta}(x) = \begin{cases} \dfrac{1}{G(\eta^{-1})}\eta^{-1}(p^n x) & \text{si } n > 0, \\ \mathbb{1}_{\mathbb{Z}_p}(x) - p^{-1}\mathbb{1}_{p^{-1}\mathbb{Z}_p}(x) & \text{si } n = 0. \end{cases}$$

En particulier, on observe que $\hat{\eta}$ est à support dans $p^{-n}\mathbb{Z}_p^\times$ (resp. $p^{-1}\mathbb{Z}_p$) si $n > 0$ (resp. si $n = 0$).

0E3. *L'espace des poids p-adiques.* On note $\mathfrak{X} = \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ l'espace des poids p-adiques. Il est un espace rigide analytique, dont les L points, pour une extension finie L de $\mathbb{Q}_p$, paramètrent les caractères continus de $\mathbb{Z}_p^\times$ à valeurs dans $L^\times$. Posons $q = p$ si $p > 2$ et $q = 4$ si $p = 2$. On a $\mathbb{Z}_p^\times = (\mathbb{Z}/q\mathbb{Z})^\times \times (1+q\mathbb{Z}_p)$. L'application logarithme induit un isomorphisme de groupes $(1 + q\mathbb{Z}_p, \times) \xrightarrow{\sim} (q\mathbb{Z}_p, +)$, d'inverse l'application exponentielle. On a un isomorphisme

$$\mathfrak{X} \xrightarrow{\sim} ((\mathbb{Z}/q\mathbb{Z})^\times)^\wedge \times \mathrm{B}(1, 1^-), \quad \eta \mapsto (\eta|_{(\mathbb{Z}/q\mathbb{Z})^\times}, \eta(\exp(q))),$$

où $\mathrm{B}(1, 1^-)$ est la boule unité ouverte centrée en 1. L'inverse de cet isomorphisme envoie $(\chi, z) \in ((\mathbb{Z}/q\mathbb{Z})^\times)^\wedge \times \mathrm{B}(1, 1^-)$ sur le caractère $x \mapsto \chi(\bar{x})z^{\log(x)/q} \in \mathfrak{X}$, ou $\bar{x}$ dénote la réduction modulo p (resp. $2p$ si $p = 2$) de x.

Si $\eta : \mathbb{Z}_p^\times \to L \in \mathfrak{X}$, on note $z_\eta = \eta(\exp(q)) \in \mathrm{B}(1, 1^-)$ la deuxième coordonné de l'image de η par l'isomorphisme ci-dessus et on note $\omega_\eta = \eta'(1) = \log(z_\eta)/q$ son *poids*.

L'espace $\mathfrak{X}$ admet un recouvrement croissant admissible $\mathfrak{X} = \bigcup_n \mathfrak{X}_n$ par des ouverts affinoïdes $\mathfrak{X}_n = ((\mathbb{Z}/q\mathbb{Z})^\times)^\wedge \times \mathrm{B}(1, p^{-1/n})$, ce qui fait de $\mathfrak{X}$ un espace quasi Stein. On note $\mathcal{O}(\mathfrak{X})$ et $\mathcal{O}(\mathfrak{X}_n)$ les anneaux des fonctions analytiques de ces espaces. On a

$$\mathcal{O}(\mathfrak{X}) = \varprojlim_n \mathcal{O}(\mathfrak{X}_n).$$

On dispose [Amice et Vélu 1975] d'un isomorphisme, dû à Amice,

$$\mathscr{D}(\mathbb{Z}_p^\times, L) \xrightarrow{\sim} \mathcal{O}_L(\mathfrak{X}),$$

envoyant $\mu \in \mathscr{D}(\mathbb{Z}_p^\times, L)$ sur la fonction analytique $F_\mu \in \mathcal{O}_L(\mathfrak{X})$ définie par $F_\mu(\eta) = \int_{\mathbb{Z}_p^\times} \eta(x) \cdot \mu$. Comme les polynômes sont denses dans l'espace des fonctions localement analytiques, l'isomorphisme précédent montre qu'une fonction $F \in \mathcal{O}(\mathfrak{X})$ s'annulant sur les caractères $x \mapsto x^k,\ k \in \mathbb{Z}$, est identiquement nulle. On exprime ceci en disant que *les caractères de la forme $x \mapsto x^k,\ k \in \mathbb{Z}$, sont Zariski denses dans $\mathfrak{X}$.*

On définit

$$\mathcal{O}(\mathfrak{X}) \mathbin{\widehat{\otimes}} \mathscr{R} = \varprojlim_{n>0} \varinjlim_{s>0} \varprojlim_{0<r<s} \mathcal{O}(\mathfrak{X}_n) \mathbin{\widehat{\otimes}} \mathscr{E}^{[r,s]},$$

où le produit tensoriel à droite c'est le produit tensoriel complété usuel entre deux espaces de Banach. On peut, plus généralement, considérer des distributions à valeurs dans une limite inductive d'espaces de Fréchet quelconque par les mêmes formules. On dira que f est une fonction analytique sur $\mathfrak{X}$ à valeurs dans $\mathscr{R}$ si elle appartient à $\mathcal{O}(\mathfrak{X}) \mathbin{\widehat{\otimes}} \mathscr{R}$.

I. La fonction L locale d'un (φ,Γ)-module de de Rham

Le présent chapitre contient le premier résultat de cet article, encadré dans l'étude des fonctions L p-adiques des (φ,Γ)-modules de de Rham associées à un système d'Euler, comme démarré par Perrin-Riou [1994]. On construit (théorème I.27), pour un (φ,Γ)-module D de Rham sur $\mathscr{R}$, une extension analytique de l'application logarithme de Perrin-Riou fournissant, à partir d'un élément dans la cohomologie d'Iwasawa de D, une fonction L p-adique, définie sur un ouvert de l'espace de poids.

IA. *Résultat principal.* On a besoin d'introduire quelques notations pour énoncer le résultat principal de cet article. Si $\xi, \delta \in \mathfrak{X}$ sont deux caractères, on définit leur distance par $v_p(\xi - \delta) = v_p(z_\xi - z_\delta)$ si $\xi|_{(\mathbb{Z}/q\mathbb{Z})^\times} = \delta|_{(\mathbb{Z}/q\mathbb{Z})^\times}$ et $v_p(\xi - \delta) = -\infty$ si non, où $z_\xi, z_\delta \in B(1, 1^-)$ sont définis dans la section 0E3. Si $\eta \in \mathfrak{X}$ est un caractère d'ordre fini, on note $\mathrm{c}(\eta)$ la p-partie de son conducteur (de sorte que $\mathrm{cond}(\eta) = p^{\mathrm{c}(\eta)}$) et, pour $N \geq -\infty$, on définit

$$\mathfrak{B}(\eta, N) = \{\xi \in \mathfrak{X} : v_p(\xi - \eta) > p^{N-\mathrm{c}(\eta)}\} \subseteq \mathfrak{X}.$$

Rappelons que $\Gamma^*(j)$ dénote le coefficient principal de la série de Laurent de la fonction $\Gamma(s)$ en $s = j$. Notons, pour $D \in \Phi\Gamma(\mathscr{R})$ de Rham et $\mu \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, D)$,

$$\log\Bigl(\int_\Gamma \eta\chi^{-j} \cdot \mu\Bigr) = \begin{cases} \exp^*\bigl(\int_\Gamma \eta\chi^{-j} \cdot \mu\bigr) & \text{si } j \geq 0, \\ \exp^{-1}\bigl(\int_\Gamma \eta\chi^{-j} \cdot \mu\bigr) & \text{si } j \ll 0. \end{cases}$$

Enfin, on note $r_{m(D)},\ m(D) \in \mathbb{N}$, le rayon de surconvergence de D (cf. la section IB1 ci-dessous). Le résultat principal de ce chapitre est le suivant :

Théorème I.1. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham à poids de Hodge–Tate positifs, $z \in D^{\psi=1}$, et notons $\mu_z = \mathrm{Exp}^*(z) \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, D)$. Il existe un entier $N(D)$,*[9] *et une unique fonction rigide analytique $\Lambda_{D,z} \in \mathcal{O}(\mathfrak{U}_D) \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$, où l'on a posé $\mathfrak{U}_D = \bigcup_{\mathrm{c}(\eta)>m(D)} \mathfrak{B}(\eta, N(D))$, telle que, pour tout $\eta\chi^j \in \mathfrak{U}_D$, où η est*

9. L'entier $N(D)$ ne dépend que du module $\boldsymbol{D}_{\mathrm{pst}}(D)$ et est borné en termes du conducteur de la plus petite extension galoisienne K de $\mathbb{Q}_p$ tel que l'action de $\mathscr{G}_{\mathbb{Q}_p}$ sur $\boldsymbol{D}_{\mathrm{pst}}(D)$ se factorise à travers $\mathscr{G}_K$. Si $D = \boldsymbol{D}_{\mathrm{rig}}(V)$, où $V \in \mathrm{Rep}_L\, \mathscr{G}_{\mathbb{Q}_p}$, la constante $N(D)$ est donc bornée en termes du conducteur de la plus petite extension $K/\mathbb{Q}_p$ sur laquelle V dévient semi-stable.

un caractère de conducteur p^n, $n > 0$, et $j \in \mathbb{Z}$ est tel que $j \geq 0$ ou $j \ll 0$, [10] *on a*

$$\Lambda_{D,z}(\eta\chi^j) = \Gamma^*(j+1)p^{n(j+1)} \cdot \log\left(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\right) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j}.$$

Voici quelques remarques :

- On obtiendra ce théorème (théorème I.27) comme conséquence d'un résultat un peu plus général énoncé dans le théorème I.15.
- Comme les caractères ηx^j sont Zariski denses dans $\mathfrak{X}$, la fonction $\Lambda_{D,z}$ est unique.
- Si D est cristallin, alors l'application $\Delta_{D,z}$ provient par restriction d'une fonction rigide analytique définie sur tout l'espace des poids $\mathfrak{X}$.
- Comme corollaire de ce théorème, on obtiendra une construction partielle de la fonction L p-adique associée à un système d'Euler d'un (φ,Γ)-module de de Rham. Si $D = \boldsymbol{D}_{\mathrm{rig}}(V)$ est le (φ,Γ)-module associé à la représentation p-adique d'une forme modulaire, on verra comment ces valeurs s'interprètent en termes de valeurs spéciales de la fonction L complexe de la forme modulaire.

Voici un bref résumé de ce chapitre. On commence par rappeler les outils et notations nécessaires dont on aura besoin pour la preuve du théorème. On pourra consulter [Nakamura 2014] et [Berger 2003], qui sont les références principales. La structure de la preuve du théorème I.1 est la suivante : Dans la section IB1, on rappelle des généralités sur les (φ,Γ)-modules de de Rham sur l'anneau de Robba. Ensuite (section IC2) on définit la multiplication analytique par un caractère sur un (φ,Γ)-module. Dans la section IC4, on définit la fonction $\Lambda_{D,z}$. Le lemme I.17 calcule le rayon de convergence de $\Lambda_{D,z}$, ce qui explique la définition de l'ouvert $\mathfrak{B}(N)$ du théorème. Les propositions I.22 et I.23 montrent, en utilisant la loi de réciprocité explicite de Perrin-Riou comme démontrée par Nakamura, les propriétés d'interpolation de $\Lambda_{D,z}$, introduisant l'opérateur différentiel auxiliaire ∇_h dans les formules. Enfin, quand D est à poids de Hodge–Tate positifs et $z \in D^{\psi=1} \subseteq \mathbb{N}_{\mathrm{rig}}(D)^{\psi=1}$ on se débarrasse (section IC8) de l'opérateur différentiel ∇_h pour obtenir ainsi l'interpolation voulue.

IB. *Généralités sur les (φ,Γ)-modules.* Notons $\Phi\Gamma(\mathscr{R})$ la catégorie de (φ,Γ)-modules sur $\mathscr{R}$.

IB1. *Sous-modules naturels de D.* Soit $D \in \Phi\Gamma(\mathscr{R})$ de rang d. L'algèbre de Lie de Γ agit sur D (cf. [Berger 2002, §5.1]) via l'opérateur L-linéaire

$$\nabla = \lim_{a\to 1} \frac{\sigma_a - 1}{a-1} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = \frac{1}{\log(\chi(\gamma))}\sum_{i=1}^{+\infty}(-1)^{i+1}\frac{(\gamma-1)^i}{i},$$

où $\gamma \in \Gamma$ dénote n'importe quel élément sans torsion de Γ, ce qui définit un opérateur différentiel au-dessus de l'opérateur $\nabla = t(1+T)\frac{d}{dT}$ agissant sur $\mathscr{R}$.

Lemme I.2 [Berger 2008, théorème I.3.3]. *Il existe $\varepsilon > 0$ et, pour tout $0 < s \leq \varepsilon$, des uniques sous-$\mathscr{E}^{]0,s]}$-modules $D^{]0,s]}$ de D, satisfaisant*

- $D = \mathscr{R} \otimes_{\mathscr{E}^{]0,s]}} D^{]0,s]}$.
- $\varphi(D^{]0,s]}) \subseteq D^{]0,s/p]}$, *et on a un isomorphisme* $\mathscr{E}^{]0,s/p]} \otimes_{\mathscr{E}^{]0,s]}} D^{]0,s]} \to D^{]0,s/p]}$, $f \otimes x \mapsto f\varphi(x)$, *pour tout n tel que $r_n \leq \varepsilon$.*

10. Plus précisément, j doit être suffisamment petit de sorte que $\exp_{D(-j)} : \boldsymbol{D}_{\mathrm{dR}}(D(-j)) \to H^1_{\varphi,\gamma}(D(-j))$ soit un isomorphisme. Il suffirait donc de demander $j < h_0 - 2$, où h_0 dénote le plus petit poids de Hodge–Tate de D (cf. [Berger 2002, Theorem 0.9]).

De plus, les modules $D^{]0,s]}$ *sont stables par* Γ *et* ∇.

On définit $m(D)$ comme le plus petit entier tel que le lemme I.2 est vrai avec $r_{m(D)} \le \varepsilon$ et on dit que $r_{m(D)}$ est le rayon de surconvergence de D. Pour $0 < r < s \le r_{m(D)}$, on pose

$$D^{[r,s]} = \mathscr{E}^{[r,s]} \otimes_{\mathscr{E}^{]0,s]}} D^{]0,s]}.$$

On a alors

$$D = \varinjlim_{s>0} \varprojlim_{0<r<s} D^{[r,s]},$$

ce qui montre que D est un espace de type LF (i.e., limite inductive d'espaces de Fréchet).

Rappelons que l'on a des morphismes de localisation

$$\varphi^{-n} : \mathscr{E}^{]0,r_n]} \hookrightarrow L_n[\![t]\!]$$

envoyant T sur $\zeta_{p^n} e^{t/p^n} - 1$. Pour $n \ge m(D)$ on définit

$$\boldsymbol{D}^+_{\mathrm{dif},n}(D) = L_n[\![t]\!] \otimes_{\varphi^{-n},\mathscr{E}^{]0,r_n]}} D^{]0,r_n]},$$
$$\boldsymbol{D}_{\mathrm{dif},n}(D) = L_n(\!(t)\!) \otimes_{L_n[\![t]\!]} \boldsymbol{D}^+_{\mathrm{dif},n}(D),$$

qui sont des $L_n[\![t]\!]$ et $L(\!(t)\!)$-modules, respectivement, libres de rang d et munis d'une action semi-linéaire de Γ. Finalement, on définit

$$\boldsymbol{D}_{\mathrm{dif}}(D) = \varinjlim_n \boldsymbol{D}_{\mathrm{dif},n}(D), \quad \boldsymbol{D}^+_{\mathrm{dif}}(D) = \varinjlim_n \boldsymbol{D}^+_{\mathrm{dif},n}(D),$$

qui sont, respectivement, des $L_\infty(\!(t)\!) = \bigcup_n L_n(\!(t)\!)$ et $L_\infty[\![t]\!] = \bigcup_n L_n[\![t]\!]$-modules libres de rang d.

IB2. *Théorie de Hodge p-adique.* Comme l'on a déjà remarqué, la plupart des objets de la théorie de Hodge p-adique peuvent être exprimés purement en terme des (φ,Γ)-modules. Suivant ce programme, commencé par Fontaine [1990], on définit les invariants suivants :

Définition 1. Soit $D \in \Phi\Gamma(\mathscr{R})$ de rang d. On définit

$$\boldsymbol{D}_{\mathrm{cris}}(D) = (D[1/t])^\Gamma = (D \otimes_{\mathscr{R}} \mathscr{R}[1/t])^\Gamma, \quad \boldsymbol{D}_{\mathrm{dR}}(D) = (\boldsymbol{D}_{\mathrm{dif}}(D))^\Gamma,$$

qui sont des L-espaces vectoriels de dimension finie.

On munit $\boldsymbol{D}_{\mathrm{dR}}(D)$ de sa filtration de Hodge, donnée par $\mathrm{Fil}^i\, \boldsymbol{D}_{\mathrm{dR}}(D) = \boldsymbol{D}_{\mathrm{dR}}(D) \cap t^i \boldsymbol{D}^+_{\mathrm{dif}} = (t^i \boldsymbol{D}^+_{\mathrm{dif}})^\Gamma$. On observe que $\boldsymbol{D}_{\mathrm{cris}}(D)$ est muni d'une action bijective du Frobenius φ ainsi que d'une filtration induite par l'inclusion $\boldsymbol{D}_{\mathrm{cris}}(D) \subseteq \boldsymbol{D}_{\mathrm{dR}}(D)$ définie par $x \in \boldsymbol{D}_{\mathrm{cris}}(D) \mapsto \iota_n(\varphi^n(x)) \in \boldsymbol{D}_{\mathrm{dR}}(D)$, où on a noté $\iota_n = \varphi^{-n} : D^{]0,r_n]}[1/t] \to \boldsymbol{D}_{\mathrm{dif}}(D)$ l'application de localisation.[11] On a

$$\dim_L \boldsymbol{D}_{\mathrm{cris}}(D) \le \dim_L \boldsymbol{D}_{\mathrm{dR}}(D) \le \mathrm{rang}_{\mathscr{R}}\, D,$$

où la première inégalité est évidente par ce qui précède et la dernière suit en remarquant que $\boldsymbol{D}_{\mathrm{dif}}(D)$ est un $L_\infty(\!(t)\!)$-espace vectoriel de rang $d = \mathrm{rang}_{\mathscr{R}}\, D$ et $\boldsymbol{D}_{\mathrm{dR}}(D) = (\boldsymbol{D}_{\mathrm{dif}}(D))^\Gamma$ est donc un $(L_\infty(\!(t)\!))^\Gamma = L$-espace vectoriel de rang $\le d$.

11. Il existe ici un petit abus évident en notant par φ^{-n} deux applications différentes, l'une étant l'application de localisation notée usuellement ι_n et l'autre l'inverse de l'opérateur φ agissant sur $\boldsymbol{D}_{\mathrm{cris}}(D)$, mais cela ne devrait pas causer de problèmes de lecture.

Définition 2. Soit D un (φ,Γ)-module sur $\mathscr{R}$. On dit que D est *cristallin* (resp. *de Rham*) si l'inégalité $\dim_{\mathbb{Q}_p} \boldsymbol{D}_{\mathrm{cris}}(D) \leq \mathrm{rang}_{\mathscr{R}} D$ (resp. $\dim_{\mathbb{Q}_p} \boldsymbol{D}_{\mathrm{dR}}(D) \leq \mathrm{rang}_{\mathscr{R}} D$) est une égalité.

Si D est de Rham, on définit ses *poids de Hodge–Tate* comme les opposés des entiers où la filtration change, c'est-à-dire l'ensemble $\{-h \in \mathbb{Z} : \mathrm{Fil}^h \boldsymbol{D}_{\mathrm{dR}}(D)/\mathrm{Fil}^{h+1} \boldsymbol{D}_{\mathrm{dR}}(D) \neq 0\}$.

IB3. *L'équation différentielle p-adique* $\mathbb{N}_{\mathrm{rig}}(D)$. Rappelons la construction de Berger [2002; 2008] de l'équation différentielle p-adique $\mathbb{N}_{\mathrm{rig}}(D)$ associée à un (φ,Γ)-module de de Rham D sur $\mathscr{R}$.

Proposition I.3 [Berger 2008, théorème III.2.3]. *Soit $D \in \Phi\Gamma(\mathscr{R})$ de rang d, de Rham, et, pour chaque $n \geq m(D)$, posons*

$$\mathbb{N}_{\mathrm{rig}}^{]0,r_n]}(D) = \left\{x \in D^{]0,r_n]}[1/t] : \ \varphi^{-m}(x) \in L_m[\![t]\!] \otimes_L \boldsymbol{D}_{\mathrm{dR}}(D) \text{ pour tout } m \geq n\right\},$$

et $\mathbb{N}_{\mathrm{rig}}(D) = \varinjlim_n \mathbb{N}_{\mathrm{rig}}^{]0,r_n]}(D)$. *Alors*, $\mathbb{N}_{\mathrm{rig}}(D)$ *est un* (φ,Γ)*-module sur* $\mathscr{R}$, *de rang* d, *qui satisfait*
— $\mathbb{N}_{\mathrm{rig}}(D)[1/t] = D[1/t]$.
— $\boldsymbol{D}^{+}_{\mathrm{dif},n}(\mathbb{N}_{\mathrm{rig}}(D)) = L_n[\![t]\!] \otimes_L \boldsymbol{D}_{\mathrm{dR}}(D)$ *pour tout* $n \geq m(D)$.
— $\nabla(\mathbb{N}_{\mathrm{rig}}(D)) \subseteq t\mathbb{N}_{\mathrm{rig}}(D)$.

Le (φ,Γ)-module $\mathbb{N}_{\mathrm{rig}}(D)$ ainsi obtenu est de Rham à poids de Hodge–Tate tous nuls. Remarquons que l'on peut reconstruire D, à partir de la donnée de $\mathbb{N}_{\mathrm{rig}}(D)$ et de la filtration de Hodge sur $\boldsymbol{D}_{\mathrm{dR}}(D)$, en utilisant la formule

$$D = \left\{x \in \mathbb{N}_{\mathrm{rig}}(D)[1/t] : \ \varphi^{-n}(x) \in \mathrm{Fil}^0\big(L_n((t)) \otimes_{\mathbb{Q}_p} \boldsymbol{D}_{\mathrm{dR}}(D)\big) \quad \text{pour tout } n \gg 0\right\}.$$

Si les poids de Hodge–Tate de D sont contenus dans $[a,b]$, on a des inclusions $t^{-a}D \subseteq \mathbb{N}_{\mathrm{rig}}(D) \subseteq t^{-b}D$.

La troisième propriété de la proposition I.3 caractérisant $\mathbb{N}_{\mathrm{rig}}(D)$ permet de définir un opérateur différentiel

$$\partial := \frac{1}{t}\nabla : \mathbb{N}_{\mathrm{rig}}(D) \to \mathbb{N}_{\mathrm{rig}}(D),$$

satisfaisant les identités $\partial \circ \varphi = p\,\varphi \circ \partial$ et $\partial \circ \sigma_a = a\,\sigma_a \circ \partial$. Si D est un (φ,Γ)-module de de Rham sur $\mathscr{R}$, on notera $\Delta = \mathbb{N}_{\mathrm{rig}}(D)$.

IB4. *Les anneaux de Fontaine.* Rappelons la construction des anneaux de Fontaine associés à une extension galoisienne finie K de $\mathbb{Q}_p$. Notons $K_n = KF_n = K(\mu_{p^n})$, $n \geq 1$, $K_\infty = KF_\infty = \bigcup_{n\geq 1} K_n$, $K_0 = K \cap \mathbb{Q}_p^{\mathrm{nr}}$ la plus grande sous-extension de K non ramifiée et K_0' la plus grande extension non ramifiée de K_0 dans K_∞. Notons $\mathscr{H}_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$, $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.

La théorie du corps des normes (cf., par exemple, [Colmez 2008] ou [Berger 2008, §I.2]) permet de construire des extensions étales $\mathscr{E}_K^\dagger/\mathscr{E}_{\mathbb{Q}_p}^\dagger$ de degré $[K_\infty : F_\infty]$, munies d'une action du Frobenius φ et du groupe Γ_K. Plus précisément, soit $\widetilde{\mathbb{E}} = \varprojlim_{x\mapsto x^p} \mathbb{C}_p = \mathbb{C}_p^\flat$ le corps basculé de $\mathbb{C}_p$. Il est un corps de caractéristique p, algébriquement clos et muni d'une valuation $v_{\mathrm{E}}(x) = v_p(x^{(0)})$ pour laquelle il est complet. Notons $1 \neq \varepsilon = (1, \varepsilon^{(1)}, \ldots) \in \widetilde{\mathbb{E}}$, et $\widetilde{\mathbb{E}}^+$ l'anneau des entiers de $\widetilde{\mathbb{E}}$, qui s'identifie à la limite projective $\varprojlim_{x\mapsto x^p} \mathscr{O}_{\mathbb{C}_p}/\mathfrak{a}$ pour n'importe quel idéal $\mathfrak{a} \subseteq \mathscr{O}_{\mathbb{C}_p}$ contenant p et différent de l'idéal maximal de $\mathscr{O}_{\mathbb{C}_p}$. Soient $\mathbb{E}_{\mathbb{Q}_p} = \mathbb{F}_p((\varepsilon - 1))$ et $\mathbb{E}$ la clôture séparable de $\mathbb{E}_{\mathbb{Q}_p}$ dans $\widetilde{\mathbb{E}}$. La théorie du corps des normes permet aussi de montrer que $\mathrm{Gal}(\mathbb{E}/\mathbb{E}_{\mathbb{Q}_p}) \cong \mathscr{H}_{\mathbb{Q}_p}$. Si $K/\mathbb{Q}_p$ est une extension finie galoisienne de $\mathbb{Q}_p$, on pose $\mathbb{E}_K = \mathbb{E}^{\mathscr{H}_K}$. On dispose d'une application bien définie et injective $\varprojlim_{x\mapsto N_{K_n/K_{n-1}}(x)} \mathscr{O}_{K_n} \to \widetilde{\mathbb{E}}^+$ d'image l'anneau des entiers $\mathbb{E}_K^+$ de $\mathbb{E}_K$, qui fournit une uniformisante $\bar{\pi}_K$

de l'extension $\mathbb{E}_K/\mathbb{E}_{\mathbb{Q}_p}$ (l'image d'un système compatible $(\omega_n)_{n\in\mathbb{N}}$ où ω_n est une uniformisante de K_n pour n assez grand). Soit $\bar{P}(X) = X^{d_{K_\infty}} + \bar{a}_{d_{K_\infty}-1}X^{d_{K_\infty}-1} + \cdots + \bar{a}_0 \in \mathbb{E}_{\mathbb{Q}_p}[X]$, où $d_{K_\infty} = [K_\infty : F_\infty]$ et $\bar{a}_i \in \mathbb{E}_{\mathbb{Q}_p}$, le polynôme minimal de $\bar{\pi}_K$ sur $\mathbb{E}_{\mathbb{Q}_p}$. Enfin, si $P \in \mathbb{Z}_p[\![T]\!][X]$ est tel que sa réduction modulo p évaluée en $T = \varepsilon - 1$ est $\bar{P}(X)$, alors $\mathscr{E}_K^\dagger = \mathscr{E}_{\mathbb{Q}_p}^\dagger[X]/P(X)$ et on note π_K l'image de X dans ce quotient, dont la réduction modulo p est $\bar{\pi}_K$.

L'anneau $\mathscr{E}_K^\dagger$ s'identifie à l'anneau des séries formelles $f(T_K) = \sum_{k\in\mathbb{Z}} a_k T_K^k$, $a_k \in K_0'$, à coefficients bornés, qui convergent sur une couronne $0 < v_p(T_K) \le r$ pour un r assez petit (qui dépend de f). On peut de la sorte définir les anneau $\mathscr{E}_K^{[r,s]}$, qui s'identifient à l'espace des séries de Laurent à coefficients dans K_0', convergentes sur la couronne $C_{[r/e,s/e]} = \{z \in \mathbb{C}_p : r/e \le v_p(z) \le s/e\}$, où e dénote l'indice de ramification de l'extension K_∞/F_∞, muni de la norme spectrale $v^{[r,s]}$, ainsi que $\mathscr{E}_K^{]0,s]}$, $\mathscr{E}_K^{\dagger,r}$, etc. En complétant $\mathscr{E}_K^\dagger$ pour la famille de normes $v^{[r,s]}$, on obtient une extension étale $\mathscr{R}_K/\mathscr{R}_{\mathbb{Q}_p}$ et on a

$$\mathrm{Gal}(\mathscr{R}_K/\mathscr{R}_{\mathbb{Q}_p}) = \mathrm{Gal}(\mathscr{E}_K^\dagger/\mathscr{E}_{\mathbb{Q}_p}^\dagger) = \mathrm{Gal}(K_\infty/F_\infty).$$

On a un opérateur de dérivation ∂ agissant sur $\mathscr{R}_K$: si $K/\mathbb{Q}_p$ est une extension non ramifiée, on a $T_K = T$ et $\partial f(T) = (1+T)f'(T)$ pour $f(T) \in \mathscr{R}_K$; si $K/\mathbb{Q}_p$ est ramifiée et $P \in \mathbb{Z}_p[\![T]\!][X]$ dénote le polynôme minimal de $T_K \in \mathscr{E}_K^\dagger$ sur $\mathscr{E}_{\mathbb{Q}_p}^\dagger$ comme ci-dessus, l'identité $P(T_K) = 0$ et le fait que ∂ est une dérivation donnent la formule

$$\partial T_K = -P'(T_K)^{-1}(\partial P)(T_K),$$

où, si $P = \sum f_i X^i \in \mathbb{Z}_p[\![T]\!][X]$, $\partial P = \sum \partial f_i \cdot X^i$.

Enfin, soit $\ell_T = \log(T)$ une variable formelle et étendons les actions φ et Γ_K sur $\mathscr{R}_K$ à des actions sur $\mathscr{R}_K[\ell_T]$ par les formules

$$\varphi(\ell_T) = p\ell_T + \log(\varphi(T)/T^p), \quad \gamma(\ell_T) = \ell_T + \log(\gamma(T)/T).$$

La dérivation ∂ agit sur ℓ_T par $\partial\ell_T = T^{-1}\partial T = 1 + T^{-1}$. On définit un opérateur de monodromie sur $\mathscr{R}_K[\ell_T]$ comme la dérivation $\mathscr{R}_K$-linéaire N telle que $N(\ell_T) = -p/(p-1)$.

IB5. *Théorème de monodromie p-adique et surconvergence.* Soit

$$\mathfrak{d}_{\mathbb{E}_K/\mathbb{E}_{\mathbb{Q}_p}} \subseteq \mathbb{E}_K$$

la différente de l'extension $\mathbb{E}_K/\mathbb{E}_{\mathbb{Q}_p}$ et posons $\delta_K = d_{K_\infty}\cdot v_{\mathbb{E}}(\mathfrak{d}_{\mathbb{E}_K/\mathbb{E}_{\mathbb{Q}_p}}) \in \mathbb{N}$, où $d_{K_\infty} = [\mathbb{E}_K : \mathbb{E}_{\mathbb{Q}_p}] = [K_\infty : F_\infty]$ comme plus haut. Rappelons (cf. [Colmez 2008, Proposition 4.12]) que, si $c(K)$ dénote le conducteur de K [12] et $n \ge c(K)+1$ est un entier, alors $[K_n : F_n] = d_{K_\infty}$ et $\delta_K = d_{K_\infty} p^n v_p(\mathfrak{d}_{K_n/F_n})$. En particulier, si K a suffisamment de racines de l'unité, dans la terminologie de [Colmez et Nizioł 2017], alors $v_K = [K : \mathbb{Q}_p] v_p(\mathfrak{d}_{K/\mathbb{Q}_p})$. On aura besoin des faits suivants :

Lemme I.4 ([Colmez et Nizioł 2017, Lemma 2.17]). *Si* $s < (\delta_K + 1)^{-1}$, *alors* $\partial T_K \in T^{-\delta_K}\mathscr{O}^\times_{\mathscr{E}_K^{\dagger,s}}$ *et on a* $v^{[r,s]}(\partial T_K) \ge -1$ *pour tout* $0 < r < s$.

Soit $D \in \Phi\Gamma(\mathscr{R})$ de Rham et notons $\Delta = \mathbb{N}_{\mathrm{rig}}(D)$. D'après le théorème de monodromie p-adique (cf. [Berger 2008, théorème III.2.1]), il existe une extension galoisienne finie K de $\mathbb{Q}_p$ tel que l'on a un

12. Le conducteur de K est la borne inférieure de l'ensemble des t tels que le groupe de ramification supérieure $\mathscr{G}_{\mathbb{Q}_p}^{(t)}$ agit trivialement sur K.

isomorphisme

$$\mathscr{R}_K[\ell_T]\otimes_{K_0}(\mathscr{R}_K[\ell_T]\otimes_{\mathscr{R}}\Delta)^{\Gamma_K}\cong\mathscr{R}_K[\ell_T]\otimes_{\mathscr{R}}\Delta.$$

On note

$$D_{\mathrm{pst}}=(\mathscr{R}_K[\ell_T]\otimes_{\mathscr{R}}\Delta)^{\Gamma_K}$$

l'espace des Γ_K-solutions de Δ. C'est un $(\varphi, N, \mathrm{Gal}(K/\mathbb{Q}_p))$-module filtré (la filtration dépend de la donnée de D). Le résultat suivant permet de reconstruire Δ à partir de D_{pst}.

Proposition I.5 [Berger 2008, théorèmes III.2.1 et C]. *Soient D et D_{pst} comme ci-dessus. Si K est une extension galoisienne finie de $\mathbb{Q}_p$ telle que $\mathscr{G}_{\mathbb{Q}_p}$ agit sur D_{pst} à travers* $\mathrm{Gal}(K/\mathbb{Q}_p)$, *alors*

$$\Delta=(\mathscr{R}_K[\ell_T]\otimes_{K_0}D_{\mathrm{pst}})^{\mathrm{Gal}(K_\infty/F_\infty),N=0},$$

le groupe $\mathrm{Gal}(K_\infty/F_\infty)$ *agissant sur D_{pst} à travers* $\mathrm{Gal}(K/(K\cap F_\infty))\subseteq\mathrm{Gal}(K/\mathbb{Q}_p)$, *et sur $\mathscr{R}_K$ à travers* $\mathrm{Gal}(K_\infty/F_\infty)=\mathrm{Gal}(\mathscr{R}_K/\mathscr{R}_{\mathbb{Q}_p})$ *(il agit trivialement sur l'élément ℓ_T).*

On récupère l'action de Γ via l'action résiduelle du groupe Γ_K sur le module Δ, l'action de φ via son action (diagonale) sur le produit tensoriel et l'action de l'opérateur ∂ à travers celle sur $\mathscr{R}_K[l_K]$. Le lemme suivant sera très utile pour nos constructions futures.

Lemme I.6. *Il existe, pour tout $0<s<(\delta_K+1)^{-1}$, des sous-$\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}$-modules $\Delta^{]0,s]}$ de Δ, munis d'une famille de valuations $(v^{[r,s]})_{0<r<s}$, satisfaisant les conditions du lemme I.2, et de sorte que, pour tout $0<r<s$, la norme $v^{[r,s]}$ de l'opérateur ∂ sur $\Delta^{]0,s]}$ soit $\geq -s-1$.*

Démonstration. Débarrassons-nous d'abord de l'opérateur N. Le module $M=K_0[\ell_T]\otimes_{K_0}D_{\mathrm{pst}}$ est un $K_0[\ell_T]$-module libre de rang d muni d'une action de $G=\mathrm{Gal}(K_\infty/F_\infty)$ (agissant sur le facteur de droite) et de N (agissant diagonalement) et on affirme qu'il contient un système libre de générateurs $w_1,\ldots,w_d$ tués par N. En effet, notons $\alpha=-p/(p-1)$ la valeur de N sur ℓ_T et posons, pour $v_1,\ldots,v_d$ une base quelconque de D_{pst},

$$w_i=\sum_{k=0}^{+\infty}(-1)^k\frac{\alpha^{-k}}{k!}\ell_T^k\otimes N^k v_i,\quad 1\leq i\leq d,$$

chaque somme étant une somme finie car N est nilpotent. Alors les w_i sont par construction tués par N et ils forment un système libre de générateurs de M, car l'opérateur $\sum_{k=0}^{+\infty}(-1)^k(\alpha^{-k}/k!)\ell_T^k\otimes N^k$ est inversible, N étant nilpotent. Le module $W=(K_0[\ell_T]\otimes D_{\mathrm{pst}})^{N=0}$ est donc un $(K_0[\ell_T])^{N=0}=K_0$-espace vectoriel de dimension d, muni d'une action de G, et on a

$$K_0[\ell_T]\otimes_{K_0}W=K_0[\ell_T]\otimes_{K_0}D_{\mathrm{pst}}.$$

Soit $v_1,\ldots,v_d$ une base de D_{pst} de sorte que la matrice de l'opérateur N soit à coefficients entiers, [13] et soient $w_1,\ldots,w_d$ les éléments associés à cette base formant une base de W fournis par le paragraphe précédent. Posons, pour $0<s<(\delta_K+1)^{-1}$,

$$\Delta^{]0,s]}=(\mathscr{E}_K^{]0,s]}\otimes W)^G\subseteq\mathscr{E}_K^{]0,s]}\otimes W\subseteq\mathscr{R}_K\otimes W.$$

13. Ceci est possible grâce au fait que N est nilpotent : si $v_1,\ldots,v_d$ est une base de D_{pst} de sorte que la matrice de N soit triangulaire supérieure avec des zéros sous la diagonale, alors on peut choisir des entiers $0<n_2<n_3<\cdots<n_d$ de sorte que la matrice de N dans la base $v_1, p^{n_2}v_2,\ldots,p^{n_d}v_d$ soit à coefficients entiers comme on voulait.

On a $G \cong \mathrm{Gal}(\mathscr{R}_K/\mathscr{R}_{\mathbb{Q}_p})$, et le théorème de Hilbert 90 implique que $H^1(G, \mathrm{GL}_d(\mathscr{R}_K)) = 1$. Le module W est donc $\mathscr{R}_K$-admissible au sens de Fontaine et on a un isomorphisme

$$\mathscr{R}_K \otimes_{\mathscr{R}_{\mathbb{Q}_p}} (\mathscr{R}_K \otimes_{K_0} W)^G \cong \mathscr{R}_K \otimes_{K_0} W,$$

induit par $f \otimes (g \otimes x) \mapsto fg \otimes x$, de $\mathscr{R}_K$-modules topologiques de rang d munis d'une action de G. Cet isomorphisme est en outre compatible avec les structures présentes sur $\mathscr{R}_K$. Cela veut dire en particulier que le $\mathscr{R}_{\mathbb{Q}_p}$-module $(\mathscr{R}_K \otimes_{K_0} W)^G$ est de dimension d.

Si $s < (\delta_K + 1)^{-1}$, alors $\mathscr{R}_K = \mathscr{R}_{\mathbb{Q}_p} \otimes_{\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}} \mathscr{E}_K^{]0,s]}$,[14] d'où

$$(\mathscr{R}_K \otimes_{K_0} W)^G = \big(\mathscr{R}_{\mathbb{Q}_p} \otimes_{\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}} \mathscr{E}_K^{]0,s]} \otimes_{K_0} W\big)^G = \mathscr{R}_{\mathbb{Q}_p} \otimes_{\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}} (\mathscr{E}_K^{]0,s]} \otimes_{K_0} W)^G,$$

ce qui montre que

$$\Delta = \mathscr{R} \otimes_{\mathscr{E}^{]0,s]}} \Delta^{]0,s]}.$$

De plus, le fait que $\mathscr{E}_K^{]0,s/p]} \otimes_{\mathscr{E}_K^{]0,s]}} \varphi(\mathscr{E}_K^{]0,s]}) = \mathscr{E}_K^{]0,s/p]}$ et les formules pour l'action de φ sur ℓ_T montrent que la deuxième condition du lemme I.2 est satisfaite.

Les modules $\mathscr{E}_K^{]0,s]} \otimes W$ sont des $\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}$-modules libres, dont une base est donnée par les éléments $(T_K^i \otimes w_j)_{0 \le i \le d_\infty,\, 1 \le j \le d}$, et ils sont munis d'une famille $(v^{[r,s]})_{0<r<s}$ de valuations définies par

$$v^{[r,s]}\Big(\sum_{i,j} f_{ij} \cdot T_k^i \otimes w_j\Big) = \inf_{i,j} v^{[r,s]}(f_{ij}) \quad (f_{ij} \in \mathscr{E}_{\mathbb{Q}_p}^{]0,s]}).$$

Calculons la norme de l'opérateur ∂ sur la base décrite ci-dessus pour une de ces valuations. On a $\partial(\ell_T) = 1 + T^{-1}$ et donc

$$\begin{aligned}
\partial(w_i) &= \partial\Big(\sum_{k=0}^{+\infty} (-1)^k \frac{\alpha^{-k}}{k!} \ell_T^k \otimes N^k v_i\Big)\\
&= (1+T^{-1}) \sum_{k=1}^{+\infty} (-1)^k \frac{\alpha^{-k}}{(k-1)!} \ell_T^{k-1} \otimes N^k v_i\\
&= -\alpha^{-1}(1+T^{-1}) \sum_{k=0}^{+\infty} (-1)^k \frac{\alpha^{-k}}{k!} \ell_T^k \otimes N^k (N v_i)\\
&= \sum_{l=1}^{d} -a_l \alpha^{-1} (1+T^{-1}) \sum_{k=0}^{+\infty} (-1)^k \frac{\alpha^{-k}}{k!} \ell_T^k \otimes N^k v_l\\
&= \sum_{l=1}^{d} -a_l \alpha^{-1} (1+T^{-1}) w_l,
\end{aligned}$$

où $N v_i = \sum_{l=1}^d a_l \cdot v_l$, avec $v_p(a_l) \ge 0$ pour tout l, par le choix de la base $v_1, \dots, v_d$ de D_{pst}. On en déduit que, si A dénote l'anneau des entiers de K_0 et W_0 dénote le A-module engendré par les éléments

14. Comme on le voit, par exemple, en notant que, si $s < (\delta_K + 1)^{-1}$, $\mathscr{E}_K^{]0,s]}$ est un $\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}$-module libre de rang d_{K_∞}, dont une base est formée par $1, T_K, \dots, T^{d_{K_\infty}-1}$, et donc $\mathscr{R}_{\mathbb{Q}_p} \otimes_{\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}} \mathscr{E}_K^{]0,s]} = \bigoplus_i \mathscr{R}_{\mathbb{Q}_p} \otimes_{\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}} \mathscr{E}_{\mathbb{Q}_p}^{]0,s]} \cdot T_K^i = \bigoplus_i \mathscr{R}_{\mathbb{Q}_p} \cdot T_K^i = \mathscr{R}_K$.

$w_1, \dots, w_d$, alors (rappelons que $\alpha = -p/(p-1)$)

$$\partial(W_0) \subseteq p^{-1}(1+T^{-1}) \cdot W_0.$$

On en déduit

$$\begin{aligned} v^{[r,s]}(\partial(T_K^i \otimes w_j)) &= v^{[r,s]}(i\partial(T_K)T_K^{i-1} \otimes w_j + T_K^i \otimes \partial w_j) \\ &\geq \inf\{v^{[r,s]}(i\partial(T_K)T_K^{i-1} \otimes w_j),\ v^{[r,s]}(p^{-1}(1+T^{-1}))\} \\ &\geq \inf\{-1, -1-s\} \\ &= -s-1, \end{aligned}$$

où dans la première égalité on a utilisé la définition de ∂, dans la deuxième l'inégalité triangulaire et l'inclusion $\partial(W_0) \subseteq p^{-1}(1+T^{-1}) \cdot W_0$ et dans la troisième l'estimation du lemme I.4 et le fait que $v^{[r,s]}(p^{-1}(1+T^{-1})) = -1-s$.

Vu que l'on a borné la norme de l'opérateur ∂ sur une base de $\mathscr{E}_K^{]0,s]} \otimes W$, un petit calcul montre que

$$v^{[r,s]}(\partial z) \geq v^{[r,s]}(z) - s - 1$$

pour tout $z \in \mathscr{E}_K^{]0,s]} \otimes W$. En effet, écrivons $z = \sum_{i,j} f_{ij} \cdot T_k^i \otimes w_j$, $f_{ij} \in \mathscr{E}_{\mathbb{Q}_p}^{]0,s]}$, de sorte que $\partial(z) = \sum_{i,j} \partial f_{ij} \cdot T_k^i \otimes w_j + f_{ij} \cdot \partial(T_k^i \otimes w_j)$. On a alors

$$\begin{aligned} v^{[r,s]}(\partial z) &\geq \inf\left\{v^{[r,s]}\left(\sum_{ij}(\partial f_{ij})(T_K^i \otimes w_j)\right),\ v^{[r,s]}\left(\sum_{ij} f_{ij}\partial(T_K^i \otimes w_j)\right)\right\} \\ &= \inf\left\{\inf_{ij} v^{[r,s]}(\partial f_{ij}),\ v^{[r,s]}\left(\sum_{ij} f_{ij}\partial(T_K^i \otimes w_j)\right)\right\} \\ &\geq \inf\left\{\inf_{ij} v^{[r,s]}(f_{ij}) - s, \inf_{ij} v^{[r,s]}(f_{ij}) - s - 1\right\} \\ &= \inf_{ij} v^{[r,s]}(f_{ij}) - s - 1 = v^{[r,s]}(z) - s - 1, \end{aligned}$$

où la première ligne suit de l'inégalité triangulaire, la deuxième suit de la définition de $v^{[r,s]}$ sur $\mathscr{E}_K^{]0,s]} \otimes W$, la troisième de l'inégalité (sur $\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}$) $v^{[r,s]}(\partial f_{ij}) \geq v^{[r,s]}(f_{ij}) - s$ et du calcul du paragraphe précédent et la dernière suit en appliquant la définition de $v^{[r,s]}$ une autre fois.

Enfin, $\Delta^{]0,s]} \subseteq \mathscr{E}_K^{]0,s]} \otimes W$ est stable par ∂, il est muni par restriction d'une famille de valuations $v^{[r,s]}$ et, par ce qui précède, la norme $v^{[r,s]}$ de ∂ agissant sur $\Delta^{]0,s]}$ est bornée par $-s-1$, ce qui permet de conclure. □

Remarque I.7. — Si $0 < r < s < (\delta_K + 1)^{-1}$, les modules

$$\Delta^{[r,s]} = \mathscr{E}_{\mathbb{Q}_p}^{[r,s]} \otimes_{\mathscr{E}_{\mathbb{Q}_p}^{]0,s]}} \Delta^{]0,s]} = (\mathscr{E}_K^{[r,s]} \otimes W)^G,$$

complétés de $\Delta^{]0,s]}$ pour la valuation $v^{[r,s]}$, sont des $\mathscr{E}^{[r,s]}$-modules de rang d, munis de la valuation $v^{[r,s]}$ pour les quelles la norme de ∂ est $\geq -s-1$. On a

$$\Delta = \varinjlim_{s>0} \varprojlim_{0<r<s} \Delta^{[r,s]}.$$

— Le lemme I.6 ci-dessus montre que le rayon de surconvergence du module Δ peut être majoré en termes du conducteur de la plus petite extension galoisienne $K/\mathbb{Q}_p$ telle que $\mathscr{G}_K$ agit trivialement sur D_{pst}. Si $V \in \mathrm{Rep}_L\, \mathscr{G}_{\mathbb{Q}_p}$ est de Rham, ceci permet de borner le rayon de surconvergence de $\Delta = \mathbb{N}_{\mathrm{rig}}(V)$ en termes de la plus petite extension (galoisienne) K de $\mathbb{Q}_p$ sur laquelle V dévient semi-stable.

IB6. (φ,Γ)-*modules relatifs.* Soit A une algèbre affinoïde sur $\mathbb{Q}_p$. L'anneau de Robba $\mathscr{R}_A$ relatif à A est défini en posant, pour $0 < r < s$,

$$\mathscr{R}_A^{[r,s]} = \mathscr{R}^{[r,s]} \mathbin{\widehat{\otimes}} A, \quad \mathscr{R}_A^{]0,s]} = \varprojlim_{0<r<s} \mathscr{R}_A^{[r,s]}, \quad \mathscr{R}_A = \varinjlim_{s>0} \mathscr{R}_A^{]0,s]},$$

où le premier produit tensoriel est le produit tensoriel complété entre deux espaces de Banach. On peut montrer que $\mathscr{R}_A = \mathscr{R}_{\mathbb{Q}_p} \mathbin{\widehat{\otimes}} A$ (produit tensoriel complété inductif ou projectif entre deux espaces topologiques localement convexes). Ceci s'interprète en termes de fonctions analytiques sur des espaces rigides : si $I \subseteq [0,1[$ est un intervalle d'extremités dans $p^{\mathbb{Q}}$ et $\boldsymbol{A}^1 = \mathrm{Sp}(\mathbb{Q}_p\langle T\rangle)$ dénote la droite affine rigide de paramètre T, en notant B_I l'ouvert admissible de $\boldsymbol{A}^1$ défini par $v_A(T) \in I$ ($v_A = -\log_p |\cdot|_A$ dénote la valuation de l'algèbre affinoïde A), on a un isomorphisme naturel

$$\mathscr{R}_A^I \cong \mathcal{O}(\mathrm{Sp}(A) \times B_I).$$

On a aussi une interprétation en termes de séries de Laurent (resp. de puissances si $0 \in I$) à coefficients dans A de la manière évidente. On a un endomorphisme A-linéaire d'anneaux $\varphi : \mathscr{R}_A^{]0,s]} \to \mathscr{R}_A^{]0,s/p]}$, qui envoie T sur $(1+T)^p - 1$, induisant une action de φ sur $\mathscr{R}_A$ et on a une action continue du groupe Γ, agissant par $\sigma_a(T) = (1+T)^a - 1$, $a \in \mathbb{Z}_p^\times$, sur tous les anneaux définis ci-dessus.

