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Generalized Fourier coefficients of multiplicative functions

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We introduce and analyze a general class of not necessarily bounded multiplicative functions, examples of which include the function $n \mapsto \delta^{\omega(n)}$, where $\delta \in \mathbb{R} \setminus \{0\}$ and where ω counts the number of distinct prime factors of n, as well as the function $n \mapsto |\lambda_f(n)|$, where $\lambda_f(n)$ denotes the Fourier coefficients of a primitive holomorphic cusp form.

For this class of functions we show that after applying a W-trick, their elements become orthogonal to polynomial nilsequences. The resulting functions therefore have small uniformity norms of all orders by the Green–Tao–Ziegler inverse theorem, a consequence that will be used in a separate paper in order to asymptotically evaluate linear correlations of multiplicative functions from our class. Our result generalizes work of Green and Tao on the Möbius function.

1. Introduction

Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. Daboussi showed (see [Daboussi and Delange 1974]) that if $|f|$ is bounded by 1, then

$$\frac{1}{x} \sum_{n \leqslant x} f(n) e^{2\pi i \alpha n} = o(x) \tag{1-1}$$

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for every irrational α. A detailed proof of the following slightly strengthened version may be found in [Daboussi and Delange 1982]: Suppose that f satisfies

$$\sum_{n\leqslant x}|f(n)|^2 = O(x); \tag{1-2}$$

then (1-1) holds for every irrational α. Montgomery and Vaughan [1977] give explicit error terms for the decay in (1-1) for multiplicative functions that satisfy, in addition to (1-2), a uniform bound at all primes, in the sense that $|f(p)| \leqslant H$ holds for some constant $H \geqslant 1$ and all primes p.

In this paper we will study the closely related question of bounding correlations of multiplicative functions with polynomial nilsequences in place of the exponential function $n \mapsto e^{2\pi i\alpha n}$. A chief concern in this work is to include unbounded multiplicative functions in the analysis. To this end we shall significantly weaken the moment condition (1-2) by decomposing f into a suitable Dirichlet convolution $f = f_1 * \cdots * f_t$ and analyzing the correlations of the individual factors with exponentials, or rather nilsequences. The benefit of such a decomposition is that we merely require control on the second moments of the individual factors of the Dirichlet convolution and not of f itself. This essentially allows us to replace (1-2) by the condition that there exists $\theta_f \in (0, 1]$ such that

$$\sqrt{\frac{1}{x}\sum_{n\leqslant x}|f_i(n)|^2} \ll (\log x)^{1-\theta_f}\frac{1}{x}\sum_{n\leqslant x}|f_i(n)| \tag{1-3}$$

for all $i \in \{1, \dots, t\}$. To illustrate the difference between these two moment conditions, let us consider a simple example of a function that satisfies (1-3), but neither (1-2) nor

$$\sum_{n\leqslant x}|f(n)|^2 \ll \sum_{n\leqslant x}|f(n)|. \tag{1-4}$$

Example 1.1. For any $t \in \mathbb{N}$, let $d_t(n) = \mathbf{1} * \cdots * \mathbf{1}(n)$ denote the general divisor function, which arises as a t-fold convolution of $\mathbf{1}$. Choosing $f_i = \mathbf{1}$ for each $1 \leqslant i \leqslant t$, it is clear that (1-3) holds with $\theta_f = 1$. If $t > 1$, then neither (1-2) nor (1-4) hold, since

$$\frac{1}{x}\sum_{n\leqslant x} d_t(n) \asymp_t (\log x)^{t-1}, \quad \text{but} \quad \frac{1}{x}\sum_{n\leqslant x} d_t^2(n) \asymp_t (\log x)^{t^2-1}.$$

Thus, the second moment is not controlled by the first.

In order to describe the three classes of multiplicative functions that we will be working with here, let us introduce some notation. Throughout this paper, we write

$$S_f(x) = \frac{1}{x}\sum_{n\leqslant x} f(n) \quad \text{and} \quad S_f(x; q, r) = \frac{q}{x}\sum_{\substack{n\leqslant x\\ x\equiv r \pmod q}} f(n)$$

for $x \geqslant 1$ and integers $q, r \in \mathbb{N}$. We furthermore require the following functions w and W:

Definition 1.2. Let $w : \mathbb{N} \to \mathbb{R}$ be an increasing function such that

$$\frac{\log\log x}{\log\log\log x} < w(x) \leqslant \log\log x$$

for all sufficiently large x, and set

$$W(x) = \prod_{p \leqslant w(x)} p.$$

The basic class of function we will be interested in is the following:

Definition 1.3. Given a positive integer $H \geqslant 1$, we let $\mathscr{M}_H$ denote the class of multiplicative arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ such that:

(1) $|f(p^k)| \leqslant H^k$ for all prime powers p^k.

(2) There is a positive constant α_f such that

$$\frac{1}{x}\sum_{p \leqslant x} |f(p)| \log p \geqslant \alpha_f$$

for all sufficiently large x.

For the purpose of our main result, Theorem 6.1, it will be necessary to restrict attention to those functions f that admit a so-called W-trick (see Section 5). For this reason, we introduce the subset of elements of $\mathscr{M}_H$ that have stable mean values in certain arithmetic progressions:

Definition 1.4. Let $\mathscr{F}_H \subset \mathscr{M}_H$ be the subset of multiplicative functions f with the following property. Let $x > 1$ be a parameter. Given any constant $C > 0$, there exists a function φ_C with $\varphi_C(x) \to 0$ as $x \to \infty$ such that, whenever $1 \leqslant Q < (\log x)^C$ is a multiple of $W(x)$ and when $A \pmod{Q}$ is a reduced residue, then

$$S_f(x'; Q, A) = S_f(x; Q, A) + O\biggl(\varphi_C(x)\frac{Q}{\phi(Q)}\frac{1}{\log x}\prod_{\substack{p \leqslant x \\ p \nmid Q}}\Bigl(1 + \frac{|f(p)|}{p}\Bigr)\biggr) \tag{1-5}$$

for all $x' \in (x(\log x)^{-C}, x)$.

We will discuss this class of functions in detail in Section 4, where we prove several sufficient conditions for $f \in \mathscr{M}_H$ to belong to $\mathscr{F}_H$, or to a related class that will be introduced below. These sufficient conditions, recorded in Propositions 4.4 and 4.10 and Lemmas 4.16 and 4.17, prove to be much easier to verify in practice than the one given in the above definition, not at least because they take a form that allows for applications of the Selberg–Delange method as presented in [Tenenbaum 1995]. As an application of Lemmas 4.16 and 4.17 (see the remarks following their statements), we obtain the following simple criterion applicable to real-valued elements of $\mathscr{M}_H$:

Proposition 1.5. *Suppose that $f \in \mathscr{M}_H$ is real-valued and that it is bounded away from zero at primes, in the sense that there exists $\delta > 0$ and a sign $\epsilon \in \{+, -\}$ such that*

$$\#\{p \leqslant x : \epsilon f(p) \geqslant \delta\} \geqslant \frac{(1+o(1))x}{\log x}, \quad (\text{as } x \to \infty).$$

Then $f \in \mathscr{F}_H$ if f is nonnegative or if, for every given $C > 0$, there exists a function $\psi_C : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $\psi_C(x) \to 0$ as $x \to \infty$ such that

$$S_{f\chi_0}(x) = O\biggl(\frac{\psi_C(x)}{\log x} \exp\biggl(\sum_{p \leqslant x, p \nmid Q} \frac{|f(p)|}{p}\biggr)\biggr), \quad (x > 1)$$

for all trivial characters $\chi_0 \pmod{Q}$ with $Q \in (1, (\log x)^C)$ and $W(x) \mid Q$.

Observe, in particular, that this criterion may be applied to functions that take negative values at all primes, such as the Möbius function. In the latter case, the prime number theorem-type estimate $S_\mu(x) \ll_B (\log x)^{-B}$, which holds for all $x \geqslant 2$ and $B > 0$, implies that all conditions are satisfied; see Example 4.18(i) for details. As an easy consequence of the above proposition, it further follows that any function of the form $f(n) = \delta^{\omega(n)}$ for fixed $\delta > 0$ belongs to $\mathscr{F}_H$. In Section 4D we will show that the function $n \mapsto |\lambda_f(n)|$ belongs to $\mathscr{F}_H$, where $\lambda_f(n)$ denotes the normalized Fourier coefficients of a primitive holomorphic cusp form. This is an example which cannot be deduced from the above proposition.

In Section 6, we will see that in the context of our main result condition (1-5) only needs to hold for slowly varying twists of f. This allows us to slightly weaken the above definition and introduce the following intermediate class of functions $\mathscr{F}_H \subset \mathscr{F}_{H,n^{it}} \subset \mathscr{M}_H$, which will also be discussed in Section 4.

Definition 1.6. Let $\mathscr{F}_{H,n^{it}} \subset \mathscr{M}_H$ denote the subset of functions f with the following property. For every constant $C > 0$ and every sufficiently large $x > 1$, there exists $t_x \in \mathbb{R}$ with $|t_x| \leqslant 2\log x$ such that the function $f_x : n \mapsto f(n)n^{-it_x}$ satisfies (1-5) for all $x' \in (x(\log x)^{-C}, x)$, all $1 \leqslant Q < (\log x)^C$, $W(x) \mid Q$, and all reduced residues $A \pmod{Q}$. Observe that $\mathscr{F}_H \subset \mathscr{F}_{H,n^{it}}$ since we may take $t_x = 0$ for all x.

Twists of the form $f(n)n^{-it}$ play an important role in the study of multiplicative functions as their behavior is closely linked to that of the mean value of f through Halász's theorem [1968]; see also [Tenenbaum 1995, §III.4.3]. While Halász's theorem concerns bounded functions that are closely related to the constant function **1**, an analogue to this result, applicable to our basic class $\mathscr{M}_H$, has recently been proved independently by Elliott [2017, Theorems 2 and 4] and Tenenbaum [2017, Théorème 1.2]. The next lemma, which we chiefly include for comparison of the error terms in (1-5) and in later results, is a straightforward consequence of their result. The first part is due to Elliott and Kish [2016, Lemma 21].

Lemma 1.7 (Elliott, Kish, Tenenbaum). *Suppose $f \in \mathscr{M}_H$ and that*

$$\sum_{p \leqslant H} \sum_{k \geqslant 2} |f(p^k)|\, p^{-k} < \infty.$$

Then

$$S_{|f|}(x) \gg \frac{1}{\log x} \exp\biggl(\sum_{p \leqslant x} \frac{|f(p)|}{p}\biggr).$$

Furthermore, we have $|S_f(x)| = o(S_{|f|}(x))$ *unless there exists* $t \in \mathbb{R}$ *such that*

$$\sum_{p \text{ prime}} \frac{|f(p)| - \Re(f(p)p^{it})}{p} < \infty,$$

in which case $|S_f(x)| \asymp S_{|f|}(x)$.

Returning to the basic class $\mathscr{M}_H$, let us record the lemma that shows that every element of $\mathscr{M}_H$ does indeed admit a Dirichlet decomposition with the properties described at the beginning of this introduction. To be precise, the lemma below corresponds to $\theta_f = \frac{1}{2}$ in (1-3). We will prove this lemma in Section 3. In accordance with the earlier discussion, this lemma will only be needed in the case where f is unbounded, i.e., when $H > 1$.

Lemma 1.8. *(Dirichlet decomposition) Let* $f \in \mathscr{M}_H$ *and let* h *be the multiplicative function defined as*

$$h(p^k) = \begin{cases} f(p)/H & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \tag{1-6}$$

Let h^{*H} *denote the* H*-fold convolution of* h *with itself. Then*

$$f = h^{*H} * h',$$

where h' *is a multiplicative function that satisfies* $h'(p) = 0$ *at primes and* $|h'(p^k)| \leqslant (2H)^k$ *at prime powers.*

Let $f = f_1 * \cdots * f_H$ *with* $f_i = h$ *for all but one of the factors and* $f_i = h * h'$ *for the remaining one. If* $x > 1$ *and if* $Q \leqslant x^{1/2}$ *is an integer multiple of* $W(x)$, *then the following bound holds for all* $A \in (\mathbb{Z}/Q\mathbb{Z})^*$:

$$\sum_{\substack{D \leqslant x^{1-1/H} \\ \gcd(D,Q)=1}} \sum_{d_1 \cdots d_{H-1} = D} \frac{|f_1(d_1) \cdots f_{H-1}(d_{H-1})|}{D} \sqrt{\frac{DQ}{x} \sum_{\substack{n \leqslant x/D \\ nD \equiv A \ (\mathrm{mod}\ Q)}} |f_H(n)|^2} \\ \ll (\log x)^{1/2} \frac{Q}{\phi(Q)} \frac{1}{\log x} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \biggl(1 + \frac{|f(p)|}{p}\biggr). \tag{1-7}$$

Aim and motivation. As mentioned before, the purpose of this paper is to study correlations of multiplicative functions, more specifically of functions from $\mathscr{M}_H$, with polynomial nilsequences. In general, such correlations can only be shown to be small if either the nilsequence is highly equidistributed or else if the multiplicative function is equidistributed in progressions with short common difference. We will consider both cases, the former in Proposition 6.4 and the latter in Theorem 6.1. In accordance with this restriction, the latter result only applies to the subsets $\mathscr{F}_H$ and $\mathscr{F}_{H,n^{it}}$ whose elements admit a W-trick as we will establish in Section 5. Restricting attention to the class $\mathscr{F}_H$ for now, then "W-trick" roughly means the following here. For every $f \in \mathscr{F}_H$ there is a product $\widetilde{W} = \widetilde{W}(x)$ of small prime powers such that f has a constant average value in all suitable subprogressions of $\{n \equiv A \pmod{\widetilde{W}}\}$ for every fixed residue $A \in (\mathbb{Z}/\widetilde{W}\mathbb{Z})^*$. Instead of bounding Fourier coefficients of f as in (1-1), we aim to show that every $f \in \mathscr{F}_H$ satisfies[1]

$$\frac{\widetilde{W}}{x} \sum_{n \leqslant x/\widetilde{W}} (f(\widetilde{W}n + A) - S_f(x; \widetilde{W}, A)) F(g(n)\Gamma) = o_{G/\Gamma}\biggl(\frac{1}{\log x} \frac{\widetilde{W}}{\phi(\widetilde{W})} \prod_{p \leqslant x,\ p \nmid \widetilde{W}} \biggl(1 + \frac{|f(p)|}{p}\biggr)\biggr) \tag{1-8}$$

for all 1-bounded polynomial nilsequences $F(g(n)\Gamma)$ of bounded degree and bounded Lipschitz constant that are defined with respect to a nilmanifold G/Γ of bounded step and bounded dimension. The precise statement will be given in Section 6. This result can be viewed as a generalization of work of Green and Tao [2012a] who were the first to study correlations of the form (1-8) and who prove (1-8) for the Möbius function. In fact, we borrow their approach to reduce Theorem 6.1 to Proposition 6.4 in Section 6 and we work with their techniques in Sections 7 and 8.

Note carefully that the bound proposed in (1-8) is nontrivial even in the case where the function f satisfies $S_f(x) = o(1)$, i.e., even for a function like $f(n) = \delta^{\omega(n)}$ with $\delta \in (0, 1)$, which satisfies

$$S_f(x) \sim (\log x)^{\delta - 1} \asymp \frac{1}{\log x} \prod_{p \leqslant x} \biggl(1 + \frac{|f(p)|}{p}\biggr) = o(1).$$

To see this, we observe that Lemma 1.7 and Shiu's lemma [1980, Theorem 1] imply that the error term in (1-8) is, at least for a positive proportion of the reduced residues $A \pmod{\widetilde{W}}$, of the form o("the trivial upper bound"), which is the bound obtained by inserting absolute values everywhere.

The interest in estimates of the form (1-8) lies in the fact that the Green–Tao–Ziegler inverse theorem [Green et al. 2012] allows one to deduce that $f(\widetilde{W}n + A) - S_f(x; \widetilde{W}, A)$ has small U^k-norms of all orders, where "small" may depend on k. Employing the nilpotent Hardy–Littlewood method of Green and Tao [2010], this in turn allows one to deduce asymptotic formulae for expressions of the form

$$\sum_{\boldsymbol{x} \in K \cap \mathbb{Z}^s} f(\varphi_1(\boldsymbol{x}) + a_1) \cdots f(\varphi_r(\boldsymbol{x}) + a_r), \tag{1-9}$$

for $a_1, \dots, a_r \in \mathbb{Z}$, pairwise nonproportional linear forms $\varphi_1, \dots, \varphi_r : \mathbb{Z}^s \to \mathbb{Z}$ and convex $K \subset \mathbb{R}^s$, *provided* that f has a sufficiently pseudorandom majorant function. We construct such pseudorandom majorants in the companion paper [Matthiesen 2016], which also addresses the question of evaluating (1-9) for functions $f \in \mathscr{F}_{H,n^{it}}$ with the property that $|f(n)| \ll_\varepsilon n^\varepsilon$ for all $\varepsilon > 0$.

Strategy and related work. Our overall strategy is to decompose the given multiplicative function via Dirichlet decomposition in such a way that we can employ the Montgomery–Vaughan approach to the individual factors. This approach reduces matters to bounding correlations of sequences defined in terms of primes. One type of correlation that appears will be handled with the help of Green and Tao's bound [2010, Proposition 10.2] on the correlation of the "W-tricked von Mangoldt function" with nilsequences.

[1] This statement needs to be slightly adapted if $f \in \mathscr{F}_{H,n^{it}}$.

Carrying out the Montgomery–Vaughan approach in the nilsequences setting makes it necessary to understand the equidistribution properties of certain families of product nilsequences which result from an application of the Cauchy–Schwarz inequality. These product sequences are studied in Section 8 refining techniques introduced in [Green and Tao 2012a]. More precisely, we show that most of these products are equidistributed provided the original sequence that these products are derived from was equidistributed. The latter can be achieved by the Green–Tao factorization theorem for nilsequences from [Green and Tao 2012b].

The question studied in this paper is in spirit related to that of Bourgain–Sarnak–Ziegler [Bourgain et al. 2013], who use an orthogonality criterion that can be proved employing ideas that go back to Daboussi and Delange [1974] (see also [Harper 2011] and [Tao 2011]). Invoking the orthogonality criterion in the form it is presented in [Kátai 1986], recent and very substantial work of Frantzikinakis and Host [2017] shows that every bounded multiplicative function can be decomposed into the sum of a Gowers-uniform function, a structured part and an error term. This error term is small in the sense that the integral of the error term over the space of all 1-bounded multiplicative functions is small. While their result provides no information on the quality of the error term of individual functions, it allows one to study simultaneously all bounded multiplicative functions.

The point of view taken in the present work is a different one: we have applications to explicit multiplicative functions in mind. For many multiplicative functions f that appear naturally in number theoretic contexts, the mean value $\frac{1}{x}\sum_{n\leqslant x} f(x)$ is described by a reasonably nice function in x, and one can hope to be able to verify the conditions from Definitions 1.3 and 1.4 (or 1.6) for such functions. In order to deduce asymptotic formulae for expressions as in (1-9), it is important that the bound on the correlation (1-8) improves at least on the trivial bound given by the average value of $|f|$. Thus, we need to be able to understand these bounds for individual functions f. We establish a noncorrelation result (Theorem 6.1) with an explicit bound that preserves information on $|f|$ just as in (1-8). An important feature of this work is that it applies to a large class of unbounded functions.

Notation. The following, perhaps unusual, piece of notation will be used throughout the paper: Suppose $\delta \in (0, 1)$, we write $x = \delta^{-O(1)}$ instead of $x = (1/\delta)^{O(1)}$ to indicate that there is a constant $0 \leqslant C \ll 1$ such that $x = (1/\delta)^{C}$.

Convention. If the statement of a result contains Vinogradov or O-notation in the assumptions, the implied constants in the conclusion may depend on all implied constants from the assumptions.

2. Brief outline of some ideas

In this section we give a very rough outline of the ideas behind the application of the Montgomery–Vaughan approach in the nilsequences setting, making a number of simplifications for the benefit of the exposition. The main idea of Montgomery and Vaughan [1977] is to introduce a log factor into the Fourier coefficient that we wish to analyze. Let $f : \mathbb{N} \to \mathbb{R}$ be a multiplicative function that satisfies $|f(p)| \leqslant H$

for some constant $H \geqslant 1$ and all primes p and suppose (1-4) holds. Then we have

$$\sum_{n\leqslant N} f(n)e(n\alpha)\log\frac{N}{n} \leqslant \biggl(\sum_{n\leqslant N}\Bigl(\log\frac{N}{n}\Bigr)^2\biggr)^{1/2}\biggl(\sum_{n\leqslant N}|f(n)|^2\biggr)^{1/2} \ll N^{1/2}\biggl(\sum_{n\leqslant N}|f(n)|^2\biggr)^{1/2},$$

and thus

$$\log N\biggl(\frac{1}{N}\sum_{n\leqslant N} f(n)e(n\alpha)\biggr) \ll \biggl(\frac{1}{N}\sum_{n\leqslant N}|f(n)|^2\biggr)^{1/2} + \biggl|\frac{1}{N}\sum_{n\leqslant N} f(n)e(n\alpha)\log n\biggr|.$$

The first term in the bound is handled by the assumptions on f, that is, by assuming that (1-4) holds. To bound the second term, one invokes the identity $\log n = \sum_{d|n}\Lambda(d)$, which reduces the task to bounding the expression

$$\sum_{nm\leqslant N} f(nm)\Lambda(m)e(nm\alpha).$$

This in turn may be reduced to the task of bounding

$$\sum_{np\leqslant N} f(n)f(p)\Lambda(p)e(pn\alpha),$$

where p runs over primes. Applying the Cauchy–Schwarz inequality and smoothing, it furthermore suffices to estimate expressions of the form

$$\sum_{p,p'} f(p)f(p')\log(p)\log(p')\sum_{n} w(n)e((p-p')n\alpha),$$

where p and p' run over primes and where w is a smooth weight function. One employs a standard sieve estimate to bound $\#\{(p,p') : p-p'=h\}$ for fixed h. Standard exponential sum estimates and a delicate decomposition of the summation ranges for n, p, p' yield an explicit bound on $(1/N)\sum_{n\leqslant N} f(n)e(n\alpha)$.

We seek to employ the above approach to correlations of the form

$$\frac{1}{N}\sum_{n\leqslant N}\biggl(f(n)-\frac{1}{N}\sum_{m\leqslant N} f(m)\biggr)F(g(n)\Gamma)$$

for multiplicative f. One problem we face is that the above approach makes substantial use of the strong equidistribution properties of the exponential functions $e((p-p')n\alpha)$ for distinct primes p, p'. A general polynomial sequence $(g(n)\Gamma)_{n\leqslant N}$ on a nilmanifold G/Γ may, on the other hand, not even be equidistributed. This problem is resolved by an application of the factorization theorem for polynomial sequences from [Green and Tao 2012b], which allows us to assume that $(g(n)\Gamma)_{n\leqslant N}$ is equidistributed in G/Γ if f is equidistributed in progressions to small moduli. The latter will be arranged for by employing a W-trick. As above, we then consider the following expression, which we split into sums over large and small primes, respectively, with respect to a suitable cutoff parameter X:

$$\frac{1}{N}\sum_{mp\leqslant N} f(m)f(p)\Lambda(p)F(g(mp)\Gamma)$$
$$=\frac{1}{N}\sum_{m\leqslant X}\sum_{p\leqslant N/m} f(m)f(p)\Lambda(p)F(g(mp)\Gamma)+\frac{1}{N}\sum_{m>X}\sum_{p\leqslant N/m} f(m)f(p)\Lambda(p)F(g(mp)\Gamma).$$

Applying Cauchy–Schwarz to both terms shows that it suffices to understand correlations of the form

$$\sum_{m,m'} f(m)f(m')\sum_{p}\Lambda(p)F(g(mp)\Gamma)\overline{F(g(m'p)\Gamma)}$$

and

$$\sum_{p,p'} f(p)f(p')\Lambda(p)\Lambda(p')\sum_{m}F(g(pm)\Gamma)\overline{F(g(p'm)\Gamma)}.$$

Choosing X suitably, only the first of these correlations matters. We shall bound this correlation by employing Green and Tao's result that the W-tricked von Mangoldt function is orthogonal to nilsequences. The necessary equidistribution properties of the sequences $n\mapsto F(g(mn)\Gamma)\overline{F(g(m'n)\Gamma)}$ will be established in Sections 7 and 8. The problem of extending the above method to functions from $\mathscr{M}_H$ will be addressed at the beginning of Section 9. For this purpose the moment condition (1-4) will be replaced by Lemma 1.8.

3. A suitable Dirichlet decomposition for $f\in\mathscr{M}_h$

In this section we prove Lemma 1.8, which shows that every function $f\in\mathscr{M}_H$ has a decomposition $f=f_1*\cdots*f_H$ into multiplicative functions f_i such that the L^2-norms of the f_i are controlled on average by the mean value of f. This lemma will replace the much more restrictive condition (1-4) in our application of the Montgomery–Vaughan approach outlined in the previous section. Before we prove Lemma 1.8, let us record a straightforward consequence of [Shiu 1980, Theorem 1] that will be used.

Lemma 3.1 (Shiu). *Let H be a positive integer and suppose $f:\mathbb{N}\to\mathbb{R}$ is a nonnegative multiplicative function satisfying $f(p^k)\leqslant H^k$ at all prime powers p^k. Let $W=W(x)$ be as before, let $q>0$ be an integer and let $A'\in(\mathbb{Z}/Wq\mathbb{Z})^*$. Then*

$$\sum_{\substack{x-y<n\leqslant x\\ n\equiv A' \ (\mathrm{mod}\ Wq)}} f(n)\ll\frac{y}{\phi(Wq)}\frac{1}{\log x}\exp\biggl(\sum_{\substack{w(x)<p\leqslant x\\ p\nmid q}}\frac{f(p)}{p}\biggr), \tag{3-1}$$

uniformly in A', q and y, provided that $q\leqslant y^{1/2}$ and $x^{1/2}\leqslant y\leqslant x$.

Proof. This lemma differs from [Shiu 1980, Theorem 1] in that it does not concern short intervals but at the same time it does not require f to satisfy $f(n)\ll_\varepsilon n^\varepsilon$. Shiu's result works with a summation range of the form $x-y<n\leqslant x$, where $x^\beta<y\leqslant x$, $\beta\in\bigl(0,\frac{1}{2}\bigr)$. Thus, in our case the parameter β can be regarded as fixed. As observed in [Nair and Tenenbaum 1998], the proof of [Shiu 1980, Theorem 1] only requires the condition $f(n)\ll_\varepsilon n^\varepsilon$ to hold for one fixed value of ε once β is fixed.

Note that any integer $n \equiv A' \pmod{Wq}$ is free from prime divisors $p < w(x)$. Thus, $f(n) \leqslant H^{\Omega(n)} \leqslant n^{\log H/\log w(x)}$. Given any $\varepsilon > 0$, we deduce that $n \equiv A' \pmod{Wq}$ implies $f(n) \leqslant n^{\varepsilon}$ provided x is sufficiently large. □

Proof of Lemma 1.8. Let h and h' be as in the statement of the lemma. We begin by showing that

$$|h'(p^k)| \leqslant (2H)^k, \tag{3-2}$$

using induction. Since $h'(p) = 0$, the inequality holds for $k = 1$. To analyze the general case, note that, since $h(p^k) = 0$ whenever $k \geqslant 2$, we have

$$h^{*H}(p^k) = \binom{H}{k} h^k(p) = \binom{H}{k} \frac{f^k(p)}{H^k}$$

for $1 \leqslant k \leqslant H$, and $h^{*H}(p^k) = 0$ if $k > H$. Thus, $f = h' * h^{*H}$ implies that

$$h'(p^k) = f(p^k) - \sum_{j=1}^{\min(k,H)} h'(p^{k-j}) h^{*H}(p^j).$$

Suppose now that $k \geqslant 2$ and that the inequality holds for all $j < k$. Then, invoking also (1) of Definition 1.3, we have

$$|h'(p^k)| < H^k + \sum_{j=1}^{\min(k,H)} (2H)^{k-j} \binom{H}{j} < (2H)^k \biggl(2^{-k} + \sum_{j=1}^{\min(k,H)} \frac{1}{j!2^j}\biggr) < (2H)^k,$$

as claimed.

To prove (1-7), suppose that $f_H = h$ or $h * h'$. By Shiu's bound, we have

$$\frac{DQ}{x} \sum_{\substack{n \leqslant x/D \\ n \equiv A \pmod{Q}}} f_H^2(n) \ll \frac{1}{\log(x/D)} \frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \biggl(1 + \frac{|f(p)|}{Hp}\biggr),$$

where we used the trivial inequality $f_H(p)^2 \leqslant |f_H(p)| = |f(p)|/H$ and extended the product over primes up to x. Multiplying the right-hand side with

$$\frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \biggl(1 + \frac{|f(p)|}{Hp}\biggr) \gg 1,$$

and observing that $\log(x/D) \asymp_H \log x$, we obtain

$$\sqrt{\frac{DQ}{x} \sum_{\substack{n \leqslant x/D \\ n \equiv A \pmod{Q}}} f_i^2(n)} \ll_H \frac{1}{(\log x)^{1/2}} \frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \biggl(1 + \frac{|f(p)|}{Hp}\biggr).$$

Thus, the left-hand side of (1-7) is bounded by

$$
\begin{aligned}
&\ll_H \frac{1}{(\log x)^{1/2}} \frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \Big(1 + \frac{|f(p)|}{Hp}\Big) \sum_{\substack{D \leqslant x^{1-1/H} \\ \gcd(D,Q)=1}} \sum_{d_1 \cdots d_{H-1} = D} \frac{|f_1(d_1) \cdots f_{H-1}(d_{H-1})|}{D} \\
&\ll_H \frac{1}{(\log x)^{1/2}} \frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \Big(1 + \frac{|f(p)|}{Hp}\Big)\Big(1 + \frac{(H-1)|f(p)|}{Hp}\Big)\Big(1 + \sum_{k \geqslant 2} \frac{|f_1 * \cdots * f_{H-1}(p^k)|}{p^k}\Big) \\
&\ll_H \frac{1}{(\log x)^{1/2}} \frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \Big(1 + \frac{|f(p)|}{p}\Big)\Big(1 + \frac{(H-1)}{p^2}\Big)\Big(1 + \sum_{k \geqslant 2} \frac{|f_1 * \cdots * f_{H-1}(p^k)|}{p^k}\Big).
\end{aligned}
$$

The above is now seen to have the claimed bound

$$
\ll_H \frac{1}{(\log x)^{1/2}} \frac{Q}{\phi(Q)} \prod_{\substack{p \leqslant x \\ p \nmid Q}} \Big(1 + \frac{|f(p)|}{p}\Big)
$$

for all sufficiently large x, providing

$$
\sum_{w(x) < p \leqslant x} \sum_{k \geqslant 2} \frac{|f_1 * \cdots * f_{H-1}(p^k)|}{p^k} \ll_H 1.
$$

To show the latter, note that $f_1 * \cdots * f_{H-1}$ equals either $h^{*(H-1)}$ or $h^{*(H-1)} * h'$. Similarly as in the first part of this proof, we have

$$
|h^{*(H-1)}(p^k)| \leqslant \binom{H-1}{k} \leqslant \frac{H^k}{k!}
$$

for $k < H$ and $h^{*(H-1)}(p^k) = 0$ for $k \geqslant H$, and, consequently,

$$
\begin{aligned}
|(h^{*(H-1)} * h')(p^k)| &= \sum_{j=0}^{\min(k,H-1)} |h'(p^{k-j}) h^{*(H-1)}(p^j)| \\
&\leqslant \sum_{j=0}^{\min(k,H-1)} (2H)^{k-j} H^j \leqslant 2(2H)^k.
\end{aligned}
$$

Thus, if x is large enough that $w(x) > 4H$, then

$$
\begin{aligned}
\sum_{w(x) < p \leqslant x} \sum_{k \geqslant 2} \frac{|f_1 * \cdots * f_{H-1}(p^k)|}{p^k} &\leqslant 2 \sum_{w(x) < p \leqslant x} \sum_{k \geqslant 2} \Big(\frac{2H}{p}\Big)^k \leqslant 8H^2 \sum_{w(x) < p \leqslant x} \frac{1}{p^2}\Big(1 - \frac{2H}{p}\Big)^{-1} \\
&\leqslant 16H^2 \sum_{w(x) < p \leqslant x} \frac{1}{p^2} \ll_H 1,
\end{aligned}
$$

which completes the proof. □

4. Multiplicative functions in progressions: The class of functions $\mathscr{F}_H$

Both of the conditions that define $\mathscr{M}_H$ are natural and simple conditions on the behavior of $|f|$ at prime powers. Our aim in this section is to discuss the more complicated stability condition (1-5) on mean values in progressions, that defines the class $\mathscr{F}_H$. We will prove two sufficient conditions, recorded in Propositions 4.4 and 4.10, for the bound (1-5) to hold and apply these to provide several examples of natural functions that belong to $\mathscr{F}_H$. In particular, we deduce a simple criterion, see Lemma 4.16, for a nonnegative function to belong to $\mathscr{F}_H$.

4A. ***A sufficient condition for $f \in \mathscr{F}_H$.*** The main tool in our analysis of (1-5) will be the following consequence of the "pretentious large sieve", which lets one bound the tail of the character sum expansion of $S_h(x; q, a)$ for any bounded multiplicative function h and thereby simplifies the task of analyzing the expression $S_h(x; q, a)$.

Proposition 4.1 ([Granville and Soundararajan ≥ 2018]; cf. [Balog et al. 2013, Lemma 3.1; Granville 2009, Theorem 2; Granville et al. 2017, Theorem 1.8). *Let $C > 0$ be fixed and let h be a bounded multiplicative function. For any given x, consider the set of primitive characters of conductor at most $(\log x)^C$ and enumerate them as $\chi_1, \chi_2, \dots$ in such a way that $|S_{h\overline{\chi}_1}(x)| \geqslant |S_{h\overline{\chi}_2}(x)| \geqslant \cdots$. If x is sufficiently large, then the following holds for all $x^{1/2} \leqslant X \leqslant x$ and $q \leqslant (\log x)^C$. Let $\mathscr{C}$ be any set of characters modulo q, $q \leqslant (\log x)^C$, which does not contain characters induced by $\chi_1, \dots, \chi_k$, where $k \geqslant 2$. Then*

$$\left|\frac{1}{\phi(q)}\sum_{\chi\in\mathscr{C}}\chi(a)\sum_{n\leqslant X}h(n)\overline{\chi}(n)\right| \ll_C \frac{e^{O_C(\sqrt{k})}X}{q}\left(\frac{\log\log x}{\log x}\right)^{1-\frac{1}{\sqrt{k}}}\log\frac{\log x}{\log\log x}\prod_{p\leqslant q,\, p\nmid q}\left(1+\frac{|h(p)|-1}{p}\right).$$

To deduce a sufficient condition for (1-5) we first extend this result to all unbounded elements of $\mathscr{M}_H$.

Corollary 4.2. *Let $f \in \mathscr{M}_H$ and set $h = f$ if $H = 1$. If $H > 1$, let h be the multiplicative function defined in (1-6) so that $f = h^{*H} * h'$ for a multiplicative function h' with support in the square-full numbers. Let $C > 0$ be a constant, let $\varepsilon = \frac{1}{2}\min(1, \alpha_f/H)$, and set $k = \lceil \varepsilon^{-2}\rceil \geqslant 2$ and $k' = \lceil \log_2(4H)\rceil$. For each $j \in \{0, \dots, k'\}$, let $\mathscr{E}_j = \{\chi_1^{(j)}, \dots, \chi_k^{(j)}\}$ denote the set consisting of the first k primitive characters of conductor at most $(\log x^{1/2^j})^C$ defined by Proposition 4.1 when applied to h and with x replaced by $x^{1/2^j}$.*

If x is sufficiently large, the following holds for all $x^{1/2} \leqslant y \leqslant x$ and all integer multiples $0 < Q \leqslant (\log x^{1/(8H)})^C$ of $W(x)$. Let $\mathscr{C}$ be any set of characters modulo Q which does not contain characters induced by any $\chi \in \mathscr{E} := \mathscr{E}_0 \cup \cdots \cup \mathscr{E}_{k'}$, then

$$\left|S_f(y; Q, a) - \frac{Q}{y}\frac{1}{\phi(Q)}\sum_{\substack{\chi \pmod{Q}\\ \chi\notin\mathscr{C}}}\chi(a)\sum_{n\leqslant y}f(n)\overline{\chi}(n)\right|$$
$$= \frac{Q}{\phi(Q)}\left|\sum_{\chi\in\mathscr{C}}\chi(a)\frac{1}{y}\sum_{n\leqslant y}f(n)\overline{\chi}(n)\right| \ll_{C,H,\alpha_f} \frac{1}{(\log x)^{1+\alpha_f/(3H)}}\frac{Q}{\phi(Q)}\exp\left(\sum_{p\leqslant x,\, p\nmid Q}\frac{|f(p)|}{p}\right).$$

The proof of this corollary makes use of the following lemma about the contribution of the sparse function h'.

Lemma 4.3. *Let $H > 1$, $f \in \mathscr{M}_H$ and let $f = h^{*H} * h'$ be the decomposition from* (1-6). *Let $g : \mathbb{N} \to \mathbb{C}$ be a bounded completely multiplicative function that vanishes at all primes $p \leqslant w$ for a fixed $w \geqslant (2H)^{16}$, and let $\delta \in (0, 1)$. Then, if $x^{1/2} \leqslant y \leqslant x$, we have*

$$\left|\frac{1}{y}\sum_{n\leqslant y} f(n)b(n)\right| \leqslant \sum_{n_1\leqslant y^\delta} \frac{|h'(n_1)b(n_1)|}{n_1}\left|\frac{n_1}{y}\sum_{n_2\leqslant \frac{y}{n_1}} h^{*H}(n_2)b(n_2)\right| + O(x^{-\frac{\delta}{8}}(\log y)^{O(H)}).$$

Proof. Recall that h' is supported on square-full numbers only and that $|h'(p^k)| \leqslant (2H)^k$ by (3-2). Since b is completely multiplicative, we have

$$\begin{aligned}\sum_{n\leqslant y} f(n)b(n) &= \sum_{n_1 n_2\leqslant y} h'(n_1)h^{*H}(n_2)b(n_1)b(n_2)\\ &\leqslant \sum_{n_1\leqslant y^\delta}|h'(n_1)b(n_1)|\left|\sum_{n_2\leqslant y/n_1} h^{*H}(n_2)b(n_2)\right| + y\sum_{n_1>y^\delta}\frac{|h'(n_1)b(n_1)|}{n_1}\sum_{n_2\leqslant y/n_1}\frac{|h^{*H}(n_2)|}{n_2}\\ &\leqslant \sum_{n_1\leqslant y^\delta}|h'(n_1)b(n_1)|\left|\sum_{n_2\leqslant y/n_1} h^{*H}(n_2)b(n_2)\right| + y(\log y)^{O(H)}\sum_{\substack{n_1\geqslant y^\delta\\ p\mid n_1\Rightarrow p>w}}\frac{|h'(n_1)|}{n_1}.\end{aligned}$$

By decomposing every square-full number n_1 as $m^2 d$ with $d \mid m$, we obtain the following bound for the sum in the final term:

$$\begin{aligned}\sum_{\substack{n_1\geqslant y^\delta\\ p\mid n_1\Rightarrow p>w}}\frac{|h'(n_1)|}{n_1} &\leqslant \sum_{\substack{m\geqslant y^{\delta/3}\\ p\mid m\Rightarrow p>w}}\frac{(2H)^{\Omega(m^2)}}{m^2}\sum_{d\mid m}\frac{(2H)^{\Omega(d)}}{d} \leqslant \sum_{\substack{m\geqslant y^{\delta/3}\\ p\mid m\Rightarrow p>w}}\frac{(2H)^{\frac{2\log m}{\log w}}}{m^2}d(m)\\ &\ll \sum_{\substack{m\geqslant y^{\delta/3}\\ p\mid m\Rightarrow p>w}} m^{-2+\frac{2\log(2H)}{\log w}+\frac{1}{8}} \ll \sum_{\substack{m\leqslant y^{\delta/3}\\ p\mid m\Rightarrow p>w}} m^{-2+\frac{1}{4}} \ll y^{-\frac{\delta}{4}},\end{aligned} \tag{4-1}$$

where we used the bound $d(n) \ll n^{1/8}$. Combining the two bounds above completes the proof. □

Proof of Corollary 4.2. To start with, we consider the bounded multiplicative function h. Note that Proposition 4.1 applies to values of X with $x^{1/2} \leqslant X \leqslant x$. Our application, will, however, require a range of the form $x^{1/(4H)} \leqslant X \leqslant x$. For this reason, we will apply Proposition 4.1 once with x replaced by $x^{1/2^j}$ for each $j \in \{0, \dots, \lceil \log_2(4H)\rceil\}$. If $\mathscr{C}$ is as in the statement of the corollary, then Proposition 4.1 shows that for all $Q \leqslant (\log x^{1/(8H)})^C$ and for all $x^{1/(4H)} < X \leqslant x$, we have

$$\begin{aligned}\frac{1}{X}\frac{Q}{\phi(Q)}\left|\sum_{\chi\in\mathscr{C}}\chi(A)\sum_{n\leqslant X} h(n)\overline{\chi}(n)\right| &\ll_{C,H,\alpha_f} \left(\frac{\log\log x}{\log x}\right)^{1-1/\sqrt{k}}\log\left(\frac{\log x}{\log\log x}\right)\\ &\ll_{C,H,\alpha_f} (\log x)^{-1+\alpha_f/(2H)}(\log\log x)^2,\end{aligned}$$

since $1/\sqrt{k} = 1/\sqrt{[\varepsilon^{-2}]} \leqslant \varepsilon = \frac{1}{2}\min(1, \alpha_f/H) \leqslant \alpha_f/(2H)$.

By property (2) of Definition 1.3, we have

$$\frac{Q}{\phi(Q)} \exp\biggl(\sum_{\substack{p\leqslant x\\ p\nmid Q}} \frac{|h(p)|}{p}\biggr) \geqslant \exp\biggl(\sum_{\substack{p\leqslant x\\ p\nmid Q}} \frac{|f(p)|}{Hp}\biggr) \geqslant \Bigl(\frac{\log x}{C\log\log x}\Bigr)^{\alpha_f/H},$$

and, thus,

$$\begin{aligned}\frac{1}{X}\frac{Q}{\phi(Q)}\biggl|\sum_{\chi\in\mathscr{C}}\chi(A)\sum_{n\leqslant X} h(n)\overline{\chi}(n)\biggr| &\ll_{C,H,\alpha_f} \frac{(\log\log x)^{2+\alpha_f/H}}{(\log x)^{1+\alpha_f/(2H)}}\frac{Q}{\phi(Q)}\exp\biggl(\sum_{\substack{p\leqslant x\\ p\nmid Q}}\frac{|h(p)|}{p}\biggr)\\ &\ll_{C,H,\alpha_f} \frac{1}{(\log x)^{1+\alpha_f/(3H)}}\frac{Q}{\phi(Q)}\exp\biggl(\sum_{\substack{p\leqslant x\\ p\nmid Q}}\frac{|h(p)|}{p}\biggr). \qquad (4\text{-}2)\end{aligned}$$

To handle the case where $H > 1$, consider the decomposition $f = h^{*H} * h'$ with h as in (1-6). If $x^{1/2} \leqslant y \leqslant x$, then Lemma 4.3 implies that for any $\delta \in (0, 1)$,

$$\frac{1}{y}\sum_{\chi\in\mathscr{C}}\chi(A)\sum_{n\leqslant y} f(n)\overline{\chi}(n) \leqslant \frac{1}{y}\sum_{\chi\in\mathscr{C}}\sum_{d_0\leqslant y^\delta}|h'(d_0)\overline{\chi}(d_0)|\biggl|\sum_{d\leqslant y/d_0} h^{*H}(d)\overline{\chi}(d)\biggr| + O(x^{-\delta/8}(\log x)^{O(H)}). \quad (4\text{-}3)$$

The error term in this bound is acceptable. A generalization of the hyperbola method applied to the sum over d (see Section 9A for a deduction) shows that the main term satisfies

$$\begin{aligned}&\frac{1}{y}\sum_{\chi\in\mathscr{C}}\sum_{d_0\leqslant y^\delta}|h'(d_0)\overline{\chi}(d_0)|\biggl|\sum_{d\leqslant y/d_0} h^{*H}(d)\overline{\chi}(d)\biggr|\\ &\qquad\leqslant \sum_{\substack{d_0\leqslant y^\delta\\ p\mid d_0\Rightarrow p>w(x)}}\frac{|h'(d_0)|}{d_0}\sum_{D\leqslant(y/d_0)^{1-1/H}}\sum_{d_1\cdots d_{H-1}=D}\frac{|h(d_1)\cdots h(d_{H-1})||\overline{\chi}(D)|}{D}\\ &\qquad\qquad\times\sum_{i=1}^{H}\frac{Dd_0}{y}\biggl|\sum_{\chi\in\mathscr{C}}\chi(A)\sum_{\substack{n:\\ (y/d_0)^{1-1/H}\max(d_1,\dots,d_{i-1})\\ \leqslant Dn\leqslant y/d_0}} h(n)\overline{\chi}(n)\biggr|. \qquad (4\text{-}4)\end{aligned}$$

Observe that the upper bound on n in the inner sum satisfies $y/(Dd_0) \in [y^{1/H-\delta}, x]$. By choosing $\delta = 1/(4H)$, this interval is contained in $[x^{1/(2H)-\delta/2}, x] = [x^{3/(8H)}, x]$. An application of the triangle inequality shows that the inner sum is bounded by

$$\frac{Dd_0}{y}\biggl|\sum_{\chi\in\mathscr{C}}\chi(A)\sum_{n\leqslant y/(d_0D)} h(n)\overline{\chi}(n)\biggr| + \frac{Dd_0}{y}\biggl|\sum_{\chi\in\mathscr{C}}\chi(A)\sum_{n\leqslant y'} h(n)\overline{\chi}(n)\biggr|,$$

where $y' = \min(y/(d_0D), (y/d_0)^{1-1/H}D^{-1}\max(d_1, \dots, d_{i-1}))$. We are now in a position to apply (4-2) to bound the first of these terms by

$$\ll_{C,H,\alpha_f} \frac{1}{(\log x)^{1+\alpha_f/(3H)}} \exp\biggl(\sum_{p\leqslant x,\, p\nmid Q} \frac{|h(p)|}{p}\biggr).$$

If $y' > x^{1/(4H)}$, then the same bound applies to the second term. If, on the other hand, $y' \leqslant x^{1/(4H)} \leqslant y^{1/(2H)}$, then the second term may trivially be bounded by

$$\phi(Q) y^{1/(4H)-1} d_0 D \leqslant \phi(Q) y^{1/(4H)-1+1/(4H)+1-1/H} \leqslant (\log x)^C x^{-1/(4H)}.$$

Inserting these bounds into (4-4) and completing the outer sums, we deduce that (4-4) is bounded by

$$\ll_{C,H,\alpha_f} \frac{1}{(\log x)^{1+\alpha_f/(3H)}} \exp\biggl(\sum_{\substack{p\leqslant x,\\ p\nmid Q}} \frac{|h(p)|}{p}\biggr) \sum_{\substack{d_0 : p \mid d_0\\ \Rightarrow p > w(x)}} \frac{|h'(d_0)|}{d_0} \biggl(\sum_{d\leqslant x} \frac{|h(d)\chi(d)|}{d}\biggr)^{H-1}.$$

The sum over d in this bound satisfies

$$\biggl(\sum_{d\leqslant x} \frac{|h(d)\chi(d)|}{d}\biggr)^{H-1} \leqslant \prod_{p\leqslant x} \biggl(1 + \frac{|h(p)\chi(p)|}{p}\biggr)^{H-1} \leqslant \exp\biggl((H-1) \sum_{p\leqslant x, p\nmid Q} \frac{|h(p)|}{p}\biggr),$$

and the sum over d_0 converges by (4-1), applied with $y = 1$, provided x is sufficiently large for $w(x) \geqslant (2H)^{16}$ to hold. Collecting all information together, it follows from (4-3) that

$$\frac{1}{x} \frac{Q}{\phi(Q)} \sum_{\chi \in \mathscr{C}} \chi(A) \sum_{n\leqslant x} f(n)\bar{\chi}(n) \ll_{C,H,\alpha_f} \frac{1}{(\log x)^{1+\frac{\alpha_f}{(3H)}}} \frac{Q}{\phi(Q)} \exp\biggl(\sum_{\substack{p\leqslant x,\\ p\nmid Q}} \frac{|f(p)|}{p}\biggr),$$

which competes the proof. □

With Corollary 4.2 in place, we obtain the following sufficient condition for $f \in \mathscr{M}_H$ to belong to $\mathscr{F}_H$:

Proposition 4.4 (sufficient condition). *Suppose that $f \in \mathscr{M}_H$. Then $f \in \mathscr{F}_H$ if the following holds. For every $C > 0$, there exists a function $\psi_C : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, with the property that $\psi_C(x) \to 0$ as $x \to \infty$, such that*

$$S_{f\chi}(x') = S_{f\chi}(x) + O\biggl(\frac{\psi_C(x)}{\log x} \exp\biggl(\sum_{p\leqslant x,\, p\nmid Q} \frac{|f(p)|}{p}\biggr)\biggr), \quad (x \geqslant 2), \tag{4-5}$$

uniformly for all $x' \in (x(\log x)^{-C}, x]$ and all characters $\chi \pmod Q$ with $1 < Q \leqslant (\log x)^C$ and $W(x) \mid Q$.

Proof. Recall from Definition 1.4 that we have to show that there exists $\varphi_C = o(1)$ such that

$$|S_f(x'; Q, A) - S_f(x; Q, A)| = O\biggl(\frac{\varphi_C(x)}{\log x} \frac{Q}{\phi(Q)} \prod_{p\leqslant x,\, p\nmid Q} \Bigl(1 + \frac{|f(p)|}{p}\Bigr)\biggr) \tag{4-6}$$

uniformly for all $x' \in (x(\log x)^{-C}, x]$, all $1 \leqslant Q \leqslant (\log x)^C$ with $W(x) \mid Q$ and all reduced $A \pmod Q$. This will be a straightforward consequence of the fact that by Corollary 4.2 there are only finitely many characters in the character sum expansions of $S_f(x'; Q, A)$ and $S_f(x; Q, A)$ that matter. Using the notation from the corollary, let $\mathscr{E}(Q)$ denote the set of characters modulo Q that are induced by the elements of $\mathscr{E}_0 \cup \cdots \cup \mathscr{E}_{k'}$. Then

$$S_f(x'; Q, A) = \frac{Q}{\phi(Q)} \sum_{\substack{\chi \pmod Q \\ \chi \in \mathscr{E}(Q)}} \chi(A) S_{f\overline{\chi}}(x') + O\biggl(\frac{\psi(x)}{\log x} \frac{Q}{\phi(Q)} \exp\biggl(\sum_{\substack{p \leqslant x \\ p \nmid Q}} \frac{|f(p)|}{p}\biggr)\biggr),$$

where $\psi(x) = O_{C,H,\alpha_f}((\log x)^{-\alpha_f/(3H)})$, uniformly in x', Q and A as above. Thus, (4-6) follows from our assumptions with $\psi_C(x) = \psi(x) + \#\mathscr{E} \cdot \varphi_C(x)$. □

Example 4.5 (applications using Selberg–Delange-type arguments). The conditions required by Proposition 4.4 are of a type that can usually be checked by means of the Selberg–Delange method (see, e.g., [Tenenbaum 1995, Section II.5]) provided the function f is closely related to a ζ- or L- function. The range of the modulus Q of the characters χ that appear is small enough to ensure that exceptional characters can be handled. Examples of functions suitable for this approach include:

(i) the function $\frac{1}{4}r(n) = \frac{1}{4}\#\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}$,

(ii) the indicator function of the set of sums of two squares,

(iii) the characteristic function of set of numbers composed of primes that split completely in a given Galois extension $K/\mathbb{Q}$ of finite degree.

In the following subsection, we will further analyze the Lipschitz condition (4-5) and prove another sufficient condition, in this case for an element $f \in \mathscr{M}_H$ to belong to $\mathscr{F}_{H,n^{it}}$.

4B. ***Lipschitz estimates for elements of $\mathscr{M}_H$ and another sufficient condition.*** For applications of Proposition 4.4 or Corollary 4.2, the following four lemmas, which we all prove in Section 4C, are very useful. The first lemma is a slight generalization of the Lipschitz estimate for bounded multiplicative functions and a related decay estimate that Granville and Soundararajan established in Theorems 3 and 4 of [Granville and Soundararajan 2003].

Lemma 4.6 (Lipschitz estimates). *Let $f_0 \in \mathscr{M}_1$ and let $x \geqslant 3$. Suppose that $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, bounded in absolute value by* 1 *and satisfies $|f(p^k)| = |f_0(p^k)|$ for all primes $p \geqslant \exp((\log\log x)^2)$ and $k \geqslant 1$. Define*

$$F(s) = \prod_{p \leqslant x} \Bigl(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\Bigr).$$

If the maximum of $\max_{|y| \leqslant 2\log x} |F(1 + iy)|$ *is attained at $y = t_{x,f}$, then, uniformly in x and f as above, we have*

$$\biggl|\frac{1}{x}\sum_{n \leqslant x} f(n) n^{-it_{x,f}} - \frac{1}{x'}\sum_{n \leqslant x'} f(n) n^{-it_{x,f}}\biggr| \ll_{f_0} \frac{1}{(\log x)^{1+C_0}} \exp\biggl(\sum_{p \leqslant x} \frac{|f(p)|}{p}\biggr) \tag{4-7}$$

for all $x' \in [x \exp(-(\log\log x)^{-4}), x]$, *where* $C_0 \in \left(0, \frac{1}{2}\alpha_{f_0}\right)$ *is a positive constant that only depends on* α_{f_0}. *Furthermore, we have, for any* $t_{x,f}$ *as above,*

$$\left|\frac{1}{x}\sum_{n\leqslant x} f(n)\right| \ll_{f_0} \frac{1}{|t_{x,f}|+1} + \frac{\log\log x}{\log x} + \frac{1}{(\log x)^{1+C_0}} \exp\biggl(\sum_{p\leqslant x} \frac{|f(p)|}{p}\biggr). \tag{4-8}$$

The conditions on f above will allow us to apply the lemma to twists $h\chi$ where $h \in \mathscr{M}_1$ and χ is a character modulo Q with $Q \leqslant (\log x)^C$ for any given constant $C > 0$. In order to extend this to twists $f\chi$ for $f \in \mathscr{M}_H$ and $H > 1$, we note that the function h associated to f via (1-6) belongs to $\mathscr{M}_1$. The following lemma will enable us to employ Lemma 4.6 in general.

Lemma 4.7. *Let* $f \in \mathscr{M}_H$ *and let* h *be as in* (1-6). *Let* $C > 0$ *be a fixed constant and let* $\psi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *be a function that satisfies* $\psi(x) \to 0$ *as* $x \to \infty$. *Let* $x > 1$ *and suppose that* $\chi \pmod Q$, *with* $Q \leqslant (\log x)^C$ *and* $W(x) \mid Q$, *is a character such that*

$$|S_{h\chi}(y) - S_{h\chi}(y')| \leqslant \frac{\psi(x)}{\log y} \exp\biggl(\sum_{p\leqslant y,\ p\nmid Q} \frac{|h(p)|}{p}\biggr)$$

for all $y \in (x^{1/(2H)}, x]$ *and* $y' \in (y(\log x)^{-C}, y]$. *Then, for all* $x' \in (x(\log x)^{-C}, x]$, *we have*

$$|S_{f\chi}(x) - S_{f\chi}(x')| \leqslant \frac{\psi'(x)}{\log x} \exp\biggl(\sum_{p\leqslant x,\ p\nmid Q} \frac{|f(p)|}{p}\biggr),$$

where ψ' *is independent of* χ *and* Q *and satisfies* $\psi'(x) \to 0$ *as* $x \to \infty$; *more precisely,*

$$\psi'(x) = O_{H,C}\bigl(\psi(x) + (\log x)^{-\min(1,\alpha_f/(2H))} + x^{-1/8}(\log x)^{O(H)}\bigr).$$

The next lemma shows that if χ is a character that is negligible in the application of Proposition 4.4 or Corollary 4.2 to the function h, then χ is also negligible in an application of the result to f.

Lemma 4.8. *Let* $f \in \mathscr{M}_H$, *let* h *be as in* (1-6), *and let* $C > 0$. *Let* $\psi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *be a function that satisfies* $\psi(x) \to 0$ *as* $x \to \infty$. *Let* $x > 1$ *and suppose that* $\chi \pmod Q$, *with* $Q \leqslant (\log x)^C$ *and* $W(x) \mid Q$, *is any character such that*

$$|S_{h\chi}(x')| \leqslant \frac{\psi(x)}{\log y} \exp\biggl(\sum_{p\leqslant y,\ p\nmid Q} \frac{|h(p)|}{p}\biggr)$$

for all $x' \in [x^{1/(4H)}, x]$. *Then*

$$|S_{f\chi}(y)| \leqslant \frac{\psi'(x)}{\log x} \exp\biggl(\sum_{p\leqslant x,\ p\nmid Q} \frac{|f(p)|}{p}\biggr), \qquad (y \in [x^{1/2}, x]),$$

where ψ' *is independent of* χ *and* Q *and satisfies*

$$\psi'(x) = O(\psi(x) + x^{-1/(32H)}(\log x)^{O(H)}).$$

Finally, we observe that (4-7) holds *uniformly* in f for some $t = t_x$ that only depends on x and f_0. This proves particularly valuable when dealing with families of induced characters.

Lemma 4.9. *Let x, f_0 and f be as in Lemma 4.6, and let $x' \in \bigl[x\exp\bigl(-\frac{1}{2}(\log\log x)^4\bigr), x\bigr]$. Then there exists $|t_x| \leqslant 2\log x$, only dependent on x and f_0, but not on f or x', such that, uniformly in x, x' and f as before,*

$$\left|\frac{1}{x}\sum_{n\leqslant x} f(n)n^{-it_x} - \frac{1}{x'}\sum_{n\leqslant x'} f(n)n^{-it_x}\right| \ll \frac{1}{(\log x)^{1+C_0}} \exp\biggl(\sum_{p\leqslant x}\frac{|f(p)|}{p}\biggr), \tag{4-9}$$

where $C_0 > 0$ is a positive constant that only depends on α_{f_0}.

As a consequence of Lemmas 4.6–4.9 and Corollary 4.2, we obtain the following sufficient condition for testing whether a function belongs to $\mathscr{F}_{H,n^{it}}$.

Proposition 4.10 (another sufficient condition). *Let $f \in \mathscr{M}_H$ and h as in* (1-6). *For every $C > 0$, let $\psi_C : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a function that satisfies $\psi_C(x) \to 0$ as $x \to \infty$. Suppose that for every sufficiently large x there exists $\tau_x \in \mathbb{R}$ with $|\tau_x| \leqslant 2\log x$ such that the following holds: If $1 \leqslant Q \leqslant (\log x)^C$ with $W(x) \mid Q$ and if $\chi \pmod Q$ is a character then either the bound*

$$|S_{g_x\chi}(x')| \leqslant \frac{\psi_C(x')}{\log x'} \exp\biggl(\sum_{p\leqslant x', p\nmid Q}\frac{|g(p)|}{p}\biggr), \quad (x^{1/(8H)} \leqslant x' \leqslant x), \tag{4-10}$$

holds for either $g = h$ or $g = f$ and for $g_x : n \mapsto g(n)n^{-i\tau_x}$, or else we have $t_{x,f} = \tau_x$ or $t_x = \tau_x$ in the statement of Lemmas 4.6 or 4.9 when applied with $f_0 = h$ and with f replaced by $h\chi$.

Then the function $n \mapsto f(n)n^{-i\tau_x}$ satisfies (1-5) *and $f \in \mathscr{F}_{H,n^{it}}$.*

Example 4.11. The above proposition applies to:

(i) the Möbius function $f = \mu$. In this case we may take $\tau_x = 0$ for all x since $S_{\mu\chi}(x) \ll_B q^{1/2}(\log x)^{-B}$ for all $B > 0$ and all $\chi \pmod q$.[2] Indeed, if χ is a trivial character this estimate follows from prime number theorem-type bounds on $S_\mu(x)$; see Example 4.18(i) for details. If $\chi \pmod q$ is nontrivial, then its conductor, q' say, is at least 2 and one may deduce the estimate from [Iwaniec and Kowalski 2004, Corollary 5.29], which proves the claimed bound for nontrivial primitive characters. In fact, if $q \leqslant x$, then it follows from [loc. cit., (5.79)] that

$$\sum_{p\leqslant x}\chi(p) \ll_B {q'}^{1/2}x(\log x)^{-B} + \omega(q) \ll_B q^{1/2}x(\log x)^{-B},$$

since $\omega(q) \ll \log x$. If $q > x$, then

$$\sum_{p\leqslant x}\chi(p) \ll_B q^{1/2}x(\log x)^{-B}$$

holds trivially. Thus, [loc. cit., (5.79)] generalizes to all nontrivial χ and, by following the original proof from [loc. cit.], so does [loc. cit., (5.80)].

[2]This is the same information about μ as was used in [Green and Tao 2012a, Proposition A.1] to handle the "major arcs".

(ii) every multiplicative function f that takes values on the unit circle, i.e., for which $|f(n)| = 1$ for all n. This, in turn, follows from [Balog et al. 2013, Theorem 2], which provides the bound

$$|S_{f\chi}(x)| \ll \big((\log\log x)^2/\log x\big)^{1/20}, \tag{4-11}$$

valid for all characters χ of conductor $Q \leqslant \exp((\log\log x)^2)$, except perhaps for those induced by one exceptional character, ξ say. By Lemma 4.9, there exists $|t_x| \leqslant 2\log x$ such that (4-9) holds for all $\chi \pmod Q$ induced by ξ. Suppose now that $|t_x| > (\log x)^{1/100}$. Then Lemma 4.9, combined with the bound (4-8), implies that the above bound on $|S_{f\chi}(x)|$ also holds for characters induced by ξ. In this case, we may take $\tau_x = 0$. If, however, $|t_x| \leqslant (\log x)^{1/100}$, then we may use partial summation to deduce from (4-11) that

$$|S_{f_x\chi}(x)| \ll \big((\log\log x)^2/\log x\big)^{3/100}$$

for all χ not induced by ξ. In this case, we may set $\tau_x = t_x$.

Proof of Proposition 4.10. This result follows from Corollary 4.2 in a similar way as Proposition 4.4 does. To show that $n \mapsto f_x(n) := f(n)n^{-i\tau_x}$ satisfies (1-5), let $1 \leqslant Q \leqslant (\log x)^C$ be such that $W(x) \mid Q$ and let $\mathscr{E}(Q)$ denote the set of characters modulo Q that are induced by the elements of $\mathscr{E}_0 \cup \cdots \cup \mathscr{E}_{k'}$ from Corollary 4.2, when applied to the function f_x. Then

$$\begin{aligned} S_{f_x}(y; Q, A) - S_{f_x}(x; Q, A) &= \frac{Q}{\phi(Q)} \sum_{\substack{\chi \pmod Q \\ \chi \in \mathscr{E}(Q)}} \chi(A)(S_{f_x\overline{\chi}}(y) - S_{f_x\overline{\chi}}(x)) \\ &\quad + O_{C,H,\alpha_f}\biggl((\log x)^{-\alpha_f/(3H)} \frac{Q}{\phi(Q)} \frac{1}{\log x} \exp\biggl(\sum_{p\leqslant x,\, p\nmid Q} \frac{|f(p)|}{p}\biggr)\biggr), \end{aligned} \tag{4-12}$$

whenever $x^{1/2} \leqslant y \leqslant x$.

We begin with the contribution from those characters $\overline{\chi}$ to which the first alternative from the statement applies. Observe that (4-10) implies that

$$|S_{g_x\chi}(x')| \leqslant \frac{\psi_C'(x)}{\log x} \exp\biggl(\sum_{p\leqslant x, p\nmid Q} \frac{|g(p)|}{p}\biggr), \quad (x^{1/(8H)} \leqslant x' \leqslant x),$$

where $\psi_C'(x) = O_H(1) \max_{x^{1/(8H)} \leqslant x' \leqslant x} \psi_C(x')$. To see this, note that for all x' as above,

$$\begin{aligned} \exp\biggl(\sum_{p\leqslant x,\, p\nmid Q} \frac{|g(p)|}{p}\biggr) \exp\biggl(-\sum_{p\leqslant x',\, p\nmid Q} \frac{|g(p)|}{p}\biggr) &\leqslant \exp\biggl(\sum_{x^{1/(8H)} < p \leqslant x} \frac{H}{p}\biggr) \\ &\leqslant \exp\bigl(H(\log\log x + \log(8H) - \log\log x + o(1))\bigr) \\ &\leqslant (8H)^{H(1+o(1))} \ll_H 1. \end{aligned}$$

Thus, by Lemma 4.8 it follows that all characters $\overline{\chi}$ with $\chi \in \mathscr{E}(Q)$ to which the first alternative from the statement applies satisfy

$$|S_{f_x\overline{\chi}}(y)| \leqslant \frac{\psi''_C(x)}{\log x} \exp\biggl(\sum_{p\leqslant x,\, p\nmid Q} \frac{|f(p)|}{p}\biggr), \quad (x^{1/2} \leqslant y \leqslant x), \tag{4-13}$$

for a suitable function $\psi''_C = o(1)$.

For all remaining $\chi \in \mathscr{E}(Q)$, Lemma 4.6 or 4.9 provides the Lipschitz estimate

$$S_{f_x\overline{\chi}}(y) = S_{f_x\overline{\chi}}(x) + O\biggl(\frac{\psi'''(x)}{\log x} \exp\biggl(\sum_{p\leqslant x,\, p\nmid Q} \frac{|f(p)|}{p}\biggr)\biggr),$$

for $y \in [x\exp(-(\log\log x)^4/2), x]$, with $\psi'''(x) = (\log x)^{-C_0}$. Thus, the result follows from (4-12). □

4C. *Proofs of Lemmas 4.6–4.9.* We prove Lemmas 4.6, 4.9, 4.7 and 4.8, in this order.

Proof of Lemma 4.6. The proof of this lemma is almost identical to the proofs of the original results of Granville and Soundararajan [2003, Theorem 3 and 4], except for one ingredient: their Lemma 2.3 needs to be replaced by Lemma 4.12 below. The estimate (4-8) follows immediately from [loc. cit., §5] and Lemma 4.12. Concerning the Lipschitz estimate (4-7), we replace the application of [loc. cit., Theorem 3] at the beginning of [loc. cit., §6] by the estimate (4-8). The bound in [loc. cit., equation (6.2)] continues to apply. The first term in this bound is acceptable since in our case $w \leqslant \exp((\log\log x)^4)$, and since $C_0 < \alpha_{f_0}$. To bound the integrand in the second term, we use the bound [loc. cit., equation (6.5)] if α is large, which in our situation means that $\alpha > \exp\bigl(\sum_{p\leqslant x} |f(p)|/p\bigr)^{-1} (\log x)^{C_0}$ with $C_0 = C_0(\alpha_{f_0}, 1)$ as in the lemma below. If $\alpha \leqslant \exp\bigl(\sum_{p\leqslant x} |f(p)|/p\bigr)^{-1} (\log x)^{C_0}$, we proceed as in the small-α-case from the original proof but, again, apply our Lemma 4.12 instead of [loc. cit., Lemma 2.3]. □

Lemma 4.12 ("new Lemma 2.3"). *Let $x \geqslant 3$, $f_0 \in \mathscr{M}_H$ and let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $|f(p^k)| \leqslant |f_0(p^k)|$ at all prime powers p^k, and such that $|f(p^k)| = |f_0(p^k)|$ whenever $p \geqslant \exp((\log\log x)^2)$ and $k \in \mathbb{N}$. Let*

$$F(s) = \prod_{p\leqslant x} \Bigl(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\Bigr).$$

Then there exists a positive constant $C_0 = C_0(\alpha_{f_0}, H) \in (0, \alpha_f/2)$ such that for all real numbers y and $1/\log x \leqslant |\beta| \leqslant \log x$, we have

$$|F(1+iy)F(1+i(y+\beta))| \ll \exp\biggl(2\sum_{p\leqslant x} \frac{|f(p)|}{p}\biggr)(\log x)^{-2C_0}.$$

Remark. Observe that we actually only use this lemma in the case where $H = 1$.

Proof. Suppose $|f(p)| = g(p) + h(p)$ for two nonnegative functions g and h. Then

$$
\begin{aligned}
|F(1+iy)F(1+i(y+\beta))| &\ll \exp\biggl(\Re \sum_{p \leqslant x} \frac{f(p)p^{-iy} + f(p)p^{-i(y+\beta)}}{p}\biggr) \\
&\ll \exp\biggl(\sum_{p \leqslant x} \frac{(g(p)+h(p))|1+p^{-i\beta}|}{p}\biggr) \\
&\ll \exp\biggl(2\sum_{p \leqslant x} \frac{g(p)|\cos\bigl(\frac{1}{2}|\beta|\log p\bigr)|}{p}\biggr) \exp\biggl(2\sum_{p \leqslant x} \frac{h(p)}{p}\biggr).
\end{aligned} \tag{4-14}
$$

The aim is to exploit the fact that $|\cos|$ is not the constant function 1 in order to bound this expression. We begin by decomposing the set of primes less than x into subsets on which $\bigl|\cos\bigl(\frac{1}{2}|\beta|\log p\bigr)\bigr|$ is almost constant. For this purpose, let $\delta = 1/(\log x)^3$ and consider the decomposition of $[0, 2\pi)$ into intervals of the form $\bigl(\frac{1}{2}(n-1)\log(1+\delta)|\beta|, \frac{1}{2}n\log(1+\delta)|\beta|\bigr]$. Thus, in order to cover the interval $[0, 2\pi)$, the parameter n runs over the range $1 \leqslant n \leqslant N$, for some $N \asymp (\delta|\beta|)^{-1}$, and, in particular, we have $(\log x)^2 \ll N \ll (\log x)^4$. By changing δ slightly, we can insure that $\frac{1}{2}N\log(1+\delta)|\beta| = 2\pi$ so that the decomposition of $[0, 2\pi)$ has exactly N full intervals and no smaller or larger ones. Next, we set $Y = \exp((\log\log x)^2)$ and decompose the set of primes in the interval $[Y, x]$ into N sets of the form

$$
P_n(x) = \bigcup_{m \equiv n \pmod N} \{p \in (Y(1+\delta)^{m-1}, Y(1+\delta)^m] \cap (Y, x]\}, \quad (1 \leqslant n \leqslant N).
$$

If $M = \log(x/Y)/\log(1+\delta)$, the Brun–Titchmarsh inequality implies that for each $n \leqslant N$:

$$
\begin{aligned}
\sum_{p \in P_n(x)} \frac{1}{p} &\leqslant \sum_{\substack{0 \leqslant m \leqslant M \\ m \equiv n \pmod N}} \frac{\pi(Y(1+\delta)^m) - \pi(Y(1+\delta)^{m-1})}{Y(1+\delta)^{m-1}} \\
&\ll \sum_{m \equiv n \pmod N} \frac{\delta}{\log(Y\delta(1+\delta)^{m-1})} \\
&\ll \delta \sum_{0 \leqslant k \leqslant M/N} \frac{1}{\log(x(1+\delta)^{-kN})} \\
&\ll \delta \sum_{0 \leqslant k \leqslant M/N} \frac{1}{\log x - kN\log(1+\delta)} \\
&\ll \frac{\delta}{N\log(1+\delta)} \log\biggl(\frac{\log x}{N\log(1+\delta)}\biggr) \ll \frac{1}{N}\log\log x.
\end{aligned} \tag{4-15}
$$

Now suppose that g satisfies

$$
\sum_{p \leqslant x} \frac{g(p)}{p} \sim \alpha \log\log x \tag{4-16}
$$

for some $\alpha > 0$ and let $S \subset \{1, \dots, N\}$ denote the set of all indices n such that

$$\sum_{p \in P_n(x)} \frac{g(p)}{p} \geqslant \frac{\alpha}{2} \sum_{p \in P_n(x)} \frac{1}{p}. \tag{4-17}$$

Then, taking our choice of Y into account and since $g(p) \leqslant |f(p)| \leqslant H$, we have

$$\sum_{n \in S} \sum_{p \in P_n(x)} \frac{1}{p} \geqslant \frac{1}{H} \sum_{n \in S} \sum_{p \in P_n(x)} \frac{g(p)}{p} \geqslant \frac{1}{H} \biggl(\sum_{Y \leqslant p \leqslant x} \frac{g(p)}{p} - \frac{\alpha}{2} \sum_{p \leqslant x} \frac{1}{p} \biggr) \sim \frac{\alpha}{2H} \log \log x. \tag{4-18}$$

Comparing this bound with (4-15) shows that S contains a positive proportion of the integers up to N.

Our next aim is to find a subset $T \subset S$ that satisfies

$$\sum_{n \in T} \sum_{p \in P_n(x)} \frac{1}{p} \geqslant \frac{1}{2} \sum_{n \in S} \sum_{p \in P_n(x)} \frac{1}{p}, \tag{4-19}$$

and for which $\bigl|\cos\bigl(\frac{1}{2}|\beta| \log p\bigr)\bigr|$ is bounded away from 1 as p ranges over $\bigcup_{n \in T} P_n$.

By (4-15) and (4-18), we can choose a positive proportion of all $n \leqslant N$ not to belong to T. In particular, we can exclude all n from T for which

$$\bigl(\tfrac{1}{2}(n-1)\log(1+\delta)|\beta|, \tfrac{1}{2}n \log(1+\delta)|\beta|\bigr]$$

intersects $[0, c) \cup (\pi - c, \pi + c) \cup (2\pi - c, 2\pi)$ for some small constant $c > 0$ that only depends on α, H and on the implied constant in (4-15). By doing so, we ensure that

$$\bigl|\cos\bigl(\tfrac{1}{2}|\beta| \log p\bigr)\bigr| < \cos c < 1$$

for all $p \in P_n(x)$ with $n \in T$. Writing $c' := \cos c$ and considering the cosine sum in the final expression of (4-14), the above yields

$$\sum_{p \in P_n(x)} \frac{g(p)\bigl|\cos\bigl(\frac{1}{2}|\beta|\log p\bigr)\bigr|}{p} \leqslant c' \sum_{p \in P_n(x)} \frac{g(p)}{p} \leqslant \sum_{p \in P_n(x)} \frac{g(p)}{p} - (1-c') \sum_{p \in P_n(x)} \frac{g(p)}{p}$$

for $n \in T$. If $n \notin T$, we have the trivial bound

$$\sum_{p \in P_n(x)} \frac{g(p)\bigl|\cos\bigl(\frac{1}{2}|\beta| \log p\bigr)\bigr|}{p} \leqslant \sum_{p \in P_n(x)} \frac{g(p)}{p}.$$

By combining these two bounds with (4-17), (4-18) and (4-19), it follows that

$$\begin{aligned}
\sum_{p \leqslant x} \frac{g(p)\bigl|\cos\bigl(\frac{1}{2}|\beta| \log p\bigr)\bigr|}{p} &\leqslant \sum_{p \leqslant x} \frac{g(p)}{p} - (1-c') \sum_{n \in T} \sum_{p \in P_n(x)} \frac{g(p)}{p} \\
&\leqslant \sum_{p \leqslant x} \frac{g(p)}{p} - \frac{\alpha(1-c')}{2} \sum_{n \in T} \sum_{p \in P_n(x)} \frac{1}{p} \\
&\leqslant \sum_{p \leqslant x} \frac{g(p)}{p} - (C_0 + o(1)) \log \log x
\end{aligned}$$

for some constant $C_0 > 0$ that only depends on α and H. By (4-14), we thus deduce that

$$|F(1+iy)F(1+i(y+\beta))| \ll \exp\biggl(2\sum_{p\leqslant x}\frac{|f(p)|}{p}\biggr)(\log x)^{-2C_0}.$$

It remains to show that there exists a decomposition of $|f(p)|$ into nonnegative functions g and h such that (4-16) holds. This will follow from [Elliott 2017, Lemma 5]. To apply this result, we observe that the two conditions from Definition 1.3 and partial summation show that every $f_0 \in \mathscr{M}_H$ has the property that

$$\liminf_{x\to\infty}\frac{1}{\varepsilon\log x}\sum_{x^{1-\varepsilon}<p\leqslant x}\frac{|f_0(p)|\log p}{p} \geqslant \alpha_{f_0}. \tag{4-20}$$

for every $\varepsilon \in (0, 1)$. Thus, the assumptions of [Elliott 2017, Lemma 5] are met and the lemma implies that there exists a nonnegative completely multiplicative function $g_0 \leqslant |f_0|$, which satisfies

$$\lim_{x\to\infty}(\log x)^{-1}\sum_{p\leqslant x}\frac{g_0(p)\log p}{p} = \frac{\alpha_{f_0}}{2}.$$

The function g_0 arises from $|f_0|$ as the result of a simple greedy-type argument that decides one by one for each prime p if $g_0(p) = 0$ or $g_0(p) = |f_0(p)|$. Partial summation yields

$$\sum_{p\leqslant z}\frac{g_0(p)}{p} \sim \frac{\alpha_{f_0}}{2}\log\log z.$$

If we let $g(p) = g_0(p)$ for all $p \geqslant Y$ and $g(p) = 0$ otherwise, then

$$\sum_{p\leqslant x}\frac{g(p)}{p} = \sum_{p\leqslant x}\frac{g_0(p)}{p} + O(H\log\log Y) \sim \frac{\alpha_{f_0}}{2}\log\log x + O(H\log\log\log x),$$

as required. Thus, we may set $\alpha = \frac{1}{2}\alpha_{f_0}$ in the first part of the proof and, hence, C_0 only depends on α_{f_0} and H. □

Proof of Lemma 4.9. Let f^* denote the multiplicative function that satisfies $f^*(p^k) = 0$ whenever $k \geqslant 1$ and $p \leqslant \exp((\log\log x)^2)$, and $f^*(p^k) = f_0(p^k)$ whenever $k \geqslant 1$ and $p > \exp((\log\log x)^2)$. Then, by applying Lemma 4.6 twice, we have

$$\biggl|\frac{1}{y}\sum_{n\leqslant y}f^*(n)n^{-it} - \frac{1}{y'}\sum_{n\leqslant y'}f^*(n)n^{-it}\biggr| \ll \frac{1}{(\log x)^{1+C_0}}\exp\biggl(\sum_{p\leqslant x}\frac{|f(p)|}{p}\biggr),$$

for all $y, y' \in [x\exp(-(\log\log x)^{-4}), x]$ and some $t = t_{x,f^*}$ with $|t| \leqslant 2\log x$.

Observe that any f may be decomposed as $f = f' * f^*$, where $f'(p^k) = 0$ for all $p > \exp((\log\log x)^2)$. Thus, if $w := \exp\bigl(\frac{1}{2}(\log\log x)^{-4}\bigr)$ and $x' \in [x/w, x]$, then

$$\begin{aligned}\biggl|\frac{1}{x}\sum_{n\leqslant x} f(n)n^{-it} - \frac{1}{x'}\sum_{n\leqslant x'} f(n)n^{-it}\biggr| &\ll \sum_{d\leqslant w}\frac{|f'(d)|}{d}\biggl|\frac{d}{x}\sum_{n\leqslant x/d} f^*(n)n^{-it} - \frac{d}{x'}\sum_{n\leqslant x'/d} f^*(n)n^{-it}\biggr| \\ &\qquad + \frac{1}{x}\sum_{d>w}\sum_{m<x/d}|f'(d)f^*(m)| + \frac{1}{x'}\sum_{d>w}\sum_{m<x'/d}|f'(d)f^*(m)| \\ &\ll \sum_{d\leqslant w}\frac{|f'(d)|}{d}\frac{1}{(\log x)^{1+C_0}}\exp\biggl(\sum_{p\leqslant x}\frac{|f^*(p)|}{p}\biggr) \\ &\qquad + \sum_{m\leqslant x}\frac{1}{m}\frac{m}{x}\sum_{w<d\leqslant x/m}|f'(d)| + \sum_{m\leqslant x'}\frac{1}{m}\frac{m}{x'}\sum_{m\leqslant x'}\sum_{w<d\leqslant x'/m}|f'(d)|.\end{aligned}$$

To bound the last two terms, recall that $|f'(n)| \leqslant 1$ and that, see, e.g., [Tenenbaum 1995, Theorem III.5.1],

$$\Psi(z, y) := \#\{n \leqslant z : p \mid n \Rightarrow p \leqslant y\} \ll ze^{-u/2} \quad (z \geqslant y \geqslant 2), \tag{4-21}$$

where $u = \log z/\log y$. In particular

$$\Psi\bigl(z, \exp((\log\log x)^2)\bigr) \ll ze^{-(\log\log x)^2/4} = z(\log x)^{-(\log\log x)/4}$$

whenever $z \geqslant \exp\bigl(\frac{1}{2}(\log\log x)^4\bigr)$. Thus, the above is bounded by

$$\begin{aligned}&\ll \frac{1}{(\log x)^{1+C_0}}\exp\biggl(\sum_{p\leqslant x}\frac{|f(p)|}{p} + \sum_{p\leqslant d}\frac{1}{p^2(1-p^{-1})}\biggr) + \sum_{m\leqslant x}\frac{1}{m}(\log x)^{-(\log\log x)/2} \\ &\ll \frac{1}{(\log x)^{1+C_0}}\exp\biggl(\sum_{p\leqslant x}\frac{|f(p)|}{p}\biggr),\end{aligned}$$

which completes the proof. □

Proof of Lemma 4.8. By Lemma 4.3 and (4-4) it follows that

$$\begin{aligned}|S_{f\chi}(y)| \leqslant x^{-\delta/8}(\log x)^{O(H)} + \sum_{d_0\leqslant y^\delta}\ \sum_{D\leqslant\left(\frac{y}{d_0}\right)^{1-\frac{1}{H}}}\ \sum_{d_1\cdots d_{H-1}=D}\frac{|h'(d_0)h(d_1)\cdots h(d_{H-1})||\chi(d_0D)|}{d_0D} \\ \times\sum_{i=1}^{H}\frac{Dd_0}{y}\biggl(\biggl|\sum_{n\leqslant\frac{y}{(Dd_0)}} h(n)\chi(n)\biggr| + \biggl|\sum_{n<\left(\frac{y}{d_0}\right)^{1-\frac{1}{H}}\frac{\max(d_1,\ldots,d_{i-1})}{D}} h(n)\chi(n)\biggr|\biggr).\end{aligned}$$

As in the proof of Corollary 4.2, we set $\delta = 1/(4H)$. Then the inner sums may be bounded using either the assumption or, if $(y/d_0)^{1-1/H}\max(d_1, \ldots, d_{i-1})/D < x^{1/(4H)}$, by the trivial estimate

$$\frac{Dd_0}{y}x^{1/(4H)} \leqslant y^{1-1/H+1/(4H)-1}x^{1/(4H)} = y^{-3/(4H)}x^{1/(4H)} \leqslant x^{-1/(8H)}.$$

The lemma follows by bounding the sums over D and d_0 as in the proof of Corollary 4.2. □

Proof of Lemma 4.7. We first use Lemma 4.3 to remove the contribution of the function h' defined in Lemma 1.8. Given $\delta \in (0, 1)$, let δ' be such that $x^{\delta} = {x'}^{\delta'}$. Then

$$|S_{f\chi}(x) - S_{f\chi}(x')| \\ \leqslant x^{-\delta/4}(\log x)^{O(H)} + \sum_{d_0 \leqslant x^{\delta}} \frac{|h'(d_0)\chi(d_0)|}{d_0} \left| \frac{d_0}{x} \sum_{n \leqslant x/d_0} h^{*H}(n)\chi(n) - \frac{d_0}{x'} \sum_{n \leqslant x'/d_0} h^{*H}(n)\chi(n) \right|. \quad (4\text{-}22)$$

To analyze the difference above, we seek to decompose h^{*H} using $H-1$ applications of the hyperbola trick[3],

$$\sum_{nm \leqslant Y} = \sum_{n \leqslant X} \sum_{m \leqslant Y/n} + \sum_{m \leqslant Y/X} \sum_{X \leqslant n \leqslant Y/m}.$$

Fix d_0 and let $X = (x'/d_0)^{1/H}$. If $y \in \{x'/d_0, x/d_0\}$, then applying the hyperbola trick with the chosen cutoff X and with $Y = y$, $n = d_1$ and $m = d_2 \cdots d_H$, we obtain

$$\sum_{d_1 \cdots d_H \leqslant y} = \sum_{d_1 \leqslant X} \sum_{d_2 \cdots d_H \leqslant y/d_1} + \sum_{d_2 \cdots d_H \leqslant y/X} \sum_{X \leqslant d_1 \leqslant y/(d_2 \cdots d_H)}.$$

We keep the second term and decompose the first term again, using the same cutoff X, and $Y = y/d_1$, $n = d_2$ and $m = d_3 \cdots d_H$. This leads to

$$\sum_{d_1 \cdots d_H \leqslant y} = \sum_{d_1 \leqslant X} \sum_{d_2 \leqslant X} \sum_{d_3 \cdots d_H \leqslant y/(d_1 d_2)} + \sum_{d_1 \leqslant X} \sum_{d_3 \cdots d_H \leqslant y/(d_1 X)} \sum_{X \leqslant d_2 \leqslant y/(d_1 d_3 \cdots d_H)} + \sum_{d_2 \cdots d_H \leqslant y/X} \sum_{X \leqslant d_1 \leqslant y/(d_2 \cdots d_H)}.$$

Continuing in this manner, i.e., keeping every time the second new term and further decomposing the first, we arrive at

$$\sum_{d_1 \cdots d_H \leqslant y} = \sum_{d_1, d_2, \ldots, d_{H-1} \leqslant X} \sum_{d_H \leqslant y/(d_1 d_2 \cdots d_{H-1})} + \sum_{i=1}^{H-1} \sum_{d_1, \ldots, d_{i-1} \leqslant X} \sum_{\substack{d_{i+1}, \ldots, d_H: \\ d_1 \cdots \hat{d}_i \cdots d_{i-1} \leqslant y/X}} \sum_{X \leqslant d_i \leqslant y/(d_1 \cdots \hat{d}_i \cdots d_H)}. \quad (4\text{-}23)$$

In order to apply this decomposition to (4-22), let us consider the difference of the normalized sums (4-23) for $y = x/d_0$ and $y = x'/d_0$. Recall that $x \geqslant x'$. By splitting the third sum of the second term of the decomposition into two sums when $y = x/d_0$, we obtain the following:

[3]This proof requires a different decomposition from the one used in (4-4) and Section 9D in order to be able to collect together terms in (4-24) below.

$$
\begin{aligned}
&\frac{d_0}{x}\sum_{d_1\cdots d_H\leqslant x/d_0} - \frac{d_0}{x'}\sum_{d_1\cdots d_H\leqslant x'/d_0}\\
&\quad= \sum_{d_1,d_2,\dots,d_{H-1}\leqslant X}\Biggl(\frac{d_0}{x}\sum_{d_H\leqslant x/(d_0d_1\cdots d_{H-1})} - \frac{d_0}{x'}\sum_{d_H\leqslant x'/(d_0d_1\cdots d_H-1)}\Biggr)\\
&\qquad+\sum_{i=1}^{H-1}\sum_{d_1,\dots,d_{i-1}\leqslant X}\sum_{\substack{d_{i+1},\dots,d_H:\\ d_0\cdots\hat{d}_i\cdots d_{i-1}\leqslant x'/X}}\Biggl(\frac{d_0}{x}\sum_{d_i\leqslant x/(d_1\cdots\hat{d}_i\cdots d_H)} - \frac{d_0}{x'}\sum_{d_i\leqslant x'/(d_1\cdots\hat{d}_i\cdots d_H)} + \frac{d_0}{x'}\sum_{d_i\leqslant X} - \frac{d_0}{x}\sum_{d_i\leqslant X}\Biggr)\\
&\qquad+\sum_{i=1}^{H-1}\sum_{d_1,\dots,d_{i-1}\leqslant X}\sum_{\substack{d_{i+1},\dots,d_H:\\ x'/X\leqslant d_0\cdots\hat{d}_i\cdots d_H\leqslant x/X}}\frac{d_0}{x}\sum_{X\leqslant d_i\leqslant x/(d_0\cdots\hat{d}_i\cdots d_H)}.
\end{aligned}
\tag{4-24}
$$

When we apply this decomposition with the summation argument $g(d_1)\cdots g(d_H)$, where $g(n)=h(n)\chi(n)$, then the first and the second term above contain expressions of the form $S_g(z)-S_g(z')$ for suitable z and z'. These will be estimated using the assumptions of the lemma. Before turning towards these, let us consider the remaining terms.

The second term contains two short sums up to X that will be estimated using Shiu's bound (3-1) in the following form. For every fixed $j\in\mathbb{N}$, every $q\in\mathbb{N}$ for which the interval $((W(x)q)^2,x]$ is nonempty and every $y\in((W(x)q)^2,x]$, we have

$$
\sum_{n\leqslant y}|g^{*j}(n)| = \sum_{A\in(\mathbb{Z}/qW(x)\mathbb{Z})^*}\sum_{\substack{n\leqslant y\\ n\equiv A \pmod{qW(x)}}}|h^{*j}(n)| \ll \frac{x}{\log x}\exp\Biggl(j\sum_{\substack{p\leqslant y\\ p\nmid qW(x)}}\frac{|h(p)|}{p}\Biggr),
\tag{4-25}
$$

since $W(x)q\leqslant y^{1/2}$, and thus $W(x)q=W(y)q'$ for some $q'\leqslant y^{1/2}$.

By applying this bound twice with $W(x)q=Q$, we obtain

$$
\begin{aligned}
&\Biggl|\frac{d_0}{x'}\sum_{d_1,\dots,d_{i-1}\leqslant X} g(d_1)\cdots g(d_{i-1})\sum_{\substack{d_{i+1}\cdots d_H\leqslant\\ x'/(d_0\cdots d_{i-1}X)}} g(d_{i+1})\cdots g(d_H)\sum_{d_i\leqslant X} g(d_i)\Biggr|\\
&\qquad\leqslant \frac{d_0X}{x'}\frac{1}{\log X}\exp\Biggl(\sum_{\substack{p\leqslant X\\ p\nmid Q}}\frac{|h(p)|}{p}\Biggr)\sum_{d_1,\dots,d_{i-1}\leqslant X}|g(d_1)\cdots g(d_{i-1})|\sum_{d\leqslant x'/(d_0\cdots d_{i-1}X)}|g^{*(H-i)}(d)|\\
&\qquad\leqslant \frac{1}{(\log X)^2}\exp\Biggl((H-i+1)\sum_{\substack{p\leqslant x'\\ p\nmid Q}}\frac{|h(p)|}{p}\Biggr)\sum_{d_1,\dots,d_{i-1}\leqslant X}\frac{|g(d_1)\cdots g(d_{i-1})|}{d_1\cdots d_{i-1}}\\
&\qquad\ll_{H,C}\frac{1}{(\log x)^2}\exp\Biggl(\sum_{\substack{p\leqslant x'\\ p\nmid Q}}\frac{|f(p)|}{p}\Biggr),
\end{aligned}
$$

which saves $(\log x)^{-1}$.

In the final term of (4-24), we will take advantage of the fact that the third sum is short. Starting off with another application of (4-25), we get

$$\sum_{d_1,\dots,d_{i-1}\leqslant X} g(d_1)\cdots g(d_{i-1}) \sum_{\substack{d_{i+1},\dots,d_H:\\ x'/X\leqslant d_0\cdots\hat{d}_i\cdots d_H\leqslant x/X}} g(d_{i+1})\cdots g(d_H)\frac{d_0}{x} \sum_{X\leqslant d_i\leqslant x/(d_0\cdots\hat{d}_i\cdots d_H)} g(d_i)$$

$$\leqslant \frac{1}{\log x}\exp\biggl(\sum_{\substack{p\leqslant x'\\ p\nmid Q}}\frac{|h(p)|}{p}\biggr)\sum_{d_1,\dots,d_{i-1}\leqslant X}\frac{|g(d_1)\cdots g(d_{i-1})|}{d_1\cdots d_{i-1}}\sum_{\substack{d_{i+1},\dots,d_H:\\ x'/X\leqslant d_0\cdots\hat{d}_i\cdots d_H\leqslant x/X}}\frac{|g(d_{i+1})\cdots g(d_H)|}{d_{i+1}\cdots d_H}$$

$$\leqslant \frac{1}{\log x}\exp\biggl(\sum_{\substack{p\leqslant x'\\ p\nmid Q}}\frac{|h(p)|}{p}\biggr)\sum_{\substack{d_1,\dots,d_{i-1}\leqslant X\\ d_{i+1},\dots d_{H-1}\leqslant x}}\frac{|g(d_1)\cdots\widehat{g(d_i)}\cdots g(d_{H-1})|}{d_1\cdots\hat{d}_i\cdots d_{H-1}}\sum_{\substack{d_H:\\ x'/X\leqslant d_0\cdots\hat{d}_i\cdots d_H\leqslant x/X}}\frac{1}{d_H}$$

$$\leqslant \frac{1}{\log x}\exp\biggl(\sum_{\substack{p\leqslant x'\\ p\nmid Q}}\frac{|h(p)|}{p}\biggr)\sum_{\substack{d_1,\dots,d_{i-1}\leqslant X\\ d_{i+1},\dots,d_{H-1}\leqslant x}}\frac{|g(d_1)\cdots\widehat{g(d_i)}\cdots g(d_{H-1})|}{d_1\cdots\hat{d}_i\cdots d_{H-1}}\Bigl(\log\frac{x}{x'}+O(1)\Bigr)$$

$$\leqslant \frac{\log\log X+\log C+O(1)}{\log x}\exp\biggl((H-1)\sum_{\substack{p\leqslant x'\\ p\nmid Q}}\frac{|h(p)|}{p}\biggr),$$

which saves a factor $(\log x)^{-\alpha_f/H+\varepsilon}$.

To summarize our progress so far, note that the decomposition (4-24) and the previous two bounds yield

$$\biggl|\frac{d_0}{x}\sum_{n\leqslant x/d_0}h^{*H}(n)\chi(n)-\frac{d_0}{x'}\sum_{n\leqslant x'/d_0}h^{*H}(n)\chi(n)\biggr|$$

$$=\sum_{\substack{d_1,\dots,d_{H-1}\leqslant\\ (x'/d_0)^{1/H}}}\frac{|g(d_1)\cdots g(d_{H-1})|}{d_1\cdots d_{H-1}}\biggl|S_g\Bigl(\frac{x}{d_0\cdots d_{H-1}}\Bigr)-S_g\Bigl(\frac{x'}{d_0\cdots d_{H-1}}\Bigr)\biggr|$$

$$+\sum_{i=1}^{H-1}\sum_{\substack{d_1,\dots,d_{i-1}\leqslant\\ (x'/d_0)^{1/H}}}\frac{|g(d_1)\cdots g(d_{i-1})|}{d_1\cdots d_{i-1}}\sum_{\substack{d_{i+1},\dots,d_H:\\ d_1\cdots\hat{d}_i\cdots d_H\leqslant\\ (x'/d_0)^{1-1/H}}}\frac{|g(d_{i+1})\cdots g(d_H)|}{d_{i+1}\cdots d_H}$$

$$\times\biggl|S_g\Bigl(\frac{x}{d_0\cdots\hat{d}_i\cdots d_H}\Bigr)-S_g\Bigl(\frac{x'}{d_0\cdots\hat{d}_i\cdots d_H}\Bigr)\biggr|$$

$$+O_{H,C}\biggl((\log x)^{-\min(1,\alpha_f/(2H))}\frac{1}{\log x}\exp\biggl(\sum_{p\leqslant x',\,p\nmid Q}\frac{|f(p)|}{p}\biggr)\biggr).$$

Choosing $\delta=\frac{1}{2}$ to ensure that $d_0\leqslant x^{1/2}$, it follows that the terms $x/(d_0\cdots d_{H-1})$ and $x/(d_0\cdots\hat{d}_i\cdots d_H)$ in the above expression are at least as large as $x^{1/(2H)}$. Since $g=h\chi$, we may thus apply the assumptions of the lemma to deduce that the above is bounded by:

$$\ll \frac{\psi(x)}{\log((x'/d_0)^{1/H})} \exp\biggl(\sum_{p\leqslant x',\, p\nmid Q} \frac{|h(p)|}{p}\biggr)\biggl(\prod_{d\leqslant x} \frac{g(d)}{d}\biggr)^{H-1} \\ + (\log x)^{-\min(1,\alpha_f/(2H))} \frac{1}{\log x} \exp\biggl(\sum_{p\leqslant x',\, p\nmid Q} \frac{|f(p)|}{p}\biggr)$$

$$\ll (\psi(x) + (\log x)^{-\min(1,\alpha_f/(2H))}) \frac{1}{\log x} \exp\biggl(\sum_{p\leqslant x',\, p\nmid Q} \frac{|f(p)|}{p}\biggr)$$

The lemma then follows from (4-22) since by (4-1), applied with $y = 1$ and $w = w(x)$, the completed outer sum over d_0 converges, i.e., $\sum_{d_0=1}^{\infty} |h'(d_0)\chi(d_0)|/d_0 < \infty$, provided x is sufficiently large for $w(x) \geqslant (2H)^{16}$ to hold. □

4D. ***Applications to functions bounded away from zero at primes.*** In this subsection, we will discuss a concrete example of an element of $\mathscr{F}_H$ and prove a criterion for real-valued f to belong to $\mathscr{F}_H$ that is just based on the values of f at primes. Let us begin by stating a special case of Proposition 4.10 for nonnegative $f \in \mathscr{M}_H$.

Lemma 4.13 (sufficient condition for nonnegative functions). *Let $f \in \mathscr{M}_H$ be a nonnegative function. Then there exists a constant $c > 0$, only depending on f, such that the following holds: If $x > 3$, if $1 < Q \leqslant \exp((\log\log x)^2)$ is a multiple of $W(x)$, and if $\chi_0 \pmod{Q}$ denotes the trivial character, then*

$$S_{f\chi_0}(x) = S_{f\chi_0}(x') + O\biggl((\log x)^{-c} \frac{1}{\log x} \prod_{p\leqslant x,\, p\nmid Q} \biggl(1 + \frac{|f(p)|}{p}\biggr)\biggr)$$

uniformly for all $x > 3$, $x' \in [x\exp(-(\log\log x)^2, x]$ and all Q as above. If, furthermore, for either $g = h$ or $g = f$ and for any $C > 0$, we have a uniform bound of the form

$$S_{g\chi}(x) = O\biggl(\frac{\psi_C(x)}{\log x} \prod_{p\leqslant x,\, p\nmid Q} \biggl(1 + \frac{|g(p)|}{p}\biggr)\biggr), \tag{4-26}$$

valid for all $x > 3$, all nontrivial $\chi \pmod{Q}$ and all $1 \leqslant Q \leqslant (\log x)^C$ with $W(x) \mid Q$, and where $\psi_C = o(1)$ may depend on C but is otherwise independent of χ and Q, then $f \in \mathscr{F}_H$.

Remark 4.14. Note that in the context of this corollary, the main term in the character sum expansion of $S_f(x; Q, A)$ always comes from the trivial character.

Proof. The first part follows from Lemmas 4.6 and 4.7 provided we can show that for all sufficiently large x we have $t_{x,h\chi_0} = 0$ in the statement of Lemma 4.6 when applied with f replaced by $h\chi_0$. This, however, is immediate since h is nonnegative. The second part is a consequence of Proposition 4.10. □

The following three lemmas all arise as (nontrivial) applications of Lemma 4.13.

Lemma 4.15 (coefficients of cusp forms). *Let f be a primitive holomorphic cusp form*[4] *of weight $k \in 2\mathbb{N}$ and level $N \in \mathbb{N}$ and let*

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz),$$

be its Fourier expansion, where the $\lambda_f(n)$ are the normalized Fourier coefficients. Then the function $n \mapsto |\lambda_f(n)|$ belongs to $\mathscr{F}_2$.

Lemma 4.16 (nonnegative f). *For every $H \geqslant 1$ and $\alpha > 0$, there exists $c = c(H, \alpha) > 0$ such that the following holds. If $f \in \mathscr{M}_H$ is nonnegative with $\alpha_f \geqslant \alpha$, and if there exists $\delta > 0$ such that*

$$\#\{p \leqslant x : f(p) > \delta\} \geqslant \frac{(1-c)x}{\log x}$$

for all sufficiently large x, then $f \in \mathscr{F}_H$.

Remark. As a special case, Lemma 4.16 yields the following simple criterion, which also proves one part of Proposition 1.5:

A nonnegative function $f \in \mathscr{M}_H$ belongs to $\mathscr{F}_H$ if it is bounded away from zero on the primes, i.e., if there exists $\delta > 0$ such that $f(p) > \delta$ for all p. The same holds true if the latter condition is replaced by $\#\{p \leqslant x : f(p) > \delta\} \geqslant (1 + o(1))x/\log x$ as $x \to \infty$.

The following variant of Lemma 4.16 will follow with minor changes in the proof.

Lemma 4.17 (real-valued f). *For every $H \geqslant 1$ and $\alpha > 0$, there exists $c = c(H, \alpha) > 0$ such that the following holds. If $f \in \mathscr{M}_H$ is a real-valued function with $\alpha_f \geqslant \alpha$, and if there exists $\delta > 0$ and a sign $\epsilon \in \{+, -\}$ such that*

$$\#\{p \leqslant x : \epsilon f(p) > \delta\} \geqslant \frac{(1-c)x}{\log x}$$

for all sufficiently large x, then $f \in \mathscr{F}_{H,n^{it}}$. If, furthermore, for every $C > 0$ there exists a function $\psi_C = o(1)$ such that

$$S_{f\chi_0}(x) = O\biggl(\frac{\psi_C(x)}{\log x} \prod_{p \leqslant x,\, p \nmid Q} \biggl(1 + \frac{|f(p)|}{p}\biggr)\biggr),$$

whenever χ_0 is the trivial character modulo Q for any $Q \in (1, (\log x)^C)$ with $W(x) \mid Q$, then $f \in \mathscr{F}_H$.

Remark. As a particular consequence, we deduce that $f \in \mathscr{F}_{H,n^{it}}$ for any function $f \in \mathscr{M}_H$ for which there exists $\delta > 0$ such that $f(p) < -\delta < 0$ at all primes p.

Example 4.18. Examples of functions the above results apply to include:

(i) The Möbius function $f(n) = \mu(n)$. Here, the full statement of Lemma 4.17 applies. We may deduce this from the estimate $S_\mu(x) \ll_B (\log x)^{-B}$ for $B > 0$ and $x \geqslant 2$. In fact, writing $d \mid Q^\infty$ to indicate that

[4]See [Iwaniec and Kowalski 2004, §14.1 and §14.7] for definitions.

$p \mid d$ implies $p \mid Q$, it follows via repeated Möbius inversion that

$$\sum_{\substack{n \leqslant x \\ (n,Q)=1}} \mu(n) = \sum_{d \mid Q^\infty} \sum_{n \leqslant x/d} \mu(n)$$

Recalling (4-21), the above is seen to be bounded by

$$\begin{aligned}
&\ll \sum_{\substack{d \mid Q^\infty \\ d \leqslant x^{1/2}}} \sum_{n \leqslant x/d} \mu(n) + \sum_{x^{1/2} < 2^k \leqslant x} \Psi(2^k, (\log x)^C) x 2^{-k} \\
&\ll_B \sum_{\substack{d \mid Q^\infty \\ d \leqslant x^{1/2}}} \frac{x}{d} (\log x)^{-B} + \sum_{x^{1/2} < 2^k \leqslant x} x \exp\Bigl(-\frac{\log x}{4C \log\log x}\Bigr) \\
&\ll_B x (\log x)^{-B} \prod_{p \mid Q} (1 - p^{-1})^{-1} \ll_B x (\log x)^{-B+1},
\end{aligned}$$

which yields the required decay estimate.

(ii) The function $f(n) = \delta^{\omega(n)}$ for any nonzero real number δ and where $\omega(n)$ counts the number of distinct prime factors of n. If $\delta > 0$, then Lemma 4.16 applies. For $\delta < 0$ we will now show that the full statement of Lemma 4.17 applies. Since $W(x) \mid Q$, we may simplify our task by removing finitely many primes from consideration to start with: let $A \geqslant |\delta|$ be a constant to be chosen later, let $Q_0 = \prod_{p > A} p^{v_p(Q)}$ and let $h(n) = \delta^{\omega(n)} \mathbf{1}_{\gcd(n, \prod_{p \leqslant A} p) = 1}$ denote the restriciton of f to integers free from primes factors $p \leqslant A$. For this function, the Selberg–Delange method as stated in [Montgomery and Vaughan 2006, Theorem 7.18] implies $S_h(x) \ll (\log x)^{\delta - 1}$ and $S_{|h|}(x) \asymp (\log x)^{|\delta| - 1}$ for all $x \geqslant 2$, while Lemma 1.7 and Shiu's lemma in its original form [Shiu 1980, Theorem 1] yield $S_{|h|}(x) \asymp (1/\log x) \prod_{p \leqslant x} (1 + |\delta|/p)$. Proceeding in a similar way as in (i), repeated Möbius inversion shows that

$$\begin{aligned}
\sum_{\substack{n \leqslant x \\ (n,Q)=1}} \delta^{\omega(n)} &= \sum_{k \geqslant 1} \sum_{\substack{d_1 \mid Q_0^\infty \\ d_1 \geqslant 1}} \sum_{\substack{d_2 \mid d_1^\infty \\ d_2 > 1}} \cdots \sum_{\substack{d_k \mid d_{k-1}^\infty \\ d_k > 1}} |\delta|^{\omega(d_1) + \cdots + \omega(d_k)} \sum_{\substack{n \leqslant x/d_k \\ p \mid n \Rightarrow p > A}} \delta^{\omega(n)} \\
&\leqslant \sum_{d \mid Q_0^\infty} |\delta|^{\varpi(d)} \wp_m(d) \Bigl| \sum_{n \leqslant x/d} h(n) \Bigr|, \qquad (4\text{-}27)
\end{aligned}$$

where $\varpi(d) = \omega(d)$ if $|\delta| < 1$ and $\varpi(d) = \Omega(d)$ if $|\delta| \geqslant 1$, and where $\wp_m$ counts factorizations of the following form:

$$\wp_m(d) = \#\bigl\{d = d_1 \cdots d_k,\ k \geqslant 1 : d_j > 1 \text{ and } (p \mid d_j \Rightarrow p \mid d_i \text{ for all } i < j) \text{ for all } 1 \leqslant j \leqslant k\bigr\}.$$

If $\wp(n)$ denotes the partition number of n as defined in [Hardy and Ramanujan 1918], then

$$\wp_m(d) \leqslant \prod_{p \mid d} \wp(\nu_p(d)).$$

To bound (4-27), we will use the fact that there exists a constant $B \geqslant 1$ such that $\wp(n) \leqslant B^{\sqrt{n}}$, as proved in [loc. cit., §2]. Further, we require a bound corresponding to the one recalled in (4-21) but for sums over $|\delta|^{\varpi(n)}\wp_m(n)$ restricted to smooth numbers that are coprime to all $p \leqslant A$. For such sums we have

$$\Psi^*(x,y) := \sum_{\substack{n\leqslant x\\ p|n\Rightarrow p\in(A,y]}} |\delta|^{\varpi(n)}\wp_m(n) \leqslant C_\varepsilon x^{1/2+\varepsilon} + \sum_{\substack{n\leqslant x\\ p|n\Rightarrow p\in(A,y]}} (nx^{-1/2})^\alpha \prod_{p|n} |\delta|^{\varpi(p^{v_p(n)})} B^{\sqrt{v_p(n)}}$$

for any $\alpha > 0$. Let $\alpha = (\log y)^{-1}$ and suppose that $y \geqslant 2$. By [Tenenbaum 1995, Corollary III.3.5.1], the final sum in the expression above is bounded by

$$\begin{aligned}
&\ll x^{1-\alpha/2} \prod_{A<p\leqslant y} (1-p^{-1}) \sum_{k\geqslant 0} \frac{p^{\alpha k}|\delta|^{\varpi(p^k)}B^k}{p^k}\\
&\ll x^{1-\alpha/2} \prod_{A<p\leqslant y} (1-p^{-1})(1-e|\delta|Bp^{-1})^{-1} \ll x^{1-\alpha/2}(\log y)^{O(1)},
\end{aligned}$$

provided $A \geqslant eB\max(1,|\delta|)$. Thus, in total, we obtain

$$\Psi^*(x,y) \ll x^{1-\alpha/2}(\log y)^{O(1)} = x\exp\biggl(-\frac{\log x}{2\log y} + O(1)\log\log y\biggr) \ll x\exp\biggl(-\frac{\log x}{4\log y}\biggr)$$

for all $2 \leqslant y \leqslant x$, provided $A \geqslant eB\max(1,|\delta|)$.

Returning to (4-27), we choose $A = eB\max(1,|\delta|)$ in the definition of h and Q_0, and recall that $Q_0 \leqslant Q \leqslant (\log x)^C$. With the above bound on $\Psi^*(x,y)$ in place, the expression (4-27) can now be bounded by

$$\begin{aligned}
&\ll \sum_{\substack{d|Q_0^\infty\\ d\leqslant x^{1/2}}} |\delta|^{\varpi(d)}\wp_m(d) \sum_{n\leqslant x/d} \delta^{\omega(n)} + \sum_{x^{1/2}<2^k\leqslant x} \Psi^*(2^k,(\log x)^C)x2^{-k}(\log(x2^{-k}))^{|\delta|-1}\\
&\ll \sum_{\substack{d|Q_0^\infty\\ d\leqslant x^{1/2}}} x\frac{|\delta|^{\varpi(d)}\wp_m(d)}{d}(\log x)^{\delta-1} + \sum_{x^{1/2}<2^k\leqslant x} x\exp\biggl(-\frac{\log x}{4C\log\log x}\biggr)(\log x)^{|\delta|}\\
&\ll x(\log x)^{\delta-1}\prod_{p|Q_0}\sum_{k\geqslant 0}\frac{|\delta|^{\varpi(p^k)}B^{\sqrt{k}}}{p^k} + O_E(x(\log x)^{-E}),
\end{aligned}$$

which is further bounded by

$$\begin{aligned}
&\ll x(\log x)^{\delta-1} \prod_{\substack{p|Q\\ p>\max(|\delta|,B)}} \sum_{k\geqslant 0} \frac{|\delta|^{\varpi(p^k)}B^k}{p^k} + O_E(x(\log x)^{-E})\\
&\ll x(\log x)^{\delta-1}(\log Q)^{B|\delta|} \ll \frac{(C\log\log x)^{O(|\delta|)}}{(\log x)^{2|\delta|}}\frac{x}{\log x}\prod_{p\leqslant x,\ p\nmid Q}\biggl(1+\frac{|\delta|}{p}\biggr).
\end{aligned}$$

Thus, the required decay estimate holds.

(iii) The general divisor functions $d_k(n) = \mathbf{1}^{(*k)}(n)$ for $k \geqslant 2$, i.e., the k-fold convolution of $\mathbf{1}$ with itself. In this case Lemma 4.16 applies.

The remainder of this subsection contains the proofs of Lemmas 4.15–4.17. We begin with the proof of Lemma 4.15, which is the least technical case. Lemmas 4.16 and 4.17 will follow with small modifications from the same proof.

