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Torsion in the 0-cycle group with modulus

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We show, for a smooth projective variety X over an algebraically closed field k with an effective Cartier divisor D , that the torsion subgroup $\mathrm{CH}_0(X|D)\{l\}$ can be described in terms of a relative étale cohomology for any prime $l \neq p = \mathrm{char}(k)$. This extends a classical result of Bloch, on the torsion in the ordinary Chow group, to the modulus setting. We prove the Roitman torsion theorem (including p -torsion) for $\mathrm{CH}_0(X|D)$ when D is reduced. We deduce applications to the problem of invariance of the prime-to- p torsion in $\mathrm{CH}_0(X|D)$ under an infinitesimal extension of D .

1. Introduction	1431
2. Review of 0-cycle groups and Albanese varieties	1434
3. Torsion in the Chow group of a singular surface	1440
4. Roitman's torsion for separably weakly normal surfaces	1447
5. Bloch's torsion theorem for 0-cycles with modulus	1453
6. Roitman's torsion theorem for 0-cycles with modulus	1461
Acknowledgments	1467
References	1467

1. Introduction

One of the fundamental problems in the study of the theory of motives, as envisioned by Grothendieck, is to construct a universal cohomology theory of varieties and describe it in terms of algebraic cycles. When X is a smooth quasiprojective scheme over a base field k , such a motivic cohomology theory of X is known to exist; see, for example, [Voevodsky 2000; Levine 1998]. Moreover, Bloch [1986] showed that this cohomology theory has an explicit description in terms of groups of algebraic cycles, called the higher Chow groups. These groups have all the properties that one expects, including Chern classes and a Chern character isomorphism from the higher K -groups of Quillen.

However, the situation becomes much complicated when we go beyond the world of smooth varieties. There is no theory of motivic cohomology of singular schemes known to date which could be a universal cohomology theory, which could be described in terms of algebraic cycles, and which could be rationally isomorphic to the algebraic K -theory.

With the aim of understanding the algebraic K -theory of the ring $k[t]/(t^n)$ in terms of algebraic cycles, Bloch and Esnault introduced the idea of algebraic cycles with modulus, called additive Chow groups.

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This idea was substantially expanded in the further works of Park [2009], Rilling [2007] and Krishna and Levine [2008].

In recent works of Binda and Saito [2017] and Kerz and Saito [2016], a theory of higher Chow groups with modulus was introduced, which generalizes the construction of the additive higher Chow groups. These groups, denoted $\mathrm{CH}^*(X|D, *)$, are designed to study the arithmetic and geometric properties of a smooth variety X with fixed conditions along an effective (possibly nonreduced) Cartier divisor D on it (see [Kerz and Saito 2016]), and are supposed to give a cycle-theoretic description of the relative K -groups $K_*(X, D)$, defined as the homotopy groups of the homotopy fiber of the restriction map $K(X) \rightarrow K(D)$ (see [Iwasa and Kai 2016]).

In order to provide evidence that the Chow groups with modulus are the right motivic cohomology groups to compute the relative K -theory of a smooth scheme with respect to an effective divisor, one would like to know if these groups share enough of the known structural properties of the Chow groups without modulus, and, if these groups could be related to other cohomological invariants of a pair (X, D) . In particular, one would like to know if the torsion part of the Chow group of 0-cycles with modulus could be described in terms of a relative étale cohomology or, in terms of an Albanese variety with modulus.

1A. The main results. For the Chow group of 0-cycles without modulus on a smooth projective scheme, Bloch [1979] showed that its prime-to- p torsion is completely described in terms of an étale cohomology. Roitman [1980] and Milne [1982] showed that the torsion in the Chow group is completely described in terms of the Albanese of the underlying scheme. When D is an effective Cartier divisor on a smooth projective scheme X , the Albanese variety with modulus $A^d(X|D)$ with appropriate universal property and the Abel–Jacobi map $\rho_{X|D} : \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ were constructed in [Binda and Krishna 2018].

The goal of this work is to extend the torsion results of Bloch and those of Roitman and Milne to 0-cycles with modulus. Our main results thus broadly look as follows. Their precise statements and underlying hypotheses and notations will be explained at appropriate places in the text.

Theorem 1.1 (Bloch’s torsion theorem). *Let k be an algebraically closed field of exponential characteristic p . Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Then, for any prime $l \neq p$, there is an isomorphism*

$$\lambda_{X|D} : \mathrm{CH}_0(X|D)\{l\} \xrightarrow{\sim} H_{\text{ét}}^{2d-1}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)).$$

Theorem 1.2 (Roitman’s torsion theorem). *Let k be an algebraically closed field of exponential characteristic p . Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Assume that D is reduced. Then, the Albanese variety with modulus $A^d(X|D)$ is a semiabelian variety and the Abel–Jacobi map $\rho_{X|D} : \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ is an isomorphism on the torsion subgroups (including p -torsion).*

Note that the isomorphism of [Theorem 1.1](#) is very subtle because one does not have any obvious map in either direction. The construction of such a map is an essential part of the problem. In the case of the Chow group without modulus, the construction of λ_X by Bloch makes essential use of the Bloch–Ogus theory

and, more importantly, it uses the Weil conjecture. The prime-to- p part of [Theorem 1.2](#) was earlier proven in [\[Binda and Krishna 2018\]](#). The new contribution here is the description of the more challenging p -part.

As part of the proof of [Theorem 1.2](#), we prove the Roitman torsion theorem (including p -torsion) for singular separably weakly normal surfaces (see [Definition 4.5](#)). This provides the first evidence that the Roitman torsion theorem may be true for nonnormal varieties in positive characteristic.

