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We prove that any complex or real analytic set or function germ is topologically equivalent to a germ defined by polynomial equations whose coefficients are algebraic numbers.

The problem of the algebraicity of analytic sets or mappings is an old subject of study. It is known that the germ of a coherent analytic set with an isolated singularity is analytically equivalent to the germ of an algebraic set [Kucharz 1986; Tougeron 1976]. But in the general case the germ of an analytic set is not even locally diffeomorphic to the germ of an algebraic set [Whitney 1965]. On the other hand, considering a weaker equivalence relation, T. Mostowski [1984] proved that the germ of an analytic set is always homeomorphic to the germ of an algebraic set and this has been generalized to analytic function germs [Bilski et al. 2017]. For practical, effective and sometimes even theoretical purposes (for instance see [Budur and Wang 2017]) it is often not possible to handle coefficients that are transcendental numbers and so it is important to work with polynomial equations whose coefficients are rational or algebraic numbers. But it is well known that a small perturbation of the coefficients of polynomial equations defining an algebraic set germ or an algebraic function germ can drastically change the topology of the germ.

The goal of this paper is to extend the results of [Bilski et al. 2017] by proving that any complex or real analytic set or function germ is homeomorphic to an algebraic germ defined over the algebraic numbers. Our main result is the following one:

Theorem 1. *Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be an analytic set germ and $g : (V, 0) \rightarrow (\mathbb{K}, 0)$ be an analytic function germ. Then there is a homeomorphism*

$$h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$$

such that

- (i) $(h(V), 0)$ is the germ of an algebraic subset of \mathbb{K}^n defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$,
- (ii) $g \circ h^{-1}$ is the germ of a polynomial function defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$.

Moreover when we consider the particular case where there is no function germ g but only the set germ $(V, 0)$ we can be more precise about the nature of the homeomorphism:

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Theorem 2. *Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be an analytic set germ. Then there is a homeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that*

- (i) *$h(V)$ is the germ of an algebraic subset of \mathbb{K}^n defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$,*
- (ii) *$(V, 0)$ is Whitney equisingular with $(h(V), 0)$,*
- (iii) *h is subanalytic and arcanalytic.*

The proof of our main result is based on the approach introduced in [Mostowski 1984] and extended in [Bilski et al. 2017]. For instance the idea to prove Theorem 2 in the case where $(V, 0)$ is a hypersurface germ is to use a version of the nested Artin–Płoski–Popescu approximation theorem, which we prove in this paper (see Theorem 12), in order to construct a regular Zariski equisingular deformation of $(V, 0)$ such that one of the fibers is the germ of a Nash hypersurface defined over $\overline{\mathbb{Q}}$. By a refinement of a theorem of Varčenko [1972] due to A. Parusiński and L. Păunescu [2017] such a deformation is a Whitney equisingular deformation so it is topologically trivial and the trivialization is subanalytic and arcanalytic. Then we use the Artin–Mazur theorem to transform our germ of a Nash set into the germ of an algebraic set (still defined over $\overline{\mathbb{Q}}$) by a local diffeomorphism. For Theorem 1 the idea is to apply essentially the same procedure to the graph of g , and the main difference concerns the part where we transform a Nash function germ into an algebraic function germ since the Artin–Mazur theorem is not sufficient to do this transformation. This part requires the construction of a particular deformation of the Nash set germ which is topologically trivial thanks to the Thom–Mather isotopy lemma. The paper is organized as follows: The first and main part is devoted to giving an algebraic statement concerning complex-coefficient algebraic power series that are solutions of algebraic equations with coefficients in $\overline{\mathbb{Q}}$. It shows that such solutions are \mathbb{C} -points of a family of algebraic solutions defined over $\overline{\mathbb{Q}}$ (see Theorem 7). In the next parts we apply this statement to prove Theorem 2 and then Theorem 1, essentially by proving that the approach used in [Bilski et al. 2017] remains valid in our situation.

Remark 3. Let us mention that B. Teissier [1990] provided an example of the germ of a complex algebraic surface in $(\mathbb{C}^3, 0)$ defined by a polynomial equation with coefficients in $\mathbb{Q}[\sqrt{5}]$ which is not Whitney equisingular to the germ of an algebraic set defined over \mathbb{Q} . So we cannot replace $\overline{\mathbb{Q}}$ by \mathbb{Q} in the statement of Theorem 2.

Remark 4. It is known that the germ of an analytic set is not always diffeomorphic to the germ of an algebraic set (see [Whitney 1965]). Let us mention that in general the germ of an algebraic set is neither diffeomorphic to the germ of an algebraic set defined over $\overline{\mathbb{Q}}$. For instance let us consider the germ of the curve $(V, 0) \subset (\mathbb{R}^2, 0)$ defined by the equation

$$xy(x - y)(x - \xi y),$$

where $\xi \in \mathbb{R}$ is a transcendental number. Indeed $(V, 0)$ is the union of four lines whose cross-ratio is ξ . If $(V, 0)$ were diffeomorphic to the germ of an algebraic set $(W, 0)$ defined over $\overline{\mathbb{Q}}$, the differential of the diffeomorphism germ would induce a bijective linear map between the tangent spaces at 0 of V and W .

Such a linear map preserves the cross-ratio, so the tangent space of $(W, 0)$ at 0 would be the union of four lines whose cross-ratio is equal to ξ and this would not be possible since $(W, 0)$ would be defined by algebraic equations with coefficients in $\overline{\mathbb{Q}}$. This example extends to the case where $\mathbb{K} = \mathbb{C}$ much as was done in [Bilski et al. 2017, Example 6.2].

Remark 5. In general for a given analytic map germ $g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^m, 0)$ there is no germ of a homeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that $g \circ h$ is the germ of a polynomial map: just take $g : (\mathbb{K}, 0) \rightarrow (\mathbb{K}^2, 0)$ given by $g(x) = (x, e^x)$ (see [Bilski et al. 2017, Example 6.3]). In particular **Theorem 1** cannot be extended to analytic map germs $(\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^m, 0)$. But in general, even if $g : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^m, 0)$ is the germ of a polynomial map there is no germ of a homeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that $g \circ h$ is the germ of a polynomial map defined over $\overline{\mathbb{Q}}$. Indeed let $g : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ be defined by $g(x) = (x, \xi x)$ where $\xi \in \mathbb{C}$ is a transcendental number. If there were a homeomorphism germ $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that $g \circ h$ is the germ of a polynomial map defined over $\overline{\mathbb{Q}}$ then both $h(x)$ and $\xi h(x)$ would be algebraic over $\overline{\mathbb{Q}}[x]$. But this would imply that ξ is algebraic over \mathbb{Q} which is not possible.

Remark 6. In **Theorem 1** we do not know if the germ of a homeomorphism can be chosen to be arcanalytic or subanalytic. Indeed the proof of this result goes as follows: First we construct a Zariski equisingular deformation of the graphs of g and of the function germs defining $(V, 0)$ with the graphs of a Nash function germ \tilde{g} and of function germs defining the germ of a Nash set $(\tilde{V}, 0)$. Using the Artin–Mazur theorem we can reduce the situation to the case where $(\tilde{V}, 0)$ is the germ of an algebraic set and \tilde{g} is a unit u times a polynomial function germ P . Then, in [Bilski et al. 2017] a Thom stratification of the deformation

$$(t, x) \rightarrow (1 - t)u(0)P(x) + tu(x)P(x)$$

is constructed which shows (by the Thom–Mather isotopy lemma) that the function germ $\tilde{g} = uP$ is homeomorphic to the function germ P . But the Thom–Mather isotopy lemma does not provide an arcanalytic or subanalytic homeomorphism in general.

