A blowup algebra for hyperplane arrangements

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It is shown that the Orlik–Terao algebra is graded isomorphic to the special fiber of the ideal I generated by the $(n-1)$-fold products of the members of a central arrangement of size n. This momentum is carried over to the Rees algebra (blowup) of I and it is shown that this algebra is of fiber-type and Cohen–Macaulay. It follows by a result of Simis and Vasconcelos that the special fiber of I is Cohen–Macaulay, thus giving another proof of a result of Proudfoot and Speyer about the Cohen–Macaulayness of the Orlik–Terao algebra.

Introduction

The central theme of this paper is to study the ideal-theoretic aspects of the blowup of a projective space along a certain scheme of codimension 2. To be more precise, let $\mathcal{A} = \{\ker(\ell_1), \dots, \ker(\ell_n)\}$ be an arrangement of hyperplanes in $\mathbb{P}^{k-1}$ with coordinate ring $R = \Bbbk[x_1, \dots, x_k]$, and consider the closure of the graph of the following rational map

$$\mathbb{P}^{k-1} \dashrightarrow \mathbb{P}^{n-1}, \quad x \mapsto (1/\ell_1(x) : \cdots : 1/\ell_n(x)).$$

Rewriting the coordinates of the map as forms of the same positive degree in the source $\mathbb{P}^{k-1} = \mathrm{Proj}(R)$, we are led to consider the corresponding graded R-algebra, namely, the Rees algebra of the ideal of R generated by the $(n-1)$-fold products of $\ell_1, \dots, \ell_n$.

This construction is significant in the theory of hyperplane arrangements as it provides a method of compactifying the complement of an arrangement complement. In [Huh and Katz 2012] and under a slightly different setup, it is shown that the cohomology class of the blowup (in the Chow ring of a product of projective spaces) is determined by the underlying combinatorics of $\mathcal{A}$. (See Remark 2.1(v) for details).

It is our view that bringing into the related combinatorics a limited universe of gadgets and numerical invariants from commutative algebra may be of help, especially regarding the typical operations with ideals and algebras. This point of view favors at the outset a second look at the celebrated Orlik–Terao algebra $\Bbbk[1/\ell_1, \dots, 1/\ell_n]$ which is regarded as a commutative counterpart to the combinatorial Orlik–Solomon algebra. The fact that the former, as observed by some authors, has a model as a finitely generated graded

k-subalgebra of a finitely generated purely transcendental extension of the field k, makes it possible to recover it as the homogeneous coordinate ring of the image of a certain rational map.

This is our departing step to naturally introduce other commutative algebras into the picture. As shown in Theorem 2.4, the Orlik–Terao algebra now becomes isomorphic, as a graded k-algebra, to the special fiber algebra (also called fiber cone algebra or central algebra) of the ideal I generated by the $(n-1)$-fold products of the members of the arrangement $\mathcal{A}$. This algebra is in turn defined as a residue algebra of the Rees algebra of I, so it is only natural to look at this and related constructions. One of these constructions takes us to the symmetric algebra of I, and hence to the syzygies of I. Since I turns out to be a perfect ideal of codimension 2, its syzygies are rather simple and allow us to further understand these algebras.

As a second result along this line of approach, we show that a presentation ideal of the Rees algebra of I can be generated by the syzygetic relations and the Orlik–Terao ideal (see Theorem 4.2). This property has been coined the *fiber type* property in the recent literature; see, e.g., [Herzog et al. 2005, page 808].

A very recent development in this area is the main theorem of Fink, Speyer and Woo in [Fink et al. 2018] who independently recover a variant of this result by obtaining a Gröbner basis under a certain term order. Their result is utilized to compute the initial ideal and consequently the Hilbert series of the presentation ideal which is the general form of our Proposition 4.1(d).

The third main result of this work, as an eventual outcome of these methods, is a proof of the Cohen–Macaulay property of the Rees algebra of I (see Theorem 4.9).

The typical argument in the proofs is induction on the size or rank of the arrangements. Here we draw heavily on the operations of deletion and contraction of an arrangement. In particular, we introduce a variant of a multiarrangement that allows repeated linear forms to be tagged with arbitrarily different coefficients. Then the main breakthrough consists in getting a precise relation between the various ideals or algebras attached to the original arrangement and those attached to the minors.

One of the important facts about the Orlik–Terao algebra is that it is Cohen–Macaulay, as proven by Proudfoot and Speyer [2006]. Using a recent result of W. Vasconcelos and one of us, we recover this result as a consequence of the Cohen–Macaulay property of the Rees algebra.

The structure of this paper is as follows. The first section is an account of the needed preliminaries from commutative algebra. The second section expands on highlights of the settled literature about the Orlik–Terao ideal as well as a tangential discussion on the so-called nonlinear invariants of our ideals such as the reduction number and analytic spread. The third section focuses on the ideal of $(n-1)$-fold products and the associated algebraic constructions. The last section is devoted to the statements and proofs of the main theorems where we draw various results from the previous sections to establish the arguments.

1. Ideal theoretic notions and blowup algebras

The *blow up algebra* of an ideal I in a ring R is the R-algebra

$$\mathcal{R}(I) := \bigoplus_{i \geq 0} I^i.$$

This is a standard R-graded algebra with $\mathcal{R}(I)_0 = R$, where multiplication is induced by the internal multiplication rule $I^r I^s \subset I^{r+s}$. One can see that there is a graded isomorphism $R[It] \simeq \mathcal{R}(I)$, where $R[It]$ is the homogeneous R-subalgebra of the standard graded algebra of polynomials $R[t]$ in one variable over R, generated by the elements at, $a \in I$, of degree 1. The algebra $R[It]$ is known as the *Rees algebra* of the ideal I. Because of the mentioned isomorphism between them, we will often identify these two algebras.

Quite generally, fixing a set of generators of I determines a surjective homomorphism of R-algebras from a polynomial ring over R to $R[It]$. The kernel of such a map is called a *presentation ideal* of $R[It]$. In this generality, even if R is Noetherian (so I is finitely generated) the notion of a presentation ideal is quite loose.

In this work we deal with a special case in which $R = \Bbbk[x_1, \ldots, x_k]$ is a standard graded polynomial ring over a field $\Bbbk$ and $I = \langle g_1, \ldots, g_n \rangle$ is an ideal generated by forms $g_1, \ldots, g_n$ of the same degree. Let $T = R[y_1, \ldots, y_n] = \Bbbk[x_1, \ldots, x_k; y_1, \ldots, y_n]$, a standard bigraded $\Bbbk$-algebra with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$. Using the given generators to obtain an R-algebra homomorphism

$$\varphi : T = R[y_1, \ldots, y_n] \to R[It], \quad y_i \mapsto g_i t,$$

yields a presentation ideal $\mathcal{I}$ which is bihomogeneous in the bigrading of T. Therefore, $R[It]$ acquires the corresponding bigrading.

Changing $\Bbbk$-linearly independent sets of generators in the same degree amounts to effecting an invertible $\Bbbk$-linear map, so the resulting effect on the corresponding presentation ideal is pretty much under control. For this reason, we will by abuse talk about *the* presentation ideal of I by fixing a particular set of homogeneous generators of I of the same degree. Occasionally, we may need to bring in a few superfluous generators into a set of minimal generators.

Since the given generators have the same degree, they span a linear system defining a rational map

$$\Phi : \mathbb{P}^{k-1} \dashrightarrow \mathbb{P}^{n-1}, \tag{1}$$

by the assignment $x \mapsto (g_1(x) : \cdots : g_n(x))$, when some $g_i(x) \neq 0$.

The ideal I is often called the base ideal (to agree with the base scheme) of Φ. Asking when Φ is birational onto its image is of interest and we will briefly deal with it as well. Again note that changing to another set of generators in the same degree will not change the linear system thereof, defining the same rational map up to a coordinate change at the target.

The Rees algebra brings along other algebras of interest. In the present setup, one of them is the *special fiber* $\mathcal{F}(I) := R[It] \otimes_R R/\mathfrak{m} \simeq \bigoplus_{s \geq 0} I^s / \mathfrak{m} I^s$, where $\mathfrak{m} = \langle x_1, \ldots, x_k \rangle \subset R$. The Krull dimension of the special fiber $\ell(I) := \dim \mathcal{F}(I)$ is called the *analytic spread* of I.

The analytic spread is a significant notion in the theory of *reductions* of ideals. An ideal $J \subset I$ is said to be a *reduction of I* if $I^{r+1} = J I^r$ for some r. Most notably, this is equivalent to the condition that the natural inclusion $R[Jt] \hookrightarrow R[It]$ is a finite morphism. The smallest such r is the *reduction number $r_J(I)$ with respect to J*. The *reduction number of I* is the infimum of all $r_J(I)$ for all minimal reductions J of I; this number is denoted by $r(I)$.

Geometrically, the relevance of the special fiber lies in the following result, which we isolate for easy reference:

Lemma 1.1. *Let Φ be as in* (1) *and I its base ideal. Then the homogeneous coordinate ring of the image of Φ is isomorphic to the special fiber $\mathcal{F}(I)$ as graded $\Bbbk$-algebras.*

To see this, note that the Rees algebra defines a biprojective subvariety of $\mathbb{P}^{k-1} \times \mathbb{P}^{n-1}$, namely the closure of the graph of Φ. Projecting down to the second coordinate recovers the image of Φ. At the level of coordinate rings this projection corresponds to the inclusion $\Bbbk[I_d t] = \Bbbk[g_1 t, \ldots, g_n t] \subset R[I t]$, where $g_1, \ldots, g_n$ are forms of the degree d; this inclusion is a split $\Bbbk[I_d t]$-module homomorphism with $\mathfrak{m}R[I t]$ as direct complement. Therefore, one has an isomorphism of $\Bbbk$-graded algebras $\Bbbk[I_d] \simeq \Bbbk[I_d t] \simeq \mathcal{F}(I)$.

As noted before, the presentation ideal of $R[I t]$

$$\mathcal{I} = \bigoplus_{(a,b) \in \mathbb{N} \times \mathbb{N}} \mathcal{I}_{(a,b)},$$

is a bihomogeneous ideal in the standard bigrading of T. Two basic subideals of $\mathcal{I}$ are $\langle \mathcal{I}_{(0,-)} \rangle$ and $\langle \mathcal{I}_{(-,1)} \rangle$, and they come in as follows.

Consider the natural surjections

$$T \xrightarrow[\psi]{\varphi} R[I t] \xrightarrow{\otimes_R R/\mathfrak{m}} \mathcal{F}(I) \,,$$

where the kernel of the leftmost map is the presentation ideal $\mathcal{I}$ of $R[I t]$. Then we have

$$\mathcal{F}(I) \simeq \frac{T}{\ker \psi} \simeq \frac{T}{\langle \ker \varphi |_{(0,-)}, \mathfrak{m} \rangle} \simeq \frac{\Bbbk[y_1, \ldots, y_n]}{\langle \mathcal{I}_{(0,-)} \rangle}.$$

Thus, $\langle \mathcal{I}_{(0,-)} \rangle$ is the homogeneous defining ideal of the special fiber (or, as explained in Lemma 1.1, of the image of the rational map Φ).

As for the second ideal $\langle \mathcal{I}_{(-,1)} \rangle$, one can see that it coincides with the ideal of T generated by the biforms $s_1 y_1 + \cdots + s_n y_n \in T$, whenever $(s_1, \ldots, s_n)$ is a syzygy of $g_1, \ldots, g_n$ of certain degree in R. Thinking about the one-sided grading in the y's, there is no essential harm in denoting this ideal simply by $\mathcal{I}_1$. Thus, $T/\mathcal{I}_1$ is a presentation of the symmetric algebra $\mathcal{S}(I)$ of I. It yields a natural surjective map of R-graded algebras

$$\mathcal{S}(I) \simeq T/\mathcal{I}_1 \twoheadrightarrow T/\mathcal{I} \simeq \mathcal{R}(I).$$

As a matter of calculation, one can easily show that $\mathcal{I} = \mathcal{I}_1 : I^\infty$, the saturation of $\mathcal{I}_1$ with respect to I.

The ideal I is said to be of *linear type* provided $\mathcal{I} = \mathcal{I}_1$, i.e., when the above surjection is injective. It is said to be of *fiber type* if $\mathcal{I} = \mathcal{I}_1 + \langle \mathcal{I}_{(0,-)} \rangle = \langle \mathcal{I}_1, \mathcal{I}_{(0,-)} \rangle$.

A basic homological obstruction for an ideal to be of linear type is the so-called G_∞ condition of Artin and Nagata [1972], also known as the F_1 condition [Herzog et al. 1983]. A weaker condition is the so-called G_s condition, for a suitable integer s. All these conditions can be stated in terms of the Fitting ideals of the given ideal or, equivalently, in terms of the various ideals of minors of a syzygy matrix of the ideal.

In this work we will have a chance to use condition G_k, where $k = \dim R < \infty$. Given a free presentation

$$R^m \xrightarrow{\varphi} R^n \to I \to 0$$

of an ideal $I \subset R$, the G_k condition for I means that

$$\operatorname{ht}(I_p(\varphi)) \geq n - p + 1, \quad \text{for } p \geq n - k + 1, \tag{2}$$

where $I_t(\varphi)$ denotes the ideal generated by the t-minors of φ. Note that nothing is required about the values of p strictly smaller than $n - k + 1$, since for such values one has $n - p + 1 > k = \dim R$, which makes the same bound impossible.