Theorem 1.3. *Let X be a reduced projective separably weakly normal surface over an algebraically closed field k of exponential characteristic p . Then, the Albanese variety $A^2(X)$ is a semiabelian variety and the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X)_{\mathrm{deg}0} \rightarrow A^2(X)$ is an isomorphism on the torsion subgroups (including p -torsion).*

Note that [Theorem 1.2](#) makes no assumption on $\dim(X)$ but this is not the case for [Theorem 1.3](#). The reason for this is that our proof of [Theorem 1.2](#) uses a weaker version of the Lefschetz type theorem for separably weakly normal varieties. We do not know if a general hyperplane section of an arbitrary separably weakly normal variety is separably weakly normal. This is known to be false for weakly normal varieties in positive characteristic; see [\[Cumino et al. 1989\]](#). We shall investigate this question in a future project.

1B. Applications. As an application of [Theorem 1.1](#), we obtain the following result about the prime-to- p torsion in the Chow group with modulus.

Corollary 1.4. *Let k be an algebraically closed field of exponential characteristic p . Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Then, the restriction map ${}_n\mathrm{CH}_0(X|D) \rightarrow {}_n\mathrm{CH}_0(X|D_{\mathrm{red}})$ is an isomorphism for every integer n prime to p .*

In [\[Miyazaki 2016, Theorem 1.3\]](#), an isomorphism of similar indication has been very recently shown after inverting $\mathrm{char}(k)$. But it has no implication on [Corollary 1.4](#). Another application of [Theorem 1.1](#) is the following extension of the results of Bloch [\[1980, Chapter 5\]](#) to the relative K -theory.

Corollary 1.5. *Let X be a smooth projective surface over an algebraically closed field k of exponential characteristic p and let $l \neq p$ be a prime. Let $D \subset X$ be an effective Cartier divisor. Then the following hold.*

- (1) $H^1(X, \mathcal{K}_{2,(X,D)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$.
- (2) $H^2(X, \mathcal{K}_{2,(X,D)})\{l\} \simeq H_{\mathrm{ét}}^3(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2))$.

Analogous results are also obtained for surfaces with arbitrary singularities (see [Theorems 3.4](#) and [3.6](#)).

1C. Outline of proofs. Our proofs of the above results are based on the decomposition theorem of [\[Binda and Krishna 2018\]](#) which relates the Chow group of 0-cycles with modulus on a smooth variety to the Levine–Weibel Chow group of 0-cycles on a singular variety. Using this decomposition theorem, our main task becomes extending the torsion results of Bloch and those of Roitman and Milne to singular varieties. We prove these results for the Chow group of 0-cycles on singular surfaces in the first part (Sections [3](#) and [4](#)) of this text. These results are of independent interest in the study of 0-cycles on singular varieties.

In Sections 5 and 6, we prove our results about 0-cycles with modulus by using the analogous results for singular surfaces and an induction on dimension. This induction is based on some variants of the Lefschetz hyperplane section theorem for the Albanese variety and the weak Lefschetz theorem for the étale cohomology with modulus. Some applications are deduced in Section 5.

1D. Notations. Throughout (except in Section 4A), k will denote an algebraically closed field of exponential characteristic $p \geq 1$. We shall denote the category of separated schemes of finite type over k by \mathbf{Sch}_k . The subcategory of \mathbf{Sch}_k consisting of smooth schemes over k will be denoted by \mathbf{Sm}_k . An undecorated product $X \times Y$ in \mathbf{Sch}_k will mean that it is taken over the base scheme $\mathrm{Spec}(k)$. An undecorated cohomology group will be assumed to be with respect to the Zariski site. We shall let $w : \mathbf{Sch}_k/\acute{e}t \rightarrow \mathbf{Sch}_k/\mathrm{zar}$ denote the canonical morphism between the étale and the Zariski sites of \mathbf{Sch}_k . For $X \in \mathbf{Sch}_k$, the notation X^N will usually mean the normalization of X_{red} unless we use a different notation in a specific context. For any abelian group A and an integer $n \geq 1$, the subgroup of n -torsion elements in A will be denoted by ${}_n A$ and, for any prime l , the l -primary torsion subgroup will be denoted by $A\{l\}$.

2. Review of 0-cycle groups and Albanese varieties

To prove the main results of Section 1, our strategy is to use the decomposition theorem for 0-cycles from [Binda and Krishna 2018]. This theorem asserts that the Chow group of 0-cycles with modulus on a smooth scheme is a direct summand of the Chow group of 0-cycles (in the sense of [Levine and Weibel 1985]) on a singular scheme. We therefore need to prove analogues of Theorems 1.1 and 1.2 for singular schemes. In the first few sections of this paper, our goal is to prove such results for arbitrary singular surfaces. These results are of independent interest and answer some questions in the study of 0-cycles on singular schemes. The higher dimensional cases of Theorems 1.1 and 1.2 will be deduced from the case of surfaces.

In this section, we review the definitions and some basic facts about the Chow group of 0-cycles on singular schemes, the Chow group with modulus and their universal quotients, called the Albanese varieties.

2A. 0-cycles on singular schemes. We recall the definition of the cohomological Chow group of 0-cycles for singular schemes from [Binda and Krishna 2018] and [Levine and Weibel 1985]. Let X be a reduced quasiprojective scheme of dimension $d \geq 1$ over k . Let X_{sing} and X_{reg} denote the loci of the singular and the regular points of X . Given a nowhere dense closed subscheme $Y \subset X$ such that $X_{\mathrm{sing}} \subseteq Y$ and no component of X is contained in Y , we let $\mathcal{Z}_0(X, Y)$ denote the free abelian group on the closed points of $X \setminus Y$. We write $\mathcal{Z}_0(X, X_{\mathrm{sing}})$ in short as $\mathcal{Z}_0(X)$.