Proof of Lemma 4.15. The function λ_f that describes the normalized Fourier coefficients of f is a multiplicative function and satisfies Deligne's bound

$$|\lambda_f(n)| \leqslant d(n),$$

where d is the divisor function. This shows that part (1) of Definition 1.3 holds with $H = 2$. Condition (2) of the definition follows from [Rankin 1973, Theorem 2], since

$$\sum_{p \leqslant x} |\lambda_f(p)| \log p \geqslant \frac{1}{2} \sum_{p \leqslant x} \lambda_f(p)^2 \log p \sim \frac{x}{2},$$

which allows us to take $\alpha_{\lambda_f} = \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$. Hence, $g = |\lambda_f|$ belongs to $\mathscr{M}_2$.

To show that $g \in \mathscr{F}_2$, let h be the bounded multiplicative functions defined, as in Lemma 1.8, by

$$h(p^k) = \begin{cases} \frac{1}{2}|\lambda_f(p)| & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

and note that, by Lemma 4.13, it suffices to show that

$$|S_{h\chi}(x)| = o\biggl(\frac{1}{\log x} \prod_{\substack{p \leqslant x \\ p \nmid q W(x)}} \biggl(1 + \frac{|h(p)|}{p}\biggr)\biggr) \tag{4-28}$$

for all nontrivial $\chi \pmod Q$ with $Q \leqslant (\log x)^C$ and $W(x) \mid Q$. We begin this task by invoking Halász's theorem. Since g is bounded, the Halász–Granville–Soundararajan bound [Granville and Soundararajan 2003, Corollary 1] implies that

$$|S_{h\chi}(x)| = \frac{1}{x} \biggl| \sum_{n \leqslant x} \chi(n) h(n) \biggr| \ll (M+1) e^{-M} + \frac{1}{Y} + \frac{\log \log x}{\log x}, \tag{4-29}$$

where

$$M = M(x, Y) = \min_{|y| \leqslant 2Y} \sum_{p \leqslant x} \frac{1 - \Re(h(p)\chi(p)p^{iy})}{p}.$$

Note that

$$\begin{aligned} M(x, Y) &= \min_{|y| \leqslant 2Y} \sum_{p \leqslant x} \frac{1 - h(p) + h(p) - \Re(h(p)\chi(p)p^{iy})}{p} \\ &= \sum_{p \leqslant x} \frac{1 - h(p)}{p} + \min_{|y| \leqslant 2Y} \sum_{p \leqslant x} \frac{h(p)(1 - \Re(\chi(p)p^{iy}))}{p}; \end{aligned} \tag{4-30}$$

we abbreviate the second term in this expression as

$$M_{h\chi}(x, Y) := \min_{|y| \leqslant 2Y} \sum_{p \leqslant x} \frac{h(p)(1 - \Re(\chi(p)p^{iy}))}{p}.$$

Observe that the product in the bound (4-28) satisfies

$$\prod_{p\leqslant x p\nmid qW(x)}\left(1+\frac{|h(p)|}{p}\right)\gg \exp\left(\sum_{(\log x)^{C+2}<p\leqslant x}\frac{|h(p)|}{p}\right)\gg_\varepsilon (\log x)^{-\varepsilon}\exp\left(\sum_{p\leqslant x}\frac{|h(p)|}{p}\right)\gg_\varepsilon (\log x)^{\alpha_h-\varepsilon} \tag{4-31}$$

with $\alpha_h=\alpha_g/H=\alpha_g/2$. Thus, if we let $Y=(\log x)^{1-\alpha_h/2}$, then the last two terms in (4-29) are negligible compared with the bound (4-28). Combining (4-29), (4-30) and (4-31) it follows that

$$\begin{aligned}|S_{h\chi}(x)| &\ll (1+M)e^{-M_{h\chi}(x,Y)}\exp\left(\sum_{p\leqslant x}\frac{|h(p)|-1}{p}\right)+(\log x)^{-1+\alpha_h/2}\\ &\ll \frac{\log\log x}{\log x}e^{-M_{h\chi}(x,Y)}\exp\left(\sum_{p\leqslant x}\frac{|h(p)|}{p}\right)+(\log x)^{-1+\alpha_h/2}\\ &\ll_\varepsilon \frac{(\log x)^{\varepsilon}e^{-M_{h\chi}(x,Y)}}{\log x}\prod_{\substack{p\leqslant x\\ p\nmid qW(x)}}\left(1+\frac{|h(p)|}{p}\right)+(\log x)^{-1+\alpha_h/2}.\end{aligned} \tag{4-32}$$

This reduces our task to that of finding a sufficiently good lower bound on $M_{h\chi}(x,Y)$. To achieve this, we aim to show that there are positive constants $\delta_0,\delta_1,\delta_2>0$ such that for all nontrivial $\chi \pmod Q$ with $Q\leqslant(\log x)^C$ and $W(x)\mid Q$, for all $0\leqslant t\leqslant 2Y$ and for all $y\in(\exp((\log x)^{1-\alpha_h/4}),x]$, the set

$$\mathscr{P}_{\delta_1,\delta_2}(y)=\{p\leqslant y: h(p)>\delta_1\}\cap\{p\leqslant y: 1-\Re(\chi(p)p^{it})>\delta_2\} \tag{4-33}$$

has positive relative density at least δ_0 in the set of primes up to y, i.e.,

$$\#\mathscr{P}_{\delta_1,\delta_2}(y)\geqslant\frac{\delta_0 y}{\log y}. \tag{4-34}$$

The restriction to nonnegative t is justified here since we consider together with every nontrivial χ $(\operatorname{mod} qW(x))$ also its conjugate character $\overline{\chi}$.

Assuming (4-34) for the moment, we then have

$$\sum_{p\in\mathscr{P}_{\delta_1,\delta_2}(x)}\frac{1}{p}\geqslant\frac{\#\mathscr{P}_{\delta_1,\delta_2}(x)}{x}+\int_2^x\frac{\#\mathscr{P}_{\delta_1,\delta_2}(t)}{t^2}\,dt\geqslant\frac{\delta_0}{\log x}+\delta_0\int_{\exp((\log x)^{1-\alpha_h/4})}^x\frac{dt}{t\log t}\,dt\geqslant\frac{\delta_0\alpha_h}{4}\log\log x,$$

and, hence,

$$e^{M_{h\chi}(x,Y)}\gg\exp\left(\delta_1\delta_2\sum_{p\in\mathscr{P}_{\delta_1,\delta_2}(x)}\frac{1}{p}\right)\gg(\log x)^{\delta_0\delta_1\delta_2\alpha_h/4}.$$

Combined with (4-32), this shows, in particular, that

$$|S_{h\chi}(x)|\ll_\varepsilon(\log x)^{-\delta_0\delta_1\delta_2\alpha_h/4+\varepsilon}\frac{1}{\log x}\prod_{\substack{p\leqslant x\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)+(\log x)^{-1+\alpha_h/2},$$

and, hence, that (4-28) holds. Thus, it remains to establish (4-34).

The set of primes $\mathscr{P}_{\delta_1,\delta_2}(y)$ is determined by two conditions involving the behavior of h, χ and n^{it} at these primes. To find a lower bound on the cardinality of $\mathscr{P}_{\delta_1,\delta_2}(y)$, our first step is to remove the condition that $h(p) > \delta_1$ from consideration. To do so, recall that the Sato–Tate law [Barnet-Lamb et al. 2011] implies that

$$\#\{p \leqslant y : 0 \leqslant |\lambda_p| \leqslant \alpha\} \sim \frac{\mu(\alpha)y}{\log y}$$

for every $\alpha \in [0, 2]$, where

$$\mu(\alpha) = \frac{2\arcsin\left(\frac{1}{2}\alpha\right) + \sin\left(2\arcsin\left(\frac{1}{2}\alpha\right)\right)}{\pi}.$$

This shows, in particular, that for every $c_1 \in (0, 1)$ there exists a $\delta(c_1) > 0$ such that

$$\#\{p \leqslant y : g(p) > \delta(c_1)\} \geqslant \frac{c_1 y}{\log y} \tag{4-35}$$

for all sufficiently large y. Thus, to prove (4-34) for $\delta_2 = \frac{1}{12}$, say, it suffices to show that for every $0 \leqslant t \leqslant 2Y$ and every $y \in \left(\exp((\log x)^{1-\alpha_h/4}), x\right]$, the set

$$\mathscr{P}_{\chi,t}(y) := \left\{p \leqslant y : \Re(\chi(p)p^{it}) < \tfrac{11}{12}\right\} \tag{4-36}$$

has positive relative density in the set of primes up to y. Indeed, if

$$\#\mathscr{P}_{\chi,t}(y) \geqslant \frac{c_2 y}{\log y} \tag{4-37}$$

for some $c_2 > 0$, then, setting $c_1 = 1 - \frac{1}{2}c_2$ in (4-35) and letting $\delta_1 = \delta(c_1)$, we find that $\#\mathscr{P}_{\delta_1,\delta_2}(y)$ is at least $c_2 y/(2\log y)$, i.e., that (4-34) holds with $\delta_0 = \frac{1}{2}c_2 > 0$, as required.[5]

Having simplified our problem to that of establishing (4-37) for a set of primes only defined by the behavior of $\chi(p)$ and p^{it}, our next step is to also remove χ from consideration and to essentially turn the problem into a question about the distribution of $(t \log p/(2\pi))_{p \leqslant y}$ modulo one. Let us begin by decomposing the set of primes into classes on which $\chi(p)$ is constant and consider the primes in each progression $A \pmod Q$ for $\gcd(A, Q) = 1$ separately. Let $\{z\} = z - \lfloor z \rfloor$ denote the fractional part of a real number z, let $T = t/(2\pi)$ and consider for each A as above the set

$$\mathscr{N}_A(y) = \{p < y : \{T \log p\} \in I_{T \log y} \text{ and } p \equiv A \pmod Q)\} \tag{4-38}$$

where $I_{T \log y} = \left[T \log y - \frac{1}{9}, T \log y\right] \pmod 1$ is an interval of fixed length $\frac{1}{9}$, the position of which only depends on the parameters y and t, but not on the residue class A. Our aim is to show that there exists a constant $c_3 > 0$ such that for every reduced residue class $A \pmod Q$ and every $y \in (\exp((\log x)^{1-\alpha_h/4}), x]$, we have

$$\#\mathscr{N}_A(y) \geqslant \frac{c_3 y}{\phi(Q)\log y}. \tag{4-39}$$

[5] In view of the reduction to (4-37), it becomes clear that we will only require (4-35) to hold for one specific value of c_1 in the end. This will later allow us to deduce Lemma 4.16 from this proof and, with some further modifications, also Lemma 4.17. For this reason we will track the information gathered on c_2 throughout the rest of the proof.

Since this bound clearly holds for all invertible residue classes if $t = T = 0$ and if $c_3 = 1-\varepsilon,\ \varepsilon > 0$, we may restrict attention to the case $t \in (0, 2Y]$ below.

Assuming (4-39) for the moment, let us first show how to deduce the claimed bound (4-37). In view of (4-39), it suffices to show that for a positive proportion of the reduced residues $A \pmod Q$ we have $\mathscr{N}_A(y) \subset \mathscr{P}_{\chi,t}(y)$ for all $y \in \big(\exp((\log x)^{1-\alpha_h/4}), x\big]$.

If χ is a nontrivial real character, then each of the preimages $\chi^{-1}(1)$ and $\chi^{-1}(-1)$ contains $\frac{1}{2}\phi(Q)$ residue classes $A \pmod Q$. If the distance of $T\log y$ to the closest integer satisfies $\|T\log y\| > \frac{1}{6}$, then we have $\Re e(z) < \cos\big(\frac{2\pi}{6}\big) + \frac{1}{9} = \frac{1}{2} + \frac{1}{9} < \frac{3}{4}$ for every $z \in I_{T\log y}$, and, hence,

$$\Re(\chi(p)p^{it}) = \Re(e^{it\log p}) < \tfrac{3}{4} < \tfrac{11}{12}$$

for all $p \in \mathscr{N}_A(y)$ and all $\frac{1}{2}\phi(Q)$ classes $A \in \chi^{-1}(1)$. If, on the other hand, $\|T\log y\| \leqslant \frac{1}{6}$, then we have $\Re\, e(z) \geqslant \cos\frac{2\pi}{6} - \frac{1}{9} = \frac{1}{2} - \frac{1}{9} > 0$ for every $z \in I_{T\log y}$, and, thus,

$$\Re(\chi(p)p^{it}) = -\Re(e^{it\log p}) < 0 < \tfrac{11}{12}$$

for all $p \in \mathscr{N}_A(y)$ and all $\frac{1}{2}\phi(Q)$ classes $A \in \chi^{-1}(-1)$. Thus, (4-37) holds with $c_2 = \frac{1}{2}c_3$ if χ is nontrivial and real.

Turning towards the case where χ is not real, recall that the nonzero values of any Dirichlet character $\chi \pmod Q$ are the k-th roots of unity if χ has order k in the group of characters modulo Q and recall also that $\chi(A)$ assumes each k-th root of unity equally often as A runs over the reduced residue classes modulo Q. Thus, if $\chi \pmod Q$ is not a real character, then each of the four sets

$$\mathscr{R}_> = \{A \pmod Q : \Re(\chi(A)) > 0\}, \quad \mathscr{R}_< = \{A \pmod Q : \Re(\chi(A)) < 0\},$$
$$\mathscr{I}_> = \{A \pmod Q : \Im(\chi(A)) > 0\}, \quad \mathscr{I}_< = \{A \pmod Q : \Im(\chi(A)) < 0\}$$

is nonempty and contains a positive proportion of the reduced residues $A \pmod Q$. To see this, note that $k \geqslant 3$, since χ is not real. By the symmetry of the set of k-th roots of unity, we have $\#\mathscr{I}_< = \#\mathscr{I}_>$ and, if i is a k-th root of unity, then $\#\mathscr{R}_< = \#\mathscr{R}_>$ as well. If i is not a k-th root of unity, then $|\#\mathscr{R}_< - \#\mathscr{R}_>| \leqslant \phi(Q)/k$. Since $\mathscr{I}_< \cup \mathscr{I}_>$ and $\mathscr{R}_< \cup \mathscr{R}_>$ both exclude at most two of the k k-th roots and since the latter set excludes none if i is not a k-th root, we have

$$\#\mathscr{S} \geqslant \frac{k-2}{2}\frac{\phi(Q)}{k} = \left(\frac{1}{2} - \frac{1}{k}\right)\phi(Q) \geqslant \frac{\phi(Q)}{6}$$

for each set $\mathscr{S} \in \{\mathscr{R}_>, \mathscr{R}_<, \mathscr{I}_>, \mathscr{I}_<\}$. This proves the claim.

For each of the above sets $\mathscr{S}$, the product set

$$\big\{\chi(A)e^{2\pi i\tau} : A \in \mathscr{S}, \tau \in \big[T\log y - \tfrac{1}{9}, T\log y\big]\big\}$$

is contained in an arc of length $2\pi\big(\frac{1}{2} + \frac{1}{9}\big)$ on the unit circle and a rotation by $\frac{\pi}{2}$ maps each of these four arcs onto another one of them. This configuration has the property that no arc of length at most $\frac{\pi}{4}$ meets more than three of the product sets. Thus, for each choice of the endpoint $T\log y$ there is one set $\mathscr{S}$ for which the above product set avoids $\{e^{iz} : \|z\| < \pi/8\}$, and for that particular set $\mathscr{S}$, we then have

$$\Re(\chi(p)p^{it}) = \Re(\chi(A)e^{it\log p}) < \cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}} < \frac{11}{12}$$

for all $A \in \mathscr{S}$ and all $p \in \mathscr{N}_A(y)$. Thus, (4-37) holds with $c_2 = \frac{1}{6}c_3$. This completes the proof of the claim that (4-39) implies (4-37).

It finally remains to analyze the set $\mathscr{N}_A(y)$ that was defined in (4-38) and we will do this by borrowing an approach from Wintner's work [1935] on the distribution of $(\log p_n)_{n\leqslant x}$ modulo one.

Let us fix a reduced residue class $A \pmod Q$ and let $(p_n^{(A)})_{n\in\mathbb{N}}$ denote the sequence of primes congruent to $A \pmod Q$, ordered in increasing order. Adapting Wintner's notation to our setting, let $N_A(\tau)$ denote the largest index m for which $\log p_m^{(A)} < \tau$, if such an m exists, and let $N_A(\tau) = 0$ otherwise. By the prime number theorem in arithmetic progressions, we then have

$$N_A(\tau) = \frac{e^\tau}{\phi(Q)\tau}(1 + O(\tau^{-1})), \quad (\tau > 0). \tag{4-40}$$

Observe that $N_A(\tau/T)$ counts the number of $m > 0$ such that $T \log p_m \leqslant \tau$. Thus, if we set $\xi := \{T \log y\}$, so that $T \log y = [T \log y] + \xi$, then, in analogy to [Wintner 1935, equation (3)], we may express the quantity $\#\mathscr{N}_A(y)$ as

$$\begin{aligned}\#\mathscr{N}_A(y) &= \sum_{n=1}^{[T\log y]} \Big(N_A\Big(\frac{n+\xi}{T}\Big) - N_A\Big(\frac{n+\xi-1/9}{T}\Big)\Big)\\ &= \sum_{n=T}^{[T\log y]} N_A\Big(\frac{n+\xi}{T}\Big) - \sum_{n=T}^{[T\log y]} N_A\Big(\frac{n+\xi-1/9}{T}\Big).\end{aligned} \tag{4-41}$$

If $T \in (0, C']$ for any fixed constant $C' \geqslant 1$, then

$$\begin{aligned}\#\mathscr{N}_A(y) &> N_A\Big(\frac{[T\log y]+\xi}{T}\Big) - N_A\Big(\frac{[T\log y]+\xi-1/9}{T}\Big)\\ &= N_A(\log y) - N_A\Big(\log y - \frac{1}{9T}\Big)\\ &= \pi(y; Q, A) - \pi(ye^{-1/(9T)}; Q, A)\\ &> \pi(y; Q, A) - \pi(ye^{-1/(9C')}; Q, A)\\ &\gg_{C'} \pi(y; Q, A),\end{aligned} \tag{4-42}$$

and $c_3 \gg_{C'} 1$ in (4-39). This leaves us to establish (4-39) for $T \in \big(C', \frac{1}{\pi}(\log x)^{1-\alpha_h/2}\big]$.

To bound (4-41) below, note that the prime number theorem (4-40) implies that

$$\sum_{n=T}^{[\tau]} N_A\Big(\frac{n+\xi}{T}\Big) = \frac{T}{\phi(Q)}\sum_{n=T}^{[\tau]}\frac{e^{(n+\xi)/T}}{n+\xi} + O\Big(\frac{T^2}{\phi(Q)}\sum_{n=T}^{[\tau]}\frac{e^{(n+\xi)/T}}{(n+\xi)^2}\Big). \tag{4-43}$$

A corresponding expansion for the second sum in (4-41) is obtained on replacing ξ by $\xi-\frac{1}{9}$. The sum in the main term above may be asymptotically evaluated, using induction:

$$(e^{1/T}-1)\sum_{n=T}^{N-1}\frac{e^{n/T}}{n+\xi}=\frac{e^{N/T}}{N}+O\biggl(\frac{1}{T}+\sum_{n=T}^{N}\frac{e^{n/T}}{(n+1)^2}\biggr),\quad (N\geqslant T+1). \tag{4-44}$$

Indeed, if $N=T+1$, then

$$\left|\frac{e}{T+\xi}-\frac{e^{1+1/T}}{T+1}\right|=\left|e^{1+1/T}\frac{1-\xi}{(T+1)(T+\xi)}-\frac{e}{T+\xi}\right|=O\biggl(\frac{e^{1+1/T}}{(T+1)^2}+\frac{1}{T}\biggr),$$

and if we assume that (4-44) holds for $N=M$, then it follows for $N=M+1$, since

$$\begin{aligned}\frac{e^{M/T}}{M}+\frac{(e^{1/T}-1)e^{M/T}}{M+\xi}&=\frac{e^{M/T}}{M}+\frac{(e^{1/T}-1)e^{M/T}}{M}+\frac{\xi e^{M/T}(e^{1/T}-1)}{M(M+\xi)}\\&=\frac{e^{(M+1)/T}}{M}+O\Bigl(\frac{e^{(M+1)/T}}{M^2}\Bigr)=\frac{e^{(M+1)/T}}{M+1}+O\Bigl(\frac{e^{(M+1)/T}}{M^2}\Bigr).\end{aligned}$$

Thus, evaluating main term in (4-43) by means of (4-44), we obtain

$$\sum_{n=T}^{[\tau]}N_A\Bigl(\frac{n+\xi}{T}\Bigr)=\frac{T}{\phi(Q)}\frac{e^{\xi/T}e^{([\tau]+1)/T}}{(e^{1/T}-1)([\tau]+1)}+O\biggl(\frac{T^2}{\phi(Q)}\sum_{n=T}^{[\tau]}\frac{e^{(n+1)/T}}{(n+1)^2}+\frac{1}{\phi(Q)}\biggr). \tag{4-45}$$

Since $\int 1/(\log x)^2\,\mathrm{d}x=\mathrm{li}(x)-x/\log x\ll x/(\log x)^2$, the sum in the error term satisfies

$$\sum_{n=T}^{[\tau]}\frac{e^{(n+1)/T}}{(n+1)^2}\leqslant\int_T^{\tau+1}\frac{e^{t/T}}{t^2}\,\mathrm{d}t=\frac{1}{T}\int_e^{e^{(\tau+1)/T}}\frac{\mathrm{d}u}{(\log u)^2}=O\biggl(\frac{Te^{(\tau+1)/T}}{(\tau+1)^2}+1\biggr).$$

Inserting this information, (4-45) and the analogues expression with ξ replaced by $\xi-\frac{1}{9}$ into (4-41), we obtain

$$\#\mathscr{N}_A(y)=\frac{T}{\phi(Q)}\frac{e^{\frac{[T\log y]+1}{T}}e^{\frac{\xi}{T}}}{[T\log y]+1}\frac{1-e^{-\frac{1}{9T}}}{e^{\frac{1}{T}}-1}+O\biggl(\frac{T}{\phi(Q)}\frac{e^{\frac{T\log y+1}{T}}}{(T\log y+1)^2/T^2}+\frac{T^2}{\phi(Q)}\biggr).$$

Recalling that $1<C'<T\leqslant(\log x)^{1-\alpha_h/2}/\pi$ and that $(1-\alpha_h/4)\log x<\log y\leqslant\log x$, this yields

$$\begin{aligned}\#\mathscr{N}_A(y)&=\frac{T}{\phi(Q)}\frac{ey}{[T\log y]+1}\frac{1-e^{-1/9T}}{e^{1/T}-1}+O\biggl(\frac{T}{\phi(Q)}\frac{y}{(\log y)^2}\biggr)\\&=\frac{T}{\phi(Q)}\frac{ey}{[T\log y]+1}\frac{1-e^{-1/9T}}{e^{1/T}-1}+O_{\alpha_h}\bigl(y(\log y)^{-1-\alpha_h/2}/\phi(Q)\bigr)\\&\gg_{\alpha_h}\frac{1-e^{-1/9T}}{e^{1/T}-1}\frac{1}{\phi(Q)}\frac{y}{\log y}.\end{aligned} \tag{4-46}$$

Thus, it remains to bound below the leading fraction in this bound. To this end, note that

$$e^{-\tau}=1-\frac{\tau}{2}-\frac{\tau-\tau^2}{2}-\sum_{k=1}^{\infty}\frac{\tau^{2k+1}}{(2k+1)!}\Bigl(1-\frac{\tau}{2k+2}\Bigr)\leqslant 1-\frac{\tau}{2}$$

for every $\tau \in [0, 1]$, and that

$$e^{\tau} \leqslant 1+\tau+\frac{\tau^2}{2}\sum_{k=0}^{\infty} 2^{-k} = 1+\tau+\tau^2 \leqslant 1+2\tau$$

for all $\tau \in \left[0, \frac{1}{2}\right]$. Thus, if $T \geqslant 2$, then the leading factor in the lower bound (4-46) satisfies

$$\frac{1-e^{-1/9T}}{e^{1/T}-1} > \frac{1-1+1/18T}{1+2/T-1} = \frac{1}{36},$$

and it follows that $c_3 \gg_{\alpha_h} 1$ in this case. Choosing $C' = 2$ in (4-42), this completes the proof of (4-39) and of the lemma. □

Proof of Lemma 4.16. This lemma follows from the proof above, observing that the information (4-35) gained from the Sato–Tate law is now included as an assumption in the statement of the lemma. More precisely, (4-35) is only required for $c_1 = 1 - \frac{1}{2}c_2$, with $c_2 = \frac{1}{6}c_3 \gg_{\alpha_h} 1$. Thus, $c_1 = 1 - c$ for some $c > 0$ only depending on $\alpha_h = \alpha/H$. □

Proof of Lemma 4.17. To deduce this lemma, we need to apply Proposition 4.10 instead of the special case recorded in Lemma 4.13. We restrict attention to the first part of this lemma, the second being a simplification. Let h be the function associated to f via (1-6). Let $x > 1$ and let t_x be a real number as in Lemma 4.9, applied with $f_0 = h$. By arguing as in the proof of Lemma 4.9, it follows from (4-8) that

$$|S_{h\chi_0}(x)| \ll \frac{1}{|t_x|+1} + \frac{\log\log x}{\log x} + \frac{1}{(\log x)^{1+C_0}} \exp\Biggl(\sum_{p\leqslant x} \frac{|h(p)\chi_0(p)|}{p}\Biggr)$$

whenever $\chi_0 \pmod{Q}$, $Q \leqslant \exp((\log\log x)^2)$, is a trivial character. If $|t_x| > (\log x)^{1-\alpha_h/2}$, then $|S_{h\chi_0}(x)|$ is small, and we set $\tau_x = 0$. If $|t_x| \leqslant (\log x)^{1-\alpha_h/2}$, we instead set $\tau_x = t_x$.

The rest of the proof proceeds almost exactly as that of Lemma 4.15, but with the following changes. Instead $|S_{h\chi}(x)|$, we now seek to bound $|S_{h(n)\chi(n)n^{-it_x}}(x)|$, or even

$$\max_{|t|\leqslant(\log x)^{1-\alpha_h/2}} |S_{h(n)\chi(n)n^{-it}}(x)|. \tag{4-47}$$

Since the parameter Y is chosen as $Y = (\log x)^{1-\alpha_h/2}$ in the proof of Lemma 4.15, we may readily turn the bound (4-29) into one on (4-47) by redefining M as $M = M(x, Y')$ with $Y' = 2Y$, a change which does not affect the rest of the argument. Continuing from here, we replace the decomposition (4-30) by

$$\begin{aligned} M(x, Y') &= \min_{|y|\leqslant 2Y'} \sum_{p\leqslant x} \frac{1-|h(p)|+|h(p)|-\Re(h(p)\chi(p)p^{iy})}{p} \\ &= \sum_{p\leqslant x} \frac{1-|h(p)|}{p} + \min_{|y|\leqslant 2Y'} \sum_{p\leqslant x} \frac{|h(p)|(1-\operatorname{sgn}(h(p))\Re(\chi(p)p^{iy}))}{p}, \end{aligned}$$

and let

$$M_{h\chi}(x, Y') = \min_{|y|\leqslant 2Y'} \sum_{p\leqslant x} \frac{|h(p)|(1-\operatorname{sgn}(h(p))\Re(\chi(p)p^{iy}))}{p}$$

denote the second term from this new expression. As in the proof of Lemma 4.16, we need to replace (4-35) by our new assumptions, which will also allow us to fix $\operatorname{sgn}(h(p)) = \epsilon$. The set of primes in (4-36) now takes the form

$$\mathscr{P}_{\chi,t}(y) = \left\{p \leqslant y : \epsilon\Re(\chi(p)p^{iy})) < \tfrac{11}{12}\right\}.$$

The deduction of (4-37) from (4-39) remains, apart from obvious changes taking into account the additional sign ϵ, unchanged. □

5. The W-trick

Generalising the fact that the bound (1-1) only applies to Fourier coefficients $(1/x)\sum_{n\leqslant x} f(n)e(\alpha n)$ at an irrational phase α, it is the case that an arbitrary multiplicative function f may *correlate* with a given nilsequence, unless this sequence itself is sufficiently equidistributed. Thus, statements of the form

$$\frac{1}{N}\sum_{n\leqslant N} h(n)F(g(n)\Gamma) = o_{G/\Gamma}(1)$$

with $h = f$ or $h = f - S_f(N; 1, 1)$ cannot be expected to hold in general. On the other hand, it turns out to be sufficient to ensure that h is equidistributed in progressions to small moduli in order to resolve this problem. For arithmetic applications such as establishing a result of the form (1-9), this can be achieved with the help of the W-trick from [Green and Tao 2008]. The basic idea is to decompose f into a sum of functions that are equidistributed in progressions to small moduli. This is achieved by decomposing the range $\{1, \dots, N\}$ into subprogressions modulo a product $W(N)$ of small primes, which has the effect of fixing or eliminating the contribution from small primes on each of the subprogressions.

For multiplicative functions some minor modifications are necessary. Our aim is to decompose the interval $\{1, \dots, N\}$ into subprogressions $r \pmod q$ in such a way that

$$S_f(N; q, r) = (1 + o(1))S_f(N; qq', r + qr') \tag{5-1}$$

for small q' and $0 \leqslant r' < q$. Thus, f should essentially have a constant average value when decomposing one of the given subprogressions into further subprogressions of small moduli q'. The example of the characteristic function of sums of two squares shows that we cannot in general choose q to be a product of small primes (consider the case where $r \equiv 1 \pmod 2$, $q' = 2$ and $r + qr' \equiv 3 \pmod 4$), but rather need to allow q to be a product of small prime powers. Further, if f is a function for which Shiu's bound on $S_f(N; q, r)$ is correct in the sense that

$$S_f(N; q, r) \sim \frac{q}{\phi(q)}\frac{1}{\log N}\prod_{\substack{p\leqslant N\\ p\nmid q}}\Bigl(1 + \frac{f(p)}{p}\Bigr),$$

then, in order for (5-1) to hold, we must have $p \mid q$ whenever $p \mid q'$ and p is small.

Our aim in this section is to show that for every $f \in \mathscr{F}_H$ we may, instead of $q = W(N)$ as in [Green and Tao 2008], take $q = \widetilde{W}(N) := q^*(N)W(N)$ for some integer-valued function $q^* : \mathbb{N} \to \mathbb{N}$ that satisfies the bound $q^*(x) \leqslant (\log x)^{O(1)}$. For comparison, recall that $W(x) = \prod_{p \leqslant w(x)} p$, with w as in Definition 1.2. Thus,

$$\log W(x) = \sum_{p \leqslant w(x)} \log p \sim w(x) \quad \text{and} \quad W(x) \leqslant (\log x)^{1+o(1)}.$$

For such a function $\widetilde{W}$, we may decompose the range $[1, N]$ into subprogressions of the form

$$\{1 \leqslant m \leqslant N : m \equiv w_1 A \pmod{w_1 \widetilde{W}(N)}\},$$

where $A \in (\mathbb{Z}/\widetilde{W}(N)\mathbb{Z})^*$ and where $w_1 \geqslant 1$ is composed entirely of primes dividing $\widetilde{W}(N)$. Abbreviating $\widetilde{W} = \widetilde{W}(N)$, we have $\gcd(w_1, \widetilde{W}n + A) = 1$ and hence $f(w_1(\widetilde{W}n + A)) = f(w_1) f(\widetilde{W}n + A)$. Thus, it suffices to study the family of functions

$$\{n \mapsto f(\widetilde{W}n + A) : 0 < A < \widetilde{W}(N),\ \gcd(A, \widetilde{W}) = 1\}.$$

Our first concern is to discard the set of large values of w_1 from consideration, as by doing so we can insure that the range on which each function $n \mapsto f(\widetilde{W}n + A)$ needs to be considered is always large. Since large values of w_1 form a sparse set, their contribution in any arithmetic application can usually be bounded by just using the Cauchy–Schwarz inequality and a bound on the second moment of f as in [Browning and Matthiesen 2017, Lemma 7.9]. More precisely, one can show that if, for $C_1 > 1$,

$$\mathscr{S}_{C_1}(N) = \{w_1 \in \mathbb{N} : w_1 > (\log N)^{C_1},\ p \mid w_1 \Rightarrow p \mid \widetilde{W}(N)\},$$

then

$$\frac{1}{N} \sum_{n \leqslant N} \sum_{w_1 \in \mathscr{S}_{C_1}(N)} \mathbf{1}_{w_1 \mid n} |f(n)| \ll (\log N)^{-C_1/3},$$

provided $q^*(N) < (\log N)^{C_1/3}$ and C_1 is sufficiently large with respect to H; see [Matthiesen 2016, §5] for details. By choosing $C_1 > 3\alpha_f$, we can for instance ensure that this bound is $o\big(\frac{1}{N} \sum_{n \leqslant N} |f(n)|\big)$. As shown in [loc. cit., §5], the contribution of $\mathscr{S}_{C_1}(N)$ to correlations of the form (1-9) is negligible.

Thus, for the purpose of arithmetic applications, it suffices to consider $n \mapsto f(\widetilde{W}n + A)$ for $n \in \{1, \dots, T\}$ with

$$T = \frac{N - A w_1}{w_1 \widetilde{W}(N)} \gg \frac{N}{(\log N)^{C_1} \widetilde{W}(N)}.$$

The next proposition shows that every function $f \in \mathscr{F}_H$ admits a W-trick. More precisely, any finite collection $f_1, \dots, f_r$ of elements from $\mathscr{F}_H$ simultaneously admits a W-trick and we moreover have control over the size of $\widetilde{W} (= q)$ and over the level of q' up to which (a weakened form of) the relation (5-1) holds. Below, $\widetilde{W}$ plays the role of q and q plays the role of q'.

Proposition 5.1 (the elements of $\mathscr{F}_{H,n^{it}}$ admit a W-trick). *Let $E, H \geqslant 1$ be constants and let $f_1, \dots, f_r \in \mathscr{F}_{H,n^{it}}$. Then there exists a constant κ, depending on E, H, r and $\alpha = \min_{1 \leqslant j \leqslant r} \alpha_{f_j}$, and functions $\varphi' : \mathbb{N} \to \mathbb{R}$ and $q^* : \mathbb{N} \to \mathbb{N}$ such that the following holds*:

(1) *$\varphi'(x) \to 0$ as $x \to \infty$.*

(2) $q^*(x) \leqslant (\log x)^\kappa$ *for all sufficiently large* $x \in \mathbb{N}$.

(3) *If* $x \in \mathbb{N}$ *is sufficiently large, if we set* $\widetilde{W}(x) := q^*(x)W(x)$, *and if we define*

$$f_x : n \mapsto f(n)n^{-it_x} \quad \textit{for any } f \in \{f_1, \dots, f_r\}$$

and with t_x *as in Definition 1.6 with* $C = 2E + \kappa + 4$, *then the estimate*

$$\frac{q\widetilde{W}(x)}{|I|} \sum_{\substack{m \in I \\ m \equiv A \ (q\widetilde{W}(x))}} f_x(m) - S_{f_x}(x; \widetilde{W}(x), A) = O_{E,H,\kappa}\biggl(\varphi'(x) \frac{1}{\log x} \frac{\widetilde{W}(x)}{\phi(\widetilde{W}(x))} \prod_{\substack{p<x \\ p \nmid \widetilde{W}(x)}} \biggl(1 + \frac{|f(p)|}{p}\biggr)\biggr) \tag{5-2}$$

holds uniformly for all intervals $I \subseteq \{1, \dots, x\}$ *with* $|I| > x(\log x)^{-E}$, *for all integers* $0 < q \leqslant (\log x)^E$ *and for all* $A \in (\mathbb{Z}/q\widetilde{W}(x)\mathbb{Z})^*$.

Remarks 5.2. (1) If $f \in \mathscr{F}_H$, then $f_x = f$.

(2) We will show that (5-2) holds with $\varphi'(x) = \varphi_C(x) + (\log w(x))^{-1} + (\log x)^{-\alpha_f/(3H)} + (\log x)^{-E}$, where φ_C is as in Definition 1.4 with $C = 2E + \kappa + 4$.

The rest of this section is devoted to a proof of Proposition 5.1. Our strategy is to first relate the left-hand side of (5-2) to a restricted character sum, which we will then attempt to bound by means of the "pretentious large sieve"-consequence recorded in Corollary 4.2.

We begin with a technical lemma that will at various points in the argument allow us to control the contribution of the prime divisors $p \mid \widetilde{W}(N)$ that are larger than $w(N)$.

Lemma 5.3. *Let* $1 \leqslant a \leqslant (\log N)^E$ *be an integer that is free from prime factors* $p < w(N)$ *and suppose that* $0 \leqslant g(p) \leqslant H$ *for all* p. *Then*

$$\prod_{p \mid a}\biggl(1 + \frac{g(p)}{p}\biggr) = 1 + O_{E,H}\biggl(\frac{1}{\log w(N)}\biggr).$$

Proof. The assumptions on a imply the bound $\Omega(a) \leqslant E \log\log N / \log w(N)$ on the total number of prime factors of a. Let $m = [w(N)/\log w(N) + \Omega(a)]$ and recall that the n-th prime p_n satisfies $p_n \sim n \log n$. Then,

$$p_m \sim m \log m \leqslant \frac{w(N) + E \log\log N}{\log w(N)} \log m \sim w(N) + E \log\log N.$$

Using the bounds on $w(N)$ from Definition 1.2 and Mertens' estimate, we obtain

$$\begin{aligned}
\prod_{p \mid a}\biggl(1 + \frac{g(p)}{p}\biggr) &\leqslant \prod_{p \mid a}(1 + p^{-1})^H \leqslant \prod_{w(N) < p < p_m} (1 + p^{-1})^H \\
&\leqslant \biggl(\frac{\log(w(N) + E \log\log N) + O(1)}{\log w(N) + O(1)}\biggr)^H \\
&= \biggl(1 + O_E\biggl(\frac{1}{\log w(N)}\biggr)\biggr)^H = 1 + O_{E,H}\biggl(\frac{1}{\log w(N)}\biggr),
\end{aligned}$$

as claimed. □

Corollary 5.4. *If x and q are as in Proposition 5.1, then*

$$\frac{q\widetilde{W}(x)}{\phi(q\widetilde{W}(x))} \leqslant \biggl(1+O_{E,H}\Bigl(\frac{1}{\log w(N)}\Bigr)\biggr)\frac{\widetilde{W}(x)}{\phi(\widetilde{W}(x))}.$$

Proof. Let $a=\prod_{p\mid q,\,p\nmid\widetilde{W}(x)} p$. Then

$$\begin{aligned}\prod_{p\mid a}(1-p^{-1})^{-1} &= \prod_{p\mid a}(1+p^{-1})(1-p^{-2})^{-1}\\ &\leqslant \exp\Bigl(\sum_{p\mid a}\frac{2}{p^2}\Bigr)\prod_{p\mid a}(1+p^{-1}) = \Bigl(1+O\Bigl(\frac{1}{w(x)}\Bigr)\Bigr)\prod_{p\mid a}(1+p^{-1}). \qquad \square\end{aligned}$$

The next lemma replaces the general interval I from (5-2) by one of the form $\{1,\dots,y\}$.

Lemma 5.5. *If E, H, x, f and f_x are as in Proposition 5.1, if $\kappa \geqslant 0$ is a given constant and if $\widetilde{W}(x) \leqslant (\log x)^{\kappa+2}$ is a multiple of $W(x)$, then* (5-2) *follows if there exists a function $\varphi''=o(1)$ such that*

$$S_{f_x}(x; q\widetilde{W}(x), A) = S_{f_x}(x;\widetilde{W}(x),A) + O_{E,H,\kappa}\biggl(\frac{\varphi''(x)}{\log x}\frac{\widetilde{W}q}{\phi(\widetilde{W}q)}\prod_{\substack{p<x\\ p\nmid\widetilde{W}q}}\Bigl(1+\frac{|f(p)|}{p}\Bigr)\biggr) \tag{5-3}$$

for all $q\in(0,(\log x)^{-E}]$ and $A\in(\mathbb{Z}/q\widetilde{W}(x)\mathbb{Z})^*$. More precisely, we may take

$$\varphi'(x)=\varphi_C(x)+\varphi''(x)+(\log w(x))^{-1}$$

in (5-2), *where φ_C is as in Definition 1.4 with $C=2E+\kappa+4$.*

Proof. In view of (5-3), it suffices to relate the first term in (5-2) to $S_{f_x}(x; q\widetilde{W}(x), A)$. Let $y_1, y_2 \in \mathbb{Z}_{\geqslant 0}$ and suppose that $I=(y_1, y_1+y_2]\subset[1,x]$ with $y_2 \geqslant x(\log x)^{-E}$. Writing $\widetilde{W}=\widetilde{W}(x)$, an application of (1-5) with $C:=2E+\kappa+4>E$ shows that the first term in (5-2) satisfies

$$\begin{aligned}\frac{q\widetilde{W}}{|I|}\sum_{\substack{m\in I\\ m\equiv A\,(q\widetilde{W})}} f_x(m) = \frac{q\widetilde{W}}{y_2}\sum_{\substack{y_1<m\leqslant y_1+y_2\\ m\equiv A\,(q\widetilde{W})}} f_x(m) &= \frac{y_1+y_2}{y_2}S_{f_x}(y_1+y_2; q\widetilde{W},A) - \frac{y_1}{y_2}S_{f_x}(y_1;q\widetilde{W},A)\\ &= \frac{y_1+y_2}{y_2}S_{f_x}(x;q\widetilde{W},A)-\frac{y_1}{y_2}S_{f_x}(y_1;q\widetilde{W},A)\\ &\quad + O\biggl(\frac{\varphi_C(x)}{\log x}\frac{\widetilde{W}q}{\phi(\widetilde{W}q)}\prod_{\substack{p<x\\ p\nmid\widetilde{W}q}}\Bigl(1+\frac{|f(p)|}{p}\Bigr)\biggr).\end{aligned} \tag{5-4}$$

We now split into two cases. If, on the one hand,

$$x>y_1>y_2(\log x)^{-E-\kappa-4}>x(\log x)^{-2E-\kappa-4},$$

then (1-5) shows that $S_{f_x}(y_1; q\widetilde{W}, A)$ can be replaced by $S_{f_x}(x; q\widetilde{W}, A)$ in the final expression in (5-4), so that (5-4) is seen to equal

$$S_{f_x}(x; q\widetilde{W}, A) + O\biggl(\frac{\varphi_C(x)}{\log x}\frac{\widetilde{W}q}{\phi(\widetilde{W}q)}\prod_{\substack{p<x\\ p\nmid \widetilde{W}q}}\biggl(1+\frac{|f(p)|}{p}\biggr)\biggr).$$

In this case, (5-2) follows with $\varphi' = \varphi_C + \varphi'' + (\log w(x))^{-1}$ from (5-3) and Corollary 5.4.

If, on the other hand, $y_1 \leqslant y_2(\log x)^{-E-\kappa-4}$, then

$$\frac{y_1+y_2}{y_2} = (1 + O((\log x)^{-E-\kappa-4})).$$

Since $\phi(q\widetilde{W}) \leqslant q\widetilde{W} \leqslant (\log x)^{E+\kappa+2}$, we further have

$$\begin{aligned} S_{f_x}(y_1; q\widetilde{W}, A) &= \frac{q\widetilde{W}}{y_1}\sum_{\substack{n\leqslant y_1\\ n\equiv A \pmod{q\widetilde{W}}}} f(n) \leqslant q\widetilde{W}\sum_{\substack{n\leqslant y_1\\ n\equiv A \pmod{q\widetilde{W}}}}\frac{f(n)}{n}\\ &\leqslant q\widetilde{W}\prod_{\substack{p\leqslant y\\ p\nmid q\widetilde{W}}}\biggl(1+\frac{|f(p)|}{p}+\frac{H^2}{p^2(1-H/p)}\biggr) \ll q\widetilde{W}\prod_{\substack{p\leqslant y\\ p\nmid q\widetilde{W}}}\biggl(1+\frac{|f(p)|}{p}\biggr)\\ &\ll (\log x)^{E+\kappa+2}\frac{q\widetilde{W}}{\phi(q\widetilde{W})}\prod_{\substack{p\leqslant y\\ p\nmid q\widetilde{W}}}\biggl(1+\frac{|f(p)|}{p}\biggr), \end{aligned}$$

which implies that

$$\frac{y_1}{y_2}S_{f_x}(y_1; q\widetilde{W}, A) \leqslant (\log x)^{-E-\kappa-4}S_{f_x}(y_1; q\widetilde{W}, A) \ll (\log x)^{-2}\frac{\widetilde{W}q}{\phi(\widetilde{W}q)}\prod_{p<x,\, p\nmid \widetilde{W}q}\biggl(1+\frac{|f(p)|}{p}\biggr).$$

Thus, in this case, (5-4) equals

$$S_{f_x}(x; q\widetilde{W}, A) + O\biggl(\frac{\varphi_C(x)+(\log x)^{-1}}{\log x}\frac{\widetilde{W}q}{\phi(\widetilde{W}q)}\prod_{\substack{p<x\\ p\nmid \widetilde{W}q}}\biggl(1+\frac{|f(p)|}{p}\biggr)\biggr),$$

and an application of (5-3) yields (5-2) with $\varphi'(x) = \varphi_C(x) + \varphi''(x) + (\log w(x))^{-1}$, when taking into account Corollary 5.4. □

Following the above reduction, we now proceed to analyze the difference of the two mean values that appear in (5-3).

Lemma 5.6 (restricted character sum). *Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, not necessarily multiplicative, let $\widetilde{W}, q, A \geqslant 1$ be integers, and suppose that* $\gcd(A, q\widetilde{W}) = 1$. *If $y \geqslant 1$, then*

$$S_g(y; \widetilde{W}, A) - S_g(y; q\widetilde{W}, A) = \frac{q\widetilde{W}}{y}\frac{1}{\phi(q\widetilde{W})}\sideset{}{^*}\sum_{\chi \pmod{q\widetilde{W}}}\chi(A)\sum_{n\leqslant y} g(n)\overline{\chi}(n), \tag{5-5}$$

where $\sum^*$ indicates the restriction of the sum to characters that are not induced from characters mod $\widetilde{W}$.

Proof. We have

$$\begin{aligned}
S_g(y;\widetilde{W},A)-S_g(y;q\widetilde{W},A) &= \frac{\widetilde{W}}{y}\Biggl(\sum_{\substack{n\leqslant y\\ n\equiv A \ (\mathrm{mod}\ \widetilde{W})}} g(n) - q\sum_{\substack{n\leqslant y\\ n\equiv A \ (\mathrm{mod}\ q\widetilde{W})}} g(n)\Biggr)\\
&= \frac{1}{y}\frac{\widetilde{W}}{\phi(q\widetilde{W})}\sum_{\chi \ (\mathrm{mod}\ q\widetilde{W})}\Biggl(\sum_{\substack{A' \ (\mathrm{mod}\ q\widetilde{W})\\ A\equiv A' \ (\mathrm{mod}\ \widetilde{W})}} \chi(A') - q\chi(A)\Biggr)\sum_{n\leqslant y} g(n)\overline{\chi}(n)\\
&= \frac{1}{y}\frac{\widetilde{W}}{\phi(q\widetilde{W})}\sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q\widetilde{W})}\Biggl(\sum_{\substack{A' \ (\mathrm{mod}\ q\widetilde{W})\\ A\equiv A' \ (\mathrm{mod}\ \widetilde{W})}} \chi(A') - q\chi(A)\Biggr)\sum_{n\leqslant y} g(n)\overline{\chi}(n), \qquad (5\text{-}6)
\end{aligned}$$

where $\sum^*$ indicates the restriction of the sum to characters that are not induced from characters mod $\widetilde{W}$; for all other characters we have $\chi(A')=\chi(A)$ and the difference in the brackets above is zero. It remains to show that the sum over A' in (5-6) vanishes. However,

$$\sum_{\substack{A' \ (\mathrm{mod}\ q\widetilde{W})\\ A\equiv A' \ (\mathrm{mod}\ \widetilde{W})}} \chi(A') = \frac{1}{\phi(\widetilde{W})}\sum_{\chi' \ (\mathrm{mod}\ \widetilde{W})}\overline{\chi'}(A)\sum_{A' \ (\mathrm{mod}\ q\widetilde{W})}\chi(A')\chi'(A')=0,$$

since $\chi\chi'$ is a nontrivial character modulo $q\widetilde{W}$. Thus the lemma follows. □

Finally, we aim to exploit the fact that the character sum on the right-hand side of (5-5) is restricted by invoking Corollary 4.2.

Proof of Proposition 5.1. Let $\varepsilon := \frac{1}{2}\min(1,\alpha/(2H))$, $k := \lceil \varepsilon^{-2}\rceil$ and $k' = k\lceil \log_2(4H)\rceil$, as in the statement of Corollary 4.2. Setting $C'=(E+1)3^{rk'+1}$, we let $\mathscr{E}$ denote the union of the sets of characters defined by Corollary 4.2 when applied with $C=C'$ to each of the r functions $f_x\in\mathscr{M}_H$ for $f\in\{f_1,\ldots,f_r\}$.

Our aim is to find a suitable integer $\widetilde{W}(x)$ so that, if $\widetilde{W}=\widetilde{W}(x)$ and $q\leqslant(\log x)^E$, then none of the characters that appear in the restricted character sum (5-5) is induced by a character from the set $\mathscr{E}$. To do so, we construct a finite sequence of integers $W_0(x), W_1(x),\ldots$ with the property

$$W_i(x)\leqslant W(x)^{2^i}(\log x)^{3^i E}$$

as follows. Let $W_0(x)=W(x)$ and suppose we have already defined $W_i(x)$ for all $0\leqslant i\leqslant j$. Consider the set of integers in the interval $I_j=[W_j(x), W_j(x)(\log x)^E]$. If there exists a character $\chi\in\mathscr{E}$ whose conductor c_χ satisfies $c_\chi\nmid W_j(x)$ but $c_\chi<W_j(x)(\log x)^E$, then we choose one such character χ and define $W_{j+1}(x):=c_\chi W_j(x)$. Note that

$$W_{j+1}(x)<W_j(x)^2(\log x)^E<W(x)^{2\cdot 2^j}(\log x)^{(2\cdot 3^j+1)E}<W(x)^{2^{j+1}}(\log x)^{3^{j+1}E}.$$

If there is no such $\chi\in\mathscr{E}$, then we stop and set $\widetilde{W}(x)=W_j(x)$. Since $\#\mathscr{E}\leqslant rk'$, this process stops after at most rk' steps and, thus, $\widetilde{W}(x)\leqslant W(x)^{2^{rk'}}(\log x)^{3^{rk'}E}\leqslant(\log x)^{2^{rk'+1}+3^{rk'}E}$ and

$$\widetilde{W}(x)q<(\log x^{1/(8H)})^C$$

for all $q \leqslant (\log x)^E$ and sufficiently large x.

Our construction ensures that there exists no character $\chi \pmod{q\widetilde{W}(x)}$ with $q \leqslant (\log x)^E$ that is induced by an element from $\mathscr{E}$ but *not* induced from a character $\pmod{\widetilde{W}(x)}$. Since the sum (5-5) is restricted to those characters modulo $q\widetilde{W}(x)$ that are not induced from characters modulo $\widetilde{W}(x)$, we may apply Corollary 4.2 with $\mathscr{C}$ given by this restricted set of characters and with $Q = q\widetilde{W}(x)$. This application shows that whenever $1 \leqslant q \leqslant (\log x)^E$ and $x^{1/2} < y \leqslant x$, then

$$\frac{1}{y} \sideset{}{^*}\sum_{\chi \pmod{q\widetilde{W}}} \chi(A) \sum_{n \leqslant y} f_x(n)\overline{\chi}(n) \ll_{C,H,\alpha} \frac{1}{(\log x)^{1+\alpha/(3H)}} \exp\biggl(\sum_{\substack{p \leqslant x \\ p \nmid q\widetilde{W}}} \frac{|f(p)|}{p} \biggr).$$

In combination with Lemma 5.6 for $g = f_x$, this yields (5-3) for $\kappa = C' - 2$ and with $\varphi''(x) = (\log x)^{-\alpha/(3H)}$. Hence, Lemma 5.5 implies the result with $\kappa = C' - 2 \ll_{E,H,r,\alpha} 1$. □

We will refer to (5-2) as *the major arc estimate*. We will show in Section 6B that despite the restriction to invertible residues $A \in (\mathbb{Z}/q\widetilde{W}\mathbb{Z})^*$, the estimate (5-2) implies that $f(\widetilde{W}n + A) - S_f(x; \widetilde{W}, A)$ is orthogonal to periodic sequences of period at most $(\log x)^E$, for every $A \in (\mathbb{Z}/\widetilde{W}\mathbb{Z})^*$. This information will be used in combination with a factorization theorem to reduce the task of proving noncorrelation for $(f(\widetilde{W}n + A) - S_f(x; \widetilde{W}, A))$ with general nilsequences to the case where the nilsequence enjoys certain equidistribution properties and the Lipschitz function satisfies, in particular, $\int_{G/\Gamma} F = 0$.

6. The noncorrelation result

This section contains a precise statement of the main result, which, informally speaking, shows the following. Given $E \geqslant 1$ and a multiplicative function $f \in \mathscr{F}_H$, let $\widetilde{W}(x)$ be the function from Proposition 5.1. Then for every residue $A \in (\mathbb{Z}/\widetilde{W}(N)\mathbb{Z})^*$ and for parameters N and T such that $N^{1-o(1)} \ll T \ll N$, the sequence $\bigl(f(\widetilde{W}n + A) - S_f(N; \widetilde{W}, A)\bigr)_{n \leqslant T}$ is orthogonal to any given polynomial nilsequence, provided E is sufficiently large with respect to H, α_f and data related to the nilsequence. In Section 6B we carry out a standard reduction of the main result to an equidistributed version, modeled on [Green and Tao 2012a, §2].

6A. ***Statement of the main result.*** We begin by recalling the definition of a polynomial nilsequence and related notions from [Green and Tao 2012b]. Let G be a connected, simply connected, nilpotent Lie group. By definition, a filtration $G_\bullet$ on G is a finite sequence of closed connected subgroups

$$G = G_0 = G_1 \geqslant G_2 \geqslant \cdots \geqslant G_d \geqslant G_{d+1} = \{\mathrm{id}_G\}$$

with the property that for all pairs (i, j) with $0 \leqslant i, j \leqslant d$, the commutator group $[G_i, G_j]$ is a subgroup of G_{i+j}, where we set $G_{i+j} = \{\mathrm{id}_G\}$ if $i + j > d + 1$. The degree of $G_\bullet$ is defined to be the largest index j for which G_j is nontrivial. Since G is nilpotent, the lower central series, defined by $G_1 = G$ and $G_{i+1} = [G, G_i]$ for $i \geqslant 1$, terminates after finitely many steps. Setting $G_0 = G$, this series defines a filtration. If s denotes the degree of this filtration, then the Lie group G is called s-step nilpotent. One can show that s is the smallest possible degree that a filtration of G can have.

Let $g : \mathbb{Z} \to G$ be a sequence with values in G and define for every $h \in \mathbb{Z}$, the discrete derivative $\partial_h g(n) = g(n+h)g(n)^{-1}$. Then following [Green and Tao 2012b, Definition 1.8], the set $\mathrm{poly}(\mathbb{Z}, G_\bullet)$ of polynomial sequences with coefficients in $G_\bullet$ is defined to be the set of all sequences $g : \mathbb{Z} \to G$ for which every i-th derivative takes values in G_i, i.e., for which $\partial_{h_i} \cdots \partial_{h_1} g(n) \in G_i$ for all $i \in \{0, \dots, d+1\}$ and for all $n, h_1, \dots, h_i \in \mathbb{Z}$.

To define polynomial *nil*sequences, let $\Gamma < G$ be a discrete cocompact subgroup. Then the compact quotient G/Γ is called a nilmanifold. Any Malcev basis $\mathscr{X}$ (see [loc. cit., §2] for a definition) for G/Γ gives rise to a metric $d_{\mathscr{X}}$ on G/Γ as described in [loc. cit., Definition 2.2]. This metric allows us to define Lipschitz functions on G/Γ as the set of functions $F : G/\Gamma \to \mathbb{C}$ for which the Lipschitz norm (see [loc. cit., Definition 1.2])

$$\|F\|_{\mathrm{Lip}} = \|F\|_\infty + \sup_{x,y \in G/\Gamma} \frac{|F(x) - F(y)|}{d_{\mathscr{X}}(x, y)}$$

is finite. If F is a 1-bounded Lipschitz function, then $(F(g(n)\Gamma))_{n\in\mathbb{Z}}$ is called a (polynomial) nilsequence.

We are now ready to state the main result:

Theorem 6.1. *Let $E, H, d, m_G \geqslant 1$ be integers and let $f \in \mathscr{F}_{H,n^{it}}$. Let N be a positive integer parameter and let $\widetilde{W} = \widetilde{W}(N)$ be the integer produced by Proposition 5.1 for the function f when applied with the given values of E, H and with $x = N$. Let $A \in \mathbb{N}$ be such that $0 < A < \widetilde{W}$ and $\gcd(\widetilde{W}, A) = 1$. Suppose further that T satisfies $N/(\log N)^{E/2} \ll T \ll N$ and that $T, N > e^e$. Let G/Γ be a nilmanifold of dimension m_G together with a filtration $G_\bullet$ of G of degree d and let $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ a polynomial sequence. Suppose that G/Γ has a M_0-rational Malcev basis adapted to $G_\bullet$ for some $M_0 \geqslant 2$ and let G/Γ be equipped with the metric defined by this basis. Let $F : G/\Gamma \to \mathbb{C}$ be a 1-bounded Lipschitz function. Then, provided $E \geqslant 1$ is sufficiently large with respect to d, m_G, α_f and H, we have*

$$\begin{aligned} \biggl| \frac{\widetilde{W}}{T} \sum_{n \leqslant T/\widetilde{W}} \bigl(f(\widetilde{W}n + A) - (\widetilde{W}n + A)^{it_N} S_{f(n)n^{-it_N}}(N; \widetilde{W}, A) \bigr) F(g(n)\Gamma) \biggr| &\ll_{d,m_G,\alpha_f,H} \\ \biggl\{ \varphi'(N) + \frac{1}{\log w(N)} + \frac{M_0^{O_{d,m_G}(1)}}{(\log\log T)^{1/(4^{d+1}\dim G)}} \biggr\} \frac{1 + \|F\|_{\mathrm{Lip}}}{\log T} \frac{\widetilde{W}}{\phi(\widetilde{W})} \prod_{\substack{p \leqslant N \\ p \nmid \widetilde{W}(N)}} \biggl(1 + \frac{|f(p)|}{p} \biggr), \end{aligned} \tag{6-1}$$

where $t_N \in [-2\log N, 2\log N]$ is, as in Proposition 5.1, given by Definition 1.6 with $C = 2E + \kappa + 4$ (in particular, $t_N = 0$ if $f \in \mathscr{F}_H$), and where φ' is given by (5-2).

Remark. Partial summation, when combined with the estimate (1-5), which holds with the same value of C as above for the function $n \mapsto f(n)n^{-it_N}$, shows that

$$S_{f(n)n^{-it_N}}(N;\widetilde{W},A)=(1+it_N)N^{-it_N}S_f(N;\widetilde{W},A)\\ +O\biggl(|t_N|(1+|t_N|)\biggl((\log N)^{-E+O(H)}+\frac{\varphi_C(N)}{\log N}\frac{\widetilde{W}}{\phi(\widetilde{W})}\prod_{p<N,\,p\nmid\widetilde{W}}\Bigl(1+\frac{|f(p)|}{p}\Bigr)\biggr)\biggr),$$

where φ_C is as in (1-5). Thus, if $E\gg_{H,\alpha_f}1$ is sufficiently large and $|t_N|^2\varphi_C(N)=o(1)$, we may replace the term $(\widetilde{W}n+A)^{it_N}S_{f(n)n^{-it_N}}(N;\widetilde{W},A)$ in the statement above by $(1+it_N)((\widetilde{W}n+A)/N)^{it_N}S_f(N;\widetilde{W},A)$.

6B. ***Reduction of Theorem 6.1 to the equidistributed case.*** Proceeding similarly as in §2 of [Green and Tao 2012a], we will reduce Theorem 6.1 to a special case that involves only equidistributed polynomial sequences. Let us begin by recalling the quantitative notion of equidistribution and total equidistribution for polynomial sequences that was introduced in [Green and Tao 2012b, Definition 1.2].

Definition 6.2. Let G/Γ be a nilmanifold equipped with Haar measure, let $\delta>0$ and let $N\in\mathbb{N}$. A finite sequence $g:\{1,\dots,N\}\to G$ is called δ-equidistributed in G/Γ if

$$\biggl|\frac{1}{N}\sum_{n\leqslant N}F(g(n)\Gamma)-\int_{G/\Gamma}F\biggr|\leqslant\delta\|F\|_{\mathrm{Lip}}$$

for all Lipschitz functions $F:G/\Gamma\to\mathbb{C}$. It is called totally δ-equidistributed if, moreover,

$$\biggl|\frac{1}{\#P}\sum_{n\in P}F(g(n)\Gamma)-\int_{G/\Gamma}F\biggr|\leqslant\delta\|F\|_{\mathrm{Lip}}$$

for all Lipschitz functions $F:G/\Gamma\to\mathbb{C}$ and progressions $P\subset\{1,\dots,N\}$ of length $\#P\geqslant\delta N$.

The tool that makes a reduction to equidistributed polynomial sequences work is the Green and Tao's factorisation theorem [2012b, Theorem 1.19], which we recall for completeness:

Lemma 6.3 (factorization lemma). *Let m and d be positive integers, and let $M_0,N,B>1$ be real numbers. Let G/Γ be an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d. Suppose that $\mathscr{X}$ is an M_0-rational Malcev basis adapted to $G_\bullet$ and let $g\in\mathrm{poly}(\mathbb{Z},G_\bullet)$ be a polynomial sequence. Then there is an integer M with $M_0\leqslant M\ll M_0^{O_{B,m,d}(1)}$, a rational subgroup $G'\subseteq G$, a Malcev basis $\mathscr{X}'$ for G'/Γ' in which each element is an M-rational combination of the elements of $\mathscr{X}$, and a decomposition $g=\varepsilon g'\gamma$ into polynomial sequences $\varepsilon,g',\gamma\in\mathrm{poly}(\mathbb{Z},G_\bullet)$ with the following properties*:

(1) *$\varepsilon:\mathbb{Z}\to G$ is (M,N)-smooth.*[6]

(2) *$g':\mathbb{Z}\to G'$ takes values in G' and the finite sequence $(g'(n)\Gamma')_{n\leqslant T}$ is totally M^{-B}-equidistributed in $G'\Gamma/\Gamma$ using the metric $d_{\mathscr{X}'}$ on $G'\Gamma/\Gamma$.*

(3) *$\gamma:\mathbb{Z}\to G$ is an M-rational sequence*[7] *and the sequence $(\gamma(n)\Gamma)_{n\in\mathbb{Z}}$ is periodic with period at most M.*

[6]The notion of smoothness was defined in [Green and Tao 2012b, Definition 1.18]. A sequence $(\varepsilon(n))_{n\in\mathbb{Z}}$ is said to be (M,N)-*smooth* if both $d_{\mathscr{X}}(\varepsilon(n),\mathrm{id}_G)\leqslant N$ and $d_{\mathscr{X}}(\varepsilon(n),\varepsilon(n-1))\leqslant M/N$ hold for all $1\leqslant n\leqslant N$.

[7]A sequence $\gamma:\mathbb{Z}\to G$ is said to be M-*rational* if for each n there is $0<r_n\leqslant M$ such that $(\gamma(n))^{r_n}\in\Gamma$; see [Green and Tao 2012b, Definition 1.17].

The following proposition handles the special case of Theorem 6.1 where the polynomial sequence is equidistributed.

Proposition 6.4 (noncorrelation, equidistributed case). *Let E, H, m_G, $d \geqslant 1$ be integers and suppose that $f \in \mathscr{M}_H$. Let N and T be integer parameters satisfying $N^{1-o(1)} \ll T \ll N$ and let $\delta = \delta(N) \in \left(0, \frac{1}{2}\right)$ depend on N in such a way that*

$$\log N \leqslant \delta(N)^{-1} \leqslant (\log N)^E.$$

Let G/Γ be a nilmanifold of dimension m_G together with a filtration $G_\bullet$ of degree d, and suppose that $\mathscr{X}$ is a $1/\delta(N)$-rational Malcev basis adapted to $G_\bullet$. This basis gives rise to the metric $d_{\mathscr{X}}$. Let $Q = Q(N) \leqslant (\log N)^E$ be an integer that is divisible by $W(N)$ and let $0 \leqslant A < Q$ be an integer such that $A \in (\mathbb{Z}/Q\mathbb{Z})^*$.

Then there is $E_0 \geqslant 1$, depending on d, m_G and H, such that the following holds provided E is sufficiently large with respect to d, m_G and H:

Let $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ be any polynomial sequence such that the finite sequence

$$(g(n)\Gamma)_{n \leqslant T/Q}$$

is totally $\delta(N)^{E_0}$-equidistributed. Let $F : G/\Gamma \to \mathbb{C}$ be any 1-bounded Lipschitz function such that $\int_{G/\Gamma} F = 0$, and let $I \subset \{1, \dots, T/Q\}$ be any discrete interval of length at least $T/(Q(\log N)^E)$. Then

$$\left|\frac{Q}{T}\sum_{n\in I} f(Qn+A)F(g(n)\Gamma)\right| \ll_{d,m_G,\alpha_f,H,E} \left\{(\log\log T)^{-1/(2^{2d+3}\dim G)} + \frac{\delta(N)^{-10^d \dim G}}{(\log\log T)^{1/2^{d+2}}}\right\}\frac{1+\|F\|_{\mathrm{Lip}}}{\log N}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant N\\ p\nmid Q}}\left(1+\frac{|f(p)|}{p}\right). \tag{6-2}$$

Proof of Theorem 6.1 assuming Proposition 6.4. We loosely follow the strategy of [Green and Tao 2012a, §2]. In view of the final error term in (6-1), we may assume that $M_0 \leqslant \log N$, as the theorem holds trivially otherwise. This implies that $\mathscr{X}$ is a $(\log N)$-rational Malcev basis. Applying the factorization lemma from above with T replaced by $T/\widetilde{W}$, with $M_0 = \log N$, and with a parameter $B > 1$ that will be determined in course of the proof (as parameter E_0 in an application of Proposition 6.4), we obtain a factorization of g as $\varepsilon g'\gamma$ with properties (1)–(3) from Lemma 6.3. In particular, there is M such that $\log N \leqslant M \leqslant (\log N)^{O_{B,m_G,d}(1)}$ and such that g' takes values in a M-rational subgroup G' of G and is M^{-B}-equidistributed in $G'\Gamma/\Gamma$. Our first aim is to decompose the summation range of n in (6-1) into subprogressions on which the three functions γ, ε and $(\widetilde{W}n+A)^{it_N}$ are all almost constant.

Since γ is periodic with some period $a \leqslant M$, the function $n \mapsto \gamma(an+b)$ is constant for every b, that is, γ is constant on every progression

$$P_{a,b} := \{n \in [1, T/\widetilde{W}] : n \equiv b \pmod{a}\},$$

where $0 \leqslant b < a$. Let γ_b denote the value that γ takes on $P_{a,b}$ and note that

$$|P_{a,b}| \geqslant T/(2a\widetilde{W}) \geqslant T/(2M\widetilde{W}).$$

Let $g'_{a,b} : \mathbb{Z} \to G'$ be defined via

$$g'_{a,b}(n) = g'(an+b).$$

Since $(g'(n)\Gamma)_{n \leqslant T/\widetilde{W}}$ is totally M^{-B}-equidistributed in $G'\Gamma/\Gamma$, it is clear that every finite subsequence $(g'_{a,b}(n)\Gamma)_{n \leqslant T/(Ca\widetilde{W})}$ is $M^{-B/2}$-equidistributed if a, b and C are such that both $0 \leqslant b < a \leqslant M$ and $C > 0$ and, furthermore, $M^{B/2} > Ca$ hold.

Let $R \geqslant 1$ be an integer that will be chosen later depending on d and $\dim G$. By splitting each progression $P_{a,b}$ into $\ll M(\log\log N)^{1/R}$ pieces $P^{(j)}_{a,b}$ of diameter bounded by $\ll T/(M\widetilde{W}(\log\log N)^{1/R})$, we may also arrange for ε and, simultaneously, for $(\widetilde{W}n + A)^{it_N}$ to be almost constant. More precisely, the fact that ε is $(M, T/\widetilde{W})$-smooth implies that

$$d_{\mathscr{X}}(\varepsilon(n), \varepsilon(n')) \leqslant |n - n'| M\widetilde{W}T^{-1} \ll (\log\log N)^{-1/R}$$

for all $n, n' \leqslant T/\widetilde{W}$ with $|n - n'| \ll T/(M\widetilde{W}(\log\log N)^{1/R})$. By choosing B sufficiently large, we may ensure that $M^{B/2} \geqslant M \log\log N$ and, hence, that the equidistribution properties of $g'_{a,b}$ are preserved on the new bounded diameter pieces of $P_{a,b}$. Let $\mathscr{P}$ denote the collection of all progressions $P^{(j)}_{a,b}$ in our decomposition.

Since F is a Lipschitz function and since $d_{\mathscr{X}}$ is right-invariant (see [Green and Tao 2012b, Appendix A]), we deduce that

$$\begin{aligned}\bigl|F(\varepsilon(n)g'(n)\gamma(n)) - F(\varepsilon(n')g'(n)\gamma(n))\bigr| &\leqslant (1 + \|F\|)d(\varepsilon(n), \varepsilon(n')) \\ &\ll (1 + \|F\|)(\log\log N)^{-1/R} \end{aligned} \tag{6-3}$$

for all $n, n' \in P_{a,b}$ with $|n - n'| \ll \dfrac{T}{M\widetilde{W}(\log\log N)^{1/R}}$. Thus, this bound holds in particular for any $n, n' \in P^{(j)}_{a,b}$.

To ensure that $(\widetilde{W}n + A)^{it_N}$ is almost constant on the bounded parameter progressions $P^{(j)}_{a,b}$ that we consider, let $\mathscr{P}' \subset \mathscr{P}$ denote the subset of progressions $P^{(j)}_{a,b}$ that are completely contained in the interval $[T/(\widetilde{W}(\log\log N)^{1/(2R)}), T/\widetilde{W}]$. Observe that the contribution of all other progressions $P^{(j)}_{a,b} \in \mathscr{P} \setminus \mathscr{P}'$ to (6-1) may be bounded by

$$(\log\log N)^{-1/(2R)} \frac{\widetilde{W}(N)}{\phi(\widetilde{W}(N))} \frac{2}{\log T} \prod_{\substack{p \leqslant N \\ p \nmid \widetilde{W}(N)}} \Bigl(1 + \frac{|f(p)|}{p}\Bigr),$$

where we used Shiu's bound (3-1) together with fact that we are only summing over $n \leqslant \dfrac{T}{\widetilde{W}(\log\log N)^{1/(2R)}}$. Since

$$\frac{\log\log N}{\log\log\log N} < w(N) \leqslant \log\log N$$

and since $1 \leqslant R \ll_{d,\dim G} 1$, we have

$$(\log\log N)^{-1/(2R)} \ll_{d,\dim G} (\log w(N))^{-1}, \tag{6-4}$$

which implies that the above contribution is negligible when compared to the bound in (6-1). For every remaining progression $P_{a,b}^{(j)} \in \mathscr{P}'$, the diameter is now short compared to the size of the endpoints and we have

$$\begin{aligned}\log(\widetilde{W}n + A) &= \log(\widetilde{W}n' + A) + \log \frac{\widetilde{W}(n'+n-n')+A}{\widetilde{W}n'+A}\\ &= \log(\widetilde{W}n' + A) + \log\biggl(1 + O\biggl(\frac{1}{M(\log\log N)^{1/(2R)}}\biggr)\biggr)\\ &= \log(\widetilde{W}n' + A) + O\biggl(\frac{1}{M(\log\log N)^{1/(2R)}}\biggr)\end{aligned}$$

for all $n, n' \in P_{a,b}^{(j)}$. Since $|t_N| \leqslant 2\log N$ and $M \geqslant \log N$, we deduce that

$$\begin{aligned}(\widetilde{W}n + A)^{it_N} &= (\widetilde{W}n' + A)^{it_N} \exp\biggl(O\biggl(\frac{\log N}{M(\log\log N)^{1/(2R)}}\biggr)\biggr)\\ &= (\widetilde{W}n' + A)^{it_N}(1 + O((\log\log N)^{-1/(2R)}))\\ &= (\widetilde{W}n' + A)^{it_N} + O((\log\log N)^{-1/(2R)})\end{aligned} \tag{6-5}$$

for all $n, n' \in P_{a,b}^{(j)}$.

Let us fix one element $n_{b,j}$ for each progression $P_{a,b}^{(j)} \in \mathscr{P}'$. As we will show next, it will be sufficient to bound the correlation

$$\begin{aligned}&\biggl|\sum_{n\in P_{a,b}^{(j)}} \bigl(f(\widetilde{W}n + A) - (\widetilde{W}n_{b,j} + A)^{it_N} S_{f(n)n^{-it_N}}(N;\, \widetilde{W}, A)\bigr) F(\varepsilon(n_{b,j})g'(n)\gamma_b\Gamma)\biggr|\\ &= \biggl|\sum_{\substack{n:\\ an+b\in P_{a,b}^{(j)}}} \bigl(f(\widetilde{W}(an+b) + A) - (\widetilde{W}n_{b,j} + A)^{it_N} S_{f(n)n^{-it_N}}(N;\, \widetilde{W}, A)\bigr) F(\varepsilon(n_{b,j})g'_{a,b}(n)\gamma_b\Gamma)\biggr|\end{aligned} \tag{6-6}$$

for each bounded diameter piece $P_{a,b}^{(j)} \in \mathscr{P}'$. Indeed, the estimates (6-3) and (6-5) applied with $n' = n_{b,j}$ to each such progression, show that the error term incurred from this reduction satisfies

$$\begin{aligned}&\biggl|\sum_{P_{a,b}^{(j)}\in\mathscr{P}'} \sum_{n\in P_{a,b}^{(j)}} \bigl\{\bigl(f(\widetilde{W}n + A) - (\widetilde{W}n + A)^{it_N} S_{f(n)n^{-it_N}}(N;\, \widetilde{W}, A)\bigr) F(\varepsilon(n)g'(n)\gamma(n)\Gamma)\\ &\qquad\qquad - \bigl(f(\widetilde{W}n + A) - (\widetilde{W}n_{b,j} + A)^{it_N} S_{f(n)n^{-it_N}}(N;\, \widetilde{W}, A)\bigr) F(\varepsilon(n_{b,j})g'(n)\gamma_b\Gamma)\bigr\}\biggr|\\ &\quad\leqslant \sum_{P_{a,b}^{(j)}\in\mathscr{P}'} \sum_{n\in P_{a,b}^{(j)}} |f(\widetilde{W}n + A)|\, \bigl|F(\varepsilon(n)g'(n)\gamma(n)\Gamma) - F(\varepsilon(n_{b,j})g'(n)\gamma_b\Gamma)\bigr|\\ &\qquad + \sum_{P_{a,b}^{(j)}\in\mathscr{P}'} \sum_{n\in P_{a,b}^{(j)}} |(\widetilde{W}n + A)^{it_N}|\, \bigl|F(\varepsilon(n)g'(n)\gamma(n)\Gamma) - F(\varepsilon(n_{b,j})g'(n)\gamma_b\Gamma)\bigr| S_{|f|}(N;\, \widetilde{W}, A)\\ &\qquad + \sum_{P_{a,b}^{(j)}\in\mathscr{P}'} \sum_{n\in P_{a,b}^{(j)}} |(\widetilde{W}n + A)^{it_N} - (\widetilde{W}n_{b,j} + A)^{it_N}|\, \bigl|F(\varepsilon(n_{b,j})g'(n)\gamma_b\Gamma)\bigr| S_{|f|}(N;\, \widetilde{W}, A)\end{aligned}$$

$$\ll \frac{T}{\widetilde{W}(N)} \frac{(1+\|F\|)S_{|f|}(N;\widetilde{W},A)}{(\log\log N)^{1/R}}.$$

By Shiu's bound (3-1), this in turn is bounded above by

$$\ll \frac{T}{\widetilde{W}(N)} \frac{(1+\|F\|)}{(\log\log N)^{1/R}} \frac{\widetilde{W}(N)}{\phi(\widetilde{W}(N))} \frac{1}{\log N} \exp\biggl(\sum_{w(N)<p\leqslant N} \frac{|f(p)|}{p}\biggr). \tag{6-7}$$

Taking into account (6-4), the error term (6-7) is acceptable in view of the bound in (6-1).

We aim to estimate the correlation (6-6) with the help of Proposition 6.4. This task will be carried out in four steps, the first of which will be to bound the contribution from noninvertible residues $\widetilde{W}b+A \pmod{\widetilde{W}a}$ to which Proposition 6.4 does not apply. The two subsequent steps consist of checking the various assumptions of Proposition 6.4, while the fourth step contains the actual application of the proposition.

Before we start, we record a final estimate that will be used throughout the rest of the proof. Note that the common difference of $P^{(j)}_{a,b}$ satisfies $a \leqslant M \ll (\log N)^{O_{d,m_G,B}(1)}$, which is bounded above by $(\log N)^E$, provided E is sufficiently large in terms of d, m_G and B.

Step 1: *Noninvertible residues.* We seek to bound the contribution to (6-1) of all progressions $P^{(j)}_{a,b} \in \mathscr{P}'$ with $\gcd(\widetilde{W}b+A,\widetilde{W}a) > 1$. Let $a' = \prod_{p\nmid \widetilde{W}(N)} p^{v_p(a)}$, so that $\widetilde{W}(N)$ is invertible modulo a'. Since $\gcd(A,\widetilde{W}) = 1$, it suffices to check whether b satisfies $\gcd(\widetilde{W}b+A,a') > 1$. Thus, the contribution we seek to bound takes the form

$$\frac{\widetilde{W}}{T} \sum_{d|a',\, d>1} \sum_{\substack{b<a:\\ \gcd(\widetilde{W}b+A,a')=d}} \sum_{\substack{n<T/\widetilde{W}\\ n\equiv b \pmod a}} \{|f(\widetilde{W}n+A)| + S_{|f|}(N;\widetilde{W},A)\}.$$

The contribution from the terms involving $S_{|f|}(N;\widetilde{W},A)$ is bounded by

$$\begin{aligned} &\ll S_{|f|}(N;\widetilde{W},A) \sum_{d|a',\, d>1} \sum_{\substack{b<a:\\ \gcd(\widetilde{W}b+A,a')=d}} \frac{1}{a} \\ &\ll S_{|f|}(N;\widetilde{W},A) \sum_{d|a',\, d>1} \frac{1}{a}\frac{a}{a'}\phi\Bigl(\frac{a'}{d}\Bigr) \ll S_{|f|}(N;\widetilde{W},A) \sum_{d|a',\, d>1} \frac{1}{d}, \end{aligned} \tag{6-8}$$

where we used the fact that $\widetilde{W}(N)$ is invertible modulo a'. In a similar fashion, we may bound the contribution from those terms involving $|f(\widetilde{W}n+A)|$ as follows:

$$\frac{\widetilde{W}}{T} \sum_{d|a',\, d>1} \sum_{\substack{b<a:\\ \gcd(\widetilde{W}b+A,a')=d}} \sum_{\substack{n<T/\widetilde{W}\\ n\equiv b \pmod a}} |f(\widetilde{W}n+A)| \leqslant \sum_{d|a',\, d>1} \sum_{\substack{b<a:\\ \gcd(\widetilde{W}b+A,a')=d}} \frac{|f(d)|}{a} S_{|f|}\Bigl(\frac{T}{d};\frac{\widetilde{W}a}{d},\frac{\widetilde{W}b+A}{d}\Bigr)$$

$$\begin{aligned}
&\leqslant \sum_{d|a',\, d>1} \frac{|f(d)|}{a}\frac{a}{a'}\phi\Big(\frac{a'}{d}\Big)\, S_{|f|}\Big(\frac{T}{d};\, \frac{\widetilde{W}a}{d},\, \frac{\widetilde{W}b+A}{d}\Big)\\
&\ll \sum_{d|a',\, d>1} \frac{|f(d)|}{a'}\phi\Big(\frac{a'}{d}\Big)\frac{\widetilde{W}a/d}{\phi(\widetilde{W}a/d)}\,\frac{1}{\log(T/d)}\exp\Big(\sum_{\substack{p<T/d\\ p\nmid \widetilde{W}a'/d}}\frac{|f(p)|}{p}\Big)\\
&\leqslant \sum_{d|a',\, d>1} \frac{|f(d)|}{a'}\phi\Big(\frac{a'}{d}\Big)\frac{a'/d}{\phi(a'/d)}\,\frac{\widetilde{W}}{\phi(\widetilde{W})}\,\frac{1}{\log(T/d)}\exp\Big(\sum_{\substack{p<T\\ p\nmid \widetilde{W}}}\frac{|f(p)|}{p}\Big)\\
&\ll \frac{\widetilde{W}}{\phi(\widetilde{W})}\,\frac{1}{\log T}\exp\Big(\sum_{\substack{p<T\\ p\nmid \widetilde{W}}}\frac{|f(p)|}{p}\Big)\sum_{d|a',\, d>1}\frac{|f(d)|}{d},
\end{aligned}$$

where we made use of (3-1) and of the fact that $d \leqslant a \leqslant (\log N)^E$ so that $\log(T/d) \geqslant \frac{1}{2}\log T$ once N and, hence, T are sufficiently large. Observe that the final sums in each of the two bounds above are similar. We restrict attention to bounding the latter of them. Assuming that the lower bound $w(N)$ on prime divisors $p \mid a'$ is sufficiently large with respect to H, we have

$$\begin{aligned}
\sum_{d|a',\, d>1}\frac{|f(d)|}{d} &\leqslant \prod_{p|a'}\Big(1+\frac{H}{p}+\frac{H^2}{p^2}+\cdots\Big)-1 \leqslant \prod_{p|a'}\Big(1+\frac{H}{p}\Big)\Big(1+\frac{H^2}{p^2(1-\frac{H}{p})}\Big)-1\\
&\leqslant \exp\Big(\sum_{p|a'}\frac{2H^2}{p^2}\Big)\prod_{p|a'}\Big(1+\frac{H}{p}\Big)-1 \leqslant \Big(1+\frac{4H^2}{w(N)}\Big)\prod_{p|a'}\Big(1+\frac{H}{p}\Big)-1\\
&\ll_{E,H} \frac{4H^2}{w(N)}+\frac{1}{\log w(N)} \ll_{E,H} \frac{1}{\log w(N)},
\end{aligned}$$

where we applied Lemma 5.3 with a replaced by a' to estimate the product over $p \mid a'$.

Bounding the inner sum in (6-8) in a similar fashion and applying (3-1) to estimate $S_{|f|}(N;\, \widetilde{W},\, A)$, we deduce that the total contribution of noninvertible residues $\widetilde{W}b + A \pmod{\widetilde{W}a}$ to (6-1) is at most

$$O_{d,m_G,B,H}\Big(\frac{1}{\log w(N)}\frac{\widetilde{W}}{\phi(\widetilde{W})}\frac{1}{\log T}\exp\Big(\sum_{w(N)<p<T}\frac{|f(p)|}{p}\Big)\Big),$$

which has been taken care of in (6-1). This leaves us to considering the case where the value of b does not impose an obstruction to applying Proposition 6.4.

Step 2: *Checking the initial conditions of Proposition 6.4.* The central assumption of Proposition 6.4 concerns the equidistribution of the polynomial sequence it is applied to. To verify this assumption for the sequence that appears in (6-6), it is necessary to show that the conjugated sequence $h^* : n \mapsto \gamma_b^{-1} g'_{a,b}(n)\gamma_b$ is, in fact, a polynomial sequence and that it inherits the equidistribution properties of $g'_{a,b}(n)$. Both these questions have been addressed in [Green and Tao 2012a, §2] in a way we can directly build on:

Let $H = \gamma_b^{-1} G' \gamma_b$ and define $H_\bullet = \gamma_b^{-1}(G')_\bullet \gamma_b$. Let $\Lambda = \Gamma \cap H$ and define $F_{b,j} : H/\Lambda \to \mathbb{R}$ via

$$F_{b,j}(x\Lambda) = F(\varepsilon(n_{b,j})\gamma_b x\Gamma).$$

Then $h^* \in \mathrm{poly}(\mathbb{Z}, H_\bullet)$ and the correlation (6-6) that we seek to bound takes the form

$$\left|\sum_n \bigl(f(\widetilde{W}(an+b)+A) - (\widetilde{W}n_{b,j}+A)^{it_N} S_{f(n)n^{-it_N}}(N; \widetilde{W}, A)\bigr) F_{b,j}(h^*(n)\Lambda)\right|, \tag{6-9}$$

where the sum is over the n such that $(an+b) \in P_{a,b}^{(j)}$. The Claim from the end of [Green and Tao 2012a, §2] guarantees the existence of a Malcev basis $\mathscr{Y}$ for H/Λ adapted to $H_\bullet$ such that each basis element Y_i is a $M^{O(1)}$-rational combination of basis elements X_i. Thus, there is $C' = O(1)$ such that $\mathscr{Y}$ is $M^{C'}$-rational. Furthermore, it implies that there is $c' > 0$, depending only on the dimension of G and the degree of $G_\bullet$, such that whenever B is sufficiently large the sequence

$$(h^*(n)\Lambda)_{n \leqslant T/(a\widetilde{W})} \tag{6-10}$$

is totally $M^{-c'B/2+O(1)}$-equidistributed in H/Λ, equipped with the metric $d_{\mathscr{Y}}$ induced by $\mathscr{Y}$. Taking B sufficiently large, we may assume that the sequence (6-10) is totally $M^{-c'B/4}$-equidistributed. Finally, the "Claim" also provides the bound $\|F_{b,j}\|_{\mathrm{Lip}} \leqslant M^{C''} \|F\|_{\mathrm{Lip}}$ for some $C'' = O(1)$. This shows that all conditions of Proposition 6.4 are satisfied except for $\int_{H/\Lambda} F_{b,j} = 0$.

Step 3: *The final condition.* The final condition that needs to be arranged for before we can apply Proposition 6.4 to (6-9) is that $\int_{H/\Lambda} F_{b,j} = 0$. This is where the major arc condition (5-2) is needed, which in turn requires that $\gcd(\widetilde{W}b + A, \widetilde{W}a) = 1$. To ensure that the integral over the test function is zero, we decompose $F_{b,j}(x\Lambda)$ as $(F_{b,j}(x\Lambda) - \mu_{b,j}) + \mu_{b,j}$, where $\mu_{b,j} := \int_{H/\Lambda} F_{b,j}$. The expression in parentheses represents a new test function that we can apply the proposition with, and we will show next that the contribution from the constant term $\mu_{b,j}$ is small provided $f(\widetilde{W}n + A) - (\widetilde{W}n_{b,j}+A)^{it_N} S_{f(n)n^{-it_N}}(N; \widetilde{W}, A)$ does not correlate with the characteristic function $\mathbf{1}_{P_{a,b}^{(j)}}$ of the corresponding progression $P_{a,b}^{(j)}$.

To start with, recall that $T \geqslant N/(\log N)^{E/2}$, that the common difference of $P_{a,b}^{(j)}$ satisfies $a \leqslant (\log N)^E$, and that the length of $P_{a,b}^{(j)}$ is bounded below by

$$\begin{aligned} |P_{a,b}^{(j)}| \geqslant T/(2aM\widetilde{W}(\log\log N)^{1/R}) &\gg T/(a\widetilde{W}(\log N)^{E/2}) \\ &\gg N/(a\widetilde{W}(\log N)^E), \end{aligned}$$

provided E is sufficiently large in terms of d, m_G and B. Observe that condition (5-2) applies to the function $n \mapsto f(n)n^{-it_N}$ and to all discrete intervals $I \subset \{1, \dots, T/\widetilde{W}\}$ of length $|I| \gg T/(\log T)^E$. In particular, we may choose $q = a$, $r = b$ and let I be a discrete interval of length $a\widetilde{W}(N)|P_{a,b}^{(j)}|$ that contains the set $\{\widetilde{W}(N)m + A : m \in P_{a,b}^{(j)}\}$. To relate $f(n)$ to $f(n)n^{-it_N}$, we observe that (6-5) and (6-4) imply that

$$f(\widetilde{W}n + A) = (\widetilde{W}n_{b,j} + A)^{it_N} f(\widetilde{W}n + A)(\widetilde{W}n + A)^{-it_N} + O\biggl(\frac{|f(\widetilde{W}n + A)|}{\log w(N)}\biggr)$$

for all $n, n_{b,j} \in P_{a,b}^{(j)} \in \mathscr{P}'$.

By applying condition (5-2) to the main term below and Shiu's bound (3-1) in combination with Corollary 5.4 to the error term, we obtain the following uniform estimate valid for all $P^{(j)}_{a,b} \in \mathscr{P}'$:

$$\begin{aligned}
&\frac{1}{|P^{(j)}_{a,b}|} \sum_{m \in P^{(j)}_{a,b}} f(\widetilde{W}m + A) \\
&= (\widetilde{W}m_{b,j} + A)^{it_N} \frac{a\widetilde{W}}{|I|} \sum_{\substack{m \in I \\ m \equiv \widetilde{W}b + A\ (a\widetilde{W})}} f(m)m^{-it_N} + O\biggl(\frac{1}{\log w(N)} \frac{a\widetilde{W}}{|I|} \sum_{\substack{m \in I \\ m \equiv \widetilde{W}b + A\ (a\widetilde{W})}} |f(m)|\biggr) \\
&= (\widetilde{W}m_{b,j} + A)^{it_N} S_{f(n)n^{-it_N}}(N; \widetilde{W}, A) + O\biggl(\biggl(\varphi'(N) + \frac{1}{\log w(N)}\biggr) \frac{1}{\log N} \frac{\widetilde{W}}{\phi(\widetilde{W})} \prod_{\substack{p<N \\ p \nmid \widetilde{W}}} \biggl(1 + \frac{|f(p)|}{p}\biggr)\biggr).
\end{aligned}$$

Let, as above, $\mu_{b,j} = \int_{H/\Lambda} F_{b,j}$, and note that $\mu_{b,j} \ll 1$. Thus, the error term incurred by replacing for each $P^{(j)}_{a,b}$ with $\gcd(\widetilde{W}b + A, \widetilde{W}a) = 1$ the factor $F_{b,j}(h(n)\Lambda)$ in (6-9) by $(F_{b,j}(h(n)\Lambda) - \mu_{b,j})$ is bounded as follows:

$$\begin{aligned}
&\biggl| \frac{\widetilde{W}}{T} \sum_{\substack{P^{(j)}_{a,b} \in \mathscr{P}' \\ \gcd(\widetilde{W}b + A, a) = 1}} \mu_{b,j} \sum_{n \in P^{(j)}_{a,b}} \bigl(f(\widetilde{W}n + A) - (\widetilde{W}m_{b,j} + A)^{it_N} S_{f(n)n^{-it_N}}(N; \widetilde{W}, A)\bigr) \biggr| \\
&\qquad \ll \frac{\widetilde{W}}{T} \sum_{P^{(j)}_{a,b} \in \mathscr{P}'} |P^{(j)}_{a,b}| \biggl(\varphi'(N) + \frac{1}{\log w(N)}\biggr) \frac{1}{\log N} \frac{\widetilde{W}}{\phi(\widetilde{W})} \prod_{\substack{p<N \\ p \nmid \widetilde{W}}} \biggl(1 + \frac{|f(p)|}{p}\biggr) \\
&\qquad \ll \biggl(\varphi'(N) + \frac{1}{\log w(N)}\biggr) \frac{1}{\log N} \frac{\widetilde{W}}{\phi(\widetilde{W})} \prod_{\substack{p<N \\ p \nmid \widetilde{W}}} \biggl(1 + \frac{|f(p)|}{p}\biggr),
\end{aligned}$$

where φ' is the function defined in Remarks 5.2. This error term has been taken care of in the bound (6-1).

Step 4: *Application of Proposition 6.4* The application of Proposition 6.4 to (6-9) will give rise to the third error term in (6-1). In view of the work carried out in Steps 1–3, we may now assume that $\gcd(\widetilde{W}b + A, \widetilde{W}a) = 1$ and that $\int_{H/\Lambda} F_{b,j} = 0$ holds, and apply Proposition 6.4 with

- $g = h,\ Q = \widetilde{W}a,\ I = \{n : an + b \in P^{(j)}_{a,b}\}$,
- a function $\delta : \mathbb{N} \to \mathbb{R}$ such that $\delta(N) = M^{-C'} (= M_0^{O_{d,m_G,B}(1)})$, which ensures that $\mathscr{Y}$ is $1/\delta(N)$-rational,
- E sufficiently large to ensure that $M^{C'} < (\log N)^E$, which in particular means that E depends on B, and
- $E_0 = \frac{1}{4}c'B = O_{d,m_G}(B)$ for some value of B that is sufficiently large to ensure that (6-10) is totally $M^{-c'B/4}$-equidistributed in H/Λ (see Step 2) and that is also sufficiently large for Proposition 6.4 to apply with the above choice of E_0.

Since there are $\ll aM(\log\log N)^{1/R}$ intervals $P_{a,b}^{(j)}$ in the decomposition $\mathscr{P}' \subset \mathscr{P}$, this yields the bound

$$\sum_{P_{a,b}^{(j)}\in\mathscr{P}'}\Biggl|\sum_{\substack{n:\,(an+b)\\ \in P_{a,b}^{(j)}}}\bigl(f(\widetilde{W}(an+b)+A)-(\widetilde{W}n_{b,j}+A)^{it_N}S_{f(n)n^{-it_N}}(N;\widetilde{W},A)\bigr)F_{b,j}(h^*(n)\Lambda)\Biggr|$$
$$\ll aM(\log\log N)^{1/R}\frac{1+M^{O(1)}\|F\|}{\log T}\frac{T}{\widetilde{W}a}\frac{\widetilde{W}a}{\phi(\widetilde{W}a)}\prod_{\substack{p\leqslant N\\ p\nmid\widetilde{W}a}}\biggl(1+\frac{|f(p)|}{p}\biggr)\mathscr{N}$$
$$\ll M^{O(1)}(\log\log N)^{1/R}\frac{1+\|F\|}{\log T}\frac{T}{\widetilde{W}}\frac{\widetilde{W}a}{\phi(\widetilde{W}a)}\prod_{\substack{p\leqslant N\\ p\nmid\widetilde{W}}}\biggl(1+\frac{|f(p)|}{p}\biggr)\mathscr{N}, \quad (6\text{-}11)$$

where the implied constant depends on d, m_G, α_f, H and B, and where

$$\mathscr{N}=(\log\log T)^{-1/(2^{2d+3}\dim G)}+\frac{M^{10^d\dim G}}{(\log\log T)^{1/2^{d+2}}}\ll\frac{M^{10^d\dim G}}{(\log\log T)^{1/(2^{2d+3}\dim G)}}.$$

Finally, we invoke Corollary 5.4 to remove the dependence on a from (6-11). We complete the deduction of Theorem 6.1 by setting $R=2^{2d+3}\dim G$ and comparing the bound arising from (6-11) with the third term in (6-1). □

It remains to establish Proposition 6.4.

7. Linear subsequences of equidistributed nilsequences

Our aim in this section is to study the equidistribution properties of families

$$\{(g(Dn+D')\Gamma)_{n\leqslant T/D} : D\in[K,2K)\}$$

of linear subsequences of an equidistributed sequence $(g(n)\Gamma)_{n\leqslant T}$, where D runs through dyadic intervals $[K,2K)$ for $K\leqslant T^{1-1/H}$. This result will only be needed in the case of unbounded multiplicative functions, which allows us to assume that $H>1$ in this section.

We begin by recalling some essential definitions and notation. Let $P:\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ be a polynomial of degree at most d and let $\alpha_0,\dots,\alpha_d\in\mathbb{R}/\mathbb{Z}$ be defined via

$$P(n)=\alpha_0+\alpha_1\binom{n}{1}+\cdots+\alpha_d\binom{n}{d}.$$

Then the *smoothness norm* of g with respect to T is defined (cf. [Green and Tao 2012b, Definition 2.7]) as

$$\|P\|_{C^\infty[T]}=\sup_{1\leqslant j\leqslant d}T^j\|\alpha_j\|_{\mathbb{R}/\mathbb{Z}}.$$

If $\beta_0,\dots,\beta_d\in\mathbb{R}/\mathbb{Z}$ are defined via

$$P(n)=\beta_d n^d+\cdots+\beta_1 n+\beta_0,$$

then, compare with [Matthiesen 2012, equation (14.3)], the smoothness norm is bounded above by a similar expression in terms of the β_i, namely

$$\|P\|_{C^\infty[T]} \ll_d \sup_{1\leqslant j\leqslant d} T^j \|j!\beta_j\|_{\mathbb{R}/\mathbb{Z}} \ll_d \sup_{1\leqslant j\leqslant d} T^j \|\beta_j\|_{\mathbb{R}/\mathbb{Z}}. \tag{7-1}$$

On the other hand, Lemma 3.2 of [Green and Tao 2012b] shows that there is a positive integer $q \ll_d 1$ such that

$$\|q\beta_j\|_{\mathbb{R}/\mathbb{Z}} \ll T^{-j} \|P\|_{C^\infty[T]}.$$

Apart from smoothness norms, we also require the notion of a horizontal character as given in [loc. cit., Definition 1.5]. A continuous additive homomorphism $\eta : G \to \mathbb{R}/\mathbb{Z}$ is called a *horizontal character* if it annihilates Γ. In order to formulate quantitative results, one defines a height function $|\eta|$ for these characters. A definition of this height, called the *modulus* of η, may be found in [Green and Tao 2012b, Definition 2.6]. All that we require to know about these heights is that there are at most $M^{O(1)}$ horizontal characters $\eta : G \to \mathbb{R}/\mathbb{Z}$ of modulus $|\eta| \leqslant M$.

The interest in smoothness norms and horizontal characters lies in Green and Tao's "quantitative Leibman Theorem":

Proposition 7.1 [Green and Tao 2012b, Theorem 2.9]. *Let m_G and d be nonnegative integers, let $0 < \delta < \frac{1}{2}$ and let $N \geqslant 1$. Suppose that G/Γ is an m_G-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d and that $\mathscr{X}$ is a $1/\delta$-rational Malcev basis adapted to $G_\bullet$. Suppose that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. If $(g(n)\Gamma)_{n\leqslant N}$ is not δ-equidistributed, then there is a nontrivial horizontal character η with $0 < |\eta| \ll \delta^{-O_{d,m_G}(1)}$ such that*

$$\|\eta \circ g\|_{C^\infty[N]} \ll \delta^{-O_{d,m_G}(1)}.$$

The following lemma shows that for polynomial sequences the notions of equidistribution and total equidistribution are equivalent with a polynomial dependence in the equidistribution parameter.

Lemma 7.2. *Let N and A be positive integers and let $\delta : \mathbb{N} \to [0, 1]$ be a function that satisfies $\delta(x)^{-t} \ll_t x$ for all $t > 0$. Suppose that G has a $1/\delta(N)$-rational Malcev basis adapted to the filtration $G_\bullet$. Suppose that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ is a polynomial sequence such that $(g(n)\Gamma)_{n\leqslant N}$ is $\delta(N)^A$-equidistributed. Then there is $1 \leqslant B \ll_{d,m_G} 1$ such that $(g(n)\Gamma)_{n\leqslant N}$ is totally $\delta(N)^{A/B}$-equidistributed, provided $A/B > 1$ and provided N is sufficiently large.*

Remark 7.3. The Green–Tao factorization theorem (see property (2) of Lemma 6.3) usually allows one to arrange for $A > B$ to hold.

Proof. We allow all implied constants to depend on d and m_G. Let $B \geqslant 1$ and suppose that $(g(n)\Gamma)_{n\leqslant N}$ fails to be totally $\delta(N)^{A/B}$-equidistributed. Then there is a subprogression $P = \{\ell n + b : 0 \leqslant n \leqslant m - 1\}$ of $\{1, \dots, N\}$ of length $m > \delta(N)^{A/B} N$ such that the sequence $(\tilde{g}(n))_{0\leqslant n<m}$, where $\tilde{g}(n) = g(\ell n + b)$, fails to be $\delta(N)^{A/B}$-equidistributed. Provided $A > B$, Proposition 7.1 implies that there is a nontrivial horizontal character $\eta : G \to \mathbb{R}/\mathbb{Z}$ of modulus $|\eta| < \delta(N)^{-O(A/B)}$ such that

$$\|\eta \circ \tilde{g}\|_{C^\infty[m]} \ll \delta(N)^{-O(A/B)}.$$

The lower bound on m implies that this is equivalent to the assertion

$$\|\eta \circ \tilde{g}\|_{C^\infty[N]} \ll \delta(N)^{-O(A/B)},$$

where we recall that the implied constant may depend on d.

Observing that $\eta \circ g$ is a polynomial of degree at most d, let $\eta \circ g(n) = \beta_d n^d + \cdots + \beta_0$. Then

$$\eta \circ \tilde{g}(n) = \sum_{i=0}^{d} n^i \sum_{j=i}^{d} \beta_j \binom{j}{i} \ell^i b^{j-i},$$

and, hence,

$$\sup_{1 \leqslant i \leqslant d} N^i \left\| \sum_{j=i}^{d} \beta_j \binom{j}{i} \ell^i b^{j-i} \right\| \ll \delta(N)^{-O(A/B)}.$$

This yields the bound

$$\left\| \sum_{j=i}^{d} \beta_j \binom{j}{i} \ell^i b^{j-i} \right\| \ll N^{-i} \delta(N)^{-O(A/B)} \tag{7-2}$$

for $1 \leqslant i \leqslant d$. Note that the lower bound on m implies that $\ell < \delta(N)^{-A/B}$. Using a downwards induction argument, we aim to show that

$$\|\ell^d \beta_j\| \ll N^{-j} \delta(N)^{-O(A/B)} \tag{7-3}$$

for all $1 \leqslant j \leqslant d$. For $j = d$, this is clear from the above. Suppose (7-3) holds for all $j > i$. For each $i < j$ we then, in particular, have that

$$\left\| \ell^d \beta_j \binom{j}{i} b^{j-i} \right\| \ll_d \|\ell^d \beta_j\| b^{j-i} \ll_d N^{-j} \delta(N)^{-O(A/B)} b^{j-i} \ll_d N^{-i} \delta(N)^{-O(A/B)}.$$

Using the fact that $\delta(N)^{-t} \ll_t N$ for all $t > 0$, we deduce that (7-3) holds for $j = i$ from the above bounds and from (7-2). This shows that there is a nontrivial horizontal character, namely $\ell^d \eta$, of modulus at most $\delta(N)^{-O(A/B)}$, such that

$$\|\ell^d \eta \circ g\|_{C^\infty[N]} \ll \sup_{1 \leqslant i \leqslant d} N^i \|\ell^d \beta_i\|_{\mathbb{R}/\mathbb{Z}} \ll \delta(N)^{-O(A/B)},$$

where we made use of (7-1). Choosing B sufficiently large in terms of m and d, [Matthiesen 2012, Proposition 14.2(b)] implies that g is not $\delta(N)^A$-equidistributed, which is a contradiction. □

We are now ready to address the equidistribution properties of linear subsequences.

Proposition 7.4. *Let $H > 1$, let N and T be as before and let $E_1 \geqslant 1$. Let $(A_D)_{D \in \mathbb{N}}$ be a sequence of integers such that $|A_D| \leqslant D$ for every $D \in \mathbb{N}$. Further, let $\delta : \mathbb{N} \to (0, 1)$ be a function that satisfies $\delta(x)^{-t} \ll_t x$ for all $t > 0$. Suppose G/Γ has a $1/\delta(N)$-rational Malcev basis adapted to a filtration $G_\bullet$ of* *degree d. Let $g \in \mathrm{poly}(G_\bullet, \mathbb{Z})$ be a polynomial sequence and suppose that the finite sequence $(g(n)\Gamma)_{n\leqslant T}$ is totally $\delta(T)^{E_1}$-equidistributed in G/Γ. Then there is a constant $c_1 \in (0,1)$, depending only on d and $m_G := \dim G$, such that the following assertion holds for all integers $K \in [(\log T)^{\log\log T}, T^{1-1/H}]$, provided $c_1 E_1 \geqslant 1$.*

Write $g_D(n) = g(Dn + A_D)$ and let $\mathscr{B}_K$ denote the set of integers $D \in [K, 2K)$ for which

$$(g_D(n)\Gamma)_{n\leqslant T/D}$$

fails to be totally $\delta(T)^{c_1 E_1}$-equidistributed. Then

$$\#\mathscr{B}_K \ll K\delta(T)^{c_1 E_1}.$$

Proof. Let $K \in [(\log T)^{\log\log T}, T^{1-1/H}]$ be a fixed integer and let $c_1 > 0$ to be determined in the course of the proof. Suppose that $E_1 > 1/c_1$. Lemma 7.2 implies that for every $D \in \mathscr{B}_K$, the sequence $(g_D(n)\Gamma)_{n\leqslant T/D}$ fails to be $\delta(T)^{c_1 E_1 B}$-equidistributed on G/Γ for some $B > 0$ only depending on d and m_G. We continue to allow implied constants to depend on d and m_G. By Proposition 7.1, there is a nontrivial horizontal character $\eta_D : G \to \mathbb{R}/\mathbb{Z}$ of modulus $|\eta_D| \ll \delta(T)^{-O(c_1 E_1)}$ such that

$$\|\eta_D \circ g_D\|_{C^\infty[T/D]} \ll \delta(T)^{-O(c_1 E_1)}. \tag{7-4}$$

For each nontrivial horizontal character $\eta : G \to \mathbb{R}/\mathbb{Z}$ we define the set

$$\mathscr{D}_\eta = \{D \in \mathscr{B}_K : \eta_D = \eta\}.$$

Note that this set is empty unless $|\eta| \ll \delta(T)^{-O(c_1 E_1)}$. Suppose that

$$\#\mathscr{B}_K \geqslant K\delta(T)^{c_1 E_1}.$$

By the pigeonhole principle, there is some η of modulus $|\eta| \ll \delta(T)^{-O(c_1 E_1)}$ such that

$$\#\mathscr{D}_\eta \geqslant K\delta(T)^{O(c_1 E_1)}.$$

Suppose

$$\eta \circ g(n) = \beta_d n^d + \ldots \beta_1 n + \beta_0$$

and let

$$\eta \circ g_D(n) = \alpha_d^{(D)} n^d + \cdots + \alpha_1^{(D)} n + \alpha_0^{(D)}$$

for any $D \in \mathscr{B}_K$. The quantities $\alpha_j^{(D)}$ and β_j are linked through the relation

$$\alpha_j^{(D)} = D^j \sum_{i=j}^{d} \binom{i}{j} A_D^{i-j} \beta_i \tag{7-5}$$

for each $1 \leqslant j \leqslant d$. Thus, the bound (7-4) on the smoothness norm asserts that

$$\sup_{1\leqslant j\leqslant d} \frac{T^j}{K^j} \|\alpha_j^{(D)}\| \ll \delta(T)^{-O(c_1 E_1)}. \tag{7-6}$$

With a downwards induction we deduce from (7-6) and (7-5) that

$$\sup_{1\leqslant j\leqslant d}\frac{T^j}{K^j}\|D^j\beta_j\|\ll\delta(T)^{-O(c_1E_1)}.\tag{7-7}$$

The bound (7-7) provides information on rational approximations of $D^j\beta_j$ for many values of D. Our next aim is to use this information in order to deduce information on rational approximations of the β_j themselves. To achieve this, we employ the Waring trick that appeared in the *Type I* sums analysis in [Green and Tao 2012a, §3], and begin by recalling the two lemmas that this trick rests upon. The first one is a recurrence result:

Lemma 7.5 [Green and Tao 2012b, Lemma 3.2]. *Let $\alpha\in\mathbb{R}$, $0<\delta<\frac{1}{2}$ and $0<\sigma<\frac{1}{2}\delta$, and let $I\subseteq\mathbb{R}/\mathbb{Z}$ be an interval of length σ such that $\alpha n\in I$ for at least δN values of n, $1\leqslant n\leqslant N$. Then there is some $k\in\mathbb{Z}$ with $0<|k|\ll\delta^{-O(1)}$ such that $\|k\alpha\|\ll\sigma\delta^{-O(1)}/N$.*

The second is a consequence of the asymptotic formula in Waring's problem:

Lemma 7.6 [Green and Tao 2012a, Lemma 3.3]. *Let $K\geqslant 1$ be an integer, and suppose that $S\subseteq\{1,\dots,K\}$ is a set of size αK. Suppose that $t\geqslant 2^j+1$. Then $\gg_{j,t}\alpha^{2t}K^j$ integers in the interval $[1,tK^j]$ can be written in the form $k_1^j+\cdots+k_t^j$, $k_1,\dots,k_t\in S$.*

Returning to the proof of Proposition 7.4, let us consider the set

$$\overline{\mathscr{D}}_j=\{m\leqslant s(2K)^j: m=D_1^j+\cdots+D_s^j,\ D_1,\dots,D_s\in\mathscr{D}_\eta\}$$

for some $s\geqslant 2^j+1$. Each element m of this set satisfies

$$\|\beta_j m\|\ll\delta(T)^{-O(c_1)}(K/T)^j,\quad 1\leqslant j\leqslant d,\tag{7-8}$$

in view of (7-7). Thus, Lemma 7.6 implies that there are

$$\#\overline{\mathscr{D}}_j\gg\delta(T)^{O(c_1E_1)}K^j$$

elements in this set. In view of the restrictions on K and the assumptions on the function $\delta(x)$, the conditions of Lemma 7.5 (on σ and δ) are satisfied provided T is sufficiently large. We conclude that there is an integer k_j such that

$$1\leqslant k_j\ll\delta(T)^{-O(c_1E_1)}$$

and such that

$$\|k_j\beta_j\|\ll\delta(T)^{-O(c_1E_1)}T^{-j}.$$

Thus

$$\beta_j=\frac{a_j}{\kappa_j}+\tilde{\beta}_j,\tag{7-9}$$

where $\kappa_j\mid k_j$, $\gcd(a_j,\kappa_j)=1$ and

$$0\leqslant\tilde{\beta}_j\ll\delta(T)^{-O(c_1E_1)}T^{-j}.$$

Hence,

$$\|\kappa_j \beta_j\| \ll \delta(T)^{-O(c_1 E_1)} T^{-j}. \tag{7-10}$$

Let $\kappa = \operatorname{lcm}(\kappa_1, \dots, \kappa_d)$ and set $\tilde{\eta} = \kappa \eta$. We proceed as in [Green and Tao 2012a, §3]: The above implies that

$$\|\tilde{\eta} \circ g(n)\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta(T)^{-O(c_1 E_1)} n}{T},$$

which is small provided n is not too large. Indeed, if $T' = \delta(T)^{c_1 E_1 C} T$ for some sufficiently large constant $C \geqslant 1$, only depending on d and m_G, and if $n \in \{1, \dots, T'\}$, then

$$\|\tilde{\eta} \circ g(n)\|_{\mathbb{R}/\mathbb{Z}} \leqslant \tfrac{1}{10}.$$

Let $\chi : \mathbb{R}/\mathbb{Z} \to [-1, 1]$ be a function of bounded Lipschitz norm that equals 1 on $\left[-\frac{1}{10}, \frac{1}{10}\right]$ and satisfies $\int_{\mathbb{R}/\mathbb{Z}} \chi(t)\, \mathrm{d}t = 0$. Then, by setting $F := \chi \circ \tilde{\eta}$, we obtain a Lipschitz function $F : G/\Gamma \to [-1, 1]$ that satisfies $\int_{G/\Gamma} F = 0$ and $\|F\|_{\mathrm{Lip}} \ll \delta(T)^{-O(c_1 E_1)}$. Finally, choosing c_1 sufficiently small, only depending on d and m_G, we may ensure that

$$\|F\|_{\mathrm{Lip}} < \delta(T)^{-E_1}$$

and, moreover, that

$$T' > \delta(T)^{E_1} T.$$

This choice of T', F and c_1 implies that

$$\left| \frac{1}{T'} \sum_{1 \leqslant n \leqslant T'} F(g(n)\Gamma) \right| = 1 > \delta(T)^{E_1} \|F\|_{\mathrm{Lip}},$$

which contradicts the fact that $(g(n)\Gamma)_{n \leqslant T}$ is totally $\delta(T)^{E_1}$-equidistributed. This completes the proof of the proposition. □

8. Equidistribution of product nilsequences

In this section we prove, building on material and techniques from [Green and Tao 2012a, §3], a result on the equidistribution of products of nilsequences which will allow us to perform applications of the Cauchy–Schwarz inequality in Section 9. The specific form of the result is adjusted to the requirements of Section 9.