A useful method to obtain new generators of $\mathcal{I}$ from old generators (starting from generators of $\mathcal{I}_1$) is via Sylvester forms (see [Hong et al. 2012, Proposition 2.1]), which has classical roots as the name indicates. It can be defined quite generally as follows: Let $R := \Bbbk[x_1, \ldots, x_k]$, and let $T := R[y_1, \ldots, y_n]$ as above. Given $F_1, \ldots, F_s \in \mathcal{I}$, let J be the ideal of R generated by all the coefficients of the F_i, the so-called R-content ideal. Suppose $J = \langle a_1, \ldots, a_q \rangle$, where a_i are forms of the same degree. Then we have the matrix equation

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{bmatrix} = A \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \end{bmatrix},$$

where A is an $s \times q$ matrix with entries in T.

If $q \geq s$ and if the syzygies on F_i's are in $\mathfrak{m}T$, then the determinant of any $s \times s$ minor of A is an element of $\mathcal{I}$. These determinants are called *Sylvester forms*. The main use in this work is to show that the Orlik–Terao ideal is generated by such forms (Proposition 3.5).

The last invariant we wish to comment on is the reduction number $r(I)$. For convenience, we state the following result:

Proposition 1.2. *With the above notation, suppose that the special fiber $\mathcal{F}(I)$ is Cohen–Macaulay. Then the reduction number $r(I)$ of I coincides with the Castelnuovo–Mumford regularity $\operatorname{reg}(\mathcal{F}(I))$ of $\mathcal{F}(I)$.*

Proof. By [Vasconcelos 2005, Proposition 1.85], when the special fiber is Cohen–Macaulay, one can read $r(I)$ off the Hilbert series. Write

$$HS(\mathcal{F}(I), s) = \frac{1 + h_1 s + h_2 s^2 + \cdots + h_r s^r}{(1 - s)^d},$$

with $h_r \neq 0$ and $d = \ell(I)$, the dimension of the fiber (analytic spread). Then, $r(I) = r$.

Since $\mathcal{F}(I) \simeq S/\langle \mathcal{I}_{(0,-)} \rangle$, where $S := \Bbbk[y_1, \ldots, y_n]$, we have that $\mathcal{F}(I)$ has a minimal graded S-free resolution of length equal to $m := \operatorname{ht}\langle \mathcal{I}_{(0,-)} \rangle$, and $\operatorname{reg}(\mathcal{F}(I)) = \alpha - m$, where α is the largest shift in the minimal graded free resolution, occurring also at the end of this resolution. These last two statements mentioned here come from the Cohen–Macaulayness of $\mathcal{F}(I)$.

The additivity of Hilbert series under short exact sequences of modules, together with the fact that $HS(S^u(-v), s) = us^v/(1 - s)^n$ gives that $r + m = \alpha = m + \operatorname{reg}(\mathcal{F}(I))$, so $r(I) = \operatorname{reg}(\mathcal{F}(I))$. $\qquad\square$

2. Hyperplane arrangements

Let $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{P}^{k-1}$ be a central hyperplane arrangement of size n and rank k. Here $H_i = \ker(\ell_i)$, $i = 1, \ldots, n$, where each ℓ_i is a linear form in $R := \mathbb{k}[x_1, \ldots, x_k]$ and $\langle \ell_1, \ldots, \ell_n \rangle = \mathfrak{m} := \langle x_1, \ldots, x_k \rangle$. From the algebraic viewpoint, there is a natural emphasis on the linear forms ℓ_i and the associated ideal theoretic notions.

Deletion and contraction are useful operations on $\mathcal{A}$. Fixing an index $1 \leq i \leq n$, one introduces two new minor arrangements:

$$\mathcal{A}' = \mathcal{A} \setminus \{H_i\} \text{ (deletion)}, \quad \mathcal{A}'' = \mathcal{A}' \cap H_i := \{H_j \cap H_i \mid 1 \leq j \leq n, \ j \neq i\} \text{ (contraction)}.$$

Clearly, $\mathcal{A}'$ is a subarrangement of $\mathcal{A}$ of size $n - 1$ and rank at most k, while $\mathcal{A}''$ is an arrangement of size $\leq n - 1$ and rank $k - 1$. Contraction comes with a natural multiplicity given by counting the number of hyperplanes of $\mathcal{A}'$ that give the same intersection. A modified version of such a notion will be thoroughly used in this work.

The following notion will play a substantial role in some inductive arguments throughout the paper: ℓ_i is called a *coloop* if the rank of the deletion $\mathcal{A}'$ with respect to ℓ_i is $k - 1$, i.e., drops by one. This simply means that $\bigcap_{j \neq i} H_j$ is a line rather than the origin in $\mathbb{A}^k$. Otherwise, we say that ℓ_i is a *noncoloop*.

2A. *The Orlik–Terao algebra.* One of our motivations is to clarify the connections between the Rees algebra and the Orlik–Terao algebra which is an important object in the theory of hyperplane arrangements. We state the definition and review some of its basic properties below.

Let $\mathcal{A} \subset \mathbb{P}^{k-1}$ be a hyperplane arrangement as above. Suppose $c_{i_1} \ell_{i_1} + \cdots + c_{i_m} \ell_{i_m} = 0$ is a linear dependency among m of the linear forms defining $\mathcal{A}$, denoted D. Consider the following homogeneous polynomial in $S := \mathbb{k}[y_1, \ldots, y_n]$:

$$\partial D := \sum_{j=1}^{m} c_{i_j} \prod_{j \neq k=1}^{m} y_{i_k}. \tag{3}$$

Note that $\deg(\partial D) = m - 1$.

The *Orlik–Terao algebra of $\mathcal{A}$* is the standard graded $\mathbb{k}$-algebra

$$\mathrm{OT}(\mathcal{A}) := S/\partial(\mathcal{A}),$$

where $\partial(\mathcal{A})$ is the ideal of S generated by $\{\partial D \mid D \text{ a dependency of } \mathcal{A}\}$, with ∂D as in (3), called the *Orlik–Terao ideal*. This algebra was introduced in [Orlik and Terao 1994] as a commutative analog of the classical combinatorial Orlik–Solomon algebra, in order to answer a question of Aomoto. The following remark states a few important properties of this algebra and related constructions.

Remark 2.1. (i) Recalling that a circuit is a minimally dependent set, one has that $\partial(\mathcal{A})$ is generated by ∂C, where C runs over the circuits of $\mathcal{A}$ [Orlik and Terao 1994]. In addition, these generators form an universal Gröbner basis for $\partial(\mathcal{A})$ [Proudfoot and Speyer 2006].

(ii) $\mathrm{OT}(\mathcal{A})$ is Cohen–Macaulay [Proudfoot and Speyer 2006].

(iii) $OT(\mathcal{A}) \simeq \Bbbk[1/\ell_1, \ldots, 1/\ell_n]$, a k-dimensional $\Bbbk$-subalgebra of the field of fractions $\Bbbk(x_1, \ldots, x_k)$ [Schenck and Tohăneanu 2009; Terao 2002]. The corresponding projective variety is called the *reciprocal plane* and it is denoted by $\mathcal{L}_{\mathcal{A}}^{-1}$.

(iv) Although the Orlik–Terao algebra is sensitive to the linear forms defining $\mathcal{A}$, its Hilbert series only depends on the underlying combinatorics [Terao 2002]. Let

$$\pi(\mathcal{A}, s) = \sum_{F \in L(\mathcal{A})} \mu_{\mathcal{A}}(F)(-s)^{r(F)}$$

be the *Poincaré polynomial* where $\mu_{\mathcal{A}}$ denotes the Möbius function, r is the rank function and F runs over the flats of $\mathcal{A}$. Then we have

$$HS(OT(\mathcal{A}), s) = \pi\left(\mathcal{A}, \frac{s}{1-s}\right).$$

See [Orlik and Terao 1994] for details and [Terao 2002; Berget 2010] for proofs of the above statement.

(v) Let $V \subset \mathbb{P}^{n-1}$ be a $(k-1)$-dimensional projective subspace that realizes $\mathcal{A}$, in the sense that $H_i \in \mathcal{A}$ is identified with the intersection of the i-th coordinate hyperplane in $\mathbb{P}^{n-1}$ with V.

Consider the Cremona map $\mathrm{Crem} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$, $(z_1, \ldots, z_n) \mapsto (z_1^{-1}, \ldots, z_n^{-1})$ and let $U(\mathcal{A}) = V \setminus \bigcup_{i=1}^{n} H_i$ be the complement of $\mathcal{A}$ in V. Under this setup, one obtains a related formulation of the blowup, here denoted by $\widetilde{V}$, as the closure of the graph of the restriction of the Cremona map to $U(\mathcal{A})$. Huh and Katz [2012] give a formula for the cohomology class of $\widetilde{V}$ as an element of the Chow ring $CH(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \mathbb{Z}[a, b]/\langle a^n, b^n \rangle$, where the coefficients come from the Poincaré polynomial after a change of variables:

$$\frac{s^k \pi(\mathcal{A}, -1/s)}{s-1} = \sum_{i=0}^{k-1} (-1)^i \mu^i s^{k-1-i}, \quad [\widetilde{V}] = \sum_{i=0}^{k} \mu^i [\mathbb{P}^{k-1-i} \times \mathbb{P}^i].$$

2B. *Ideals of products from arrangements.* Let $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ denote a central arrangement in $\mathbb{P}^{k-1}$, $n \geq k$, and let (as always throughout this paper) $R := \Bbbk[x_1, \ldots, x_k]$. Denoting $[n] := \{1, \ldots, n\}$, if $S \subset [n]$, then we set $\ell_S := \prod_{i \in S} \ell_i$, $\ell_{\varnothing} := 1$. Also set $S^c := [n] \setminus S$.

Let $\mathfrak{S} := \{S_1, \ldots, S_m\}$, where $S_j \subseteq [n]$ are subsets of the same size e. We are interested in studying the Rees algebras of ideals of the form

$$I_{\mathfrak{S}} := \langle \ell_{S_1}, \ldots, \ell_{S_m} \rangle \subset R. \tag{4}$$

Example 2.2. (i) (The Boolean case) Let $n = k$ and $\ell_i = x_i$, $i = 1, \ldots, k$. Then the ideal $I_{\mathfrak{S}}$ is monomial for any $\mathfrak{S}$. In the simplest case where $e = n - 1$, it is the ideal of the partial derivatives of the monomial $x_1 \cdots x_k$ — also the base ideal of the classical Möbius involution. For $e = 2$ the ideal becomes the edge ideal of a simple graph with k vertices. In general, it gives a subideal of the ideal of paths of a given length on the complete graph and, as such, it has a known combinatorial nature.

(ii) ($(n-1)$-fold products) Here one takes $S_1 := [n] \setminus \{1\}, \ldots, S_n := [n] \setminus \{n\}$. We will designate the corresponding ideal by $I_{n-1}(\mathcal{A})$. This case will be the main concern of the paper and will be fully examined in the following sections.

(iii) (a-fold products) This is a natural extension of (ii), where $I_a(\mathcal{A})$ is the ideal generated by all distinct a-products of the linear forms defining $\mathcal{A}$. The commutative algebraic properties of these ideals connect strongly to properties of the linear code built on the defining linear forms; see [Anzis et al. 2017]. In addition, the dimensions of the vector spaces generated by a-fold products give a new interpretation to the Tutte polynomial of the matroid of $\mathcal{A}$; see [Berget 2010].

We can naturally introduce the following algebra

$$OT(\mathfrak{S}, \mathcal{A}) := \Bbbk\left[\frac{1}{\ell_{S_1^c}}, \ldots, \frac{1}{\ell_{S_m^c}}\right] \tag{5}$$

as a generalized version of the notion mentioned in Remark 2.1(iii).

Proposition 2.3. *In the above setup there is a graded isomorphism of $\Bbbk$-algebras*

$$\Bbbk[\ell_{S_1}, \ldots, \ell_{S_m}] \simeq \Bbbk\left[\frac{1}{\ell_{S_1^c}}, \ldots, \frac{1}{\ell_{S_m^c}}\right].$$

Proof. Consider both algebras as homogeneous $\Bbbk$-subalgebras of the homogeneous total quotient ring of the standard polynomial ring R, generated in degrees e and $-(d-e)$, respectively. Then multiplication by the total product $\ell_{[d]}$ gives the required isomorphism:

$$\Bbbk\left[\frac{1}{\ell_{S_1^c}}, \ldots, \frac{1}{\ell_{S_m^c}}\right] \xrightarrow{\cdot \ell_{[d]}} \Bbbk[\ell_{S_1}, \ldots, \ell_{S_m}] \qquad \square$$

A neat consequence is the following result:

Theorem 2.4. *Let $\mathcal{A}$ denote a central arrangement of size n, let $\mathfrak{S} := \{S_1, \ldots, S_m\}$ be a collection of subsets of $[n]$ of the same size and let $I_{\mathfrak{S}}$ be as in (4). Then the algebra $OT(\mathfrak{S}, \mathcal{A})$ is isomorphic to the special fiber of the ideal $I_{\mathfrak{S}}$ as graded $\Bbbk$-algebras. In particular, the Orlik–Terao algebra $OT(\mathcal{A})$ is graded isomorphic to the special fiber $\mathcal{F}(I)$ of the ideal $I = I_{n-1}(\mathcal{A})$ of $(n-1)$-fold products of $\mathcal{A}$.*

Proof. It follows immediately from Proposition 2.3 and Lemma 1.1. $\qquad \square$

Remark 2.5. In the case of the Orlik–Terao algebra, the above result gives an answer to the third question at the end of [Schenck 2011]. Namely, let $k \geq 3$ and consider the rational map Φ as in (1). Then Theorem 2.4 says that the projection of the graph of Φ onto the second factor coincides with the reciprocal plane $\mathcal{L}_{\mathcal{A}}^{-1}$ (see Remark 2.1(iii)). In addition, the ideal $I := I_{n-1}(\mathcal{A})$ has a similar primary decomposition as obtained in [Schenck 2011, Lemmas 3.1 and 3.2], for arbitrary $k \geq 3$. By [Anzis et al. 2017, Proposition 2.2], one gets

$$I = \bigcap_{Y \in L_2(\mathcal{A})} I(Y)^{\mu_{\mathcal{A}}(Y)}.$$

Theorem 2.4 contributes additional information on certain numerical invariants and properties in the strict realm of commutative algebra and algebraic geometry.