Definition 2.1. Let C be a pure dimension-one reduced scheme in \mathbf{Sch}_k . We shall say that a pair (C, Z) is a *good curve relative to X* if there exists a finite morphism $\nu : C \rightarrow X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

- (1) No component of C is contained in Z .
- (2) $\nu^{-1}(X_{\mathrm{sing}}) \cup C_{\mathrm{sing}} \subseteq Z$.
- (3) ν is local complete intersection at every point $x \in C$ such that $\nu(x) \in X_{\mathrm{sing}}$.

Let (C, Z) be a good curve relative to X and let $\{\eta_1, \dots, \eta_r\}$ be the set of generic points of C . Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \dots, \eta_r\}$. Let $k(C)$ denote the ring of total quotients of C and write $\mathcal{O}_{C,Z}^\times$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \emptyset$. As C is Cohen–Macaulay, $\mathcal{O}_{C,Z}^\times$ is the subgroup of $k(C)^\times$ consisting of those f which are regular and invertible in the local rings $\mathcal{O}_{C,x}$ for every $x \in Z$.

Given any $f \in \mathcal{O}_{C,Z}^\times \hookrightarrow k(C)^\times$, we denote by $\text{div}_C(f)$ (or $\text{div}(f)$ in short) the divisor of zeros and poles of f on C , which is defined as follows. If C_1, \dots, C_r are the irreducible components of C , and f_i is the factor of f in $k(C_i)$, we set $\text{div}(f)$ to be the 0-cycle $\sum_{i=1}^r \text{div}(f_i)$, where $\text{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [Fulton 1984]. As f is an invertible regular function on C along Z , $\text{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to X , we have a pushforward map $\mathcal{Z}_0(C, Z) \xrightarrow{v_*} \mathcal{Z}_0(X)$. We shall write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X)$ generated by the set $\{v_*(\text{div}(f)) \mid f \in \mathcal{O}_{C,Z}^\times\}$. Let $\mathcal{R}_0(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the image of the map $\mathcal{Z}_0(C, Z, X) \rightarrow \mathcal{Z}_0(X)$, where (C, Z) runs through all good curves relative to X . We let $\text{CH}_0(X) = \mathcal{Z}_0(X)/\mathcal{R}_0(X)$.

If we let $\mathcal{R}_0^{LW}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $v : C \rightarrow X$ is a closed immersion, then the resulting quotient group $\mathcal{Z}_0(X)/\mathcal{R}_0^{LW}(X)$ is denoted by $\text{CH}_0^{LW}(X)$. Such curves on X are called the *Cartier curves*. There is a canonical surjection $\text{CH}_0^{LW}(X) \rightarrow \text{CH}_0(X)$. The Chow group $\text{CH}_0^{LW}(X)$ was discovered by Levine and Weibel [1985] in an attempt to describe the Grothendieck group of a singular scheme in terms of algebraic cycles. The modified version $\text{CH}_0(X)$ was introduced in [Binda and Krishna 2018]. As an application of our main results, we shall be able to identify the two Chow groups for certain singular schemes (see Theorem 6.6).

2B. The Chow group of the double. Let X be a smooth quasiprojective scheme of dimension d over k and let $D \subset X$ be an effective Cartier divisor. Recall from [Binda and Krishna 2018, § 2.1] that the double of X along D is a quasiprojective scheme $S(X, D) = X \amalg_D X$ so that

$$\begin{array}{ccc}
 D & \xrightarrow{\iota} & X \\
 \downarrow \iota & & \downarrow \iota_+ \\
 X & \xrightarrow{\iota_-} & S(X, D)
 \end{array} \tag{2-1}$$

is a co-Cartesian square in \mathbf{Sch}_k . In particular, the identity map of X induces a finite map $\Delta : S(X, D) \rightarrow X$ such that $\Delta \circ \iota_\pm = \text{Id}_X$ and $\pi = \iota_+ \amalg \iota_- : X \amalg X \rightarrow S(X, D)$ is the normalization map. We let $X_\pm = \iota_\pm(X) \subset S(X, D)$ denote the two irreducible components of $S(X, D)$. We shall often write $S(X, D)$ as S_X when the divisor D is understood. S_X is a reduced quasiprojective scheme whose singular locus is $D_{\text{red}} \subset S_X$. It is projective whenever X is so. The following easy algebraic lemma shows that (2-1) is also a Cartesian square.

Lemma 2.2. *Let*

$$\begin{array}{ccc}
 R & \xrightarrow{\psi_1} & A_1 \\
 \psi_2 \downarrow & & \downarrow \phi_1 \\
 A_2 & \xrightarrow{\phi_2} & B
 \end{array} \tag{2-2}$$

be a Cartesian square of commutative Noetherian rings such that $\text{Ker}(\phi_i) = (a_i)$ for $i = 1, 2$. Then $A_1 \otimes_R A_2 \xrightarrow{\sim} B$.

Proof. If we let $\alpha_1 = (0, a_2)$ and $\alpha_2 = (a_1, 0)$ in $R \subset A_1 \times A_2$, then it is easy to check that for $i = 1, 2$, the map ψ_i is surjective and $\text{Ker}(\psi_i) = (\alpha_i)$. This implies that $A_1 \otimes_R A_2 \simeq R/(\alpha_1) \otimes_R A_2 \simeq A_2/(a_2) = B$. \square

We shall later consider a more general situation than (2-1) where we allow the two components of the double to be distinct. This general case is a nonaffine version of (2-2) and is used in the proof of [Theorem 1.2](#).