We begin by introducing the product sequences we shall be interested in. Suppose $g \in \operatorname{poly}(G_\bullet, \mathbb{Z})$ is a polynomial sequence. This is equivalent to the assertion that there exists an integer k, elements $a_1, \dots, a_k$ of G, and integral polynomials $P_1, \dots, P_k \in \mathbb{Z}[X]$ such that

$$g(n) = a_1^{P_1(n)} a_2^{P_2(n)} \cdots a_k^{P_k(n)}.$$

Then, for any pair of integers (m, m'), the sequence $n \mapsto (g(mn), g(m'n)^{-1})$ is a polynomial sequence on $G \times G$ that may be represented by

$$(g(mn), g(m'n)^{-1}) = \left(\prod_{i=1}^{k}(a_i, 1)^{P_i(mn)}\right)\left(\prod_{i=1}^{k}(1, a_i)^{P_i(m'n)}\right)^{-1}.$$

The horizontal torus of $G \times G$ arises as the direct product $G/\Gamma[G, G] \times G/\Gamma[G, G]$ of horizontal tori for G. Let $\pi : G \to G/\Gamma[G, G]$ be the natural projection map. Any horizontal character on $G \times G$ restricts to a horizontal character on each of its factors. Thus, it takes the form $\eta \oplus \eta'(g_1, g_2) := \eta(g_1) + \eta'(g_2)$ for horizontal characters η, η' of G. The following proposition will be applied in the proof of Proposition 6.4 to sequences $g = g_D$ for unexceptional D in the sense of Proposition 7.4.

Proposition 8.1. *Let N and T be as before and let $E_2 \geqslant 1$. Let $(\tilde{D}_m)_{m\in\mathbb{N}}$ be a sequence of integers satisfying $|\tilde{D}_m| < m$ for every $m \in \mathbb{N}$. Further, let $\delta : \mathbb{N} \to (0, 1)$ be a function that satisfies $\delta(x)^{-t} \ll_t x$ for all $t > 0$. Suppose G/Γ has a $1/\delta(T)$-rational Malcev basis adapted to a filtration $G_\bullet$ of degree d. Let $P \subset \{1, \ldots, T\}$ be a discrete interval. Suppose $F : G/\Gamma \to \mathbb{C}$ is a 1-bounded function of bounded Lipschitz norm $\|F\|_{\mathrm{Lip}}$ and suppose that $\int_{G/\Gamma} F = 0$. Let $g \in \mathrm{poly}(G_\bullet, \mathbb{Z})$ and suppose that the finite sequence $(g(n)\Gamma)_{n\leqslant T}$ is totally $\delta(T)^{E_2}$-equidistributed in G/Γ. Then there is a constant $c_2 \in (0, 1)$, only depending on d and $m_G := \dim G$, such that the following assertion holds for all integers K satisfying*

$$\exp((\log\log T)^2) \le K \le \exp\Big(\frac{1}{H}(\log T - (\log T)^{1/U})\Big),$$

where $1 < U \ll 1$, provided $c_2 E_2 \geqslant 1$.

Let $\mathscr{E}_K$ denote the set of integer pairs $(m, m') \in (K, 2K]^2$ such that the discrete interval

$$I_{m,m'} = \{n \in \mathbb{N} : nm + \tilde{D}_m \in P,\ nm' + \tilde{D}_{m'} \in P\}$$

has length at least

$$\#I_{m,m'} > \delta(N)^{c_2E_2} T/K,$$

and such that

$$\Big|\sum_{n\in I_{m,m'}} F(g(mn + \tilde{D}_m)\Gamma)\overline{F(g(m'n + \tilde{D}_{m'})\Gamma)}\Big| > (1 + \|F\|_{\mathrm{Lip}})\delta(T)^{c_2E_2}\,\#I_{m,m'}$$

holds. Then,

$$\#\mathscr{E}_K < K^2\delta(T)^{O(c_2E_2)},$$

uniformly for all K as above.

Remark 8.2. Using a trivial bound when $\#I_{m,m'} \leqslant \delta(N)^{c_2E_2}T/K$, we deduce that

$$\Big|\sum_{n\in I_{m,m'}} F(g(mn + \tilde{D}_m)\Gamma)\overline{F(g(m'n + \tilde{D}_{m'})\Gamma)}\Big| < \frac{(1 + \|F\|_{\mathrm{Lip}})\delta(T)^{c_2E_2}T}{K}$$

for all $(m, m') \in (K, 2K]^2 \setminus \mathscr{E}_K$.

Remark 8.3. Proposition 8.1 essentially continues to hold when the variables (m, m') are restricted to pairs of primes. Thanks to a suitable choice of a cutoff parameter X that appears in Section 9C, we will not need this variant of the proposition (cf. Section 9G) and only provide a very brief account of it at the very end of this section.

Proof. To begin with, we endow $G/\Gamma \times G/\Gamma$ with a metric by setting

$$d((x, y), (x', y')) = d_{G/\Gamma}(x, x') + d_{G/\Gamma}(y, y').$$

Let $\tilde{F} : G/\Gamma \times G/\Gamma \to \mathbb{C}$ be defined via $\tilde{F}(\gamma, \gamma') = F(\gamma)\overline{F(\gamma')}$. This is a Lipschitz function. Indeed, the fact that F and $\bar{F}$ are 1-bounded Lipschitz functions allows us to deduce that $\|\tilde{F}\|_{\mathrm{Lip}} \leqslant \|F\|_{\mathrm{Lip}}$. Let $g_{m,m'} : \mathbb{N} \to G \times G$ be the polynomial sequence defined by

$$g_{m,m'}(n) = (g(nm + \tilde{D}_m), g(nm' + \tilde{D}_{m'})).$$

Furthermore, we write $\Gamma' = \Gamma \times \Gamma$. Then $\tilde{F}$ satisfies

$$\int_{G/\Gamma \times G/\Gamma} \tilde{F}(\gamma, \gamma')\, \mathrm{d}(\gamma, \gamma') = \int_{G/\Gamma} F(\gamma) \int_{G/\Gamma} \overline{F(\gamma')}\, \mathrm{d}\gamma'\, \mathrm{d}\gamma = 0.$$

Now, suppose that

$$K \in \bigl[\exp\bigl((\log\log T)^2\bigr), \exp\bigl(H^{-1}(\log T - (\log T)^{1/U})\bigr)\bigr]$$

and that

$$\mathscr{E}_K \geqslant K^2 \delta(T)^{c_2 E_2}.$$

For each pair $(m, m') \in \mathscr{E}_K$, we have

$$\Bigl|\sum_{n \in I_{m,m'}} \tilde{F}(g_{m,m'}(n)\Gamma)\Bigr| > (1 + \|F\|_{\mathrm{Lip}})\delta(T)^{c_2 E_2} \# I_{m,m'}. \tag{8-1}$$

Thus, for every pair $(m, m') \in \mathscr{E}_K$, the corresponding sequence

$$(\tilde{F}(g_{m,m'}(n)\Gamma))_{n \leqslant T/\max(m,m')}$$

fails to be totally $\delta(T)^{c_2 E_2}$-equidistributed. Lemma 7.2 implies that this finite sequence also fails to be $\delta(T)^{c_2 E_2 B}$-equidistributed for some $B \geqslant 1$ that only depends on d and m_G. All implied constants in the sequel will be allowed to depend on d and m_G, without explicit mentioning. By [Green and Tao 2012b, Theorem 2.9][8], there is for each pair $(m, m') \in \mathscr{E}_K$ a nontrivial horizontal character

$$\tilde{\eta}_{m,m'} = \eta_{m,m'} \oplus \eta'_{m,m'} : G \times G \to \mathbb{R}/\mathbb{Z}$$

of modulus $\ll \delta(T)^{-O(c_2 E_2)}$ such that

$$\|\tilde{\eta}_{m,m'} \circ \tilde{g}_{m,m'}\|_{C^\infty[T/\max(m,m')]} \ll \delta(T)^{-O(c_2 E_2)}. \tag{8-2}$$

[8]The $1/\delta(T)$-rational Malcev basis for G/Γ induces one for $G/\Gamma \times G/\Gamma$. Thus [Green and Tao 2012b, Theorem 2.9] is applicable.

Given any nontrivial horizontal character $\tilde{\eta} : G \times G \to \mathbb{R}/\mathbb{Z}$, we define the set

$$\mathscr{M}_\eta = \{(m, m') \in \mathscr{E}_K \mid \tilde{\eta}_{m,m'} = \tilde{\eta}\}.$$

This set is empty unless $|\tilde{\eta}| \ll \delta(T)^{-O(c_2E_2)}$. Pigeonholing over all nontrivial $\tilde{\eta}$ of modulus bounded by $\delta(T)^{-O(c_2E_2)}$, we find that there is some $\tilde{\eta}$ amongst them for which

$$\#\mathscr{M}_{\tilde{\eta}} > K^2\delta(T)^{O(c_2E_2)}.$$

Let us fix such a character $\tilde{\eta} = \eta \oplus \eta'$ and suppose without loss of generality that the component η is nontrivial. Suppose

$$\tilde{\eta} \circ (g(n), g(n')) = (\alpha_d n^d + \alpha'_d n'^d) + \cdots + (\alpha_1 n + \alpha'_1 n') + (\alpha_0 + \alpha'_0)$$

and define for $(m, m') \in \mathscr{E}_K$ the coefficients $\alpha_j(m, m')$, $1 \leqslant j \leqslant d$, via

$$\tilde{\eta} \circ g_{m,m'}(n) = \alpha_d(m, m')n^d + \cdots + \alpha_1(m, m')n + \alpha_0(m, m').$$

Then the bound (8-2) on the smoothness norm asserts that

$$\sup_{1 \leqslant j \leqslant d} \frac{T^j}{K^j} \|\alpha_j(m, m')\| \ll \delta(T)^{-O(c_2E_2)}. \tag{8-3}$$

Observe that each $\alpha_j(m, m')$, $1 \leqslant j \leqslant d$, satisfies

$$\alpha_j(m, m') = \sum_{i=j}^{d} \binom{i}{j} (\tilde{D}_m^{i-j} \alpha_i m^j + \tilde{D}_{m'}^{i-j} \alpha'_i m'^j). \tag{8-4}$$

We now aim to show with a downwards induction starting from $j = d$ that

$$\alpha_j = \frac{a_j}{\kappa_j} + \tilde{\alpha}_j, \tag{8-5}$$

where $1 \leqslant \kappa_j \ll \delta(T)^{-O(c_2E_2)}$, $\gcd(a_j, \kappa_j) = 1$, and

$$\tilde{\alpha}_j \ll \delta(T)^{-O(c_2E_2)} T^{-j}. \tag{8-6}$$

Suppose $j_0 \leqslant d$ and that the above holds for all $j > j_0$. Set $k_{j_0} = \operatorname{lcm}(\kappa_{j_0+1}, \ldots, \kappa_d)$ if $j_0 < d$, and $k_{j_0} = 1$ when $j_0 = d$. Note that $k_{j_0} \ll \delta(T)^{-O(c_2E_2)}$.

Pigeonholing, we find that there is $\tilde{m}'$ such that $m' = \tilde{m}'$ for $\gg K\delta(T)^{O(c_2E_2)}$ pairs $(m, m') \in \mathscr{M}_{\tilde{\eta}}$. Amongst these there are furthermore $\gg K\delta(T)^{O(c_2E_2)}$ values of m that belong to the same fixed residue class modulo k_{j_0}. Denote this set of integers m by $\mathscr{M}'$. Suppose $m = k_{j_0}m_1 + m_0 \in \mathscr{M}'$. Letting $\{x\}$ denote the fractional part of $x \in \mathbb{R}$, we then have

$$\{\tilde{D}_m^{i-j_0} \alpha_i m^{j_0}\} = \left\{\tilde{D}_m^{i-j_0} \tilde{\alpha}_i m^{j_0} + \frac{a_i m_0^{j_0}}{\kappa_i}\right\}, \quad (i \geqslant j_0),$$

where, in view of (8-6),

$$\tilde{D}_m^{i-j_0} \tilde{\alpha}_i m^{j_0} \ll \delta(T)^{-O(c_2E_2)} K^i T^{-i}.$$

Since m_0 is fixed, it thus follows from (8-3), (8-4), (8-5) and the above bound that as m varies over $\mathscr{M}'$, the value of

$$\|\alpha_{j_0} m^{j_0}\|$$

lies in a fixed interval of length $\ll \delta(T)^{-O(c_2 E_2)} K^{j_0} T^{-j_0}$.

We aim to make use of this information in combination with the Waring trick from [Green and Tao 2012a, §3] that was already employed in Section 7. For this purpose, we consider the set of integers

$$\mathscr{M}^* = \{m \leqslant s(2K)^{j_0} : m = m_1^{j_0} + \cdots + m_s^{j_0},\ m_1, \ldots, m_s \in \mathscr{M}'\}$$

with $s \geqslant 2^{j_0} + 1$. For each element $m \in \mathscr{M}^*$ of this set, $\|\alpha_{j_0} m\|$ lies in an interval of length $\ll_s \delta(T)^{-O(c_2 E_2)} K^{j_0} T^{-j_0}$. Furthermore, Lemma 7.6 implies that $\#\mathscr{M}^* \gg \delta(T)^{O(c_2 E_2)} K^{j_0}$. The restrictions on the size of K and the assumptions on the function δ imply that the conditions of Lemma 7.5 are satisfied once T is sufficiently large. Thus, assuming T is sufficiently large, there is an integer $1 \leqslant \kappa'_{j_0} \ll \delta(T)^{-O(c_2 E_2)}$ such that

$$\|\kappa'_{j_0} \alpha_{j_0}\| \ll \delta(T)^{-O(c_2 E_2)} T^{-j_0},$$

i.e.,

$$\alpha_{j_0} = \frac{a_{j_0}}{\kappa_{j_0}} + \tilde{\alpha}_{j_0},$$

where $\kappa_{j_0} \mid \kappa'_{j_0}$, $\gcd(a_{j_0}, \kappa_{j_0}) = 1$ and $\tilde{\alpha}_{j_0} \ll \delta(T)^{-O(c_2 E_2)} T^{-j_0}$, as claimed.

In particular, we have

$$\|\kappa_j \alpha_j\| \ll \delta(T)^{-O(c_2 E_2)} T^{-j} \tag{8-7}$$

for $1 \leqslant j \leqslant d$. Proceeding as in [Green and Tao 2012a, §3], let $\kappa = \operatorname{lcm}(\kappa_1, \ldots, \kappa_d)$ and set $\tilde{\eta} = \kappa\eta$. Then (8-7) implies that

$$\|\tilde{\eta} \circ g\|_{C^\infty[T]} = \sup_{1 \leqslant j \leqslant d} T^j \|\kappa \alpha_j\| \ll \delta(T)^{-O(c_2 E_2)},$$

which in turn shows that

$$\|\tilde{\eta} \circ g(n)\|_{\mathbb{R}/\mathbb{Z}} \ll \delta(T)^{-O(c_2 E_2)} n/T$$

for every $n \in \{1, \ldots, T\}$. Note that the latter bound can be controlled by restricting n to a smaller range. For this, set $T' = \delta(T)^{c_2 E_2 C} T$ for some constant $C \geqslant 1$ depending only on d and m_G, chosen sufficiently large to guarantee that

$$\|\tilde{\eta} \circ g(n)\|_{\mathbb{R}/\mathbb{Z}} \leqslant 1/10,$$

whenever $n \in \{1, \ldots, T'\}$. Let $\chi : \mathbb{R}/\mathbb{Z} \to [-1, 1]$ be a function of bounded Lipschitz norm that equals 1 on $\left[-\frac{1}{10}, \frac{1}{10}\right]$ and satisfies $\int_{\mathbb{R}/\mathbb{Z}} \chi(t)\,\mathrm{d}t = 0$. Then, by setting $F := \chi \circ \tilde{\eta}$, we obtain a function $F : G/\Gamma \to [-1, 1]$ such that $\int_{G/\Gamma} F = 0$ and $\|F\|_{\mathrm{Lip}} \ll \delta(T)^{-O(c_2 E_2)}$. Choosing c_2 sufficiently small, we may ensure that

$$\|F\|_{\mathrm{Lip}} < \delta(T)^{-E_2}$$

and, moreover, that

$$T' > \delta(T)^{E_2} T.$$

The quantities T', F and c_2 are chosen in such a way that

$$\left|\frac{1}{T'} \sum_{1 \leqslant n \leqslant T'} F(g(n)\Gamma)\right| = 1 > \delta(T)^{E_2} \|F\|_{\mathrm{Lip}},$$

This contradicts the fact that $(g(n)\Gamma)_{n \leqslant T}$ is totally $\delta(T)^{E_2}$-equidistributed and completes the proof of the proposition. □

8A. ***Restriction of Proposition 8.1 to pairs of primes.*** We end this section by making the contents of Remark 8.3 more precise. The variables (m, m') in Proposition 8.1 can without much additional effort be restricted to range over pairs of primes. It is clear that in the above proof all applications of the pigeonhole principle that involve the parameters m and m' have to be restricted to the set of primes. The only true difference lies in the application of Waring's result: Lemma 7.6 needs to be replaced by the following one.

Lemma 8.4. *Let $K \geqslant 1$ be an integer and let $S \subset \{1, \dots, K\} \cap \mathscr{P}$ be a subset of the primes less than K. Suppose $\#S = \alpha K/\log K$. Let $s \geqslant 2^k + 1$. Let $X \subset \{1, \dots, sK^k\}$ denote the set of integers that are representable as $p_1^k + \cdots + p_s^k$ with $p_1, \dots, p_s \in S$. Then*

$$|X| \gg_{k,s} \alpha^{2s} K^k,$$

as $K \to \infty$.

Proof. Let $I_s(N)$ denote the number of solutions to the equation

$$p_1^k + \cdots + p_s^k = N$$

in positive prime numbers $p_1, \dots, p_s$. Hua's asymptotic formula [Hua 1965, Theorem 11] for the Waring–Goldbach problem implies that

$$I_s(N) \ll_{k,s} \frac{N^{s/k-1}}{(\log N)^s}.$$

Thus, for $1 \leqslant n \leqslant sK^k$, we have

$$I_s(n) \ll_{k,s} \frac{K^{s-k}}{(\log K)^s}.$$

Hence,

$$\begin{aligned} \alpha^{2s} \frac{K^{2s}}{(\log K)^{2s}} = \left(\sum_{n=1}^{sK^k} I_s(n)\right)^2 &\leqslant |X| \sum_n I_s^2(n) \\ &\ll_{k,s} |X| K^k \frac{K^{2s-2k}}{(\log K)^{2s}} \ll_{k,s} |X| \frac{K^{2s-k}}{(\log K)^{2s}}. \end{aligned}$$

Rearranging completes the proof of the lemma. □

9. Proof of Proposition 6.4

In this section we prove Proposition 6.4 by invoking the possibly trivial Dirichlet decomposition from Lemma 1.8. Let $f \in \mathscr{M}_H$, let h, h' be as in Lemma 1.8 and let $f = f_1 * \cdots * f_H$ with $f_i = h$ for $i < H$ and $f_H = h * h'$. We are given integers Q and A such that $0 \leqslant A < Q \leqslant (\log N)^E$ and such that $A \in (\mathbb{Z}/Q\mathbb{Z})^*$. Recall that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ is a polynomial sequence with the property that $(g(n)\Gamma)_{n \leqslant T/Q}$ is totally $\delta(N)^{E_0}$-equidistributed in G/Γ. Let $I \subset \{1, \dots, T/Q\}$ be a discrete interval of length at least $T/(Q(\log N)^E)$. Our aim is to bound above the expression

$$\left| \frac{Q}{T} \sum_{n \in I} f(Qn + A) F(g(n)\Gamma) \right|. \tag{9-1}$$

If $H = 1$, then we may write this expression as

$$\left| \frac{Q}{T} \sum_{n \in I} f(Qn + D') F(g(Dn + D'')\Gamma) \right|, \tag{9-2}$$

where $D = 1$, $D' = A$ and $D'' = 0$. The aim of the next two sections is to show that in the case where $H > 1$ and the Dirichlet decomposition is nontrivial, the task of bounding (9-1) can be reduced to that of bounding an expression similar to (9-2), but with f replaced by one of the f_i.

9A. ***Reduction by hyperbola method.*** Taking into account that $f = f_1 * \cdots * f_H$, the correlation from Proposition 6.4 may be written as

$$\begin{aligned}\frac{Q}{T} \sum_{n \leqslant T/Q} &\mathbf{1}_I(n) f(Qn + A) F(g(n)\Gamma) \\ &= \frac{Q}{T} \sum_{\substack{d_1 \cdots d_H \leqslant T \\ d_1 \cdots d_H \equiv A \\ \pmod{Q}}} f_1(d_1) f_2(d_2) \cdots f_H(d_H) F\left(g\left(\frac{d_1 \cdots d_H - A}{Q} \right) \Gamma \right) \mathbf{1}_P(d_1 \cdots d_H),\end{aligned} \tag{9-3}$$

where P is the finite progression defined via $P = QI + A$. Our first step is to split this summation via inclusion-exclusion into a finite sum of weighted correlations of individual factors f_i with a nilsequence. To describe these weighted correlations, let $i \in \{1, \dots, H\}$. For every $j \neq i$, let d_j be a fixed positive integer and write $D_i := \prod_{j \neq i} d_j$. Let $a_i \in [0, T/D_i)$ be an integer. Weighted correlations involving f_i will then take the form

$$\begin{aligned}\frac{Q}{T} \Bigl(\prod_{j \neq i} f_j(d_j) \Bigr) &\sum_{\substack{a_i < d_i \leqslant T/D_i \\ d_i D_i \equiv A \pmod{Q}}} \mathbf{1}_P(d_i D_i) f_i(d_i) F\left(g\left(\frac{d_i D_i - A}{Q} \right) \Gamma \right) \\ &= \frac{Q}{T} \Bigl(\prod_{j \neq i} f_j(d_j) \Bigr) \sum_{\frac{a_i - D_i'}{Q} < n \leqslant \frac{T - D_i'}{D_i Q}} f_i(Qn + D_i') F(g(D_i n + D_i'')\Gamma) \mathbf{1}_I(D_i n + D_i''),\end{aligned} \tag{9-4}$$

for suitable integers D_i', D_i'', determined by the values of $D_i \pmod{Q}$ and A. In order to bound correlations of the form (9-4), we need to ensure that d_i runs over a sufficiently long range, which will be achieved by arranging for $D_i \leqslant T^{1-1/H}$ to hold.

Let $\tau = T^{1-1/H}$ and note that $D_i = D_j d_j / d_i$. Hence,

$$D_i > \tau \iff d_j > \frac{\tau d_i}{D_j}.$$

With the help of this equivalence, the function $\mathbf{1} : \mathbb{Z}^H \to 1$ can be decomposed as follows. Suppose $d_1 \cdots d_H \leqslant T$. Then

$$\begin{aligned}
\mathbf{1}(d_1, \dots, d_H) &= \mathbf{1}_{D_1 \leqslant \tau} + \mathbf{1}_{D_1 > \tau}\big(\mathbf{1}_{D_2 \leqslant \tau} + \mathbf{1}_{D_2 > \tau}(\mathbf{1}_{D_3 \leqslant \tau} + \cdots (\mathbf{1}_{D_H \leqslant \tau} + \mathbf{1}_{D_H > \tau}) \cdots)\big) \\
&= \mathbf{1}_{D_1 \leqslant \tau} + \mathbf{1}_{D_1 > \tau}\big(\mathbf{1}_{D_2 \leqslant \tau}\mathbf{1}_{d_2 > \tau d_1/D_2} + \mathbf{1}_{D_2 > \tau}(\mathbf{1}_{D_3 \leqslant \tau}\mathbf{1}_{d_3 > \tau \max(d_1,d_2)/D_3} + \cdots \\
&\qquad\qquad \cdots + \mathbf{1}_{D_{H-1} > \tau}(\mathbf{1}_{D_H \leqslant \tau} + \mathbf{1}_{D_H > \tau}) \cdots)\big) \\
&= \mathbf{1}_{D_1 \leqslant \tau} + \mathbf{1}_{D_2 \leqslant \tau}\mathbf{1}_{d_2 > \tau d_1/D_2} + \mathbf{1}_{D_3 \leqslant \tau}\mathbf{1}_{d_3 > \tau \max(d_1,d_2)/D_3} + \cdots + \mathbf{1}_{D_H \leqslant \tau}\mathbf{1}_{d_H > \tau \max(d_1,\dots,d_{H-1})/D_H}.
\end{aligned}$$

Thus,

$$\sum_{d_1 \cdots d_H < T} = \sum_{i=1}^{H} \sum_{D \leqslant T^{1-1/H}} \sum_{\substack{d_1,\dots,\widehat{d_i},\dots,d_H \\ D_i = D}} \sum_{\substack{d_i \leqslant T/D_i \\ d_i > \tau \max(d_1,\dots,d_{i-1})/D_i}}.$$

This shows that the original summation (9-3) may be decomposed as a sum of summations of the shape (9-4) while only increasing the total number of terms by a factor of order $O(H)$. Expressing, if necessary, the summation range

$$\left(\frac{\tau \max(d_1, \dots, d_{i-1})}{D_i}, \frac{T}{D_i}\right)$$

of d_i as the difference of two intervals starting from 1, we can ensure that d_i runs over an interval of length $\gg T/D_i \gg T^{1/H}$. The correlation now decomposes as

$$\begin{aligned}
&\frac{Q}{T} \sum_{\substack{d_1 \cdots d_H \leqslant T \\ d_1 \cdots d_H \equiv A \\ (\mathrm{mod}\ Q)}} f_1(d_1) f_2(d_2) \cdots f_H(d_H) F\left(g\left(\frac{d_1 \cdots d_H - A}{Q}\right)\Gamma\right)\mathbf{1}_P(d_1 \cdots d_H) \\
&\leqslant \sum_{i=1}^{H} \sum_{k=0}^{\frac{1-1/H}{\log 2}\log T} \sum_{\substack{D \sim 2^k \\ (D,Q)=1}} \sum_{\substack{d_1,\dots,\widehat{d_i},\dots,d_H \\ D_i = D}} \left(\prod_{j \neq i} \frac{|f_j(d_j)|}{d_i}\right) \left| \frac{DQ}{T} \sum_{\substack{n \leqslant T/D \\ n > \tau \max(d_1,\dots,d_{i-1})/D \\ Dn + D'' \in I}} f_i(Qn + D') F(g(Dn + D'')) \right|. \quad (9\text{-}5)
\end{aligned}$$

Observe that (9-2) can be regarded as the special case $H = 1$ and $D = 1$ of this bound. Our next aim is to analyze the innermost sum of (9-5) as $D \sim 2^k$ varies. Setting $E_1 = E_0$, we deduce from Proposition 7.4 that whenever $2^k \in [\exp((\log\log T)^2), (\log T)^{1-1/H}]$ then there is a set $\mathscr{B}_{2^k}$ of cardinality at most $O(\delta(N)^{c_1 E_0} 2^k)$ such that for each $D \sim 2^k$ with $D \notin \mathscr{B}_{2^k}$ the sequence

$$(g_D(n)\Gamma)_{n \leqslant T/Q}, \quad g_D(n) := g(Dn + D''),$$

is totally $\delta(N)^{c_1 E_0}$-equidistributed. Before turning to the case of $D \notin \mathscr{B}_{2^k}$, we bound the total contribution from exceptional D, that is, from $D \in \mathscr{B}_{2^k}$ and from $D \leqslant \exp((\log\log T)^2)$.

9B. ***Contribution from exceptional D.*** Let $\mathscr{B}_{2^k}$ denote the exceptional set from the previous section.

Lemma 9.1. *Whenever E_0 is sufficiently large in terms of d, m_G and H, we have*

$$\sum_{\frac{(\log\log T)^2}{\log 2} < k \leqslant \frac{(1-1/H)\log T}{\log 2}} \sum_{D \in \mathscr{B}_{2^k}} \sum_{\substack{d_1 \cdots d_H \leqslant T \\ d_1 \cdots d_H \equiv A \pmod{Q} \\ D_i = D}} |f_1(d_1) f_2(d_2) \cdots f_H(d_H)| \mathbf{1}_P(d_1 \cdots d_H) \ll_t \frac{T}{Q} \frac{1}{(\log T)^2}$$

and

$$\sum_{\substack{D \leqslant \exp((\log\log T)^2) \\ \gcd(D,W)=1}} \sum_{\substack{d_1 \cdots d_H \leqslant T \\ d_1 \cdots d_H \equiv A \pmod{Q} \\ D_i = D}} |f_1(d_1) f_2(d_2) \cdots f_H(d_H)| \mathbf{1}_P(d_1 \cdots d_H) \\ \ll (\log\log T)^{2H} \frac{T}{Q} \frac{Q}{\phi(Q)} \frac{1}{\log T} \prod_{\substack{p \leqslant T \\ p \nmid Q}} \left(1 + \frac{|f(p)|}{Hp}\right).$$

Before we prove this lemma, let us consider its contribution to the bound in Proposition 6.4. The contribution from the first part is easily seen to be negligible. Regarding the second part, recall that $H > 1$ and note that by property (2) of Definition 1.3, we have

$$\prod_{Q < p \leqslant T} \left(1 + \frac{(H-1)|f(p)|}{Hp}\right) \gg \left(\frac{\log T}{E \log\log T}\right)^{(H-1)\alpha_f/H},$$

where the exponent is positive. Thus, the bound in the second part saves a power of $\log x$ when compared with the bound in (6-2) and is therefore also negligible.

Proof. Set

$$\bar{f}_i(n) := |f_1 * \ldots \widehat{f_i} \cdots * f_H(n)|.$$

Then $\bar{f}_i = |h^{*(H-1)} * h'|$ or $|h^{*(H-1)}|$ and it follows from (3-2) and the properties of h that $\bar{f}_i(n) \leqslant (CH)^{\Omega(n)}$ for some constant C. This implies a second moment bound of the form

$$\sum_{\substack{n \leqslant x \\ \gcd(n,Q)=1}} \bar{f}_i(n)^2 \leqslant x \sum_{\substack{n \leqslant x \\ \gcd(n,Q)=1}} \frac{\bar{f}_i(n)^2}{n} \leqslant x \prod_{w(N) < p \leqslant x} \left(1 - \frac{(CH)^2}{p}\right)^{-1} \leqslant x (\log x)^{O(H^2)}.$$

Similarly, we have

$$\sum_{\substack{n \leqslant x \\ \gcd(n,Q)=1}} |f_i(n)| \ll x (\log x)^{O(H)}.$$

Since Proposition 7.4 provides the bound $\#\mathscr{B}^*_{2^k} \ll \delta(N)^{c_1 E_0} 2^k$, a trivial application of the Cauchy–Schwarz inequality yields

$$\sum_{D\in\mathscr{B}^*_{2^k}} \bar{f}_i(D) \sum_{\substack{n\leqslant T/D\\ nD\equiv A \pmod Q\\ nD\in P}} |f_i(n)| \leqslant \sum_{\substack{n\leqslant T/2^k\\ \gcd(n,Q)=1}} |f_i(n)| \sum_{\substack{D\in\mathscr{B}^*_{2^k}\\ \gcd(n,Q)=1}} \bar{f}_i(D)$$

$$\leqslant \sum_{\substack{n\leqslant T/2^k\\ \gcd(n,Q)=1}} |f_i(n)| 2^k \delta(N)^{c_1E_0} k^{O(H^2)} \leqslant T(\log T)^{O(H)}\delta(N)^{c_1E_0}k^{O(H^2)}.$$

Recall that c_1 only depends on d and m_G, and that by the assumptions of Proposition 6.4 we have $\delta(N) \ll (\log T)^{-1}$. Since the summation in k has length at most $\log T$ and since $k^{O(H^2)} < (\log T)^{O_H(1)}$ for each k, the first part of the lemma follows by choosing E_0 sufficiently large in terms of d, m_G and H.

Concerning the second part, we have

$$\sum_{\substack{D\leqslant\exp((\log\log T)^2)\\ \gcd(D,Q)=1}} \bar{f}_i(D) \sum_{\substack{n\leqslant T/D\\ nD\equiv A \pmod Q\\ nD\in P}} |f_i(n)| \leqslant \sum_{\substack{D\leqslant\exp((\log\log T)^2)\\ \gcd(D,Q)=1}} \bar{f}_i(D) \sum_{\substack{n\leqslant T/D\\ n\equiv A\bar{D}\\ \pmod Q}} |f_i(n)|,$$

where $\bar{D}D \equiv 1 \pmod Q$. Since $\log(T/D) \asymp \log T$ and $T/D < T$, Shiu's bound (3-1) yields the upper bound

$$\ll \sum_{\substack{D\leqslant\exp((\log\log T)^2)\\ \gcd(D,Q)=1}} \frac{\bar{f}_i(D)}{D}\frac{T}{Q}\frac{Q}{\phi(Q)}\frac{1}{\log T}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\left(1+\frac{|f_i(p)|}{p}\right). \tag{9-6}$$

The outer sum satisfies

$$\sum_{\substack{D\leqslant\exp((\log\log T)^2)\\ \gcd(D,Q)=1}} \frac{\bar{f}_i(D)}{D} \ll \prod_{w(N)<p\leqslant\exp((\log\log T)^2)} \left(1+\frac{|f(p)|}{Hp}\right)^{H-1}$$

$$\ll \exp\Bigg((H-1)\sum_{p\leqslant\exp((\log\log T)^2)}\frac{1}{p}\Bigg) \ll (\log\log T)^{2H},$$

which completes the proof of the second part. □

9C. ***Montgomery–Vaughan approach.*** Since $\mathscr{M}_1 \subset \mathscr{M}_H$ for all $H > 1$, it suffices to prove Proposition 6.4 for $H > 1$. Since the task of bounding (9-2) for $D = 1$ presents an easier special case of the task of bounding the inner sum of (9-5) for unexceptional D when $H > 1$, a proof for the $H = 1$ case may, however, be extracted from the argument below. In fact, most of the argument directly applies when setting $H = D = 1$. The main differences leading to simplifications are that

(1) if $H = D = 1$, one can, instead of later referring to the results from Section 7, directly work with the equidistribution properties of the given polynomial sequence g, and

(2) the extra work of handling the outer sums in (9-5) is not required when $H = D = 1$.

From now on we assume that $H > 1$ and that D is unexceptional, that is $D \sim 2^k$ for $k \geqslant (\log\log T)^2/\log 2$ and $D \notin \mathscr{B}_{2^k}$, where $\mathscr{B}_{2^k}$ is the exceptional set from Section 9A. To bound the inner sum of (9-5) for unexceptional D, we employ the strategy of Montgomery and Vaughan [1977] outlined in Section 2, and begin by introducing a factor $\log n$ into the average. This will later allow us to reduce matters to understanding equidistribution along sequences defined in terms of primes. We set $h = f_i$. We caution that this is *not* the function h from Lemma 1.8, but could either be h or $h * h'$ in the notation of the lemma.

Cauchy–Schwarz and several integral comparisons show that

$$\begin{aligned}\sum_{n \leqslant T/(DQ)} \mathbf{1}_I(Dn+D'')h(Qn+D')F(g(Dn+D'')\Gamma)\log\left(\frac{T/D}{Qn+D'}\right) \\ \leqslant \left(\sum_{n \leqslant T/(DQ)}\left(\log\left(\frac{T}{DQ}\right)-\log n\right)^2\right)^{1/2}\left(\sum_{n \leqslant T/(DQ)} h^2(Qn+D')\right)^{1/2} \\ \ll \frac{T}{DQ}\sqrt{\frac{DQ}{T}\sum_{n \leqslant T/(DQ)} h^2(Qn+D')},\end{aligned}$$

and hence, invoking $D \leqslant T^{1-1/H}$,

$$\begin{aligned}\frac{DQ}{T}\sum_{\substack{n \leqslant T/(DQ)\\ Dn+D'' \in I}} h(Qn+D')F(g(Dn+D'')\Gamma) \\ \ll_H \frac{1}{\log T}\sqrt{\frac{DQ}{T}\sum_{n \leqslant T/(DQ)} h^2(Qn+D')}+\frac{1}{\log T}\left|\frac{DQ}{T}\sum_{\substack{n \leqslant T/(DQ)\\ Dn+D'' \in I}} h(Qn+D')F(g(Dn+D'')\Gamma)\log(Qn+D')\right|.\end{aligned} \tag{9-7}$$

Lemma 1.8 shows that the contribution of the first term in this bound to (9-5) is at most

$$O_H\left(\frac{1}{(\log T)^{1/2}}\frac{Q}{\phi(Q)}\frac{1}{\log x}\prod_{\substack{p \leqslant x\\ p \nmid Q}}\left(1+\frac{|f(p)|}{p}\right)\right),$$

which is negligible in view of the bound stated in Proposition 6.4. It remains to estimate the second term from (9-7). For this, it will be convenient to abbreviate

$$g_D(n) := g(Dn+D''),$$

and to introduce the two finite progressions

$$I_D = \{n : Dn+D'' \in I\} \quad \text{and} \quad P_D = \left\{n : \frac{n-D'}{Q} \in I_D\right\}. \tag{9-8}$$

Since $\log n = \sum_{m|n}\Lambda(m)$, our task is to bound

$$\frac{DQ}{T\log T}\left|\sum_{\substack{mn \leqslant T/D\\ mn \equiv D' \ (\mathrm{mod}\ Q)}} \mathbf{1}_{P_D}(nm)h(nm)\Lambda(m)F\left(g_D\left(\frac{nm-D'}{Q}\right)\Gamma\right)\right|. \tag{9-9}$$

To further simplify this expression we now show that one can, at the expense of a small error term, restrict the summation in (9-9) to pairs (m, n) of coprime integers for which $m = p$ is prime. To see this, recall that F is 1-bounded and observe that

$$\sum_{\substack{nm\leqslant T/D\\ \Omega(m)\geqslant 2 \text{ or } \gcd(n,m)>1\\ mn\equiv D' \ (\mathrm{mod}\ Q)}} |h(nm)|\Lambda(m) \leqslant 2\sum_{p}\sum_{k\geqslant 2} k\log p \sum_{\substack{n\leqslant T/D,\, p^k\| n\\ n\equiv D' \ (\mathrm{mod}\ Q)}} |h(n)|$$
$$\leqslant 2\sum_{p>w(N)}\sum_{k\geqslant 2} H^k k\log p \sum_{\substack{n\leqslant T/(Dp^k)\\ p^k n\equiv D' \ (\mathrm{mod}\ Q)}} |h(n)|.$$

If $p^k \leqslant (T/D)^{1/2}$, then Shiu's bound (3-1) implies for the inner sum:

$$\sum_{\substack{n\leqslant T/(Dp^k)\\ p^k n\equiv D' \ (\mathrm{mod}\ Q)}} |h(n)| \ll \frac{1}{p^k}\frac{T}{D}\frac{1}{\phi(Q)}\frac{1}{\log T}\prod_{\substack{p\leqslant T/D\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right).$$

If N is sufficiently large, then $H\log p \ll p^{1/4}$ for all $p > w(N)$ and thus

$$\sum_{p>w(N)}\sum_{\substack{k\geqslant 2\\ p^k\leqslant (T/D)^{1/2}}} \frac{H^k\log p^k}{p^k} \ll \sum_{p>w(N)}\frac{1}{p^{2-1/2}} \ll \frac{1}{w(N)^{1/2}}.$$

Combining the last three steps, the contribution to (9-9) from the terms $p^k \leqslant (T/D)^{1/2}$ is seen to be bounded by

$$\ll \frac{1}{w(N)^{1/2}\log T}\frac{Q}{\phi(Q)}\frac{1}{\log T}\prod_{\substack{p\leqslant T/D\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right).$$

Turning towards the case of $p^k > (T/D)^{1/2}$, note first that, provided N is large enough that $w(N) > H$, then

$$\sum_{\substack{n\leqslant T/(Dp^k)\\ p^k n\equiv D' \ (\mathrm{mod}\ Q)}} |h(n)| \leqslant \frac{T}{Dp^k}\sum_{\substack{n\leqslant T/(Dp^k)\\ \gcd(n,Q)=1}}\frac{|h(n)|}{n} \leqslant \frac{T}{Dp^k}\prod_{w(N)<p'\leqslant T/(Dp^k)}\left(1-\frac{H}{p'}\right)^{-1} \leqslant \frac{T}{Dp^k}\left(\log_+\frac{T}{Dp^k}\right)^{O(H)},$$

where $\log_+(x) = \max\{\log x, 0\}$ for $x > 0$, as usual. Assuming, again, that $H\log p \ll p^{1/4}$ for all $p > w(N)$, the remaining sum over $p^k > (T/D)^{1/2}$ therefore satisfies

$$\frac{DQ}{T\log T}\sum_{p>w(N)}\sum_{\substack{k\geqslant 2\\ p^k>(T/D)^{1/2}}} H^k\log p^k \sum_{\substack{n\leqslant T/(Dp^k)\\ p^k n\equiv D' \ (\mathrm{mod}\ Q)}} |h(n)|$$
$$\leqslant \frac{DQ}{T\log T}\sum_{p>w(N)}\sum_{\substack{k\geqslant 2\\ p^k>(T/D)^{1/2}}} H^k\log p^k\,\frac{T}{Dp^k}\left(\log_+\frac{T}{Dp^k}\right)^{O(H)},$$

which is further bounded by

$$\begin{aligned}
&\ll \frac{Q}{\log T}\Big(\log\frac{T}{D}\Big)^{O(H)} \sum_{p>w(N)} \sum_{\substack{k\geqslant 2\\ p^k>(T/D)^{1/2}}} \frac{H^k \log p^k}{p^k}\\
&\ll \frac{Q}{\log T}\Big(\log\frac{T}{D}\Big)^{O(H)} \sum_{p>w(N)} \sum_{\substack{k\geqslant 2\\ p^k>(T/D)^{1/2}}} p^{-k(1-1/4)}\\
&\ll \frac{Q}{\log T}\Big(\log\frac{T}{D}\Big)^{O(H)} \sum_{p>w(N)} p^{-2+1/2}p^{1/4}\Big(\frac{T}{D}\Big)^{-1/4}\\
&\ll \frac{Q}{\log T}\Big(\frac{T}{D}\Big)^{-1/4}\Big(\log\frac{T}{D}\Big)^{O(H)} \ll T^{-1/8H}.
\end{aligned}$$

This contribution is dominated by that of the smaller prime powers above.

Thus, the total contribution to (9-5) of pairs (m, n) that are not of the form (m, p), where p is prime and does not divide m, is bounded by

$$\frac{1}{\log T}\sum_{i=1}^{t}\sum_{k}\sum_{D\sim 2^k}\sum_{d_1\cdots\hat{d}_i\cdots d_t=D}\Big(\prod_{j\neq i}\frac{|f_j(d_j)|}{d_i}\Big)\frac{Q}{\phi(Q)}\frac{1}{\log T}\prod_{\substack{p\leqslant T/D\\ p\nmid Q}}\Big(1+\frac{|h(p)|}{p}\Big)$$
$$\leqslant \frac{1}{w(N)^{1/2}\log T}\frac{Q}{\phi(Q)}\frac{1}{\log T}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\Big(1+\frac{|f(p)|}{p}\Big),$$

which is negligible in view of the bound claimed in Proposition 6.4.

This reduces the task of proving Proposition 6.4 to that of bounding the expression

$$\frac{DQ}{T}\Bigg|\sum_{\substack{mp\leqslant T/D\\ mp\equiv D' \ (\mathrm{mod}\ Q)}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\Big(g\Big(\frac{pm-D'}{Q}\Big)\Gamma\Big)\Bigg|. \tag{9-10}$$

9D. ***Decomposing the summation range.*** We prepare the analysis of (9-10) by first splitting the summation into large and small divisors with respect to the parameter

$$X = X(D) = \Big(\frac{T}{D}\Big)^{1-1/(\log\frac{T}{D})^{(U-1)/U}},$$

for a fixed integer $U \geqslant 4$. With this choice of X we obtain

$$\frac{QD}{T}\sum_{\substack{m<X\\ \gcd(m,Q)=1}}\sum_{\substack{p\leqslant T/(mD)\\ p\equiv D'\overline{m} \ (\mathrm{mod}\ Q)}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\Big(g\Big(\frac{pm-D'}{Q}\Big)\Gamma\Big)$$
$$+\frac{QD}{T}\sum_{\substack{m>X\\ \gcd(m,Q)=1}}\sum_{\substack{p\leqslant T/(mD)\\ p\equiv D'\overline{m} \ (\mathrm{mod}\ Q)}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\Big(g\Big(\frac{pm-D'}{Q}\Big)\Gamma\Big). \tag{9-11}$$

In order to analyze these expressions, we dyadically decompose in each of the two terms the sum with shorter summation range. The cutoff parameter X is chosen in such a way that one of the dyadic decompositions is of short length, depending on U. Indeed, we have $\log_2 X \sim \log_2(T/D)$, while

$$\log_2 \frac{T}{DX} = \frac{\left(\log \frac{T}{D}\right)^{1/U}}{\log 2}.$$

Define

$$T_0 = \exp((\log\log T)^2).$$

Then the two sums from (9-11) decompose as

$$\frac{QD}{T} \sum_{\substack{m<T_0 \\ \gcd(m,Q)=1}} \sum_{\substack{p\leqslant \frac{T}{(mD)} \\ p\equiv D'\bar{m} \pmod Q}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right)$$
$$+\frac{QD}{T} \sum_{j=1}^{\log_2 \frac{X}{T_0}} \sum_{\substack{m\sim 2^{-j}X \\ \gcd(m,Q)=1}} \sum_{\substack{p\leqslant \frac{T}{(mD)} \\ p\equiv D'\bar{m} \pmod Q}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right) \tag{9-12}$$

and

$$\frac{QD}{T}\Bigg\{ \sum_{\substack{m>X \\ \gcd(m,Q)=1}} \sum_{\substack{p\leqslant \min(\frac{T}{mD},T_0) \\ p\equiv D'\bar{m} \pmod Q}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right)$$
$$+\sum_{j=1}^{\log_2 \frac{T}{XDT_0}} \sum_{\substack{m>X \\ \gcd(m,Q)=1}} \sum_{\substack{p\sim 2^{-j}\frac{T}{XD} \\ p\equiv D'\bar{m} \pmod Q}} \mathbf{1}_{pm<\frac{T}{D}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right)\Bigg\}. \tag{9-13}$$

We now analyze the contribution from these four sums to (9-5) in turn, beginning with the two short sums up to T_0, which are both straightforward to bound. The main work goes into handling the large primes case corresponding to the long sum in (9-12). Here we will make use of the results from Sections 7 and 8. The long sum from (9-13) will, again, be straightforward to handle due to the above choice of the parameter X.

9E. *Short sums.* The following lemma provides straightforward bounds on the contribution of the short sums in (9-12) and (9-13) to (9-5).

Lemma 9.2. *Writing $\bar{f}_i(n) = |f_1 * \cdots * \widehat{f_i} * \cdots * f_H(n)|$, we have*

$$\sum_{\substack{D\leqslant T^{1-1/H} \\ (D,Q)=1}} \frac{\bar{f}_i(D)}{\log T} \left| \frac{Q}{T} \sum_{\substack{m<T_0 \\ \gcd(m,Q)=1}} \sum_{\substack{p\leqslant T/(mD) \\ p\equiv D'\bar{m} \\ \pmod Q}} \mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right)\right|$$
$$\ll (\log\log T)^2 \frac{1}{\log T}\frac{Q}{\phi(Q)} \exp\left(\frac{H-1}{H}\sum_{\substack{p\leqslant T \\ p\nmid Q}} \frac{|f(p)|}{p}\right) \tag{9-14}$$

and

$$\sum_{\substack{D\leqslant T^{1-1/H}\\(D,Q)=1}}\frac{\bar{f}_i(D)}{\log T}\left|\frac{Q}{T}\sum_{\substack{m>X\\ \gcd(m,Q)=1}}\sum_{\substack{p\leqslant\min(\frac{T}{Dm},T_0)\\ p\equiv D\bar{m}\pmod Q}}\mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\Big(g_D\Big(\frac{pm-D'}{Q}\Big)\Gamma\Big)\right|$$
$$\ll\frac{(\log\log T)^2}{\log T}\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\Big(1+\frac{|f(p)|}{p}\Big). \quad (9\text{-}15)$$

Remark. Both these bounds are negligible when compared to (6-2). In the first case this follows from property (2) of Definition 1.3.

Proof. The short sum in (9-12) satisfies

$$\left|\frac{QD}{T}\sum_{\substack{m<T_0\\ \gcd(m,Q)=1}}\sum_{\substack{p\leqslant T/(mD)\\ p\equiv D\bar{m}\pmod Q}}\mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\Big(g_D\Big(\frac{pm-D'}{Q}\Big)\Gamma\Big)\right|$$
$$\ll\sum_{\substack{m<T_0\\ \gcd(m,Q)=1}}|h(m)|\frac{QD}{T}\sum_{\substack{p\leqslant T/(mD)\\ p\equiv D\bar{m}\pmod Q}}\Lambda(p)\ll\frac{Q}{\phi(Q)}\sum_{\substack{m<T_0\\ \gcd(m,Q)=1}}\frac{|h(m)|}{m}$$
$$\ll\frac{Q}{\phi(Q)}\frac{1}{\log T}\exp\Big(\sum_{w(N)<p<T_0}\frac{1}{p}\Big)\ll(\log\log T)^2\frac{Q}{\phi(Q)}.$$

Thus, the left-hand side of (9-14) is bounded by

$$(\log\log T)^2\frac{Q}{\phi(Q)}\frac{1}{\log T}\sum_{\substack{D\leqslant T^{1-1/H}\\(D,Q)=1}}\frac{\bar{f}_i(D)}{D}.$$

The claimed bound now follows since

$$\sum_{D\leqslant T^{1-1/H},(D,Q)=1}\frac{\bar{f}_i(D)}{D}\ll\exp\Big(\sum_{p\leqslant T,p\nmid Q}\frac{|f_1(p)+\cdots+f_H(p)-f_i(p)|}{p}\Big)=\exp\Big(\frac{H-1}{H}\sum_{p\leqslant T,p\nmid Q}\frac{|f(p)|}{p}\Big),$$

recalling the definition of the functions f_j from (1-6).

The short sum in (9-13) is bounded by

$$\left|\frac{QD}{T}\sum_{\substack{m>X\\ \gcd(m,Q)=1}}\sum_{\substack{p\leqslant\min(T/(mD),T_0)\\ p\equiv D\bar{m}\pmod Q}}\mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\Big(g_D\Big(\frac{pm-D'}{Q}\Big)\Gamma\Big)\right|$$
$$\ll\sum_{w(N)<p<T_0}\frac{\Lambda(p)}{p}\max_{A'\in(\mathbb{Z}/Q\mathbb{Z})^*}S_{|h|}\Big(\frac{T}{pD};Q,A'\Big)$$
$$\ll\sum_{w(N)<p<T_0}\frac{\Lambda(p)}{p}\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\Big(1+\frac{|h(p)|}{p}\Big)\ll\frac{\log T_0}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\Big(1+\frac{|h(p)|}{p}\Big),$$

where we used (3-1). This shows that left-hand side of (9-15) is bounded by

$$\frac{\log T_0}{(\log T)^2}\frac{Q}{\phi(Q)}\sum_{\substack{D\leqslant T^{1-1/H}\\(D,Q)=1}}\frac{\bar{f}_i(D)}{D}\prod_{\substack{p\leqslant T\\p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right).$$

Recall from Section 9D that $\log T_0=(\log\log T)^2$. To finish the proof of (9-15), recall also that $\bar{f}_i(p)=(H-1)f(p)/H$ and $h(p)=f(p)/H$, that $|f(p)|\leqslant H$, and that $|\bar{f}_i(p^k)|\leqslant (CH)^k$ for some positive constant C. Assuming that N is sufficiently large to ensure that $w(N)>2CH$, we then have

$$\begin{aligned}\sum_{\substack{D\leqslant T^{1-1/H}\\(D,Q)=1}}\frac{\bar{f}_i(D)}{D}\prod_{\substack{p\leqslant T\\p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)&\leqslant\prod_{\substack{w(N)<p\leqslant T\\p\nmid q}}\left(1+\frac{|f(p)|}{Hp}\right)\left(1+\frac{(H-1)|f(p)|}{Hp}+\frac{(CH)^2}{p^2}\left(1-\frac{CH}{p}\right)^{-1}\right)\\&\leqslant\exp\left(\sum_{w(N)<p\leqslant T}\frac{2(CH)^2}{p^2}+\frac{H-1}{p^2}\right)\prod_{\substack{w(N)<p\leqslant T\\p\nmid q}}\left(1+\frac{|f(p)|}{p}\right)\\&\ll\prod_{\substack{p\leqslant T\\p\nmid Q}}\left(1+\frac{|f(p)|}{p}\right),\end{aligned}$$

which completes the proof. □

9F. *Large primes.* In this subsection we finally apply the results from Section 8 to bound the contribution of the dyadic parts of (9-12) to (9-5). More precisely, we prove:

Lemma 9.3 (contribution from large primes). *Keep the assumptions of Proposition 6.4. Let Q be as in Proposition 6.4, recall the definition of P_D from* (9-8), *and let $E_h^\sharp(T,D,j)$ denote the expression*

$$\left|\frac{DQ}{T}\sum_{\substack{m\sim 2^{-j}X\\\gcd(m,Q)=1}}\sum_{\substack{p\leqslant T/(mD)\\p\equiv D'\bar{m}\ (\mathrm{mod}\ Q)}}\mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right)\right|.$$

Then, provided the parameter E_0 from Proposition 6.4 is sufficiently large depending on d, m_G and H, we have

$$\sum_{i=1}^{H}\sum_{k=\frac{(\log\log T)^2}{\log 2}}^{(1-\frac{1}{H})\frac{\log T}{\log 2}}\sum_{\substack{D\sim 2^k\\(D,Q)=1}}\mathbf{1}_{D\notin\mathscr{B}_{2^k}}\sum_{\substack{d_1,\dots,\widehat{d_i},\dots,d_H\\D_i=D}}\left(\prod_{i'\neq i}\frac{|f_{i'}(d_{i'})|}{d_{i'}}\right)\sum_{j=0}^{\log_2\frac{X}{T_0}}\frac{E_{f_i}^\sharp(T,D,j)}{\log T}$$
$$\ll\left((\log\log T)^{-1/(2^{s+2}\dim G)}+\frac{\delta(N)^{-10^s\dim G}}{(\log\log T)^{1/2}}\right)\frac{1+\|F\|_{\mathrm{Lip}}}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\p\nmid Q}}\left(1+\frac{|f(p)|}{p}\right),\quad\text{(9-16)}$$

where the implied constant may depend on d, m_G, α_f, E and H.

Remark. This contribution agrees with the bound (6-2).

The remainder of this subsection is concerned with the proof of (9-16). Considering $E_h^{\sharp}(T, D, j)$ for a fixed value of j, $1 \leqslant j \leqslant \log_2(X/T_0)$, the Cauchy–Schwarz inequality yields

$$\sum_{\substack{m\sim 2^{-j}X \\ \gcd(m,Q)=1}} \sum_{\substack{p\leqslant 2^j\frac{T}{XD} \\ p\equiv D\bar{m} \pmod Q \\ mp\in P_D}} \mathbf{1}_{mp\leqslant N} h(m)h(p)\Lambda(p)F\Bigl(g_D\Bigl(\frac{pm-D'}{Q}\Bigr)\Gamma\Bigr)$$

$$\leqslant \biggl(\sum_{p\leqslant 2^j\frac{T}{XD}} |h(p)|^2\Lambda(p)\biggr)^{\frac{1}{2}} \biggl(\frac{Q}{\phi(Q)}\sum_{A'\in(\mathbb{Z}/Q\mathbb{Z})^*} \sum_{\substack{m,m'\sim 2^{-j}X \\ m\equiv m'\equiv A' \pmod Q \\ \gcd(Q,m)=1}} h(m)h(m')$$

$$\times \frac{\phi(Q)}{Q}\sum_{\substack{p\leqslant\frac{T}{D\max(m,m')} \\ pA'\equiv D' \pmod Q \\ pm,pm'\in P_D}} \Lambda(p)F\Bigl(g_D\Bigl(\frac{pm-D'}{Q}\Bigr)\Gamma\Bigr)\overline{F\Bigl(g_D\Bigl(\frac{pm'-D'}{Q}\Bigr)\Gamma\Bigr)}\biggr)^{\frac{1}{2}}. \tag{9-17}$$

The first factor is easily seen to equal $O(2^jT/(XD))$, since $h(p)\ll_H 1$ at primes. To estimate the second factor, we seek to employ the orthogonality of the "W-tricked von Mangoldt function" with nilsequences, combined with the fact that for most pairs (m, m') the product nilsequence that appears in the above expression is equidistributed (see Proposition 8.1). For this purpose, let us make the change of variables $p = Qn + D'_m$ in the inner sum of the second factor, where D'_m is such that $D'_m \equiv D'\bar{m} \pmod Q$. This yields

$$\frac{\phi(Q)}{Q}\sum_{\substack{p\leqslant T/(D\max(m,m')) \\ pA'\equiv D' \pmod Q \\ pm,pm'\in P_D}} \Lambda(p)F\Bigl(g_D\Bigl(\frac{pm-D'}{Q}\Bigr)\Gamma\Bigr)\overline{F\Bigl(g_D\Bigl(\frac{pm'-D'}{Q}\Bigr)\Gamma\Bigr)}$$
$$= \sum_{\substack{n\leqslant T/(QD\max(m,m')) \\ nm+\widetilde{D}_m,nm'+\widetilde{D}_{m'}\in I_D}} \frac{\phi(Q)}{Q}\Lambda(Qn+D'_m)F(g_D(nm+\widetilde{D}_m)\Gamma)\overline{F(g_D(nm'+\widetilde{D}_{m'})\Gamma)}, \tag{9-18}$$

for suitable values of $0\leqslant \widetilde{D}_m < m,\ 0\leqslant \widetilde{D}_{m'} < m'$ and with $I_D = \{n : Dn + D'' \in I\}$ as defined in (9-8) and I as in the statement of Proposition 6.4. Let us consider the summation range

$$I_{m,m'} = \{n\in\mathbb{N} : nm+\widetilde{D}_m \in I_D,\ nm'+\widetilde{D}_{m'}\in I_D\}$$

in the above expression more closely. Since I is a discrete interval, I_D is a discrete interval too and, for $m, m' \sim 2^{-j}X$, we have

$$\#\{n\in\mathbb{N} : nm+\widetilde{D}_m\in I_D\} \ll |I_D|2^j/X \ll |I|2^j/(DX) \leqslant T2^j/(DXQ)$$

and, similarly, $\#\{n\in\mathbb{N} : nm'+\widetilde{D}_{m'}\in I_D\} \ll T2^j/(DXQ)$. We will now split the set

$$\{(m,m') : m,m'\sim 2^{-j}X, m\equiv m'\equiv A' \pmod Q)\}$$

into two subsets, one containing all pairs (m,m') for which $\#I_{m,m'} \leqslant \delta(N)2^j T/(DXQ)$, and one containing those pairs for which

$$\#I_{m,m'} > \delta(N)2^j T/(DXQ). \tag{9-19}$$

In the former case, the trivial bound of (9-18) asserts that

$$\Bigg| \sum_{\substack{n \leqslant T/QD\max(m,m') \\ n \in I_{m,m'}}} \frac{\phi(Q)}{Q} \Lambda(Qn + D'_m) F(g_D(nm + \widetilde{D}_m)\Gamma) \overline{F(g_D(nm' + \widetilde{D}_{m'})\Gamma)} \Bigg| \leqslant \frac{\delta(N)T2^j}{DXQ}.$$

This leaves us to bound (9-18) in the case where (9-19) holds.

To start with, recall our assumption from the start of Section 9C that all values of D are unexceptional in the sense that $D \sim 2^k$ for some $k \geqslant (\log\log T)^2/\log 2$ and $D \notin \mathscr{B}_{2^k}$, where $\mathscr{B}_K$ was defined in Proposition 7.4. Thus, for any fixed unexceptional value of D, the finite sequence

$$(g_D(n)\Gamma)_{n \leqslant T/(Dq)}$$

is totally $\delta(N)^{c_1E_0}$-equidistributed. Thus, applying Proposition 8.1 with $g = g_D$ and with $E_2 = c_1E_0$, we obtain for every integer

$$K \in [T_0, X]$$

an exceptional set $\mathscr{E}_K$ of size

$$\#\mathscr{E}_K \ll \delta(T)^{O(c_1c_2E_0)} K^2 \tag{9-20}$$

such that for all pairs of integers $(m,m') \in (K,2K]^2 \setminus \mathscr{E}_K$ the following estimate holds:

$$\Bigg| \sum_{\substack{n \leqslant T/(QD\max(m,m')) \\ nm+\widetilde{D}_m,\, nm'+\widetilde{D}_{m'} \in I_D}} F(g_D(nm + \widetilde{D}_m)\Gamma) \overline{F(g_D(nm' + \widetilde{D}_{m'})\Gamma)} \Bigg| < \frac{(1+\|F\|_{\mathrm{Lip}})\delta(N)^{c_1c_2E_0}T}{KQD}.$$

Before we continue with the analysis of (9-18), we prove a quick lemma that will allow us to handle the contribution of exceptional sets $\mathscr{E}_K$ in the proof of Lemma 9.3.

Lemma 9.4. *Suppose $j \leqslant \log_2(X/T_0)$ and let $\mathscr{E}_K$ be the exceptional set obtained from Proposition 8.1 when applied with $g = g_D$. Then, provided E_0 is sufficiently large, we have*

$$\frac{1}{\phi(Q)} \sum_{A' \in (\mathbb{Z}/Q\mathbb{Z})^*} \sum_{\substack{m,m' \sim 2^{-j}X \\ m \equiv m' \equiv A' \pmod Q}} |h(m)h(m')| \mathbf{1}_{(m,m') \in \mathscr{E}_{D,2^{-j}X}} \\ \ll \delta(N)^{O(c_1c_2E_0)} \Bigg(\frac{2^{-j}X}{\phi(Q)} \frac{1}{\log(2^{-j}X)} \prod_{\substack{p \leqslant 2^{-j}X \\ p \nmid Q}} \Big(1 + \frac{|h(p)|}{p}\Big)\Bigg)^2,$$

where c_1 and c_2 are the constants defined in Proposition 7.4 and Proposition 8.1, respectively.

Proof. In view of (9-20), Cauchy–Schwarz yields

$$\frac{1}{\phi(Q)}\sum_{A'\in(\mathbb{Z}/Q\mathbb{Z})^*}\sum_{\substack{m,m'\sim 2^{-j}X\\ m\equiv m'\equiv A' \ (\mathrm{mod}\ Q)}}|h(m)h(m')|\mathbf{1}_{(m,m')\in\mathscr{E}_{D,2^{-j}X}}$$

$$\ll \frac{2^{-j}X}{\phi(Q)}\delta(N)^{O(c_1c_2E_0)}\biggl(\sum_{\substack{m,m'\sim 2^{-j}X\\ m\equiv m'\equiv A' \ (\mathrm{mod}\ Q)}}|h(m)|^2|h(m')|^2\biggr)^{1/2} \ll \frac{2^{-j}X}{\phi(Q)}\delta(N)^{O(c_1c_2E_0)}\sum_{\substack{m\sim 2^{-j}X\\ m\equiv A' \ (\mathrm{mod}\ Q)}}|h(m)|^2.$$

Since $2^{-j}X \geqslant T_0 = \exp((\log\log T)^2) \gg Q^2$, we may apply Shiu's bound (3-1) and the trivial inequality $h(p)^2 \leqslant |h(p)|$ to obtain the upper bound

$$\ll \left(\frac{2^{-j}X}{\phi(Q)}\right)^2 \delta(N)^{O(c_1c_2E_0)}\frac{1}{\log(2^{-j}X)}\prod_{\substack{p\leqslant 2^{-j}X\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)$$

$$\ll \delta(N)^{O(c_1c_2E_0)}\log(2^{-j}X)\left(\frac{2^{-j}X}{\phi(Q)}\frac{1}{\log(2^{-j}X)}\prod_{\substack{p\leqslant 2^{-j}X\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)\right)^2.$$

Recall that X was defined in Section 9D and satisfies $X \leqslant T \ll N$. Since furthermore $\delta(N) \leqslant (\log N)^{-1}$, any sufficiently large choice of E_0 guarantees that

$$\delta(N)^{O(c_1c_2E_0)}\log(2^{-j}X) \leqslant \delta(N)^{O(c_1c_2E_0)}$$

holds. This completes the proof. □

As a final tool for the proof of Lemma 9.3, we require an explicit bound on the correlation of the "W-tricked von Mangoldt function" with nilsequences. The following lemma provides such bounds in our specific setting. We include a proof building on that of Green and Tao [2010, Proposition 10.2] in the Appendix.

Lemma 9.5. *Let G/Γ be an s-step nilmanifold, let $G_\bullet$ be a filtration of G of degree d and let $\mathscr{X}$ be an M-rational Malcev basis adapted to it. Let $\Lambda' : \mathbb{N} \to \mathbb{R}$ be the restriction of the ordinary von Mangoldt function to primes, that is, $\Lambda'(p^k) = 0$ whenever $k > 1$. Let $W = W(x)$, let q' and b' be integers such that $0 < b' < Wq' \leqslant (\log x)^E$ and $\gcd(Wq', b') = 1$ hold. Let $\alpha \in (0, 1)$. Then, for every $y \in [\exp((\log x)^\alpha), x]$ and for every polynomial sequence $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$, the following estimate holds*:

$$\left|\sum_{n\leqslant y}\frac{\phi(Wq')}{Wq'}\Lambda'(Wq'n+b')F(g(n)\Gamma)\right| \ll_{\alpha,d,\dim G,E,\|F\|_{\mathrm{Lip}}} \left|\sum_{n\leqslant y}F(g(n)\Gamma)\right| + y\mathscr{E}(x),$$

where

$$\mathscr{E}(x) := (\log\log x)^{-1/(2^{2d+3}\dim G)} + \frac{M^{O(10^d\dim G)}}{(\log\log x)^{1/2^{d+2}}}.$$

Employing Lemma 9.5 for the upper endpoint of an interval $[y_0, y_1]$, and either a trivial estimate or the lemma for the lower endpoint, say, depending on whether or not $y_0 \leqslant y_1^{1/2}$, we obtain as an immediate consequence that

$$\Big|\sum_{y_0\leqslant n\leqslant y_1}\frac{\phi(Wq')}{Wq'}\Lambda'(Wq'n+b')F(g(n)\Gamma)\Big| \\ \ll_{\alpha,s,E,\|F\|_{\mathrm{Lip}}} \Big|\sum_{n\leqslant y_0}F(g(n)\Gamma)\Big|+\Big|\sum_{n\leqslant y_1}F(g(n)\Gamma)\Big|+y_1^{1/2}+y_1\mathscr{E}(x) \quad (9\text{-}21)$$

for any $0<y_0<y_1\leqslant x$ such that $y_1^{1/2}\geqslant \exp((\log x)^{\alpha})$.

This brings us back to the task of bounding (9-18) under the assumption of (9-19). We shall start by applying (9-21) with $[y_0, y_1]=I_{m,m'}$ and $x=N=T^{1+o(1)}$. To do so, note that (9-19) implies that

$$\begin{aligned}\frac{1}{2}\log y_1 &\geqslant \log\frac{\delta(N)T2^j}{DXQ}\\ &\geqslant \log\frac{T}{DX}+j\log 2+\log\delta(N)-\log Q\\ &\geqslant \Big(\log\frac{T}{D}\Big)^{1/U}+j\log 2-\log\log N-2E\log\log T\\ &\geqslant \Big(\frac{\log T}{H}\Big)^{1/4}-\log\log N-2E\log\log T\\ &\gg_{E,H}(\log T)^{1/4},\end{aligned}$$

where we used the definition of X and the assumptions that Proposition 6.4 makes on δ. Thus, choosing $\alpha=\frac{1}{5}$, say, the conditions of Lemma 9.5 are satisfied for every T that is sufficiently large with respect to E and H. Hence, (9-21) yields the following estimate for the interval $[y_0, y_1]=I_{m,m'}$:

$$\Big|\sum_{\substack{n\leqslant T/(QD\max(m,m'))\\ n\in I_{m,m'}}}\frac{\phi(Q)}{Q}\Lambda(Qn+D'_m)F(g_D(nm+\widetilde{D}_m)\Gamma)\overline{F(g_D(nm'+\widetilde{D}_{m'})\Gamma)}\Big| \\ \ll_{s,E,H,\|F\|_{\mathrm{Lip}}}\Big|\sum_{n\leqslant y_0}F(g_D(nm+\widetilde{D}_m)\Gamma)\overline{F(g_D(nm'+\widetilde{D}_{m'})\Gamma)}\Big| \\ +\Big|\sum_{n\leqslant y_1}F(g_D(nm+\widetilde{D}_m)\Gamma)\overline{F(g_D(nm'+\widetilde{D}_{m'})\Gamma)}\Big|+\frac{T2^j}{DXQ}\mathscr{E}(T).$$

Proposition 8.1 shows that the right-hand side is small for most pairs (m, m'). Indeed, together with Proposition 8.1, the above implies that (9-17) is bounded above by

$$\ll_{s,E,H,\|F\|_{\mathrm{Lip}}}\Big(\frac{T2^j}{DX}\Big)^{1/2} \\ \times\Big(\frac{Q}{\phi(Q)}\sum_{A'\in(\mathbb{Z}/Q\mathbb{Z})^*}\sum_{\substack{m,m'\sim 2^{-j}X\\ m\equiv m'\equiv A' \ (\mathrm{mod}\ Q)}}|h(m)h(m')|\frac{T2^j}{QDX}\big(\delta(N)^{O(c_1c_2E_0)}+\mathbf{1}_{(m,m')\in\mathscr{E}_{D,2^{-j}X}}+\mathscr{E}(T)\big)\Big)^{1/2}.$$

Treating the part of this expression to which Lemma 9.4 applies separately and rewriting in the remaining part the sum over m, m' as a square, we obtain after collecting together all the normalization factors:

$$\ll_{s,E,H,\|F\|_{\mathrm{Lip}}} \frac{T}{QD}\Bigg\{\max_{A'\in(\mathbb{Z}/Q\mathbb{Z})^*}\Bigg(\frac{Q2^j}{X}\sum_{\substack{m\sim 2^{-j}X\\ m\equiv A'(Q)}}|h(m)|\Bigg)^2(\delta(N)^{O(c_1c_2E_0)}+\mathscr{E}(T)) \\ +\delta(N)^{O(c_1c_2E_0)}\Bigg(\frac{Q}{\phi(Q)}\frac{1}{\log(2^{-j}X)}\prod_{\substack{p\leqslant 2^{-j}X\\ p\nmid Q}}\Bigg(1+\frac{|h(p)|}{p}\Bigg)\Bigg)^2\Bigg\}^{\frac{1}{2}}$$

$$\ll_{s,E,H,\|F\|_{\mathrm{Lip}}} \frac{T}{QD}(\delta(N)^{O(c_1c_2E_0)}+\mathscr{E}(T))\Bigg(\frac{Q}{\phi(Q)}\frac{1}{\log(2^{-j}X)}\prod_{\substack{p\leqslant 2^{-j}X\\ p\nmid Q}}\Bigg(1+\frac{|h(p)|}{p}\Bigg)\Bigg),$$

where we applied Shiu's bound in the last step. Summing the above expression over $j\leqslant \log_2(X/T_0)$ and taking into account the factor $(\log T)^{-1}$, we deduce that the inner sum in (9-16) is bounded by

$$\ll_{s,E,H,\|F\|_{\mathrm{Lip}}} (\delta(N)^{O(c_1c_2E_0)}+\mathscr{E}(T)) \\ \times\Bigg(\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\Bigg(1+\frac{|h(p)|}{p}\Bigg)\Bigg)\sum_{j=1}^{\log_2(X/T_0)}\frac{1}{\log(2^{-j}X)}\prod_{2^{-j}X<p'<T}\Bigg(1-\frac{|h(p')|}{p'}\Bigg)$$

Since $\delta(N)\leqslant(\log N)^{-1}$, choosing E_0 sufficiently large in terms of d and m_0 ensures that

$$\delta(N)^{O(c_1c_2E_0)}+\mathscr{E}(T)\ll(\log N)^{-1}+\mathscr{E}(T)\ll\mathscr{E}(T).$$

To complete the proof of Lemma 9.3, it thus remains to show that the inner sum over j in the expression above is $O_{\alpha_f}(1)$. To see this, observe that property (2) of Definition 1.3 yields

$$\prod_{X2^{-j}<p\leqslant T}\Bigg(1-\frac{|h(p)|}{p}\Bigg)\ll\Bigg(\frac{\log(2^{-j}X)}{\log T}\Bigg)^{\alpha_f/H}.$$

Thus,

$$\sum_{j=1}^{\log_2(X/T_0)}\frac{1}{\log(2^{-j}X)}\prod_{2^{-j}X<p'<T}\Bigg(1-\frac{|h(p')|}{p'}\Bigg)\ll\frac{1}{(\log T)^{\alpha_f/H}}\sum_{j=1}^{\log_2(X/T_0)}\frac{1}{(\log X-j\log 2)^{1-\alpha_f/H}} \\ \ll_{\alpha_f}\frac{(\log X)^{\alpha_f/H}}{(\log T)^{\alpha_f/H}}\ll_{\alpha_f}1,$$

as required.

9G. *Small primes.* To complete the proof of Proposition 6.4, it remains to bound the contribution of the dyadic parts of (9-13) to (9-5). This is achieved by the following lemma, which will be proved by a combination of Cauchy–Schwarz, Lemma 1.8 and the choice of the parameter X from Section 9D.

Lemma 9.6 (contribution from small primes). *Let $E_h^\flat(T, D, j)$ denote the expression*

$$\left|\frac{DQ}{T}\sum_{\substack{m>X\\ \gcd(m,Q)=1}}\ \sum_{\substack{p\sim 2^{-j}\frac{T}{XD}\\ p\equiv D'\overline{m}\ (\mathrm{mod}\ Q)}}\mathbf{1}_{pm<\frac{T}{D}}\mathbf{1}_{P_D}(mp)h(m)h(p)\Lambda(p)F\left(g_D\left(\frac{pm-D'}{Q}\right)\Gamma\right)\right|.$$

Then

$$\sum_{i=1}^{H}\ \sum_{k=(\log\log T)^2/\log 2}^{(1-1/H)\log T/\log 2}\ \sum_{\substack{D\sim 2^k\\ (D,Q)=1}}\mathbf{1}_{D\notin\mathscr{B}_{2^k}}\sum_{\substack{d_1,\dots,\widehat{d_i},\dots,d_H\\ D_i=D}}\left(\prod_{j\neq i}\frac{|f_j(d_j)|}{d_j}\right)\sum_{j=0}^{\log_2(T/(XDT_0))}\frac{E_{f_i}^\flat(T,D,j)}{\log T}$$

$$\ll(\log T)^{-1/4}\frac{1}{\log T}\frac{\phi(Q)}{Q}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\left(1+\frac{|f(p)|}{p}\right).$$

Proof. Applying Cauchy–Schwarz to the expression $E_h^\flat(T, D, j)$ for a fixed value of j satisfying $0\leqslant j\leqslant\log_2(T/(XDT_0))$, we obtain

$$\left|\frac{QD}{T}\sum_{\substack{m>X\\ (m,Q)=1}}\ \sum_{\substack{p\sim 2^{-j}\frac{T}{XD}\\ p\equiv D'\overline{m}\ (\mathrm{mod}\ Q)}}\mathbf{1}_{pm<\frac{T}{D}}h(m)h(p)\Lambda(p)F\left(g\left(\frac{pm-D'}{Q}\right)\Gamma\right)\mathbf{1}_{P_D}(mp)\right|$$

$$\leqslant\left(\frac{Q}{\phi(Q)}\frac{1}{2^jX}\sum_{\substack{X<m<2^jX\\ \gcd(m,Q)=1}}|h(m)|^2\right)^{\frac12}\left(\phi(Q)\left(\frac{2^jXD}{T}\right)^2\sum_{\substack{p,p'\sim 2^{-j}\frac{T}{XD}\\ p\equiv p'\ (\mathrm{mod}\ Q)}}h(p)h(p')\Lambda(p)\Lambda(p')\right.$$

$$\left.\times\frac{Q}{X2^j}\sum_{\substack{X<m<T/(D\max(p,p'))\\ mp\equiv D'\ (\mathrm{mod}\ Q)\\ pm\in I_D}}F\left(g\left(\frac{pm-D'}{Q}\right)\Gamma\right)\overline{F\left(g\left(\frac{p'm-D'}{Q}\right)\Gamma\right)}\right)^{\frac12}. \quad (9\text{-}22)$$

We estimate the second factor trivially as $O(1)$ by using the bounds $|h(p)h(p')|\ll 1$ and $\|F\|_\infty=\|\bar{F}\|_\infty\leqslant 1$. Thus, (9-22) is bounded by

$$\ll\left(\frac{Q}{\phi(Q)}\frac{1}{2^jX}\sum_{\substack{X<m<2^jX\\ \gcd(m,Q)=1}}|h(m)|^2\right)^{1/2}.$$

This expression can be handled as that in Lemma 1.8: Note that $X\leqslant 2^jX\leqslant T/(DT_0)$, where

$$X=\left(\frac{T}{D}\right)^{1-1/(\log\frac{T}{D})^{(U-1)/U}}\gg(T/D)^{1/2}$$

and

$$\frac{T}{DT_0}=\left(\frac{T}{D}\right)^{1-(\log\log\frac{T}{D})^2/\log\frac{T}{D}}.$$

Thus, Shiu's bound (3-1) and the trivial inequality $|h(p)|^2 \leqslant |h(p)|$ imply that

$$\left(\frac{Q}{\phi(Q)}\frac{1}{2^j X}\sum_{\substack{X<m<2^j X\\ \gcd(m,Q)=1}} |h(m)|^2\right)^{1/2} \ll \left(\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)\right)^{1/2}.$$

The right-hand side is bounded below by $(\log T)^{-1/2}$, thus the above is bounded by

$$\ll (\log T)^{1/2}\left(\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)\right).$$

Finally, note that the summation range in j is short: it is bounded by

$$\log_2(T/(XDT_0)) \ll (\log T)^{1/U} \ll (\log T)^{1/4}.$$

This shows that

$$\sum_{i=1}^{H}\sum_{k=1}^{(1-1/H)\log T/\log 2}\sum_{\substack{D\sim 2^k\\ (D,Q)=1}} \mathbf{1}_{D\notin\mathscr{B}_{2^k}} \sum_{\substack{d_1,\dots,\widehat{d_i},\dots,d_H\\ D_i=D}}\left(\prod_{j\neq i}\frac{|f_j(d_j)|}{d_j}\right)\sum_{j=0}^{\log_2(T/(XDT_0))}\frac{E^{\flat}_{f_i}(T,D,j)}{\log T}$$
$$\ll (\log T)^{-1+\frac{1}{2}+\frac{1}{4}}\sum_{i=1}^{H}\sum_{\substack{D\leqslant T^{1-1/H}\\ (D,Q)=1}}\sum_{\substack{d_1,\dots,\widehat{d_i},\dots,d_t\\ D_i=D}}\left(\prod_{j\neq i}\frac{|f_j(d_j)|}{d_j}\right)\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\left(1+\frac{|h(p)|}{p}\right)$$
$$\ll (\log T)^{-\frac{1}{4}}\frac{1}{\log T}\frac{Q}{\phi(Q)}\prod_{\substack{p\leqslant T\\ p\nmid Q}}\left(1+\frac{|f(p)|}{p}\right).$$

This completes the proof of Lemma 9.6 as well as the proof of Proposition 6.4. □

Appendix: Explicit bounds on the correlation of Λ with nilsequences

The aim of this appendix is to provide a proof of Lemma 9.5. This result is due to Green and Tao and we expect that a statement like Lemma 9.5 will eventually appear in [Green 2014]. The author is grateful to Ben Green for very helpful discussions.

The proof of Lemma 9.5 rests upon the decomposition of Λ' that already appeared in the proof of the original result, [Green and Tao 2010, Proposition 10.2]. To be precise, let $\gamma \in (0,1)$ be a small positive real number that will later be chosen depending on the degree d of the given filtration $G_\bullet$. Further, let $\chi^\flat + \chi^\sharp = \mathrm{id}_{\mathbb{R}}$ be a smooth decomposition of the identity function $\mathrm{id}_{\mathbb{R}} : \mathbb{R}\to\mathbb{R}$, $\mathrm{id}_{\mathbb{R}}(t) := t$, that is such that $\mathrm{supp}(\chi^\sharp) \subset (-1,1)$ and $\mathrm{supp}(\chi^\flat) \subset \mathbb{R}\setminus\left[-\frac{1}{2},\frac{1}{2}\right]$. This decomposition of $\mathrm{id}_{\mathbb{R}}$ induces the following decomposition of Λ':

$$\frac{\phi(Wq')}{Wq'}\Lambda'(Wq'n+b')-1 = \frac{\phi(Wq')}{Wq'}\Lambda^\flat(Wq'n+b') + \left(\frac{\phi(Wq')}{Wq'}\Lambda^\sharp(Wq'n+b')-1\right),$$

where (cf. [Green and Tao 2010, (12.2)])

$$\Lambda^\sharp(n) = -\log x^\gamma \sum_{d|n} \mu(d)\chi^\sharp\Big(\frac{\log d}{\log x^\gamma}\Big) \quad \big(|t| \geqslant 1 \Rightarrow \chi^\sharp(t) = 0\big)$$

is a truncated divisor sum, and where

$$\Lambda^\flat(n) = -\log x^\gamma \sum_{d|n} \mu(d)\chi^\flat\Big(\frac{\log d}{\log x^\gamma}\Big) \quad \big(|t| \leqslant \tfrac{1}{2} \Rightarrow \chi^\flat(t) = 0\big)$$

is an average of $\mu(d)$ running over large divisors of n. This decomposition in turn splits the correlation from Lemma 9.5 into two correlations that shall be bounded separately.

The correlation estimate of the $\Lambda^\flat$ term with nilsequences follows as in [Green and Tao 2010, §12] from the noncorrelation of Möbius with nilsequences and inherits an error term which saves a factor $O_A(\log x)^{-A}$ for any given $A \geqslant 1$ when compared to the trivial bound. In [Green and Tao 2010, Conjecture 8.5], it was conjectured that the Möbius function is orthogonal to linear nilsequences. Since [Green and Tao 2012a, Theorem 1.1] proves this conjecture, not just for linear, but for polynomial nilsequences, it follows without any essential changes in the proof, that the correlation estimate [Green and Tao 2010, equation (12.10)] continues to hold for polynomial sequences. That is to say, we have an estimate of the form

$$\Big|\sum_{n\leqslant N} \Lambda^\flat(n) F(g(n)\Gamma)\Big| \ll_{\|F\|_{\mathrm{Lip}},G/\Gamma,s,A} N(\log N)^{-A}. \tag{A-1}$$

In our setting, we may express the congruence condition modulo Wq' as a character sum

$$\frac{\phi(Wq')}{Wq'}\Lambda^\flat(n)\mathbf{1}_{n\equiv b' \pmod{Wq'}} = \mathbb{E}_{\chi \pmod{Wq'}} \frac{\phi(Wq')}{Wq'}\Lambda^\flat(n)\chi(n)\overline{\chi}(b').$$

As with equation (12.8) of [Green and Tao 2010], the factor $F(g(n)\Gamma)$ from the statement of Lemma 9.5 may be reinterpreted as $F(g'(Wq'n+b')\Gamma)$ for a new polynomial sequence g'. Reinterpreting the product $\chi(n)F(g'(n)\Gamma)$ of a character χ with the given nilsequence as a nilsequence itself allows us to employ the correlation estimate (A-1) with N given by $xq'W \ll x(\log x)^E$ to handle the correlation for $\Lambda^\flat$. Thanks to the saving of an arbitrary power of $\log x$ in (A-1), we can compensate the factor of Wq', which is bounded above by $(\log x)^E$, that we lose when passing to the character sums. In total, we obtain

$$\frac{1}{y}\sum_{n\leqslant y}\frac{\phi(Wq')}{Wq'}\Lambda^\flat(Wq'n+b')F(g(n)\Gamma) \ll_{\|F\|_{\mathrm{Lip}},s,G/\Gamma,B} (\log y)^{-B} \ll_{\|F\|_{\mathrm{Lip}},s,G/\Gamma,B'} (\log x)^{-B'}.$$

It remains to analyze the contribution of the function $\lambda^\sharp : \mathbb{N} \to \mathbb{R}$, defined via

$$\lambda^\sharp(n) := \frac{\phi(Wq')}{Wq'}\Lambda^\sharp(Wq'n+b') - 1.$$

This contribution satisfies the general bound

$$\left|\frac{1}{y}\sum_{n\leqslant y}\left(\frac{\phi(Wq')}{Wq'}\Lambda^{\sharp}(Wq'n+b')-1\right)F(g(n)\Gamma)\right| \leqslant \|\lambda^{\sharp}\|_{U^{k+1}[y]}\|F(g(\cdot)\Gamma)\|_{U^{k+1}[y]^*}$$

for every $k \geqslant 1$, where the dual uniformity norm is defined via

$$\|F(g(\,\cdot\,)\Gamma)\|_{U^{k+1}[N]^*} := \sup\left\{\left|\frac{1}{N}\sum_{n\leqslant N} f(n)F(g(n)\Gamma)\right| : \|f\|_{U^k[N]} \leqslant 1\right\}.$$

The main task that remains is to obtain control on the above dual uniformity norm for at least one value of k. In [Green and Tao 2010], this is achieved through their Proposition 11.2, which decomposes a general nilsequence into an averaged nilsequence of bounded dual uniformity norm plus an error term that is small in the L^∞ norm. The proof of this decomposition uses a compactness argument and, as such, does not provide explicit error terms. Central ideas for a new approach not working with compactness were indirectly provided by work of Eisner and Zorin-Kranich [2013] on a different question. They replace in their work the Lipschitz function in the definition of a nilsequence by a smooth function and the Lipschitz norm by a Sobolev norm. Moreover, they show that certain constructions that play a central role in [Green and Tao 2012b] have counterparts in the Sobolev norm setting. Building on these observations, Green [2014] proves that in the Sobolev norm setting the dual U^{s+1} norm of an s-step nilsequence *is* in fact bounded. The statement of the latter result involves the following notion of Sobolev norms.

Definition A.1 [Green 2014]. Let G/Γ be an m-dimension nilmanifold together with a Malcev basis $\mathscr{X} = \{X_1, \dots, X_m\}$. For any $\psi \in C^\infty(G/\Gamma)$, set

$$\|\psi\|_{W^m,\mathscr{X}} = \sup_{m'\leqslant m}\ \sup_{1\leqslant i_1,\dots,i_{m'}\leqslant m} \|D_{X_{i_1}}\cdots D_{X_{i_{m'}}}\psi\|_\infty,$$

where $D_X\psi(g\Gamma) = \lim_{t\to 0}(\mathrm{d}/\mathrm{d}t)\psi(\exp(tX)g\Gamma)$.

Lemma A.2 [Green 2014, Theorem 5.3.1]. *Let G/Γ be a k-step nilmanifold together with a filtration $G_\bullet$ of degree $d \geqslant k$ and an M-rational Malcev basis adapted to it. Let $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ and suppose $\tilde{F} \in C^\infty(G/\Gamma)$. Then*

$$\|\tilde{F}(g(\cdot)\Gamma)\|_{U^{d+1}[N]^*} := \sup\left\{\left|\frac{1}{N}\sum_{n\leqslant N} f(n)\tilde{F}(g(n)\Gamma)\right| : \|f\|_{U^{d+1}[N]} \leqslant 1\right\} \ll M^{10^d \dim G}\|\tilde{F}\|_{W^{2^d \dim G},\mathscr{X}}.$$

In order to apply Lemma A.2 in our situation, an auxiliary result is needed that allows one to pass from the Lipschitz setting to the Sobolev setting, i.e., to write any Lipschitz function on G/Γ as the sum of a smooth function, to which Lemma A.2 can be applied, and a small L^∞ error. This is the content of the following lemma which will be proved using a standard smoothing trick; the author thanks Ben Green for pointing out this approach.

Lemma A.3. *Suppose that $F : G/\Gamma \to \mathbb{C}$ is a Lipschitz function and let m be a positive integer. Then there is a constant $c \in (0, 1)$, only depending on G, such that for every $\varepsilon \in (0, c)$ there exists a function*

$\psi_m \in C^\infty(G/\Gamma)$ *such that*

$$\|F - F * \psi_m\|_\infty \leqslant \varepsilon(1 + \|F\|_{\mathrm{Lip}}) \tag{A-2}$$

and

$$\|F * \psi_m\|_{W^m,\mathscr{X}} \ll (m/\varepsilon)^{2m} M^{O(m)}. \tag{A-3}$$

Taking Lemma A.3 on trust for the moment, we first complete the proof of Lemma 9.5 before providing that of Lemma A.3. Recall that the filtration $G_\bullet$ of the nilmanifold G/Γ from Lemma 9.5 is of degree d. The previous two lemmas allow us to reduce the proof of Lemma 9.5 to a bound on the U^{d+1}-norm of $\lambda^\sharp : \mathbb{N} \to \mathbb{R}$. More precisely, we have

$$\begin{aligned} \frac{1}{y}\sum_{n\leqslant y}\Big(\frac{\phi(Wq')}{Wq'}\Lambda^\sharp(Wq'n+b')-1\Big)F(g(n)\Gamma) \\ &\ll \varepsilon(1+\|F\|_{\mathrm{Lip}}) + \frac{1}{y}\sum_{n\leqslant y}\Big(\frac{\phi(Wq')}{Wq'}\Lambda^\sharp(Wq'n+b')-1\Big)(F*\psi_m)(g(n)\Gamma) \\ &\ll \varepsilon(1+\|F\|_{\mathrm{Lip}}) + \|\lambda^\sharp\|_{U^{d+1}[y]}\|(F*\psi_m)(g(\cdot)\Gamma)\|_{U^{d+1}[y]^*}. \end{aligned} \tag{A-4}$$

Since $\Lambda^\sharp$ is a truncated divisor sum, one can analyze its U^{d+1}-norm with the help of Theorem D.3 in Appendix D of [Green and Tao 2010]. We will follow the final section of that appendix ("The correlation estimate for $\Lambda^\sharp$") of closely.

For each nonempty subset $\mathscr{B} \subset \{0,1\}^{d+1}$, let

$$\Psi_{\mathscr{B}}(n,\boldsymbol{h}) = (Wq'(n+\boldsymbol{\omega}\cdot\boldsymbol{h})+b')_{\omega\in\mathscr{B}}, \quad (n,\boldsymbol{h}) \in \mathbb{Z}\times\mathbb{Z}^{d+1},$$

denote the relevant system of forms. The set of exceptional primes for this system, denoted by $\mathscr{P}_{\Psi_{\mathscr{B}}}$, is defined to be the set of all primes p such that the reduction modulo p of $\mathscr{P}_{\Psi_{\mathscr{B}}}$ contains two linearly dependent forms or a form that degenerates to a constant. It is clear that whenever x is sufficiently large, the set $\mathscr{P}_{\Psi_{\mathscr{B}}}$ consists of all prime factors of $W(x)q'$ and, in particular, it contains all primes up to $w(x)$. For each prime p, the local factor $\beta_p^{(\mathscr{B})}$ corresponding to $\Psi_{\mathscr{B}}$ is defined to be

$$\beta_p^{(\mathscr{B})} = \frac{1}{p^{d+2}}\sum_{(n,\boldsymbol{h})\in(\mathbb{Z}/p\mathbb{Z})^{d+2}}\prod_{\omega\in\mathscr{B}}\frac{p}{\phi(p)}\mathbf{1}_{p\nmid Wq'(n+\boldsymbol{\omega}\cdot\boldsymbol{h})+b'}.$$

By [Green and Tao 2010, Lemma 1.3], we have $\beta_p^{(\mathscr{B})} = 1 + O_d(1/p^2)$ for all $p \notin \mathscr{P}_{\Psi_{\mathscr{B}}}$, and hence

$$\prod_{p\notin\mathscr{P}_{\Psi_{\mathscr{B}}}}\beta_p^{(\mathscr{B})} = 1 + O_d\Big(\frac{1}{w(x)}\Big) = 1 + O_d\Big(\frac{1}{\log\log x}\Big),$$

while the product of exceptional local factors satisfies

$$\prod_{p\in\mathscr{P}_\Psi}\beta_p^{(\mathscr{B})} = \prod_{p\mid W(x)q'}\beta_p^{(\mathscr{B})} = \Big(\frac{W(x)q'}{\phi(W(x))q'}\Big)^{|\mathscr{B}|},$$

since $\gcd(W(x)q', b') = 1$.

Let K_y be a convex body that is contained in the hypercube $[-y, y]^{d+2}$. Then, Theorem D.3 of [Green and Tao 2010], applied with $a_i = 1$ and $\chi_i = \chi_\#$, implies that if $\gamma > 0$ is sufficiently small depending on d, then

$$\frac{1}{y^{d+2}} \sum_{(n,\boldsymbol{h})\in K_y} \prod_{\omega\in\mathscr{B}} \Lambda^\sharp(Wq'(n+\boldsymbol{\omega}\cdot\boldsymbol{h})+b') \\ = \frac{\operatorname{vol}(K_y)}{y^{d+2}} \prod_p \beta_p^{(\mathscr{B})} + O_d\biggl((\log y^\gamma)^{-1/20} \exp\biggl(O_d\biggl(\sum_{p\in P_{\Psi_{\mathscr{B}}}} p^{-1/2}\biggr)\biggr)\biggr).$$

Since $Wq' \leqslant (\log x)^E$, we have $|\mathscr{P}_{\Psi_{\mathscr{B}}}| \ll w(x)/\log x + E \log\log x/\log w(x)$. Recall that $w(x) \leqslant \log\log x$ and that $\log y \in [(\log x)^\alpha, \log x]$. Thus,

$$(\log y^\gamma)^{-1/20} \exp\biggl(O_d\biggl(\sum_{p\in\mathscr{P}_{\Psi_{\mathscr{B}}}} p^{-1/2}\biggr)\biggr) \ll (\gamma(\log x)^\alpha)^{-1/20} \exp(O_d(|P_{\Psi_{\mathscr{B}}}|)) \\ \ll_d (\log x)^{-\alpha/20} (\log x)^{O_d(E)/\log w(x)},$$

which is $o(1)$ as $x \to \infty$.

Choosing $K_y = \{(n, \boldsymbol{h}) : 0 < n + \boldsymbol{\omega}\cdot\boldsymbol{h} \leqslant y \text{ for all } \boldsymbol{\omega} \in \{0, 1\}^{d+1}\}$, we obtain

$$\|\lambda^\sharp\|_{U^{d+1}[y]}^{2^{d+1}} = \frac{\operatorname{vol}(K_y)}{y^{d+2}} \sum_{\mathscr{B}\subseteq\{0,1\}^{d+1}} (-1)^{|\mathscr{B}|} \prod_{p\notin P_{\Psi_{\mathscr{B}}}} \beta_p^{(\mathscr{B})} + O_d\bigl((\log x)^{-\alpha/20+O_d(E)/\log w(x)}\bigr) \\ \ll_d \frac{\operatorname{vol}(K_y)}{y^{d+2}} \frac{1}{\log\log x} + (\log x)^{-\alpha/20+O(E)/\log w(x)} \ll_{d,\alpha,E} \frac{1}{\log\log x}.$$

Returning to (A-4), it follows from the above bound, Lemma A.2 and an application of Lemma A.3 with $m = 2^d \dim G$ and $\varepsilon = (\log\log x)^{-1/(m2^{d+3})}$, that for $\exp((\log x)^\alpha) \leqslant y \leqslant x$

$$\frac{1}{y} \sum_{n\leqslant y} \Bigl(\frac{\phi(Wq')}{Wq'} \Lambda'(Wq'n+b') - 1\Bigr) F(g(n)\Gamma) \\ \ll_{d,\alpha,E} \frac{1+\|F\|_{\mathrm{Lip}}}{(\log\log x)^{1/(2^{2d+3}\dim G)}} + \|\lambda_\sharp\|_{U^{d+1}[y]} \|(F*\psi_m)(g(\cdot)\Gamma)\|_{U^{d+1}[y]^*} \\ \ll_{d,\alpha,E} \frac{1+\|F\|_{\mathrm{Lip}}}{(\log\log x)^{1/(2^{2d+3}\dim G)}} + \frac{M^{10^d \dim G} \|F*\psi_m\|_{W^{2^d \dim G},\mathscr{X}}}{(\log\log x)^{1/2^{d+1}}} \\ \ll_{d,\dim G,\alpha,E} \frac{1+\|F\|_{\mathrm{Lip}}}{(\log\log x)^{1/(2^{2d+3}\dim G)}} + \frac{(\log\log x)^{1/2^{d+2}} M^{O(10^d \dim G)}}{(\log\log x)^{1/2^{d+1}}},$$

which reduces the proof of Lemma 9.5 to that of Lemma A.3.

Proof of Lemma A.3. Let $d_{\mathscr{X}}$ denote the metric on G/Γ that was introduced in [Green and Tao 2012b, Definition 2.2] and define for every $\varepsilon' > 0$ the following ε'-neighborhood

$$\mathscr{B}_{\varepsilon'} = \{x \in G/\Gamma : d_{\mathscr{X}}(x, \mathrm{id}_G\,\Gamma) < \varepsilon'\}.$$

Let $\varepsilon \in (0,1)$. Since F is Lipschitz, we have $|F(x)-F(y)| \leqslant \varepsilon(1+\|F\|_{\mathrm{Lip}})$ whenever both x and y belong to the neighborhood $\mathscr{B}_\varepsilon$ of $\mathrm{id}_G \Gamma$. To ensure that (A-2) holds, it thus suffices to ensure that ψ_m is nonnegative, supported in $\mathscr{B}_\varepsilon$ and that $\int_{G/\Gamma} \psi_m = 1$. Indeed, these assumptions imply that

$$\begin{aligned}\left|F(x) - \int_{G/\Gamma} F(y)\psi_m(x-y)\,\mathrm{d}y\right| &= \left|\int_{G/\Gamma} (F(y)-F(x))\psi_m(x-y)\,\mathrm{d}y\right| \\ &\leqslant \varepsilon(1+\|F\|_{\mathrm{Lip}}) \int_{G/\Gamma} \psi_m(x-y)\,\mathrm{d}y \\ &= \varepsilon(1+\|F\|_{\mathrm{Lip}}).\end{aligned}$$

The function ψ_m will be constructed as the m-fold convolution of a smooth bump function. For this purpose, observe that

$$m\mathscr{B}_{\varepsilon/m} \subseteq \mathscr{B}_\varepsilon.$$

If $g = \exp(s_1 X_1)\cdots\exp(s_{\dim G} X_{\dim G})$, then the (unique) coordinates

$$\psi(g) := (s_1, \dots, s_{\dim G})$$

are called Malcev coordinates, while the unique coordinates

$$\psi_{\exp}(g) := (t_1, \dots, t_{\dim G})$$

for which $g = \exp(t_1 X_1 + \cdots + t_{\dim G} X_{\dim G})$ are called exponential coordinates. Proceeding as in the proof of Lemma A.14 in [Green and Tao 2012b], one can identify G/Γ with the fundamental domain $\left\{g \in G : \psi(g) \in \left[-\frac{1}{2}, \frac{1}{2}\right)\right\} \subset G$. Furthermore, their Lemma A.2 shows that the change of coordinates between exponential and Malcev coordinates, i.e., $\psi \circ \psi_{\exp}^{-1}$ or $\psi_{\exp} \circ \psi^{-1}$, is in either direction a polynomial mapping with $M^{O(1)}$-rational coefficients. Thus, $\mathscr{B}_\varepsilon$ lies within the fundamental domain provided $\varepsilon < c_0$ for some sufficiently small constant c_0. This embedding of $\mathscr{B}_\varepsilon$ in G allows us to define log on $\mathscr{B}_\varepsilon$. Let us equip $\mathfrak{g}$ with the maximum norm associated to $\mathscr{X}$, that is $\|X\| := \max_i |t_i|$ for $X = \sum_i t_i X_i$. Then the definition of $d_{\mathscr{X}}$ and Green and Tao's Lemma A.2 imply that

$$\{X \in \mathfrak{g} : \|X\| < \delta\} \subseteq \log \mathscr{B}_{\varepsilon/m}$$

for some δ of the form $\delta = (\varepsilon/m)M^{-O(1)}$. Following the above preparation, we now choose a nonnegative smooth function $\chi_1 : \mathbb{R}^{\dim G} \to \mathbb{R}_{\geqslant 0}$ with support in $\{\boldsymbol{t} \in \mathbb{R}^{\dim G} : \|\boldsymbol{t}\|_\infty < 1\}$ that satisfies $\int_{\mathbb{R}^{\dim G}} \chi_1(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} = 1$. Then, by setting $\chi(\boldsymbol{t}) = \delta \cdot \chi_1(\delta \boldsymbol{t})$, we obtain a function $\chi : \mathbb{R}^{\dim G} \to \mathbb{R}_{\geqslant 0}$ that is supported on $\{\boldsymbol{t} \in \mathbb{R}^{\dim G} : \|\boldsymbol{t}\|_\infty < \delta\}$, satisfies $\int_{\mathbb{R}^{\dim G}} \chi(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} = 1$ and has furthermore the property that

$$\left\|\frac{\partial}{\partial t_i}\chi(t_1, \dots, t_{\dim G})\right\|_\infty \ll (m/\varepsilon)^2 M^{O(1)} \tag{A-5}$$

for $1 \leqslant i \leqslant \dim G$. We may identify χ with a function defined on the vector space $\mathfrak{g}$ equipped with the basis $\{X_1, \dots, X_{\dim G}\}$, by setting $\chi(t_1 X_1 + \cdots + t_{\dim G} X_{\dim G}) = \chi(t_1, \dots, t_{\dim G})$.

To obtain a smooth bump function on G/Γ, we consider the composition $\chi \circ \log : G/\Gamma \to \mathbb{R}$, which is supported in $\mathscr{B}_{\varepsilon/m}$. Since the differential $\mathrm{d}\log_{\mathrm{id}_G} : \mathfrak{g} \to \mathfrak{g}$ is the identity, there are positive constants C_0, C_1 and c_1, such that

$$C_0 \leqslant \int_{G/\Gamma} \chi \circ \log \leqslant C_1,$$

provided $\varepsilon < c_1$. Hence there is a constant C such that $\int_{G/\Gamma} \psi = 1$ for $\psi = C\chi \circ \log$.

With this function ψ at hand, let $\psi_m = \psi^{*m}$ be the m-th convolution power of ψ. It is clear that for every $0 < k \leqslant m$, the function ψ^{*k} is supported in $\mathscr{B}_\varepsilon$ and that $\int_{G/\Gamma} \psi^{*k} = 1$. Setting $\psi^{*0} = \delta_0$, where δ_0 denotes the Kronecker δ-function with weight 1 at 0, we furthermore have

$$D_{X_{i_1}} \cdots D_{X_{i_k}} (F * \psi_m) = F * D_{X_{i_1}} \psi * \cdots * D_{X_{i_k}} \psi * \psi^{*(m-k)}$$

and, hence,

$$\| D_{X_{i_1}} \cdots D_{X_{i_k}} (F * \psi_m) \|_\infty \leqslant \|F\|_\infty \cdot \| D_{X_{i_1}} (C\chi \circ \log) \|_\infty \cdots \| D_{X_{i_k}} (C\chi \circ \log) \|_\infty$$

for any $k \leqslant m$. Our final task is to bound $\| D_{X_j} (C\chi \circ \log) \|_\infty$ for every $j \leqslant \dim G$. Writing $[\cdot]_i : \mathfrak{g} \to \mathbb{R}$ for the i-th co-ordinate map with respect to the basis $\mathscr{X}$, we have

$$D_{X_j} (\chi \circ \log)(g) = \sum_{i=1}^{\dim G} \frac{\partial \chi}{\partial X_i} (\log g) \cdot \lim_{t \to 0} \big[\log(\exp(t X_j) g) \big]_i. \tag{A-6}$$

Since the differential $\mathrm{d}\log_{\mathrm{id}_G} : \mathfrak{g} \to \mathfrak{g}$ is the identity, there are constants $C_2 > 0$ and $c_2 > 0$, such that for every $g \in \mathscr{B}_{c_2}$ and for $1 \leqslant i \leqslant m$, the derivative

$$\Big| \lim_{t \to 0} [\log(\exp(t X_j) g)]_i \Big|$$

is bounded by C_2. Choosing $c < \min(c_0, c_1, c_2)$, the bound (A-3) now follows from (A-6) and the bounds given in (A-5). $\square$

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A blowup algebra for hyperplane arrangements

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It is shown that the Orlik–Terao algebra is graded isomorphic to the special fiber of the ideal I generated by the $(n-1)$-fold products of the members of a central arrangement of size n. This momentum is carried over to the Rees algebra (blowup) of I and it is shown that this algebra is of fiber-type and Cohen–Macaulay. It follows by a result of Simis and Vasconcelos that the special fiber of I is Cohen–Macaulay, thus giving another proof of a result of Proudfoot and Speyer about the Cohen–Macaulayness of the Orlik–Terao algebra.

Introduction

The central theme of this paper is to study the ideal-theoretic aspects of the blowup of a projective space along a certain scheme of codimension 2. To be more precise, let $\mathcal{A} = \{\ker(\ell_1), \dots, \ker(\ell_n)\}$ be an arrangement of hyperplanes in $\mathbb{P}^{k-1}$ with coordinate ring $R = \Bbbk[x_1, \dots, x_k]$, and consider the closure of the graph of the following rational map

$$\mathbb{P}^{k-1} \dashrightarrow \mathbb{P}^{n-1}, \quad x \mapsto (1/\ell_1(x) : \cdots : 1/\ell_n(x)).$$

Rewriting the coordinates of the map as forms of the same positive degree in the source $\mathbb{P}^{k-1} = \mathrm{Proj}(R)$, we are led to consider the corresponding graded R-algebra, namely, the Rees algebra of the ideal of R generated by the $(n-1)$-fold products of $\ell_1, \dots, \ell_n$.

This construction is significant in the theory of hyperplane arrangements as it provides a method of compactifying the complement of an arrangement complement. In [Huh and Katz 2012] and under a slightly different setup, it is shown that the cohomology class of the blowup (in the Chow ring of a product of projective spaces) is determined by the underlying combinatorics of $\mathcal{A}$. (See Remark 2.1(v) for details).

It is our view that bringing into the related combinatorics a limited universe of gadgets and numerical invariants from commutative algebra may be of help, especially regarding the typical operations with ideals and algebras. This point of view favors at the outset a second look at the celebrated Orlik–Terao algebra $\Bbbk[1/\ell_1, \dots, 1/\ell_n]$ which is regarded as a commutative counterpart to the combinatorial Orlik–Solomon algebra. The fact that the former, as observed by some authors, has a model as a finitely generated graded

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$\Bbbk$-subalgebra of a finitely generated purely transcendental extension of the field $\Bbbk$, makes it possible to recover it as the homogeneous coordinate ring of the image of a certain rational map.

This is our departing step to naturally introduce other commutative algebras into the picture. As shown in Theorem 2.4, the Orlik–Terao algebra now becomes isomorphic, as a graded $\Bbbk$-algebra, to the special fiber algebra (also called fiber cone algebra or central algebra) of the ideal I generated by the $(n-1)$-fold products of the members of the arrangement $\mathcal{A}$. This algebra is in turn defined as a residue algebra of the Rees algebra of I, so it is only natural to look at this and related constructions. One of these constructions takes us to the symmetric algebra of I, and hence to the syzygies of I. Since I turns out to be a perfect ideal of codimension 2, its syzygies are rather simple and allow us to further understand these algebras.

As a second result along this line of approach, we show that a presentation ideal of the Rees algebra of I can be generated by the syzygetic relations and the Orlik–Terao ideal (see Theorem 4.2). This property has been coined the *fiber type* property in the recent literature; see, e.g., [Herzog et al. 2005, page 808].

A very recent development in this area is the main theorem of Fink, Speyer and Woo in [Fink et al. 2018] who independently recover a variant of this result by obtaining a Gröbner basis under a certain term order. Their result is utilized to compute the initial ideal and consequently the Hilbert series of the presentation ideal which is the general form of our Proposition 4.1(d).

The third main result of this work, as an eventual outcome of these methods, is a proof of the Cohen–Macaulay property of the Rees algebra of I (see Theorem 4.9).

The typical argument in the proofs is induction on the size or rank of the arrangements. Here we draw heavily on the operations of deletion and contraction of an arrangement. In particular, we introduce a variant of a multiarrangement that allows repeated linear forms to be tagged with arbitrarily different coefficients. Then the main breakthrough consists in getting a precise relation between the various ideals or algebras attached to the original arrangement and those attached to the minors.

One of the important facts about the Orlik–Terao algebra is that it is Cohen–Macaulay, as proven by Proudfoot and Speyer [2006]. Using a recent result of W. Vasconcelos and one of us, we recover this result as a consequence of the Cohen–Macaulay property of the Rees algebra.

The structure of this paper is as follows. The first section is an account of the needed preliminaries from commutative algebra. The second section expands on highlights of the settled literature about the Orlik–Terao ideal as well as a tangential discussion on the so-called nonlinear invariants of our ideals such as the reduction number and analytic spread. The third section focuses on the ideal of $(n-1)$-fold products and the associated algebraic constructions. The last section is devoted to the statements and proofs of the main theorems where we draw various results from the previous sections to establish the arguments.

1. Ideal theoretic notions and blowup algebras

The *blow up algebra* of an ideal I in a ring R is the R-algebra

$$\mathcal{R}(I) := \bigoplus_{i\geq 0} I^i.$$

This is a standard R-graded algebra with $\mathcal{R}(I)_0 = R$, where multiplication is induced by the internal multiplication rule $I^r I^s \subset I^{r+s}$. One can see that there is a graded isomorphism $R[It] \simeq \mathcal{R}(I)$, where $R[It]$ is the homogeneous R-subalgebra of the standard graded algebra of polynomials $R[t]$ in one variable over R, generated by the elements at, $a \in I$, of degree 1. The algebra $R[It]$ is known as the *Rees algebra* of the ideal I. Because of the mentioned isomorphism between them, we will often identify these two algebras.