Corollary 2.6. *Let $I := I_{n-1}(\mathcal{A})$ denote the ideal generated by the $(n-1)$-fold products coming from a central arrangement of size n and rank k.*

(a) *The special fiber $\mathcal{F}(I)$ of I is Cohen–Macaulay.*

(b) *The analytic spread is $\ell(I) = k$.*

(c) *The map Φ is birational onto its image.*

(d) *The reduction number is $r(I) \leq k - 1$.*

Proof. (a) It follows from Theorem 2.4 via Remark 2.1(ii).

(b) It follows by the same token from Remark 2.1(iii).

(c) This follows from [Doria et al. 2012, Theorem 3.2] since the ideal I is linearly presented (see proof of Lemma 3.1), and $\ell(I) = k$, the maximum possible.

(d) Follows from part (a), Proposition 1.2, and [Schenck 2011, Theorem 3.7]. □

The next result is a refinement of part (d) in the corollary above.

Proposition 2.7. *Let $\mathcal{A}$ be a hyperplane arrangement of rank k and n hyperplanes. Let $I := I_{n-1}(\mathcal{A})$. Then the reduction number of I is $r(I) = k - u$, where $u \geq 1$ is the number of components of $\mathcal{A}$.*

Proof. Let $r := r(I)$, and recall that the Hilbert series of the Orlik–Terao algebra is determined by the Poincaré polynomial:

$$HS(\mathrm{OT}(\mathcal{A}), s) = \pi\left(\mathcal{A}, \frac{s}{1-s}\right) = \frac{\overbrace{1 + h_1 s + \cdots + h_r s^r}^{H(s)}}{(1-s)^k}.$$

As with any central arrangement, the Poincaré polynomial has the trivial factor $(1 + t)$ and we write $\pi(\mathcal{A}, t) = (1 + t)\bar{\pi}(\mathcal{A}, t)$, where bar denotes the reduced Poincaré polynomial. Moreover, if $\mathcal{A}$ has a decomposition as a product of two smaller arrangements, then the Poincaré polynomial splits and we get a $(1 + t)$ factor for each component, and by a result of Crapo, this is the only way for more $(1 + t)$ factors to occur. Here, we need the notion of the beta invariant[1] of an arrangement: $\beta(\mathcal{A}) := |\bar{\pi}(\mathcal{A}, -1)|$. Theorem II in [Crapo 1967] states that an arrangement is decomposable if and only if its beta invariant is zero.

We have

$$\pi\left(\mathcal{A}, \frac{s}{1-s}\right) = \bar{\pi}\left(\mathcal{A}, \frac{s}{1-s}\right)/(1-s),$$

which indicates that only $\bar{\pi}$ can contribute to the numerator $H(s)$. If $\deg H(s) < k - 1$, then by undoing the substitution, we find another $(1 + t)$ factor in the reduced Poincaré polynomial and hence $\beta(\mathcal{A}) = 0$, indicating that this happens exactly when $\mathcal{A}$ is decomposable. So, when $\mathcal{A}$ is indecomposable, the argument is complete and if it does decompose, then we can apply this argument to each component and hence the formula. □

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[1] In the case of complex arrangements, this gives the Euler characteristic of the projective complement.

3. Ideals of $(n-1)$-fold products and their blowup algebras

As mentioned in Example 2.2, a special case of the ideal $I_{\mathfrak{S}}$, extending the case of the ideal generated by the $(n-1)$-fold products, is obtained by fixing $a \in \{1, \ldots, n\}$ and considering the collection of all subsets of $[n]$ of cardinality a. Then the corresponding ideal is

$$I_a(\mathcal{A}) := \langle \ell_{i_1} \cdots \ell_{i_a} \mid 1 \leq i_1 < \cdots < i_a \leq n \rangle \subset R$$

and is called the ideal generated by the a-fold products of linear forms of $\mathcal{A}$. The projective schemes defined by these ideals are known as generalized star configuration schemes. Unfortunately, very few facts are known about these ideals: if d is the minimum distance of the linear code built from the linear forms defining $\mathcal{A}$ and if $1 \leq a \leq d$, then $I_a(\mathcal{A}) = \mathfrak{m}^a$ (see [Tohǎneanu 2010, Theorem 3.1]); and the case when $a = n$ is trivial.

In the case where $a = n-1$, some immediate properties are known already, yet the more difficult questions in regard to the blowup and related algebras have not been studied before. These facets, to be thoroughly examined in the subsequent sections, are our main endeavor in this work. Henceforth, we will be working with the following data: $\mathcal{A}$ is an arrangement with $n \geq k$ and for every $1 \leq i \leq n$, we consider the $(n-1)$-fold products of the n linear forms defining the hyperplanes of $\mathcal{A}$

$$f_i := \ell_1 \cdots \hat{\ell}_i \cdots \ell_n \in R,$$

and write

$$I := I_{n-1}(\mathcal{A}) := \langle f_1, \ldots, f_n \rangle.$$

Let $T = \Bbbk[x_1, \ldots, x_k, y_1, \ldots, y_n] = R[y_1, \ldots, y_n]$ as before and denote by $\mathcal{I}(\mathcal{A}, n-1) \subset T$ the presentation ideal of the Rees algebra $R[It]$ corresponding to the generators $f_1, \ldots, f_n$.

3A. _The symmetric algebra._ Let $\mathcal{I}_1(\mathcal{A}, n-1) \subset T$ stand for the subideal of $\mathcal{I}(\mathcal{A}, n-1)$ presenting the symmetric algebra $\mathcal{S}(I)$ of $I = I_{n-1}(\mathcal{A})$.

Lemma 3.1. _With the above notation, one has_:

(a) _The ideal $I = I_{n-1}(\mathcal{A})$ is a perfect ideal of codimension_ 2.

(b) $\mathcal{I}_1(\mathcal{A}, n-1) = \langle \ell_i y_i - \ell_{i+1} y_{i+1} \mid 1 \leq i \leq n-1 \rangle.$

(c) $\mathcal{I}_1(\mathcal{A}, n-1)$ _is an ideal of codimension k; in particular, it is a complete intersection if and only if $n = k$._

Proof. (a) This is well known, but we give the argument for completeness. Clearly, I has codimension 2. The following reduced Koszul like relations are syzygies of I: $\ell_i y_i - \ell_{i+1} y_{i+1}$, $1 \leq i \leq n-1$. They alone

form the following matrix of syzygies of I:

$$\varphi = \begin{bmatrix} \ell_1 & & & & \\ -\ell_2 & \ell_2 & & & \\ & -\ell_3 & \ddots & & \\ & & \ddots & \ell_{n-1} & \\ & & & -\ell_n & \end{bmatrix}.$$

Since the rank of this matrix is $n-1$, it is indeed a full syzygy matrix of I; in particular, I has a linear resolution

$$0 \to R(-n)^{n-1} \xrightarrow{\varphi} R(-(n-1))^n \to I \to 0.$$

(b) This is an expression of the details of (a).

(c) Clearly, $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathfrak{m}T$, hence its codimension is at most k. Assuming, as we may, that $\{\ell_1, \ldots, \ell_k\}$ is $\mathbb{k}$-linearly independent, we contend that the elements $\mathfrak{s} := \{\ell_i y_i - \ell_{i+1} y_{i+1}, 1 \leq i \leq k\}$ form a regular sequence. To see this, we first apply a $\mathbb{k}$-linear automorphism of R to assume that $\ell_i = x_i$, for $1 \leq i \leq k$ — this will not affect the basic ideal theoretic invariants associated to I. Then note that in the set of generators of $\mathcal{I}_1(\mathcal{A}, n-1)$, the elements of $\mathfrak{s}$ can be replaced by the following ones: $\{x_i y_i - \ell_{k+1} y_{k+1}, 1 \leq i \leq k\}$. Clearly, this is a regular sequence — for example, because $\langle x_i y_i, 1 \leq i \leq k \rangle$ is the initial ideal of the ideal generated by this sequence, in the revlex order. $\qquad \square$

There are two basic ideals that play a distinguished role at the outset. In order to capture both in one single blow, we consider the Jacobian matrix of the generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ given in Lemma 3.1(b). Its transpose turns out to be the stack of two matrices, the first is the Jacobian matrix with respect to the variables $y_1, \ldots, y_n$ — which coincides with the syzygy matrix ϕ of I as described in the proof of Lemma 3.1(a) — while the second is the Jacobian matrix $B = B(\phi)$ with respect to the variables $x_1, \ldots, x_k$ — the so-called *Jacobian dual matrix* of [Simis et al. 1993]. The offspring are the respective ideals of maximal minors of these stacked matrices, the first retrieves I, while the second gives an ideal $I_k(B) \subset S = \mathbb{k}[y_1, \ldots, y_n]$ that will play a significant role below (see also Proposition 4.1) as a first crude approximation to the Orlik–Terao ideal.

Proposition 3.2. *Let $\mathcal{S}(I) \simeq T/\mathcal{I}_1(\mathcal{A}, n-1)$ stand for the symmetric algebra of the ideal I of $(n-1)$-fold products. Then*:

(i) $\mathrm{depth}(\mathcal{S}(I)) \leq k+1$.

(ii) *As an ideal in T, every minimal prime of $\mathcal{S}(I)$ is either $\mathfrak{m}T$, the Rees ideal $\mathcal{I}(\mathcal{A}, n-1)$ or else has the form $(\ell_{i_1}, \ldots, \ell_{i_s}, y_{j_1}, \ldots, y_{j_t})$, where $2 \leq s \leq k-1$, $t \geq 1$, $\{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_t\} = \varnothing$, and $\ell_{i_1}, \ldots, \ell_{i_s}$ are $\mathbb{k}$-linearly independent.*

(iii) *The primary components relative to the minimal primes $\mathfrak{m} = (\mathbf{x})T$ and $\mathcal{I}(\mathcal{A}, n-1)$ are radical; in addition, with the exception of $\mathfrak{m}T$, every minimal prime of $\mathcal{S}(I)$ contains the ideal $I_k(B)$.*

Proof. (i) Since $\mathcal{I}(\mathcal{A}, n-1)$ is a prime ideal which is a saturation of $\mathcal{I}_1(\mathcal{A}, n-1)$, it is an associated prime of $\mathcal{S}(I)$. Therefore, $\mathrm{depth}(\mathcal{S}(I)) \leq \dim \mathcal{R}(I) = k+1$.

(ii) Since $\mathcal{I}(\mathcal{A}, n-1)$ is a saturation of $\mathcal{I}_1(\mathcal{A}, n-1)$ by I, one has $\mathcal{I}(\mathcal{A}, n-1)I^t \subset \mathcal{I}_1(\mathcal{A}, n-1)$, for some $t \geq 1$. This implies that any (minimal) prime of $\mathcal{S}(I)$ in T contains either I or $\mathcal{I}(\mathcal{A}, n-1)$. By the proof of (i), $\mathcal{I}(\mathcal{A}, n-1)$ is an associated prime of $\mathcal{S}(I)$, hence it must be a minimal prime thereof since a minimal prime of $\mathcal{S}(I)$ properly contained in it would have to contain I, which is absurd.

Now, suppose $P \subset T$ is a minimal prime of $\mathcal{S}(I)$ containing I. One knows by Lemma 3.1 that $\mathfrak{m} = (\mathbf{x})T$ is a minimal prime of $\mathcal{S}(I)$. Therefore, we assume that $\mathfrak{m}T \not\subset P$. Since any minimal prime of I is a complete intersection of two distinct linear forms of $\mathcal{A}$, P contains at least two, and at most $k-1$, linearly independent linear forms of $\mathcal{A}$. On the other hand, since $\mathcal{I}_1(\mathcal{A}, n-1) \subset P$, looking at the generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ as in Lemma 3.1(b), by a domino effect principle we finally reach the desired format for P as stated.