Notation and terminology. We will denote by x and y the vectors of indeterminates (x_1, \dots, x_n) and (y_1, \dots, y_m) . The notation x^i denotes the vector of indeterminates (x_1, \dots, x_i) for any $i \leq n$. When $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we denote by $\mathbb{K}\{x\}$ the ring of convergent power series with coefficients in \mathbb{K} , and by $\mathbb{K}\langle x \rangle$ the ring of algebraic power series with coefficients in \mathbb{K} . This means that $\mathbb{K}\langle x \rangle$ is the subring of $\mathbb{K}[[x]]$ whose elements are algebraic over $\mathbb{K}[x]$. We have $\mathbb{K}\langle x \rangle \subset \mathbb{K}\{x\}$; i.e., every algebraic power series is convergent.

Let $\mathbb{K} = \mathbb{C}$ or \mathbb{R} . Let Ω be an open subset of \mathbb{K}^n and let f be an analytic function on Ω . We say that f is a *Nash function* at $p \in \Omega$ if its Taylor expansion at p is an algebraic power series. An analytic function on Ω is a Nash function if it is a Nash function at every point of Ω . An analytic mapping $\varphi : \Omega \rightarrow \mathbb{K}^N$ is a *Nash mapping* if all its components are Nash functions on Ω . A subset X of Ω is called a *Nash subset* of Ω if for every $p \in \Omega$ there exist an open neighborhood U of p in Ω and Nash functions f_1, \dots, f_s on U such that $X \cap U = \{z \in U \mid f_1(z) = \dots = f_s(z) = 0\}$. A germ X_p of a set X at $p \in \Omega$ is a *Nash germ* if

there exists an open neighborhood U of p in Ω such that $X \cap U$ is a Nash subset of U . A Nash function germ is said to be defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$ if it satisfies a nontrivial polynomial equation with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$. This is equivalent to saying that its Taylor expansion at a $\overline{\mathbb{Q}} \cap \mathbb{K}$ -point is an algebraic power series whose coefficients are in $\overline{\mathbb{Q}} \cap \mathbb{K}$, i.e., an element of $(\overline{\mathbb{Q}} \cap \mathbb{K})[[x]]$. A Nash set is said to be defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$ if it is locally defined by Nash function germs defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$.

1. An approximation result

We begin by stating the main result of this part:

Theorem 7. *Let $f(x, y) \in \overline{\mathbb{Q}}\langle x \rangle[y]^p$ and let us consider a solution $y(x) \in \mathbb{C}\langle x \rangle^m$ of*

$$f(x, y(x)) = 0.$$

Then there exist a new set of indeterminates $t = (t_1, \dots, t_r)$, a vector of algebraic power series

$$y(t, x) = \sum_{\alpha \in \mathbb{N}^n} y_\alpha(t) x^\alpha \in \overline{\mathbb{Q}}\langle t, x \rangle^m$$

and $\mathbf{t} = (t_1, \dots, t_r) \in \mathbb{C}^r$ belonging to the domain of convergence of all the $y_\alpha(t)$ such that

$$y(x) = y(\mathbf{t}, x) \quad \text{and} \quad f(x, y(\mathbf{t}, x)) = 0.$$

Remark 8. This theorem is not true if we replace $\overline{\mathbb{Q}}$ by \mathbb{Q} . For instance let x and y be single indeterminates and set $f = y^2 - 2x^2$. Then there is no algebraic power series $y(x, t) \in \mathbb{Q}\langle x, t \rangle$ such that

$$y(x, t)^2 - 2x^2 = 0$$

but we have

$$f(x, \sqrt{2}x) = 0.$$

Proof of Theorem 7. If $y(x) \in \overline{\mathbb{Q}}\langle x \rangle^m$ then we take $r = 0$ and there is nothing to prove. Let us assume that $y(x) \in \mathbb{C}\langle x \rangle^m \setminus \overline{\mathbb{Q}}\langle x \rangle^m$. By Lemma 10 given below we may assume that there exist $y'(t, u, v, x) \in \overline{\mathbb{Q}}\langle t, u, v, x \rangle^m \cap \overline{\mathbb{Q}}[t, u, v][[x]]^m$, where $t = (t_1, \dots, t_r)$ and u and v are single indeterminates, and $\mathbf{t} \in \mathbb{C}^r$, $\mathbf{u} \in \mathbb{C}$, $\mathbf{v} \in \mathbb{C}$ such that

$$y(x) = y'(\mathbf{t}, \mathbf{u}, \mathbf{v}, x). \tag{1}$$

Moreover we may assume that t_1, \dots, t_r are algebraically independent over $\overline{\mathbb{Q}}$, $\mathbf{u} = 1/R(t_1, \dots, t_r)$ for some polynomial $R \in \overline{\mathbb{Q}}[t_1, \dots, t_r]$ such that $R(t_1, \dots, t_r) \neq 0$, and \mathbf{v} is finite over $\mathbb{L} := \overline{\mathbb{Q}}(t_1, \dots, t_r)$. Let $P(t_1, \dots, t_r, v) \in \overline{\mathbb{Q}}(t)[v]$ be the monic polynomial of minimal degree in v such that

$$P(t_1, \dots, t_r, \mathbf{v}) = 0.$$

Let $D \subset \mathbb{C}^r$ be the discriminant locus of $P(t, v)$ seen as a polynomial in v (i.e., D is the locus of points $q \in \mathbb{C}^r$ such that q is a pole of one of the coefficients of P or such that $P(q, v)$ has at least one multiple

root). Since $P(t_1, \dots, t_r, v)$ has no multiple roots in an algebraic closure of \mathbb{L} , the point \mathbf{t} is not in D . Then there exist $\mathcal{U} \subset \mathbb{C}^r \setminus D$ a simply connected open neighborhood of p and analytic functions

$$w_i : \mathcal{U} \rightarrow \mathbb{C}, \quad i = 1, \dots, d,$$

such that

$$P(t, v) = \prod_{i=1}^d (v - w_i(t))$$

and $w_1(t_1, \dots, t_r) = v$. Moreover the $t \mapsto w_i(t)$ are algebraic functions over $\overline{\mathbb{Q}}[t]$. In particular the Taylor series of w_1 at a point of $\mathcal{U} \cap \overline{\mathbb{Q}}^r$ is an algebraic power series with algebraic coefficients. Since the polynomial R is not vanishing at p the function

$$t \in \mathbb{C}^r \setminus \{R = 0\} \mapsto \frac{1}{R(t)}$$

is also an analytic function which is algebraic over $\overline{\mathbb{Q}}[t]$ and so its Taylor series at a point of $\overline{\mathbb{Q}}^r \setminus \{R = 0\}$ is an algebraic power series with algebraic coefficients. Let $q := (q_1, \dots, q_r) \in \overline{\mathbb{Q}}^r \cap \mathcal{U} \setminus \{R = 0\}$ such that \mathbf{t} belongs to an open polydisc Δ centered at q and such that $\Delta \subset \mathcal{U} \setminus \{R = 0\}$. We denote by $\varphi_1(t)$ and $\varphi_2(t) \in \overline{\mathbb{Q}}\langle t \rangle$ the Taylor series of $t \mapsto 1/R(t)$ and w_1 at q . For simplicity we can make a translation and assume that q is the origin of \mathbb{C}^r . In particular the series $\varphi_1(t)$ and $\varphi_2(t)$ are convergent at \mathbf{t} . We have

$$f(x, y'(t_1, \dots, t_r, \mathbf{u}, \mathbf{v}, x)) = 0$$

or equivalently

$$f(x, y'(t_1, \dots, t_r, \varphi_1(t_1, \dots, t_r), \varphi_2(t_1, \dots, t_r), x)) = 0.$$