Quite generally, fixing a set of generators of I determines a surjective homomorphism of R-algebras from a polynomial ring over R to $R[It]$. The kernel of such a map is called a *presentation ideal* of $R[It]$. In this generality, even if R is Noetherian (so I is finitely generated) the notion of a presentation ideal is quite loose.

In this work we deal with a special case in which $R = \Bbbk[x_1, \ldots, x_k]$ is a standard graded polynomial ring over a field $\Bbbk$ and $I = \langle g_1, \ldots, g_n \rangle$ is an ideal generated by forms $g_1, \ldots, g_n$ of the same degree. Let $T = R[y_1, \ldots, y_n] = \Bbbk[x_1, \ldots, x_k; y_1, \ldots, y_n]$, a standard bigraded $\Bbbk$-algebra with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$. Using the given generators to obtain an R-algebra homomorphism

$$\varphi : T = R[y_1, \ldots, y_n] \to R[It], \quad y_i \mapsto g_i t,$$

yields a presentation ideal $\mathcal{I}$ which is bihomogeneous in the bigrading of T. Therefore, $R[It]$ acquires the corresponding bigrading.

Changing $\Bbbk$-linearly independent sets of generators in the same degree amounts to effecting an invertible $\Bbbk$-linear map, so the resulting effect on the corresponding presentation ideal is pretty much under control. For this reason, we will by abuse talk about *the* presentation ideal of I by fixing a particular set of homogeneous generators of I of the same degree. Occasionally, we may need to bring in a few superfluous generators into a set of minimal generators.

Since the given generators have the same degree, they span a linear system defining a rational map

$$\Phi : \mathbb{P}^{k-1} \dashrightarrow \mathbb{P}^{n-1}, \tag{1}$$

by the assignment $x \mapsto (g_1(x) : \cdots : g_n(x))$, when some $g_i(x) \neq 0$.

The ideal I is often called the base ideal (to agree with the base scheme) of Φ. Asking when Φ is birational onto its image is of interest and we will briefly deal with it as well. Again note that changing to another set of generators in the same degree will not change the linear system thereof, defining the same rational map up to a coordinate change at the target.

The Rees algebra brings along other algebras of interest. In the present setup, one of them is the *special fiber* $\mathcal{F}(I) := R[It] \otimes_R R/\mathfrak{m} \simeq \bigoplus_{s \geq 0} I^s/\mathfrak{m}I^s$, where $\mathfrak{m} = \langle x_1, \ldots, x_k \rangle \subset R$. The Krull dimension of the special fiber $\ell(I) := \dim \mathcal{F}(I)$ is called the *analytic spread* of I.

The analytic spread is a significant notion in the theory of *reductions* of ideals. An ideal $J \subset I$ is said to be a *reduction of* I if $I^{r+1} = JI^r$ for some r. Most notably, this is equivalent to the condition that the natural inclusion $R[Jt] \hookrightarrow R[It]$ is a finite morphism. The smallest such r is the *reduction number* $r_J(I)$ *with respect to* J. The *reduction number of* I is the infimum of all $r_J(I)$ for all minimal reductions J of I; this number is denoted by $r(I)$.

Geometrically, the relevance of the special fiber lies in the following result, which we isolate for easy reference:

Lemma 1.1. *Let Φ be as in* (1) *and I its base ideal. Then the homogeneous coordinate ring of the image of Φ is isomorphic to the special fiber $\mathcal{F}(I)$ as graded $\Bbbk$-algebras.*

To see this, note that the Rees algebra defines a biprojective subvariety of $\mathbb{P}^{k-1} \times \mathbb{P}^{n-1}$, namely the closure of the graph of Φ. Projecting down to the second coordinate recovers the image of Φ. At the level of coordinate rings this projection corresponds to the inclusion $\Bbbk[I_d t] = \Bbbk[g_1 t, \dots, g_n t] \subset R[It]$, where $g_1, \dots, g_n$ are forms of the degree d; this inclusion is a split $\Bbbk[I_d t]$-module homomorphism with $\mathfrak{m}R[It]$ as direct complement. Therefore, one has an isomorphism of $\Bbbk$-graded algebras $\Bbbk[I_d] \simeq \Bbbk[I_d t] \simeq \mathcal{F}(I)$.

As noted before, the presentation ideal of $R[It]$

$$\mathcal{I} = \bigoplus_{(a,b)\in\mathbb{N}\times\mathbb{N}} \mathcal{I}_{(a,b)},$$

is a bihomogeneous ideal in the standard bigrading of T. Two basic subideals of $\mathcal{I}$ are $\langle \mathcal{I}_{(0,-)} \rangle$ and $\langle \mathcal{I}_{(-,1)} \rangle$, and they come in as follows.

Consider the natural surjections

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi"'] & R[It] \arrow[r, "\otimes_R R/\mathfrak{m}"] & \mathcal{F}(I) \end{tikzcd},$$

where the kernel of the leftmost map is the presentation ideal $\mathcal{I}$ of $R[It]$. Then we have

$$\mathcal{F}(I) \simeq \frac{T}{\ker\psi} \simeq \frac{T}{\langle \ker\varphi|_{(0,-)}, \mathfrak{m}\rangle} \simeq \frac{\Bbbk[y_1, \dots, y_n]}{\langle \mathcal{I}_{(0,-)} \rangle}.$$

Thus, $\langle \mathcal{I}_{(0,-)} \rangle$ is the homogeneous defining ideal of the special fiber (or, as explained in Lemma 1.1, of the image of the rational map Φ).

As for the second ideal $\langle \mathcal{I}_{(-,1)} \rangle$, one can see that it coincides with the ideal of T generated by the biforms $s_1 y_1 + \cdots + s_n y_n \in T$, whenever $(s_1, \dots, s_n)$ is a syzygy of $g_1, \dots, g_n$ of certain degree in R. Thinking about the one-sided grading in the y's, there is no essential harm in denoting this ideal simply by $\mathcal{I}_1$. Thus, $T/\mathcal{I}_1$ is a presentation of the symmetric algebra $\mathcal{S}(I)$ of I. It yields a natural surjective map of R-graded algebras

$$\mathcal{S}(I) \simeq T/\mathcal{I}_1 \twoheadrightarrow T/\mathcal{I} \simeq \mathcal{R}(I).$$

As a matter of calculation, one can easily show that $\mathcal{I} = \mathcal{I}_1 : I^\infty$, the saturation of $\mathcal{I}_1$ with respect to I.

The ideal I is said to be of *linear type* provided $\mathcal{I} = \mathcal{I}_1$, i.e., when the above surjection is injective. It is said to be of *fiber type* if $\mathcal{I} = \mathcal{I}_1 + \langle \mathcal{I}_{(0,-)} \rangle = \langle \mathcal{I}_1, \mathcal{I}_{(0,-)} \rangle$.

A basic homological obstruction for an ideal to be of linear type is the so-called G_∞ condition of Artin and Nagata [1972], also known as the F_1 condition [Herzog et al. 1983]. A weaker condition is the so-called G_s condition, for a suitable integer s. All these conditions can be stated in terms of the Fitting ideals of the given ideal or, equivalently, in terms of the various ideals of minors of a syzygy matrix of the ideal.

In this work we will have a chance to use condition G_k, where $k = \dim R < \infty$. Given a free presentation

$$R^m \xrightarrow{\varphi} R^n \to I \to 0$$

of an ideal $I \subset R$, the G_k condition for I means that

$$\operatorname{ht}(I_p(\varphi)) \geq n - p + 1, \quad \text{for } p \geq n - k + 1, \tag{2}$$

where $I_t(\varphi)$ denotes the ideal generated by the t-minors of φ. Note that nothing is required about the values of p strictly smaller than $n - k + 1$, since for such values one has $n - p + 1 > k = \dim R$, which makes the same bound impossible.

A useful method to obtain new generators of $\mathcal{I}$ from old generators (starting from generators of $\mathcal{I}_1$) is via Sylvester forms (see [Hong et al. 2012, Proposition 2.1]), which has classical roots as the name indicates. It can be defined quite generally as follows: Let $R := \Bbbk[x_1, \ldots, x_k]$, and let $T := R[y_1, \ldots, y_n]$ as above. Given $F_1, \ldots, F_s \in \mathcal{I}$, let J be the ideal of R generated by all the coefficients of the F_i, the so-called *R-content ideal*. Suppose $J = \langle a_1, \ldots, a_q \rangle$, where a_i are forms of the same degree. Then we have the matrix equation

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{bmatrix} = \boldsymbol{A} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \end{bmatrix},$$

where $\boldsymbol{A}$ is an $s \times q$ matrix with entries in T.

If $q \geq s$ and if the syzygies on F_i's are in $\mathfrak{m}T$, then the determinant of any $s \times s$ minor of $\boldsymbol{A}$ is an element of $\mathcal{I}$. These determinants are called *Sylvester forms*. The main use in this work is to show that the Orlik–Terao ideal is generated by such forms (Proposition 3.5).

The last invariant we wish to comment on is the reduction number $r(I)$. For convenience, we state the following result:

Proposition 1.2. *With the above notation, suppose that the special fiber $\mathcal{F}(I)$ is Cohen–Macaulay. Then the reduction number $r(I)$ of I coincides with the Castelnuovo–Mumford regularity* $\operatorname{reg}(\mathcal{F}(I))$ *of $\mathcal{F}(I)$.*

Proof. By [Vasconcelos 2005, Proposition 1.85], when the special fiber is Cohen–Macaulay, one can read $r(I)$ off the Hilbert series. Write

$$HS(\mathcal{F}(I), s) = \frac{1 + h_1 s + h_2 s^2 + \cdots + h_r s^r}{(1-s)^d},$$

with $h_r \neq 0$ and $d = \ell(I)$, the dimension of the fiber (analytic spread). Then, $r(I) = r$.

Since $\mathcal{F}(I) \simeq S/\langle \mathcal{I}_{(0,-)} \rangle$, where $S := \Bbbk[y_1, \ldots, y_n]$, we have that $\mathcal{F}(I)$ has a minimal graded S-free resolution of length equal to $m := \operatorname{ht}\langle \mathcal{I}_{(0,-)} \rangle$, and $\operatorname{reg}(\mathcal{F}(I)) = \alpha - m$, where α is the largest shift in the minimal graded free resolution, occurring also at the end of this resolution. These last two statements mentioned here come from the Cohen–Macaulayness of $\mathcal{F}(I)$.

The additivity of Hilbert series under short exact sequences of modules, together with the fact that $HS(S^u(-v), s) = us^v/(1-s)^n$ gives that $r + m = \alpha = m + \operatorname{reg}(\mathcal{F}(I))$, so $r(I) = \operatorname{reg}(\mathcal{F}(I))$. ☐

2. Hyperplane arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\} \subset \mathbb{P}^{k-1}$ be a central hyperplane arrangement of size n and rank k. Here $H_i = \ker(\ell_i), i = 1, \dots, n$, where each ℓ_i is a linear form in $R := \Bbbk[x_1, \dots, x_k]$ and $\langle \ell_1, \dots, \ell_n \rangle = \mathfrak{m} := \langle x_1, \dots, x_k \rangle$. From the algebraic viewpoint, there is a natural emphasis on the linear forms ℓ_i and the associated ideal theoretic notions.

Deletion and contraction are useful operations on $\mathcal{A}$. Fixing an index $1 \le i \le n$, one introduces two new minor arrangements:

$$\mathcal{A}' = \mathcal{A} \setminus \{H_i\} \text{ (deletion)}, \quad \mathcal{A}'' = \mathcal{A}' \cap H_i := \{H_j \cap H_i \mid 1 \le j \le n\,,\, j \ne i\} \text{ (contraction)}.$$

Clearly, $\mathcal{A}'$ is a subarrangement of $\mathcal{A}$ of size $n-1$ and rank at most k, while $\mathcal{A}''$ is an arrangement of size $\le n-1$ and rank $k-1$. Contraction comes with a natural multiplicity given by counting the number of hyperplanes of $\mathcal{A}'$ that give the same intersection. A modified version of such a notion will be thoroughly used in this work.

The following notion will play a substantial role in some inductive arguments throughout the paper: ℓ_i is called a *coloop* if the rank of the deletion $\mathcal{A}'$ with respect to ℓ_i is $k-1$, i.e., drops by one. This simply means that $\bigcap_{j \ne i} H_j$ is a line rather than the origin in $\mathbb{A}^k$. Otherwise, we say that ℓ_i is a *noncoloop*.

2A. *The Orlik–Terao algebra.* One of our motivations is to clarify the connections between the Rees algebra and the Orlik–Terao algebra which is an important object in the theory of hyperplane arrangements. We state the definition and review some of its basic properties below.

Let $\mathcal{A} \subset \mathbb{P}^{k-1}$ be a hyperplane arrangement as above. Suppose $c_{i_1}\ell_{i_1} + \dots + c_{i_m}\ell_{i_m} = 0$ is a linear dependency among m of the linear forms defining $\mathcal{A}$, denoted D. Consider the following homogeneous polynomial in $S := \Bbbk[y_1, \dots, y_n]$:

$$\partial D := \sum_{j=1}^{m} c_{i_j} \prod_{j \ne k = 1}^{m} y_{i_k}. \tag{3}$$

Note that $\deg(\partial D) = m - 1$.

The *Orlik–Terao algebra of* $\mathcal{A}$ is the standard graded $\Bbbk$-algebra

$$\mathrm{OT}(\mathcal{A}) := S/\partial(\mathcal{A}),$$

where $\partial(\mathcal{A})$ is the ideal of S generated by $\{\partial D \mid D \text{ a dependency of } \mathcal{A}\}$, with ∂D as in (3), called the *Orlik–Terao ideal*. This algebra was introduced in [Orlik and Terao 1994] as a commutative analog of the classical combinatorial Orlik–Solomon algebra, in order to answer a question of Aomoto. The following remark states a few important properties of this algebra and related constructions.

Remark 2.1. (i) Recalling that a circuit is a minimally dependent set, one has that $\partial(\mathcal{A})$ is generated by ∂C, where C runs over the circuits of $\mathcal{A}$ [Orlik and Terao 1994]. In addition, these generators form an universal Gröbner basis for $\partial(\mathcal{A})$ [Proudfoot and Speyer 2006].

(ii) $\mathrm{OT}(\mathcal{A})$ is Cohen–Macaulay [Proudfoot and Speyer 2006].

(iii) $\mathrm{OT}(\mathcal{A}) \simeq \Bbbk[1/\ell_1, \ldots, 1/\ell_n]$, a k-dimensional $\Bbbk$-subalgebra of the field of fractions $\Bbbk(x_1, \ldots, x_k)$ [Schenck and Tohǎneanu 2009; Terao 2002]. The corresponding projective variety is called the *reciprocal plane* and it is denoted by $\mathcal{L}_{\mathcal{A}}^{-1}$.

(iv) Although the Orlik–Terao algebra is sensitive to the linear forms defining $\mathcal{A}$, its Hilbert series only depends on the underlying combinatorics [Terao 2002]. Let

$$\pi(\mathcal{A}, s) = \sum_{F \in L(\mathcal{A})} \mu_{\mathcal{A}}(F)(-s)^{r(F)}$$

be the *Poincaré polynomial* where $\mu_{\mathcal{A}}$ denotes the Möbius function, r is the rank function and F runs over the flats of $\mathcal{A}$. Then we have

$$HS(\mathrm{OT}(\mathcal{A}), s) = \pi\Big(\mathcal{A}, \frac{s}{1-s}\Big).$$

See [Orlik and Terao 1994] for details and [Terao 2002; Berget 2010] for proofs of the above statement.

(v) Let $V \subset \mathbb{P}^{n-1}$ be a $(k-1)$-dimensional projective subspace that realizes $\mathcal{A}$, in the sense that $H_i \in \mathcal{A}$ is identified with the intersection of the i-th coordinate hyperplane in $\mathbb{P}^{n-1}$ with V.

Consider the Cremona map $\mathrm{Crem} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$, $(z_1, \ldots, z_n) \mapsto (z_1^{-1}, \ldots, z_n^{-1})$ and let $U(\mathcal{A}) = V \setminus \bigcup_{i=1}^{n} H_i$ be the complement of $\mathcal{A}$ in V. Under this setup, one obtains a related formulation of the blowup, here denoted by $\widetilde{V}$, as the closure of the graph of the restriction of the Cremona map to $U(\mathcal{A})$. Huh and Katz [2012] give a formula for the cohomology class of $\widetilde{V}$ as an element of the Chow ring $CH(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \mathbb{Z}[a, b]/\langle a^n, b^n\rangle$, where the coefficients come from the Poincaré polynomial after a change of variables:

$$\frac{s^k \pi(\mathcal{A}, -1/s)}{s-1} = \sum_{i=0}^{k-1} (-1)^i \mu^i s^{k-1-i}, \quad [\widetilde{V}] = \sum_{i=0}^{k} \mu^i [\mathbb{P}^{k-1-i} \times \mathbb{P}^i].$$

2B. *Ideals of products from arrangements.* Let $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ denote a central arrangement in $\mathbb{P}^{k-1}$, $n \geq k$, and let (as always throughout this paper) $R := \Bbbk[x_1, \ldots, x_k]$. Denoting $[n] := \{1, \ldots, n\}$, if $S \subset [n]$, then we set $\ell_S := \prod_{i \in S} \ell_i$, $\ell_\varnothing := 1$. Also set $S^c := [n] \setminus S$.

Let $\mathfrak{S} := \{S_1, \ldots, S_m\}$, where $S_j \subseteq [n]$ are subsets of the same size e. We are interested in studying the Rees algebras of ideals of the form

$$I_{\mathfrak{S}} := \langle \ell_{S_1}, \ldots, \ell_{S_m}\rangle \subset R. \tag{4}$$

Example 2.2. **(i) (The Boolean case)** Let $n = k$ and $\ell_i = x_i$, $i = 1, \ldots, k$. Then the ideal $I_{\mathfrak{S}}$ is monomial for any $\mathfrak{S}$. In the simplest case where $e = n-1$, it is the ideal of the partial derivatives of the monomial $x_1 \cdots x_k$ — also the base ideal of the classical Möbius involution. For $e = 2$ the ideal becomes the edge ideal of a simple graph with k vertices. In general, it gives a subideal of the ideal of paths of a given length on the complete graph and, as such, it has a known combinatorial nature.

(ii) ($(n-1)$-fold products) Here one takes $S_1 := [n] \setminus \{1\}, \ldots, S_n := [n] \setminus \{n\}$. We will designate the corresponding ideal by $I_{n-1}(\mathcal{A})$. This case will be the main concern of the paper and will be fully examined in the following sections.

(iii) (a-fold products) This is a natural extension of (ii), where $I_a(\mathcal{A})$ is the ideal generated by all distinct a-products of the linear forms defining $\mathcal{A}$. The commutative algebraic properties of these ideals connect strongly to properties of the linear code built on the defining linear forms; see [Anzis et al. 2017]. In addition, the dimensions of the vector spaces generated by a-fold products give a new interpretation to the Tutte polynomial of the matroid of $\mathcal{A}$; see [Berget 2010].

We can naturally introduce the following algebra

$$OT(\mathfrak{S}, \mathcal{A}) := \Bbbk\left[\frac{1}{\ell_{S_1^c}}, \ldots, \frac{1}{\ell_{S_m^c}}\right] \tag{5}$$

as a generalized version of the notion mentioned in Remark 2.1(iii).

Proposition 2.3. *In the above setup there is a graded isomorphism of* $\Bbbk$*-algebras*

$$\Bbbk[\ell_{S_1}, \ldots, \ell_{S_m}] \simeq \Bbbk\left[\frac{1}{\ell_{S_1^c}}, \ldots, \frac{1}{\ell_{S_m^c}}\right].$$

Proof. Consider both algebras as homogeneous $\Bbbk$-subalgebras of the homogeneous total quotient ring of the standard polynomial ring R, generated in degrees e and $-(d-e)$, respectively. Then multiplication by the total product $\ell_{[d]}$ gives the required isomorphism:

$$\Bbbk\left[\frac{1}{\ell_{S_1^c}}, \ldots, \frac{1}{\ell_{S_m^c}}\right] \xrightarrow{\cdot \ell_{[d]}} \Bbbk[\ell_{S_1}, \ldots, \ell_{S_m}] \qquad \square$$

A neat consequence is the following result:

Theorem 2.4. *Let $\mathcal{A}$ denote a central arrangement of size n, let $\mathfrak{S} := \{S_1, \ldots, S_m\}$ be a collection of subsets of $[n]$ of the same size and let $I_{\mathfrak{S}}$ be as in* (4). *Then the algebra $OT(\mathfrak{S}, \mathcal{A})$ is isomorphic to the special fiber of the ideal $I_{\mathfrak{S}}$ as graded $\Bbbk$-algebras. In particular, the Orlik–Terao algebra* $\mathrm{OT}(\mathcal{A})$ *is graded isomorphic to the special fiber $\mathcal{F}(I)$ of the ideal $I = I_{n-1}(\mathcal{A})$ of $(n-1)$-fold products of $\mathcal{A}$.*

Proof. It follows immediately from Proposition 2.3 and Lemma 1.1. $\square$

Remark 2.5. In the case of the Orlik–Terao algebra, the above result gives an answer to the third question at the end of [Schenck 2011]. Namely, let $k \geq 3$ and consider the rational map Φ as in (1). Then Theorem 2.4 says that the projection of the graph of Φ onto the second factor coincides with the reciprocal plane $\mathcal{L}_{\mathcal{A}}^{-1}$ (see Remark 2.1(iii)). In addition, the ideal $I := I_{n-1}(\mathcal{A})$ has a similar primary decomposition as obtained in [Schenck 2011, Lemmas 3.1 and 3.2], for arbitrary $k \geq 3$. By [Anzis et al. 2017, Proposition 2.2], one gets

$$I = \bigcap_{Y \in L_2(\mathcal{A})} I(Y)^{\mu_{\mathcal{A}}(Y)}.$$

Theorem 2.4 contributes additional information on certain numerical invariants and properties in the strict realm of commutative algebra and algebraic geometry.

Corollary 2.6. *Let $I := I_{n-1}(\mathcal{A})$ denote the ideal generated by the $(n-1)$-fold products coming from a central arrangement of size n and rank k.*

(a) *The special fiber $\mathcal{F}(I)$ of I is Cohen–Macaulay.*

(b) *The analytic spread is $\ell(I) = k$.*

(c) *The map Φ is birational onto its image.*

(d) *The reduction number is $r(I) \leq k-1$.*

Proof. (a) It follows from Theorem 2.4 via Remark 2.1(ii).

(b) It follows by the same token from Remark 2.1(iii).

(c) This follows from [Doria et al. 2012, Theorem 3.2] since the ideal I is linearly presented (see proof of Lemma 3.1), and $\ell(I) = k$, the maximum possible.

(d) Follows from part (a), Proposition 1.2, and [Schenck 2011, Theorem 3.7]. □

The next result is a refinement of part (d) in the corollary above.

Proposition 2.7. *Let $\mathcal{A}$ be a hyperplane arrangement of rank k and n hyperplanes. Let $I := I_{n-1}(\mathcal{A})$. Then the reduction number of I is $r(I) = k-u$, where $u \geq 1$ is the number of components of $\mathcal{A}$.*

Proof. Let $r := r(I)$, and recall that the Hilbert series of the Orlik–Terao algebra is determined by the Poincaré polynomial:

$$HS(\mathrm{OT}(\mathcal{A}), s) = \pi\left(\mathcal{A}, \frac{s}{1-s}\right) = \frac{\overbrace{1 + h_1 s + \cdots + h_r s^r}^{H(s)}}{(1-s)^k}.$$

As with any central arrangement, the Poincaré polynomial has the trivial factor $(1+t)$ and we write $\pi(\mathcal{A}, t) = (1+t)\bar{\pi}(\mathcal{A}, t)$, where bar denotes the reduced Poincaré polynomial. Moreover, if $\mathcal{A}$ has a decomposition as a product of two smaller arrangements, then the Poincaré polynomial splits and we get a $(1+t)$ factor for each component, and by a result of Crapo, this is the only way for more $(1+t)$ factors to occur. Here, we need the notion of the beta invariant[1] of an arrangement: $\beta(\mathcal{A}) := |\bar{\pi}(\mathcal{A}, -1)|$. Theorem II in [Crapo 1967] states that an arrangement is decomposable if and only if its beta invariant is zero.

We have

$$\pi\left(\mathcal{A}, \frac{s}{1-s}\right) = \bar{\pi}\left(\mathcal{A}, \frac{s}{1-s}\right)/(1-s),$$

which indicates that only $\bar{\pi}$ can contribute to the numerator $H(s)$. If $\deg H(s) < k-1$, then by undoing the substitution, we find another $(1+t)$ factor in the reduced Poincaré polynomial and hence $\beta(\mathcal{A}) = 0$, indicating that this happens exactly when $\mathcal{A}$ is decomposable. So, when $\mathcal{A}$ is indecomposable, the argument is complete and if it does decompose, then we can apply this argument to each component and hence the formula. □

[1] In the case of complex arrangements, this gives the Euler characteristic of the projective complement.

3. Ideals of $(n-1)$-fold products and their blowup algebras

As mentioned in Example 2.2, a special case of the ideal $I_{\mathfrak{S}}$, extending the case of the ideal generated by the $(n-1)$-fold products, is obtained by fixing $a \in \{1, \dots, n\}$ and considering the collection of all subsets of $[n]$ of cardinality a. Then the corresponding ideal is

$$I_a(\mathcal{A}) := \langle \ell_{i_1} \cdots \ell_{i_a} \mid 1 \le i_1 < \cdots < i_a \le n \rangle \subset R$$

and is called the ideal generated by the a-fold products of linear forms of $\mathcal{A}$. The projective schemes defined by these ideals are known as generalized star configuration schemes. Unfortunately, very few facts are known about these ideals: if d is the minimum distance of the linear code built from the linear forms defining $\mathcal{A}$ and if $1 \le a \le d$, then $I_a(\mathcal{A}) = \mathfrak{m}^a$ (see [Tohăneanu 2010, Theorem 3.1]); and the case when $a = n$ is trivial.

In the case where $a = n-1$, some immediate properties are known already, yet the more difficult questions in regard to the blowup and related algebras have not been studied before. These facets, to be thoroughly examined in the subsequent sections, are our main endeavor in this work. Henceforth, we will be working with the following data: $\mathcal{A}$ is an arrangement with $n \ge k$ and for every $1 \le i \le n$, we consider the $(n-1)$-fold products of the n linear forms defining the hyperplanes of $\mathcal{A}$

$$f_i := \ell_1 \cdots \hat{\ell_i} \cdots \ell_n \in R,$$

and write

$$I := I_{n-1}(\mathcal{A}) := \langle f_1, \dots, f_n \rangle.$$

Let $T = \Bbbk[x_1, \dots, x_k, y_1, \dots, y_n] = R[y_1, \dots, y_n]$ as before and denote by $\mathcal{I}(\mathcal{A}, n-1) \subset T$ the presentation ideal of the Rees algebra $R[It]$ corresponding to the generators $f_1, \dots, f_n$.

3A. ***The symmetric algebra.*** Let $\mathcal{I}_1(\mathcal{A}, n-1) \subset T$ stand for the subideal of $\mathcal{I}(\mathcal{A}, n-1)$ presenting the symmetric algebra $\mathcal{S}(I)$ of $I = I_{n-1}(\mathcal{A})$.

Lemma 3.1. *With the above notation, one has:*

(a) *The ideal $I = I_{n-1}(\mathcal{A})$ is a perfect ideal of codimension* 2.

(b) $\mathcal{I}_1(\mathcal{A}, n-1) = \langle \ell_i y_i - \ell_{i+1} y_{i+1} \mid 1 \le i \le n-1 \rangle$.

(c) $\mathcal{I}_1(\mathcal{A}, n-1)$ *is an ideal of codimension k; in particular, it is a complete intersection if and only if $n = k$.*

Proof. (a) This is well known, but we give the argument for completeness. Clearly, I has codimension 2. The following reduced Koszul like relations are syzygies of I: $\ell_i y_i - \ell_{i+1} y_{i+1}$, $1 \le i \le n-1$. They alone form the following matrix of syzygies of I:

$$\varphi = \begin{bmatrix} \ell_1 & & & \\ -\ell_2 & \ell_2 & & \\ & -\ell_3 & \ddots & \\ & & \ddots & \ell_{n-1} \\ & & & -\ell_n \end{bmatrix}.$$

Since the rank of this matrix is $n-1$, it is indeed a full syzygy matrix of I; in particular, I has a linear resolution

$$0 \to R(-n)^{n-1} \overset{\varphi}{\to} R(-(n-1))^n \to I \to 0.$$

(b) This is an expression of the details of (a).

(c) Clearly, $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathfrak{m}T$, hence its codimension is at most k. Assuming, as we may, that $\{\ell_1, \dots, \ell_k\}$ is $\Bbbk$-linearly independent, we contend that the elements $\mathfrak{s} := \{\ell_i y_i - \ell_{i+1} y_{i+1}, 1 \le i \le k\}$ form a regular sequence. To see this, we first apply a $\Bbbk$-linear automorphism of R to assume that $\ell_i = x_i$, for $1 \le i \le k$ — this will not affect the basic ideal theoretic invariants associated to I. Then note that in the set of generators of $\mathcal{I}_1(\mathcal{A}, n-1)$, the elements of $\mathfrak{s}$ can be replaced by the following ones: $\{x_i y_i - \ell_{k+1} y_{k+1}, 1 \le i \le k\}$. Clearly, this is a regular sequence — for example, because $\langle x_i y_i, 1 \le i \le k\rangle$ is the initial ideal of the ideal generated by this sequence, in the revlex order. □

There are two basic ideals that play a distinguished role at the outset. In order to capture both in one single blow, we consider the Jacobian matrix of the generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ given in Lemma 3.1(b). Its transpose turns out to be the stack of two matrices, the first is the Jacobian matrix with respect to the variables $y_1, \dots, y_n$ — which coincides with the syzygy matrix ϕ of I as described in the proof of Lemma 3.1(a) — while the second is the Jacobian matrix $B = B(\phi)$ with respect to the variables $x_1, \dots, x_k$ — the so-called *Jacobian dual matrix* of [Simis et al. 1993]. The offspring are the respective ideals of maximal minors of these stacked matrices, the first retrieves I, while the second gives an ideal $I_k(B) \subset S = \Bbbk[y_1, \dots, y_n]$ that will play a significant role below (see also Proposition 4.1) as a first crude approximation to the Orlik–Terao ideal.

Proposition 3.2. *Let $\mathcal{S}(I) \simeq T/\mathcal{I}_1(\mathcal{A}, n-1)$ stand for the symmetric algebra of the ideal I of $(n-1)$-fold products. Then*:

(i) $\operatorname{depth}(\mathcal{S}(I)) \le k+1$.

(ii) *As an ideal in T, every minimal prime of $\mathcal{S}(I)$ is either $\mathfrak{m}T$, the Rees ideal $\mathcal{I}(\mathcal{A}, n-1)$ or else has the form $(\ell_{i_1}, \dots, \ell_{i_s}, y_{j_1}, \dots, y_{j_t})$, where $2 \le s \le k-1$, $t \ge 1$, $\{i_1, \dots, i_s\} \cap \{j_1, \dots, j_t\} = \varnothing$, and $\ell_{i_1}, \dots, \ell_{i_s}$ are $\Bbbk$-linearly independent.*

(iii) *The primary components relative to the minimal primes $\mathfrak{m} = (\mathbf{x})T$ and $\mathcal{I}(\mathcal{A}, n-1)$ are radical*; *in addition*, *with the exception of $\mathfrak{m}T$, every minimal prime of $\mathcal{S}(I)$ contains the ideal $I_k(B)$.*

Proof. (i) Since $\mathcal{I}(\mathcal{A}, n-1)$ is a prime ideal which is a saturation of $\mathcal{I}_1(\mathcal{A}, n-1)$, it is an associated prime of $\mathcal{S}(I)$. Therefore, $\operatorname{depth}(\mathcal{S}(I)) \le \dim \mathcal{R}(I) = k+1$.

(ii) Since $\mathcal{I}(\mathcal{A}, n-1)$ is a saturation of $\mathcal{I}_1(\mathcal{A}, n-1)$ by I, one has $\mathcal{I}(\mathcal{A}, n-1)I^t \subset \mathcal{I}_1(\mathcal{A}, n-1)$, for some $t \geq 1$. This implies that any (minimal) prime of $\mathcal{S}(I)$ in T contains either I or $\mathcal{I}(\mathcal{A}, n-1)$. By the proof of (i), $\mathcal{I}(\mathcal{A}, n-1)$ is an associated prime of $\mathcal{S}(I)$, hence it must be a minimal prime thereof since a minimal prime of $\mathcal{S}(I)$ properly contained in it would have to contain I, which is absurd.

Now, suppose $P \subset T$ is a minimal prime of $\mathcal{S}(I)$ containing I. One knows by Lemma 3.1 that $\mathfrak{m} = (\mathbf{x})T$ is a minimal prime of $\mathcal{S}(I)$. Therefore, we assume that $\mathfrak{m}T \not\subset P$. Since any minimal prime of I is a complete intersection of two distinct linear forms of $\mathcal{A}$, P contains at least two, and at most $k-1$, linearly independent linear forms of $\mathcal{A}$. On the other hand, since $\mathcal{I}_1(\mathcal{A}, n-1) \subset P$, looking at the generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ as in Lemma 3.1(b), by a domino effect principle we finally reach the desired format for P as stated.

(iii) With the notation prior to the statement of the proposition, we claim the following equality:

$$\mathcal{I}_1(\mathcal{A}, n-1) : I_k(B)^\infty = \mathfrak{m}T.$$

It suffices to show for the first quotient as $\mathfrak{m}T$ is a prime ideal. The inclusion $\mathfrak{m}I_k(B) \subset \mathcal{I}_1(\mathcal{A}, n-1)$ is a consequence of the Cramer rule. The reverse inclusion is obvious because $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathfrak{m}T$ implies that $\mathcal{I}_1(\mathcal{A}, n-1) : I_k(B) \subset \mathfrak{m}T : I_k(B) = \mathfrak{m}T$, as $\mathfrak{m}T$ is a prime ideal. Note that, as a very crude consequence, one has $I_k(B) \subset \mathcal{I}(\mathcal{A}, n-1)$. Now, let $\mathcal{P}(\mathfrak{m}T)$ denote the primary component of $\mathfrak{m}T$ in $\mathcal{I}_1(\mathcal{A}, n-1)$. Then

$$\mathfrak{m}T = \mathcal{I}_1(\mathcal{A}, n-1) : I_k(B)^\infty \subset \mathcal{P}(\mathfrak{m}T) : I_k(B)^\infty = \mathcal{P}(\mathfrak{m}T).$$

The same argument goes through for the primary component of $\mathcal{I}(\mathcal{A}, n-1)$ using the ideal I instead of $I_k(B)$.

To see the last statement of the proposition, let $\mathcal{P}$ denote the primary component of one of the remaining minimal primes P of $\mathcal{S}(I)$. Since $P : I_k(B)^\infty$ is P-primary and $\mathfrak{m} \not\subset P$, by the same token we get that $I_k(B) \subset P$. □

Remark 3.3. (a) It will be shown in the last section that the estimate in (i) is actually an equality.

As a consequence, every associated prime of $\mathcal{S}(I)$ viewed in T has codimension at most $n-1$. This will give a much better grip on the minimal primes of the form $\langle \ell_{i_1}, \ldots, \ell_{i_s}, y_{j_1}, \ldots, y_{j_t} \rangle$. Namely, one must have in addition that $s + t \leq n-1$ and, moreover, due to the domino effect principle, one must have $s = k-1$, hence $t \leq n-k$.

(b) We conjecture that $\mathcal{S}(I)$ is reduced. The property (R_0) of Serre's is easily verified due to the format of the Jacobian matrix as explained before the above proposition. The problem is, of course, the property (S_1), the known obstruction for the existence of embedded associated primes. The case where $n = k+1$, is easily determined. Here the minimal primes are seen to be $\mathfrak{m}$, $\langle x_1, \ldots, x_{k-1}, y_k \rangle$ and the Rees ideal $\langle \mathcal{I}_1(\mathcal{A}, k), \partial \rangle$, where ∂ is the relation corresponding to the unique circuit. A calculation will show that the three primes intersect in $\mathcal{I}_1(\mathcal{A}, k)$. As a side, this fact alone implies that the maximal regular sequence in the proof of Lemma 3.1(c) generates a radical ideal. For $n \geq k+2$ the calculation becomes sort of formidable, but we will prove later on that the Rees ideal is of fiber type.

(c) The weaker question as to whether the minimal component of $\mathcal{S}(I)$ is radical seems pliable.

If the conjectural statement in Remark 3.3(b) is true then, for any linear form $\ell = \ell_i$ the following basic formula holds:

$$\mathcal{I}_1(\mathcal{A}, n-1) : \ell = \mathcal{I}(\mathcal{A}, n-1) \cap g\Bigl(\bigcap_{\ell \notin P} P\Bigr),$$

where P denotes a minimal prime other that $\mathfrak{m}T$ and $\mathcal{I}(\mathcal{A}, n-1)$, as described in Proposition 3.2(i). Thus one would recover sectors of the Orlik–Terao generators inside this colon ideal. Fortunately, this latter virtual consequence holds true and has a direct simple proof. For convenience of later use, we state it explicitly. Let $\partial(\mathcal{A}|_\ell)$ denote the subideal of $\partial(\mathcal{A})$ generated by all polynomial relations ∂ corresponding to minimal dependencies (circuits) involving the linear form $\ell \in \mathcal{A}$.

Lemma 3.4. $$\partial(\mathcal{A}|_\ell) \subset \mathcal{I}_1(\mathcal{A}, n-1) : \ell.$$

Proof. Say, $\ell = \ell_1$. Let $D : a_1\ell_1 + a_2\ell_2 + \cdots + a_s\ell_s = 0$ be a minimal dependency involving ℓ_1, for some $3 \le s \le n$. In particular, $a_i \neq 0$, $i = 1, \ldots, s$. The corresponding generator of $\partial(\mathcal{A}|_{\ell_1})$ is

$$\partial D := a_1 y_2 y_3 \cdots y_s + a_2 y_1 y_3 \cdots y_s + \cdots + a_s y_1 y_2 \cdots y_{s-1}.$$

The following calculation is straightforward:

$$\begin{aligned}
\ell_1 \partial D &= a_1\ell_1 y_2 y_3 \cdots y_s + (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&\qquad\qquad + \ell_2 y_2 (a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&= (a_1\ell_1 + a_2\ell_2) y_2 y_3 \cdots y_s + (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&\qquad\qquad + \ell_2 y_2 (a_3 y_2 y_4 \cdots y_s + \cdots + a_s y_2 y_3 \cdots y_{s-1}) \\
&= (-a_3\ell_3 - \cdots - a_s\ell_s) y_2 y_3 \cdots y_s + (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&\qquad\qquad + \ell_2 y_2^2 (a_3 y_4 \cdots y_s + \cdots + a_s y_3 \cdots y_{s-1}) \\
&= (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) + y_2(\ell_2 y_2 - \ell_3 y_3) a_3 y_4 \cdots y_s \\
&\qquad\qquad + \cdots + y_2(\ell_2 y_2 - \ell_s y_s) a_s y_3 \cdots y_{s-1} \\
&= a_2 y_3 \cdots y_s (\ell_1 y_1 - \ell_2 y_2) + a_3 y_2 y_4 \cdots y_s (\ell_1 y_1 - \ell_3 y_3) + \cdots + a_s y_2 \cdots y_{s-1} (\ell_1 y_1 - \ell_s y_s).
\end{aligned}$$

Hence the result. ☐

3B. *Sylvester forms.* The Orlik–Terao ideal $\partial(\mathcal{A})$ has an internal structure of classical flavor, in terms of Sylvester forms.

Proposition 3.5. *The generators $\partial(\mathcal{A})$ of the Orlik–Terao ideal are Sylvester forms obtained from the generators of the presentation ideal $\mathcal{I}_1(\mathcal{A}, n-1)$ of the symmetric algebra of I.*

Proof. Let D be a dependency $c_{i_1}\ell_{i_1} + \cdots + c_{i_m}\ell_{i_m} = 0$ with all coefficients $c_{i_j} \neq 0$. Let $f = \prod_{i=1}^n \ell_i$. Evaluating the Orlik–Terao element ∂D on the products we have

$$\partial D(f_1, \ldots, f_n) = \sum_{j=1}^m c_{i_j} \frac{f^{m-1}}{\prod_{j \neq k=1}^m \ell_{i_k}} = \sum_{j=1}^m c_{i_j} \frac{f^{m-1}}{\prod_{k=1}^m \ell_{i_k}} \ell_{i_j} = \frac{f^{m-1}}{\ell_{i_1} \cdots \ell_{i_m}} g(c_{i_1}\ell_{i_1} + \cdots + c_{i_m}\ell_{i_m}) = 0.$$

Therefore, $\partial D \in \mathcal{I}(\mathcal{A}, n-1)$, and since $\partial D \in S := \Bbbk[y_1, \dots, y_n]$, then $\partial D \in \langle \mathcal{I}(\mathcal{A}, n-1)_{(0,-)} \rangle$.

For the second part, suppose that the minimal generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ are

$$\Delta_1 := \ell_1 y_1 - \ell_2 y_2, \quad \Delta_2 := \ell_2 y_2 - \ell_3 y_3, \quad \dots, \quad \Delta_{n-1} := \ell_{n-1} y_{n-1} - \ell_n y_n.$$

Without loss of generality suppose $\ell_j = c_1\ell_1 + \cdots + c_{j-1}\ell_{j-1}$ is some arbitrary dependency D. We have

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{j-1} \end{bmatrix} = \begin{bmatrix} y_1 & -y_2 & 0 & \cdots & 0 & 0 \\ 0 & y_2 & -y_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & y_{j-2} & -y_{j-1} \\ -c_1 y_j & -c_2 y_j & -c_3 y_j & \cdots & -c_{j-2} y_j & y_{j-1} - c_{j-1} y_j \end{bmatrix} \cdot \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_{j-1} \end{bmatrix}.$$

The determinant of the $(j-1) \times (j-1)$ matrix we see above is $\pm\partial D$. □

3C. ***A lemma on deletion.*** In this and the next parts we build on the main tool of an inductive procedure.

Let $\mathcal{A}' = \mathcal{A} \setminus \{\ell_1\}$, and denote $n' := |\mathcal{A}'| = n-1$. We would like to investigate the relationship between the Rees ideal $\mathcal{I}(\mathcal{A}', n'-1)$ of $I_{n'-1}(\mathcal{A}')$ and the Rees ideal $\mathcal{I}(\mathcal{A}, n-1)$ of $I_{n-1}(\mathcal{A})$, both defined in terms of the naturally given generators.

To wit, we will denote the generators of $I_{n'-1}(\mathcal{A}')$ as

$$f_{12} := \ell_{[n]\setminus\{1,2\}}, \quad \dots, \quad f_{1n} := \ell_{[n]\setminus\{1,n\}}.$$

One can move between the two ideals in a simple manner, which is easy to verify:

$$I_{n-1}(\mathcal{A}) : \ell_1 = I_{n'-1}(\mathcal{A}').$$

Note that the presentation ideal $\mathcal{I}(\mathcal{A}', n'-1)$ of the Rees algebra of $I_{n'-1}(\mathcal{A}')$ with respect to these generators lives in the polynomial subring $T' := R[y_2, \dots, y_n] \subset T := R[y_1, y_2, \dots, y_n]$. From Lemma 3.1, we know that

$$\mathcal{I}_1(\mathcal{A}', n'-1)T = \langle \ell_2 y_2 - \ell_3 y_3, \ell_3 y_3 - \ell_4 y_4, \dots, \ell_{n-1} y_{n-1} - \ell_n y_n \rangle T \subset \mathcal{I}_1(\mathcal{A}, n-1).$$

Likewise, for the Orlik–Terao ideal (which is an ideal in $S' := \Bbbk[y_2, \dots, y_n] \subset S := \Bbbk[y_1, y_2, \dots, y_n]$), one obtains via Theorem 2.4

$$\partial(\mathcal{A}')S = \langle \mathcal{I}(\mathcal{A}', n'-1)_{(0,-)} \rangle S \subset \partial(\mathcal{A}) = \langle \mathcal{I}(\mathcal{A}, n-1)_{(0,-)} \rangle.$$

Lemma 3.6. *One has*

$$\mathcal{I}(\mathcal{A}, n-1) = \langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle : \ell_1^{\infty}.$$

Proof. The inclusion $\langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle : \ell_1^{\infty} \subset I(\mathcal{A}, n-1)$ is clear since we are saturating a subideal of a prime ideal by an element not belonging to the latter. We note that the codimension of $\langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle$ exceeds by 1 that of $I(\mathcal{A}', n'-1)$ since the latter is a prime ideal even after extending to the ambient ring T. Therefore, by a codimension counting it would suffice to show that the saturation is itself a prime ideal.

Instead, we choose a direct approach. Thus, let $F \in \mathcal{I}(\mathcal{A}, n-1)$ be (homogeneous) of degree d in variables $y_1, \dots, y_n$. We can write

$$F = y_1^u G_u + y_1^{u-1} G_{u-1} + \cdots + y_1 G_1 + G_0,\ 0 \le u \le d,$$

where $G_j \in \Bbbk[x_1, \dots, x_k][y_2, \dots, y_n]$, are homogeneous of degree $d-j$ in $y_2, \dots, y_n$ for $j = 0, \dots, u$.

Evaluating $y_i = f_i$, $i = 1, \dots, n$ we obtain

$$0 = F(f_1,\dots,f_n) = \ell_2^u f_{12}^u \ell_1^{d-u} G_u(f_{12},\dots,f_{1n}) + \cdots + \ell_2 f_{12} \ell_1^{d-1} G_1(f_{12},\dots,f_{1n}) + \ell_1^d G_0(f_{12},\dots,f_{1n}).$$

This means that

$$\ell_1^{d-u}\Big[\underbrace{\ell_2^u y_2^u G_u(y_2, \dots, y_n) + \cdots + \ell_1^{u-1}\ell_2 y_2 G_1(y_2, \dots, y_n) + \ell_1^u G_0(y_2, \dots, y_n)}_{F'}\Big] \in \mathcal{I}(\mathcal{A}', n'-1).$$

By writing $\ell_1 y_1 = \ell_1 y_1 - \ell_2 y_2 + \ell_2 y_2$, it is not difficult to see that

$$\ell_1^u F \equiv F' \mod \langle \ell_1 y_1 - \ell_2 y_2 \rangle,$$

hence the result. □

3D. ***Stretched arrangements with coefficients.*** Recall the notion of contraction and the inherent idea of a multiarrangement, as mentioned in Section 2. Here we wish to consider such multiarrangements, allowing moreover the repeated individual linear functionals corresponding to repeated hyperplanes to be tagged with a nonzero element of the ground field. For lack of better terminology, we call such a new gadget a *stretched arrangement with coefficients*. Note that, by construction, a stretched arrangement with coefficients $\mathcal{B}$ has a uniquely defined (simple) arrangement $\mathcal{A}$ as support. Thus, if $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ is a simple arrangement, then a stretched arrangement with coefficients $\mathcal{B}$ is of the form

$$\{\underbrace{b_{1,1}\ell_1, \dots, b_{1,m_1}\ell_1}_{H_1=\ker \ell_1}, \underbrace{b_{2,1}\ell_2, \dots, b_{2,m_2}\ell_2}_{H_2=\ker \ell_2}, \dots, \underbrace{b_{n,1}\ell_n, \dots, b_{n,m_n}\ell_n}_{H_n=\ker \ell_n}\},$$

where $0 \ne b_{i,j} \in \Bbbk$ and $H_i = \ker(\ell_i)$ has multiplicity m_i for any $1 \le i \le n$, and for convenience, we assume that $b_{i,1} = 1$. We set $m := m_1 + \cdots + m_n$, and emphasize the ingredients of a stretched arrangement by writing $\mathcal{B} = (\mathcal{A}, \boldsymbol{b})$ where $\boldsymbol{b}$ is the vector of the above coefficients in the same order.

Proceeding as in the situation of a simple arrangement, we introduce the collection of $(m-1)$-products of elements of $\mathcal{B}$ and denote $I_{m-1}(\mathcal{B})$ the ideal of R generated by them. As in the simple case, we consider the presentation ideal $\mathcal{I}(\mathcal{B}, m-1)$ of $I_{m-1}(\mathcal{B})$ with respect to its set of generators consisting of the $(m-1)$-products. The next lemma relates this ideal to the previously considered presentation ideal $\mathcal{I}(\mathcal{A}, n-1)$ of $I_{n-1}(\mathcal{A})$ obtained by taking the set of generators consisting of the $(n-1)$-products of elements of $\mathcal{A}$.

Lemma 3.7. *Let $\mathcal{A}$ denote an arrangement and let $\mathcal{B} = (\mathcal{A}, \boldsymbol{b})$ denote a stretched arrangement supported on $\mathcal{A}$, as above. Let $G \in R$ stand for the* gcd *of the $(m-1)$-products of elements of $\mathcal{B}$. Then*:

(i) *The vector of the $(m-1)$-products of elements of $\mathcal{B}$ has the form $G \cdot P_{\mathcal{A}}$, where $P_{\mathcal{A}}$ denotes the vector whose coordinates are the $(n-1)$-products of the corresponding simple $\mathcal{A}$, each such product repeated* *as many times as the stretching in $\mathcal{B}$ of the corresponding linear form deleted in the expression of the product, and further tagged with a certain coefficient*;

(ii) $\mathcal{I}(\mathcal{B}, m-1) = \langle \mathcal{I}(\mathcal{A}, n-1), \mathcal{D}_{\mathcal{A}} \rangle$, *where* $\mathcal{D}_{\mathcal{A}}$ *denotes the* $\Bbbk$*-linear dependency relations among elements of* $P_{\mathcal{A}}$.

Proof. The first statement follows from the definition of a stretched arrangement vis-à-vis its support arrangement. Now, by (i), the Rees algebra of $I_{m-1}(\mathcal{B})$ is isomorphic to the Rees algebra of the ideal with generating set $P_{\mathcal{A}}$. By the nature of the latter, the second statement is now clear. □

4. The main theorems

We keep the previous notation as in Section 3C, where $I_{n-1}(\mathcal{A})$ is the ideal of $(n-1)$-fold products of a central arrangement $\mathcal{A}$ of size n and rank k. We had $T := R[y_1, \dots, y_n]$, with $R := \Bbbk[x_1, \dots, x_k]$, $S := \Bbbk[y_1, \dots, y_n]$, and $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathcal{I}(\mathcal{A}, n-1) \subset T$ denote, respectively, the presentation ideals of the symmetric algebra and of the Rees algebra of I. Recall that from Theorem 2.4, the Orlik–Terao ideal $\partial(\mathcal{A})$ coincides with the defining ideal $(\mathcal{I}(\mathcal{A}, n-1)_{(0,-)})S$ of the special fiber algebra of I.

4A. ***The case of a generic arrangement.*** Simple conceptual proofs can be given in the case where $\mathcal{A}$ is generic (meaning that any k of the defining linear forms are linearly independent), as follows.

Proposition 4.1. *If* $\mathcal{A} = \{\ell_1, \dots, \ell_n\} \subset R = \Bbbk[x_1, \dots, x_k]$ *is a generic arrangement, one has*:

(a) $I := I_{n-1}(\mathcal{A})$ *is an ideal of fiber type.*

(b) *The Rees algebra* $R[It]$ *is Cohen–Macaulay.*

(c) *The Orlik–Terao ideal of* $\mathcal{A}$ *is the* 0*-th Fitting ideal of the Jacobian dual matrix of* I (*i.e.*, *the ideal generated by the* $k \times k$ *minors of the Jacobian matrix of the generators of* $\mathcal{I}_1(\mathcal{A}, n-1)$ *with respect to the variables of* R).

(d) *Let* $k = n$, *i.e.*, *the case of Boolean arrangement. Under the standard bigrading* $\deg x_i = (1, 0)$ *and* $\deg y_j = (0, 1)$, *the bigraded Hilbert series of* $R[It]$ *is*

$$HS(R[It]; u, v) = \frac{(1-uv)^{k-1}}{(1-u)^k (1-v)^k}.$$

Proof. As described in the proof of Lemma 3.1, I is a linearly presented codimension 2 perfect ideal with syzygy matrix of the following shape

$$\varphi = \begin{bmatrix} \ell_1 & & & \\ -\ell_2 & \ell_2 & & \\ & -\ell_3 & \ddots & \\ & & \ddots & \ell_{n-1} \\ & & & -\ell_n \end{bmatrix}.$$

The Boolean case $n = k$ is well known, so we assume that $\mu(I) = n > k$. We claim that I satisfies the G_k condition. For this purpose we check the requirement in (2). First note that, for $p \geq n-k+1$, one has

$$I_p(\varphi) = I_p(\mathcal{A}),$$

where the rightmost ideal is the ideal generated by all p-fold products of the linear forms defining $\mathcal{A}$, as in our earlier notation. Because $\mathcal{A}$ is generic, it is the support of the codimension $(n-p+1)$-star configuration V_{n-p+1}; see [Geramita et al. 2013]. By Proposition 2.9(4) there, the defining ideal of V_{n-p+1} is a subset of $I_p(\mathcal{A})$, hence $\operatorname{ht}(I_p(\mathcal{A})) \geq n-p+1$. By [Tohăneanu 2010], any minimal prime of $I_p(\mathcal{A})$ can be generated by $n-p+1$ elements. Therefore, $\operatorname{ht}(I_p(\mathcal{A})) \leq n-p+1$, and hence equality.

The three statements now follow from [Morey and Ulrich 1996, Theorem 1.3], where (a) and (c) are collected together by saying that $R[It]$ has a presentation ideal of the expected type, quite stronger than being of fiber type. Note that, as a bonus, the same theorem also gives that $\ell(I) = k$ and $r(I) = k-1$, which are parts (b) and (d) in Corollary 2.6, when $\mathcal{A}$ is generic.

Part (d) follows from an immediate application of [Robbiano and Valla 1998, Theorem 5.11] to the $(k-1)\times k$ matrix

$$M = \begin{bmatrix} x_1 & 0 & \dots & 0 & x_k \\ 0 & x_2 & \dots & 0 & x_k \\ \vdots & & \ddots & \vdots & x_k \\ 0 & 0 & \dots & x_{k-1} & x_k \end{bmatrix}.$$

One can verify that the codimension of $I_t(M)$, the ideal of size t minors of M, is $k-t+2$. Note that their setup is different in that they set $\deg y_j = (n-1, 1)$, whereas for us $\deg y_j = (0, 1)$. To get our formula we make the substitution in their formula: $a \leftrightarrow u$, and $a^{n-1}b \leftrightarrow v$. □

4B. ***The fiber type property.*** In this part we prove one of the main assertions of the section and state a few structural consequences.

Theorem 4.2. *Let $\mathcal{A}$ be a central arrangement of rank $k \geq 2$ and size $n \geq k$. The ideal $I_{n-1}(\mathcal{A})$ of $(n-1)$-fold products of $\mathcal{A}$ is of fiber type*:

$$\mathcal{I}(\mathcal{A}, n-1) = \langle \mathcal{I}_1(\mathcal{A}, n-1)\rangle + \langle \mathcal{I}(\mathcal{A}, n-1)_{(0,-)}\rangle,$$

as ideals in T, where $\langle \mathcal{I}(\mathcal{A}, n-1)_{(0,-)}\rangle_S = \partial(\mathcal{A})$ is the Orlik–Terao ideal.

Proof. We first consider the case where $n = k$. Then $I_{n-1}(\mathcal{A})$ is an ideal of linear type by Lemma 3.1, that is to say, $\mathcal{I}(\mathcal{A}, n-1) = \mathcal{I}_1(\mathcal{A}, n-1)$. This proves the statement of the theorem since $\partial(\mathcal{A}) = 0$ in this case.

We now prove the statement by induction on the pairs (n, k), where $n > k \geq 2$. In the initial induction step, we deal with the case $k = 2$ and arbitrary $n > 2$ (the argument will even be valid for $n = 2$). Here one claims that $I_{n-1}(\mathcal{A}) = \langle x_1, x_2\rangle^{n-1}$. In fact, since no two forms of the arrangement are proportional, the generators of $I_{n-1}(\mathcal{A})$ are $\Bbbk$-linearly independent because, e.g., dehomogenizing in one of the variables yields the first n powers of the other variable up to elementary transformations. Also, since these forms have degree $n-1$, they forcefully span the power $\langle x_1, x_2\rangle^{n-1}$.

Now, any $\langle x_1, x_2\rangle$-primary ideal in $\Bbbk[x_1, x_2]$ automatically satisfies the property G_2; see (2). Therefore, the Rees ideal is of fiber type, and in fact it is of the expected type and Cohen–Macaulay by [Morey and Ulrich 1996, Theorem 1.3]. In any case, the Rees ideal has long been known in this case, with the defining ideal of the special fiber generated by the 2-minors of the generic $2 \times (n-1)$ Hankel matrix, i.e., by the homogeneous defining ideal of the rational normal curve in $\mathbb{P}^{n-1}$; see [Corsini 1967].

For the main induction step, suppose $n > k > 2$ and let $\mathcal{A}' := \mathcal{A} \setminus \{\ell_1\}$ stand for the deletion of ℓ_1, a subarrangement of size $n' := n-1$. Applying a change of variables in the base ring R — which, as already remarked, does not disturb the ideal-theoretic properties in sight — we can assume that $\ell_1 = x_1$ and $\ell_2 = x_2$. The extended ideals $\mathcal{I}(\mathcal{A}', n'-1)T$, $\partial(\mathcal{A}')S$, $\mathcal{I}_1(\mathcal{A}', n'-1)T$ will be of our concern. The following equalities of ideals of T are easily seen to hold:

$$\begin{aligned} \mathcal{I}_1(\mathcal{A}, n-1) &= \langle x_1y_1 - x_2y_2, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle \quad \text{as ideals in } T,\\ \partial(\mathcal{A}) &= \langle \partial(\mathcal{A}|_{x_1}), \partial(\mathcal{A}')\rangle, \qquad\qquad\quad\; \text{as ideals in } S. \end{aligned} \tag{6}$$

Let $F \in \mathcal{I}(\mathcal{A}, n-1)$ be bihomogeneous with $\deg_{\mathbf{y}}(F) = d$. Suppose that $M = x_1^a y_1^b N \in T$ is a monomial that appears in F, where $x_1, y_1 \nmid N$. If $a \geq b$, we can write

$$M = x_1^{a-b}(x_1y_1 - x_2y_2 + x_2y_2)^b N,$$

and hence

$$M \equiv x_1^{a-b} x_2^b y_2^b N \mod \langle x_1y_1 - x_2y_2\rangle.$$

If $a < b$, we have

$$M = (x_1y_1 - x_2y_2 + x_2y_2)^a y_1^{b-a} N,$$

and hence

$$M \equiv x_2^a y_2^a y_1^{b-a} N \mod \langle x_1y_1 - x_2y_2\rangle.$$

Denote $R' := \Bbbk[x_2, \dots, x_k] \subset R$, $T'' := R'[y_2, \dots, y_n] \subset T' := R[y_2, \dots, y_n] \subset T$. In any case, one can write

$$F = (x_1y_1 - x_2y_2)Q + x_1^{m_1}P_1 + x_1^{m_2}P_2 + \cdots + x_1^{m_u}P_u + P_{u+1},\ m_1 > \cdots > m_u \geq 1,$$

for certain forms $Q \in T$, $P_1, \dots, P_u \in T''$, and $P_{u+1} \in R'[y_1, \dots, y_n] = T''[y_1]$ of degree d in the variables $y_1, \dots, y_n$. Also

$$P_{u+1} = y_1^v G_v + y_1^{v-1}G_{v-1} + \cdots + y_1 G_1 + G_0,$$

where $G_j \in T''$ and $\deg(G_j) = d - j$, $j = 0, \dots, v$. Let us use the elements we have seen at the beginning of Section 3C, as generators for $I_{n'-1}(\mathcal{A}')$:

$$f_{12} := \ell_{[n]\setminus\{1,2\}}, \dots, f_{1n} := \ell_{[n]\setminus\{1,n\}}.$$

Since evaluating $F \in \mathcal{I}(\mathcal{A}, n-1)$ at

$$y_1 \mapsto f_1 = x_2\ell_3 \cdots \ell_n, \quad y_2 \mapsto f_2 = x_1 f_{12}, \quad \dots, \quad y_n \mapsto f_n = x_1 f_{1n}$$

vanishes, upon pulling out the appropriate powers of x_1, it yields

$$0 = x_1^{m_1+d} P_1(f_{12}, \dots, f_{1n}) + \cdots + x_1^{m_u+d} P_u(f_{12}, \dots, f_{1n}) \\ + f_1^v x_1^{d-v} G_v(f_{12}, \dots, f_{1n}) + \cdots + f_1 x_1^{d-1} G_1(f_{12}, \dots, f_{1n}) + x_1^d G_0(f_{12}, \dots, f_{1n}).$$

Suppose first that the rank of $\mathcal{A}'$ is $k-1$, i.e., x_1 is a coloop. This means that $x_2 = \ell_2, \ell_3, \dots, \ell_n$ are actually forms in the subring $R' = \Bbbk[x_2, \dots .x_k]$. Since $m_1 + d > \cdots > m_u + d > d > \cdots > d - v$,

$$P_i(f_{12}, \dots, f_{1n}) = 0, i = 1, \dots, u, \quad G_j(f_{12}, \dots, f_{1n}) = 0, j = 0, \dots, v.$$

Therefore, $P_i, G_j \in \mathcal{I}(\mathcal{A}', n'-1)$, and hence $F \in \langle x_1 y_1 - x_2 y_2, \mathcal{I}(\mathcal{A}', n'-1)\rangle$. This shows that

$$\mathcal{I}(\mathcal{A}, n-1) = \langle x_1 y_1 - x_2 y_2, \mathcal{I}(\mathcal{A}', n'-1)\rangle$$

and the required result follows by the inductive hypothesis as applied to $\mathcal{I}(\mathcal{A}', n'-1)$. Suppose now that the rank of $\mathcal{A}'$ does not drop, i.e., x_1 is a noncoloop.

Case 1: $v = 0$. In this case, after canceling x_1^d, we obtain

$$0 = x_1^{m_1} P_1(f_{12}, \dots, f_{1n}) + \cdots + x_1^{m_u} P_u(f_{12}, \dots, f_{1n}) + G_0(f_{12}, \dots, f_{1n}).$$

Thus,

$$x_1^{m_1} P_1 + x_1^{m_2} P_2 + \cdots + x_1^{m_u} P_u + P_{u+1} \in \mathcal{I}(\mathcal{A}', n'-1).$$

Case 2: $v \geq 1$. In this case we cancel the factor x_1^{d-v} in the above equation. This will give

$$x_1 \mid G_v(f_{12}, \dots, f_{1n}).$$

At this point we resort to the idea of stretched arrangements with coefficients as developed in Section 3D. Namely, we take the restriction (contraction) of $\mathcal{A}$ to the hyperplane $x_1 = 0$. Precisely, say

$$\ell_i = a_i x_1 + \bar{\ell}_i, \quad \text{where } \bar{\ell}_i \in R', a_i \in \Bbbk,$$

for $i = 2, \dots, n$. Note that $a_2 = 0$ since $\ell_2 = x_2$. Write

$$\bar{\mathcal{A}} = \{\bar{\ell}_2, \dots, \bar{\ell}_n\} \subset R',$$

a stretched arrangement of total multiplicity $\bar{n} = n - 1$ with support $\mathcal{A}''$ of size $n'' \leq \bar{n}$. Likewise, let

$$\bar{f}_{12} := \bar{\ell}_3 \cdots \bar{\ell}_n, \quad \dots, \quad \bar{f}_{1n} := \bar{\ell}_2 \cdots \bar{\ell}_{n-1}$$

denote the $(\bar{n}-1)$-products of this stretched arrangement. Then, G_v vanishes on the tuple $(\bar{f}_{12}, \dots, \bar{f}_{1n})$ and since its is homogeneous it necessarily belong to $\mathcal{I}(\bar{\mathcal{A}}, \bar{n}-1)$. From Lemma 3.7, we have

$$\mathcal{I}(\bar{\mathcal{A}}, \bar{n}-1) = \langle \mathcal{I}(\mathcal{A}'', n''-1), \mathcal{D}_{\bar{\mathcal{A}}}\rangle,$$

where $\mathcal{D}_{\bar{\mathcal{A}}}$ is a linear ideal of the form $\langle y_i - b_{i,j} y_j \rangle_{2 \leq i,j \leq n}$. Let us analyze the generators of $\mathcal{I}(\bar{\mathcal{A}}, \bar{n}-1)$.

- A generator $y_i - b_{i,j}y_j,\ i, j \geq 2$ of $\mathcal{D}_{\bar{\mathcal{A}}}$ comes from the relation $\bar{\ell}_j = b_{i,j}\bar{\ell}_i,\ b_{i,j} \in \Bbbk$. Thus, back in $\mathcal{A}$ we have the minimal dependency
$$\ell_j - a_j x_1 = b_{i,j}(\ell_i - a_i x_1),$$
yielding an element of $\partial(\mathcal{A}|_{\ell_1})$:
$$y_1(y_i - b_{i,j}y_j) + \underbrace{(b_{i,j}a_i - a_j)}_{c_{i,j}} y_i y_j.$$
- Since $\gcd(\bar{\ell}_i, \bar{\ell}_j) = 1$, for $2 \leq i < j \leq n'' + 1$, a typical generator of $\mathcal{I}_1(\mathcal{A}'', n'' - 1)$ is $\bar{\ell}_i y_i - \bar{\ell}_j y_j$, that we will rewrite as
$$\bar{\ell}_i y_i - \bar{\ell}_j y_j = (\ell_i y_i - \ell_j y_j) - x_1(a_i y_i - a_j y_j).$$
- A typical generator of $\partial(\mathcal{A}'')$ is of the form $b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}$ coming from a minimal dependency
$$b_1\bar{\ell}_{i_1} + \cdots + b_s\bar{\ell}_{i_s} = 0, \quad i_j \in \{2, \ldots, n'' + 1\}.$$
Since $\bar{\ell}_{i_j} = \ell_{i_j} - a_{i_j}x_1$, we obtain a dependency
$$b_1\ell_{i_1} + \cdots + b_s\ell_{i_s} - (\underbrace{b_1 a_{i_1} + \cdots + b_s a_{i_s}}_{\alpha})x_1 = 0.$$
If $\alpha = 0$, then
$$b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}} \in \partial(\mathcal{A}'),$$
whereas if $\alpha \neq 0$, then
$$-\alpha y_{i_2} \cdots y_{i_s} + y_1(b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}) \in \partial(\mathcal{A}|_{\ell_1}).$$

We have that
$$G_v = \sum E_{s,t}(y_s - b_{s,t}y_t) + \sum A_{i,j}(\bar{\ell}_i y_i - \bar{\ell}_j y_j) + \sum B_{i_1,\ldots,i_s}(b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}),$$
where $E_{s,t}, A_{i,j}, B_{i_1,\ldots,i_s} \in T''$ and $s, t, i, j, i_k \geq 2$. Then, by using the expressions in the three bullets above and splicing according to the equality $x_1y_1 = (x_1y_1 - x_2y_2) + x_2y_2$, we get
$$\begin{aligned} y_1^v G_v = y_1^{v-1}\Bigg(&\underbrace{\sum E_{s,t}(y_1(y_s - b_{s,t}y_t) + c_{s,t}y_s y_t)}_{\in\partial(\mathcal{A}|_{\ell_1})} - \underbrace{\sum E_{s,t}c_{s,t}y_s y_t}_{\in T''} \\ &+ \underbrace{\sum A_{i,j}y_1(\ell_i y_i - \ell_j y_j)}_{\in\mathcal{I}(\mathcal{A}', n'-1)} - \underbrace{\sum A_{i,j}(x_1y_1 - x_2y_2)(a_i y_i - a_j y_j)}_{\in\langle x_1y_1 - x_2y_2\rangle} - \underbrace{\sum A_{i,j}x_2y_2(a_i y_i - a_j y_j)}_{\in T''} \\ &+ \underbrace{\sum B_{i_1,\ldots,i_s}(y_1(b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}) - \alpha y_{i_2} \cdots y_{i_s})}_{\in\partial(\mathcal{A}|_{\ell_1})} + \underbrace{\sum B_{i_1,\ldots,i_s}\alpha y_{i_2} \cdots y_{i_s}}_{\in T''}\Bigg). \end{aligned}$$

Thus, $y_1^v G_v = y_1^{v-1} G'_{v-1} + W$, where

$$G'_{v-1} \in T'', \quad W \in \langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1)\rangle.$$

Then returning to our original F, one obtains

$$F = \Delta + x_1^{m_1} P_1 + \cdots + x_1^{m_u} P_u + y_1^{v-1}\underbrace{(G'_{v-1} + G_{v-1})}_{G''_{v-1} \in S''} + y_1^{v-2} G_{v-2} + \cdots + G_0,$$

where $\Delta \in \langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1)\rangle \subset \mathcal{I}(\mathcal{A}, n-1)$. The key is that modulo the ideal $\langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1)\rangle$, the power of y_1 dropped from v to $v-1$ in the expression of F. Iterating, with $F(f_1, \ldots, f_n) = 0 = \Delta(f_1, \ldots, f_n)$, will eventually drop further the power of y_1 to $v-2$. Recursively we end up with $v = 0$, which is Case 1 above. This way, we eventually get

$$\mathcal{I}(\mathcal{A}, n-1) = \langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1)\rangle.$$

By the inductive hypothesis as applied to $\mathcal{I}(\mathcal{A}', n'-1)$ and from the two equalities in (6), one gets the stated result. □

Corollary 4.3. $\mathcal{I}(\mathcal{A}', n'-1) = \mathcal{I}(\mathcal{A}, n-1) \cap T'$ *as ideals in* $T' = R[y_2, \ldots, y_n]$.

Proof. Recall the notation $T' := R[y_2, \ldots, y_n] \subset T := R[y_1, \ldots, y_n] = T'[y_1]$ as in the proof of the previous theorem. Denote $J := \mathcal{I}(\mathcal{A}, n-1) \cap T'$. We show that $J \subseteq \mathcal{I}(\mathcal{A}', n'-1)$, the other inclusion being obvious. Let $F \in J$. Then $F \in T'$ and $F \in \mathcal{I}(\mathcal{A}, n-1)$. By Theorem 4.2, we can write

$$F = (\ell_1 y_1 - \ell_2 y_2)P + Q + G, \quad \text{where } P \in T,\ Q \in \partial(\mathcal{A}|_{\ell_1})T,\ G \in \mathcal{I}(\mathcal{A}', n'-1)T.$$

By Lemma 3.4,

$$\ell_1 Q \in \mathcal{I}_1(\mathcal{A}, n-1) = \langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle.$$

Therefore,

$$\ell_1 F = (\ell_1 y_1 - \ell_2 y_2)P' + G', \tag{7}$$

for suitable $P' \in T$, $G' \in \mathcal{I}(\mathcal{A}', n'-1)T$. We write $P' = y_1^u P_u + \cdots + y_1 P_1 + P_0$, $P_i \in T'$, and $G' = y_1^v G_v + \cdots + y_1 G_1 + G_0$, $G_j \in T'$. Since $G' \in \mathcal{I}(\mathcal{A}', n'-1) \subset T'$, setting $y_1 = 0$ in the expression of G' gives that $G_0 \in \mathcal{I}(\mathcal{A}', n'-1)$. Therefore, $G - G_0 = y_1(y_1^{v-1} G_v + \cdots + G_1) \in \mathcal{I}(\mathcal{A}', n'-1)$, and hence $y_1^{v-1} G_v + \cdots + G_1 \in \mathcal{I}(\mathcal{A}', n'-1)$ since $\mathcal{I}(\mathcal{A}', n'-1)$ is prime. Setting again $y_1 = 0$ in this expression we obtain that $G_1 \in \mathcal{I}(\mathcal{A}', n'-1)$, and so on, eventually obtaining

$$G_j \in \mathcal{I}(\mathcal{A}', n'-1), \quad j = 0, \ldots, v.$$

Suppose $u \geq v$. Then, by grouping the powers of y_1 we have

$$\begin{aligned}\ell_1 F = (-\ell_2 y_2 P_0 + G_0) + (\ell_1 P_0 - \ell_2 y_2 P_1 + G_1) y_1 + \cdots + (\ell_1 P_{v-1} - \ell_2 y_2 P_v + G_v) y_1^v \\ + (\ell_1 P_v - \ell_2 y_2 P_{v+1}) y_1^{v+1} + \cdots + (\ell_1 P_{u-1} - \ell_2 y_2 P_u) y_1^u + \ell_1 P_u y_1^{u+1}.\end{aligned}$$

Since $F \in T'$, then $\ell_1 F \in T'$. Thus, the "coefficients" of $y_1, y_1^2, \ldots, y_1^{u+1}$ must vanish. It follows that

$$P_u = \cdots = P_v = 0 \quad \text{and} \quad \ell_1 P_{v-1} = -G_v \in \mathcal{I}(\mathcal{A}', n'-1).$$

Since $\mathcal{I}(\mathcal{A}', n'-1)$ is a prime ideal, we have $P_{v-1} \in \mathcal{I}(\mathcal{A}', n'-1)$, and therefore

$$\ell_1 P_{v-2} = \ell_2 y_2 P_{v-1} - G_{v-1} \in \mathcal{I}(\mathcal{A}', n'-1).$$

Recursively we get that

$$P_{v-1}, P_{v-2}, \ldots, P_1, P_0 \in \mathcal{I}(\mathcal{A}', n'-1).$$

If $u < v$, a similar analysis will give the same conclusion that $P' \in \mathcal{I}(\mathcal{A}', n'-1)T$. Therefore, (7) gives $\ell_1 F \in \mathcal{I}(\mathcal{A}', n'-1)T$, and hence $F \in \mathcal{I}(\mathcal{A}', n'-1)T$ by primality of the extended ideal. But then $F \in I(\mathcal{A}', n'-1)T \cap T' = I(\mathcal{A}', n'-1)$, as required. □

The next two corollaries help compute the Rees ideal from the symmetric ideal via a simple colon of ideals.

Corollary 4.4. *Let $\ell_i \in \mathcal{A}$ and y_i be the corresponding external variable. Then*

$$\mathcal{I}(\mathcal{A}, n-1) = \mathcal{I}_1(\mathcal{A}, n-1) : \ell_i y_i.$$

Proof. Without loss of generality, assume $i = 1$. The inclusion $\supseteq$ is immediate, since $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathcal{I}(\mathcal{A}, n-1)$, and the Rees ideal $\mathcal{I}(\mathcal{A}, n-1)$ is a prime ideal not containing ℓ_1 nor y_1. For the reverse inclusion, let $F \in \mathcal{I}(\mathcal{A}, n-1)$. Then, from Theorem 4.2,

$$F = G + \sum_D P_D \partial D,$$

where the sum is taken over all minimal dependencies D, and $G \in \mathcal{I}_1(\mathcal{A}, n-1)$.

Obviously, $\ell_1 y_1 G \in \mathcal{I}_1(\mathcal{A}, n-1)$. Also, if $\partial D \in \partial(\mathcal{A}|_{\ell_1})$, then, from Lemma 3.4, $\ell_1 \partial D \in \mathcal{I}_1(\mathcal{A}, n-1)$, hence $\ell_1 y_1 \partial D$ belongs to $\mathcal{I}_1(\mathcal{A}, n-1)$ as well. Suppose $\partial D \notin \partial(\mathcal{A}|_{\ell_1})$. Since D is a minimal dependency among the hyperplanes of $\mathcal{A}$, there exists $j \in \{1, \ldots, n\}$ such that $\partial D \in \partial(\mathcal{A}|_{\ell_j})$. Thus, $\ell_1 y_1 \partial D = (\ell_1 y_1 - \ell_j y_j)\partial D + \ell_j y_j \partial D$ belongs to the ideal $\mathcal{I}_1(\mathcal{A}, n-1)$ since each summand belongs to $\mathcal{I}_1(\mathcal{A}, n-1)$, the first trivially and the second due to Lemma 3.4. □

Since the rank of $\mathcal{A}$ is k, after a reordering of the linear forms $\ell_1, \ldots, \ell_n$ that define $\mathcal{A}$, we can assume that the last k linear forms $\ell_{n-k+1}, \ldots, \ell_n$ are linearly independent. With this proviso, one has:

Corollary 4.5.

$$\mathcal{I}(\mathcal{A}, n-1) = \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i.$$

Proof. Since $\ell_{n-k+1}, \ldots, \ell_n$ are k linearly independent linear forms, any minimal dependency that involves at least one of them, must involve also a linear form ℓ_j, where $j \in \{1, \ldots, n-k\}$. So

$$\partial(\mathcal{A}) = \partial(\mathcal{A}|_{\ell_1}) + \cdots + \partial(\mathcal{A}|_{\ell_{n-k}}).$$

We obviously have $\mathcal{I}_1(\mathcal{A}, n-1) \subseteq \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i$, and from Lemma 3.4,

$$\partial(\mathcal{A}|_{\ell_j}) \subset \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i, \quad \text{for all } j = 1, \dots, n-k.$$

Then, from Theorem 4.2, one has

$$\mathcal{I}(\mathcal{A}, n-1) \subseteq \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i.$$

The reverse inclusion comes from the fact that $\mathcal{I}_1(\mathcal{A}, n-1) \subseteq \mathcal{I}(\mathcal{A}, n-1)$, and from $\mathcal{I}(\mathcal{A}, n-1)$ being a prime ideal with $\ell_i \notin \mathcal{I}(\mathcal{A}, n-1)$. □

In the next statement we denote the extended ideal $(\mathcal{I}_1(\mathcal{A}', n'-1))T$ by $\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle$.

Lemma 4.6. *Let $\mathcal{A}' = \mathcal{A} \setminus \{\ell_1\}$ and $n' = |\mathcal{A}'| = n-1$. We have*

$$\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : (\ell_1 y_1 - \ell_2 y_2) = \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : \ell_1.$$

In particular, when ℓ_1 is a coloop, the biform $\ell_1 y_1 - \ell_2 y_2$ is a nonzero divisor on $\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle$.

Proof. For convenience, let us change coordinates to have $\ell_1 = x_1$ and $\ell_2 = x_2$. Let $f \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : (x_1 y_1 - x_2 y_2)$. Then $f(x_1 y_1 - x_2 y_2) \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle \subset \langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$. Since $\langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$ is a prime ideal not containing $x_1 y_1 - x_2 y_2$, we obtain $f \in \langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$, and by Theorem 4.2, we have

$$f = g + h, \quad g \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle, \quad h \in \langle \partial(\mathcal{A}')\rangle.$$

By multiplying this by $x_1 y_1 - x_2 y_2$, we get that

$$(x_1 y_1 - x_2 y_2) h \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle.$$

By Corollary 4.4, since $h \in \langle \partial(\mathcal{A}')\rangle \subset \langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$, and $x_2 \in \mathcal{A}'$, we have $x_2 y_2 h \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle$. So $h \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1 y_1$, and together with $f = g + h$ with $g \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle \subset \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1 y_1$, gives

$$f \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1 y_1.$$

Conversely, let $\Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1 y_1$. Then $x_1 y_1 \Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle \subseteq \langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$. The ideal $\langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$ is a prime ideal, and $x_1 y_1 \notin \langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$, so $\Delta \in \langle \mathcal{I}(\mathcal{A}', n'-1)\rangle$. So, by Corollary 4.4, $x_2 y_2 \Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle$. Therefore

$$(x_1 y_1 - x_2 y_2)\Delta = x_1 y_1 \Delta - x_2 y_2 \Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle.$$

Thus far, we have shown that $\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : (\ell_1 y_1 - \ell_2 y_2) = \langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1 y_1$. Clearly, the right hand side is the same as $\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1$ since y_1 is a nonzero divisor on $\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle$. □

4C. ***The Cohen–Macaulay property.*** In this part the goal is to prove that the Rees algebra is Cohen–Macaulay. Since we are in a graded setting, this is equivalent to showing that its depth with respect to the maximal graded ideal $\langle \mathfrak{m}, y_1, \dots, y_n\rangle$ is (at least) $k+1 = \dim R[It]$.

This will be accomplished by looking at a suitable short exact sequence, where two of the modules will be examined next. We state the results in terms of depth since this notion is inherent to the Cohen–Macaulay property, yet the proofs will take the approach via projective (i.e., homological) dimension. By the Auslander–Buchsbaum equality, we are home anyway. Throughout, pdim_T will denote projective dimension over the polynomial ring T. Since we are in a graded situation, this is the same as the projective dimension over the local ring $T_{(\mathfrak{m}, y_1, \dots, y_n)}$, so we may harmlessly proceed.

The first module is obtained by cutting the binomial generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ into its individual terms. The result may be of interest on its own.

Lemma 4.7. *Let $\ell_1, \dots, \ell_n \in \Bbbk[x_1, \dots, x_k]$ be linear forms, allowing some of them to be mutually proportional. Let $T := \Bbbk[x_1, \dots, x_k; y_1, \dots, y_n]$. Then*

$$\operatorname{depth}\Big(\frac{T}{\langle \ell_1 y_1, \ell_2 y_2, \dots, \ell_n y_n\rangle}\Big) \geq k.$$

Proof. If $k = 1$, the claim is clearly satisfied, since $\langle x_1y_1, \dots, x_1y_n\rangle = x_1\langle y_1, \dots, y_n\rangle$, and $\{y_1, \dots, y_n\}$ is a T-regular sequence. Assume $k \geq 2$.

We will use induction on $n \geq 1$ to show that the projective dimension is at most $n + k - k = n$. If $n = 1$, the ideal $\langle \ell_1 y_1\rangle$ is a principal ideal, hence the claim is true. Suppose $n > 1$. We may apply a $\Bbbk$-linear automorphism on the ground variables, which will not disturb the projective dimension. Thus, say, $\ell_1 = x_1$ and this form is repeated s times. Since nonzero coefficients from $\Bbbk$ tagged to x_1 will not change the ideal in question, we assume that $\ell_i = x_1$, for $1 \leq i \leq s$, and $\gcd(x_1, \ell_j) = 1$, for $s+1 \leq j \leq n$. Write $\ell_j = c_j x_1 + \bar{\ell}_j$, for $s+1 \leq j \leq n$, with $c_j \in \Bbbk$, and $0 \neq \bar{\ell}_j \in \Bbbk[x_2, \dots, x_k]$.

Denoting $J := \langle x_1y_1, \dots, x_1y_s, \ell_{s+1}y_{s+1}, \dots, \ell_n y_n\rangle$, we claim that

$$J : x_1 = \langle y_1, \dots, y_s, \ell_{s+1}y_{s+1}, \dots, \ell_n y_n\rangle. \tag{8}$$

This is certainly the expression of a more general result, but we give a direct proof here. One inclusion is obvious. For the reverse inclusion, let $F \in \langle x_1y_1, \dots, x_1y_s, \ell_{s+1}y_{s+1}, \dots, \ell_n y_n\rangle : x_1$. Then, say,

$$x_1 F = x_1 \sum_{i=1}^{s} P_i y_i + \sum_{j=s+1}^{n} P_j \ell_j y_j,$$

for certain $P_i, P_j \in T$. Rearranging we have

$$x_1\Big(F - \sum_{i=1}^{s} P_i y_i - \sum_{j=s+1}^{n} c_j P_j y_j\Big) = \sum_{j=s+1}^{n} P_j \bar{\ell}_j y_j. \tag{9}$$

Since x_1 is a nonzero divisor in $T/\langle\bar{\ell}_{s+1}y_{s+1},\dots,\bar{\ell}_n y_n\rangle$, the second factor of the left hand side in (9) must be of the form

$$\sum_{j=s+1}^{n} Q_j\bar{\ell}_j y_j,\qquad Q_j\in T.$$

Substituting in (9) we find $P_j=x_1Q_j$, $s+1\le j\le n$, and hence $F=\sum_{i=1}^{s}P_iy_i+\sum_{j=s+1}^{n}Q_j\ell_jy_j$, as claimed. Computing projective dimensions with respect to T and $T'=\Bbbk[x_1,\dots,y_{s+1},\dots,y_n]$ and applying the inductive hypothesis, one has

$$\operatorname{pdim}_T\Bigl(\frac{T}{J:x_1}\Bigr)=s+\operatorname{pdim}_{T'}\Bigl(\frac{T'}{\langle\ell_{s+1}y_{s+1},\dots,\ell_ny_n\rangle}\Bigr)\le s+(n-s)=n.$$

At the other end, we have $\langle x_1,J\rangle=\langle x_1,\bar{\ell}_{s+1}y_{s+1},\dots,\bar{\ell}_ny_n\rangle$. Applying the inductive hypothesis this time around gives

$$\operatorname{pdim}_T\Bigl(\frac{T}{\langle x_1,J\rangle}\Bigr)\le 1+(n-s)\le n.$$

From the short exact sequence of T-modules

$$0\to T/(J:x_1)\overset{\cdot x_1}{\to}T/J\to T/\langle x_1,J\rangle\to 0,$$

knowingly the projective dimension of the middle term does not exceed the maximum of the projective dimensions of the two extreme terms. Therefore, $\operatorname{pdim}_T(T/J)\le n$, as was to be shown. $\square$

The difficult result of this section is the following exact invariant of the symmetric algebra $\mathcal{S}(I)\simeq T/\langle\mathcal{I}_1(\mathcal{A},n-1)\rangle$:

Proposition 4.8. *Let $I=I_{n-1}(\mathcal{A})$ as before. Then* $\operatorname{depth}(\mathcal{S}(I))=k+1$.

Proof. By Proposition 3.2(i), it suffices to prove the lower bound $\operatorname{depth}(\mathcal{S}(I))\ge k+1$. As in the proof of Theorem 4.2, we argue by induction on all pairs n,k, with $n\ge k\ge 2$, where n and k are, respectively, the size and the rank of $\mathcal{A}$. If $k=2$ and $n>2$, let $R=\Bbbk[x,y]$. As seen in that proof, one has $I=\langle x,y\rangle^{n-1}$, and hence

$$\mathcal{I}_1(\mathcal{A},n-1)=\langle xy_1-yy_2,xy_2-yy_3,\dots,xy_{n-1}-yy_n\rangle.$$

A direct calculation shows that $\{y_1,x+y_n,y+y_{n-1}\}$ is a regular sequence modulo $\mathcal{I}_1(\mathcal{A},n-1)$. If $n=k$, the ideal $\mathcal{I}_1(\mathcal{A},n-1)$ is a complete intersection by Lemma 3.1. Thus, for the main inductive step suppose $n>k>2$. We will equivalently show that $\operatorname{pdim}_T(\mathcal{S}(I))\le n-1$. First apply a change of ground variables so as to have $\ell_1=x_1$ and $\ell_2=x_2$; let $\mathcal{A}':=\mathcal{A}\setminus\{x_1\}$ denote the deletion. Since $\mathcal{I}_1(\mathcal{A},n-1)=\langle\mathcal{I}_1(\mathcal{A}',n'-1),x_1y_1-x_2y_2\rangle$, we have the following short exact sequence of T-modules

$$0\to\frac{T}{\langle\mathcal{I}_1(\mathcal{A}',n'-1)\rangle:(x_1y_1-x_2y_2)}\xrightarrow{\cdot(x_1y_1-x_2y_2)}\frac{T}{\langle\mathcal{I}_1(\mathcal{A}',n'-1)\rangle}\to\frac{T}{\mathcal{I}_1(\mathcal{A},n-1)}\to 0.\qquad(10)$$

We consider separately the cases where ℓ_1 is a coloop or a noncoloop.

x_1 **is a coloop:** Here the rank of $\mathcal{A}'$ is $k-1$ and x_1 is altogether absent in the linear forms of the deletion. Thus, the natural ambient ring of $\mathcal{I}_1(\mathcal{A}', n'-1)$ is $T' := \Bbbk[x_2, \ldots, x_k;\, y_2, \ldots, y_n]$. In this case, by Lemma 4.6, the left most nonzero term of (10) becomes

$$T/\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle = \frac{T'}{\mathcal{I}_1(\mathcal{A}', n'-1)}[x_1, y_1],$$

hence

$$\operatorname{pdim}_T(T/\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle) = \operatorname{pdim}_{T'}(T'/\mathcal{I}_1(\mathcal{A}', n'-1)) \le n'-1,$$

by the inductive hypothesis applied to $\mathcal{S}(I_{n'-1}(\mathcal{A}')) \simeq T'/\mathcal{I}_1(\mathcal{A}', n'-1)$. Then, from (10) we have

$$\begin{aligned}\operatorname{pdim}_T(T/\mathcal{I}_1(\mathcal{A}, n-1)) &\le \max\{\operatorname{pdim}_T(T/\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle) + 1, \operatorname{pdim}_T(T/\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle)\}\\ &\le (n'-1)+1 = n' = n-1.\end{aligned}$$

x_1 **is a noncoloop:** This case will occupy us for the rest of the proof. Here $T' := \Bbbk[x_1, \ldots, x_k;\, y_2, \ldots, y_n]$ is the natural ambient ring of the deletion symmetric ideal. By Lemma 4.6, the left most nonzero term of (10) is $T/(\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1)$. Thus, multiplication by x_1 gives a similar exact sequence to (10):

$$0 \to \frac{T}{\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle : x_1} \to \frac{T}{\langle \mathcal{I}_1(\mathcal{A}', n'-1)\rangle} \to \frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle} \to 0. \tag{11}$$

Suppose for a minute that one has

$$\operatorname{pdim}_T\Big(\frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle}\Big) \le n'. \tag{12}$$

Then (11) implies

$$\operatorname{pdim}_T\Big(\frac{T}{\mathcal{I}_1(\mathcal{A}', n'-1) : x_1}\Big) \le \max\{n'-1, n'-1\} = n'-1.$$

Back to (10) would finally give

$$\operatorname{pdim}_T\Big(\frac{T}{\mathcal{I}_1(\mathcal{A}, n-1)}\Big) \le \max\{(n'-1)+1, n'-1\} = n' = n-1,$$

proving the required statement of the theorem. Thus, it suffices to prove (12). For this, one sets $\langle x_1, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle = \langle x_1, x_2y_2 - \bar{\ell}_3y_3, \ldots, \bar{\ell}_{n-1}y_{n-1} - \bar{\ell}_ny_n\rangle$, where we have written $\ell_j = c_jx_1 + \bar{\ell}_j$, with $c_j \in \Bbbk$, $\bar{\ell}_j \in \Bbbk[x_2, \ldots, x_k]$, for $3 \le j \le n$. Then,

$$\operatorname{pdim}_T\Big(\frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle}\Big) = 1 + \operatorname{pdim}_{T'}\Big(\frac{T'}{\langle x_2y_2 - \bar{\ell}_3y_3, \ldots, \bar{\ell}_{n-1}y_{n-1} - \bar{\ell}_ny_n\rangle}\Big). \tag{13}$$

Let $\bar{\mathcal{A}} = \{x_2, \bar{\ell}_3, \ldots, \bar{\ell}_n\}$ denote the corresponding stretched arrangement and set

$$J := \langle x_2y_2 - \bar{\ell}_3y_3, \ldots, \bar{\ell}_{n-1}y_{n-1} - \bar{\ell}_ny_n\rangle \subset T' := \Bbbk[x_2, \ldots, x_k;\, y_2, \ldots, y_n].$$

Claim: $\operatorname{pdim}_{T'}(T'/J) \le n'-1$.

If the size of $\bar{\mathcal{A}}$ is $= n-1 = n'$ (i.e., no two linear forms of $\bar{\mathcal{A}}$ are proportional), then $J = \mathcal{I}_1(\bar{\mathcal{A}}, n'-1)$, and by the inductive hypotheses $\operatorname{pdim}_{T'}(T'/\mathcal{I}_1(\bar{\mathcal{A}}, n'-1)) \leq n'-1$. Otherwise, suppose $s-1 \geq 2$ of the linear forms of $\bar{\mathcal{A}}$ are mutually proportional. Without loss of generality, say

$$\bar{\ell}_3 = d_3 x_2, \quad \ldots, \quad \bar{\ell}_s = d_s x_2, \quad d_i \in \Bbbk \setminus \{0\},\ 3 \leq i \leq s$$

and

$$\bar{\ell}_j = d_j x_2 + L_j, \quad d_j \in \Bbbk, 0 \neq L_j \in \Bbbk[x_3, \ldots, x_k],\ 4 \leq j \leq n.$$

Then

$$J = \langle x_2(y_2 - d_3 y_3), \ldots, x_2(y_2 - d_s y_s), x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n \rangle.$$

We now provide the following estimates:

(a) $\operatorname{pdim}_{T'}(T'/\langle x_2, J\rangle) \leq 1 + (n-s)$.

(b) $\operatorname{pdim}_{T'}(T'/\langle J : x_2\rangle) \leq n'-1$.

For (a), note that $\langle x_2, J\rangle = \langle x_2, L_{s+1} y_{s+1}, \ldots, L_n y_n\rangle$, while from the proof of Lemma 4.7 we have

$$\operatorname{pdim}_{T'}(T'/\langle x_2, J\rangle) \leq 1 + (n-s),$$

since x_2 is a nonzero divisor in $T'/\langle L_{s+1} y_{s+1}, \ldots, L_n y_n\rangle$. As for (b), we first claim that $J : x_2 = \langle y_2 - d_3 y_3, \ldots, y_2 - d_s y_s, x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n\rangle$. The proof is pretty much the same as that of the equality in (8), hence will be omitted. This equality implies that

$$\operatorname{pdim}_{T'} g\left(\frac{T'}{J : x_2}\right) = s - 2 + \operatorname{pdim}_{T''} g\left(\frac{T''}{\langle x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n\rangle}\right),$$

where $T'' := \Bbbk[x_2, \ldots, x_k;\ y_2, y_{s+1}, \ldots, y_n]$.

Let $\mathcal{B} := \{x_2, \bar{\ell}_{s+1}, \ldots, \bar{\ell}_n\}$. With same reasoning for $\mathcal{B}$ as for $\bar{\mathcal{A}}$ (i.e., removing proportional linear forms), we obtain

$$\operatorname{pdim}_{T''} g\left(\frac{T''}{\langle x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n\rangle}\right) \leq (n-s+1) - 1 = n-s,$$

and therefore

$$\operatorname{pdim}_{T'} g\left(\frac{T'}{J : x_2}\right) \leq s - 2 + n - s = n - 2 = n' - 1.$$

Drawing on the estimates (a) and (b) above, the exact sequence of T'-modules

$$0 \to T'/(J : x_2) \to T'/J \to T'/\langle x_2, J\rangle \to 0,$$

gives that

$$\operatorname{pdim}_{T'}(T'/J) \leq \max\{n'-1, 2+n'-s\} \leq n'-1,$$

since $s \geq 3$. Rolling all the way back to (13), we have proved that

$$\operatorname{pdim}_T g\left(\frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n'-1)\rangle}\right) \leq 1 + (n'-1) = n',$$

as required. □

The main result now follows quite smoothly.

Theorem 4.9. *The Rees algebra of $I_{n-1}(\mathcal{A})$ is Cohen–Macaulay.*

Proof. From Corollary4.4 we have the following short exact sequence of T-modules

$$0 \to \frac{T}{\mathcal{I}(\mathcal{A}, n-1)} \to \frac{T}{\mathcal{I}_1(\mathcal{A}, n-1)} \to \frac{T}{\langle \mathcal{I}_1(\mathcal{A}, n-1), \ell_1 y_1 \rangle} \to 0.$$

By Proposition 4.8, the depth of the middle module is $k+1$, while by Lemma 4.7 that of the right most module is at least k — in fact, by the domino effect one has $\langle \mathcal{I}_1(\mathcal{A}, n-1), \ell_1 y_1 \rangle = \langle \ell_1 y_1, \ell_2 y_2, \dots, \ell_n y_n \rangle$. By standard knowledge, the depth of the left most module is at least that of the middle module, namely, $k+1$. □

A consequence is an alternative proof of a result of Proudfoot and Speyer [2006]:

Corollary 4.10. *Let $\mathcal{A}$ be any central arrangement. Then the associated Orlik–Terao algebra is Cohen–Macaulay.*

Proof. As we have seen, the Orlik–Terao algebra is the special fiber of the ideal $I = I_{n-1}(\mathcal{A})$. Since I is a homogeneous ideal generated in one single degree, then its special fiber is identified with the $\Bbbk$-subalgebra $\Bbbk[It]$ of the Rees algebra $R[It]$ of I and, as such, it is a direct summand as a $\Bbbk[It]$-module. In this situation W. Vasconcelos and one of us have shown that the Cohen–Macaulay property of $R[It]$ transfers to $\Bbbk[It]$. A proof of the latter result is given in [Ramos and Simis 2017, Proposition 3.10] for the case where $k = 3$. The proof for arbitrary k is the same. □

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Torsion in the 0-cycle group with modulus

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We show, for a smooth projective variety X over an algebraically closed field k with an effective Cartier divisor D, that the torsion subgroup $\mathrm{CH}_0(X|D)\{l\}$ can be described in terms of a relative étale cohomology for any prime $l \neq p = \mathrm{char}(k)$. This extends a classical result of Bloch, on the torsion in the ordinary Chow group, to the modulus setting. We prove the Roitman torsion theorem (including p-torsion) for $\mathrm{CH}_0(X|D)$ when D is reduced. We deduce applications to the problem of invariance of the prime-to-p torsion in $\mathrm{CH}_0(X|D)$ under an infinitesimal extension of D.

1. Introduction

One of the fundamental problems in the study of the theory of motives, as envisioned by Grothendieck, is to construct a universal cohomology theory of varieties and describe it in terms of algebraic cycles. When X is a smooth quasiprojective scheme over a base field k, such a motivic cohomology theory of X is known to exist; see, for example, [Voevodsky 2000; Levine 1998]. Moreover, Bloch [1986] showed that this cohomology theory has an explicit description in terms of groups of algebraic cycles, called the higher Chow groups. These groups have all the properties that one expects, including Chern classes and a Chern character isomorphism from the higher K-groups of Quillen.

However, the situation becomes much complicated when we go beyond the world of smooth varieties. There is no theory of motivic cohomology of singular schemes known to date which could be a universal cohomology theory, which could be described in terms of algebraic cycles, and which could be rationally isomorphic to the algebraic K-theory.

With the aim of understanding the algebraic K-theory of the ring $k[t]/(t^n)$ in terms of algebraic cycles, Bloch and Esnault introduced the idea of algebraic cycles with modulus, called additive Chow groups.

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This idea was substantially expanded in the further works of Park [2009], Rülling [2007] and Krishna and Levine [2008].

In recent works of Binda and Saito [2017] and Kerz and Saito [2016], a theory of higher Chow groups with modulus was introduced, which generalizes the construction of the additive higher Chow groups. These groups, denoted $\mathrm{CH}^*(X|D, *)$, are designed to study the arithmetic and geometric properties of a smooth variety X with fixed conditions along an effective (possibly nonreduced) Cartier divisor D on it (see [Kerz and Saito 2016]), and are supposed to give a cycle-theoretic description of the relative K-groups $K_*(X, D)$, defined as the homotopy groups of the homotopy fiber of the restriction map $K(X) \to K(D)$ (see [Iwasa and Kai 2016]).

In order to provide evidence that the Chow groups with modulus are the right motivic cohomology groups to compute the relative K-theory of a smooth scheme with respect to an effective divisor, one would like to know if these groups share enough of the known structural properties of the Chow groups without modulus, and, if these groups could be related to other cohomological invariants of a pair (X, D). In particular, one would like to know if the torsion part of the Chow group of 0-cycles with modulus could be described in terms of a relative étale cohomology or, in terms of an Albanese variety with modulus.

1A. ***The main results.*** For the Chow group of 0-cycles without modulus on a smooth projective scheme, Bloch [1979] showed that its prime-to-p torsion is completely described in terms of an étale cohomology. Roitman [1980] and Milne [1982] showed that the torsion in the Chow group is completely described in terms of the Albanese of the underlying scheme. When D is an effective Cartier divisor on a smooth projective scheme X, the Albanese variety with modulus $A^d(X|D)$ with appropriate universal property and the Abel–Jacobi map $\rho_{X|D} : \mathrm{CH}_0(X|D)_{\deg 0} \to A^d(X|D)$ were constructed in [Binda and Krishna 2018].

The goal of this work is to extend the torsion results of Bloch and those of Roitman and Milne to 0-cycles with modulus. Our main results thus broadly look as follows. Their precise statements and underlying hypotheses and notations will be explained at appropriate places in the text.

Theorem 1.1 (Bloch's torsion theorem). *Let k be an algebraically closed field of exponential characteristic p. Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Then, for any prime $l \neq p$, there is an isomorphism*

$$\lambda_{X|D} : \mathrm{CH}_0(X|D)\{l\} \xrightarrow{\sim} H^{2d-1}_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)).$$

Theorem 1.2 (Roitman's torsion theorem). *Let k be an algebraically closed field of exponential characteristic p. Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Assume that D is reduced. Then, the Albanese variety with modulus $A^d(X|D)$ is a semiabelian variety and the Abel–Jacobi map $\rho_{X|D} : \mathrm{CH}_0(X|D)_{\deg 0} \to A^d(X|D)$ is an isomorphism on the torsion subgroups* (*including p-torsion*).

Note that the isomorphism of Theorem 1.1 is very subtle because one does not have any obvious map in either direction. The construction of such a map is an essential part of the problem. In the case of the Chow group without modulus, the construction of λ_X by Bloch makes essential use of the Bloch–Ogus theory and, more importantly, it uses the Weil conjecture. The prime-to-p part of Theorem 1.2 was earlier proven in [Binda and Krishna 2018]. The new contribution here is the description of the more challenging p-part.

As part of the proof of Theorem 1.2, we prove the Roitman torsion theorem (including p-torsion) for singular separably weakly normal surfaces (see Definition 4.5). This provides the first evidence that the Roitman torsion theorem may be true for nonnormal varieties in positive characteristic.

Theorem 1.3. *Let X be a reduced projective separably weakly normal surface over an algebraically closed field k of exponential characteristic p. Then, the Albanese variety $A^2(X)$ is a semiabelian variety and the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X)_{\deg 0} \to A^2(X)$ is an isomorphism on the torsion subgroups (including p-torsion).*

Note that Theorem 1.2 makes no assumption on $\dim(X)$ but this is not the case for Theorem 1.3. The reason for this is that our proof of Theorem 1.2 uses a weaker version of the Lefschetz type theorem for separably weakly normal varieties. We do not know if a general hyperplane section of an arbitrary separably weakly normal variety is separably weakly normal. This is known to be false for weakly normal varieties in positive characteristic; see [Cumino et al. 1989]. We shall investigate this question in a future project.

1B. ***Applications.*** As an application of Theorem 1.1, we obtain the following result about the prime-to-p torsion in the Chow group with modulus.

Corollary 1.4. *Let k be an algebraically closed field of exponential characteristic p. Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Then, the restriction map ${}_n\mathrm{CH}_0(X|D) \to {}_n\mathrm{CH}_0(X|D_{\mathrm{red}})$ is an isomorphism for every integer n prime to p.*

In [Miyazaki 2016, Theorem 1.3], an isomorphism of similar indication has been very recently shown after inverting $\mathrm{char}(k)$. But it has no implication on Corollary 1.4. Another application of Theorem 1.1 is the following extension of the results of Bloch [1980, Chapter 5] to the relative K-theory.

Corollary 1.5. *Let X be a smooth projective surface over an algebraically closed field k of exponential characteristic p and let $l \neq p$ be a prime. Let $D \subset X$ be an effective Cartier divisor. Then the following hold.*

(1) $H^1(X, \mathcal{K}_{2,(X,D)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$.

(2) $H^2(X, \mathcal{K}_{2,(X,D)})\{l\} \simeq H^3_{ét}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2))$.

Analogous results are also obtained for surfaces with arbitrary singularities (see Theorems 3.4 and 3.6).

1C. ***Outline of proofs.*** Our proofs of the above results are based on the decomposition theorem of [Binda and Krishna 2018] which relates the Chow group of 0-cycles with modulus on a smooth variety to the Levine–Weibel Chow group of 0-cycles on a singular variety. Using this decomposition theorem, our main task becomes extending the torsion results of Bloch and those of Roitman and Milne to singular varieties. We prove these results for the Chow group of 0-cycles on singular surfaces in the first part (Sections 3 and 4) of this text. These results are of independent interest in the study of 0-cycles on singular varieties.

In Sections 5 and 6, we prove our results about 0-cycles with modulus by using the analogous results for singular surfaces and an induction on dimension. This induction is based on some variants of the Lefschetz hyperplane section theorem for the Albanese variety and the weak Lefschetz theorem for the étale cohomology with modulus. Some applications are deduced in Section 5.

1D. ***Notations.*** Throughout (except in Section 4A), k will denote an algebraically closed field of exponential characteristic $p \geq 1$. We shall denote the category of separated schemes of finite type over k by $\boldsymbol{Sch}_k$. The subcategory of $\boldsymbol{Sch}_k$ consisting of smooth schemes over k will be denoted by $\boldsymbol{Sm}_k$. An undecorated product $X \times Y$ in $\boldsymbol{Sch}_k$ will mean that it is taken over the base scheme $\mathrm{Spec}(k)$. An undecorated cohomology group will be assumed to be with respect to the Zariski site. We shall let $w : \boldsymbol{Sch}_k/\acute{e}t \to \boldsymbol{Sch}_k/\mathrm{zar}$ denote the canonical morphism between the étale and the Zariski sites of $\boldsymbol{Sch}_k$. For $X \in \boldsymbol{Sch}_k$, the notation X^N will usually mean the normalization of X_{red} unless we use a different notation in a specific context. For any abelian group A and an integer $n \geq 1$, the subgroup of n-torsion elements in A will be denoted by ${}_nA$ and, for any prime l, the l-primary torsion subgroup will be denoted by $A\{l\}$.

2. Review of 0-cycle groups and Albanese varieties

To prove the main results of Section 1, our strategy is to use the decomposition theorem for 0-cycles from [Binda and Krishna 2018]. This theorem asserts that the Chow group of 0-cycles with modulus on a smooth scheme is a direct summand of the Chow group of 0-cycles (in the sense of [Levine and Weibel 1985]) on a singular scheme. We therefore need to prove analogues of Theorems 1.1 and 1.2 for singular schemes. In the first few sections of this paper, our goal is to prove such results for arbitrary singular surfaces. These results are of independent interest and answer some questions in the study of 0-cycles on singular schemes. The higher dimensional cases of Theorems 1.1 and 1.2 will be deduced from the case of surfaces.

In this section, we review the definitions and some basic facts about the Chow group of 0-cycles on singular schemes, the Chow group with modulus and their universal quotients, called the Albanese varieties.

2A. ***0-cycles on singular schemes.*** We recall the definition of the cohomological Chow group of 0-cycles for singular schemes from [Binda and Krishna 2018] and [Levine and Weibel 1985]. Let X be a reduced quasiprojective scheme of dimension $d \geq 1$ over k. Let X_{sing} and X_{reg} denote the loci of the singular and the regular points of X. Given a nowhere dense closed subscheme $Y \subset X$ such that $X_{\mathrm{sing}} \subseteq Y$ and no component of X is contained in Y, we let $\mathcal{Z}_0(X, Y)$ denote the free abelian group on the closed points of $X \setminus Y$. We write $\mathcal{Z}_0(X, X_{\mathrm{sing}})$ in short as $\mathcal{Z}_0(X)$.

Definition 2.1. Let C be a pure dimension-one reduced scheme in $\boldsymbol{Sch}_k$. We shall say that a pair (C, Z) is *a good curve relative to* X if there exists a finite morphism $\nu : C \to X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

(1) No component of C is contained in Z.

(2) $\nu^{-1}(X_{\mathrm{sing}}) \cup C_{\mathrm{sing}} \subseteq Z$.

(3) ν is local complete intersection at every point $x \in C$ such that $\nu(x) \in X_{\mathrm{sing}}$.

Let (C, Z) be a good curve relative to X and let $\{\eta_1, \dots, \eta_r\}$ be the set of generic points of C. Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \dots, \eta_r\}$. Let $k(C)$ denote the ring of total quotients of C and write $\mathcal{O}^\times_{C,Z}$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \varnothing$. As C is Cohen–Macaulay, $\mathcal{O}^\times_{C,Z}$ is the subgroup of $k(C)^\times$ consisting of those f which are regular and invertible in the local rings $\mathcal{O}_{C,x}$ for every $x \in Z$.

Given any $f \in \mathcal{O}^\times_{C,Z} \hookrightarrow k(C)^\times$, we denote by $\operatorname{div}_C(f)$ (or $\operatorname{div}(f)$ in short) the divisor of zeros and poles of f on C, which is defined as follows. If $C_1, \dots, C_r$ are the irreducible components of C, and f_i is the factor of f in $k(C_i)$, we set $\operatorname{div}(f)$ to be the 0-cycle $\sum_{i=1}^r \operatorname{div}(f_i)$, where $\operatorname{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [Fulton 1984]. As f is an invertible regular function on C along Z, $\operatorname{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to X, we have a pushforward map $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X)$. We shall write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X)$ generated by the set $\{\nu_*(\operatorname{div}(f)) \mid f \in \mathcal{O}^\times_{C,Z}\}$. Let $\mathcal{R}_0(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the image of the map $\mathcal{Z}_0(C, Z, X) \to \mathcal{Z}_0(X)$, where (C, Z) runs through all good curves relative to X. We let $\mathrm{CH}_0(X) = \mathcal{Z}_0(X)/\mathcal{R}_0(X)$.

If we let $\mathcal{R}^{LW}_0(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu : C \to X$ is a closed immersion, then the resulting quotient group $\mathcal{Z}_0(X)/\mathcal{R}^{LW}_0(X)$ is denoted by $\mathrm{CH}^{LW}_0(X)$. Such curves on X are called the *Cartier curves*. There is a canonical surjection $\mathrm{CH}^{LW}_0(X) \twoheadrightarrow \mathrm{CH}_0(X)$. The Chow group $\mathrm{CH}^{LW}_0(X)$ was discovered by Levine and Weibel [1985] in an attempt to describe the Grothendieck group of a singular scheme in terms of algebraic cycles. The modified version $\mathrm{CH}_0(X)$ was introduced in [Binda and Krishna 2018]. As an application of our main results, we shall be able to identify the two Chow groups for certain singular schemes (see Theorem 6.6).

2B. ***The Chow group of the double.*** Let X be a smooth quasiprojective scheme of dimension d over k and let $D \subset X$ be an effective Cartier divisor. Recall from [Binda and Krishna 2018, § 2.1] that the double of X along D is a quasiprojective scheme $S(X, D) = X \amalg_D X$ so that

$$\begin{array}{ccc} D & \xrightarrow{\iota} & X \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota_+} \\ X & \xrightarrow[\iota_-]{} & S(X, D) \end{array} \tag{2-1}$$

is a co-Cartesian square in $\boldsymbol{Sch}_k$. In particular, the identity map of X induces a finite map $\Delta : S(X, D) \to X$ such that $\Delta \circ \iota_\pm = \mathrm{Id}_X$ and $\pi = \iota_+ \amalg \iota_- : X \amalg X \to S(X, D)$ is the normalization map. We let $X_\pm = \iota_\pm(X) \subset S(X, D)$ denote the two irreducible components of $S(X, D)$. We shall often write $S(X, D)$ as S_X when the divisor D is understood. S_X is a reduced quasiprojective scheme whose singular locus is $D_{\mathrm{red}} \subset S_X$. It is projective whenever X is so. The following easy algebraic lemma shows that (2-1) is also a Cartesian square.