(iii) With the notation prior to the statement of the proposition, we claim the following equality:

$$\mathcal{I}_1(\mathcal{A}, n-1) : I_k(B)^\infty = \mathfrak{m}T.$$

It suffices to show for the first quotient as $\mathfrak{m}T$ is a prime ideal. The inclusion $\mathfrak{m}I_k(B) \subset \mathcal{I}_1(\mathcal{A}, n-1)$ is a consequence of the Cramer rule. The reverse inclusion is obvious because $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathfrak{m}T$ implies that $\mathcal{I}_1(\mathcal{A}, n-1) : I_k(B) \subset \mathfrak{m}T : I_k(B) = \mathfrak{m}T$, as $\mathfrak{m}T$ is a prime ideal. Note that, as a very crude consequence, one has $I_k(B) \subset \mathcal{I}(\mathcal{A}, n-1)$. Now, let $\mathcal{P}(\mathfrak{m}T)$ denote the primary component of $\mathfrak{m}T$ in $\mathcal{I}_1(\mathcal{A}, n-1)$. Then

$$\mathfrak{m}T = \mathcal{I}_1(\mathcal{A}, n-1) : I_k(B)^\infty \subset \mathcal{P}(\mathfrak{m}T) : I_k(B)^\infty = \mathcal{P}(\mathfrak{m}T).$$

The same argument goes through for the primary component of $\mathcal{I}(\mathcal{A}, n-1)$ using the ideal I instead of $I_k(B)$.

To see the last statement of the proposition, let $\mathcal{P}$ denote the primary component of one of the remaining minimal primes P of $\mathcal{S}(I)$. Since $P : I_k(B)^\infty$ is P-primary and $\mathfrak{m} \not\subset P$, by the same token we get that $I_k(B) \subset P$. □

Remark 3.3. (a) It will be shown in the last section that the estimate in (i) is actually an equality.

As a consequence, every associated prime of $\mathcal{S}(I)$ viewed in T has codimension at most $n-1$. This will give a much better grip on the minimal primes of the form $\langle \ell_{i_1}, \ldots, \ell_{i_s}, y_{j_1}, \ldots, y_{j_t} \rangle$. Namely, one must have in addition that $s + t \leq n-1$ and, moreover, due to the domino effect principle, one must have $s = k-1$, hence $t \leq n-k$.

(b) We conjecture that $\mathcal{S}(I)$ is reduced. The property (R_0) of Serre's is easily verified due to the format of the Jacobian matrix as explained before the above proposition. The problem is, of course, the property (S_1), the known obstruction for the existence of embedded associated primes. The case where $n = k+1$, is easily determined. Here the minimal primes are seen to be $\mathfrak{m}$, $\langle x_1, \ldots, x_{k-1}, y_k \rangle$ and the Rees ideal $\langle \mathcal{I}_1(\mathcal{A}, k), \partial \rangle$, where ∂ is the relation corresponding to the unique circuit. A calculation will show that the three primes intersect in $\mathcal{I}_1(\mathcal{A}, k)$. As a side, this fact alone implies that the maximal regular sequence in the proof of Lemma 3.1(c) generates a radical ideal. For $n \geq k+2$ the calculation becomes sort of formidable, but we will prove later on that the Rees ideal is of fiber type.

(c) The weaker question as to whether the minimal component of $\mathcal{S}(I)$ is radical seems pliable.

If the conjectural statement in Remark 3.3(b) is true then, for any linear form $\ell = \ell_i$ the following basic formula holds:

$$\mathcal{I}_1(\mathcal{A}, n-1) : \ell = \mathcal{I}(\mathcal{A}, n-1) \cap g\left(\bigcap_{\ell \notin P} P\right),$$

where P denotes a minimal prime other that $\mathfrak{m}T$ and $\mathcal{I}(\mathcal{A}, n-1)$, as described in Proposition 3.2(i). Thus one would recover sectors of the Orlik–Terao generators inside this colon ideal. Fortunately, this latter virtual consequence holds true and has a direct simple proof. For convenience of later use, we state it explicitly. Let $\partial(\mathcal{A}|_\ell)$ denote the subideal of $\partial(\mathcal{A})$ generated by all polynomial relations ∂ corresponding to minimal dependencies (circuits) involving the linear form $\ell \in \mathcal{A}$.

Lemma 3.4. $$\partial(\mathcal{A}|_\ell) \subset \mathcal{I}_1(\mathcal{A}, n-1) : \ell.$$

Proof. Say, $\ell = \ell_1$. Let $D : a_1\ell_1 + a_2\ell_2 + \cdots + a_s\ell_s = 0$ be a minimal dependency involving ℓ_1, for some $3 \leq s \leq n$. In particular, $a_i \neq 0, i = 1, \ldots, s$. The corresponding generator of $\partial(\mathcal{A}|_{\ell_1})$ is

$$\partial D := a_1 y_2 y_3 \cdots y_s + a_2 y_1 y_3 \cdots y_s + \cdots + a_s y_1 y_2 \cdots y_{s-1}.$$

The following calculation is straightforward:

$$
\begin{aligned}
\ell_1 \partial D &= a_1 \ell_1 y_2 y_3 \cdots y_s + (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&\qquad\qquad\qquad\qquad + \ell_2 y_2 (a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&= (a_1 \ell_1 + a_2 \ell_2) y_2 y_3 \cdots y_s + (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&\qquad\qquad\qquad\qquad + \ell_2 y_2 (a_3 y_2 y_4 \cdots y_s + \cdots + a_s y_2 y_3 \cdots y_{s-1}) \\
&= (-a_3 \ell_3 - \cdots - a_s \ell_s) y_2 y_3 \cdots y_s + (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) \\
&\qquad\qquad\qquad\qquad + \ell_2 y_2^2 (a_3 y_4 \cdots y_s + \cdots + a_s y_3 \cdots y_{s-1}) \\
&= (\ell_1 y_1 - \ell_2 y_2)(a_2 y_3 \cdots y_s + \cdots + a_s y_2 \cdots y_{s-1}) + y_2(\ell_2 y_2 - \ell_3 y_3) a_3 y_4 \cdots y_s \\
&\qquad\qquad\qquad\qquad + \cdots + y_2(\ell_2 y_2 - \ell_s y_s) a_s y_3 \cdots y_{s-1} \\
&= a_2 y_3 \cdots y_s (\ell_1 y_1 - \ell_2 y_2) + a_3 y_2 y_4 \cdots y_s (\ell_1 y_1 - \ell_3 y_3) + \cdots + a_s y_2 \cdots y_{s-1}(\ell_1 y_1 - \ell_s y_s).
\end{aligned}
$$

Hence the result. $\square$

3B. *Sylvester forms.* The Orlik–Terao ideal $\partial(\mathcal{A})$ has an internal structure of classical flavor, in terms of Sylvester forms.

Proposition 3.5. *The generators $\partial(\mathcal{A})$ of the Orlik–Terao ideal are Sylvester forms obtained from the generators of the presentation ideal $\mathcal{I}_1(\mathcal{A}, n-1)$ of the symmetric algebra of I.*

Proof. Let D be a dependency $c_{i_1}\ell_{i_1} + \cdots + c_{i_m}\ell_{i_m} = 0$ with all coefficients $c_{i_j} \neq 0$. Let $f = \prod_{i=1}^n \ell_i$. Evaluating the Orlik–Terao element ∂D on the products we have

$$\partial D(f_1, \ldots, f_n) = \sum_{j=1}^m c_{i_j} \frac{f^{m-1}}{\Pi_{j \neq k=1}^m \ell_{i_k}} = \sum_{j=1}^m c_{i_j} \frac{f^{m-1}}{\Pi_{k=1}^m \ell_{i_k}} \ell_{i_j} = \frac{f^{m-1}}{\ell_{i_1} \cdots \ell_{i_m}} g(c_{i_1}\ell_{i_1} + \cdots + c_{i_m}\ell_{i_m}) = 0.$$

Therefore, $\partial D \in \mathcal{I}(\mathcal{A}, n-1)$, and since $\partial D \in S := \Bbbk[y_1, \ldots, y_n]$, then $\partial D \in \langle \mathcal{I}(\mathcal{A}, n-1)_{(0,-)} \rangle$.

For the second part, suppose that the minimal generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ are

$$\Delta_1 := \ell_1 y_1 - \ell_2 y_2, \quad \Delta_2 := \ell_2 y_2 - \ell_3 y_3, \quad \ldots, \quad \Delta_{n-1} := \ell_{n-1} y_{n-1} - \ell_n y_n.$$

Without loss of generality suppose $\ell_j = c_1 \ell_1 + \cdots + c_{j-1} \ell_{j-1}$ is some arbitrary dependency D. We have

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{j-1} \end{bmatrix} = \begin{bmatrix} y_1 & -y_2 & 0 & \cdots & 0 & 0 \\ 0 & y_2 & -y_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & y_{j-2} & -y_{j-1} \\ -c_1 y_j & -c_2 y_j & -c_3 y_j & \cdots & -c_{j-2} y_j & y_{j-1} - c_{j-1} y_j \end{bmatrix} \cdot \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_{j-1} \end{bmatrix}.$$

The determinant of the $(j-1) \times (j-1)$ matrix we see above is $\pm \partial D$. $\qquad\square$

3C. *A lemma on deletion.* In this and the next parts we build on the main tool of an inductive procedure.

Let $\mathcal{A}' = \mathcal{A} \setminus \{\ell_1\}$, and denote $n' := |\mathcal{A}'| = n - 1$. We would like to investigate the relationship between the Rees ideal $\mathcal{I}(\mathcal{A}', n'-1)$ of $I_{n'-1}(\mathcal{A}')$ and the Rees ideal $\mathcal{I}(\mathcal{A}, n-1)$ of $I_{n-1}(\mathcal{A})$, both defined in terms of the naturally given generators.

To wit, we will denote the generators of $I_{n'-1}(\mathcal{A}')$ as

$$f_{12} := \ell_{[n] \setminus \{1,2\}}, \quad \ldots, \quad f_{1n} := \ell_{[n] \setminus \{1,n\}}.$$

One can move between the two ideals in a simple manner, which is easy to verify:

$$I_{n-1}(\mathcal{A}) : \ell_1 = I_{n'-1}(\mathcal{A}').$$

Note that the presentation ideal $\mathcal{I}(\mathcal{A}', n'-1)$ of the Rees algebra of $I_{n'-1}(\mathcal{A}')$ with respect to these generators lives in the polynomial subring $T' := R[y_2, \ldots, y_n] \subset T := R[y_1, y_2, \ldots, y_n]$. From Lemma 3.1, we know that

$$\mathcal{I}_1(\mathcal{A}', n'-1)T = \langle \ell_2 y_2 - \ell_3 y_3, \ell_3 y_3 - \ell_4 y_4, \ldots, \ell_{n-1} y_{n-1} - \ell_n y_n \rangle T \subset \mathcal{I}_1(\mathcal{A}, n-1).$$

Likewise, for the Orlik–Terao ideal (which is an ideal in $S' := \Bbbk[y_2, \ldots, y_n] \subset S := \Bbbk[y_1, y_2, \ldots, y_n]$), one obtains via Theorem 2.4

$$\partial(\mathcal{A}')S = \langle \mathcal{I}(\mathcal{A}', n'-1)_{(0,-)} \rangle S \subset \partial(\mathcal{A}) = \langle \mathcal{I}(\mathcal{A}, n-1)_{(0,-)} \rangle.$$

Lemma 3.6. *One has*

$$\mathcal{I}(\mathcal{A}, n-1) = \langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle : \ell_1^\infty.$$

Proof. The inclusion $\langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle : \ell_1^\infty \subset I(\mathcal{A}, n-1)$ is clear since we are saturating a subideal of a prime ideal by an element not belonging to the latter. We note that the codimension of $\langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle$ exceeds by 1 that of $I(\mathcal{A}', n'-1)$ since the latter is a prime ideal even after extending to the ambient ring T. Therefore, by a codimension counting it would suffice to show that the saturation is itself a prime ideal.

Instead, we choose a direct approach. Thus, let $F \in \mathcal{I}(\mathcal{A}, n-1)$ be (homogeneous) of degree d in variables $y_1, \ldots, y_n$. We can write

$$F = y_1^u G_u + y_1^{u-1} G_{u-1} + \cdots + y_1 G_1 + G_0, \quad 0 \le u \le d,$$

where $G_j \in \Bbbk[x_1, \ldots, x_k][y_2, \ldots, y_n]$, are homogeneous of degree $d - j$ in $y_2, \ldots, y_n$ for $j = 0, \ldots, u$.