The function

$$(t, x) \mapsto F(t, x) := f(x, y'(t_1, \dots, t_r, \varphi_1(t_1, \dots, t_r), \varphi_2(t_1, \dots, t_r), x))$$

is an algebraic function over $\overline{\mathbb{Q}}[t, x]$. So if $F(t, x) \not\equiv 0$ there exists an algebraic function $(t, x) \mapsto g(t, x)$ such that

$$(t, x) \mapsto g(t, x)F(t, x)$$

is a nonzero polynomial function. Indeed if

$$a_0(t, x)T^e + a_1(t, x)T^{e-1} + \dots + a_e(t, x)$$

is a polynomial of minimal degree having $F(t, x)$ as a root then $a_e(t, x) \not\equiv 0$ and we can choose

$$g(t, x) := -a_0(t, x)F(t, x)^{e-1} - a_1(t, x)F(t, x)^{e-2} + \dots - a_{e-1}(t, x)$$

so we have

$$g(t, x)F(t, x) = a_e(t, x).$$

Since $F(t, x) = 0$ we have $a_e(t, x) = 0$ but t_1, \dots, t_r, x being algebraically independent over $\overline{\mathbb{Q}}$ we obtain that $a_e(t, x) \equiv 0$ which is a contradiction. Thus we have

$$F(t, x) = f(x, y'(t_1, \dots, t_r, \varphi_1(t_1, \dots, t_r), \varphi_2(t_1, \dots, t_r), x)) = 0.$$

This proves the theorem if we define

$$y(t, x) = y'(t, \varphi_1(t), \varphi_2(t), x).$$

All the series $y_\alpha(t)$ are then convergent at t , since $\varphi_1(t)$ and $\varphi_2(t)$ are convergent power series at t and since $y'(t, u, v, x) \in \overline{\mathbb{Q}}\langle t, u, v, x \rangle^m \cap \overline{\mathbb{Q}}[t, u, v][[x]]^m$. □

Remark 9. Let us assume that $f(x, y) \in \mathbb{Q}\langle x \rangle[y]^p$. In the proof of [Theorem 7](#) let us assume that $r = 0$, i.e., the coefficients of $y(x)$ belong to a finite field extension of \mathbb{Q} . In this case the analytic function w_1 is a constant function whose value is in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$. This is why we need to work with the algebraically closed field $\overline{\mathbb{Q}}$ and not only with \mathbb{Q} .

Lemma 10. *Let $f \in \mathbb{C}\langle x \rangle^m \setminus \overline{\mathbb{Q}}\langle x \rangle^m$. Then there exist complex numbers t_1, \dots, t_r, u and v with $r \geq 1$ and $F \in \overline{\mathbb{Q}}\langle t, u, v, x \rangle^m$, where $t = (t_1, \dots, t_r)$ and u and v are single indeterminates, such that*

- $F \in \overline{\mathbb{Q}}[t, u, v][[x]]^m$,
- $f(x) = F(t_1, \dots, t_r, u, v, x)$,
- the extension $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}(t_1, \dots, t_r)$ is purely transcendental,
- $u = 1/R(t_1, \dots, t_r)$ for some polynomial $R \in \overline{\mathbb{Q}}[t_1, \dots, t_r]$ with $R(t_1, \dots, t_r) \neq 0$,
- v is finite over $\overline{\mathbb{Q}}(t_1, \dots, t_r)$.

Proof. Let \mathbb{K} be the field extension of $\overline{\mathbb{Q}}$ generated by the coefficients of the minimal polynomials of the components of f . Then the coefficients of the components of f belong to a finite field extension of \mathbb{K} (see for instance [\[Cutkosky and Kashcheyeva 2008\]](#)). Let us replace \mathbb{K} by this finite field extension. There exists a purely transcendental finitely generated field extension $\overline{\mathbb{Q}} \rightarrow \mathbb{L}$ such that $\mathbb{L} \rightarrow \mathbb{K}$ is finite. By enlarging \mathbb{K} we may assume that $\mathbb{L} \rightarrow \mathbb{K}$ is normal. By the primitive element theorem $\mathbb{K} = \mathbb{L}(a)$ for some $a \in \mathbb{C}$ algebraic over \mathbb{L} . Let us write $f(x) = (f_1(x), \dots, f_m(x))$. We can write

$$f_i(x) = \sum_{k=0}^{d-1} a^k f_{i,k}(x) \quad \text{for } i = 1, \dots, m,$$

where d is the degree of a over \mathbb{L} and the $f_{i,k}(x)$ are power series with coefficients in \mathbb{L} . Let us denote by

$$a_1 = a, a_2, \dots, a_d$$

the conjugates of a over \mathbb{L} . Since $f_i(x)$ is algebraic over $\mathbb{K}[x]$ and $\mathbb{L} \rightarrow \mathbb{K}$ is an algebraic extension, $f_i(x)$ is algebraic over $\mathbb{L}[x]$. Let $P_i(x, y) = \sum_{\alpha, l} p_{\alpha, l} x^\alpha y^l \in \mathbb{L}[x, y]$ be a nonzero vanishing polynomial of $f_i(x)$ and let σ be a \mathbb{L} -automorphism of \mathbb{K} such that $\sigma(a) = a_j$ for some j . It induces a $\mathbb{L}[[x]]$ -automorphism of

$\mathbb{K}[[x]]$ defined by $\sigma(\sum_{\alpha} c_{\alpha}x^{\alpha}) = \sum_{\alpha} \sigma(c_{\alpha})x^{\alpha}$. Then we have

$$0 = \sigma(P_i(x, f_i(x))) = \sum_{\alpha,l} \sigma(p_{\alpha,l})x^{\alpha} \sigma(f_i(x))^l = \sum_{\alpha,l} \sigma(p_{\alpha,l})x^{\alpha} \left(\sum_{k=0}^{d-1} a_j^k f_{i,k}(x) \right)^l.$$

Thus for every $i = 1, \dots, m$ and $j = 1, \dots, d$ the power series

$$\sum_{k=0}^{d-1} a_j^k f_{i,k}(x) \in \mathbb{K}[[x]]$$

is algebraic over $\mathbb{K}[x]$. Let M be the (nonsingular) $d \times d$ Vandermonde matrix associated to the a_j . Then we have

$$\tilde{f}_i(x) = M \bar{f}_i(x),$$

where $\tilde{f}_i(x)$ is the vector whose entries are the $\sum_{k=0}^{d-1} a_j^k f_{i,k}(x)$ for $j = 1, \dots, d$, and $\bar{f}_i(x)$ is the vector whose entries are the $f_{i,k}(x)$. Then $\bar{f}_i(x) = M^{-1} \tilde{f}_i(x)$; thus the $f_{i,k}(x)$ are algebraic over $\mathbb{K}[x]$ and so over $\mathbb{L}[x]$. This shows that $f_{i,k}(x) \in \mathbb{L}\langle x \rangle$ for every i and k .