Lemma 2.2. *Let*

$$\begin{array}{ccc} R & \xrightarrow{\psi_1} & A_1 \\ {\scriptstyle\psi_2}\downarrow & & \downarrow\!\!\!\!\downarrow{\scriptstyle\phi_1} \\ A_2 & \xrightarrow{\phi_2}\!\!\!\!\!\rightarrow & B \end{array} \tag{2-2}$$

be a Cartesian square of commutative Noetherian rings such that $\mathrm{Ker}(\phi_i) = (a_i)$ *for* $i = 1, 2$. *Then* $A_1 \otimes_R A_2 \xrightarrow{\simeq} B$.

Proof. If we let $\alpha_1 = (0, a_2)$ and $\alpha_2 = (a_1, 0)$ in $R \subset A_1 \times A_2$, then it is easy to check that for $i = 1, 2$, the map ψ_i is surjective and $\mathrm{Ker}(\psi_i) = (\alpha_i)$. This implies that $A_1 \otimes_R A_2 \simeq R/(\alpha_1) \otimes_R A_2 \simeq A_2/(a_2) = B$. □

We shall later consider a more general situation than (2-1) where we allow the two components of the double to be distinct. This general case is a nonaffine version of (2-2) and is used in the proof of Theorem 1.2.

2C. ***The Albanese varieties for singular schemes.*** Let X be a reduced projective scheme over k of dimension d. Let $\mathrm{CH}_0^{LW}(X)_{\deg 0}$ denote the kernel of the degree map $\deg : \mathrm{CH}_0^{LW}(X) \twoheadrightarrow H^0(X, \mathbb{Z})$. An Albanese variety $A^d(X)$ of X was constructed in [Esnault et al. 1999]. This variety is a connected commutative algebraic group over k with an Abel–Jacobi map $\rho_X : \mathrm{CH}_0^{LW}(X)_{\deg 0} \twoheadrightarrow A^d(X)$. Moreover, $A^d(X)$ is the universal object in the category of commutative algebraic groups G over k with a rational map $f : X \dashrightarrow G$ which induces a group homomorphism $f_* : \mathrm{CH}_0^{LW}(X)_{\deg 0} \to G$. Such a group homomorphism is called a regular homomorphism. For this reason, $A^d(X)$ is also called the *universal regular quotient* of $\mathrm{CH}_0^{LW}(X)_{\deg 0}$. When X is smooth, $A^d(X)$ coincides with the classical Albanese variety $\mathrm{Alb}(X)$. One says that $\mathrm{CH}_0^{LW}(X)_{\deg 0}$ is *finite-dimensional*, if ρ_X is an isomorphism.

2D. ***The universal semiabelian quotient of*** $\mathbf{CH_0^{LW}(X)_{deg\,0}}$**.** In positive characteristic, apart from the theorem about its existence, not much is known about $A^d(X)$ and almost everything that one would like to know about it is an open question. However, the universal semiabelian quotient of the algebraic group $A^d(X)$ can be described more explicitly and this is all one needs to know in order to understand the prime-to-p torsion in $A^d(X)$. The following description of this quotient is recalled from [Mallick 2009, §2].

Let $\pi : X^N \to X$ denote the normalization map. Let $\mathrm{Cl}(X^N)$ and $\mathrm{Pic}_W(X^N)$ denote the class group and the Weil Picard group of X^N. Recall from [Weil 1954] that $\mathrm{Pic}_W(X^N)$ is the subgroup of $\mathrm{Cl}(X^N)$ consisting of Weil divisors which are algebraically equivalent to zero in the sense of [Fulton 1984, Chapter 19]. Let $\mathrm{Div}(X)$ denote the free abelian group of Weil divisors on X. Let $\Lambda_1(X)$ denote the subgroup of $\mathrm{Div}(X^N)$ generated by the Weil divisors which are supported on $\pi^{-1}(X_{\mathrm{sing}})$. This gives us a map $\iota_X : \Lambda_1(X) \to \mathrm{Cl}(X^N)/\mathrm{Pic}_W(X^N)$.

Let $\Lambda(X)$ denote the kernel of the canonical map

$$\Lambda_1(X) \xrightarrow{(\iota_X, \pi_*)} \frac{\mathrm{Cl}(X^N)}{\mathrm{Pic}_W(X^N)} \oplus \mathrm{Div}(X). \tag{2-3}$$

It follows from [Mallick 2009, § 4] that the universal semiabelian quotient of $A^d(X)$ is the Cartier dual of the 1-motive $[\Lambda(X) \to \mathrm{Pic}_W(X^N)]$ and is denoted by $J^d(X)$. The composite homomorphism $\rho_X^{\mathrm{semi}} : \mathrm{CH}_0^{LW}(X)_{\deg 0} \twoheadrightarrow A^d(X) \twoheadrightarrow J^d(X)$ is universal among regular homomorphisms from $\mathrm{CH}_0^{LW}(X)_{\deg 0}$ to semiabelian varieties. $J^d(X)$ is called the *semiabelian Albanese variety* of X. By [Binda and Krishna 2018, Proposition 9.7], there is a factorization of ρ_X^{semi}:

$$\mathrm{CH}_0^{LW}(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0(X)_{\deg 0} \xrightarrow{\tilde{\rho}_X^{\mathrm{semi}}} J^d(X) \tag{2-4}$$

and $J^d(X)$ is in fact a universal regular semiabelian variety quotient of $\mathrm{CH}_0(X)_{\deg 0}$.

There is a short exact sequence of commutative algebraic groups

$$0 \to A^d_{\mathrm{unip}}(X) \to A^d(X) \to J^d(X) \to 0, \tag{2-5}$$

where $A^d_{\mathrm{unip}}(X)$ is the unipotent radical of $A^d(X)$. Since k is algebraically closed, $A^d_{\mathrm{unip}}(X)$ is uniquely n-divisible for any integer $n \in k^\times$. In particular,

$${}_nA^d(X) \xrightarrow{\sim} {}_nJ^d(X). \tag{2-6}$$

2E. ***A Lefschetz hyperplane theorem.*** Recall that for a smooth projective scheme $X \hookrightarrow \mathbb{P}^N_k$ of dimension $d \geq 3$, a general hypersurface section $Y \subset X$ has the property that the canonical map $\mathrm{Alb}(Y) \to \mathrm{Alb}(X)$ is an isomorphism. We now wish to prove a similar result for the semiabelian Albanese variety $J^d(S_X)$ of the double.

Let X be a connected smooth projective scheme of dimension $d \geq 3$ over k and let $D \subset X$ be an effective Cartier divisor. Let $\{E_1, \ldots, E_r\}$ be the set of irreducible components of D_{red} so that each E_i is integral. For each $1 \leq j \leq r$, let $P_j \in E_j \setminus \left(\bigcup_{j' \neq j} E_{j'}\right)$ be a chosen closed point, and let $S = \{P_1, \ldots, P_r\}$. We can apply [Kleiman and Altman 1979, Theorems 1, 7] to find a closed embedding $X \hookrightarrow \mathbb{P}^N_k$ (for $N \gg 0$) such that for a general set of distinct hypersurfaces $H_1, \ldots, H_{d-2} \hookrightarrow \mathbb{P}^N_k$, we have the following.

(1) The successive intersection $X \cap H_1 \cap \cdots \cap H_i$ $(1 \leq i \leq d-2)$ is integral and smooth over k.

(2) No component of $X \cap H_1 \cap \cdots \cap H_{d-2}$ is contained in D.

(3) $S \subset X \cap H_1 \cap \cdots \cap H_{d-2}$.

(4) For each $1 \leq j \leq r$, the successive intersection $E_j \cap H_1 \cap \cdots \cap H_i$ $(1 \leq i \leq d-2)$ is integral.

Setting $X_i = X \cap H_1 \cap \cdots \cap H_i$ and $D_i = X_i \cap D$, our choice of the hypersurfaces implies that $D_i \subset X_i$ is an effective Cartier divisor. We let $S_i = X_i \amalg_{D_i} X_i$ for $1 \leq i \leq d-2$. It follows from [Binda and Krishna 2018, Proposition 2.4] that each inclusion $\tau_i : S_i \hookrightarrow S_{i-1}$ is a local complete intersection (l.c.i.) and $S_i = S_{i-1} \times_{X_{i-1}} X_i$ for $1 \leq i \leq d-2$, where we let $S_0 = S_X$. Since $S_i \hookrightarrow S_{i-1}$ is an l.c.i. inclusion with $(S_i)_{\mathrm{sing}} = (D_i)_{\mathrm{red}} = (S_i \cap D)_{\mathrm{red}} = S_i \cap (S_{i-1})_{\mathrm{sing}}$, it follows that there are compatible pushforward maps $\tau^{LW}_{i,*} : \mathrm{CH}_0^{LW}(S_i) \to \mathrm{CH}_0^{LW}(S_{i-1})$ (see [Esnault et al. 1999, Lemma 1.8]) and $\tau_{i,*} : \mathrm{CH}_0(S_i) \to \mathrm{CH}_0(S_{i-1})$ (see [Binda and Krishna 2018, Proposition 3.17]). With the above setup, we have the following.

Proposition 2.3. *For $1 \le i \le d-2$, there is an isomorphism of algebraic groups $\phi_i : J^{d-i}(S_i) \to J^{d-i+1}(S_{i-1})$ and a commutative diagram*

$$\begin{array}{ccc} \mathrm{CH}_0(S_i)_{\deg 0} & \xrightarrow{\tau_{i,*}} & \mathrm{CH}_0(S_{i-1})_{\deg 0} \\ {\scriptstyle \rho_i^{\mathrm{semi}}}\downarrow & & \downarrow{\scriptstyle \rho_{i-1}^{\mathrm{semi}}} \\ J^{d-i}(S_i) & \xrightarrow[\phi_i]{} & J^{d-i+1}(S_{i-1}), \end{array} \tag{2-7}$$

whose restriction to the l-primary torsion subgroups induces isomorphism of all arrows, for every prime $l \neq p$.

Proof. Before we begin the proof, we note that when $S_X \cap H \hookrightarrow S_X$ is a general hypersurface section in a projective embedding of S_X, then an analogue of the proposition was proven by Mallick [2009, §5]. But we can not apply his result directly because $S_Y \hookrightarrow S_X$ is not a hypersurface section. Nevertheless, we shall closely follow Mallick's construction in the proof of the proposition. Also, we shall prove a stronger assertion than in Theorem 14 there because of the special nature of the double.

We have seen above that the inclusion $S_i \hookrightarrow S_{i-1}$ is l.c.i. which preserves the singular loci, and hence there is a pushforward $\tau_{i,*} : \mathrm{CH}_0(S_i) \to \mathrm{CH}_0(S_{i-1})$. We now construct the map ϕ_i. We construct this map for $i = 1$ as the method in the general case is completely identical. With this reduction in mind, we shall let $Y = X_1 = X \cap H_1$, $F = D_1 = D \cap H_1$ so that $S_1 = S_Y = S(Y, F)$.

Since $Y \hookrightarrow X$ is a general hypersurface section with $S_Y^N = Y \amalg Y$ and $S_X^N = X \amalg X$, one knows that the map $\mathrm{Pic}_W(S_Y^N)^\vee = \mathrm{Alb}(Y) \times \mathrm{Alb}(Y) \to \mathrm{Alb}(X) \times \mathrm{Alb}(X) = \mathrm{Pic}_W(S_X^N)^\vee$ is an isomorphism. We thus have to construct a pullback map $\Lambda(S_X) \to \Lambda(S_Y)$ which is an isomorphism.

We consider the commutative square

$$\begin{array}{ccc} Y \amalg Y & \xrightarrow{\psi} & X \amalg X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ S_Y & \xrightarrow[\tau_1]{} & S_X \end{array} \tag{2-8}$$

where the vertical arrows are the normalization maps and ψ is the disjoint sum of the inclusion of the hypersurface section $Y = X \cap H_1$. We claim that this square is Cartesian. To see this, note that the vertical arrows are disjoint sums of two maps. Hence, it is enough to show that (2-8) is Cartesian when we replace $X \amalg X$ by X_+ and $Y \amalg Y$ by Y_+. In this case, the vertical arrows are closed immersions and we know directly from the construction that $S_Y \times_{S_X} X_+ \simeq (Y \times_X S_X) \times_{S_X} X_+ \simeq Y_+$.

If we let $F_i := E_i \cap H_1$ for $1 \le j \le r$, it follows from the conditions (2) and (3) on page 1437 that each F_i is integral and $F_i \neq F_j$ if $E_i \neq E_j$. Now, the refined Gysin homomorphism

$$\psi^! : \Lambda_1(S_X) = \mathrm{CH}_{d-1}(D \amalg D) \to \mathrm{CH}_{d-2}(F \amalg F) = \Lambda_1(S_Y)$$

(in the sense of [Fulton 1984]) takes E_i to F_i for $i = 1, \dots, r$ in each copy of D. This map is bijective by our choice of the hypersurface sections Y and F_i's.

Now, for any $1 \le j \le r$, we have $\psi^!([E_j]) = \alpha_j[F_j]$ in $\Lambda_1(S_Y) = \mathrm{CH}_{d-2}(F \amalg F)$, where α_j is the intersection multiplicity of $E_j \times_{\mathbb{P}^N_k} H_1$ in $D \cdot H_1$. Since the composite morphism $E_j \hookrightarrow X \hookrightarrow \mathbb{P}^N_k$ is a closed immersion (and hence separable), this intersection multiplicity must be one so that we have $\alpha_j = 1$ for each j. It follows that $\psi^! : \Lambda_1(S_X) \to \Lambda_1(S_Y)$ is an isomorphism of abelian groups.

We next show that $\psi^!$ takes $\Lambda(S_X)$ onto $\Lambda(S_Y)$. This follows because for any $\Delta \in \Lambda(S_X)$, by [Fulton 1984, Theorem 6.2] we have $\pi_{1,*} \circ \psi^!(\Delta) = \tau_1^! \circ \pi_*(\Delta)$ and $\pi_*(\Delta) = 0$ in $\mathrm{CH}_{d-1}((S_X)_{\mathrm{sing}}) = Z_{d-1}((S_X)_{\mathrm{sing}})$. This means $\psi^!(\Delta) \in \Lambda(S_Y)$. If we let $\Delta = \sum_l \alpha_l[F_l] \in \Lambda(S_Y)$, then letting $\Delta' = \sum_l \alpha_l[E_l] \in \Lambda_1(S_X)$, we get $\psi^!(\Delta') = \Delta$. We also have

$$\tau_1^! \circ \pi_*(\Delta') = \pi_{1,*} \circ \psi^!(\Delta') = \pi_{1,*}(\Delta) = 0.$$

Since $\tau_1^! : \mathrm{CH}_{d-1}((S_X)_{\mathrm{sing}}) = \mathrm{CH}_{d-1}(D) \to \mathrm{CH}_{d-2}(F) = \mathrm{CH}_{d-2}((S_Y)_{\mathrm{sing}})$ is bijective, it follows that $\pi_*(\Delta') = 0$. It follows that $\psi^! : \Lambda(S_X) \to \Lambda(S_Y)$ is surjective and hence an isomorphism. We have thus constructed an isomorphism $\phi_1 : J^{d-1}(S_Y) \xrightarrow{\sim} J^d(S_X)$.

To show the commutative diagram (2-7), we can again assume $i = 1$. We can also replace $\mathrm{CH}_0(S_X)$ and $\mathrm{CH}_0(S_Y)$ by $\mathcal{Z}_0(S_X)$ and $\mathcal{Z}_0(S_Y)$, respectively.

We now need to observe that $J^d(S_X)$ is a quotient of the Cartier dual $J^d_{\mathrm{Serre}}(S_X)$ of the 1-motive $[\Lambda_1(S_X) \to \mathrm{Pic}_W(X \amalg X)]$ and this dual semiabelian variety is the universal object in the category of morphisms from $(S_X)_{\mathrm{reg}} = (X \setminus D) \amalg (X \setminus D)$ to semiabelian varieties (see [Serre 1958-1959]). Since $(S_Y)_{\mathrm{reg}} = (Y \setminus D) \amalg (Y \setminus D) = \tau_1^{-1}((S_X)_{\mathrm{reg}})$, it follows from this universality of $J^d_{\mathrm{Serre}}(S_X)$ that there is a commutative diagram as in (2-7) if we replace $J^d(S_X)$ and $J^{d-1}(S_Y)$ by $J^d_{\mathrm{Serre}}(S_X)$ and $J^d_{\mathrm{Serre}}(S_Y)$, respectively. The commutative diagram

$$\begin{tikzcd} J^d_{\mathrm{Serre}}(S_Y) \arrow[r, "\phi_{1,*}"] \arrow[d, two heads] & J^d_{\mathrm{Serre}}(S_X) \arrow[d, two heads] \\ J^d(S_Y) \arrow[r, "\phi_{1,*}"'] & J^d(S_X) \end{tikzcd}$$

now finishes the proof of the commutativity of (2-7). The final assertion about the isomorphism between the l-primary torsion subgroups follows from [Binda and Krishna 2018, Theorem 9.8] and [Mallick 2009, Theorem 16]. □

2F. *0-cycles and Albanese variety with modulus.* Given an integral normal curve C over k and an effective divisor $E \subset C$, we say that a rational function f on C has modulus E if $f \in \mathrm{Ker}(\mathcal{O}^\times_{C,E} \to \mathcal{O}^\times_E)$. Here, $\mathcal{O}_{C,E}$ is the semilocal ring of C at the union of E and the generic point of C. In particular, $\mathrm{Ker}(\mathcal{O}^\times_{C,E} \to \mathcal{O}^\times_E)$ is just $k(C)^\times$ if $|E| = \varnothing$. Let $G(C, E)$ denote the group of such rational functions.

Let X be a smooth quasiprojective scheme of dimension $d \ge 1$ over k and let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{Z}_0(X|D)$ be the free abelian group on the set of closed points of $X \setminus D$. Let C be an integral normal curve over k and let $\varphi_C : C \to X$ be a finite morphism such that $\varphi_C(C) \not\subset D$. The pushforward of cycles along φ_C gives a well-defined group homomorphism $\tau_C : G(C, \varphi_C^*(D)) \to Z_0(X|D)$.

Definition 2.4. We define the Chow group $\mathrm{CH}_0(X|D)$ of 0-cycles of X with modulus D to be the cokernel of the homomorphism $\mathrm{div} : \bigoplus_{\varphi_C : C \to X} G(C, \varphi_C^*(D)) \to Z_0(X|D)$, where the sum is taken over the set of finite morphisms $\varphi_C : C \to X$ from an integral normal curve such that $\varphi_C(C) \not\subset D$.

It is clear from the moving lemma of Bloch that there is a canonical surjection $\mathrm{CH}_0(X|D) \twoheadrightarrow \mathrm{CH}_0(X)$. If X is projective, we denote the kernel of the composite map $\mathrm{CH}_0(X|D) \twoheadrightarrow \mathrm{CH}_0(X) \xrightarrow{\deg} H^0(X, \mathbb{Z})$ by $\mathrm{CH}_0(X|D)_{\deg 0}$.

If D is reduced, it follows using Theorem 6.6 and the constructions of [Binda and Krishna 2018, §11] that there exists an Albanese variety with modulus $A^d(X|D)$ together with a surjective Abel–Jacobi map $\rho_{X|D} : \mathrm{CH}_0(X|D)_{\deg 0} \to A^d(X|D)$ which is a universal regular homomorphism in a suitable sense. One says that $\mathrm{CH}_0(X|D)_{\deg 0}$ is *finite-dimensional*, if $\rho_{X|D}$ is an isomorphism.

3. Torsion in the Chow group of a singular surface

Let X be a reduced projective surface over k. When X is smooth, Suslin [1987, p. 19] showed that there is a short exact sequence

$$0 \to H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Z}/n \to H^3_{\acute{e}t}(X, \mu_n(2)) \to {}_n\mathrm{CH}_0(X) \to 0 \tag{3-1}$$

for any integer $n \in k^\times$. When X has only isolated singularities, this exact sequence was conjectured by Pedrini and Weibel [1994, (0.4)] and proven by Barbieri-Viale, Pedrini and Weibel in [Barbieri-Viale et al. 1996, Theorem 7.7]. Expanding on their ideas, our goal in this section is to prove such an exact sequence for surfaces with arbitrary singularities. This exact sequence will be used to prove an analogue of Theorem 1.1 for singular surfaces.

3A. ***Gillet's Chern classes.*** Let $\mathcal{K} = \Omega BQP$ denote the simplicial presheaf on $\boldsymbol{Sch}_k$ such that $\mathcal{K}(X) = \Omega BQP(X)$, where $BQP(X)$ is the Quillen's construction of the simplicial set associated to the exact category $P(X)$ of locally free sheaves on X. For any $i \geq 0$, let $\mathcal{K}_i$ denote the Zariski sheaf on $\boldsymbol{Sch}_k$ associated to the presheaf $X \mapsto K_i(X) = \pi_i(\mathcal{K}(X))$.

We fix integers $d, i \geq 0$ and an integer $n \geq 1$ which is prime to the exponential characteristic p of the base field k. Let $\mu_n(d)$ denote the sheaf $\mu_n^{\otimes d}$ on $\boldsymbol{Sch}_k/\acute{e}t$ such that $\mu_n(X) = \mathrm{Ker}(\mathcal{O}^\times(X) \xrightarrow{n} \mathcal{O}^\times(X))$. Let $\mathcal{E}_i$ denote the simplicial sheaf on $\boldsymbol{Sch}_k/\mathrm{zar}$ associated, by the Dold–Kan correspondence, to the good truncation $\tau^{\leq 0}\boldsymbol{R}w_*\mu_n(i)[2i]$ of the chain complex of Zariski sheaves $\boldsymbol{R}w_*\mu_n(i)[2i]$. In particular, we have $\pi_q\mathcal{E}_i(X) = H^{2i-q}_{\acute{e}t}(X, \mu_n(i))$ for $X \in \boldsymbol{Sch}_k$ and $0 \leq q \leq 2i$.

Let $\mathcal{L}$ denote the homotopy fiber of the map of simplicial presheaves $\mathcal{K} \xrightarrow{n} \mathcal{K}$ so that $\pi_q\mathcal{L}(X) = K_{q+1}(X, \mathbb{Z}/n)$ for $q \geq -1$. Let $\mathcal{H}^i(\mu_n(d))$ denote the Zariski sheaf $\boldsymbol{R}^i w_*\mu_n(d)$ on $\boldsymbol{Sch}_k/\mathrm{zar}$. It follows from Gillet's construction of the Chern classes (see [Gillet 1981, §5]) that there is a morphism of the simplicial presheaves on $\boldsymbol{Sch}_k$:

$$c_i : \mathcal{K} \to \mathcal{E}_i \tag{3-2}$$

in the homotopy category of simplicial presheaves. This map is compatible with the local to global spectral sequences for the associated Zariski sheaves so that we get a map of spectral sequences which at the E_2 level is $H^p(X, \mathcal{K}_q) \to H^p(X, \mathcal{H}^{2i-q}(\mu_n(i)))$ and converges to the global Chern class on $\mathcal{E}_i$ given by $c_{i,X} : K_{q-p}(X) \to H^{2i-q+p}_{\acute{e}t}(X, \mu_n(i))$. These Chern classes in fact factor through the maps $\mathcal{K} \to \Sigma\mathcal{L} \to \mathcal{E}_i$ (see [Pedrini and Weibel 1994, §2]) so that there are Chern class maps $c_i : \mathcal{L} \to \Omega\mathcal{E}_i$ in the homotopy category of simplicial presheaves [Brown and Gersten 1973].

Let $\widetilde{\mathcal{K}}$ denote the simplicial presheaf on $\boldsymbol{Sch}_k$ such that $\widetilde{\mathcal{K}}(X)$ is the universal covering space of $\mathcal{K}(X)$ and let $\widetilde{\mathcal{L}}$ denote the homotopy fiber of the map $\widetilde{\mathcal{K}} \xrightarrow{n} \widetilde{\mathcal{K}}$. This yields $\pi_q(\widetilde{\mathcal{K}}(X)) = K_q(X)$ for $q \geq 2$ and $\pi_1(\widetilde{\mathcal{K}}(X)) = 0$. Furthermore, we have

$$\pi_q(\widetilde{\mathcal{L}}(X)) = \begin{cases} K_{q+1}(X, \mathbb{Z}/n) & \text{if } q \geq 2, \\ K_2(X)/n & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3-3}$$

Let $\widetilde{\mathcal{K}}^{(2)}$ denote the second layer of the Postnikov tower

$$(\cdots \to \widetilde{\mathcal{K}}^{(n)} \to \widetilde{\mathcal{K}}^{(n-1)} \to \cdots \to \widetilde{\mathcal{K}}^{(2)} \to \widetilde{\mathcal{K}}^{(1)} \to \widetilde{\mathcal{K}}^{(0)} = \star)$$

of $\widetilde{\mathcal{K}}$. There are compatible family of maps $f_n : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}^{(n)}$ inducing $\pi_q\widetilde{\mathcal{K}} \xrightarrow{\simeq} \pi_q\widetilde{\mathcal{K}}^{(n)}$ for $q \leq n$ and $\pi_q\widetilde{\mathcal{K}}^{(n)} = 0$ for $q > n$. Let $\widetilde{\mathcal{L}}^{(2)}$ denote the homotopy fiber of the map $\widetilde{\mathcal{K}}^{(2)} \xrightarrow{n} \widetilde{\mathcal{K}}^{(2)}$ so that there is a commutative diagram of simplicial presheaves

$$\begin{tikzcd} \widetilde{\mathcal{L}}^{(2)} \arrow[r] & \widetilde{\mathcal{K}}^{(2)} \arrow[r, "n"] & \widetilde{\mathcal{K}}^{(2)} \\ \widetilde{\mathcal{L}} \arrow[r] \arrow[u] \arrow[d] & \widetilde{\mathcal{K}} \arrow[r, "n"] \arrow[u] \arrow[d] & \widetilde{\mathcal{K}} \arrow[u] \arrow[d] \\ \mathcal{L} \arrow[r] & \mathcal{K} \arrow[r, "n"] & \mathcal{K}. \end{tikzcd} \tag{3-4}$$

Lemma 3.1. *Let $\mathcal{K}_2^{\bullet}$ denote the complex of presheaves $(\mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2)$ on $\boldsymbol{Sch}_k/\mathrm{zar}$. Then $\widetilde{\mathcal{L}}^{(2)}$ is weak equivalent to the simplicial presheaf obtained by applying the Dold–Kan correspondence to the chain complex $\mathcal{K}_2^{\bullet}[2]$.*

Proof. Using (3-4), it suffices to show that $\widetilde{\mathcal{K}}^{(2)}$ is an Eilenberg–Mac Lane complex of the type $(\mathcal{K}_2, 2)$. Let F^2 denote the homotopy fiber of the Kan fibration $\widetilde{\mathcal{K}}^{(2)} \to \widetilde{\mathcal{K}}^{(1)}$ in the Postnikov tower. We have $\pi_i\widetilde{\mathcal{K}}^{(1)} = 0$ for $i \geq 2$ and $\pi_1\widetilde{\mathcal{K}}^{(1)} = \pi_1\widetilde{\mathcal{K}} = 0$ by [May 1967, Theorem 8.4]. The long exact homotopy sequence now implies that $\pi_i F^2 \xrightarrow{\simeq} \pi_i\widetilde{\mathcal{K}}^{(2)}$ for all i and hence $F^2 \to \widetilde{\mathcal{K}}^{(2)}$ is a weak equivalence by the Whitehead theorem. Since F^2 is an Eilenberg–Mac Lane complex of type $(\mathcal{K}_2, 2)$ by Corollary 8.7 of the same work, we are done. □

3B. ***Cohomology of $\mathcal{K}_2$ on a surface.*** Let us now assume that X is a reduced quasiprojective surface over k. Let $\mathcal{K}_i(\mathbb{Z}/n)$ denote the Zariski sheaf on $\boldsymbol{Sch}_k$ associated to the presheaf $X \mapsto K_i(X, \mathbb{Z}/n)$.

Applying the Brown–Gersten spectral sequence (see [Brown and Gersten 1973, Theorem 3])

$$H^p(X, \pi_{-q}\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

to (3-2) and (3-3), we obtain a commutative diagram of exact sequences

$$\begin{tikzcd} H^0(X, \mathcal{K}_2/n) \arrow[r, "d"] \arrow[d, "c_2"', "\simeq"] & H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \arrow[r] \arrow[d, "c_2"] & H^0(X, \widetilde{\mathcal{L}}) \arrow[r] \arrow[d, "c_2"] & H^1(X, \mathcal{K}_2/n) \arrow[r] \arrow[d, "\simeq"', "c_2"] & 0 \\ H^0(X, \mathcal{H}^2(\mu_n(2))) \arrow[r, "d"'] & H^2(X, \mathcal{H}^1(\mu_n(2))) \arrow[r] & H^3_{ét}(X, \mu_n(2)) \arrow[r] & H^1(X, \mathcal{H}^2(\mu_n(2))) \arrow[r] & 0 \end{tikzcd} \tag{3-5}$$

where the vertical arrows are induced by the Chern class map $c_{2,X}$. The bottom row is exact because $\mathcal{H}^3(\mu_n(2)) = 0$ by [Milne 1980, Theorem VI.7.2]. The map $H^0(X, \widetilde{\mathcal{L}}) \xrightarrow{c_2} H^3_{ét}(X, \mu_n(2))$ is the one induced by the maps $H^0(X, \widetilde{\mathcal{L}}) \xrightarrow{c_2} H^0(X, \Omega\mathcal{E}_2) \simeq H^{-1}(X, \mathcal{E}_2) = \pi_1\mathcal{E}_2(X) = H^3_{ét}(X, \mu_n(2))$. The leftmost and the rightmost vertical arrows are isomorphisms by Hoobler's theorem [2006, Theorem 3].

Lemma 3.2. *There is a functorial map $H^0(X, \widetilde{\mathcal{L}}^{(2)}) \to \mathbb{H}^2(X, \mathcal{K}_2^\bullet)$ and a commutative diagram with exact rows*

$$\begin{tikzcd} H^0(X, \pi_1\widetilde{\mathcal{L}}^{(2)}) \arrow[r, "d"] \arrow[d] & H^2(X, \pi_2\widetilde{\mathcal{L}}^{(2)}) \arrow[r] \arrow[d] & H^0(X, \widetilde{\mathcal{L}}^{(2)}) \arrow[r] \arrow[d] & H^1(X, \pi_1\widetilde{\mathcal{L}}^{(2)}) \arrow[r] \arrow[d] & 0 \\ H^0(X, \mathcal{K}_2/n) \arrow[r, "d"'] & H^2(X, {}_n\mathcal{K}_2) \arrow[r] & \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \arrow[r] & H^1(X, \mathcal{K}_2/n) \arrow[r] & 0 \end{tikzcd} \tag{3-6}$$

such that all vertical arrows are isomorphisms.

Proof. The bottom exact sequence follows from the exact triangle

$${}_n\mathcal{K}_2 \to \mathcal{K}_2^\bullet \to \mathcal{K}_2/n[-1] \to {}_n\mathcal{K}_2[1] \tag{3-7}$$

in the derived category of Zariski sheaves of abelian groups on $\boldsymbol{Sch}_k$/zar. On the other hand, Lemma 3.1 says that $\widetilde{\mathcal{K}}^{(2)}$ is an Eilenberg–Mac Lane complex of the type $(\mathcal{K}_2, 2)$ and there is a homotopy equivalence $\widetilde{\mathcal{L}}^{(2)} \to \mathcal{K}_2^\bullet[2]$. This in particular implies that $\pi_1\widetilde{\mathcal{L}}^{(2)} \simeq \mathcal{K}_2/n$, $\pi_2\widetilde{\mathcal{L}}^{(2)} \simeq {}_n\mathcal{K}_2$ and $\pi_i\widetilde{\mathcal{L}}^{(2)} = 0$ for $i \neq 1, 2$. Applying the Brown–Gersten spectral sequence to $H^0(X, \widetilde{\mathcal{L}}^{(2)})$ and using Lemma 3.1, we conclude the proof. The commutativity follows because both rows are obtained by applying the hypercohomology spectral sequence to the map $\widetilde{\mathcal{L}}^{(2)} \to \mathcal{K}_2^\bullet[2]$. □

The key step in extending (3-1) to arbitrary surfaces is the following result.

Lemma 3.3. *The map $H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \to H^2(X, {}_n\mathcal{K}_2)$ induced by the universal coefficient theorem has a natural factorization*

$$H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \xrightarrow{c_2} H^2(X, \mathcal{H}^1(\mu_n(2))) \xrightarrow{\nu} H^2(X, {}_n\mathcal{K}_2)$$

such that the map ν is an isomorphism.

Proof. By [Barbieri-Viale et al. 1996, Lemma 6.2, Variant 6.3], there is a commutative diagram

$$\begin{tikzcd} \mathcal{H}^1(\mu_n(2)) \arrow[r, "\phi"] \arrow[dr] & \mathcal{K}_3(\mathbb{Z}/n) \arrow[r, "c_2"] \arrow[d, "\psi"] & \mathcal{H}^1(\mu_n(2)) \arrow[dl, dashed] \\ & {}_n\mathcal{K}_2 & \end{tikzcd} \tag{3-8}$$

such that $c_2 \circ \phi = -1$ and the composite $\psi \circ \phi$ is given by $a \mapsto \{a, \zeta\} \in {}_n\mathcal{K}_2$ (where $\zeta \in k^\times$ is a primitive n-th root of unity). This composite map is surjective and an isomorphism on the regular locus of X by [Barbieri-Viale et al. 1996, Lemma 6.2]. It follows that the induced map $\psi \circ \phi : H^2(X, \mathcal{H}^1(\mu_n(2))) \to H^2(X, {}_n\mathcal{K}_2)$ is an isomorphism (see [Pedrini and Weibel 1994, Lemma 1.3]). We thus have a diagram

$$\begin{tikzcd} H^2(X, \mathcal{H}^1(\mu_n(2))) \arrow[r, "\phi"] \arrow[dr, "\simeq"'] & H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \arrow[r, two heads, "c_2"] \arrow[d, "\psi"] & H^2(X, \mathcal{H}^1(\mu_n(2))) \arrow[dl, dashed, "\nu"] \\ & H^2(X, {}_n\mathcal{K}_2) & \end{tikzcd} \tag{3-9}$$

in which the triangle on the left is commutative. To prove the lemma, it is therefore sufficient to show that $\mathrm{Ker}(c_2) = \mathrm{Ker}(\psi)$. Equivalently, $\mathrm{Ker}(c_2) \subseteq \mathrm{Ker}(\psi)$.

We set $\mathcal{F} = \mathrm{Ker}(c_2)$ so that we have a split exact sequence

$$0 \to \mathcal{F} \to \mathcal{K}_3(\mathbb{Z}/n) \xrightarrow{c_2} \mathcal{H}^1(\mu_n(2)) \to 0. \tag{3-10}$$

By [Pedrini and Weibel 1994, Lemma 1.3], it suffices to show that the composite map $\mathcal{F} \to \mathcal{K}_3(\mathbb{Z}/n) \xrightarrow{\psi} {}_n\mathcal{K}_2$ of Zariski sheaves is zero on the smooth locus of X. Since this is a local question, it suffices to show that for a regular local ring A which is essentially of finite type over k, the map $\mathrm{Ker}(c_2) \to {}_nK_2(A)$ is zero. By the Gersten resolution of $K_2(A)$, Bloch–Ogus resolution for $H^1_{\acute{e}t}(A, \mu_n(2))$ and Gillet's resolution for $K_3(A, \mathbb{Z}/n)$ (see [Quillen 1973; Bloch and Ogus 1974; Gillet 1986]), we can replace A by its fraction field F.

We now have a commutative diagram

$$\begin{tikzcd} K_3^M(F) \arrow[r] \arrow[d] & K_3(F, \mathbb{Z}/n) \arrow[r, "c_2"] \arrow[d, equal] & H^1_{\acute{e}t}(F, \mu_n(2)) \arrow[r] \arrow[d, dashed] & 0 \\ K_3(F) \arrow[r] & K_3(F, \mathbb{Z}/n) \arrow[r, "\psi"'] & {}_nK_2(F) \arrow[r] & 0. \end{tikzcd} \tag{3-11}$$

One of the main results of [Levine 1989] (and also [Merkur'ev and Suslin 1990]) shows that the top row in (3-11) is exact. Since the bottom row is clearly exact, we get the desired conclusion. □

Our first main result of this section is the following extension of (3-1) to surfaces with arbitrary singularities.

Theorem 3.4. *Let X be a reduced quasiprojective surface over k and let $n \geq 1$ be an integer prime to p. Then, there is a short exact sequence*

$$0 \to H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Z}/n \to H^3_{\acute{e}t}(X, \mu_n(2)) \to {}_n\mathrm{CH}_0(X) \to 0.$$

Proof. It follows from Lemma 3.2 and the map between the Brown–Gersten spectral sequences associated to the morphism of simplicial presheaves $\widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}}^{(2)}$ that there is a commutative diagram

$$\begin{tikzcd}
H^0(X, \mathcal{K}_2/n) \arrow[r, "d"] \arrow[d, "\simeq"'] & H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \arrow[r] \arrow[d, "\psi"] & H^0(X, \widetilde{\mathcal{L}}) \arrow[r] \arrow[d] & H^1(X, \mathcal{K}_2/n) \arrow[r] \arrow[d, "\simeq"] & 0 \\
H^0(X, \mathcal{K}_2/n) \arrow[r, "d"'] & H^2(X, {}_n\mathcal{K}_2) \arrow[r] & H^0(X, \widetilde{\mathcal{L}}^{(2)}) \arrow[r] & H^1(X, \mathcal{K}_2/n) \arrow[r] & 0.
\end{tikzcd} \tag{3-12}$$

Combining this with (3-5) and Lemma 3.2, we obtain a commutative diagram

$$\begin{tikzcd}
H^0(X, \mathcal{K}_2/n) \arrow[r, "d"] & H^2(X, {}_n\mathcal{K}_2) \arrow[r] & \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \arrow[r] & H^1(X, \mathcal{K}_2/n) \arrow[r] & 0 \\
H^0(X, \mathcal{K}_2/n) \arrow[r, "d"] \arrow[u, "\simeq"] \arrow[d, "\simeq"', "c_2"] & H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \arrow[r] \arrow[u, "\psi"'] \arrow[d, "c_2"] & H^0(X, \widetilde{\mathcal{L}}) \arrow[r] \arrow[u] \arrow[d, "c_2"] & H^1(X, \mathcal{K}_2/n) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "c_2"', "\simeq"] & 0 \\
H^0(X, \mathcal{H}^2(\mu_n(2))) \arrow[r, "d"'] & H^2(X, \mathcal{H}^1(\mu_n(2))) \arrow[r] & H^3_{\acute{e}t}(X, \mu_n(2)) \arrow[r] & H^1(X, \mathcal{H}^2(\mu_n(2))) \arrow[r] & 0.
\end{tikzcd} \tag{3-13}$$

It follows from Lemma 3.3 that ψ and the corresponding vertical arrow downward are surjective with identical kernels. A simple diagram chase shows that the maps $H^0(X, \widetilde{\mathcal{L}}) \to \mathbb{H}^2(X, \mathcal{K}_2^\bullet)$ and $H^0(X, \widetilde{\mathcal{L}}) \to H^3_{\acute{e}t}(X, \mu_n(2))$ are surjective with identical kernels. In particular, there is a natural isomorphism $c_{2,X} : \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\simeq} H^3_{\acute{e}t}(X, \mu_n(2))$. Using the exact sequence

$$0 \to H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Z}/n \to \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \to {}_nH^2(X, \mathcal{K}_2) \to 0$$

and the isomorphisms $H^2(X, \mathcal{K}_2) \xrightarrow{\simeq} \mathrm{CH}_0^{LW}(X) \xrightarrow{\simeq} \mathrm{CH}_0(X)$ (see [Levine 1985, Theorem 7] and [Binda and Krishna 2018, Theorem 3.17]), we now conclude the proof. □

3C. ***Theorem 1.1 for singular surfaces.*** As an application of Theorem 3.4, we now prove a version of Theorem 1.1 for singular surfaces. This result was proven for normal projective surfaces in [Barbieri-Viale et al. 1996, Theorem 7.9]. Let $l \neq p$ be a prime number. Let X be a reduced projective surface over k. We shall make no distinction between $\mathrm{CH}_0^{LW}(X)$ and $\mathrm{CH}_0(X)$ in view of [Binda and Krishna 2018, Theorem 3.17].

Lemma 3.5. *$H^3_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ is divisible by l^n for every $n \geq 1$.*

Proof. The exact sequence of Theorem 3.4 is compatible with the maps $\mathbb{Z}/l^n \to \mathbb{Z}/l^{n+1}$. Taking the direct limit, we obtain a short sequence

$$0 \to H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \to H^3_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \xrightarrow{\tau_X} \mathrm{CH}_0(X)\{l\} \to 0. \tag{3-14}$$

The group on the left is divisible. It is known that $\mathrm{CH}_0(X)_{\deg 0}$ is generated by the images of the maps $\mathrm{Pic}^0(C) \to \mathrm{CH}_0(X)$, where $C \subsetneq X$ is a reduced Cartier curve. Since $\mathrm{Pic}^0(C)$ is l^n-divisible, it follows that $\mathrm{CH}_0(X)_{\deg 0}$ is l^n-divisible. In particular, $\mathrm{CH}_0(X)\{l\} = \mathrm{CH}_0(X)_{\deg 0}\{l\}$ is also l^n-divisible. The lemma follows. □

Theorem 3.6. *Given a reduced projective surface X over k and a prime $l \neq p$, the following hold.*

(1) $H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$.

(2) *The map* $\tau_X : H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to \mathrm{CH}_0(X)\{l\}$ *is an isomorphism.*

Proof. In view of (3-14), the theorem is equivalent to showing that τ_X is injective. To show this, it suffices to prove the stronger assertion that the map $\delta := \rho_X^{\mathrm{semi}} \circ \tau_X : H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to \mathrm{CH}_0(X)\{l\} \to J^2(X)\{l\}$ is an isomorphism.

In order to prove this, we first prove a stronger version of its surjectivity assertion, namely, that for every $n \geq 1$, the map $H^3_{ét}(X, \mu_{l^n}(2)) \to {}_{l^n}J^2(X)$ is surjective. In view of Theorem 3.4, it suffices to show that the map ${}_{l^n}\mathrm{CH}_0(X) \to {}_{l^n}J^2(X)$ is surjective. To prove this, we use [Mallick 2009, Theorem 14], which says that we can find a reduced Cartier curve $C \subsetneq X$ (a suitable hypersurface section in a projective embedding) such that $C \cap X_{\mathrm{reg}} \subseteq C_{\mathrm{reg}}$ and the induced map ${}_{l^n}\mathrm{Pic}^0(C) \to {}_{l^n}J^2(X)$ is surjective. The commutative diagram

$$\begin{tikzcd} {}_{l^n}\mathrm{Pic}^0(C) \arrow[r] \arrow[d, "\simeq"'] & {}_{l^n}\mathrm{CH}_0(X) \arrow[d] \\ {}_{l^n}J^1(C) \arrow[r] & {}_{l^n}J^2(X) \end{tikzcd}$$

then proves the desired surjectivity where the left vertical arrow is an isomorphism because the kernel $\mathrm{Ker}(\mathrm{Pic}^0(C) \twoheadrightarrow J^1(C))$ is unipotent, and hence uniquely l^n-divisible.

In the rest of the proof, we shall ignore the Tate twist in étale cohomology. Let δ_n denote the composite map $H^3_{ét}(X, \mu_{l^n}(2)) \to {}_{l^n}\mathrm{CH}_0(X) \to {}_{l^n}J^2(X)$ so that $\delta = \varinjlim_n \delta_n$. Using the above surjectivity, we get a direct system of short exact sequences

$$0 \to T_{l^n}(X) \to H^3_{ét}(X, \mu_{l^n}(2)) \to {}_{l^n}J^2(X) \to 0 \tag{3-15}$$

whose direct limit is the short exact sequence

$$0 \to T_{l^\infty}(X) \to H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to J^2(X)\{l\} \to 0. \tag{3-16}$$

To prove the theorem, we are only left with showing that $T_{l^\infty}(X) = 0$. Since

$$H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \varinjlim_n H^3_{ét}(X, \mu_{l^n}(2)) \simeq \varinjlim_n H^3_{ét}(X, \mathbb{Z}/l^n),$$

it follows that $H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))\{l\}$. In particular, $T_{l^\infty}(X)$ is an l-primary torsion group.

We next consider a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & T_{l^\infty}(X) \arrow[r] \arrow[d, "l_1^n"'] & H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \arrow[r, "\delta"] \arrow[d, "l_2^n"] & J^2(X)\{l\} \arrow[r] \arrow[d, "l_3^n"] & 0 \\ 0 \arrow[r] & T_{l^\infty}(X) \arrow[r] & H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \arrow[r, "\delta"] & J^2(X)\{l\} \arrow[r] & 0, \end{tikzcd}$$

in which the vertical arrows are simply multiplication by l^n. It follows from Lemma 3.5 that the middle vertical arrow is surjective. We have shown above that $H^3_{ét}(X, \mu_{l^n}(2)) \to {}_{l^n}J^2(X)$ is surjective. In particular, the map $\ker(l_2^n) \to \mathrm{Ker}(l_3^n)$ is surjective. It follows that $T_{l^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}/l^n = 0$ for every $n \geq 1$. We thus only have to show that T_{l^∞} is finite to finish the proof.

Now, an easy argument that involves dualizing the argument of [Barbieri-Viale and Srinivas 2001, Theorem 2.5.4] (see [Mallick 2009, Proof of Theorem 15, Claim 2]), shows that there is a positive integer N_X, depending only on X and not on the integer n, such that

$$|H^3_{ét}(X, \mu_{l^n}(2)))| \leq N_X \cdot |{}_{l^n}J^2(X)|. \tag{3-17}$$

Combining this with (3-15), it follows that for every $n \geq 1$, either ${}_{l^n}J^2(X) = 0$ and hence $T_{l^n}(X) = 0$ or ${}_{l^n}J^2(X) \neq 0$ and $|T_{l^n}(X)| \cdot |{}_{l^n}J^2(X)| \leq N_X \cdot |{}_{l^n}J^2(X)|$. In particular, either $T_{l^n}(X) = 0$ or $|T_{l^n}(X)| \leq N_X$ for every $n \geq 1$. But this implies that $T_{l^\infty}(X)$ is finite. This completes the proof of the theorem. □

3D. *Relation with Bloch's construction.* Bloch [1979, §2] constructed a map $\lambda_X : \mathrm{CH}_0(X)\{l\} \to H^{2d-1}_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ for a smooth projective scheme X of dimension $d \geq 1$ over k. We end this section with the following lemma that explains the relation between Bloch's construction and ours when X is a smooth surface.

Lemma 3.7. *If X is a smooth projective surface over k, then $\tau_X : H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to \mathrm{CH}_0(X)\{l\}$ coincides with the negative of the inverse of Bloch's map $\lambda_X : \mathrm{CH}_0(X)\{l\} \to H^3_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$.*

Proof. To prove the lemma, we have to go back to the construction of $\tau_X : H^3_{ét}(X, \mu_n(2)) \to {}_{l^n}\mathrm{CH}_0(X)$ in Section 3B for $n \in k^\times$. Since X is smooth, there is an isomorphism of Zariski sheaves $\mathcal{O}_X^\times/n \xrightarrow{\simeq} \mathcal{H}^1(\mu_n(2)) \xrightarrow{\simeq} {}_n\mathcal{K}_2$ by [Barbieri-Viale et al. 1996, Lemma 6.2]. Since $H^2(X, \mathcal{O}_X^\times) = 0$, we see that the top cohomology of all these sheaves vanish.

This implies that the map $\mathbb{H}^2(X, \mathcal{K}_2^\bullet) \to {}_nH^2(X, \mathcal{K}_2)$ is simply the composite

$$\mathbb{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\simeq} H^1(X, \mathcal{K}_2/n) \to H^2(X, {}_n\mathcal{K}_2) \to {}_nH^2(X, \mathcal{K}_2).$$

Moreover, there is a diagram

$$\begin{tikzcd} \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \arrow[r, "\simeq"] \arrow[d, dashed] & H^1(X, \mathcal{K}_2/n) \arrow[d, "\beta_n"', "\simeq"] \\ H^3_{ét}(X, \mu_n(2)) \arrow[r, "\simeq"'] & H^1(X, \mathcal{H}^2(\mu_n(2))) \end{tikzcd} \tag{3-18}$$

where the right vertical arrow is the Galois symbol map. This induces a unique isomorphism

$$\alpha_n : H^3_{\acute{e}t}(X, \mu_n(2)) \xrightarrow{\simeq} \mathbb{H}^2(X, \mathcal{K}_2^{\bullet}).$$

One checks from (3-13) that τ_X is simply the composite

$$H^3_{\acute{e}t}(X, \mu_n(2)) \xrightarrow{\alpha_n} \mathbb{H}^2(X, \mathcal{K}_2^{\bullet}) \xrightarrow{\simeq} H^1(X, \mathcal{K}_2/n) \xrightarrow{\delta_n} {}_nH^2(X, \mathcal{K}_2). \tag{3-19}$$

We also have a commutative diagram

$$\begin{tikzcd} K_2(k(X))/n \arrow[r] \arrow[d, "\beta_n"'] & \coprod_{x\in X^{(1)}} k(x)^{\times}/n \arrow[r] \arrow[d, "\beta_n"] & \coprod_{y\in X^{(2)}} \mathbb{Z}/n \arrow[d, "\beta_n"] \\ H^2_{\acute{e}t}(k(X), \mu_n(2)) \arrow[r] & \coprod_{x\in X^{(1)}} H^1_{\acute{e}t}(k(x), \mu_n(1)) \arrow[r] & \coprod_{y\in X^{(2)}} H^0_{\acute{e}t}(k(y), \mathbb{Z}/n) \end{tikzcd} \tag{3-20}$$

where all vertical arrows are the Galois symbols and are isomorphisms.

Since the Gersten resolution is universally exact (see [Quillen 1973; Grayson 1985]), the middle cohomology of the top row is $H^1(X, \mathcal{K}_2/n)$ and the Bloch–Ogus resolution (see [Bloch and Ogus 1974]) shows that the middle cohomology of the bottom row is $H^1(X, \mathcal{H}^2(\mu_n(2)))$. The isomorphism β_n in (3-18) is induced by the vertical arrows of (3-20).

For a fixed prime $l \neq p$, the map δ_{l^m} in (3-19) becomes an isomorphism on the limit over $m \geq 1$. Assuming this isomorphism, it follows from (3-19) that $\tau_X = \varinjlim_m \alpha_{l^m}$. On the other hand, it follows from Bloch's construction in [Bloch 1979, § 2] that $\lambda_X = \varinjlim_m \beta_{l^m}$. Since each α_n (with $n \in k^{\times}$) is the inverse of β_n by definition, the lemma follows. □

4. Roitman's torsion for separably weakly normal surfaces

The Roitman torsion theorem says that for a smooth projective scheme of dimension $d \geq 1$ over k, the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X)_{\deg 0} \to J^d(X)$ is an isomorphism on the torsion subgroups. This theorem was extended to normal projective schemes in [Krishna and Srinivas 2002]. However, when X has arbitrary singularities, this theorem is known only for the torsion prime-to-p; see [Biswas and Srinivas 1999; Mallick 2009]. In this section, we extend the Roitman torsion theorem to separably weakly normal (see below for definition) surfaces. Later in this text, we shall prove a suitable generalization of this theorem in higher dimension. This generalization will be used to prove Theorem 1.2.

4A. ***Separably weakly normal schemes.*** The weak normality is a singularity type of schemes, which in characteristic $p > 0$, is closely related to various F-singularities. These F-singularities are naturally encountered while running the minimal model program in positive characteristic. Most of the F-regularity conditions imply weak normality. In characteristic zero, weak normality coincides with the more familiar notion of seminormality. In this text, we shall study the Chow group of 0-cycles on certain singular schemes whose singularities are closely related to weak normality in positive characteristic.

Let A be a reduced commutative Noetherian ring. Let B be the integral closure of A in its total quotient ring. Recall from [Manaresi 1980] that the seminormalization of A is the largest among the subrings A' of B containing A such that

(1) for all $x \in \mathrm{Spec}(A)$, there exists exactly one $x' \in \mathrm{Spec}(A')$ over x;

(2) the canonical homomorphism $k(x) \to k(x')$ is an isomorphism.

The weak normalization of A is the largest among the subrings A' of B containing A such that

(1) for all $x \in \mathrm{Spec}(A)$, there exists exactly one $x' \in \mathrm{Spec}(A')$ over x;

(2) the field extension $k(x) \to k(x')$ is finite purely inseparable.

We let $A_s \subset B$ and $A_w \subset B$ denote the seminormalization and weak normalization of A, respectively. One says that A is seminormal (resp. weakly normal) if $A = A_s$ (resp. $A = A_w$). It is clear from the above definition that $A \subset A_s \subset A_w \subset B$. Moreover, $A_s = A_w$ if A is a $\mathbb{Q}$-algebra. To get a more geometric understanding of these singularities, we make the following definition.

Definition 4.1. Let k be a field and let R be a k-algebra which is finite as a k-vector space. We shall say that R is *weakly separable* over k if it is reduced and either $\mathrm{char}(k) = 0$, or $\mathrm{char}(k) > 0$ and there is no inclusion of rings $k \subsetneq K \subset R$ such that K is a purely inseparable field extension of k.

Recall that a commutative Noetherian ring A is called an S_2 ring if for every prime ideal $\mathfrak{p}$ of A, one has $\mathrm{depth}(A_{\mathfrak{p}}) \geq \min(\mathrm{ht}(\mathfrak{p}), 2)$. It follows easily from this definition that a Cohen–Macaulay ring is S_2.

Proposition 4.2. *Let A be reduced commutative Noetherian ring. Let B be the integral closure of A in its total quotient ring and let $I \subset B$ be the largest ideal which is contained in A. Assume that B is a finite A-module. Then the following hold.*

(1) *If A is seminormal, then B/I is reduced. If A is S_2, the converse also holds.*

(2) *If A is weakly normal, then B/I is reduced and the inclusion map $A/I \to B/I$ is generically weakly separable. If A is S_2, the converse also holds.*

Proof. If A is seminormal, then B/I is reduced by [Traverso 1970, Lemma 1.3]. Suppose now that A is S_2 and B/I is reduced. By [Greco and Traverso 1980, Theorem 2.6], it suffices to show that $A_{\mathfrak{p}}$ is seminormal for every height one prime ideal $\mathfrak{p}$ in A.

Let $\mathfrak{p} \subset A$ be a prime ideal of height one. Let $\{x_1, \dots, x_r\} \subset B$ be a subset whose image in B/A generates it as an A-module. We now note that $(A_{\mathfrak{p}} : B_{\mathfrak{p}}) = (A : B)_{\mathfrak{p}}$, where

$$I := (A : B) = \{a \in A \mid aB \subset A\} = \mathrm{ann}(B/A).$$

The first equality uses the fact that B/A is a finite A-module so that $I = \bigcap_{i=1}^{r} (A : x_i)$ and $(J \cap J')_{\mathfrak{p}} = J_{\mathfrak{p}} \cap J'_{\mathfrak{p}}$; see [Matsumura 1986, Example 4.8]. Since $B_{\mathfrak{p}}/I_{\mathfrak{p}} = (B/I)_{\mathfrak{p}}$ is reduced (by our assumption) and $A_{\mathfrak{p}}$ is 1-dimensional, it follows from [Bass and Murthy 1967, Proposition 7.2] and [Traverso 1970, Theorem 3.6] that $A_{\mathfrak{p}}$ is seminormal. This proves (1).

Suppose now that A is weakly normal. Then it is clearly seminormal. In particular, B/I is reduced by (1). We now show that $A/I \hookrightarrow B/I$ is generically weakly separable. Note that as A is reduced and generically weakly normal, I must have height at least one in A. Let $\mathfrak{p}$ be a minimal prime of I in A and let $k(\mathfrak{p})$ be its residue field. Then $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ is the Jacobson radical of $B_{\mathfrak{p}}$ and hence $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is a product of finite field extensions of $k(\mathfrak{p})$.

We consider the commutative diagram

$$\begin{tikzcd} A_{\mathfrak{p}} \arrow[r] \arrow[d] & B_{\mathfrak{p}} \arrow[d] & \\ k(\mathfrak{p}) \arrow[r] & B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \arrow[r, "\simeq"] & \prod_{i=1}^{s} k(\mathfrak{q}_i), \end{tikzcd} \tag{4-1}$$

in which the square is Cartesian, the horizontal arrows are injective finite morphisms and the vertical arrows are surjective. Since $A_{\mathfrak{p}}$ is weakly normal (see [Manaresi 1980, Theorem IV.3]) and $B_{\mathfrak{p}}$ is its integral closure, it means that $A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$. We conclude from [Yanagihara 1983, Proposition 3] that $k(\mathfrak{p})$ is weakly normal in $\prod_{i=1}^{s} k(\mathfrak{q}_i)$. We now apply Lemma 2 there to deduce that the lower horizontal arrow in (4-1) is weakly separable, as desired.

Conversely, suppose that A is S_2 and the inclusion $A/I \hookrightarrow B/I$ is generically weakly separable map of reduced schemes. By [Manaresi 1980, Corollary IV. 4], it suffices to show that $A_{\mathfrak{p}}$ is weakly normal for every height one prime ideal $\mathfrak{p}$ in A.

Let $\mathfrak{p} \subset A$ be a prime ideal of height one. Since $I_{\mathfrak{p}} = (A_{\mathfrak{p}} : B_{\mathfrak{p}})$ as shown above, it follows that $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ (hence $A_{\mathfrak{p}}$ is weakly normal) if $I \not\subset \mathfrak{p}$. We can therefore assume that $I \subset \mathfrak{p}$. Since A is generically normal (and hence weakly normal), I must have height at least one in A. It follows that $\mathfrak{p}$ must be a minimal prime ideal of I.

It follows from (1) that $A_{\mathfrak{p}}$ is seminormal. In particular, $I_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$. Furthermore, $\mathfrak{p}B_{\mathfrak{p}}$ is the Jacobson radical of $B_{\mathfrak{p}}$ such that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is a finite product of finite field extensions of $k(\mathfrak{p})$. This gives rise to a Cartesian square of rings as in (4-1). Our assumption says that the lower horizontal arrow in (4-1) is weakly separable. We conclude again from [Yanagihara 1983, Lemma 2, Proposition 3] that $A_{\mathfrak{p}}$ is weakly normal. This proves (2). □

Example 4.3. Using Proposition 4.2, we can construct many examples of seminormal rings which are not weakly normal. Let k be perfect field of characteristic $p > 0$ and consider the Cartesian square of rings

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d] & k[x,t] \arrow[d, two heads] \\ k[x] \arrow[r, "\phi"] & \dfrac{k[x,t]}{(t^p - x)} \end{tikzcd} \tag{4-2}$$

in which the right vertical arrow is the canonical surjection and ϕ is the canonical inclusion. In particular, one has $A = \{(f(x), g(x,t)) \mid f(t^p) = g(t^p, t)\}$. Since ϕ is a finite purely inseparable map of reduced rings which is not an isomorphism, it follows that ψ is not an isomorphism. It follows from Proposition 4.2 that A is seminormal but not weakly normal.

Example 4.4. We now provide an example of a weakly normal S_2 ring A with normalization B and reduced conductor I such that the map $A/I \to B/I$ is not generically separable. Let k be a perfect field of characteristic $p > 0$ and consider the Cartesian square

$$\begin{array}{ccc} A & \xrightarrow{\psi} & k[x,t] \\ \downarrow & & \downarrow \\ k[x] & \xrightarrow{\phi} & \dfrac{k[x,t]}{t(t^p - x)} \end{array} \tag{4-3}$$

where the right vertical arrow is the canonical quotient map and the lower horizontal arrow is given by $\phi(x) = x$. One can easily check (e.g., use Eisenstein's criterion) that $t^p - x$ is an irreducible polynomial in $k[x,t]$. If we let $\mathfrak{q}_1 = (t)$ and $\mathfrak{q}_2 = (t^p - x)$, then we see that $k(x) \to k(\mathfrak{q}_1)$ is an isomorphism and $k(x) \to k(\mathfrak{q}_2) \xrightarrow{\simeq} k(x^{1/p})$ is purely inseparable. In particular, ϕ induces a weakly separable map between the function fields. It follows from [Yanagihara 1983, Proposition 3] that A is weakly normal. We just saw however that ϕ is not generically separable.

Note also that the kernels of the vertical arrows in (4-3) are isomorphic and the kernel on the right is a principal ideal generated by $f(x,t) = t(t^p - x)$. Since $f(x,t)$ a nonzero-divisor in $k[x,t]$ and lies in A, it must be a nonzero-divisor in A. Since $k[x]$ is Cohen–Macaulay, it follows that A is Cohen–Macaulay too; see [Matsumura 1986, Theorem 17.3]. In particular, it is S_2.

Motivated by Example 4.4, we make the following definition.

Definition 4.5. Let A be a commutative reduced Noetherian ring with finite normalization B. Let $I \subset B$ be the largest ideal lying in A. We shall say that A is *separably weakly normal* if B/I is reduced and the induced map $A/I \to B/I$ is generically separable. We shall say that a Noetherian scheme X is (separably) weakly normal (resp. seminormal) if the coordinate ring of every affine open in X is (separably) weakly normal (resp. seminormal).

It follows from Proposition 4.2 that if A is S_2 and separably weakly normal, then it is weakly normal. On the other hand, Example 4.4 shows that a weakly normal S_2 ring may not be separably weakly normal. For $\mathbb{Q}$-algebras, seminormality implies separably weak normality by Proposition 4.2. The three singularity types coincide for $\mathbb{Q}$-algebras which are S_2. Our goal in the rest of this text is to study the torsion in the Chow group of 0-cycles on schemes which are separably weakly normal.

Remark 4.6. Since we only deal with schemes which are separably weakly normal in the rest of this text, the reader can, in principle, only read Definition 4.5 in Section 4A and move to the next subsection. Our discussion of seminormality and weak normality is primarily meant to motivate the reader to Definition 4.5, and to give a comparison of various nonnormal singularity types which are closely related yet different.

We end this subsection by recalling the notion of conducting ideals and conducting subschemes. Let A be a reduced commutative Noetherian ring and let B be a subring of the integral closure of A in its total quotient ring such that $A \subset B$. Assume that B is a finite A-module. Recall that an ideal $I \subset A$ is called a conducting ideal for the inclusion $A \subset B$ if $I = IB$. It is clear from this definition that $I \subset (A : B)$. Furthermore, one knows that $(A : B)$ is the largest conducting ideal for $A \subset B$; see [Huneke and Swanson 2006, Example 2.11]. If X is a reduced Noetherian scheme and $f : X' \to X$ is a finite birational map, then a closed subscheme $Y \subset X$ is called a conducting subscheme if $\mathcal{I}_Y \subset \mathcal{O}_X$ is a sheaf of conducting ideals for the inclusion of sheaves of rings $\mathcal{O}_X \subset f_*(\mathcal{O}_{X'})$.

4B. ***The torsion theorem for separably weakly normal surfaces.*** For the remaining part of this section, we shall identify the two Chow groups $\mathrm{CH}_0^{LW}(X)$ and $\mathrm{CH}_0(X)$ for curves and surfaces using [Binda and Krishna 2018, Theorem 3.17]. To prove the Roitman torsion theorem for a separably weakly normal projective surface, we need the following excision result for our Chow group. We fix an algebraically closed field k.

Lemma 4.7. *Let X be a reduced quasiprojective separably weakly normal surface over k and let $f : \overline{X} \to X$ denote the normalization map. Let $Y \subset X$ denote the smallest conducting closed subscheme with $\overline{Y} = Y \times_X \overline{X}$. Then there is an exact sequence*

$$SK_1(\overline{X}) \oplus SK_1(Y) \to SK_1(\overline{Y}) \to \mathrm{CH}_0(X) \to \mathrm{CH}_0(\overline{X}) \to 0. \tag{4-4}$$

Proof. We have a commutative diagram of relative and birelative K-theory exact sequences:

$$\begin{tikzcd}
 & & K_0(X,\overline{X},Y) \arrow[d] & & & K_{-1}(X,\overline{X},Y) \arrow[d] \\
K_1(X) \arrow[r] \arrow[d] & K_1(Y) \arrow[r] \arrow[d] & K_0(X,Y) \arrow[r] \arrow[d] & K_0(X) \arrow[r] \arrow[d] & K_0(Y) \arrow[r] \arrow[d] & K_{-1}(X,Y) \arrow[d] \\
K_1(\overline{X}) \arrow[r] & K_1(\overline{Y}) \arrow[r] & K_0(\overline{X},\overline{Y}) \arrow[r] \arrow[d] & K_0(\overline{X}) \arrow[r] & K_0(\overline{Y}) \arrow[r] & K_{-1}(\overline{X},\overline{Y}) \arrow[d] \\
 & & K_{-1}(X,\overline{X},Y) & & & K_{-2}(X,\overline{X},Y).
\end{tikzcd} \tag{4-5}$$

Using the Thomason–Trobaugh spectral sequence and the results of Bass [1968] that the sheaves $\mathcal{K}_{i,(X,\overline{X},Y)}$ on Y vanish for $i \le 0$, we see that $K_i(X,\overline{X},Y) = 0$ for $i \le -1$. It also follows from the spectral sequence that $K_0(X,\overline{X},Y) \simeq H^1(Y, \mathcal{K}_{1,(X,\overline{X},Y)})$. It follows from [Geller and Weibel 1983, Theorem 0.2] that $H^1(Y, \mathcal{K}_{1,(X,\overline{X},Y)}) \simeq H^1(X, \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathcal{O}_Y} \Omega^1_{\overline{Y}/Y})$. Since X is separably weakly normal, it follows from Definition 4.5 that the map $\overline{Y} \to Y$ is a finite generically étale map of reduced schemes. Hence $\Omega^1_{\overline{Y}/Y}$ has 0-dimensional support and we conclude that $H^1(X, \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathcal{O}_Y} \Omega^1_{\overline{Y}/Y}) = 0$. In particular, $K_0(X,\overline{X},Y) = 0$.

A diagram chase in (4-5) shows that there is an exact sequence

$$K_1(\overline{X}) \oplus K_1(Y) \to K_1(\overline{Y}) \to \widetilde{K}_0(X) \to \widetilde{K}_0(\overline{X}) \oplus \widetilde{K}_0(Y) \to \widetilde{K}_0(\overline{Y}),$$

where $\widetilde{K}_0(-) = \mathrm{Ker}(K_0(-) \to H^0(-, \mathbb{Z}))$. Using the map of sheaves $\mathcal{O}_X^\times \to f_*(\mathcal{O}_{\overline{X}}^\times)$, this exact sequence maps to a similar unit-Pic exact sequence

$$U(\overline{X}) \oplus U(Y) \to U(\overline{Y}) \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X}) \oplus \mathrm{Pic}(Y) \to \mathrm{Pic}(\overline{Y}).$$

Taking the kernels and using the Levine's formula $SK_0(Z) := \mathrm{Ker}(\widetilde{K}_0(Z) \to \mathrm{Pic}(Z)) \simeq H^2(Z, \mathcal{K}_2) \simeq \mathrm{CH}_0(Z)$ for a reduced surface Z, we get (4-4). □

Lemma 4.8. *Let Y be a reduced curve over k and let $r \geq 0$ denote the number of irreducible components of Y which are projective over k. If Y is affine, then $SK_1(Y)$ is uniquely divisible. If Y is projective, then $SK_1(Y)$ is divisible, $SK_1(Y)\{l\} \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^r$ for a prime $l \neq p$ and $SK_1(Y)\{p\} = 0$.*

Proof. It follows from [Barbieri-Viale et al. 1996, Theorem 5.3] that $SK_1(Y) \simeq (k^\times)^r \oplus V$, where V is uniquely divisible. The lemma follows directly from this isomorphism. □

Theorem 4.9. *Let X be a reduced projective separably weakly normal surface over an algebraically closed field k of exponential characteristic p. Then, $A^2(X)$ is a semiabelian variety and the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X)_{\deg 0} \to A^2(X)$ is an isomorphism on the torsion subgroups.*

Proof. We can find a finite collection of reduced complete intersection Cartier curves $\{C_1, \dots, C_r\}$ on X such that the induced map of algebraic groups $\prod_{i=1}^r \mathrm{Pic}^0(C_i) \to A^2(X)$ is surjective (see, for instance, [Esnault et al. 1999, (7.1), p. 657]). In characteristic zero, we can further assume by [Cumino et al. 1983, Corollary 2.5] that each of these curves is weakly normal. Since a surjective morphism of smooth connected algebraic groups restricts to a surjective map on their unipotent and semiabelian parts and since $\mathrm{Pic}^0(C)$ is semiabelian if C is a weakly normal curve (easy to check), it follows that $A^2(X)$ is a semiabelian variety in characteristic zero.

In characteristic $p \geq 2$, the surjections $\prod_{i=1}^r \mathrm{Pic}^0(C_i) \twoheadrightarrow A^2(X) \twoheadrightarrow J^2(X)$ and [Krishna 2015a, Lemma 2.7] together imply that the induced maps $(\mathrm{CH}^2(X)_{\deg 0})_{\mathrm{tors}} \to A^2(X)_{\mathrm{tors}} \to J^2(X)_{\mathrm{tors}}$ are also surjective. Since $A^2_{\mathrm{unip}}(X)$ is a p-primary torsion group of bounded exponent, the theorem will follow if we prove, in any characteristic, that the composite map $\rho_X^{\mathrm{semi}} : \mathrm{CH}_0(X)_{\deg 0} \to A^2(X) \to J^2(X)$ is injective on the torsion subgroup.

It follows from [Mallick 2009, Theorem 15] that the map $\mathrm{CH}_0(X)_{\deg 0}\{l\} \to J^2(X)\{l\}$ is an isomorphism for every prime $l \neq p$. This also follows immediately from Theorem 3.4 and the proof of Theorem 3.6. We can thus assume that $p \geq 2$ and $l = p$.

Let $f : \bar{X} \to X$ be the normalization and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0(X)_{\deg 0}\{p\} & \xrightarrow{f^*} & \mathrm{CH}_0(\bar{X})_{\deg 0}\{p\} \\ \big\downarrow & & \big\downarrow \\ J^2(X)\{p\} & \longrightarrow & J^2(\bar{X})\{p\}. \end{array} \tag{4-6}$$

The right vertical arrow is an isomorphism by [Krishna and Srinivas 2002, Theorem 1.6]. Thus, it suffices to prove the stronger assertion that the map $f^* : \mathrm{CH}_0(X)\{p\} \to \mathrm{CH}_0(\bar{X})\{p\}$ is an isomorphism. It is enough to show that $A := \mathrm{Ker}(\mathrm{CH}_0(X) \twoheadrightarrow \mathrm{CH}_0(\bar{X}))$ is uniquely p-divisible.

It follows from Lemma 4.7 that there is an exact sequence

$$SK_1(\bar{X}) \oplus SK_1(Y) \to SK_1(\bar{Y}) \to A \to 0.$$

Following the notations of Lemma 4.7, it follows from Lemma 4.8 that $SK_1(Y)$ and $SK_1(\bar{Y})$ are uniquely p-divisible. Furthermore, it follows from [Krishna 2015a, Theorem 5.6] that $SK_1(\bar{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$. An elementary argument now shows that A must be uniquely p-divisible. This finishes the proof. □

Corollary 4.10. *If X is a reduced projective separably weakly normal surface over $\bar{\mathbb{F}}_p$, then $\mathrm{CH}_0(X)_{\deg 0}$ is finite-dimensional. That is, the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X)_{\deg 0} \to A^2(X)$ is an isomorphism.*

Proof. In view of Theorem 4.9, it suffices to show that for any reduced projective scheme X of dimension $d \geq 1$ over $\bar{\mathbb{F}}_p$, the group $\mathrm{CH}_0^{LW}(X)_{\deg 0}$ is a torsion abelian group.

Given $\alpha \in \mathrm{CH}_0^{LW}(X)_{\deg 0}$, we can find a reduced Cartier curve $C \subseteq X$ such that α lies in the image of the pushforward map $\mathrm{Pic}^0(C) \to \mathrm{CH}_0^{LW}(X)_{\deg 0}$. It is therefore enough to show that $\mathrm{Pic}^0(C)$ is torsion. But this is a special case of the more general fact that $G(\bar{\mathbb{F}}_p)$ is torsion whenever G is a smooth commutative algebraic group over $\bar{\mathbb{F}}_p$. □

5. Bloch's torsion theorem for 0-cycles with modulus

We continue with our assumption that k is algebraically closed with exponential characteristic p. Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor on X. We shall prove Theorem 1.1 in this section.

5A. ***Relative étale cohomology.*** Given an étale sheaf $\mathcal{F}$ on $\boldsymbol{Sch}_k$ and a finite map $f : Y \to X$ in $\boldsymbol{Sch}_k$, let $\mathcal{F}_{(X,Y)} := \mathrm{Cone}(\mathcal{F}|_X \to f_*(\mathcal{F}|_Y))[-1]$ be the chain complex of étale sheaves on X. The exactness of f_* on étale sheaves implies that there is a long exact sequence of étale (hyper)cohomology groups

$$0 \to H^0_{\acute{e}t}(X, \mathcal{F}_{(X,Y)}) \to H^0_{\acute{e}t}(X, \mathcal{F}) \to H^0_{\acute{e}t}(Y, \mathcal{F}) \to H^1_{\acute{e}t}(X, \mathcal{F}_{(X,Y)}) \to \cdots. \tag{5-1}$$

If $Y \hookrightarrow X$ is a closed immersion with complement $j : U \hookrightarrow X$, then one checks immediately from the above definition that $H^*_{\acute{e}t}(X, \mathcal{F}_{(X,Y)})$ is canonically isomorphic to $H^*_{\acute{e}t}(X, j_!(\mathcal{F}|_U))$. We conclude that if X is projective over k and $Y \hookrightarrow X$ is closed, then $H^*_{\acute{e}t}(X, \mathcal{F}_{(X,Y)})$ is same as the étale cohomology with compact support $H^*_{c,\acute{e}t}(X \setminus Y, \mathcal{F})$ of $X \setminus Y$.

Let us now consider an abstract blow-up diagram in $\boldsymbol{Sch}_k$:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd} \tag{5-2}$$

This is a Cartesian square in which the horizontal arrows are closed immersions, π is a proper morphism such that $X' \setminus Y' \xrightarrow{\simeq} X \setminus Y$. The proper base change theorem for the étale cohomology implies that for a torsion constructible étale sheaf $\mathcal{F}$ on $\boldsymbol{Sch}_k$, the cohomology groups $H^*_{ét}(-, \mathcal{F})$ satisfy the cdh-descent. In particular, the canonical map

$$\pi^* : H^i_{ét}(X, \mathcal{F}_{(X,Y)}) \to H^i_{ét}(X', \mathcal{F}_{(X',Y')}) \tag{5-3}$$

is an isomorphism for every $i \geq 0$. We shall write the *relative étale cohomology groups* $H^*_{ét}(X, \mathcal{F}_{(X,Y)})$ for a closed immersion $Y \hookrightarrow X$ as $H^*_{ét}(X|Y, \mathcal{F})$.

5B. ***A weak Lefschetz-type theorem for the double.*** We now come back to our situation of X being a smooth projective scheme of dimension $d \geq 1$ over k and $D \subset X$ an effective Cartier divisor. Let $\{E_1, \dots, E_r\}$ be the set of irreducible components of D_{red}. If $d \geq 3$, we choose, as in Section 2E, a closed embedding $X \hookrightarrow \mathbb{P}^N_k$ and a smooth hypersurface section $\tau : Y = X \cap H_1 \hookrightarrow X$ such that Y is not contained in D and $Y \cap E_i$ is integral for every $1 \leq i \leq r$. We set $F = D \cap Y$. We shall use these notations throughout the rest of this section.

Given a prime-to-p integer n, it is clear from the definition of the relative étale cohomology and its cdh-descent (see Section 5A) associated to the Cartesian square (2-1) (see Lemma 2.2) that the pullback maps via the closed immersions $\iota_\pm : X \hookrightarrow S_X$ induce, for each $i \geq 0$, a split exact sequence of étale cohomology

$$0 \to H^i_{ét}(X|D, \mu_n(j)) \xrightarrow{p_{+,*}} H^i_{ét}(S_X, \mu_n(j)) \xrightarrow{\iota^*_-} H^i_{ét}(X, \mu_n(j)) \to 0. \tag{5-4}$$

Here, the splitting of ι^*_- is given by the pullback $\Delta^* : H^i_{ét}(X, \mu_n(j)) \to H^i_{ét}(S_X, \mu_n(j))$.

Let us next recall from Gabber's construction [Fujiwara 2002] of the Gysin morphism for étale cohomology (see, also [Navarro 2018, Definition 2.1]) that the regular closed embeddings $\tau : Y \hookrightarrow X$ and $\tau_1 : S_Y \hookrightarrow S_X$ induce Gysin morphisms $\tau_* : H^i_{ét}(Y, \mu_n(j)) \to H^{i+2}_{ét}(X, \mu_n(j+1))$ and $\tau_{1,*} : H^i_{ét}(S_Y, \mu_n(j)) \to H^{i+2}_{ét}(S_X, \mu_n(j+1))$ for $i \geq 0$. Furthermore, it follows from the Cartesian square (2-8) and [Navarro 2018, Corollary 2.12] (see also [Fujiwara 2002, Proposition 1.1.3]) that the pullback via the closed immersions $\iota_\pm : X \hookrightarrow S_X$ induces a commutative diagram

$$\begin{tikzcd} H^i_{ét}(S_Y, \mu_n(j)) \arrow[r, "\iota^*_\pm"] \arrow[d, "\tau_{1,*}"'] & H^i_{ét}(Y, \mu_n(j)) \arrow[d, "\tau_*"] \\ H^{i+2}_{ét}(S_X, \mu_n(j+1)) \arrow[r, "\iota^*_\pm"'] & H^{i+2}_{ét}(X, \mu_n(j+1)). \end{tikzcd} \tag{5-5}$$

Lemma 5.1. *If $d \geq 3$, then the Gysin maps $\tau_* : H^{2d-3}_{\acute{e}t}(Y, \mu_n(d-1)) \to H^{2d-1}_{\acute{e}t}(X, \mu_n(d))$ and $\tau_{1,*} : H^{2d-3}_{\acute{e}t}(S_Y, \mu_n(d-1)) \to H^{2d-1}_{\acute{e}t}(S_X, \mu_n(d))$ are isomorphisms.*

Proof. The first isomorphism is a well known consequence of the weak Lefschetz theorem for étale cohomology. The main problem is to prove the second isomorphism. Since k is algebraically closed, we shall replace μ_n by the constant sheaf $\Lambda = \mathbb{Z}/n$.

Recall from [SGA 4½ 1977, §2] that the line bundle $\mathcal{O}_X(Y)$ (which we shall write in short as $\mathcal{O}(Y)$) on X has a canonical class $[\mathcal{O}(Y)] \in H^1_{Y,\acute{e}t}(X, \mathbb{G}_m)$ and its image via the boundary map $H^1_{Y,\acute{e}t}(X, \mathbb{G}_m) \to H^2_{Y,\acute{e}t}(X, \Lambda)$ is Deligne's localized Chern class $c_1(Y)$. Here, $H^*_{Y,\acute{e}t}(X, -)$ denotes the étale cohomology with support in Y. On the level of the derived category $D^+(Y, \Lambda)$, this Chern class is given in terms of the map $c_1(Y) : \Lambda \to \tau^!\Lambda(1)[2]$. Using this Chern class, Gabber's Gysin morphism $\tau_* : H^*_{\acute{e}t}(Y, \Lambda(j)) \to H^{*+2}_{\acute{e}t}(X, \Lambda(j+1))$ is the one induced on the cohomology by the composite map $\tau_* : \tau_*(\Lambda_Y) \to \tau_*\tau^!(\Lambda_Y(1)[2]) \to \Lambda_X(1)[2]$ in $D^+(X, \Lambda)$.

Corresponding to the Cartesian square (2-8), we have $\pi^*(\mathcal{O}(S_Y)) = \mathcal{O}(Y \amalg Y)$ and hence we have a commutative diagram

$$\begin{tikzcd}
H^1_{S_Y,\acute{e}t}(S_X, \mathbb{G}_m) \arrow[r, "\partial"] \arrow[d, "\pi^*"'] & H^2_{S_Y,\acute{e}t}(S_X, \Lambda) \arrow[d, "\pi^*"] \\
H^1_{Y\amalg Y,\acute{e}t}(X \amalg X, \mathbb{G}_m) \arrow[r, "\partial"'] & H^2_{Y\amalg Y,\acute{e}t}(X \amalg X, \Lambda).
\end{tikzcd} \tag{5-6}$$

We thus have commutative diagrams

$$\begin{tikzcd}
\tau_{1,*}(\Lambda_{S_Y}) \arrow[r, "\tau_{1,*}"] \arrow[d, "\pi^*"'] & \Lambda_{S_X}(1)[2] \arrow[d, "\pi^*"] \\
\pi_*\psi_*(\Lambda_{Y\amalg Y}) \arrow[r, "\pi_*\psi_*"'] & \pi_*\Lambda_{X\amalg X}(1)[2]
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau_*(\Lambda_Y) \arrow[r, "\tau_*"] \arrow[d, "\iota^*_F"'] & \Lambda_X(1)[2] \arrow[d, "\iota^*_D"] \\
\tau_*\iota_{F,*}(\Lambda_F) \arrow[r, "\tau_*"'] & \iota_*(\Lambda_D(1)[2])
\end{tikzcd} \tag{5-7}$$

in $D^+(S_X, \Lambda)$ and $D^+(X, \Lambda)$.

We next observe that the canonical map $\Lambda_{X\amalg X} \to \Lambda_X \oplus \Lambda_X$ induced by the inclusions of the two components is an isomorphism. We thus have a sequence of maps

$$\Lambda_{S_X} \xrightarrow{\pi^*} \pi_*(\Lambda_{X\amalg X}) \simeq \Lambda_{X_+} \oplus \Lambda_{X_-} \longrightarrow \Lambda_D,$$

where the last map is the difference of two restrictions $\Lambda_{X_\pm} \to \iota_*(\Lambda_D)$. Furthermore, it is easy to check that the sequence

$$0 \to \Lambda_{S_X}(j) \xrightarrow{\pi^*} \pi_*(\Lambda_{X\amalg X}(j)) \to \Lambda_D(j) \to 0$$

is exact. A combination of this with (5-7) yields a commutative diagram of exact triangles in $D^+(S_X, \Lambda)$:

$$\begin{tikzcd}
0 \arrow[r] & \tau_{1,*}(\Lambda_{S_Y}) \arrow[r, "\pi^*"] \arrow[d, "\tau_{1,*}"'] & \pi_*\psi_*(\Lambda_{Y \amalg Y}) \arrow[r] \arrow[d, "\pi_*\psi_*"] & \tau_{1,*}(\Lambda_F) \arrow[r] \arrow[d, "\tau_{1,*}"] & 0 \\
0 \arrow[r] & \Lambda_{S_X}(1)[2] \arrow[r, "\pi^*"'] & \pi_*(\Lambda_{X \amalg X}(1)[2]) \arrow[r] & \Lambda_D(1)[2] \arrow[r] & 0.
\end{tikzcd} \tag{5-8}$$

Since all the underlying maps in (2-8) are finite, we obtain a commutative diagram of long exact sequence of cohomology groups

$$\begin{tikzcd}
H^{2d-4}_{\acute{e}t}(Y \amalg Y, \Lambda(d-1)) \arrow[r] \arrow[d] & H^{2d-4}_{\acute{e}t}(F, \Lambda(d-1)) \arrow[r] \arrow[d] & H^{2d-3}_{\acute{e}t}(S_Y, \Lambda(d-1)) \arrow[r] \arrow[d, "\tau_{1,*}"] & H^{2d-3}_{\acute{e}t}(Y \amalg Y, \Lambda(d-1)) \arrow[r] \arrow[d] & 0 \\
H^{2d-2}_{\acute{e}t}(X \amalg X, \Lambda(d)) \arrow[r] & H^{2d-2}_{\acute{e}t}(D, \Lambda(d)) \arrow[r] & H^{2d-1}_{\acute{e}t}(S_X, \Lambda(d)) \arrow[r] & H^{2d-1}_{\acute{e}t}(X \amalg X, \Lambda(d)) \arrow[r] & 0
\end{tikzcd} \tag{5-9}$$

Since $d \geq 3$, it follows from the weak Lefschetz theorem for étale cohomology (see [Milne 1980, Theorem VI.7.1]) that the vertical arrow on the left end of (5-9) is surjective and the one on the right end is an isomorphism. We next note that the inclusion $F = D \cap H_1 \hookrightarrow D$ induces a bijection between the irreducible components of F and D by our choice of the hypersurface H_1 (see Section 2E). It follows from this, together with the isomorphism $H^{2d-2}_{\acute{e}t}(D, \Lambda) \xrightarrow{\simeq} H^{2d-2}_{\acute{e}t}(D^N, \Lambda)$ and [Milne 1980, Lemma VI.11.3], that the second vertical arrow from the left in (5-9) is an isomorphism. A diagram chase now shows that $\tau_{1,*}$ is an isomorphism. This proves the lemma. $\square$

Remark 5.2. We should warn the reader here that Lemma 5.1 proves an analogue of the weak Lefschetz theorem only for a specific étale cohomology group. We do not expect this to be true for other cohomology groups in general and it will depend critically on D.

5C. ***The torsion in Chow group with modulus and relative étale cohomology.*** Let $D \subset X$ be an effective Cartier divisor on a smooth scheme X as above. Recall from [Binda and Krishna 2018, § 4, 5] that there are maps $p_{\pm,*} : \mathrm{CH}_0(X|D) \to \mathrm{CH}_0(S_X)$ and $\iota^*_{\pm} : \mathrm{CH}_0(S_X) \to \mathrm{CH}_0(X)$, where $p_{\pm,*}([x]) = \iota_{\pm,*}([x])$ and $\iota^*_{\pm}$ is induced by the projection map $\mathcal{Z}_0(S_X \setminus D) = \mathcal{Z}_0(X_+ \setminus D) \oplus \mathcal{Z}_0(X_- \setminus D) \twoheadrightarrow \mathcal{Z}_0(X_{\pm} \setminus D)$. It follows at once from this description of the projection maps $\iota^*_{\pm}$ that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_0(S_Y)_{\deg 0} \arrow[r, "\iota^*_{\pm}"] \arrow[d, "\tau_{1,*}"'] & \mathrm{CH}_0(Y)_{\deg 0} \arrow[d, "\tau_*"] \\
\mathrm{CH}_0(S_X)_{\deg 0} \arrow[r, "\iota^*_{\pm}"'] & \mathrm{CH}_0(X)_{\deg 0}.
\end{tikzcd} \tag{5-10}$$

We shall use the following decomposition theorem from [Binda and Krishna 2018, Theorem 7.1].

Theorem 5.3. *The projection map $\Delta : S_X \to X$ induces a flat pullback $\Delta^* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(S_X)$ such that $\iota^*_{\pm} \circ \Delta^* = \mathrm{Id}_{\mathrm{CH}_0(X)}$. Moreover, there is a split exact sequence*

$$0 \to \mathrm{CH}_0(X|D) \xrightarrow{p_{+,*}} \mathrm{CH}_0(S_X) \xrightarrow{\iota^*_-} \mathrm{CH}_0(X) \to 0. \tag{5-11}$$

Theorem 5.4. *Let X be a smooth projective scheme of dimension $d \geq 1$ over an algebraically closed field k of exponential characteristic p. Then for any prime $l \neq p$, there is an isomorphism*

$$\lambda_{X|D} : \mathrm{CH}_0(X|D)\{l\} \xrightarrow{\sim} H^{2d-1}_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)).$$

Proof. If $D = \varnothing$, we take $\lambda_{X|D}$ to be the isomorphism $\lambda_X : \mathrm{CH}_0(X)\{l\} \xrightarrow{\sim} H^{2d-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(d))$ given by Bloch [1979, §2].

We now assume $D \neq \varnothing$ and let S_X be the double of X along D. We shall first prove by induction on d that there exists an isomorphism

$$\lambda_{S_X} : \mathrm{CH}_0(S_X)\{l\} \xrightarrow{\sim} H^{2d-1}_{\acute{e}t}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \tag{5-12}$$

such that $\iota^*_{\pm} \circ \lambda_{S_X} = \lambda_X \circ \iota^*_{\pm}$.

When $d = 1$, it follows easily from the Kummer sequence and [Levine and Weibel 1985, Proposition 1.4] that there is a natural isomorphism $H^1_{\acute{e}t}(C, \mu_n(1)) \xrightarrow{\sim} {}_n\mathrm{CH}_0(C)$ for any reduced curve C over k and any integer $n \geq 1$ prime to p. The naturality of this isomorphism proves our assertion. Note that this isomorphism coincides with that of Bloch when C is smooth, as one directly checks (or see [Bloch 1979, Proposition 3.6]).

We next assume $d = 2$. In this case, we have shown in Theorem 3.4 that there is a homomorphism $\tau_Y : H^3_{\acute{e}t}(Y, \mu_n(2)) \to {}_n\mathrm{CH}_0(Y)$ for any reduced quasiprojective surface Y over k and any integer n prime to p. We claim that the following diagram commutes:

$$\begin{tikzcd} H^3_{\acute{e}t}(S_X, \mu_n(2)) \arrow[r, "\tau_{S_X}"] \arrow[d, "\iota^*_-"'] & {}_n\mathrm{CH}_0(S_X) \arrow[d, "\iota^*_-"] \\ H^3_{\acute{e}t}(X, \mu_n(2)) \arrow[r, "\tau_X"'] & {}_n\mathrm{CH}_0(X) \end{tikzcd} \tag{5-13}$$

To prove this, recall from the construction of τ_Y in Section 3 that there are isomorphisms

$$H^3_{\acute{e}t}(Y, \mu_n(2)) \xrightarrow{\sim} \mathbb{H}^2(Y, \mathcal{K}^{\bullet}_2) \quad \text{and} \quad \mathrm{CH}_0(Y) \xrightarrow{\sim} \mathrm{CH}^{LW}_0(X) \xrightarrow{\sim} H^2(Y, \mathcal{K}_2)$$

(see [Binda and Krishna 2018, Theorem 3.17]) which are clearly functorial for the normalization map $Y^N \to Y$. Since $X_{\pm}$ are two disjoint components of the normalization S^N_X, we see that these isomorphisms are functorial for the inclusions $\iota_{\pm} : X_{\pm} \hookrightarrow S_X$.

For a surface Y, the map τ_Y is then the natural map $\mathbb{H}^2(Y, \mathcal{K}^{\bullet}_2) \to {}_nH^2(Y, \mathcal{K}_2)$ obtained via the exact sequence $\mathbb{H}^2(Y, \mathcal{K}^{\bullet}_2) \to H^2(Y, \mathcal{K}_2) \xrightarrow{n} H^2(Y, \mathcal{K}_2)$. The commutativity of (5-13) then follows immediately from the naturality of the complex of Zariski sheaves on $\mathcal{K}_2[-1] \to \mathcal{K}^{\bullet}_2 \to \mathcal{K}_2$ on $\boldsymbol{Sch}_k$. This proves the claim.

Theorem 3.6 says that $\tau_Y : H^3_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(2)) \xrightarrow{\sim} \mathrm{CH}_0(Y)\{l\}$ is an isomorphism on the limit for every prime $l \neq p$. We let $\lambda_Y : \mathrm{CH}_0(Y)\{l\} \to H^3_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(2))$ be the negative of the inverse of this isomorphism. When Y is smooth over k, it follows from Lemma 3.7 that λ_Y agrees with Bloch's isomorphism. It follows then from (5-13) that

$$\iota_{\pm}^* \circ \lambda_{S_X} = \lambda_X \circ \iota_{\pm}^*. \tag{5-14}$$

We now assume $\dim(X) \geq 3$. We choose a closed embedding $X \hookrightarrow \mathbb{P}_k^N$ and hypersurfaces $H_1, \ldots, H_{d-2}$ as on page 1437. We continue to use the notations of Proposition 2.3. We set $Y = X_1 := X \cap H_1$ and assume by induction that there is an isomorphism $\lambda_{S_Y} : \mathrm{CH}_0(S_Y)\{l\} \xrightarrow{\simeq} H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1))$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CH}_0(S_Y)\{l\} \arrow[r, "\lambda_{S_Y}"] \arrow[d, "\iota_{\pm}^*"'] & H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \arrow[d, "\iota_{\pm}^*"] \\
\mathrm{CH}_0(Y)\{l\} \arrow[r, "\lambda_Y"'] & H_{\acute{e}t}^{2d-3}(Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1))
\end{tikzcd} \tag{5-15}$$

We now consider the diagram

$$\begin{tikzcd}
\mathrm{CH}_0(S_Y)\{l\} \arrow[r, "\lambda_{S_Y}"] \arrow[d, "\tau_{1,*}"'] & H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \arrow[d, "\tau_{1,*}"] \\
\mathrm{CH}_0(S_X)\{l\} \arrow[r, dotted, "\lambda_{S_X}"'] & H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d))
\end{tikzcd} \tag{5-16}$$

It follows from Proposition 2.3 that the left vertical arrow is an isomorphism and Lemma 5.1 says that the right vertical arrow is an isomorphism. Since λ_{S_Y} is an isomorphism too, it follows that there is a unique isomorphism $\lambda_{S_X} : \mathrm{CH}_0(S_X)\{l\} \xrightarrow{\simeq} H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d))$ such that (5-16) commutes.

We have to check that $\iota_{\pm}^* \circ \lambda_{S_X} = \lambda_X \circ \iota_{\pm}^*$. For this, we consider the diagram

$$\begin{tikzcd}
\mathrm{CH}_0(S_Y)\{l\} \arrow[rrr, "\iota_{\pm}^*"] \arrow[dr, "\tau_{1,*}"'] \arrow[dd, "\lambda_{S_Y}"'] & & & \mathrm{CH}_0(Y))\{l\} \arrow[dr, "\tau_*"] \arrow[dd, "\lambda_Y"] & \\
& \mathrm{CH}_0(S_X))\{l\} \arrow[rrr, "\iota_{\pm}^*"'] \arrow[dd, "\lambda_{S_X}"] & & & \mathrm{CH}_0(X))\{l\} \arrow[dd, "\lambda_X"] \\
H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \arrow[rrr, "\iota_{\pm}^*"] \arrow[dr, "\tau_{1,*}"'] & & & H_{\acute{e}t}^{2d-3}(Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \arrow[dr, "\tau_*"'] & \\
& H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \arrow[rrr, "\iota_{\pm}^*"'] & & & H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d)).
\end{tikzcd}$$

We need to show that the front face of this cube commutes. Since $\tau_{1,*} : \mathrm{CH}_0(S_Y)\{l\} \to \mathrm{CH}_0(S_X)\{l\}$ is an isomorphism, it suffices to show that all other faces of the cube commute. The top face commutes by (5-10) and the bottom face commutes by (5-5). The left face commutes by (5-16) and the right face commutes by [Bloch 1979, Proposition 3.3]. Finally, the back face commutes by (5-15) and we are done.

To finish the proof of the theorem, we use (5-4) and Theorem 5.3 and consider the diagram of split exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{CH}_0(X|D)\{l\} \arrow[r, "p_{+,*}"] \arrow[d, dashed] & \mathrm{CH}_0(S_X)\{l\} \arrow[r, "\iota_-^*"] \arrow[d, "\lambda_{S_X}"] & \mathrm{CH}_0(X)\{l\} \arrow[r] \arrow[d, "\lambda_X"] & 0 \\ 0 \arrow[r] & H^{2d-1}_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)) \arrow[r, "p_{+,*}"'] & H^{2d-1}_{\acute{e}t}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \arrow[r, "\iota_-^*"'] & H^{2d-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \arrow[r] & 0. \end{tikzcd} \tag{5-17}$$

It follows from (5-12) that the square on the right is commutative. Moreover, the maps λ_{S_X} and λ_X are both isomorphisms. We conclude that there is an isomorphism

$$\lambda_{X|D} : \mathrm{CH}_0(X|D)\{l\} \xrightarrow{\simeq} H^{2d-1}_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)). \qquad \square$$

5D. *Applications.* We now deduce two applications of Theorem 5.4. Since the étale cohomology of $\mu_n(j)$ is nilinvariant whenever $(n, p) = 1$, it follows from (5-1) that $H^*_{\acute{e}t}(X|D, \mu_n(j)) \xrightarrow{\simeq} H^*_{\acute{e}t}(X|D_{\mathrm{red}}, \mu_n(j))$. Using Theorem 5.4, we therefore obtain the following result about the prime-to-p torsion in the Chow group with modulus.

Theorem 5.5. *Let X be a smooth projective scheme of dimension $d \geq 1$ over k let $D \subset X$ be an effective Cartier divisor. Then, the restriction map ${}_n\mathrm{CH}_0(X|D) \to {}_n\mathrm{CH}_0(X|D_{\mathrm{red}})$ is an isomorphism for every integer n prime to p.*

Our second application is the following extension of Theorem 3.6 to the cohomology of relative K_2-sheaf on a smooth surface. Recall that for a closed immersion $Y \hookrightarrow X$ in $\boldsymbol{Sch}_k$, the relative K-theory sheaf $\mathcal{K}_{i,(X,Y)}$ is the Zariski sheaf on X associated to the presheaf $U \mapsto K_i(U, Y \cap U)$.

Theorem 5.6. *Let X be a smooth projective surface over an algebraically closed field k of exponential characteristic p and let $l \neq p$ be a prime. Let $D \subset X$ be an effective Cartier divisor. Then, the following hold.*

(1) $H^1(X, \mathcal{K}_{2,(X,D)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$.

(2) $H^2(X, \mathcal{K}_{2,(X,D)})\{l\} \simeq H^3_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2))$.

Proof. It follows from Theorem 5.4 that $H^3_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \mathrm{CH}_0(X|D)\{l\}$. On the other hand, a combination of [Binda and Krishna 2018, Theorem 1.7] and [Krishna 2015b, Lemma 2.1] shows that there is a canonical isomorphism $\mathrm{CH}_0(X|D) \xrightarrow{\simeq} H^2(X, \mathcal{K}_{2,(X,D)})$. This proves (2).

To prove (1), we consider the following commutative diagram for any integer $n \in k^{\times}$.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^1(S_X, \mathcal{K}_{2,(S_X,X_-)}) \otimes_{\mathbb{Z}} \mathbb{Z}/n \arrow[r] \arrow[d] & H^3_{\acute{e}t}(S_X|X_-, \mu_n(2)) \arrow[r] \arrow[d] & {}_n\mathrm{CH}_0(X|D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(S_X, \mathcal{K}_{2,S_X}) \otimes_{\mathbb{Z}} \mathbb{Z}/n \arrow[r] \arrow[d] & H^3_{\acute{e}t}(S_X, \mu_n(2)) \arrow[r] \arrow[d] & {}_n\mathrm{CH}_0(S_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(X, \mathcal{K}_{2,X}) \otimes_{\mathbb{Z}} \mathbb{Z}/n \arrow[r] \arrow[d] & H^3_{\acute{e}t}(X, \mu_n(2)) \arrow[r] \arrow[d] & {}_n\mathrm{CH}_0(X) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{5-18}$$

The columns on the left and in the middle are exact by the splitting $\Delta \circ \iota_- = \mathrm{Id}_X$ in (2-1). The column on the right is exact by Theorem 5.3. The middle and the bottom rows are exact by Theorem 3.4. It follows that the top row is also exact. Using the isomorphism $\iota_+^* : H^3_{\acute{e}t}(S_X|X_-, \mu_n(2)) \xrightarrow{\simeq} H^3_{\acute{e}t}(X|D, \mu_n(2))$ (see (5-3)) and taking the direct limit of the terms in (5-18) with respect to the direct system $\{\mathbb{Z}/l^n\}_n$, we obtain a short exact sequence

$$0 \to H^1(S_X, \mathcal{K}_{2,(S_X,X_-)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \to H^3_{\acute{e}t}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to \mathrm{CH}_0(X|D)\{l\} \to 0. \tag{5-19}$$

Using (5-19) Theorem 5.4, we conclude that $H^1(S_X, \mathcal{K}_{2,(S_X,X_-)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$ for every prime $l \neq p$. To finish the proof, it suffices now to show that the pullback map $\iota_+^* : H^1(S_X, \mathcal{K}_{2,(S_X,X_-)}) \to H^1(X, \mathcal{K}_{2,(X,D)})$ is surjective.

Given an open subset $W \subset D$, let $U = S_X \setminus (D \setminus W)$ be the open subset of S_X. Let $\mathcal{K}_{i,(S_X,X_-,D)}$ be the Zariski sheaf on D associated to the presheaf

$$W \mapsto K_i(U, X_+ \cap U, X_- \cap U) = \mathrm{hofib}((K(U, X_- \cap U) \xrightarrow{i_+^*} K(X_+ \cap U, D \cap U));$$

see [Pedrini and Weibel 1994, Proposition A.5]. There is an exact sequence of K-theory sheaves

$$\nu_*(\mathcal{K}_{2,(S_X,X_-,D)}) \to \mathcal{K}_{2,(S_X,X_-)} \to \mathcal{K}_{2,(X_+,D)} \to \nu_*(\mathcal{K}_{1,(S_X,X_-,D)}),$$

where $\nu : D \hookrightarrow S_X$ is the inclusion. We have $\mathcal{K}_{1,(S_X,X_-,D)} = \mathcal{I}_D/\mathcal{I}_D^2 \otimes_D \Omega^1_{D/X}$ by [Geller and Weibel 1983, Theorem 1.1] and the latter term is zero. We thus get an exact sequence

$$\nu_*(\mathcal{K}_{2,(S_X,X_-,D)}) \to \mathcal{K}_{2,(S_X,X_-)} \to \mathcal{K}_{2,(X_+,D)} \to 0.$$

Since $H^2(S_X, \iota_*(\mathcal{K}_{2,(S_X,X_-,D)}) = H^2(D, \mathcal{K}_{2,(S_X,X_-,D)}) = 0$, we get

$$H^1(X, \mathcal{K}_{2,(S_X,X_-)}) \twoheadrightarrow H^1(X, \mathcal{K}_{2,(X_+,D)}) \simeq H^1(X, \mathcal{K}_{2,(X,D)}). \qquad \square$$

6. Roitman's torsion theorem for 0-cycles with modulus

In this section, we prove the Roitman torsion theorem (Theorem 1.2) for the Chow group of 0-cycles with modulus. We shall deduce this result by proving a more general Roitman torsion theorem for singular varieties which are obtained by joining two smooth schemes along a common reduced Cartier divisor. As before, we assume the base field k to be algebraically closed of exponential characteristic p.

6A. ***Join of two smooth schemes along a common divisor.*** Let X_+ and X_- be two smooth connected quasiprojective schemes of dimension $d \geq 1$ over k and let $X_+ \overset{i_+}{\hookleftarrow} D \overset{i_-}{\hookrightarrow} X_-$ be two closed embeddings such that D is a reduced effective Cartier divisor on each $X_\pm$ via these embeddings. Let X be the quasiprojective scheme over k such that the square

$$\begin{array}{ccc} D & \xrightarrow{i_+} & X_+ \\ {\scriptstyle i_-}\big\downarrow & & \big\downarrow{\scriptstyle \iota_+} \\ X_- & \xrightarrow[\iota_-]{} & X \end{array} \tag{6-1}$$

is co-Cartesian in $\boldsymbol{Sch}_k$.

It is easy to check that all arrows in (6-1) are closed immersions and X is a reduced scheme with irreducible components $\{X_+, X_-\}$ with $X_{\text{sing}} = D$. In particular, the canonical map $\pi = \iota_+ \amalg \iota_- : X_+ \amalg X_- \to X$ is the normalization map. Let $U = X_{\text{reg}} = (X_+ \setminus D) \amalg (X_- \setminus D)$. We shall use the following important further properties of X.

Lemma 6.1. *The scheme X satisfies the following properties.*

(1) *It is Cohen–Macaulay.*

(2) *It is separably weakly normal.*

(3) *It is weakly normal.*

(4) *The map $D \to X_+ \times_X X_-$ is an isomorphism.*

Proof. Because X_+ and X_- are smooth of dimension d, they are Cohen–Macaulay. Since D is an effective Cartier divisor on a smooth scheme of dimension d, it is Cohen–Macaulay of dimension $d-1$. The statement (1) now follows from [Ananthnarayan et al. 2012, Lemma 1.2, Lemma 1.5, (1.5.3)]. Second statement follows because $D \subset X$ is the smallest conducting closed subscheme for the normalization $\pi : X^N \to X$ which is reduced. Furthermore, the map $D \times_X X^N \simeq D \amalg D \xrightarrow{\pi} D$ is just the collapse map and hence generically separable. The statement (3) follows from the previous two and (4) follows from Lemma 2.2. □

6B. ***The main result.*** We shall use the setup of Section 6A throughout this section. We assume from now on that $d = \dim(X) \geq 3$. Let $C \hookrightarrow X$ be a reduced Cartier curve (see Section 2A) such that each irreducible component of $C \setminus D$ is smooth and $C \setminus D$ has only double point singularities.

Lemma 6.2. *There exists a locally closed embedding $X \hookrightarrow \mathbb{P}^n_k$ such that for a general set of distinct hypersurfaces $H_1, \dots, H_{d-2} \subset \mathbb{P}^n_k$ of large degree containing C, the intersection $L = H_1 \cap \cdots \cap H_{d-2}$ satisfies the following.*

(1) *$X \cap L$ is reduced.*

(2) *$D \cap L$ is reduced.*

(3) *$X_\pm \cap L$ is an integral normal surface whose singular locus is contained in $C_{\mathrm{sing}} \cap D$.*

(4) *The inclusion $C \subset (X \cap L)$ is a local complete intersection along D.*

Proof. Since X is reduced of dimension $d \geq 3$, and since $C \subset X$ is a local complete intersection along $D = X_{\mathrm{sing}}$, it follows from [Levine 1987, Lemmas 1.3, 1.4] (see also [Biswas and Srinivas 1999, Sublemma 1]) that for all $m \gg 1$, there is an open dense subset $\mathcal{U}_n(C, m)$ of the scheme $\mathcal{H}_n(C, m)$ of hypersurfaces in $\mathbb{P}^n_k$ of degree m containing C such that the following hold.

(a) For general distinct $H_1, \dots, H_{d-2} \in \mathcal{U}_n(C, m)$, the scheme-theoretic intersection $L = H_1 \cap \cdots \cap H_{d-2}$ has the property that $X \cap L$ is reduced away from C.

(b) $D \cap L$ is reduced away from C.

(c) $X_\pm \cap L$ is integral away from C.

(d) $C \subset (X \cap L)$ is a local complete intersection along D.

Note that as $X_\pm$ is smooth and $X_\pm \cap L$ is a complete intersection in $X_\pm$, it follows that $X_\pm \cap L$ is a Cohen–Macaulay surface. In particular, it has no embedded component. Since C is nowhere dense in $X_\pm \cap L$, it follows from (c) that $X_\pm \cap L$ must be irreducible. Since C does not contain the generic point of $X_\pm \cap L$, it follows from (b) and Lemma 6.3 that $X_\pm \cap L$ is integral.

Setting $W := C \setminus (C_{\mathrm{sing}} \cap D)$ and following [Kleiman and Altman 1979, § 5], let $W(\Omega^1_C, e)$ denote the locally closed subset of points in W where the embedding dimension of W is e. It follows from our assumption on the singularities of $C \setminus D$ that $\max_e\{\dim(W(\Omega^1_C, e)) + e\} \leq 2$. We conclude from their Theorem 7 that for all $m \gg 1$, there is an open dense subset $\mathcal{W}_n(W, m)$ of the scheme $\mathcal{H}_n(W, m)$ of hypersurfaces of $\mathbb{P}^n_k$ of degree m containing W such that for general distinct $H_1, \dots, H_{d-2} \in \mathcal{W}_n(W, m)$, the scheme-theoretic intersection $L = H_1 \cap \cdots \cap H_{d-2}$ has the following properties:

(a') $X \cap L$ is a complete intersection in X of dimension two.

(b') $(X \cap L) \setminus D$ is smooth.

(c') $X_\pm \cap L$ is smooth away from $C_{\mathrm{sing}} \cap D$.

It follows from (c') that $X_\pm \cap L$ is a Cohen–Macaulay surface whose singular locus is contained in $C_{\mathrm{sing}} \cap D$. It follows from Serre's normality condition that $X_\pm \cap L$ is normal. Since C is the closure of W in X, we must have $\mathcal{H}_n(W, m) = \mathcal{H}_n(C, m)$ and in particular, $D \subset X \cap L$. Combining (a) – (d) and (a') – (c') together, we conclude that there is a closed immersion $Y \subset X$ with the following properties:

(1) $\dim(Y) = 2$.

(2) The inclusions $Y \subset X$, $(X_\pm \cap Y) \subset X_\pm$ and $(D \cap Y) \subset D$ are all complete intersections.

(3) Y is reduced away from $C \cap D$.

(4) $(D \cap Y) \subset Y$ is a Cartier divisor which is reduced away from C.

(5) $X_\pm \cap Y$ is an integral normal surface which is smooth away from $C_{\mathrm{sing}} \cap D$.

(6) $C \subset Y$ is a local complete intersection along D.

(7) $Y \setminus D$ is smooth.

If $Y \subset X$ is as above, then (6) says that the local rings of Y at $C \cap D$ contain regular elements. In particular, $C \cap D$ can contain no embedded point of Y. We conclude from (3) and Lemma 6.3 that a surface Y as above must be reduced. Since $D \subset X_\pm$ is a Cartier divisor, it is Cohen–Macaulay and hence (2) shows that $D \cap Y$ is a complete intersection Cohen–Macaulay curve inside D. Since $C \cap D$ can contain no generic point of $D \cap Y$, it follows form (4) and Lemma 6.3 that $D \cap Y$ is a reduced curve. This finishes the proof of the lemma. □

The following is a straightforward application of the prime avoidance theorem in commutative algebra and its proof is left to the reader.

Lemma 6.3. *Let R be commutative Noetherian ring such that $R_{\mathfrak{p}}$ is reduced for every associated prime ideal $\mathfrak{p}$ of R. Then R must be reduced.*

Lemma 6.4. *Let $X = X_+ \amalg_D X_-$ be as in* (6-1) *and let $Y = X \cap L$ be the complete intersection surface as obtained in Lemma 6.2. Assume that X is projective. Then, $A^2(Y)$ is a semiabelian variety and the Roitman torsion theorem holds for Y. That is, the Abel–Jacobi map $\rho_Y : \mathrm{CH}_0^{LW}(Y)_{\deg 0} \to A^2(Y)$ is an isomorphism on the torsion subgroups.*

Proof. We let $Y_\pm = X_\pm \cap Y$ and $E = D \cap Y$. Let $\iota'_\pm : Y_\pm \hookrightarrow Y$ be the inclusion maps. It follows from the construction of $Y \subset X$ and an easy variant of [Binda and Krishna 2018, Lemma 2.2] that the canonical map $Y_+ \amalg_E Y_- \to Y$ is an isomorphism. It also follows from Lemma 6.2 that $\iota'_+ \amalg \iota'_- : Y_+ \amalg Y_- \to Y$ is the normalization map. Since Y is the join of normal surfaces Y_+ and Y_- along the common closed subscheme E, it follows that the nonnormal locus of Y is the support of E. Let us denote the map $\iota'_+ \amalg \iota'_-$ by π'.