Pour un module $D^{]0,s]}$ sur $\mathscr{R}_A^{]0,s]}$ et $0 < s' < s$, on note

$$D^{]0,s']} = D^{]0,s]} \otimes_{\mathscr{R}_A^{]0,s]}} \mathscr{R}_A^{]0,s']} \quad \text{et} \quad \varphi^*(D^{]0,s]}) = D^{]0,s]} \otimes_{\mathscr{R}_A^{]0,s]},\varphi} \mathscr{R}_A^{]0,s/p]}.$$

Un (φ,Γ)-*module sur* $\mathscr{R}_A^{]0,s]}$ est un module projectif de type fini $D^{]0,s]}$ sur $\mathscr{R}_A^{]0,s]}$, muni d'un isomorphisme $\mathscr{E}^{]0,s/p]}$-linéaire $\tilde{\varphi} : \varphi^*(D^{]0,s]}) \to D^{]0,s/p]}$ et d'une action semi-linéaire de Γ, commutant avec $\tilde{\varphi}$ dans le sens évident. L'isomorphisme $\varphi^*(D^{]0,s]}) \to D^{]0,s/p]}$ induit, pour tout $0 < s' < s$, des opérateurs semi-linéaires $\varphi : D^{]0,s']} \to D^{]0,s'/p]}$, définis comme la composée

$$D^{]0,s']} = D^{]0,s]} \otimes_{\mathscr{R}_A^{]0,s]}} \mathscr{R}_A^{]0,s']} \to \varphi^*(D^{]0,s]}) \otimes_{\mathscr{R}_A^{]0,s]}} \mathscr{R}_A^{]0,s']} \to D^{]0,s/p]} \otimes_{\mathscr{R}_A^{]0,s]}} \mathscr{R}_A^{]0,s'/p]} = D^{]0,s'/p]},$$

la première flèche étant celle induite par $D^{]0,s]} \to \varphi^*(D^{]0,s]})$, $x \mapsto x \otimes 1$ et la deuxième étant $\tilde{\varphi} \otimes \varphi$.

Un (φ,Γ)-*module sur* $\mathscr{R}_A$ est un $\mathscr{R}_A$-module (projectif de type fini) D tel qu'il existe $s > 0$ et un (φ,Γ)-module $D^{]0,s]}$ sur $\mathscr{R}_A^{]0,s]}$ tel que $D \cong D^{]0,s]} \otimes_{\mathscr{R}_A^{]0,s]}} \mathscr{R}_A$.

Plus généralement, si X est un espace rigide analytique sur $\mathbb{Q}_p$ et $r \geq 0$, on définit $\mathscr{R}_X^{]0,r]}$ comme le faisceau des fonctions rigides analytiques sur $X \times B_{]0,r]}$, $\mathscr{R}_X = \varinjlim_{r<0} \mathscr{R}_X^{]0,r]}$ et un (φ,Γ)-module sur $\mathscr{R}_X$ est une collection compatible de (φ,Γ)-modules sur $\mathscr{R}_A$ pour chaque ouvert admissible A de X.

IB7. *Cohomologie des* (φ,Γ)*-modules.* Si $V \in \mathrm{Rep}_L\,\mathscr{G}_{\mathbb{Q}_p}$ est une représentation p-adique de $\mathscr{G}_{\mathbb{Q}_p}$ et γ est un générateur topologique de Γ, [15] le complexe

$$0 \to \boldsymbol{D}(V) \xrightarrow{x\mapsto((1-\varphi)x,(1-\gamma)x)} \boldsymbol{D}(V)\oplus\boldsymbol{D}(V) \xrightarrow{(x,y)\mapsto(1-\gamma)x-(1-\varphi)y} \boldsymbol{D}(V)\to 0$$

calcule la cohomologie galoisienne de V en termes de son (φ,Γ)-module $\boldsymbol{D}(V)$ sur $\mathscr{E}_{\mathbb{Q}_p}$ associé (cf. [Herr 2001]). D'après [Liu 2008; Kedlaya et al. 2014], on peut définir la cohomologie des (φ,Γ)-modules sur l'anneau de Robba $\mathscr{R}_A$ relatif à une algèbre affinoïde A sur $\mathbb{Q}_p$ (et, plus généralement, sur un espace analytique rigide sur une extension finie de $\mathbb{Q}_p$), retrouvant les constructions de Herr dans le cas d'un (φ,Γ)-module étale au-dessus d'un point.

Soient A une algèbre affinoïde sur $\mathbb{Q}_p$ et D un (φ,Γ)-module sur $\mathscr{R}_A$. On note $\Gamma'\subseteq\Gamma$ la partie de p-torsion de Γ (qui est triviale si $p\neq 2$, et cyclique d'ordre 2 quand $p=2$). Soit $\gamma\in\Gamma$ tel que son image dans Γ/Γ' en est un générateur topologique. On pose $\gamma_0=\gamma$ et, pour $n\geq 1$, γ_n un générateur topologique de Γ_n. Pour $\delta\in\{\varphi,\psi\}$ et $\gamma'\in\{\gamma_n : n\geq 0\}$, on note $D'=D^{\Gamma'}$ si $n=0$ et $D'=D$ si $n\geq 1$, et on définit le complexe

$$\mathscr{C}^{\bullet}_{\delta,\gamma'}(D) : 0\to D'\to D'\oplus D'\to D'\to 0,$$

où les flèches sont données, respectivement, par $x\mapsto((\delta-1)x,\ (\gamma'-1)x)$ et $(x,\,y)\mapsto(\gamma'-1)x-(\delta-1)y$. Les modules $H^{\bullet}_{\delta,\gamma'}(D)$ sont définis comme les groupes de cohomologie de ce complexe. Quand $n=0$, on note $H^i_{\delta,\gamma}(D)=H^i_{\delta,\gamma_0}(D)$.

Proposition I.8 [Kedlaya et al. 2014, Proposition 2.3.6, Theorem 4.4.2]. *Soient A une algèbre affinoïde sur $\mathbb{Q}_p$ et D un (φ,Γ)-module sur $\mathscr{R}_A$. Les complexes $\mathscr{C}_{\varphi,\gamma'}(D)$ et $\mathscr{C}_{\psi,\gamma'}(D)$ sont quasi-isomorphes et les groupes de cohomologie $H^i_{\varphi,\gamma'}(D)$ sont des A-modules de type fini, compatibles au changement de base. On a une dualité locale et une formule de Euler–Poincaré.*

IB8. *Cohomologie d'Iwasawa des* (φ,Γ)*-modules.* Soit $\Lambda=\mathbb{Z}_p[\![\Gamma]\!]$ l'algèbre d'Iwasawa de Γ. On décompose $\Gamma=\Gamma^{\mathrm{tors}}\times\widetilde{\Gamma}$, où Γ^{tors} désigne la partie de torsion de Γ et $\widetilde{\Gamma}\cong 1+2p\mathbb{Z}_p$ via le caractère cyclotomique, et on obtient un isomorphisme $\Lambda\cong\mathbb{Z}_p[\Gamma^{\mathrm{tors}}]\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\![\widetilde{\Gamma}]\!]$. Soit γ un générateur topologique de $\widetilde{\Gamma}$ et notons $[\gamma]$ son image dans Λ. On obtient un isomorphisme $\Lambda\cong\mathbb{Z}_p[\Gamma^{\mathrm{tors}}]\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\![T]\!]$ en envoyant $[\gamma]$ sur $1+T$.

Soit $\Lambda_\infty=\mathscr{R}^+(\Gamma)$ l'algèbre algèbre de distributions sur Γ. Précisément, on obtient $\mathscr{R}^+(\Gamma)$ en remplaçant la variable T par $[\gamma]-1$ dans la définition de $\mathscr{R}^+$. On peut de la sorte définir les anneaux $\Lambda_n=\mathscr{R}^{[r_n,+\infty]}(\Gamma)$ et on a $\Lambda_\infty=\varprojlim_n\Lambda_n$. Le choix de l'isomorphisme $\Lambda\cong\mathbb{Z}_p[\Gamma^{\mathrm{tors}}]\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\![T]\!]$ fait que Λ_n s'identifie aux fonction analytiques sur la boule $v_p(T)\geq r_n$ et Λ_∞ à celui des fonctions analytiques sur la boule ouverte unité. Ce dernier espace s'identifie à l'anneau $H^0(\mathfrak{X},\mathcal{O})$ des sections globales sur l'espace des poids p-adiques, et aussi, par le théorème d'Amice (cf. [Schneider et Teitelbaum 2003, Theorem 2.2]) au L-dual continu des fonctions localement analytiques sur $\mathbb{Z}_p^\times$.

On note (cf. [Kedlaya et al. 2014, Definition 4.4.7]) Λ_n^ι le module Λ_n muni de l'action de Γ via $\gamma(f)=[\gamma^{-1}]\cdot f,\ \gamma\in\Gamma$ et $f\in\Lambda_n$. On définit

$$\mathbf{Dfm}=\varprojlim_n\Lambda_n^\iota\,\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{R}=\varprojlim_n\varinjlim_{s>0}\varprojlim_{r<s}\mathscr{R}^{[s,r]}\,\widehat{\otimes}\,\Lambda_n^\iota.$$

15. Si $p=2$, il faut considérer $\Gamma'\subseteq\Gamma$ la p-partie du sous-groupe de torsion de Γ, $\gamma\in\Gamma$ dont l'image dans le quotient Γ/Γ' soit un générateur topologique, et prendre les invariants par Γ' dans tous les modules de la suite ci-dessous.

Plus généralement, si D est un (φ,Γ)-module sur $\mathscr{R}$, on définit sa déformation cyclotomique par

$$\mathbf{Dfm}(D) = D \,\widehat{\otimes}_{\mathbb{Q}_p} \Lambda_\infty^\iota = \varprojlim_n \varinjlim_{s>0} \varprojlim_{r<s} D^{[r,s]} \,\widehat{\otimes}\, \Lambda_n^\iota.$$

Les actions φ, ψ et Γ sont données par les formules

$$\varphi(x\otimes\lambda) = \varphi(x)\otimes\lambda, \quad \psi(x\otimes\lambda) = \psi(x)\otimes\lambda, \quad \gamma(x\otimes\lambda) = \gamma(x)\otimes[\gamma^{-1}]\lambda,$$

pour $x\in D$, $\lambda\in\Lambda_\infty$ et $\gamma\in\Gamma$. Le module $\mathbf{Dfm}(D)$ est un (φ,Γ)-module sur l'anneau $\mathscr{R}_{\mathfrak{X}} = \varprojlim_n \mathscr{R}_{\mathfrak{X}_n}$ de Robba relatif à l'espace des poids.

On définit la cohomologie d'Iwasawa de D comme

$$H^i_{\mathrm{Iw}}(D) = H^i_{\psi,\gamma}(\mathbf{Dfm}(D)).$$

Ce sont, d'après le proposition I.8, des Λ_∞-modules de type fini. On peut donc voir les groupes de cohomologie d'Iwasawa comme des groupes de cohomologie à valeurs dans $\mathscr{D}(\mathbb{Z}_p^\times, D)$.

Si $\eta : \Gamma\to L^\times$ est un caractère, le changement de base par rapport à $f_\eta : \Lambda_\infty \to L,\ [\gamma]\mapsto \eta^{-1}(\gamma)$ fournit un isomorphisme

$$\mathbf{Dfm}(D)\otimes_{\Lambda_\infty, f_\eta} L \xrightarrow{\sim} D(\eta).$$

Si $\mu\in H^i_{\mathrm{Iw}}(D)$, cet isomorphisme induit des morphismes de spécialisation

$$H^i_{\mathrm{Iw}}(D)\to H^i_{\psi,\gamma}(D(\eta)), \quad \mu\mapsto \int_\Gamma \eta\cdot\mu,$$

la notation étant justifiée par l'interprétation classique de la cohomologie d'Iwasawa en termes des distributions.

Si $D\in\Phi\Gamma(\mathscr{R})$, on définit le complexe $\mathscr{C}^\bullet_\psi(D)$, concentré en $[1,2]$, par

$$\big[D \xrightarrow{\psi-1} D\big].$$

Le complexe $\mathscr{C}^\bullet_\psi(D)$ appartient à $\boldsymbol{D}^{[0,2]}_{\mathrm{perf}}(\Lambda_\infty)$ [16] et calcule la cohomologie d'Iwasawa de D (cf. [Kedlaya et al. 2014, Theorem 4.4.8]). On a, en particulier, un isomorphisme

$$\mathrm{Exp}^* : H^1_{\mathrm{Iw}}(D)\to D^{\psi=1}$$

dont l'inverse est donnée par $z\mapsto [((p-1)/p)\log(\chi(\gamma))(z\otimes 1), 0]$. [17] Si $z\in D^{\psi=1}$, on note, afin d'alléger les notation, μ_z l'élément $(\mathrm{Exp}^*)^{-1}(z)\in H^1_{\mathrm{Iw}}(D)$. Si $n\geq 0$, on a (cf. [Colmez 2010, §VIII.1.3] ou [Nakamura 2017, §2.2.3, Equation (6)]) la formule pour la spécialisation

$$\int_{\Gamma_n}\eta\cdot\mu_z = [\tau_n(\gamma_n)(z\otimes e_\eta), 0]\in H^1_{\psi,\gamma_n}(D(\eta)),$$

où $\tau_n(\gamma_n) = p^{-n}\log(\chi(\gamma_n))$, si $n\geq 1$ et $\tau_0(\gamma_0) = ((p-1)/p)\log(\chi(\gamma_0))$.

16. Rappelons que, pour un anneau R, $\boldsymbol{D}^{[0,2]}_{\mathrm{perf}}(R)$ dénote la sous-catégorie de la catégorie dérivée de la catégorie des modules sur R dont les objets sont quasi-isomorphes à un complexe borné formé par des R-modules projectifs finis et tel que leur cohomologie est concentrée en dégrées $[0,2]$.

17. Si $p=2$, il faut appliquer à z le projecteur naturel sur le sous espace d'éléments Γ'-invariants. On évitera ce cas-ci, se traitant de la même manière mais avec une complication technique supplémentaire.

Terminons en mentionnant que η induit un automorphisme sur Λ_∞ donné par $\eta([\gamma]) = \eta(\gamma)^{-1} \cdot [\gamma]$, ce qui induit un isomorphisme de (φ, Γ)-modules $\mathbf{Dfm}(D) \cong \mathbf{Dfm}(D(\eta))$, donné par

$$x \otimes [\gamma] \mapsto (x \otimes e_\eta) \otimes \eta(\gamma)^{-1}[\gamma],$$

et donc un isomorphisme de Λ_∞-modules

$$H^i_{\mathrm{Iw}}(D) \to H^i_{\mathrm{Iw}}(D(\eta)), \quad \mu \mapsto \mu \otimes e_\eta.$$

On a, par exemple,

$$\int_\Gamma \eta \cdot \mu = \int_\Gamma 1 \cdot (\mu \otimes e_\eta).$$

IB9. *Applications exponentielles.* Si $D \in \Phi\Gamma(\mathscr{R})$ est de Rham et $n \geq 0$, on note

$$\exp_{D,F_n} : L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D) \to H^1_{\varphi,\gamma_n}(D), \quad \exp^*_{D,F_n} : H^1_{\varphi,\gamma_n}(D) \to L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D),$$

les applications exponentielle et exponentielle duale de Bloch–Kato comme définies dans [Nakamura 2014, §2.3, §2.4]. Quand cela ne pose pas de problèmes, on omettra les indices dans les notations des applications exponentielles.

Explicitement [Nakamura 2014, Lemma 2.12(1)], si $x \in \boldsymbol{D}_{\mathrm{dR}}(D) = (\boldsymbol{D}_{\mathrm{dif}}(D))^\Gamma$, alors il existe m tel que $x \in \boldsymbol{D}_{\mathrm{dif},m}(D)$, et il existe aussi $\tilde{x} \in D^{]0,r_m]}[1/t]$ tel que $\varphi^{-k}(\tilde{x}) - x \in \boldsymbol{D}^+_{\mathrm{dif},k}(D)$ pour tout $k \geq m$. Alors on a

$$\exp_{D,F_n}(x) = [(\gamma_n - 1)\tilde{x}, (\varphi - 1)\tilde{x}] \in H^1_{\varphi,\gamma_n}(D).$$

On remarquera que l'application $\exp_D$ est nulle sur $\mathrm{Fil}^0 \boldsymbol{D}_{\mathrm{dR}}(D)$.

Si M est un module muni d'une action de Γ, et $\gamma' \in \{\gamma_n : n \geq 0\}$, on pose [18]

$$C^\bullet_{\gamma'}(M) = \big[M' \xrightarrow{\gamma'-1} M'\big],$$

et on définit les groupes de cohomologie $H^i_{\gamma'}(M) = H^i(C^\bullet_{\gamma'}(M))$. Par exemple, si $D \in \Phi\Gamma(\mathscr{R})$ et $n \geq 0$, alors

$$H^0_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D)) = \boldsymbol{D}_{\mathrm{dif}}(D)^{\Gamma_n} = L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D).$$

Si $D \in \Phi\Gamma(\mathscr{R})$ est de Rham et $n \geq 0$, on a un isomorphisme $L_\infty((t)) \otimes_{L_n} (L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)) \cong \boldsymbol{D}_{\mathrm{dif}}(D)$ et l'application $L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D) \to H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$ qui envoie $x \in L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ vers la classe de cohomologie $[\log \chi(\gamma_n)(1 \otimes x)] \in H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$ est un isomorphisme (cf. [Nakamura 2014], juste avant le lemme 2.14). De plus, on a une application naturelle $H^1_{\varphi,\gamma_n}(D) \to H^1_{\gamma_n}(\boldsymbol{D}^+_{\mathrm{dif}}(D)) \to H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$ définie par $[(x, y)] \mapsto [\varphi^{-m}x]$, où $m \gg 0$ et $[\,\cdot\,]$ dénote la classe de cohomologie correspondante. On définit

$$\exp^*_{D,F_n} : H^1_{\varphi,\gamma_n}(D) \to L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$$

comme la composition $H^1_{\varphi,\gamma_n}(D) \to H^1_{\gamma_n}(\boldsymbol{D}^+_{\mathrm{dif}}(D)) \to H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D)) \xrightarrow{\sim} L_n \otimes_{\mathbb{Q}_p} \boldsymbol{D}_{\mathrm{dR}}(D)$. Remarquons que, par construction, l'image de $\exp^*_{D,F_n}$ tombe dans $L_n \otimes \mathrm{Fil}^0 \boldsymbol{D}_{\mathrm{dR}}(D)$.

18. Comme précédemment, $M' = M^{\Gamma'}$ si $\gamma' = \gamma_0$ et $M' = M$ autrement.

IB10. *Exponentielle de Perrin-Riou.* Rappelons la formulation de l'application exponentielle de Perrin-Riou pour un (φ,Γ)-module de de Rham (cf. [Nakamura 2014; Berger 2003]) et la loi de réciprocité. On définit, pour $h \geq 0$,[19] l'opérateur différentiel

$$\nabla_h = (\nabla - h + 1) \circ (\nabla - h + 2) \circ \cdots \circ \nabla \in \Lambda_\infty.$$

Lemme I.9 [Nakamura 2014, Lemma 3.6]. *Soit $D \in \Phi\Gamma(\mathscr{R})$ de Rham et soit $h \geq 1$ tel que* $\mathrm{Fil}^{-h}\, \boldsymbol{D}_{\mathrm{dR}}(D) = \boldsymbol{D}_{\mathrm{dR}}(D)$. *Alors $\nabla_h(\mathbb{N}_{\mathrm{rig}}(D)) \subseteq D$ et il induit un opérateur Λ_∞-linéaire*

$$\nabla_h : \mathbb{N}_{\mathrm{rig}}(D)^{\psi=1} \to D^{\psi=1}.$$

Si $x \in \mathbb{N}_{\mathrm{rig}}(D)^{\psi=1}$ et $n \geq 0$, l'expression $[\varphi^{-n}x]_j$,[20] pour $j \in \mathbb{N}$, n'est définie que pour n assez grand. Pourtant, l'identité $\mathrm{Tr}_{L_m/L_n} \circ \varphi^{-m} = \varphi^{-m} \circ \psi^{m-n}$, $m \geq n \geq 0$, nous permet de lui donner toujours un sens en utilisant les traces et en considérant les valeurs $p^{-m}\,\mathrm{Tr}_{L_m/L_n}[\varphi^{-m}x]_j$, où m est n'importe quel entier assez grand.

Théorème I.10 [Nakamura 2014, Theorem 3.10]. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham, $h \geq 1$ un entier tel que* $\mathrm{Fil}^{-h}\, \boldsymbol{D}_{\mathrm{dR}}(D) = \boldsymbol{D}_{\mathrm{dR}}(D)$, *$z \in \mathbb{N}_{\mathrm{rig}}(D)^{\psi=1}$ et $n \geq 0$. On a :*

(i) *Si $j > 0$ et si il existe $z_j \in \mathbb{N}_{\mathrm{rig}}(D)^{\psi=p^{-j}}$ vérifiant $\partial^j z_j = z$, ou si $-(h-1) \leq j \leq 0$ et $z_j = \partial^{-j} z$, alors, on a, pour $m \gg 0$,*

$$\int_{\Gamma_n} \chi^j \cdot \mu_{\nabla_h z} = \frac{p^{m(j-1)}}{\Gamma^*(-j-h+1)} \exp\big(\mathrm{Tr}_{L_m/L_n}[\varphi^{-m} z_j]_0 \otimes t^{-j} e_j\big).$$

(ii) *Si $j \geq h$, on a, pour $m \gg 0$,*

$$\exp^*\left(\int_{\Gamma_n} \chi^{-j} \cdot \mu_{\nabla_h z}\right) = \frac{p^{m(-j-1)}}{\Gamma^*(-j-h+1)} \mathrm{Tr}_{L_m/L_n}[\varphi^{-m}\partial^j z]_0 \otimes t^j e_{-j}.$$

Remarque I.11. — Rappelons que, pour $j \in \mathbb{Z}$, l'élément e_j dénote une base du L-espace vectoriel $L(j)$ muni d'actions de Γ et φ par les formules $\sigma_a(e_j) = a^j \cdot e_j$, $a \in \mathbb{Z}_p^\times$, et $\varphi(e_j) = e_j$. Si D est un (φ,Γ)-module, on note $D(j) = D \otimes_L L(j)$ la tordue de D par la j-ième puissance du caractère cyclotomique. Si D est de Rham, $D(j)$ l'est aussi et on a $\boldsymbol{D}_{\mathrm{dR}}(D(j)) = \boldsymbol{D}_{\mathrm{dR}}(D) \otimes \boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(j)) = \boldsymbol{D}_{\mathrm{dR}}(D) \otimes L \cdot (t^{-j} e_j) = \boldsymbol{D}_{\mathrm{dR}}(D) \otimes L \cdot \boldsymbol{e}_j^{\mathrm{dR}}$, dans la notation de section IC3. Le terme $[\varphi^{-m} z_j]_0$ appartient à $L_m \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ et $x \mapsto x \otimes t^{-j} e_j$ induit un isomorphisme $\boldsymbol{D}_{\mathrm{dR}}(D) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(D(j))$. La première égalité a lieu dans $L_n \otimes H^1_{\varphi,\gamma}(D(j))$ (ou $H_{\varphi,\gamma_n}(D(j))$) et la deuxième dans $L_n \otimes_{\mathbb{Q}_p} \boldsymbol{D}_{\mathrm{dR}}(D(-j))$.

— Notons que $\mu \mapsto \mu \otimes e_j$ induit, pour $j \in \mathbb{Z}$, un isomorphisme $H^1_{\mathrm{Iw}}(D) \xrightarrow{\sim} H^1_{\mathrm{Iw}}(D(j))$ et on a donc, pour $z \in \mathbb{N}_{\mathrm{rig}}(D)^{\psi=1}$,

$$\int_{\Gamma_n} \chi(x)^{-j} \cdot \mu_z = \int_{\Gamma_n} 1 \cdot (\mu_z \otimes e_{-j}).$$

Si $j \geq 0$, l'élément $\varphi^{-n}(z \otimes e_{-j}) \in L_n[\![t]\!] \otimes \boldsymbol{D}_{\mathrm{dR}}(D(-j))$ s'écrit sous la forme $t^{-j}\varphi^{-n} z \otimes t^j e_{-j}$ et, si on écrit $\varphi^{-n} z = \sum_l d_l t^l \in L_n[\![t]\!] \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$, l'expression $[\varphi^{-n}(z \otimes e_{-j})]_0$ dénote l'élément $d_j \otimes t^{-j} e_j$, qui n'est autre que $p^{-nj}/j![\varphi^{-n}\partial^j z]_0 \otimes t^j e_{-j}$. Le deuxième point du théorème est

19. Dans la formule ci-dessous, ∇_0 dénote simplement l'identité et $\nabla_1 = \nabla$.

20. Rappelons que, si $x \in D^{]0,r_n]}$, l'élément $\varphi^{-n}(x)$ appartient à $L_n((t)) \otimes_{\mathbb{Q}_p} \boldsymbol{D}_{\mathrm{dR}}(D)$, s'écrit donc sous la forme $\varphi^{-n} x = \sum_j d_j t^j$, $d_j \in L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ et on note $[\varphi^{-n}x]_j = d_j \in L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$.

donc une paraphrase de la loi de réciprocité de Cherbonnier–Colmez. Pour $j < 0$, le résultat est (au moins pour l'auteur) un peu plus mystérieux car l'opérateur ∂^j, devenant un opérateur d'intégration, fait apparaître des termes qu'on ne voyait pas directement dans le développement de $\varphi^{-n}z$.

— Si $j \gg 0$ l'application $\exp_{D(j),F_n}$ est bijective et on peut reformuler le résultat en disant que, si $j \geq 0$ ou $j \ll 0$, alors, pour $m \gg 0$, on a

$$\mathrm{Tr}_{L_m/L_n}[\varphi^{-m}\partial^j z]_0 = \frac{\Gamma^*(-j-h+1)}{p^{m(-j-1)}} \cdot \log\left(\int_{\Gamma_n} \chi^{-j} \cdot \nabla_h \mu_z\right) \otimes \boldsymbol{e}_{-j}^{\mathrm{dR},\vee},$$

où log dénote $\exp^{-1}$ ou $\exp^*$ selon que $j \ll 0$ ou que $j \geq 0$.

— Si $n \geq m(\Delta)$ ($\Delta = \mathbb{N}_{\mathrm{rig}}(D)$), alors l'introduction de m dans les formules est superflue et, comme $\psi(z) = z$, alors $z \in D^{]0,r_n]}$ et on a

$$p^{-m(j+1)}\,\mathrm{Tr}_{L_m/L_n}[\varphi^{-m}\partial^j z]_0 = \frac{[L_m : L_n]}{p^{m-n}}[\varphi^{-n(j+1)}\partial^j z].$$

On remarque que $[L_m : L_n]/p^{m-n} = 1$ si $n > 0$ et $[L_m : L]/p^m = p - 1/p$.

On se servira de la version suivante de la loi de réciprocité de Cherbonnier–Colmez, qui est un des ingrédients de la preuve du théorème précédent, mais pour lequel on n'a pas trouvé de référence précise.

Proposition I.12. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham, $z \in D^{\psi=1}$, $n \geq 0$ et $j \geq 0$. Alors, pour $m \gg 0$, on a l'égalité suivante dans $L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$:*

$$\exp^*\left(\int_{\Gamma_n} \chi^{-j} \cdot \mu_z\right) \otimes t^{-j} e_j = \frac{1}{j!} p^{-m(j+1)}\,\mathrm{Tr}_{L_m/L_n}([\varphi^{-m}\partial^j z]_0).$$

Démonstration. La formule $\exp^*\big(\int_{\Gamma_n} \chi^{-j} \cdot \mu_z\big) = \exp^*\big(\int_{\Gamma_n} 1 \cdot \mu_{z\otimes e_{-j}}\big)$ permet de nous ramener au cas $j = 0$. En outre, la formule $\mathrm{Tr}_{L_{n+1}/L_n}(\exp^*(x)) = \exp^*(\mathrm{cor}_{F_{n+1}/F_n}(x))$ [21] nous ramène à montrer le résultat pour n assez grand. En particulier on peut considérer $n > 0$ de sorte que $z \in D^{]0,r_n]}$ et on doit donc montrer

$$\exp^*\left(\int_{\Gamma_n} 1 \cdot \mu_z\right) = p^{-n}[\varphi^{-n}z]_0.$$

On a la formule pour la spécialisation $\int_{\Gamma_n} 1 \cdot \mu_z = [\tau_n(\gamma_n)(z \otimes 1), 0] \in H^1_{\psi,\gamma}(D)$, où $\tau_n(\gamma_n) = p^{-n}\log(\chi(\gamma_n))$ (car on suppose $n > 0$). Rappelons que l'application $\exp^*$ est définie comme la composition du morphisme $H^1_{\varphi,\gamma_n}(D) \to H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$ avec l'inverse de l'isomorphisme $L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D) \xrightarrow{\sim} H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$ donné par $x \mapsto [\log(\chi(\gamma_n))(1 \otimes x)]$.

Si $z \in D^{\psi=1}$, la classe de cohomologie $[z, 0] \in H^1_{\psi,\gamma_n}(D)$ correspond à la classe $[z, (1-\gamma_n)^{-1}(1-\varphi)z] \in H^1_{\varphi,\gamma}$ sous l'isomorphisme entre ces deux modules. L'image du cocycle $[\tau_n(\gamma_n)(z\otimes 1), 0]$ par le morphisme $H^1_{\varphi,\gamma_n}(D) \to H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$ est donc donnée par $[\tau_n(\gamma_n)\varphi^{-n}z]$.

Or, $\varphi^{-n}z = \sum_{l\geq 0} a_l t^l$, $a_l \in L_n[\![t]\!] \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ et $[\tau_n(\gamma_n)\sum_{l\geq 0} a_l t^l] = [\tau_n(\chi(\gamma_n))a_0]$ (dans $H^1_{\gamma_n}(\boldsymbol{D}_{\mathrm{dif}}(D))$), car tous les autres termes sont dans l'image de $(1-\gamma_n)$ (en effet, si $l \neq 0$, $a_l t^l = (1-\gamma_n)(1-\chi(\gamma_n)^l)^{-1}a_l t^l$), ce qui permet de conclure car $a_0 = [\varphi^{-n}z]_0$. □

21. Si $[x, y] \in H^1_{\psi,\gamma_{n+1}}(D)$, sa corestriction est définie par la formule

$$\mathrm{cor}_{F_{n+1}/F_n}([x, y]) = \left[\frac{1-\gamma_{n+1}}{1-\gamma_n}x, y\right] \in H^1_{\psi,\gamma_n}(D),$$

cf. [Cherbonnier et Colmez 1999, lemme II.2.1].

IC. ***La fonction L locale.***

IC1. *Distributions à valeurs dans un espace de type* LF. On reprend les constructions de la section 0E3. Si $M = \varinjlim_s \varprojlim_r M^{[r,s]}$ est un espace de type LF, on pose

$$\mathscr{O}(\mathfrak{X}) \widehat{\otimes} M = \varprojlim_n \varinjlim_{s>0} \varprojlim_{0<r<s} \mathscr{O}(\mathfrak{X}_n) \widehat{\otimes} M^{[r,s]},$$

où le dernier produit tensoriel est le produit tensoriel usuel entre deux espaces de Banach. On dit que f est une fonction analytique sur $\mathfrak{X}$ à valeurs dans M si f est un élément de $\mathscr{O}(\mathfrak{X}) \widehat{\otimes} M$.

IC2. *Prolongement analytique de* $k \mapsto \partial^k$. Soit $D \in \Phi\Gamma(\mathscr{R})$ de rang d, de Rham, et notons $\Delta = \mathbb{N}_{\mathrm{rig}}(D)$. Rappelons que, d'après le lemme I.6, si $K/\mathbb{Q}_p$ dénote la plus petite extension galoisienne telle que $\mathscr{G}_{\mathbb{Q}_p}$ agit sur $\boldsymbol{D}_{\mathrm{pst}}(D)$ à travers $\mathrm{Gal}(K/\mathbb{Q}_p)$ et si l'on note $r(\Delta) = (\delta_K + 1)^{-1}$, il existe, pour tout $s < r(\Delta)$, des sous-$\mathscr{E}^{]0,s]}$-modules $\Delta^{]0,s]}$ de Δ, munis d'une famille de valuations $v^{[r,s]}$, $0 < r < s < r(\Delta)$, pour lesquelles la norme de l'opérateur ∂ satisfait

$$v^{[r,s]}(\partial) \geq -s - 1 \geq -(\delta_K + 1)^{-1} - 1.$$

On supposera dorénavant, et par commodité, que $m(\Delta)$ *est tel que* $r_{m(\Delta)} < r(\Delta)$. De plus, pour $s < r(\Delta)$, les opérateurs

$$\varphi : \Delta^{]0,s]} \to \Delta^{]0,s/p]}, \quad \psi : \Delta^{]0,s/p]} \to \Delta^{]0,s]},$$

agissent de façon continue pour les valuations $v^{[r,s]}$. Rappelons aussi que l'on a

$$\Delta = \varinjlim_{s>0} \Delta^{]0,s]} = \varinjlim_{s>0} \varprojlim_{0<r<s} \Delta^{[r,s]}.$$

Si $z \in \Delta^{\psi=0}$ et $N > 0$, on peut écrire $z = \sum_{i \in (\mathbb{Z}/p^N\mathbb{Z})^\times} (1+T)^i \varphi^N(z_i)$, $z_i = \psi^N((1+T)^{-i}z)$. La formule de Leibnitz et l'identité $\partial \circ \varphi = p\,\varphi \circ \partial$ donnent, pour $k \geq 0$,

$$\partial^k z = \sum_{i \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \sum_{j=0}^{k} \binom{k}{j} i^{k-j} (1+T)^i p^{Nj} \varphi^N(\partial^j z_i).$$

Enfin, rappelons que, si $\eta \in \mathfrak{X}(L)$ alors ω_η dénote son poids (cf. la section 0E3) et que le poids du caractère $x \mapsto x^k$ est k. L'identité de la formule ci-dessus suggère la proposition suivante, qui est le point de départ des constructions de cet article.

Proposition I.13. *Soient* $\kappa \in \mathfrak{X}$ *et* $z \in \Delta^{\psi=0}$. *Alors, la série*

$$\sum_{i \in (\mathbb{Z}/p^N\mathbb{Z})^\times} \sum_{j=0}^{+\infty} \binom{\omega_\kappa}{j} \kappa(i) i^{-j} (1+T)^i p^{Nj} \varphi^N(\partial^j z_i) \tag{2}$$

converge, pour $N \gg 0$, *dans* $\Delta^{\psi=0}$, *et la somme ne dépend ni de* N *ni du choix du système de représentants de* $(\mathbb{Z}/p^N\mathbb{Z})^\times$. *L'application* $\kappa \mapsto \kappa(\partial)z$ *ainsi définie est une fonction rigide analytique sur* $\mathfrak{X}$ *à valeurs dans* $\Delta^{\psi=0}$.

Démonstration. Notons, pour $N > 0$ et $0 < r < s < r(\Delta)p^{-N}$, $C^{[r,s]}_{\varphi^N}$ et $C^{[r,s]}_{\psi^N}$ les normes, relatives aux valuations $v^{[r,s]}$ et $v^{[p^N r, p^N s]}$, des opérateurs

$$\varphi^N : \Delta^{]0,p^N s]} \to \Delta^{]0,s]}, \quad \psi^N : \Delta^{]0,s]} \to \Delta^{]0,p^N s]},$$

et, pour $N > 0$ et $0 < r < s < r(\Delta)$, $C^{[r,s]}_{\partial}$ la norme de l'opérateur ∂ pour la valuation $v^{[r,s]}$, de sorte que $C^{[r,s]}_{\varphi^N}, C^{[r,s]}_{\psi^N} > -\infty$ et $C^{[r,s]}_{\partial} \geq -s-2$.

Pour $i \in (\mathbb{Z}/p^N\mathbb{Z})^\times$, on pose

$$g_j(\kappa) = \binom{\omega_\kappa}{j}\kappa(i)i^{-j}(1+T)^i p^{Nj}\varphi^N(\partial^j z_i).$$

Supposons $s < r(\Delta)p^{-N}$, on a alors

$$\begin{aligned} v^{[r,s]}(\varphi^N(\partial^j z_i)) &\geq v^{[p^N r, p^N s]}(\partial^j \psi^N((1+T)^{-i}z)) + C^{[r,s]}_{\varphi^N} \\ &\geq v^{[p^N r, p^N s]}(\psi^N((1+T)^{-i}z)) + C^{[r,s]}_{\varphi^N} + jC^{[p^N r, p^N s]}_{\partial} \\ &\geq v^{[r,s]}(z) + C^{[r,s]}_{\varphi^N} + jC^{[p^N r, p^N s]}_{\partial} + C^{[r,s]}_{\psi^N} \\ &\geq v^{[r,s]}(z) + C^{[r,s]}_{\varphi^N} + j(-p^N s - 1) + C^{[r,s]}_{\psi^N}. \end{aligned}$$

Remarquons d'ailleurs les estimations suivantes évidentes :

— $v^{[r,s]}((T+1)^i) = 0$.

— $v^{[r,s]}\left(\binom{\omega_\kappa}{j}\right) = v_p\left(\binom{\omega_\kappa}{j}\right) \geq j(\inf(v_p(\omega_\kappa), 0) - 1/(p-1)) = jC_\kappa$. Or $\omega_\kappa = \log z_\kappa / q$ et donc

$$v_p(\omega_\kappa) \geq \inf_{k \geq 0}\left\{p^k v_p(z_\kappa - 1) - k\right\} - v_p(q).$$

Or, si $\kappa \in \mathfrak{X}_n$, la constante C_κ est bornée en fonction de n. En effet, si $v_p(z_\kappa - 1) \geq 1/n$, la formule pour ω_κ donne $C_\kappa \geq C_n = \min(\inf_{k\geq 0}\{p^k/n - k\} - v_p(q), 0) - 1/(p-1)$. On en déduit donc

$$v_p\left(\binom{\omega_\kappa}{j}\right) \geq jC_n.$$

— $v^{[r,s]}(i^{-j}) = v_p(i^{-j}) = 0$.

— $v^{[r,s]}(\kappa(i)) = v_p(\kappa(i)) = 0$.

On en déduit :

$$\inf_{\kappa \in \mathfrak{X}_n} v^{[r,s]}(g_j(\kappa)) \geq v^{[r,s]}(z) + C^{[r,s]}_{\varphi^N} + C^{[r,s]}_{\psi^N} + j(C_n + N - p^N s - 1).$$

Par définition de fonction analytique sur $\mathfrak{X}$ à valeurs dans Δ, il faut montrer que, pour tout $n > 0$, il existe $s > 0$ et $N > 0$ tel que, pour tout $r \in]0, s]$, l'expression (2) est convergente pour la valuation naturelle de $\mathcal{O}(\mathfrak{X}_n) \widehat{\otimes} \Delta^{[r,s]}$, et que les éléments dans $\mathcal{O}(\mathfrak{X}_n) \widehat{\otimes} \mathscr{R}$ ainsi définis ne dépendent ni de N ni du système de représentants de $(\mathbb{Z}/p^N\mathbb{Z})^\times$ et sont compatibles par rapport aux applications naturelles de restriction $\mathcal{O}(\mathfrak{X}_{n+1}) \widehat{\otimes} \mathscr{R} \to \mathcal{O}(\mathfrak{X}_n) \widehat{\otimes} \mathscr{R}$.

Fixons n et choisissons N assez grand et $s < r(\Delta)p^{-N}$ assez petit de sorte que

$$C_n + N - p^N s - 1 > 0.$$

On observe qu'aucun choix ne dépend de r. Ceci montre que la somme des $g_j(\kappa)$ converge sur $\mathcal{O}(\mathfrak{X}_n)\mathbin{\widehat{\otimes}}\Delta^{[r,s]}$ pour tout $r<s$ et définit, pour chaque $n\geq 0$, un élément dans $\mathcal{O}(\mathfrak{X}_n)\mathbin{\widehat{\otimes}}\Delta$.

On montre maintenant que l'expression ne dépend ni de N ni du choix du système de représentants de $(\mathbb{Z}/p^N\mathbb{Z})^\times$. Si l'on fixe $n>0$, $N>0$ et $s>0$ comme ci-dessus et un système de représentants de $(\mathbb{Z}/p^N\mathbb{Z})^\times$, l'expression (2) définit une fonction rigide analytique sur $\mathfrak{X}_n$. De plus, les valeurs de ces fonctions aux caractères $x^k\in\mathfrak{X}_n$, $k\in\mathbb{Z}$, ne dépendent pas du choix de N. On conclut en remarquant que ces caractères sont Zariski denses dans $\mathfrak{X}_n$.

Enfin, les fonctions définies ne dépendent évidement pas de n, et on conclut donc que (2) définit un élément de $\mathcal{O}(\mathfrak{X})\mathbin{\widehat{\otimes}}\Delta^{\psi=0}$. □

Corollaire I.14. *Soit $N(n)$ le plus petit N qui fait que la formule* (2) *soit bien définie sur $\mathfrak{X}_n$ et soit $\kappa\in\mathfrak{X}_n$. Alors, pour tous $0<r<s<r(\Delta)p^{-N(n)}$, $\kappa(\partial)$ stabilise $\Delta^{[r,s]}$.*

IC3. *Bases et modules de de Rham.* Les éléments définis ci-dessous permettront de simplifier considérablement les notations futures ainsi que de clarifier certains facteurs apparaissant dans les formules d'interpolation (on y reviendra dans la section ID1).

Si $\xi:\mathbb{Q}_p^\times\to L^\times$ est un caractère, on note e_ξ une base du L-espace vectoriel $L(\xi)$ de dimension 1 muni d'actions de φ et Γ via les formules $\varphi(e_\xi)=\xi(p)\cdot e_\xi$ et $\sigma_a(e_\xi)=\xi(a)\cdot e_\xi$, $a\in\mathbb{Z}_p^\times$. Si D est un (φ,Γ)-module, on note $D(\xi)=D\otimes\xi$ le module $D\otimes_L L(\xi)$ (c'est le tordu de D par ξ). Le choix de e_ξ fournit un isomorphisme de L-espaces vectoriels $D\xrightarrow{\sim}D(\xi)$, $x\mapsto x\otimes e_\xi$. Si $\eta:\mathbb{Z}_p^\times\to L^\times$ est un caractère, les mêmes constructions s'appliquent sur η en regardant η comme un caractère sur $\mathbb{Q}_p^\times$ en posant $\eta(p)=1$.

Soit $\eta:\mathbb{Z}_p^\times\to L^\times$ un caractère localement constant. L'élément $G(\eta)e_\eta\in L_\infty(\eta)$ est fixé par l'action de Γ et on a donc un isomorphisme $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta))=(L_\infty((t))\cdot e_\eta)^\Gamma=L\cdot G(\eta)e_\eta$, ce qui nous fournit un générateur $\boldsymbol{e}_\eta^{\mathrm{dR}}=G(\eta)e_\eta$ de ce module. Si $j\in\mathbb{Z}$, on rappelle que l'on a noté $e_j=e_{\chi^j}$ une base du module $L(\chi^j)$, de sorte que $\boldsymbol{e}_j^{\mathrm{dR}}=t^{-j}e_j$ est une base de $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\chi^j))$. Notons

$$\boldsymbol{e}_{\eta,j}=e_\eta\otimes e_j,$$

qui est une base de $L(\eta\chi^j)$. L'élément

$$\boldsymbol{e}_{\eta,j}^{\mathrm{dR}}=\boldsymbol{e}_\eta^{\mathrm{dR}}\otimes\boldsymbol{e}_j^{\mathrm{dR}}=G(\eta)t^{-j}\cdot\boldsymbol{e}_{\eta,j}=G(\eta)e_\eta\otimes t^{-j}e_j$$

constitue une base du module $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta\chi^j))$.

Si $D\in\Phi\Gamma(\mathscr{R})$ est de Rham, alors $D(\eta\chi^j)$ l'est aussi et on a, par ce qui précède,

$$\boldsymbol{D}_{\mathrm{dR}}(D(\eta\chi^j))=\boldsymbol{D}_{\mathrm{dif}}(D\otimes L(\eta\chi^j))^\Gamma=\boldsymbol{D}_{\mathrm{dR}}(D)\otimes L\cdot\boldsymbol{e}_{\eta,j}^{\mathrm{dR}},$$

de sorte que l'application $x\mapsto x\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR}}$ induit un isomorphisme $\boldsymbol{D}_{\mathrm{dR}}(D)\xrightarrow{\sim}\boldsymbol{D}_{\mathrm{dR}}(D(\eta\chi^j))$.

Enfin, on note $e_\eta^\vee=e_{\eta^{-1}}$, $e_j^\vee=e_{-j}$ les éléments duaux, respectivement, de e_η et e_j, ainsi que

$$\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}=G(\eta)^{-1}\cdot e_\eta^\vee\otimes t^je_j^\vee,$$

base du module $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta^{-1}\chi^{-j}))$. L'application $x\mapsto x\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}$ induit un isomorphisme $\boldsymbol{D}_{\mathrm{dR}}(D(\eta\chi^j))\xrightarrow{\sim}\boldsymbol{D}_{\mathrm{dR}}(D)$ et on a $x\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR}}\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}=x$ pour tout $x\in\boldsymbol{D}_{\mathrm{dR}}(D)$.

IC4. *Interpolation.* Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham, $\Delta = \mathbb{N}_{\mathrm{rig}}(D)$, η un caractère de Dirichlet de conducteur p^m, $m \geq m(\Delta)$ et $\kappa(x) = z_\kappa^{\log x/q}$. On pose

$$\Lambda_{D,z}(\eta\kappa) = \mathrm{G}(\eta)^{-1} \sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(a)\sigma_a[\varphi^{-m}\kappa(\partial)(1-\varphi)z]_0. \tag{3}$$

L'application φ^{-m} est définie sur $\Delta^{]0,r_m]}$ et l'expression ci-dessus n'a donc un sens que pour les caractères κ tels que $\kappa(\partial)(1-\varphi)z \in \Delta^{]0,r_m]}$ et, dans ce cas, on a $\Lambda_{D,z}(\eta\kappa) \in L_m \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$. Nous allons montrer que $\Lambda_{D,z}$ est bien définie dès que η est un caractère de Dirichlet assez ramifié et κ est un caractère vivant dans un certain voisinage ouvert du caractère trivial et que les valeurs interpolées par $\Lambda_{D,z}$ aux caractères spéciaux $\eta\chi^j$, $j \in \mathbb{Z}$, sont reliées à des valeurs arithmétiquement intéressantes. Le théorème principal de ce chapitre est le suivant :

Théorème I.15. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham, $z \in \Delta^{\psi=1}$ et $h \in \mathbb{Z}$ tel que* $\mathrm{Fil}^{-h}\, \boldsymbol{D}_{\mathrm{dR}}(D) = \boldsymbol{D}_{\mathrm{dR}}(D)$. *Il existe une constante $N(D)$ (ne dépendant que de l'extension K fournie par le théorème de monodromie p-adique et de p),*[22] *telle que la formule* (3) *définit une fonction rigide analytique $\Lambda_{D,z} \in \mathcal{O}(\mathfrak{U}_D) \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$, où $\mathfrak{U}_D = \bigcup_{\mathrm{c}(\eta) > m(\Delta)} \mathfrak{B}(\eta, N(D))$* (cf. la section IA). *De plus, pour tout caractère $\eta\chi^j \in \mathfrak{U}_D$, où η est un caractère de conducteur p^n, $n > 0$, et $j \in \mathbb{Z}$, on a*

$$\Lambda_{D,z}(\eta\chi^j) = \Gamma^*(j-h+1)p^{n(j+1)} \cdot \begin{cases} \exp^*\left(\int_\Gamma \eta\chi^{-j} \cdot \mu_{\nabla_h z}\right) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j} & \textit{si } j \geq h, \\ \exp^{-1}\left(\int_\Gamma \eta\chi^{-j} \cdot \mu_{\nabla_h z}\right) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j} & \textit{si } j \ll 0, \end{cases}$$

De plus, si D est cristallin, l'application $\Lambda_{D,z}$ provient par restriction d'une fonction rigide analytique définie sur tout l'espace des poids.

Remarque I.16. — L'application Exp^* étant Λ_∞-linéaire, on a $\mu_{\nabla_h z} = \nabla_h \mu_z$.

— La preuve du théorème consiste de trois parties. Dans la section IC5, on calcule la constante $N(D)$, qui fournit l'ouvert de définition de $\Lambda_{D,z}$, et, dans les sections IC6 et IC7, on montre les propriétés d'interpolation. Notons que, si l'on définit log comme $\exp^{-1}$ ou $\exp^*$ selon le cas, on obtient un énoncé d'interpolation uniforme pour tous les entiers $j \in \mathbb{Z}$.

IC5. *Calcul du rayon de convergence.* Si η est un caractère de conducteur p^m, $\kappa(x) = z_\kappa^{\log x/q}$ alors, pour que la formule (3) définissant l'application $\Lambda_{D,z}(\eta\kappa)$ ait un sens, il suffit que $\kappa(\partial)(1-\varphi)z \in \Delta^{]0,r_m]}$. Ceci impose des conditions sur la valeur de N dans la définition de $\kappa(\partial)$, comme le montre le corollaire I.14. Le lemme suivant calcule, pour un η fixe, le rayon de convergence autour η de cette formule, ce qui décrit l'ouvert $\mathfrak{U}_D$ de définition de l'application $\Lambda_{D,z}$.

Lemme I.17. *Soient $z \in \Delta^{\psi=1}$ et $\eta : \mathbb{Z}_p^\times \to L^\times$ un caractère de conducteur p^m, $m > m(\Delta)$. Il existe une constante $N(D) = C_p + m(\Delta)$, où C_p est une constante qui ne dépend que de p, telle que la formule définissant $\Lambda_{D,z}(\eta\kappa)$ est bien définie dès que $v_p(z_\kappa - 1) > p^{N(D)-m}$.*

Démonstration. L'élément $\kappa(\partial)(1-\varphi)z$ est défini comme somme des

$$g_j(\kappa) = \binom{\omega_\kappa}{j}\kappa(a)a^{-j}(1+T)^a p^{Nj}\varphi^N(\partial^j z_a),$$

22. cf. le lemme I.17 ci-dessous pour le calcul de la constante.

pour N assez grand. Ces éléments appartiennent[23] à $\Delta^{]0,r_m]}$ si $N < m - m(\Delta)$. Prenons donc $N = m - m(\Delta) - 1$.

Soit $N(D) = C_p + m(\Delta)$, avec

$$C_p = -1/\log p - \log\log p + v_p(q) + \frac{1}{p-1} + 3,$$

et montrons que, si $v_p(z_\kappa - 1) > p^{N(D)-m}$, la somme définissant $\kappa(\partial)(1-\varphi)z$ converge.