Let t_1, \dots, t_r be a transcendence basis of $\mathbb{L}/\overline{\mathbb{Q}}$. Then by [Lemma 11](#) given below we have

$$f_{i,k} = \sum_{\alpha \in \mathbb{N}^n} \frac{S_{i,k,\alpha}(t_1, \dots, t_r)}{R_{i,k}(t_1, \dots, t_r)^{|\alpha|}} x^{\alpha}$$

for some polynomials $S_{i,k,\alpha}$ and $R_{i,k} \in \overline{\mathbb{Q}}[t_1, \dots, t_r]$. By replacing each $R_{i,k}$ by $\prod_{j,l} R_{j,l}$ and multiplying every $S_{i,k,\alpha}$ by $\prod_{(j,l) \neq (i,k)} R_{j,l}^{|\alpha|}$ we may assume that $R_{i,k} = R_{i',k'} = R$ for every (i, k) and (i', k') . The power series

$$f_{i,k}^*(x) = f_{i,k}(R(t_1, \dots, t_r)x_1, \dots, R(t_1, \dots, t_r)x_n) = \sum_{\alpha \in \mathbb{N}^n} S_{i,k,\alpha}(t_1, \dots, t_r) x^{\alpha}$$

belongs to $\overline{\mathbb{Q}}(t_1, \dots, t_r)\langle x \rangle$ since $f_{i,k}(x) \in \overline{\mathbb{Q}}(t_1, \dots, t_r)\langle x \rangle$. Thus we have

$$f_{i,k}^* = F_{i,k}(t_1, \dots, t_r, x)$$

with, for every i and k ,

$$F_{i,k} := \sum_{\alpha \in \mathbb{N}^n} S_{k,\alpha}(t_1, \dots, t_r) x^{\alpha} \in \overline{\mathbb{Q}}[t_1, \dots, t_r][[x]]$$

where the t_i are new indeterminates. Moreover let $P_{i,k}(t_1, \dots, t_r, x, y) \in \overline{\mathbb{Q}}[t_1, \dots, t_r, x, y]$, where y is a new indeterminate, be a nonzero polynomial with $P_{i,k}(t, x, f_{i,k}^*(x)) = 0$. Since $F_{i,k} \in \overline{\mathbb{Q}}[t_1, \dots, t_r][[x]]$ for every k , we can write

$$P_{i,k}(t_1, \dots, t_r, x, F_{i,k}(t_1, \dots, t_r, x)) = \sum_{\beta \in \mathbb{N}^n} P_{i,k,l}(t_1, \dots, t_r) x^{\beta}$$

for some polynomials $P_{i,k,\beta} \in \overline{\mathbb{Q}}[t_1, \dots, t_r]$. Thus $P_{i,k,\beta}(t_1, \dots, t_r) = 0$ for every i, k and β , but since t_1, \dots, t_r are algebraically independent over $\overline{\mathbb{Q}}$ we have

$$P_{i,k,\beta}(t_1, \dots, t_r) = 0$$

for every i, k and β so

$$P_{i,k}(t_1, \dots, t_r, x, F_{i,k}(t_1, \dots, t_r, x)) = 0$$

and this implies that $F_{i,k} \in \overline{\mathbb{Q}}\langle t_1, \dots, t_r, x \rangle$. In particular if u denotes a new indeterminate we have

$$F_{i,k}(t_1, \dots, t_r, ux_1, \dots, ux_n) \in \overline{\mathbb{Q}}\langle t_1, \dots, t_r, u, x \rangle \cap \overline{\mathbb{Q}}[t_1, \dots, t_r, u][[x]] \quad \text{for all } k.$$

Finally we set $t = (t_1, \dots, t_r)$ and

$$F_i(t, x) = \sum_{k=0}^{d-1} v^k F_{i,k}(t_1, \dots, t_r, ux_1, \dots, ux_n)$$

where v denotes a new indeterminate. Thus the result is proven with F the vector whose components are the F_i and $\mathbf{u} = 1/R(t_1, \dots, t_r)$ and $\mathbf{v} = a$. □

The following version of Eisenstein lemma is essentially [Tougeron 1990, Lemma 2.2] and the proof is the same—but we give it here for the convenience of the reader:

Lemma 11 (Eisenstein lemma). *Let $f \in \overline{\mathbb{Q}}\langle t_1, \dots, t_r \rangle\langle x \rangle$ be an algebraic power series where the $t_i \in \mathbb{C}$ are algebraically independent over $\overline{\mathbb{Q}}$. Then there exist a polynomial $R(t) \in \overline{\mathbb{Q}}[t]$ and polynomials $S_\alpha(t) \in \overline{\mathbb{Q}}[t]$ for every $\alpha \in \mathbb{N}^n$, where $t = (t_1, \dots, t_r)$ is a vector of new indeterminates, such that*

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{S_\alpha(t_1, \dots, t_r)}{R(t_1, \dots, t_r)^{|\alpha|}} x^\alpha.$$

Proof of Lemma 11. Let $P(x, y) \in \overline{\mathbb{Q}}\langle t_1, \dots, t_r \rangle[x, y]$ be a minimal polynomial of f , i.e., a generator of the kernel of the ring morphism:

$$\begin{aligned} \overline{\mathbb{Q}}\langle t_1, \dots, t_r \rangle[x, y] &\rightarrow \overline{\mathbb{Q}}\langle t_1, \dots, t_r \rangle[[x]], \\ p(x, y) &\mapsto p(x, f(x)). \end{aligned}$$

Let us set

$$e := \text{ord}_x \left(\frac{\partial P}{\partial y}(x, f(x)) \right).$$

We have $e < \infty$ since $P(x, y)$ is a minimal polynomial of $f(x)$. Let us write $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha(t)x^\alpha$, where $t = (t_1, \dots, t_r)$ and $f_\alpha(t) \in \overline{\mathbb{Q}}\langle t \rangle$. Let $b(t) \in \overline{\mathbb{Q}}[t]$ be a common denominator of the $f_\alpha(t)$ for $|\alpha| \leq 2e + 1$. Then Lemma 11 is satisfied by f if and only if it is satisfied by $b(t)f$. Thus we may replace f by $b(t)f$. In this case a minimal polynomial of $b(t)f$ is

$$P'(x, y) := b(t)^{\text{deg}_y(P)} P\left(x, \frac{y}{b(t)}\right).$$

Moreover by multiplying $P'(x, y)$ by an element of $\overline{\mathbb{Q}}(t_1, \dots, t_r)$ we may assume that $P'(x, y) \in \overline{\mathbb{Q}}[t_1, \dots, t_r][x, y]$. Then we have

$$e = \text{ord}_x \left(\frac{\partial P'}{\partial y}(x, b(t)f(x)) \right).$$

Thus we may replace f by $b(t)f$ and assume that $f_\alpha(t) \in \overline{\mathbb{Q}}[t]$ for $|\alpha| \leq 2e + 1$.