Let $\mathcal{I}_\pm \subset \mathcal{O}_{Y_\pm}$ denote the defining ideal sheaf for the inclusion $E \subset Y_\pm$. It is then immediate from (6-1) that $\pi'^*(\mathcal{I}_E) = \mathcal{I}_+ \times \mathcal{I}_-$ which is actually in $\mathcal{O}_Y$ under the inclusion $\pi'^* : \mathcal{O}_Y \hookrightarrow \pi'_*(\mathcal{O}_{Y_+ \amalg Y_-})$. In other words, $E \hookrightarrow Y$ is a conducting subscheme for π'. Since $\pi'^*(E) = E \amalg E$ is reduced by Lemma 6.2 and since the support of E is the nonnormal locus of Y, we see that π' is the normalization map for which E is the smallest conducting subscheme and is reduced. Since $\pi' : E \amalg E \to E$ is just the collapse map, it is clearly generically separable. We conclude (see Definition 4.5) that Y is a reduced projective surface which is separably weakly normal. The lemma now follows from Theorem 4.9. □

Theorem 6.5. *Let X_+ and X_- be two smooth projective schemes of dimension $d \geq 1$ over k and let $X = X_+ \amalg_D X_-$ be as in* (6-1). *Then, the Albanese variety $A^d(X)$ is a semiabelian variety and the Abel–Jacobi map $\rho_X : \mathrm{CH}_0^{LW}(X)_{\deg 0} \to A^d(X)$ is isomorphism on the torsion subgroups.*

Proof. It follows from Lemma 6.1 that X is separably weakly normal and weakly normal. Hence, in characteristic zero, a general reduced Cartier curve on X containing a 0-cycle is weakly normal by [Cumino et al. 1983, Corollary 2.5]. We can thus repeat the argument of the proof of Theorem 4.9, which shows that ρ_X is surjective and reduces the remaining proof to showing only that the restriction of ρ_X to the p-primary torsion subgroup of $\mathrm{CH}_0^{LW}(X)_{\deg 0}$ is injective when $p \geq 2$.

When $d = 1$, the scheme X is a weakly normal curve, and it is well known in this case that $\mathrm{Pic}^0(X) \simeq A^1(X) \simeq J^1(X)$. The case $d = 2$ is Lemma 6.4. So we assume $d \geq 3$.

Let $\alpha \in \mathcal{Z}_0(X)$ be such that $n\alpha = 0$ in $\mathrm{CH}_0^{LW}(X)$ for some integer $n = p^m$. Equivalently, $n\alpha \in \mathcal{R}_0^{LW}(X)$. Let us assume further that $\rho_X^{\mathrm{semi}}(\alpha) = 0$.

We can now use [Biswas and Srinivas 1999, Lemma 2.1] to find a reduced Cartier curve C on X and a function $f \in \mathcal{O}_{C,S}^{\times}$ such that $n\alpha = \mathrm{div}(f)$, where $S = C \cap X_{\mathrm{sing}} = C \cap D$. Since part of $n\alpha$ supported on any connected component of C is also of the form $n\alpha' = \mathrm{div}_{C'}(f')$ for some Cartier curve C' and some f', we can assume that C is connected. Let $\{C_1, \dots, C_r\}$ denote the set of irreducible components of C.

We set $U = X \setminus D = (X_+ \setminus D) \amalg (X_- \setminus D)$. Let $\phi : X' \to X$ be a successive blow-up at smooth points such that the following hold.

(1) The strict transform D_i of each C_i is smooth along $\phi^{-1}(U)$.

(2) $D_i \cap D_j \cap \phi^{-1}(U) = \varnothing$ for $i \neq j$.

(3) Each D_i intersects the exceptional divisor E (which is reduced) transversely at smooth points.

It is clear that there exists a finite set of blown-up closed points $T = \{x_1, \dots, x_n\} \subset U$ such that $\phi : \phi^{-1}(X \setminus T) \to X \setminus T$ is an isomorphism. In particular, $\phi : X'_{\mathrm{sing}} = \phi^{-1}(D) \to D$ is an isomorphism. If we identify $\phi^{-1}(D)$ with D and let $X'_{\pm} = \mathrm{Bl}_{T \cap X_{\pm}}(X_{\pm})$, it becomes clear that $X' = X'_+ \amalg_D X'_-$. We set $U' = X' \setminus D = \phi^{-1}(U)$. Let C' denote the strict transform of C with components $\{C'_1, \dots, C'_r\}$.

Since ϕ is an isomorphism over an open neighborhood of D, it follows that $\phi^{-1}(S) \simeq S$ and the map $C' \to C$ is an isomorphism along D. In particular, we have $f \in \mathcal{O}_{C',S}^{\times}$ and $\phi_*(\mathrm{div}_{C'}(f)) = \mathrm{div}_C(f) = n\alpha$. Since $\mathrm{Supp}(\alpha) \subset C'$, we can find $\alpha' \in \mathcal{Z}_0(X')$ supported on C' such that $\phi_*(\alpha') = \alpha$. This implies that $\phi_*(n\alpha' - \mathrm{div}_{C'}(f)) = 0$. Setting $\beta = n\alpha' - \mathrm{div}_{C'}(f)$, it follows that β must be a 0-cycle on the exceptional divisor such that $\phi_*(\beta) = 0$.

We can now write $\beta = \sum_{i=0}^{n} \beta_i$, where β_i is a 0-cycle on X' supported on $\phi^{-1}(x_i)$ for $1 \leq i \leq n$ and β_0 is supported on the complement E. We must then have $\phi_*(\beta_i) = 0$ for all $0 \leq i \leq n$. Since ϕ is an isomorphism away from $T = \{x_1, \dots, x_n\}$, we must have $\beta_0 = 0$. We can therefore assume that β is a 0-cycle on E.

Next, we note that each $\phi^{-1}(\{x_i\})$ is a $(d-1)$-dimensional projective variety whose irreducible components are point blow-ups of $\mathbb{P}_k^{d-1}$, intersecting transversely in X'_{reg}. Moreover, we have $\phi_*(\beta_i) = 0$

for the pushforward map $\phi_*: \mathcal{Z}_0(\pi^{-1}(\{x_i\})) \to \mathbb{Z}$, induced by $\phi: \phi^{-1}(\{x_i\}) \to \operatorname{Spec}(k(x_i)) = k$. But this means that $\deg(\beta) = \sum_{i=0}^{n} \deg(\beta_i) = 0$. In particular, there are finitely many smooth projective rational curves $L_j \subset E$ and rational functions $g_j \in k(L_j)$ such that $\beta = \sum_{j=1}^{s} (g_j)_{L_j}$.

Using the argument of [Bloch 1980, Lemma 5.2], we can further choose L_j's so that $C'' := C' \cup \big(\bigcup_j L_j\big)$ is a connected reduced curve with following properties.

(1) Each component of C'' is smooth along U'.

(2) $C'' \cap U'$ has only ordinary double point singularities.

In particular, the embedding dimension of C'' at each of its singular points lying over U is two. Furthermore, $C'' \cap D = \big(C'' \setminus \big(\bigcup_j L_j\big)\big) \cap D = C' \cap D$. This implies that C'' is a Cartier curve on X'.

We now fix a closed embedding $X' = X'_+ \amalg_D X'_- \hookrightarrow \mathbb{P}^N_k$ and choose a complete intersection surface $j' : Y' \subset X'$ as in Lemma 6.2. Let $\pi' : X'_+ \amalg X'_- \to X'$ denote the normalization map with $\pi'^{-1}(Y') = Y'_+ \amalg Y'_-$. Note that $E' = Y' \cap D$ is a reduced curve and $Y'_{\mathrm{sing}} = E'$. Since $C'' \cap E' = C' \cap E'$ and since C'' is Cartier on Y', it follows that C' and L_j's are also Cartier curves on Y'. Furthermore, α' is an element of $\mathcal{Z}_0(Y', E')$ such that $n\alpha' = \operatorname{div}_{C'}(f) + \sum_j \operatorname{div}_{L_j}(g_j)$. In particular, $\alpha' \in \mathrm{CH}_0^{LW}(Y')$ and $n\alpha' = 0$ in $\mathrm{CH}_0^{LW}(Y')$. Note that this also implies that $\alpha' \in \mathrm{CH}_0^{LW}(Y')_{\deg 0}$.

It follows from [Mallick 2009, Theorem 14] that there exists a commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}_0^{LW}(Y')_{\deg 0} & \xrightarrow{\rho^{\mathrm{semi}}_{Y'}} & J^2(Y') & \xrightarrow{\pi'^*} & J^2(Y'_+) \times J^2(Y'_-) \\ \downarrow{\scriptstyle j'_*} & & \downarrow{\scriptstyle j'_*} & & \downarrow{\scriptstyle j'_*} \\ \mathrm{CH}_0^{LW}(X')_{\deg 0} & \xrightarrow[\rho^{\mathrm{semi}}_{X'}]{} & J^d(X') & \xrightarrow[\pi'^*]{} & J^d(X'_+) \times J^d(X'_-). \end{array} \tag{6-2}$$

Since $\phi : X' \to X$ is a blow-up at smooth closed points and since the Albanese variety of a smooth projective scheme is a birational invariant, it follows from the construction of $J^d(X)$ in Section 2D and [Esnault et al. 1999, Lemma 2.5] that the canonical pushforward map $\phi_* : \mathcal{Z}_0(X') \to \mathcal{Z}_0(X)$ gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0^{LW}(X')_{\deg 0} & \xrightarrow{\rho^{\mathrm{semi}}_{X'}} & J^d(X') \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{CH}_0^{LW}(X)_{\deg 0} & \xrightarrow[\rho^{\mathrm{semi}}_{X}]{} & J^d(X), \end{array} \tag{6-3}$$

in which the two vertical arrows are isomorphisms. It follows that $\alpha' \in \mathrm{CH}_0^{LW}(Y')_{\deg 0}$ is a 0-cycle such that $\rho^{\mathrm{semi}}_{X'} \circ j'_*(\alpha') = 0$ in $J^d(X')$. Using (6-2), we conclude that

$$j'_* \circ \pi'^* \circ \rho^{\mathrm{semi}}_{Y'}(\alpha') = \pi'^* \circ \rho^{\mathrm{semi}}_{X'} \circ j'_*(\alpha') = 0. \tag{6-4}$$

Since $Y'_+ \amalg Y'_- \to Y'$ is the normalization map (see Lemma 6.4), we know that $J^2(Y'_+) \times J^2(Y'_-)$ is the universal abelian variety quotient of $J^2(Y')$. In particular, the map $J^2(Y')\{p\} \to J^2(Y'_+)\{p\} \times J^2(Y'_-)\{p\}$ is an isomorphism. On the other hand, $Y'_\pm \subset X'_\pm$ are the iterated hypersurface sections of normal projective schemes and hence it follows from [Lang 1959, Chapter 8, § 2, Theorem 5] that the right vertical arrow in (6-2) is an isomorphism of abelian varieties. Note here that X' or Y' need not be smooth for this isomorphism. It follows therefore from (6-4) that $\rho^{\mathrm{semi}}_{Y'}(\alpha') = 0$. Lemma 6.4 now implies that $\alpha' = 0$ and we finally get $\alpha = \phi_*(\alpha') = 0$. □

6C. ***Applications of Theorem 6.5.*** We now obtain some applications of Theorem 6.5. Our first result is the following comparison theorem.

Theorem 6.6. *Let X_+ and X_- be two smooth projective schemes of dimension $d \geq 1$ over k and let $X = X_+ \amalg_D X_-$ be as in* (6-1). *Then, the canonical map $\mathrm{CH}^{LW}_0(X) \to \mathrm{CH}_0(X)$ is an isomorphism.*

Proof. Let L denote the kernel of the surjection $\mathrm{CH}^{LW}_0(X) \twoheadrightarrow \mathrm{CH}_0(X)$. Using the factorization $\mathrm{CH}^{LW}_0(X) \twoheadrightarrow \mathrm{CH}_0(X) \to K_0(X)$ (see [Binda and Krishna 2018, Lemma 3.13] and [Levine 1987, Corollary 2.7]), we know that L is a torsion subgroup of $\mathrm{CH}^{LW}_0(X)_{\deg 0}$ of bounded exponent. On the other hand, we also have a factorization $\mathrm{CH}^{LW}_0(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0(X)_{\deg 0} \twoheadrightarrow J^d(X)$ by (2-4). Since $J^d(X)$ is the universal semiabelian variety quotient of $A^d(X)$, we can use Theorem 6.5 to replace $J^d(X)$ by $A^d(X)$. We therefore have a factorization $\mathrm{CH}^{LW}_0(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0(X)_{\deg 0} \twoheadrightarrow A^d(X)$ of ρ^{semi}_X. Another application of Theorem 6.5 now shows that L is torsion-free. Hence, it must be zero. □

As second application of Theorem 6.5, we now prove the Roitman torsion theorem for the Chow group of 0-cycles with modulus when the underlying divisor is reduced.

Theorem 6.7. *Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Assume that D is reduced. Then, the Albanese variety with modulus $A^d(X|D)$ is a semiabelian variety and the Abel–Jacobi map $\rho_{X|D} : \mathrm{CH}_0(X|D)_{\deg 0} \to A^d(X|D)$ is isomorphism on the torsion subgroups.*

Proof. By Theorem 6.6 and [Binda and Krishna 2018, (11.2)], there is a commutative diagram of split exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}_0(X|D)_{\deg 0} \arrow[r, "p_{+,*}"] \arrow[d, "\rho_{X|D}"'] & \mathrm{CH}^{LW}_0(S_X)_{\deg 0} \arrow[r, "i^*_-"] \arrow[d, "\rho_{S_X}"] & \mathrm{CH}_0(X)_{\deg 0} \arrow[r] \arrow[d, "\rho_X"] & 0 \\
0 \arrow[r] & A^d(X|D) \arrow[r, "p_{+,*}"'] & A^d(S_X) \arrow[r, "i^*_-"'] & A^d(X) \arrow[r] & 0.
\end{tikzcd}$$

We note here that the constructions of § 11 there are based on the assumption that Theorem 6.6 holds. Since X is smooth over k, we know that $A^d(X)$ is an abelian variety. Moreover, the right vertical arrow is isomorphism on the torsion subgroups by Roitman [1980] and Milne [1982]. We conclude that the theorem is equivalent to showing that $A^d(S_X)$ is a semiabelian variety and ρ_{S_X} is isomorphism on the torsion subgroups. But this follows from Theorem 6.5. □

Corollary 6.8. *Let X be a smooth projective scheme of dimension $d \geq 1$ over $\bar{\mathbb{F}}_p$ and let $D \subset X$ be a reduced effective Cartier divisor. Then*, $\mathrm{CH}_0(X|D)_{\deg 0}$ *is finite-dimensional. That is*, *the Abel–Jacobi map* $\rho_X : \mathrm{CH}_0(X|D)_{\deg 0} \to A^d(X|D)$ *is an isomorphism.*

Proof. In view of Theorem 6.7, we only need to show that $\mathrm{CH}_0(S_X)$ is a torsion abelian group. But this is already shown in the proof of Corollary 4.10. □

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Continuity of the Green function in meromorphic families of polynomials

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We prove that along any marked point the Green function of a meromorphic family of polynomials parametrized by the punctured unit disk is the sum of a logarithmic term and a continuous function.

Introduction

Our aim is to analyze in detail the degeneration of the Green function of a meromorphic family of polynomials. Our main result is somewhat technical but is the key for applications in the study of algebraic curves in the parameter space of polynomials using techniques from arithmetic geometry. In particular it applies to the dynamical André–Oort conjecture for algebraic curves in the moduli space of polynomials [Baker and De Marco 2013; Ghioca and Ye 2017; Favre and Gauthier 2018] and to the problem of unlikely intersection [Baker and DeMarco 2011]. We postpone to another paper these applications.

Let us describe our setup. We fix any algebraically closed complete metrized field $(k, |\cdot|)$. In the applications we have in mind, the field k is either the field of complex numbers, or the p-adic field $\mathbb{C}_p$. In particular, the norm $|\cdot|$ may be either Archimedean or non-Archimedean.

Let P be any polynomial of degree $d \geq 2$ with coefficients in k. Recall that the sequence of functions $\frac{1}{d^n}\log\max\{1, |P^n|\}$ converges uniformly on k to a continuous function g_P which satisfies the invariance property $g_P \circ P = d g_P$. The function g_P is also continuous as a function of P when the polynomial ranges over the set of polynomials of degree d. Our analysis gives precise information on the behavior of g_P when P degenerates.

More precisely, denote by $\mathbb{D} = \{|z| < 1\}$ the open unit disk in the affine line over k, and let $\mathcal{O}(\mathbb{D})$ be the set of analytic functions on $\mathbb{D}$. A function f belongs to $\mathcal{O}(\mathbb{D})$ if it can be expanded as a power series $f(t) = \sum_{i\geq 0} a_i t^i$ with the condition that $\sum_{i\geq 0}|a_i|\rho^i < \infty$ when k is Archimedean and $|a_i|\rho^i \to 0$ as $i \to \infty$ when k is non-Archimedean, for all $\rho < 1$. Observe that a function f belongs to $\mathcal{O}(\mathbb{D})[t^{-1}]$ if and only if it is a meromorphic function on $\mathbb{D}$ with (at worst) a single pole at 0.

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Given any meromorphic family $P_t \in \mathcal{O}(\mathbb{D})[t^{-1}][z]$ of polynomials of degree $d \geq 2$, and any marked point $a(t) \in \mathcal{O}(\mathbb{D})[t^{-1}]$, it follows from [DeMarco 2016, §3] that $g_{P_t}(a(t)) \sim \alpha \log|t|^{-1}$ for some nonnegative constant α (see also [Favre 2016] for a generalization of this fact to higher dimension).

Main Theorem. *For any meromorphic family $P \in \mathcal{O}(\mathbb{D})[t^{-1}][z]$ of polynomials of degree $d \geq 2$ and for any function $a(t) \in \mathcal{O}(\mathbb{D})[t^{-1}]$, there exists a nonnegative rational number $\alpha \in \mathbb{Q}_+$ such that the function*

$$h(t) := g_{P_t}(a(t)) - \alpha \log|t|^{-1}$$

on $\mathbb{D}^*$ extends continuously across the origin. Moreover, one of the following occurs:

(1) *There exists an affine change of coordinates depending analytically on t conjugating P_t to Q_t such that the family Q is analytic and $\deg(Q_0) = d$, and the constant α vanishes.*

(2) *The constant α is strictly positive and h is harmonic in a neighborhood of 0.*

(3) *The constant α vanishes and $h(0) = 0$.*

This result was previously known only for polynomials of degree 3 with a marked critical point, see [Ghioca and Ye 2017, Theorem 3.3]. Indeed the core of the proof is the continuity of $h(t)$ when the constant α is zero and we follow their line of arguments at this crucial step.

Observe that our main theorem fails for meromorphic families of rational maps. DeMarco and Okuyama have recently constructed a meromorphic family of rational maps of degree 2 with a critical marked point for which the continuity statement does not hold. The rationality of the coefficient α also fails for rational maps, as shown by DeMarco and Ghioca [2016].

Suppose that k is of characteristic zero. Recall that the equilibrium measure μ_P of a polynomial P of degree $d \geq 2$ is the limit of the sequence of averaged pull-backs $\mu_P := \lim_{n\to\infty} d^{-n} P^{n*}\delta_x$ on the Berkovich projective line over k for all x but at most two exceptions, see [Favre and Rivera-Letelier 2010] in the non-Archimedean case and [Brolin 1965] in the complex case. It is a P-invariant probability measure whose support is the Julia set, and it integrates the logarithm of the modulus of any nonzero polynomial. In particular, one can define the Lyapunov exponent $L(P)$ as the integral $L(P) = \int \log|P'|\, d\mu_P$. By the Manning–Przytycki's formula (obtained by Okuyama [2015, §2] in the non-Archimedean case), the Lyapunov exponent satisfies the formula

$$L(P_t) = \log|d| + \sum_{i=1}^{d-1} g_{P_t}(c_i(t)),$$

where $c_1(t), \ldots, c_{d-1}(t)$ denote the critical points of P_t in k counted with multiplicity.

Our Main Theorem then implies:

Corollary 1. *For any meromorphic family $P_t \in \mathcal{O}(\mathbb{D})[t^{-1}][z]$ of polynomials of degree $d \geq 2$ defined over a field of characteristic zero, there exists a nonnegative rational number λ such that the function $t \mapsto L(P_t) - \lambda \log|t|^{-1}$ extends continuously through the origin.*

Moreover we have $\lambda > 0$ unless there exists an affine change of coordinates depending on t conjugating P_t to an analytic family of polynomials.

Let C be a smooth connected affine curve defined over a number field $\mathbb{K}$. An *algebraic family P* parametrized by C is determined by $(d+1)$-regular maps $\alpha_i \in \mathbb{K}[C]$ where α_0 is invertible (i.e., has no zero on C) so that $P_t(z) = \alpha_0(t)z^d + \cdots + \alpha_d(t)$.

A pair (P, a) with $a \in \mathbb{K}[C]$ is said to be *isotrivial* if there exists a finite field extension $\mathbb{L}/\mathbb{K}$, a finite branched cover $p : C' \to C$ defined over $\mathbb{L}$ and a map $\phi : C' \times \mathbb{A}^1 \to C' \times \mathbb{A}^1$ of the form $\phi(t, z) = (t, \phi_t(z))$ where ϕ_t is an affine map for all t such that both $\phi_t \circ P_t \circ \phi_t^{-1}$ and $\phi_t(a(t))$ are independent of t. Finally a is *persistently preperiodic* on C if there exist two integers $n > m \geq 0$ such that $P^n(a) = P^m(a)$ (as regular functions on C).

Recall from [Silverman 2007] that for any t in the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, one can build a canonical height function $\hat{h}_{P_t} : \overline{\mathbb{K}} \to \mathbb{R}_+$ such that $\hat{h}_{P_t} \circ P_t = d\hat{h}_{P_t}$ and $\hat{h}_{P_t}(b) = 0$ if and only if b is preperiodic.

Let us define the height function $h_{P,a}$ on $C(\overline{\mathbb{K}})$ by setting

$$h_{P,a}(t) := \hat{h}_{P_t}(a(t)), \text{ for all } t \in C(\overline{\mathbb{K}}).$$

Note that $h_{P,a}(t) = 0$ if and only if $a(t)$ is preperiodic under iteration of P_t.

Our next result shows that this height function is in fact determined by nice geometric data in the sense of Arakelov theory. We refer to, e.g., the survey [Chambert-Loir 2011] for basics on metrizations on line bundles and their associated height function.

Denote by $\overline{C}$ the (unique up to marked isomorphism) smooth projective curve containing C as an open Zariski dense subset.

Corollary 2. *Let C be an irreducible affine curve defined over a number field $\mathbb{K}$. Let P be an algebraic family parametrized by C and pick any marked point $a \in \mathbb{K}[C]$. Assume that the pair (P, a) is not isotrivial and that a is not persistently preperiodic on C.*

Then there exists an integer $q \geq 1$ such that the height function $q \cdot h_{P,a}$ is induced by an adelic semipositive continuous metrization on some ample line bundle $\mathcal{L} \to \overline{C}$.

Moreover, the global height of the curve $\overline{C}$ is zero, i.e., $h_{P,a}(\overline{C}) = 0$.

Let us explain how we prove our Main Theorem. Basic estimates using the Nullstellensatz imply the existence of a constant $C > 0$ such that

$$\left|\tfrac{1}{d}\log\max\{1, |P_t(z)|\} - \log\max\{1, |z|\}\right| \leq C \log|t|^{-1}, \tag{1}$$

for all $0 < |t| < \frac{1}{2}$ and for all $z \in k$. Using (1), it is not difficult to see that $g_{P_t}(a(t)) = \alpha \log|t|^{-1} + o(\log|t|^{-1})$ for some $\alpha \in \mathbb{R}_+$ [DeMarco 2016, §3]. To get further, we shall interpret the constant α in terms of the dynamics of the polynomial with coefficients in $k((t))$ naturally induced by the family P_t.

We endow the ring $\mathcal{O}(\mathbb{D})[t^{-1}]$ with the t-adic norm $|\cdot|$ whose restriction to k is trivial and normalized by $|t| = e^{-1}$. Observe that the completion of its field of fraction is the field of Laurent series $(k((t)), |\cdot|)$. We may then view the family P_t as a polynomial P with coefficients in the complete metrized $(k((t)), |\cdot|)$

and consider its dynamical Green function $g_{\mathsf{P}} : k((t))) \to \mathbb{R}_+$. The marked point a gives rise to a point $\mathsf{a} \in k((t))$, and it follows from the analysis developed in [Favre 2016] that $\alpha = g_{\mathsf{P}}(\mathsf{a})$.

Let us first consider the case $\alpha > 0$. The point a then belongs to the basin of attraction at infinity of P which implies $a(t)$ to also belong to the basin of attraction at infinity of P_t for all t small enough. To conclude one then uses the fact that the Green function is the logarithm of the modulus of the Böttcher coordinate and expand this coordinate as an analytic function in the two parameters z and t. This strategy was made precise in degree 3 in [Favre and Gauthier 2018; Ghioca and Ye 2017], and we write the details here in arbitrary degree for the convenience of the reader.

When $\alpha = 0$ the point a lies in the filled-in Julia set of P. Since $k((t))$ is discretely valued, results of Trucco [2014] give strong restrictions on the orbit of a. In Section 1, we give a direct argument showing that either a lies in the Fatou set of P and belongs to a preperiodic ball under iteration of P; or a lies in the Julia set of P and the closure of its orbit under P is compact in $k((t))$.

In the former case, one can make a change of coordinates (depending on t) and assume that $P_t(z) = Q(z) + tR_t(z)$ where $\delta = \deg(Q) \leq d$. When $\delta = d$ the family of polynomial P_t degenerates to a polynomial of degree d and the Green function $g_{P_t}(z)$ is continuous both in z and t. Otherwise $\delta < d$, and direct estimates show that $g_{P_t}(a(t)) = o(1)$ as required.

In the latter case, the estimates are more delicate and we follow the arguments of Ghioca and Ye [2017, Theorem 3.1]. The key observation is the following. Since the closure of the orbit of a is compact, for any integer l there exists a finite collection of polynomials $Q_1, \dots, Q_N$ such that for all n, one has $P_t^n(a(t)) = Q_{i_n}(t) + o(t^{l+1})$ for some $i_n \in \{1, \dots, N\}$. The proof of $g_{P_t}(a(t)) = o(1)$ uses in a subtle way this approximation result together with (1).

1. Compact orbits of polynomials

In this section we fix a discrete valued complete field $(L, |\cdot|)$. In our applications we shall take $L = k((t))$ endowed with the t-adic norm normalized by $|t| = e^{-1} < 1$ where k is an arbitrary field.

1.1. ***A criterion for the compactness of polynomial orbits.*** Let P be any polynomial of degree $d \geq 2$ with coefficients in L. Recall that one can find a positive constant $C' > 0$ such that

$$\frac{1}{C'} \leq \frac{\max\{1, |P(z)|\}}{\max\{1, |z|\}^d} \leq C', \tag{2}$$

for all $z \in L$. It follows that the sequence $\frac{1}{d^n} \log \max\{1, |P^n|\}$ converges uniformly on L to a continuous function $g_P : L \to \mathbb{R}_+$ such that $g_P \circ P = d g_P$ and $g_P(z) = \log|z| + O(1)$.

Theorem 3. *Suppose $P \in L[z]$ is a polynomial of degree $d \geq 2$, and a is a point in L. Then $g_P(a) \in \mathbb{Q}_+$ and one of the following holds*:

(1) *The iterates of a tend to infinity in L, and $g_P(a) > 0$.*

(2) *The point a lies in a preperiodic ball B under iteration of P whose radius lies in $|L^*|$, and $g_P(a) = 0$.*

(3) *The closure of the orbit of a in L is compact in $(L, |\cdot|)$, and $g_P(a) = 0$.*

Remark. This fact is a direct consequence of the results of Trucco [2014, Proposition 6.7] when the characteristic of k is zero. We present here a simple proof which does not rely on the delicate combinatorial analysis done by Trucco in his paper and does not use any assumption on k.

1.2. ***The tree of closed balls.*** Let $\mathcal{T}$ be the space of closed balls in L; a point in $\mathcal{T}$ is a set of the form $\overline{B(z_0, r)} := \{z \in L, |z - z_0| \le r\}$ for some $z_0 \in L$ and some $r \in |L^*|$. The map sending a point $z \in L$ to the ball of center z and radius 0 identifies L with a subset of $\mathcal{T}$. We endow $\mathcal{T}$ with the weakest topology making all evaluation maps $Q \mapsto |Q(x)| := \sup_x |Q|$ continuous for all $Q \in L[T]$. By Tychonov, for any $r \ge 0$ the set $\{x \in \mathcal{T}, |T(x)| \le r\}$ is compact for this (weak) topology.

Observe that for any closed ball $x = \overline{B(z, r)} \in \mathcal{T}$ with $r \in |L^*|$, we have $\operatorname{diam}(x) := \sup_{z,z' \in x} |z - z'| = r$. Since $(L, |\cdot|)$ is non-Archimedean, any point $z' \in x$ is a center for x so that $x = \overline{B(z', \operatorname{diam}(x))}$. Also any two balls $x, x' \in \mathcal{T}$ are either contained one into the other or disjoint. When x is contained in x', we may write $x = \overline{B(z, r)}$ and $x' = \overline{B(z, r')}$ for some $r, r' \in |L^*|$, and one sets $d(x', x) = |r' - r| = |\operatorname{diam}(x') - \operatorname{diam}(x)|$.

Denote by $\le$ the partial order relation on $\mathcal{T}$ induced by the inclusion, i.e., $x \le x'$ if and only if the ball x is included in x'. For any two balls $x = \overline{B(z, r)}$ and $x' = \overline{B(z', r')} \in \mathcal{T}$, we let $x \vee x'$ be the smallest closed ball containing both x and x', so that

$$x \vee x' = B(z, \max\{r, r', |z - z'|\}) = B(z', \max\{r, r', |z - z'|\}).$$

The distance between any two closed balls x and x' is now defined as

$$d(x, x') = \max\{|\operatorname{diam}(x \vee x') - \operatorname{diam}(x)|, |\operatorname{diam}(x \vee x') - \operatorname{diam}(x')|\} \in |L^*|.$$

The restriction of the distance d to L is the ultradistance induced by the norm $|\cdot|$.

In the sequel the strong topology refers to the topology on $\mathcal{T}$ induced by the ultradistance d. It is not locally compact unless the residue field $\tilde{L} := \{|z| \le 1\}/\{|z| < 1\}$ is finite.

Let x_n be a sequence of points in L of norm ≤ 1. Its residue classes $\tilde{x}_n$ are by definition their images in $\tilde{L}$ under the natural projection $\{|z| \le 1\} \to \tilde{L}$. If the points $\tilde{x}_n$ are all distinct then for any polynomial $Q = \sum_{k \le d} a_k T^k \in L[T]$ we have $|Q(x_n)| = \max\{|a_k|\}$ for all n large enough so that x_n converges in the weak topology to the point $\overline{B(0, 1)}$.

Proposition 4. *Let F be any bounded infinite subset of L so that $\sup_F |T| < \infty$. Then*

(1) *either the weak closure of F is strongly compact,*

(2) *or one can find a closed ball of positive radius $x \in \mathcal{T}$ containing infinitely many points $x_n \in F$ such that $x_n \to x$.*

Proof. Let $\overline{F}$ be the weak closure of F inside $\mathcal{T}$. Since F is supposed to be bounded, $\overline{F}$ is weakly compact. Observe that for any $z \in L$ and for any $\epsilon > 0$, the set $B_{\mathcal{T}}(z, \epsilon) := \{x \in \mathcal{T}, d(x, z) < \epsilon\}$ is weakly open since it coincides with $B_{\mathcal{T}}(z, \epsilon) = \{x \in \mathcal{T}, |(T - z)(x)| < \epsilon\}$.

Assume first that $\bar{F}$ is included in L, and take $\epsilon > 0$. The previous observation implies that by weak compactness the covering of $\bar{F}$ by the family of balls for all $z \in \bar{F}$ admits a finite subcover. It follows that $\bar{F}$ is strongly precompact. Since L is complete, $\bar{F}$ is also complete hence strongly compact.

Assume now that $\bar{F}$ contains a point x of positive diameter. Up to making an affine change of coordinates, we may suppose that $x = \overline{B(0,1)}$. Since $(L, |\cdot|)$ is discrete, we may find $\epsilon > 0$ such that $1-\epsilon < |z| < 1+\epsilon$ implies $|z| = 1$. It follows that the set $U = \{y \in \mathcal{T}, |T(y)| < 1+\epsilon\}$ is an open neighborhood of x, hence it contains infinitely many points of F.

Let us now prove that one can find a sequence of points $x_n \in F \cap U$ such that $x_n \to x$. We construct the sequence x_n by induction. Choose any $x_1 \in x$, and consider the open set

$$U_1 := \{y \in \mathcal{T}, |T(y)| < 1+\epsilon\} \cap \{y \in \mathcal{T}, |(T-x_1)(y)| > 1-\epsilon\}.$$

It is an open neighborhood of x which does not contain x_1. We may thus find a point $x_2 \in U_1 \cap F$. Proceeding inductively, we find a sequence of points x_n and open neighborhoods U_n of x such that

$$x_n \in U_n := \{y \in \mathcal{T}, |T(y)| < 1+\epsilon\} \bigcap_{i=1}^{n-1} \{y \in \mathcal{T}, |(T-x_i)(y)| > 1-\epsilon\}.$$

The choice of ϵ implies that the residue classes of x_n and x_m are all distinct which implies $x_n \to x$ as required. $\square$

Let $P(T) = a_0 T^d + \cdots + a_d$ be any polynomial with coefficients in L. We define the image of $x = \overline{B(z,r)} \in \mathcal{T}$ by the formula

$$P(x) = \overline{B(P(z), \max_{i \geq 1}\{|a_i| r^i\})}$$

This map is weakly continuous since one has $|Q(P(x))| = |(Q \circ P)(x)|$ for all polynomials Q. It coincides with P on L. Observe that $P : \mathcal{T} \to \mathcal{T}$ preserves the order relation.

Remark. There is a canonical continuous and injective map from $\mathcal{T}$ into the (Berkovich) analytification of $\mathbb{A}^1_L$ sending a closed ball $x \in \mathcal{T}$ to the multiplicative seminorm on $L[T]$ defined by $P \mapsto |P(x)|$. This map identifies $\mathcal{T}$ with the smallest closed subset of $\mathbb{A}^1_L$ whose intersection with the set of rigid points in $\mathbb{A}^1_L$ is equal to L. In the terminology of Berkovich [1990, §1], it consists only of type 1 and type 2 points.

1.3. ***Proof of Theorem 3.*** When $g_P(a)$ is positive, then $\log|P^n(a)| \geq \left(\frac{1}{2} g_P(a)\right)^{d^n}$ for n large enough so that $P^n(a)$ tends to infinity. In that case one can refine (2) since by the non-Archimedean inequality one has $|P(z)| = (r|z|)^d$ for all $|z|$ large enough where r^d is the norm of the leading coefficient of P hence belongs to $e^{\mathbb{Z}}$. If follows that $g_P(z) = \log(r|z|)$ for $|z|$ large enough, hence

$$g_P(a) = \frac{1}{d^N} g_P(P^N(a)) = \frac{1}{d^N} \log(r|a|) \in \mathbb{Q}_+.$$

When $g_P(a) = 0$, all iterates of a belong to $\{g_P = 0\}$ which is a bounded set of L. We consider the weak closure Ω of the orbit of a in the space of balls $\mathcal{T}$. If Ω consists only of points in L, then it is compact in $(L, |\cdot|)$ by Proposition 4(1), and we are in case (3).

Otherwise by Proposition 4(2) there exists a closed ball x with radius in $|L^*|$, and a strictly increasing sequence $n_i \to \infty$ such that $P^{n_i}(a) \to x$, and $P^{n_i}(a) \in x$ for all i. Observe that there exists a closed ball containing the orbit of a, hence the orbit of x is also bounded. Replacing x by a suitable iterate we may assume that $\operatorname{diam}(x) = \max\{\operatorname{diam}(P^n(x))\}_{n\in\mathbb{N}}$.

Since x contains both $P^{n_0}(a)$ and $P^{n_1}(a)$, the ball $P^{n_1-n_0}(x)$ intersects x and thus is included in x. When $P^{n_1-n_0}(x) = x$, we are in case (2). When $P^{n_1-n_0}(x)$ is strictly included in x, then $P^{n_1-n_0}$ admits an attracting fixed point lying in $L \cap P^{n_1-n_0}(x)$ whose basin of attraction contains x. This is however not possible because x lies in the closure of the orbit of a.

2. The point a has an unbounded orbit under P

Recall the setting of the statement of the Main Theorem. Let $P_t(z) = \alpha_0(t)z^d + \alpha_1(t)z^{d-1} + \cdots + \alpha_d(t)$ be a meromorphic family of polynomials of degree d with $\alpha_i \in \mathcal{O}(\mathbb{D})[t^{-1}]$ and $\alpha_0(t) \neq 0$ for all $t \in \mathbb{D}^*$. Take also any meromorphic map $a \in \mathcal{O}(\mathbb{D})[t^{-1}]$.

Recall from, e.g., [DeMarco 2016, Lemma 3.3] or [Favre 2016, Proposition 4.4] that the Nullstellensatz implies the existence of a constant $\beta > 0$ such that

$$|t|^{\beta} \leq \frac{\max\{1, |P_t(z)|\}}{\max\{1, |z|^d\}} \leq |t|^{-\beta} \tag{3}$$

for all $0 < |t| \leq \frac{1}{2}$ and for all $z \in k$. Observe that in particular we get

$$\left| g_{P_t}(z) - \log\max\{1, |z|\} \right| \leq C'' \log|t|^{-1} \tag{4}$$

for all $0 < |t| \leq \frac{1}{2}$, and all $z \in k$, for some constant $C'' > 0$.

Since the base field k is supposed to be algebraically closed conjugating P_t by a suitable homothety with coefficient in k, we may write $\alpha_0(t) = t^N(1 + o(1))$ for some $N \in \mathbb{Z}$. One can then consider the family of monic polynomials $\tilde{P}_t(z) = \phi_t^{-1} \circ P_{t^{d-1}} \circ \phi_t$ with $\phi_t(z) = \alpha_0(t^{d-1})^{-1/(d-1)} z$. Observe that $g_{P_{t^{d-1}}}(a(t^{d-1})) = g_{\tilde{P}_t}(\phi_t^{-1} \circ a(t^{d-1}))$ so that the proof of the Main Theorem is reduced to the case of a family of monic polynomials. From now on we shall therefore assume that $\alpha_0 \equiv 1$.

Proof of the Main Theorem in the case $|\mathsf{P}^n(\mathsf{a})| \to \infty$. Recall that P denotes the monic polynomial of degree d with coefficients in $k((t))$ induced by the family $P_t(z) = z^d + \alpha_1(t)z^{d-1} + \cdots + \alpha_d(t)$, and a is the point in $k((t))$ defined by the meromorphic function $a(t) \in \mathcal{O}(\mathbb{D})[t^{-1}]$. We endow $k((t))$ with the t-adic norm $|\cdot|$ normalized by $|t| = e^{-1}$.

If a has an unbounded orbit under P, then replacing a by $P^{n_0} \circ a$ for n_0 sufficiently large we may suppose that a has a pole of order l which can be taken as large as we wish.

If l is strictly larger than the constant C'' appearing in (4), then we get

$$g_{P_t}(a(t)) \geq \log\max\{1, |a(t)|\} - C'' \log|t|^{-1} \geq (l - C'') \log|t|^{-1} + O(1).$$

In particular $g_{P_t}(a(t))$ is positive for any $0 < |t| \leq \varepsilon$ with $\varepsilon > 0$ small enough. And by (3) the convergence $g_{P_t}(a(t)) = \lim_n \frac{1}{d^n} \log|P_t^n(a(t))|$ holds uniformly on compact subsets on $\mathbb{D}_\varepsilon^*$.

Lemma 5. *There exists an integer $N \in \mathbb{N}^*$ such that for any meromorphic function $a(t) = t^{-l}(1+h)$ with $l \geq N$ and $h \in \mathcal{O}(\mathbb{D})$ such that $h(0) = 0$ and $\sup_{\mathbb{D}}|h| \leq \frac{1}{2}$, then we have*

$$\sup_{0<|t|\leq\frac{1}{2}} \left|\tfrac{1}{d}\log|P_t(a(t))| - \log|a(t)|\right| < +\infty.$$

Now suppose $l \geq N$ so that we may apply Lemma 5. It follows that the sequence of functions $\frac{1}{d^n}\log|P^n \circ a| - \log|a|$ converges uniformly in $\mathbb{D}^*_{1/2}$ to a function φ which is necessarily harmonic and bounded. By [Ransford 1995, Corollary 3.6.2], this function thus extends to the origin and remains harmonic. We conclude that $g_{P_t}(a(t)) = \varphi(t) + \log|a(t)|$ which shows that we are in case 2 of the Main Theorem. □

Proof of Lemma 5. Pick N large enough such that $\sup_{1\leq i\leq d, |t|\leq\frac{1}{2}} |t|^N|\alpha_i(t)| \leq 1$. For any $l \geq N$, we may then write

$$P_t(a(t)) = t^{-ld}((1+h)^d + \alpha_1 t^l (1+h)^{d-1} + \cdots + \alpha_0 t^{ld})$$

so that

$$\tfrac{1}{d}\log|P_t(a(t))| - \log|a(t)| = \log\left|\frac{(1+h)^d + \alpha_1 t^k(1+h)^{d-1} + \cdots + \alpha_0 t^{kd}}{1+h}\right|$$

is bounded by $\log((d+1)2^{d+1})$. □

3. The point a has a bounded orbit under P and lies in the Fatou set of P

We suppose now that $\sup_{m\geq 0}|\mathsf{P}^m(\mathsf{a})| < \infty$ and a belongs to a closed ball $B := \overline{B(\mathsf{b}, \rho)}$ fixed by P with $\mathsf{b} \in k((t))$ and $\rho = e^{-n}$ for some $n \in \mathbb{Z}$. Observe that we may take b to be a Laurent polynomial $b(t)$, and conjugating P_t by $\phi_t(z) = t^n(z + b(t))$, we may thus suppose that B is the closed unit ball.

In that case, we can write $P_t(z) = Q(z) + t R_t(z)$ for some polynomial $Q \in k[z]$ with $1 \leq \delta := \deg(Q) \leq d$ and $R_t(z) \in k((t))[z]$. When $\delta = d$ then we are in the case 1 of the Main Theorem.

When $\delta < d$, we shall prove that we fall in case 3. To see this, observe first that there exists $C_1 \geq 1$ such that

$$\max\{1, |P_t(z)|\} \leq C_1 \cdot \max\{1, |z|^\delta, |t| \cdot |z|^d\},$$

for all $z \in k$ and all $t \in \mathbb{D}$.

Lemma 6. *There exists a constant $A \geq 1$ independent of n such that*

$$\max\{1, |P^n_t(a(t))|\} \leq C_1^{1+\cdots+\delta^{n-1}} \cdot A^{\delta^n} \tag{5}$$

if $|t| = (C_1^{1+\cdots+\delta^{n-2}} \cdot A^{\delta^{n-1}})^{\delta-d}$.

We fix some integer $N \geq 2$ and suppose $|t| = R_N := (C_1^{1+\cdots+\delta^{N-2}} \cdot A^{\delta^{N-1}})^{\delta-d}$. Under these assumptions (4) and (5) imply

$$0 \leq g_t(a(t)) \leq \tfrac{1}{d^N}\log^+|P^N_t(a(t))| - C''\tfrac{1}{d^N}\log|t| \leq \left(\tfrac{\delta}{d}\right)^N\left(1 + \tfrac{d-\delta}{\delta}C''\right)\log(C_1^{1/(\delta-1)}A).$$

For any $m \geq 0$, since $\mathsf{P}^m(\mathsf{a})$ belongs to the closed unit ball for all m by assumption, the functions $P^m(a(t))$ are analytic at 0 so that $\frac{1}{d^m}\log|P^m(a(t))|$ is subharmonic on $\mathbb{D}$ (in the sense of Thuillier when $(k, |\cdot|)$ is non-Archimedean, see e.g., [Thuillier 2005, §3]). It follows that $g_t(a(t))$ is also subharmonic on $\mathbb{D}$, and the maximum principle implies

$$0 \leq g_t(a(t)) \leq \left(\tfrac{\delta}{d}\right)^N \cdot B,$$

for all $|t| \leq R_N$ with $B := (1 + (d-\delta)C''/\delta)\log(C_1^{1/(\delta-1)}A)$.

Fix $\varepsilon > 0$ and pick $N \geq 1$ large enough such that $\left(\frac{\delta}{d}\right)^N \cdot B \leq \varepsilon$. We have proved that $0 \leq g_t(a(t)) \leq \epsilon$ when $|t| \leq \eta := R_N$ so that $g_t(a(t))$ is a nonnegative subharmonic function on $\mathbb{D}$ and $\lim_{t\to 0} g_t(a(t)) = 0$. In particular, it is continuous at $t = 0$.

Proof of Lemma 6. We argue by induction on $n \geq 2$. Note that since a is analytic as seen above, there exists $A \geq 2$ such that $|a(t)| \leq A$ for all $|t| \leq \frac{1}{2}$. Assume that $|t| = A^{\delta-d}$. Then we have

$$\max\{1, |P_t(a(t))|\} \leq C_1 \cdot \max\{1, A^\delta, |t| \cdot A^d\} = C_1 \cdot A^\delta.$$

Now let $|t| = (C_1 \cdot A^\delta)^{\delta-d}$. We obtain

$$\max\{1, |P_t^2(a(t))|\} \leq C_1 \cdot \max\{1, (C_1 \cdot A^\delta)^\delta, |t| \cdot (C_1 \cdot A^\delta)^d\} = C_1^{1+\delta} \cdot A^{\delta^2}.$$

Suppose (5) holds for some $n \geq 2$. When $|t| = (C_1^{1+\cdots+\delta^{n-1}} \cdot A^{\delta^n})^{\delta-d}$, we have

$$\begin{aligned}\max\{1, |P_t^{n+1}(a(t))|\} &\leq C_1 \max\{1, |P_t^n(a(t))|^\delta, |t| \cdot |P_t^n(a(t))|^d\}\\ &\leq C_1 \max\{1, (C_1^{1+\cdots+\delta^{n-1}} \cdot A^{\delta^n})^\delta, |t|(C_1^{1+\cdots+\delta^{n-1}} \cdot A^{\delta^n})^d\}\\ &\leq C_1^{1+\cdots+\delta^n} \cdot A^{\delta^{n+1}},\end{aligned}$$

which concludes the proof. □

4. The point a lies in the Julia set of P

In this section, we complete the proof of the Main Theorem. We apply Theorem 3 and discuss the situation case by case.

- In case 1 of Theorem 3, the arguments of Section 2 enable us to conclude directly.
- In case 2 of Theorem 3, we replace the marked point by $P^m(a)$ for a suitable m, and P by a suitable iterate so that a belongs to a ball fixed by P. This case was treated in the previous section.

It thus remains to treat case 3 of Theorem 3: the orbit a is compact in $k((t))$.

Conjugating the family by linear maps $\phi_t(z) = t^{-M}z$ with M sufficiently large, we may suppose that the fixed ball (under P) containing a, and thus the orbit of a, is the closed unit ball. It follows that

$$g_m(t) := \tfrac{1}{d^m}\log\max\{|u_m(t)|, 1\} \text{ with } u_m(t) := P_t^m(a(t))$$

is subharmonic on $\mathbb{D}$ and $g_t(a(t))$ also.

Replacing P_t by its second iterate, we may and shall assume that $d \geq 3$.

As $(g_n(t))_n$ converges locally uniformly to $g_t(a(t))$ on $\mathbb{D}^*$ and as $g_t(a(t))$ is bounded on $\left\{|t| = \frac{1}{2}\right\}$, there exists a constant $M \geq 1$ such that, $\sup_{|t|=1/2} g_n(t) \leq M$ for all $n \geq 1$. By the maximum principle, this gives

$$\sup_{|t|\leq 1/2} g_n(t) \leq M.$$

Fix any integer $l \geq C'' \cdot d$, where $C'' > 0$ is the constant given by (4). Recall that the orbit of a lies in the unit ball in $k((t))$, hence $P_t^n(a(t))$ is analytic at 0 for all n. Observe that the set of balls of radius r^{-l} centered at polynomials covers the unit ball in $k((t))$. Since the orbit of a is compact in k, we may thus find a finite collection of polynomials $Q_1, \dots, Q_N$ such that for any n one can find $i_n \in \{1, \dots, N\}$ such that $P_t^n(a(t)) - Q_{i_n}(t) = O(t^l)$.

Let

$$A := \max_{1\leq j\leq N} \{\sup_{|t|<1} |Q_j(t)| + 2\}.$$

Fix a very large integer $n_0 \geq 1$ once and for all. We may thus find $r_0 > 0$ small enough such that

$$\sup_{|t|<r_0} |u_{n_0}(t)| \leq A.$$

Set $r_j := r_0^{2^j}$ for any $j \geq 0$, so that $0 < r_{j+1} < r_j$ and $r_j \to 0$ as $j \to \infty$.

Lemma 7. *For all $j \geq 0$, one has*

$$\sup_{|t|<r_j} g_{n_0+j}(t) \leq \frac{C_1}{d^{n_0}},$$

with $C_1 = d\log(3A)/(d-1)$.

Using (3), an easy induction gives a constant $C'' > 0$ such that

$$0 \leq g_{n+\ell}(t) \leq g_n(t) + \frac{C''}{d^n}\log|t|^{-1}$$

for all $t \in \mathbb{D}^*_{1/2}$ and all $n, \ell \geq 1$. For $|t| = r_j$ and $n = n_0 + j$, this reads as

$$0 \leq g_{n_0+j+\ell}(t) \leq \sup_{|\tau|=r_j} g_{n_0+j}(\tau) - \frac{C''}{d^{n_0+j}}\log r_j.$$

By the maximum principle applied to $g_{n_0+j+\ell}$ and g_{n_0+j} and by Lemma 7, we find

$$0 \leq g_{n_0+j+\ell}(t) \leq \sup_{|\tau|<r_j} g_{n_0+j}(\tau) - \frac{C''}{d^{n_0+j}}\log r_j \leq \frac{C_2}{d^{n_0}}\left(1 - \left(\frac{2}{d}\right)^j \log r_0\right),$$

for any $|t| < r_j$ and any $\ell \geq 1$, where $C_2 := \max(C'', C_1)$.

Pick now $\varepsilon > 0$. We may choose n_0 be large enough such that $C_2/d^{n_0} \leq \varepsilon/2$. We then fix $j \geq 0$ large enough (depending on r_0, hence on n_0) so that $\left(1 - \left(\frac{2}{d}\right)^j \log r_0\right) \leq 2$. The previous estimate implies

$$0 \leq g_n(t) \leq \varepsilon$$

for all $|t| < r_j$ and all $n \geq n_0 + j$. Letting n tend to infinity we finally obtain

$$0 \leq g_t(a(t)) \leq \varepsilon,$$

for all $|t| < r_j$ which concludes the proof.

Proof of Lemma 7. It is sufficient to show by induction on $j \geq 0$ that

$$\sup_{|t|<r_{j+1}} g_{n_0+j+1}(t) \leq \frac{\log(3A)}{d^{n_0+j+1}} + \sup_{|s|<r_j} g_{n_0+j}(s).$$

By assumption, for all $j \geq 1$, there exists $1 \leq i_j \leq N$ such that the function

$$\frac{u_{n_0+j}(t) - Q_{i_j}(t)}{t^l}$$

is analytic on $\mathbb{D}$. The maximum principle applied for the analytic function $(u_{n_0+j}(t) - Q_{i_j}(t))/t^l$ on $\mathbb{D}(0, r)$ for $0 < r < 1$ gives

$$\left|\frac{u_{n_0+j}(t) - Q_{i_j}(t)}{t^l}\right| \leq \big(A + \sup_{|s|<r}\big|u_{n_0+j}(s)\big|\big) \cdot \frac{1}{r^l}, \quad \text{for all } |t| < r.$$

This implies for any $0 < r < 1$ the estimate

$$\big|u_{n_0+j}(t) - Q_{i_j}(t)\big| \leq \big(A + \sup_{|s|<r}\big|u_{n_0+j}(s)\big|\big) \cdot \left(\frac{|t|}{r}\right)^l, \quad \text{for all } |t| < r.$$

In particular, we find

$$\sup_{|t|<r_{j+1}} \big|u_{n_0+j+1}(t)\big| \leq A + \big(A + \sup_{|s|<r_j}\big|u_{n_0+j+1}(s)\big|\big) \cdot \left(\frac{r_{j+1}}{r_j}\right)^l \leq 2A + \big(\sup_{|s|<r_j}\big|u_{n_0+j+1}(s)\big|\big)r_j^l,$$

hence

$$\sup_{|t|<r_{j+1}} \max\big\{1, \big|u_{n_0+j+1}(t)\big|\big\} \leq (3A) \sup_{|s|<r_j} \max\big\{1, \big|u_{n_0+j+1}(s)\big|r_j^l\big\}.$$

When $\sup_{|s|<r_j}\big|u_{n_0+j+1}(s)\big|r_j^l \leq 1$, we get

$$\sup_{|t|<r_{j+1}} g_{n_0+j+1}(t) \leq \frac{\log(3A)}{d^{n_0+j+1}} \leq \frac{\log(3A)}{d^{n_0+j+1}} + \sup_{|s|<r_j} g_{n_0+j}(s),$$

as required. Otherwise, we have

$$\begin{aligned}
\sup_{|t|<r_{j+1}} g_{n_0+j+1}(t) &\leq \frac{\log(3A)}{d^{n_0+j+1}} + \frac{l}{d^{n_0+j+1}} \log r_j + \sup_{|s|<r_j} g_{n_0+j+1}(s) \\
&\leq \frac{\log(3A)}{d^{n_0+j+1}} + \left(\frac{l}{d^{n_0+j+1}} - \frac{C''}{d^{n_0+j}}\right) \log r_j + \sup_{|s|<r_j} g_{n_0+j}(s) \\
&\leq \frac{\log(3A)}{d^{n_0+j+1}} + \sup_{|s|<r_j} g_{n_0+j}(s),
\end{aligned}$$

since $r_j < 1$ and $l \geq C''d$, where the middle inequality follows from (1). The lemma follows. $\square$

Remark. We claim that

$$g_{P_t}(a(t)) = g_{\mathsf{P}}(\mathsf{a}) \log|t|^{-1} + h(t), \tag{6}$$

with h continuous.

When $|\mathsf{P}^n(\mathsf{a})|$ is bounded, then $g_{\mathsf{P}}(\mathsf{a}) = 0$ and the equation follows from the arguments in Section 3 and Section 4. When $|\mathsf{P}^n(\mathsf{a})| \to \infty$ by the invariance of the Green function under iteration it is sufficient to prove (6) when $a(t) = t^{-l}h$ with $h(0) \neq 0$, $l \geq C''$ as in the proof of the Main Theorem in the case $|\mathsf{P}^n(\mathsf{a})| \to \infty$ on page 1477.

In that case we have

$$\tfrac{1}{d^n} \log|P_t^n(a(t))| = \log|a(t)| + \varphi_n(t) = \log|\mathsf{a}| \log|t|^{-1} + \log|h(t)| + \varphi_n(t)$$

where φ_n is a sequence of harmonic functions converging uniformly on $\mathbb{D}_{1/2}$. We also have

$$\tfrac{1}{d^n} \log|P_t^n(a(t))| = \tfrac{1}{d^n} \log|\mathsf{P}^n(\mathsf{a})| \log|t|^{-1} + \psi(t),$$

where ψ is harmonic, hence $\frac{1}{d^n} \log|\mathsf{P}^n(\mathsf{a})| = \log|\mathsf{a}|$ for all n and $g_{\mathsf{P}}(\mathsf{a}) = \log|\mathsf{a}|$ which implies (6).

5. Degeneration of the Lyapunov exponent

In this section we prove Corollary 1. Assume $(k, |\cdot|)$ is an algebraically closed complete metrized field of characteristic zero. Fix a meromorphic family $P_t(z) = a_d(t)z^d + \cdots + a_0(t) \in \mathcal{O}(\mathbb{D})[t^{-1}][z]$ of degree $d \geq 2$ polynomials defined over k. Recall that the Lyapunov exponent of P_t is equal to

$$L(P_t) = \log|d| + \sum_{i=1}^{d-1} g_{P_t}(c_i), \tag{7}$$

where $c_1, \ldots, c_{d-1}$ denote the critical points of P_t in k counted with multiplicity.

To control these critical points when t varies, we observe that the polynomial $P_t'(z)$ is a polynomial of degree $d-1$ with coefficients in $\mathcal{O}(\mathbb{D})[t^{-1}]$ and dominant term $da_d(t)z^{d-1}$. It splits over the field of Puiseux series so that one can find Puiseux series $c_1(t), \ldots, c_{d-1}(t)$ such that $P_t'(z) = da_d(t) \prod_{i=1}^{d-1}(z - c_i(t))$. Pick any sufficiently divisible integer N such that all series $c_i(t^N)$ become formal power series. By Artin's approximation theorem [1968], they are necessarily analytic in a neighborhood of 0.

Our Main Theorem applied to the meromorphic family P_{t^N} and the marked points $c_i(t^N)$ shows that we can write $g_{P_{t^N}}(c_i(t^N)) = \lambda_i \log|t|^{-1} + h_i(t)$ where h_i is a continuous function and $\lambda_i \in \mathbb{Q}_+$. By (7), we infer

$$L(P_{t^N}) = \log|d| + \left(\sum_{i=1}^{d-1} \lambda_i\right) \log|t|^{-1} + \sum_{i=1}^{d-1} h_i(t).$$

Now observe that by definition $\tilde{h}(t) = \sum_{i=1}^{d-1} h_i(t)$ is a continuous function on the unit disk which is invariant by the multiplication by any N-th root of unity. It follows that one may find a continuous function h on the unit disk such that $h(t^N) = \tilde{h}(t)$, and we get $L(P_t) = \lambda \log|t|^{-1} + \log|d| + h(t)$ with $\lambda := \frac{1}{N} \sum_{i=1}^{d-1} \lambda_i \in \mathbb{Q}_+$ as required.

Suppose now that $\lambda = 0$. Denote by P the polynomial with coefficients in $k((t))$ defined by the family P_t, and by c_i the point in $k((t^{1/N}))$ defined by the Puiseux series $c_i(t)$. By [Favre 2016, Theorem C] and [Okuyama 2015, §5], it follows that $g_{\mathsf{P}}(\mathsf{c}_i) = 0$ for all i so that all critical points of P belongs to the filled-in Julia set.

We claim that there exists an affine transformation ϕ with coefficients in $k((t))$ such that $\mathsf{Q} := \phi^{-1} \circ \mathsf{P} \circ \phi$ leaves the closed unit ball totally invariant.

Granting this claim we conclude the proof of the corollary. We write $\phi(z) = b_0(t)z + b_1(t)$ with $b_i(t) = t^{-n_i}(b_{i0} + \sum_{i\geq 1} b_{ij}t^j)$, $b_{i0} \neq 0$ and $b_{ij} \in k$, and we define for all $M \geq 1$ the affine transformation

$$\phi_M = b_0^M(t)z + b_1^M(t),$$

where $b_i^M = t^{-n_i}(b_{i0} + \sum_{M\geq i\geq 1} b_{ij}t^j)$. For M large enough the difference $\mathsf{Q} - \mathsf{Q}_M$ is a polynomial with coefficients in $tk[\![t]\!]$ so that the polynomial $\mathsf{Q}_M := \phi_M^{-1} \circ \mathsf{P} \circ \phi_M$ leaves the closed unit ball totally invariant too.

In particular Q_M is a polynomial of degree d with coefficients in $k[\![t]\!]$ and dominant term $\mathsf{b}z^d$ with b invertible (in $k[\![t]\!]$). Together with the fact that the family P_t is meromorphic and the coefficients of ϕ_M are Laurent polynomials, we conclude that b determines an analytic function $b(t)$ with $b(0) \neq 0$, and Q_M determines an analytic family of polynomials of the form $Q_t(z) = b(t)z^d + \text{l.o.t.}$ conjugated to P_t, as required.

It remains to prove our claim. Denote by $\mathbb{L}$ the completion of the field of Puiseux series (i.e., of the algebraic closure of $k((t))$).

By [Kiwi 2006, Corollary 2.11] the fact that all critical points of P belong to the filled-in Julia set implies that P is simple over $\mathbb{L}$. This means that the filled-in Julia set of P in $\mathbb{A}^1_{\mathbb{L}}$ is equal to a closed ball B. This ball B contains all fixed points, and this set of fixed points is defined by a polynomial equation of the form $a_0z^l + a_1z^{l-1} + \cdots + a_l = 0$ of degree l with $a_i \in k((t))$ hence B contains the point $-a_1/(la_0) \in k((t))$. The radius of B also belongs to $|k((t))^*|$ because B is fixed by P, so that we can find an affine transformation with coefficients in $k((t))$ sending B to the closed unit ball.

6. The adelic metric associated to a pair (P, a)

In this section, we prove Corollary 2. Let us first recall the setting. Let C be any smooth connected affine curve C defined over a number field $\mathbb{K}$. Assume P is an algebraic family parametrized by C and $a \in \mathbb{K}[C]$ is a marked point such that (P, a) is not isotrivial, and a is not persistently preperiodic.

Denote by $M_{\mathbb{K}}$ be the set of places of $\mathbb{K}$.

Step 1: construction of a suitable line bundle $\mathcal{L}$ on $\bar{C}$ the (smooth) projective compactification of C.

To any branch at infinity $\mathfrak{c} \in \bar{C} \setminus C$ we associate a nonnegative rational number $\alpha(\mathfrak{c})$ as follows.

We fix a projective embedding of $\bar{C}$ into the projective space $\mathbb{P}^3_{\mathbb{K}}$ such that $\mathfrak{c}$ is the homogeneous point $[0:0:0:1]$. By, e.g., [Favre and Gauthier 2018, Proposition 3.1] there exist a number field $\mathbb{L} \supset \mathbb{K}$, a finite set of places S of $\mathbb{L}$ and adelic series $\beta_1, \beta_2, \beta_3 \in t\mathcal{O}_{\mathbb{L},S}[\![t]\!]$ such that the following holds.

For each place $v \in M_{\mathbb{L}}$ the series $\beta_j(t)$ are convergent in some neighborhood $\{|t| < c_v\}$ of the origin. The map $\beta(t) = [\beta_1(t) : \beta_2(t) : \beta_3(t)) : 1]$ induces an analytic isomorphism from $\{|t| < c_v\}$ to a neighborhood of $\mathfrak{c}$ in $\overline{C}(\mathbb{C}_v)$. And the constants c_v equal 1 for all but finitely many places. In the sequel we refer to an adelic parametrization of $\overline{C}$ near $\mathfrak{c}$ for such a data.

Our family of polynomials is determined by $d+1$ rational functions on $\overline{C}$

$$P(z) = \alpha_0 z^d + \alpha_1 z^{d-1} + \cdots + \alpha_d$$

with $\alpha_i \in \mathbb{K}(C)$, so that $P_\mathfrak{c} := (\alpha_0 \circ \beta) z^d + (\alpha_1 \circ \beta) z^{d-1} + \cdots + (\alpha_d \circ \beta)$ belongs to $\mathcal{O}_{\mathbb{L},S}((t))[z] \subset \mathbb{L}((t))[z]$. Write $a_\mathfrak{c} = a \circ \beta \in \mathbb{L}((t))$.

Working over the non-Archimedean field $L = \mathbb{L}((t))$ endowed with the t-adic norm, we may define $\alpha(\mathfrak{c}) := g_{P_\mathfrak{c}}(a_\mathfrak{c})$ which is a nonnegative rational number by Theorem 3.

We finally define the effective divisor with rational coefficients

$$\mathsf{D} := \sum_{\overline{C} \setminus C} \alpha(\mathfrak{c})[\mathfrak{c}]$$

and set $\mathcal{L} := \mathcal{O}_{\overline{C}}(q\mathsf{D})$ for a sufficiently divisible $q \in \mathbb{N}^*$. Observe that since C is defined over $\mathbb{K}$, its projective compactification $\overline{C}$ is also defined over $\mathbb{K}$ and the divisor D too since it is invariant by the absolute Galois group of $\mathbb{K}$.

Step 2: we build a semipositive and continuous metrization $|\cdot|_{\mathcal{L},v}$ on the line bundle induced by $\mathcal{L}$ on $\overline{C}_{\mathbb{K}_v}$ for any place $v \in M_\mathbb{K}$.

Fix a place $v \in M_\mathbb{K}$. We let $\mathbb{C}_v$ be the completion of the algebraic closure $\overline{\mathbb{K}}_v$ of the completion $\mathbb{K}_v$ of $(\mathbb{K}, |\cdot|_v)$, and define

$$g_{P_t,v}(z) := \lim_{n\to\infty} \tfrac{1}{d^n} \log^+ |P_t^n(z)|_v, \ z \in \mathbb{C}_v.$$

Pick a branch at infinity $\mathfrak{c} \in \overline{C} \setminus C$ and choose a local adelic parametrization β of $\overline{C}$ centered at $\mathfrak{c}$ as in the previous step. It is given by formal power series with coefficients in a number field $\mathbb{L}$.

According to the Main Theorem, there exists $\alpha_v(\mathfrak{c}) \in \mathbb{Q}_+$ such that

$$g_{a,v}(t) := g_{P_{\beta(t)},v}(a(t)) = \alpha_v(\mathfrak{c}) \cdot \log|t|^{-1} + h_{\mathfrak{c},v}(t), \tag{8}$$

where $h_{\mathfrak{c},v}$ extends continuously across 0. Moreover by (6) the constant $\alpha_v(\mathfrak{c})$ is equal to $g_{P_\mathfrak{c}}(a_\mathfrak{c}) = \alpha(\mathfrak{c})$. In particular, $\alpha_v(\mathfrak{c})$ is independent of v.

Pick an open subset U of the Berkovich analytification $\overline{C}^{v,\mathrm{an}}$ of $\overline{C}$ over the field $\mathbb{K}_v$ and a section σ of the line bundle $\mathcal{L}$ over U. By definition, σ is a meromorphic function on U whose divisor of poles and zeroes satisfies $\mathrm{div}(\sigma) + q\mathsf{D} \geq 0$. We set

$$|\sigma|_{a,v} := |\sigma|_v e^{-q \cdot g_{a,v}}.$$

According to (8), the function $|\sigma|_{a,v}$ is continuous. Moreover, since $g_{a,v}$ is subharmonic on $C^{v,\mathrm{an}}$ and since the function $-\log|\sigma|_{a,v}$ extends continuously to U, [Favre and Gauthier 2018, Lemma 3.7] implies that $-\log|\sigma|_{a,v}$ is subharmonic on U. This implies the metrization $|\cdot|_{a,v}$ is continuous and semipositive in the sense of Zhang (by definition in the Archimedean case and by [Favre and Gauthier 2018, Lemma 3.11] in the non-Archimedean case).

Step 3: the line bundle $\mathcal{L}$ is (very) ample (if q is large enough).

Since $\mathcal{L}$ is determined by the effective divisor $\mathsf{D} = \sum_{\overline{C}\setminus C} \alpha(\mathfrak{c}) \cdot [\mathfrak{c}]$ it is sufficient to show that $\alpha(\mathfrak{c}) > 0$ for at least one branch at infinity. Suppose to the contrary that $\mathsf{D} = 0$, and choose any Archimedean place $v_0 \in M_{\mathbb{K}}$. Observe first that the function g_{a,v_0} extends continuously to $\overline{C}(\mathbb{C})$ as a subharmonic function which is thus constant since $\overline{C}(\mathbb{C})$ is compact.

By [Dujardin and Favre 2008] the family of analytic maps $t \mapsto P_t^n(a(t))$ is hence normal locally near any point $t \in C$. Since (P, a) not isotrivial, [DeMarco 2016, Theorem 1.1] implies a is persistently preperiodic, which is a contradiction.

Step 4: the collection of metrizations $|\cdot|_{\mathcal{L},v}$ equips $\mathcal{L}$ with an adelic semipositive continuous metrization whose induced height function $h_{\hat{\mathcal{L}}}$ satisfies

$$h_{\hat{\mathcal{L}}}(t) = q \cdot h_{P,a}(t), \text{ for all } t \in C(\overline{\mathbb{K}}). \tag{9}$$

Let us prove the first assertion. Since for any place v the metrization $|\cdot|_{\mathcal{L},v}$ is semipositive and continuous, one only needs to show that the collection $\{|\cdot|_{a,v}\}_{v\in M_{\mathbb{K}}}$ is adelic. Following exactly the proof of [Ghioca and Ye 2017, Lemma 4.2], we get the existence of $g \in \mathbb{K}(\overline{C})$ such that $q \cdot g_{a,v}(z) = \log|g(z)|_v$ for all but finitely many places $v \in M_{\mathbb{K}}$ and the conclusion follows.

To get (9), we follow closely [Favre and Gauthier 2018, §4.1]. If t is a point in C that is defined over a finite extension $\mathbb{K}$, denote by $\mathsf{O}(t)$ its orbit under the absolute Galois group of $\mathbb{K}$, and let $\deg(t) := \mathrm{Card}(\mathsf{O}(t))$. Fix a rational function ϕ on $\overline{C}$ with $\mathrm{div}(\phi) + q\mathsf{D} \geq 0$ that is not vanishing at t. By [Chambert-Loir 2011, §3.1.3], since $\phi(t) \neq 0$ we have

$$\begin{aligned} h_{\hat{\mathcal{L}}}(t) &= \frac{1}{\deg(t)} \sum_{t'\in\mathsf{O}(t)} \sum_{v\in M_{\mathbb{K}}} -\log|\phi|_{a,v}(t') \\ &= \frac{1}{\deg(t)} \sum_{t'\in\mathsf{O}(t)} \sum_{v\in M_{\mathbb{K}}} (q \cdot g_{a,v}(t') - \log|\phi|_v(t')) \\ &= \frac{1}{\deg(t)} \sum_{t'\in\mathsf{O}(t)} \sum_{v\in M_{\mathbb{K}}} q \cdot g_{P_{t'},v}(a(t')) = q \cdot \hat{h}_{P_t}(a(t)) \geq 0, \end{aligned}$$

where the last line follows from the product formula and the definition of $\hat{h}_{P_t}$.

Step 5: the total height of $\overline{C}$ is $h_{\hat{\mathcal{L}}}(\overline{C}) = 0$.

We use [Chambert-Loir 2011, (1.2.6) and (1.3.10)]. Choose any two meromorphic functions ϕ_0 and ϕ_1 such that $\mathrm{div}(\phi_0) + q\mathsf{D}$ and $\mathrm{div}(\phi_1) + q\mathsf{D}$ are both effective with disjoint support included in C. Let σ_0 and σ_1 be the associated sections of $\mathcal{O}_{\overline{C}}(q\mathsf{D})$. Let $\sum n_i[t_i]$ be the divisor of zeroes of σ_0 and $\sum n'_j[t'_j]$ be the divisor of zeroes of σ_1. Then

$$\begin{aligned} h_{\hat{\mathcal{L}}}(\overline{C}) &= \sum_{v\in M_{\mathbb{K}}} (\widehat{\mathrm{div}}(\sigma_0)\cdot\widehat{\mathrm{div}}(\sigma_1)|\overline{C})_v \\ &= \sum_i n_i\cdot q\cdot \hat{h}_{P_{t_i}}(a(t_i)) - \sum_{v\in M_{\mathbb{K}}} \int_{\overline{C}} \log\sigma_0|_{a,v}\Delta(q\cdot g_{a,v}) \\ &= \sum_{v\in M_{\mathbb{K}}} \int_{\overline{C}} q\cdot g_{a,v}\Delta(q\cdot g_{a,v}) \geq 0, \end{aligned}$$

where the third equality follows from Poincaré–Lelong formula and writing $\log|\sigma_0|_{a,v} = \log|\phi_0|_v - q\cdot g_{a,v}$.

Pick any Archimedean place v_0. The total mass on $\overline{C}$ of the positive measure $\Delta g_{a,v_0}$ is the degree of $\mathcal{L}$, hence is nonzero. It follows from, e.g., [Dujardin and Favre 2008, Lemma 2.3] that any point t_0 in the support of $\Delta g_{a,v_0}$ is accumulated by parameters $t_* \in C(\mathbb{K})$ such that $P^n_{t_*}(a(t_*)) = P^m_{t_*}(a(t_*))$ for some $n > m \geq 0$. For any such point (9) implies $h_{\hat{\mathcal{L}}}(t_*) = 0$. In particular, the essential minimum of $h_{\hat{\mathcal{L}}}$ is nonpositive. By the arithmetic Hilbert–Samuel theorem (see [Thuillier 2005, Théorème 4.3.6], [Autissier 2001, Proposition 3.3.3], or [Zhang 1995, Theorem 5.2]), we get $h_{\hat{\mathcal{L}}}(\overline{C}) = 0$, ending the proof.

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Local root numbers and spectrum of the local descents for orthogonal groups: p-adic case

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We investigate the local descents for special orthogonal groups over p-adic local fields of characteristic zero, and obtain explicit spectral decomposition of the local descents at the first occurrence index in terms of the local Langlands data via the explicit local Langlands correspondence and explicit calculations of relevant local root numbers. The main result can be regarded as a refinement of the local Gan–Gross–Prasad conjecture (2012).

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1. Introduction

Let G be a group and H be a subgroup of G. For any representation π of G, it is a classical problem to look for the spectral decomposition of the restriction of π from G to H. The spectral decomposition problem can also be formulated in a different way. For a given π of G and a subgroup H, which representation σ of H has the property that

$$\mathrm{Hom}_H(\pi, \sigma) \neq 0? \tag{1-1}$$

And what is the dimension of this Hom-space? When π or H is given arbitrarily, it is hard for such a spectral decomposition to be well understood, and those questions may not have reasonable answers.

When G is a Lie group or more generally a locally compact topological group defined by a reductive algebraic group, one may seek geometric conditions on the pair (G, H) such that the multiplicity $m(\pi, \sigma)$, which is the dimension of the Hom-space in (1-1), is bounded and at most one. In such a circumstance, one may seek invariants attached to π and σ that detect the multiplicity $m(\pi, \sigma)$. The local Gan–Gross–Prasad conjecture [2012] for classical groups G defined over a local field F is one of the most successful examples concerning those general questions. When the local field F is a finite extension of the p-adic number field $\mathbb{Q}_p$ for some prime p, the local Gan–Gross–Prasad conjecture for orthogonal groups has been completely resolved by the work of J.-L. Waldspurger [2010; 2012a; 2012b] and of C. Mœglin and Waldspurger [2012].

One of the basic notions in the local Gan–Gross–Prasad conjecture for orthogonal groups G are the so called *Bessel models*. Over a p-adic local field F, Bessel models are defined in terms of a special family of twisted Jacquet functors. It is proved through the work of Aizenbud et al. [2010], Sun and Zhu [2012], Gan et al. [2012], and Jiang et al. [2010b] that the Bessel models over any local fields of characteristic zero are of multiplicity at most one. The local Gan–Gross–Prasad conjecture is to detect the multiplicity (which is either 1 or 0) in terms of the sign of the relevant local ε-factors.

Meanwhile, the Bessel models have been widely used in the theory of the Rankin–Selberg method to study families of automorphic L-functions and to define the corresponding local L-factors and local γ-factors. In terms of representation theory and the local Langlands functoriality, the Bessel models produce the local descent method, which has been successfully used in the explicit construction of certain local Langlands functorial transfers for classical groups [Jiang and Soudry 2003; 2012]. In the spirit of the Bernstein–Zelevinsky derivatives for irreducible admissible representations of general linear groups over p-adic local fields [Bernstein and Zelevinsky 1977], the Bessel models can be regarded as a tool to investigate basic properties of irreducible admissible representations of $G(F)$ in general.

For instance, if G is an odd special orthogonal group SO_{2n+1}, then the local descents constructed via the family of Bessel models may produce representations on the family of even special orthogonal groups SO_{2m}, whose F-ranks should be controlled (up to ± 1) by the F-rank δ of SO_{2n+1}, with $m = n-\delta, n-\delta+1, \dots, n-1, n$. When $m=n$, it is the restriction from SO_{2n+1} to SO_{2n}, which is the case of the classical problem of *symmetric breaking*. Hence the explicit spectral decomposition when a representation π of $\mathrm{SO}_{2n+1}(F)$ descends, via the twisted Jacquet functors of Bessel type, to $\mathrm{SO}_{2m}(F)$ is an interesting and important problem, and may be considered as a refinement of the local Gan–Gross–Prasad conjecture. One of the problems in our mind is to understand the spectral decomposition of the local descent for special orthogonal groups over p-adic local fields in terms of the local Langlands parameters.

We explain our approach below with more details. The method is applicable to other classical groups. In some cases, we have to replace the Bessel models by the Fourier–Jacobi models, following the formulation of the local Gan–Gross–Prasad conjecture in [Gan et al. 2012]. The connection of the results in this paper to automorphic forms is considered in the work of the authors [Jiang and Zhang 2015].

1A. ***Local descents.*** Let F be a nonarchimedean local field of characteristic zero, which is a finite extension of the p-adic number field $\mathbb{Q}_p$ for some prime p. As in [Jiang and Zhang 2015; Arthur 2013, Chapter 9], we use $G_n^* = \mathrm{SO}(V^*, q_{V^*}^*)$ to denote an F-quasisplit special orthogonal group that is defined by a nondegenerate, $\mathfrak{n}$-dimensional quadratic space (V^*, q^*) over F with $n = \left[\frac{\mathfrak{n}}{2}\right]$ and use $G_n = \mathrm{SO}(V, q)$ to denote a pure inner F-form of G_n^*. This means that both quadratic spaces (V^*, q^*) and (V, q) have the same dimension and the same discriminant, as discussed in [Gan et al. 2012], for instance.

Let $\Pi(G_n)$ be the set of equivalence classes of irreducible smooth representations of $G_n(F)$. It is well-known that any $\pi \in \Pi(G_n)$ is also admissible. Let $\mathfrak{r}$ be the F-rank of G_n. Take X^+ to be an $\mathfrak{r}$-dimensional totally isotropic subspace of (V, q), and take X^- to be the dual subspace of X^+. Then one has a polar decomposition of (V, q): $V = X^- \oplus V_0 \oplus X^+$, where (V_0, q_{V_0}) is the F-anisotropic kernel of (V, q). With a suitable choice of the order of the dual bases in X^- and X^+, one must have a minimal parabolic subgroup P_0 of G_n, whose unipotent radical can be realized in the upper triangular matrix form. For any standard F-parabolic subgroup $P = MN$, containing P_0, of G_n, take a character ψ_N of the unipotent radical $N(F)$ of $P(F)$, which is defined through a nontrivial additive character ψ_F of F. One may define the *twisted Jacquet module* for any $\pi \in \Pi(G_n)$ with respect to (N, ψ_N) to be the quotient

$$\mathcal{J}_{N,\psi_N}(V_\pi) := V/V(N, \psi_N),$$

where $V(N, \psi_N)$ is the span of the subset

$$\{\pi(n)v - \psi_N(n)v \mid \forall v \in V_\pi, \forall n \in N(F)\}.$$

Let M_{ψ_N} is the stabilizer of ψ_N in M. Then the twisted Jacquet module $\mathcal{J}_{N,\psi_N}(V_\pi)$ is a smooth representation of $M_{\psi_N}(F)$. In such a generality, one may not have much information about the twisted Jacquet module $\mathcal{J}_{N,\psi_N}(V_\pi)$, as a representation of $M_{\psi_N}(F)$. Following the inspiration of the Bernstein–Zelevinsky theory of derivatives for representations of p-adic GL_n [Bernstein and Zelevinsky 1977], the theory of the local descents is to obtain more explicit information about the twisted Jacquet module $\mathcal{J}_{N,\psi_N}(V_\pi)$ in terms of the given π and its local Langlands parameter, for a family of specially chosen data (N, ψ_N).

To introduce the twisted Jacquet modules of Bessel type, we take a family of partitions of the form:

$$\underline{p}_\ell := [(2\ell+1)1^{\mathfrak{n}-2\ell-1}], \tag{1-2}$$

with $0 \le \ell \le \mathfrak{r}$. Those partitions $\underline{p}_\ell$ are G_n*-relevant* in the sense that they correspond to F-rational unipotent orbits of $G_n(F)$. As in [Jiang and Zhang 2015], the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ corresponding to the partition $\underline{p}_\ell$ defines a unipotent subgroup $V_{\underline{p}_\ell}$ of G_n over F, and each F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ defines a *generic* character $\psi_{\mathcal{O}_\ell}$ of $V_{\underline{p}_\ell}(F)$.

More precisely, let $\{e_{\pm 1}, e_{\pm 2}, \dots, e_{\pm \mathfrak{r}}\}$ be a basis of $X^{\pm}$, respectively such that $q(e_i, e_{-j}) = \delta_{i,j}$ for all $1 \le i, j \le \mathfrak{r}$. Then we may choose the minimal parabolic subgroup P_0 to fix the following totally isotropic flag in (V, q):

$$V_1^+ \subset V_2^+ \subset \cdots \subset V_{\mathfrak{r}}^+ \quad \text{where } V_i^{\pm} = \mathrm{Span}\{e_{\pm 1}, \dots, e_{\pm i}\}. \tag{1-3}$$

For the partition $\underline{p}_\ell$ in (1-2), we consider the standard parabolic subgroup $P_{1^\ell} = M_{1^\ell} N_{1^\ell}$, containing P_0, with the Levi subgroup $M_{1^\ell} \cong \mathrm{GL}_1^{\times \ell} \times G_{n-\ell}$ and $V_{\underline{p}_\ell} = N_{1^\ell}$. Here $V_{\underline{p}_\ell}$ consists of elements of form:

$$V_{\underline{p}_\ell} = \left\{ v = \begin{pmatrix} z & y & x \\ & I_{\mathfrak{n}-2\ell} & y' \\ & & z^* \end{pmatrix} \in G_n : z \in Z_\ell \right\}, \tag{1-4}$$

where Z_ℓ is the standard maximal (upper-triangular) unipotent subgroup of GL_ℓ. Then the F-rational nilpotent orbits $\mathcal{O}_\ell$ in the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ correspond to the $\mathrm{GL}_1(F) \times G_{n-\ell}(F)$-orbits of F-anisotropic vectors in $(F^{\mathfrak{n}-2\ell}, q)$. The generic character $\psi_{\mathcal{O}_\ell}$ of $V_{\underline{p}_\ell}(F)$ may be explicitly defined as follows: Fix a nontrivial additive character ψ_F of F. For an anisotropic vector w_0 in $((V_\ell^+ \oplus V_\ell^-)^\perp, q)$ associated to the F-rational orbit $\mathcal{O}_\ell$ in $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$, define a character ψ_{ℓ, w_0} of $V_{\underline{p}_\ell}(F)$ by

$$\psi_{\mathcal{O}_\ell}(v) = \psi_{\ell, w_0} := \psi\left(\sum_{i=1}^{\ell-1} z_{i,i+1} + q(y_\ell, w_0) \right) \tag{1-5}$$

where $z_{i,j}$ is the entry of the matrix z in the i-th row and j-th column and y_ℓ is the last row of the matrix y in (1-4).

The Levi subgroup M_{1^ℓ} acts on the set of those generic characters ψ_{ℓ, w_0} via the adjoint action on $V_{\underline{p}_\ell}$. We denote by $G_n^{\mathcal{O}_\ell}$ the identity component of the stabilizer in M_{1^ℓ} of the character $\psi_{\mathcal{O}_\ell}$, viewed as a subgroup of $G_{n-\ell}$. By Proposition 2.5 of [Jiang and Zhang 2015], the algebraic group $G_n^{\mathcal{O}_\ell}$ is a special orthogonal group defined over F by a nondegenerate quadratic subspace (W_ℓ, q) of (V, q) with dimension $\mathfrak{n} - 2\ell - 1$. Here if $\psi_{\mathcal{O}_\ell}$ is of form ψ_{ℓ, w_0}, we have that

$$W_\ell = (V_\ell^+ \oplus \mathrm{Span}\{w_0\} \oplus V_\ell^-)^\perp \tag{1-6}$$

and $G_n^{\mathcal{O}_\ell}$ can be identified as the special orthogonal group $\mathrm{SO}(W_\ell, q)$. We refer to [Jiang and Zhang 2015, Proposition 2.5] for more structures of $G_n^{\mathcal{O}_\ell}$.

We define the following subgroup of G_n, which is a semidirect product,

$$R_{\mathcal{O}_\ell} := G_n^{\mathcal{O}_\ell} \ltimes V_{\underline{p}_\ell}. \tag{1-7}$$

For any $\pi \in \Pi(G_n)$, the twisted Jacquet module with respect to the pair $(V_{\underline{p}_\ell}, \psi_{\mathcal{O}_\ell})$ is defined by

$$\mathcal{J}_{\mathcal{O}_\ell}(\pi) = \mathcal{J}_{\mathcal{O}_\ell}(V_\pi) = \mathcal{J}_{V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell}}(V_\pi) := V_\pi / V_\pi(V_{\underline{p}_\ell}, \psi_{\mathcal{O}_\ell}), \tag{1-8}$$

which may also be called the *twisted Jacquet module of Bessel type* of π.

For an irreducible admissible representation σ of $G_n^{\mathcal{O}_\ell}(F)$, the linear functionals that belong to the following Hom-space

$$\mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}), \tag{1-9}$$

where $\sigma^\vee$ is the admissible dual of σ, are called the *local Bessel functionals* for the pair (π, σ). The uniqueness of local Bessel functionals asserts that

$$\dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}) \leq 1. \tag{1-10}$$

This was proved in [Aizenbud et al. 2010; Sun and Zhu 2012; Gan et al. 2012; Jiang et al. 2010b]. It is clear that

$$\mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}) \cong \mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma).$$

It is a natural problem to understand possible irreducible quotients σ of the module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ of $G_n^{\mathcal{O}_\ell}(F)$. The local Gan–Gross–Prasad conjecture determines such a quotient σ by means of the sign of the epsilon factor for the pair (π, σ). One of the main results of this paper is to determine all possible quotients σ for a given π with explicit description of their local Langlands parameters (Theorem 1.7), when the index ℓ is the first occurrence index (given in Definition 1.3).

In order to understand all possible irreducible quotients, we introduce a notion of the ℓ-th *maximal quotient* of π if $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero. Define

$$\mathcal{S}_{\mathcal{O}_\ell}(\pi) := \cap_{\mathfrak{l}_\sigma} \ker(\mathfrak{l}_\sigma), \tag{1-11}$$

where the intersection is taken over all local Bessel functionals $\mathfrak{l}_\sigma$ in $\mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma)$ for all $\sigma \in \Pi(G_n^{\mathcal{O}_\ell})$. The ℓ-th maximal quotient of π is defined to be

$$\mathcal{Q}_{\mathcal{O}_\ell}(\pi) := \mathcal{J}_{\mathcal{O}_\ell}(\pi)/\mathcal{S}_{\mathcal{O}_\ell}(\pi). \tag{1-12}$$

Of course, if $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is zero, we define $\mathcal{Q}_{\mathcal{O}_\ell}(\pi)$ to be zero. In this paper, we study this quotient for irreducible admissible representations π of $G_n(F)$ with generic local L-parameters, the definition of which will be explicitly given in Section 2A. Since we mainly discuss the local situation, we may call the local L-parameters *the L-parameters* for simplicity.

Proposition 1.1. *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, if the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero, then there exists a $\sigma \in \Pi(G_n^{\mathcal{O}_\ell})$ such that*

$$\mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma) \neq 0.$$

Namely, the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ of π is nonzero if and only if the ℓ-th maximal quotient $\mathcal{Q}_{\mathcal{O}_\ell}(\pi)$ of π is nonzero.

Proposition 1.1 follows from Lemma 3.1 in Section 3. By Proposition 1.1, if $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero, then the ℓ-th maximal quotient $\mathcal{Q}_{\mathcal{O}_\ell}(\pi)$ is nonzero. In this situation, we set

$$\mathcal{D}_{\mathcal{O}_\ell}(\pi) := \mathcal{Q}_{\mathcal{O}_\ell}(\pi) \tag{1-13}$$

and call $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ the ℓ-th *local descent* of π (with respect to the given F-rational orbit $\mathcal{O}_\ell$). Note that the group $G_n^{\mathcal{O}_\ell}(F)$ and the representation $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ depend on the F-rational structure of orbit $\mathcal{O}_\ell$.

The *theory of the local descents* is to understand the structure of the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ as a representation of $G_n^{\mathcal{O}_\ell}(F)$, in particular, in the situation when ℓ is the first occurrence index. In order to define the notion of the *first occurrence*, we prove the following *stability* of the local descents.

Proposition 1.2. *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, if there exists an ℓ_1 such that the ℓ_1-th local descent $\mathcal{D}_{\mathcal{O}_{\ell_1}}(\pi)$ is nonzero for some F-rational orbit $\mathcal{O}_{\ell_1}$, then the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ is nonzero for every $\ell \leq \ell_1$ with a certain compatible $\mathcal{O}_\ell$.*

The details of the compatibility of $\mathcal{O}_{\ell_1}$ and $\mathcal{O}_\ell$ will be given in Proposition 3.3. Proposition 1.2 follows essentially from the relation between multiplicity and parabolic induction as discussed in Section 2D and will be included in Proposition 3.3 on the stability of the local descents in Section 3.

Definition 1.3 (first occurrence index). For $\pi \in \Pi(G_n)$, the first occurrence index $\ell_0 = \ell_0(\pi)$ of π is the integer ℓ_0 in $\{0, 1, \dots, \mathfrak{r}\}$ where $\mathfrak{r}$ is the F-rank of G_n, such that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\ell_0}}(\pi)$ is nonzero for some F-rational orbit $\mathcal{O}_{\ell_0}$, but for any $\ell \in \{0, 1, \dots, \mathfrak{r}\}$ with $\ell > \ell_0$, the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is zero for every F-rational orbit $\mathcal{O}_\ell$ associated to the partition $\underline{p}_\ell$ as defined in (1-2).

It is clear that the definition of the first occurrence index is also applicable to the representations π that may not be irreducible. At the first occurrence index, we define the notion of the local descent of π.

Definition 1.4 (local descent). For any given $\pi \in \Pi(G_n)$, assume that ℓ_0 is the first occurrence index of π. The ℓ_0-th local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ of π is called the first local descent of π (with respect to $\mathcal{O}_{\ell_0}$) or simply the local descent of π, which is the ℓ_0-th nonzero maximal quotient

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) := \mathcal{Q}_{\mathcal{O}_{\ell_0}}(\pi)$$

for the F-rational orbit $\mathcal{O}_{\ell_0}$ associated to the partition $\underline{p}_{\ell_0}$ as defined in (1-2).

It is clear that such an $\mathcal{O}_{\ell_0}$ always exists by the definition of ℓ_0, but may not be unique. Also, when $G_n = G_n^*$ is F-quasisplit and π is generic, i.e., has a nonzero Whittaker model, the first occurrence index is clearly $\ell_0 = n = \left[\frac{\mathfrak{n}}{2}\right]$, where $\mathfrak{n} = \dim V^*$. The discussion related to the first occurrence index in this paper will exclude this trivial case.

1B. ***Main results.*** The main results of the paper are about the spectral properties of the ℓ-th local descents of the irreducible smooth representations of $G_n(F)$ with generic L-parameters. At the first occurrence index, the spectral properties of the local descents are explicit.

Theorem 1.5 (square integrability). *Assume that $\pi \in \Pi(G_n)$ has a generic L-parameter. Then, at the first occurrence index $\ell_0 = \ell_0(\pi)$ of π, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is square-integrable and admissible. Moreover, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a multiplicity-free direct sum of irreducible square-integrable representations, with all its irreducible summands belonging to different Bernstein components.*

The proof of Theorem 1.5 depends on the following weaker result about the ℓ-th local descent for general ℓ. Because it can be deduced from the work on the local Gan–Gross–Prasad conjecture for orthogonal groups of Waldspurger [2010; 2012a; 2012b] for tempered L-parameters, and of Mœglin and Waldspurger [2012] with generic L-parameters, we state it as a corollary and refer to Section 3 for the details.

Proposition 1.6 (irreducible tempered quotient). *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, and for any $\ell \in \{0, 1, \dots, \mathfrak{r}\}$ with $\mathfrak{r}$ being the F-rank of G_n, if the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero, then $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ has a tempered irreducible quotient as a representation of $G_n^{\mathcal{O}_\ell}(F)$, and so does the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$. In other words, if $\mathcal{J}_{\mathcal{O}_\ell}(\pi) \neq 0$, then there exists a $\sigma \in \Pi_{\mathrm{temp}}(G_n^{\mathcal{O}_\ell})$ such that*

$$\mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma) = \mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{D}_{\mathcal{O}_\ell}(\pi), \sigma) \neq 0.$$

It is worthwhile to mention an analogy of Proposition 1.6 to the *generic summand conjecture* (Conjecture 2.3 in [Jiang and Zhang 2015] and Conjecture 2.4 in [Jiang and Zhang 2017]). In such generality, if the representation π in Proposition 1.6 is unramified, then for any index ℓ, the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ has a tempered, unramified irreducible quotient. While the generic summand conjecture [Jiang and Zhang 2015; 2017] asserts that for any irreducible cuspidal automorphic representation of G_n with a generic global Arthur parameter, its global descent at the first occurrence index is cuspidal and contains at least one irreducible summand that has a generic global Arthur parameter. This assertion serves as a base for the construction of explicit modules for irreducible cuspidal automorphic representations of general classical groups with generic global Arthur parameters as developed in [Jiang and Zhang 2015; 2017]. We will discuss the global impact of the results obtained in this paper to the generic summand conjecture in our future work.

From the setup of the local descents, the theory is closely related to the local Gan–Gross–Prasad conjecture. By applying the theorems of Mœglin and Waldspurger on the local Gan–Gross–Prasad conjecture for generic L-parameters, and by explicit calculations of local L-parameters and the relevant local root numbers via the local Langlands correspondence in the situation considered here, we are able to obtain the following explicit spectral decomposition for the local descent at the first occurrence index. In order to state the result, we briefly explain the notation used in Theorem 1.7 below, and leave the details for Sections 2 and 5.

Write $\mathcal{Z} := F^\times/F^{\times 2}$. After fixing a rationality of the local Langlands correspondence ι_a as in Section 2B, when a $\pi \in \Pi(G_n)$ is determined by the parameter (φ, χ), the abelian group $\mathcal{Z}$ acts on the dual $\hat{\mathcal{S}}_\varphi$ of $\mathcal{S}_\varphi$. Denote by $\mathcal{O}_{\mathcal{Z}}(\pi)$ the $\mathcal{Z}$-orbit in $\hat{\mathcal{S}}_\varphi$ determined by π (see (2-17)). Denote by $[\varphi]_c$ the pair of the local L-parameters which are conjugate to each other via an element c in the complex group $\mathrm{O}(V)(\mathbb{C})$ with $\det(c) = -1$. In Definition 4.1, we introduce the notion of the descent $\mathfrak{D}_\ell(\varphi, \chi)$ and that of the first occurrence index $\ell_0 = \ell_0(\varphi, \chi)$ for the local parameters (φ, χ). The basic structure of the descent $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ at the first occurrence index is given in Theorem 4.6.

Theorem 1.7 (spectral decomposition). *Assume that $\pi \in \Pi(G_n(V_{\mathfrak{n}}))$ is associated to an equivalence class $[\varphi]_c$ of generic L-parameters.*

(1) *The first occurrence index of π is determined by the first occurrence index of the local parameters via the formula*
$$\ell_0(\pi) = \max_{\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)} \{\ell_0([\varphi]_c, \chi)\}.$$

(2) *For each F-rational orbit $\mathcal{O}_{\ell_0}$, the local descent of π at the first occurrence index $\ell_0 = \ell_0(\pi)$ is a multiplicity-free, direct sum of irreducible, square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$, which can be explicitly given below.*

   (a) *When $\mathfrak{n} = 2n$ is even and for $\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)$*
$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \bigoplus_{\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi)} \pi(\phi, \chi^{\star}_{\phi}(\varphi, \phi)),$$
   where the local Langlands correspondence ι_a for π is given by $a = \operatorname{disc}(\mathcal{O}_{\ell_0})$ and $\chi_a(\pi) = \chi$, and the quadratic space W defining $G_n^{\mathcal{O}_{\ell_0}}$ is given by $\operatorname{disc}(W) = -\operatorname{disc}(\mathcal{O}_{\ell_0}) \cdot \operatorname{disc}(V_{\mathfrak{n}})$ and (2-25).

   (b) *When $\mathfrak{n} = 2n + 1$ is odd,*
$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \bigoplus_{\substack{\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi) \\ \det(\phi) = \operatorname{disc}(\mathcal{O}_{\ell_0}) \cdot \operatorname{disc}(V_{\mathfrak{n}})}} \pi_{-\operatorname{disc}(\mathcal{O}_{\ell_0})}(\phi, \chi^{\star}_{\phi}(\varphi, \phi)),$$
   where the quadratic space defining $G_n^{\mathcal{O}_{\ell_0}}$ is given by (2-24) *and* (2-26).

Theorems 1.5 and 1.7 will be proved in Section 5, not only using the result of Proposition 1.6, but also using the proof of Proposition 1.6 in Section 3, with refinement. Moreover, in order to keep tracking the behavior of the local L-parameters and the sign of the local root numbers with the local descents, we need explicit information about the descent of local L-parameters $\mathfrak{D}_{\ell_0}(\varphi, \chi)$, given in Theorem 4.6.

We note that it is possible that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = 0$ for some F-rational orbit $\mathcal{O}_{\ell_0}$. But there exists at least one F-rational orbit $\mathcal{O}_{\ell_0}$ such that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) \neq 0$. Also an explicit formula for the character $\chi^{\star}_{\phi}(\varphi, \phi)$ can be found in Corollary 5.4.

Some more comments on Theorem 1.7 are in order.

First of all, in terms of the local Gan–Gross–Prasad conjecture, the spectral decomposition as given in Theorem 1.7 can be interpreted as follows: For any $\pi \in \Pi(G_n)$ with a generic L-parameter, at the first occurrence index, the spectral decomposition explicitly determines in terms of the local Langlands data of all possible irreducible representations σ of $G_n^{\mathcal{O}_{\ell_0}}(F)$ that form the distinguished pair with the given π as required by the local Gan–Gross–Prasad conjecture. Meanwhile, this spectral decomposition indicates that for such a given π, if a $\sigma \in \Pi(G_n^{\mathcal{O}_{\ell_0}})$ can be paired with π as in the local Gan–Gross–Prasad conjecture, then σ must be square integrable. Hence Theorem 1.7 and Corollary 5.4 can be regarded as a refinement of the local Gan–Gross–Prasad conjecture.

Secondly, it is interesting to compare briefly Theorem 1.7 with the local descent of the first named author and Soudry [Jiang and Soudry 2003; 2012], see also [Jiang et al. 2010a]. For instance, one takes G_n^* to be the F-split SO_{2n+1}. Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(F)$, which is of symplectic type, i.e., the local exterior square L-function of τ has a pole at $s=0$. The local descent in [Jiang and Soudry 2003; Jiang et al. 2010a] is to take $\pi=\pi(\tau,2)$ to be the unique Langlands quotient of the induced representation of the F-split $\mathrm{SO}_{4n}(F)$ with supercuspidal support (GL_{2n},τ). According to the endoscopic classification of Arthur [2013], $\pi=\pi(\tau,2)$ has a nongeneric (nontempered) local Arthur parameter $(\tau,1,2)$, and has the local L-parameter $\phi_\tau|\cdot|^{\frac{1}{2}}\oplus\phi_\tau|\cdot|^{-\frac{1}{2}}$, where ϕ_τ is the local L-parameter of τ. Now the local descent in [Jiang and Soudry 2003; Jiang et al. 2010a] shows that $\mathcal{D}_{n-1}(\pi)$ with $\ell_0=n-1$ being the first occurrence index is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(F)$ with the generic local Arthur parameter $(\tau,1,1)$ or the local L-parameter ϕ_τ. In this case, τ is the image of $\mathcal{D}_{n-1}(\pi)$ under the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}. The result is even simpler than Theorem 1.7, as expected. However, from the point of view of the local Gan–Gross–Prasad conjecture, the result in [Jiang and Soudry 2003; Jiang et al. 2010a] can be viewed as a case of the local Gan–Gross–Prasad conjecture for nontempered local Arthur parameters. Hence the work to extend Theorem 1.7 to the representations with general local Arthur parameters is closely related to the local Gan–Gross–Prasad conjecture for nontempered local Arthur parameters. This is definitely a very interesting topic, but we will not discuss it with any more details in this paper.

Finally, we would like to elaborate an application of Theorem 1.7. For a generic local L-parameter ϕ of an F-quasisplit G_n^*, the local L-packet $\Pi_\phi(G_n^*)$ as defined in [Mœglin and Waldspurger 2012] contains a generic member, i.e., a member with a nonzero Whittaker model. From the relation between unipotent orbits of $G_n^*(F)$ and the twisted Jacquet modules for $G_n^*(F)$. The Whittaker model corresponds to the twisted Jacquet module associated to the regular unipotent orbit of G_n^*. In general, other members in the local L-packet $\Pi_\phi(G_n^*)$ may not have a nonzero twisted Jacquet module associated to the regular unipotent orbit. Hence it is desirable to know that for any member π in the local L-packet $\Pi_\phi(G_n^*)$, what kind twisted Jacquet modules does π have?

Let $\mathcal{P}(G_n^*)$ be the set of orthogonal partitions $\underline{p}=[p_1p_2\cdots p_r]$ associated to the F-stable unipotent orbits of $G_n^*(F)$. As defined in [Jiang 2014, Section 4] and similar to the definition of the twisted Jacquet modules of Bessel type, we may construct a twisted Jacquet module associated to any F-rational unipotent orbit $\mathcal{O}_{\underline{p}}$ in the corresponding F-stable orbit $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$. More precisely, we may construct, for any $\mathcal{O}_{\underline{p}}$ in $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$, a unipotent subgroup $V_{\mathcal{O}_{\underline{p}}}$ of G_n^* and a character $\psi_{\mathcal{O}_{\underline{p}}}$, and define the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\underline{p}}}(\pi)$ for any irreducible smooth representation π of $G_n^*(F)$. Now, we define $\mathfrak{p}(\pi)$ to be the subset of $\mathcal{P}(G_n^*)$, consisting of partitions $\underline{p}$ with the property that there exists an F-rational $\mathcal{O}_{\underline{p}}$ in the F-stable $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$ such that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\underline{p}}}(\pi)$ is nonzero. Let $\mathfrak{p}^m(\pi)$ be the subset of $\mathfrak{p}(\pi)$ consisting of all maximal members in $\mathfrak{p}(\pi)$. Following [Kawanaka 1987; Mœglin 1996], one may take $\mathfrak{p}^m(\pi)$ to be the algebraic version of the *wave-front set* of π. It is generally believed [Jiang and Liu 2016, Conjecture 3.1] that if π is tempered, then the set $\mathfrak{p}^m(\pi)$ contains only one partition. One expects that this property holds for general π. Assume that $\mathfrak{p}^m(\pi)=\{\underline{p}=[p_1p_2\cdots p_r]\}$ with $p_1\geq p_2\geq\cdots\geq p_r>0$. In order to determine the algebraic wave-front set $\mathfrak{p}^m(\pi)$, it is an important step to understand how the largest part p_1 in the partition $\underline{p}\in\mathfrak{p}^m(\pi)$ is determined by the local Langlands data associated to π, via the local Langlands correspondence for $G_n^*(F)$. Here is our conjecture.

Conjecture 1.8. *Assume that $\pi\in\Pi_\phi(G_n^*)$ has a generic local L-parameter ϕ. Then the largest part p_1 in the partition $\underline{p}=[p_1p_2\cdots p_r]$ that belongs to $\mathfrak{p}^m(\pi)$ is equal to $2\ell_0+1$, where $\ell_0=\ell_0(\pi)$ is the first occurrence index in the local descents.*

Assume that Conjecture 1.8 holds. Then part (1) of Theorem 1.7 asserts that the largest part p_1 of the partition $\underline{p}$ in the algebraic wave-front set $\mathfrak{p}^m(\pi)$ of π is completely determined by the property of the local Langlands data associated to π, if π has a generic local L-parameter. From this point of view, Theorem 1.7 serves as a base of an induction argument to determine the remaining parts $p_2,\ldots,p_r$. In Section 6, we will discuss Conjecture 1.8 via some examples. However, we expect that the induction argument is long and complicate, and hence leave it to our future work. Waldspruger [2018] confirmed this conjecture for a special family of generic local L-parameters of special odd orthogonal groups.

1C. ***The structure of the proofs and the paper.*** The local Gan–Gross–Prasad conjecture as proved in [Waldspurger 2010; 2012a; 2012b; Mœglin and Waldspurger 2012] is the starting point and the technical backbone of this paper; we state it in Section 2. In order to understand the F-rationality of the local L-parameters and the F-rationality of the local descents, we reformulate the local Langlands correspondence and the local Gan–Gross–Prasad conjecture in terms of the basic rationality data given by the underlying quadratic forms and quadratic spaces. This is discussed in Section 2B. It is clear that one might formulate such rationality in terms of rigid inner forms as discussed by T. Kaletha [2016]. Due to the nature of the current paper, the authors thought it more convenient and direct to use the formulation in Section 2B. In addition to the local Langlands correspondence as proved by Arthur [2013], we need the result for even special orthogonal groups as discussed by H. Atobe and W. T. Gan [2017].

We start to prove Proposition 1.6 in Section 3. First we show (Lemma 3.1) that for any $\pi\in\Pi(G_n)$, if the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ of Bessel type is nonzero, then it has an irreducible quotient. Then by applying the relation of multiplicity with parabolic induction (Proposition 2.6 as proved by Mœglin and Waldspurger [2012]), we obtain (Corollary 3.2) the relation of the first occurrence index with parabolic induction and the result that for any $\pi\in\Pi(G_n)$ with generic L-parameters, every irreducible quotient of the local descent at the first occurrence index is square-integrable. It is clear that Corollary 3.2 is one step towards Theorem 1.5. Finally, Proposition 1.6 follows from the stability of local descents (Proposition 3.3) and its proof.

In order to prove Theorems 1.5 and 1.7, we have to work on the local L-parameters. We define in Section 4 the *descent of local L-parameters*. In order to explicitly determine the structure of the descent of local L-parameters, we first calculate explicitly the local epsilon factors associated to a local L-parameter or a pair of local L-parameters, and keep tracking the local Langlands data through the process of local descents. With help from the theorem of Mœglin and Waldspurger [2012] on the local Gan–Gross–Prasad conjecture for orthogonal groups with generic L-parameters and the explicit local Langlands correspondence, we undertake a long and tedious calculation of the characters that parametrize the distinguished pair of representations as given by the local Gan–Gross–Prasad conjecture and determine the descent of the local L-parameters. Theorem 4.6 describes explicitly all the local L-parameters occurring in the descent of the local L-parameter associated to the initially given representation π. The point is that all the local L-parameters occurring in the descent of local L-parameters are discrete local L-parameters.

With Theorem 4.6 in hand, we are able to determine in Section 5 the local L-parameters for all the irreducible quotients of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$. We then use the structure of the Bernstein components of the local descent to prove Theorem 5.2; the irreducible quotients of local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ belong to different Bernstein components and are square integrable. Hence the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a multiplicity free direct sum of irreducible square-integrable representations and hence is square-integrable, which is Theorem 1.5. Theorem 1.7, which gives an explicit spectral decomposition of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$, can now be deduced from Theorems 1.5 and 4.6.

In Section 6 we provide even more explicit results for two special families of generic local L-parameters: the local cuspidal L-parameters as discussed by A.-M. Aubert, A. Moussaoui, and M. Solleveld [2015] (where they are called the local cuspidal Langlands parameters) and the local discrete unipotent L-parameters that correspond to the discrete unipotent representations in the sense of G. Lusztig [1995]. Moreover, we discuss Conjecture 1.8 via examples.

2. On the local Gan–Gross–Prasad conjecture

2A. ***Generic local L-parameters and local Vogan packets.*** We recall from [Mœglin and Waldspurger 2012] the notion of the generic local L-parameters and their structure. Without the assumption of the generalized Ramanujan conjecture, the localization of the generic global Arthur parameters [2013] will be examples of the generic local L-parameters defined and discussed in [Mœglin and Waldspurger 2012] for a p-adic local field F of characteristic zero. Following [loc. cit.], we denote by $\Phi_{\mathrm{gen}}(G_n^*)$ the set of conjugacy classes of the generic local L-parameters of G_n^*. We may simply call $\phi\in\Phi_{\mathrm{gen}}(G_n^*)$ the *generic L-parameters* since we only consider the local situation in this paper.

Now we recall from [loc. cit.] the definition of generic local L-parameters for orthogonal groups. We denote by $\mathcal{W}_F$ the local Weil group of F. The local Langlands group of F, which is denoted by $\mathcal{L}_F$, is equal to the local Weil–Deligne group $\mathcal{W}_F\times \mathrm{SL}_2(\mathbb{C})$. The local L-parameters for $G_n^*(F)$ are of the form

$$\phi:\mathcal{L}_F\to {}^LG_n^* \tag{2-1}$$

with the property that the restriction of ϕ to the local Weil group $\mathcal{W}_F$ is Frobenius semisimple and trivial on an open subgroup of the inertia group $\mathcal{I}_F$ of F, the restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic, and ϕ is compatible with the projections of $\mathcal{L}_F$ and ${}^LG_n^*$ to the Weil group $\mathcal{W}_F$ in the definition. Then, for each given local L-parameter ϕ, there exists a datum $(L^*,\phi^{L^*},\underline{\beta})$ such that:

(1) L^* is a Levi subgroup of $G_n^*(F)$ of the form

$$L^* = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t} \times G_{n_0}^*.$$

(2) ϕ^{L^*} is a local L-parameter of L^* given by

$$\phi^{L^*} := \phi_1 \oplus \cdots \oplus \phi_t \oplus \phi_0 : \mathcal{L}_F \to {}^L L^*,$$

where ϕ_j is a local tempered L-parameter of GL_{n_j} for $j = 1, 2, \ldots, t$, and ϕ_0 is a local tempered L-parameter of $G_{n_0}^*$.

(3) $\underline{\beta} := (\beta_1, \ldots, \beta_t) \in \mathbb{R}^t$, such that $\beta_1 > \beta_2 > \cdots > \beta_t > 0$.