Par la démonstration de la proposition I.13, on a

$$v^{[r,r_m]}(g_j(\kappa)) \geq C + j\Big(\frac{1}{j} v_p\Big(\binom{\omega_\kappa}{j}\Big)\Big) + N + C_\partial^{[p^N r, p^N r_m]}\Big),$$

où C est une constante qui ne dépend pas de j et $C_\partial^{[p^N r, p^N r_m]} \geq -p^N r_m - 1 = -r_{m(\Delta)+1} - 1$. On se ramène donc à montrer que

$$\frac{1}{j} v_p\Big(\binom{\omega_\kappa}{j}\Big) > m(\Delta) - m + r_{m(\Delta)+1} + 2.$$

On a

$$\omega_\kappa = \frac{\log \kappa(\exp(q))}{q} = \frac{\log z_\kappa}{q},$$

donc $v_p(\omega_\kappa) = v_p(\log z_\kappa) - v_p(q) \geq \inf\{v_p(z_\kappa - 1)p^k - k\} - v_p(q)$. On a deux cas :

— Si $v_p(\omega_\kappa) \geq 0$, alors la condition est automatiquement satisfaite dès que $m > m(\Delta) + 3$.

— Supposons que $v_p(\omega_\kappa) < 0$. On a $v_p\big(\binom{\omega_\kappa}{j}\big) = j v_p(\omega_\kappa) - v_p(j!) \geq j(v_p(\omega_\kappa) - 1/(p-1))$. Pour montrer l'inégalité ci-dessus, il suffit donc de montrer que

$$\inf_{k \geq 0}\{v_p(z_\kappa - 1)p^k - k\} > v_p(q) + \frac{1}{p-1} + m(\Delta) - m + r_{m(\Delta)+1} + 2.$$

Ceci revient à montrer que, pour tout $k \geq 0$, on a

$$v_p(z_\kappa - 1)p^k - k > v_p(q) + \frac{1}{p-1} + m(\Delta) - m + r_{m(\Delta)+1} + 2,$$

ou, de manière équivalente,

$$v_p(z_\kappa - 1) > p^{-k}\Big(v_p(q) + \frac{1}{p-1} + m(\Delta) - m + r_{m(\Delta)+1} + 2 + k\Big) = p^{-k}(C + k).$$

La fonction d'une variable réelle $f(x) = p^{-x}(C + x)$ atteint son maximum absolu en $x = 1/\log p - C$. On a donc

$$p^{-k}(v_p(q) + \frac{1}{p-1} + m(\Delta) - m + r_{m(\Delta)+1} + 2 + k) \leq p^{m(\Delta)+C_p-m},$$

ce qui permet de conclure. □

Remarque I.18. — Le lemme ci-dessus nous permet de décrire l'ouvert $\mathfrak{U}_D \subseteq \mathfrak{X}$ de définition de $\Lambda_{D,z}$: il est une union de boules centrées sur les points ζ_{p^m}, $m > m(\Delta)$, de rayon $p^{-p^{N(D)-m}}$ (qui tend vers 1 quand $\mathrm{c}(\eta) \to +\infty$).

23. Rappelons que, comme $z \in \Delta^{\psi=1}$, alors $z \in \Delta^{]0,r_{m(\Delta)}]}$, et donc $\varphi^N(\partial^j z_a) \in \Delta^{]0,r_{m(\Delta)+N+1}]}$.

— Si $\zeta_{p^m}^b, \zeta_{p^m}^a,\ 1 \neq a, b \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, sont deux racines primitives de l'unité (correspondant au choix de deux caractères d'ordre fini du même conducteur), alors

$$v_p(\zeta_{p^m}^b - \zeta_{p^m}^a) = v_p(\zeta_{p^m}^{b-a} - 1) = \frac{1}{p^{m-1-v_p(b-a)}(p-1)} = p^{v_p(b-a)} r_m.$$

On en déduit que les boules sont disjointes dès que $v_p(b-a) < N(D)$.

— Un entier $j \in \mathbb{Z}$, $j \equiv 0 \pmod{p-1}$, correspond au caractère χ^j et $z_j = z_{\chi^j} = (1+2p)^j$.[24] On en déduit que $v_p(z_j - 1) > p^{N-\mathrm{c}(\eta)}$ si $v_p(j) \geq p^{N-\mathrm{c}(\eta)}$. L'ouvert $\mathfrak{B}(\eta, N)$ contient alors tous les caractères de la forme $\eta\chi^j,\ v_p(j) \geq p^{N-\mathrm{c}(\eta)}$. En particulier, si $N - \mathrm{c}(\eta) < 0$, il contient tous les caractères $\eta\chi^j$, avec $(p-1)p \mid j$.

IC6. *Interpolation des applications exponentielles duales.* On commence par quelques résultats préliminaires. On pourra comparer le lemme suivant avec [Berger 2003, Lemma II.1] (pour le cas cristallin et le caractère trivial).

Lemme I.19. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham, $z \in \Delta^{\psi=1}$ et $\eta : \mathbb{Z}_p^\times \to L^\times$ un caractère de conducteur p^n, $n \geq m(\Delta)$ et $m \geq 0$. Alors :*

(i) *Si $m(\Delta) \leq m < n$, on a*

$$\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta(a)\sigma_a([\partial^j \varphi^{-m} z]_0) = 0,$$

(ii) *L'expression $\sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(a)\sigma_a(p^{-m}\,\mathrm{Tr}_{L_m/L_n}[\partial^j \varphi^{-m} z]_0)$ ne dépend pas de $m \geq n$.*

Démonstration. Par définition de Δ, on a $\varphi^{-m} z \in L_m[\![t]\!] \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ et $p^{-k}\,\mathrm{Tr}_{L_{m+k}/L_m} \varphi^{-(m+k)} z = \varphi^{-m} z$ car z est fixé par ψ, d'où $\varphi^{-m} z$ peut être exprimé comme $\sum_{h \geq 0} \sum_{b \in (\mathbb{Z}/p^m\mathbb{Z})} (\zeta_{p^m}^b \otimes d_{h,b}) t^h$, avec $d_{h,b} \in \boldsymbol{D}_{\mathrm{dR}}(D)$ (l'expression n'est évidement pas unique).

Montrons le premier point. Il suffit, par linéarité, de montrer le résultat sous l'hypothèse que

$$[\partial^j \varphi^{-m} z]_0 = \zeta_{p^m}^b \otimes d, \quad \text{avec } 0 \leq b \leq p^m - 1,\ d \in \boldsymbol{D}_{\mathrm{dR}}(D).$$

Or,

$$\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta(a)\sigma_a(\zeta_{p^m}^b \otimes d) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta(a)\zeta_{p^m}^{ab} \otimes d = p^n \hat{\eta}(-b/p^m) \otimes d.$$

On en déduit le résultat en remarquant que $\hat{\eta}$ est une fonction à support dans $p^{-n}\mathbb{Z}_p^\times$ et que $v_p(-b/p^m) > -n$.

En ce qui concerne le dernier point, on peut supposer par linéarité que $\mathrm{Tr}_{L_m/L_n}[\partial^j \varphi^{-m} z]_0 = \zeta_{p^n}^b \otimes d$, où $b \in (\mathbb{Z}/p^n\mathbb{Z})$ et $d \in \boldsymbol{D}_{\mathrm{dR}}(D)$. On a

$$p^{-m} \sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(a)\sigma_a(\zeta_{p^n}^b \otimes d) = p^{-m} \sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(a)\zeta_{p^n}^{ab} \otimes d = \hat{\eta}(-b/p^n) \otimes d = p^{-n} G(\eta)\eta^{-1}(b) \otimes d.$$

L'expression ci-dessus ne dépend pas de m, d'où le résultat. □

Remarque I.20. Si D est cristallin, $z = \sum_k a_k T^k \in (\mathscr{R}^+ \otimes \boldsymbol{D}_{\mathrm{cris}}(D))^{\psi=1} \subseteq (\Delta[1/t])^{\psi=1}$ et $n = 0$ dans le lemme ci-dessus, on a, d'après [Berger 2003, Lemma II.1],

$$p^{-m}\,\mathrm{Tr}_{L_m/L}[\varphi^{-m} z]_0 = (1 - p^{-1}\varphi^{-1}) a_0.$$

24. On a posé $z_{\chi^j} = (\exp q)^j$, mais cela ne change rien si l'on choisit un autre générateur de $1 + 2p\mathbb{Z}_p$, comme par exemple $1 + 2p$.

Lemme I.21. *Soient $z \in D^{\psi=1}$, η un caractère constant modulo p^n avec $n \geq m(\Delta)$ et $m \geq n$. On a alors l'égalité suivante dans $L_m \otimes H^1_{\varphi,\gamma}(D(-j))$:*

$$\int_\Gamma \eta\chi^{-j} \cdot \mu_z = \sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(a) a^{-j} \int_{\Gamma_m} \chi^{-j} \cdot \mu_{\sigma_a(z)}.$$

De plus, si D est de Rham, on a l'égalité (dans $L_m \otimes \boldsymbol{D}_{\mathrm{dR}}(D(-j))$)

$$\exp^*\Big(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\Big) \otimes \boldsymbol{e}_\eta^{\mathrm{dR},\vee} = G(\eta)^{-1} \sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(a) a^{-j} \exp^*_{D(-j)}\Big(\int_{\Gamma_m} \chi^{-j} \cdot \mu_{\sigma_a(z)}\Big).$$

Démonstration. Montrons seulement le deuxième point, vu que les mêmes techniques seront utilisées plus tard. Par la proposition I.12 (appliquée à $D(\eta\chi^{-j})$, $j = 0$ et $n = 0$), on a

$$\begin{aligned}\exp^*\Big(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\Big) &= p^{-m} \operatorname{Tr}_{L_m/L}\big([\varphi^{-m}(z \otimes e_\eta \otimes e_{-j})]_0\big) \\ &= p^{-m} \operatorname{Tr}_{L_m/L}\big(G(\eta)^{-1}[t^{-j}\varphi^{-m} z]_0\big) \otimes G(\eta) e_\eta \otimes t^j e_{-j},\end{aligned}$$

où on a utilisé que l'élément $\boldsymbol{e}^{\mathrm{dR}}_{\eta,-j} = G(\eta) e_\eta \otimes t^j e_{-j}$ est fixé par Γ et commute donc à la trace. Notons que $[t^{-j}\varphi^{-m} z]_0 = [\varphi^{-m} z]_j$. La somme de Gauss s'écrit comme

$$G(\eta)^{-1} = p^{-n}\eta(-1) G(\eta^{-1}) = p^{-n}\eta(-1) \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta^{-1}(a) \zeta_{p^n}^a$$

et on a

$$\begin{aligned}\operatorname{Tr}_{L_m/L}\big(G(\eta)^{-1}[\varphi^{-m} z]_j\big) &= \eta(-1) p^{-n} \sum_{b \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(b) \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \eta^{-1}(ab) \zeta_{p^n}^{ab} \sigma_b [\varphi^{-m} z]_j \\ &= G(\eta)^{-1} \sum_{b \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(b) \sigma_b [\varphi^{-m} z]_j,\end{aligned}$$

où on a utilisé la formule pour la trace et encore une fois la formule reliant $G(\eta)$ et $G(\eta^{-1})$. On en déduit

$$\exp^*\Big(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\Big) = p^{-m} G(\eta)^{-1} \sum_{b \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(b) \sigma_b([\varphi^{-m} z]_j) \otimes G(\eta) e_\eta \otimes t^j e_{-j}.$$

Or $\sigma_b[\varphi^{-m} z]_j = b^{-j}[\varphi^{-m}\sigma_b z]_j$ et, par la loi de réciprocité encore une fois (pour D, j et m), on sait que $p^{-m}[\varphi^{-m}\sigma_b z]_j \otimes t^j e_{-j} = \exp^*\big(\int_{\Gamma_m} \chi^{-j} \cdot \mu_{\sigma_b(z)}\big)$, d'où

$$\exp^*\Big(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\Big) \otimes \boldsymbol{e}_\eta^{\mathrm{dR},\vee} = G(\eta)^{-1} \sum_{b \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \eta(b) b^{-j} \exp^*\Big(\int_{\Gamma_m} \chi^{-j} \mu_{\sigma_b(z)}\Big),$$

ce qui permet de conclure. □

Proposition I.22. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham, $z \in \Delta^{\psi=1}$, $h \in \mathbb{Z}$ tel que $\mathrm{Fil}^{-h}\, \boldsymbol{D}_{\mathrm{dR}}(D) = \boldsymbol{D}_{\mathrm{dR}}(D)$, $\eta : \mathbb{Z}_p^\times \to L^\times$ un caractère de conducteur p^n, $n \geq m(\Delta)$, et $j \geq h$ un entier. On a alors l'égalité suivante dans $L_n \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$:*

$$\Lambda_{D,z}(\eta\chi^j) = (-h+j)!\, p^{n(j+1)} \cdot \exp^*\Big(\int_\Gamma \eta\chi^{-j} \cdot \mu_{\nabla_h z}\Big) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j}.$$

Démonstration. La preuve du lemme I.21 appliqué à $\nabla_h z \in D^{\psi=1}$ donne

$$\exp^*\left(\int_\Gamma \eta\chi^{-j}\cdot\mu_{\nabla_h z}\right)\otimes \boldsymbol{e}_{\eta,-j}^{\mathrm{dR},\vee} = G(\eta)^{-1}p^{-n}\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}\nabla_h z]_j.$$

Or, on remarque que $(\nabla-h+1)\circ\cdots\circ(\nabla-1)\circ\nabla\big(\sum_{j\geq 0}a_jt^j\big)=\sum_{j\geq h}a_j j(j-1)(j-2)\cdots(j-h+1)t^j$. L'opérateur $(\nabla-h+1)\circ\cdots\circ(\nabla-1)$ a donc l'effet de tuer les coefficients plus petits que $h-1$. Comme $j\geq h$, on a

$$[\varphi^{-n}\nabla_h z]_j = \frac{j!}{(j-h)!}[\varphi^{-n}z]_j = \frac{p^{-nj}}{(j-h)!}[\varphi^{-n}\partial^j z]_0.$$

Par ailleurs, en utilisant, respectivement, le lemme I.19 et la définition de $\Lambda_{D,z}$ (cf. la section IC4, (3)), on obtient

$$\begin{aligned}\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}\partial^j z]_0 &= \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}\partial^j z - p\varphi^{-(n-1)}\partial^j z]_0\\ &= \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}\partial^j(1-\varphi)z]_0\\ &= \mathrm{G}(\eta)\cdot\Lambda_{D,z}(\eta\chi^j),\end{aligned}$$

ce qui permet de conclure. □

IC7. *Interpolation des applications exponentielles.* Des calculs du même genre montrent que

Proposition I.23. *Soient $D\in\Phi\Gamma(\mathscr{R})$ de Rham, $z\in\Delta^{\psi=1}$, $h\in\mathbb{Z}$ tel que $\mathrm{Fil}^{-h}\,\boldsymbol{D}_{\mathrm{dR}}(D)=\boldsymbol{D}_{\mathrm{dR}}(D)$, $\eta:\mathbb{Z}_p^\times\to L^\times$ un caractère de conducteur p^n tel que $n\geq m(\Delta)$ et $j\geq -h+1$ un entier tel que l'équation $\partial^j z_j=z$ a une solution dans Δ et tel que $\exp_{D(j)}:\boldsymbol{D}_{\mathrm{dR}}(D(j))\to H^1_{\varphi,\gamma}(D(j))$ soit un isomorphisme. On a alors l'égalité suivante dans $L_n\otimes\boldsymbol{D}_{\mathrm{dR}}(D)$:*

$$\Lambda_{D,z}(\eta\chi^{-j})=\Gamma^*(-h-j+1)p^{n(-j+1)}\cdot\exp^{-1}\left(\int_\Gamma\eta\chi^j\cdot\mu_{\nabla_h z}\right)\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}.$$

Démonstration. De la même façon que dans le lemme I.21, on montre que

$$\exp^{-1}\left(\int_\Gamma\eta\chi^j\cdot\mu_{\nabla_h z}\right)\otimes\boldsymbol{e}_{\eta}^{\mathrm{dR},\vee}=G(\eta)^{-1}\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)a^j\exp^{-1}\left(\int_{\Gamma_n}\chi^j\cdot\mu_{\nabla_h\sigma_a(z)}\right).$$

Observons que, si $\partial^j z_j=z$, alors $\partial^j(a^{-j}\sigma_a z_j)=\sigma_a z$. En appliquant la loi de réciprocité (théorème I.10), on voit que l'expression ci-dessus est égale à

$$G(\eta)^{-1}\frac{p^{n(j-1)}}{\Gamma^*(-j-h+1)}\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)[\varphi^{-n}(\sigma_a z_j)]_0\otimes t^{-j}e_j.$$

D'où

$$\Gamma^*(-h-j+1)p^{n(-j+1)}\cdot\exp^{-1}\left(\int_\Gamma\eta\chi^j\cdot\mu_{\nabla_h z}\right)\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}=G(\eta)^{-1}\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)[\varphi^{-n}(\sigma_a z_j)]_0.$$

Par le lemme I.19 et par le fait que ∂ est inversible sur $\Delta^{\psi=0}$ et qu'on a donc le droit d'écrire $(1-p^{-j}\varphi)z_j=\partial^{-j}(1-\varphi)z$, on a

$$\begin{aligned}\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}(z_j)]_0&=\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}((1-p^{-j}\varphi)z_j)]_0\\&=\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\eta(a)\sigma_a[\varphi^{-n}(\partial^{-j}(1-\varphi)z)]_0\\&=\mathrm{G}(\eta)\cdot\Lambda_{D,z}(\eta\chi^{-j}),\end{aligned}$$

ce qui permet de conclure. □

Ceci finit la preuve du théorème I.15.

IC8. *Le cas des poids de Hodge–Tate positifs.* Soit $D\in\Phi\Gamma(\mathscr{R})$ de Rham à poids de Hodge–Tate positifs et notons toujours $\Delta=\mathbb{N}_{\mathrm{rig}}(D)$. Fixons un entier h tel que $\mathrm{Fil}^{-h}\,\boldsymbol{D}_{\mathrm{dR}}(D)=\boldsymbol{D}_{\mathrm{dR}}(D)$. On a donc $D\subseteq\Delta$ et, si $z\in D^{\psi=1}$, on peut appliquer la construction faite ci-dessus à l'élément z (et h) pour obtenir une application $\Lambda_{D,z}$. On commence avec quelques remarques.

Lemme I.24. *Soient $z\in D^{\psi=1}$, $\eta:\Gamma\to L^\times$ un caractère localement analytique et $j\in\mathbb{Z}$. On a*

$$\int_\Gamma\eta\chi^j\cdot\mu_{\nabla_h z}=(-\omega_\eta-j-h+1)(-\omega_\eta-j-h+2)\cdots(-\omega_\eta-j)\int_\Gamma\eta\chi^j\cdot\mu_z.$$

En particulier, si η est localement constant, on a

$$\int_\Gamma\eta\chi^j\cdot\mu_{\nabla_h z}=(-j-h+1)(-j-h+2)\cdots(-j)\int_\Gamma\eta\chi^j\cdot\mu_z.$$

Démonstration. Si $a\in\mathbb{Z}_p^\times$, par définition de l'action de Γ sur $D^{\psi=1}$, on a

$$\int_\Gamma\eta\chi^j\cdot\sigma_a\mu_z=\eta(a)^{-1}a^{-j}\int_\Gamma\eta\chi^j\cdot\mu_z,$$

et, si $i\in\mathbb{Z}$, la formule $\nabla=\lim_{a\to1}(\sigma_a-1)/(a-1)$ donne

$$\int_\Gamma\eta\chi^j(\nabla-i)\cdot\mu_z=\left(\lim_{a\to1}\frac{\eta(a)^{-1}a^{-j}-1}{a-1}-i\right)\int_\Gamma\eta\chi^j\cdot\mu_z=(-\eta'(1)-j-i)\int_\Gamma\eta\chi^j\cdot\mu_z,$$

ce qui permet de conclure car $\nabla_h=(\nabla-h+1)\circ\cdots\circ(\nabla-1)\circ\nabla$. □

Lemme I.25. *Soient $z\in D^{\psi=1}$, η un caractère de Dirichlet de conducteur p^n et $j\geq0$ assez grand.*[25] *Alors*

$$\Lambda_{D,z}(\eta\chi^{-j})=\Gamma^*(-j+1)p^{n(-j+1)}\cdot\exp^{-1}\left(\int_\Gamma\eta\chi^j\cdot\mu_z\right)\otimes\boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee}.$$

Démonstration. C'est une conséquence directe de la proposition I.23 et du lemme I.24 ci-dessus. □

Lemme I.26. *Soient $z\in D^{\psi=1}$, η et $j\geq0$ comme dans la proposition I.22. Alors*

$$\Lambda_{D,z}(\eta\chi^j)=\Gamma^*(j+1)p^{n(j+1)}\cdot\exp^*\left(\int_\Gamma\eta\chi^{-j}\cdot\mu_z\right)\otimes\boldsymbol{e}_{\eta,-j}^{\mathrm{dR},\vee}.$$

25. Il suffit que j soit plus grand que le plus grand poids de Hodge–Tate et assez grand de sorte que $\exp_{D(j)}$ soit bijective.

Démonstration. Si l'on part de $z \in D^{\psi=1}$, les calculs faits dans la preuve de la proposition I.22 marchent en posant $h = 0$ et donnent exactement le résultat cherché. $\square$

En notant, comme précédemment, par log l'application $\exp^{-1}$ ou $\exp^*$ selon le cas, on peut résumer ces résultats dans la forme énoncée au début de du chapitre :

Théorème I.27. *Soient $D \in \Phi\Gamma(\mathscr{R})$ de Rham à poids de Hodge–Tate positifs et $z \in D^{\psi=1}$. Il existe une fonction rigide analytique $\Lambda_{D,z} \in \mathcal{O}(\mathfrak{U}_D) \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ telle que, si $\eta\chi^j \in \mathfrak{U}_D$, où η est un caractère de conducteur p^n et $j \in \mathbb{Z}$ est tel que $j \geq 0$ ou $j \ll 0$, alors*

$$\Lambda_{D,z}(\eta\chi^j) = \Gamma^*(j+1)p^{n(j+1)} \cdot \log\left(\int_\Gamma \eta\chi^{-j} \cdot \mu_z\right) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j}.$$

IC9. *Le cas cristallin.* Soit $D \in \Phi\Gamma(\mathscr{R})$ cristallin de rang d. Supposons que les pentes de φ sur $\boldsymbol{D}_{\mathrm{cris}}(D)$ sont < 0. On a $\Delta \cong \mathscr{R} \otimes \boldsymbol{D}_{\mathrm{cris}}(D)$ (cf. [Nakamura 2014, Lemma 3.16]) et, si $z \in \Delta^{\psi=1}$, alors $z \in \mathscr{R}^+ \otimes \boldsymbol{D}_{\mathrm{cris}}(D)$ (cf. [Berger et Breuil 2010, proposition 2.5.2]). Notons $\alpha_1, \ldots, \alpha_d \in L$ les valeurs propres de φ et choisissons une base e_i, $1 \leq i \leq d$, de $\boldsymbol{D}_{\mathrm{cris}}(D)$ telle que $\varphi(e_i) = \alpha_i e_i$ pour tout i. On peut alors écrire

$$z = \sum_{i=1}^{d} \mathscr{A}_{\lambda_i} \otimes e_i$$

pour certaines distributions $\lambda_i \in \mathscr{D}(\mathbb{Z}_p, L)$ vérifiant $\psi(\lambda_i) = \alpha_i \lambda_i$, $1 \leq i \leq d$. Quelques calculs classiques nous permettent de montrer le résultat suivant :

Proposition I.28. *Soient $\eta : \mathbb{Z}_p^\times \to L^\times$ un caractère de conducteur p^n, $n > 0$, et $j > 0$. On a alors*

$$\Lambda_{D,z}(\eta\chi^j) = \sum_{i=1}^{d} \left(\int_{\mathbb{Z}_p^\times} \eta^{-1}x^j \cdot \lambda_i\right) \otimes \alpha_i^{-n} e_i.$$

En regardant le terme de droite de cette proposition, on en déduit :

Corollaire I.29. *Soit $D \in \Phi\Gamma(\mathscr{R})$ cristallin, alors la fonction $\Lambda_{D,z}$ provient par restriction d'une fonction définie sur tout l'espace des poids.*

ID. ***Équation fonctionnelle en dimension 2.*** Dans cette section, nous utilisons le résultat principal de [Rodrigues Jacinto 2018] (où on pourra trouver une exposition plus détaillée des résultats ainsi que ses preuves) pour en déduire une équation fonctionnelle satisfaite par la fonction L locale quand le (φ, Γ)-module est de dimension 2, de Rham et non triangulin. On commence par fixer un certain nombre de notations.

ID1. *Notations.* Soit $D \in \Phi\Gamma^{\text{ét}}(\mathscr{R})$ de dimension 2, de Rham à poids de Hodge–Tate 0 et k que l'on suppose non triangulin [26] et notons $\Delta = \mathbb{N}_{\mathrm{rig}}(D)$, qui est à poids de Hodge–Tate tous nuls.

Étendons un peu les notations de la section IC3. Si $\delta : \mathbb{Q}_p^\times \to L^\times$ est un caractère, on note e_δ une base du module $L(\delta)$ muni d'actions de φ et Γ via les formules $\varphi(e_\delta) = \delta(p) \cdot e_\delta$ et $\sigma_a(e_\delta) = \delta(a) \cdot e_\delta$, $a \in \mathbb{Z}_p^\times$. On note $D(\delta) = D \otimes \delta$ le module $D \otimes_L L(\delta)$. Le choix de e_δ fournit un isomorphisme de L-espaces vectoriels $D \xrightarrow{\sim} D(\delta)$, $x \mapsto x \otimes e_\delta$.

26. D'après Kedlaya, tout (φ, Γ)module sur $\mathscr{R}$ est étale à torsion près par un caractère ou triangulin. Dans ce dernier cas les calculs qui suivent sont déjà connus et l'hypothèse de non triangularité n'est donc pas vraiment restrictive.

Soit $\check{D} = \mathrm{Hom}_{\varphi,\Gamma}\big(D, \mathscr{R}\frac{dT}{1+T}\big)$ le dual de Tate de D, où $\mathscr{R}\frac{dT}{1+T}$ est le (φ,Γ)-module étale libre de rang 1 de base $\frac{dT}{1+T}$ sur lequel φ et Γ agissent par les formules $\gamma\big(\frac{dT}{1+T}\big) = \chi(\gamma)\frac{dT}{1+T}$ si $\gamma \in \Gamma$, $\varphi\big(\frac{dT}{1+T}\big) = \frac{dT}{1+T}$. On note

$$\langle\ ,\ \rangle : \check{D}_{\mathrm{rig}} \times D_{\mathrm{rig}} \to \mathscr{R}\frac{dT}{1+T}$$

l'accouplement naturel. Soient $\omega_D = (\det_D)\chi^{-1}$ et $\omega_\Delta = (\det_\Delta)\chi^{-1}$. Le fait que D soit de dimension 2 nous permet d'identifier $\check{D} = D \otimes \omega_D^{-1}$. Comme les poids de Hodge–Tate de D sont 0 et k, et ceux de Δ sont nuls, le caractère $\det_\Delta$ est localement constant et $\omega_D = \omega_\Delta x^k$ (et $\det_D = x^k \det_\Delta$). Notons $e_D = e_{\det_D}$.

Soit $\eta : \mathbb{Z}_p^\times \to L^\times$ est un caractère localement constant, vu comme un caractère de $\mathbb{Q}_p^\times$ en posant $\eta(p) = 1$. Rappelons que l'on a un générateur $\boldsymbol{e}_\eta^{\mathrm{dR}} = G(\eta)e_\eta$ du module $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta))$, que $e_\eta^\vee = e_{\eta^{-1}}$ dénote la base de $L(\eta)^*$ duale de e_η, et que $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta)^*) = L\cdot G(\eta)^{-1}e_\eta^\vee = L\cdot G(\eta^{-1})e_{\eta^{-1}}$. Ceci fournit deux bases $\boldsymbol{e}_\eta^{\mathrm{dR},\vee} = G(\eta)^{-1}e_\eta^\vee$ et $\boldsymbol{e}_{\eta^{-1}}^{\mathrm{dR}} = G(\eta^{-1})e_{\eta^{-1}}$ du module $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta)^*) = \boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta^{-1}))$, reliées par la formule $p^n\eta(-1)\boldsymbol{e}_\eta^{\mathrm{dR},\vee} = \boldsymbol{e}_{\eta^{-1}}^{\mathrm{dR}}$.

On aura besoin de jongler un peu avec des éléments habitant dans le module de de Rham des différents tordus de D et de son dual de Tate et les identifications suivantes permettent de voir tous ces éléments dans $\boldsymbol{D}_{\mathrm{dR}}(D)$. Fixons une base f_1, f_2 de $\boldsymbol{D}_{\mathrm{dR}}(D)$ et notons

$$\langle\ ,\ \rangle_{\mathrm{dR}} : \boldsymbol{D}_{\mathrm{dR}}(D) \times \boldsymbol{D}_{\mathrm{dR}}(D) \to L,$$

le produit scalaire défini par la formule $\langle a_1 f_1 + a_2 f_2,\ b_1 f_1 + b_2 f_2\rangle_{\mathrm{dR}} = a_1b_1 + a_2b_2$.

L'isomorphisme $\bigwedge^2 D = \big(\mathscr{R}\frac{dT}{1+T}\big)\otimes\omega_D$ induit un isomorphisme $\bigwedge^2 \boldsymbol{D}_{\mathrm{dR}}(D) = \boldsymbol{D}_{\mathrm{dR}}\big(\big(\mathscr{R}\frac{dT}{1+T}\big)\otimes\omega_D\big) = (t^{-k}L_\infty e_D)^\Gamma$. On définit $\Omega \in L_\infty$ par la formule $f_1 \wedge f_2 = (t^k\Omega)^{-1}e_D$, ce qui nous permet de fixer les bases $(t^k\Omega)^{-1}e_D$ et $t^k\Omega e_D^\vee$ du module $\boldsymbol{D}_{\mathrm{dR}}\big(\bigwedge^2 D\big)$ et de son dual. On fixe aussi les bases $\boldsymbol{e}_{\omega_D}^{\mathrm{dR}} = (t^{k-1}\Omega)^{-1}e_{\omega_D}$ et $\boldsymbol{e}_{\omega_D}^{\mathrm{dR},\vee} = (t^{k-1}\Omega)e_{\omega_D}^\vee$ du module $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\omega_D))$ et de son dual.

Enfin, afin d'alléger les notations dans les calculs futurs, notons, pour η comme ci-dessus et $j \in \mathbb{Z}$,

$$\boldsymbol{e}_{\eta,j,\omega_D^\vee} := e_\eta \otimes e_j \otimes e_{\omega_D}^\vee, \quad \boldsymbol{e}_{\eta,j} := e_\eta \otimes e_j,$$

bases de $L(\eta\chi^j\omega_D^{-1})$ et $L(\eta\chi^j)$ respectivement, et leurs duales

$$\boldsymbol{e}_{\eta,j,\omega_D^\vee}^\vee := e_\eta^\vee \otimes e_{-j} \otimes e_{\omega_D}, \quad \boldsymbol{e}_{\eta,j}^\vee := e_\eta^\vee \otimes e_{-j},$$

ainsi que des bases des module $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta\chi^j\omega_D^{-1}))$ et $\boldsymbol{D}_{\mathrm{dR}}(\mathscr{R}(\eta\chi^j))$

$$\begin{aligned}\boldsymbol{e}_{\eta,j,\omega_D^\vee}^{\mathrm{dR}} &:= G(\eta)e_\eta \otimes t^{-j}e_j \otimes t^{k-1}\Omega e_{\omega_D}^\vee = \boldsymbol{e}_\eta^{\mathrm{dR}} \otimes \boldsymbol{e}_j^{\mathrm{dR}} \otimes \boldsymbol{e}_{\omega_D}^{\mathrm{dR},\vee},\\ \boldsymbol{e}_{\eta,j}^{\mathrm{dR}} &:= G(\eta)e_\eta \otimes t^{-j}e_j = \boldsymbol{e}_\eta^{\mathrm{dR}} \otimes \boldsymbol{e}_j^{\mathrm{dR}}\end{aligned}$$

et leurs duales

$$\begin{aligned}\boldsymbol{e}_{\eta,j,\omega_D^\vee}^{\mathrm{dR},\vee} &:= G(\eta)^{-1}e_\eta^\vee \otimes t^j e_{-j} \otimes (t^{k-1}\Omega)^{-1}e_{\omega_D} = \boldsymbol{e}_\eta^{\mathrm{dR},\vee} \otimes \boldsymbol{e}_{-j}^{\mathrm{dR}} \otimes \boldsymbol{e}_{\omega_D}^{\mathrm{dR}},\\ \boldsymbol{e}_{\eta,j}^{\mathrm{dR},\vee} &:= G(\eta)^{-1}e_\eta^\vee \otimes t^j e_{-j} = \boldsymbol{e}_\eta^{\mathrm{dR},\vee} \otimes \boldsymbol{e}_{-j}^{\mathrm{dR}},\end{aligned}$$

et les variantes évidentes que l'on puisse imaginer.

Par exemple, si $\eta : \mathbb{Z}_p^\times \to L^\times$ est un caractère d'ordre fini, si $j \geq 0$ et si $x \in \boldsymbol{D}_{\mathrm{dR}}(\check{D}(\eta\chi^{-j}))$, on écrira $x \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j,\omega_D^\vee} \in \boldsymbol{D}_{\mathrm{dR}}(D)$ l'image de x par l'isomorphisme

$$\boldsymbol{D}_{\mathrm{dR}}(\check{D}(\eta\chi^{-j})) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(D), \quad x \mapsto x \otimes G(\eta)^{-1} e_\eta^\vee \otimes t^{-j} e_j \otimes (t^{k-1}\Omega)^{-1} e_{\omega_D}$$

et de même, si $x \in \boldsymbol{D}_{\mathrm{dR}}(D(\eta^{-1}\chi^j))$, on notera $x \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta^{-1},j} \in \boldsymbol{D}_{\mathrm{dR}}(D)$ l'image de x par l'isomorphisme

$$\boldsymbol{D}_{\mathrm{dR}}(D(\eta^{-1}\chi^j)) \xrightarrow{\sim} \boldsymbol{D}_{\mathrm{dR}}(D), \quad x \mapsto x \otimes G(\eta^{-1})^{-1} e^\vee_{\eta^{-1}} \otimes t^j e_{-j}.$$

Remarque I.30. — Les bases des modules de de Rham ainsi définies héritent une action de l'opérateur φ. On a, par exemple,

$$\varphi(\boldsymbol{e}^{\mathrm{dR}}_\eta) = \boldsymbol{e}^{\mathrm{dR}}_\eta, \quad \varphi(\boldsymbol{e}^{\mathrm{dR}}_j) = p^{-j} \boldsymbol{e}^{\mathrm{dR}}_j,$$
$$\varphi(\boldsymbol{e}^{\mathrm{dR}}_{\eta,j,\omega_D^\vee}) = p^{-j+k-1}\omega_D^{-1}(p) \cdot \boldsymbol{e}^{\mathrm{dR}}_{\eta,j,\omega_D^\vee} = p^{-j-1}\omega_\Delta^{-1}(p) \cdot \boldsymbol{e}^{\mathrm{dR}}_{\eta,j,\omega_D^\vee}.$$

— Il faut faire un peu d'attention et distinguer le caractère identité x et le caractère cyclotomique $\chi = x|x|$. Les deux coïncident sur $\mathbb{Z}_p^\times$ mais $\chi(p) = 1$, tandis que le premier prend la valeur p. Par exemple, Γ agit trivialement sur l'élément $e_j \otimes e^\vee_{x^j}$ mais $\varphi(e_j \otimes e^\vee_{x^j}) = p^{-j}\, e_j \otimes e^\vee_{x^j}$.

ID2. *Facteurs epsilon pour* GL_1. Commençons par rappeler la définition des facteurs locaux associés à un caractère. Soit $\eta : \mathbb{Q}_p^\times \to L^\times$ un caractère continu. On dit que η est non ramifié si sa restriction à $\mathbb{Z}_p^\times$ est triviale et il est ramifié dans le cas contraire. On définit son conducteur par 0 s'il est non ramifié, et par p^n, où n est le plus petit entier tel que la restriction $\eta|_{1+p^n\mathbb{Z}_p}$ soit triviale, dans le cas contraire. Notons (cf. [Bushnell et Henniart 2006, §6.23 ; Schmidt 2002, §1.1])

$$\varepsilon(\eta, s) = \begin{cases} 1 & \text{si } \eta \text{ n'est pas ramifié,} \\ p^{-ns}\eta(p)^n G(\eta^{-1}) & \text{si } \eta \text{ est ramifié} \end{cases}$$

le facteur epsilon associé au caractère η. Il satisfait l'équation fonctionnelle

$$\varepsilon(\eta, s)\varepsilon(\eta^{-1}, 1-s) = \eta(-1).$$

On notera dans la suite $\varepsilon(\eta) := \varepsilon\big(\eta, \tfrac{1}{2}\big) = p^{-n/2}\eta(p)^n G(\eta^{-1})$.

ID3. *Facteurs epsilon pour* GL_2. Soit π une représentation lisse irréductible de $\mathrm{GL}_2(\mathbb{Q}_p)$ et notons $\check{\pi}$ sa contragrédiente. On note (cf. [Bushnell et Henniart 2006, §6]) $\varepsilon(\pi) \in L$ le facteur epsilon de la représentation π, ainsi que $\varepsilon(\pi, s) = p^{-c(\pi)(s-\frac{1}{2})}\varepsilon(\pi)$, où $c(\pi)$ est le conducteur de π.[27] Observons que, si $j \in \mathbb{Z}$, alors $\varepsilon\big(\pi, j + \tfrac{1}{2}\big) = p^{-c(\pi)j}\varepsilon(\pi) = \varepsilon(\pi \otimes |\cdot|^j)$.[28] Le facteur epsilon satisfait une équation fonctionnelle

$$\varepsilon(\pi, s)\varepsilon(\check{\pi}, 1-s) = \omega_\pi(-1),$$

où ω_π est le caractère central de la représentation π.

27. Le conducteur $c(\pi)$ est défini comme le plus petit entier n tel que π possède un élément fixe par les matrices de la forme $K_n = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}_p) : c = d - 1 = 0 \bmod p^n \right\}$. On a $\dim_L \pi^{K_{c(\pi)}} = 1$.

28. On note $|\cdot|^j := |\det(\cdot)|^j$ le caractère non ramifié de $\mathrm{GL}_2(\mathbb{Q}_p)$ envoyant $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ vers $|ad - bc|^j$.

ID4. *Une équation fonctionnelle locale.* Soit $D \in \Phi\Gamma^{\text{ét}}(\mathscr{R})$ étale de dimension 2, de Rham non triangulin à poids de Hodge–Tate 0 et $k \geq 0$. Soit $\omega_D = \det_D \chi^{-1}$ de sorte que $\check{D} = D \otimes \omega_D^{-1}$. Rappelons que la correspondance de Langlands permet (cf. [Colmez 2010, proposition V.2.1 ; Colmez et Dospinescu 2014, remarque V.14 ; Rodrigues Jacinto 2018, §2.5]) de construire une involution

$$w_D : D^{\psi=1} \to \check{D}^{\psi=1}.$$

Plus précisement, notre (φ,Γ)-module D peut être vu comme un faisceau $\left(\begin{smallmatrix} \mathbb{Z}_p-\{0\} & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$-équivariant [29] $U \mapsto D \boxtimes U$ sur $\mathbb{Z}_p$ dont les sections globales sont données par $D \boxtimes \mathbb{Z}_p = D$ et la construction (cf. [Colmez 2010]) de la représentaiton $\Pi(D)$ de $\mathrm{GL}_2(\mathbb{Q}_p)$ associée à D par la correspondance de Langlands p-adique est fondée sur l'extension de ce faisceaux en un faisceau G-équivariant $U \mapsto D \boxtimes U$ sur $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{Q}_p)$. On a un accouplement parfait et G-équivariant $[\ ,\ ]_{\mathbb{P}^1}$ sur $D \boxtimes \mathbb{P}^1$ et la suite exacte fondamentale de G-modules suivante :

$$0 \to \Pi(D)^* \otimes \omega_D \to D \boxtimes \mathbb{P}^1 \to \Pi(D) \to 0.$$

De plus, en notant $u = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, on a un isomorphisme

$$\mathrm{Res}_{\mathbb{Z}_p} : (\Pi(D)^* \otimes \omega_D)^{u=1} \xrightarrow{\sim} D^{\psi=1}.$$

Si $z \in D^{\psi=1}$, on note $\tilde{z}$ l'image inverse de z par cet isomorphisme. En notant $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, l'élément $w \cdot \tilde{z}$ appartient donc à $(\Pi(D)^* \otimes \omega_D)^{u=\omega_D(p)}$. On pose alors

$$w_D(z) := \mathrm{Res}_{\mathbb{Z}_p}(w_D(\tilde{z})) \otimes e_{\omega_D}^{\vee} \in \check{D}^{\psi=1}.$$

L'équation fonctionnelle suivante est le résultat principal de [Rodrigues Jacinto 2018].

Théorème I.31. *Soit $z \in D^{\psi=1}$ et notons $\check{z} = w_D(z) \in \check{D}^{\psi=1}$. On a*

$$\exp^*\left(\int_\Gamma \eta\chi^{-j} \cdot \mu_{\check{z}}\right) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j,\omega_D^\vee} = C(D,\eta,j) \cdot \exp^{-1}\left(\int_\Gamma \eta^{-1}\chi^{j} \cdot \mu_{z}\right) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta^{-1},j},$$

où

$$C(D,\eta,j) = -\Omega^{-1}\,\frac{\Gamma^*(-j+1)}{\Gamma^*(j+k)}\,\varepsilon(\eta^{-1})^{-2}\varepsilon(\pi \otimes \eta^{-1} \otimes |\cdot|^{j})$$

pour tout $j \geq 1$, où π dénote la représentation lisse de $\mathrm{GL}_2(\mathbb{Q}_p)$ associée à (*la représentation galoisienne associée à*) *D par la correspondance de Langlands classique.*

ID5. *Équation fonctionnelle de la fonction L locale.* Le théorème I.31 nous permet de montrer une équation fonctionnelle pour la fonction $\Lambda_{D,z}$.

Théorème I.32. *Soit $D \in \Phi\Gamma^{\text{ét}}(\mathscr{R})$ étale de dimension 2, de Rham à poids de Hodge–Tate 0 et $k \geq 0$. Soient $z \in D^{\psi=1}$ et $\check{z} = w_D(z) \in \check{D}^{\psi=1}$. Soit $\eta : \mathbb{Z}_p^\times \to L^\times$ un caractère de conducteur p^n avec $n \geq m(\Delta)$. Soient $\mathfrak{U}_\Delta \subseteq \mathfrak{X}$ l'ouvert fourni par le théorème I.15 et κ tel que $\eta\kappa \in \mathfrak{U}_\Delta$. Alors*

$$\Lambda_{\check{D}(k-1),\check{z}(k-1)}(\eta\chi^{j}) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{k-1,\omega_D^\vee} = -\Omega^{-1} p^{n(k-1)}\varepsilon(\eta^{-1}\otimes|\cdot|^{j-k+1})^{-2}\varepsilon(\pi\otimes\eta^{-1}\otimes|\cdot|^{j-k+1}) \cdot \Lambda_{D,z}(\eta^{-1}\chi^{-j+k-1}).$$

29. L'action de φ, σ_a et la multiplication par $(1+T)^b$, $a \in \mathbb{Z}_p^\times$, $b \in \mathbb{Z}_p$, correspondant à $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ et $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ respectivement.

Démonstration. Par le théorème I.27, on sait que

$$\Lambda_{\check{D}(k-1),\check{z}(k-1)}(\eta\chi^{j})\otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{k-1,\omega_D^\vee} = \Gamma^*(j+1)p^{n(j+1)}\cdot \exp^*\left(\int_\Gamma \eta\chi^{-j+k-1}\cdot \mu_{\check{z}}\right)\otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j+k-1,\omega_D^\vee}.$$

L'équation fonctionnelle du théorème I.31 nous dit que

$$\exp^*\left(\int_\Gamma \eta\chi^{-j+k-1}\cdot \mu_{\check{z}}\right)\otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta,-j+k-1,\omega_D^\vee} = C(D,\eta,j-k+1)\cdot \exp^{-1}\left(\int_\Gamma \eta^{-1}\chi^{j-k+1}\cdot \mu_z\right)\otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta^{-1},j-k+1}.$$

Finalement, en appliquant encore une fois le théorème I.27, on obtient

$$\exp^{-1}\left(\int_\Gamma \eta^{-1}\chi^{j-k+1}\cdot \mu_z\right)\otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{\eta^{-1},j-k+1} = \left(\Gamma^*(-j+k)\, p^{n(-j+k)}\right)^{-1}\cdot \Lambda_{D,z}(\eta^{-1}\chi^{-j+k-1}).$$

Ces trois équations et un petit calcul permettent de conclure. □

II. Fonction L p-adique d'une forme modulaire

Pour terminer, on applique les résultats obtenus à la représentation associée à une forme modulaire et au système d'Euler de Kato pour obtenir une construction (partielle) de la fonction L p-adique de la forme modulaire en question.

Soit

$$f=\sum_{n=1}^{+\infty} a_n q^n \in \mathrm{S}_k(\Gamma_1(N),\omega_f)\otimes\mathbb{C}$$

une forme primitive (cuspidale, propre pour les opérateurs de Hecke, nouvelle et normalisée) de poids $k\geq 2$, niveau N et caractère $\omega_f:(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$. Les opérateurs de Hecke T_n agissent sur f par $T_n f=a_n f$. On note $F=\mathbb{Q}(a_1,a_2,\ldots)$ le corps de nombres engendré par les coefficients de f et $\check{f}=\sum_{n=1}^{+\infty}\bar{a}_n q^n\in \mathrm{S}_k(\Gamma_1(N),\omega_f^{-1})\otimes\mathbb{C}$ la forme conjuguée à f. On note

$$\Lambda_\infty(f,\eta^{-1},s)=\frac{\Gamma(s)}{(2i\pi)^s}L(f,\eta^{-1},s)$$

la normalisation de la fonction L complexe de la forme f. Soit v une place de F au-dessus de p et soit $L=F_v$. Notons $V(f)\in \mathrm{Rep}_L\,\mathscr{G}_\mathbb{Q}$ la représentation galoisienne de dimension 2 attachée à f [Kato 2004, §6.3] et $D=\boldsymbol{D}_{\mathrm{rig}}(V(f)|_{\mathscr{G}_{\mathbb{Q}_p}})(k-1)\in\Phi\Gamma^{\text{ét}}(\mathscr{R})$, qui est de Rham à poids de Hodge–Tate 0 et $k-1$.

En appliquant une version p-adique de la conjecture de Bloch–Kato, on construit (cf. la section IIB), pour tout $\eta:\mathbb{Z}_p^\times\to L$ (que l'on voit comme un caractère de Dirichlet de conducteur une puissance de p) et tout $j\geq 0$, un plongement p-adique naturel

$$\Lambda_\infty(f,\eta^{-1},-j)\mapsto \iota_p(\Lambda_\infty(f,\eta^{-1},-j))\in \boldsymbol{D}_{\mathrm{dR}}(D)$$

des valeurs spéciales aux entiers négatifs de la fonction L complexe (normalisée) de la forme modulaire. Rappelons que, dans la bande critique $1\leq j\leq k-1$, les valeurs $\Lambda_\infty(f,\eta^{-1},j)$ sont naturellement interprétés (cf., par exemple, [Kato 2004, Theorem 16.2]) p-adiquement en les multipliant par les périodes complexes de la forme f. Le théorème final (annoncé dans l'introduction) de ce texte peut être énoncé sous la forme suivante :

Théorème II.1. *Il existe un ouvert $\mathfrak{U}_f \subseteq \mathfrak{X}$, ne dépendant que de l'extension sur laquelle la représentation galoisienne associée à f dévient semi-stable, et contenant tous les caractères d'ordre fini assez ramifiés, et une fonction rigide analytique $L_p(f) \in \mathcal{O}(\mathfrak{U}_f) \otimes \boldsymbol{D}_{\mathrm{dR}}(D)$ telle que, si $\eta\chi^j \in \mathfrak{U}_f$, où $\eta : \mathbb{Z}_p^\times \to L^\times$ est un caractère de conducteur p^n et $j \in \mathbb{Z}$ est tel que $0 \le j \le k-2$ ou $j \ll 0$, alors*

$$L_p(f)(\eta\chi^j) = p^{n(j+1)} G(\eta)^{-1} \cdot \iota_p(\Lambda_\infty(f, \eta^{-1}, j+1)).$$

De plus, la fonction $L_p(f)$ satisfait une équation fonctionnelle de la forme

$$L_p(f)(\eta\chi^j) = C(f, \eta, j) \cdot L_p(\check{f})(\eta^{-1}\chi^{-j+k-2}) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{k-1,\omega_D^{-1}},$$

où

$$C(f,\eta,j) = \Omega p^n \varepsilon\big(\eta\otimes|\cdot|^{-l+(k-1)/2}\big)^2 \varepsilon\big(\pi_p(\check{f})\otimes\eta\otimes|\cdot|^{-j+k-1}\big)^{-1} \cdot \prod_{\ell \mid N'} \varepsilon\big(\pi_\ell(\check{f})\otimes\eta^{-1}\otimes|\cdot|^{-j+(k-1)/2}\big)^{-1}.$$

Finalement, si $p \nmid N$,[30] *α est une valeur propre du polynôme de Hecke de p de f et $e_\alpha \in \boldsymbol{D}_{\mathrm{cris}}(V(f)) = \boldsymbol{D}_{\mathrm{cris}}(V)^*$ est un vecteur propre du Frobenius cristallin de valeur propre α, alors $\mathfrak{U}_f = \mathfrak{X}$ et on a*

$$L_{p,\alpha}(f) = \langle L_p(f), e_\alpha \rangle.$$

Remarque II.2. — Si $N = N'p^r$, $(N, N') = 1$, on devrait pouvoir trouver un lien entre l'exposant r et le discriminant de l'extension K sur laquelle la L-représentation $V_L(f)$ dévient semi-stable, c'est-à-dire, entre r et le rayon de surconvergence de l'équation différentielle p-adique associée à $V(f)$. L'ouvert du théorème ne devrait donc dépendre que de r.

— La dernière affirmation suit de la construction de Kato de la fonction L p-adique de f en utilisant le logarithme de Perrin-Riou, et du fait que la construction menée dans ce travail est une généralisation directe de celui-ci.