Evaluating $y_i = f_i$, $i = 1, \ldots, n$ we obtain

$$0 = F(f_1, \ldots, f_n) = \ell_2^u f_{12}^u \ell_1^{d-u} G_u(f_{12}, \ldots, f_{1n}) + \cdots + \ell_2 f_{12} \ell_1^{d-1} G_1(f_{12}, \ldots, f_{1n}) + \ell_1^d G_0(f_{12}, \ldots, f_{1n}).$$

This means that

$$\ell_1^{d-u} \left[\underbrace{\ell_2^u y_2^u G_u(y_2, \ldots, y_n) + \cdots + \ell_1^{u-1} \ell_2 y_2 G_1(y_2, \ldots, y_n) + \ell_1^u G_0(y_2, \ldots, y_n)}_{F'} \right] \in \mathcal{I}(\mathcal{A}', n'-1).$$

By writing $\ell_1 y_1 = \ell_1 y_1 - \ell_2 y_2 + \ell_2 y_2$, it is not difficult to see that

$$\ell_1^u F \equiv F' \mod \langle \ell_1 y_1 - \ell_2 y_2 \rangle,$$

hence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3D. *Stretched arrangements with coefficients.* Recall the notion of contraction and the inherent idea of a multiarrangement, as mentioned in Section 2. Here we wish to consider such multiarrangements, allowing moreover the repeated individual linear functionals corresponding to repeated hyperplanes to be tagged with a nonzero element of the ground field. For lack of better terminology, we call such a new gadget a *stretched arrangement with coefficients*. Note that, by construction, a stretched arrangement with coefficients $\mathcal{B}$ has a uniquely defined (simple) arrangement $\mathcal{A}$ as support. Thus, if $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ is a simple arrangement, then a stretched arrangement with coefficients $\mathcal{B}$ is of the form

$$\{ \underbrace{b_{1,1}\ell_1, \ldots, b_{1,m_1}\ell_1}_{H_1 = \ker \ell_1}, \underbrace{b_{2,1}\ell_2, \ldots, b_{2,m_2}\ell_2}_{H_2 = \ker \ell_2}, \ldots, \underbrace{b_{n,1}\ell_n, \ldots, b_{n,m_n}\ell_n}_{H_n = \ker \ell_n} \},$$

where $0 \ne b_{i,j} \in \Bbbk$ and $H_i = \ker(\ell_i)$ has multiplicity m_i for any $1 \le i \le n$, and for convenience, we assume that $b_{i,1} = 1$. We set $m := m_1 + \cdots + m_n$, and emphasize the ingredients of a stretched arrangement by writing $\mathcal{B} = (\mathcal{A}, \boldsymbol{b})$ where $\boldsymbol{b}$ is the vector of the above coefficients in the same order.

Proceeding as in the situation of a simple arrangement, we introduce the collection of $(m-1)$-products of elements of $\mathcal{B}$ and denote $I_{m-1}(\mathcal{B})$ the ideal of R generated by them. As in the simple case, we consider the presentation ideal $\mathcal{I}(\mathcal{B}, m-1)$ of $I_{m-1}(\mathcal{B})$ with respect to its set of generators consisting of the $(m-1)$-products. The next lemma relates this ideal to the previously considered presentation ideal $\mathcal{I}(\mathcal{A}, n-1)$ of $I_{n-1}(\mathcal{A})$ obtained by taking the set of generators consisting of the $(n-1)$-products of elements of $\mathcal{A}$.

Lemma 3.7. *Let $\mathcal{A}$ denote an arrangement and let $\mathcal{B} = (\mathcal{A}, \boldsymbol{b})$ denote a stretched arrangement supported on $\mathcal{A}$, as above. Let $G \in R$ stand for the gcd of the $(m-1)$-products of elements of $\mathcal{B}$. Then:*

  (i) *The vector of the $(m-1)$-products of elements of $\mathcal{B}$ has the form $G \cdot P_{\mathcal{A}}$, where $P_{\mathcal{A}}$ denotes the vector whose coordinates are the $(n-1)$-products of the corresponding simple $\mathcal{A}$, each such product repeated*

as many times as the stretching in $\mathcal{B}$ *of the corresponding linear form deleted in the expression of the product, and further tagged with a certain coefficient;*

(ii) $\mathcal{I}(\mathcal{B}, m - 1) = \langle \mathcal{I}(\mathcal{A}, n - 1), \mathcal{D}_{\mathcal{A}} \rangle$, *where* $\mathcal{D}_{\mathcal{A}}$ *denotes the* $\Bbbk$*-linear dependency relations among elements of* $P_{\mathcal{A}}$.

Proof. The first statement follows from the definition of a stretched arrangement vis-à-vis its support arrangement. Now, by (i), the Rees algebra of $I_{m-1}(\mathcal{B})$ is isomorphic to the Rees algebra of the ideal with generating set $P_{\mathcal{A}}$. By the nature of the latter, the second statement is now clear.   □

4. The main theorems

We keep the previous notation as in Section 3C, where $I_{n-1}(\mathcal{A})$ is the ideal of $(n - 1)$-fold products of a central arrangement $\mathcal{A}$ of size n and rank k. We had $T := R[y_1, \ldots, y_n]$, with $R := \Bbbk[x_1, \ldots, x_k]$, $S := \Bbbk[y_1, \ldots, y_n]$, and $\mathcal{I}_1(\mathcal{A}, n - 1) \subset \mathcal{I}(\mathcal{A}, n - 1) \subset T$ denote, respectively, the presentation ideals of the symmetric algebra and of the Rees algebra of I. Recall that from Theorem 2.4, the Orlik–Terao ideal $\partial(\mathcal{A})$ coincides with the defining ideal $(\mathcal{I}(\mathcal{A}, n - 1)_{(0,-)})S$ of the special fiber algebra of I.

4A. *The case of a generic arrangement.* Simple conceptual proofs can be given in the case where $\mathcal{A}$ is generic (meaning that any k of the defining linear forms are linearly independent), as follows.

Proposition 4.1. *If* $\mathcal{A} = \{\ell_1, \ldots, \ell_n\} \subset R = \Bbbk[x_1, \ldots, x_k]$ *is a generic arrangement, one has*:

(a) $I := I_{n-1}(\mathcal{A})$ *is an ideal of fiber type.*

(b) *The Rees algebra* $R[It]$ *is Cohen–Macaulay.*

(c) *The Orlik–Terao ideal of* $\mathcal{A}$ *is the* 0*-th Fitting ideal of the Jacobian dual matrix of* I *(i.e., the ideal generated by the* $k \times k$ *minors of the Jacobian matrix of the generators of* $\mathcal{I}_1(\mathcal{A}, n - 1)$ *with respect to the variables of* R*).*

(d) *Let* $k = n$, *i.e., the case of Boolean arrangement. Under the standard bigrading* $\deg x_i = (1, 0)$ *and* $\deg y_j = (0, 1)$, *the bigraded Hilbert series of* $R[It]$ *is*

$$HS(R[It]; u, v) = \frac{(1 - uv)^{k-1}}{(1 - u)^k (1 - v)^k}.$$

Proof. As described in the proof of Lemma 3.1, I is a linearly presented codimension 2 perfect ideal with syzygy matrix of the following shape

$$\varphi = \begin{bmatrix} \ell_1 & & & & \\ -\ell_2 & \ell_2 & & & \\ & -\ell_3 & \ddots & & \\ & & \ddots & \ell_{n-1} & \\ & & & -\ell_n & \end{bmatrix}.$$

The Boolean case $n = k$ is well known, so we assume that $\mu(I) = n > k$. We claim that I satisfies the G_k condition. For this purpose we check the requirement in (2). First note that, for $p \geq n - k + 1$, one has

$$I_p(\varphi) = I_p(\mathcal{A}),$$

where the rightmost ideal is the ideal generated by all p-fold products of the linear forms defining $\mathcal{A}$, as in our earlier notation. Because $\mathcal{A}$ is generic, it is the support of the codimension $(n - p + 1)$-star configuration V_{n-p+1}; see [Geramita et al. 2013]. By Proposition 2.9(4) there, the defining ideal of V_{n-p+1} is a subset of $I_p(\mathcal{A})$, hence $\mathrm{ht}(I_p(\mathcal{A})) \geq n - p + 1$. By [Tohǎneanu 2010], any minimal prime of $I_p(\mathcal{A})$ can be generated by $n - p + 1$ elements. Therefore, $\mathrm{ht}(I_p(\mathcal{A})) \leq n - p + 1$, and hence equality.

The three statements now follow from [Morey and Ulrich 1996, Theorem 1.3], where (a) and (c) are collected together by saying that $R[It]$ has a presentation ideal of the expected type, quite stronger than being of fiber type. Note that, as a bonus, the same theorem also gives that $\ell(I) = k$ and $r(I) = k - 1$, which are parts (b) and (d) in Corollary 2.6, when $\mathcal{A}$ is generic.

Part (d) follows from an immediate application of [Robbiano and Valla 1998, Theorem 5.11] to the $(k - 1) \times k$ matrix

$$M = \begin{bmatrix} x_1 & 0 & \ldots & 0 & x_k \\ 0 & x_2 & \ldots & 0 & x_k \\ \vdots & & \ddots & \vdots & x_k \\ 0 & 0 & \ldots & x_{k-1} & x_k \end{bmatrix}.$$

One can verify that the codimension of $I_t(M)$, the ideal of size t minors of M, is $k - t + 2$. Note that their setup is different in that they set $\deg y_j = (n - 1, 1)$, whereas for us $\deg y_j = (0, 1)$. To get our formula we make the substitution in their formula: $a \leftrightarrow u$, and $a^{n-1}b \leftrightarrow v$. $\qquad\square$

4B. *The fiber type property.* In this part we prove one of the main assertions of the section and state a few structural consequences.

Theorem 4.2. *Let $\mathcal{A}$ be a central arrangement of rank $k \geq 2$ and size $n \geq k$. The ideal $I_{n-1}(\mathcal{A})$ of $(n-1)$-fold products of $\mathcal{A}$ is of fiber type:*

$$\mathcal{I}(\mathcal{A}, n - 1) = \langle \mathcal{I}_1(\mathcal{A}, n - 1) \rangle + \langle \mathcal{I}(\mathcal{A}, n - 1)_{(0,-)} \rangle,$$

as ideals in T, where $\langle \mathcal{I}(\mathcal{A}, n - 1)_{(0,-)} \rangle_S = \partial(\mathcal{A})$ is the Orlik–Terao ideal.

Proof. We first consider the case where $n = k$. Then $I_{n-1}(\mathcal{A})$ is an ideal of linear type by Lemma 3.1, that is to say, $\mathcal{I}(\mathcal{A}, n - 1) = \mathcal{I}_1(\mathcal{A}, n - 1)$. This proves the statement of the theorem since $\partial(\mathcal{A}) = 0$ in this case.

We now prove the statement by induction on the pairs (n, k), where $n > k \geq 2$. In the initial induction step, we deal with the case $k = 2$ and arbitrary $n > 2$ (the argument will even be valid for $n = 2$). Here one claims that $I_{n-1}(\mathcal{A}) = \langle x_1, x_2 \rangle^{n-1}$. In fact, since no two forms of the arrangement are proportional, the generators of $I_{n-1}(\mathcal{A})$ are $\Bbbk$-linearly independent because, e.g., dehomogenizing in one of the variables yields the first n powers of the other variable up to elementary transformations. Also, since these forms have degree $n - 1$, they forcefully span the power $\langle x_1, x_2 \rangle^{n-1}$.

Now, any $\langle x_1, x_2 \rangle$-primary ideal in $\Bbbk[x_1, x_2]$ automatically satisfies the property G_2; see (2). Therefore, the Rees ideal is of fiber type, and in fact it is of the expected type and Cohen–Macaulay by [Morey and Ulrich 1996, Theorem 1.3]. In any case, the Rees ideal has long been known in this case, with the defining ideal of the special fiber generated by the 2-minors of the generic $2 \times (n-1)$ Hankel matrix, i.e., by the homogeneous defining ideal of the rational normal curve in $\mathbb{P}^{n-1}$; see [Corsini 1967].

For the main induction step, suppose $n > k > 2$ and let $\mathcal{A}' := \mathcal{A} \setminus \{\ell_1\}$ stand for the deletion of ℓ_1, a subarrangement of size $n' := n - 1$. Applying a change of variables in the base ring R — which, as already remarked, does not disturb the ideal-theoretic properties in sight — we can assume that $\ell_1 = x_1$ and $\ell_2 = x_2$. The extended ideals $\mathcal{I}(\mathcal{A}', n' - 1)T$, $\partial(\mathcal{A}')S$, $\mathcal{I}_1(\mathcal{A}', n' - 1)T$ will be of our concern. The following equalities of ideals of T are easily seen to hold:

$$\mathcal{I}_1(\mathcal{A}, n - 1) = \langle x_1 y_1 - x_2 y_2, \mathcal{I}_1(\mathcal{A}', n' - 1) \rangle \quad \text{as ideals in } T,$$
$$\partial(\mathcal{A}) = \langle \partial(\mathcal{A}|_{x_1}), \partial(\mathcal{A}') \rangle, \quad\quad\quad\quad \text{as ideals in } S. \tag{6}$$