2C. The Albanese varieties for singular schemes. Let X be a reduced projective scheme over k of dimension d . Let $\text{CH}_0^{LW}(X)_{\text{deg } 0}$ denote the kernel of the degree map $\text{deg} : \text{CH}_0^{LW}(X) \rightarrow H^0(X, \mathbb{Z})$. An Albanese variety $A^d(X)$ of X was constructed in [\[Esnault et al. 1999\]](#). This variety is a connected commutative algebraic group over k with an Abel–Jacobi map $\rho_X : \text{CH}_0^{LW}(X)_{\text{deg } 0} \rightarrow A^d(X)$. Moreover, $A^d(X)$ is the universal object in the category of commutative algebraic groups G over k with a rational map $f : X \dashrightarrow G$ which induces a group homomorphism $f_* : \text{CH}_0^{LW}(X)_{\text{deg } 0} \rightarrow G$. Such a group homomorphism is called a regular homomorphism. For this reason, $A^d(X)$ is also called the *universal regular quotient* of $\text{CH}_0^{LW}(X)_{\text{deg } 0}$. When X is smooth, $A^d(X)$ coincides with the classical Albanese variety $\text{Alb}(X)$. One says that $\text{CH}_0^{LW}(X)_{\text{deg } 0}$ is *finite-dimensional*, if ρ_X is an isomorphism.

2D. The universal semiabelian quotient of $\text{CH}_0^{LW}(X)_{\text{deg } 0}$. In positive characteristic, apart from the theorem about its existence, not much is known about $A^d(X)$ and almost everything that one would like to know about it is an open question. However, the universal semiabelian quotient of the algebraic group $A^d(X)$ can be described more explicitly and this is all one needs to know in order to understand the prime-to- p torsion in $A^d(X)$. The following description of this quotient is recalled from [\[Mallick 2009, §2\]](#).

Let $\pi : X^N \rightarrow X$ denote the normalization map. Let $\text{Cl}(X^N)$ and $\text{Pic}_W(X^N)$ denote the class group and the Weil Picard group of X^N . Recall from [\[Weil 1954\]](#) that $\text{Pic}_W(X^N)$ is the subgroup of $\text{Cl}(X^N)$ consisting of Weil divisors which are algebraically equivalent to zero in the sense of [\[Fulton 1984, Chapter 19\]](#). Let $\text{Div}(X)$ denote the free abelian group of Weil divisors on X . Let $\Lambda_1(X)$ denote the subgroup of $\text{Div}(X^N)$ generated by the Weil divisors which are supported on $\pi^{-1}(X_{\text{sing}})$. This gives us a map $\iota_X : \Lambda_1(X) \rightarrow \text{Cl}(X^N)/\text{Pic}_W(X^N)$.

Let $\Lambda(X)$ denote the kernel of the canonical map

$$\Lambda_1(X) \xrightarrow{(\iota_X, \pi_*)} \frac{\text{Cl}(X^N)}{\text{Pic}_W(X^N)} \oplus \text{Div}(X). \tag{2-3}$$

It follows from [Mallick 2009, § 4] that the universal semiabelian quotient of $A^d(X)$ is the Cartier dual of the 1-motive $[\Lambda(X) \rightarrow \text{Pic}_W(X^N)]$ and is denoted by $J^d(X)$. The composite homomorphism $\rho_X^{\text{semi}} : \text{CH}_0^{LW}(X)_{\text{deg } 0} \rightarrow A^d(X) \rightarrow J^d(X)$ is universal among regular homomorphisms from $\text{CH}_0^{LW}(X)_{\text{deg } 0}$ to semiabelian varieties. $J^d(X)$ is called the *semiabelian Albanese variety* of X . By [Binda and Krishna 2018, Proposition 9.7], there is a factorization of ρ_X^{semi} :

$$\text{CH}_0^{LW}(X)_{\text{deg } 0} \rightarrow \text{CH}_0(X)_{\text{deg } 0} \xrightarrow{\tilde{\rho}_X^{\text{semi}}} J^d(X) \tag{2-4}$$

and $J^d(X)$ is in fact a universal regular semiabelian variety quotient of $\text{CH}_0(X)_{\text{deg } 0}$.

There is a short exact sequence of commutative algebraic groups

$$0 \rightarrow A_{\text{unip}}^d(X) \rightarrow A^d(X) \rightarrow J^d(X) \rightarrow 0, \tag{2-5}$$

where $A_{\text{unip}}^d(X)$ is the unipotent radical of $A^d(X)$. Since k is algebraically closed, $A_{\text{unip}}^d(X)$ is uniquely n -divisible for any integer $n \in k^\times$. In particular,

$${}_n A^d(X) \xrightarrow{\sim} {}_n J^d(X). \tag{2-6}$$

2E. A Lefschetz hyperplane theorem. Recall that for a smooth projective scheme $X \hookrightarrow \mathbb{P}_k^N$ of dimension $d \geq 3$, a general hypersurface section $Y \subset X$ has the property that the canonical map $\text{Alb}(Y) \rightarrow \text{Alb}(X)$ is an isomorphism. We now wish to prove a similar result for the semiabelian Albanese variety $J^d(S_X)$ of the double.

Let X be a connected smooth projective scheme of dimension $d \geq 3$ over k and let $D \subset X$ be an effective Cartier divisor. Let $\{E_1, \dots, E_r\}$ be the set of irreducible components of D_{red} so that each E_i is integral. For each $1 \leq j \leq r$, let $P_j \in E_j \setminus (\bigcup_{j' \neq j} E_{j'})$ be a chosen closed point, and let $S = \{P_1, \dots, P_r\}$. We can apply [Kleiman and Altman 1979, Theorems 1, 7] to find a closed embedding $X \hookrightarrow \mathbb{P}_k^N$ (for $N \gg 0$) such that for a general set of distinct hypersurfaces $H_1, \dots, H_{d-2} \hookrightarrow \mathbb{P}_k^N$, we have the following.