IIA. ***Notations et compléments.***

IIA1. *Conjecture de Bloch–Kato pour les formes modulaires.* Soit $Y_1(N)$ la courbe modulaire de niveau $\Gamma_1(N)$ et notons $\mathcal{KS}^{k-2}_{\Gamma_1(N)}$ la $k-2$-ième variété de Kuga–Sato de niveau $\Gamma_1(N)$ et ε l'idempotent usuel (cf. [Kato 2004, §1.1, §11.1] ou [Scholl 1990]). Soit $M = M(f \otimes \eta^{-1})$ le motif associé à la forme $f \otimes \eta^{-1}$ (cf. [Gealy 2006; Scholl 1990]) et considérons

$$M^*(1+j) = M(\check{f} \otimes \eta)(k+j),$$

dont $V(\eta\chi^{j+1})$ est la réalisation p-adique.

Notons $\boldsymbol{T}$ l'algèbre engendrée par les opérateurs de Hecke de niveau premier à N et $\bar{\lambda} : \boldsymbol{T} \to L$ le caractère associé à $\check{f} \otimes \eta$. On a une description (cf. [Scholl 1990; Deninger et Scholl 1991; Gealy 2006])

$$H^1(M^*(1+j)) = H^k_{\mathscr{M}}(\mathcal{KS}^{k-2}_{\Gamma_1(N)}, k+j)(\varepsilon) \otimes_{\boldsymbol{T}} \bar{\lambda},$$
$$H^1_{\mathscr{D}}(M^*(1+j)) = H^k_{\mathscr{D}}(\mathcal{KS}^{k-2}_{\Gamma_1(N)}, \mathbb{R}(k+j))(\varepsilon) \otimes_{\boldsymbol{T}} \bar{\lambda},$$
$$H^1_{\text{ét}}(M^*(1+j)) = H^1\big(\mathbb{Q}, H^{k-1}_{\text{ét}}(\mathcal{KS}^{k-2}_{\Gamma_1(N),\bar{\mathbb{Q}}}, \mathbb{Q}_p)(k+j)(\varepsilon) \otimes_{\boldsymbol{T}} \bar{\lambda}\big),$$

30. Plus généralement, si la représentation associée à f est cristabéline.

des groupes de cohomologie motivique, de Deligne et étale, respectivement, du motif $M^*(1+j)$, ainsi que des régulateurs

$$r_\infty : H^1(M^*(1+j)) \otimes_{\mathbb{Q}} \mathbb{R} \to H^1_{\mathscr{D}}(M^*(1+j)),$$
$$r_{\text{ét}} : H^1(M^*(1+j)) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_{\text{ét}}(M^*(1+j)).$$

En utilisant les symboles d'Eisenstein définis par Beilinson [1986], on construit,pour $1 \le r \le k-1$, des éléments (cf. [Gealy 2003, §3.2], où les éléments sont notés $\bar{\xi}_r$)

$$\mathscr{Z}(k, j, r) \in H^k_{\mathscr{M}}(\mathcal{KS}^{k-2}_{\Gamma_1(N)}, k+j)(\varepsilon).$$

Si $\xi \in \mathrm{SL}_2(\mathbb{Z})$, on pose

$$\mathscr{Z}(\check{f} \otimes \eta, j, r, \xi) = \xi^*(\mathscr{Z}(k, j, r)) \otimes_{\boldsymbol{T}} \bar{\lambda} \in H^1(M^*(1+j)).$$

Enfin, on sait [Kato 2004, Theorem 13.6] que les symboles modulaires $\delta(\check{f} \otimes \eta, r, \xi) \in V_F(\check{f} \otimes \eta)$, $1 \le r \le k-1$, $\xi \in \mathrm{SL}_2(\mathbb{Z})$ engendrent $V_F(\check{f} \otimes \eta))$ sur F, ce qui nous permet, en prenant des combinaisons linéaires des éléments $\mathscr{Z}(\check{f} \otimes \eta, j, r, \xi)$, de définir, pour tout $\gamma \in V_F(\check{f})$,

$$\mathscr{Z}(\check{f} \otimes \eta, j, \gamma) \in H^1(M^*(1+j)).$$

Proposition II.3 [Gealy 2006, Theorem 4.1.1]. *Soit $\gamma \in V_F(\check{f})$. Alors*

$$r_\infty(\mathscr{Z}(\check{f} \otimes \eta, j, \gamma)) = L^{(N),*}(f, \eta^{-1}, -j) \cdot \gamma,$$

où $L^{(N),*}(f, \eta^{-1}, -j)$ dénote le coefficient principal de la série de Taylor en $s = -j$ de la fonction L de f sans ses facteurs en les places divisant N.

IIA2. *Cohomologie syntomique.* Les groupes de cohomologie syntomique $H^k_{\text{syn}}(X_h, r)$ d'un schema séparé de type fini X sur un corps p-adique, ainsi que des morphismes de périodes syntomiques

$$\rho_{\text{syn}} : \mathbb{R}\Gamma_{\text{syn}}(X_h, r) \to \mathbb{R}\Gamma_{\text{ét}}(X_{\text{ét}}, \mathbb{Q}_p(r)),$$

et des morphismes de réalisation p-adiques (ou syntomiques)

$$r_p : H^i_{\mathscr{M}}(X, r) \to H^i_{\text{syn}}(X_h, r),$$

de la cohomologie motivique vers la cohomologie syntomique compatibles avec les morphismes de périodes syntomiques et les réalisations étales, ont été définis dans [Nekovář et Nizioł 2016, Theorem A].

Soient $X = \mathcal{KS}^{k-2}_{\Gamma_1(N)}$, $j \ge 0$ et $r = k+j$ (et donc $\mathrm{Fil}^r\, H^{k-1}_{\text{dR}}(\mathcal{KS}^{(k-2)}_{\Gamma_1(N)}) = 0$). On a $H^k_{\text{syn}}(X_h, r) \cong H^{k-1}_{\text{dR}}(X) = \boldsymbol{D}_{\text{dR}}(H^{k-1}_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p))$ (cf. [Nekovář et Nizioł 2016, Remark 4.14] et le diagramme qui le précède pour la première égalité et [Kato 2004, Equation 11.3.3] pour la deuxième) et, en appliquant le projecteur ε, en projetant sur la partie correspondante à la forme $\check{f} \otimes \eta$ et en tordant, on obtient des régulateurs p-adiques

$$r_p : H^1(M^*(1+j)) \to \boldsymbol{D}_{\text{dR}}(V(\eta\chi^{j+1})).$$

La proposition 4.13 de [Nekovář et Nizioł 2016], avec $q = k-1$ (et $r = j+k$) se traduit alors en la relation

$$\exp \circ r_p = r_{\text{ét}},$$

qui nous sera très utile dans la suite.

IIB. ***Plongements p-adiques des valeurs spéciales.*** Soient f comme ci-dessus et $\eta : \mathbb{Z}_p^\times \to L^\times$ d'ordre fini, vu comme un caractère de Dirichlet en fixant un isomorphisme entre $\overline{\mathbb{Q}}_p$ et $\mathbb{C}$. La proposition II.3 nous permet, en utilisant les régulateurs p-adiques, de donner un sens p-adique aux valeurs spéciales de la fonction L complexe associée à f en dehors de la bande critique.

Notons

$$D = D(\check{f})(k-1),$$

qui est de Rham à poids de Hodge–Tate 0 et $k-1$. On a, inspirés de la proposition II.3, envie de voir les éléments $r_p(\mathscr{Z}_{\text{Kato}}(\check{f} \otimes \eta, j, r, \xi)) \in \boldsymbol{D}_{\text{dR}}(D(\eta\chi^{j+1}))$ comme les transmutations des valeur spéciales $L(f, \eta^{-1}, j)$ en p-adique. Or, comme on l'a déjà remarqué, afin de construire une fonction interpolant ces valeurs, il faut les voir tous dans un même module. Rappelons que, pour η un caractère de Dirichlet, on a

$$\Lambda_\infty(f, \eta^{-1}, s) = \frac{\Gamma(s)}{(2i\pi)^s} \cdot L(f, \eta^{-1}, s).$$

Définition 3. On pose,[31] pour $j \geq 0$,

$$\iota_p(\Lambda_\infty(f, \eta^{-1}, -j)) = \Gamma^*(-j) \cdot G(\eta) \cdot r_p(\mathscr{Z}(\check{f} \otimes \eta, j, r, \xi)) \otimes \boldsymbol{e}_{\eta, j+1}^{\text{dR},\vee} \in \boldsymbol{D}_{\text{dR}}(D).$$

IIC. ***Interpolation.*** Dans cette section, on démontre que les constructions faites dans les chapitres précédents nous permettent d'interpoler les plongements p-adiques des valeurs spéciales de la fonction L complexe de f définis dans IIB. Ceci constitue la preuve du théorème II.1 annoncé au début du chapitre. On démontre d'abord, en utilisant un deuxième résultat de Gealy reliant les classes de cohomologie motiviques $\mathscr{Z}(\check{f} \otimes \eta, j, r, \xi)$ au système d'Euler de Kato et le théorème I.27, les propriétés d'interpolation des valeurs spéciales aux entiers négatifs. On sait déjà, d'après les résultats de Kato, que les valeurs interpolées par notre fonction dans la bande critique s'interprètent bien en termes des valeurs spéciales complexes. Finalement, en utilisant une équation fonctionnelle du système d'Euler de Kato établie par Nakamura et l'équation fonctionnelle du théorème I.32, on obtiendra dans la section suivante une interprétation des valeurs spéciales aux entiers positifs $j \geq k-1$, ce qui donne une image complète des valeurs interpolées par la fonction L p-adique d'une forme modulaire.

IIC1. *Relèvement motivique des éléments de Kato.* Rappelons que, pour chaque $\gamma \in V_L(\check{f})$, on a des éléments dans la cohomologie d'Iwasawa

$$z_\gamma^{(p)}(\check{f})(k+j) = (z_{p^n}^{(p)}(\check{f}, -j, \gamma))_{n \geq 1} \in H^1_{\text{Iw}}\big(\mathbb{Q}, V(\check{f})(k+j)\big)$$

construits par Kato [2004, Theorem 12.5].

Proposition II.4 [Gealy 2006, Proposition 9.1.1]. *Soit* $\gamma \in V_L(\check{f})$. *Alors*

$$r_{\text{ét}}(\mathscr{Z}(\check{f} \otimes \eta, j, \gamma)) = \int_\Gamma 1 \cdot z_\gamma^{(p)}(\check{f} \otimes \eta)(k+j).$$

Remarque II.5. — La proposition 9.1.1 de [Gealy 2006] montre le résultat pour $\gamma = \delta(\check{f}, r, \text{id})$. Le cas $\gamma = \delta(\check{f}, r, \xi)$ quelconque s'en déduit des compatibilités des réalisations par des correspondances

31. Notons que, dans la formule, on "mutiplie" et "divise" par la somme de Gauss de η, de sorte qu'elle n'a moralement aucun effet. Le terme $2i\pi$ correspond, dans le monde p-adique, à l'élément t de Fontaine, apparaissant dans le facteur $\boldsymbol{e}_{\eta, j+1}^{\text{dR},\vee}$. Enfin, le facteur $\Gamma^*(j)$ est le coefficient principal de la série de Laurent de $\Gamma(s)$ en $s = -j$, où elle a un pôle simple.

algébriques et de la définition des éléments zêta, et le cas d'un élément γ quelconque suit par linéarité.

— Remarquons que, par construction, $z_\gamma^{(p)}(\check{f} \otimes \eta)(k+j) = z_\gamma^{(p)}(\check{f})(k) \otimes e_\eta \otimes e_j$ et la proposition ci-dessus s'exprime donc aussi comme

$$r_{\text{ét}}(\mathscr{Z}(\check{f} \otimes \eta, j, \gamma)) = \int_\Gamma \eta\chi^j \cdot \boldsymbol{z}_\gamma^{(p)}(\check{f})(k).$$

IIC2. *Interpolation aux entiers négatifs.* Notons

$$D(f) = \boldsymbol{D}_{\text{rig}}(V_L(f)|_{\mathscr{G}_{\mathbb{Q}_p}}), \quad D(\check{f}) = \boldsymbol{D}_{\text{rig}}(V_L(\check{f})|_{\mathscr{G}_{\mathbb{Q}_p}}) \in \Phi\Gamma^{\text{ét}}(\mathscr{R})$$

les (φ,Γ)-modules associés aux formes f et $\check{f}$. Rappelons que l'on a posé $D = D(\check{f})(k-1)$, qui est de Rham à poids de Hodge–Tate 0 et $k-1$, et notons

$$z_{\text{Kato}} = \text{Exp}^*(z_\gamma^{(p)}(\check{f})(k-1)) \in \boldsymbol{D}_{\text{rig}}(V(\check{f})(k-1)))^{\psi=1} = D^{\psi=1}.$$

Lemme II.6. *Soit* $j \geq 0$. *On a*

$$\Lambda_{D,z_{\text{Kato}}}(\eta\chi^{-j-1}) = p^{-nj}\, G(\eta)^{-1} \cdot \iota_p(\Lambda_\infty(f, \eta^{-1}, -j)).$$

Démonstration.

En utilisant la proposition II.4 et la remarque qui le suit, on obtient

$$\exp^{-1}\big(r_{\text{ét}}(\mathscr{Z}(\check{f} \otimes \eta, j, \gamma))\big) = \exp^{-1}\bigg(\int_\Gamma \eta\chi^{j+1} \cdot \mu_{z_{\text{Kato}}}\bigg),$$

d'où, par la compatibilité entre le régulateur p-adique et le régulateur étale, on en déduit

$$\begin{aligned}\iota_p(\Lambda_\infty(f, \eta^{-j}, -j)) &= \Gamma^*(-j)\, G(\eta) \cdot r_p(\mathscr{Z}(\check{f} \otimes \eta, j, \gamma)) \otimes \boldsymbol{e}_{\eta,j+1}^{\text{dR},\vee} \\ &= \Gamma^*(-j)\, G(\eta) \cdot \exp^{-1}\bigg(\int_\Gamma \eta\chi^{j+1} \cdot \boldsymbol{z}_{\text{Kato}}\bigg) \otimes \boldsymbol{e}_{\eta,j+1}^{\text{dR},\vee}.\end{aligned}$$

Par ailleurs, le théorème I.27 affirme que

$$\Lambda_{D,z_{\text{Kato}}}(\eta\chi^{-j-1}) = p^{-nj}\, \Gamma^*(-j) \cdot \exp^{-1}\bigg(\int_\Gamma \eta\chi^{j+1} \cdot \mu_{z_{\text{Kato}}}\bigg) \otimes \boldsymbol{e}_{\eta,j+1}^{\text{dR},\vee}. \qquad \square$$

IIC3. *La bande critique.* Si $0 \leq j \leq k-1$, on pose

$$\iota_p(\Lambda_\infty(f, \eta^{-1}, j+1)) = \Gamma^*(j+1)\, G(\eta) \cdot \exp^*\bigg(\int_\Gamma \eta\chi^{-j} \cdot \mu_{z_{\text{Kato}}}\bigg) \otimes \boldsymbol{e}_{\eta,-j}^{\text{dR},\vee}.$$

On sait, d'après [Kato 2004, Theorem 12.5], que les images de ces valeurs par l'application de périodes sont reliées aux valeurs spéciales de la fonction L de f. Le lemme suivant est immédiat

Lemme II.7. *Soit* $0 \leq j \leq k-1$. *Alors*

$$\Lambda_{D,z_{\text{Kato}}}(\eta\chi^j) = p^{n(j+1)}\, G(\eta)^{-1} \cdot \iota_p(\Lambda_\infty(f, \eta^{-1}, j+1)).$$

IID. ***Équation fonctionnelle et valeurs aux entiers positifs.*** Pour l'interprétation p-adique des valeurs $L(f,\eta,j)$, $j\geq k$, on fera appel à l'équation fonctionnelle de la fonction L locale, qui s'avéra fortement ressemblante à l'équation fonctionnelle complexe.

IID1. *L'équation fonctionnelle complexe.* Notons $\mathbb{A}_{\mathbb{Q}}$ le groupe des adèles de $\mathbb{Q}$. Soit $\eta:\mathbb{Z}_p^\times\to L^\times$ un caractère d'ordre fini. On regarde η comme un caractère de Dirichlet (via $L^\times\subseteq\overline{\mathbb{Q}}_p^\times\cong\mathbb{C}^\times$) ainsi que comme un caractère de Hecke de la façon usuelle.[32]

La forme $f\otimes\eta$ est supercuspidale et on note

$$\pi(f\otimes\eta)=\bigotimes\nolimits'_\ell\pi_\ell(f\otimes\eta)$$

la représentation automorphe de $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associée à $f\otimes\eta$. On a $\pi(f\otimes\eta)=\pi(f)\otimes\eta=\bigotimes'_\ell\pi_\ell(f)\otimes\eta$. Notons $\varepsilon(\pi(f)\otimes\eta,s)$ le facteur epsilon global de la représentation $\pi(f\otimes\eta)$ défini par[33]

$$\varepsilon(\pi(f)\otimes\eta,s)=\varepsilon(\pi_\infty(f)\otimes\eta,s)\cdot\prod_\ell\varepsilon(\pi_\ell(f)\otimes\eta,s),$$

où $\varepsilon(\pi_\ell(f)\otimes\eta,s)$ est le facteur epsilon de la représentation $\pi_\ell(f)\otimes\eta$ de $\mathrm{GL}_2(\mathbb{Q}_l)$, comme décrit dans la section ID2, et $\varepsilon(\pi_\infty(f)\otimes\eta,s)=i^k$.

La fonction L complexe satisfait l'équation fonctionnelle[34]

$$\frac{\Gamma(s)}{(2\pi)^s}\cdot L(f,\eta^{-1},s)=\varepsilon\Big(\pi(f)\otimes\eta^{-1},s-\frac{k-1}{2}\Big)\cdot\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\cdot L(\check{f},\eta,k-s).$$

Si $j\geq k$ est un entier, on peut écrire l'équation fonctionnelle sous la forme

$$\frac{\Gamma(j)}{(2i\pi)^j}\cdot L(f,\eta^{-1},j)=i^k(-1)^{-j}\varepsilon\big(\pi(f)\otimes\eta^{-1}\otimes|\cdot|^{j-(k-1)/2}\big)\cdot\frac{\Gamma(k-j)}{(2i\pi)^{k-j}}\cdot L(\check{f},\eta,k-j),$$

ou bien

$$\Lambda_\infty(f,\eta^{-1},j)=i^k(-1)^{-j}\varepsilon\big(\pi(f)\otimes\eta^{-1}\otimes|\cdot|^{j-(k-1)/2}\big)\cdot\Lambda_\infty(\check{f},\eta,k-j).$$

En décomposant le facteur epsilon, on peut réécrire l'équation fonctionnelle sous la forme suivante :

$$\Lambda_\infty(f,\eta^{-1},j)\\ =(-1)^{k-j}\varepsilon\big(\pi_p(f)\otimes\eta^{-1}\otimes|\cdot|^{j-(k-1)/2}\big)\cdot\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(f)\otimes\eta^{-1}\otimes|\cdot|^{j-(k-1)/2}\big)\cdot\Lambda_\infty(\check{f},\eta,k-j).$$

Remarquons pour finir que, si l'on écrit $N=N'p^r$, alors, pour $p\neq\ell\mid N$, on a l'égalité

$$\varepsilon\big(\pi_\ell(f)\otimes\eta^{-1}\otimes|\cdot|^{j-(k-1)/2}\big)=\varepsilon(\pi_\ell(f))\eta(\ell)^{c(\pi_\ell(f))}\ell^{-c(\pi_\ell(f))(j-(k-1)/2)},$$

32. Si $\eta:\mathbb{Z}_p^\times\to L^\times$ est de conducteur p^n, il est vu comme un caractère des idèles en utilisant la décomposition $\mathbb{A}_{\mathbb{Q}}=\mathbb{Q}^\times\times\mathbb{R}^{>0}\times\hat{\mathbb{Z}}^\times$. Le caractère de $\mathbb{Q}_p^\times$ induit par η est η (avec $\eta(p)=1$) et, si $\ell\neq p$, celui de $\mathbb{Q}_\ell^\times$ est l'unique caractère non ramifié caractère prenant la valeur $\eta^{-1}(\bar{\ell})$ en ℓ, où $\bar{\ell}\in\mathbb{Z}_p^\times$ est n'importe quel relèvement de la classe de ℓ modulo p^n.

33. Le produit étant fini car $\pi_\ell(f\otimes\eta)$ est non ramifiée en presque toute place.

34. Le décalage en $(k-1)/2$ provient du fait que les facteurs locaux des représentations de GL_2 sont normalisés de sorte que le centre de symétrie de l'équation fonctionnelle des fonctions L soit situé en $\frac{1}{2}$, tandis que celui des fonctions L automorphes l'est en $\frac{k}{2}$.

et, en utilisant le fait que le conducteur de $\pi_\ell(f)$ est $\ell^{v_\ell(N)}$, on en déduit

$$\prod_{\ell\mid N'} \varepsilon\big(\pi_\ell(f)\otimes\eta^{-1}\otimes|\cdot|^{j-(k-1)/2}\big)=\eta(N')(N')^{(k-1)/2-j}\prod_{\ell\mid N'}\varepsilon(\pi_\ell(f)).$$

IID2. *L'équation fonctionnelle du système d'Euler de Kato, d'après Nakamura.* Écrivons $N=N'p^r$, $(N',N)=1$. Si $\gamma\in V(\check{f})$, notons $\check{\gamma}\in V(f)$ l'élément dual à $\gamma\otimes e_k$ sous l'accouplement parfait $V(\check{f})(k)\times V(f)\to L$ donné par la dualité de Poincaré.

On note

$$\begin{aligned} z_{\mathrm{Kato}} &= \mathrm{Exp}^*(z_\gamma^{(p)}(\check{f})(k-1))\in D(\check{f})(k-1)^{\psi=1}=D^{\psi=1},\\ \check{z}_{\mathrm{Kato}} &= \mathrm{Exp}^*(z_{\check{\gamma}}^{(p)}(f)(k-1))\in D(f)(k-1)^{\psi=1}=\check{D}(k-2)^{\psi=1},\\ z^*_{\mathrm{Kato}} &= w_D(z_{\mathrm{Kato}})\in\check{D}^{\psi=1}, \end{aligned}$$

les tordus des systèmes d'Euler associés aux formes $\check{f}$ et f et l'image par l'involution de z_{Kato} (cf. la section ID4), respectivement.

Proposition II.8 [Nakamura 2017, Conjecture 4.4, Theorem 4.6, Proposition 4.7]. *Notons* $[N']=\prod_{\ell\mid N'}[\sigma_\ell]^{v_\ell(N')}\in\Lambda$. *Alors*

$$z^*_{\mathrm{Kato}}=-\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes|\cdot|^{(k-1)/2}\big)^{-1}\cdot\big([N']\cdot(\check{z}_{\mathrm{Kato}}\otimes e_{2-k})\big).$$

Remarque II.9. — [Nakamura 2017, Conjecture 4.4] est énoncé en termes de facteurs epsilon des représentations de Weil–Deligne. Sa démonstration (dans le cas de Rham non triangulin) est basée sur la compatibilité locale-globale dans la correspondance de Langlands p-adique et [Nakamura 2017, Proposition 3.14]. Cette dernière proposition peut être énoncée naturellement (cf. le théorème I.31) en termes de facteurs locaux des représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$. La preuve de la proposition ne ferait donc pas usage de la compatibilité locale-globale et elle resterait donc purement locale.

— Il faut faire un peu d'attention car les normalisations des facteurs locaux dans ce travail ne coïncident pas avec celles de [Nakamura 2017]. Comme on l'a remarqué, dans le texte présent, les facteurs locaux des représentations lisses sont normalisés de sorte que l'équation fonctionnelle de la fonction L soit centrée en $s=\frac{1}{2}$, tandis que, dans [Nakamura 2017], elle est centrée en $s=\frac{k}{2}$. La différence entre les facteurs locaux est donc un twist par $|\cdot|^{(k-1)/2}$.

IID3. *L'équation fonctionnelle de la fonction L p-adique.* L'équation fonctionnelle du système d'Euler de Kato et l'équation fonctionnelle du théorème I.32 nous permettent d'interpréter les valeurs aux entiers positifs de la fonction $\Lambda_{D,z_{\mathrm{Kato}}}$.

Théorème II.10. *Soit $j>0$ un entier. Alors*

$$\Lambda_{D,z_{\mathrm{Kato}}}(\eta\chi^j)=C(f,\eta,j)\cdot\Lambda_{\check{D}(k-2),\check{z}_{\mathrm{Kato}}}(\eta^{-1}\chi^{-j+k-2})\otimes\boldsymbol{e}^{\mathrm{dR},\vee}_{k-1,\omega_D^{-1}},$$

où

$$C(f,\eta,j)=\Omega p^n\varepsilon\big(\eta\otimes|\cdot|^{-j+(k-1)/2}\big)^2\varepsilon\big(\pi_p(\check{f})\otimes\eta\otimes|\cdot|^{-j+k-1}\big)^{-1}\cdot\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes\eta^{-1}\otimes|\cdot|^{-j+(k-1)/2}\big)^{-1}.$$

Démonstration. En appliquant le théorème I.32 (avec $k-1$ au lieu de k, η^{-1} au lieu de η et $-j+k-2$ au lieu de j), on obtient

$$\Lambda_{D,z_{\mathrm{Kato}}}(\eta\chi^j) = C_1 \cdot \Lambda_{\check{D}(k-2),z^*_{\mathrm{Kato}}(k-2)}(\eta^{-1}\chi^{-j+k-2}) \otimes \boldsymbol{e}^{\mathrm{dR},\vee}_{k-2,\omega_D^\vee}, \tag{4}$$

où

$$C_1 = -\Omega\, p^{-n(k-2)}\varepsilon(\eta\otimes|\cdot|^{-j})^2\varepsilon(\pi\otimes\eta\otimes|\cdot|^{-j})^{-1}.$$

Observons que

$$p^{-n(k-2)}\varepsilon(\eta\otimes|\cdot|^{-j})^2 = p^n\big(p^{-n(k-1)/2}\varepsilon(\eta\otimes|\cdot|^{-j})\big)^2 = p^n\varepsilon\big(\eta\otimes|\cdot|^{-l+(k-1)/2}\big)^2.$$

Comme $\pi = \pi(D) = \pi_p(\check{f})\otimes|\cdot|^{k-1}$, on en déduit

$$C_1 = -\Omega p^n\varepsilon\big(\eta\otimes|\cdot|^{-l+(k-1)/2}\big)^2\varepsilon\big(\pi_p(\check{f})\otimes\eta\otimes|\cdot|^{-j+k-1}\big)^{-1}.$$

D'après le théorème I.27, on a

$$\Lambda_{\check{D}(k-2),z^*_{\mathrm{Kato}}(k-2)}(\eta^{-1}\chi^{-j+k-2}) = \Gamma^*(-j+k-1)\,p^{n(-j+k-1)}\cdot\log\bigg(\int_\Gamma \eta^{-1}\chi^{j-k+2}\cdot\mu_{z^*_{\mathrm{Kato}}(k-2)}\bigg)\otimes\boldsymbol{e}^{\mathrm{dR},\vee}_{\eta^{-1},j-k+2}.$$

On a

$$\begin{aligned}\int_\Gamma \eta^{-1}\chi^{j-k+2}\cdot\mu_{z^*_{\mathrm{Kato}}(k-2)} &= \int_\Gamma \eta^{-1}\chi^{j}\cdot\mu_{z^*_{\mathrm{Kato}}}\\ &= -\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes|\cdot|^{(k-1)/2}\big)^{-1}\cdot\int_\Gamma \eta^{-1}\chi^{j}\cdot\mu_{[N'](\check{z}_{\mathrm{Kato}}(2-k))}\\ &= -\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes|\cdot|^{(k-1)/2}\big)^{-1}\eta(N')(N')^{-j}\cdot\int_\Gamma \eta^{-1}\chi^{j-k+2}\cdot\mu_{\check{z}_{\mathrm{Kato}}},\end{aligned}$$

où on a utilisé la proposition II.8 dans la deuxième égalité, et la définition de l'action de $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ sur $D^{\psi=1}$ dans la troisième. En utilisant l'égalité

$$\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes|\cdot|^{(k-1)/2}\big)\eta^{-1}(N')(N')^j = \prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes\eta^{-1}\otimes|\cdot|^{-j+(k-1)/2}\big),$$

on en déduit

$$\begin{aligned}\Lambda_{\check{D}(k-2),z^*_{\mathrm{Kato}}(k-2)}&(\eta^{-1}\chi^{-j+k-2})\\ &= -\prod_{\ell\mid N'}\varepsilon\big(\pi_\ell(\check{f})\otimes\eta^{-1}\otimes|\cdot|^{-j+(k-1)/2}\big)^{-1}\cdot\Lambda_{\check{D}(k-2),\check{z}_{\mathrm{Kato}}}(\eta^{-1}\chi^{-j+k-2}).\end{aligned}\tag{5}$$

En rassemblant les formules (4) et (5), on déduit le résultat. □

Remarque II.11. Notons que, pour $j > k-2$, les valeurs du côté droite de la formule du théorème s'interprètent en termes des valeurs spéciales de la fonction L complexe de $\check{f}$. En utilisant l'équation fonctionnelle complexe on peut traduire ceci et donner une formule d'interpolation de la fonction L p-adique en termes de valeurs spéciales complexes en tout entier $j\in\mathbb{Z}$.

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Invariant theory of $\bigwedge^3(9)$ and genus-2 curves

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Previous work established a connection between the geometric invariant theory of the third exterior power of a 9-dimensional complex vector space and the moduli space of genus-2 curves with some additional data. We generalize this connection to arbitrary fields, and describe the arithmetic data needed to get a bijection between both sides of this story.

1. Introduction

This paper is a companion to our previous paper [Rains and Sam 2016]. We begin by briefly recalling what was done there. Given a genus-2 curve C over a field $\boldsymbol{k}$, let $\mathrm{SU}_3(C)$ be the coarse moduli space of rank 3 semistable vector bundles on C. It admits a degree 2 map $\mathrm{SU}_3(C) \to \mathbb{P}^8$ which is branched along a sextic hypersurface. Remarkably, the singular locus of the projective dual of this sextic is a surface which is isomorphic to the Jacobian of C over the algebraic closure of $\boldsymbol{k}$. This story has been developed over algebraically closed fields of characteristic 0 in [Ortega 2005; Nguyễn 2007] and connected to the invariant theory of the action of $\mathrm{SL}_9(\boldsymbol{k})$ on $\bigwedge^3 \boldsymbol{k}^9$ in [Gruson et al. 2013; Gruson and Sam 2015]. In [Rains and Sam 2016], the setting is generalized to arbitrary fields, and the purpose of this paper is to extend the invariant-theoretic aspects.

More precisely, let V be a 9-dimensional vector space over an arbitrary field $\boldsymbol{k}$ and consider the action of $\mathrm{SL}(V)$ on $\bigwedge^3 V$. Given a stable (in the sense of geometric invariant theory) element $\gamma \in \bigwedge^3 V$, we generalize the constructions in [Gruson et al. 2013; Gruson and Sam 2015] to produce

- a genus-2 curve C with a Weierstrass point $P \in C(\boldsymbol{k})$ and
- a cubic hypersurface in $\mathbb{P}(V^*)$ whose singular locus is a smooth surface X,

such that X is isomorphic to the Jacobian $\mathrm{J}(C)$ of C over the algebraic closure of $\boldsymbol{k}$. In fact, we also get some interesting arithmetic data:

- a 3-covering $X \to \mathrm{J}(C)$ which becomes the multiplication by 3 map over $\bar{\boldsymbol{k}}$, i.e., an element in $\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3])$; furthermore, it lies in the kernel of a map $\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3]) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}_9 /\boldsymbol{\mu}_3)$.

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Conversely, given this data, we show how to construct a stable element in $\bigwedge^3 V$ (which is only well-defined up to scalar multiple and the action of $\mathrm{SL}(V)$). A bulk of the work in this paper is to show that these two constructions are inverse to one another.

Our work is partially motivated by recent work in *arithmetic invariant theory* (see [Bhargava and Gross 2014], for example). One goal is to count arithmetic objects of interest, and the first step in many of these cases is to parametrize them by orbits in a linear space. This first step is achieved here; when $\boldsymbol{k}$ is a global field, we show that the 3-Selmer group of $\mathrm{J}(C)$ is a subgroup of the kernel of $\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3]) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}_9/\boldsymbol{\mu}_3)$, so that, in fact, they are parametrized by special kinds of orbits in $\bigwedge^3 V$. Following the analogies of previous work in the area, we may hope to count the average size of this 3-Selmer group using $\bigwedge^3 V$.

Here is a brief overview of the contents. In Section 2, we work out the aspects of the invariant theory of $\bigwedge^3 V$ which are needed in the rest of the paper. In Section 3, we generalize the construction of [Gruson et al. 2013; Gruson and Sam 2015] to arbitrary fields, i.e., we produce the data above starting from a stable element γ. In Section 4, we provide a construction in the reverse direction; starting from the data above, we produce a stable element γ. In Section 5, we show that these two constructions are inverse to one another. Finally, in Section 6, we discuss a few additional topics: Selmer groups, ordinary curves, and an explicit model for the 3-torsion of $\mathrm{J}(C)$ given γ above.

1A. ***Notation.*** Throughout, let $\boldsymbol{k}$ be a field and R be a complete discrete valuation ring (DVR) of characteristic 0 whose residue field is $\boldsymbol{k}$. The quotient field of R is denoted $\boldsymbol{K}$. Write $\boldsymbol{k}^{\mathrm{sep}}$ for a separable closure of $\boldsymbol{k}$.

If G is a group scheme defined over $\boldsymbol{k}$, we let $\mathrm{H}^*(\boldsymbol{k}; G)$ denote the flat cohomology of G. When G is smooth, this coincides with the Galois cohomology of G, but we will not have any use for Galois cohomology of nonsmooth group schemes.

2. Invariant theory preliminaries

2A. ***Geometric invariant theory review.*** Let G be a reductive group acting linearly on a vector space V. A point $u \in V$ is *stable* if its stabilizer subgroup in G is finite and its orbit is closed, and it is *semistable* if 0 is not in the closure of its orbit. If u is not semistable, then it is *unstable*. Hence, an element is *nonstable* if it is unstable or if it is semistable, but not stable.

The Hilbert–Mumford criterion says that u is stable if and only if $\lim_{t\to 0} \rho(t).u$ does not exist for any 1-parameter subgroup $\rho\colon \boldsymbol{G}_m \to G$, and that u is semistable if and only if $\lim_{t\to 0} \rho(t).u$ does not exist, or it is nonzero whenever the limit exists. Note that we will work over arbitrary fields, but one must consider all 1-parameter subgroups which are defined over the algebraic closure.

The set of unstable points form a Zariski closed set, and is the zero locus of all positive degree G-invariant homogeneous polynomials on V. Similarly, the set of nonstable points form a Zariski closed set. Finally, two points $x, y \in V$ are *S-equivalent* if $f(x) = f(y)$ for all homogeneous G-invariant polynomials f on V, and they are *projectively S-equivalent* if αx is S-equivalent to y for some $\alpha \neq 0$. If

x and y are S-equivalent semistable points, then their orbit closures have a semistable point in common. Furthermore, the orbit closure of any semistable point x contains a unique closed orbit of semistable points, and if x is not stable, then neither are the points of this closed orbit.

Let V_9 denote a vector space of dimension 9 with basis $e_1, \dots, e_9$. The group $G = \mathrm{SL}(V_9)/\boldsymbol{\mu}_3$ acts on $\bigwedge^3(V_9)$, and the invariant theory of this representation is the main focus of this paper (see also [Elashvili and Vinberg 1978] for earlier work). It has a natural basis of monomials $e_i \wedge e_j \wedge e_k$ (with $1 \le i < j < k \le 9$), and we will use $[ijk]$ as shorthand for this monomial.

The following result is due to Skip Garibaldi, Robert Guralnick and the first author (work in preparation).

Proposition 2.1. *Over an algebraically closed field, every element of $\bigwedge^3(V_9)$ is S-equivalent to an element γ_c of the form*

$$[267]+[258]+[348]+[169]+[357]+[249]+[178]+[456] \\ -c_3[257]-c_6[247]+c_9[148]-c_{12}[147]+c_{15}[235]+c_{18}[145]+c_{24}[134]+c_{30}[123],$$

where if 2 is invertible we may take $c_3 = c_9 = c_{15} = 0$ and if 5 is invertible we may take $c_6 = 0$. Two such elements are projectively S-equivalent if and only if the corresponding pairs (C_c, P_c) are isomorphic, where C_c is the curve

$$C_c : x^2 + z^5 + c_3 x z^2 + c_6 z^4 + c_9 x z + c_{12} z^3 + c_{15} x + c_{18} z^2 + c_{24} z + c_{30} = 0,$$

and P_c is the point at infinity.

Remark 2.2. Above, we see that projective S-equivalence classes classify pairs (C, P) where C is a genus-2 curve and $P \in C(\boldsymbol{k})$ is a rational Weierstrass point. In fact, one can show that the S-equivalence classes themselves classify triples (C, P, φ) where $\varphi\colon \omega_C \otimes \mathcal{O}_P \cong \mathcal{O}_P$ specifies a nonzero tangent vector at P. We omit the details, as we have not been able to figure out how to build φ into the construction below, and can thus only work at the level of projective S-equivalence.

Remark 2.3. The only way in which the results below will logically depend on Proposition 2.1 is the claim that elements corresponding to isomorphic (C_c, P_c) pairs are projectively equivalent. This is quite easy to check computationally; any isomorphism of pairs has the form

$$(x, z) \mapsto (\alpha^5 x + b_3 z^2 + b_9 z + b_{15},\ \alpha^2 z + b_6),$$

and it is easy to find an equivalence between the corresponding trivectors given the ansatz that the element of GL_9 be upper-triangular. As for the other claims of Proposition 2.1, not only will they not be used below, but in fact for stable γ, they are easy consequences of our results! (Nonetheless an explicit derivation is given by Garibaldi, Guralnick and Rains in the work mentioned above.)

Proposition 2.4. *If F be an SL_9-invariant section of $\mathcal{O}_{\mathbb{P}(\bigwedge^3(V_9))}(d)$. Then $F(\gamma_c)$ is a homogeneous polynomial of degree d in $c_3, \dots, c_{30}$, with $\deg(c_i) = i$.*

Proof. The 1-parameter subgroup of GL_9 of weight $(15, 9, 6, 3, 0, -3, -6, -9, -12)$ preserves the space of elements γ_c and acts on each c_i by t^i. Since F is an SL_9-invariant of degree d, $F(g\gamma) = \det(g)^{d/3} F(\gamma)$ for any $g \in GL_9$, and thus the 1-parameter subgroup multiplies F by t^d, so that $F(\gamma_{t\cdot c}) = t^d F(\gamma_c)$, implying the desired homogeneity. □

2B. *Cartan subspaces.* Assume the characteristic of $\boldsymbol{k}$ is different from 3. Let $G = SL(V_9)/\boldsymbol{\mu}_3$ and let $\mathfrak{e}_8$ be the split Lie algebra of type E_8 and let Γ be its simply connected group. We have a $\mathbb{Z}/3$-graded decomposition

$$\mathfrak{e}_8 = \mathfrak{sl}(V_9) \oplus \bigwedge^3 V_9 \oplus \bigwedge^6 V_9. \tag{2.5}$$

The decomposition (2.5) corresponds to an order 3 automorphism θ of Γ such that $G = \Gamma^\theta$ and $\bigwedge^3 V_9$ is one of the nontrivial eigenspaces of θ acting on $\mathfrak{e}_8$. More explicitly, pick a set of simple roots $\alpha_1, \dots, \alpha_8$ for the root system of $\mathfrak{e}_8$. Then the height of a root is the sum of its coefficients when expressed as a sum of the α_i, and the $\mathbb{Z}/3$-grading comes from taking the height modulo 3.

The 4 dimensional subspace $\mathfrak{h}$ of $\bigwedge^3 V_9$ spanned by

$$[123]+[456]+[789], \quad [147]+[258]+[369], \quad [159]+[267]+[348], \quad [168]+[249]+[357] \tag{2.6}$$

is the *standard Cartan subspace*. It may be helpful to visualize this in terms of the finite geometry $\mathbb{P}^2_{\mathbb{F}_3}$, namely, each basis vector is a sum over all lines in a direction of the following table:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

This carries the action of the Weyl group $W = N(\mathfrak{h})/Z(\mathfrak{h})$ (normalizer modulo centralizer).

Proposition 2.7. *If $\boldsymbol{k}$ has characteristic* 0, *the restriction map*

$$\boldsymbol{k}\big[\bigwedge^3 V_9\big]^G \xrightarrow{\sim} \boldsymbol{k}[\mathfrak{h}]^W$$

is an isomorphism, and both are polynomial rings generated by elements of degrees 12, 18, 24, *and* 30.

Proof. See [Vinberg 1976, Theorem 7] for the isomorphism, and see [Vinberg 1976, §9] for the degrees of the invariants. □

When $\boldsymbol{k} = \mathbb{C}$, the quotient space $\mathfrak{h}/W$ is classically known to parametrize genus-2 curves together with a choice of Weierstrass point, see [Dolgachev and Lehavi 2008, §4].

W is a complex reflection group (the reflections have order 3), and there are 40 reflection hyperplanes. With respect to the 4 basis vectors in (2.6) for the standard Cartan subspace, the matrix representation of the reflection group in characteristic 0 is given in [Gruson and Sam 2015, §3.1]. Each reflection hyperplane is in the orbit of the hyperplane spanned by the first 3 basis vectors, see [Gruson and Sam 2015, Table 1]. As an abstract finite group, we have an isomorphism $W \cong \mathbb{Z}/3 \times Sp_4(\mathbb{F}_3)$.

Lemma 2.8. *Suppose $\boldsymbol{k}$ has characteristic* 0. *If x is semistable and Gx is closed*, *then $Gx \cap \mathfrak{h} \neq 0$.*

Proof. Combine Proposition 4 and the corollary of Theorem 1 of [Vinberg 1976]. ☐

Proposition 2.9. *Any element of a reflection hyperplane in the standard Cartan subspace has a positive-dimensional stabilizer subgroup in G.*

Proof. In positive characteristic, lift our element over the DVR R to characteristic 0 and use semicontinuity of stabilizer dimension to reduce the proof to the case of characteristic 0.

The reflection hyperplanes form a single orbit under the reflection group, so it suffices to consider a single one. From the discussion above, we may assume that this hyperplane is the span of $[123]+[456]+[789]$, $[147]+[258]+[369]$, $[159]+[267]+[348]$. Then for any t, the diagonal matrix with entries $(t^{-2}, t, t, t, t, t^{-2}, t, t^{-2}, t)$ stabilizes each of these 3 basis vectors, and hence any element in this hyperplane. So the stabilizer of any element has positive dimension. ☐

Proposition 2.10. *An element u in the standard Cartan subspace is stable if and only if it does not lie in any reflection hyperplane.*

Proof. The standard Cartan subspace is the intersection of a Cartan subalgebra of $\mathfrak{e}_8$ with $\bigwedge^3 V_9$ and none of the reflection hyperplanes of the Cartan subalgebra of $\mathfrak{e}_8$ contain the standard Cartan subspace (this follows from the discussion in [Elkies 1999, §3]), so u is contained in the complement of reflection hyperplanes in a Cartan subalgebra of $\mathfrak{e}_8$, which means that it is stable under the action of Γ. The Hilbert–Mumford criterion implies that u is stable as an element of $\bigwedge^3 V_9$ under the action of G. Conversely, we have already seen that any element in a reflection hyperplane has a positive-dimensional stabilizer, so cannot be stable. ☐

2C. *Stable elements.*

Lemma 2.11. *In characteristic* 0, *the locus of nonstable elements of $\bigwedge^3 V_9$ is contained in an irreducible G-invariant hypersurface of degree* 120.

Proof. Let x be a semistable, but not stable point, and let y be a point in its orbit closure such that Gy is closed. Then $Gy \cap \mathfrak{h} \neq 0$ by Lemma 2.8, and we may assume $y \in \mathfrak{h}$. By Proposition 2.10, y lies on a reflection hyperplane. Let f be the product of the linear forms vanishing on the reflection hyperplanes of $\mathfrak{h}$, so $\deg f = 40$. The reflections transform f by a cube root of unity, so f^3 is the lowest degree W-invariant vanishing on each reflection hyperplane. Let δ be the G-invariant function on $\bigwedge^3 V_9$ which corresponds to f^3 under the isomorphism in Proposition 2.7. Then δ vanishes on y since it restricts to f^3, and hence δ also vanishes on x. Finally, δ is irreducible: if not, then each component is cut out by a G-invariant since G is connected, and would restrict to a W-invariant function of degree < 120 vanishing on some of the reflection hyperplanes, but no such function exists. ☐

Proposition 2.12. (a) *An element $u \in \bigwedge^3 V_9$ is nonstable if and only if there exists a* 6*-dimensional subspace $U \subset V_9$ such that $\gamma \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$.*

(b) *The set of nonstable elements in $\bigwedge^3 V_9$ is an irreducible hypersurface which is set-theoretically defined by a polynomial of degree* 120. *This hypersurface is reduced in characteristic* 0.

Proof. Let Z be the set of u such that there exists a 6-dimensional subspace $U \subset V_9$ such that $\gamma \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$.

Pick $\gamma \in Z$ with U as above. Pick a basis $u_1, \dots, u_6$ for U and extend it to a basis $u_1, \dots, u_9$ for V_9. Then γ is a sum of trivectors $[ijk]$ where $|\{i, j, k\} \cap \{7, 8, 9\}| \leq 1$. In particular, given the diagonal 1-parameter subgroup $\rho(t) = (t^3, t^3, t^3, t^3, t^3, t^3, t^{-6}, t^{-6}, t^{-6})$, we have $\lim_{t\to 0} \rho(t) \cdot \gamma$ exists, and is the result of throwing away the $[ijk]$ where $|\{i, j, k\} \cap \{7, 8, 9\}| = 0$. By the Hilbert–Mumford criterion, γ is nonstable, so Z is contained in the nonstable locus.

Let P be the stabilizer in $\mathrm{GL}(V_9)$ of the subspace $e_1, \dots, e_6$ and let E be the span of $e_1, \dots, e_6$. Then the span of $\bigwedge^3 E$ and $\bigwedge^2 E \otimes (V_9/E)$ is a P-submodule of $\bigwedge^3 V_9$ and by algebraic induction this P-submodules becomes a rank 65 vector bundle $\mathcal{E}$ which is a subbundle of $\bigwedge^3 V_9 \times \mathrm{Gr}(6, V_9)$ where $\mathrm{Gr}(6, V_9)$ is the Grassmannian of 6-dimensional subspaces of V_9. By the discussion above, the image of the projection $\pi : \mathcal{E} \to \bigwedge^3 V_9$ is Z. In particular, Z is irreducible. Let $\xi \subset \bigwedge^3 V_9^* \times \mathrm{Gr}(6, V_9)$ be the annihilator of $\mathcal{E}$, then the Koszul complex $\bigwedge^\bullet \xi$ is a locally free resolution of $\mathcal{E}$ as a subscheme of $\bigwedge^3 V_9 \times \mathrm{Gr}(6, V_9)$, and so its derived pushforward with respect to π has the same Euler characteristic as $\mathrm{R}\pi_* \mathcal{O}_{\mathcal{E}}$ (for a discussion of this, see [Weyman 2003, Chapter 5]). More specifically, everything respects the natural $\mathbb{Z}$-grading, so we can calculate the Hilbert series of $\mathrm{R}\pi_* \mathcal{O}_{\mathcal{E}}$ as

$$\sum_{i\geq 0} (-1)^i \mathrm{H}_{\mathrm{R}^i \pi_* \mathcal{O}_{\mathcal{E}}}(t) = \sum_{i=0}^{19} (-1)^i \chi\big(\mathrm{Gr}(6, V_9); \textstyle\bigwedge^i \xi\big) \frac{t^i}{(1-t)^{84}}.$$

The right-hand side can be computed using Borel–Weil–Bott [Weyman 2003, Corollary 4.1.7] and yields

$$\frac{1 + t^6 + t^{12} + 81t^{18} - 84t^{19}}{(1-t)^{84}} = \frac{h(t)}{(1-t)^{83}},$$

where

$$h(t) = 84t^{18} + 3t^{17} + 3t^{16} + 3t^{15} + 3t^{14} + 3t^{13} + 3t^{12} + 2t^{11} + 2t^{10} + 2t^9 + 2t^8 + 2t^7 + 2t^6 + t^5 + t^4 + t^3 + t^2 + t + 1.$$

In particular, the support of $\mathrm{R}\pi_* \mathcal{O}_{\mathcal{E}}$ has dimension 83, and this support is Z. This matches the dimension of the total space of $\mathcal{E}$, so generically, the map π has 0-dimensional fibers. Since π is projective, this implies that the support of $\mathrm{R}^i \pi_* \mathcal{O}_{\mathcal{E}}$ for each $i > 0$ has dimension ≤ 82. In particular, the multiplicity of $\pi_* \mathcal{O}_{\mathcal{E}}$ is $h(1) = 120$ and the degree of Z divides 120.

In characteristic 0, we know that Z is contained in an irreducible hypersurface of degree 120 by Lemma 2.11, so we conclude that Z coincides with this hypersurface. This proves (a) and (b) in characteristic 0.

Now we prove (a) in general. What remains is to show that every nonstable element belongs to Z. Let γ be a nonstable element. Let R be a complete DVR with residue field $\boldsymbol{k}$ and fraction field $\boldsymbol{K}$ of characteristic 0. Let ρ be a 1-parameter subgroup of $G(\boldsymbol{k})$ such that $\lim_{t\to 0} \rho(t) \cdot \gamma$ exists. By changing basis, we may assume that the image of ρ is contained in the diagonal matrices, and hence ρ can be lifted to a 1-parameter subgroup $\tilde{\rho}$ of $G(R)$. The action of $\tilde{\rho}$ on $\bigwedge^3 R^9$ decomposes it into weight spaces which are free R-submodules, we are interested in the negative versus nonnegative subspaces. The nonnegative subspace corresponds to all elements which have a limit under the action of $\tilde{\rho}(t)$ for $t \to 0$ and its reduction to $\boldsymbol{k}$ is the nonnegative subspace of the action of ρ on $\bigwedge^3 V_9$. So we can lift γ to a nonstable element $\tilde{\gamma} \in \bigwedge^3 R^9$ such that $\tilde{\gamma}_{\boldsymbol{K}} \in \bigwedge^3 \boldsymbol{K}^9$ is also nonstable. By what we just showed, there exists a 6-dimensional subspace $U \subset \boldsymbol{K}^9$ such that $\tilde{\gamma}_{\boldsymbol{K}} \in \bigwedge^3 U + \bigwedge^2 U \otimes (\boldsymbol{K}^9/U)$. Since the Grassmannian is proper, U can be lifted to a rank 6 R-submodule $\tilde{U} \subset R^9$ such that $R^9/\tilde{U}$ is free. In particular, $\tilde{\gamma} \in \bigwedge^3 \tilde{U} + \bigwedge^2 \tilde{U} \otimes R^9/\tilde{U}$ since this is a closed condition on the fibers of R and it is true generically. In particular, the special fiber of $\tilde{U}$ gives a subspace which shows that $\gamma \in Z$.