(4) The parameter ϕ can be expressed as

$$\phi = (\phi_1 \otimes |\cdot|^{\beta_1} \oplus \phi_1^\vee \otimes |\cdot|^{-\beta_1}) \oplus \cdots \oplus (\phi_t \otimes |\cdot|^{\beta_t} \oplus \phi_t^\vee \otimes |\cdot|^{-\beta_t}) \oplus \phi_0.$$

Following [Arthur 2013; Mœglin and Waldspurger 2012], the local L-packets can be formed for all L-parameters ϕ as displayed above, and are denoted by $\Pi_\phi(G_n^*)$. Now a local L-parameter ϕ is called *generic* if the associated local L-packet $\Pi_\phi(G_n^*)$ contains a generic member, i.e., a member with a nonzero Whittaker model with respect to a certain Whittaker data for G_n^*. The set of all generic local L-parameters of G_n^* is denoted by $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$. It was proved in [Mœglin and Waldspurger 2012] that all the members in such local L-packets are given by irreducible standard modules. Note that the situation here is more general than that considered in [Arthur 2013] and hence the members in a generic local L-packet may not be unitary. By [Mœglin and Waldspurger 2012], the local Gan–Gross–Prasad conjecture, which we call the local GGP conjecture for short, holds for all generic L-parameters $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$.

Recall that an F-quasisplit special orthogonal group $G_n^* = \mathrm{SO}(V^*, q^*)$ and its pure inner F-forms $G_n = \mathrm{SO}(V, q)$ share the same L-group ${}^L G_n^*$. As explained in [Gan et al. 2012, §7], if the dimension $\mathfrak{n} = \dim V = \dim V^*$ is odd, one may take $\mathrm{Sp}_{\mathfrak{n}-1}(\mathbb{C})$ to be the L-group ${}^L G_n^*$, and if the dimension $\mathfrak{n} = \dim V = \dim V^*$ is even, one may take $\mathrm{O}_{\mathfrak{n}}(\mathbb{C})$ to be ${}^L G_n^*$ when $\mathrm{disc}(V^*)$ is not a square in $F^\times$ and take $\mathrm{SO}_{\mathfrak{n}}(\mathbb{C})$ to be ${}^L G_n^*$ when $\mathrm{disc}(V^*)$ is a square in $F^\times$. Let S_ϕ be the centralizer of the image of ϕ in $\mathrm{SO}_{\mathfrak{n}}(\mathbb{C})$ or $\mathrm{Sp}_{\mathfrak{n}-1}(\mathbb{C})$, and S_ϕ° be its identity connected component group. Define the component group $\mathcal{S}_\phi := S_\phi / S_\phi^\circ$, which is an abelian 2-group.

By Theorem 1.5.1 of [Arthur 2013], and its extension to the generic L-parameters in $\Phi_{\mathrm{gen}}(G_n^*)$ in [Mœglin and Waldspurger 2012], the local L-packets $\Pi_\phi(G_n^*)$ are of multiplicity free and there exists a bijection

$$\pi \mapsto <\cdot, \pi> = \chi_\pi(\cdot) = \chi(\cdot) \tag{2-2}$$

between the finite set $\Pi_\phi(G_n^*)$ and the dual of $\mathcal{S}_\phi / Z({}^L G_n^*)$ of the finite abelian 2-group $\mathcal{S}_\phi$ associated to ϕ. Following the Whittaker normalization of Arthur, the trivial character χ corresponds to a generic member in the local L-packet $\Pi_\phi(G_n^*)$ with a chosen Whittaker character. There is an F-rationality issue on which the bijection may depend. We will discuss this issue explicitly in Section 2B. Under the bijection in (2-2), we may write

$$\pi = \pi(\phi, \chi) \tag{2-3}$$

in a unique way for each member $\pi \in \Pi_\phi(G_n^*)$ with $\chi = \chi_\pi$ as in (2-2). For any pure inner F-forms G_n of G_n^*, the same formulation works [Arthur 2013; Kaletha 2016]. The local Vogan packet associated to any generic L-parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$ is defined to be

$$\Pi_\phi[G_n^*] := \bigcup_{G_n} \Pi_\phi(G_n), \tag{2-4}$$

where G_n runs over all pure inner F-forms of the given F-quasisplit G_n^*. The L-packet $\Pi_\phi(G_n)$ is defined to be *empty* if the parameter ϕ is not G_n-relevant.

According to the structure of the generic L-parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$, one may easily figure out the structure of the abelian 2-group $\mathcal{S}_\phi$. Write

$$\phi = \bigoplus_{i \in \mathrm{I}} m_i \phi_i, \tag{2-5}$$

which is the decomposition of ϕ into simple and generic ones. The simple, generic local L-parameter ϕ_i can be written as $\rho_i \boxtimes \mu_{b_i}$, where ρ_i is an a_i-dimensional irreducible representation of $\mathcal{W}_F$ and μ_{b_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension b_i. We denote by $\Phi_{\mathrm{sg}}(G_n^*)$ the set of all simple, generic local L-parameters of G_n^*. In the decomposition (2-5), ϕ_i is called *of good parity* if $\phi_i \in \Phi_{\mathrm{sg}}(G_{n_i}^*)$ with $G_{n_i}^*$ being the same type as G_n^* where $n_i := \left[\frac{a_i b_i}{2}\right]$. We denote by I_{gp} the subset of I consisting of indices i such that ϕ_i is of good parity and by I_{bp} the subset of I consisting of indices i such that ϕ_i is self-dual, but not of good parity. We set $\mathrm{I}_{\mathrm{nsd}} := \mathrm{I} - (\mathrm{I}_{\mathrm{gp}} \cup \mathrm{I}_{\mathrm{bp}})$ for the indices of non-self-dual ϕ_i. Hence we may write $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$ in the following more explicit way:

$$\phi = \Big(\bigoplus_{i \in \mathrm{I}_{\mathrm{gp}}} m_i \phi_i\Big) \oplus \Big(\bigoplus_{j \in \mathrm{I}_{\mathrm{bp}}} 2m'_j \phi_j\Big) \oplus \Big(\bigoplus_{k \in \mathrm{I}_{\mathrm{nsd}}} m_k(\phi_k \oplus \phi_k^\vee)\Big), \tag{2-6}$$

where $2m'_j = m_j$ in (2-5) for $j \in \mathrm{I}_{\mathrm{bp}}$. According to this explicit decomposition, it is easy to know that

$$\mathcal{S}_\phi \cong \mathbb{Z}_2^{\#\mathrm{I}_{\mathrm{gp}}} \quad \text{or} \quad \mathbb{Z}_2^{\#\mathrm{I}_{\mathrm{gp}}-1}. \tag{2-7}$$

The latter case occurs if G_n is even orthogonal, and some orthogonal summand ϕ_i for $i \in \mathrm{I}_{\mathrm{gp}}$ has odd dimension.

In all cases, when G_n is even or odd orthogonal, for any $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$, we write elements of $\mathcal{S}_\phi$ in the following form

$$(e_i)_{i \in \mathrm{I}_{\mathrm{gp}}} \in \mathbb{Z}_2^{\#\mathrm{I}_{\mathrm{gp}}}, \quad \text{(or simply denoted by } (e_i)) \tag{2-8}$$

where each e_i corresponds to ϕ_i-component in the decomposition (2-6) for $i \in \mathrm{I}_{\mathrm{gp}}$. The component group $\mathrm{Cent}_{\mathrm{O_n}(\mathbb{C})}(\phi)/\mathrm{Cent}_{\mathrm{O_n}(\mathbb{C})}(\phi)^\circ$ is denoted by A_ϕ. Then $\mathcal{S}_\phi$ consists of elements in A_ϕ with determinant 1, which is a subgroup of index 1 or 2. Also write elements in A_ϕ of form (e_i) where $e_i \in \{0, 1\}$ corresponds to the ϕ_i-component in the decomposition (2-6) for $i \in \mathrm{I}_{\mathrm{gp}}$. When G_n is even orthogonal and some ϕ_i for $i \in \mathrm{I}_{\mathrm{gp}}$ has odd dimension, then $(e_i)_{i\in\mathrm{I}_{\mathrm{gp}}}$ is in $\mathcal{S}_\phi$ if and only if $\sum_{i\in\mathrm{I}_{\mathrm{gp}}} e_i \dim\phi_i$ is even.

An L-parameter ϕ is of orthogonal type or symplectic type if its image $\mathrm{Im}(\phi)$ lies in $\mathrm{O}_{\mathfrak{n}}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$, respectively. In this paper, a self-dual L-parameter refers to be of either orthogonal type or symplectic type. Let ρ be an irreducible smooth representation of $\mathcal{W}_F$, which is Frobenius semisimple and trivial on an open subgroup of the inertia group $\mathcal{I}_F$ of F. Similarly, ρ is of orthogonal type or symplectic type if $\rho\boxtimes 1$ is of orthogonal type or symplectic type, respectively. In this case, $\rho\boxtimes 1$ is a discrete L-parameter.

2B. ***Rationality and the local Langlands correspondence.*** As explained in Section 1B, one of our motivations is to study the algebraic version of the wave-front set $\mathfrak{p}^m(\pi)$ of π via this local descent method. Our main approach is to perform an induction argument on the parts of partitions $\underline{p}$ in $\mathfrak{p}^m(\pi)$. In this inductive argument, the representations are descended to the ones of special orthogonal groups with different parity alternatively. We need to keep tracking the F-rational nilpotent orbits $\mathcal{O}_\ell$, which give the nonzero local descents. Meanwhile, $\mathcal{O}_\ell$ also determines the quadratic forms of the descendant special orthogonal groups. On the other hand, one needs to fix a normalization of the Whittaker datum in order to fix the local Langlands correspondence. Such a normalization depends also on the quadratic forms of the special orthogonal groups. Hence, we will fix the normalization for the parent representations and track the F-rational forms $\mathcal{O}_\ell$ for their descendants, then the normalization for their descendants will be determined. In this sense, we refer those normalizations as the F-rationality of the local Langlands correspondence for $G_n = \mathrm{SO}(V_{\mathfrak{n}})$ in terms of the F-rationality of the underlying quadratic space $(V_{\mathfrak{n}}, q_{\mathfrak{n}})$. When more quadratic spaces get involved in the discussion, we denote by $q_{\mathfrak{n}}$ the quadratic form of $V_{\mathfrak{n}}$, which was simply denoted by q before.

Define the discriminant of the quadratic space $V_{\mathfrak{n}}$ by

$$\mathrm{disc}(V_{\mathfrak{n}}) = (-1)^{\mathfrak{n}(\mathfrak{n}-1)/2}\det(V_{\mathfrak{n}}) \in F^\times/F^{\times 2}$$

and, similar to [O'Meara 2000, p. 167], define the Hasse invariant of $V_{\mathfrak{n}}$ by

$$\mathrm{Hss}(V_{\mathfrak{n}}) = \prod_{1\le i\le j\le \mathfrak{n}} (\alpha_i, \alpha_j),$$

where $V_{\mathfrak{n}}$ is decomposed orthogonally as $Fv_1 \oplus Fv_2 \oplus \cdots \oplus Fv_{\mathfrak{n}}$ with $q_{\mathfrak{n}}(v_i, v_i) = \alpha_i \in F^\times$. According to this definition, if $V_{\mathfrak{n}}$ is decomposed orthogonally as $V_{\mathfrak{n}} = W \oplus U$, we have, by [O'Meara 2000, Remark 58:3], the following formulas:

$$\mathrm{disc}(V_{\mathfrak{n}}) = (-1)^{ab}\,\mathrm{disc}(W)\,\mathrm{disc}(U) \tag{2-9}$$

$$\mathrm{Hss}(V_{\mathfrak{n}}) = \mathrm{Hss}(W)\,\mathrm{Hss}(U)((-1)^{a(a-1)/2}\,\mathrm{disc}(W), (-1)^{b(b-1)/2}\,\mathrm{disc}(U)) \tag{2-10}$$

where $a = \dim W$ and $b = \dim U$.

Recall from Section 1A that for each F-rational orbit $\mathcal{O}_\ell$ the nondegenerate character $\psi_{\mathcal{O}_\ell}$ is given by ψ_{ℓ,w_0} defined in (1-5) where w_0 is an anisotropic vector w_0. Define

$$\operatorname{disc}(\mathcal{O}_\ell) := \operatorname{disc}(V_\ell^+ \oplus Fw_0 \oplus V_\ell^-) \quad \text{and} \quad \operatorname{Hss}(\mathcal{O}_\ell) := \operatorname{Hss}(V_\ell^+ \oplus Fw_0 \oplus V_\ell^-),$$

where $V_\ell^\pm$ is defined in (1-3). Since the quadratic space $V_\ell^+ \oplus Fw_0 \oplus V_\ell^-$ is split, one has

$$\operatorname{disc}(\mathcal{O}_\ell) = q_{\mathfrak{n}}(w_0, w_0) \quad \text{and} \quad \operatorname{Hss}(\mathcal{O}_\ell) = (-1,-1)^{\ell(\ell+1)/2}((-1)^{\ell+1}, \operatorname{disc}(\mathcal{O}_\ell)).$$

Let W_ℓ be defined in (1-6). We have the decomposition

$$V_{\mathfrak{n}} = (V_\ell^+ \oplus Fw_0 \oplus V_\ell^-) \oplus W_\ell. \tag{2-11}$$

Then one has $\operatorname{disc}(W_\ell) = (-1)^{\mathfrak{n}-1} \operatorname{disc}(V_{\mathfrak{n}}) \operatorname{disc}(\mathcal{O}_\ell)$, and it is easy to check that

$$\operatorname{Hss}(W_\ell) = (-1,-1)^{\ell(\ell+1)/2} \operatorname{Hss}(V_{\mathfrak{n}})((-1)^\ell \operatorname{disc}(\mathcal{O}_\ell), (-1)^{\mathfrak{n}(\mathfrak{n}-1)/2+\ell} \operatorname{disc}(V_{\mathfrak{n}})). \tag{2-12}$$

In order to fix the F-rationality of the local Langlands correspondence, we adopt the (QD) *condition* of Waldspurger [2012a, p. 119] for the even special orthogonal group $(V_{\mathfrak{n}}, q_{\mathfrak{n}})$:

(QD) *The special orthogonal group of* $(V_{\mathfrak{n}}, q_{\mathfrak{n}}) \oplus (F, q_0)$ *is split, where* (F, q_0) *is the one-dimensional quadratic space with* $q_0(x,y) = -a \cdot xy$.

Here $a \in \mathcal{Z}$ will be specified later. In [Waldspurger 2012a], a is denoted $2\nu_0$.

Lemma 2.1. *Assume that* $\mathfrak{n}$ *is even. Then* $(V_{\mathfrak{n}}, q_{\mathfrak{n}})$ *satisfies* (QD) *if and only if*

$$\operatorname{Hss}(V_{\mathfrak{n}}) = (-1,-1)^{n(n+1)/2}((-1)^n a, \operatorname{disc}(V_{\mathfrak{n}})), \tag{2-13}$$

where $n = \frac{\mathfrak{n}}{2}$.

Proof. Write $V' = (V_{\mathfrak{n}}, q_{\mathfrak{n}}) \oplus (F, q_0)$. Since $\operatorname{disc}(V') = -a \cdot \operatorname{disc}(V_{\mathfrak{n}})$, we deduce

$$\operatorname{Hss}(V') = \operatorname{Hss}(V_{\mathfrak{n}})((-1)^n \operatorname{disc}(V_{\mathfrak{n}}), -a)(-a,-a) = \operatorname{Hss}(V_{\mathfrak{n}})((-1)^{n+1} \operatorname{disc}(V_{\mathfrak{n}}), -a). \tag{2-14}$$

Note that $\mathrm{SO}(V')$ is split if and only if V' is isometric to the quadratic space $\mathbb{H}^n \oplus Fv_0$ for some v_0, where $\mathbb{H}$ is a hyperbolic plane, equivalently $q_{\mathfrak{n}}(v_0, v_0) = \operatorname{disc}(V')$ and $\operatorname{Hss}(V') = \operatorname{Hss}(\mathbb{H}^n \oplus Fv_0)$. Under the assumption that $q_{\mathfrak{n}}(v_0, v_0) = \operatorname{disc}(V')$, by (2-10), we have

$$\begin{aligned} \operatorname{Hss}(\mathbb{H}^n \oplus Fv_0) &= \operatorname{Hss}(\mathbb{H}^n)((-1)^n, \operatorname{disc}(V'))(\operatorname{disc}(V'), \operatorname{disc}(V')) \\ &= (-1,-1)^{n(n+1)/2}(-1, \operatorname{disc}(V'))^{n+1}. \end{aligned} \tag{2-15}$$

$\mathrm{SO}(V')$ is split if and only if (2-14) equals (2-15). By using the relation that $\operatorname{disc}(V') = -a \cdot \operatorname{disc}(V_{\mathfrak{n}})$ and by simplifying the equality, we obtain this lemma. □

For example, suppose $\dim V_0^* = 2$, that is, $\mathrm{SO}(V_{\mathfrak{n}}^*)$ is a quasisplit and nonsplit even special orthogonal group, whose pure inner form $\mathrm{SO}(V_{\mathfrak{n}})$ satisfies $\operatorname{disc}(V_{\mathfrak{n}}) = \operatorname{disc}(V_{\mathfrak{n}}^*)$ and $\operatorname{Hss}(V_{\mathfrak{n}}) = -\operatorname{Hss}(V_{\mathfrak{n}}^*)$. Note that $\mathrm{SO}(V_{\mathfrak{n}})$ and $\mathrm{SO}(V_{\mathfrak{n}}^*)$ are F-isomorphic. Recall that $\Pi_\phi[G_n^*]$ is a generic Vogan packet of $\mathrm{SO}(V_{\mathfrak{n}}^*)$. The issue is which orthogonal group shall be assigned to $\chi \in \hat{\mathcal{S}}_\phi$ with $\chi((1)) = -1$ [Gan et al. 2012, §10]. We fix the F-rationality of the local Langlands correspondence following [Waldspurger 2012a, §4.6]. This means that for $\pi(\phi,\chi)\in\Pi_\phi[G_n^*]$ of $\mathrm{SO}(V')$ where $V'\in\{V_{\mathfrak{n}},V_{\mathfrak{n}}^*\}$, the quadratic space V' is determined by (2-13).

For the even special orthogonal groups, the local Langlands correspondence needs more explanation. Define c to be an element in $\mathrm{O}(V_{\mathfrak{n}})\setminus\mathrm{SO}(V_{\mathfrak{n}})$ with $\det(c)=-1$. For instance, when $\mathfrak{n}$ is odd, we can take $c=-I_{\mathfrak{n}}$. Consider the conjugate action of c on $\Pi(G_n)$, from which arises an equivalence relation $\sim_c$ on $\Pi(G_n)$. Obviously, when $\mathfrak{n}$ is odd, the c-conjugation is trivial. We only discuss the even special orthogonal case here. Denote by $\Pi(G_n)/\sim_c$ the set of equivalence classes. For $\sigma\in\Pi(G_n)$, let $[\sigma]_c$ denote the equivalence class of σ. Similarly, one has an analogous equivalence relation on the set $\Phi(G_n)$ of all L-parameters of G_n, which is also denoted by $\sim_c$.

Let us recall the desiderata of the weak local Langlands correspondence for even special orthogonal groups $\mathrm{SO}(V_{\mathfrak{n}})$ from [Atobe and Gan 2017, Desideratum 3.2]. For the needs of this paper, we only recall some partial facts from their desiderata, which has been verified in [loc. cit.], in order to fix the rationality of the local Langlands correspondence.

A Weak Local Langlands Correspondence for $G_n=\mathrm{SO}(V_{2n})$:

(1) There exists a canonical surjection

$$\bigsqcup_{G_n}\Pi(G_n)/\sim_c\to\Phi(G_n^*)/\sim_c,$$

where G_n runs over all pure inner forms of G_n^*. Note that the preimage of ϕ under the above map is the Vogan packet $\Pi_\phi[G_n^*]$ associated to ϕ.

(2) Let $\Phi^c(G_n^*)/\sim_c$ be the subset of $\Phi(G_n^*)/\sim_c$ consisting of the ϕ which contain an irreducible orthogonal subrepresentation of $\mathcal{L}_F$ with odd dimension. The following are equivalent:

- $\phi\in\Phi^c(G_n^*)/\sim_c$.
- Some $[\sigma]_c\in\Pi_\phi[G_n^*]$ satisfies $\sigma\circ\mathrm{Ad}(c)\cong\sigma$.
- All $[\sigma]_c\in\Pi_\phi[G_n^*]$ satisfy $\sigma\circ\mathrm{Ad}(c)\cong\sigma$.

(3) For each $a\in\mathcal{Z}$, there exists a bijection

$$\iota_a\colon\Pi_\phi[G_n^*]\longrightarrow\hat{\mathcal{S}}_\phi,$$

which satisfies the endoscopic and twisted endoscopic character identities (refer to [Arthur 2013; Kaletha 2016] for instance).

(4) For $[\sigma]_c\in\Pi_\phi[G_n^*]$ and $a\in\mathcal{Z}$, the following are equivalent:

- $\sigma\in\Pi(\mathrm{SO}(V_{2n}))$.
- $\iota_a([\sigma]_c)((1))$ and $\mathrm{Hss}(V_{2n})$ satisfy the following equation

$$\mathrm{Hss}(V_{2n})=\iota_a([\sigma]_c)((1))(-1,-1)^{n(n+1)/2}((-1)^na,\mathrm{disc}(V_{2n})).\tag{2-16}$$

Note that the subscript a in ι_a is used to indicate the F-rationality of the Whittaker datum. The details can be found in [Atobe and Gan 2017, §3].

By the above weak local Langlands correspondence for $\mathrm{SO}(V_{2n})$ and the local Langlands correspondence for $\mathrm{SO}(V_{2n+1})$, each irreducible admissible representation $\pi \in \Pi(G_n)$ is associated to an equivalence class $[\phi]_c$ of L-parameters under c-conjugation. Following [Gan et al. 2012, §9 and §10] and [Atobe and Gan 2017, Proposition 3.5], define the action of $\mathcal{Z}$ on $\hat{\mathcal{S}}_\phi$ via an one-dimensional twist given by the local Langlands correspondence ι_a, which is

$$a \cdot \chi \to \chi \otimes \eta_a$$

where $\eta_a((e_i)_{i \in \mathrm{I_{gp}}}) = (\det \phi_i, a)_F^{e_i}$ and $(\,\cdot\,,\,\cdot\,)_F$ is the Hilbert symbol defined over F. Denote by $\mathcal{O}_{\mathcal{Z}}(\pi)$ the orbit in $\hat{\mathcal{S}}_\phi$ corresponding to π. More precisely, if $\pi = \pi(\phi, \chi)$ under the local Langlands correspondence ι_a for some a, one has

$$\mathcal{O}_{\mathcal{Z}}(\pi) = \{\chi \otimes \eta_\alpha \in \hat{\mathcal{S}}_\phi : \alpha \in \mathcal{Z}\}. \tag{2-17}$$

Note that the set $\mathcal{O}_{\mathcal{Z}}(\pi)$ is uniquely determined by π and independent of the choice of the local Langlands correspondence ι_a.

In the rest of this paper, the local Langlands correspondence refers to the weak local Langlands correspondence for even special orthogonal groups $\mathrm{SO}(V_{2n})$*, and the local Langlands correspondence for odd special orthogonal groups* $\mathrm{SO}(V_{2n+1})$.

Under the local Langlands correspondence ι_a of $G_n(F)$ with some $a \in \mathcal{Z}$, for an L-parameter ϕ and a character $\chi \in \hat{\mathcal{S}}_\phi$, denote by $\pi_a(\phi, \chi)$ the corresponding irreducible admissible representation of $G_n(F)$. Conversely, given an irreducible admissible representation π of $G_n(F)$, denote by $\phi_a(\pi)$ and $\chi_a(\pi)$ the associated L-parameter and its corresponding character in $\hat{\mathcal{S}}_\phi$, respectively.

When $G_n = \mathrm{SO}(V_{2n+1})$, the local Langlands correspondence is unique and independent of the choice of a. We denote the trivial action by $\mathcal{Z}$, then $\mathcal{O}_{\mathcal{Z}}(\pi)$ contains only π, and we simply write $\pi(\phi, \chi)$ and $(\phi(\pi), \chi(\pi))$, respectively.

Remark 2.2. Let ϕ be an L-parameter of $\mathrm{SO}(V_{2n})$. Suppose that all irreducible orthogonal summands of ϕ are even dimensional. Then the c-conjugate L-parameter ϕ^c is different from ϕ because ϕ^c is not $G_n^\vee(\mathbb{C}) = \mathrm{SO}_{2n}(\mathbb{C})$-conjugate to ϕ. It follows that $\Pi_\phi[G_n^*]$ and $\Pi_{\phi^c}[G_n^*]$ are two different Vogan packets. However, the conjugation $\mathrm{Ad}(c) \colon \pi_a(\phi, \chi) \mapsto \pi_a(\phi^c, \chi)$ gives a bijection between $\Pi_\phi[G_n^*]$ and $\Pi_{\phi^c}[G_n^*]$. According to [Atobe and Gan 2017, §3], the c-conjugation stabilizes the Whittaker datum associated to ι_a. Thus, the corresponding characters of $\mathcal{S}_\phi$ associated to π and $\pi \circ \mathrm{Ad}(c)$ under the same local Langlands correspondence are identical.

2C. ***On the local GGP conjecture: multiplicity one.*** The local GGP conjecture was explicitly formulated in [Gan et al. 2012] for general classical groups. We recall the case of orthogonal groups here.

Let $\mathfrak{n}$ and $\mathfrak{m}$ be two positive integers with different parity. For a relevant pair $G_n = \mathrm{SO}(V, q_V)$ and $H_m = \mathrm{SO}(W, q_W)$, and an F-quasisplit relevant pair $G_n^* = \mathrm{SO}(V^*, q_V^*)$ and $H_m^* = \mathrm{SO}(W^*, q_W^*)$ in the sense of [Gan et al. 2012], where $m = \left[\frac{\mathfrak{m}}{2}\right]$ with $\mathfrak{m} = \dim W = \dim W^*$, we are going to discuss the local

L-parameters for the group $G_n^* \times H_m^*$ and its relevant pure inner F-form $G_n \times H_m$. Consider admissible group homomorphism:

$$\phi : \mathcal{L}_F = \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_n^* \times {}^L H_m^*, \tag{2-18}$$

with the properties described for the local L-parameters in (2-1). We consider those L-parameters analogous to $\Phi_{\mathrm{gen}}(G_n^*)$, and denote the set of those L-parameters by $\Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$. To each parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$, one defines the associated local L-packet $\Pi_\phi(G_n^* \times H_m^*)$, as in [Mœglin and Waldspurger 2012]. For any relevant pure inner F-form $G_n \times H_m$, if a parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$ is $G_n \times H_m$-relevant, it defines a local L-packet $\Pi_\phi(G_n \times H_m)$, following [Arthur 2013; Mœglin and Waldspurger 2012]. If a parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$ is not $G_n \times H_m$-relevant, the corresponding local L-packet $\Pi_\phi(G_n \times H_m)$ is defined to be the empty set. The local Vogan packet associated to a parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$ is defined to be the union of the local L-packets $\Pi_\phi(G_n \times H_m)$ over all relevant pure inner F-forms $G_n \times H_m$ of the F-quasisplit group $G_n^* \times H_m^*$, which is denoted by

$$\Pi_\phi[G_n^* \times H_m^*].$$

The local GGP conjecture is formulated in terms of the local Bessel functionals as introduced in Section 1A. For a given relevant pair (G_n, H_m), assuming that $\mathfrak{n} \geq \mathfrak{m}$, take a partition of the form:

$$\underline{p}_\ell := [(2\ell+1)1^{\mathfrak{n}-2\ell-1}],$$

where $2\ell+1 = \dim W^\perp = \mathfrak{n} - \mathfrak{m}$. As in Section 1A, the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ corresponding to the partition $\underline{p}_\ell$ defines a unipotent subgroup $V_{\underline{p}_\ell}$ and a generic character $\psi_{\mathcal{O}_\ell}$ associated to any F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$. Following [Jiang and Zhang 2015], there is an F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ such that the subgroup $H_m = G_n^{\mathcal{O}_\ell}$ normalizes the unipotent subgroup $V_{\underline{p}_\ell}$ and stabilizes the character $\psi_{\mathcal{O}_\ell}$. As in Section 1A again, the uniqueness of local Bessel functionals asserts that

$$\dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}) \leq 1,$$

as proved in [Aizenbud et al. 2010; Sun and Zhu 2012; Gan et al. 2012; Jiang et al. 2010b]. The stronger version in terms of Vogan packets is formulated as follows.

Theorem 2.3 (Mœglin–Waldspurger). *Let G_n^* and H_m^* be a relevant pair as given above. For any local L-parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$, the following identity holds*:

$$\sum_{\pi \otimes \sigma \in \Pi_\phi[G_n^* \times H_m^*]} \dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma, \psi_{\mathcal{O}_\ell}) = 1. \tag{2-19}$$

Theorem 2.3 is the orthogonal group case of the general local GGP conjecture over p-adic local fields. It was proved by Waldspurger [2010; 2012a; 2012b] for tempered local L-parameters, and by Mœglin and Waldspurger [2012] for generic local L-parameters.

2D. ***On multiplicity and parabolic induction.*** We now discuss the relation between multiplicity in the local GGP conjecture and parabolic induction as given in the work of Mœglin and Waldspurger in a series of papers [Waldspurger 2010; 2012a; 2012b; Mœglin and Waldspurger 2012] for orthogonal groups.

Let ϕ and φ be generic L-parameters of different type and of even dimension. Denote by $\pi^{\mathrm{GL}}(\phi)$ and $\pi^{\mathrm{GL}}(\varphi)$ the two irreducible representations of general linear groups via the local Langlands functoriality, which is independent with the choice of φ in $[\varphi]_c$. Define

$$\mathcal{E}(\varphi,\phi)=\det(\varphi)(-1)^n\cdot\varepsilon\big(\tfrac{1}{2},\pi^{\mathrm{GL}}(\varphi)\times\pi^{\mathrm{GL}}(\phi),\psi_F\big), \tag{2-20}$$

where $\varepsilon(s,\cdot,\psi_F)$ is the ε-factor defined by H. Jacquet, I. Piatetski-Shapiro and J. Shalika [1983]. Recall that $\det(\varphi)(-1)=(\det(\varphi),-1)_F$ and $(\cdot,\cdot)_F$ is the Hilbert symbol defined over F. Decompose φ and ϕ as in (2-6), their index sets are denoted by $\mathrm{I}^{\phi}_{\mathrm{gp}}$ and $\mathrm{I}^{\varphi}_{\mathrm{gp}}$ respectively, and define the character $\chi^\star(\phi,\varphi)$ to be the pair $(\chi^\star_\phi(\phi,\varphi),\chi^\star_\varphi(\phi,\varphi))$ (or simply $(\chi^\star_\phi,\chi^\star_\varphi)$), where

$$\chi^\star_\phi((e_{i'})_{i'\in\mathrm{I}^{\phi}_{\mathrm{gp}}})=\prod_{i'\in\mathrm{I}^{\phi}_{\mathrm{gp}}}\mathcal{E}(\varphi,\phi_{i'})^{e_{i'}} \tag{2-21}$$

and

$$\chi^\star_\varphi((e_{i})_{i\in\mathrm{I}^{\varphi}_{\mathrm{gp}}})=\prod_{i\in\mathrm{I}^{\varphi}_{\mathrm{gp}}}\mathcal{E}(\varphi_{i},\phi)^{e_{i}}. \tag{2-22}$$

By convention, if φ or ϕ equals 0, then $\chi^\star(\phi,\varphi)=(1,1)$.

Note that $\chi^\star_\phi$ and $\chi^\star_\varphi$ belong to $\hat{\mathcal{S}}_\phi$ and $\hat{\mathcal{S}}_\varphi$, respectively, and are independent of the choice of ψ_F defining local root numbers (see [Gan et al. 2012, §6]). It is easy to see that

$$\chi^\star_\phi((1))=\chi^\star_\varphi((1))=\mathcal{E}(\varphi,\phi).$$

By [Gan et al. 2012, §6 and §18], the character $\chi^\star(\phi,\varphi)$ of $\mathcal{S}_\phi\times\mathcal{S}_\varphi$ only depends on ϕ and $[\varphi]_c$.

For $\pi\in\Pi(G_n)$ with $G_n=\mathrm{SO}(V_{2n})$ and $\sigma\in\Pi(H_m)$ with $H_m=\mathrm{SO}(V_{2m+1})$, define the multiplicity for the pair (π,σ) by

$$m(\pi,\sigma)=\begin{cases}\dim\mathrm{Hom}_{R_{\mathcal{O}_\ell}}(\pi,\sigma\otimes\psi_{\mathcal{O}_\ell}) & \text{if } n>m,\\ \dim\mathrm{Hom}_{R_{\mathcal{O}_\ell}}(\sigma,\pi\otimes\psi_{\mathcal{O}_\ell}) & \text{if } n\le m.\end{cases} \tag{2-23}$$

Theorem 2.4 [Waldspurger 2012a, Theorem in §4.9]. *Assume that the twisted endoscopic identities and twisted endoscopic character identities as described in* [Waldspurger 2012a, §4.2 and §4.3] *hold. Suppose that φ and ϕ are generic L-parameters of $G_n=\mathrm{SO}(V_{2n})$ and $H_m=\mathrm{SO}(W_{2m+1})$. There exists an isometry class of quadratic spaces $V\times W$, unique up to a scalar multiplication, that satisfies the following conditions*:

$$\mathrm{disc}(V)=\det(\varphi). \tag{2-24}$$

$$\mathrm{Hss}(W)=(-1,-1)^{m(m+1)/2}((-1)^{m+1},\mathrm{disc}(W))\mathcal{E}(\varphi,\phi). \tag{2-25}$$

$$\mathrm{Hss}(V)=(-1,-1)^{n(n+1)/2}((-1)^{n}.\,\mathrm{disc}(W),\mathrm{disc}(V))\mathcal{E}(\varphi,\phi). \tag{2-26}$$

Moreover, under the local Langlands correspondence ι_a

$$m(\pi_a(\varphi, \chi_\varphi^\star), \pi_a(\phi, \chi_\phi^\star)) = 1,$$

where $a = -\operatorname{disc}(W)\operatorname{disc}(V)$.

It must be mentioned that the assumption on the t.e. identities and t.e. character identities for the quasisplit groups in Theorem 2.4 is removed by the work of Mœglin and Waldspurger [2016] on the stabilization of the twisted trace formula.

We note that Atobe and Gan [2017, Theorem 5.2] give a version of local GGP conjecture in terms of the weak local Langlands correspondence.

Remark 2.5. Suppose that $V \times W$ satisfies the conditions in Theorem 2.4, so $\lambda V \times \lambda W$ for any $\lambda \in \mathcal{Z}$. More precisely, after choosing $\operatorname{disc}(W) \in \mathcal{Z}$, one can choose an F-rational orbit $\mathcal{O}_\ell$ with $\operatorname{disc}(\mathcal{O}_\ell) = (-1)^{\min\{2m+1,2n\}} \operatorname{disc}(V)\operatorname{disc}(W)$. Then there is a unique isometry class $V \times W$ satisfying (2-24), (2-25) and (2-26), which is associated with $\mathcal{O}_\ell$.

The relation between the multiplicity and the parabolic induction is given by the following proposition of Mœglin and Waldspurger.

Proposition 2.6 [Mœglin and Waldspurger 2012, Proposition 1.3]. *Assume that* π *is the induced representation*

$$\operatorname{Ind}_P^{G_n} \tau_1|\det|^{\alpha_1} \otimes \cdots \otimes \tau_r|\det|^{\alpha_r} \otimes \pi_0$$

and σ *is the induced representation*

$$\operatorname{Ind}_Q^{H_m} \tau_1'|\det|^{\beta_1} \otimes \cdots \otimes \tau_t'|\det|^{\beta_t} \otimes \sigma_0$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t \geq 0$, *all* τ_i *and* τ_j' *are unitary tempered irreducible representations of general linear groups, and* π_0 *and* σ_0 *are tempered irreducible representations of classical groups of smaller rank. Then the following equation of multiplicities holds*:

$$m(\pi, \sigma^\vee) = m(\pi_0, \sigma_0).$$

3. Proof of Proposition 1.6

The proof of Proposition 1.6 takes a few steps. We first prove the following lemma, which implies Proposition 1.1.

Lemma 3.1. *For any* $\pi \in \Pi(G_n)$, *the following hold*:

(1) *Each Bernstein component of the* ℓ*-th twisted Jacquet module* $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ *is finitely generated.*

(2) *If* $\mathcal{J}_{\mathcal{O}_\ell}(\pi) \neq 0$, *then there exists an irreducible representation* $\sigma \in \Pi(G_n^{\mathcal{O}_\ell})$ *such that*

$$\operatorname{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma) \neq 0.$$

(3) *If* $\mathcal{J}_{\mathcal{O}_\ell}(\pi) \neq 0$, *then the* ℓ*-th local descent* $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ *of* π *is not zero.*

Proof. It is clear that (3) follows from (2). Assume that (1) holds. Then (2) follows, since any smooth representation of finite length has an irreducible quotient. Hence we only need to show that (1) holds.

For any $\pi \in \Pi(G_n)$, let $\mathrm{End}(\pi)$ be the space of continuous endomorphisms of the space of π. We have the following $G_n(F) \times G_n(F)$-equivariant homomorphism:

$$f \in \mathcal{S}(G_n(F)) := C_c^\infty(G_n(F)) \mapsto \pi(f) \in \mathrm{End}(\pi) \simeq \pi \otimes \pi^\vee,$$

where $\pi(f)(v) := \int_{G_n(F)} f(g)\pi(g)v\,\mathrm{d}g$ is the convolution operator induced by π for all Bruhat–Schwartz functions $f \in \mathcal{S}(G_n(F))$. It is not hard to check that this homomorphism is surjective. By the separation of variables, the homomorphism induces a surjective homomorphism

$$\mathcal{S}(G_n(F) \times G_n^{\mathcal{O}_\ell}(F)) = \mathcal{S}(G_n(F)) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)) \twoheadrightarrow \pi \otimes \pi^\vee \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)).$$

By taking the $(V_{\underline{p}_\ell}, \psi_{\mathcal{O}_\ell})$-coinvariant for the action by the left translation of $\mathcal{S}(G_n(F))$, one has the projection

$${}_{(V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell})}(\mathcal{S}(G_n)(F)) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)) \twoheadrightarrow \mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \pi^\vee \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)).$$

Then taking $G_n^{\mathcal{O}_\ell}$-coinvariant for the action by the left translation, one obtains a surjective homomorphism

$${}_{G_n^{\mathcal{O}_\ell}}[{}_{(V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell})}(\mathcal{S}(G_n(F))) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))] \twoheadrightarrow \pi^\vee \otimes {}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))].$$

Note that $G_n^{\mathcal{O}_\ell}(F)$ acts on ${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))]$ via the left translation on the second variable. As smooth representations of $G_n^{\mathcal{O}_\ell}(F)$, one has a surjection

$${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))] \twoheadrightarrow \mathcal{J}_{\mathcal{O}_\ell}(\pi).$$

We need to show that the map

$$\mathfrak{p}\colon \mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)) \to \mathcal{J}_{\mathcal{O}_\ell}(\pi)$$

defined by $v \otimes f \mapsto \int_{G_n^{\mathcal{O}_\ell}(F)} f(h^{-1})\mathcal{J}_{\mathcal{O}_\ell}(\pi)(h)v\,\mathrm{d}h$ factors through the quotient ${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))]$ and is surjective. First, for any smooth vector $v \in \mathcal{J}_{\mathcal{O}_\ell}(\pi)$, suppose that v is fixed by a compact open subgroup K_0 of $G_n^{\mathcal{O}_\ell}(F)$. Let 1_{K_0} be a characteristic function of K_0 in $\mathcal{S}(G_n^{\mathcal{O}_\ell}(F))$. By choosing a suitable nonzero constant c_0, we have $v \otimes c_0 \cdot 1_{K_0} \mapsto v$. It follows that this map is surjective. Then, if $v \otimes f$ is of form $\sum(\pi(h_i)v_i \otimes L(h_i)f_i - v_i \otimes f_i)$ for some h_i, v_i and f_i, one must have that $\mathfrak{p}(v \otimes f) = 0$. Thus, the map $\mathfrak{p}$ factors through the quotient ${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))]$.

Because

$$\mathcal{S}((G_n^{\mathcal{O}_\ell} \ltimes V_{\underline{p}_\ell})\backslash G_n \times G_n^{\mathcal{O}_\ell}, \psi_{\mathcal{O}_\ell}) = {}_{G_n^{\mathcal{O}_\ell}}({}_{(V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell})}(\mathcal{S}(G_n)) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell})),$$

we obtain a projection

$$\mathcal{S}((G_n^{\mathcal{O}_\ell}(F) \ltimes V_{\underline{p}_\ell}(F))\backslash G_n(F) \times G_n^{\mathcal{O}_\ell}(F), \psi_{\mathcal{O}_\ell}) \twoheadrightarrow \pi^\vee \otimes \mathcal{J}_{\mathcal{O}_\ell}(\pi).$$

By Theorem A and the subsequent remark in [Aizenbud et al. 2012], as a representation of $G_n(F)\times G_n^{\mathcal{O}_\ell}(F)$, each Bernstein component of

$$\mathcal{S}((G_n^{\mathcal{O}_\ell}(F)\ltimes V_{\underline{p}_\ell}(F))\backslash G_n(F)\times G_n^{\mathcal{O}_\ell}(F),\psi_{\mathcal{O}_\ell})$$

is finitely generated, and so is each Bernstein component of $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$. This finishes the proof of (1). □

Assume that $\pi\in\Pi(G_n)$ has a generic L-parameter. By the corollary in [Mœglin and Waldspurger 2012, §2.14], π can be written as the irreducible induced representation (standard module)

$$\mathrm{Ind}_{P(F)}^{G_n(F)}\tau_1|\det|^{\alpha_1}\otimes\cdots\otimes\tau_t|\det|^{\alpha_t}\otimes\pi_0, \tag{3-1}$$

where $\alpha_1>\alpha_2>\cdots>\alpha_t>0$, and all τ_i and π_0 are irreducible unitary tempered representations. One may write the Levi subgroup of P as $\mathrm{GL}_{a_1}\times\cdots\times\mathrm{GL}_{a_t}\times G_{n_0}$ and has that $\pi_0\in\Pi_{\mathrm{temp}}(G_{n_0})$.

As a corollary to Proposition 2.6, one has:

Corollary 3.2. *For any $\pi\in\Pi(G_n)$ with a generic L-parameter and written as in* (3-1), *the following hold*:

(1) *The first occurrence indices $\ell_0(\pi)$ and $\ell_0(\pi_0)$, with π_0 being as in* (3-1), *enjoy the relation*:

$$\ell_0(\pi)=n-n_0+\ell_0(\pi_0).$$

(2) *Every irreducible quotient of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi)$ of π is square-integrable.*

Finally we prove the following proposition, which proves Proposition 1.2 and finishes the proof of Proposition 1.6.

Let $\pi\in\Pi(G_n)$ of a generic L-parameter. Assume that the character $\psi_{\mathcal{O}_{\ell_0}}$ is given by ψ_{ℓ_0,w_0} (defined in (1-5)). Then the quadratic space W_{ℓ_0} defining $G^{\mathcal{O}_{\ell_0}}$ is of form (1-6). For each $\ell\le\ell_0$, we may choose the *compatible* F-rational orbit $\mathcal{O}_\ell$ such that its corresponding character is ψ_{ℓ,w_0}. Then its stabilizer $G^{\mathcal{O}_\ell}=\mathrm{SO}(W_\ell,q)$, where $W_\ell=X^+_{\ell_0-\ell}\oplus W_{\ell_0}\oplus X^-_{\ell_0-\ell}$ and $X^\pm_{\ell_0-\ell}=\mathrm{Span}\{e^\pm_{\ell+1},e^\pm_{\ell+2},\dots,e^\pm_{\ell_0}\}$.

Proposition 3.3 (stability of local descent). *For any $\pi\in\Pi(G_n)$ with a generic L-parameter, then the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ is nonzero for $\ell\le\ell_0$ and the above compatible $\mathcal{O}_\ell$.*

Proof. By Corollary 3.2, there exists an F-rational orbit $\mathcal{O}_{\ell_0}$ associated to the partition $\underline{p}_{\ell_0}$ such that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\ell_0}}(\pi)$ has an irreducible quotient σ belonging to $\Pi_{\mathrm{temp}}(G_n^{\mathcal{O}_{\ell_0}})$ and hence we have that $m(\pi,\sigma)\neq 0$.

For any occurrence index $\ell<\ell_0(\pi)$, we want to show that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero for the F-rational orbit $\mathcal{O}_\ell$, which is compatible with $\mathcal{O}_{\ell_0}$. Note that the quadratic spaces defining both orthogonal groups $G_n^{\mathcal{O}_\ell}$ and $G_n^{\mathcal{O}_{\ell_0}}$ have the same anisotropic kernel if $\mathcal{O}_\ell$ is compatible with $\mathcal{O}_{\ell_0}$. Let τ be a unitary non-self-dual irreducible supercuspidal representation of $\mathrm{GL}_{\ell_0-\ell}(F)$. Define

$$\sigma_1=\mathrm{Ind}_{P_{\ell_0-\ell}(F)}^{G_n^{\mathcal{O}_\ell}(F)}\tau\otimes\sigma,$$

where $P_{\ell_0-\ell}$ is a parabolic subgroup of $G_n^{\mathcal{O}_\ell}$, whose Levi subgroup is isomorphic to $\mathrm{GL}_{\ell_0-\ell}\times G_n^{\mathcal{O}_{\ell_0}}$. It is clear that σ_1 is an irreducible tempered representation of $G_n^{\mathcal{O}_\ell}(F)$. By applying Proposition 2.6, we obtain an identity: $m(\pi, \sigma_1^\vee) = m(\pi, \sigma) \neq 0$. Hence the ℓ-th twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ has a quotient $\sigma_1^\vee$, as representations of $G_n^{\mathcal{O}_\ell}(F)$. In particular, the ℓ-local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ is nonzero, and has the irreducible tempered representation $\sigma_1^\vee$ as a quotient. This finishes the proof. $\square$

4. Descent of local L-parameters

We introduce a notion of the *descent of the local L-parameters* and determine the structure of the descent of local L-parameters, which forms one of the technical cores of the proofs of the main results in the paper. To do so, we have to calculate explicitly the relevant local root numbers.

Let $\varphi \in \Phi_{\mathrm{gen}}(G_n^*)$, which is a $2n$-dimensional, self-dual local L-parameter, either of orthogonal type or of symplectic type. For $\chi \in \hat{\mathcal{S}}_\varphi$, define the *$\ell$-th descent* of (φ, χ) for any $\ell \in \{0, 1, \ldots, n\}$, which is denoted by $\mathfrak{D}_\ell(\varphi, \chi)$, to be the set of generic self-dual local L-parameters ϕ, satisfying the following conditions:

(1) ϕ are local L-parameters of dimension $2(n-\ell)$ for $G_n^{\mathcal{O}_\ell}(F)$, which have the type different from that of φ.

(2) The equation $\chi_\varphi^\star(\varphi, \phi) = \chi$ holds.

Definition 4.1 (descent for L-parameters). For a parameter φ in $\Phi_{\mathrm{gen}}(G_n)$ and $\chi \in \hat{\mathcal{S}}_\varphi$, the first occurrence index $\ell_0 := \ell_0(\varphi, \chi)$ of (φ, χ) is the integer ℓ_0 in $\{0, 1, \ldots, n\}$, such that $\mathfrak{D}_{\ell_0}(\varphi, \chi) \neq \varnothing$ but for any $\ell \in \{0, 1, \ldots, n\}$ with $\ell > \ell_0$, $\mathfrak{D}_\ell(\varphi, \chi) = \varnothing$. The ℓ_0-th descent $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ of (φ, χ) is called the first descent of (φ, χ) or simply the descent of (φ, χ).

Note that $\ell_0(\varphi, \chi) = n$ if and only if $\chi = 1$ by convention. As in the local descent on the representation side, this case will be excluded from the discussion at the first occurrence index, because in this case, the Bessel model becomes the Whittaker model.

Note that by definition $\mathfrak{D}_\ell(\varphi, \chi) = \mathfrak{D}_\ell(\varphi^c, \chi)$, and hence $\mathfrak{D}_\ell(\varphi, \chi)$ is stable under c-conjugate. It follows that $\mathfrak{D}_\ell([\varphi]_c, \chi)$ is well defined, and that $\phi^c \in \mathfrak{D}_\ell(\varphi, \chi)$ if and only if $\phi \in \mathfrak{D}_\ell(\varphi, \chi)$. In fact, the c-conjugation preserves the Bessel models, and hence the local descent of representations is stable under c-conjugate and any two c-conjugate L-parameters have the same local descent.

4A. ***Local root number.*** Let ϕ be a local L-parameter of G_n and ψ_F be a nontrivial additive character of F, as before. The *local epsilon factor* defined by P. Deligne and J. Tate [1979] is given by

$$\varepsilon(s, \phi, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big) q^{a(\phi,\psi)(1/2-s)}, \tag{4-1}$$

where $a(\phi, \psi)$ is the Artin conductor of ϕ and $\varepsilon\big(\frac{1}{2}, \phi, \psi_F\big)$ is the *local root number*. By [Gross and Reeder 2006, Proposition 15.1] and [Tate 1979], one also has the following properties of the local epsilon factors:

- $\varepsilon(s, \phi, \psi_a) = \det\phi(a)|a|^{-\dim\phi(1/2-s)}\varepsilon(s, \phi, \psi_F)$, where ψ_a is defined by $\psi_a(x) = \psi_F(ax)$.
- $\varepsilon(s, \phi \otimes |\cdot|^{s_0}, \psi_F) = q_F^{-s_0 a(\phi,\psi)}\varepsilon(s, \phi, \psi_F)$.
- $\varepsilon(s, \phi, \psi_F)\varepsilon(1-s, \phi^\vee, \psi_F) = \det(\phi)(-1)$.

- $\varepsilon(s, \phi \oplus \phi', \psi_F) = \varepsilon(s, \phi, \psi_F)\varepsilon(s, \phi', \psi_F)$.

Denote by μ_n the algebraic n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ in this paper. Let us decompose ϕ as

$$\phi = \oplus_{n\geq 0}\rho_n \otimes \mu_{n+1},$$

where (ρ_n, V_{ρ_n}) is a semisimple complex representation of $\mathcal{W}_F$, which may possibly be zero. By the work of B. Gross and M. Reeder [2010] for instance, one has the local epsilon factor

$$\varepsilon(s, \phi, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big) q^{a(\phi)(1/2-s)},$$

where $a(\phi) = \sum_{n\geq 0}(n+1)a(\rho_n) + \sum_{n\geq 1} n \cdot \dim V_{\rho_n}^{\mathcal{I}}$, and

$$\varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big) = \prod_{n\geq 0} \varepsilon\big(\tfrac{1}{2}, \rho_n, \psi_F\big)^{n+1} \prod_{n\geq 1} \det(-\rho_n(\mathrm{Fr})|V_{\rho_n}^{\mathcal{I}})^n, \tag{4-2}$$

with $\mathcal{I} = \mathcal{I}_F$ being the inertia group and V_{ρ_n} being the vector space defining ρ_n, and with $a(\rho_n)$ being the Artin conductor of ρ_n. More details on the normalization of ψ and the Haar measure defining the Artin conductor can be found in [Gross and Reeder 2010, §2].

By [Gan et al. 2012, Propositions 5.1 and 5.2], if ϕ is a self-dual L-parameter and $\det(\phi) = 1$, then $\varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big)$ is independent of the choice of ψ_F and is simply denoted by $\varepsilon(\phi)$. Moreover, $\varepsilon(\phi) = \pm 1$.

For example, if τ is a character of $F^\times$, we may rewrite τ as $|\cdot|^{s_0}\omega$, where ω is a unitary character of $F^\times$. Then

$$\varepsilon(s, \tau, \psi_F) = q^{n(1/2-s-s_0)} \frac{g(\bar{\omega}, \psi_{\varpi^{-n}})}{|g(\bar{\omega}, \psi_{\varpi^{-n}})|},$$

where n is the conductor of ω (i.e., ω is trivial on $1+\varpi^n\mathfrak{o}_F$ but not trivial on $1+\varpi^{n-1}\mathfrak{o}_F$) and the Gauss sum is defined by

$$g(\omega, \psi_a) = \int_{\mathfrak{o}_F^\times} \omega(u)\psi_F(au)\,\mathrm{d}u.$$

In particular, if τ is unramified, then $\varepsilon(s, \tau, \psi_F) = 1$. If τ is a ramified quadratic character, then it is of conductor 1. Thus $\varepsilon(s, \tau, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \tau, \psi_F\big) q^{1/2-s}$ and $\varepsilon^2\big(\tfrac{1}{2}, \tau, \psi_F\big) = \tau(-1)$.

More generally, let $\pi = [\tau|\cdot|^{1-r/2}, \tau|\cdot|^{r-1/2}]$ be the square-integrable representation of a general linear group determined by the line segment of Bernstein and Zelevinsky [1977], with the local L-parameter $\varphi_\tau \boxtimes \mu_r$. Here φ_τ is the associated irreducible representation of $\mathcal{W}_F$ via the local Langlands correspondence for general linear groups. Denote by ω_π the central character of π. If τ is a quadratic character of $\mathrm{GL}_1(F)$ and $\omega_\pi = 1$ (that is, if τ is nontrivial, then r is even), then

$$\varepsilon(s, \varphi_\tau \boxtimes \mu_r, \psi_F) = \begin{cases} q^{(r-1)(1/2-s)} & \text{if } \tau = 1 \text{ and } r \text{ is odd,} \\ -\tau(\varpi) q^{(r-1)(1/2-s)} & \text{if } \tau \text{ is unramified and } r \text{ is even,} \\ \tau(-1)^{r/2} q^{r(1/2-s)} & \text{if } \tau \text{ is ramified.} \end{cases} \tag{4-3}$$

We remark that under the assumption $\varphi_\tau \boxtimes \mu_r$ is self-dual and $\det(\varphi_\tau \boxtimes \mu_r) = 1$. If τ is a self-dual supercuspidal representation of $\mathrm{GL}_a(F)$ with $a > 1$, then

$$\varepsilon(s, \varphi_\tau \boxtimes \mu_r, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \tau, \psi_F\big)^r q^{ra(\tau)(1/2-s)}. \tag{4-4}$$

Lemma 4.2. *Let $\varphi_1 = \varphi_\pi \boxtimes \mu_n$ and $\varphi_2 = \varphi_\tau \boxtimes \mu_m$ be two irreducible local L-parameters, with φ_π and φ_τ being self-dual and of dimensions a and b, respectively.*

(1) *If m and n are even, then $\varepsilon(\varphi_1 \otimes \varphi_2) = 1$.*

(2) *If $\varphi_\pi \otimes \varphi_\tau$ is of symplectic type, then*

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \begin{cases} 1 & \text{when } m+n \text{ is odd}, \\ \varepsilon(\pi \times \tau) & \text{when } mn \text{ is odd}. \end{cases}$$

(3) *If $\varphi_\pi \otimes \varphi_\tau$ is of orthogonal type and $\pi \ncong \tau$, then*

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2},$$

when $m+n$ is odd, and $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$, when mn is odd.

(4) *If $\varphi_\pi \otimes \varphi_\tau$ is of orthogonal type and $\pi \cong \tau$, then*

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2}(-1)^{\min\{m,n\}} = (-1)^{\min\{m,n\}},$$

when $m+n$ is odd, and $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$, when mn is odd.

Remark 4.3. If mn is odd, and φ_π and φ_τ are of orthogonal type, then $\det(\varphi_\pi \otimes \varphi_\tau) = \pm 1$ and $\varepsilon(\varphi_\pi \otimes \varphi_\tau)$ depends possibly on the additive character ψ_F. Thus, we add ψ_F in the ε for this case. In this case the local root number is calculated in [Gan et al. 2012, Theorem 6.2(2)]. However, after the normalization in (2-20), the local root number is independent on ψ_F (see [Gan et al. 2012, Theorem 6.2(1)]).

Proof. Since $\mu_n \otimes \mu_m = \bigoplus_{i=1}^{\min\{m,n\}} \mu_{n+m+1-2i}$, as representations of $\mathrm{SL}_2(\mathbb{C})$, one has

$$\varphi_1 \otimes \varphi_2 = \bigoplus_{i=1}^{\min\{m,n\}} (\varphi_\pi \otimes \varphi_\tau) \boxtimes \mu_{n+m+1-2i}.$$

If $\pi \ncong \chi\tau$ for any unramified character χ, there is no $\varphi_\pi \otimes \varphi_\tau(\mathcal{I})$-invariant vector. Following (4-4), one has

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \prod_{i=1}^{\min\{m,n\}} \varepsilon(\pi \times \tau, \psi_F)^{n+m+1-2i} = \varepsilon(\pi \times \tau, \psi_F)^{mn}. \tag{4-5}$$

Suppose that $\pi \cong \chi\tau$ for some unramified character χ. Following (4-2) and (6.2.5) in [Bushnell and Kutzko 1993], one has

$$\varepsilon((\varphi_\pi \otimes \varphi_\tau) \boxtimes \mu_r) = \varepsilon(\pi \times \tau, \psi_F)^r (-\chi(\varpi)^{d(\pi)})^{r-1},$$

where $d(\pi)$ is the number of all unramified characters χ such that $\pi \cong \chi\pi$. And an unramified character χ' satisfies $\pi \cong \chi'\pi$ if and only if the order of χ' divides $d(\pi)$. It follows that

$$\begin{aligned}\varepsilon(\varphi_1 \otimes \varphi_2) &= \prod_{i=1}^{\min\{m,n\}} \varepsilon(\pi \times \tau, \psi_F)^{n+m+1-2i}(-\chi(\varpi)^{d(\pi)})^{n+m-2i} \\ &= \varepsilon(\pi \times \tau, \psi_F)^{mn}(-\chi(\varpi)^{d(\pi)})^{mn-\min\{m,n\}}. \end{aligned} \tag{4-6}$$

Since π and τ are self-dual, the unramified character χ has the property that $\pi \cong \chi^2\pi$. Hence, the order of χ^2 divides $d(\pi)$, equivalently $\chi(\varpi)^{d(\pi)} = \pm 1$. If $\chi(\varpi)^{d(\pi)} = -1$, then the order of χ equals $2d(\pi)$, which implies $\pi \ncong \chi\pi$ and then $\pi \ncong \tau$. In this case, $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)^{mn}$.

If $\chi(\varpi)^{d(\pi)} = 1$, then the order χ divides of $d(\pi)$, which implies $\pi \cong \chi\tau \cong \tau$ and

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)^{mn}(-1)^{mn-\min\{m,n\}}. \tag{4-7}$$

By (4-6), one obtains (4-7).

Combining with the case $\pi \ncong \chi\tau$ for any unramified χ, we summarize

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \begin{cases} \varepsilon(\pi \times \tau, \psi_F)^{mn} & \text{if } \pi \ncong \tau, \\ \varepsilon(\pi \times \tau, \psi_F)^{mn}(-1)^{mn-\min\{m,n\}} & \text{if } \pi \cong \tau. \end{cases}$$

Recall that $\varepsilon(\varphi)^2 = (\det\varphi)(-1) = \pm 1$ if φ is self-dual. Since $\varphi_\pi \otimes \varphi_\tau$ is self-dual, we have

$$\varepsilon(\varphi_\pi \otimes \varphi_\tau)^2 = \det(\varphi_\pi)(-1)^b \det(\varphi_\tau)(-1)^a. \tag{4-8}$$

If m and n are even, then 4 divides mn and $\min\{m,n\}$ is even. By (4-5), (4-6) and (4-8), we have $\varepsilon(\varphi_1 \otimes \varphi_2) = 1$.

Suppose that $\varphi_\pi \otimes \varphi_\tau$ is of symplectic type. Then $\det(\varphi_\pi \otimes \varphi_\tau) = 1$ and φ_π and φ_τ are of different type. If $m+n$ is odd, then mn is even and $\varepsilon(\pi \times \tau, \psi_F)^{mn} = 1$. As φ_π and φ_τ are irreducible and of different type, there is no unramified character χ satisfying $\pi \cong \chi\tau$. Therefore, $\varepsilon(\varphi_1 \otimes \varphi_2) = 1$ if $m+n$ is odd, and $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau)$ if mn is odd.

Suppose that $\varphi_\pi \otimes \varphi_\tau$ is of orthogonal type. If $\pi \ncong \tau$, then, when mn is even,

$$\varepsilon(\varphi_1 \otimes \varphi_2) = (\varepsilon(\pi \times \tau, \psi_F)^2)^{mn/2} = \det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2},$$

which is independent of the choice of ψ_F; and when mn is odd, $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$, which depends on the choice of ψ_F.

If $\pi \cong \tau$, then, when mn is even, $\varepsilon(\varphi_1 \otimes \varphi_2)$ equals

$$\det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2}(-1)^{\min\{m,n\}} = (-1)^{\min\{m,n\}}$$

and when mn is odd, $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$ as $mn - \min\{m,n\}$ is even. □

In some special cases, the result is simple. The following example is about the quadratic unipotent L-parameters, which will be more explicitly discussed in Section 6.

Example 4.4. Let $\varphi_\pi = \chi \boxtimes \mu_n$ and $\varphi_\tau = \xi \boxtimes \mu_m$, where χ and ξ are quadratic characters. If $m+n$ is odd, then

$$\varepsilon(\pi \times \tau) = \begin{cases} (-\chi\xi(\varpi))^{\min\{m,n\}} & \text{if } \chi\xi \text{ is unramified,} \\ \chi\xi(-1)^{mn/2} & \text{if } \chi\xi \text{ is ramified.} \end{cases}$$

Remark 4.5. Let φ and ϕ be discrete L-parameters of different type. Then $\chi^\star(\varphi, \phi) = (\chi^\star_\varphi, \chi^\star_\phi)$, with $\chi^\star_\varphi$ and $\chi^\star_\phi$ as defined in (2-21) and (2-22), yields a pair of characters on A_φ and A_ϕ, which are independent of ψ_F. In fact, one may decompose φ and ϕ as $\varphi = \boxplus_{i=1}^r \varphi_i$ and $\phi = \boxplus_{j=1}^s \phi_j$, where φ_i and ϕ_j are irreducible and of different type for all i and j. Since $\varphi_i \otimes \phi_j$ is of symplectic type, $\mathcal{E}(\varphi_i, \phi)$ and $\mathcal{E}(\varphi, \phi_j)$ are ± 1 and independent of ψ_F.

In the remark, we extend the character $\chi^\star(\varphi, \phi)$ to be a character of $A_\varphi \times A_\phi$, which is well-defined and independent of ψ_F. And it is allowed that the orthogonal parameter φ or ϕ is of odd dimension. Then the character $\chi^\star(\varphi, \phi)$ is still well defined.

4B. ***Descent of local L-parameters.*** The main result of this subsection is Theorem 4.6, which explicitly determines the descent of local L-parameters.

Let φ be a local L-parameter of G_n for a square-integrable representation of $G_n(F)$. It can be decomposed as

$$\varphi = \boxplus_{i=1}^r \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2\alpha_{i,j}} \boxplus \boxplus_{i=1}^s \boxplus_{j=1}^{s_i} \varrho_i \boxtimes \mu_{2\beta_{i,j}+1}, \tag{4-9}$$

where all ρ_i and ϱ_i are irreducible self-dual distinct representations of $\mathcal{W}_F$ of dimension a_i and b_i, respectively, and $1 \le \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,r_i}$ and $0 \le \beta_{i,1} < \beta_{i,2} < \cdots < \beta_{i,s_i}$ for all i are integers. For such a local L-parameter φ as in (4-9), we define the even and odd parts according to the dimension of the μ_k:

$$\varphi_e := \boxplus_{i=1}^r \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2\alpha_{i,j}} \tag{4-10}$$

$$\varphi_o := \boxplus_{i=1}^s \boxplus_{j=1}^{s_i} \varrho_i \boxtimes \mu_{2\beta_{i,j}+1}. \tag{4-11}$$

For an irreducible representation ρ of $\mathcal{W}_F$, denote by $\varphi(\rho)$ the ρ-isotypic component of φ when restricted to $\mathcal{W}_F$. It is clear that $\varphi(\rho)$ is still a local L-parameter. For instance, $\varphi(\rho_i) = \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2\alpha_{i,j}}$. Hence $\varphi(\rho) = 0$ if ρ is not isomorphic to any of the ρ_i or ϱ_i for all i.

For a local L-parameter φ of G_n for a square-integrable representation of $G_n(F)$, we decompose it as $\varphi = \boxplus_{i \in \mathrm{I}} \varphi_i$. Let φ' be a subrepresentation of φ, i.e., $\varphi' = \boxplus_{i \in \mathrm{I}'} \varphi_i$ where $\mathrm{I}' \subseteq \mathrm{I}$. For an element $(e_i) \in \mathcal{S}_\varphi$, denote by $(e_i)|_{\varphi'}$ the elements in $\mathcal{S}_\varphi$ such that $e_i = 0$ for $i \in \mathrm{I}'$. For example, the element $1_{\varphi'}$ is given by $e_i = 1$ for $i \in \mathrm{I}'$ and $e_i = 0$ for $i \notin \mathrm{I}'$.

Define the *sign alternative index set,* $\mathrm{sgn}_{*,\rho}(\chi)$, as follows: when $\rho \cong \rho_i$ for some i,

$$\mathrm{sgn}_{o,\rho_i}(\chi) := \{ j : 0 \le j < r_i,\ \chi(1_{\rho_i \boxtimes \mu_{2\alpha_{i,j}} \boxplus \rho_i \boxtimes \mu_{2\alpha_{i,j+1}}}) = -1 \} \tag{4-12}$$

and when $\rho \cong \varrho_i$ for some i,

$$\mathrm{sgn}_{e,\varrho_i}(\chi) = \{ j : 1 \le j < s_i,\ \chi(1_{\varrho_i \boxtimes \mu_{2\beta_{i,j}+1} \boxplus \varrho_i \boxtimes \mu_{2\beta_{i,j+1}+1}}) = -1 \}. \tag{4-13}$$

By convention, $\alpha_{i,0}=0$. For each (φ,χ), define

$$\varphi_{\mathrm{sgn}}=\boxplus_{i=1}^{r}\boxplus_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\rho_i\boxtimes\mu_{2\alpha_{i,j}+1}\boxplus\boxplus_{i=1}^{s}\boxplus_{j\in\mathrm{sgn}_{e,\varrho_i}(\chi)}\varrho_i\boxtimes\mu_{2\beta_{i,j}+2}. \tag{4-14}$$

Note that φ_{sgn} is uniquely determined by the given local Langlands data (φ,χ), but it may possibly be zero. If φ_{sgn} is not zero, then φ_{sgn} and φ are of different type. It is worthwhile to mention that φ_{sgn} is a common component of each elements in the local descent $\mathfrak{D}_\ell(\varphi,\chi)$ (see Definition 4.1), whose dimension gives an upper bound for the index of the first occurrence, i.e., $2\ell_0\le\dim\varphi-\dim\varphi_{\mathrm{sgn}}$. According to Section 6A, such types of L-parameters are closely related to the cuspidal local L-parameters, as discussed in [Aubert et al. 2015], which, by definition, have the property that their L-packets contain at least one irreducible supercuspidal representation.

From φ, we define two new parameters $\lceil\varphi_o\rceil$ and $\varphi^\dagger$ by

$$\lceil\varphi_o\rceil=\boxplus_{i=1}^{s}\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}\quad\text{and}\quad\varphi^\dagger=\boxplus_{i=1}^{s}\varrho_i\boxtimes 1. \tag{4-15}$$

Note that for each i, the piece $\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}$ is the one with maximal dimension among the summands $\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}$'s for $j=1,2,\dots,s_i$. Both $\lceil\varphi_o\rceil$ and $\varphi^\dagger$ are of the same type as φ, and are possibly of orthogonal type and have odd dimension. Define $\chi_{\lceil\varphi_o\rceil}=\chi|_{\lceil\varphi_o\rceil}$, the restriction of χ on the elements $((e_i)|_{\lceil\varphi_o\rceil})$, to be a character of $A_{\lceil\varphi_o\rceil}$. Also $\chi_{\lceil\varphi_o\rceil}$ is considered as a character of $\mathcal{S}_{\lceil\varphi_o\rceil}$ by restriction.

Note that there is an isomorphism from $A_{\lceil\varphi_o\rceil}$ to $A_{\varphi^\dagger}$ given by $(e_i)\in A_{\lceil\varphi_o\rceil}\mapsto(e_i')\in A_{\varphi^\dagger}$ and $e_i=e_i'$, where e_i and e_i' correspond to the component $\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}$ in $A_{\lceil\varphi_o\rceil}$ and $\varrho_i\boxtimes 1$ in $A_{\varphi^\dagger}$, respectively. Hence we have that $\mathcal{S}_{\lceil\varphi_o\rceil}\cong\mathcal{S}_{\varphi^\dagger}$.

For a local generic L-parameter φ with the decomposition (2-6), define its discrete part by

$$\varphi_\square=\boxplus_{i\in\mathrm{I_{gp}}}\varphi_i, \tag{4-16}$$

which is a discrete L-parameter.

Theorem 4.6. *Let φ be a generic L-parameter of even dimension with the discrete part $\varphi_\square$ of the decomposition* (4-9). *Then for $\chi\in\hat{\mathcal{S}}_\varphi$, the descent of the parameter (φ,χ) at the first occurrence index ℓ_0 can be completely determined by the following*

$$\mathfrak{D}_{\ell_0}(\varphi,\chi)=\bigcup_{\psi}[\psi\boxplus\varphi_{\mathrm{sgn}}]_c,$$

where ψ runs over all discrete L-parameters satisfying the following conditions with minimal dimension:

(1) $\dim\psi\equiv\sum_{i=1}^{r}\#\,\mathrm{sgn}_{o,\rho_i}(\chi)\cdot\dim\rho_i\pmod 2$.

(2) $\psi=(\boxplus_{i=1}^{k}\delta_i\boxtimes 1)\boxplus(\boxplus_{i=1}^{r}m_i\rho_i\boxtimes\mu_{2\alpha_{i,r_i}+1})\boxplus(\boxplus_{j=1}^{s}n_j\varrho_j\boxtimes\mu_{2\beta_{j,s_j}+2})$ *with multiplicity $m_i,n_j\in\{0,1\}$, which are multiplicity-free.*

(3) *All δ_i and ρ_i are of the same type as ψ and*

$$\{\delta_i:1\le i\le k\}\cap\{\rho_i:1\le i\le r\}=\varnothing.$$

(4) $\chi_{\lceil\varphi_o\rceil}|_{\mathcal{S}_{\lceil\varphi_o\rceil}}=\chi^\star_{\varphi^\dagger}|_{\mathcal{S}_{\varphi^\dagger}}$ *via $\mathcal{S}_{\lceil\varphi_o\rceil}\cong\mathcal{S}_{\varphi^\dagger}$, where $\chi^\star_{\varphi^\dagger}$ is the character in $\chi^\star(\varphi^\dagger,\psi\boxplus\varphi_{\mathrm{sgn},o})$.*

Note that φ_{sgn}, $\varphi^{\dagger}$ and $\lceil\varphi_o\rceil$ are associated to $\varphi_{\square}$ and $\varphi_{\mathrm{sgn},o}=\boxplus_{i=1}^{r}\boxplus_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\rho_i\boxtimes\mu_{2\alpha_{i,j}+1}$. Recall that $[\phi]_c=\{\phi,\phi^c\}$ is the c-conjugacy class of ϕ, and $\chi^{\star}(\varphi^{\dagger},\psi)$ is well defined even though $\varphi^{\dagger}$ or ψ is of orthogonal type of odd dimension, following from Remark 4.5. In addition, $\psi=0$ is allowed. We will see some examples in Section 6.

Now, let us sketch the proof of Theorem 4.6. First, we only need to calculate the character $\chi^{\star}(\varphi,\phi)$ defined in (2-21) and (2-22) for general discrete parameters φ and ϕ by Corollary 3.2. Following from Lemma 4.2 and their decompositions of form (4-9), we may reduce $\chi^{\star}(\varphi,\phi)$ to the symplectic root numbers of type $\mathcal{E}(\varrho_i,\varrho'_{\ell})$, where ϱ_i and ϱ'_{ℓ} are irreducible self-dual distinct representations of $\mathcal{W}_F$ and are of different type. The formula is stated in Lemma 4.7 below.

Next, at the first occurrence ℓ_0, we have the property that the descent of the parameter (φ,χ) has the minimal dimension such that $\mathfrak{D}_{\ell_0}(\varphi,\chi)$ is not empty. Following this property, we apply Lemma 4.7 repeatedly on the descent parameters in $\mathfrak{D}_{\ell_0}(\varphi,\chi)$ and then we give a refined description of those descent parameters in Theorem 4.6. Its proof will be given in Section 4C.

The rest of Section 4B is devoted to calculating the character $\chi^{\star}_{\varphi}(\varphi,\phi)$ for two discrete parameters φ and ϕ. Due to symmetry, the formula for $\chi^{\star}_{\phi}(\varphi,\phi)$ is similar. In order to state Lemma 4.7, we introduce more notation first.

For an L-parameter φ with the decomposition (4-9), define

$$\varphi_*^{\diamond m}(\rho):=\begin{cases}\boxplus_{\{j:\,1\le j\le r_i,\ \alpha_{i,j}\diamond m\}}\rho_i\boxtimes\mu_{2\alpha_{i,j}} & \text{if } \rho\cong\rho_i \text{ for some } i,\\ \boxplus_{\{j:\,1\le j\le s_i,\ \beta_{i,j}\diamond m\}}\varrho_i\boxtimes\mu_{2\beta_{i,j}+1} & \text{if } \rho\cong\varrho_i \text{ for some } i,\\ 0 & \text{otherwise,}\end{cases}\tag{4-17}$$

where $\diamond\in\{>,<,\ge,\le\}$, $*=e$ if φ and ρ are of the different type and $*=o$, otherwise. Also define

$$\varphi_*^{-}(\rho)=\varphi_*\boxminus\varphi_*(\rho),\tag{4-18}$$

which is the remaining part of the parameter φ_* without the ρ-isotypic component $\varphi_*(\rho)$. For a subrepresentation φ' of φ, write $\#\varphi'$ for the number of the irreducible summands in φ'.

We decompose ϕ as in (4-9), i.e.,

$$\phi=\boxplus_{i=1}^{r'}\boxplus_{j=1}^{r'_i}\rho'_i\boxtimes\mu_{2\alpha'_{i,j}}\boxplus\boxplus_{i=1}^{s'}\boxplus_{j=1}^{s'_i}\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1},\tag{4-19}$$

where ρ'_i and ϱ'_i are of dimension a'_i and b'_i, respectively.

Lemma 4.7. *Let φ and ϕ be discrete L-parameters decomposed as in* (4-9) *and* (4-19) *respectively*, *and be of different type. Then as the character of $\mathcal{S}_{\varphi}$*,

$$\chi^{\star}_{\varphi}((e_{i,j})|_{\varphi_e})=\prod_{i=1}^{r}\prod_{j=1}^{r_i}(\det(\rho_i)(-1)^{\alpha_{i,j}\dim\phi}(-1)^{\#\phi_o^{<\alpha_{i,j}}(\rho_i)})^{e_{i,j}}\tag{4-20}$$

$$\chi^{\star}_{\varphi}((e_{i,j})|_{\varphi_o})=\prod_{i=1}^{s}\prod_{j=1}^{s_i}\left(\left(\prod_{l=1}^{s'}\mathcal{E}(\varrho_i,\varrho'_l)^{s'_l}\right)(-1)^{\#\phi_e^{>\beta_{i,j}}(\varrho_i)}\right)^{e_{i,j}},\tag{4-21}$$

where $\mathcal{E}(\cdot,\cdot)$ is defined in (2-20).

It is clear that $A_\varphi \cong A_{\varphi_e} \times A_{\varphi_o}$ and $\mathcal{S}_\varphi \cong A_{\varphi_e} \times \mathcal{S}_{\varphi_o}$. The isomorphism is given by $(e_i) \mapsto ((e_i)|_{\varphi_e}, (e_i)|_{\varphi_o})$. Equations (4-20) and (4-21) give a formula for all values of $\chi_\varphi^\star$ on $\mathcal{S}_\varphi$. Over A_φ, the formula (4-21) will be slightly different. Note that ρ_i in (4-9) and ρ_i' in (4-19) are of different types, so are ϱ_i and ϱ_i'.

Proof. To prove this lemma, we evaluate $\chi_\varphi^\star$ at three types of elements in $\mathcal{S}_\varphi$:

Type (1): $1_{\rho\boxtimes\mu_{2\alpha}}$.

Type (2): $1_{\varrho\boxtimes\mu_{2\beta+1}}$ where $\dim\varrho$ is even.

Type (3): $1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}}$ where $\dim\varrho$ and $\dim\varrho_0$ are odd.

For each type, the evaluation is reduced to the calculation of the symplectic roots of form $\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi)$, where $\varsigma\boxtimes\mu_\kappa$ is the summand occurring in the subscript of above types. From (4-10), (4-11), and (4-18), we may write that $\phi = \phi_e \boxplus \phi_o = \phi_e \boxplus \phi_o(\varsigma) \boxplus \phi_o^-(\varsigma)$. Then

$$\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi) = \varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi_e)\cdot\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi_o(\varsigma))\cdot\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi_o^-(\varsigma)). \tag{4-22}$$

Finally, we apply Lemma 4.2 to calculate each factor on the right hand side of (4-22).

Type (1): We verify (4-20). Consider a summand of form $\rho\boxtimes\mu_{2\alpha}$ in φ. Write $a = \dim\rho$. Since the dimension of $\rho\boxtimes\mu_{2\alpha}$ is even, $\langle (e_i)|_{\rho\boxtimes\mu_{2\alpha}}\rangle \cong \mathbb{Z}_2$ is a subgroup of $\mathcal{S}_\varphi$ in both symplectic and orthogonal types. For all types, it is sufficient to show

$$\chi_\varphi^\star(1_{\rho\boxtimes\mu_{2\alpha}}) = \det(\rho)(-1)^{\alpha\dim\phi}(-1)^{\#\phi_o^{<\alpha}(\rho)}. \tag{4-23}$$

Here $1_{\rho\boxtimes\mu_{2\alpha}}$ corresponds to the nontrivial element in $\langle (e_i)|_{\rho\boxtimes\mu_{2\alpha}}\rangle$.

By definition, if φ is of symplectic type, then

$$\chi_\varphi^\star(1_{\rho\boxtimes\mu_{2\alpha}}) = \det(\phi)(-1)^{a\alpha}\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi) \tag{4-24}$$

and if φ is of orthogonal type, then $\chi_\varphi^\star(1_{\rho\boxtimes\mu_{2\alpha}}) = \varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi)$, as $\det(\rho) = 1$. Similar to (4-22), we have $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi)$ equal to

$$\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_e)\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho))\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^-(\rho)).$$

As discussed in Remark 4.3, the local root number in this case is independent of the choice of the additive character ψ_F. We omit ψ_F in the calculation. By Lemma 4.2, $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes(\rho_i'\boxtimes\mu_{2\alpha_{i,j}'})) = 1$. It follows that $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_e) = 1$.

Next, we calculate $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho))$. As defined in (4-17), we may write $\phi_o(\rho) = \phi_o^{<\alpha}(\rho)\boxplus\phi_o^{\geq\alpha}(\rho)$. It follows that

$$\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho)) = \varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{<\alpha}(\rho))\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{\geq\alpha}(\rho)).$$

Since $\phi_o^{<\alpha}(\rho)=\boxplus_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}<\alpha\}}\varrho'_{i_0}\boxtimes\mu_{2\beta'_{i_0,j}+1}$, we have, by part (4) of Lemma 4.2, that

$$\begin{aligned}\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{<\alpha}(\rho)) &= \prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}<\alpha\}} \det(\rho)(-1)^{b'_{i_0}\alpha(2\beta'_{i_0,j}+1)}\det(\varrho'_{i_0})(-1)^{a\alpha(2\beta'_{i_0,j}+1)}(-1)^{2\beta'_{i_0,j}+1}\\ &=(-1)^{\#\phi_o^{<\alpha}(\rho)}\times\prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}<\alpha\}}\det(\rho)(-1)^{b'_{i_0}\alpha}\det(\varrho'_{i_0})(-1)^{a\alpha},\end{aligned}\tag{4-25}$$

where i_0 is the index of ϱ'_{i_0} with $\varrho'_{i_0}=\rho$ and $\#\phi_o^{<\alpha}(\rho)$ is the number of irreducibles in $\phi_o^{<\alpha}(\rho)$, i.e., the cardinality of $\{1\le j\le s'_{i_0}:\beta'_{i_0,j}<\alpha_{i,j}\}$.

Since $\phi_o^{\ge\alpha}(\rho)=\boxplus_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}\ge\alpha\}}\varrho'_{i_0}\boxtimes\mu_{2\beta'_{i_0,j}+1}$, we have, by part (4) of Lemma 4.2 again, that

$$\begin{aligned}\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{\ge\alpha}(\rho)) &= \prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}\ge\alpha\}} \det(\rho)(-1)^{b'_{i_0}\alpha(2\beta'_{i_0,j}+1)}\det(\varrho'_{i_0})(-1)^{a\alpha(2\beta'_{i_0,j}+1)}(-1)^{2\alpha}\\ &=\prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}\ge\alpha\}}\det(\rho)(-1)^{b'_{i_0}\alpha}\det(\varrho'_{i_0})(-1)^{a\alpha}.\end{aligned}\tag{4-26}$$

On the other hand, because $\phi_o^{-}(\rho)=\boxplus_{\substack{i=1\\ i\ne i_0}}^{s'}\boxplus_{j=1}^{s'_i}\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1}$, we have, by part (3) of Lemma 4.2, that

$$\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{-}(\rho))=\prod_{\substack{i=1\\ i\ne i_0}}^{s'}\prod_{j=1}^{s'_i}\det(\rho)(-1)^{b'_i\alpha(2\beta'_{i,j}+1)}\det(\varrho'_i)(-1)^{a\alpha(2\beta'_{i,j}+1)}.\tag{4-27}$$

Finally, by taking the product of (4-25), (4-26), and (4-27), we obtain that

$$\begin{aligned}&\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho))\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{-}(\rho))\\ &\qquad=\left(\prod_{i=1}^{s'}\prod_{j=1}^{s'_i}\det(\rho)(-1)^{b'_i\alpha}\det(\varrho'_i)(-1)^{a\alpha}\right)\times(-1)^{\#\phi_o^{<\alpha}(\rho)}\\ &\qquad=\det(\rho)(-1)^{(\sum_{i=1}^{s'}b'_is'_i)\alpha}\times\left(\prod_{i=1}^{s'}\det(\varrho'_i)(-1)^{s'_i}\right)^{a\alpha}\times(-1)^{\#\phi_o^{<\alpha}(\rho)}\\ &\qquad=\det(\rho)(-1)^{\alpha\dim\phi}\times\det(\phi)(-1)^{a\alpha}\times(-1)^{\#\phi_o^{<\alpha}(\rho)}.\end{aligned}$$

Indeed, since ϕ_e is of even dimension, $\sum_{i=1}^{s'}b'_is'_i$ and $\dim\phi$ are of the same parity.

If φ is of symplectic type, continuing with (4-24), we obtain (4-23). If φ is of orthogonal type, then ϕ is of symplectic type, which implies that $\det(\phi)=1$. One also has (4-23).

Next, we show (4-21) by considering summands of form $\varrho\boxtimes\mu_{2\beta+1}$ in φ. Write $b=\dim\varrho$.

Type (2): Assume that ϱ is of even dimension (that is, b is even). In this case, since $\varrho\boxtimes\mu_{2\beta+1}$ has even dimension, $\langle(e_i)|_{\varrho\boxtimes\mu_{2\beta+1}}\rangle\cong\mathbb{Z}_2$ is a subgroup of $\mathcal{S}_\varphi$ regardless to the type of φ. Similarly, denote by

$1_{\varrho\boxtimes\mu_{2\beta+1}}$ the nontrivial element in $\langle(e_i)|_{\varrho\boxtimes\mu_{2\beta+1}}\rangle$. By definition, $\chi_\varphi^\star(1_{\varrho\boxtimes\mu_{2\beta+1}})$ equals

$$\begin{cases}\det(\phi)(-1)^{b(2\beta+1)/2}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi) & \text{if } \varphi \text{ is of symplectic type,}\\ \det(\varrho)(-1)^{m}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi) & \text{if } \varphi \text{ is of orthogonal type,}\end{cases} \tag{4-28}$$

where $m=\frac{\dim\phi}{2}$. Under the assumption, we need to show that

$$\chi_\varphi^\star(1_{\varrho\boxtimes\mu_{2\beta+1}})=\prod_{i=1}^{s'}\mathcal{E}(\varrho,\varrho_i')^{s_i'}(-1)^{\#\phi_e^{>\beta}(\varrho)}. \tag{4-29}$$

Recall from (2-20) that

$$\mathcal{E}(\varrho,\varrho_i')=\begin{cases}\det(\varrho_i')(-1)^{b/2}\varepsilon(\varrho\otimes\varrho_i') & \text{if } \varphi \text{ is of symplectic type,}\\ \det(\varrho)(-1)^{b_i'/2}\varepsilon(\varrho\otimes\varrho_i') & \text{if } \varphi \text{ is of orthogonal type.}\end{cases}$$

We may write that $\phi=\phi_e\boxplus\phi_o=\phi_e(\varrho)\boxplus\phi_e^-(\varrho)\boxplus\phi_o$, following from (4-10), (4-11), and (4-18) again. It follows that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi)=\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e)\cdot\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_o,\psi_F),$$
$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e)=\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e(\varrho))\cdot\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho)).$$

Here only the term $\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_o,\psi_F)$ is possibly dependent on ψ_F. By part (2) of Lemma 4.2, when $\varrho\otimes\varrho_i'$ is of symplectic type, we have that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes(\varrho_i'\boxtimes\mu_{2\beta_{i,j}'+1}),\psi_F)=\varepsilon(\varrho\otimes\varrho_i'),$$

which is independent of ψ_F. As $\phi_o=\boxplus_{i=1}^{s'}\boxplus_{j=1}^{s_i'}\varrho_i'\boxtimes\mu_{2\beta_{i,j}'+1}$, we have that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_o,\psi_F)=\prod_{i=1}^{s'}\prod_{j=1}^{s_i'}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes(\varrho_i'\boxtimes\mu_{2\beta_{i,j}'+1}),\psi_F)=\prod_{i=1}^{s'}\varepsilon(\varrho\otimes\varrho_i')^{s_i'},$$

which is independent of ψ_F.

Now, let us calculate the first two terms: $\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e(\varrho))$ and $\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho))$.

Since $\phi_e^-(\varrho)=\boxplus_{\substack{i=1\\ i\neq i_0}}^{r'}\boxplus_{j=1}^{r_i'}\rho_i'\boxtimes\mu_{2\alpha_{i,j}'}$, by part (3) of Lemma 4.2, we have that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho))=\prod_{\substack{i=1\\ i\neq i_0}}^{r'}\prod_{j=1}^{r_i'}\det(\varrho)(-1)^{a_i'\alpha_{i,j}'(2\beta+1)}\det(\rho_i')(-1)^{b\alpha_{i,j}'(2\beta+1)}, \tag{4-30}$$

where i_0 is the index of ρ_{i_0}' with $\rho_{i_0}'=\varrho$.

Following (4-25) and (4-26), we write $\phi_e(\varrho)=\phi_e^{\le\beta}(\varrho)\boxplus\phi_e^{>\beta}(\varrho)$. By part (4) of Lemma 4.2, since $\phi_e^{\le\beta}(\varrho)=\boxplus_{\{1\le j\le r'_{i_0}\,:\,\alpha'_{i_0,j}\le\beta\}}\rho'_{i_0}\boxtimes\mu_{2\alpha'_{i_0,j}}$, we have that

$$\begin{aligned}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^{\le\beta}(\varrho)) &= \prod_{\{1\le j\le r'_{i_0}\,:\,\alpha'_{i_0,j}\le\beta\}} \det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}(2\beta+1)}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}(2\beta+1)}\cdot(-1)^{2\alpha'_{i_0,j}}\\ &= \prod_{\{1\le j\le r'_{i_0}\,:\,\alpha'_{i_0,j}\le\beta\}} \det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}}. \qquad (4\text{-}31)\end{aligned}$$

Since $\phi_e^{>\beta}(\varrho)=\boxplus_{\{1\le j\le r'_{i_0}\,:\,\alpha'_{i_0,j}>\beta\}}\rho'_{i_0}\boxtimes\mu_{2\alpha'_{i_0,j}}$, we have, by part (4) of Lemma 4.2, that

$$\begin{aligned}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^{>\beta}(\varrho)) &= \prod_{\{1\le j\le r'_{i_0}\,:\,\alpha'_{i_0,j}>\beta\}} \det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}(2\beta+1)}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}(2\beta+1)}\cdot(-1)^{2\beta+1}\\ &= (-1)^{\#\phi_e^{>\beta}(\varrho)}\times\prod_{\{1\le j\le r'_{i_0}\,:\,\alpha'_{i_0,j}>\beta\}} \det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}}, \qquad (4\text{-}32)\end{aligned}$$

where $\#\phi_e^{>\beta}(\varrho)$ is the number of irreducibles in $\phi_e^{>\beta}(\varrho)$.

Finally, by taking the product of (4-30), (4-31), and (4-32), we obtain that

$$\begin{aligned}&\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e(\varrho))\cdot\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho))\\ &\qquad= \left(\prod_{i=1}^{r'}\prod_{j=1}^{r'_i}\det(\varrho)(-1)^{a'_i\alpha'_{i,j}}\det(\rho'_i)(-1)^{b\alpha'_{i,j}}\right)\times(-1)^{\#\phi_e^{>\beta}(\varrho)}\\ &\qquad= \det(\varrho)(-1)^{\sum_{i=1}^{r'}\sum_{j=1}^{r'_i}a'_i\alpha'_{i,j}}\cdot(-1)^{\#\phi_e^{>\beta}(\varrho)}.\end{aligned}$$

Recall that b is even and $\det\rho'_i(-1)^b=1$.

If φ is of symplectic type, then ϱ is of symplectic type, which implies that $\det(\varrho)(-1)=1$. Continuing with (4-28) and by

$$\det(\phi)(-1)=\det(\phi_o)(-1)=\prod_{i=1}^{s'}\det(\varrho_i)(-1)^{s'_i},$$

one has (4-29).

If φ is of orthogonal type and b is even, then ϱ and ρ'_i are of orthogonal type and ϱ'_i is of symplectic type. Because

$$m\equiv\sum_{i=1}^{r'}\sum_{j=1}^{r'_i}a'_i\alpha'_{i,j}+\sum_{i=1}^{s'}s'_i\tfrac{b'_i}{2}\ \bmod 2,$$

continuing with (4-28), one obtains (4-29).

Type (3): Assume that $b=\dim\varrho$ is odd, which implies that φ is of orthogonal type. Let $\varrho_0\boxtimes\mu_{2\beta_0+1}$ be any different summand in φ such that $b_0:=\dim\varrho_0$ is odd. Consider the following subgroup of $\mathcal{S}_\varphi$

$$\langle (e_i)|_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}}:(e_i)\in\mathcal{S}_\varphi\rangle\cong\mathbb{Z}_2.$$

Denote by $1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}}$ the nontrivial element in the above subgroup. If $\varrho_0\boxtimes\mu_{2\beta_0+1}$ does not exist, we do not need to consider this case as $\chi_\varphi^\star$ is a character of $\mathcal{S}_\varphi$. Then we have

$$\chi_\varphi^\star(1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}})=\det(\varrho)(-1)^m\det(\varrho_0)(-1)^m\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi)\varepsilon((\varrho_0\boxtimes\mu_{2\beta_0+1})\otimes\phi).$$

Following the above calculation, one obtains that

$$\begin{aligned}
&\chi_\varphi^\star(1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}})\\
&\quad=\det(\varrho)(-1)^m\cdot\det(\varrho_0)(-1)^m\times\prod_{i=1}^{r'}\prod_{j=1}^{r'_i}(\det(\varrho)(-1)\det(\varrho_0)(-1))^{\alpha'_i\alpha'_{i,j}}\prod_{i=1}^{r'}\prod_{j=1}^{r'_i}\det(\rho'_i)(-1)^{(b+b_0)\alpha'_{i,j}}\\
&\qquad\qquad\times\prod_{i=1}^{s'}\varepsilon(\varrho\otimes\varrho'_i)^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho)}\prod_{i=1}^{s'}\varepsilon(\varrho_0\otimes\varrho'_i)^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho_0)}\\
&\quad=\prod_{i=1}^{s'}(\det(\varrho)(-1)^{b'_i/2}\varepsilon(\varrho\otimes\varrho'_i))^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho)}\times\prod_{i=1}^{s'}(\det(\varrho_0)(-1)^{b'_i/2}\varepsilon(\varrho_0\otimes\varrho'_i))^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho_0)}.
\end{aligned}$$

Finally, by putting all the calculations above together, we obtain (4-21). □

At the end of this section, we present an example of Lemma 4.7, which also will be used in the proof of Theorem 4.6. Let $\varphi^\dagger$ be a discrete L-parameter of the decomposition defined in (4-15).

Example 4.8. Let ψ be a discrete L-parameter of the different type from $\varphi^\dagger$ and ψ be given in item (2) of Theorem 4.6. Applying Lemma 4.7 to both $\varphi^\dagger$ and $\psi\boxplus\varphi_{\mathrm{sgn},o}$, one has

$$\chi_{\varphi^\dagger}^\star((e_i))=\prod_{i=1}^{s}\left(\left(\prod_{l=1}^{k}\mathcal{E}(\varrho_i,\delta_l)\right)\cdot\left(\prod_{j=1}^{r}\mathcal{E}(\varrho_i,\rho_j)^{m_j+\#\mathrm{sgn}_{o,\rho_j}(\chi)}\right)\cdot(-1)^{n_i}\right)^{e_i},\tag{4-33}$$

where $\mathrm{sgn}_{o,\rho_i}(\chi)$ is defined in (4-12).

Note that from this example the explicit description of $\mathfrak{D}_{\ell_0}(\varphi,\chi)$ is reduced to finding a ψ of minimal dimension satisfying (4-33).

4C. ***Proof of Theorem 4.6.*** Following Corollary 3.2, we have

$$\mathfrak{D}_{\ell_0}(\varphi,\chi)=\mathfrak{D}_{\ell_0}(\varphi_\square,\chi),$$

and all local L-parameters ϕ in $\mathfrak{D}_{\ell_0}(\varphi_\square,\chi)$ are discrete. The proof is reduced to the case where φ is discrete. It is enough to show that items (1), (2), (3), and (4) are *necessary* and *sufficient* to characterize the set $\mathfrak{D}_{\ell_0}(\varphi,\chi)$. First, assume that $\chi_\varphi^\star=\chi$. Applying Lemma 4.7, we conclude that items (2) and (3) are the necessary conditions for the descent parameters in $\mathfrak{D}_{\ell_0}(\varphi,\chi)$. Note that item (1) holds by the definition. Then assume that ϕ is of the decomposition $\psi\boxplus\varphi_{\mathrm{sgn}}$, where ψ is given in items (2) and satisfies (3). We show that for such ϕ, $\chi_\varphi^\star(\varphi,\phi)=\chi$ is equivalent to $\chi_{\lceil\varphi_o\rceil}=\chi_{\varphi^\dagger}^\star$, which is item (4). The calculation of $\chi_{\varphi^\dagger}^\star$ is given in Example 4.8. Finally, by the minimality condition on $\dim\phi$, items (1), (2), (3), and (4) are sufficient.

Necessity. Take ϕ in $\mathfrak{D}_{\ell_0}(\varphi,\chi)$. Then ϕ is of the minimal even dimension such that $\chi^\star_\varphi=\chi$ for the pair (φ,ϕ). By Corollary 3.2, ϕ is discrete.

First, we consider the subrepresentation $\phi_o(\rho_i)$. By (4-20) in Lemma 4.7, the parity of $\#\phi_o^{<\alpha_{i,j}}(\rho_i)$ is determined by $\chi(1_{\rho_i\boxtimes\mu_{\alpha_{i,j}}})$. By (4-21), the value $\chi^\star_\varphi((e_{i,j})|_{\varphi_o})$ is partially affected by $\varepsilon(\varrho\otimes\varrho'_{i'})^{s'_{i'}}$ with $\varrho'_{i'}\cong\rho_i$ where $s'_{i'}=\#\phi_o(\rho_i)$, and more precisely by the parity of $s'_{i'}$. By the minimality of $\dim\phi$, one has that

$$\phi_o^{<\alpha_{i,r_i}}(\rho_i)=\boxplus_{i=1}^{r}\boxplus_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\rho_i\boxtimes\mu_{2\alpha_{i,j}+1},$$

where $\mathrm{sgn}_{o,\rho_i}(\chi)$ is defined in (4-12).

Next, we consider the component $\phi_e(\varrho_i)$. For $1\le j_1<j_2\le s_i$, by Lemma 4.7, we have that

$$\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j_1}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j_2}+1}})=(-1)^{\#\phi_e^{>\beta_{i,j_1}}(\varrho_i)+\#\phi_e^{>\beta_{i,j_2}}(\varrho_i)}.\tag{4-34}$$

The parity of $\#\phi_e^{>\beta_{i,j_1}}(\varrho_i)\pm\#\phi_e^{>\beta_{i,j_2}}(\varrho_i)$ is uniquely determined by $\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j_1}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j_2}+1}})$. In addition, $\chi^\star_\varphi((e_i)|_{\varphi^-(\varrho)})$ is independent of $\phi_e(\varrho_i)$. The only requirement on $\phi_e^{<\beta_{i,s_i}}(\varrho_i)$ is that (4-34) holds for all $1\le j_1<j_2\le s_i$. Then by the minimality of $\dim\phi$ one has, for $1\le j<r_i$, that

$$\phi_e^{>\beta_{i,j}}(\varrho_i)\boxminus\phi_e^{>\beta_{i,j+1}}(\varrho_i)=\begin{cases}\varrho_i\boxtimes\mu_{2(\beta_{i,j}+1)} & \text{if } \chi(\cdot)=-1,\\ 0 & \text{if } \chi(\cdot)=1.\end{cases}$$

Here $\chi(\cdot)=\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j+1}+1}})$ for simplicity. Thus, we obtain that

$$\phi_e^{<\beta_{i,s_i}}(\varrho_i)=\boxplus_{i=1}^{s}\boxplus_{j\in\mathrm{sgn}_{e,\varrho_i}(\chi)}\varrho_i\boxtimes\mu_{2\beta_{i,j}+2},$$

where $\mathrm{sgn}_{e,\varrho_i}(\chi)$ is defined in (4-13).

Now we rewrite $\phi=\psi\boxplus\varphi_{\mathrm{sgn}}$, where ψ and φ_{sgn} (see (4-14) for definition) have no common irreducibles. By the above discussion, Example 4.8 and Lemma 4.7, we may assume that $\psi_o^{\le\alpha_{i,r_i}}(\rho_i)$ and $\psi_e^{\le\beta_{i,s_i}}(\varrho_i)$ are zero for all ρ_i and ϱ_i. In the rest of the proof, we will repeatedly apply Lemma 4.7 and the minimality of $\dim\phi$ to obtain the requirements on ψ. First, no matter what $\psi_e(\rho)$ for $\rho\notin\{\varrho_1,\dots,\varrho_s\}$ are, $\chi^\star_\varphi$ does not change in (4-20) and (4-21). It follows that $\psi_e(\rho)=0$ for $\rho\notin\{\varrho_1,\dots,\varrho_s\}$. Next, note that for all ϱ_i only the parity of $\#\psi_e^{>\beta_{i,s_i}}(\varrho_i)$ has nontrivial contribution in (4-20) and (4-21), which implies, for all $1\le j\le s$, that

$$\psi_e(\varrho_i)=n_j\varrho_j\boxtimes\mu_{2\beta_{j,s_j}+2},\tag{4-35}$$

where $n_j\in\{0,1\}$. Finally, let us consider $\psi_o(\rho)$ in two cases: $\rho\notin\{\rho_1,\dots,\rho_r\}$ or $\rho\cong\rho_i$ for some i. If $\rho\notin\{\rho_1,\dots,\rho_r\}$, only the parity of $\#\psi_o(\rho)$ is involved in (4-21). When $\psi_o(\rho)\ne0$, one has that

$$\psi_o(\rho)=\rho\boxtimes1.\tag{4-36}$$

If $\rho\cong\rho_i$ for some i, only the parity of $\#\,\mathrm{sgn}_{o,\rho_i}(\chi)-\#\psi_o^{\ge\alpha_{i,r_i}}(\rho_i)$ possibly changes the value of $\chi^\star_\varphi$ in (4-21). Thus, we obtain that

$$\psi_o(\rho_i)=m_i\rho_i\boxtimes\mu_{2\alpha_{i,r_i}+1},\tag{4-37}$$

where $m_i \in \{0, 1\}$. Combining (4-35), (4-36) and (4-37), we obtain items (2) and (3), which are necessary conditions on ψ.

Sufficiency. Assume that $\phi = \psi \boxplus \varphi_{\mathrm{sgn}}$ and ψ is of the form in item (2) satisfying item (3). By the above discussion and (4-20), $\chi|_{\varphi_e} = \chi^\star_\varphi|_{\varphi_e}$. In order to complete the proof, it is sufficient to show that $\chi|_{\varphi_o} = \chi^\star_\varphi|_{\varphi_o}$ if and only if $\chi_{\lceil\varphi_o\rceil} = \chi^\star_{\varphi^\dagger}$. In the rest of the proof, all the characters are on the corresponding subgroups $\mathcal{S}_{\varphi_o}$, $\mathcal{S}_{\lceil\varphi_o\rceil}$ and $\mathcal{S}_{\varphi^\dagger}$.

By applying (4-21) in Lemma 4.7 to φ and $\varphi^\dagger$, and by (4-33) in Example 4.8, we have

$$\chi^\star_\varphi((e_{i,j})|_{\varphi_o}) = \chi^\star_{\varphi^\dagger}\left(\left(\sum_{k=1}^{s_i} e_{i,k}\right)\right) \cdot \prod_{i=1}^{s}\prod_{j=1}^{s_i}((-1)^{\#\varphi^{>\beta_{i,j}}_{\mathrm{sgn},e}(\varrho_i)})^{e_{i,j}}. \tag{4-38}$$

Define $f\colon (e_{i,j})|_{\varphi_o} \in \mathcal{S}_{\varphi_o} \mapsto \left(\sum_{j=1}^{s_i} e_{i,j}\right) \in A_{\varphi^\dagger}$, where $\mathcal{S}_{\varphi_o}$ is considered as a subgroup of $\mathcal{S}_\varphi$. It is easy to check that f is surjective on $\mathcal{S}_{\varphi^\dagger}$. (In general it is not surjective on $A_{\varphi^\dagger}$.) Denote $f|_{\lceil\varphi_o\rceil}$ to be the restriction map into $\mathcal{S}_{\lceil\varphi_o\rceil}$. Then $f|_{\lceil\varphi_o\rceil}$ is an isomorphism. Following (4-38) and by $\#\varphi^{>\beta_{i,s_i}}_{\mathrm{sgn},e}(\varrho_i) = 0$ for all $1 \le i \le s$, one has that

$$\chi^\star_\varphi((e_{i,j})|_{\lceil\varphi_o\rceil}) = \chi^\star_{\varphi^\dagger}((e_{i,s_i})) = \chi^\star_{\varphi^\dagger}(f((e_{i,j})|_{\lceil\varphi_o\rceil})). \tag{4-39}$$

By (4-39), if $\chi|_{\varphi_o} = \chi^\star_\varphi|_{\varphi_o}$ then $\chi_{\lceil\varphi_o\rceil} = \chi^\star_{\varphi^\dagger}$ as $f|_{\lceil\varphi_o\rceil}$ is an isomorphism.

Suppose that $\chi_{\lceil\varphi_o\rceil} = \chi^\star_{\varphi^\dagger}$. Let $\varrho_i \boxtimes \mu_{2\beta_{i,j}+1}$ be an irreducible subrepresentation of φ_o. We consider two cases: $\dim\varrho_i$ is even and $\dim\varrho_i$ is odd, respectively, as $\mathcal{S}_{\varphi_o}$ is generated by the elements of two types $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}$ and $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+1}}$ with $i \ne i'$, or $i = i'$ and $j \ne j'$.

Assume that $\dim\varrho_i$ is even. In this case, $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}$ is in $\mathcal{S}_{\varphi_o}$. Set

$$j_0 = \#\{j \le l < s_i : l \in \mathrm{sgn}_{e,\varrho_i}(\chi)\} = \#\varphi^{>\beta_{i,j}}_{\mathrm{sgn},e}(\varrho_i).$$

By the definition of $\mathrm{sgn}_{e,\varrho_i}(\chi)$, one has that

$$\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}) = (-1)^{j_0}\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}}) = (-1)^{j_0}\chi_{\lceil\varphi_o\rceil}(1_{\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}}).$$

By (4-38), we have that $\chi^\star_\varphi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}) = \chi^\star_{\varphi^\dagger}(1_{\varrho_i\boxtimes 1})(-1)^{\#\varphi^{>\beta_{i,j}}_{\mathrm{sgn},e}(\varrho_i)}$. Thus, we obtain that $\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}) = \chi^\star_\varphi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}})$.

Assume that $\dim\varrho_i$ is odd. Consider the elements in $\mathcal{S}_{\varphi_o}$ of form $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+1}}$ with $i \ne i'$, or $i = i'$ and $j \ne j'$. Suppose that $i = i'$ and $j < j'$. Set

$$j_0 = \#\{j \le l < j' : l \in \mathrm{sgn}_{e,\varrho_i}(\chi)\} = \#\varphi^{>\beta_{i,j}}_{\mathrm{sgn},e}(\varrho_i) - \#\varphi^{>\beta_{i,j'}}_{\mathrm{sgn},e}(\varrho_i).$$

One has that

$$\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}}) = (-1)^{j_0} = (-1)^{\#\varphi^{>\beta_{i,j}}_{\mathrm{sgn},e}(\varrho_i) - \#\varphi^{>\beta_{i,j'}}_{\mathrm{sgn},e}(\varrho_i)}.$$

Since $f(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}}) = 0 \in \mathcal{S}_{\varphi^\dagger}$, one has, by (4-38), that

$$\chi^\star_\varphi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}}) = \chi^\star_{\varphi^\dagger}(0)(-1)^{\#\varphi^{>\beta_{i,j}}_{\mathrm{sgn},e}(\varrho_i) + \#\varphi^{>\beta_{i,j'}}_{\mathrm{sgn},e}(\varrho_i)},$$

which is equal to $\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}})$.

Suppose that $i\neq i'$. Denote $1_{(i,j),(l,k)}=1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_l\boxtimes\mu_{2\beta_{l,k}+1}}$. When $i=l$ and $j=k$, define that $1_{(i,j),(l,k)}=0$. Then $1_{(i,j),(l,k)}$ is in $\mathcal{S}_{\varphi_o}$. We rewrite that

$$1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+1}}=1_{(i,j),(i,s_i)}+1_{(i,s_i),(i',s_{i'})}+1_{(i',s_{i'}),(i',j')}.$$

In the above case, we proved that $\chi(1_{(i,j),(i,s_i)})=\chi^\star_\varphi(1_{(i,j),(i,s_i)})$ and $\chi(1_{(i',s_{i'}),(i',j')})=\chi^\star_\varphi(1_{(i',s_{i'}),(i',j')})$.

It remains to show that $\chi(1_{(i,s_i),(i',s_{i'})})=\chi^\star_\varphi(1_{(i,s_i),(i',s_{i'})})$ with $i\neq i'$. By definition, one has that $\chi(1_{(i,s_i),(i',s_{i'})})=\chi_{\lceil\varphi_o\rceil}(1_{(i,s_i),(i',s_{i'})})$. Note that $\#\varphi^{>\beta_{i,s_i}}_{\mathrm{sgn},e}(\varrho_i)=\#\varphi^{>\beta_{i',s_{i'}}}_{\mathrm{sgn},e}(\varrho_{i'})=0$. It then follows from (4-38) that $\chi^\star_\varphi(1_{(i,s_i),(i',s_{i'})})=\chi^\star_{\varphi^\dagger}(1_{\varrho_i\boxtimes 1\boxplus\varrho_{i'}\boxtimes 1})$, which implies that $\chi(1_{(i,s_i),(i',s_{i'})})=\chi^\star_\varphi(1_{(i,s_i),(i',s_{i'})})$. Therefore, we complete the proof of Theorem 4.6.

5. Proof of Theorems 1.5 and 1.7

In this section, we will apply our main results from Theorem 4.6 to prove Theorems 1.5 and 1.7. First, we use the descents of discrete parameters studied in Section 4 to recover the local descents of representations via the local Langlands corresponding.

Proposition 5.1 (part (1) of Theorem 1.7). *For a $\pi\in\Pi(G_n)$ with a generic L-parameter φ, the first occurrence index of π can be calculated by*

$$\ell_0(\pi)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}=n-\tfrac{1}{2}\dim(\varphi_\square)+\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\},$$

where $\mathcal{O}_{\mathcal{Z}}(\pi)$ is defined in (2-17) *and $\varphi_\square$ is defined in* (4-16). *Moreover, suppose that χ_0 is a character such that*

$$\ell_0(\varphi_\square,\chi_0)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\}$$

holds, and that $\pi=\pi_a(\varphi,\chi_0)$ under the local Langlands correspondence ι_a. Define $\pi_\square:=\pi_a(\varphi_\square,\chi_0)\in\Pi(G_{n_0})$ where $n_0=\frac{1}{2}\dim\varphi_\square$. If $\mathrm{disc}(\mathcal{O}_{\ell_0(\pi)})=\mathrm{disc}(\mathcal{O}_{\ell_0(\pi_\square)})$, then an irreducible square integrable quotient occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi)$ if and only if it occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi_\square)}}(\pi_\square)$.

Proof. For each $\chi\in\mathcal{S}_\varphi\cong\mathcal{S}_{\varphi_\square}$ and a generic L-parameter ϕ of type different from that of φ, since $\chi^\star(\varphi,\phi)=\chi^\star(\varphi_\square,\phi)$, we have $\mathfrak{D}_\ell(\varphi,\chi)=\mathfrak{D}_\ell(\varphi_\square,\chi)$. Hence $\ell_0(\varphi,\chi)=n-\frac{1}{2}\dim(\varphi_\square)+\ell_0(\varphi_\square,\chi)$. It suffices to show that $\ell_0(\pi)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}$.

By Corollary 3.2, $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is nonzero for some rational orbit $\mathcal{O}_{\ell_0}$ if and only if there exists a nonzero irreducible square-integrable representation σ of $G_n^{\mathcal{O}_{\ell_0}}$ such that $m(\pi,\sigma)\neq 0$. If $m(\pi,\sigma)\neq 0$, assume that $\pi=\pi_a(\varphi,\chi_\pi)$ and $\sigma=\pi_a(\phi,\chi_\sigma)$. By Theorem 2.4, $(\chi_\pi,\chi_\sigma)=\chi^\star(\varphi,\phi)$ and then $[\phi]_c$ is in $\mathfrak{D}_{\ell_0(\pi)}(\varphi,\chi_\pi)$. It follows that

$$\ell_0(\pi)\le\ell_0(\varphi,\chi_\pi)\le\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}.$$

On the other hand, let χ_0 be in $\mathcal{O}_{\mathcal{Z}}(\pi)$ such that

$$\ell_0(\varphi,\chi_0)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}=n-\tfrac{1}{2}\dim(\varphi_\square)+\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\}.$$

Take an equivalence class $[\phi]_c$ of the local L-parameters in $\mathfrak{D}_{\ell_0^{\max}}(\varphi, \chi_0)$ at the first occurrence index $\ell_0^{\max} = \ell_0(\varphi, \chi_0)$. Choose the local Langlands correspondence ι_a such that $\pi = \pi_a(\varphi, \chi_0)$. By Definition 4.1, $\chi_0 = \chi_\varphi^\star(\varphi, \phi)$. Denote $\chi_\phi = \chi_\phi^\star(\varphi, \phi)$ and $\sigma = \pi_a(\phi, \chi_\phi)$ under the same Langlands correspondence ι_a. By Theorem 2.4, we have $m(\pi, \sigma) = 1$ and σ is an irreducible quotient in $\mathcal{D}_{\mathcal{O}_{\ell_0^{\max}}}(\pi)$. Then $\mathcal{D}_{\mathcal{O}_{\ell_0^{\max}}}(\pi) \neq 0$ and $\ell_0(\pi) \geq \ell_0^{\max}$. Hence, we have that $\ell_0(\pi) = \ell_0^{\max}$.