Let $F \in \mathcal{I}(\mathcal{A}, n-1)$ be bihomogeneous with $\deg_{\mathbf{y}}(F) = d$. Suppose that $M = x_1^a y_1^b N \in T$ is a monomial that appears in F, where $x_1, y_1 \nmid N$. If $a \geq b$, we can write

$$M = x_1^{a-b}(x_1 y_1 - x_2 y_2 + x_2 y_2)^b N,$$

and hence

$$M \equiv x_1^{a-b} x_2^b y_2^b N \mod \langle x_1 y_1 - x_2 y_2 \rangle.$$

If $a < b$, we have

$$M = (x_1 y_1 - x_2 y_2 + x_2 y_2)^a y_1^{b-a} N,$$

and hence

$$M \equiv x_2^a y_2^a y_1^{b-a} N \mod \langle x_1 y_1 - x_2 y_2 \rangle.$$

Denote $R' := \Bbbk[x_2, \ldots, x_k] \subset R$, $T'' := R'[y_2, \ldots, y_n] \subset T' := R[y_2, \ldots, y_n] \subset T$. In any case, one can write

$$F = (x_1 y_1 - x_2 y_2)Q + x_1^{m_1} P_1 + x_1^{m_2} P_2 + \cdots + x_1^{m_u} P_u + P_{u+1}, \, m_1 > \cdots > m_u \geq 1,$$

for certain forms $Q \in T$, $P_1, \ldots, P_u \in T''$, and $P_{u+1} \in R'[y_1, \ldots, y_n] = T''[y_1]$ of degree d in the variables $y_1, \ldots, y_n$. Also

$$P_{u+1} = y_1^v G_v + y_1^{v-1} G_{v-1} + \cdots + y_1 G_1 + G_0,$$

where $G_j \in T''$ and $\deg(G_j) = d - j$, $j = 0, \ldots, v$. Let us use the elements we have seen at the beginning of Section 3C, as generators for $I_{n'-1}(\mathcal{A}')$:

$$f_{12} := \ell_{[n] \setminus \{1,2\}}, \ldots, f_{1n} := \ell_{[n] \setminus \{1,n\}}.$$

Since evaluating $F \in \mathcal{I}(\mathcal{A}, n - 1)$ at

$$y_1 \mapsto f_1 = x_2 \ell_3 \cdots \ell_n, \quad y_2 \mapsto f_2 = x_1 f_{12}, \quad \ldots, \quad y_n \mapsto f_n = x_1 f_{1n}$$

vanishes, upon pulling out the appropriate powers of x_1, it yields

$$0 = x_1^{m_1+d} P_1(f_{12}, \ldots, f_{1n}) + \cdots + x_1^{m_u+d} P_u(f_{12}, \ldots, f_{1n})$$
$$+ f_1^v x_1^{d-v} G_v(f_{12}, \ldots, f_{1n}) + \cdots + f_1 x_1^{d-1} G_1(f_{12}, \ldots, f_{1n}) + x_1^d G_0(f_{12}, \ldots, f_{1n}).$$

Suppose first that the rank of $\mathcal{A}'$ is $k-1$, i.e., x_1 is a coloop. This means that $x_2 = \ell_2, \ell_3, \ldots, \ell_n$ are actually forms in the subring $R' = \Bbbk[x_2, \ldots .x_k]$. Since $m_1 + d > \cdots > m_u + d > d > \cdots > d - v$,

$$P_i(f_{12}, \ldots, f_{1n}) = 0, i = 1, \ldots, u, \quad G_j(f_{12}, \ldots, f_{1n}) = 0, j = 0, \ldots, v.$$

Therefore, $P_i, G_j \in \mathcal{I}(\mathcal{A}', n'-1)$, and hence $F \in \langle x_1 y_1 - x_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle$. This shows that

$$\mathcal{I}(\mathcal{A}, n-1) = \langle x_1 y_1 - x_2 y_2, \mathcal{I}(\mathcal{A}', n'-1) \rangle$$

and the required result follows by the inductive hypothesis as applied to $\mathcal{I}(\mathcal{A}', n'-1)$. Suppose now that the rank of $\mathcal{A}'$ does not drop, i.e., x_1 is a noncoloop.

Case 1: $v = 0$. In this case, after canceling x_1^d, we obtain

$$0 = x_1^{m_1} P_1(f_{12}, \ldots, f_{1n}) + \cdots + x_1^{m_u} P_u(f_{12}, \ldots, f_{1n}) + G_0(f_{12}, \ldots, f_{1n}).$$

Thus,

$$x_1^{m_1} P_1 + x_1^{m_2} P_2 + \cdots + x_1^{m_u} P_u + P_{u+1} \in \mathcal{I}(\mathcal{A}', n'-1).$$

Case 2: $v \geq 1$. In this case we cancel the factor x_1^{d-v} in the above equation. This will give

$$x_1 \mid G_v(f_{12}, \ldots, f_{1n}).$$

At this point we resort to the idea of stretched arrangements with coefficients as developed in Section 3D. Namely, we take the restriction (contraction) of $\mathcal{A}$ to the hyperplane $x_1 = 0$. Precisely, say

$$\ell_i = a_i x_1 + \bar{\ell}_i, \quad \text{where } \bar{\ell}_i \in R', a_i \in \Bbbk,$$

for $i = 2, \ldots, n$. Note that $a_2 = 0$ since $\ell_2 = x_2$. Write

$$\bar{\mathcal{A}} = \{\bar{\ell}_2, \ldots, \bar{\ell}_n\} \subset R',$$

a stretched arrangement of total multiplicity $\bar{n} = n - 1$ with support $\mathcal{A}''$ of size $n'' \leq \bar{n}$. Likewise, let

$$\bar{f}_{12} := \bar{\ell}_3 \cdots \bar{\ell}_n, \quad \ldots, \quad \bar{f}_{1n} := \bar{\ell}_2 \cdots \bar{\ell}_{n-1}$$

denote the $(\bar{n}-1)$-products of this stretched arrangement. Then, G_v vanishes on the tuple $(\bar{f}_{12}, \ldots, \bar{f}_{1n})$ and since its is homogeneous it necessarily belong to $\mathcal{I}(\bar{\mathcal{A}}, \bar{n}-1)$. From Lemma 3.7, we have

$$\mathcal{I}(\bar{\mathcal{A}}, \bar{n}-1) = \langle \mathcal{I}(\mathcal{A}'', n''-1), \mathcal{D}_{\bar{\mathcal{A}}} \rangle,$$

where $\mathcal{D}_{\bar{\mathcal{A}}}$ is a linear ideal of the form $\langle y_i - b_{i,j} y_j \rangle_{2 \leq i,j \leq n}$. Let us analyze the generators of $\mathcal{I}(\bar{\mathcal{A}}, \bar{n}-1)$.

- A generator $y_i - b_{i,j} y_j$, $i, j \geq 2$ of $\mathcal{D}_{\bar{A}}$ comes from the relation $\bar{\ell}_j = b_{i,j} \bar{\ell}_i$, $b_{i,j} \in \Bbbk$. Thus, back in $\mathcal{A}$ we have the minimal dependency

$$\ell_j - a_j x_1 = b_{i,j}(\ell_i - a_i x_1),$$

yielding an element of $\partial(\mathcal{A}|_{\ell_1})$:

$$y_1(y_i - b_{i,j} y_j) + \underbrace{(b_{i,j} a_i - a_j)}_{c_{i,j}} y_i y_j.$$

- Since $\gcd(\bar{\ell}_i, \bar{\ell}_j) = 1$, for $2 \leq i < j \leq n'' + 1$, a typical generator of $\mathcal{I}_1(\mathcal{A}'', n'' - 1)$ is $\bar{\ell}_i y_i - \bar{\ell}_j y_j$, that we will rewrite as

$$\bar{\ell}_i y_i - \bar{\ell}_j y_j = (\ell_i y_i - \ell_j y_j) - x_1(a_i y_i - a_j y_j).$$

- A typical generator of $\partial(\mathcal{A}'')$ is of the form $b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}$ coming from a minimal dependency

$$b_1 \bar{\ell}_{i_1} + \cdots + b_s \bar{\ell}_{i_s} = 0, \quad i_j \in \{2, \ldots, n'' + 1\}.$$

Since $\bar{\ell}_{i_j} = \ell_{i_j} - a_{i_j} x_1$, we obtain a dependency

$$b_1 \ell_{i_1} + \cdots + b_s \ell_{i_s} - \underbrace{(b_1 a_{i_1} + \cdots + b_s a_{i_s})}_{\alpha} x_1 = 0.$$

If $\alpha = 0$, then

$$b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}} \in \partial(\mathcal{A}'),$$

whereas if $\alpha \neq 0$, then

$$-\alpha y_{i_2} \cdots y_{i_s} + y_1(b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}) \in \partial(\mathcal{A}|_{\ell_1}).$$

We have that

$$G_v = \sum E_{s,t}(y_s - b_{s,t} y_t) + \sum A_{i,j}(\bar{\ell}_i y_i - \bar{\ell}_j y_j) + \sum B_{i_1,\ldots,i_s}(b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}),$$

where $E_{s,t}, A_{i,j}, B_{i_1,\ldots,i_s} \in T''$ and $s, t, i, j, i_k \geq 2$. Then, by using the expressions in the three bullets above and splicing according to the equality $x_1 y_1 = (x_1 y_1 - x_2 y_2) + x_2 y_2$, we get

$$y_1^v G_v = y_1^{v-1} \left(\underbrace{\sum E_{s,t}(y_1(y_s - b_{s,t} y_t) + c_{s,t} y_s y_t)}_{\in \partial(\mathcal{A}|_{\ell_1})} - \underbrace{\sum E_{s,t} c_{s,t} y_s y_t}_{\in T''} \right.$$

$$+ \underbrace{\sum A_{i,j} y_1(\ell_i y_i - \ell_j y_j)}_{\in \mathcal{I}(\mathcal{A}', n'-1)} - \underbrace{\sum A_{i,j}(x_1 y_1 - x_2 y_2)(a_i y_i - a_j y_j)}_{\in \langle x_1 y_1 - x_2 y_2 \rangle} - \underbrace{\sum A_{i,j} x_2 y_2 (a_i y_i - a_j y_j)}_{\in T''}$$

$$\left. + \underbrace{\sum B_{i_1,\ldots,i_s}(y_1(b_1 y_{i_2} \cdots y_{i_s} + \cdots + b_s y_{i_1} \cdots y_{i_{s-1}}) - \alpha y_{i_2} \cdots y_{i_s})}_{\in \partial(\mathcal{A}|_{\ell_1})} + \underbrace{\sum B_{i_1,\ldots,i_s} \alpha y_{i_2} \cdots y_{i_s}}_{\in T''} \right).$$

Thus, $y_1^v G_v = y_1^{v-1} G'_{v-1} + W$, where

$$G'_{v-1} \in T'', \quad W \in \langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1) \rangle.$$

Then returning to our original F, one obtains

$$F = \Delta + x_1^{m_1} P_1 + \cdots + x_1^{m_u} P_u + y_1^{v-1} \underbrace{(G'_{v-1} + G_{v-1})}_{G''_{v-1} \in S''} + y_1^{v-2} G_{v-2} + \cdots + G_0,$$

where $\Delta \in \langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1) \rangle \subset \mathcal{I}(\mathcal{A}, n-1)$. The key is that modulo the ideal $\langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1) \rangle$, the power of y_1 dropped from v to $v-1$ in the expression of F. Iterating, with $F(f_1, \ldots, f_n) = 0 = \Delta(f_1, \ldots, f_n)$, will eventually drop further the power of y_1 to $v-2$. Recursively we end up with $v = 0$, which is Case 1 above. This way, we eventually get

$$\mathcal{I}(\mathcal{A}, n-1) = \langle x_1 y_1 - x_2 y_2, \partial(\mathcal{A}|_{\ell_1}), \mathcal{I}(\mathcal{A}', n'-1) \rangle.$$

By the inductive hypothesis as applied to $\mathcal{I}(\mathcal{A}', n'-1)$ and from the two equalities in (6), one gets the stated result. $\square$

Corollary 4.3. $\mathcal{I}(\mathcal{A}', n'-1) = \mathcal{I}(\mathcal{A}, n-1) \cap T'$ *as ideals in* $T' = R[y_2, \ldots, y_n]$.