By what was shown already, we know that Z is an irreducible hypersurface whose degree divides 120, so we conclude that the same is true for the nonstable locus. □

Proposition 2.13. *γ_c is stable if and only if C_c is smooth.*

Proof. If the curve C_c is singular, then translating the singular point to (0, 0) gives a curve

$$C_{c'} : x^2 + z^5 + c'_3 x z^2 + c'_6 z^4 + c'_9 x z + c'_{12} z^3 + c'_{15} x + c'_{18} z^2 + c'_{24} z + c'_{30} = 0.$$

In particular, $c'_{30} = 0$ (since (0, 0) is a point) and $c'_{15} = c'_{24} = 0$ (since the partial derivatives of x and z vanish at (0, 0)). If we take $U = \langle e_4, e_5, e_6, e_7, e_8, e_9 \rangle$, then $\gamma_{c'} \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$, so $\gamma_{c'}$ is nonstable by Proposition 2.12, so the same is true for γ_c since they are projectively S-equivalent by Proposition 2.1.

Consider the set of all pairs (γ_c, U) with $\gamma_c \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$. This is a closed subscheme of $\mathbb{A}^8 \times \mathrm{Gr}(6, 9)$, and is thus proper over $\mathbb{A}^8$. We claim that in any characteristic, the total space is smooth of dimension 7 and irreducible.

The 1-parameter subgroup of Proposition 2.4 acts on this scheme, and since the limit $t \to 0$ always exists in $\mathbb{A}^8$, properness implies that it exists in the scheme of pairs. We claim that the limit must, in fact, be $(\gamma_0, \langle e_4, \dots, e_9 \rangle)$. Indeed, since we are taking a limit along a diagonal 1-parameter subgroup with distinct eigenvalues, the limiting subspace is a coordinate subspace, and there is only one coordinate subspace that destabilizes γ_0. Since the limit point is independent of the starting point, we can bound the dimension of every tangent space by computing its dimension at the limit. This is straightforward linear algebra, and we find that it is indeed 7-dimensional. Since we already know a 7-dimensional component and every component meets the limit point, there can be no other components, and the component we know is smooth. In particular, the image of this scheme in $\mathbb{A}^8$ must be precisely the locus where C_c is singular, as required. □

3. Parametrizing 3-coverings of abelian surfaces

Let $\left(\bigwedge^3 V_9\right)_{\mathrm{st}}$ be the set of stable elements of $\bigwedge^3 V_9$ with respect to the $\mathrm{SL}(V_9)/\boldsymbol{\mu}_3$-action.

Fix $u \in \left(\bigwedge^3 V_9\right)_{\mathrm{st}}$. From this data, we will construct

- a genus-2 curve C with a marked Weierstrass point $P \in C(\boldsymbol{k})$,

- a 3-covering $\psi : X \to J$ (where $J = \mathrm{J}(C)$ is the Jacobian of C) such that $[\psi] \in \ker(\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3]) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}(V_9)/\boldsymbol{\mu}_3))$.

Recall that $\psi : X \to J$ is a 3-covering if X is a torsor for J and ψ can be identified with the multiplication-by-3 map over an algebraic closure of $\boldsymbol{k}$; 3-coverings are classified by cohomology classes in $\mathrm{H}^1(\boldsymbol{k}; J[3])$ [Skorobogatov 2001, Proposition 3.3.2].

To simplify notation, we will not label the objects by u, but we emphasize that all constructions depend on the $\mathrm{PGL}(V_9)$-orbit of $[u] \in \mathbb{P}((\bigwedge^3 V_9)_{\mathrm{st}})$.

Let $\mathbb{P}(V_9^*)$ denote the space of lines in V_9^*. Then V_9 is the space of linear functions on $\mathbb{P}(V_9^*)$, so we can treat $e_1, \dots, e_9$ as coordinate functions. Following [Gruson and Sam 2015, §3.2], we interpret $u \in \bigwedge^3 V_9$ as a family of 9×9 skew-symmetric matrices

$$\Phi : V_9^* \to V_9 \otimes \mathcal{O}_{\mathbb{P}(V_9^*)}(1)$$

over $\mathbb{P}(V_9^*)$. In more details, given $u \in \bigwedge^3 V_9$, apply the comultiplication map $\bigwedge^3 V_9 \to \bigwedge^2 V_9 \otimes V_9$, use the natural surjection $V_9 \otimes \mathcal{O}_{\mathbb{P}(V_9^*)} \to \mathcal{O}_{\mathbb{P}(V_9^*)}(1)$, and interpret $\bigwedge^2 V_9$ as the space of skew-symmetric matrices $V_9^* \to V_9$. In particular, this construction is $\mathrm{GL}(V_9)$-equivariant, so acting by $\mathrm{GL}(V_9)$ amounts to a projective linear change of coordinates in $\mathbb{P}(V_9^*)$.

Let $Y \subset \mathbb{P}(V_9^*)$ be the locus where $\operatorname{rank} \Phi \le 6$. Let $X \subset \mathbb{P}(V_9^*)$ be the locus where $\operatorname{rank} \Phi \le 4$.

Lemma 3.1. *X is smooth of dimension* 2, *and the locus where* $\operatorname{rank} \Phi \le 2$ *is empty.*

Proof. If there is a point in the rank 2 locus, then we can choose a basis so that it is $[1:0:0:0:0:0:0:0:0]$. So $u = [123] + u'$ where no monomial in u' contains e_1. Then $(e_1 \mapsto e_1 + \alpha e_2 + \beta e_3)$ is a 2-dimensional subgroup of the stabilizer of u, which contradicts that u has a finite stabilizer group. A similar argument works if there is a point in the rank 0 locus.

Let $\mathcal{V}_1 = \mathcal{O}(-1)|_X$ be the restriction of the tautological subbundle of lines to X. Also let $\mathcal{V}_9 = V_9^*|_X$ and $\mathcal{V}_5 = \ker \Phi|_X$. Then $\mathcal{V}_5$ is a rank 5 vector bundle on X satisfying $\mathcal{V}_1 \subset \mathcal{V}_5 \subset \mathcal{V}_9$.

We now compute the tangent space of $x \in X$. Do a change of basis so that $e_i(x) = 0$ for $i > 1$ and so that $(\mathcal{V}_5)_x$ is defined by $e_i = 0$ for $i > 5$. Let R be the local ring of $\mathbb{P}(V_9^*)$ at x, and let $\mathfrak{m}$ be its maximal ideal. Over the fiber of x, i.e., working modulo $\mathfrak{m}$, the matrix Φ has rank 4 and its kernel is the fiber of $\mathcal{V}_5$, so looks like $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \psi \end{smallmatrix}\right)$ where ψ is an invertible 4×4 skew-symmetric matrix. In particular, the determinant of the corresponding 4×4 block over R does not belong to $\mathfrak{m}$, so is a unit, and hence that block is invertible. So after a change of basis over R, we can assume that the matrix over R is of the form $A = \left(\begin{smallmatrix} \Phi' & 0 \\ 0 & \Psi \end{smallmatrix}\right)$, where Φ' is 5×5 and all of its entries belong to $\mathfrak{m}$, and

$$\Psi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The 6×6 Pfaffians of A are the equations that locally cut out X at x, and their partial derivatives are the 4×4 Pfaffians. So any 6×6 Pfaffian that uses at least 3 rows from Φ' has identically 0 partial derivatives. Hence the only 6×6 Pfaffians that have nonzero derivatives are those that use 2 rows from Φ' together with the last 4 rows of A. The partial derivatives of these Pfaffians are the entries of Φ', so the tangent space of $x \in X$ is the kernel of the Jacobian map $\mathcal{V}_9/\mathcal{V}_1 \to \bigwedge^2(\mathcal{V}_5/\mathcal{V}_1)$ restricted to x.

Hence it suffices to prove that the Jacobian map is surjective at all points of X. Suppose there is a point $x \in X$ so that the map is not surjective. Choose a nonzero linear functional λ that annihilates the image. The calculation is equivariant under $\mathrm{SL}((\mathcal{V}_5/\mathcal{V}_1)_x)$, so we only need to check what happens for a single representative in each orbit in $\bigwedge^2(\mathcal{V}_5/\mathcal{V}_1)^*$.

If λ has rank 2 (say $\lambda(m)$ is the coefficient of $e_2 \wedge e_3$), it induces a subspace V_3 of $(\mathcal{V}_5)_x$ containing $(\mathcal{V}_1)_x$, and the 1-parameter subgroup with weight $(-2, -2, -2, 1, 1, 1, 1, 1, 1)$ is destabilizing. Indeed, before imposing the condition that λ annihilates the map to $\bigwedge^2(V_5/V_1)$, the only monomials preventing that weight from destabilizing are $[23i]$ for $4 \le i \le 9$; let α_i be the coefficient of $[23i]$. So the $e_2 \wedge e_3$ entry, which is 0, is $\sum_{i=4}^{9} \pm\alpha_i e_i$, so $\alpha_i = 0$ for all i.

If λ has rank 4, we similarly find that the weight $(-4, -1, -1, -1, -1, 2, 2, 2, 2)$ is destabilizing. Here the only monomials preventing that weight from destabilizing are $[ijk]$ where $\{i, j, k\} \subset \{2, 3, 4, 5\}$, and it is easy to see that they cannot appear. □

Lemma 3.2. *If u is not stable, then X is singular.*

Proof. If u is not stable, let $U \subset V_9$ be the corresponding destabilizing subspace as in Proposition 2.12, i.e., $\dim U = 6$ and $u \in \bigwedge^3 U + \bigwedge^2 U \otimes (V_9/U)$. Then U cuts out a $\mathbb{P}^2 \subset \mathbb{P}(V_9^*)$, and the restriction of Φ to that plane is supported on U. Thus the intersection of X and that plane is cut out by a single 6×6 Pfaffian. So either X contains the plane, or it meets it in a cubic curve.

Let p be a point of the intersection. If $\mathrm{rank}(\Phi|_p) \le 2$, then the 6×6 Pfaffians of any matrix $\Phi|_p + \varepsilon\Psi$ (here $\varepsilon^2 = 0$) all vanish, and thus the tangent space of X at p is 8-dimensional. If $\mathrm{rank}(\Phi|_p) = 4$, then the tangent space consists of v in $V_9^*/\langle p\rangle$ such that the restriction of $\Phi(v)$ to the kernel of $\Phi|_p$ is 0. Since this restriction is contained in the 5-dimensional space $\bigwedge^2(U/\operatorname{image}\Phi|_p) \oplus (U/\operatorname{image}\Phi_p) \otimes (\ker(p)/U)$, it follows that the tangent space is at least 3-dimensional. Either way, X cannot be a smooth surface. □

Lemma 3.3. *Y is a cubic hypersurface whose singular locus is X.*

Proof. The fact that Y is a cubic hypersurface follows from [Gruson et al. 2013, §5]. We remark that while that paper works over the complex numbers, the particular calculation that Y is a cubic hypersurface is independent of the field since it only relies on knowing that the determinant of the tautological quotient bundle on projective space is the line bundle $\mathcal{O}(1)$.

It follows from the chain rule that all partial derivatives of the cubic defining Y vanish on X, so we just need to show that Y is smooth away from X. Let $\mathcal{V}_1$ denote the restriction of the tautological subbundle of lines to $Y \setminus X$. Note that $\mathcal{V}_1 = \mathcal{O}(-1)|_{Y\setminus X}$. Also let $\mathcal{V}_9 = V_9^*|_{Y\setminus X}$ and $\mathcal{V}_3 = \ker\Phi|_{Y\setminus X}$. Then $\mathcal{V}_3$ is a rank 3 vector bundle on $Y \setminus X$ satisfying $\mathcal{V}_1 \subset \mathcal{V}_3 \subset \mathcal{V}_9$.

The tangent space at a point $x \in Y \setminus X$ is the kernel of the Jacobian map $\mathcal{V}_9/\mathcal{V}_1 \to \bigwedge^2(\mathcal{V}_3/\mathcal{V}_1)$ restricted to x (this is similar to the argument in the previous proof). So x is smooth if and only if this map is nonzero. Suppose that the map is zero at x and do a change of basis so that $e_i(x) = 0$ for $i > 1$ and so that $(\mathcal{V}_3)_x$ is defined by $e_i = 0$ for $i > 3$. The entries of the Jacobian matrix are given by the coefficients of $[23i]$ for $i = 4, \dots, 9$, and so those coefficients are 0. This means that the 1-parameter subgroup with weight $(-2, -2, -2, 1, 1, 1, 1, 1, 1)$ destabilizes u, which contradicts that u is stable. So Y is indeed a smooth hypersurface away from X. □

Recall that given a variety X, its Albanese variety is an abelian variety satisfying a certain universal property (which will not relevant for our purposes).

Proposition 3.4. *X is a torsor over its Albanese variety J and $\mathcal{O}_X(1)$ is a $(3, 3)$-polarization, i.e., becomes a $(3, 3)$-polarization upon passing to the algebraic closure of $\boldsymbol{k}$.*

Furthermore, J is indecomposable as a polarized variety, i.e., is not a product of two elliptic curves upon passing to the algebraic closure of $\boldsymbol{k}$.

Proof. Let R be a DVR whose residue field is $\boldsymbol{k}$ and whose fraction field K is of characteristic 0. Pick a lift u_R of u to $\bigwedge^3(R^9)$; then u_K is a stable element of $\bigwedge^3(K^9)$ since being nonstable is a closed condition. The construction that we just discussed gives a surface $\mathcal{X}_R$ over R whose generic fiber $\mathcal{X}_K$ is a torsor over its Albanese variety [Gruson et al. 2013, Theorem 5.5] and whose special fiber is $\mathcal{X}_{\boldsymbol{k}} = X$. Let ℓ be a prime different from the characteristic of $\boldsymbol{k}$. Then the ℓ-adic Betti numbers of $\mathcal{X}_K$ and X are the same [Milne 1980, Corollary VI.4.2]. We also know that $\omega_X = \mathcal{O}_X$ (from the locally free resolution of $\mathcal{O}_X$ in [Gruson et al. 2013, §5.2]). So over $\boldsymbol{k}^{\text{sep}}$, X is isomorphic to an abelian surface [Bombieri and Mumford 1977]. In particular, X is a torsor over its Albanese variety (see the proof of [Gruson et al. 2013, Theorem 3.1]).

The statement about $\mathcal{O}_X(1)$ is proven in [Gruson et al. 2013, Proposition 5.6] for a field of characteristic 0. In particular, after base changing to a finite extension of R, we can find a cube root of $\mathcal{O}_X(1)$ over the generic fiber. This can be extended to a line bundle over the whole family whose cube is $\mathcal{O}_{\mathcal{X}}(1)$ (using properness of the Picard variety), which means that it is a $(3, 3)$-polarization over the special fiber as well.

For the last statement, note that if J is isomorphic to a product of elliptic curves E and E' as a polarized variety (after passing to the algebraic closure of $\boldsymbol{k}$), then the embedding of X into $\mathbb{P}^8$ is the Segre embedding of the product of E and E' in their plane embeddings. But X the singular locus of a cubic hypersurface, and hence can be set-theoretically cut out by its partial derivatives (quadrics) together with the equation of the cubic. The Segre embedding of two plane cubics requires two cubic equations to be cut out set-theoretically, so they cannot be the same. □

Since $\mathcal{O}_X(1)$ is a $(3, 3)$-polarization, the action of $J[3]$ on X extends to an action of $J[3]$ on $\mathbb{P}(V_9^*)$. Let X^i be the Picard variety of line bundles on X whose polarization is of type (i, i). By [Gruson and Sam 2015, Theorem 3.6] (although it is stated in characteristic 0, the proof does not rely on this assumption, except for the reference to [Gruson et al. 2013, Proposition 5.6], but see the last paragraph of the previous proof to work around this), we have an isomorphism

$$X(\boldsymbol{k}^{\text{sep}}) \to X^1(\boldsymbol{k}^{\text{sep}}), \quad x \mapsto \mathbb{P}(\ker \Phi(x)) \cap X(\boldsymbol{k}^{\text{sep}}).$$

Since Φ is defined over $\boldsymbol{k}$, this map descends to an isomorphism $X \to X^1$ defined over $\boldsymbol{k}$. Furthermore, we have a cubing map $X^1 \to X^3$ and $\mathcal{O}_X(1) \in X^3$ gives us an isomorphism $X^3 \cong J$. Combining this, we have a map $\psi\colon X \to J$ which gives X the structure of a 3-covering of J.

The preimage of $\mathcal{O}_X(1)$ under the cubing map $X^1 \to X^3$ is a torsor for $J[3]$. Each geometric point represents a line bundle $\mathcal{L}$ such that $\mathrm{h}^0(X; \mathcal{L}) = 1$, and the zero locus $Z(\mathcal{L})$ of the unique, up to scalar multiple, section is a theta divisor of X. So $Z(\mathcal{L})$ is a genus-2 curve whose Jacobian is X.

Lemma 3.5. *Under the isomorphism $X \to X^1$, the image of $Z(\mathcal{L})$ contains the point representing $\mathcal{L}$. Furthermore, this point is a Weierstrass point of $Z(\mathcal{L})$.*

Proof. The first statement is equivalent to $x \in \ker \Phi(x)$. But this follows from the fact that $\Phi(x)$ is the contraction of an alternating trilinear form on V_9 by x.

For the second statement, let P be the point on $Z(\mathcal{L})$. First assume that the characteristic of $\boldsymbol{k}$ is 0. Then we can check more generally that for any point $x \in X$, we have that x is a Weierstrass point of $\mathbb{P}(\ker \Phi(x)) \cap X$. For this, it suffices to check a single point since the property is invariant under translation, and this is done in [Gruson and Sam 2015, Remark 3.15].

For the general case, pick a DVR R as in the proof of Proposition 3.4 and a lift u_R of u to $\bigwedge^3(R^9)$. Our construction is valid in families, so we get a curve $\mathcal{C}$ over R together with a section $\mathcal{P}\colon \operatorname{Spec}(R) \to \mathcal{C}$. Since $\mathcal{M} = \mathcal{O}_{\mathcal{C}}(\mathcal{P})^{\otimes 2}$ extends the canonical bundle on $\mathcal{C}_K$, we see that $\mathcal{M} = \Omega^1_{\mathcal{C}/R}$. In particular, $\mathcal{M}_{\boldsymbol{k}} = \omega_{Z(\mathcal{L})}$, and so P is a Weierstrass point. □

For any two choices $\mathcal{L}$ and $\mathcal{L}'$, $Z(\mathcal{L})$ and $Z(\mathcal{L}')$ differ by translation by an element of $J[3]$, so they have the same image under ψ. So the reduced image of the union of these curves under ψ is a genus-2 curve $C \subset J$ (defined over $\boldsymbol{k}$) whose Jacobian is J and $P := \mathcal{O}_X(1) \in C(\boldsymbol{k})$ is a Weierstrass point.

Using basic properties of finite Heisenberg group schemes, we know that the inclusion $J[3] \subset \mathrm{PGL}(V_9)$ coming from the translation action of $J[3]$ on $\mathbb{P}(V_9^*)$ lifts to an inclusion $J[3] \subset \mathrm{SL}(V_9)/\boldsymbol{\mu}_3$.

Lemma 3.6. *The kernel of the map of pointed sets $\mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}(V_9)/\boldsymbol{\mu}_3) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{PGL}(V_9))$ is trivial, i.e., nontrivial cohomology classes map to nontrivial cohomology classes.*

Proof. We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_3 & \longrightarrow & \mathrm{SL}_9 & \longrightarrow & \mathrm{SL}_9/\boldsymbol{\mu}_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{G}_m & \longrightarrow & \mathrm{GL}_9 & \longrightarrow & \mathrm{PGL}_9 & \longrightarrow & 1 \end{array}$$

which gives the commutative diagram.

$$\begin{array}{ccc} \mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}_9/\boldsymbol{\mu}_3) & \longrightarrow & \mathrm{H}^2(\boldsymbol{k}; \boldsymbol{\mu}_3) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\boldsymbol{k}; \mathrm{PGL}_9) & \longrightarrow & \mathrm{H}^2(\boldsymbol{k}; \boldsymbol{G}_m) \end{array}$$

The horizontal maps have trivial kernel since $\mathrm{H}^1(\boldsymbol{k};\mathrm{SL}_9)=\mathrm{H}^1(\boldsymbol{k};\mathrm{GL}_9)=1$ and the right vertical map has trivial kernel since $\boldsymbol{G}_m/\boldsymbol{\mu}_3\cong\boldsymbol{G}_m$ and $\mathrm{H}^1(\boldsymbol{k};\boldsymbol{G}_m)=1$. So we conclude that the map $\mathrm{H}^1(\boldsymbol{k};\mathrm{SL}_9/\boldsymbol{\mu}_3)\to\mathrm{H}^1(\boldsymbol{k};\mathrm{PGL}_9)$ has trivial kernel. □

Recall that 3-coverings $\psi\colon X\to J$ are classified by cohomology classes $[\psi]\in\mathrm{H}^1(\boldsymbol{k};J[3])$. To get the cohomology class, note that $\psi^{-1}(0)$ is a torsor under $J[3]$.

Lemma 3.7. $$[\psi]\in\ker(\mathrm{H}^1(\boldsymbol{k};J[3])\to\mathrm{H}^1(\boldsymbol{k};\mathrm{SL}(V_9)/\boldsymbol{\mu}_3)).$$

Proof. By Lemma 3.6, it suffices to show that $[\psi]$ is in the kernel of the composition $\mathrm{H}^1(\boldsymbol{k};\mathrm{J}(C)[3])\to\mathrm{H}^1(\boldsymbol{k};\mathrm{PGL}(V_9))$. The map sends the $J[3]$-torsor $\psi^{-1}(0)$ to the $\mathrm{PGL}(V_9)$-torsor $\psi^{-1}(0)\times^{J[3]}\mathrm{PGL}(V_9)$. The data of this $\mathrm{PGL}(V_9)$-torsor is equivalent to the embedding $\psi^{-1}(0)\subset\mathbb{P}(V_9^*)$. Projective space represents the trivial PGL-torsor, so the image of $[\psi]$ in $\mathrm{H}^1(\boldsymbol{k};\mathrm{PGL}(V_9))$ is trivial. □

The trivectors γ_c described in Proposition 2.1 are particularly nice for this construction. Recall from Proposition 2.13 that γ_c is stable whenever the corresponding curve C_c is smooth.

Proposition 3.8. *Suppose that C_c is smooth. Then the pair (C,P) corresponding to γ_c is isomorphic to (C_c,P_c), and the corresponding torsor $\psi_c^{-1}(0)$ is trivial.*

Proof. Let C' be the image of C_c in $\mathbb{P}^8$ under the embedding

$$f\colon(x,z)\mapsto[0:0:-1:0:z:0:-z^2:x:z^3].$$

The point $P':=[0:0:0:0:0:0:0:0:1]$ is a Weierstrass point of the closure of C', and C' is contained in $\mathbb{P}(\ker(\Phi(P')))$, so the first claim will follow if we can show that C' is contained in the rank 4 locus. (Indeed, then C' is contained in a theta divisor of $X(\gamma_c)$, so must be that theta divisor.) We may verify that the subspace with basis

$$\begin{pmatrix}1&0&0&z^2&x&-c_{12}z-c_{18}&0&-c_9x-c_{24}&0\\0&1&c_3&-z&0&-z^2-c_6z&-x-c_{15}&-c_3x&0\\0&0&1&0&-z&0&z^2&-x&0\\0&0&0&0&0&1&0&-z&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}$$

is in the kernel of Φ restricted to the point $f(x,z)$, and thus that $\Phi|_{C'}$ has rank at most 4 as required.

To see that $\psi_c(P')=0$, we need to show that $3\overline{C'}$ is a section of $\mathcal{O}_{\mathbb{P}^8}(1)$. Since $\overline{C'}$ induces a principal polarization, the restriction map $\mathrm{Pic}^0(X(\gamma_c))\to\mathrm{Pic}^0(\overline{C'})$ is an isomorphism, and thus it suffices to show that $\mathcal{O}_{\mathbb{P}^8}(1)$ and $3\overline{C'}$ have the same restriction to $\overline{C'}$. In fact, both restrictions are isomorphic to $\mathcal{L}_{\overline{C'}}(3K_{\overline{C'}})$: the first because $\overline{C'}$ is tricanonically embedded in $\mathbb{P}^4$ and the second by adjunction and the fact that $K_{X(\gamma_c)}=0$. □

4. A construction of trivectors

Let C be a smooth genus-2 curve with a marked Weierstrass point $P \in C(\boldsymbol{k})$. Let $\mathrm{J}^1(C)$ be the Picard variety of degree 1 line bundles, and let $\mathrm{J}(C)$ be the Jacobian of degree 0 line bundles. We identify $\mathrm{J}^1(C) \cong \mathrm{J}(C)$ via $\mathcal{L} \mapsto \mathcal{L}(-P)$.

Define $V_9 = \mathrm{H}^0(\mathrm{J}^1(C); 3\Theta)$. Then $\mathrm{J}(C) \subset \mathbb{P}(V_9^*)$ is embedded by a (3, 3)-polarization, denoted $\mathcal{O}(1)$. Define a codimension 1 subvariety (Poincaré divisor) of $\mathrm{J}(C) \times \mathrm{J}(C)$ by

$$X = X_{C,P} = \{(\mathcal{L}_1, \mathcal{L}_2) \mid \hom_C(\mathcal{L}_1, \mathcal{L}_2(P)) \neq 0\}.$$

The line bundle $\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)$ has divisor class $3\pi_1^*\Theta + 3\pi_2^*\Theta - \Theta_{\mathrm{diag}}$. This is the pullback of a principal polarization on $\mathrm{J}(C) \times \mathrm{J}(C)$ via the endomorphism

$$\mathrm{J}(C) \times \mathrm{J}(C) \to \mathrm{J}(C) \times \mathrm{J}(C), \quad (a, b) \mapsto (2a + b, a + 2b).$$

The kernel of this map is the diagonal copy of $\mathrm{J}(C)[3]$ which has degree 81. In particular, $\mathcal{O}(1, 1)\otimes\mathcal{O}(-X)$ has a single cohomology group of dimension $9 = \sqrt{81}$.

Lemma 4.1. $\mathrm{h}^0(\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)) = 9$ *and all other cohomology groups vanish.*

Proof. It suffices to show that $\mathrm{h}^0(\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)) \neq 0$. Define a divisor of $\mathrm{J}(C) \times \mathrm{J}(C)$ by

$$D = \{(\mathcal{L}_1, \mathcal{L}_2) \mid \mathrm{h}^0(\mathcal{L}_1 \otimes \mathcal{L}_2(-P)) \neq 0 \text{ or } \mathrm{h}^0(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2(P)) \neq 0\}.$$

Then D is linearly equivalent to $2\pi_1^*\Theta \otimes 2\pi_2^*\Theta$. In particular, $\mathcal{O}(1, 1) \otimes \mathcal{O}(-D)$ has a nonzero section. But $X \subset D$, so we see that $\mathcal{O}(1, 1) \otimes \mathcal{O}(-X)$ also has a nonzero section. □

Define

$$W = \mathrm{H}^0(\mathrm{J}(C) \times \mathrm{J}(C); \mathcal{O}(1, 1) \otimes \mathcal{O}(-X)) \subset V_9 \times V_9.$$

By Serre duality and Riemann–Roch, $\hom_C(\mathcal{L}_1, \mathcal{L}_2(P)) \neq 0$ if and only if $\hom_C(\mathcal{L}_2, \mathcal{L}_1(P)) \neq 0$, so X is preserved under the involution that swaps the two copies of V_9.

Let H denote the finite Heisenberg group scheme, i.e., the extension

$$1 \to \boldsymbol{\mu}_3 \to H \to \mathrm{J}(C)[3] \to 1.$$

Then H acts diagonally on $V_9 \otimes V_9$ preserving W. Note that V_9 is the unique irreducible representation of H of weight 1 (see [Sekiguchi 1977, Appendix]), and W has weight 2, so V_9 and W^* are isomorphic as representations of H. So the inclusion gives an H-equivariant map (well-defined up to scalar multiple) $V_9^* \to V_9 \otimes V_9$.

Lemma 4.2. *The image of* V_9^* *is contained in* $\bigwedge^2 V_9$.

Proof. By irreducibility, it suffices to show that a single nonzero element in W is alternating under the involution swapping the two copies of V_9. Define D as in the proof of Lemma 4.1. Pick a bilinear equation that vanishes on D, i.e., a section of $\mathcal{O}(1, 1) \otimes \mathcal{O}(-D)$. Since the diagonal $\mathrm{J}(C)$ is contained in D, if we restrict this equation to the diagonal, we get a section of 4Θ that vanishes on $\mathrm{J}(C)$. But we know that such equations are alternating since the Kummer variety has no quadratic polynomials vanishing on it in its 2Θ embedding. □

So we can represent this map by an element $\gamma = \gamma_{(C,P)} \in V_9 \otimes \bigwedge^2 V_9$.

Lemma 4.3. $$\gamma_{(C,P)} \in \bigwedge^3 V_9.$$

Proof. Note that γ is an H-invariant element. Furthermore, $\bigwedge^2 V_9$ is a weight 2 representation of dimension 36, and hence it is a direct sum of 4 copies of V_9^* [Sekiguchi 1977, Theorem A.6], so the space of H-invariant vectors in $V_9 \otimes \bigwedge^2 V_9$ is 4-dimensional. The space of H-invariant vectors in $\bigwedge^3 V_9$ is also 4-dimensional (we can do this calculation in characteristic 0 and then specialize to get $\geq$ 4-dimensional), so $\gamma \in \bigwedge^3 V_9$. □

Lemma 4.4. *The projection of $X_{C,P}$ to either copy of $\mathbb{P}^8$ lies in the rank 4 locus $X(\gamma)$ constructed in Section 3.*

Proof. Pick a point x in the projection of $X_{C,P}$ to $\mathbb{P}(V_9^*)$. Evaluating γ on x, we get a skew-symmetric matrix $V_9^* \to V_9$ whose image is the set of linear equations vanishing on the fiber of $X_{C,P}$ over x. This fiber is a translate of a theta divisor. As an embedded variety, the theta divisor is a genus-2 curve under its tricanonical embedding, and hence satisfies 4 linear equations, so this skew-symmetric map has rank 4. □

Proposition 4.5. *Let $G = \mathrm{SL}(V_9)/\boldsymbol{\mu}_3$.*

(a) *$\gamma_{(C,P)} \in \bigwedge^3 V_9$ is stable with respect to the action of G.*

(b) *The stabilizer of $[\gamma_{(C,P)}] \in \mathbb{P}(\bigwedge^3 V_9)$ in G is isomorphic to $\mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P)$ where $\mathrm{Aut}(C, P)$ is the group scheme of $\boldsymbol{k}^{\mathrm{sep}}$-automorphisms of C which fix P.*

(c) *If the characteristic is different from 2 and 5, then the stabilizer of $\gamma_{(C,P)}$ in G is isomorphic to $\mathrm{J}(C)[3]$.*

(d) *In characteristic 2, the stabilizer of $\gamma_{(C,P)}$ is isomorphic to $\mathrm{J}(C)[3] \rtimes \mathbb{Z}/2$ if (C, P) is generic, where the $\mathbb{Z}/2$ comes from the hyperelliptic involution on C and acts by the automorphism $g \mapsto g^{-1}$.*

(e) *In characteristic 5, the stabilizer of $\gamma_{(C,P)}$ is isomorphic to $\mathrm{J}(C)[3]$ if (C, P) is generic.*

Proof. Recall the notation from Proposition 2.1. By Proposition 2.13, taking any smooth curve (C_c, P_c) guarantees that γ_c is stable. The element γ_c induces a system of 9 bilinear equations in the above way, and we may consider the resulting (symmetric) subscheme of $X(\gamma_c) \times X(\gamma_c)$. By the proof of Lemma 3.5, the fiber over any point $x \in X(\gamma_c)$ is a theta divisor C_x on which x is a Weierstrass point. Since the associated torsor of γ_c is trivial (Proposition 3.8), this becomes the Poincaré divisor on $\mathrm{J}(C_x) \times \mathrm{J}(C_x)$ associated to x, and it follows that γ_c is (projectively) equivalent to $\gamma_{(C_x,x)}$. Furthermore, by Proposition 3.8, the curve (C_x, x) is isomorphic to (C_c, P_c). Over $\boldsymbol{k}^{\mathrm{sep}}$, every smooth pair (C, P) is isomorphic to (C_c, P_c) for some c, and stability is insensitive to enlarging the field, so we conclude that $\gamma_{(C,P)}$ is always stable when (C, P) is smooth, which proves (a).

Now we handle (b). First we calculate the stabilizer $G_{[\gamma]}$ of $[\gamma] \in \mathbb{P}(\bigwedge^3 V_9)$. By functoriality, it is clear that $\mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P) \subseteq G_{[\gamma]}$. Conversely, let S be a $\boldsymbol{k}$-scheme and consider an element $g \in G_{[\gamma]}(S)$.

Since $\gamma_{(C,P)}$ is stable, $\mathrm{J}(C) \subset \mathbb{P}(V_9^*)$ is the rank 4 locus of Φ and hence is preserved by g (this is true for γ_c and $\gamma_{(C,P)}$ is equivalent to γ_c over $\boldsymbol{k}^{\mathrm{sep}}$). So g preserves the embedding of $(\mathrm{J}(C) \times \mathrm{J}(C))(S)$ in $(\mathbb{P}(V_9^*) \times \mathbb{P}(V_9^*))(S)$ and the subvariety $X(S)$. In particular, g acts on $\mathrm{J}(C)(S)$ and preserves the relation $\hom_{C(S)}(\mathcal{L}_1, \mathcal{L}_2(P)) \neq 0$, which implies that g permutes the elements of $\mathrm{J}(C)[3](S)$. So $G_{[\gamma]}$ is generated by $\mathrm{J}(C)[3]$ and a subgroup of the automorphisms of $\mathrm{J}(C)$ which fixes the identity. Using Torelli's theorem, an automorphism of $\mathrm{J}(C)$ that fixes the identity and the embedding of $\mathrm{J}(C)$ comes from an automorphism of C which fixes P; since $G_{[\gamma]}$ contains $\mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P)$, we deduce that they are equal. This proves (b).

In particular, $G_{[\gamma]}$ is finite. Let $\lambda\colon G_{[\gamma]} \to \boldsymbol{G}_m$ be the eigenvalue associated with the action of $G_{[\gamma]}$ on $[\gamma]$. The stabilizer of γ is $\ker\lambda$. First note that $\mathrm{J}(C)[3] \subseteq \ker\lambda$ since the projective action of $\mathrm{J}(C)[3]$ lifts to a linear action of the Heisenberg group scheme H in $\mathrm{SL}(V_9)$, and we have already explained why H acts trivially on γ.

Now we prove (c), so we assume that the characteristic is different from 2 and 5. Recall that $G_\gamma = \ker\lambda$, so we need to show that $\mathrm{Aut}(C, P)$ is mapped faithfully via λ. Put (C, P) into Weierstrass normal form

$$y^2 = x^5 + c_{12}x^3 + c_{18}x^2 + c_{24}x + c_{30}. \tag{4.5.1}$$

By degree considerations (where $\deg(y) = 5$ and $\deg(x) = 2$), any automorphism of (C, P) must be of the form

$$y \mapsto a_1^5 y + a_2x^2 + a_3x + a_4, \quad x \mapsto a_1^2 x + a_5$$

for some scalars a_1, a_2, a_3, a_4, a_5 and $a_1 \neq 0$. When we do these substitutions to (4.5.1) and subtract (4.5.1), we get a relation on x, y which is of degree < 10, so which must be identically 0. The coefficients of x^2y, xy, y on the left side are $2a_1^5a_2, 2a_1^5a_3, 2a_1^5a_4$, respectively, so we conclude that $a_2 = a_3 = a_4 = 0$. Similarly, the coefficient of x^4 on the right side is $5a_1^8a_5$, so we conclude that $a_5 = 0$.

In particular, the automorphism takes the form

$$y \mapsto a_1^5 y \quad \text{and} \quad x \mapsto a_1^2 x,$$

for some ℓ-th root of unity a_1 (since the automorphism has finite order). Again, do the substitution to (4.5.1), divide by a_1^{10} and subtract (4.5.1). Then we get

$$c_{12}(a_1^{-4} - 1)x^3 + c_{18}(a_1^{-6} - 1)x^2 + c_{24}(a_1^{-8} - 1)x + c_{30}(a_1^{-10} - 1) = 0,$$

so the left hand side must be identically 0. If $\ell \notin \{1, 2, 4, 5, 8, 10\}$, then $c_{12} = c_{24} = c_{30} = 0$. But then (4.5.1) is $y^2 = x^2(x^3 + c_{18})$, which is a singular curve. So we only need to show that λ maps $\boldsymbol{\mu}_\ell \subset \mathrm{Aut}(C, P)$ faithfully where $\ell \in \{1, 2, 4, 5, 8, 10\}$; it suffices to consider the cases $\ell = 2$ and $\ell = 5$.

For $\ell \in \{2, 5\}$, let $\mathcal{M}_\ell$ be the space of curves with an action of $\boldsymbol{\mu}_\ell$ as described above. Then $\mathcal{M}_\ell$ is an irreducible stack over $\mathbb{Z}[1/\ell]$. Indeed, the action of $\boldsymbol{\mu}_\ell$ must survive completing the ℓ-th power in the curve, and this forces the action to be diagonal in the variables. Thus $\mathcal{M}_5$ is the irreducible stack of curves of the form $x^2 + z^5 + c_{15}x + c_{30} = 0$ modulo $x \mapsto x + a$ (with $\boldsymbol{\mu}_5$ acting by $z \mapsto \zeta_5 z$) and $\mathcal{M}_2$ is the irreducible stack of curves of the form

$$x^2 + z^5 + c_6 z^4 + c_{12} z^3 + c_{18} z^2 + c_{24} z + c_{30},$$

modulo $z \mapsto z + a$ (with $\boldsymbol{\mu}_2$ acting by $x \mapsto -x$).

Let $\mathscr{C}$ be the universal curve over $\mathscr{M}_\ell$. By composing λ with the natural morphism $\boldsymbol{\mu}_\ell \to \mathrm{Aut}(\mathscr{C})$, we obtain a scheme morphism from $\mathscr{M}_\ell$ to the dual group $\boldsymbol{\mu}_\ell^\vee \cong \mathbb{Z}/\ell$. Since $\mathscr{M}_\ell$ is irreducible, this morphism must be constant, and thus may be computed in characteristic 0. In this case, any point in the Cartan subspace which is not in the union of the reflection hyperplanes has a trivial stabilizer. In particular, the stabilizer of γ is isomorphic to $\mathrm{J}(C)[3]$. So faithfulness of λ in characteristic 0 implies faithfulness of the restriction to $\boldsymbol{\mu}_\ell$ over $\mathbb{Z}[1/\ell]$.

If (C, P) is generic, then $\mathrm{Aut}(C, P) \cong \mathbb{Z}/2$ and is generated by the hyperelliptic involution ι_C (via Torelli's theorem, this is equivalent to the statement that the generic principally polarized Jacobian has automorphism group $\mathbb{Z}/2$, which is [Katz and Sarnak 1999, Lemma 11.2.6]). The induced action of ι_C on $\mathrm{J}(C)[3]$ is the inverse map and, if $\boldsymbol{k}$ has characteristic 0, we can calculate explicitly in a standard Cartan (see, for example, [Gruson and Sam 2015, (3.2)]) that $\lambda(\iota_C) = -1$, so $\iota_C \notin \ker\lambda$. By semicontinuity, the same is true in any characteristic different from 2. In characteristic 2, the restriction of λ to ι_C is trivial since its image is in $\boldsymbol{\mu}_2 \subset \boldsymbol{G}_m$, which is nonreduced, while ι_C generates a subgroup isomorphic to $\mathbb{Z}/2$. So $\iota_C \in \ker\lambda$. This proves (d) and (e). □

5. Putting it all together

Let $G = \mathrm{SL}(V_9)/\boldsymbol{\mu}_3$ and let G_γ be the stabilizer subgroup of $\gamma \in \bigwedge^3 V_9$.

Proposition 5.1. *Pick (C, P) and (C', P') so that we have elements*

$$\gamma = \gamma_{(C,P)} \in \textstyle\bigwedge^3 V_9 \quad \textit{and} \quad \gamma' = \gamma_{(C',P')} \in \textstyle\bigwedge^3 V_9'.$$

Suppose that there is a linear isomorphism $\varphi\colon V_9 \cong V_9'$ that sends the line generated by $\gamma_{(C,P)}$ to the line generated by $\gamma_{(C',P')}$. Then there exists an isomorphism $(C, P) \cong (C', P')$.

Proof. Using φ, we can embed $X_{C,P}$ and $X_{C',P'}$ in the same $\mathbb{P}^8 \times \mathbb{P}^8$, in such a way that their images satisfy the same 9 bilinear equations $W_\gamma = W_{\gamma'}$. Now, consider the projection π onto the first $\mathbb{P}^8$. By Lemma 4.4, the image of $X_{C,P}$ in $\mathbb{P}^8$ maps into the rank 4 locus $X(\gamma)$, which is a torsor over an abelian surface (Proposition 3.4). The fibers of π are curves, and so the image of π is a surface. Since $X(\gamma)$ is irreducible, the image must be equal to $X(\gamma)$. In particular, π gives an identification $\mathrm{J}(C) = X(\gamma)$. The same applies to $X_{C',P'}$, so in particular, we find that φ defines an isomorphism $\mathrm{J}(C) \cong \mathrm{J}(C')$ which identifies the respective 3Θ line bundles. Finally, we can recover C as $\pi^{-1}(0)$ under $\pi : X_{C,P} \to \mathrm{J}(C)$, and P as the point $(0, 0) \in X_{C,P} \subset \mathrm{J}(C) \times \mathrm{J}(C)$, and similarly for (C', P'). □

Proposition 5.2. *If we apply the construction of Section 3 to $\gamma_{(C,P)}$, then the torsor $X(\gamma_{(C,P)})$ is trivial, and (C, P) is the marked curve that comes from the construction in Section 3.*

Proof. This was shown in the proof of Proposition 5.1. □

Lemma 5.3. *Let $\gamma \in \bigwedge^3 V_9$ be a stable element. Then γ can be recovered from the 9-dimensional space of bilinear forms $W_\gamma \subset \bigwedge^2 V_9$ up to scalar multiple.*

Proof. We represent this space as an injective map $f\colon W_\gamma \to \bigwedge^2 V_9$. Since γ is stable, the locus $Y(\gamma) = \{x \in \mathbb{P}(W_\gamma) \mid \operatorname{rank} f(x) \le 6\}$ is a cubic hypersurface (Lemma 3.3), and so the generic element in W_γ has rank 8, and hence its kernel is a line in V_9^*. This gives a rational map $\rho\colon \mathbb{P}(W_\gamma) \dashrightarrow \mathbb{P}(V_9^*)$. Furthermore, ρ is the projectivization of a linear map $\varphi\colon W_\gamma \to V_9^*$ since there exists an identification $W_\gamma = V_9^*$ coming from $\gamma\colon V_9^* \to \bigwedge^2 V_9$. This linear map is unique up to scalar multiple, and our goal is to reconstruct it from W_γ.

Pick 10 elements in W_γ with rank 8 such that any 9 of them are linearly independent. Pick a basis $e_1, \dots, e_9$ for W_γ. Up to projective equivalence, we may assume that the points are the projectivizations of $e_1, \dots, e_9, e_1 + \cdots + e_9$. For $i = 1, \dots, 9$, choose $x_i \in \rho(e_i)$ such that $x_1 + \cdots + x_9 \in \rho(e_1 + \cdots + e_9)$. This can be used to define a linear map $\varphi'\colon W_\gamma \to V_9^*$ which is well-defined up to a global choice of scalar. In particular, there must be scalars α_i such that $\varphi'(e_i) = \alpha_i \varphi(e_i)$ for $i = 1, \dots, 9$. However, $\varphi'(e_1 + \cdots + e_9) = \alpha_1\varphi(e_1) + \cdots + \alpha_9\varphi(e_9)$, and it must generate the same line as $\varphi(e_1 + \cdots + e_9)$, so we conclude that $\alpha_1 = \cdots = \alpha_9$ and hence φ and φ' agree up to scalar multiple. □

Proposition 5.4. *Pick stable elements $\gamma, \gamma' \in \bigwedge^3 V_9$ with trivial cohomology class, i.e., $[\psi] = [\psi'] = 0$. Let (C, P) and (C', P') be the marked curves constructed in Section 3 and assume that there is an isomorphism $(C, P) \cong (C', P')$ defined over $\boldsymbol{k}$. Then the lines spanned by γ and γ' are in the same* $\mathrm{PGL}(V_9)$*-orbit.*

Proof. Via the construction in Section 3, we have torsors $X(\gamma), X(\gamma') \subset \mathbb{P}(V_9^*)$. Since its cohomology class is trivial, we can find a $\boldsymbol{k}$-rational point in the preimage of 0 under the 3-covering $X(\gamma) \to \mathrm{J}(C)$. Use this point, call it 0, to identify $X(\gamma)$ with $\mathrm{J}(C)$. The construction shows that $X(\gamma)$ has a $\boldsymbol{k}$-rational theta divisor $C \subset X(\gamma)$ such that $0 \in X(\gamma)$ is a Weierstrass point on C. These remarks also apply to $C' \subset X(\gamma')$.

The embedding $X(\gamma) \subset \mathbb{P}(V_9^*)$ can be reconstructed from the data of (C, P). In particular, the isomorphism $(C, P) \cong (C', P')$ that is assumed to exist induces an isomorphism $X(\gamma) \cong X(\gamma')$ that preserves their embeddings into $\mathbb{P}(V_9^*)$. So, up to a linear change of coordinates for one of the embeddings, we have $X(\gamma) = X(\gamma')$. In particular, there is an identification of their Poincaré divisors, which then satisfy the same 9 bilinear equations, i.e., $W_\gamma = W_{\gamma'}$. Lemma 5.3 implies that γ and γ' are equal up to scalar multiple after the change of coordinates. □

Theorem 5.5. *The construction in Section 3 is a bijection between the stable orbits of $\mathbb{P}\big(\bigwedge^3 V_9\big)$ under the action of* $\mathrm{PGL}(V_9)$ *and the set of $\boldsymbol{k}$-isomorphism classes of triples (C, P, ψ) where C is a smooth genus-2 curve, $P \in C(\boldsymbol{k})$ is a Weierstrass point, and* $\psi \in \ker(\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3]) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{PGL}(V_9)))$.

Proof. Given a smooth genus-2 curve with Weierstrass point P, we have constructed a stable element in $\mathbb{P}\big(\bigwedge^3 \mathrm{H}^0(\mathrm{J}(C); 3\Theta)^*\big)$ in Section 4. If we pick a linear isomorphism $\mathrm{H}^0(\mathrm{J}(C); 3\Theta)^* \cong V_9$, we hence get an element of $\mathbb{P}(\bigwedge^3 V_9)$. The $\mathrm{PGL}(V_9)$-orbit of this element does not depend on the choice of isomorphism. So we have a well-defined map Φ from the set of $\boldsymbol{k}$-isomorphism classes of (C, P) to $\mathrm{PGL}(V_9)$-orbits in $\mathbb{P}(\bigwedge^3 V_9)$. Furthermore, by Proposition 5.2, $\Phi(C, P)$ has trivial cohomology class. By Proposition 5.1, this map is injective on $\boldsymbol{k}$-isomorphism classes of (C, P).

In Section 3, we constructed a map from $\mathrm{PGL}(V_9)$-orbits of $\mathbb{P}((\bigwedge^3 V_9)_{\mathrm{st}})$ to the set of $\boldsymbol{k}$-isomorphism classes of (C, P); let Ψ be the restriction to the orbits with trivial cohomology class. By Proposition 5.2, $\Psi \circ \Phi$ is the identity, so Ψ is surjective. By Proposition 5.4, Ψ is injective, so Φ is a bijection between $\boldsymbol{k}$-isomorphism classes of marked curves (C, P) and $\mathrm{PGL}(V_9)$-orbits of stable elements in $\mathbb{P}(\bigwedge^3 V_9)$ with trivial cohomology class.

By Proposition 4.5, the stabilizer of any element in $\Phi(C, P)$ is isomorphic to $\mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P)$. In particular,

$$\ker(\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P)) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{PGL}(V_9)))$$

is in bijection with the $\mathrm{PGL}(V_9)$-orbits in $\mathbb{P}(\bigwedge^3 V_9)$ which are in the same orbit as $\Phi(C, P)$ over a separable closure of $\boldsymbol{k}$. Now consider the map

$$\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P)) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{Aut}(C, P)).$$

The latter group parametrizes $\boldsymbol{k}$-forms of C, so each such orbit is naturally associated to a $\boldsymbol{k}$-form of C. In particular, the orbits that correspond to C itself, i.e., $\boldsymbol{k}$-forms that are actually isomorphic to C over $\boldsymbol{k}$, are in bijection with

$$\ker(\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3]) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{PGL}(V_9))).$$

In particular, Φ extends to a map on triples (C, P, ψ) and gives an isomorphism to all stable $\mathrm{PGL}(V_9)$-orbits in $\mathbb{P}(\bigwedge^3 V_9)$. ☐

Corollary 5.6. *If $\boldsymbol{k}$ is algebraically closed of characteristic different from* 3, *then every stable element of $\bigwedge^3 V_9$ is in the standard Cartan subspace up to the action of G.*

Proof. By the construction in Section 4, every stable element of the form $\gamma_{(C,P)}$ is in the standard Cartan subspace up to the action of G. By Theorem 5.5, they all arise in this way. ☐

6. Complements

6A. *Selmer groups.* For this section, suppose that $\boldsymbol{k}$ is a global field, and let B be an abelian variety defined over $\boldsymbol{k}$. Let $\alpha \in \mathrm{H}^1(\boldsymbol{k}; B[n])$ be a torsor for $B[n]$. We can use this to twist the multiplication by n map $B \xrightarrow{\cdot n} B$ to get $B' \to B$ where B' is a B-torsor. We say that α is an element of the *n-Selmer group* of B if, for all completions $\boldsymbol{k}_v$ of $\boldsymbol{k}$, the corresponding torsor B' has a $\boldsymbol{k}_v$-rational point. We denote this subgroup by $\mathrm{Sel}_n(B) \subset \mathrm{H}^1(\boldsymbol{k}; B[n])$.