We define

$$P^*(u, x, y) := P(ux_1, \dots, ux_n, y) \in \overline{\mathbb{Q}}[t_1, \dots, t_r, x_1, \dots, x_n][u, y]$$

and

$$f^*(u, x) := f(ux_1, \dots, ux_n)$$

where u is a new indeterminate. Then $P^*(u, x, f^*(u, x)) = 0$ so $f^* \in \overline{\mathbb{Q}}(t, x)\langle u \rangle$. Let us denote by $f^{*(2e+1)}(u)$ the $(2e + 1)$ -truncation of $f^*(u)$ (i.e., we remove from $f^*(u)$ all the monomials which are divisible by u^{2e+2}). Then

$$P^*(u, x, f^{*(2e+1)}) \in (u)^{2e+2} \quad \text{and} \quad \frac{\partial P^*}{\partial y}(u, x, f^{*(2e+1)}) \in (u)^e \setminus (u)^{e+1}.$$

Let us set

$$y = u^{e+1}y' + f^{*(2e+1)}$$

where y' is a new indeterminate. Then

$$P^*(u, x, y) = P^*(u, x, f^{*(2e+1)}) + \frac{\partial P^*}{\partial y}(u, x, f^{*(2e+1)})u^{e+1}y' + u^{2e+2}y'^2Q(u, y')$$

for some polynomial Q . Thus the equation $P^*(u, x, y) = 0$ is equivalent to

$$\frac{P^*(u, x, f^{*(2e+1)})}{u^{2e+1}} + \frac{\frac{\partial P^*}{\partial y}(u, x, f^{*(2e+1)})}{u^e}y' + uy'^2Q(u, y') = 0. \tag{2}$$

Since $f_\alpha(t) \in \overline{\mathbb{Q}}[t]$ for $|\alpha| \leq 2e + 1$ we have $f^{*(2e+1)} \in \overline{\mathbb{Q}}[t, x, u]$. Since the coefficients in (2) are polynomials in u, x and the $f_\alpha(t)$ for $|\alpha| \leq 2e + 1$, they belong to $\overline{\mathbb{Q}}[t, x, u]$. Let $R(t, x) \in \overline{\mathbb{Q}}[t, x]$ be defined by

$$R(t, x) = \left(\frac{\frac{\partial P^*}{\partial y}(u, x, f^{*(2e+1)})}{u^e} \right)_{|u=0}.$$

Since $\text{ord}_u(P^*(u, x, f^{*(2e+1)})/u^{2e+1}) \geq 1$ we see that $R(t, x)^2$ divides

$$\frac{P^*(R(t, x)^2u', x, f^{*(2e+1)}(R(t, x)^2u'))}{(R(t, x)^2u')^{2e+1}}$$

where u' is a new indeterminate. Thus by replacing y' by $R(t, x)y''$ and u by $R(t, x)^2u'$ in (2), and dividing by $R(t, x)^2$ we conclude that (2) is equivalent to

$$A_1(t, x, u') + (1 + u'A_2(t, x, u'))y'' + u'y''^2A_3(t, x, u') = 0, \tag{3}$$

where the A_i belongs to $\overline{\mathbb{Q}}[\mathbf{t}, x, u']$. By the implicit function theorem (or Hensel’s lemma) this equation has a unique solution in $\overline{\mathbb{Q}}\langle \mathbf{t}, x, u' \rangle$ which is necessarily $g^* := f^*(R(\mathbf{t}, x)^2 u')/R(\mathbf{t}, x)$. Moreover we can also apply Hensel’s lemma to this equation to see that it has a unique solution in the completion of $\overline{\mathbb{Q}}[\mathbf{t}, x, u']$ with respect to the ideal generated by u' , i.e., in the ring $\overline{\mathbb{Q}}[\mathbf{t}, x][[u']]$. Thus

$$g^* \in \overline{\mathbb{Q}}[\mathbf{t}, x][[u']] \cap \overline{\mathbb{Q}}\langle \mathbf{t}, x, u' \rangle.$$

In particular the coefficients g_k^* defined by $g^*(u') = \sum_{k \geq 0} g_k^* u'^k$ are polynomials over $\overline{\mathbb{Q}}$ depending on the t_i and the x_j . Moreover

$$f^*(u) = \sum_{k \geq 0} \frac{g_k^*(\mathbf{t}, x)}{R(\mathbf{t}, x)^{2k-1}} u^k.$$

On the other hand we have

$$f^*(u) = \sum_{k \geq 0} \left(\sum_{|\alpha|=k} f_\alpha(\mathbf{t}) x^\alpha \right) u^k;$$

hence

$$\sum_{|\alpha|=k} f_\alpha(\mathbf{t}) x^\alpha = \frac{g_k^*(\mathbf{t}, x)}{R(\mathbf{t}, x)^{2k-1}} \quad \text{for all } k \in \mathbb{N}. \tag{4}$$

For every $\alpha \in \mathbb{N}^n$, let us write $f_\alpha(\mathbf{t}) = h_\alpha(\mathbf{t})/l_{|\alpha|}(\mathbf{t})$ where $h_\alpha(\mathbf{t}), l_{|\alpha|}(\mathbf{t}) \in \overline{\mathbb{Q}}[\mathbf{t}]$ and $l_{|\alpha|}(\mathbf{t})$ is coprime with $\sum_{|\alpha|=k} h_\alpha(\mathbf{t}) x^\alpha$. Then $l_{|\alpha|}(\mathbf{t})$ divides $R(\mathbf{t}, x)^{2|\alpha|-1}$. Let $r(\mathbf{t})$ be the greatest divisor of $R(\mathbf{t}, x)$ belonging to $\overline{\mathbb{Q}}[\mathbf{t}]$. Then there exists $d_{|\alpha|}(\mathbf{t}) \in \overline{\mathbb{Q}}[\mathbf{t}]$ such that $l_{|\alpha|}(\mathbf{t}) d_{|\alpha|}(\mathbf{t}) = r(\mathbf{t})^{2|\alpha|-1}$. Thus

$$f_\alpha(\mathbf{t}) = \frac{h_\alpha(\mathbf{t}) d_{|\alpha|}(\mathbf{t})}{r(\mathbf{t})^{2|\alpha|-1}}.$$

This proves the lemma. □

Theorem 7 allows us to prove the following version of the nested Artin–Płoski–Popescu approximation theorem:

Theorem 12. *Let $f(x, y) \in \overline{\mathbb{Q}}\langle x \rangle[y]^p$ and let us consider a solution $y(x) \in \mathbb{C}\langle x \rangle^m$ of*

$$f(x, y(x)) = 0.$$

Let us assume that $y_i(x)$ depends only on $(x_1, \dots, x_{\sigma(i)})$ where $i \mapsto \sigma(i)$ is an increasing function. Then there exist two sets of indeterminates $z = (z_1, \dots, z_s)$ and $t = (t_1, \dots, t_r)$, an increasing function τ , convergent power series $z_i(x) \in \mathbb{C}\langle x \rangle$ vanishing at 0 such that $z_1(x), \dots, z_{\tau(i)}(x)$ depend only on $(x_1, \dots, x_{\sigma(i)})$, complex numbers $t_1, \dots, t_r \in \mathbb{C}$ and an algebraic power series vector solution $y(t, x, z) \in \overline{\mathbb{Q}}\langle t, x, z \rangle^m$ of

$$f(x, y(t, x, z)) = 0$$

such that

$$y_i(t, x, z) \in \overline{\mathbb{Q}}\langle t, x_1, \dots, x_{\sigma(i)}, z_1, \dots, z_{\tau(i)} \rangle \quad \text{for every } i,$$

$y(t, x, z)$ is well defined and $y(x) = y(t, x, z(x))$.