Proof. Recall the notation $T' := R[y_2, \ldots, y_n] \subset T := R[y_1, \ldots, y_n] = T'[y_1]$ as in the proof of the previous theorem. Denote $J := \mathcal{I}(\mathcal{A}, n-1) \cap T'$. We show that $J \subseteq \mathcal{I}(\mathcal{A}', n'-1)$, the other inclusion being obvious. Let $F \in J$. Then $F \in T'$ and $F \in \mathcal{I}(\mathcal{A}, n-1)$. By Theorem 4.2, we can write

$$F = (\ell_1 y_1 - \ell_2 y_2) P + Q + G, \quad \text{where } P \in T, \ Q \in \partial(\mathcal{A}|_{\ell_1}) T, \ G \in \mathcal{I}(\mathcal{A}', n'-1) T.$$

By Lemma 3.4,

$$\ell_1 Q \in \mathcal{I}_1(\mathcal{A}, n-1) = \langle \ell_1 y_1 - \ell_2 y_2, \mathcal{I}_1(\mathcal{A}', n'-1) \rangle.$$

Therefore,

$$\ell_1 F = (\ell_1 y_1 - \ell_2 y_2) P' + G', \tag{7}$$

for suitable $P' \in T$, $G' \in \mathcal{I}(\mathcal{A}', n'-1) T$. We write $P' = y_1^u P_u + \cdots + y_1 P_1 + P_0$, $P_i \in T'$, and $G' = y_1^v G_v + \cdots + y_1 G_1 + G_0$, $G_j \in T'$. Since $G' \in \mathcal{I}(\mathcal{A}', n'-1) \subset T'$, setting $y_1 = 0$ in the expression of G' gives that $G_0 \in \mathcal{I}(\mathcal{A}', n'-1)$. Therefore, $G - G_0 = y_1(y_1^{v-1} G_v + \cdots + G_1) \in \mathcal{I}(\mathcal{A}', n'-1)$, and hence $y_1^{v-1} G_v + \cdots + G_1 \in \mathcal{I}(\mathcal{A}', n'-1)$ since $\mathcal{I}(\mathcal{A}', n'-1)$ is prime. Setting again $y_1 = 0$ in this expression we obtain that $G_1 \in \mathcal{I}(\mathcal{A}', n'-1)$, and so on, eventually obtaining

$$G_j \in \mathcal{I}(\mathcal{A}', n'-1), \quad j = 0, \ldots, v.$$

Suppose $u \geq v$. Then, by grouping the powers of y_1 we have

$$\ell_1 F = (-\ell_2 y_2 P_0 + G_0) + (\ell_1 P_0 - \ell_2 y_2 P_1 + G_1) y_1 + \cdots + (\ell_1 P_{v-1} - \ell_2 y_2 P_v + G_v) y_1^v$$
$$+ (\ell_1 P_v - \ell_2 y_2 P_{v+1}) y_1^{v+1} + \cdots + (\ell_1 P_{u-1} - \ell_2 y_2 P_u) y_1^u + \ell_1 P_u y_1^{u+1}.$$

Since $F \in T'$, then $\ell_1 F \in T'$. Thus, the "coefficients" of $y_1, y_1^2, \ldots, y_1^{u+1}$ must vanish. It follows that

$$P_u = \cdots = P_v = 0 \quad \text{and} \quad \ell_1 P_{v-1} = -G_v \in \mathcal{I}(\mathcal{A}', n'-1).$$

Since $\mathcal{I}(\mathcal{A}', n'-1)$ is a prime ideal, we have $P_{v-1} \in \mathcal{I}(\mathcal{A}', n'-1)$, and therefore

$$\ell_1 P_{v-2} = \ell_2 y_2 P_{v-1} - G_{v-1} \in \mathcal{I}(\mathcal{A}', n'-1).$$

Recursively we get that

$$P_{v-1}, P_{v-2}, \ldots, P_1, P_0 \in \mathcal{I}(\mathcal{A}', n'-1).$$

If $u < v$, a similar analysis will give the same conclusion that $P' \in \mathcal{I}(\mathcal{A}', n'-1)T$. Therefore, (7) gives $\ell_1 F \in \mathcal{I}(\mathcal{A}', n'-1)T$, and hence $F \in \mathcal{I}(\mathcal{A}', n'-1)T$ by primality of the extended ideal. But then $F \in I(\mathcal{A}', n'-1)T \cap T' = I(\mathcal{A}', n'-1)$, as required. $\square$

The next two corollaries help compute the Rees ideal from the symmetric ideal via a simple colon of ideals.

Corollary 4.4. *Let $\ell_i \in \mathcal{A}$ and y_i be the corresponding external variable. Then*

$$\mathcal{I}(\mathcal{A}, n-1) = \mathcal{I}_1(\mathcal{A}, n-1) : \ell_i y_i.$$

Proof. Without loss of generality, assume $i = 1$. The inclusion $\supseteq$ is immediate, since $\mathcal{I}_1(\mathcal{A}, n-1) \subset \mathcal{I}(\mathcal{A}, n-1)$, and the Rees ideal $\mathcal{I}(\mathcal{A}, n-1)$ is a prime ideal not containing ℓ_1 nor y_1. For the reverse inclusion, let $F \in \mathcal{I}(\mathcal{A}, n-1)$. Then, from Theorem 4.2,

$$F = G + \sum_D P_D \partial D,$$

where the sum is taken over all minimal dependencies D, and $G \in \mathcal{I}_1(\mathcal{A}, n-1)$.

Obviously, $\ell_1 y_1 G \in \mathcal{I}_1(\mathcal{A}, n-1)$. Also, if $\partial D \in \partial(\mathcal{A}|_{\ell_1})$, then, from Lemma 3.4, $\ell_1 \partial D \in \mathcal{I}_1(\mathcal{A}, n-1)$, hence $\ell_1 y_1 \partial D$ belongs to $\mathcal{I}_1(\mathcal{A}, n-1)$ as well. Suppose $\partial D \notin \partial(\mathcal{A}|_{\ell_1})$. Since D is a minimal dependency among the hyperplanes of $\mathcal{A}$, there exists $j \in \{1, \ldots, n\}$ such that $\partial D \in \partial(\mathcal{A}|_{\ell_j})$. Thus, $\ell_1 y_1 \partial D = (\ell_1 y_1 - \ell_j y_j)\partial D + \ell_j y_j \partial D$ belongs to the ideal $\mathcal{I}_1(\mathcal{A}, n-1)$ since each summand belongs to $\mathcal{I}_1(\mathcal{A}, n-1)$, the first trivially and the second due to Lemma 3.4. $\square$

Since the rank of $\mathcal{A}$ is k, after a reordering of the linear forms $\ell_1, \ldots, \ell_n$ that define $\mathcal{A}$, we can assume that the last k linear forms $\ell_{n-k+1}, \ldots, \ell_n$ are linearly independent. With this proviso, one has:

Corollary 4.5. $$\mathcal{I}(\mathcal{A}, n-1) = \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i.$$

Proof. Since $\ell_{n-k+1}, \ldots, \ell_n$ are k linearly independent linear forms, any minimal dependency that involves at least one of them, must involve also a linear form ℓ_j, where $j \in \{1, \ldots, n-k\}$. So

$$\partial(\mathcal{A}) = \partial(\mathcal{A}|_{\ell_1}) + \cdots + \partial(\mathcal{A}|_{\ell_{n-k}}).$$

We obviously have $\mathcal{I}_1(\mathcal{A}, n-1) \subseteq \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i$, and from Lemma 3.4,

$$\partial(\mathcal{A}|_{\ell_j}) \subset \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i, \quad \text{for all } j = 1, \ldots, n-k.$$

Then, from Theorem 4.2, one has

$$\mathcal{I}(\mathcal{A}, n-1) \subseteq \mathcal{I}_1(\mathcal{A}, n-1) : \prod_{i=1}^{n-k} \ell_i.$$

The reverse inclusion comes from the fact that $\mathcal{I}_1(\mathcal{A}, n-1) \subseteq \mathcal{I}(\mathcal{A}, n-1)$, and from $\mathcal{I}(\mathcal{A}, n-1)$ being a prime ideal with $\ell_i \notin \mathcal{I}(\mathcal{A}, n-1)$. $\qquad\square$

In the next statement we denote the extended ideal $(\mathcal{I}_1(\mathcal{A}', n'-1))T$ by $\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle$.

Lemma 4.6. *Let $\mathcal{A}' = \mathcal{A} \setminus \{\ell_1\}$ and $n' = |\mathcal{A}'| = n - 1$. We have*

$$\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : (\ell_1 y_1 - \ell_2 y_2) = \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : \ell_1.$$

In particular, when ℓ_1 is a coloop, the biform $\ell_1 y_1 - \ell_2 y_2$ is a nonzero divisor on $\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle$.

Proof. For convenience, let us change coordinates to have $\ell_1 = x_1$ and $\ell_2 = x_2$. Let $f \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : (x_1 y_1 - x_2 y_2)$. Then $f(x_1 y_1 - x_2 y_2) \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle \subset \langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$. Since $\langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$ is a prime ideal not containing $x_1 y_1 - x_2 y_2$, we obtain $f \in \langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$, and by Theorem 4.2, we have

$$f = g + h, \quad g \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle, \quad h \in \langle \partial(\mathcal{A}') \rangle.$$

By multiplying this by $x_1 y_1 - x_2 y_2$, we get that

$$(x_1 y_1 - x_2 y_2) h \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle.$$

By Corollary 4.4, since $h \in \langle \partial(\mathcal{A}') \rangle \subset \langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$, and $x_2 \in \mathcal{A}'$, we have $x_2 y_2 h \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle$. So $h \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : x_1 y_1$, and together with $f = g + h$ with $g \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle \subset \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : x_1 y_1$, gives

$$f \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : x_1 y_1.$$

Conversely, let $\Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : x_1 y_1$. Then $x_1 y_1 \Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle \subseteq \langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$. The ideal $\langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$ is a prime ideal, and $x_1 y_1 \notin \langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$, so $\Delta \in \langle \mathcal{I}(\mathcal{A}', n'-1) \rangle$. So, by Corollary 4.4, $x_2 y_2 \Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle$. Therefore

$$(x_1 y_1 - x_2 y_2) \Delta = x_1 y_1 \Delta - x_2 y_2 \Delta \in \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle.$$

Thus far, we have shown that $\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : (\ell_1 y_1 - \ell_2 y_2) = \langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : x_1 y_1$. Clearly, the right hand side is the same as $\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : x_1$ since y_1 is a nonzero divisor on $\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle$. $\qquad\square$

4C. *The Cohen–Macaulay property.* In this part the goal is to prove that the Rees algebra is Cohen–Macaulay. Since we are in a graded setting, this is equivalent to showing that its depth with respect to the maximal graded ideal $\langle \mathfrak{m}, y_1, \ldots, y_n \rangle$ is (at least) $k + 1 = \dim R[It]$.

This will be accomplished by looking at a suitable short exact sequence, where two of the modules will be examined next. We state the results in terms of depth since this notion is inherent to the Cohen–Macaulay property, yet the proofs will take the approach via projective (i.e., homological) dimension. By the Auslander–Buchsbaum equality, we are home anyway. Throughout, pdim_T will denote projective dimension over the polynomial ring T. Since we are in a graded situation, this is the same as the projective dimension over the local ring $T_{(\mathfrak{m}, y_1, \ldots, y_n)}$, so we may harmlessly proceed.

The first module is obtained by cutting the binomial generators of $\mathcal{I}_1(\mathcal{A}, n-1)$ into its individual terms. The result may be of interest on its own.

Lemma 4.7. *Let* $\ell_1, \ldots, \ell_n \in \Bbbk[x_1, \ldots, x_k]$ *be linear forms, allowing some of them to be mutually proportional. Let* $T := \Bbbk[x_1, \ldots, x_k; y_1, \ldots, y_n]$. *Then*

$$\mathrm{depth}\left(\frac{T}{\langle \ell_1 y_1, \ell_2 y_2, \ldots, \ell_n y_n \rangle} \right) \geq k.$$

Proof. If $k = 1$, the claim is clearly satisfied, since $\langle x_1 y_1, \ldots, x_1 y_n \rangle = x_1 \langle y_1, \ldots, y_n \rangle$, and $\{y_1, \ldots, y_n\}$ is a T-regular sequence. Assume $k \geq 2$.

We will use induction on $n \geq 1$ to show that the projective dimension is at most $n + k - k = n$. If $n = 1$, the ideal $\langle \ell_1 y_1 \rangle$ is a principal ideal, hence the claim is true. Suppose $n > 1$. We may apply a $\Bbbk$-linear automorphism on the ground variables, which will not disturb the projective dimension. Thus, say, $\ell_1 = x_1$ and this form is repeated s times. Since nonzero coefficients from $\Bbbk$ tagged to x_1 will not change the ideal in question, we assume that $\ell_i = x_1$, for $1 \leq i \leq s$, and $\gcd(x_1, \ell_j) = 1$, for $s+1 \leq j \leq n$. Write $\ell_j = c_j x_1 + \bar{\ell}_j$, for $s+1 \leq j \leq n$, with $c_j \in \Bbbk$, and $0 \neq \bar{\ell}_j \in \Bbbk[x_2, \ldots, x_k]$.