Proposition 6.1. *Let C be a genus-*2 *curve with rational Weierstrass point. The* 3*-Selmer group* $\mathrm{Sel}_3(\mathrm{J}(C))$ *is contained in* $\ker(\mathrm{H}^1(\boldsymbol{k}; \mathrm{J}(C)[3]) \to \mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}_9/\boldsymbol{\mu}_3))$.

Proof. Pick $\psi \in \mathrm{Sel}_3(\mathrm{J}(C))$. Then ψ gives an embedding $X \subset S$ where S is a Brauer–Severi variety of dimension 8 and X is the corresponding twist of $\mathrm{J}(C)$. By assumption, $X(\boldsymbol{k}_v) \neq \varnothing$ for all completions $\boldsymbol{k}_v$ of $\boldsymbol{k}$. Brauer–Severi varieties satisfy the Hasse principle, so we conclude that $S \cong \mathbb{P}^8$ and that the image of ψ in $\mathrm{H}^1(\boldsymbol{k}; \mathrm{PGL}(9))$ is trivial. By Lemma 3.6, its image in $\mathrm{H}^1(\boldsymbol{k}; \mathrm{SL}(9)/\boldsymbol{\mu}_3)$ is also trivial. □

In particular, triples (C, P, ψ) where C is a genus-2 curve, $P \in C(\boldsymbol{k})$ is a Weierstrass point, and $\psi \in \mathrm{Sel}_3(\mathrm{J}(C))$ are parametrized by certain $\mathrm{PGL}(V_9)$-orbits in $\mathbb{P}\big(\bigwedge^3 V_9\big)$.

6B. *Ordinary curves.* Let $\boldsymbol{k}$ be a field of characteristic 3. Given a smooth curve of genus-g, then $|\mathrm{J}(C)(\boldsymbol{k}^{\mathrm{sep}})| = 3^r$ where $0 \leq r \leq g$. The quantity r is the 3-*rank* of the curve. If $r = g$, then C is *ordinary*.

The Lie algebra of type E_8 has a cubing map $x \mapsto x^{[3]}$ which induces a cubing map $\bigwedge^3 V_9 \to \mathfrak{sl}(V_9)$.

Set γ_0 to be the principal nilpotent element with all $c_i = 0$ in Proposition 2.1:

$$\gamma_0 = [267] + [258] + [348] + [169] + [357] + [249] + [178] + [456].$$

Lemma 6.2. *Let C be the genus-2 curve associated with a stable element $\gamma \in \bigwedge^3 V_9$. The Lie algebra of the stabilizer of γ (equivalently, the Lie algebra of $\mathrm{J}(C)[3]$) is 2-dimensional, and is spanned by $\gamma^{[3]}$ and $\gamma^{[9]}$.*

Proof. Consider the height grading on $\mathfrak{e}_8$ discussed in Section 2B. The principal nilpotent element in $\mathfrak{e}_8$ restricts to γ_0 and [Springer 1966, Theorem 2.6] shows that, outside of degrees -1 and 0, multiplication by γ_0 has a single kernel element in characteristic 3 in degrees 3, 9, -4, and -10. Reducing these degrees modulo 3, we see that $\ker(\operatorname{ad}\gamma_0 \cap \mathfrak{sl}(V_9))$ has two elements coming from degrees 3 and 9, which are $\gamma_0^{[3]}$ and $\gamma_0^{[9]}$, together with whatever comes from degree 0. However, the latter is 0 since the structure constants of the Lie algebra $\mathfrak{e}_8$ are all ± 2.

By semicontinuity, the Lie algebra of $\mathrm{J}(C)[3]$ coming from γ is at most 2-dimensional. However, a generic γ (for example, take a stable element in the Cartan subspace) comes from an ordinary curve, in which case the Lie algebra is 2-dimensional, so the dimension is always 2, and agrees with the $\geq$ 2-dimensional span of $\gamma^{[3]}$ and $\gamma^{[9]}$. □

Corollary 6.3. *Pick a stable element $\gamma \in \bigwedge^3 V_9$. The 3-rank of the associated curve is*

(a) 2 *if $\gamma^{[3]}$ is semisimple*,

(b) 1 *if $\gamma^{[3]}$ is not semisimple*, *but $\gamma^{[9]}$ is semisimple*,

(c) 0 *if neither $\gamma^{[3]}$ nor $\gamma^{[9]}$ is semisimple.*

Proof. The Weil pairing shows that $\mathrm{J}(C)[3]^\vee \cong \mathrm{J}(C)[3]$. In particular, the 3-rank r appears in the reduced quotient $(\mathbb{Z}/3)^r$ of $\mathrm{J}(C)[3]$ and hence appears in the largest diagonalizable subgroup $\boldsymbol{\mu}_3^r \subset \mathrm{J}(C)[3]$. So the 3-rank of the curve C is the dimension of the largest semisimple subalgebra of the Lie algebra of $\mathrm{J}(C)[3]$. □

Remark 6.4. We can write our curve C in Weierstrass normal form

$$y^2 = x^5 + c_{12}x^3 + c_{18}x^2 + c_{24}x + c_{30}.$$

According to [Elkin and Pries 2007, Lemma 2.2], the 3-rank of C is

$$\begin{cases} 2 & \text{if } c_{24} \neq 0, \\ 1 & \text{if } c_{24} = 0,\ c_{18} \neq 0, \\ 0 & \text{if } c_{24} = c_{18} = 0. \end{cases}$$

Furthermore, by Lemma 6.2, we know that $\gamma^{[27]}$ is a linear combination of $\gamma^{[3]}$ and $\gamma^{[9]}$; in Weierstrass normal form, a computer calculation shows that

$$\gamma^{[27]} = c_{24}\gamma^{[3]} - c_{18}\gamma^{[9]}.$$

6C. ***Model for 3-torsion.*** Let $\gamma \in \bigwedge^3 V_9$ be a stable vector. By Proposition 4.5, the stabilizer of $[\gamma] \in \mathbb{P}\big(\bigwedge^3 V_9\big)$ in $\mathrm{SL}(V_9)/\boldsymbol{\mu}_3$ is isomorphic to $\mathrm{J}(C)[3] \rtimes \mathrm{Aut}(C, P)$ where (C, P) is the marked curve associated to γ, and there is also an associated torsor of $\mathrm{J}(C)[3]$. Here is a more direct construction for this torsor.

The split Lie algebra of type E_8 has a graded direct sum decomposition

$$\mathfrak{sl}(V_9) \oplus \textstyle\bigwedge^3 V_9 \oplus \bigwedge^6 V_9.$$

Pick a flag of subspaces $F_1 \subset F_3 \subset F_6 \subset F_8 \subset V_9$ (the subscripts indicate the dimension of the subspace). Via the embedding $\mathrm{Flag}(1, 8; V_9) \subset \mathbb{P}(\mathfrak{sl}(V_9))$, the subspaces $F_1 \subset F_8$ determine (up to scalar multiple) an element $v_0 \in \mathfrak{sl}(V_9)$, F_3 determines an element $v_1 \in \bigwedge^3 V_9$, and F_6 determines an element $v_2 \in \bigwedge^6 V_9$. We say that $F_\bullet$ is *compatible with* γ if

(a) $[v_0, \gamma] \in \bigwedge^3 V_9$ is a scalar multiple of v_1,

(b) $[v_1, \gamma] \in \bigwedge^6 V_9$ is a scalar multiple of v_2, and

(c) $[v_2, \gamma] \in \mathfrak{sl}(V_9)$ contains F_6 in its kernel and its image is contained in F_1.

The conditions above are algebraic, so determine a subscheme $F(\gamma)$ of compatible flags. To be precise, let $F_\bullet$ be the standard flag defined by $F_i = \langle e_1, \dots, e_i \rangle$. If it is compatible with γ, then it implies that the coefficient of $e_i \wedge e_j \wedge e_k$ vanishes where ijk is

$$\begin{aligned} &ij9, && 4 \le i < j \le 8; \\ &ij9, && i = 2, 3; \quad 4 \le j \le 8; \\ &i78, && 2 \le i \le 6; \\ &ij7, ij8, && 4 \le i < j \le 6. \end{aligned} \tag{6.5}$$

Let P be the stabilizer in $\mathrm{GL}(V_9)$ of the standard flag $F_1 \subset F_3 \subset F_6 \subset F_8$. The span of the monomials which are not listed above forms a P-submodule of $\bigwedge^3 V_9$, and via algebraic induction from P to $\mathrm{GL}(V_9)$, we get a subbundle $\xi \subset \bigwedge^3 V_9 \times \mathrm{Flag}(1, 3, 6, 8; V_9)$. Hence, γ is a section of the quotient bundle η, which is of rank 31, and $F(\gamma)$ is the zero locus of this section, and this can be used to define it as a scheme.

Define a $\mathrm{GL}(V_9)$-equivariant map $\pi\colon \mathrm{Flag}(1, 3, 6, 8; V_9) \to \mathbb{P}(V_9^*)$ by sending $F_\bullet$ to the annihilator of F_8 in V_9^*.

Lemma 6.6. *$\pi(F(\gamma))$ is the underlying set of the torsor for* $\mathrm{J}(C)[3]$ *constructed previously. In fact, π gives a bijection between the underlying sets.*

Proof. Fix a compatible flag $F_\bullet$. Pick nonzero $u \in F_1$ and pick nonzero $x \in V_9^*$ which annihilates F_8. The action of v_0 on γ can be obtained by first contracting γ by x and then multiplying by u. By assumption, the result is a pure trivector, say equal to $u \wedge \ell_1 \wedge \ell_2$ for $\ell_1, \ell_2 \in V_9$. Let $\Phi(x)$ be the contraction of γ by x. Then $\Phi(x) = \ell_1 \wedge \ell_2 + \ell_3 \wedge u$ for some $\ell_3 \in V_9$ and so $x \in X(\gamma)$. So $\ker \Phi(x) \cap X$ gives a divisor D on X by [Gruson and Sam 2015, Theorem 3.6], and we want to show that $3D$ is the divisor corresponding to $\mathcal{O}_X(1)$.

To do this, it suffices to show that there is a hyperplane $H \subset \mathbb{P}(V_9^*)$ such that $H \cap X = \ker \Phi(x) \cap X$ as sets. We claim that this works if H is the zero locus of u. It follows from the definition that $\ker \Phi(x) \cap X \subseteq H \cap X$. The correctness of this statement is unaffected if we do a change of basis and if we pass to an algebraic closure of $\boldsymbol{k}$. So we do both and assume that $F_\bullet$ is the standard flag. This implies that $\Phi(x) = e_1 \wedge \ell + e_2 \wedge e_3$ where ℓ is in the span of $e_2, \dots, e_8$ but is not contained in the span of e_2 and e_3. In particular, doing a further change of basis using the stabilizer of $F_\bullet$, we may assume that $\ell = e_6$ or $\ell = e_8$. In both cases we can verify, for generic γ, using a computer algebra system, that if $y \in H \cap X$, then $y \in \ker \Phi(x)$. The general case follows because $H \cap X$ contains C so cannot possibly degenerate any further unless it increases in dimension (but X is not contained in a hyperplane).

For the last statement, let x be a $\boldsymbol{k}^{\mathrm{sep}}$-point in the image of π. From the proof above, we see that x determines the subspace F_8 in the flag. Also, there is a hyperplane $H \subset \mathbb{P}(V_9^*)$ such that $H \cap X = \ker \Phi(x) \cap X$ as sets. Since X is not contained in a hyperplane, H is unique with this property, and it determines F_1. If $F_\bullet$ is a compatible flag, then F_3 is determined by $F_1 \subset F_8$ and F_6 is determined by F_3, so we are done. □

Theorem 6.7. *$F(\gamma)$ is a degree 81 scheme of dimension 0. In particular, π restricts to a* $\mathrm{J}(C)[3]$*-equivariant isomorphism between $F(\gamma)$ and the torsor for* $\mathrm{J}(C)[3]$*, so $F(\gamma)$ is reduced outside of characteristic 3.*

Proof. Note that $\dim \mathrm{Flag}(1, 3, 6, 8; V_9) = 31 = \mathrm{rank}(\eta)$, and Lemma 6.6 shows that $F(\gamma)$ is 0-dimensional whenever γ is stable. Hence the degree of $F(\gamma)$ can be calculated as the top Chern class of η, which can be shown to be 81 as follows. The Borel presentation for the (rational) Chow ring of $\mathrm{Flag}(1, 3, 6, 8; V_9)$ describes it as the subring of $S_1 \times S_2 \times S_3 \times S_2 \times S_1$-invariants inside the quotient ring $\mathbb{Q}[x_1, \dots, x_9]/I$ where I is generated by all positive degree homogeneous S_9-invariants (S_9 is the symmetric group on 9 letters, and acts by permuting the x_i; $S_1 \times S_2 \times S_3 \times S_2 \times S_1$ is the subgroup where the first S_2 permutes x_2, x_3, S_3 permutes x_4, x_5, x_6, and the second S_2 permutes x_7, x_8). Over the full flag variety of V_9, the bundle η is filtered by line bundles, one for each monomial in (6.5) and the (rational) Chow ring of the full flag variety is $\mathbb{Q}[x_1, \dots, x_9]/I$. In the Borel presentation, the Chern class of the line bundle corresponding to the monomial ijk is represented by $x_i + x_j + x_k$. So the top Chern class of η is the product of these linear forms, which is $81m$ modulo I, where m is a nonzero monomial of degree 31. Doing this in Macaulay2 (http://www.math.uiuc.edu/Macaulay2), we get $81 x_2 x_3 x_4^3 x_5^3 x_6^3 x_7^6 x_8^6 x_9^8$.

The $\mathrm{J}(C)[3]$-equivariance of π comes from the fact that $\mathrm{J}(C)[3]$ is a subgroup of the stabilizer of γ. □

In particular, in every $G(\boldsymbol{k}^{\mathrm{sep}})$-orbit of a point in $\bigwedge^3 V_9$, there is a distinguished $G(\boldsymbol{k})$-orbit corresponding to elements which have a compatible flag defined over $\boldsymbol{k}$.

Corollary 6.8. *In characteristics different from* 2 *and* 5, $\mathrm{SL}(V_9)/\boldsymbol{\mu}_3$ *acts freely on the scheme of pairs* $(\gamma, F_\bullet)$ *where* $\gamma \in \bigwedge^3 V_9$ *is stable and* $F_\bullet$ *is a compatible flag for* γ.

Remark 6.9. If we permute the basis via 974852631, then the family in Proposition 2.1 becomes

$$[348]-[357]+[267]-[189]+[456]+[239]-[147]-[258] \\ +c_3[345]-c_6[234]+c_9[127]-c_{12}[124]-c_{15}[356]-c_{18}[236]+c_{24}[126]-c_{30}[136],$$

and the standard coordinate flag is compatible with the entire family.

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Sums of two cubes as twisted perfect powers, revisited

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We sharpen earlier work (2011) of the first author, Luca and Mulholland, showing that the Diophantine equation

$$A^3 + B^3 = q^\alpha C^p, \quad ABC \neq 0, \quad \gcd(A, B) = 1,$$

has, for "most" primes q and suitably large prime exponents p, no solutions. We handle a number of (presumably infinite) families where no such conclusion was hitherto known. Through further application of certain *symplectic criteria*, we are able to make some conditional statements about still more values of q; a sample such result is that, for all but $O(\sqrt{x}/\log x)$ primes q up to x, the equation

$$A^3 + B^3 = qC^p.$$

has no solutions in coprime, nonzero integers A, B and C, for a positive proportion of prime exponents p.

1. Introduction

The problem of classifying perfect powers that are representable as a sum of two coprime integer cubes has a long history. The nonexistence of cubes $C^3 > 1$ with this property, a special case of Fermat's last theorem, was essentially proven by Euler. For higher powers, we have a substantial amount of recent work; at the time of writing, this can be summarized in the following theorem.

Theorem 1.1 [Bruin 2000; Chen and Siksek 2009; Dahmen 2008; Freitas 2016; Kraus 1998]. *There are no solutions in relatively prime nonzero integers A, B and C to the equation*

$$A^3 + B^3 = C^n \tag{1-1}$$

with exponent n satisfying one of $3 \le n \le 10^9$, $n \equiv 2 \pmod 3$, $n \equiv 2, 3 \pmod 5$, $n \equiv 61 \pmod{78}$, $n \equiv 51, 103, 105 \pmod{106}$, *or*

$$n \equiv 43, 49, 61, 79, 97, 151, 157, 169, 187, 205, 259, 265, 277, 295, 313, 367, 373, 385, 403, 421, 475, 481, 493, 511, 529, 583, 601, 619, 637, 691, 697, 709, 727, 745, 799, 805, 817, 835, 853, 907, 913, 925, 943, 961, 1015, 1021, 1033, 1051, 1069, 1123, 1129, 1141, 1159, 1177, 1231, 1237, 1249, 1267, 1285 \pmod{1296}.$$

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Underlying each of these results is an appeal to a particular Frey–Hellegouarch elliptic curve, defined over $\mathbb{Q}$. Just as in the case of Fermat's last theorem, with analogous equation $A^n + B^n = C^n$, this curve corresponds to a particular weight 2, cuspidal newform f. In the latter case, Wiles [1995] showed that f necessarily has level 2 (whereby the absence of such newforms implies an immediate contradiction). In the case of (1-1), however, one finds a corresponding f at one of levels 18, 36 or 72. The first two of these are readily handled, but the last is not. The obstruction to completely resolving (1-1) is the existence of a particular elliptic curve over $\mathbb{Q}$ with conductor 72 which, on some level, "mimics" a solution to (1-1) (the curve in question is labeled $72A1$ in Cremona's tables [1997]).

In an earlier paper [Bennett et al. 2011], the first author, jointly with Luca and Mulholland, considered a modification of (1-1), where the right-hand side is replaced by a "twisted" version of the shape $q^\alpha C^p$, for q prime (the replacement of the exponent n by a prime one, p, loses no generality). The question we wished to answer there was whether or not a similar obstruction exists in this new situation. Here and henceforth, let us assume that we have a solution in nonzero integers (A, B, C) to the equation

$$A^3 + B^3 = q^\alpha C^p, \tag{1-2}$$

where α is a positive integer. To avoid, trivialities, we will always without comment assume further that A, B and C are pairwise relatively prime. Write S for the set of primes $q \geq 5$ for which there exists an elliptic curve $E/\mathbb{Q}$ with conductor $N(E) \in \{18q, 36q, 72q\}$ and at least one nontrivial rational 2-torsion point. The two main results of Bennett, Luca and Mulholland [2011] are the following:

Theorem 1.2. *If p and $q \geq 5$ are primes with $p \geq q^{2q}$ such that there exist coprime, nonzero integers A, B and C, and a positive integer α, satisfying equation* (1-2), *then $q \in S$.*

Theorem 1.3. *Let $\pi_S(x) = \#\{q \leq x : q \in S\}$. Then*

$$\pi_S(x) \ll \sqrt{x}\log^2(x). \tag{1-3}$$

This latter result may be reasonably easily sharpened, through sieve methods, but, even as stated, demonstrates that $\pi_S(x) = o(\pi(x))$ and hence that we may "solve" (1-2) for "almost all" primes q (i.e., for almost all primes, there is no analogous obstruction to that provided by the curve $72A1$ for (1-1)).

Our goal in the paper at hand is to improve this result by treating (1-2) for a significant number of the primes in S. We begin by defining S_0 to be the subset of S consisting of those primes $q \geq 5$ for which there exist an elliptic curve $E/\mathbb{Q}$ with conductor $N(E) \in \{18q, 36q, 72q\}$, nontrivial rational 2-torsion and the additional property that discriminant $\Delta(E) = T^2$ or $\Delta(E) = -3T^2$ for some integer T. The first main result of this paper is the following sharpening of Theorem 1.2.

Theorem 1.4. *If p and $q \geq 5$ are primes with $p \geq q^{2q}$ such that there exist coprime, nonzero integers A, B and C, and a positive integer α, satisfying equation* (1-2), *then $q \in S_0$.*

It is by no means clear that the set S_0 is appreciably "smaller" than S. In fact, our expectation is that their counting functions satisfy

$$\pi_S(x) \sim c_1\sqrt{x}\log x \quad \text{and} \quad \pi_{S_0}(x) \sim c_2\sqrt{x}\log x,$$

for positive constants c_1 and c_2, where $c_2 < c_1$. A cursory check of Cremona's elliptic curve database [1997] reveals that the primes $5 \le q < 1000$ lying outside S are precisely

$$q = 197, 317, 439, 557, 653, 677, 701, 773, 797 \text{ and } 821,$$

while, in the same range, the primes in S but not S_0 are

$$q = 53, 83, 149, 167, 173, 199, 223, 227, 233, 263, 281, 293, 311, 347, 353, 359, 389, 401, 419, 443, 449, 461, 467, 479, 487, 491, 563, 569, 571, 587, 599, 617, 641, 643, 659, 719, 727, 739, 743, 751, 809, 811, 823, 827, 829, 839, 859, 881, 887, 907, 911, 929, 941, 947, 953, 977 \text{ and } 983.$$

It is, in fact, possible to give a much more concrete characterization of S_0. Let us define sets

$$\begin{aligned}
S_1 &= \{q \text{ prime} : q = 2^a 3^b \pm 1,\ a \in \{2,3\} \text{ or } a \ge 5,\ b \ge 0\},\\
S_2 &= \{q \text{ prime} : q = |2^a \pm 3^b|,\ a \in \{2,3\} \text{ or } a \ge 5,\ b \ge 1\},\\
S_3 &= \{q \text{ prime} : q = \tfrac{1}{3}(2^a + 1),\ a \ge 5 \text{ odd}\},\\
S_4 &= \{q \text{ prime} : q = d^2 + 2^a 3^b,\ a \in \{2,4\} \text{ or } a \ge 8 \text{ even},\ b \text{ odd}\},\\
S_5 &= \{q \text{ prime} : q = 3d^2 + 2^a,\ a \in \{2,4\} \text{ or } a \ge 8 \text{ even},\ d \text{ odd}\},\\
S_6 &= \{q \text{ prime} : q = \tfrac{1}{4}(d^2 + 3^b),\ d \text{ and } b \text{ odd}\},\\
S_7 &= \{q \text{ prime} : q = \tfrac{1}{4}(3d^2 + 1),\ d \text{ odd}\}, \quad \text{and}\\
S_8 &= \{q \text{ prime} : q = \tfrac{1}{2}(3v^2 - 1),\ u^2 - 3v^2 = -2\}.
\end{aligned}$$

Here, a, b, u, v and d are integers.

Proposition 1.5. *We have*

$$S_0 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8.$$

An advantage of this characterization is that it makes it a routine matter to check if a given prime is in S_0 (something that is far from being true for S). It also allows one to rather easily find, via local conditions, sets of primes outside S_0; simply checking that S_0 contains no primes which are simultaneously 5 mod 8, 2 mod 3 and 3 mod 5, yields that if $q \equiv 53 \bmod 120$, then $q \notin S_0$. More generally, from Theorem 1.4, we deduce the following.

Corollary 1.6. *If p and q are primes with either $q \equiv 53 \bmod D_1$ for $D_1 \in \{96, 120, 144\}$ or $q \equiv 65 \bmod D_2$ for $D_2 \in \{81, 84\}$, and $p \ge q^{2q}$, then there are no coprime, nonzero integers A, B and C, and positive integer α, satisfying equation* (1-2).

For primes in S_0, we are often still able to say something about solutions to (1-2), in many cases eliminating a positive proportion of the possible prime exponents p. Indeed, let us define

$$T = S_7 \cup \{q \text{ prime} : q = 3d^2 + 16,\, d \in \mathbb{Z}\},$$

and, to simplify matters, suppose that $\alpha = 1$ in (1-2), focusing our attention on the equation

$$A^3 + B^3 = qC^p. \tag{1-4}$$

We have the following.

Theorem 1.7. *If q is a prime with $q \notin T$, then, for a positive proportion of primes p, there are no solutions to* (1-4) *in coprime nonzero integers A, B and C.*

We note that, defining $\pi_T(x)$ to be the counting function for primes in T, it is not difficult to show that

$$\pi_T(x) \ll \frac{\sqrt{x}}{\log x},$$

whereby standard heuristics suggest that the set T is genuinely of smaller order than S_0 (though, in point of fact, it would be remarkably difficult to prove that either set is even infinite).

As a sampling of more explicit work along these lines, we mention the following results for certain primes in S_0 (see also Theorem 7.2).

Theorem 1.8. *Suppose that $q = 2^a 3^b - 1$ is prime, where $a \geq 5$ and $b \geq 1$ integers. If $p > q^{2q}$ is prime and there exist a positive integer α and coprime, nonzero integers A, B and C satisfying equation* (1-2), *then*

$$\left(\frac{\alpha}{p}\right) = \left(\frac{4-a}{p}\right) = \left(\frac{-6b}{p}\right).$$

Theorem 1.9. *If p is prime with $p \equiv 13$, 19 or $23 \bmod 24$, then there are no coprime, nonzero integers A, B and C satisfying*

$$A^3 + B^3 = 5C^p. \tag{1-5}$$

These results all follow from applying the modular method, together with a somewhat elaborate blend of techniques from algebraic and analytic number theory, and Diophantine approximation, with a variety of *symplectic criteria* (see Section 6) to (1-2). This last approach was developed initially by Halberstadt and Kraus [2002] and has recently been refined and generalized by the third author together with Naskręcki, Stoll, and Kraus [Freitas 2016; Freitas et al. 2017; Freitas and Kraus 2016]. One of the justifications for the current paper is to provide a number of examples which, on some level, utilize the full power of these recently developed symplectic tools.

As a final comment, we note that it should be possible to apply techniques based upon quadratic reciprocity, as in, say, [Chen and Siksek 2009], to say something further about (1-2) for certain primes q and certain exponents. We will not undertake this here.

The outline of this paper is as follows. In Section 2, we restate a number of results from [Bennett et al. 2011] pertaining to Frey–Hellegouarch curves that we require in the sequel. In Section 3, we characterize isomorphism classes of elliptic curves over $\mathbb{Q}$ with nontrivial rational 2-torsion and conductor $18q$, $36q$ or $72q$, for q prime. Section 4 contains the proof of Theorem 1.4. In Section 5, we make a number of remarks about the sets S_i comprising S_0. In Section 6, we apply several symplectic criteria to the Frey–Hellegouarch curve and the elliptic curves corresponding to the primes in S_0. In Section 7, we prove Theorems 1.7, 1.8 and 1.9 (and somewhat more besides). The tables in the Appendix contains information on the invariants $c_4(E)$ and $c_6(E)$ for elliptic curves of conductor $18q$, $36q$ and $72q$, corresponding to the primes in S.

2. Frey–Hellegouarch curves

Let us suppose that $q \geq 5$ is prime, α is a positive integer, and that we have a solution to (1-2) in coprime nonzero integers A, B and C where, without loss of generality, AC is even and $B \equiv (-1)^{C+1} \bmod 4$. Following Darmon and Granville [1995, p. 530], we associate to such a solution a *Frey–Hellegouarch elliptic curve* $F = F^{(i)}_{A,B}$ given by

$$F^{(0)}_{A,B} : y^2 + xy = x^3 + \frac{3(B-A)+2}{8}x^2 + \frac{3(A+B)^2}{64}x + \frac{9(B-A)(A+B)^2}{512}$$

or

$$F^{(1)}_{A,B} : y^2 = x^3 + 3ABx + B^3 - A^3,$$

depending on whether C is even or odd, respectively (the first of these is just a minimal model of the curve given by Darmon and Granville; indeed both $F^{(0)}_{A,B}$ and $F^{(1)}_{A,B}$ are minimal). The standard invariants $c_4(F)$, $c_6(F)$ and $\Delta(F)$ attached to $F = F^{(i)}_{A,B}$ are

$$c_4(F) = -2^{4i}3^2AB, \quad c_6(F) = 2^{6i-1}3^3(A^3 - B^3), \quad \Delta(F) = -2^{12i-8}3^3q^{2\alpha}C^{2p}. \tag{2-1}$$

Let $\mathcal{R}$ denote the product of the primes ℓ satisfying $\ell \mid C$ and $\ell \nmid 6q$. A standard application of Tate's algorithm leads to the following.

Lemma 2.1. *If $F = F^{(i)}_{A,B}$, then the conductor N_F satisfies*

$$N_F = \begin{cases} 18q\mathcal{R} & \text{if } C \text{ even, } B \equiv -1 \pmod 4, \text{ or} \\ 36q\mathcal{R} & \text{if } C \text{ odd, } v_2(A) \geq 2 \text{ and } B \equiv 1 \pmod 4, \text{ or} \\ 72q\mathcal{R} & \text{if } C \text{ odd, } v_2(A) = 1 \text{ and } B \equiv 1 \pmod 4. \end{cases}$$

In particular, F has multiplicative reduction at the prime q.

Arguing as in [Bennett et al. 2011] and [Kraus 1998] we find that, for $p \geq 17$, there necessarily exists a newform f in $S_2^+(N_F/\mathcal{R})$ (the space of weight 2 cuspidal newforms for the congruence subgroup $\Gamma_0(N_F/\mathcal{R})$), whose Taylor expansion is

$$f = q + \sum_{m \geq 2} a_m(f)q^m,$$

and a place $\mathfrak{p}$ of $\overline{\mathbb{Q}}$ lying above p, such that

$$\bar{\rho}_{F,p} \sim \bar{\rho}_{f,\mathfrak{p}}, \tag{2-2}$$

where $\bar{\rho}_{F,p}$ and $\bar{\rho}_{f,\mathfrak{p}}$ denote, respectively, the mod $\mathfrak{p}$ Galois representations attached to F and f. In particular, for all prime numbers $\ell \nmid pN_F$, we have

$$a_\ell(f) \equiv a_\ell(F) \pmod{\mathfrak{p}},$$

where $a_\ell(F)$ denotes the trace of Frobenius of F at the prime ℓ. Therefore,

$$p \mid \mathrm{Norm}_{K_f/\mathbb{Q}}(a_\ell(f) - a_\ell(F)), \tag{2-3}$$

for K_f the field of definition of the coefficients of f. Furthermore, the level lowering condition implies

$$p \mid \mathrm{Norm}_{K_f/\mathbb{Q}}(a_\ell(f) \pm (\ell + 1)), \tag{2-4}$$

for each prime $\ell \neq p$ dividing $\mathcal{R}$.

From the arguments of [Bennett et al. 2011], under the assumption that $p > q^{2p}$, we may conclude that the form f has rational integer Fourier coefficients $a_m(f)$ for all $m \geq 1$, whereby f corresponds to an isogeny class of elliptic curves over $\mathbb{Q}$ with conductor $N = 18q$, $36q$ or $72q$, and further that the corresponding elliptic curve E has a rational 2-torsion point. This, in essence, is Theorem 1.2. To complete the proof of Theorem 1.4, it remains to eliminate the possibility of the Frey–Hellegouarch curve F "arising mod p" from an elliptic curve E that fails to be isogenous to a curve with discriminant of the shape T^2 or $-3T^2$. To do this, we first require a very precise characterization of elliptic curves of conductor $N = 18q$, $36q$ or $72q$, with nontrivial rational 2-torsion.

3. Classification results for primes of conductor $18q$, $36q$ and $72q$

In this section, we will state theorems that provide an explicit classification for primes q of the corresponding isomorphism classes of elliptic curves $E/\mathbb{Q}$ with conductor $18q$, $36q$ or $72q$ and nontrivial rational 2-torsion. The following results are mild sharpenings and simplifications of special cases of Theorems 3.13, 3.14 and 3.15 of Mulholland [2006] (see also Theorems 4.0.8, 4.0.10 and 4.0.12 of [Bruni 2015]), where analogous results are derived more generally for elliptic curves with nontrivial rational 2-torsion and conductor of the shape $2^\alpha 3^\beta q^\gamma$. In each case, all elliptic curves which we label as, say, $18q.i.\alpha$ for a positive integer i and a letter α belong to a fixed isogeny class (similarly for $36q.i.\alpha$ and $72q.i.\alpha$). By way of example, each of $18q.1.a1$, $18q.1.a2$, $18q.1.a3$ and $18q.1.a4$ are isogenous.

In the next statement we use the notation from [Cremona 2006].

Theorem 3.1. *If $q > 3$ is prime, then there exists an elliptic curve $E/\mathbb{Q}$ of conductor $18q$ with at least one rational 2-torsion point precisely when either E is isogenous to one of*

$90a$, $90b$, $90c$, $126a$, $126b$, $198b$, $198c$, $198d$, $198e$, $306a$, $306b$, $306c$, $342c$, $342f$, $414a$, $1314a$ *or* $1314f$,

or E is $\mathbb{Q}$-isomorphic to

$$\tilde{E} : y^2 + xy = x^3 + a_2x^2 + a_4x + a_6$$

and at least one of the following occurs:

(1) *There exist integers* $a \geq 5$ *and* $b \geq 0$ *such that*

$$q = 2^a 3^b + (-1)^\delta, \quad \textit{for } \delta \in \{0, 1\},$$

and $\tilde{E}$ *is one of the following*:

curve	a_2	a_4	a_6	Δ
$18q.1.a1$	$(-1)^{\delta+1}2^{a-1}3^{b+1}-1$	$2^{a-4}3^{b+2}q$	0	$2^{2a-8}3^{2b+6}q^2$
$18q.1.a2$	$(-1)^{\delta+1}2^{a-1}3^{b+1}-1$	$-2^{a-2}3^{b+2}q$	$(-1)^\delta 2^{a-4}3^{b+3}(2q-(-1)^\delta)q$	$2^{a-4}3^{b+6}q$
$18q.1.a3$	$(-1)^{\delta+1}2^{a-3}3^{b+1}-1$	$2^{2a-8}3^{2b+2}$	0	$(-1)^\delta 2^{4a-16}3^{4b+6}q$
$18q.1.a4$	$(-1)^{\delta}2^{a-2}3^{b+1}-1$	$(-1)^\delta 2^{a-2}3^{b+2}$	$2^{a-4}3^{b+3}(q-2(-1)^\delta)$	$(-1)^{\delta+1}2^{a-4}3^{b+6}q^4$

(2) *There exists an odd integer* $a \geq 5$ *such that*

$$q = \tfrac{1}{3}(2^a + 1)$$

and $\tilde{E}$ *is one of the following*:

curve	a_2	a_4	a_6	Δ
$18q.2.a1$	$-3 \cdot 2^{a-1} - 1$	$2^{a-4}3^3q$	0	$2^{2a-8}3^8q^2$
$18q.2.a2$	$-3 \cdot 2^{a-1} - 1$	$-2^{a-2}3^3q$	$2^{a-4}3^4(2^{a+1}+1)q$	$2^{a-4}3^7q$
$18q.2.a3$	$-3 \cdot 2^{a-3} - 1$	$2^{2a-8}3^2$	0	$2^{4a-16}3^7q$
$18q.2.a4$	$3 \cdot 2^{a-2} - 1$	$2^{a-2}3^2$	$2^{a-4}3^3(2^a-1)$	$-2^{a-4}3^{10}q^4$

(3) *There exist integers* $a \geq 5$ *and* $b \geq 1$, *and* $\delta_1, \delta_2 \in \{0, 1\}$ *such that*

$$q = (-1)^{\delta_1}2^a + (-1)^{\delta_2}3^b$$

and, writing $\delta = b + \delta_1 + \delta_2 + 1$, $\tilde{E}$ *is one of the following*:

curve	a_2	a_4
$18q.3.a1$	$-\frac{1}{4} + (-1)^\delta\big(3 \cdot 2^{a-2} + (-1)^{\delta_1}\frac{3q}{4}\big)$	$(-1)^{\delta_1}2^{a-4}3^2q$
$18q.3.a2$	$-\frac{1}{4} + (-1)^\delta\big(3 \cdot 2^{a-2} + (-1)^{\delta_1}\frac{3q}{4}\big)$	$(-1)^{\delta_1+1}2^{a-2}3^2q$
$18q.3.a3$	$-\frac{1}{4} + (-1)^\delta 3 \cdot 2^{a-3} - (-1)^b\frac{3^{b+1}}{4}$	$2^{2a-8}3^2$
$18q.3.a4$	$-\frac{1}{4} + (-1)^{\delta+1}\big(3 \cdot 2^{a-1} + (-1)^{\delta_1+1}\frac{3q}{4}\big)$	$(-1)^{\delta_1+\delta_2}2^{a-2}3^{b+2}$

curve	a_6	Δ
$18q.3.a1$	0	$2^{2a-8}3^{2b+6}q^2$
$18q.3.a2$	$(-1)^{\delta+\delta_1+1}3^3 \cdot 2^{a-4}(2^{a+1}+(-1)^{\delta_1+\delta_2}3^b)q$	$2^{a-4}3^{4b+6}q$
$18q.3.a3$	0	$(-1)^{\delta_2}2^{4a-16}3^{b+6}q$
$18q.3.a4$	$(-1)^\delta 3^{b+3} \cdot 2^{a-4}(3^b + (-1)^{1+\delta_1+\delta_2}2^a)$	$(-1)^{b+\delta}2^{a-4}3^{b+6}q^4$

(4) *There exist integers* $a \geq 7$, $b \geq 0$, $\delta_1, \delta_2 \in \{0, 1\}$ *and* $d \equiv 1 \pmod 4$, *such that* $(\delta_1, \delta_2) \neq (1, 1)$ *and, if we have* $a \equiv b \equiv 0 \pmod 2$, *then* $(\delta_1, \delta_2) = (0, 0)$, *with*

$$q = (-1)^{\delta_1}d^2 + (-1)^{\delta_2}2^a 3^b,$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	a_6	Δ
$18q.4.a1$	$-\left(\frac{3d+1}{4}\right)$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^{b+2}$	0	$(-1)^{\delta_1}2^{2a-12}3^{2b+6}q$
$18q.4.a2$	$-\left(\frac{3d+1}{4}\right)$	$(-1)^{\delta_1+\delta_2}2^{a-4}3^{b+2}$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^{b+3}d$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^{b+6}q^2$

(5) *There exist integers $a \geq 7$, $\delta_1, \delta_2 \in \{0, 1\}$ and $d \equiv 1 \pmod 4$, such that $(\delta_1, \delta_2) \neq (1, 1)$,*

$$q = (-1)^{\delta_1}3d^2 + (-1)^{\delta_2}2^a,$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	a_6	Δ
$18q.5.a1$	$-\left(\frac{3d+1}{4}\right)$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3$	0	$(-1)^{\delta_1}2^{2a-12}3^3q$
$18q.5.a2$	$-\left(\frac{3d+1}{4}\right)$	$(-1)^{\delta_1+\delta_2}2^{a-4}3$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^2d$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^3q^2$
$18q.5.b1$	$\frac{9d-1}{4}$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^3$	0	$(-1)^{\delta_1}2^{2a-12}3^9q$
$18q.5.b2$	$\frac{9d-1}{4}$	$(-1)^{\delta_1+\delta_2}2^{a-4}3^3$	$(-1)^{\delta_1+\delta_2}2^{a-6}3^5d$	$(-1)^{\delta_1+\delta_2+1}2^{a-6}3^9q^2$

(6) *There exist integers $a \geq 7$, $b \geq 1$ and $d \equiv 1 \pmod 4$, with a odd, such that*

$$q = \frac{d^2 + 2^a}{3^b},$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	a_6	Δ
$18q.6.a1$	$-\left(\frac{3d+1}{4}\right)$	$-2^{a-6}3^2$	0	$2^{2a-12}3^{b+6}q$
$18q.6.a2$	$-\left(\frac{3d+1}{4}\right)$	$2^{a-4}3^2$	$-2^{a-6}3^3d$	$-2^{a-6}3^{2b+6}q^2$

Theorem 3.2. *If $q > 3$ is prime, then there exists an elliptic curve $E/\mathbb{Q}$ of conductor $36q$ with at least one rational 2-torsion point precisely when either E is isogenous to one of (in Cremona's notation)*

$$180a, \quad 252a \quad \text{or} \quad 468d,$$

or E is $\mathbb{Q}$-isomorphic to

$$\tilde{E} : y^2 = x^3 + a_2x^2 + a_4x$$

and at least one of the following occurs:

(1) *There exist integers u and v with $u \equiv v \equiv 1 \pmod 4$ and $u^2 - 3v^2 = -2$, such that*

$$q = \tfrac{1}{2}(3v^2 - 1)$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.1.a1$	$-3uv$	$3q^2$	$-2^4 3^3 q^4$
$36q.1.a2$	$6uv$	-3	$2^8 3^3 q^2$
$36q.1.b1$	$9uv$	$3^3 q^2$	$-2^4 3^9 q^4$
$36q.1.b2$	$-18uv$	-3^3	$2^8 3^9 q^2$

(2) *There exists an integer* $d \equiv 1 \bmod 8$, *such that*

$$q = \tfrac{1}{4}(3d^2 + 1)$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.2.a1$	$-3d$	$3q$	$-2^4 3^3 q^2$
$36q.2.a2$	$6d$	-3	$2^8 3^3 q$
$36q.2.b1$	$9d$	$3^3 q$	$-2^4 3^9 q^2$
$36q.2.b2$	$-18d$	-3^3	$2^8 3^9 q$

(3) *There exists an odd integer $b \geq 1$ and an integer $d \equiv 1 \pmod 4$ such that*

$$q = \tfrac{1}{4}(d^2 + 3^b) \equiv 3 \pmod 4$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.3.a1$	$-3d$	$3^2 q$	$-2^4 3^{b+6} q^2$
$36q.3.a2$	$6d$	-3^{b+2}	$2^8 3^{2b+6} q$

(4) *There exist integers $b \geq 1$, $\delta \in \{0, 1\}$ and $d \equiv 1 \pmod 4$, such that b is odd*, $d \equiv 1 \bmod 4$,

$$q = (-1)^\delta (d^2 - 4 \cdot 3^b)$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.4.a1$	$-3d$	3^{b+2}	$(-1)^\delta 2^4 3^{2b+6} q$
$36q.4.a2$	$6d$	$(-1)^\delta 3^2 q$	$2^8 3^{b+6} q^2$

(5) *There exist integers $b \geq 1$, $\delta \in \{0, 1\}$, $n \geq 7$ and $d \equiv 1 \pmod 4$, such that b is odd, every prime factor of n is at least* 7,

$$q^n = (-1)^\delta (d^2 - 4 \cdot 3^b)$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.5.a1$	$-3d$	3^{b+2}	$(-1)^{\delta}2^4 3^{2b+6} q^n$
$36q.5.a2$	$6d$	$(-1)^{\delta}3^2 q^n$	$2^8 3^{b+6} q^{2n}$

(6) *There exists an integer* $d \equiv 1 \mod 4$, *such that*

$$q = 3d^2 - 4$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.6.a1$	$-3d$	3	$2^4 3^3 q$
$36q.6.a2$	$6d$	$3q$	$2^8 3^3 q^2$
$36q.6.b1$	$9d$	3^3	$2^4 3^9 q$
$36q.6.b2$	$-18d$	$3^3 q$	$2^8 3^9 q^2$

(7) *There exists an integer* $d \equiv 1 \mod 4$, *and an even integer* $b \geq 0$ *such that*

$$q = d^2 + 4 \cdot 3^b,$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$36q.7.a1$	$-3d$	-3^{b+2}	$2^4 3^{2b+6} q$
$36q.7.a2$	$6d$	$3^2 q$	$-2^8 3^{b+6} q^2$

Theorem 3.3. *If $q > 3$ is prime, then there exists an elliptic curve $E/\mathbb{Q}$ of conductor $72q$ with at least one rational 2-torsion point precisely when either E is isogenous to one of* (*in Cremona's notation*)

$$360a, 360b, 360c, 360d, 936a, 936d, 936f, 2088b, 2088h, 3384a, 5256e, 13896f \text{ or } 83016c,$$

or E is $\mathbb{Q}$-isomorphic to

$$\tilde{E} : y^2 = x^3 + a_2 x^2 + a_4 x$$

and at least one of the following occurs:

(1) *There exists an odd integer* $b \geq 1$ *such that*

$$q = \tfrac{1}{4}(3^b + 1)$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.1.a1$	$24q-3$	$2^2 3^{b+2} q$	$2^8 3^{2b+6} q^2$
$72q.1.a2$	$-48q+6$	3^2	$2^{10} 3^{b+6} q$
$72q.1.a3$	$24q+6$	3^{2b+2}	$2^{10} 3^{4b+6} q$
$72q.1.a4$	$6q-3$	$3^2 q^2$	$-2^4 3^{b+6} q^4$

(2) *There exist integers $a \in \{2, 3\}$, $b \geq 0$ and $\delta \in \{0, 1\}$ such that*

$$q = 2^a \cdot 3^b + (-1)^\delta$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.2.a1$	$(-1)^{\delta+1} 2^{a+1} 3^{b+1} - 3$	$2^a 3^{b+2} q$	$2^{2a+4} 3^{2b+6} q^2$
$72q.2.a2$	$(-1)^{\delta} 2^{a+2} 3^{b+1} + 6$	3^2	$2^{a+8} 3^{b+6} q$
$72q.2.a3$	$(-1)^{\delta+1} 2^{a-1} 3^{b+1} - 3$	$2^{2a-4} 3^{2b+2}$	$(-1)^{\delta} 2^{4a-4} 3^{4b+6} q$
$72q.2.a4$	$(-1)^{\delta+1} 2^{a+1} 3^{b+1} + 6$	$3^2 q^2$	$(-1)^{\delta+1} 2^{a+8} 3^{b+6} q^4$

(3) *There exist integers $a \in \{2, 3\}$, $b \geq 0$ and $\delta \in \{0, 1\}$ such that*

$$q = 3^b + (-1)^\delta 2^a$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.3.a1$	$(-1)^{b+1} 3(3^b - (-1)^\delta 2^a)$	$(-1)^{\delta+1} 2^a 3^{b+2}$	$2^{2a+4} 3^{2b+6} q^2$
$72q.3.a2$	$(-1)^b 6(3^b - (-1)^\delta 2^a)$	$3^2 q^2$	$(-1)^{\delta+1} 2^{a+8} 3^{b+6} q^4$
$72q.3.a3$	$(-1)^b 6(3^b + (-1)^\delta 2^{a+1})$	3^{2b+2}	$(-1)^{\delta} 2^{a+8} 3^{4b+6} q$
$72q.3.a4$	$(-1)^{b+1} 3(3^b + (-1)^\delta 2^{a-1})$	$2^{2a-4} 3^2$	$2^{4a-4} 3^{b+6} q$

(4) *There exists an integer $d \equiv 1 \pmod 4$ such that*

$$q = 3d^2 + 4$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.4.a1$	$3d$	-3	$2^4 3^3 q$
$72q.4.a2$	$-6d$	$3q$	$-2^8 3^3 q^2$
$72q.4.b1$	$-9d$	-3^3	$2^4 3^9 q$
$72q.4.b2$	$18d$	$3^3 q$	$-2^8 3^9 q^2$

(5) *There exist integers $a \in \{4, 5\}$, $\delta \in \{0, 1\}$ and $d \equiv 1 \pmod 4$, such that*

$$q = 3d^2 + (-1)^\delta 2^a$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.5.a1$	$-3d$	$(-1)^{\delta+1}2^{a-2}3$	$2^{2a}3^3q$
$72q.5.a2$	$6d$	$3q$	$(-1)^{\delta+1}2^{a+6}3^3q^2$
$72q.5.b1$	$9d$	$(-1)^{\delta+1}2^{a-2}3^3$	$2^{2a}3^9q$
$72q.5.b2$	$-18d$	3^3q	$(-1)^{\delta+1}2^{a+6}3^9q^2$

(6) *There exists an integer* $d \equiv 5 \mod 8$ *such that*

$$q = \frac{3d^2+1}{4}$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.6.a1$	$3d$	$3q$	$-2^43^3q^2$
$72q.6.a2$	$-6d$	-3	2^83^3q
$72q.6.b1$	$-9d$	3^3q	$-2^43^9q^2$
$72q.6.b2$	$18d$	-3^3	2^83^9q

(7) *There exist odd integers* $b \geq 1$ *and* $d \equiv 1 \mod 4$ *such that*

$$q = \frac{d^2+3^b}{4} \equiv 1 \mod 4$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.7.a1$	$3d$	3^2q	$-2^43^{b+6}q^2$
$72q.7.a2$	$-6d$	-3^{b+2}	$2^83^{2b+6}q$

(8) *There exist odd integers* $b \geq 1$ *and* $d \equiv 1 \pmod 4$ *such that*

$$q = d^2 + 4 \cdot 3^b$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.8.a1$	$3d$	-3^{b+2}	$2^43^{2b+6}q$
$72q.8.a2$	$-6d$	3^2q	$-2^83^{b+6}q^2$

(9) *There exist integers* $a \in \{4, 5\}$, $b \geq 0$, $\delta_1, \delta_2 \in \{0, 1\}$ *and* $d \equiv 1 \pmod 4$, *such that* $(\delta_1, \delta_2) \neq (1, 1)$, b *is odd if* $a = 4$ *and* $\delta_1 \neq \delta_2$,

$$q = (-1)^{\delta_1}d^2 + (-1)^{\delta_2}2^a3^b,$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.9.a1$	$-3d$	$(-1)^{\delta_1+\delta_2+1}2^{a-2}3^{b+2}$	$(-1)^{\delta_1}2^{2a}3^{2b+6}q$
$72q.9.a2$	$6d$	$(-1)^{\delta_1}3^2q$	$(-1)^{\delta_1+\delta_2+1}2^{a+6}3^{b+6}q^2$

(10) *There exist integers $a \in \{4, 5\}$, $b \geq 0$, $\delta \in \{0, 1\}$, $d \equiv 1 \bmod 4$ and n, such that the least prime divisor of n is at least 7, b is odd if $a = 4$,*

$$q^n = (-1)^\delta(d^2 - 2^a 3^b),$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.10.a1$	$-3d$	$2^{a-2}3^{b+2}$	$(-1)^\delta 2^{2a}3^{2b+6}q^n$
$72q.10.a2$	$6d$	$(-1)^\delta 3^2 q^n$	$2^{a+6}3^{b+6}q^{2n}$

(11) *There exist integers $b \geq 0$ and $d \equiv 1 \bmod 4$ such that*

$$q = \frac{d^2 + 32}{3^b}$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.11.a1$	$-3d$	$-2^3 3^2$	$2^{10}3^{b+6}q$
$72q.11.a2$	$6d$	$3^{b+2}q$	$-2^{11}3^{2b+6}q^2$

(12) *There exist integers $b \geq 0$, $d \equiv 1 \bmod 4$ and n, such that the least prime divisor of n is at least 7,*

$$q^n = \frac{d^2 + 32}{3^b}$$

and $\tilde{E}$ is one of the following:

curve	a_2	a_4	Δ
$72q.12.a1$	$-3d$	$-2^3 3^2$	$2^{10}3^{b+6}q^n$
$72q.12.a2$	$6d$	$3^{b+2}q^n$	$-2^{11}3^{2b+6}q^{2n}$

We should mention that while we are currently unable to rule out the existence of primes in families (10) and (12) in Theorem 3.3, we strongly suspect that there are no such primes. Further, we must confess that our notation can admit a certain amount of ambiguity as, for a given prime q, we could have multiple representations of q giving rise to nonisogenous curves with the same labels. By way of example,

$$q = 10369 = 1^2 + 2^7 \cdot 3^4 = 65^2 + 2^{11} \cdot 3 \tag{3-1}$$

and the curves denoted $18q.4.a$ corresponding to these two representations are nonisogenous. For $q \in S_i$ for a fixed $1 \leq i \leq 8$, however, it is straightforward to show that there are at most finitely many such distinct representations — for all except $i = 2$, the parametrizations are monotonically increasing in the variables a, b, v and d. For $q \in S_2$, the same is easily seen to be true except, possibly, for the cases with $q = |2^a - 3^b|$. In this last situation, via a result of Tijdeman [1973], we have

$$|2^a - 3^b| \geq 3^b b^{-\kappa},$$

for some effectively computable absolute positive constant κ, at least provided $b > 2$, and hence, again, q has only finitely many such representations (at most 3, in fact, by a result of the first author [Bennett 2003]).