- (1) The successive intersection $X \cap H_1 \cap \dots \cap H_i$ ($1 \leq i \leq d - 2$) is integral and smooth over k .
- (2) No component of $X \cap H_1 \cap \dots \cap H_{d-2}$ is contained in D .
- (3) $S \subset X \cap H_1 \cap \dots \cap H_{d-2}$.
- (4) For each $1 \leq j \leq r$, the successive intersection $E_j \cap H_1 \cap \dots \cap H_i$ ($1 \leq i \leq d - 2$) is integral.

Setting $X_i = X \cap H_1 \cap \dots \cap H_i$ and $D_i = X_i \cap D$, our choice of the hypersurfaces implies that $D_i \subset X_i$ is an effective Cartier divisor. We let $S_i = X_i \amalg_{D_i} X_i$ for $1 \leq i \leq d - 2$. It follows from [Binda and Krishna 2018, Proposition 2.4] that each inclusion $\tau_i : S_i \hookrightarrow S_{i-1}$ is a local complete intersection (l.c.i.) and $S_i = S_{i-1} \times_{X_{i-1}} X_i$ for $1 \leq i \leq d - 2$, where we let $S_0 = S_X$. Since $S_i \hookrightarrow S_{i-1}$ is an l.c.i. inclusion with $(S_i)_{\text{sing}} = (D_i)_{\text{red}} = (S_i \cap D)_{\text{red}} = S_i \cap (S_{i-1})_{\text{sing}}$, it follows that there are compatible pushforward maps $\tau_{i,*}^{LW} : \text{CH}_0^{LW}(S_i) \rightarrow \text{CH}_0^{LW}(S_{i-1})$ (see [Esnault et al. 1999, Lemma 1.8]) and $\tau_{i,*} : \text{CH}_0(S_i) \rightarrow \text{CH}_0(S_{i-1})$ (see [Binda and Krishna 2018, Proposition 3.17]). With the above setup, we have the following.

Proposition 2.3. *For $1 \leq i \leq d - 2$, there is an isomorphism of algebraic groups $\phi_i : J^{d-i}(S_i) \rightarrow J^{d-i+1}(S_{i-1})$ and a commutative diagram*

$$\begin{array}{ccc}
 \mathrm{CH}_0(S_i)_{\mathrm{deg} 0} & \xrightarrow{\tau_{i,*}} & \mathrm{CH}_0(S_{i-1})_{\mathrm{deg} 0} \\
 \rho_i^{\mathrm{semi}} \downarrow & & \downarrow \rho_{i-1}^{\mathrm{semi}} \\
 J^{d-i}(S_i) & \xrightarrow{\phi_i} & J^{d-i+1}(S_{i-1}),
 \end{array} \tag{2-7}$$

whose restriction to the l -primary torsion subgroups induces isomorphism of all arrows, for every prime $l \neq p$.

Proof. Before we begin the proof, we note that when $S_X \cap H \hookrightarrow S_X$ is a general hypersurface section in a projective embedding of S_X , then an analogue of the proposition was proven by Mallick [2009, §5]. But we can not apply his result directly because $S_Y \hookrightarrow S_X$ is not a hypersurface section. Nevertheless, we shall closely follow Mallick’s construction in the proof of the proposition. Also, we shall prove a stronger assertion than in Theorem 14 there because of the special nature of the double.

We have seen above that the inclusion $S_i \hookrightarrow S_{i-1}$ is l.c.i. which preserves the singular loci, and hence there is a pushforward $\tau_{i,*} : \mathrm{CH}_0(S_i) \rightarrow \mathrm{CH}_0(S_{i-1})$. We now construct the map ϕ_i . We construct this map for $i = 1$ as the method in the general case is completely identical. With this reduction in mind, we shall let $Y = X_1 = X \cap H_1$, $F = D_1 = D \cap H_1$ so that $S_1 = S_Y = S(Y, F)$.

Since $Y \hookrightarrow X$ is a general hypersurface section with $S_Y^N = Y \sqcup Y$ and $S_X^N = X \sqcup X$, one knows that the map $\mathrm{Pic}_W(S_Y^N)^\vee = \mathrm{Alb}(Y) \times \mathrm{Alb}(Y) \rightarrow \mathrm{Alb}(X) \times \mathrm{Alb}(X) = \mathrm{Pic}_W(S_X^N)^\vee$ is an isomorphism. We thus have to construct a pullback map $\Lambda(S_X) \rightarrow \Lambda(S_Y)$ which is an isomorphism.

We consider the commutative square

$$\begin{array}{ccc}
 Y \sqcup Y & \xrightarrow{\psi} & X \sqcup X \\
 \pi_1 \downarrow & & \downarrow \pi \\
 S_Y & \xrightarrow{\tau_1} & S_X
 \end{array} \tag{2-8}$$

where the vertical arrows are the normalization maps and ψ is the disjoint sum of the inclusion of the hypersurface section $Y = X \cap H_1$. We claim that this square is Cartesian. To see this, note that the vertical arrows are disjoint sums of two maps. Hence, it is enough to show that (2-8) is Cartesian when we replace $X \sqcup X$ by X_+ and $Y \sqcup Y$ by Y_+ . In this case, the vertical arrows are closed immersions and we know directly from the construction that $S_Y \times_{S_X} X_+ \simeq (Y \times_X S_X) \times_{S_X} X_+ \simeq Y_+$.

If we let $F_i := E_i \cap H_1$ for $1 \leq j \leq r$, it follows from the conditions (2) and (3) on page 1437 that each F_i is integral and $F_i \neq F_j$ if $E_i \neq E_j$. Now, the refined Gysin homomorphism

$$\psi^! : \Lambda_1(S_X) = \mathrm{CH}_{d-1}(D \sqcup D) \rightarrow \mathrm{CH}_{d-2}(F \sqcup F) = \Lambda_1(S_Y)$$

(in the sense of [Fulton 1984]) takes E_i to F_i for $i = 1, \dots, r$ in each copy of D . This map is bijective by our choice of the hypersurface sections Y and F_i ’s.