Proof. By [Bilski et al. 2017, Theorem 1.2] there exist a new set of indeterminates $z = (z_1, \dots, z_s)$, an increasing function τ , convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0 such that $z_1(x), \dots, z_{\tau(i)}(x)$ depend only on $(x_1, \dots, x_{\sigma(i)})$, and an algebraic power series vector solution $y(x, z) \in \mathbb{C}\langle x, z \rangle^m$ of

$$f(x, y(x, z)) = 0$$

such that

$$y_i(x, z) \in \mathbb{C}\langle x_1, \dots, x_{\sigma(i)}, z_1, \dots, z_{\tau(i)} \rangle \quad \text{for every } i$$

and $y(x) = y(x, z(x))$. Then we apply Theorem 7 to the vector $y(x, z)$. □

2. Proof of Theorem 2

The proof is similar to the proof of [Bilski et al. 2017, Theorem 1.2] and so we will refer several times to this paper for details. For convenience x^{n-1} will denote the vector of indeterminates (x_1, \dots, x_{n-1}) and, more generally, x^i will denote the vector of indeterminates (x_1, \dots, x_i) . Firstly we consider the case $\mathbb{K} = \mathbb{C}$. Let $g_1, \dots, g_k \in \mathbb{C}\{x\}$ be the defining equations of $(V, 0)$. By a linear change of coordinates we may assume that the g_i are Weierstrass polynomials in x_n :

$$g_s(x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1})x_n^{r_s-j} \quad \text{for all } s = 1, \dots, k$$

and

$$\text{mult}_0(g_s) = r_s \quad \text{for all } s = 1, \dots, k. \tag{5}$$

Then the $a_{n-1,s,j}$ are arranged in a row vector $a_{n-1} \in \mathbb{C}\{x^{n-1}\}^{p_n}$ with $p_n = \sum_s r_s$. Let f_n denote the product of the g_s . Let $\Delta_{n,i}$ denote the i -th generalized discriminant of f_n seen as a polynomial in x_n (see [Bilski et al. 2017, 4.2]). This is a polynomial depending on a_{n-1} . Then let $\Delta_{n,j_n}(a_{n-1})$ be the first nonvanishing generalized discriminant. After a linear change of coordinates in x_1, \dots, x_{n-1} we may assume, by the Weierstrass preparation theorem, that

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1}) \left(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j} \right),$$

where $u_{n-1}(0) \neq 0$ and for all j , $a_{n-2,j}(0) = 0$. We carry on with this construction (exactly as in [Bilski et al. 2017, 4.2]) and define a sequence of Weierstrass polynomials $f_i(x^i)$ for $i = 1, \dots, n-1$ such that $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}$ is the Weierstrass polynomial associated to the first nonidentically zero generalized discriminant $\Delta_{i+1,j_{i+1}}(a_i)$ of f_{i+1} , where a_i denotes the vector $(a_{i,1}, \dots, a_{i,p_{i+1}})$:

$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i) \left(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j} \right), \quad i = 0, \dots, n-1. \tag{6}$$

Thus the vector of power series a_i satisfies

$$\Delta_{i+1,k}(a_i) \equiv 0 \quad \text{for } k < j_{i+1} \text{ and } i = 0, \dots, n-1. \tag{7}$$

In particular $\Delta_{1,j_1}(a_0)$ is a constant. Then we use [Theorem 12](#) to see that there exist two sets of indeterminates $z = (z_1, \dots, z_s)$ and $t = (t_1, \dots, t_r)$, an increasing function τ , convergent power series $z_i(x) \in \mathbb{C}\langle x \rangle$ vanishing at 0, complex numbers $t_1, \dots, t_r \in \mathbb{C}$, algebraic power series $u_i(t, x^i, z) \in \overline{\mathbb{Q}}\langle t, x^i, z_1, \dots, z_{\tau(i)} \rangle$ and vectors of algebraic power series

$$a_i(t, x^i, z) \in \overline{\mathbb{Q}}\langle t, x^i, z_1, \dots, z_{\tau(i)} \rangle^{p_i}$$

such that the following hold:

- (a) $z_1(x), \dots, z_{\tau(i)}(x)$ depend only on (x_1, \dots, x_i) .
- (b) $a_i(t, x^i, z)$ and $u_i(t, x^i, z)$ are solutions of [\(6\)](#) and [\(7\)](#).
- (c) $a_i(x^i) = a_i(t, x^i, z(x^i))$ and $u_i(x^i) = u_i(t, x^i, z(x^i))$.

Let \bar{u}_i be the constant coefficient of $u_i(x^i)$. Because $z(0) = 0$ we have $\bar{u}_i = u_i(t, 0, 0)$ and $u_i(t, 0, 0) \in \overline{\mathbb{Q}}\langle t \rangle$. In particular $u_i(t, 0, 0) \neq 0$. Let $\gamma : \mathcal{U} \rightarrow \mathbb{C}^r$ be the analytic map defined by

$$\gamma(\lambda) = (1 - \lambda)\mathbf{q} + \lambda\mathbf{t}$$

where \mathcal{U} is an open connected neighborhood of the closed unit disc in \mathbb{C} and $\mathbf{q} \in \overline{\mathbb{Q}}^r$. Because $u_i(t, 0, 0) \neq 0$ and $\overline{\mathbb{Q}}$ is dense in \mathbb{C} we may choose \mathbf{q} close enough to \mathbf{t} such that

$$u_i(\gamma(\lambda), 0, 0) \neq 0 \quad \text{for all } i \text{ and } \lambda \in \mathcal{U}.$$

Again because $\overline{\mathbb{Q}}$ is dense in \mathbb{C} we can find $\mathbf{q} \in \overline{\mathbb{Q}}$ close enough to \mathbf{t} such that the following are, for all $\lambda \in \mathcal{U}$, well defined convergent power series in x :

$$F_n(\lambda, x) := \prod_s G_s(\lambda, x), \quad \text{for } G_s(\lambda, x) := x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(\gamma(\lambda), x^{n-1}, \lambda z(x^{n-1}))x_n^{r_s-j}$$

and, for $i = 1, \dots, n - 1$,

$$F_i(\lambda, x) := x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(\gamma(\lambda), x^{i-1}, \lambda z(x^{i-1}))x_i^{p_i-j} \quad \text{and} \quad u_i(\gamma(\lambda), x^i, z(x^i)).$$