Combining Theorems 3.1, 3.2 and 3.3, together with the definition of S_0, yields the following.

Corollary 3.4. *An elliptic curve $E/\mathbb{Q}$ corresponds to a prime in S_0 precisely if E is either in one of the isogeny classes* (*in Cremona's notation*)

$$90c,\ 126b,\ 252a,\ 306c,\ 342f,\ 360a,\ 360d,\ 936d \textit{ or } 5256e,$$

or E is one of the curves in the isogeny classes

$18q.1.a$, $18q.2.a$, $18q.3.a$, $18q.4.a$ (*with $\delta_1 = \delta_2 = 0$, a even, b odd*),
$18q.5.a$ *and* $18q.5.b$ (*with, in both cases, $\delta_1 = \delta_2 = 0$ and a even*),

$36q.1.a$, $36q.1.b$, $36q.2.a$, $36q.2.b$, $36q.3.a$,

$72q.1.a$, $72q.2.a$, $72q.3.a$, $72q.4.a$, $72q.4.b$,
$72q.5.a$ *and* $72q.5.b$ (*with, in both cases, $\delta = 0$ and $a = 4$*),
$72q.6.a$, $72q.6.b$, $72q.7.a$, $72q.8.a$ *and* $72q.9.a$ (*with $\delta_1 = \delta_2 = 0$, $a = 4$, b odd*).

4. Finishing the proof of Theorem 1.4

From the classification results of the preceding section, we need to show only that, for suitably large primes p, (1-2) has no solutions in coprime nonzero integers, with Frey–Hellegouarch curve F corresponding (in the sense of Section 2) to an elliptic curve E in one of the isogeny classes

$$\begin{gathered} 90a,\ 90b,\ 126a,\ 180a,\ 198b,\ 198c,\ 198d,\ 198e,\ 306a,\ 306b,\ 342b,\ 342c,\ 360b,\ 360c, \\ 414a,\ 468d,\ 936a,\ 936f,\ 1314a,\ 1314f,\ 2088b,\ 2088h,\ 3384a,\ 13896f \text{ or } 83016c, \end{gathered} \tag{4-1}$$

or

$18q.4.a$ (with $\delta_1 \neq \delta_2$, or $\delta_1 = \delta_2 = 0$ and either a odd, or b even),
$18q.5.a$ and $18q.5.b$ (with, in both cases, $\delta_1 \neq \delta_2$, or $\delta_1 = \delta_2 = 0$ and a odd),
$18q.6.a$,

$36q.4.a$, $36q.5.a$, $36q.6.a$, $36q.6.b$, $36q.7.a$,

$72q.5.a$and $72q.5.b$ (with, in both cases, $\delta = 1$ or $a = 5$),
$72q.9.a$ (with $\delta_1 \neq \delta_2$, or $\delta_1 = \delta_2 = 0$ and either $a = 5$, or $a = 4$ and b even),
$72q.10.a$, $72q.11.a$ or $72q.12.a$. (4-2)

Our key observation to start is that, from (2-1), the Frey–Hellegouarch curve $F = F_{A,B}^{(i)}$ has minimal discriminant of the shape $-3T^2$ for $T = 2^{6i-4}3q^{\alpha}C^p$. It follows that $F(\mathbb{F}_\ell)$ contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}2\mathbb{Z}$ for every prime $\ell \nmid 6q$ for which $\left(\frac{-3}{\ell}\right) = 1$; i.e., for $\ell \equiv 1 \bmod 6$. We thus have that

$$a_\ell(F) \equiv \ell + 1 \bmod 4 \tag{4-3}$$

for every such prime ℓ. If, for each curve E in the isogeny classes (4-1) and (4-2), we are able to find a prime $\ell \equiv 1 \bmod 6$ with $\ell \nmid 6q$, for which $a_\ell(E) \not\equiv \ell + 1 \bmod 4$, it follows from (2-3), (2-4), (4-3) and the Hasse bounds that

$$p \leq \ell + 1 + 2\sqrt{\ell}. \tag{4-4}$$

For curves E in the isogeny classes (4-1), we may check that it suffices to choose, in all cases,

$$\ell \in \{7, 13, 19, 31, 37\}.$$

We will now show that we can always find a suitable prime ℓ for E in the isogeny classes (4-2). We prove

Lemma 4.1. *Let $E/\mathbb{Q}$ be an elliptic curve with a nontrivial rational 2-torsion point, say, $(0, 0)$, given by the model*

$$E : y^2 = f(x) = x^3 + ux^2 + vx, \tag{4-5}$$

where $u, v \in \mathbb{Z}$, and let $\ell \geq 5$ be a prime of good reduction for E. Then the Fourier coefficient $a_\ell(E)$ satisfies $a_\ell(E) \equiv \ell + 1 \bmod 4$ precisely when either

$$\left(\frac{\Delta(E)}{\ell}\right) = \left(\frac{u^2 - 4v}{\ell}\right) = 1 \quad \text{or} \quad \left(\frac{v}{\ell}\right) = 1.$$

Proof. If $\ell \geq 5$ is a prime of good reduction for E, it follows that the Fourier coefficient $a_\ell(E)$ satisfies $a_\ell(E) \equiv \ell + 1 \bmod 4$ exactly when $E(\mathbb{F}_\ell)$ contains a subgroup isomorphic to either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or to $\mathbb{Z}/4\mathbb{Z}$. The first case occurs if and only if the cubic $x^3 + ux^2 + vx$ splits completely modulo ℓ, i.e., when

$$\left(\frac{\Delta(E)}{\ell}\right) = \left(\frac{u^2 - 4v}{\ell}\right) = 1.$$

Indeed, if we have $u^2 - 4v \equiv t^2 \bmod \ell$, then necessarily $t \not\equiv \pm u \bmod \ell$ and $t \not\equiv 0 \bmod \ell$ (since otherwise $v \equiv 0 \bmod \ell$ or $\Delta(E) \equiv 0 \bmod \ell$, respectively, contradicting the fact that E has good reduction at ℓ) and

$$f(x) = x^3 + ux^2 + vx \equiv x(x - 2^{-1}(u - t))(x - 2^{-1}(u + t)) \bmod \ell,$$

whence $(0, 0)$, $(2^{-1}(u - t), 0)$ and $(2^{-1}(u + t), 0)$ are distinct 2-torsion points in $\mathbb{F}_\ell$.

Suppose next that $E(\mathbb{F}_\ell)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$, but that $(u^2 - 4v)/\ell = -1$. It follows that there must exist a point P in $E(\mathbb{F}_\ell)$ with the property that $P \neq (0, 0)$ but $2P = (0, 0)$. From the standard duplication formula, if $P = (x, y)$ lies on a curve E with a model as given in (4-5), the

x-coordinate of the point $2P$ on E is just

$$\frac{x^4 - 2vx^2 + v^2}{4y^2} = \frac{(x^2 - v)^2}{4y^2}$$

and hence there can exist a point P on E for which this coordinate is zero only when v is a square modulo ℓ. Conversely, if $v \equiv t^2 \bmod \ell$ for some integer t, then we claim that either $f(t)$ or $f(-t)$ is a square modulo ℓ. Indeed, if this fails to be the case, then we would have

$$1 = \left(\frac{f(t)}{\ell}\right)\left(\frac{f(-t)}{\ell}\right) = \left(\frac{u+2t}{\ell}\right)\left(\frac{u-2t}{\ell}\right) = \left(\frac{u^2-4v}{\ell}\right) = -1,$$

a contradiction. □

For the curves of conductor $36q$ and $72q$ ($36q.4.a$, $36q.5.a$, $36q.6.a$, $36q.6.b$, $72q.4.a$, $72q.4.b$, $72q.8.a$, $72q.9.a$, $72q.10.a$ and $72q.11.a$.), our given models are already of the form $y^2 = x^3 + ux^2 + vx$, with $u = a_2(E)$ and $v = a_4(E)$. For our families of conductor $18q$ ($18q.4.a$, $18q.5.a$, $18q.5.b$ and $18q.6.a$), we need to move our nontrivial rational 2-torsion point to $(0, 0)$ to obtain a (nonminimal) model of the shape (4-5) (the discriminant remaining invariant modulo squares). We summarize our results in the following table.

E	additional conditions	$\{(\frac{v}{\ell}), (\frac{\Delta(E)}{\ell})\}$
$18q.4.a$	unless $\delta_1 = \delta_2 = 0$, a even and b odd	$\{(\frac{(-1)^{\delta_1}q}{\ell}), (\frac{(-1)^{\delta_1+\delta_2+1}2^a3^b}{\ell})\}$
$18q.5.a$ and b	unless $\delta_1 = \delta_2 = 0$, a even	$\{(\frac{(-1)^{\delta_1}3q}{\ell}), (\frac{(-1)^{\delta_1+\delta_2+1}2^a3}{\ell})\}$
$18q.6.a$	none	$\{(\frac{3^bq}{\ell}), (\frac{-2}{\ell})\}$
$36q.4.a$ and $5.a$	none	$\{(\frac{3}{\ell}), (\frac{(-1)^{\delta}q}{\ell})\}$
$36q.6.a$ and b	none	$\{(\frac{3q}{\ell}), (\frac{3}{\ell})\}$
$36q.7.a$	none	$\{(\frac{-1}{\ell}), (\frac{q}{\ell})\}$
$72q.5.a$ and b	$(\delta, a) = (0, 5), (1, 4)$ or $(1, 5)$	$\{(\frac{3q}{\ell}), (\frac{(-1)^{\delta+1}2^a3}{\ell})\}$
$72q.9.a$	unless $\delta_1 = \delta_2 = 0$, $a = 4$ and b odd	$\{(\frac{(-1)^{\delta_1}q}{\ell}), (\frac{(-1)^{\delta_1+\delta_2+1}2^a3^b}{\ell})\}$
$72q.10.a$	none	$\{(\frac{(-1)^{\delta}q}{\ell}), (\frac{2^a3^b}{\ell})\}$
$72q.11.a$ and $12.a$	none	$\{(\frac{3^bq}{\ell}), (\frac{-2}{\ell})\}$

For example, in case $E = 72q.12.a1$, we have, for $\ell \nmid 6q$,

$$\left(\frac{\Delta(E)}{\ell}\right) = \left(\frac{3^bq}{\ell}\right) \quad \text{and} \quad \left(\frac{v}{\ell}\right) = \left(\frac{a_4(E)}{\ell}\right) = \left(\frac{-2}{\ell}\right).$$

In particular, if we assume, say, that b is odd, for any prime $\ell \equiv 7 \bmod 24$ such that q is a quadratic residue modulo ℓ, or prime $\ell \equiv 13 \bmod 24$ with q a quadratic nonresidue modulo ℓ, we have

$$\left(\frac{\Delta(E)}{\ell}\right) = \left(\frac{v}{\ell}\right) = -1 \tag{4-6}$$

and hence for such a prime ℓ, both $\ell \equiv 1 \bmod 6$ and

$$a_\ell(E) \not\equiv \ell + 1 \bmod 4. \tag{4-7}$$

In both cases, we therefore obtain inequality (4-4). For the other isogeny classes in the above table, in each case there exists at least one pair of integers (ℓ_0, t), with $\ell_0 \in \{7, 13, 19\}$ and $t \in \{0, 1\}$, such that if

$$\ell \equiv \ell_0 \bmod 24 \quad \text{and} \quad \left(\frac{q}{\ell}\right) = (-1)^t, \tag{4-8}$$

then (4-6) and (4-7) hold. Specifically, we have

E	(ℓ_0, t)
$18q.4.a$	(7, 0), (7, 1), (13, 1), (19, 0) or (19, 1)
$18q.5.a$ and b	(7, 0), (7, 1), (13, 1), (19, 0) or (19, 1)
$18q.6.a$	(7, 0), (7, 1) or (13, 1)
$36q.4.a$ and $5.a$	(7, 0), (7, 1), (19, 0) or (19, 1)
$36q.6.a$ and b	(7, 0) or (19, 0)
$36q.7.a$	(7, 1) or (19, 1)
$72q.5.a$ and b	(7, 0), (13, 1) or (19, 0)
$72q.9.a$	(7, 0), (7, 1), (13, 1), (19, 0) or (19, 1)
$72q.10.a$	(7, 0), (7, 1), (13, 1), (19, 0) or (19, 1)
$72q.11.a$ and $12.a$	(7, 0), (7, 1) or (13, 1)

To complete the proof of Theorem 1.4, from (4-4), we require a suitably strong upper bound for the smallest ℓ satisfying (4-8). Such a bound would follow from either a modified version of the arguments traditionally used to find smallest nonresidues modulo q (though the additional constraint that $\ell \equiv \ell_0 \bmod 24$ causes some complications), or from an explicit version of Linnik's theorem on the smallest prime in a given arithmetic progression (see, e.g., [Heath-Brown 1990] for an effective but inexplicit result along these lines). For our purposes (and since we require something completely explicit), we will instead appeal to a recent result of the first author, Martin, O'Bryant and Rechnitzer, which we now state, with $\theta(x; k, a)$ denoting the sum of the logarithms of the primes $p \equiv a \bmod k$ with $p \le x$.

Theorem 4.2 [Bennett et al. 2018]. *Let k and a be integers with $k \ge 3$ and $\gcd(a, k) = 1$. Then*

$$\left|\theta(x; k, a) - \frac{x}{\phi(k)}\right| < \frac{1}{180} \frac{x}{\log x},$$

for all $x \ge x_0(k)$, where $\phi(k)$ is the Euler phi function and

$$x_0(q) = \begin{cases} 4.1 \times 10^9 & \text{if } 3 \le q \le 16, \\ 6.7 \times 10^{10}/q & \text{if } 17 \le q \le 10^5, \\ \exp(0.03\sqrt{q}\log^3 q) & \text{if } q > 10^5. \end{cases} \tag{4-9}$$

Proposition 4.3. *Let $q \ge 5$ be prime and suppose that $\ell_0 \in \{7, 13, 19\}$ and $t \in \{0, 1\}$. Then there exists a prime $\ell \ne q$ satisfying (4-8) with $\ell < e^q$.*

Proof. Given $\ell_0 \in \{7, 13, 19\}$ and $t \in \{0, 1\}$, conditions (4-8) are equivalent, via the Chinese remainder theorem, to a congruence of the shape $\ell \equiv a \mod 24q$ for some integer $7 \le a < 24q$ with $\gcd(a, 24q) = 1$. For $5 \le q \le 23$ and each of the 6 pairs (ℓ_0, t), we verify by direct computation that we can always find an $\ell < e^q$ with (4-8). If $q > 23$, then $e^q > x_0(24q)$ and hence we may apply Theorem 4.2 to conclude that

$$\left|\theta(e^q; 24q, a) - \frac{e^q}{8(q-1)}\right| < \frac{e^q}{180q},$$

whereby

$$\theta(e^q; 24q, a) > \frac{0.11\,e^q}{q-1} > \log q.$$

It follows that there exists a prime $\ell \equiv a \mod 24q$ (which necessarily also satisfies (4-8)) with $\ell \neq q$ and $\ell < e^q$, as desired. □

For $q \ge 5$, we apply Proposition 4.3 to (4-4) to conclude that $p < e^q + 1 + 2\sqrt{e^q} = (e^{q/2}+1)^2 < q^{2q}$. This completes the proof of Theorem 1.4.

5. Sets of primes and trivial solutions

5A. *Intersections of the S_i.* We would like to make a few remarks on the sets S_i. Firstly, we note that some of the S_i overlap substantially. Obviously, primes of the form $(3^b+1)/4$ belong to both S_6 and S_7, while many primes in S_1 are also in S_4 (taking $d = 1$). Additionally, every prime $q \in S_5$ of the shape $q = 3d^2 + 2^a$ with $a = 2$ or $a \ge 8$, and $d = \pm 3^k$ for k an integer, is necessarily also in S_2.

For many other $i \neq j$, the intersection $S_i \cap S_j$ is rather small. For future use, it will be helpful for us to record an explicit statement along these lines.

Proposition 5.1. *We have*

$$S_1 \cap S_2 = \{5, 7, 11, 13, 23, 31, 37, 73\}, \quad S_1 \cap S_3 = \{11\}, \quad S_1 \cap S_5 = \{7, 31\},$$
$$S_1 \cap S_7 = \{7, 37, 127\}, \quad S_1 \cap S_8 = \{13\}, \quad S_2 \cap S_3 = \{11\},$$
$$S_3 \cap S_4 = \varnothing, \quad S_3 \cap S_5 = \{43\}, \quad S_3 \cap S_7 = S_3 \cap S_8 = S_4 \cap S_5 = S_5 \cap S_8 = S_7 \cap S_8 = \varnothing.$$

To prove this, we will have use of a pair of results on polynomial-exponential Diophantine equations.

Lemma 5.2. *If x, y and z are nonnegative integers such that $z^2 = 2^x 3^y + 1$, then*

$$(x, y, z) \in \{(0, 1, 2), (3, 0, 3), (3, 1, 5), (4, 1, 7), (5, 2, 17)\}.$$

Proof. This follows from straightforward factoring and local arguments. □

Lemma 5.3. *If x and y are positive integers such that $2^x = 3y^2 + 5$, then*

$$(x, y) \in \{(3, 1), (5, 3), (9, 13)\}.$$

Proof. Writing $x = 3x_1 + x_0$ for $x_0 \in \{0, 1, 2\}$, we have that a solution to the equation $2^x = 3y^2 + 5$ necessarily corresponds to an integer point on the (Mordell) elliptic curve $Y^2 = X^3 - 2^{2x_0} \cdot 3^3 \cdot 5$ (with $Y = 2^{x_0} \cdot 3^2 \cdot y$ and $X = 3 \cdot 2^{x_0+x_1}$). The integer points for each of these curves can be found at http://www.math.ubc.ca/ bennett/BeGa-data.html (see [Bennett and Ghadermarzi 2015] for more details), whereby the stated conclusion obtains. □

Proof of Proposition 5.1. The desired conclusions for $S_1 \cap S_2$, $S_1 \cap S_3$ and $S_2 \cap S_3$ all follow from combining Theorems 1, 2 and 3 of Tijdeman and Wang [1988] with Theorems 3, 4 and 5 of Wang [1989]. Further, the fact that

$$S_3 \cap S_4 = S_3 \cap S_8 = S_4 \cap S_5 = S_5 \cap S_8 = \varnothing$$

is immediate from considering the corresponding equations modulo 8.

If $q \in S_1 \cap S_5$, then there exist integers a, b, δ, d and a_5 with $a \in \{2, 3\}$ or $a \geq 5$, $b \geq 0$, $\delta \in \{0, 1\}$, $d \geq 1$ and $a_5 \in \{2, 4\}$ or $a_5 \geq 8$ even, such that $2^a 3^b + (-1)^\delta = 3d^2 + 2^{a_5}$. Modulo 4, we have that $\delta = 1$ and so, modulo 3, $b = 0$. It follows, modulo 8, that $a_5 = 2$, so that $2^a = 3d^2 + 5$. From Lemma 5.3, we therefore have $S_1 \cap S_5 = \{7, 31\}$, as desired. If instead $q \in S_1 \cap S_7$, then we have integers a, b, δ and d, with $a \in \{2, 3\}$ or $a \geq 5$, $b \geq 0$, $\delta \in \{0, 1\}$, $d \geq 1$ and $2^a 3^b + (-1)^\delta = (3d^2 + 1)/4$. If $b = 0$, then, modulo 3, $\delta = 1$, whence $2^{a+2} = 3d^2 + 5$ and so, from Lemma 5.3, $a \in \{1, 3, 7\}$, giving rise to $q = 7$ and $q = 127$. If $b \geq 1$, then, again modulo 3, $\delta = 0$ and hence $d^2 = 2^{a+2} 3^{b-1} + 1$. Lemma 5.2 thus implies that $(a, b, d) = (1, 2, 5), (2, 2, 7)$ or $(3, 3, 17)$, yielding $q = 37$ (the first triple fails to have $a \in \{2, 3\}$ while the third leads to a composite value of q).

Suppose next that we have $q \in S_1 \cap S_8$, so that there exist integers a, b, δ, v with $a \in \{2, 3\}$ or $a \geq 5$, $b \geq 0$, $\delta \in \{0, 1\}$ and $2^a 3^b + (-1)^\delta = (3v^2 - 1)/2$. Modulo 4, $\delta = 0$ and hence $2^{a+1} 3^{b-1} + 1 = v^2$; again Lemma 5.2 implies, after a little work, that $|v| = 3$ and $q = 13$. If $q \in S_3 \cap S_5$, there are integers a, d and a_5 with $a \geq 5$ odd, $d \geq 1$, $a_5 \in \{2, 4\}$ or $a_5 \geq 8$ even, and $(2^a + 1)/3 = 3d^2 + 2^{a_5}$. Modulo 8, we have that $a_5 \geq 4$ and so

$$9d^2 = 2^a - 3 \cdot 2^{a_5} + 1. \tag{5-1}$$

We thus have $a \geq a_5 + 3$ and there must exist integers $\delta \in \{0, 1\}$ and positive $d_1 \equiv \pm 1 \bmod 6$ such that $3d = 2^{a_5 - 1} d_1 + (-1)^\delta$, whence

$$2^{a_5 - 2} d_1^2 + (-1)^\delta d_1 = 2^{a - a_5} - 3.$$

If $d_1 = 1$, then $\delta = 0$ and we have $2^{a_5 - 4} + 1 = 2^{a - a_5 - 2}$, so that $a_5 = 4$, $a = 7$, $d = 3$ and $q = 43$. If $d_1 > 1$, then $d_1 \geq 5$ and so $2^{a_5 - 2} 5^2 - 5 \leq 2^{a - a_5} - 3$. It follows that $a \geq 2a_5 + 2$ and hence $2^{a_5 - 2} \mid (-1)^\delta d_1 + 3$, say $(-1)^\delta d_1 + 3 = 2^{a_5 - 2} d_2$, for $d_2 \in \mathbb{Z}$. We thus have

$$2^{2a_5 - 4} d_2^2 - 3 \cdot 2^{a_5 - 1} d_2 + 9 + d_2 = 2^{a - 2a_5 + 2}.$$

Since $a_5 \geq 4$ and $a \geq 2a_5 + 2$, we have that $9 + d_2 \equiv 0 \bmod 8$. If $d_2 = -1$, $2^{2a_5 - 7} + 3 \cdot 2^{a_5 - 4} + 1 = 2^{a - 2a_5 - 1}$, contradicting $a_5 = 4$ or $a_5 \geq 8$. We thus have $d_2 = 7$ or $|d_2| \geq 9$, so that

$$2^{a - 2a_5 + 2} \geq 2^{2a_5 - 4} \cdot 7^2 - 3 \cdot 2^{a_5 - 1} \cdot 7 + 16 > 2^{2a_5 + 1},$$

and so $a \geq 4a_5 + 4$. Applying Corollary 1.7 of [Bauer and Bennett 2002], since $a \geq 4a_5 + 4 \geq 24$, we have from (5-1) that

$$3 \cdot 2^{a_5} - 1 = |9d^2 - 2^a| > 2^{0.26a} \geq 2^{1.04a_5 + 1.04},$$

whereby $a_5 \leq 13$. A short check confirms that $S_3 \cap S_5 = \{43\}$, as stated.

For $q \in S_3 \cap S_7$, $q = (2^a + 1)/3 = (3d^2 + 1)/4$ for $a \geq 5$ and d odd integers, so that $2^{a+2} + 1 = 9d^2$ and yet another elementary argument implies that $a = 1$, a contradiction. Let us therefore suppose, finally, that $q \in S_7 \cap S_8$. We thus have

$$4q = 3d^2 + 1 = 6v^2 - 2 = 2u^2 + 2,$$

for integers d, u and v, and so

$$(3dv)^2 = (2u^2 + 1)(u^2 + 2) = 2u^4 + 5u^2 + 2.$$

From Magma's *IntegralQuarticPoints* routine, we find that the only integer solution to the latter equation is with $|dv| = |u| = 1$. This completes the proof of Proposition 5.1. □

It should also be noted that representations within a given set S_i are sometimes unique, but not always. In particular, it is straightforward to show that a given prime $q \in S_1$ has a single representation of the form $q = 2^a 3^b \pm 1$, with $b \geq 0$ and $a \in \{2, 3\}$ or $a \geq 5$, while a similar conclusion is immediate for primes $q \in S_i$ for $i \in \{3, 7, 8\}$. The situation in S_2 is slightly more complicated; combining work of Pillai [1945] with Stroeker and Tijdeman [1982], the only primes with multiple representations of the form $q = |2^a \pm 3^b|$, with $b \geq 1$ and $a \in \{2, 3\}$ or $a \geq 5$, are $q \in \{5, 13, 17, 23, 73\}$, corresponding to the identities

$$5 = 2^3 - 3 = 2^5 - 3^3 = 3^2 - 2^2, \quad 13 = 2^8 - 3^5 = 2^2 + 3^2, \quad 17 = 3^4 - 2^6 = 2^3 + 3^2,$$
$$23 = 2^5 - 3^2 = 3^3 - 2^2, \quad \text{and} \quad 73 = 3^4 - 2^3 = 2^6 + 3^2.$$

5B. ***Limitations due to trivial solutions.*** Notice that we have the identity

$$\left(\frac{d+1}{2}\right)^3 + \left(\frac{1-d}{2}\right)^3 = \frac{3d^2+1}{4}$$

and hence, for all exponents n, a coprime integer solution with $C = 1$ to the equation

$$A^3 + B^3 = q^\alpha \cdot C^n, \quad \text{whenever } q^\alpha = \frac{3d^2+1}{4}.$$

We expect q^α to be of this shape infinitely often for $\alpha = 1$ and $\alpha = 2$ (these are precisely the primes in S_7 and S_8, respectively), though both of these results are a long way from provable with current technology.

We will term a solution to (1-2) with $C = 1$ *trivial*, whereby, for primes q as above, there exists a trivial solution for all prime exponents p. In particular, this means that one of the newforms $f \in S_2^+(N_F/\mathcal{R})$ (see Section 2) will correspond (via modularity) to the Frey curve F evaluated at the trivial solution. This is a major obstruction to the modular method; the techniques of this paper are unlikely to provide further information about (1-2) with $\alpha = 1$ for $q \in S_7$ and $\alpha = 2$ for $q \in S_8$.

A similar relation is the identity

$$(d+4)^3+(4-d)^3=8(3d^2+16).$$

While this does not actually give trivial solutions to (1-2) in case $\alpha=1$ and $q=3d^2+16$ (a subset of the primes in S_5), it does appear to provide an obstruction to solving (1-4) for such primes, leading to Frey–Hellegouarch curves that play the role of the curve $72A1$ for (1-1).

6. Applying the symplectic criteria

Let E and F be elliptic curves over $\mathbb{Q}$ and suppose there exists an isomorphism $\phi : F[p] \to E[p]$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. Here, $F[p]$ and $E[p]$ are the p-torsion modules attached to F and E, respectively. Write $e_{E,p}$ and $e_{F,p}$ for the Weil pairings on $E[p]$ and $F[p]$, respectively. Then there exists an element $r(\phi) \in \mathbb{F}_p^\times$ such that

$$e_{E,p}(\phi(P),\phi(Q)) = e_{F,p}(P,Q)^{r(\phi)} \quad \text{for all } P, Q \in F[p].$$

If $r(\phi)$ is a square in $\mathbb{F}_p^\times$, we call the isomorphism ϕ *symplectic*; if $r(\phi)$ is a nonsquare, we call it *antisymplectic*. We say that $E[p]$ and $F[p]$ are *symplectically (antisymplectically) isomorphic* if there exists a symplectic (antisymplectic) isomorphism ϕ between them. It is possible that $E[p]$ and $F[p]$ are both symplectically and antisymplectically isomorphic, but this situation will not occur in the applications of these techniques in this paper (as we shall see in Proposition 6.1).

6A. ***The symplectic argument.*** To treat (1-2) for certain primes $q \in S_0$ and exponents $p \geq q^{2q}$ we need to use a number of local symplectic criteria to describe the symplectic type of the isomorphisms between the p-torsion modules $E[p]$ and $F[p]$, where F is our Frey–Hellegouarch curve and E is one of the curves in Corollary 3.4 (see Section 2 and Theorem 1.4). The idea is to use local information at different primes ℓ to obtain congruence conditions on the exponent p for which $E[p]$ and $F[p]$ are symplectically and antisymplectically isomorphic. Then, our desired contradictions will arise each time we are able to prove that these constraints are incompatible. This is, in essence, what is sometimes called the *symplectic argument*. One advantage we have here, working with (1-2) as opposed to (1-1), is that we will be able to apply the (local) criteria at the primes $\ell \in \{2, 3, q\}$ rather than just $\ell \in \{2, 3\}$.

6B. ***Notation.*** Let ℓ be a prime and, for a nonzero integer t, define $\nu_\ell(t)$ to be the largest nonnegative integer such that $\ell^{\nu_\ell(t)}$ divides t. Let $E/\mathbb{Q}_\ell$ be an elliptic curve and write $c_4(E)$, $c_6(E)$ and $\Delta(E)$ for the usual invariants attached to a minimal model of E. Further, with slight abuse of notation since we define $\nu_\ell(t)$ over $\mathbb{Z}$, we introduce the quantities

$$c_4(E) = \ell^{\nu_\ell(c_4(E))} c_4(E)_\ell, \quad c_6(E) = \ell^{\nu_\ell(c_6(E))} c_6(E)_\ell \quad \text{and} \quad \Delta(E) = \ell^{\nu_\ell(\Delta(E))} \Delta(E)_\ell.$$

Fix an algebraic closure of $\mathbb{Q}_\ell$ and let $\mathbb{Q}_\ell^{\mathrm{un}}$ to be the maximal unramified extension of $\mathbb{Q}_\ell$. For an elliptic curve $E/\mathbb{Q}$ with potentially good reduction at ℓ we write $e(E,\ell)$ to denote the order of $\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{un}}(E[p])/\mathbb{Q}_\ell^{\mathrm{un}})$ for $p \geq 3$ different from ℓ. It is well known that $e(E,\ell)$ is independent of p.

6C. ***The curves.*** Except for the few isogeny classes given in Corollary 3.4 by their Cremona label, from Theorem 1.4, we are primarily interested in applying symplectic criteria to our Frey–Hellegouarch curve and curves in the following isogeny classes:

$18q.1.a,\ 18q.2.a,\ 18q.3.a,\ 18q.4.a$ (with $\delta_1 = \delta_2 = 0,\ a$ even, b odd),
$18q.5.a$ and $18q.5.b$ (with, in both cases, $\delta_1 = \delta_2 = 0$ and a even),

$36q.1.a,\ 36q.1.b,\ 36q.2.a,\ 36q.2.b,\ 36q.3.a,$

$72q.1.a,\ 72q.2.a,\ 72q.3.a,\ 72q.4.a,\ 72q.4.b,$
$72q.5.a$ and $72q.5.b$ (with, in both cases, $\delta = 0$ and $a = 4$),
$72q.6.a,\ 72q.6.b,\ 72q.7.a,\ 72q.8.a,\ 72q.9.a$ (with $\delta_1 = \delta_2 = 0,\ a = 4,\ b$ odd).

The relevant arithmetic data $c_4(E)$, $c_6(E)$ and $\Delta(E)$ is available in Tables 1–3 in the Appendix and in the statements of Theorems 3.1, 3.2 and 3.3. In the remainder of this section we will apply the criteria to the curves listed above to obtain congruence conditions on p. Then, in Section 7, we complete the symplectic argument by deriving contradictions from these conditions, allowing us to finish the proofs of our main Diophantine statements. We start by proving the following proposition which holds for all our choices of E, independently of whether E has conductor $N_E = 18q$, $36q$ or $72q$.

Proposition 6.1. *Let* (A, B, C) *be a nontrivial primitive solution to* (1-2) *so that there is a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-*modules isomorphism* $\phi : F[p] \to E[p]$, *where F is the Frey–Hellegouarch curve and E is any elliptic curve in one of the isogeny classes above. Then*

$$\phi \text{ is symplectic} \iff \alpha \text{ is a square mod } p.$$

Proof. We have $\nu_q(N_F) = \nu_q(N_E) = 1$, so q is a prime of multiplicative reduction of both curves. We can always choose E such that $\nu_q(\Delta(E)) = 2$; moreover, we have $p \nmid \alpha$ and $\nu_q(\Delta(F)) = 2\alpha + 2p\nu_q(C)$. The conclusion now follows from a direct application of [Kraus and Oesterlé 1992, Proposition 2] with the prime q. □

6D. ***Curves of conductor* 18*q*.** We summarize the necessary information about the invariants of the relevant elliptic curves.

curve	$\nu_2(c_4)$	$\nu_2(c_6)$	$\nu_2(\Delta)$	$\nu_3(c_4)$	$\nu_3(c_6)$	$\nu_3(\Delta)$
$F_{A,B}^{(0)}$	0	0	$2p\nu_2(C) - 8$	$2 + \nu_3(AB)$	$3 + \nu_3(A^3 - B^3)$	$2p\nu_3(C) + 3$
$18q.1.a1$ ($b = 0$)	0	0	$2a - 8$	3	≥ 7	6
$18q.1.a1$ ($b \geq 1$)	0	0	$2a - 8$	2	3	$2b + 6$
$18q.2.a1$	0	0	$2a - 8$	2	3	8
$18q.3.a1$	0	0	$2a - 8$	2	3	$2b + 6$
$18q.4.a2$	0	0	$a - 6$	2	3	$b + 6$
$18q.5.a2$	0	0	$a - 6$	2	$3 + \nu_3(d)$	3
$18q.5.b2$	0	0	$a - 6$	4	$6 + \nu_3(d)$	9

Suppose (A, B, C) is a nontrivial primitive solution to (1-2) and the Frey–Hellegouarch curve F satisfies isomorphism (2-2) where f is the newform corresponding to one of the isogeny classes

$$18q.1.a, \quad 18q.2.a, \quad 18q.3.a, \quad 18q.4.a, \quad 18q.5.a \quad \text{or} \quad 18q.5.b.$$

In particular, $F = F^{(0)}_{A,B}$, C is even and $B \equiv -1 \bmod 4$. Moreover, there is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules isomorphism $\phi : F[p] \to E[p]$, where E is one of the elliptic curves

$$18q.1.a1, \quad 18q.2.a1, \quad 18q.3.a1, \quad 18q.4.a2, \quad 18q.5.a2 \quad \text{or} \quad 18q.5.b2.$$

6D1. *Applying the criteria at $\ell = 2$.* Since $\nu_2(N_E) = 1$ the prime $\ell = 2$ is of multiplicative reduction for E. From [Kraus and Oesterlé 1992, Proposition 2] and the valuations given in the preceding table, it follows that either $p \nmid a - 4$ and

$$\phi \text{ is symplectic} \iff 4 - a \text{ is a square mod } p,$$

in case $E = 18q.1.a1$, $18q.2.a1$ or $18q.3.a1$, or that $p \nmid a - 6$ and

$$\phi \text{ is symplectic} \iff 12 - 2a \text{ is a square mod } p,$$

in the other cases.

6D2. *Applying the criteria at $\ell = 3$.* We first consider E one of $18q.1.a1$ with $b = 0$, $18q.5.a2$ or $18q.5.b2$. We have that the corresponding j-invariant satisfies $\nu_3(j_E) > 0$, and hence E has potentially good reduction at 3. Indeed, for $E = 18q.1.a1$ (with $b = 0$), we have $\nu_3(\Delta(E)) = 6$ and $\nu_3(c_6(E)) \geq 7$ so that, from [Kraus 1990, p. 356], we conclude that $e(E, 3) = 2$.

For $E = 18q.5.a2$ and $E = 18q.5.b2$, we have $\nu_3(\Delta(E)) \in \{3, 9\}$ and the results of [Kraus 1990, p. 356] imply that $e(E, 3) \in \{4, 12\}$. Since $\nu_3(N_E) = 2$, we are in a case of tame reduction and so the inertia must be of order coprime to $\ell = 3$, whereby $e(E, 3) = 4$. On the other hand, for our Frey–Hellegouarch curve F to have potentially good reduction at 3, we require that $3\nu_3(c_4(F)) \geq \nu_3(\Delta(F))$, or, equivalently, $\nu_3(C) = 0$. In this situation, $\nu_3(\Delta(F)) = 3$ and arguing exactly as for the previous curves we also conclude that $e(F, 3) = 4$. This contradicts $E = 18q.1.a1$ (with $b = 0$). We will now apply [Freitas and Kraus 2016, Theorem 5] with F and $E = 18q.5.a2$ or $18q.5.b2$ (with, in both cases, $\delta_1 = \delta_2 = 0$ and a even). Let r and t be the quantities defined in the statement of that theorem. We have, since $3 \nmid C$,

$$\nu_3(\Delta(F)) = \nu_3(\Delta(18q.5.a2)) = 3 \quad \text{and} \quad \nu_3(\Delta(18q.5.b2)) = 9,$$

whereby $r = 0$ if $E = 18q.5.a2$ and $r = 1$ if $E = 18q.5.b2$. Moreover, since $3 \nmid C$ and a is even, we may check that $\Delta(F)_3 \equiv \Delta(E)_3 \equiv 2 \pmod 3$, i.e., $t = 0$ for both E. Finally, applying [Freitas and Kraus 2016, Theorem 5], we conclude that ϕ is symplectic when $E = 18q.5.a2$ and, if $E = 18q.5.b2$, then ϕ is symplectic if and only if $(3/p) = 1$.

We now consider the remaining curves E of conductor $18q$ under consideration. We have, in all cases,

$$\nu_3(c_4(E)) = 2, \quad \nu_3(c_6(E)) = 3, \quad \nu_3(\Delta(E)) \geq 7 \quad \text{and} \quad \nu_3(j_E) < 0,$$

and hence E has potentially multiplicative reduction at 3; after a quadratic twist (with corresponding elliptic curve denoted E_t) the reduction becomes multiplicative and we have

$$\nu_3(\Delta(E_t)) = \begin{cases} 2 & \text{if } E = 18q.2.a1, \\ b & \text{if } E = 18q.4.a2 \text{ (with } b \geq 1), \\ 2b & \text{if } E = 18q.1.a1 \text{ (and } b \geq 1) \text{ or } E = 18q.3.a1. \end{cases}$$

Furthermore, 3 must divide C (since otherwise F would have potentially good reduction) and twisting the Frey curve F by the same element (to obtain F_t), we find that $\nu_3(\Delta(F_t)) = -3 + 2p\nu_3(C)$.

If $E = 18q.1.a1$ with $b \geq 1$ or $E = 18q.3.a1$, it follows from [Kraus and Oesterlé 1992, Proposition 2] applied to E_t and F_t that $p \nmid b$ and

$$\phi \text{ is symplectic} \iff -6b \text{ is a square mod } p.$$

Similarly, if $E = 18q.4.a2$ then $p \nmid b$ and

$$\phi \text{ is symplectic} \iff -3b \text{ is a square mod } p.$$

If $E = 18q.2.a1$, then

$$\phi \text{ is symplectic} \iff -6 \text{ is a square mod } p.$$

6D3. *Conclusions for level* $18q$. From the calculations above and Proposition 6.1 we can extract the following relations. If $E = 18q.1.a1$ or $18q.3.a1$ then $b \geq 1$ and

$$\left(\frac{4-a}{p}\right) = \left(\frac{\alpha}{p}\right) = \left(\frac{-6b}{p}\right), \tag{6-1}$$

while if $E = 18q.4.a2$ or $E = 18q.2.a1$, then, respectively,

$$\left(\frac{12-2a}{p}\right) = \left(\frac{\alpha}{p}\right) = \left(\frac{-3b}{p}\right) \quad \text{or} \quad \left(\frac{4-a}{p}\right) = \left(\frac{\alpha}{p}\right) = \left(\frac{-6}{p}\right).$$

If $E = 18q.5.a2$, we have that

$$\left(\frac{12-2a}{p}\right) = \left(\frac{\alpha}{p}\right) = 1.$$

Finally, if $E = 18q.5.b2$,

$$\left(\frac{12-2a}{p}\right) = \left(\frac{\alpha}{p}\right) = \left(\frac{3}{p}\right).$$

6E. *Curves of conductor* 36*q*. We next proceed with the case of elliptic curves of conductor $36q$. We encounter the following invariants.

curve	$\nu_2(c_4)$	$\nu_2(c_6)$	$\nu_2(\Delta)$	$\nu_3(c_4)$	$\nu_3(c_6)$	$\nu_3(\Delta)$
$F^{(1)}_{A,B}$	$4+\nu_2(A)$	5	4	$2+\nu_3(AB)$	$3+\nu_3(A^3-B^3)$	$2p\nu_3(C)+3$
$36q.1.a2\ (3\mid s)$	4	6	8	2	$4+\nu_3(v)$	3
$36q.1.a2\ (3\nmid s)$	4	6	8	2	3	3
$36q.1.b2\ (3\mid s)$	4	6	8	4	$7+\nu_3(v)$	9
$36q.1.b2\ (3\nmid s)$	4	6	8	4	6	9
$36q.2.a1\ (3\mid d)$	≥ 6	5	4	2	$4+\nu_3(d)$	3
$36q.2.a1\ (3\nmid d)$	≥ 6	5	4	≥ 3	3	3
$36q.2.b1\ (3\mid d)$	≥ 6	5	4	4	$7+\nu_3(d)$	9
$36q.2.b1\ (3\nmid d)$	≥ 6	5	4	≥ 5	6	9
$36q.3.a1$	≥ 6	5	4	2	3	$b+6$

Suppose (A, B, C) is a nontrivial primitive solution to (1-2) and the Frey–Hellegouarch curve F satisfies isomorphism (2-2) where f is the newform corresponding to one of the isogeny classes

$$36q.1.a, \quad 36q.1.b, \quad 36q.2.a, \quad 36q.2.b \quad \text{or} \quad 36q.3.a.$$

In particular, $F = F^{(1)}_{A,B}$, C is odd and $B \equiv 1 \bmod 4$. Moreover, there is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules isomorphism $\phi : F[p] \to E[p]$, where E is one of the elliptic curves

$$36q.1.a2, \quad 36q.1.b2, \quad 36q.2.a1, \quad 36q.2.b1 \quad \text{or} \quad 36q.3.a1.$$

6E1. *Applying the criteria at $\ell = 2$.* The table shows that $\nu_2(j(E)) > 0$ for all E, so that the curves have potentially good reduction. Since $\nu_2(N_E) = 2$ the reduction is tame and hence $e(E, 2) = 3$ for all E.