Now, for any $1 \leq j \leq r$, we have $\psi^!([E_j]) = \alpha_j[F_j]$ in $\Lambda_1(S_Y) = \text{CH}_{d-2}(F \amalg F)$, where α_j is the intersection multiplicity of $E_j \times_{\mathbb{P}_k^N} H_1$ in $D \cdot H_1$. Since the composite morphism $E_j \hookrightarrow X \hookrightarrow \mathbb{P}_k^N$ is a closed immersion (and hence separable), this intersection multiplicity must be one so that we have $\alpha_j = 1$ for each j . It follows that $\psi^! : \Lambda_1(S_X) \rightarrow \Lambda_1(S_Y)$ is an isomorphism of abelian groups.

We next show that $\psi^!$ takes $\Lambda(S_X)$ onto $\Lambda(S_Y)$. This follows because for any $\Delta \in \Lambda(S_X)$, by [Fulton 1984, Theorem 6.2] we have $\pi_{1,*} \circ \psi^!(\Delta) = \tau_1^! \circ \pi_*(\Delta)$ and $\pi_*(\Delta) = 0$ in $\text{CH}_{d-1}((S_X)_{\text{sing}}) = \text{Z}_{d-1}((S_X)_{\text{sing}})$. This means $\psi^!(\Delta) \in \Lambda(S_Y)$. If we let $\Delta = \sum_l \alpha_l[F_l] \in \Lambda(S_Y)$, then letting $\Delta' = \sum_l \alpha_l[E_l] \in \Lambda_1(S_X)$, we get $\psi^!(\Delta') = \Delta$. We also have

$$\tau_1^! \circ \pi_*(\Delta') = \pi_{1,*} \circ \psi^!(\Delta') = \pi_{1,*}(\Delta) = 0.$$

Since $\tau_1^! : \text{CH}_{d-1}((S_X)_{\text{sing}}) = \text{CH}_{d-1}(D) \rightarrow \text{CH}_{d-2}(F) = \text{CH}_{d-2}((S_Y)_{\text{sing}})$ is bijective, it follows that $\pi_*(\Delta') = 0$. It follows that $\psi^! : \Lambda(S_X) \rightarrow \Lambda(S_Y)$ is surjective and hence an isomorphism. We have thus constructed an isomorphism $\phi_1 : J^{d-1}(S_Y) \xrightarrow{\sim} J^d(S_X)$.

To show the commutative diagram (2-7), we can again assume $i = 1$. We can also replace $\text{CH}_0(S_X)$ and $\text{CH}_0(S_Y)$ by $\mathcal{Z}_0(S_X)$ and $\mathcal{Z}_0(S_Y)$, respectively.

We now need to observe that $J^d(S_X)$ is a quotient of the Cartier dual $J_{\text{Serre}}^d(S_X)$ of the 1-motive $[\Lambda_1(S_X) \rightarrow \text{Pic}_W(X \amalg X)]$ and this dual semiabelian variety is the universal object in the category of morphisms from $(S_X)_{\text{reg}} = (X \setminus D) \amalg (X \setminus D)$ to semiabelian varieties (see [Serre 1958-1959]). Since $(S_Y)_{\text{reg}} = (Y \setminus D) \amalg (Y \setminus D) = \tau_1^{-1}((S_X)_{\text{reg}})$, it follows from this universality of $J_{\text{Serre}}^d(S_X)$ that there is a commutative diagram as in (2-7) if we replace $J^d(S_X)$ and $J^{d-1}(S_Y)$ by $J_{\text{Serre}}^d(S_X)$ and $J_{\text{Serre}}^d(S_Y)$, respectively. The commutative diagram

$$\begin{CD} J_{\text{Serre}}^d(S_Y) @>\phi_{1,*}>> J_{\text{Serre}}^d(S_X) \\ @VVV @VVV \\ J^d(S_Y) @>\phi_{1,*}>> J^d(S_X) \end{CD}$$

now finishes the proof of the commutativity of (2-7). The final assertion about the isomorphism between the l -primary torsion subgroups follows from [Binda and Krishna 2018, Theorem 9.8] and [Mallick 2009, Theorem 16]. □

2F. 0-cycles and Albanese variety with modulus. Given an integral normal curve C over k and an effective divisor $E \subset C$, we say that a rational function f on C has modulus E if $f \in \text{Ker}(\mathcal{O}_{C,E}^\times \rightarrow \mathcal{O}_E^\times)$. Here, $\mathcal{O}_{C,E}$ is the semilocal ring of C at the union of E and the generic point of C . In particular, $\text{Ker}(\mathcal{O}_{C,E}^\times \rightarrow \mathcal{O}_E^\times)$ is just $k(C)^\times$ if $|E| = \emptyset$. Let $G(C, E)$ denote the group of such rational functions.

Let X be a smooth quasiprojective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{Z}_0(X|D)$ be the free abelian group on the set of closed points of $X \setminus D$. Let C be an integral normal curve over k and let $\varphi_C : C \rightarrow X$ be a finite morphism such that $\varphi_C(C) \not\subset D$. The pushforward of cycles along φ_C gives a well-defined group homomorphism $\tau_C : G(C, \varphi_C^*(D)) \rightarrow \mathcal{Z}_0(X|D)$.

Definition 2.4. We define the Chow group $\text{CH}_0(X|D)$ of 0-cycles of X with modulus D to be the cokernel of the homomorphism $\text{div} : \bigoplus_{\varphi_C : C \rightarrow X} G(C, \varphi_C^*(D)) \rightarrow Z_0(X|D)$, where the sum is taken over the set of finite morphisms $\varphi_C : C \rightarrow X$ from an integral normal curve such that $\varphi_C(C) \not\subset D$.