Consider the local descents given by the rational orbits $\mathcal{O}_{\ell_0(\pi)}$ and $\mathcal{O}_{\ell_0(\pi_\square)}$ with $\operatorname{disc}(\mathcal{O}_{\ell_0(\pi)}) = \operatorname{disc}(\mathcal{O}_{\ell_0(\pi_\square)})$. Referring to Theorem 2.4 and Remark 2.5, the local Langlands correspondence ι_b for π is determined by $\operatorname{disc}(\mathcal{O}_{\ell_0(\pi)})$, so is the same choice for $\pi_\square$. Recall that the corresponding character of $\mathcal{S}_\varphi \cong \mathcal{S}_{\varphi_\square}$ is denoted by $\chi_b := \chi_b(\pi) = \chi_b(\pi_\square)$.

By the definition in (2-17), $\mathcal{O}_{\mathcal{Z}}(\pi) = \mathcal{O}_{\mathcal{Z}}(\pi_\square)$ and then $\ell_0(\pi_\square) = \max_{\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square, \chi)\}$. By the above proof, $\ell_0(\pi) = n - \frac{1}{2}\dim \varphi_\square + \ell_0(\pi_\square)$. If $\operatorname{disc}(\mathcal{O}_{\ell_0(\pi)}) = \operatorname{disc}(\mathcal{O}_{\ell_0(\pi_\square)})$, then $G_n^{\mathcal{O}_{\ell_0(\pi)}} \cong G_{n_0}^{\mathcal{O}_{\ell_0(\pi_\square)}}$.

If $\ell_0(\varphi, \chi_b(\pi)) < \ell_0(\pi)$ (equivalently, if $\ell_0(\varphi_\square, \chi_b(\pi_\square)) < \ell_0(\pi_\square)$), then both $\mathfrak{D}_{\ell_0(\pi)}(\varphi, \chi_b)$ and $\mathfrak{D}_{\ell_0(\pi_\square)}(\varphi_\square, \chi_b)$ are empty. By Theorem 2.4, $\mathcal{D}_{\ell_0(\pi)}(\pi) = \mathcal{D}_{\ell_0(\pi_\square)}(\pi_\square) = 0$.

Assume that $\ell_0(\varphi, \chi_a(\pi)) = \ell_0(\pi)$ for some ι_a as denoted in the proposition. With the above notation, an irreducible square-integrable representation σ of $G_n^{\mathcal{O}_{\ell_0}}$ occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi)$ if and only if $(\chi_a(\pi), \chi_\sigma) = \chi^\star(\varphi, \phi)$ by Theorem 2.4. Recall that by the definition of $\pi_\square$, $\chi_a(\pi) = \chi_a(\pi_\square)$ under the same local Langlands correspondence. By $\chi^\star(\varphi, \phi) = \chi^\star(\varphi_\square, \phi)$, we have $\chi^\star(\varphi_\square, \phi) = (\chi_a(\pi_\square), \chi_\sigma)$, which is equivalent that σ is also an irreducible quotient of $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi_\square)$. □

In order to prove Theorem 1.5, it remains to show that the irreducible quotients of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ (at the first occurrence index $\ell_0 = \ell_0(\pi)$) belong to different Bernstein components of $G_n^{\mathcal{O}_{\ell_0}}(F)$.

Theorem 5.2 (Theorem 1.5). *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, irreducible quotients of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ at the first occurrence index $\ell_0 = \ell_0(\pi)$ of π belong to different Bernstein components. Moreover, $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ can be written as a multiplicity-free direct sum of irreducible square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$, and hence is square-integrable and admissible.*

Remark 5.3. In general, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ at the first occurrence index could be a direct sum of infinitely many irreducible nonsupercuspidal square-integrable representations. When $\ell < \ell_0$, the descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ may not be completely reducible.

Proof. Assume that $\pi = \pi_a(\varphi, \chi)$ under local Langlands correspondence ι_a for both even and odd special orthogonal groups discussed in Section 2B. The local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a smooth representation of $G_n^{\mathcal{O}_{\ell_0}}(F)$. By the Bernstein decomposition, $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a direct sum of its Bernstein components. Thus, it is sufficient to show that if σ_1 and σ_2 are nonisomorphic irreducible quotients of $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$, i.e.,

$$m(\pi_a(\varphi, \chi), \sigma_1) = m(\pi_a(\varphi, \chi), \sigma_2) = 1,$$

then σ_1 and σ_2 have different cuspidal supports. Corollary 3.2 asserts that σ_1 and σ_2 are square-integrable.

By Proposition 5.1, we may assume, without loss of generality, that φ is discrete. Then $\sigma_1 = \pi_a(\phi_1, \xi_1)$ and $\sigma_2 = \pi_a(\phi_2, \xi_2)$, where ϕ_1 and ϕ_2 are of the form in Theorem 4.6. We have decompositions

$\phi_i = \psi_i \boxplus \varphi_{\mathrm{sgn}}$ for $i = 1, 2$. Let λ_{ϕ_i} be the infinitesimal characters of ϕ_i (as explained in [Arthur 2013, p. 69], for instance). That is,

$$\lambda_{\phi_i}(w) = \phi_i\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

Referring to [Aubert et al. 2015], if $\lambda_{\phi_1} \neq \lambda_{\phi_2}$, then $\Pi_{\phi_1}(G_n^{\mathcal{O}_{\ell_0}})$ and $\Pi_{\phi_2}(G_n^{\mathcal{O}_{\ell_0}})$ belong to different Bernstein components. Because $\phi_1 \neq \phi_2$, we have that $\psi_1 \neq \psi_2$, which implies that $\lambda_{\psi_1} \neq \lambda_{\psi_2}$ by the definition of ψ_i. Then $\lambda_{\phi_1} \neq \lambda_{\phi_2}$. Hence each Bernstein component of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ has a unique irreducible quotient, which is square-integrable by Corollary 3.2. By the definition of the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$, which is the ℓ-th maximal quotient of the ℓ-th twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$, it follows that each Bernstein component of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is an irreducible square-integrable representation of $G_n^{\mathcal{O}_{\ell_0}}(F)$. Therefore, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a multiplicity-free direct sum of irreducible square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$, and hence is itself square-integrable. Moreover, for any compact open subgroup K of $G_n^{\mathcal{O}_{\ell_0}}(F)$, there are only finitely many square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$ with K-fixed vectors. Thus, the K-invariant subspace of $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is finite and hence the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is admissible. $\square$

Now, let us finish the proof of part (2) of Theorem 1.7. Following Theorem 5.2, for each F-rational orbit $\mathcal{O}_{\ell_0}$, $\mathcal{D}_{\ell_0}(\pi)$ is either zero or a direct sum of square-integrable representations. Recall that an irreducible square-integrable representation σ is a subrepresentation of $\mathcal{D}_{\ell_0}(\pi)$ if and only if the data associated to σ belong to $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ for some $\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)$ by Theorem 2.4. To prove this theorem, we will characterize σ by the descents of L-parameters in Theorem 4.6.

Fix a choice of F-rational orbit $\mathcal{O}_{\ell_0}$. Then the quadratic space W defining $G_n^{\mathcal{O}_{\ell_0}}$ is determined by $\operatorname{disc}(W) = (-1)^{\mathrm{n}-1} \operatorname{disc}(\mathcal{O}_{\ell_0}) \operatorname{disc}(V_{\mathrm{n}})$ (refer to Remark 2.5). Following Theorem 2.4, we choose the local Langlands correspondence ι_a where $a = (-1)^{\mathrm{n}} \operatorname{disc}(\mathcal{O}_\ell)$ for the even special orthogonal group. For the odd special orthogonal group, the normalization is unique.

Assume that G_n is a even special orthogonal group. Let $\pi = \pi_a(\varphi, \chi_a)$ where $\chi_a = \chi_a(\pi)$ via ι_a. If $\ell_0(\varphi, \chi_a) < \ell_0(\pi)$, then $\mathfrak{D}_{\ell_0}(\varphi, \chi_a)$ is empty and the local descent $\mathcal{D}_{\ell_0}(\pi)$ is zero for the F-rational orbit $\mathcal{O}_{\ell_0}$. If $\ell_0(\varphi, \chi_a) = \ell_0(\pi)$, then $\mathfrak{D}_{\ell_0}(\varphi, \chi_a)$ is not empty. By the definition of $\mathfrak{D}_{\ell_0}(\varphi, \chi_a)$, for $\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi_a)$, $\chi_a = \chi^\star(\varphi, \phi)$ and denote χ_ϕ to $\chi^\star(\varphi, \phi)$. Under ι_a, by (2-25), the corresponding representation $\pi_a(\phi, \chi_\phi)$ of $G_n^{\mathcal{O}_{\ell_0}}$ occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$. This gives the decomposition (a) in Theorem 1.7.

Assume that G_n is an odd special orthogonal group. In this case, the set $\mathcal{O}_{\mathcal{Z}}(\pi)$ is a singleton and $\pi = \pi(\varphi, \chi)$. Thus $\ell_0(\varphi, \chi_\pi) = \ell_0(\pi)$ and $\ell_0(\pi) = \ell_0(\varphi, \chi)$. Let W be the quadratic space defining $G_n^{\mathcal{O}_{\ell_0}}$. By (2-24), only the L-parameters ϕ satisfying $\det \phi = \operatorname{disc}(\mathcal{O}_{\ell_0}) \operatorname{disc}(V_{\mathrm{n}})$ correspond representations of $G_n^{\mathcal{O}_{\ell_0}}$. Thus if $\det \phi = \operatorname{disc}(\mathcal{O}_{\ell_0}) \operatorname{disc}(V_{\mathrm{n}})$, by the definition of $\mathfrak{D}_{\ell_0}(\varphi, \chi)$, W satisfies (2-24) and (2-26). Then $\pi_a(\phi, \chi_\phi^\star(\varphi, \phi))$ is a representation of $G_n^{\mathcal{O}_{\ell_0}}$, where $a = -\operatorname{disc}(\mathcal{O}_{\ell_0})$. This gives the decomposition (b) in Theorem 1.7.

With Proposition 5.1 together, we finally complete the proof of Theorem 1.7

Furthermore, by following Theorem 4.6 and Lemma 4.7, we will give a formula for $\chi^\star_\phi(\varphi,\phi)$ (simply written as $\chi^\star_\phi$) in Theorem 1.7. For L-parameters $\phi\in\mathfrak{D}_{\ell_0}(\varphi,\chi)$, $(\phi,\chi^\star_\phi)$ determine the irreducible square-integrable representations $\sigma=\pi_a(\phi,\chi^\star_\phi(\varphi,\phi))$ of $G_n^{\mathcal{O}_{\ell_0}}(F)$, via the given local Langlands correspondence ι_a. These σ occur as irreducible summands in the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi))$.

Assume that $\phi\in\mathfrak{D}_{\ell_0}(\varphi,\chi)$ is as given in Theorem 4.6, which can be written as

$$\phi=(\boxplus_{l=1}^k\delta_l\boxtimes 1)\boxplus(\boxplus_{\mathrm{i}=1}^r m_{\mathrm{i}}\rho_{\mathrm{i}}\boxtimes\mu_{2\alpha_{\mathrm{i},r_{\mathrm{i}}}+1})\boxplus(\boxplus_{\mathrm{i}'=1}^s n_{\mathrm{i}'}\varrho_{\mathrm{i}'}\boxtimes\mu_{2\beta_{\mathrm{i}',s_{\mathrm{i}'}}+2})\boxplus\varphi_{\mathrm{sgn}}.$$

Here the notation follows from Theorem 4.6. We may use more self-explanatory notation for the elements in A_ϕ to make the formula clearer. A general element of A_ϕ is written as

$$((e_{\delta,l}),(e_{\rho,\mathrm{i}}),(e_{\varrho,\mathrm{i}'}),(e_{i,j}),(\mathfrak{e}_{i',j'})).$$

Those components correspond to the components determined by the summands: $\delta_{\mathrm{i}}\boxtimes 1$, $m_{\mathrm{i}}\rho_{\mathrm{i}}\boxtimes\mu_{2\alpha_{\mathrm{i},r_{\mathrm{i}}}+1}$, $n_{\mathrm{i}'}\varrho_{\mathrm{i}'}\boxtimes\mu_{2\beta_{\mathrm{i}',s_{\mathrm{i}'}}+2}$, $\rho_i\boxtimes\mu_{2\alpha_{i,j}+1}$, and $\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+2}$, respectively. Note that $(e_{i,j})$ and $(\mathfrak{e}_{i',j'})$ are indexed by $1\le i\le r$ and $j\in\mathrm{sgn}_{o,\rho_i}(\chi)$, and by $1\le i'\le s$ and $j'\in\mathrm{sgn}_{e,\varrho_{i'}}(\chi)$, respectively.

Corollary 5.4. *With the notation as in Theorem 1.7 and Theorem 4.6, for $\phi\in\mathfrak{D}_{\ell_0}(\varphi,\chi)$, the character $\chi^\star_\phi((e_{\delta,l}),(e_{\rho,\mathrm{i}}),(e_{\varrho,\mathrm{i}'}),(e_{i,j}),(\mathfrak{e}_{i',j'}))$ can be explicitly written as the following product*:

$$\prod_{l=1}^k\Big(\prod_{l'=1}^s\mathcal{E}(\delta_l,\varrho_{l'})^{s_{l'}}\Big)^{e_{\delta,l}}\times\prod_{\mathrm{i}=1}^r\Big(\prod_{l'=1}^s\mathcal{E}(\rho_{\mathrm{i}},\varrho_{l'})^{s_{l'}}\Big)^{m_{\mathrm{i}}e_{\rho,\mathrm{i}}}\times\prod_{i=1}^r\prod_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\Big(\prod_{l'=1}^s\mathcal{E}(\rho_i,\varrho_{l'})^{s_{l'}}\cdot(-1)^{r_i-j}\Big)^{e_{i,j}}\\ \times\prod_{\mathrm{i}'=1}^s(-1)^{n_{\mathrm{i}'}s_{\mathrm{i}'}e_{\mathrm{i}'}}\times\prod_{i'=1}^s\prod_{j'\in\mathrm{sgn}_{e,\varrho_{i'}}(\chi)}(-1)^{j'\mathfrak{e}_{i',j'}},$$

which can also be written as the following product:

$$\prod_{l=1}^k\Big(\prod_{l'=1}^s\mathcal{E}(\delta_l,\varrho_{l'})^{s_{l'}}\Big)^{e_{\delta,l}}\times\Big(\prod_{l'=1}^s\mathcal{E}(\rho_{\mathrm{i}},\varrho_{l'})^{s_{l'}}\Big)^{\sum_{\mathrm{i}=1}^r m_{\mathrm{i}}e_{\rho,\mathrm{i}}+\sum_{i=1}^r\sum_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}e_{i,j}}\\ \times\prod_{i=1}^r\prod_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}(-1)^{(r_i-j)e_{i,j}}\times\prod_{\mathrm{i}'=1}^s(-1)^{n_{\mathrm{i}'}s_{\mathrm{i}'}e_{\mathrm{i}'}}\times\prod_{i'=1}^s\prod_{j'\in\mathrm{sgn}_{e,\varrho_{i'}}(\chi)}(-1)^{j'\mathfrak{e}_{i',j'}}.$$

6. Examples

We will consider the descents for two special families of the discrete local L-parameters. The spectral decomposition of the local descents in these cases can be even more explicitly described. Also, we will discuss Conjecture 1.8 via some examples.

6A. ***Cuspidal Local L-parameters.*** In order to understand the summand φ_{sgn} defined in (4-14) occurring in the local descent $\mathfrak{D}_{\ell_0}(\varphi,\chi)$, we give an example on such summands for cuspidal local L-parameters, which is called *cuspidal Langlands parameters* in [Aubert et al. 2015, Definition 6.8], and their descents.

Aubert, Moussaoui, and Solleveld conjectured [2015, Conjecture 7.5] that an L-packet $\Pi_\varphi(G_n)$ of a reductive group contains a supercuspidal representation if and only if φ is a cuspidal L-parameter. For the split orthogonal groups and symplectic group cases, Moussaoui [2017] verified this conjecture and gave a description of cuspidal L-parameters. Take

$$\varphi = \boxplus_{i=1}^{r} \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2j} \boxplus \boxplus_{i=1}^{s} \boxplus_{j=1}^{s_i} \varrho_i \boxtimes \mu_{2j-1},$$

where $\sum_{i=1}^{s} s_i b_i$ (here $b_i = \dim \varrho_i$) is even. By [Moussaoui 2017, Proposition 3.7], φ is a cuspidal L-parameter for split even orthogonal group G_n when φ is of orthogonal type and $\prod_{i=1}^{s} \det \varrho_i = 1$.

Following (2-8), we write the elements in $\mathcal{S}_\varphi$ in the form $((e_{i,j}), (\mathfrak{e}_{i,j}))$, indexed by the set

$$\{e_{i,j} : 1 \le i \le r,\ 1 \le j \le r_i\} \cup \{\mathfrak{e}_{i,j} : 1 \le i \le s,\ 1 \le j \le s_i\}.$$

The index sets correspond to the summands $\rho_i \boxtimes \mu_{2j}$ and $\varrho_i \boxtimes \mu_{2j-1}$ respectively. If ϱ_i has even dimension for all $1 \le i \le s$, then

$$\mathcal{S}_\varphi = \{((e_{i,j}), (\mathfrak{e}_{i,j})) \in \mathbb{Z}_2^r \times \mathbb{Z}_2^s : e_{i,j}, \mathfrak{e}_{i,j} \in \{0, 1\}\}.$$

Otherwise, $\mathcal{S}_\varphi$ is the subgroup of $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ consisting of elements with the condition that $\sum_{i,j} \mathfrak{e}_{i,j} \dim \varrho_i$ is even. Define the character χ in $\hat{\mathcal{S}}_\varphi$ by

$$\chi((e_{i,j}), (\mathfrak{e}_{i,j})) = \prod_{i=1}^{r} \prod_{j=1}^{r_i} (-1)^{(j+r_i)e_{i,j}} \cdot \prod_{i=1}^{s} \prod_{j=1}^{s_i} (-1)^{(j+s_i)\mathfrak{e}_{i,j}}.$$

Remark 6.1. In some cases, the associated representation $\pi_a(\varphi, \chi)$ is supercuspidal. For instance, take $\varphi = \boxplus_{i=1}^{r} \boxplus_{j=1}^{2r_i} \rho_i \boxtimes \mu_{2j}$. By definition, $\chi(1_{\rho_i \boxtimes \mu_2}) = -1$ for all $1 \le i \le r$ and $\chi((1)) = 1$ where (1) is the element with $e_{i,j} = 1$ for all i and j. Thus, $\pi_a(\varphi, \chi)$ is a representation of a symplectic group or a split even special orthogonal group, because $\det(\varphi) = 1$. By [Moussaoui 2017], $\pi_a(\varphi, \chi)$ is supercuspidal.

Under the above assumption, by (4-14), we have that

$$\varphi_{\mathrm{sgn}} = \boxplus_{i=1}^{r} \boxplus_{j=1}^{2[r_i/2]} \rho_i \boxtimes \mu_{2(r_i-j)+1} \boxplus \boxplus_{i=1}^{s} \boxplus_{j=1}^{s_i-1} \varrho_i \boxtimes \mu_{2j}.$$

By convention, the summand $\varrho_i \boxtimes \mu_{2j}$ is empty if $s_i = 1$. As $\sum_{j=1}^{s} s_j b_j$ is even and

$$\dim \varphi - \dim \varphi_{\mathrm{sgn}} = \sum_{i=1}^{r} 2a_i\left[\tfrac{1}{2}(r_i+1)\right] + \sum_{j=1}^{s} s_j b_j,$$

we deduce that $\dim \varphi_{\mathrm{sgn}}$ is even. By the definition in (4-15), one has that

$$\lceil \varphi_o \rceil = \boxplus_{i=1}^{s} \varrho_i \boxtimes \mu_{2s_i-1} \text{ and } \chi_{\lceil \varphi_o \rceil} = 1.$$

According to Theorem 4.6, if $\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi)$, we may decompose ϕ as $\phi = \psi \boxplus \varphi_{\mathrm{sgn}}$, where ψ satisfies the conditions in Theorem 4.6. Then $\psi = 0$ is the unique choice. In fact, $\dim \psi = 0$ is even and of the minimal dimension. By convention, $\chi^\star_{\varphi^\dagger}(\varphi^\dagger, \psi) = 1$. Thus,

$$\mathfrak{D}_{\ell_0}(\varphi, \chi) = [\varphi_{\mathrm{sgn}}]_c = \{\varphi_{\mathrm{sgn}}, \varphi_{\mathrm{sgn}}^c\} \quad \text{and} \quad \ell_0 = \sum_{i=1}^{r} a_i\left[\tfrac{1}{2}(r_i+1)\right] + \sum_{j=1}^{s} \tfrac{1}{2} s_j b_j. \tag{6-1}$$

Similarly, assume that the elements in $\mathcal{S}_\phi$ are of form $((e'_{i,j}),(\mathfrak{e}'_{i,j}))$, where $e'_{i,j}$ and $\mathfrak{e}'_{i,j}$ correspond to a $\rho_i\boxtimes\mu_{2(r_i-j)+1}$-component and $\varrho_i\boxtimes\mu_{2j}$-component respectively. Define the character $\zeta\in\hat{\mathcal{S}}_\phi$ with the following conditions for all i:

- $\zeta(1_{\rho_i\boxtimes\mu_{2(r_i-j)+1}})=1$ when $j=2\left[\frac{r_i}{2}\right]$ and $\zeta(1_{\varrho_i\boxtimes\mu_2})=-1$.
- $\zeta(1_{\rho_i\boxtimes\mu_{2(r_i-j)-1}\boxplus\rho_i\boxtimes\mu_{2(r_i-j)+1}})=-1$ for $1\le j<2\left[\frac{r_i}{2}\right]$.
- $\zeta(1_{\varrho_i\boxtimes\mu_{2j}\boxplus\varrho_i\boxtimes\mu_{2(j+1)}})=-1$ for $1\le j<s_i-1$.

By Lemma 4.7, one has that $(\chi,\zeta)=\chi^\star(\varphi,\phi)$. By choosing the quadratic spaces and the Langlands correspondence as in Theorem 2.4, we have that $m(\pi_a(\varphi,\chi),\pi_a(\phi,\zeta))=1$.

For example, when $G_n=\mathrm{SO}(V_{2n+1})$ is an odd special orthogonal group, take $\mathcal{O}_{\ell_0}$ with $\mathrm{disc}(\mathcal{O}_{\ell_0})=\mathrm{disc}(V_{2n+1})$ as $\det(\varphi_{\mathrm{sgn}})=1$, which is the unique F-rational orbit such that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi))\neq 0$ and

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi))=\begin{cases}\pi_a(\varphi_{\mathrm{sgn}},\zeta) & \text{if some }\dim\rho_i\text{ is odd},\\ \pi_a(\varphi_{\mathrm{sgn}},\zeta)\oplus\pi_a(\varphi^c_{\mathrm{sgn}},\zeta) & \text{otherwise},\end{cases}$$

where $a=-\mathrm{disc}(V_{2n+1})$. Note that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi))$ is a representation of a pure inner form of split even orthogonal group as $\det(\varphi_{\mathrm{sgn}})=1$.

6B. ***Discrete unipotent representations.*** We follow the definition of unipotent representations given by Lusztig [1995]: $\pi(\phi,\chi)$ is a unipotent representation if and only if ϕ is trivial on the inertia subgroup $\mathcal{I}=\mathcal{I}_F$ of the local Weil group $\mathcal{W}_F$, and such a ϕ is called a unipotent local L-parameter. We apply the local descent method to give an explicit description on the descent of discrete unipotent representations.

Denote by $\xi_{\mathrm{un}}=(\,\cdot\,,\epsilon)_F$ the nontrivial unramified quadratic character of $F^\times$, which is also regarded as a character $\mathcal{W}_F$ via the local class field theory. Here ϵ is a nonsquare element in F with absolute value 1. Let φ be a discrete unipotent L-parameter. Then it can be written as

$$\varphi=\begin{cases}\boxplus_{i=1}^r 1\boxtimes\mu_{2a_i}\boxplus\boxplus_{j=1}^s\xi_{\mathrm{un}}\boxtimes\mu_{2b_j} & \text{if } G_n=\mathrm{SO}_{2n+1},\\ \boxplus_{i=1}^r 1\boxtimes\mu_{2a_i+1}\boxplus\boxplus_{j=1}^s\xi_{\mathrm{un}}\boxtimes\mu_{2b_j+1} & \text{if } G_n=\mathrm{SO}_{2n}.\end{cases}\tag{6-2}$$

Recall the calculation on the local root number from Example 4.4. We obtain the following.

Corollary 6.2. *Let $\pi(\varphi,\eta)$ be a discrete unipotent representation of $G_n=\mathrm{SO}(V_{2n+1})$. Then $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is irreducible and*

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)=\begin{cases}\pi_a(\varphi_{\mathrm{sgn}},\zeta) & \textit{if }\ \mathrm{disc}(\mathcal{O}_{\ell_0})=\det(\varphi_{\mathrm{sgn}})\,\mathrm{disc}(V_{2n+1}),\\ 0 & \textit{otherwise},\end{cases}$$

where $\zeta=\chi^\star_{\varphi_{\mathrm{sgn}}}(\varphi,\varphi_{\mathrm{sgn}})$ and $a=-\det(\varphi_{\mathrm{sgn}})\,\mathrm{disc}(V_{2n+1})$.

Note that when $G_n=\mathrm{SO}(V_{2n+1})$, the descent φ_{sgn} is invariant under c-conjugate. And the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is an irreducible unipotent discrete representation.

Now, let us consider the case $G_n=\mathrm{SO}(V_{2n})$. Suppose that an irreducible unipotent discrete representation π in the L-packet $\Pi_\varphi(G_n)$. One can choose a local Langlands correspondence ι_a such that

$\pi = \pi_a(\varphi, \eta)$ with the condition that $\eta(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$. To prove this, let us consider the action of $\mathcal{Z}$ on $\hat{\mathcal{S}}_\varphi$. In this case, as $\det\varphi = \epsilon^s$, the character η_ϖ (defined in Section 2B) is equal to

$$\eta_\varpi(((e_i), (\mathfrak{e}_j))) = \prod_{j=1}^{s}(-1)^{\mathfrak{e}_j},$$

where ϖ is a uniformizer in F. Then $\chi(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$ or $\chi\otimes\eta_\varpi(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$ for any $\chi \in \hat{\mathcal{S}}_\varphi$. Hence, by the definition of $\mathcal{O}_{\mathcal{Z}}(\pi)$ in (2-17), there exists a character χ in $\mathcal{O}_{\mathcal{Z}}(\pi)$ such that $\chi(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$. Then we can choose ι_a such that $\chi_a(\pi) = \chi$, which is the desired normalization for the local Langlands correspondence.

Corollary 6.3. *Let π be a discrete unipotent representation of $G_n = \mathrm{SO}(V_{2n})$ in $\Pi_\varphi[G_n^*]$. Choose the local Langlands correspondence ι_a such that $\pi = \pi_a(\pi, \eta)$ with $\eta(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$. Then $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is an irreducible representation of $\mathrm{SO}(V_{2m+1})$ with $m = \dim\varphi_{\mathrm{sgn}}/2$ and*

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \begin{cases} \pi(\varphi_{\mathrm{sgn}}, \zeta) & \text{if } \mathrm{disc}(\mathcal{O}_{\ell_0}) = a, \\ 0 & \text{otherwise}, \end{cases}$$

where $\zeta = \chi^\star_{\varphi_{\mathrm{sgn}}}(\varphi, \varphi_{\mathrm{sgn}})$, $\mathrm{disc}(V_{2m+1}) = -a\epsilon^s$ and

$$\mathrm{Hss}(V_{2m+1}) = (-1, -1)^{m(m+1)/2}((-1)^{m+1}, \mathrm{disc}(V_{2m+1}))\eta((1)).$$

6C. ***On Conjecture 1.8.*** We will show that Conjecture 1.8 holds for certain representations of SO_7^*.

Referring to [Collingwood and McGovern 1993], for SO_7^*, all stable unipotent orbits are parametrized by the following partitions $\underline{p}$, respectively

$$[7], \quad [5, 1^2], \quad [3^2, 1], \quad [3, 2^2], \quad [3, 1^4], \quad [2^2, 1^3], \quad [1^7], \tag{6-3}$$

where the powers indicate the multiplicities in the partitions, and the corresponding unipotent orbits are listed following the topological order. In particular, [7] is for the regular unipotent orbit and $[1^7]$ is for the trivial orbit. In this case, this topological order is a total order. Hence, for an irreducible smooth representation π, the set $\mathfrak{p}^m(\pi)$ is a singleton. We may assume that

$$\mathfrak{p}^m(\pi) = \{\underline{p} = [p_1 p_2 \cdots p_r]\}, \quad \text{where } p_1 \geq p_2 \geq \cdots \geq p_r > 0. \tag{6-4}$$

Let π be an irreducible square-integrable representation of $\mathrm{SO}(V_7, F)$ and φ be its L-parameter. We are going to apply our main results to the following two types of φ

(1) $\varphi = \chi_1 \boxtimes \mu_4 \boxplus \chi_2 \boxtimes \mu_2$ or

(2) $\varphi = \chi_1 \boxtimes \mu_2 \boxplus \chi_2 \boxtimes \mu_2 \boxplus \chi_3 \boxtimes \mu_2$,

where χ_i are quadratic characters, and to verify that Conjecture 1.8 holds for the representations in the corresponding Vogan packets. For simplicity, we assume that -1 is a square in $F^\times$. Then $(\det\varphi, -1)_F = 1$ in (2-20) for all L-parameters φ. We take V_7 satisfying $\mathrm{disc}(V_7) = -1 = 1 \bmod F^{\times 2}$. For the odd special orthogonal group, the local Langland correspondence is unique and denote χ_π to be the corresponding character in $\hat{\mathcal{S}}_\varphi$.

To verify Conjecture 1.8, we will construct an L-parameter ϕ in $\mathfrak{D}_\ell(\varphi, \chi_\pi)$ for some $\ell \in \{1, 2\}$, which implies the descent $\mathfrak{D}_\ell(\varphi, \chi_\pi)$ is not empty. By Theorem 1.7, for such ϕ the F-rational orbit $\mathcal{O}_\ell$ and the quadratic space W defining $G_n^{\mathcal{O}_\ell}$ are determined by $\mathrm{disc}(\mathcal{O}_\ell) = \mathrm{disc}(W) = \det\phi \mod F^{\times 2}$. Following (2-25) and (2-26), after simplification, $\mathrm{Hss}(V_7) = \mathrm{Hss}(W) = \chi_\pi((1))$. The local Langlands correspondence ι_a for the even special orthogonal group $\mathrm{SO}(W)$ is normalized by $a = \det\phi \mod F^{\times 2}$. Hence we obtain that the following irreducible square-integrable representation of $\mathrm{SO}(W, F)$

$$\sigma = \pi_{\det\phi}(\phi, \chi^\star(\varphi, \phi))$$

occurs in $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ for the above chosen F-rational orbit, where

$$\mathrm{disc}(W) = \det\phi \text{ and } \mathrm{Hss}(W) = \mathrm{Hss}(V_7).$$

It follows that p_1 in (6-4) is greater than or equal to $2\ell + 1$. Because $\ell \in \{1, 2\}$, we have a lower bound $p_1 \geq 3$ by the total order in (6-3).

Now, if $p_1 = 7$, π is generic and then $\ell_0 = 3$. Conjecture 1.8 holds. If $p_1 = 5$, then the nonvanishing of the twisted Jacquet module associated $\underline{p} = [51^2]$ is equivalent to the nonvanishing of the local descent $\mathcal{D}_{\mathcal{O}_2}(\pi)$ by Lemma 3.1. By definition, the first occurrence index ℓ_0 equals 2. If $p_1 = 3$, by the above lower bound $p_1 \geq 3$ we have $p_1 = 3$. Then $\ell_0 = 1$ in this case, which implies the conjecture.

Furthermore, for these two types of parameters, our results may explicitly determine $\mathfrak{p}^m(\pi)$ in terms of (φ, χ_π) associated to π. The detailed calculation will be given in the remainder of this paper.

Type (1): Assume that $\varphi = \chi_1 \boxtimes \mu_4 \boxplus \chi_2 \boxtimes \mu_2$. Then $\mathcal{S}_\varphi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Denote by ζ_+ and ζ_- the trivial and nontrivial characters of $\mathbb{Z}_2$, respectively. Then we may write the characters of $\mathcal{S}_\varphi$ as $\zeta_\pm \otimes \zeta_\pm$. Its Vogan packet $\Pi_\varphi[\mathrm{SO}_7^*]$ contains four representations $\pi(\varphi, \zeta_\pm \otimes \zeta_\pm)$.

If $\pi = \pi(\varphi, \zeta_+ \otimes \zeta_+)$, then π is the unique generic representation in $\Pi_\varphi[\mathrm{SO}_7^*]$. We have $\ell_0 = 3$ and $\underline{p} = [7]$.

For $\pi = \pi(\varphi, \zeta_- \otimes \zeta_-)$, choose

$$\phi = \begin{cases} \chi_1 \boxplus \chi_2 & \text{if } \chi_1 \neq \chi_2, \\ \chi_1 \boxplus \chi' & \text{if } \chi_1 = \chi_2, \end{cases}$$

where χ' is a quadratic character not isomorphic to χ_1 or χ_2. Since π is nongeneric, we have $\ell_0(\pi) \leq 2$. By Lemma 4.7, $\phi \in \mathfrak{D}_2(\varphi, \zeta_- \otimes \zeta_-)$ and then $\ell_0(\pi) \geq 2$. It follows that $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

When $\pi = \pi(\varphi, \zeta_+ \otimes \zeta_-)$ or $\pi(\varphi, \zeta_- \otimes \zeta_+)$, $\mathrm{Hss}(V_7) = -1$ (i.e., SO_7 is nonsplit) and $\ell_0 \leq 2$. Similarly, we have $\chi_2 \boxplus \chi' \in \mathfrak{D}_2(\varphi, \zeta_+ \otimes \zeta_-)$ and $\phi = \chi_1 \boxplus \chi' \in \mathfrak{D}_2(\varphi, \zeta_- \otimes \zeta_+)$. In both cases, $\ell_0 \geq 2$ and then $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

Type (2): Assume that $\varphi = \chi_1 \boxtimes \mu_2 \boxplus \chi_2 \boxtimes \mu_2 \boxplus \chi_3 \boxtimes \mu_2$. Since π is square-integrable, χ_1, χ_2 and χ_3 are distinct. Then $\mathcal{S}_\varphi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the character of $\mathcal{S}_\varphi$ is of form $\zeta_1 \otimes \zeta_2 \otimes \zeta_3$, where ζ_i for $1 \leq i \leq 3$ are characters of $\mathbb{Z}_2$.

Let $\pi = \pi(\varphi, \zeta_1 \otimes \zeta_2 \otimes \zeta_3)$. If $\zeta_i = \zeta_+$ for all $1 \le i \le 3$, then π is generic and $\ell_0(\pi) = 3$. All the other representations are nongeneric. For the remaining cases, we always have $\ell_0(\pi) \le 2$, i.e., $\underline{p} \neq [7]$.

As $F^\times/(F^\times)^2$ contains at least 4 elements, there exists a character χ' of $F^\times$ such that $\chi'^2 = 1$ and χ' is not isomorphic to any χ_i for $1 \le i \le 3$. When $\zeta_{i_0} = \zeta_-$ for some i_0 and $\zeta_i = \zeta_+$ for all $i \neq i_0$, we may take $\phi = \chi_{i_0} \boxplus \chi'$. In this case, SO_7 is nonsplit. It follows that $\phi \in \mathfrak{D}_2(\varphi, \otimes_{i=1}^3 \zeta_i)$ and then $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

If $\zeta_{i_0} = \zeta_+$ for some i_0 and $\zeta_i = \zeta_-$ for all $i \neq i_0$, we may take $\phi = \chi_i \boxplus \chi_j$ where $\{i, j\} = \{1, 2, 3\} \smallsetminus \{i_0\}$. One has that $\phi \in \mathfrak{D}_2(\varphi, \otimes_{i=1}^3 \zeta_i)$ and hence $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

If $\zeta_i = \zeta_-$ for all $1 \le i \le 3$, we may take $\phi = \boxplus_{i=1}^3 \chi_i \boxplus \chi'$, which is in $\mathfrak{D}_1(\varphi, \otimes_{i=1}^3 \zeta_i)$. We have the lower bound $\ell_0(\pi) \ge 1$. By the above discussion, this implies Conjecture 1.8 for this representation. In this case, one may also explicitly determine $\mathfrak{p}^m(\pi)$ by calculating the symplectic root number $\varepsilon(\tau \times \chi_i)$ where τ is a supercuspidal representation of $\mathrm{GL}_2(F)$ with the central character $\omega_\tau = 1$ (i.e., of symplectic type). We omit the details here.

Remark 6.4. Beside the above *Type* (1) and *Type* (2), for any irreducible smooth representation π of $\mathrm{SO}(V_7, F)$ in a generic L-packet, we may obtain the lower bound $\ell_0(\pi) \ge 1$ by using an alternative global argument. Then Conjecture 1.8 holds for all pure inner forms of SO_7^*. However, such global arguments only work for the special orthogonal groups of lower rank.

Remark 6.5 (counter example). We give an example to show that Conjecture 1.8 may not be true for nontempered representations. Let $G_n^* = \mathrm{SO}_4^*$ be the split even orthogonal group. Note that SO_4^* only has 4 stable unipotent orbits, whose corresponding partitions are

$$[3, 1], [2^2]^I, [2^2]^{II}, [1^4].$$

Here $[2^2]^I$ and $[2^2]^{II}$ are the same partitions of 4 but give two different unipotent orbits. We take π to be the irreducible nongeneric nontempered infinite dimensional representation of SO_4^*. Then it is not generic and has a nonzero twisted Jacquet model associated to $[2^2]^I$ or $[2^2]^{II}$. In this case, the largest part p_1 is even and not equal to $2\ell + 1$ for all ℓ. In general, one can find a family of nontempered representations π of SO_{2n}^*, whose largest part p_1 in the partitions of $\mathfrak{p}^m(\pi)$ are even. Hence Conjecture 1.8 fails for those nontempered representations.

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Bases for quasisimple linear groups

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Let V be a vector space of dimension d over $\mathbb{F}_q$, a finite field of q elements, and let $G \le \mathrm{GL}(V) \cong \mathrm{GL}_d(q)$ be a linear group. A *base* for G is a set of vectors whose pointwise stabilizer in G is trivial. We prove that if G is a quasisimple group (i.e., G is perfect and $G/Z(G)$ is simple) acting irreducibly on V, then excluding two natural families, G has a base of size at most 6. The two families consist of alternating groups Alt_m acting on the natural module of dimension $d = m-1$ or $m-2$, and classical groups with natural module of dimension d over subfields of $\mathbb{F}_q$.

1. Introduction

Let G be a permutation group on a finite set Ω of size n. A subset of Ω is said to be a *base* for G if its pointwise stabilizer in G is trivial. The minimal size of a base for G is denoted by $b(G)$ (or sometimes $b(G, \Omega)$ if we wish to emphasize the action). It is easy to see that $|G| \le n^{b(G)}$, so that $b(G) \ge \log|G|/\log n$. A well known conjecture of Pyber [1993] asserts that there is an absolute constant c such that if G is primitive on Ω, then $b(G) < c \log|G|/\log n$. Following substantial contributions by a number of authors, the conjecture was finally established in [Duyan et al. 2018] in the following form: there is an absolute constant C such that for every primitive permutation group G of degree n,

$$b(G) < 45\frac{\log|G|}{\log n} + C. \tag{1}$$

To obtain a more explicit, usable bound, one would like to reduce the multiplicative constant 45 in the above, and also estimate the constant C.

Most of the work in [Duyan et al. 2018] was concerned with affine groups contained in $\mathrm{AGL}(V)$, acting on the set of vectors in a finite vector space V (since the conjecture had already been established for nonaffine groups elsewhere). For these, one needs to bound the base size for a linear group $G \le \mathrm{GL}(V)$ that acts irreducibly on V. One source for the undetermined constant C in the bound (1) comes from a key result in this analysis, namely Proposition 2.2 of [Liebeck and Shalev 2002], in which quasisimple linear groups are handled. This result says that there is a constant C_0 such that if G is a quasisimple group acting irreducibly on a finite vector space V, then either $b(G) \le C_0$, or G is a classical or alternating

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group and V is the natural module for G; here by the natural module for an alternating group Alt_m over $\mathbb{F}_{p^e}$ (p prime) we mean the irreducible "deleted permutation module" of dimension $m-\delta(p,m)$, where $\delta(p,m)$ is 2 if $p \mid m$ and is 1 otherwise. This result played a major role in the proof of Pyber's conjecture for primitive linear groups in [Liebeck and Shalev 2002; 2014], which was heavily used in the final completion of the conjecture in [Duyan et al. 2018].

The main result in this paper shows that the constant C_0 just mentioned can be taken to be 6. Recall that for a finite group G, we denote by $E(G)$ the subgroup generated by all quasisimple subnormal subgroups of G. Also write $V_d(q)$ to denote a d-dimensional vector space over $\mathbb{F}_q$.

Theorem 1. *Let $V = V_d(q)$ ($q = p^e$, p prime) and $G \le \mathrm{GL}(V)$, and suppose that $E(G)$ is quasisimple and absolutely irreducible on V. Then one of the following holds*:

(i) *$E(G) = \mathrm{Alt}_m$ and V is the natural Alt_m-module over $\mathbb{F}_q$ of dimension $d = m-\delta(p,m)$.*

(ii) *$E(G) = \mathrm{Cl}_d(q_0)$, a classical group with natural module of dimension d over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$.*

(iii) *$b(G) \le 6$.*

This result has been used in [Halasi et al. 2018] to improve the bound (1), replacing the multiplicative constant 45 by 2, and the constant C by 24.

With substantially more effort, it should be possible to reduce the constant 6 in part (iii) of theorem, and work on this by the first author is in progress.

The paper is organized as follows. In Section 2 we present some preliminary results needed for the proof of Theorem 1. Section 3 contains the proof of Theorem 3.1, a result that bounds the base size for various actions of classical groups on orbits of nondegenerate subspaces. The proof of Theorem 1 follows in Section 4, where crucial use of Theorem 3.1 is made in Lemmas 4.7 and 4.8.

2. Preliminary lemmas

If G is a finite classical group with natural module V, we define a *subspace action* of G to be an action on an orbit of subspaces of V, or, in the case where $G = \mathrm{Sp}_{2m}(q)$ with q even, the action on the cosets of a subgroup $O^{\pm}_{2m}(q)$.

Lemma 2.1. *Let G be an almost simple group with socle G_0 and suppose G acts transitively on a set Ω.*

(i) *If G_0 is exceptional of Lie type, or sporadic, then $b(G) \le 7$, with equality only if $G = M_{24}$.*

(ii) *If G_0 is classical, and the action of G on Ω is primitive and not a subspace action, then $b(G) \le 5$, with equality if and only if $G = U_6(2).2$, $\Omega = (G : U_4(3).2^2)$.*

Proof. Part (i) follows from [Burness et al. 2009, Corollary 1] and [Burness et al. 2010, Corollary 1]. Part (ii) is [Burness 2007c, Theorem 1.1]. □

For a simple group G_0, and $1 \ne x \in \mathrm{Aut}(G_0)$, define $\alpha(x)$ to be the minimal number of G_0-conjugates of x required to generate the group $\langle G_0, x\rangle$, and define

$$\alpha(G_0) = \max\{\alpha(x) \mid 1 \ne x \in \mathrm{Aut}(G_0)\}.$$

Lemma 2.2. *Let $G_0 = \mathrm{Cl}_n(q)$, a simple classical group over $\mathbb{F}_q$ with natural module of dimension n. Then one of the following holds*:

(i) $\alpha(G_0) \le n$.

(ii) $G_0 = \mathrm{PSp}_n(q)$ (q *even*) *and* $\alpha(G_0) \le n+1$.

(iii) $G_0 = L_2(q)$ *and* $\alpha(G_0) \le 4$.

(iv) $G_0 = L_3(q)$ *and* $\alpha(G_0) \le 4$.

(v) $G_0 = L_4^{\epsilon}(q)$ *and* $\alpha(G_0) \le 6$.

(vi) $G_0 = \mathrm{PSp}_4(q)$ *and* $\alpha(G_0) \le 5$.

(vii) $G_0 = L_2(9), U_3(3)$ *or* $L_4^{\epsilon}(2)$.

Proof. This is [Guralnick and Saxl 2003, 3.1 and 4.1]. □

To state the next result, let $\overline{G}$ be a simple algebraic group over an algebraically closed field K of characteristic p, and let $V = V(\lambda)$ be an irreducible $K\overline{G}$-module of p-restricted highest weight λ. Let Φ be the root system of $\overline{G}$, with simple roots $\alpha_1, \dots, \alpha_l$, and let $\lambda_1, \dots, \lambda_l$ be the corresponding fundamental dominant weights. Denote by Φ_S and Φ_L the set of short and long roots in Φ, respectively, and if all roots have the same length, just write $\Phi_S = \Phi$ and $\Phi_L = \varnothing$. Let $W = W(\Phi)$ be the Weyl group, and for $\alpha \in \Phi$, let $U_\alpha = \{u_\alpha(t) : t \in K\}$ be a corresponding root subgroup with respect to a fixed maximal torus.

Now let μ be a dominant weight of $V = V(\lambda)$, write $\mu = \sum_{j=1}^{l} c_j \lambda_j$, and let $\Psi = \langle \alpha_i \mid c_i = 0 \rangle_{\mathbb{Z}} \cap \Phi$, a subsystem of Φ. Define

$$r_\mu = \frac{|W : W(\Psi)| \cdot |\Phi_S \setminus \Psi_S|}{2|\Phi_S|} \quad \text{and} \quad r'_\mu = \frac{|W : W(\Psi)| \cdot |\Phi_L \setminus \Psi_L|}{2|\Phi_L|}$$

(the latter only if $\Phi_L \neq \varnothing$). Let

$$s_\lambda = \sum_\mu r_\mu \quad \text{and} \quad s'_\lambda = \sum_\mu r'_\mu \ \text{ (if } \Phi_L \neq \varnothing\text{)},$$

where each sum is over the dominant weights μ of $V(\lambda)$.

For $g \in \overline{G} \setminus Z(\overline{G})$ and $\gamma \in K^*$, let $V_\gamma(g) = \{v \in V : vg = \gamma v\}$, and write

$$\operatorname{codim} V_\gamma(g) = \dim V - \dim V_\gamma(g).$$

Lemma 2.3. *Let $V = V(\lambda)$ be as above.*

(i) *If $g \in \overline{G} \setminus Z(\overline{G})$ is semisimple and $\gamma \in K^*$, then* $\operatorname{codim} V_\gamma(g) \ge s_\lambda$.

(ii) *If $\alpha \in \Phi_S$, then* $\operatorname{codim} V_1(u_\alpha(1)) \ge s_\lambda$.

(iii) *If $\Phi_L \neq \varnothing$ and $\beta \in \Phi_L$, then* $\operatorname{codim} V_1(u_\beta(1)) \ge s'_\lambda$.

(iv) *For any nonidentity unipotent element $u \in \overline{G}$, we have* $\operatorname{codim} V_1(u) \ge \min(s_\lambda, s'_\lambda)$.

Proof. Parts (i)–(iii) are [Guralnick and Lawther ≥ 2018, Proposition 2.2.1]. For part (iv), note that [Guralnick and Malle 2004, Corollary 3.4] shows that $\dim V_1(u)$ is bounded above by the maximum of $\dim V_1(u_\alpha(1))$ and $\dim V_1(u_\beta(1))$; hence (iv) follows from (ii) and (iii). □

For $\bar{G}$ of type D_5 or D_6 and V a half-spin module for $\bar{G}$, we shall need the following sharper result. Note that the root system $D_n (n \geq 5)$ has two subsystems of type A_1^2 (up to conjugacy in the Weyl group); with the usual labeling of fundamental roots, we denote these by $(A_1^2)^{(1)} = \langle \alpha_1, \alpha_3 \rangle$ and $(A_1^2)^{(2)} = \langle \alpha_{n-1}, \alpha_n \rangle$.

Lemma 2.4. *Let $\bar{G} = D_n$ with $n \in \{5, 6\}$, and let $V = V(\lambda)$ be a half-spin module for $\bar{G}$ with $\lambda = \lambda_n$ or λ_{n-1}. Let $s \in \bar{G} \setminus Z(\bar{G})$ be a semisimple element, and $u \in \bar{G}$ a unipotent element of order p.*

(i) *Suppose $n = 6$. Then* $\operatorname{codim} V_\gamma(s) \geq 12$ *for any $\gamma \in K^*$; and* $\operatorname{codim} V_1(u) \geq 12$ *provided u is not a root element.*

(ii) *Suppose $n = 5$.*

  (a) *If $C_{\bar{G}}(s)' \neq A_4$ then* $\operatorname{codim} V_\gamma(s) \geq 8$ *for any $\gamma \in K^*$ and if $C_{\bar{G}}(s)' = A_4$ then* $\operatorname{codim} V_\gamma(s) \geq 6$.

  (b) *Provided u is not a root element and also does not lie in a subsystem subgroup $(A_1^2)^{(1)}$, we have* $\operatorname{codim} V_1(u) \geq 8$.

Proof. For semisimple elements s, we follow the method of [Guralnick and Lawther ≥ 2018, §2.6] (originally in [Kenneally 2010]). Let Ψ be a closed subsystem of the root system Φ of $\bar{G}$, and define an equivalence relation on the set of weights of $V(\lambda)$ by saying that two weights are related if their difference is a sum of roots in Ψ. Call the equivalence classes Ψ-*nets*.

Now define $\Phi_s = \{\alpha \in \Phi \mid \alpha(s) = 1\}$, the root system of $C_{\bar{G}}(s)$. If $\Phi_s \cap \Psi = \varnothing$, then any two weights in a given Ψ-net that differ by a root in Ψ correspond to different eigenspaces for s.

The subsystem Φ_s is contained in a proper subsystem spanned by a subset of the nodes of the extended Dynkin diagram of $\bar{G}$. Suppose $\Phi_s \neq A_{n-1}$. Then it is straightforward to check that there is a subsystem Ψ that is W-conjugate to $(A_1^2)^{(2)}$ such that $\Phi_s \cap \Psi = \varnothing$. For this Ψ there are 2^{n-2} Ψ-nets of size 2, and so it follows from the observation in the previous paragraph that $\operatorname{codim} V_\gamma(s) \geq 2^{n-2}$ for any $\gamma \in K^*$.

Now suppose $\Phi_s = A_{n-1}$. Here there is a subsystem Ψ that is W-conjugate to $(A_1^2)^{(1)}$ such that $\Phi_s \cap \Psi = \varnothing$. For this Ψ there are 2^{n-5}, 2^{n-3} or 2^{n-3} Ψ-nets of size 4, 2 or 1, respectively, and hence $\operatorname{codim} V_\gamma(s) \geq 2^{n-4} + 2^{n-3}$ for any $\gamma \in K^*$. This lower bound is 12 when $n = 6$, and 6 when $n = 5$. This proves (i) and (ii) for semisimple elements.

Now consider unipotent elements $u \in \bar{G}$ of order p. Assume first that p is odd. Recall that the Jordan form of a unipotent element $u \in D_n$ on the natural module determines a partition ϕ of $2n$ having an even number of parts of each even size; moreover, each such partition corresponds to a single conjugacy class, except when all parts of ϕ are even, in which case there are two classes, interchanged by a graph automorphism of D_n (see [Liebeck and Seitz 2012, Chapter 3]). Denote by u_ϕ (and by u_ϕ, u'_ϕ for the exceptional partitions) representatives of the unipotent classes in $\bar{G}$. By [Spaltenstein 1982, §4], if μ and ϕ are partitions and $\mu < \phi$ in the usual dominance order, then u_μ lies in the closure of the class $u_\phi^{\bar{G}}$ (or $u_\phi'^{\bar{G}}$).

Suppose u is not a root element, and also is not in a subsystem subgroup $(A_1^2)^{(1)}$ when $n=5$. Then it follows from the above that the closure of $u^{\bar{G}}$ contains $u'=u_\mu$ with $\mu=(3,1^{2n-3})$ or $(2^4,1^{2n-8})$, the latter only if $n=6$. Moreover, $\operatorname{codim} V_1(u)\geq \operatorname{codim} V_1(u')$ (see the proof of [Guralnick and Malle 2004, 3.4]). If $\mu=(3,1^{2n-3})$, then u' lies in the B_1 factor of a subgroup $B_1\times B_{n-2}$ of $\bar{G}$, and the restriction of V to this subgroup is given by [Liebeck and Seitz 2012, 11.15(ii)]; it follows that u' acts on V with Jordan form $J_2^{2^{n-2}}$, giving the conclusion in this case. And if $\mu=(2^4,1^4)$ with $n=6$, then u' is in $(A_1^2)^{(1)}$, which is contained in a subsystem A_4, and the restriction of the half-spin module V to A_4 can be deduced from [Liebeck and Seitz 2012, 11.15(i)]; the lower bound on $\operatorname{codim} V_1(u')$ in (i) follows easily from this.

It remains to consider unipotent involutions with $p=2$. The conjugacy classes of these in $\bar{G}$ are described in [Aschbacher and Seitz 1976, §7] (alternatively in [Liebeck and Seitz 2012, Chapter 6]). Adopting the notation of [Aschbacher and Seitz 1976], representatives are a_l, c_l (l even, $2\leq l\leq n$), and also a_6' in D_6 (which is conjugate to a_6 under a graph automorphism). These are regular elements of Levi subsystem subgroups S, as follows:

u	a_2	c_2	a_4	c_4	a_6	a_6'	c_6
S	A_1	$(A_1^2)^{(2)}$	$(A_1^2)^{(1)}$	$A_1(A_1^2)^{(2)}$	$(A_1^3)^{(1)}$	$(A_1^3)^{(2)}$	A_1^4

where $(A_1^3)^{(1)}=\langle\alpha_1,\alpha_3,\alpha_5\rangle$ and $(A_1^3)^{(2)}=\langle\alpha_1,\alpha_3,\alpha_6\rangle$. The restrictions $V\downarrow S$ can be worked out using [Liebeck and Seitz 2012, 11.15], from which we calculate $\dim C_V(u)$ for all the representatives:

u	a_2	c_2	a_4	c_4	a_6	a_6'	c_6
$\dim C_V(u), n=5$	12	8	10	8	–	–	–
$\dim C_V(u), n=6$	24	16	20	16	20	16	16

The conclusion of the lemma follows. □

3. Bases for some subspace actions

Let $G=\mathrm{Cl}(V)$ be a simple symplectic, unitary or orthogonal group over $\mathbb{F}_q$, with natural module V of dimension n. For $r<n$, denote by $\mathcal{N}_r$ an orbit of G on the set of nondegenerate r-subspaces of V. The main result of this section gives an upper bound for the base size of the action of G on $\mathcal{N}_r$ when r is very close to $\frac{n}{2}$. This will be used in the next section in the proof of Theorem 1 (see Lemmas 4.7 and 4.8).

Theorem 3.1. *Let $G_0=\mathrm{PSp}_n(q)(n\geq 6)$, $\mathrm{PSU}_n(q)(n\geq 4)$ or $P\Omega_n^\epsilon(q)(n\geq 7, q$ odd$)$, and let G be a group with socle G_0 such that $G\leq \mathrm{PGL}(V)$, where V is the natural module for G_0. Define*

$$r=\begin{cases}\frac{1}{2}(n-(n,4)) & \text{if } G_0=\mathrm{PSp}_n(q),\\ \frac{1}{2}(n-(n,2)) & \text{if } G_0=\mathrm{PSU}_n(q) \text{ or } P\Omega_n^\epsilon(q).\end{cases}$$

Then $b(G,\mathcal{N}_r)\leq 5$.

Theorem 3.1 will follow quickly from the following result. The deduction is given in Section 3B.

Type of element x	N_x
semisimple of odd prime order	$\frac{1}{2}\dim x^{\overline{G}}+\frac{1}{4}(n-l_0)+m^2$
semisimple involutions	$\left(\frac{1}{2}+\frac{2m}{n}\right)\dim x^{\overline{G}}$
unipotent of odd prime order	$\frac{1}{2}\dim x^{\overline{G}}+\frac{1}{4}\left(n-\sum_{i\text{ odd}} r_i\right)+m^2$
unipotent involutions of types b_l, c_l	$\left(\frac{1}{2}+\frac{2m+1}{n+2}\right)\dim x^{\overline{G}}$
unipotent involutions of type a_l	$\left(\frac{1}{2}+\frac{3m}{2n}\right)\dim x^{\overline{G}}$

Table 3.1. Bounds on $\dim(x^{\overline{G}}\cap\overline{H})$ for elements x of prime order. Here, l_0 is the multiplicity of the eigenvalue 1 in the action of x on V, and r_i is the number of Jordan blocks of size i in the Jordan form of x.

Theorem 3.2. *Let G and r be as in Theorem 3.1, and let H be the stabilizer in G of a nondegenerate r-subspace in $\mathcal{N}_r$. Let $x\in G$ be an element of prime order. Then one of the following holds*:

(i) $\log|x^G\cap H|/\log|x^G|<\frac{1}{2}+\frac{7}{30}$.

(ii) $G_0=\mathrm{PSp}_8(q)$ *and x is a unipotent element with Jordan form* $(2,1^6)$.

Our proof is modeled on that of [Burness 2007b, Theorem 1.1], where a similar conclusion is obtained for the action of G on the set of pairs $\{U,U^{\perp}\}$ of nondegenerate $n/2$-spaces.

3A. ***Proof of Theorem 3.2.*** We shall give a proof of the theorem just for the case where G_0 is a symplectic group $\mathrm{PSp}_n(q)$. The proofs for the orthogonal and unitary groups run along entirely similar lines.

We begin with a lemma on the corresponding algebraic groups. Let $K=\overline{\mathbb{F}}_q$ and $\overline{G}=\mathrm{PSp}_n(K)$, and let $V=V_n(K)$ be the underlying symplectic space. As in Theorem 3.2, write $r=\frac{1}{2}(n-(n,4))=\frac{1}{2}n-m$, where $m=\frac{1}{2}(n,4)$. Let $\overline{H}$ be the stabilizer in $\overline{G}$ of a nondegenerate r-subspace, so that $\overline{H}=(\mathrm{Sp}_{n/2-m}(K)\times\mathrm{Sp}_{n/2+m}(K))/\{\pm I\}$.

Write $p=\mathrm{char}(K)$. When $p=2$, the classes of involutions in $\overline{G}$ are determined by [Aschbacher and Seitz 1976]: For any odd $l\le n/2$, there is one class with Jordan form of type $(2^l,1^{n-2l})$, with representative denoted by b_l. For any nonzero even $l\le n/2$ there are two such classes, with representatives denoted by a_l, c_l. These are distinguished by the fact that $(v,va_l)=0$ for all $v\in V$.

Lemma 3.3. *With the above notation, if x is an element of prime order in $\overline{H}$, then* $\dim(x^{\overline{G}}\cap\overline{H})\le N_x$, *where N_x is given in Table 3.1.*

Proof. Denote by V_1 and $V_2=V_1^{\perp}$ the $(n/2-m)$- and $(n/2+m)$-dimensional subspaces of V preserved by $\overline{H}$. First suppose $x\in\overline{H}$ is a semisimple element of odd prime order t. Define ω to be a t-th root of unity and let l_i be the multiplicity of ω^i $(0\le i\le t-1)$ as an eigenvalue of x in its action on V. Then

$$\dim x^{\overline{G}}=\frac{n^2+n}{2}-\left(\frac{l_0}{2}+\frac{1}{2}\sum_{i=0}^{t-1}l_i^2\right),$$

and furthermore, $x^{\bar{G}} \cap \bar{H}$ is a union of a finite number of $\bar{H}$-classes, from which we see that

$$\begin{aligned}\dim(x^{\bar{G}} \cap \bar{H}) &\leq \tfrac{1}{4}(n^2+2n) + m^2 - \left(\tfrac{1}{2}l_0 + \tfrac{1}{4}\sum_{i=0}^{t-1} l_i^2\right)\\ &= \tfrac{1}{2}\dim x^{\bar{G}} + \tfrac{1}{4}(n-l_0) + m^2\\ &\leq \left(\tfrac{1}{2} + \tfrac{1}{n+2}\right)\dim x^{\bar{G}} + m^2.\end{aligned}$$

Now suppose that x is a semisimple involution. Here $C_{\bar{G}}(x)^0$ is the image modulo $\pm I$ of either $\mathrm{GL}_{n/2}(K)$ or $\mathrm{Sp}_l(K) \times \mathrm{Sp}_{n-l}(K)$, for some even $l \leq n/2$. In the first case, $\dim x^{\bar{G}} = n^2/4 + n/2$ and so

$$\dim(x^{\bar{G}} \cap \bar{H}) = \tfrac{1}{2}\dim x^{\bar{G}} + \tfrac{n}{4} + \tfrac{m^2}{2} = \left(\tfrac{1}{2} + \tfrac{1}{n}\right)\dim x^{\bar{G}} + \tfrac{1}{2}(m^2-1) \leq \left(\tfrac{1}{2} + \tfrac{2}{n}\right)\dim x^{\bar{G}}.$$

Now consider the second case, where $C_{\bar{G}}(x)^0 = \mathrm{Sp}_l(K) \times \mathrm{Sp}_{n-l}(K)$. Here x is $\bar{G}$-conjugate to $[-I_l, I_{n-l}]$, and $\dim x^{\bar{G}} = nl - l^2 = l(n-l)$. For $j = 1, 2$, the restriction of x to V_j is $\mathrm{Sp}(V_j)$-conjugate to $[-I_{l_j}, I_{d_j - l_j}]$ for some even integer $l_j \geq 0$, where $d_j = \dim V_j$. Noting that $l = l_1 + l_2$, we then have

$$\dim(x^{\bar{G}} \cap \bar{H}) = l_1\left(\tfrac{n}{2} - m - l_1\right) + l_2\left(\tfrac{n}{2} + m - l_2\right) \leq \tfrac{1}{2}\dim x^{\bar{G}} + m(l_2 - l_1) \leq \left(\tfrac{1}{2} + \tfrac{2m}{n}\right)\dim x^{\bar{G}}.$$

Now suppose that x is a unipotent element of odd prime order p and that x has Jordan form on V corresponding to the partition $(p^{r_p}, \dots, 1^{r_1}) \vdash n$. By [Lawther et al. 2002, 1.10],

$$\dim x^{\bar{G}} = \tfrac{1}{2}(n^2+n) - \tfrac{1}{2}\sum_{i=1}^{p}\left(\sum_{k=i}^{p} r_k\right)^2 - \tfrac{1}{2}\sum_{i \text{ odd}} r_i.$$

Hence, using [Burness 2007b, p.698], we have

$$\dim(x^{\bar{G}} \cap \bar{H}) \leq \tfrac{1}{2}\dim x^{\bar{G}} + \tfrac{1}{4}\left(n - \sum_{i \text{ odd}} r_i\right) + m^2 \leq \left(\tfrac{1}{2} + \tfrac{1}{n+2}\right)\dim x^{\bar{G}} + m^2.$$

Finally, we consider the case where x is a unipotent involution. First suppose that x is $\bar{G}$-conjugate to either b_l or c_l (as described in the preamble to the lemma). Then [Lawther et al. 2002, 1.10] implies that $\dim x^{\bar{G}} = l(n-l+1)$. Let x act on V_i with associated partition $(2^{l_i}, 1^{d_i - 2l_i})$ for $i = 1, 2$, where $d_1 = n/2 - m$ and $d_2 = n/2 + m$. Then

$$\dim(x^{\bar{G}} \cap \bar{H}) \leq \tfrac{1}{2}\dim x^{\bar{G}} + \tfrac{l}{2} + m(l_2 - l_1) \leq \left(\tfrac{1}{2} + \tfrac{2m+1}{n+2}\right)\dim x^{\bar{G}}.$$

Lastly, if x is $\bar{G}$-conjugate to a_l for some $2 \leq l \leq n/2$, then by [Lawther et al. 2002, 1.10], $\dim x^{\bar{G}} = l(n-l)$. By the definition of an a-type involution, if $y \in x^{\bar{G}} \cap \bar{H}$ fixes a subspace V_i, then the restriction of y to V_i is conjugate to a_{l_i} for some even integer $l_i \geq 0$. Therefore

$$\dim(x^{\bar{G}} \cap \bar{H}) \leq \tfrac{1}{2}\dim x^{\bar{G}} + m(l_2 - l_1).$$

Since $l_2 \leq \frac{d_2}{2}$ and $l_1 = l - l_2$, we see that $l_2 - l_1 \leq 3l(n-l)/(2n)$, so

$$\dim(x^{\bar{G}} \cap \bar{H}) \leq \left(\tfrac{1}{2} + \tfrac{3m}{2n}\right)\dim x^{\bar{G}}.$$

This completes the proof of the lemma. □

Now we embark on the proof of Theorem 3.2, considering in turn the various types of elements x of prime order in the symplectic group G. We shall frequently use the notation for such elements given in [Burness and Giudici 2016, §3.4]. Our approach in general is to find a function $\kappa(n)$ such that

$$\frac{\log|x^G \cap H|}{\log|x^G|} < \frac{1}{2} + \kappa(n), \tag{2}$$

where $\kappa(n) < \frac{7}{30}$ except possibly for some small values of n; these small values are then handled separately, usually by direct computation.

Lemma 3.4. *The conclusion of Theorem 3.2 holds when x is a semisimple element of odd prime order.*

Proof. Suppose $x \in H$ is a semisimple element of odd prime order r. Let $\mu = (l, a_1, \dots, a_k)$ be the tuple associated to x (as defined in [Burness 2007a, Definition 3.27]), and define i to be the smallest natural number such that $r \mid q^i - 1$. According to [Burness 2007a, 3.30] this means that

$$|C_G(x)| = \begin{cases} |\mathrm{Sp}_l(q)| \prod_{j=1}^{k} |\mathrm{GL}_{a_j}(q^i)| & i \text{ odd}, \\ |\mathrm{Sp}_l(q)| \prod_{j=1}^{k} |\mathrm{GU}_{a_j}(q^{i/2})| & i \text{ even}. \end{cases}$$

Let d be the number of nonzero a_j, and further define e to be equal to 1 or 2 when i is even or odd, respectively. By Lemma 3.3 and adapting the argument given in [Burness 2007b, p.720], we have

$$|x^G \cap H| < \left(\frac{n-l}{di} + 1\right)^{d/e} 2^{d(e-1)} q^{\dim x^{\bar{G}}/2 + (n-l)/4 + m^2}. \tag{3}$$

Furthermore, [Burness 2007a, 3.27] implies that

$$|x^G| \geq \frac{1}{2}\left(\frac{q}{q+1}\right)^{d(2-e)} q^{\dim x^{\bar{G}}}, \tag{4}$$

and [Burness 2007a, 3.33] gives the lower bound

$$\dim x^{\bar{G}} \geq \tfrac{1}{2}\big(n^2 + n - l^2 - l - \tfrac{1}{ei}(n-l-i(d-e))^2 - i(d-e)\big). \tag{5}$$

First suppose $m = 1$ (so that $n \equiv 2 \bmod 4$). Then (3)–(5) imply that the inequality (2) holds with $\kappa(n) = \frac{3}{n} + \frac{1}{n+1}$. Note that $\kappa(n) < \frac{7}{30}$ for $n \geq 18$. For $n = 6, 10, 14$, we must either adjust our value of $\kappa(n)$ or compute $|x^G \cap H|$ and $|x^G|$ explicitly, since here $\frac{3}{n} + \frac{1}{n+1} > \frac{7}{30}$. For $n = 14$, we find that (2) holds with $\kappa(n) = \frac{7}{30}$ for all choices of (l, i, d) except $(l, i, d) = (0, 1, 2)$. In the latter case, $H = (\mathrm{Sp}_8(q) \times \mathrm{Sp}_6(q))/\{\pm I\}$ and $|C_G(x)| = |\mathrm{GL}_{a_1}(q)||\mathrm{GL}_{a_2}(q)|$ with $a_1 + a_2 = 7$. Hence

$$|x^G \cap H| = \sum_{\substack{b_i \leq a_i \\ b_1 + b_2 = 4}} |\mathrm{Sp}_8(q) : \mathrm{GL}_{b_1}(q) \times \mathrm{GL}_{b_2}(q)| + |\mathrm{Sp}_6(q) : \mathrm{GL}_{a_1 - b_1}(q) \times \mathrm{GL}_{a_2 - b_2}(q)|,$$

and explicit computation gives $\log|x^G \cap H| / \log|x^G| < \frac{1}{2} + \frac{7}{30}$. For $n = 10$, (2) holds with $\kappa(n) = \frac{7}{30}$ for all valid choices of (l, i, d) except $(l, i, d) = (0, 1, 2)$ or $(0, 1, 4)$, and again explicit calculations as above give $\log|x^G \cap H|/\log|x^G| < \frac{1}{2}+\frac{7}{30}$. Finally, for $n=6$, we find that $\log|x^G \cap H|/\log|x^G| < \frac{1}{2}+\frac{7}{30}$ for all choices of x with associated parameters (l,i,d).

Now suppose $m=2$. Then (3)–(5) imply that (2) holds with $\kappa(n)=\frac{79}{20(n+1)}$ (when $e=1$), and with $\kappa(n)=\frac{22}{5(n+2)}$ (when $e=2$). We have $\kappa(n)<\frac{7}{30}$ for $n\geq 20$. For $n<20$, explicit calculations of $|x^G\cap H|$ as above yield the conclusion. □

Lemma 3.5. *The conclusion of Theorem 3.2 holds when x is a semisimple involution.*

Proof. Suppose that $x\in H$ is a semisimple involution. Denote by s the codimension of the largest eigenspace of x on $V=V_n(K)$. By [Burness 2007a, 3.37], $|C_G(x)|$ is equal to $|\mathrm{Sp}_s(q)||\mathrm{Sp}_{n-s}(q)|$, $|\mathrm{Sp}_{n/2}(q)|^2.2$, $|\mathrm{Sp}_{n/2}(q^2)|.2$ or $|\mathrm{GL}^{\epsilon}_{n/2}(q)|.2$, with $s<\frac{n}{2}$ in the first case, and $s=\frac{n}{2}$ in the latter three cases. Suppose x is as in one of the first two cases. Adapting the analogous argument given in [Burness 2007b, p.720], we deduce that

$$|x^G\cap H|<4\left(\frac{q^2+1}{q^2-1}\right)q^{s(n-s)/2-m(1-m)} \quad\text{and}\quad |x^G|>\tfrac{1}{2}q^{s(n-s)}$$

(the constant $\frac{1}{2}$ in the second inequality should be replaced by $\frac{1}{4}$ when $s=\frac{n}{2}$). These bounds imply that (2) holds with

$$\kappa(n)=\begin{cases}\frac{2}{n} & \text{if } s<\frac{n}{2}, m=1,\\ \frac{3}{n+1} & \text{if } s<\frac{n}{2}, m=2,\\ \frac{3}{2n} & \text{if } s=\frac{n}{2}, n\geq 12.\end{cases}$$

For $n\geq 12$ we have $\kappa(n)<\frac{7}{30}$, giving the conclusion. And for smaller values of n, we obtain the conclusion by explicit calculation of the values of $|x^G\cap H|$ and $|x^G|$.

Next suppose $|C_G(x)|=2|\mathrm{Sp}_{n/2}(q^2)|$. Then $|x^G|>\frac{1}{4}q^{n^2/4}$ by [Burness 2007a, 3.37]. If $\frac{n}{4}$ is even then $x^G\cap H=\varnothing$, so assume $\frac{n}{4}$ is odd. An argument analogous to that at the top of p.722 of [Burness 2007b] for this case gives $|x^G\cap H|<\frac{1}{4}q^{(n^2/8)+2}$. These bounds imply that (2) holds with $\kappa(n)=\frac{2}{n}$, and this is less than $\frac{7}{30}$ for all $n\geq 12$.

Finally, suppose that $|C_G(x)|=2|\mathrm{GL}^{\epsilon}_{n/2}(q)|$. Again [Burness 2007a, 3.37] and arguments of [Burness 2007b, p.722] give

$$|x^G|>\tfrac{1}{4}\left(\tfrac{q}{q+1}\right)q^{n(n+2)/4} \quad\text{and}\quad |x^G\cap H|<\tfrac{1}{4}q^{n^2/8+n/2+m^2/2}.$$

Hence (2) holds with $\kappa(n)=\frac{5}{2n}$, which is less than $\frac{7}{30}$ for $n>10$, and for $n\leq 10$ we obtain the conclusion as usual by explicit calculation of $|x^G\cap H|$ and $|x^G|$. □

Lemma 3.6. *The conclusion of Theorem 3.2 holds when x is a unipotent element of odd prime order.*

Proof. Let $x\in H$ be a unipotent element of order p, and suppose p is odd. Let the Jordan form of x on V correspond to the partition $\lambda\vdash n$. By Lemma 3.3,

$$\dim x^{\bar{H}}\leq \tfrac{1}{2}\dim x^{\bar{G}}+\tfrac{1}{4}(n-e)+m^2, \tag{6}$$

where e is the number of odd parts in λ.

Case $\lambda = (k^{n/k})$: Since k must divide both $\frac{n}{2} - m$ and $\frac{n}{2} + m$, we have $k = 2$ or 4 (the latter only if $m = 2$). Arguing as at the bottom of p.722 of [Burness 2007b], we have $\dim x^{\bar{G}} \geq \frac{1}{4}n(n+2)$, and also

$$|x^G| > \frac{q}{q+1} q^{\dim x^{\bar{G}}} \quad \text{and} \quad |x^G \cap H| = |x^H| < 4q^{\dim x^{\bar{H}}} \leq 4q^{\dim x^{\bar{G}}/2 + (n-e)/4 + m^2}.$$

These bounds imply that (2) holds with $\kappa(n) = \frac{3}{n+1}$, which is less than $\frac{7}{30}$ for $n \geq 12$. As usual, for smaller values of n we obtain the result by explicit computation of $|x^G \cap H|$ and $|x^G|$.

Case $\lambda = (2^j, 1^{n-2j})$, $n - 2j > 0$: First suppose $j = 1$. Then $|x^G| > \frac{1}{4}q^n$ and $|x^G \cap H| < q^{n/2+m} + q^{n/2-m}$. This implies that $\log|x^G \cap H|/\log|x^G| < \frac{1}{2} + \frac{7}{30}$ for all values of $n \geq 6$ except $n = 8$. The case $n = 8$ is the exception in part (ii) of Theorem 3.2.

Next suppose that $j = 2$. Here $|x^G| > \frac{1}{4(q+1)} q^{2n-1}$. Since the two Jordan blocks of size 2 can lie in the two different subspaces V_1 and V_2, or in the same one, we have

$$|x^G \cap H| < q^{(n-2m)/2 + (n+2m)/2} + 2q^{n-4+m(m-1)} + 2q^{n+m(m-1)}.$$

Hence (2) holds with $\kappa(n) = \frac{3}{n+1}$, which is less than $\frac{7}{30}$ for $n \geq 12$. For smaller values of n we obtain the conclusion by explicit computations of $|x^G \cap H|$ and $|x^G|$.

Finally, assume $j \geq 3$ (and so $n \geq 8$ since $n - 2j > 0$). The number of ways to distribute the j Jordan blocks of size 2 amongst the subspaces V_1 and V_2 is at most $j + 1$. Then, adapting the analogous bound in [Burness 2007b, p.723] and making use of Lemma 3.3, we have

$$|x^G \cap H| < 4(j+1) q^{\dim x^{\bar{G}}/2 + j/2 + m^2}$$

and as in [Burness 2007b, p.723], we have $|x^G| > \frac{1}{4} q^{\dim x^{\bar{G}}} = \frac{1}{4} q^{j(n-j+1)}$. This yields (2) with $\kappa(n) = \frac{4}{n+2}$, which is less than $\frac{7}{30}$ for $n \geq 16$. As usual, smaller values of n are handled by direct computation.

Case $\lambda = (k^{a_k}, \ldots, 2^{a_2}, 1^l)$, $k \leq n/2 + m$: In the computations below, we adapt the arguments on p.723 of [Burness 2007b]. Let d be the number of nonzero a_i. Then

$$|x^G| > \frac{1}{2^{d+1}} \left(\frac{q}{q+1} \right)^d q^{\dim x^{\bar{G}}}.$$

If $d = 1$ then $\lambda = (k^{(n-l)/k}, 1^l)$, and we can take $k > 2$ by the previous case. By [Lawther et al. 2002, 1.10], we have

$$\dim x^{\bar{G}} = \frac{n^2}{2} + \frac{n}{2} - \frac{l(n-l)}{k} - \frac{l^2}{2} - \frac{1}{2k}(n-l)^2 - \frac{l}{2} - \frac{\alpha}{2k}(n-l),$$

where α is zero if k is even and one if k is odd. Arguing as in [Burness 2007b, p.723] we also have

$$|x^G \cap H| < \left(\frac{n-l}{k} + 1 \right) 2^2 q^{\dim x^{\bar{G}}/2 + (n-l)(1-\alpha/k)/4 + m^2}.$$

These bounds imply (2) with $\kappa(n) = \frac{3}{n-3}$, which is less than $\frac{7}{30}$ for $n \geq 16$, and smaller values of n are handed by explicit computation.

Now suppose that $d \geq 2$. By [Burness 2007b, p.723],

$$\dim x^{\bar{G}} \geq \tfrac{1}{4}n^2 + \tfrac{1}{4}(d^2 - d + 2) - \tfrac{1}{16}d^4 - \tfrac{1}{24}d^3 + \tfrac{3}{16}d^2 - \tfrac{1}{3}d - \tfrac{1}{4}l^2 - \tfrac{1}{2},$$

and adapting the analogous bound given in [Burness 2007b, p.723] and referring to Lemma 3.3, we have

$$|x^G \cap H| < 4^d \left(\frac{\frac{n}{2} - \frac{d^2}{4} + \frac{d}{4} - \frac{l}{2} - 1}{d} + 1 \right)^d q^{\dim x^{\bar{G}}/2 + (n-l)/4 + m^2}.$$

These bounds give (2) with $\kappa(n) = \frac{4}{n}$, which is less than $\frac{7}{30}$ for $n \geq 18$, and smaller values of n are handed by explicit computation. □

Lemma 3.7. *The conclusion of Theorem 3.2 holds when x is a unipotent involution.*

Proof. Let $p = 2$, and recall the description of the involution class representatives a_l, b_l, c_l of G in the preamble to Lemma 3.3.

First assume that x is conjugate to a_l for some even integer l with $2 \leq l \leq \frac{n}{2}$. If $l = 2$, then by [Lawther et al. 2002, 1.10] and [Burness 2007a, Proposition 3.9] we have

$$|x^G \cap H| < 2q^{2(n/2-m-2)} + 2q^{2(n/2+m-2)}. \tag{7}$$

If $l \geq 4$ then we may adapt the analogous equation in [Burness 2007b, p.723] and obtain

$$|x^G \cap H| < \left(\tfrac{l}{2} + 1\right) 2^2 q^{(1/2+3m/(2n))l(n-l)}.$$

Furthermore, for all l, by [Burness 2007b, p.723]

$$|x^G| > \tfrac{1}{2} q^{l(n-l)}.$$

These bounds imply that $\log|x^G \cap H| / \log|x^G| < \frac{1}{2} + \frac{7}{30}$, provided $n \geq 14$ when $l = 2$, and $n \geq 24$ when $l \geq 4$. Smaller values of n can be dealt with by explicit computation of $|x^G \cap H|$ and $|x^G|$.

Now suppose that x is conjugate to either a b_l- or c_l-type involution. If $l = 1$ then by [Lawther et al. 2002, 1.10] and [Burness 2007a, Proposition 3.9]

$$|x^G \cap H| < q^{n/2-m} + q^{n/2+m}, \tag{8}$$

and if $l = 2$, then

$$|x^G \cap H| < q^n + q^{2(n/2-m-1)} + q^{2(n/2+m-1)}. \tag{9}$$

If $l \geq 3$, then by adapting the analogous argument in [Burness 2007b, p.724], we deduce

$$|x^G \cap H| < 4\left(\frac{q^2+1}{q^2-1}\right)(q^{\dim x^{\bar{G}}/2+2m-1} + q^{\dim x^{\bar{G}}/2+m-1}) + 4\left(\frac{q^2+1}{q^2-1}\right) q^{\dim x^{\bar{G}}/2+l/2+m}$$

where $\dim x^{\bar{G}} = l(n-l+1)$. Lastly, [Burness 2007b, p.724] gives

$$|x^G| > \tfrac{1}{2} q^{l(n-l+1)}.$$

As usual, these bounds imply that $\log|x^G \cap H|/\log|x^G| < \frac{1}{2} + \frac{7}{30}$ for $n \geq 14$, and explicit computations give the same conclusion for smaller values of n. □

This completes the proof of Theorem 3.2.

3B. ***Deduction of Theorem 3.1.*** The deduction of Theorem 3.1 from Theorem 3.2 proceeds along the lines of the proof of [Burness 2007c, 1.1].

First we shall require a small extension of [Burness 2007c, Proposition 2.2]. For a finite group G, define

$$\eta_G(t) = \sum_{C \in \mathcal{C}} |C|^{-t},$$

where $\mathcal{C}$ is the set of conjugacy classes of elements of prime order in G.

Lemma 3.8. *Let G be a finite classical group as in Theorem 3.1, with $n \geq 6$.*

(i) *Then $\eta_G\left(\frac{1}{3}\right) < 1$.*

(ii) *Let $G = \mathrm{PGSp}_8(q)$. Then $\eta_G\left(\frac{1}{3}\right) < 0.396$.*

Proof. (i) This is [Burness 2007c, Proposition 2.2].

(ii) We compute the sizes of the conjugacy classes with each centralizer type using [Burness and Giudici 2016, Table B.7], and bound the number of classes with each centralizer type using the same arguments as those given in the proof of [Burness 2007c, Lemma 3.2]. The result follows from these computations. □

We also need to cover separately the two cases of Theorem 3.1 for dimensions less than 6.

Lemma 3.9. *Theorem 3.1 holds for $G_0 = \mathrm{PSU}_4(q)$ or $\mathrm{PSU}_5(q)$.*

Proof. Consider the first case. Here $G = \mathrm{PGU}_4(q)$ acting on $\mathcal{N}_1$, the set of nondegenerate 1-spaces. Let $v_1, \ldots, v_4$ be an orthonormal basis of the natural module for G. If q is odd, then $\langle v_1 \rangle$, $\langle v_2 \rangle$, $\langle v_3 \rangle$, $\langle v_1 + v_2 + v_3 + v_4 \rangle$ is a base for the action of G; and if q is even, then $\langle v_1 \rangle$, $\langle v_2 \rangle$, $\langle v_3 \rangle$, $\langle v_1 + v_2 + v_3 \rangle$, $\langle v_2 + v_3 + v_4 \rangle$ is a base.

Now let $G = \mathrm{PGU}_5(q)$ acting on $\mathcal{N}_2$. Let $v_1, \ldots, v_5$ be an orthonormal basis. Any element of G that fixes the three nondegenerate 2-spaces $\langle v_1, v_2 \rangle$, $\langle v_2, v_3 \rangle$ and $\langle v_3, v_4 \rangle$ also fixes $\langle v_1, v_5 \rangle$ and $\langle v_4, v_5 \rangle$ (as these are $\langle v_2, v_3, v_4 \rangle^{\perp}$ and $\langle v_1, v_2, v_3 \rangle^{\perp}$), hence fixes all the 1-spaces $\langle v_1 \rangle, \ldots, \langle v_5 \rangle$. Hence adding two further nondegenerate 2-spaces intersecting in $\langle v_1 + \cdots + v_5 \rangle$ to the first three gives a base of size 5. □

Proof of Theorem 3.1. Let G and r be as in the statement of Theorem 3.1, and let H be the stabilizer of a nondegenerate r-subspace in $\mathcal{N}_r$. In view of Lemma 3.9, we may assume that the dimension $n \geq 6$.

For a positive integer c, let $Q(G, c)$ be the probability that a randomly chosen c-tuple of elements of $\mathcal{N}_r$ does not form a base for G. Then

$$Q(G, c) \leq \sum_{x \in X} |x^G| \left(\frac{\mathrm{fix}_{\mathcal{N}_r}(x)}{|\mathcal{N}_r|} \right)^c = \sum_{x \in X} |x^G| \left(\frac{|x^G \cap H|}{|x^G|} \right)^c, \tag{10}$$

where X is a set of conjugacy class representatives of the elements of G of prime order. Clearly G has a base of size c if and only if $Q(G, c) < 1$.

Assume for the moment that $G_0 \neq \mathrm{PSp}_8(q)$. Then by Theorem 3.2 we have

$$\frac{|x^G \cap H|}{|x^G|} < |x^G|^{-1/2+7/30}$$

for all elements $x \in G$ of prime order. Hence it follows from (10) that

$$Q(G,5) < \sum_{x \in X} |x^G|^{1+5(-1/2+7/30)} = \eta_G\big(\tfrac{1}{3}\big).$$

Therefore by Lemma 3.8(i), G has a base of size 5, as required.

It remains to consider the case where $G_0 = \mathrm{PSp}_8(q)$. Here Theorem 3.2(ii) gives $|x^G \cap H|/|x^G| < |x^G|^{-1/2+7/30}$ for all elements $x \in G$ of prime order, except when x is a unipotent element with Jordan form $(2, 1^6)$. In the latter case $|x^G| = q^8 - 1$ and $|x^G \cap H| = q^6 + q^2 - 2$. Hence

$$Q(G,5) < \eta_G\big(\tfrac{1}{3}\big) + (q^8 - 1)\left(\frac{q^6 + q^2 - 2}{q^8 - 1}\right)^5,$$

and this is less than 1 for all q, by Lemma 3.8(ii).

This completes the proof of Theorem 3.1. □

4. Proof of Theorem 1

Assume the hypotheses of Theorem 1. Thus $G \leq \mathrm{GL}(V) = \mathrm{GL}_d(q)$, and $E(G)$ is quasisimple and absolutely irreducible on V. Then the group $Z := Z(G)$ consists of scalars, and G/Z is almost simple. Let G_0 be the socle of G/Z. Note that $G_0 = E(G)/(Z \cap E(G))$.

Lemma 4.1. *If G_0 is exceptional of Lie type or sporadic, then $b(G) \leq 6$.*

Proof. Pick $v \in V \setminus \{0\}$, and consider the action of G on the orbit $\Delta = v^G$. By Lemma 2.1(i), if $G_0 \neq M_{24}$ then there exist Z-orbits $\delta_1, \ldots, \delta_6$ such that $G_{\delta_1 \cdots \delta_6} \leq Z$. Hence $b(G) \leq 6$. The case where $G_0 = M_{24}$ is taken care of in Remark 4.3 below. □

Lemma 4.2. Theorem 1(i) *or* (iii) *holds if G_0 is an alternating group.*

Proof. This follows from [Fawcett et al. 2016, Theorem 1.1]. □

In view of the previous two lemmas, we can suppose from now on that G_0 is a classical simple group. Assume that

$$b(G) \geq 7. \tag{11}$$

We aim to show that conclusion (ii) of Theorem 1 must hold. By the above assumption, the dimension $d \geq 7$, and also every element of V^6 is fixed by some element of prime order in $G \setminus Z$, and so

$$V^6 = \bigcup_{g \in \mathcal{P}} C_{V^6}(g), \tag{12}$$

where $\mathcal{P}$ denotes the set of elements of prime order in $G \setminus Z$. Now $|C_{V^6}(g)| = |C_V(g)|^6$, and

$$\dim C_V(g) \leq \left\lfloor \left(1 - \frac{1}{\alpha(g)}\right) \dim V \right\rfloor, \tag{13}$$

where $\alpha(g)$ is as defined in the preamble to Lemma 2.2 (strictly speaking, it is $\alpha(gZ)$ for $gZ \in G/Z$). Writing $\alpha = \alpha(G_0)$, it follows that

$$|V|^6 = q^{6d} \leq |\mathcal{P}| q^{6\lfloor d(1-1/\alpha) \rfloor}.$$

Since $|G| = |Z||G/Z| \leq (q-1)|\mathrm{Aut}(G_0)|$, we therefore have

$$q^{6\lceil d/\alpha \rceil} \leq |\mathcal{P}| < |G| \leq (q-1)|\mathrm{Aut}(G_0)|. \tag{14}$$

Remark 4.3. Using (14) we can handle the case $G_0 = M_{24}$ as follows, completing the proof of Lemma 4.1: we have $\alpha(M_{24}) \leq 4$ by [Goodwin 2000, 2.4], so (14) yields $\frac{6}{4}d < \log_2|M_{24}|$, hence $d \leq 18$. By [Hiss and Malle 2001], this forces $d = 11$ and $q = 2$, so $G = M_{24} < \mathrm{GL}_{11}(2)$. Here V or V^* is a quotient of the binary Golay code of length 24, dimension 12, by a trivial submodule, and we see from [Conway et al. 1985, p.94] that there is a G-orbit on V of size 276 or 759 on which G acts primitively. The base sizes of these actions of M_{24} are less than 7, by [Burness et al. 2010], and the conclusion follows. Similar, much simpler, computations also rule out the cases where G_0 is one of the three small groups in the conclusion of Lemma 2.2(vii).

Let $q = p^a$, where p is prime. The analysis divides naturally, according to whether or not the underlying characteristic of G_0 is equal to p — that is, whether or not G_0 is in the set $\mathrm{Lie}(p)$.

Lemma 4.4. *Under assumption* (11), G_0 *is not in* $\mathrm{Lie}(p')$.

Proof. Suppose $G_0 \in \mathrm{Lie}(p')$. Lower bounds for $d = \dim V$ are given by [Landazuri and Seitz 1974; Seitz and Zalesskii 1993], and the values of α by Lemma 2.2. Plugging these into (14) (and also using the fact that $d \geq 7$), we see that G_0 must be one of the following:

$$\mathrm{PSp}_4(3), \mathrm{PSp}_4(5), \mathrm{Sp}_6(2), \mathrm{PSp}_6(3), \mathrm{PSp}_8(3), \mathrm{PSp}_{10}(3),$$
$$U_3(3), U_4(3), U_5(2),$$
$$\Omega_7(3), \Omega_8^+(2).$$

At this point we use [Hiss and Malle 2001], which gives the dimensions and fields of definition of all the irreducible projective representations of the above groups of dimension up to 250. Combining this information with (14) leaves just the following possibilities:

G_0	d	q
$U_5(2)$	10	3
$U_4(3)$	20	2
$\mathrm{Sp}_6(2)$	7, 8	$q \leq 11$
	14	3
$\Omega_8^+(2)$	8	$q \leq 29$

Consider first $G_0 = U_5(2)$. Here $G = \langle -I \rangle \times U_5(2).2 < \mathrm{GL}_{10}(3)$, and the Brauer character of this representation of G is given in [Conway et al. 1985]. From this we can read off the dimensions of the fixed point spaces of $3'$-elements of prime order. These are as follows, using Atlas notation:

g	$2A, -2A$	$2B, -2B$	$2C, -2C$	$5A$	$11AB$
$\dim C_V(g)$	$2, 8$	$6, 4$	$5, 5$	2	0

Also $\alpha \le 5$ by Lemma 2.2, so (13) gives $\dim C_V(g) \le 8$ for all elements $g \in G$ of order 3. At this point, the inequality $|V|^6 \le \sum_{g\in\mathcal{P}} |C_V(g)|^6$ implied by (12) gives

$$3^{60} \le |2A| \cdot (3^{12} + 3^{48}) + |2B| \cdot (3^{24} + 3^{36}) + |2C| \cdot (3^{30} + 3^{30}) + |5A| \cdot 3^{12} + |3ABCDEF| \cdot 3^{48},$$

where $|2A|$ denotes the size of the conjugacy class of $2A$-elements, and so on. This is a contradiction.

This method works for all the cases in the above table, except $(G_0, d, q) = (\Omega_8^+(2), 8, 3)$; in this case the crude inequality $|V|^6 \le \sum_{g\in\mathcal{P}} |C_V(g)|^6$ implied by (12) does not yield a contradiction. Here we have $G \le 2.\Omega_8^+(2).2 < \mathrm{GL}(V) = \mathrm{GL}_8(3)$. Observe that $\Omega_8^+(2).2$ has a subgroup $N = S_3 \times \Omega_6^-(2).2$, and N is the normalizer of $\langle x \rangle$, where x is an element of order 3. Then $C_V(x) \neq 0$, and N must fix a 1-space in $C_V(x)$. Moreover, we compute that the minimal base size of $\Omega_8^+(2).2$ acting on the cosets of N is equal to 4. It follows that there are four 1-spaces in V whose pointwise stabilizer in G is Z. Hence $b(G) \le 4$ in this case. □

In view of the previous lemmas, from now on we may assume that $G_0 = \mathrm{Cl}_n(q_0)$, a classical simple group over a field $\mathbb{F}_{q_0}$ of characteristic p, with natural module of dimension n. There are various standard isomorphisms between classical groups of low dimensions (e.g., $L_4(q_0) \cong P\Omega_6^+(q_0)$); in such cases we adopt the notation $\mathrm{Cl}_n(q_0)$ taking n to be the minimal possible value. Recall that $G \le \mathrm{GL}(V) = \mathrm{GL}_d(q)$ and $G_0 = \mathrm{soc}(G/Z) = E(G)/(Z \cap E(G))$. The next lemma identifies the possible highest weights for V as a module for the quasisimple classical group $E(G)$.

Lemma 4.5. *Suppose as above that $G_0 = \mathrm{Cl}_n(q_0)$, a classical group in $\mathrm{Lie}(p)$. Then $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$, and one of the following holds*:

(1) $V = V(\lambda)$, *where λ is one of the weights $\lambda_1, \lambda_2, 2\lambda_1, \lambda_1 + p^i\lambda_1$, or $\lambda_1 + p^i\lambda_{n-1} (i > 0)$ (listed up to automorphisms of G_0, the last one only for $G_0 = L_n^\epsilon(q_0)$).*

(2) $G_0 = L_n^\epsilon(q_0) (n \ge 3)$ *and* $V = V(\lambda_1 + \lambda_{n-1})$.

(3) $G_0 = L_n(q_0) (7 \le n \le 21)$ *and* $V = V(\lambda_3)$.

(4) $G_0 = L_6^\epsilon(q_0)$ *and* $V = V(\lambda_3)$.

(5) $G_0 = L_8^\epsilon(q_0)$ *and* $V = V(\lambda_4)$.

(6) $G_0 = \mathrm{PSp}_6(q_0)$ *and* $V = V(\lambda_3)$ *(p odd).*

(7) $G_0 = \mathrm{PSp}_8(q_0)$ *and* $V = V(\lambda_3)$ *(p odd) or* $V(\lambda_4)$ *(p odd).*

(8) $G_0 = \mathrm{PSp}_{10}(q_0)$ *and* $V = V(\lambda_3)$ $(p = 2)$.

(9) $G_0 = P\Omega_n^\epsilon(q_0) (7 \le n \le 20, n \neq 8)$ *and V is a spin or half-spin module.*

Proof. Assume first that $q_0 > q$. Then by [Kleidman and Liebeck 1990, 5.4.6], there is an integer $s \geq 2$ such that $q_0 = q^s$ and $d = m^s$, where m is the dimension of an irreducible module for $E(G)$. Note that $m \geq n$ (by the minimal choice of n). By (14),

$$q^{6m^s/\alpha} \leq (q-1)|\mathrm{Aut}(\mathrm{Cl}_n(q^s))|.$$

Lemma 2.2 shows that $\alpha \leq n+2$ (excluding the small groups in Lemma 2.2(vii) which were ruled out in Remark 4.3), and hence

$$q^{6m^s/(n+2)} \leq (q-1)|\mathrm{Aut}(\mathrm{Cl}_n(q^s))| < (q-1)q^{s(n^2-1)}(2s\log_p q).$$

Since $m \geq n$, it follows from this that $s = 2$ and

$$m^2 < \frac{(n+2)(2n^2+1)}{6}.$$

Now using [Lübeck 2001], we deduce that $m = n$ and so

$$E(G) \leq \mathrm{SL}_n(q^2) < \mathrm{SL}_{n^2}(q).$$

As in [Liebeck and Shalev 2002, p.104], we see that there is a vector v such that $E(G)_v \leq \mathrm{SU}_n(q)$. By Lemma 2.1, the base size of an almost simple group with socle $L_n(q^2)$ acting on the cosets of a subgroup containing $U_n(q)$ is at most 4. Hence there are 1-spaces $\delta_1, \ldots, \delta_4$ whose pointwise stabilizer in G is equal to Z, and so $b(G) \leq 4$ in this case. This contradicts our initial assumption that $b(G) \geq 7$.

Hence we may assume now that $q_0 \leq q$, so that $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$ by [Kleidman and Liebeck 1990, 5.4.6]. Now (14) gives

$$d < \tfrac{\alpha}{6}(1 + \log_q|\mathrm{Aut}(G_0)|). \tag{15}$$

Noting that apart from the case where $G_0 = P\Omega_8^+(q_0)$, we have $|\mathrm{Out}(G_0)| \leq q$, it now follows using Lemma 2.2 that $d < N$, where N is as defined in Table 4.1.

G_0	N
$L_n^\epsilon(q_0)$	$\frac{1}{6}n(1+n^2),\ n > 4$
	$\frac{1}{6}(n+2)(1+n^2),\ n \leq 4$
$\mathrm{PSp}_n(q_0),\ n \geq 4$	$\frac{1}{6}(n+1)\big(2+\frac{1}{2}n(n+1)\big),\ n > 4$
	$10,\ n = 4$
$P\Omega_n^\epsilon(q_0),\ n \geq 7$	$\frac{1}{6}n\big(2+\frac{1}{2}n(n-1)\big)+\delta$

Table 4.1. Where δ is $\log_q 6$ if $G_0 = P\Omega_8^+(q_0)$, and $\delta = 0$ otherwise.

Now applying the bounds in [Lübeck 2001] (and also the improved bound for type A in [Martínez 2017]), we see that with one possible exception, one of the cases (1)–(9) in the conclusion holds. The possible exception is $G_0 = L_4^\epsilon(q_0)$ with $p = 3$ and $V = V(\lambda_1 + \lambda_2)$, of dimension 16. But in this case G does not contain a graph automorphism of G_0 (since the weight $\lambda_1 + \lambda_2$ is not fixed by a graph automorphism), and so [Guralnick and Saxl 2003, 4.1] implies that we can take $\alpha = 4$ in (15), and this rules out this case. □

Lemma 4.6. *Under the above assumption* (11), *G_0 is not as in* (3)–(9) *of Lemma 4.5.*

Proof. Suppose G_0 is as in (3)–(9) of Lemma 4.5. First we consider the actions of the simple algebraic groups $\bar{G}$ over $K = \bar{\mathbb{F}}_q$ corresponding to G_0 on the $K\bar{G}$-modules $\bar{V} = V \otimes K = V_{\bar{G}}(\lambda)$. Define

$$M_\lambda = \min\{\operatorname{codim} V_\gamma(g) \mid \gamma \in K^*,\ g \in \bar{G} \setminus Z(\bar{G})\}.$$

By Lemma 2.3, a lower bound for M_λ is given by $\min(s_\lambda, s'_\lambda)$, and simple calculations give the following lower bounds:

$\bar{G}$	λ	$M_\lambda \geq$
$A_n (n \geq 5)$	λ_3	$\frac{1}{2}(n-1)(n-2)$
A_7	λ_4	20
C_3	$\lambda_3 (p > 2)$	4
C_4	$\lambda_3 (p > 2)$	13
	$\lambda_4 (p > 2)$	13
C_5	$\lambda_3 (p = 2)$	25
$D_n (n \geq 5)$	λ_{n-1}, λ_n	2^{n-3}
$B_n (n \geq 3)$	λ_n	2^{n-2}

Apart from cases (4) and (5) of Lemma 4.5, the group G/Z is contained in $\bar{G}/Z$; in cases (4) and (5), a graph automorphism of $\bar{G}$ may also be present. Thus excluding (4) and (5), we see that (12) gives

$$q^{6M_\lambda} \leq |G|. \tag{16}$$

The bounds for M_λ in the above table now give a contradiction, except when $\bar{G} = D_n (n \leq 6)$ or $B_n (n \leq 5)$.

We now consider the cases $\bar{G} = D_n (n \leq 6)$ or $B_n (n \leq 5)$. Since $B_{n-1}(q) < D_n(q) < \mathrm{GL}(V)$, it suffices to deal with $\bar{G} = D_6$, D_5 or B_3.

Suppose $G_0 = D_6^\epsilon(q_0)$ with $\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$. By Lemma 2.4(i), for any element $g \in G$ that is not a scalar multiple of a root element, we have $\operatorname{codim} C_V(g) \geq 12$; and for root elements u, from the above table we have $\operatorname{codim} C_V(u) \geq 8$. The number of root elements in G_0 is less than $2q^{18}$. Hence (12) gives

$$|V|^6 = q^{32\times 6} \leq 2q^{18}(q-1) \cdot q^{24\times 6} + |G|q^{20\times 6},$$

which is a contradiction.

Now suppose $G_0 = D_5^\epsilon(q_0)$. We perform a similar calculation, using Lemma 2.4(ii). The number of semisimple elements s of G for which $C_{\bar{G}}(s)' = A_4$ is at most $|Z| \cdot (q-1)|D_5^\epsilon(q) : A_4^\epsilon(q).(q-1)| < 2q^{22}$. The number of root elements in G_0 is less than $2q^{14}$, and the number of unipotent elements in the class

$(A_1^2)^{(1)}$ is less than $2q^{20}$ (these have centralizer in $D_5^\epsilon(q)$ of order $q^{14}|\mathrm{Sp}_4(q)|(q-\epsilon)$, see [Liebeck and Seitz 2012, Table 8.6a]). Hence (12) together with Lemma 2.4(ii) gives

$$q^{16\times 6} \le 2(q^{14}+q^{20})(q-1)q^{12\times 6} + 2q^{22}q^{10\times 6} + |G|q^{8\times 6}.$$

This is a contradiction.

Next consider $G_0 = B_3(q_0)$. In the action on the spin module V, there is a vector v with stabilizer $G_2(q_0)$ in $B_3(q_0)$. Hence $b(G) \le 4$ in this case, by Lemma 2.1(ii).

It remains to handle cases (4) and (5), where G may contain graph automorphisms of $\bar{G}$. For $G_0 = L_6^\epsilon(q_0)$ or $L_8^\epsilon(q_0)$, the conjugacy classes of involutions in the coset of a graph automorphism are given by [Aschbacher and Seitz 1976, §19] for q even and by [Gorenstein et al. 1998, 4.5.1] for q odd. It follows that the number of such involutions is less than $2q^{21}$ or $2q^{36}$ in case (4) or (5), respectively. For such an involution g, by (13) we have $\dim C_V(g) \le 16$ or 60, respectively. All other elements of prime order in G lie in $\bar{G}Z$, hence have fixed point space of codimension at least M_λ. Hence we see that (12) gives

$$|V|^6 = \begin{cases} q^{20\times 6} \le |G|\cdot q^{14\times 6} + 2q^{21}\cdot q^{16\times 6} & \text{in case (4)},\\ q^{70\times 6} \le |G|\cdot q^{50\times 6} + 2q^{36}\cdot q^{60\times 6} & \text{in case (5)}.\end{cases}$$

Both of these yield contradictions.

This completes the proof of the lemma. □

Lemma 4.7. *The group G_0 is not as in* (2) *of Lemma 4.5.*

Proof. Here $G_0 = L_n^\epsilon(q_0)$ with $n \ge 3$, and $V = V(\lambda_1+\lambda_{n-1})$. Suppose first that $\epsilon = +$. Then $G/Z \le \mathrm{PGL}_n(q)$, and V can be identified with T/T_0, where

$$T = \{A \in M_{n\times n}(q) : \mathrm{Tr}(A) = 0\} \quad \text{and} \quad T_0 = \{\lambda I_n : n\lambda = 0\},$$

and the action of $\mathrm{GL}_n(q)$ is by conjugation. By [Steinberg 1962], we can choose $X, Y \in \mathrm{SL}_{n-1}(q_0)$ generating $\mathrm{SL}_{n-1}(q_0)$. Let $x = \mathrm{Tr}(X)$, $y = \mathrm{Tr}(Y)$, and define

$$A_1 = \begin{pmatrix} X & 0\\ 0 & -x\end{pmatrix},\quad A_2 = \begin{pmatrix} Y & 0\\ 0 & -y\end{pmatrix},\quad A_3 = \begin{pmatrix} -x & 0\\ 0 & X\end{pmatrix},\quad A_4 = \begin{pmatrix} -y & 0\\ 0 & Y\end{pmatrix}.$$

Then $\{A_1, \dots, A_4\}$ is a base for the action of $\mathrm{GL}_n(q)$, and hence $b(G) \le 4$.

Now suppose $\epsilon = -$, so that $G/Z \le \mathrm{PGU}_n(q)$, where we take $\mathrm{GU}_n(q) = \{g \in \mathrm{GL}_n(q^2) : g^T g^{(q)} = I\}$. Then we can identify V with the $\mathbb{F}_q$-space S modulo scalars, where

$$S = \{A \in M_{n\times n}(q^2) : \mathrm{Tr}(A) = 0,\ A^T = A^{(q)}\},$$

with $\mathrm{GU}_n(q)$ acting by conjugation. As in [Liebeck and Shalev 2002, p.104], there is a vector $A \in V$ such that $\mathrm{GU}_n(q)_A \le N_r$, where N_r is the stabilizer of a nondegenerate r-space and $r = \frac{1}{2}n$ or $\frac{1}{2}(n-(n,2))$. In the first case, the base size of $\mathrm{PGU}_n(q)$ acting on $\mathcal{N}_r$ is at most 5, by Lemma 2.1(ii) (since in this case N_r is contained in a nonsubspace subgroup of type $\mathrm{GU}_{n/2}(q) \wr S_2$); and the same holds in the second case, by Theorem 3.1. It follows that $b(G) \le 5$, contradicting our assumption (11). □

The proof of Theorem 1 is completed by the following lemma.

Lemma 4.8. *If G_0 is as in* (1) *of Lemma 4.5, then conclusion* (ii) *of Theorem 1 holds.*

Proof. Here $G_0 = \mathrm{Cl}_n(q_0)$, and $V = V(\lambda)$ with $\lambda = \lambda_1, \lambda_2, 2\lambda_1, \lambda_1 + p^i\lambda_1$ or $\lambda_1 + p^i\lambda_{n-1}$.

If $\lambda = \lambda_1$, then $d = n$ and $E(G) = \mathrm{Cl}_d(q_0)$ is as in part (ii) of Theorem 1.

Now consider $\lambda = \lambda_2$. Here we argue as in the proof of [Liebeck and Shalev 2002, 2.2] (see p.102). Assume first that $V = \wedge^2 W$ where W is the natural module for $\mathrm{Cl}_n(q_0)$ (with scalars extended to $\mathbb{F}_q$). Then $E(G)$ lies in the action of $\mathrm{SL}(W)$ on this space. If n is even, then the argument in [loc. cit.] provides a vector $v \in V$ such that $\mathrm{SL}(W)_v = \mathrm{Sp}(W)$, and so $b(G) \le b(\mathrm{PGL}(W)/\mathrm{PSp}(W))$. By Lemma 2.1(ii), this is at most 4, provided $n \ge 6$; for $n = 4$, the action $\mathrm{PGL}_4/\mathrm{PSp}_4$ is a subspace action (it is O_6/N_1), so Lemma 2.1 does not apply — but an easy argument shows that the base size is at most 5 in this case. And if n is odd, say $n = 2k+1$, then the argument in [loc. cit.] gives three vectors with stabilizer normalizing a subgroup Sp_{2k}, and then adding three further vectors gives a base — so $b(G) \le 6$ (again, a slightly different argument is needed for the case $2k = 4$, but this is straightforward). Now assume $V \ne \wedge^2 W$. Then V is equal to $(\wedge^2 W)^+$ (which is $f^\perp$ or $f^\perp/\langle f\rangle$ in the notation of [loc. cit., p.103]), and $E(G)$ lies in the action of $\mathrm{Sp}(W)$ on this space; the argument in [loc. cit.] gives

$$b(G) \le b(\mathrm{PSp}(W), \mathcal{N}_r),$$

where $\mathcal{N}_r$ is the set of nondegenerate subspaces of dimension r and $r = \frac{1}{2}n$ or $\frac{1}{2}(n-(n,4))$. As before, Lemma 2.1(ii) (in the first case) and Theorem 3.1 (in the second) now give $b(G) \le 5$.

The case where $\lambda = 2\lambda_1$ is similar to the λ_2 case, arguing as in [loc. cit., p.103]. Note that p is odd here. If G_0 is not an orthogonal group, then $E(G) \le \mathrm{SL}(W)$ acting on $V = S^2W$, and there is a vector v such that $\mathrm{SL}(W)_v = \mathrm{SO}(W)$; hence $b(G) \le b(\mathrm{SL}(W)/\mathrm{SO}(W)) \le 4$, by Lemma 2.1(ii). And if G_0 is orthogonal, then $V = (S^2W)^+$ (of dimension $\dim S^2W - \delta$, $\delta \in \{1, 2\}$), and we see as in the previous case that $b(G) \le b(PGO(W), \mathcal{N}_r)$ with $r = \frac{1}{2}(n-(n,2))$. Hence Theorem 3.1 gives $b(G) \le 5$ again.

Finally, suppose $\lambda = \lambda_1 + p^i\lambda_1$ or $\lambda_1 + p^i\lambda_{n-1}$. Here as in [loc. cit., p.103], we have $E(G) \le \mathrm{SL}(W) = \mathrm{SL}_n(q)$ acting on $V = W \otimes W^{(p^i)}$ or $W \otimes (W^*)^{(p^i)}$. We can think of the action of $\mathrm{SL}(W)$ on V as the action on $n \times n$ matrices, where $g \in \mathrm{SL}(W)$ sends

$$A \to g^T A g^{(p^i)} \text{ or } g^{-1} A g^{(p^i)}.$$

Hence we see that the stabilizer of the identity matrix I is contained in $\mathrm{SU}_n(q^{1/2})$ or $\mathrm{SL}_n(q^{1/r})$ for some $r > 1$, and so as usual Lemma 2.1(ii) gives $b(G) \le 5$. □

This completes the proof of Theorem 1.

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