Finally we set $F_0 \equiv 1$. Because $u_i(\gamma(\lambda), 0, z(0)) \neq 0$, the family $F_i(\lambda, x)$ satisfies the assumptions of [\[Parusiński and Păunescu 2017, Theorem 3.3\]](#) with $|\lambda| \leq 1$, i.e., the family is Zariski equisingular. Moreover by [\(5\)](#) we have

$$\text{mult}_0(G_s) = r_s \quad \text{for all } s = 1, \dots, k,$$

so the family is Zariski equisingular with transverse projections (see [\[Parusiński and Păunescu 2017, Definition 4.1\]](#)). So by [Theorem 4.3](#) of that work this family is a regular Zariski equisingular family and by [Theorem 7.1](#) of that work it is Whitney equisingular. Thus $\{F_n(0, x) = 0\}$ and $\{F_n(1, x) = 0\} = \{f_n(x) = 0\}$ are homeomorphic and the homeomorphism between them can be chosen to be subanalytic and arcanalytic. We have $F_n(0, x) \in \overline{\mathbb{Q}}\langle x \rangle$ thus, by [\[Bilski et al. 2017, Theorem 3.2\]](#), we may assume that $(V, 0)$ is the germ of a Nash set defined over $\overline{\mathbb{Q}}$. When $\mathbb{k} = \mathbb{R}$ we may also assume that $(V, 0)$ is the germ of a Nash

set defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$. This follows from the complex case by the same argument used in the proof of [Bilski et al. 2017, Corollary 4.1]. Then we conclude with the following theorem:

Theorem 13. *Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be a Nash set germ defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$. Then there exists a local Nash diffeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that $h(V)$ is the germ of an algebraic subset of \mathbb{K}^n defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$.*

Proof. This follows from Proposition 14 given below which is a slight modification of [Bochnak and Kucharz 1984, Proposition 2]. Indeed let $f : U \rightarrow \mathbb{K}^m$ be a Nash function such that $f^{-1}(0) = V$. Then by Proposition 14 we have $V = s^{-1}(\varphi^{-1}(0))$. But $s : U \rightarrow s(U)$ is a Nash diffeomorphism by Proposition 14ii. So we set $h = s$ and $h(V)$ is an algebraic set equal to $\varphi^{-1}(0)$, again by using the notations of Proposition 14. \square

Proposition 14. *Let $f : U \rightarrow \mathbb{K}^m$ be a Nash map defined on an open connected set $U \subset \mathbb{K}^n$ by algebraic power series with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$. Then there exist an algebraic set $X \subset \mathbb{K}^n \times \mathbb{K}^N$, a polynomial map $\varphi : X \rightarrow \mathbb{K}^m$ and a Nash map $s : U \rightarrow \mathbb{K}^n \times \mathbb{K}^N$ satisfying the following properties:*

- (i) $s(U) \subset \text{Reg}(X)$ is a connected component of $p^{-1}(U) \cap X$, where $p : \mathbb{K}^n \times \mathbb{K}^N \rightarrow \mathbb{K}^n$ is the first projection.
- (ii) $p \circ s = \text{Id}_U$.
- (iii) $f = \varphi \circ s$.
- (iv) the coefficients of the polynomials defining X and φ are in $\overline{\mathbb{Q}} \cap \mathbb{K}$.

Proof. The existence of X , φ and s satisfying (i), (ii) and (iii) are given by [Bochnak and Kucharz 1984, Proposition 2] in the general case where f is defined by algebraic power series with coefficients in \mathbb{K} . In fact X is the normalization of the Zariski closure of the graph of f and φ is the restriction to X of a generic linear map $\mathbb{K}^{n+N} \rightarrow \mathbb{K}^m$. In particular, since f is assumed to be defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$, we have that X is defined by polynomial equations with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$. Because φ is generic we can choose such a φ with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$ since this field is dense in \mathbb{K} . \square

3. Proof of Theorem 1

The proof is similar to the proof of [Bilski et al. 2017, Theorem 1.3] and so once again we will refer several times to this paper for some details. We begin by considering the case $\mathbb{K} = \mathbb{C}$. Let g_1, \dots, g_p be power series defining $(V, 0)$. Let us replace n by $n - 1$ to assume that $(V, 0) \subset (\mathbb{C}^{n-1}, 0)$ and let (x_2, \dots, x_n) denote the coordinates in \mathbb{C}^{n-1} . Let us set $g_0 := g$. After a linear change of coordinates in x_2, \dots, x_n (i.e., preserving x_1) we have

$$\prod_{m=0}^p (x_1 - g_m(x_2, \dots, x_n))$$

is x_n -regular. Thus we may write

$$\prod_{m=0}^p (x_1 - g_m(x_2, \dots, x_n)) = u_n(x) \left(x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j} (x^{n-1}) x_n^{p_n-j} \right),$$

where $u_n(0) \neq 0$ and $a_{n-1,j}(0) = 0$. We set

$$f_n(x) = x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j} (x^{n-1}) x_n^{p_n-j}$$

so that

$$u_n(x) f_n(x) = \prod_{m=0}^p \left(x_1 - \sum_{k=2}^n x_k b_{m,k} (x_2, \dots, x_n) \right), \tag{8}$$

with $g_m = \sum_{k=2}^n x_k b_{m,k}$ for some power series $b_{m,k}$ since $g_m(0) = 0$ for every m . We denote by $b \in \mathbb{C}\{x\}^{p(n-1)}$ and $a_{n-1} \in \mathbb{C}\{x^{n-1}\}^{p_n}$ the vector of the coefficients $b_{m,k}$ and $a_{n-1,j}$, respectively. Again we denote by $\Delta_{n,i}$ the generalized discriminants of f_n which are polynomials in a_{n-1} . Let j_n be the positive integer such that

$$\Delta_{n,i}(a_{n-1}) \equiv 0 \quad \text{for } i < j_n,$$

and $\Delta_{n,j_n}(a_{n-1}) \neq 0$. After a linear change of coordinates (x_2, \dots, x_{n-1}) we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1}) x_1^{q_{n-1}} \left(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j} (x^{n-2}) x_{n-1}^{p_{n-1}-j} \right),$$

where $u_{n-1}(0) \neq 0$ and $a_{n-2,j}(0) = 0$. We set

$$f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j} (x^{n-2}) x_{n-1}^{p_{n-1}-j}$$

and the vector of its coefficients $a_{n-2,j}$ is denoted by $a_{n-2} \in \mathbb{C}\{x^{n-2}\}^{p_{n-1}}$. Let j_{n-1} be the positive integer such that

$$\Delta_{n-1,k}(a_{n-2}) \equiv 0 \quad \text{for all } k < j_{n-1} \quad \text{and} \quad \Delta_{n-1,j_{n-1}}(a_{n-2}) \neq 0.$$

Then again we divide $\Delta_{n-1,j_{n-1}}$ by the maximal power of x_1 and, after a linear change of coordinates (x_2, \dots, x_{n-2}) , we denote by $f_{n-2}(x^{n-2})$ the associated Weierstrass polynomial.