It is clear from the moving lemma of Bloch that there is a canonical surjection $\text{CH}_0(X|D) \twoheadrightarrow \text{CH}_0(X)$. If X is projective, we denote the kernel of the composite map $\text{CH}_0(X|D) \twoheadrightarrow \text{CH}_0(X) \xrightarrow{\text{deg}} H^0(X, \mathbb{Z})$ by $\text{CH}_0(X|D)_{\text{deg } 0}$.

If D is reduced, it follows using [Theorem 6.6](#) and the constructions of [\[Binda and Krishna 2018, §11\]](#) that there exists an Albanese variety with modulus $A^d(X|D)$ together with a surjective Abel–Jacobi map $\rho_{X|D} : \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow A^d(X|D)$ which is a universal regular homomorphism in a suitable sense. One says that $\text{CH}_0(X|D)_{\text{deg } 0}$ is *finite-dimensional*, if $\rho_{X|D}$ is an isomorphism.

3. Torsion in the Chow group of a singular surface

Let X be a reduced projective surface over k . When X is smooth, Suslin [\[1987, p. 19\]](#) showed that there is a short exact sequence

$$0 \rightarrow H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Z}/n \rightarrow H_{\acute{e}t}^3(X, \mu_n(2)) \rightarrow {}_n\text{CH}_0(X) \rightarrow 0 \tag{3-1}$$

for any integer $n \in k^\times$. When X has only isolated singularities, this exact sequence was conjectured by Pedrini and Weibel [\[1994, \(0.4\)\]](#) and proven by Barbieri-Viale, Pedrini and Weibel in [\[Barbieri-Viale et al. 1996, Theorem 7.7\]](#). Expanding on their ideas, our goal in this section is to prove such an exact sequence for surfaces with arbitrary singularities. This exact sequence will be used to prove an analogue of [Theorem 1.1](#) for singular surfaces.

3A. Gillet’s Chern classes. Let $\mathcal{K} = \Omega BQP$ denote the simplicial presheaf on \mathbf{Sch}_k such that $\mathcal{K}(X) = \Omega BQP(X)$, where $BQP(X)$ is the Quillen’s construction of the simplicial set associated to the exact category $P(X)$ of locally free sheaves on X . For any $i \geq 0$, let \mathcal{K}_i denote the Zariski sheaf on \mathbf{Sch}_k associated to the presheaf $X \mapsto K_i(X) = \pi_i(\mathcal{K}(X))$.

We fix integers $d, i \geq 0$ and an integer $n \geq 1$ which is prime to the exponential characteristic p of the base field k . Let $\mu_n(d)$ denote the sheaf $\mu_n^{\otimes d}$ on $\mathbf{Sch}_k/\acute{e}t$ such that $\mu_n(X) = \text{Ker}(\mathcal{O}^\times(X) \xrightarrow{n} \mathcal{O}^\times(X))$. Let \mathcal{E}_i denote the simplicial sheaf on $\mathbf{Sch}_k/\text{zar}$ associated, by the Dold–Kan correspondence, to the good truncation $\tau^{\leq 0} \mathbf{R}w_*\mu_n(i)[2i]$ of the chain complex of Zariski sheaves $\mathbf{R}w_*\mu_n(i)[2i]$. In particular, we have $\pi_q \mathcal{E}_i(X) = H_{\acute{e}t}^{2i-q}(X, \mu_n(i))$ for $X \in \mathbf{Sch}_k$ and $0 \leq q \leq 2i$.

Let \mathcal{L} denote the homotopy fiber of the map of simplicial presheaves $\mathcal{K} \xrightarrow{n} \mathcal{K}$ so that $\pi_q \mathcal{L}(X) = K_{q+1}(X, \mathbb{Z}/n)$ for $q \geq -1$. Let $\mathcal{H}^i(\mu_n(d))$ denote the Zariski sheaf $\mathbf{R}^i w_*\mu_n(d)$ on $\mathbf{Sch}_k/\text{zar}$. It follows from Gillet’s construction of the Chern classes (see [\[Gillet 1981, §5\]](#)) that there is a morphism of the simplicial presheaves on \mathbf{Sch}_k :

$$c_i : \mathcal{K} \rightarrow \mathcal{E}_i \tag{3-2}$$

in the homotopy category of simplicial presheaves. This map is compatible with the local to global spectral sequences for the associated Zariski sheaves so that we get a map of spectral sequences which at the E_2 level is $H^p(X, \mathcal{K}_q) \rightarrow H^p(X, \mathcal{H}^{2i-q}(\mu_n(i)))$ and converges to the global Chern class on \mathcal{E}_i given by $c_{i,X} : K_{q-p}(X) \rightarrow H_{\acute{e}t}^{2i-q+p}(X, \mu_n(i))$. These Chern classes in fact factor through the maps $\mathcal{K} \rightarrow \Sigma\mathcal{L} \rightarrow \mathcal{E}_i$ (see [Pedrini and Weibel 1994, §2]) so that there are Chern class maps $c_i : \mathcal{L} \rightarrow \Omega\mathcal{E}_i$ in the homotopy category of simplicial presheaves [Brown and Gersten 1973].