We will now apply Theorem 1 of [Freitas and Kraus 2016] at $\ell = 2$ with F and $E = 36q.1.a2$ or $36q.1.b2$. Let t and r be as in that theorem. Since $\nu_2(\Delta(F)) = 4$ and $\nu_2(\Delta(E)) = 8$, we have $r = 1$ for both E. Now, to determine the value of t, we must first appeal to Theorem 2 of the same work. Indeed, the curve $36q.1.a2$ has

$$c_4(E)_2 = 3\left(16\left(\frac{3v^2-1}{2}\right)^2 - 1\right) \quad \text{and} \quad c_6(E)_2 = -3^2uv\left(32\left(\frac{3v^2-1}{2}\right)^2 + 1\right),$$

while for $36q.1.b2$,

$$c_4(E)_2 = 3^3\left(16\left(\frac{3v^2-1}{2}\right)^2 - 1\right) \quad \text{and} \quad c_6(E)_2 = 3^5uv\left(32\left(\frac{3v^2-1}{2}\right)^2 + 1\right).$$

We thus have, respectively,

$$c_4(E)_2 \equiv 13 \bmod 32 \quad \text{and} \quad c_6(E)_2 \equiv 7uv \bmod 16,$$

and

$$c_4(E)_2 \equiv 21 \bmod 32 \quad \text{and} \quad c_6(E)_2 \equiv 3uv \bmod 16.$$

Since $u^2 - 3v^2 = -2$ with $u \equiv v \equiv 1 \mod 4$, the u and v are terms in binary recurrence sequences. To be specific, the positive integers $u = u_k$ and $v = v_k$ satisfying this equation also satisfy the binary recurrences

$$u_{k+1} = 4u_k - u_{k-1}, \quad \text{for } k \geq 1, \text{ where } u_0 = 1, u_1 = 5, \tag{6-2}$$

and

$$v_{k+1} = 4v_k - v_{k-1}, \quad \text{for } k \geq 1, \text{ where } v_0 = 1, v_1 = 3. \tag{6-3}$$

We may readily prove by induction that $uv \equiv 1 \mod 16$ (recall that we are assuming that $u \equiv v \equiv 1 \mod 4$, so that we choose $u = \pm u_k$ and $v = \pm v_k$ as necessary), whereby it follows from [Freitas and Kraus 2016, Theorem 2] that the curve $36q.1.a2$ has a 3-torsion point over $\mathbb{Q}_2$, while $36q.1.b2$ does not. For $F = F^{(1)}_{A,B}$, since $\nu_2(A) \geq 2$, we have

$$\nu_2(c_4(F)) \geq 6 \quad \text{and} \quad c_6(F)_2 = 3^3(A^3 - B^3) \equiv -3B \mod 8,$$

whereby, from part (B2) of [Freitas and Kraus 2016, Theorem 2], F has a 3-torsion point over $\mathbb{Q}_2$ precisely when we have $B \equiv 1 \mod 8$. We thus conclude that, if $B \equiv 1 \mod 8$ and $E = 36q.1.a2$ or $B \equiv 5 \mod 8$ and $E = 36q.1.b2$, then $t = 0$ and

$$\phi \text{ is symplectic} \iff 2 \text{ is a square mod } p.$$

If $B \equiv 5 \mod 8$ and $E = 36q.1.a2$, or $B \equiv 1 \mod 8$ and $E = 36q.1.b2$, then $t = r = 1$ and so

$$\phi \text{ is symplectic} \iff \left(\frac{2}{p}\right) = \left(\frac{3}{p}\right).$$

Next, suppose that E is one of $36q.2.a1$, $36q.2.b1$ or $36q.3.a1$, so that we always have $r = 0$. Then

$$c_6(E)_2 = -3^3 d\frac{(d^2+3)}{4} \equiv \begin{cases} 5 \mod 8 & \text{if } d \equiv 1 \mod 16, \\ 1 \mod 8 & \text{if } d \equiv 9 \mod 16, \end{cases}$$

$$c_6(E)_2 = 3^6 d\frac{(d^2+3)}{4} \equiv \begin{cases} 5 \mod 8 & \text{if } d \equiv 9 \mod 16, \\ 1 \mod 8 & \text{if } d \equiv 1 \mod 16, \end{cases}$$

$$c_6(E)_2 = -3^3 d\frac{(d^2+3^{b+2})}{4} \equiv \begin{cases} 5 \mod 8 & \text{if } q \equiv d+2 \mod 8, \\ 1 \mod 8 & \text{if } q \equiv d-2 \mod 8, \end{cases}$$

respectively. Thus ϕ is always symplectic if any of the following conditions hold:

$$\begin{array}{llll} B \equiv 1 \mod 8, & E = 36q.2.a1 & \text{and} \quad d \equiv 1 \mod 16, & \text{or} \\ B \equiv 5 \mod 8, & E = 36q.2.a1 & \text{and} \quad d \equiv 9 \mod 16, & \text{or} \\ B \equiv 1 \mod 8, & E = 36q.2.b1 & \text{and} \quad d \equiv 9 \mod 16, & \text{or} \\ B \equiv 5 \mod 8, & E = 36q.2.b1 & \text{and} \quad d \equiv 1 \mod 16, & \text{or} \\ B \equiv 1 \mod 8, & E = 36q.3.a1 & \text{and} \quad q \equiv d+2 \mod 8, & \text{or} \\ B \equiv 5 \mod 8, & E = 36q.3.a1 & \text{and} \quad q \equiv d-2 \mod 8. & \end{array}$$

If, however, we have either

$$\begin{array}{llll} B \equiv 1 \bmod 8, & E = 36q.2.a1 & \text{and} \quad d \equiv 9 \bmod 16, & \text{or} \\ B \equiv 5 \bmod 8, & E = 36q.2.a1 & \text{and} \quad d \equiv 1 \bmod 16, & \text{or} \\ B \equiv 1 \bmod 8, & E = 36q.2.b1 & \text{and} \quad d \equiv 1 \bmod 16, & \text{or} \\ B \equiv 5 \bmod 8, & E = 36q.2.b1 & \text{and} \quad d \equiv 9 \bmod 16, & \text{or} \\ B \equiv 1 \bmod 8, & E = 36q.3.a1 & \text{and} \quad q \equiv d-2 \bmod 8, & \text{or} \\ B \equiv 5 \bmod 8, & E = 36q.3.a1 & \text{and} \quad q \equiv d+2 \bmod 8, & \end{array}$$

then we may conclude that

$$\phi \text{ is symplectic} \iff 3 \text{ is a square mod } p.$$

6E2. *Applying the criteria at $\ell = 3$.* For $E = 36q.3.a1$, we have $\nu_3(j(E)) < 0$ and so E has potentially multiplicative reduction at 3. After a suitable quadratic twist (denoted E_t) the reduction becomes multiplicative and $\nu_3(\Delta(E_t)) = b$. Therefore, the twisted Frey curve F_t must also have multiplicative reduction at 3 (since $p \geq 5$) and it satisfies $\nu_3(\Delta(F_t)) = 2p\nu_3(C) - 3$. Since $p \nmid \nu_3(\Delta(F_t))$, it follows from [Kraus and Oesterlé 1992, Proposition 2] that $p \nmid b$ and

$$\phi \text{ is symplectic} \iff -3b \text{ is a square mod } p.$$

For all other cases of E we have $\nu_3(j(E)) \geq 0$ and $\nu_3(\Delta(E)) \neq 6$, whence E has potentially good reduction which does not become good after a quadratic twist. As before, since $\nu_3(N_E) = 2$ the reduction is tame, whereby $e(E, 3) = 4$. A similar argument guarantees that $e(F, 3) = 4$ when $3 \nmid C$, in which case, $\nu_3(\Delta(F)) = 3$ and $\Delta(F)_3 \equiv 2 \pmod 3$. To apply [Freitas and Kraus 2016, Theorem 5] at $\ell = 3$ with F and each of the curves $E = 36q.1.a2, 36q.1.b2, 36q.2.a1$ or $36q.2.b1$, we first compute that $(r, t) = (0, 1), (1, 1), (0, 0)$ and $(1, 0)$, respectively. We conclude that if $E = 36q.2.a1$ then ϕ is symplectic, while, if $E = 36q.1.a2$,

$$\phi \text{ is symplectic} \iff 2 \text{ is a square mod } p.$$

If $E = 36q.1.b2$, then

$$\phi \text{ is symplectic} \iff \left(\frac{2}{p}\right) = \left(\frac{3}{p}\right)$$

and if $E = 36q.2.b1$, then

$$\phi \text{ is symplectic} \iff 3 \text{ is a square mod } p.$$

6E3. *Conclusions for level* $36q$. From the calculations above and Proposition 6.1 we can extract the following relations. If $E = 36q.1.a2$ and $B \equiv 1 \bmod 8$, we have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right)$$

while $E = 36q.1.a2$ and $B \equiv 5 \mod 8$ implies that either

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = 1, \quad \text{or} \quad \left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = -1, \quad \left(\frac{3}{p}\right) = 1.$$

If $E = 36q.1.b2$ and $B \equiv 1 \mod 8$, we have either

$$\left(\frac{\alpha}{p}\right) = 1, \quad \left(\frac{2}{p}\right) = \left(\frac{3}{p}\right), \quad \text{or} \quad \left(\frac{\alpha}{p}\right) = -1, \quad \left(\frac{2}{p}\right) \neq \left(\frac{3}{p}\right).$$

If $E = 36q.1.b2$ and $B \equiv 5 \mod 8$, we either have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = 1 \quad \text{or} \quad \left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = -1, \quad \left(\frac{3}{p}\right) = 1.$$

If $E = 36q.2.a1$ and either $B \equiv 1 \mod 8$, $d \equiv 1 \mod 16$, or $B \equiv 5 \mod 8$, $d \equiv 9 \mod 16$, we have

$$\left(\frac{\alpha}{p}\right) = 1.$$

If $E = 36q.2.a1$ and either $B \equiv 1 \mod 8$, $d \equiv 9 \mod 16$, or $B \equiv 5 \mod 8$, $d \equiv 1 \mod 16$, we have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{3}{p}\right) = 1.$$

If $E = 36q.2.b1$ and either $B \equiv 1 \mod 8$, $d \equiv 9 \mod 16$, or $B \equiv 5 \mod 8$, $d \equiv 1 \mod 16$, we have, again,

$$\left(\frac{\alpha}{p}\right) = \left(\frac{3}{p}\right) = 1,$$

while, if $E = 36q.2.b1$ and either $B \equiv 1 \mod 8$, $d \equiv 1 \mod 16$, or $B \equiv 5 \mod 8$, $d \equiv 9 \mod 16$, we have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{3}{p}\right) = 1.$$

If $E = 36q.3.a1$ and either $B \equiv 1 \mod 8$, $q \equiv d + 2 \mod 8$, or $B \equiv 5 \mod 8$, $q \equiv d - 2 \mod 8$, we have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{-3b}{p}\right) = 1,$$

while, if $E = 36q.3.a1$ and either $B \equiv 1 \mod 8$, $q \equiv d - 2 \mod 8$, or $B \equiv 5 \mod 8$, $q \equiv d + 2 \mod 8$, we have that

$$\left(\frac{\alpha}{p}\right) = \left(\frac{3}{p}\right) = \left(\frac{-3b}{p}\right).$$

6F. ***Curves of conductor 72q.*** We have the following data.

curve	$\nu_2(c_4)$	$\nu_2(c_6)$	$\nu_2(\Delta)$	$\nu_3(c_4)$	$\nu_3(c_6)$	$\nu_3(\Delta)$
$F^{(1)}_{A,B}$	5	5	4	$2+\nu_3(AB)$	$3+\nu_3(A^3-B^3)$	$2p\nu_3(C)+3$
$72q.1.a1$	4	6	8	2	3	$2b+6$
$72q.2.a1$	4	6	8 or 10	2	3	$2b+6$
$72q.3.a1$	4	6	8 or 10	2	3	$2b+6$
$72q.4.a2$ $(3 \mid d)$	4	6	8	2	$4+\nu_3(d)$	3
$72q.4.a2$ $(3 \nmid d)$	4	6	8	≥ 3	3	3
$72q.4.b2$ $(3 \mid d)$	4	6	8	4	$7+\nu_3(d)$	9
$72q.4.b2$ $(3 \nmid d)$	4	6	8	≥ 5	6	9
$72q.5.a2$ $(3 \mid d)$	4	6	10	2	$4+\nu_3(d)$	3
$72q.5.a2$ $(3 \nmid d)$	4	6	10	≥ 3	3	3
$72q.5.b2$ $(3 \mid d)$	4	6	10	4	$7+\nu_3(d)$	9
$72q.5.b2$ $(3 \nmid d)$	4	6	10	≥ 5	6	9
$72q.6.a1$ $(3 \mid d)$	5	5	4	2	$4+\nu_3(d)$	3
$72q.6.a1$ $(3 \nmid d)$	5	5	4	≥ 3	3	3
$72q.6.b1$ $(3 \mid d)$	5	5	4	4	$7+\nu_3(d)$	9
$72q.6.b1$ $(3 \nmid d)$	5	5	4	≥ 5	6	9
$72q.7.a1$	5	5	4	2	3	$b+6$
$72q.8.a2$	4	6	8	2	3	$b+6$
$72q.9.a2$	4	6	10	2	3	$b+6$

Suppose (A, B, C) is a nontrivial primitive solution to (1-2) and the Frey–Hellegouarch curve F satisfies isomorphism (2-2) where f is the newform corresponding to one of the isogeny classes

$$72q.1.a,\ 72q.2.a,\ 72q.3.a,\ 72q.4.a,\ 72q.4.b,\ 72q.5.a,\ 72q.5.b,\ 72q.6.a,\ 72q.6.b,\ 72q.7.a,\ 72q.8.a \text{ or } 72q.9.a.$$

In particular, for this case we have $F = F^{(1)}_{A,B}$,

$$C \text{ is odd}, \quad A \equiv 2 \bmod 4 \quad \text{and} \quad B \equiv 1 \pmod 4),$$

and there is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module isomorphism

$$\phi : F[p] \to E[p],$$

where E is one of the elliptic curves labeled

$$72q.1.a1,\ 72q.2.a1,\ 72q.3.a1,\ 72q.4.a2,\ 72q.4.b2,\ 72q.5.a2,\ 72q.5.b2,$$
$$72q.6.a1,\ 72q.6.b1,\ 72q.7.a1,\ 72q.8.a2 \text{ or } 72q.9.a2.$$

6F1. *Applying the criteria at $\ell = 2$.* Note that all the curves in the preceding table have potentially good reduction at $\ell = 2$ since their j-invariants satisfy $\nu_2(j) \geq 0$. We see, from [Kraus 1990, p. 358], that the

Frey curve F satisfies $e(F, 2) = 24$; the same is immediately seen to be true also for E satisfying

$$(\nu_2(c_4(E)), \nu_2(c_6(E)), \nu_2(\Delta(E))) \in \{(4, 6, 10), (5, 5, 4)\}.$$

For the curves E in the table with

$$(\nu_2(c_4(E)), \nu_2(c_6(E)), \nu_2(\Delta(E))) = (4, 6, 8),$$

we further check that $\Delta(E)_2 \equiv 1 \bmod 4$ and hence we also have $e(E, 2) = 24$. We may therefore, in all cases, apply [Freitas 2016, Theorem 4] to find that, if $\nu_2(\Delta(E)) \in \{4, 10\}$, then ϕ is always symplectic, while, if $\nu_2(\Delta(E)) = 8$, then

$$\phi \text{ is symplectic} \iff 2 \text{ is a square mod } p.$$

6F2. *Applying the criteria at $\ell = 3$.* If $E = 72q.1.a1$, $72q.2.a1$, $72q.3.a1$, $72q.7.a1$, $72q.8.a2$ or $72q.9.a2$, then E has potentially multiplicative reduction at 3 and so, after a suitable quadratic twist (denoted E_t) the reduction becomes multiplicative and $\nu_3(\Delta(E_t)) = b$ or $2b$. Therefore, $3 \mid C$ and the twisted Frey curve F_t must also have multiplicative reduction at 3 and satisfy $\nu_3(\Delta(F_t)) = 2p\nu_3(C) - 3$. Since $p \nmid \nu_3(\Delta(F_t))$, it follows from [Kraus and Oesterlé 1992, Proposition 2] that $p \nmid b$ and

$$\phi \text{ is symplectic} \iff -3b \text{ is a square mod } p,$$

for $E = 72q.7.a1$, $72q.8.a2$ and $72q.9.a2$, while

$$\phi \text{ is symplectic} \iff -6b \text{ is a square mod } p,$$

for $E = 72q.1.a1$, $72q.2.a1$, and $72q.3.a1$.

For the curves $E = 72q.4.a2$, $72q.4.b2$, $72q.5.a2$, $72q.5.b2$, $72q.6.a1$ or $72q.6.b1$, the reduction at $\ell = 3$ is potentially good and tame (because $\nu_3(N_E) = 2$) and since $\nu_3(\Delta(E)) \neq 6$ we have $e(E, 3) = 4$. As before, it follows that $e(F, 3) = 4$ (so that $3 \nmid C$), and we may apply [Freitas and Kraus 2016, Theorem 5]. Let r and t be as in that theorem. In all cases we have $t = 0$; furthermore, we have $r = 0$ for $E = 72q.4.a2$, $E = 72q.5.a2$ or $E = 72q.6.a1$, and $r = 1$ for $E = 72q.4.b2$, $E = 72q.5.b2$ or $E = 72q.6.b1$. It follows that ϕ is always symplectic in the first cases, while

$$\phi \text{ is symplectic} \iff 3 \text{ is a square mod } p,$$

in the latter three.

6F3. *Conclusions for level $72q$.* From the calculations above we extract the following relations. For $E = 72q.1.a1$, or either of $E = 72q.2.a1$ or $E = 72q.3.a1$ with $a = 2$, it follows that

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{-6b}{p}\right),$$

while, for $E = 72q.2.a1$ or $E = 72q.3.a1$ with $a = 3$,

$$\left(\frac{\alpha}{p}\right) = \left(\frac{-6b}{p}\right) = 1.$$

If $E = 72q.4.a2$, we have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = 1,$$

while $E = 72q.5.a2$ or $E = 72q.6.a1$ give

$$\left(\frac{\alpha}{p}\right) = 1.$$

Taking $E = 72q.4.b2$ yields

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{3}{p}\right),$$

while $E = 72q.5.b2$ or $E = 72q.6.b1$ give

$$\left(\frac{\alpha}{p}\right) = \left(\frac{3}{p}\right) = 1.$$

If $E = 72q.7.a1$ or $72q.9.a2$, we have

$$\left(\frac{\alpha}{p}\right) = \left(\frac{-3b}{p}\right) = 1.$$

Finally, if $E = 72q.8.a2$,

$$\left(\frac{\alpha}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{-3b}{p}\right).$$

7. Some applications of symplectic criteria

As the preceding section reveals, there are many results we could state now for the various families of primes S_i comprising the set S_0. For simplicity, we limit ourselves to the three statements we have mentioned in our introduction (Theorems 1.7, 1.8 and 1.9) and one result valid for small values of q (Theorem 7.2).

7A. ***Proof of Theorem 1.7.*** If $q \notin S_0$, the desired conclusion is immediate from Theorem 1.4. Suppose, then, that $q \in S_0 \setminus T$ and that there exists a solution to (1-4) in coprime nonzero integers A, B and C and prime $p \geq q^{2q}$. In particular, we note, without further mention, that the primes p under consideration all satisfy $\gcd(p, 6q) = 1$. Also, we have that $p \nmid (4-a)b$, whenever these parameters appear in the sequel. From Section 2 and Theorem 1.4, it follows there exists an isomorphism $\phi : F[p] \to E[p]$, where F is the Frey–Hellegouarch curve and E is one of the curves in Corollary 3.4. Since $\alpha = 1$, we see from Proposition 6.1 that ϕ is symplectic. Furthermore, the shape of the primes in S_7 implies that $7, 19 \in S_7$ and E does not correspond to the isogeny classes $36q.2.a$, $36q.2.b$, $72q.5.a$ or $72q.5.b$. In conclusion, we need to consider E in the remaining conjugacy classes; in particular, we can either take E isogenous to one of

$$90c, \quad 306c, \quad 360a, \quad 360d, \quad 936d \quad \text{or} \quad 5256e,$$

whereby $q \in \{5, 13, 17, 73\}$, or E isomorphic to a curve in the following set:

$$\begin{aligned} E_1 = \{&18q.1.a1, 18q.2.a1, 18q.3.a1, 18q.4.a2 \quad (\text{with } \delta_1 = \delta_2 = 0, a \text{ even}, b \text{ odd}), \\ &18q.5.a2 \text{ or } 18q.5.b2 \quad (\text{with, in both cases, } \delta_1 = \delta_2 = 0 \text{ and } a \text{ even}), \\ &36q.1.a2, 36q.1.b2, 36q.3.a1, 72q.1.a1, \\ &72q.2.a1, 72q.3.a1, 72q.4.a2, 72q.4.b2, \\ &72q.7.a1, 72q.8.a2, 72q.9.a2 \quad (\text{with } \delta_1 = \delta_2 = 0, a = 4, b \text{ odd})\}. \end{aligned} \tag{7-1}$$

For $q \le 73$, the desired conclusion will follow immediately from our Theorem 7.2, which we will prove later in this section. For the remaining possible types for q, we will place a number of conditions upon p to guarantee that, in each case, ϕ is antisymplectic, providing the desired contradiction. These conditions will be of the form $\left(\frac{\kappa_i}{p}\right) = -1$, for, in each case, a finite collection of integers κ_i, and hence are each equivalent to p lying in certain residue classes modulo $8|\kappa_i|$. We remind the reader that a given prime q has at most finitely many (isogeny classes of) curves E associated to it. This will prove Theorem 1.7 provided we can show that these conditions are compatible, i.e., that we do not have three distinct indices i, say $i = 1, 2$ and 3, with $\kappa_1\kappa_2\kappa_3$ an integer square. In particular, compatibility is immediate if we have κ_i negative for each i. Our goal will be to show that, for a given prime in $q \in S_0 \setminus T$, we can always find a corresponding set of κ_i with either

(i) κ_i negative for all i, or

(ii) κ_i either positive and $\kappa_i \equiv 2 \bmod 4$, or κ_i negative and odd, or

(iii) $\kappa_i \equiv 2 \bmod 4$ for all i.

Combining the conclusions of subsections 6D3, 6E3 and 6F3, we can choose κ_i for which we require $\left(\frac{\kappa_i}{p}\right) = -1$, to contradict the fact that ϕ is symplectic, as follows.

E	κ_i	E	κ_i
$18q.1.a1$	$4-a$ or $-6b$	$72q.1.a1$	2 or $-6b$
$18q.2.a1$	$4-a$ or -6	$72q.2.a1$	$-6b$
$18q.3.a1$	$4-a$ or $-6b$	$72q.3.a1$	$-6b$
$18q.4.a2$	$12-2a$ or $-3b$	$72q.4.a2$	2
$18q.5.a2$	$12-2a$	$72q.4.b2$	2 or 3
$18q.5.b2$	$12-2a$	$72q.7.a1$	$-3b$
$36q.1.a2$	2	$72q.8.a2$	2 or $-3b$
$36q.1.b2$	6	$72q.9.a2$	$-3b$
$36q.3.a1$	$-3b$		

Here, the integers a and b are as given in the definitions of the curves E in Section 3. It is important to remember that, for a given q and corresponding type of curve E, we have not ruled out the possibility of there being more than one nonisogenous curve involved. As example (3-1) illustrates, there can certainly be nonisogenous curves associated to a fixed pair (q, E); in the case of (3-1), neither curve of the shape $E = 18q.4.a$ satisfies $a \equiv 0 \bmod 2$, $b \equiv 1 \bmod 2$.

From the preceding table, the only cases where we cannot choose κ_i to be negative are the primes q corresponding to $E = 36q.1.a2$, $36q.1.b2$, $72q.4.a2$ or $72q.4.b2$. The first two of these require $q \in S_8$, while the latter two arise from $q \in S_5$ of the form $q = 3d^2 + 4$ for integer d. In each of these cases, we can choose $\kappa_i \equiv 2 \bmod 4$ positive (see the table above).

Noting that representations of a prime q as $q = (3v^2 - 1)/2$ or $q = 3d^2 + 4$ are unique (and that, modulo 4, we cannot have both simultaneously), to conclude the proof of Theorem 1.7, it remains, then, to treat those primes $q \in S_0 \setminus T$ for which we have a solution to (1-4), which correspond to a curve in the set

$$E_2 = \{36q.1.a2, 36q.1.b2, 72q.4.a2, 72q.4.b2\}, \tag{7-2}$$

and which, further, are associated with at least one curve in the set $E_1 \setminus E_2$. We will show that, in each situation, we are in case (ii), i.e., we can find a set of κ_i with either κ_i positive and $\kappa_i \equiv 2 \bmod 4$, or κ_i negative and odd.

7A1. *The case $q \in S_8$.* In this subsection, we will show that if $q \in S_i \cap S_8$ for some $1 \leq i \leq 7$ corresponds to some $E \in E_1 \setminus E_2$, then necessarily

$$E \in \{18q.3.a1, 18q.4.a2, 72q.3.a1, 72q.7.a1, 72q.9.a2\}, \tag{7-3}$$

with q correspondingly represented in one or more of the following ways.

$$\begin{aligned}
&q = 3^{b_1} - 2^{a_1}, \text{ with } a_1 \equiv 3 \bmod 6,\ b_1 \equiv 2 \bmod 12, \text{ if } E = 18q.3.a1 \text{ or } 72q.3.a1,\\
&q = d_2^2 + 2^{a_2} 3^{b_2}, \text{ with } a_2 \geq 4 \text{ even, and } b_2 \text{ odd, if } E = 18q.4.a2 \text{ or } 72q.9.a2, \text{ or}\\
&q = (d_3^2 + 3^{b_3})/4, \text{ with } b_3 \text{ odd, if } E = 72q.7.a1.
\end{aligned}$$

Note that here, there might possibly exist more than one representation of a given prime q, with, say, distinct d_2, a_2 and b_2. From this and applying the preceding table, our set of κ_i (modulo squares) can thus be chosen to be contained in

$$\{2, 6\} \cup \{-3b_1/2\} \cup \{-3b_2\} \cup \{-3b_3\},$$

where, as desired, each integer is either positive and $\equiv 2 \bmod 4$, or negative and odd.

Suppose that $q \in S_8$. It follows that there exist integers u and v such that $q = (3v^2 - 1)/2$ where $u^2 - 3v^2 = -2$ (and hence $q \equiv 1 \bmod 4$). As noted earlier, the positive integers $v = v_k$ satisfying this latter equation also satisfy the binary recurrence (6-3). In particular, we have that $v_k \equiv 0 \bmod 3$ precisely when $k \equiv 1 \bmod 4$. For such k, we may readily show via induction that $v_k \equiv \pm 3 \bmod 13$ and hence that $3v_k^2 - 1 \equiv 0 \bmod 13$. It follows that, in order to have $q = (3v^2 - 1)/2$ prime with $u^2 - 3v^2 = -2$ for some integer u, we require that either $q = 13$, or that $v \equiv \pm 1 \bmod 3$ (whereby $q \equiv 1 \bmod 9$).

Next suppose that $q \in S_i$ for some $1 \leq i \leq 7$. From Proposition 5.1, necessarily $i \in \{2, 4, 6\}$. In particular, if our prime q is associated to some elliptic curve E in $E_1 \setminus E_2$, then, since $q \equiv 1 \bmod 36$, we have (7-3). To complete the proof of Theorem 1.7 in case $q \in S_8$, it remains to show that if we write $q = (-1)^{\delta_1} 2^a + (-1)^{\delta_2} 3^b$ with $a \geq 5$, then $\delta_1 = 1$, $\delta_2 = 0$ and we have $a \equiv 3 \mod 6$, $b \equiv 2 \mod 12$.

If, for our $q \in S_8$ with $q \neq 13$, we have $q = 2^a + 3^b$ for integers $a \geq 2$ and $b \geq 1$, then, modulo 4, b is necessarily even, so that we require $2^a \equiv 1 \mod 9$, whence $a \equiv 0 \mod 6$. It follows that $2^a + 3^b \equiv 2, 3$ or $5 \mod 7$. On the other hand, again from considering the recursion (6-3), we find that $q \equiv \pm 1 \mod 7$, a contradiction. If, instead, we have $q = 2^a - 3^b$, for $a \geq 2$ and $b \geq 1$, then, modulo 12, a is even and b is odd. If $b = 1$, then we have $2^{a+1} = 3v^2 + 5$ and so, from Lemma 5.3, since $q > 73$, a contradiction. If we suppose that $b \geq 2$, then, modulo 9, we again require that $a \equiv 0 \mod 6$, so that $2^a - 3^b \equiv 0, 3, 5, 9$ or $11 \mod 13$. On the other hand, from (6-3), we have that $q \equiv \pm 1 \mod 13$, a contradiction.

It follows that, if $q \in S_2 \cap S_8$ with $q \neq 13$, then there exist integers $a \geq 2$ and $b \geq 1$, with $q = 3^b - 2^a$. Arguing as previously, modulo $2^2 \cdot 3^3 \cdot 7$, necessarily $a \equiv 3 \mod 6$ and $b \equiv 2 \mod 6$. Working modulo 73, we find from (6-3) that $q \equiv \pm 1, \pm 34, \pm 35 \mod 73$ which shows that, in fact, $b \equiv 2 \mod 12$, as desired.

7A2. *The case $q = 3d^2 + 4$ in S_5.* In this subsection, we will show that if a prime q which can be written as $q = 3d^2 + 4$ for $d \in \mathbb{Z}$ corresponds to some $E \in E_1 \setminus E_2$, then

$$E \in \{36q.3.a1, 72q.3.a1\}, \tag{7-4}$$

with q correspondingly represented in one or more of the following ways.

$$q = \frac{d_1^2 + 3^{b_1}}{4}, \text{ with } b_1 \text{ odd, if } E = 36q.3.a1, \tag{7-5}$$

or

$$q = 3^{b_2} + (-1)^{\delta} 2^{a_2}, \text{ with } a_2 \in \{2, 3\}, b_2 \text{ odd, if } E = 72q.3.a1. \tag{7-6}$$

To see this, begin by supposing that $q = 3d^2 + 4$ for an (odd) integer d. Modulo 8, we cannot have $q = 3d_1^2 + 2^\alpha$ for integer d_1 and $\alpha \geq 4$. Further, applying Proposition 5.1, $q \notin S_i$ for each $i \in \{1, 3, 4, 8\}$, while the assumption that $q \in S_0 \setminus T$ implies $q \notin S_7$. It follows that if $q \in S_i$ for some $i \neq 5$, then we must have $i \in \{2, 6\}$ and hence that if q corresponds to some $E \in E_1 \setminus E_2$, then E is one of $18q.3.a1$, $36q.3.a1$, $72q.3.a1$ or $72q.7.a1$. The last of these possibilities is eliminated modulo 4.

If we can write $q = (-1)^{\delta_1} 2^a + (-1)^{\delta_2} 3^b$, for $a \in \{2, 3\}$ or $a \geq 5$, and $b \geq 1$, then, modulo 3, $\delta_1 \equiv a \mod 2$, while, modulo 8, either $a = 2$, $\delta_1 = \delta_2 = 0$, $b \equiv 1 \mod 2$ and $d = 3^{(b-1)/2}$, or we have $a \geq 3$, $b \equiv 0 \mod 2$ and $\delta_2 = 1$. In this latter case, we also have $\delta_1 = 0$, $a \equiv 0 \mod 2$ and hence

$$2^{a/2} - 3^{b/2} = 1 \quad \text{and} \quad 2^{a/2} + 3^{b/2} = q.$$

If $a = 4$, we find that $q = 7$. Otherwise, we have $a \geq 6$ and hence the first equation here has no solutions modulo 8, eliminating the possibility that $E = 18q.3.a1$. We therefore conclude that E satisfies (7-4) (and, additionally, that if $q = 3d^2 + 4 \in S_2$, then $d = 3^{(b-1)/2}$ for some odd integer b).

A priori, at this point, all we can conclude is that our set of κ_i is contained in

$$\{2\} \cup \{-3b_1\} \cup \{-6b_2\},$$

where the exponents b_1 and b_2 are as in (7-5) and (7-6). Since both b_1 and b_2 are odd, we cannot immediately conclude that our set of κ_i satisfies any of (i), (ii) or (iii). To show that it is indeed compatible, we will appeal to the following result:

Lemma 7.1. *If d is an integer such that $q = 3d^2 + 4$ is prime with, additionally, $q \in S_2 \cap S_6$, then $q \in \{7, 31\}$.*

Proof of Lemma 7.1. Let us suppose that $q = 3d^2 + 4$ is prime with $q \in S_2 \cap S_6$. Then, from our prior work, we can write

$$q = 3^{b_2} + 4 = \frac{d_6^2 + 3^{b_6}}{4},$$

for odd positive integers b_2, d_6 and b_6, so that

$$d_6^2 = 4 \cdot 3^{b_2} - 3^{b_6} + 16. \tag{7-7}$$

In general, this equation has precisely the solutions

$$(d_6, b_2, b_6) = (1, 1, 3),\ (5, 1, 1),\ (11, 3, 1) \text{ and } (31, 5, 3)$$

in odd positive integers; none of these correspond to a prime values of $q > 73$. To prove this, note that an elementary argument easily yields that $b_2 > b_6$ unless $|d_6| \le 5$. We may thus write $d_6 = 3^{b_6} \cdot k_1 + (-1)^{\delta} 4$, for some $\delta \in \{0, 1\}$ and $k_1 \equiv \pm 1 \bmod 6$ a positive integer. Substituting into (7-7), we have

$$3^{b_6} k_1^2 + (-1)^{\delta} 8 \cdot k_1 = 4 \cdot 3^{b_2 - b_6} - 1.$$

If $k_1 = 1$, then, modulo 3, we have $3^{b_6 - 1} + 3 = 4 \cdot 3^{b_2 - b_6 - 1}$, corresponding to $(d_6, b_2, b_6) = (31, 5, 3)$. If $k_1 > 1$ then $k_1 \ge 5$ and necessarily $b_2 > 2b_6$. It follows that we can write $(-1)^{\delta} 8 \cdot k_1 + 1 = 3^{b_6} \cdot k_2$ for a (nonzero) integer $k_2 \equiv 3 \bmod 8$, so that

$$(3^{b_6} k_2 - 1)^2 + 64 k_2 = 256 \cdot 3^{b_2 - 2b_6}. \tag{7-8}$$

We check that the only solution to this equation with $k_2 \in \{-13, -5, 3, 11\}$ corresponds to $(d_6, b_2, b_6) = (11, 3, 1)$; otherwise, after a little work, we may suppose that $b_2 > 4b_6$ and hence that $-2k_2 3^{b_6} + 64k_2 + 1$ is divisible by 3^{2b_6} (and hence $|2k_2 3^{b_6} - 64k_2 - 1| \ge 3^{2b_6}$). It follows that either $b_6 \le 3$, or that we have $|k_2| > 3^{b_6 - 1}$. From (7-8), after a little more work, we may thus conclude that either $b_6 \in \{1, 3\}$, or that $b_2 \ge 6b_6 - 7$.

On the other hand, applying Theorem 1.5 of [Bauer and Bennett 2002], with (in the notation of that theorem)

$$(a, y, x_0, m_0, \Delta, \alpha, s) = (1, 3, 3788, 15, -37, 3.1, 2),$$

we find that

$$\left| \sqrt{3} - \frac{p}{2 \cdot 3^k} \right| > e^{-170} 3^{-1.64281k},$$

for p and k positive integers with $k \geq 4775$. It follows that

$$|p^2 - 4 \cdot 3^{2k+1}| > 4 \cdot e^{-170} 3^{0.35719k},$$

provided $k \geq 4775$. Applying this with $p = d_6$ and $b_2 = 2k+1$, (7-7) thus implies that either $b_6 \in \{1, 3\}$ or we have

$$3^{(b_2+7)/6} \geq 3^{b_6} > 4 \cdot e^{-170} 3^{0.35719(b_2-1)/2},$$

whence $b_2 \leq 12979$. A brute-force search confirms that (7-7) has only the listed solutions.

We thus have $q = 3^{b_2} + 4$ for $b_2 \in \{1, 3, 5\}$, whereby, since we assume that q is prime, $q \in \{7, 31\}$. □

Applying Lemma 7.1 and assuming that $q > 73$, we can therefore conclude that if $q = 3d^2 + 4$, then our set of κ_i is contained in either

$$\{2\} \cup \{-3b_1\} \quad \text{or} \quad \{2\} \cup \{-6b_2\},$$

where, again, the exponents b_1 and b_2 are as in (7-5) and (7-6). Since both b_1 and b_2 are odd, it follows that our set of κ_i is compatible of type either (ii) or (iii), respectively. This completes the proof of Theorem 1.7.

7B. ***Proof of Theorem 1.8.*** Let $q = 2^a 3^b - 1$ with $a \geq 5$ and $b \geq 1$ be a prime. Then $q \in S_1$ and hence, from Proposition 5.1, $q \notin S_i$ for $i \in \{2, 3, 5, 7, 8\}$. On the other hand, $q \notin S_i$ for $i \in \{4, 6\}$, since $q \equiv 2 \bmod 3$. It follows that, in this case, a solution to (1-2) with $p \geq q^{2q}$ necessarily corresponds to an elliptic curve in the isogeny class $18q.1.a$. The result now follows from the equalities in (6-1).

7C. ***Proof of Theorem 1.9.*** Suppose that A, B and C are coprime, nonzero integers satisfying (1-5) with $p \geq 17$, and write F for the corresponding Frey–Hellegouarch curve. Note that, for $q = 5$, we are led to consider levels 90, 180 and 360. For these levels, each weight 2, cuspidal newform f corresponds to one of the 9 isogeny classes of elliptic curves $E/\mathbb{Q}$ given in Cremona's notation by

$$90a, 90b, 90c, 180a, 360a, 360b, 360c, 360d \text{ and } 360e.$$

For E in the isogeny classes $90a$, $90b$, $180a$, $360b$ and $360c$, we find that $a_7(E) = 2$ and hence, it follows from (4-3), the Hasse bound and the level lowering condition that

$$2 \equiv 0, \pm 4, \pm 8 \bmod p.$$

This gives a contradiction with $p \geq 17$.

Next, we treat the isogeny class $360e$. Taking $E = 360e2$, we find that $e(E, 3) = 2$. In the beginning of Section 6D2, it is explained that either F has potentially multiplicative reduction at $\ell = 3$ or potentially good reduction with $e(F, 3) = 4$, a contradiction in either cases.

Finally, suppose that E is in one of the isogeny classes $90c$, $360a$ and $360d$, say, $E = 90c2$, $360a2$ or $360d2$. We will apply [Freitas 2016, Theorem 4] and [Kraus and Oesterlé 1992, Proposition 2] with $\ell \in \{2, 3, q\}$. In all cases, from [Kraus and Oesterlé 1992, Proposition 2] with $\ell = q$, we have that our isomorphism between $F[p]$ and $E[p]$ is necessarily symplectic. If $E = 90c2$, we may thus further appeal to [Kraus and Oesterlé 1992, Proposition 2] with $\ell = 2$ and $\ell = 3$ (after suitable twist) to conclude that

$$\left(\frac{-1}{p}\right) = \left(\frac{-2}{p}\right) = 1. \tag{7-9}$$

For $E = 360a2$, we apply [Freitas 2016, Theorem 4] and [Kraus and Oesterlé 1992, Proposition 2] with $\ell = 3$, whereby

$$\left(\frac{2}{p}\right) = \left(\frac{-3}{p}\right) = 1. \tag{7-10}$$

If $E = 360d2$, we apply [Kraus and Oesterlé 1992, Proposition 2] with $\ell = 3$ to conclude that

$$\left(\frac{-6}{p}\right) = 1. \tag{7-11}$$

We reach our desired conclusion upon observing that, if $p \equiv 13, 19$ or $23 \bmod 24$, then each of (7-9), (7-10) and (7-11) fails to hold.

7D. ***Further results for small primes q.*** To conclude this paper, we will provide some more explicit results for small values of q. We obtain these by proceeding in a similar fashion to the proof of Theorem 1.9. Making the further assumption that $p \geq q^{2q}$, we reduce the calculation to consideration of elliptic curves E with nontrivial rational 2-torsion, conductor in the set $\{18q, 36q, 72q\}$ and such that $\Delta(E)$ is of the shape T^2 or $-3T^2$ for some integer T (i.e., those corresponding to primes in S_0). We summarize our results as follows.

Theorem 7.2. *If p and q are primes with $p \geq q^{2q}$, then there are no coprime, nonzero integers A, B and C satisfying equation* (1-4) *with q in the following table and p satisfying the listed conditions.*

q	p	q	p
5	13, 19, 23 mod 24	47	5, 11, 13, 17, 19, 23 mod 24
11	13, 17, 19, 23 mod 24	59	5, 7, 11, 13, 19, 23 mod 24
13	11 mod 12	67	7, 11, 13, 29, 37, 41, 43, 59, 67, 71, 89, 101, 103 mod 120
17	5, 17, 23 mod 24	71	5 mod 6
23	19, 23 mod 24	73	41, 71, 89 mod 120
29	7, 11, 13, 17, 19, 23 mod 24	79	5, 7, 11, 13, 19, 23 mod 24
31	5, 11 mod 24	89	13, 17, 19, 23 mod 24
41	5, 7, 11, 17, 19, 23 mod 24	97	11 mod 12

Here, we have omitted both primes for which Theorem 1.4 applies directly (i.e., $q = 53$ and 83, according to Corollary 1.6) and also primes for which the symplectic method fails to eliminate exponents, i.e., $q \in \{7, 19, 37, 43, 61\}$. For these latter primes, observe that, in each case, q is of the shape $(3d^2+1)/4$ or $3d^2 + 16$ for an integer d; as explained in Section 5B, these are those primes for which there exists a solution to (1-4) (with $C = 1$) for every exponent p (whereby we expect our techniques to fail), together with those for which we have a "trivial" solution to the related equation $A^3 + B^3 = 8qC^p$, again for every p.

Appendix: c-invariants

curve	c_4	c_6
$18q.1.a1$	$3^2(2^{2a}3^{2b}+(-1)^\delta 2^a 3^b+1)$	$(-1)^{\delta+1}3^3(2^{a+1}3^b+(-1)^\delta)(2^a3^b+(-1)^{\delta+1})(2^{a-1}3^b+(-1)^\delta)$
$18q.1.a2$	$3^2(2^{2a+4}3^{2b}+(-1)^\delta 2^{a+4}3^b+1)$	$3^3(2^{a+1}3^b+(-1)^\delta)(2^{2a+5}3^{2b}+(-1)^\delta 2^{a+5}3^b-1)$
$18q.1.a3$	$3^2(2^{2a-4}3^{2b}+(-1)^\delta 2^a 3^b+1)$	$(-1)^{\delta+1}3^3(2^{a-1}3^b+(-1)^\delta)(2^{2a-5}3^{2b}+(-1)^{\delta+1}2^a3^b-1)$
$18q.1.a4$	$3^2(2^{2a}3^{2b}+(-1)^{\delta+1}7\cdot 2^{a+1}3^b+1)$	$(-1)^{\delta+1}3^3(2^a3^b+(-1)^{\delta+1})(2^{2a}3^{2b}+(-1)^\delta 17\cdot 2^{a+1}3^b+1)$
$18q.2.a1$	$3^2(2^{2a}+2^a+1)$	$-3^3(2^{a+1}+1)(2^a-1)(2^{a-1}+1)$
$18q.2.a2$	$3^2(2^{2a+4}+2^{a+4}+1)$	$-3^3(2^{a+1}+1)(2^{2a+5}+2^{a+5}-1)$
$18q.2.a3$	$3^2(2^{2a-4}+2^a+1)$	$-3^3(2^{a-1}+1)(2^{2a-5}-2^a-1)$
$18q.2.a4$	$3^2(2^{2a}-7\cdot 2^{a+1}+1)$	$-3^3(2^a-1)(2^{2a}+17\cdot 2^{a+1}+1)$
$18q.3.a1$	$3^2(2^{2a}+(-1)^{\delta_1+\delta_2}2^a3^b+3^{2b})$	$(-1)^\delta 3^3(2^{a+1}+(-1)^{\delta_1+\delta_2}3^b)(2^a-(-1)^{\delta_1+\delta_2}3^b)(2^{a-1}+(-1)^{\delta_1+\delta_2}3^b)$
$18q.3.a2$	$3^2(2^{2a+4}+(-1)^{\delta_1+\delta_2}2^{a+4}3^b+3^{2b})$	$(-1)^\delta 3^3(2^{a+1}+(-1)^{\delta_1+\delta_2}3^b)(2^{2a+5}+(-1)^{\delta_1+\delta_2}2^{a+5}3^b-3^{2b})$
$18q.3.a3$	$3^2(2^{2a-4}+(-1)^{\delta_1+\delta_2}2^a3^b+3^{2b})$	$(-1)^{b+1}3^3((-1)^{\delta_1+\delta_2}2^{a-1}+3^b)(2^{2a-5}-(-1)^{\delta_1+\delta_2}2^a3^b-3^{2b})$
$18q.3.a4$	$3^2(2^{2a}+(-1)^{1+\delta_1+\delta_2}7\cdot 2^{a+1}3^b+3^{2b})$	$(-1)^b 3^3(3^b+(-1)^{b+\delta}2^a)(2^{2a}+(-1)^{\delta_1+\delta_2}17\cdot 2^{a+1}3^b+3^{2b})$
$18q.4.a1$	$(-1)^{\delta_1}3^2(q-(-1)^{\delta_2}2^{a-2}3^b)$	$3^3d(d^2+(-1)^{\delta_1+\delta_2}2^{a-3}3^{b+2})$
$18q.4.a2$	$(-1)^{\delta_1}3^2(q-(-1)^{\delta_2}2^{a+2}3^b)$	$3^3d(d^2+(-1)^{\delta_1+\delta_2}2^a3^{b+2})$
$18q.5.a1$	$3^2(d^2+(-1)^{\delta_1+\delta_2}2^{a-2})$	$3^3d(d^2+(-1)^{\delta_1+\delta_2}2^{a-3}3)$
$18q.5.a2$	$3^2(d^2-(-1)^{\delta_1+\delta_2}2^a)$	$3^3d(d^2+(-1)^{\delta_1+\delta_2}2^a3)$
$18q.5.b1$	$3^4(d^2+(-1)^{\delta_1+\delta_2}2^{a-2})$	$-3^6d(d^2+(-1)^{\delta_1+\delta_2}2^{a-3}3)$
$18q.5.b2$	$3^4(d^2-(-1)^{\delta_1+\delta_2}2^a)$	$-3^6d(d^2+(-1)^{\delta_1+\delta_2}2^a3)$
$18q.6.a1$	$3^2(d^2+3\cdot 2^{a-2})$	$3^3d(d^2+2^{a-3}3^2)$
$18q.6.a2$	$3^2(d^2-3\cdot 2^a)$	$3^3d(d^2+2^a3^2)$

Table 1. Data for curves with conductor $18q$.

curve	c_4	c_6	curve	c_4	c_6
$36q.1.a1$	$2^43(q^2-1)$	$-2^53^2rs(q^2+2)$	$36q.4.a1$	$2^43^2(d^2-3^{b+1})$	$2^53^3d(2d^2-3^{b+2})$
$36q.1.a2$	$2^43(16q^2-1)$	$-2^63^2rs(32q^2+1)$	$36q.4.a2$	$2^43^2(d^2+4\cdot 3^{b+1})$	$2^63^3d(d^2-4\cdot 3^{b+2})$
$36q.1.b1$	$2^43^3(q^2-1)$	$2^53^5rs(q^2+2)$	$36q.5.a1$	$2^43^2(d^2-3^{b+1})$	$2^53^3d(2d^2-3^{b+2})$
$36q.1.b2$	$2^43^3(16q^2-1)$	$2^63^5rs(32q^2+1)$	$36q.5.a2$	$2^43^2(d^2+4\cdot 3^{b+1})$	$2^63^3d(d^2-4\cdot 3^{b+2})$
$36q.2.a1$	$2^23^2(d^2-1)$	$-2^33^3d(d^2+3)$	$36q.6.a1$	$2^43^2(d^2-1)$	$2^53^3d(2d^2-3)$
$36q.2.a2$	$2^43^2(4d^2+1)$	$-2^63^3d(8d^2+3)$	$36q.6.a2$	$2^43^2(d^2+4)$	$2^63^3d(d^2-12)$
$36q.2.b1$	$2^23^4(d^2-1)$	$2^33^6d(d^2+3)$	$36q.6.b1$	$2^43^4(d^2-1)$	$-2^53^6d(2d^2-3)$
$36q.2.b2$	$2^43^4(4d^2+1)$	$2^63^6d(8d^2+3)$	$36q.6.b2$	$2^43^4(d^2+4)$	$-2^63^6d(d^2-12)$
$36q.3.a1$	$2^23^2(d^2-3^{b+1})$	$-2^33^3d(d^2+3^{b+2})$	$36q.7.a1$	$2^43^2(d^2+3^{b+1})$	$2^53^3d(2d^2+3^{b+2})$
$36q.3.a2$	$2^43^2(4d^2+3^{b+1})$	$-2^63^3d(8d^2+3^{b+2})$	$36q.7.a2$	$2^43^2(d^2-4\cdot 3^{b+1})$	$2^63^3d(d^2+4\cdot 3^{b+2})$

Table 2. Data for curves with conductor $36q$.

curve	c_4	c_6
$72q.1.a1$	$2^4 3^2(3^{2b}+3^b+1)$	$2^5 3^3(3^b-1)(3^b+2)(2\cdot 3^b+1)$
$72q.1.a2$	$2^4 3^2(16\cdot 3^{2b}+16\cdot 3^b+1)$	$2^6 3^3(2\cdot 3^b+1)(32\cdot 3^{2b}+32\cdot 3^b-1)$
$72q.1.a3$	$2^4 3^2(16\cdot 3^{2b}+16\cdot 3^b+1)$	$2^6 3^3(2\cdot 3^b+1)(32\cdot 3^{2b}+32\cdot 3^b-1)$
$72q.1.a4$	$2^4 3^2(16\cdot 3^{2b}+16\cdot 3^b+1)$	$2^6 3^3(2\cdot 3^b+1)(32\cdot 3^{2b}+32\cdot 3^b-1)$
$72q.2.a1$	$2^4 3^2(2^{2a}3^{2b}+(-1)^\delta 2^a 3^b+1)$	$-2^6 3^3((-1)^\delta 2^{a+1}3^b+1)(2^{2a-1}3^{2b}+(-1)^\delta 2^{a-1}3^b-1)$
$72q.2.a2$	$2^4 3^2(2^{2a+4}3^{2b}+(-1)^\delta 2^{a+4}3^b+1)$	$-2^6 3^3((-1)^\delta 2^{a+1}3^b+1)(2^{2a+5}3^{2b}+(-1)^\delta 2^{a+5}3^b-1)$
$72q.2.a3$	$2^4 3^2(2^{2a-4}3^{2b}+(-1)^\delta 2^a 3^b+1)$	$2^5 3^3((-1)^\delta 2^{a-1}3^b+1)(-2^{2a-4}3^{2b}+(-1)^\delta 2^{a+1}3^b+2)$
$72q.2.a4$	$2^4 3^2(2^{2a}3^{2b}+(-1)^\delta 7\cdot 2^{a+1}3^b+1)$	$-2^6 3^3((-1)^\delta 2^a 3^b-1)(2^{2a}3^{2b}+(-1)^\delta 17\cdot 2^{a+1}3^b+1)$
$72q.3.a1$	$2^4 3^2(3^{2b}+(-1)^\delta 2^a 3^b+2^{2a})$	$2^6 3^3(-1)^b(3^b-(-1)^\delta 2^a)(2^{2a}+(-1)^\delta 5\cdot 2^{a-1}3^b+3^{2b})$
$72q.3.a2$	$2^4 3^2(3^{2b}-(-1)^\delta 7\cdot 2^{a+1}3^b+2^{2a})$	$2^6 3^3(-1)^b(3^b-(-1)^\delta 2^a)(2^{2a}+(-1)^\delta 17\cdot 2^{a+1}3^b+3^{2b})$
$72q.3.a3$	$2^4 3^2(3^{2b}+(-1)^\delta 2^{a+4}3^b+2^{2a+4})$	$-2^6 3^3(-1)^b(3^b+(-1)^\delta 2^{a+1})(2^{2a+5}+(-1)^\delta 2^{a+5}3^b-3^{2b})$
$72q.3.a4$	$2^4 3^2(3^{2b}+(-1)^\delta 2^a 3^b+2^{2a-4})$	$2^5 3^3(-1)^b(3^b+(-1)^\delta 2^{a-1})(-2^{2a-4}+(-1)^\delta 2^{a+1}3^b+2\cdot 3^{2b})$
$72q.4.a1$	$2^4 3^2(d^2+1)$	$-2^5 3^3 d(2d^2+3)$
$72q.4.a2$	$2^4 3^2(d^2-4)$	$-2^6 3^3 d(d^2+12)$
$72q.4.b1$	$2^4 3^4(d^2+1)$	$2^5 3^6 d(2d^2+3)$
$72q.4.b2$	$2^4 3^4(d^2-4)$	$2^6 3^6 d(d^2+12)$
$72q.5.a1$	$2^4 3^2(d^2+(-1)^\delta 2^{a-2})$	$-2^5 3^3 d((-1)^\delta 2d^2+3\cdot 2^{a-2})$
$72q.5.a2$	$2^4 3^2(d^2-(-1)^\delta 2^a)$	$-2^6 3^3 d((-1)^\delta d^2+3\cdot 2^a)$
$72q.5.b1$	$2^4 3^4(d^2+(-1)^\delta 2^{a-2})$	$2^5 3^6 d((-1)^\delta 2d^2+3\cdot 2^{a-2})$
$72q.5.b2$	$2^4 3^4(d^2-(-1)^\delta 2^a)$	$2^6 3^6 d((-1)^\delta d^2+3\cdot 2^a)$
$72q.6.a1$	$2^2 3^2(d^2-1)$	$2^3 3^3 d(d^2+3)$
$72q.6.a2$	$2^4 3^2(4d^2+1)$	$2^6 3^3 d(8d^2+3)$
$72q.6.b1$	$2^2 3^4(d^2-1)$	$-2^3 3^6 d(d^2+3)$
$72q.6.b2$	$2^4 3^4(4d^2+1)$	$-2^6 3^6 d(8d^2+3)$
$72q.7.a1$	$2^2 3^2(d^2-3^{b+1})$	$2^3 3^3 d(d^2+3^{b+2})$
$72q.7.a2$	$2^4 3^2(4d^2-3^{b+1})$	$2^6 3^3 d(8d^2+3^{b+2})$
$72q.8.a1$	$2^4 3^2(d^2+3^{b+1})$	$-2^5 3^3 d(2d^2+3^{b+2})$
$72q.8.a2$	$2^4 3^2(d^2-4\cdot 3^{b+1})$	$-2^6 3^3 d(d^2+4\cdot 3^{b+2})$
$72q.9.a1$	$2^4 3^2(d^2+(-1)^{\delta_1+\delta_2}2^{a-2}3^{b+1})$	$2^6 3^3 d(d^2+(-1)^{\delta_1+\delta_2}2^{a-3}3^{b+2})$
$72q.9.a2$	$2^4 3^2(d^2+(-1)^{\delta_1+\delta_2+1}2^a\cdot 3^{b+1})$	$2^6 3^3 d(d^2+(-1)^{\delta_1+\delta_2}2^a\cdot 3^{b+2})$
$72q.10.a1$	$2^4 3^2(d^2-2^{a-2}3^{b+1})$	$2^6 3^3 d(d^2-2^{a-3}3^{b+2})$
$72q.10.a2$	$2^4 3^2(d^2+2^a\cdot 3^{b+1})$	$2^6 3^3 d(d^2-2^a\cdot 3^{b+2})$
$72q.11.a1$	$2^4 3^2(d^2+24)$	$2^6 3^3 d(d^2+36)$
$72q.11.a2$	$2^4 3^2(d^2-96)$	$2^6 3^3 d(d^2+288)$
$72q.12.a1$	$2^4 3^2(d^2+24)$	$2^6 3^3 d(d^2+36)$
$72q.12.a2$	$2^4 3^2(d^2-96)$	$2^6 3^3 d(d^2+288)$

Table 3. Data for curves with conductor $72q$.

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