We carry on with this construction and define a sequence of Weierstrass polynomials $f_i(x^i)$, for $i = 1, \dots, n-1$, such that $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j} (x^{i-1}) x_i^{p_i-j}$ is the Weierstrass polynomial associated to the first nonidentically zero generalized discriminant $\Delta_{i,j_i}(a_{i+1})$ of f_{i+1} , divided by the maximal power of x_1 , where $a_i = (a_{i,1}, \dots, a_{i,p_i})$:

$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i) x_1^{q_i} \left(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j} (x^{i-1}) x_i^{p_i-j} \right) \quad \text{for } i = 0, \dots, n-1. \tag{9}$$

Thus the vector of power series a_i satisfies

$$\Delta_{i+1,k}(a_{i-1}) \equiv 0 \quad \text{for } k < j_{i+1} \text{ and } i = 0, \dots, n-1. \tag{10}$$

Then we use [Theorem 12](#) to see that there exist two sets of indeterminates $z = (z_1, \dots, z_s)$ and $t = (t_1, \dots, t_r)$, an increasing function τ , convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0, complex numbers $\mathbf{t}_1, \dots, \mathbf{t}_r \in \mathbb{C}$, algebraic power series $u_i(t, x^i, z) \in \overline{\mathbb{Q}}\langle t, x^i, z_1, \dots, z_{\tau(i)} \rangle$ and vectors of algebraic power series

$$b(t, x, z) \in \overline{\mathbb{Q}}\langle t, x, z \rangle^{p(n-1)} \quad \text{and} \quad a_i(t, x^i, z) \in \overline{\mathbb{Q}}\langle t, x^{(i)}, z_1, \dots, z_{\tau(i)} \rangle^{p_i},$$

such that the following hold:

- (a) $z_1(x), \dots, z_{\tau(i)}(x)$ depend only on (x_1, \dots, x_i) .
- (b) $a_i(t, x^i, z)$, $u_i(t, x^i, z)$ and $b(t, x, z)$ are solutions of [\(8\)](#), [\(9\)](#) and [\(10\)](#).
- (c) $a_i(x^i) = a_i(\mathbf{t}, x^i, z(x^i))$, $u_i(x^i) = u_i(\mathbf{t}, x^i, z(x^i))$ and $b(x) = b(\mathbf{t}, x, z(x))$.

Then we repeat what we did in the proof of [Theorem 2](#). Let \bar{u}_i be the constant coefficient of $u_i(x^i)$. Because $z(0) = 0$ we have $\bar{u}_i = u_i(\mathbf{t}, 0, 0)$ and $u_i(\mathbf{t}, 0, 0) \in \overline{\mathbb{Q}}\langle t \rangle$. In particular $u_i(\mathbf{t}, 0, 0) \neq 0$. Let $\gamma : \mathcal{U} \rightarrow \mathbb{C}^r$ be the analytic map defined by

$$\gamma(\lambda) = (1 - \lambda)\mathbf{q} + \lambda\mathbf{t}$$

where \mathcal{U} is an open connected neighborhood of the closed unit disc in \mathbb{C} and $\mathbf{q} \in \overline{\mathbb{Q}}^r$. Because $u_i(\mathbf{t}, 0, 0) \neq 0$ and $\overline{\mathbb{Q}}$ is dense in \mathbb{C} we may choose \mathbf{q} close enough to \mathbf{t} such that

$$u_i(\gamma(\lambda), 0, 0) \neq 0 \quad \text{for all } i \text{ and } \lambda \in \mathcal{U}.$$

Again because $\overline{\mathbb{Q}}$ is dense in \mathbb{C} we can find $\mathbf{q} \in \overline{\mathbb{Q}}$ close enough to \mathbf{t} such that the following are, for all $\lambda \in \mathcal{U}$, well defined convergent power series in x :

$$F_i(\lambda, x) := x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(\gamma(\lambda), x^{i-1}, \lambda z(x^{i-1}))x_i^{p_i-j} \quad \text{for } i = 0, \dots, n,$$

$$u_i(\gamma(\lambda), x^i, z(x^i)) \quad \text{for } i = 1, \dots, n-1.$$

We have

$$u_n(\gamma(\lambda), x, \lambda z(x))F_n(\lambda, x) = \prod_{m=0}^p \left(x_1 - \sum_{k=2}^n x_k b_{m,k}(\gamma(\lambda), x, \lambda z(x)) \right).$$

By the implicit function theorem or the Weierstrass preparation theorem we have

$$x_1 - \sum_{k=2}^n x_k b_{m,k}(\gamma(\lambda), x, \lambda z(x)) = v_m(\lambda, x)(x_1 - G_m(\lambda, x_2, \dots, x_n)),$$

where $v_m(\lambda, x) \in \mathbb{C}\{\lambda, x\}$, $G_m(\lambda, x_2, \dots, x_n) \in \mathbb{C}\{\lambda, x_2, \dots, x_n\}$ and $v_m(0, 0) \neq 0$. Because

$$x_1 - \sum_{k=2}^n x_k b_{m,k}(\gamma(0), x, 0) \in \overline{\mathbb{Q}}\langle x \rangle$$

we have

$$v_m(0, x), G_m(0, x_2, \dots, x_n) \in \overline{\mathbb{Q}}\langle x \rangle$$

by unicity in the Weierstrass preparation theorem. We set

$$\hat{g}_m(y) := G_m(0, y) \quad \text{for } m = 0, \dots, p$$

where $y = (y_1, \dots, y_{n-1})$ is a new vector of indeterminates. Then, for both cases $\mathbb{K} = \mathbb{C}$ or \mathbb{R} , we conclude exactly as in [Bilski et al. 2017] (see the end of 5.4 and Proposition 5.3 in that work) to show that there is a homeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that $(h(V), 0)$ is a germ of a Nash subset of \mathbb{K}^n defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$ and $g \circ h$ is the germ of a Nash function defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$.

Then we deduce from Proposition 14 the following analogue of [Bilski et al. 2017, Theorem 5.4]:

Corollary 15. *Let g_i be algebraic powers series with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$ defining Nash function germs $g_i : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}, 0)$. Then there exist a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ and Nash units $u_i : (\mathbb{K}^n, 0) \rightarrow \mathbb{K}$, $u_i(0) \neq 0$, such that, for every i , $u_i(x)g_i(h(x))$ are polynomial function germs defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$.*

Proof. We have the following fact: let $(Y, 0) \subset (\mathbb{K}^n, 0)$ be a Nash set germ defined by algebraic power series with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$. Then there exists a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that for every irreducible analytic component W of $(Y, 0)$, the ideal of functions vanishing on $h(W)$ is generated by polynomials with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$. This follows from Proposition 14 by applying word for word the proof of “(i) \implies (iv)” in [Bochnak and Kucharz 1984, Theorem 5]. Thus when $\mathbb{K} = \mathbb{C}$ this fact applied to the germ $(Y, 0)$ defined by the products of the g_i proves the theorem. When $\mathbb{K} = \mathbb{R}$ we conclude as done for this case in the proof of [Bilski et al. 2017, Theorem 5.4]. \square

Let us recall that we have shown that there is a homeomorphism $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ such that $(h(V), 0)$ is the germ of a Nash subset of \mathbb{K}^n defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$ and $g \circ h$ is the germ of a Nash function defined over $\overline{\mathbb{Q}} \cap \mathbb{K}$. So we conclude the proof of Theorem 1 by using Theorem 13 and [Bilski et al. 2017, Theorem 5.5].

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