Denoting $J := \langle x_1 y_1, \ldots, x_1 y_s, \ell_{s+1} y_{s+1}, \ldots, \ell_n y_n \rangle$, we claim that

$$J : x_1 = \langle y_1, \ldots, y_s, \ell_{s+1} y_{s+1}, \ldots, \ell_n y_n \rangle. \tag{8}$$

This is certainly the expression of a more general result, but we give a direct proof here. One inclusion is obvious. For the reverse inclusion, let $F \in \langle x_1 y_1, \ldots, x_1 y_s, \ell_{s+1} y_{s+1}, \ldots, \ell_n y_n \rangle : x_1$. Then, say,

$$x_1 F = x_1 \sum_{i=1}^{s} P_i y_i + \sum_{j=s+1}^{n} P_j \ell_j y_j,$$

for certain $P_i, P_j \in T$. Rearranging we have

$$x_1 \left(F - \sum_{i=1}^{s} P_i y_i - \sum_{j=s+1}^{n} c_j P_j y_j \right) = \sum_{j=s+1}^{n} P_j \bar{\ell}_j y_j. \tag{9}$$

Since x_1 is a nonzero divisor in $T/\langle \bar{\ell}_{s+1} y_{s+1}, \ldots, \bar{\ell}_n y_n \rangle$, the second factor of the left hand side in (9) must be of the form

$$\sum_{j=s+1}^{n} Q_j \bar{\ell}_j y_j, \quad Q_j \in T.$$

Substituting in (9) we find $P_j = x_1 Q_j, s+1 \leq j \leq n$, and hence $F = \sum_{i=1}^{s} P_i y_i + \sum_{j=s+1}^{n} Q_j \ell_j y_j$, as claimed. Computing projective dimensions with respect to T and $T' = \Bbbk[x_1, \ldots, y_{s+1}, \ldots, y_n]$ and applying the inductive hypothesis, one has

$$\mathrm{pdim}_T \left(\frac{T}{J : x_1} \right) = s + \mathrm{pdim}_{T'} \left(\frac{T'}{\langle \ell_{s+1} y_{s+1}, \ldots, \ell_n y_n \rangle} \right) \leq s + (n - s) = n.$$

At the other end, we have $\langle x_1, J \rangle = \langle x_1, \bar{\ell}_{s+1} y_{s+1}, \ldots, \bar{\ell}_n y_n \rangle$. Applying the inductive hypothesis this time around gives

$$\mathrm{pdim}_T \left(\frac{T}{\langle x_1, J \rangle} \right) \leq 1 + (n - s) \leq n.$$

From the short exact sequence of T-modules

$$0 \to T/(J : x_1) \xrightarrow{\cdot x_1} T/J \to T/\langle x_1, J \rangle \to 0,$$

knowingly the projective dimension of the middle term does not exceed the maximum of the projective dimensions of the two extreme terms. Therefore, $\mathrm{pdim}_T(T/J) \leq n$, as was to be shown. $\square$

The difficult result of this section is the following exact invariant of the symmetric algebra $\mathcal{S}(I) \simeq T/\langle \mathcal{I}_1(\mathcal{A}, n-1) \rangle$:

Proposition 4.8. *Let $I = I_{n-1}(\mathcal{A})$ as before. Then $\mathrm{depth}(\mathcal{S}(I)) = k + 1$.*

Proof. By Proposition 3.2(i), it suffices to prove the lower bound $\mathrm{depth}(\mathcal{S}(I)) \geq k + 1$. As in the proof of Theorem 4.2, we argue by induction on all pairs n, k, with $n \geq k \geq 2$, where n and k are, respectively, the size and the rank of $\mathcal{A}$. If $k = 2$ and $n > 2$, let $R = \Bbbk[x, y]$. As seen in that proof, one has $I = \langle x, y \rangle^{n-1}$, and hence

$$\mathcal{I}_1(\mathcal{A}, n-1) = \langle xy_1 - yy_2, xy_2 - yy_3, \ldots, xy_{n-1} - yy_n \rangle.$$

A direct calculation shows that $\{y_1, x + y_n, y + y_{n-1}\}$ is a regular sequence modulo $\mathcal{I}_1(\mathcal{A}, n-1)$. If $n = k$, the ideal $\mathcal{I}_1(\mathcal{A}, n-1)$ is a complete intersection by Lemma 3.1. Thus, for the main inductive step suppose $n > k > 2$. We will equivalently show that $\mathrm{pdim}_T(\mathcal{S}(I)) \leq n - 1$. First apply a change of ground variables so as to have $\ell_1 = x_1$ and $\ell_2 = x_2$; let $\mathcal{A}' := \mathcal{A} \setminus \{x_1\}$ denote the deletion. Since $\mathcal{I}_1(\mathcal{A}, n-1) = \langle \mathcal{I}_1(\mathcal{A}', n'-1), x_1 y_1 - x_2 y_2 \rangle$, we have the following short exact sequence of T-modules

$$0 \to \frac{T}{\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle : (x_1 y_1 - x_2 y_2)} \xrightarrow{\cdot (x_1 y_1 - x_2 y_2)} \frac{T}{\langle \mathcal{I}_1(\mathcal{A}', n'-1) \rangle} \to \frac{T}{\mathcal{I}_1(\mathcal{A}, n-1)} \to 0. \quad (10)$$

We consider separately the cases where ℓ_1 is a coloop or a noncoloop.

x_1 **is a coloop:** Here the rank of $\mathcal{A}'$ is $k - 1$ and x_1 is altogether absent in the linear forms of the deletion. Thus, the natural ambient ring of $\mathcal{I}_1(\mathcal{A}', n' - 1)$ is $T' := \Bbbk[x_2, \ldots, x_k; y_2, \ldots, y_n]$. In this case, by Lemma 4.6, the left most nonzero term of (10) becomes

$$T/\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle = \frac{T'}{\mathcal{I}_1(\mathcal{A}', n' - 1)}[x_1, y_1],$$

hence

$$\mathrm{pdim}_T(T/\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle) = \mathrm{pdim}_{T'}(T'/\mathcal{I}_1(\mathcal{A}', n' - 1)) \leq n' - 1,$$

by the inductive hypothesis applied to $\mathcal{S}(I_{n'-1}(\mathcal{A}')) \simeq T'/\mathcal{I}_1(\mathcal{A}', n' - 1)$. Then, from (10) we have

$$\mathrm{pdim}_T(T/\mathcal{I}_1(\mathcal{A}, n - 1)) \leq \max\{\mathrm{pdim}_T(T/\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle) + 1, \mathrm{pdim}_T(T/\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle)\}$$
$$\leq (n' - 1) + 1 = n' = n - 1.$$

x_1 **is a noncoloop:** This case will occupy us for the rest of the proof. Here $T' := \Bbbk[x_1, \ldots, x_k; y_2, \ldots, y_n]$ is the natural ambient ring of the deletion symmetric ideal. By Lemma 4.6, the left most nonzero term of (10) is $T/(\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle : x_1)$. Thus, multiplication by x_1 gives a similar exact sequence to (10):

$$0 \to \frac{T}{\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle : x_1} \to \frac{T}{\langle\mathcal{I}_1(\mathcal{A}', n' - 1)\rangle} \to \frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n' - 1)\rangle} \to 0. \qquad (11)$$

Suppose for a minute that one has

$$\mathrm{pdim}_T\left(\frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n' - 1)\rangle}\right) \leq n'. \qquad (12)$$

Then (11) implies

$$\mathrm{pdim}_T\left(\frac{T}{\mathcal{I}_1(\mathcal{A}', n' - 1) : x_1}\right) \leq \max\{n' - 1, n' - 1\} = n' - 1.$$

Back to (10) would finally give

$$\mathrm{pdim}_T\left(\frac{T}{\mathcal{I}_1(\mathcal{A}, n - 1)}\right) \leq \max\{(n' - 1) + 1, n' - 1\} = n' = n - 1,$$

proving the required statement of the theorem. Thus, it suffices to prove (12). For this, one sets $\langle x_1, \mathcal{I}_1(\mathcal{A}', n' - 1)\rangle = \langle x_1, x_2 y_2 - \bar{\ell}_3 y_3, \ldots, \bar{\ell}_{n-1} y_{n-1} - \bar{\ell}_n y_n\rangle$, where we have written $\ell_j = c_j x_1 + \bar{\ell}_j$, with $c_j \in \Bbbk$, $\bar{\ell}_j \in \Bbbk[x_2, \ldots, x_k]$, for $3 \leq j \leq n$. Then,

$$\mathrm{pdim}_T\left(\frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n' - 1)\rangle}\right) = 1 + \mathrm{pdim}_{T'}\left(\frac{T'}{\langle x_2 y_2 - \bar{\ell}_3 y_3, \ldots, \bar{\ell}_{n-1} y_{n-1} - \bar{\ell}_n y_n\rangle}\right). \qquad (13)$$

Let $\bar{\mathcal{A}} = \{x_2, \bar{\ell}_3, \ldots, \bar{\ell}_n\}$ denote the corresponding stretched arrangement and set

$$J := \langle x_2 y_2 - \bar{\ell}_3 y_3, \ldots, \bar{\ell}_{n-1} y_{n-1} - \bar{\ell}_n y_n\rangle \subset T' := \Bbbk[x_2, \ldots, x_k; y_2, \ldots, y_n].$$

Claim: $\mathrm{pdim}_{T'}(T'/J) \leq n' - 1.$

If the size of $\bar{\mathcal{A}}$ is $= n - 1 = n'$ (i.e., no two linear forms of $\bar{\mathcal{A}}$ are proportional), then $J = \mathcal{I}_1(\bar{\mathcal{A}}, n' - 1)$, and by the inductive hypotheses $\operatorname{pdim}_{T'}(T'/\mathcal{I}_1(\bar{\mathcal{A}}, n' - 1)) \le n' - 1$. Otherwise, suppose $s - 1 \ge 2$ of the linear forms of $\bar{\mathcal{A}}$ are mutually proportional. Without loss of generality, say

$$\bar{\ell}_3 = d_3 x_2, \quad \ldots, \quad \bar{\ell}_s = d_s x_2, \quad d_i \in \Bbbk \setminus \{0\}, \ 3 \le i \le s$$

and

$$\bar{\ell}_j = d_j x_2 + L_j, \quad d_j \in \Bbbk, 0 \ne L_j \in \Bbbk[x_3, \ldots, x_k], \ 4 \le j \le n.$$

Then

$$J = \langle x_2(y_2 - d_3 y_3), \ldots, x_2(y_2 - d_s y_s), x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n \rangle.$$

We now provide the following estimates:

(a) $\operatorname{pdim}_{T'}(T'/\langle x_2, J \rangle) \le 1 + (n - s)$.

(b) $\operatorname{pdim}_{T'}(T'/\langle J : x_2 \rangle) \le n' - 1$.

For (a), note that $\langle x_2, J \rangle = \langle x_2, L_{s+1} y_{s+1}, \ldots, L_n y_n \rangle$, while from the proof of Lemma 4.7 we have

$$\operatorname{pdim}_{T'}(T'/\langle x_2, J \rangle) \le 1 + (n - s),$$

since x_2 is a nonzero divisor in $T'/\langle L_{s+1} y_{s+1}, \ldots, L_n y_n \rangle$. As for (b), we first claim that $J : x_2 = \langle y_2 - d_3 y_3, \ldots, y_2 - d_s y_s, x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n \rangle$. The proof is pretty much the same as that of the equality in (8), hence will be omitted. This equality implies that

$$\operatorname{pdim}_{T'} g \left(\frac{T'}{J : x_2} \right) = s - 2 + \operatorname{pdim}_{T''} g \left(\frac{T''}{\langle x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n \rangle} \right),$$

where $T'' := \Bbbk[x_2, \ldots, x_k; y_2, y_{s+1}, \ldots, y_n]$.

Let $\mathcal{B} := \{x_2, \bar{\ell}_{s+1}, \ldots, \bar{\ell}_n\}$. With same reasoning for $\mathcal{B}$ as for $\bar{\mathcal{A}}$ (i.e., removing proportional linear forms), we obtain

$$\operatorname{pdim}_{T''} g \left(\frac{T''}{\langle x_2 y_2 - \bar{\ell}_{s+1} y_{s+1}, \ldots, x_2 y_2 - \bar{\ell}_n y_n \rangle} \right) \le (n - s + 1) - 1 = n - s,$$

and therefore

$$\operatorname{pdim}_{T'} g \left(\frac{T'}{J : x_2} \right) \le s - 2 + n - s = n - 2 = n' - 1.$$

Drawing on the estimates (a) and (b) above, the exact sequence of T'-modules

$$0 \to T'/(J : x_2) \to T'/J \to T'/\langle x_2, J \rangle \to 0,$$

gives that

$$\operatorname{pdim}_{T'}(T'/J) \le \max\{n' - 1, 2 + n' - s\} \le n' - 1,$$

since $s \ge 3$. Rolling all the way back to (13), we have proved that

$$\operatorname{pdim}_T g \left(\frac{T}{\langle x_1, \mathcal{I}_1(\mathcal{A}', n' - 1) \rangle} \right) \le 1 + (n' - 1) = n',$$

as required. $\qquad\qquad\square$

The main result now follows quite smoothly.

Theorem 4.9. *The Rees algebra of $I_{n-1}(\mathcal{A})$ is Cohen–Macaulay.*

Proof. From Corollary 4.4 we have the following short exact sequence of T-modules

$$0 \to \frac{T}{\mathcal{I}(\mathcal{A}, n-1)} \to \frac{T}{\mathcal{I}_1(\mathcal{A}, n-1)} \to \frac{T}{\langle \mathcal{I}_1(\mathcal{A}, n-1), \ell_1 y_1 \rangle} \to 0.$$

By Proposition 4.8, the depth of the middle module is $k+1$, while by Lemma 4.7 that of the right most module is at least k — in fact, by the domino effect one has $\langle \mathcal{I}_1(\mathcal{A}, n-1), \ell_1 y_1 \rangle = \langle \ell_1 y_1, \ell_2 y_2, \ldots, \ell_n y_n \rangle$. By standard knowledge, the depth of the left most module is at least that of the middle module, namely, $k+1$. □

A consequence is an alternative proof of a result of Proudfoot and Speyer [2006]:

Corollary 4.10. *Let $\mathcal{A}$ be any central arrangement. Then the associated Orlik–Terao algebra is Cohen–Macaulay.*

Proof. As we have seen, the Orlik–Terao algebra is the special fiber of the ideal $I = I_{n-1}(\mathcal{A})$. Since I is a homogeneous ideal generated in one single degree, then its special fiber is identified with the $\Bbbk$-subalgebra $\Bbbk[It]$ of the Rees algebra $R[It]$ of I and, as such, it is a direct summand as a $\Bbbk[It]$-module. In this situation W. Vasconcelos and one of us have shown that the Cohen–Macaulay property of $R[It]$ transfers to $\Bbbk[It]$. A proof of the latter result is given in [Ramos and Simis 2017, Proposition 3.10] for the case where $k = 3$. The proof for arbitrary k is the same. □

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