Let $\tilde{\mathcal{K}}$ denote the simplicial presheaf on \mathbf{Sch}_k such that $\tilde{\mathcal{K}}(X)$ is the universal covering space of $\mathcal{K}(X)$ and let $\tilde{\mathcal{L}}$ denote the homotopy fiber of the map $\tilde{\mathcal{K}} \xrightarrow{n} \tilde{\mathcal{K}}$. This yields $\pi_q(\tilde{\mathcal{K}}(X)) = K_q(X)$ for $q \geq 2$ and $\pi_1(\tilde{\mathcal{K}}(X)) = 0$. Furthermore, we have

$$\pi_q(\tilde{\mathcal{L}}(X)) = \begin{cases} K_{q+1}(X, \mathbb{Z}/n) & \text{if } q \geq 2, \\ K_2(X)/n & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3-3}$$

Let $\tilde{\mathcal{K}}^{(2)}$ denote the second layer of the Postnikov tower

$$(\dots \rightarrow \tilde{\mathcal{K}}^{(n)} \rightarrow \tilde{\mathcal{K}}^{(n-1)} \rightarrow \dots \rightarrow \tilde{\mathcal{K}}^{(2)} \rightarrow \tilde{\mathcal{K}}^{(1)} \rightarrow \tilde{\mathcal{K}}^{(0)} = \star)$$

of $\tilde{\mathcal{K}}$. There are compatible family of maps $f_n : \tilde{\mathcal{K}} \rightarrow \tilde{\mathcal{K}}^{(n)}$ inducing $\pi_q \tilde{\mathcal{K}} \xrightarrow{\sim} \pi_q \tilde{\mathcal{K}}^{(n)}$ for $q \leq n$ and $\pi_q \tilde{\mathcal{K}}^{(n)} = 0$ for $q > n$. Let $\tilde{\mathcal{L}}^{(2)}$ denote the homotopy fiber of the map $\tilde{\mathcal{K}}^{(2)} \xrightarrow{n} \tilde{\mathcal{K}}^{(2)}$ so that there is a commutative diagram of simplicial presheaves

$$\begin{array}{ccccc} \tilde{\mathcal{L}}^{(2)} & \longrightarrow & \tilde{\mathcal{K}}^{(2)} & \xrightarrow{n} & \tilde{\mathcal{K}}^{(2)} \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{\mathcal{L}} & \longrightarrow & \tilde{\mathcal{K}} & \xrightarrow{n} & \tilde{\mathcal{K}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L} & \longrightarrow & \mathcal{K} & \xrightarrow{n} & \mathcal{K}. \end{array} \tag{3-4}$$

Lemma 3.1. *Let \mathcal{K}_2^\star denote the complex of presheaves $(\mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2)$ on $\mathbf{Sch}_k/\text{zar}$. Then $\tilde{\mathcal{L}}^{(2)}$ is weak equivalent to the simplicial presheaf obtained by applying the Dold–Kan correspondence to the chain complex $\mathcal{K}_2^\star[2]$.*

Proof. Using (3-4), it suffices to show that $\tilde{\mathcal{K}}^{(2)}$ is an Eilenberg–Mac Lane complex of the type $(\mathcal{K}_2, 2)$. Let F^2 denote the homotopy fiber of the Kan fibration $\tilde{\mathcal{K}}^{(2)} \rightarrow \tilde{\mathcal{K}}^{(1)}$ in the Postnikov tower. We have $\pi_i \tilde{\mathcal{K}}^{(1)} = 0$ for $i \geq 2$ and $\pi_1 \tilde{\mathcal{K}}^{(1)} = \pi_1 \tilde{\mathcal{K}} = 0$ by [May 1967, Theorem 8.4]. The long exact homotopy sequence now implies that $\pi_i F^2 \xrightarrow{\sim} \pi_i \tilde{\mathcal{K}}^{(2)}$ for all i and hence $F^2 \rightarrow \tilde{\mathcal{K}}^{(2)}$ is a weak equivalence by the Whitehead theorem. Since F^2 is an Eilenberg–Mac Lane complex of type $(\mathcal{K}_2, 2)$ by Corollary 8.7 of the same work, we are done. \square

3B. Cohomology of \mathcal{K}_2 on a surface. Let us now assume that X is a reduced quasiprojective surface over k . Let $\mathcal{K}_i(\mathbb{Z}/n)$ denote the Zariski sheaf on \mathbf{Sch}_k associated to the presheaf $X \mapsto K_i(X, \mathbb{Z}/n)$.

Applying the Brown–Gersten spectral sequence (see [Brown and Gersten 1973, Theorem 3])

$$H^p(X, \pi_{-q}\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

to (3-2) and (3-3), we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 H^0(X, \mathcal{K}_2/n) & \xrightarrow{d} & H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) & \longrightarrow & H^0(X, \tilde{\mathcal{L}}) & \longrightarrow & H^1(X, \mathcal{K}_2/n) & \longrightarrow & 0 \\
 \downarrow c_2 \simeq & & \downarrow c_2 & & \downarrow c_2 & & \simeq \downarrow c_2 & & \\
 H^0(X, \mathcal{H}^2(\mu_n(2))) & \xrightarrow{d} & H^2(X, \mathcal{H}^1(\mu_n(2))) & \longrightarrow & H_{\acute{e}t}^3(X, \mu_n(2)) & \longrightarrow & H^1(X, \mathcal{H}^2(\mu_n(2))) & \longrightarrow & 0
 \end{array} \tag{3-5}$$

where the vertical arrows are induced by the Chern class map $c_{2,X}$. The bottom row is exact because $\mathcal{H}^3(\mu_n(2)) = 0$ by [Milne 1980, Theorem VI.7.2]. The map $H^0(X, \tilde{\mathcal{L}}) \xrightarrow{c_2} H_{\acute{e}t}^3(X, \mu_n(2))$ is the one induced by the maps $H^0(X, \tilde{\mathcal{L}}) \xrightarrow{c_2} H^0(X, \Omega\mathcal{E}_2) \simeq H^{-1}(X, \mathcal{E}_2) = \pi_1\mathcal{E}_2(X) = H_{\acute{e}t}^3(X, \mu_n(2))$. The leftmost and the rightmost vertical arrows are isomorphisms by Hoobler’s theorem [2006, Theorem 3].

Lemma 3.2. *There is a functorial map $H^0(X, \tilde{\mathcal{L}}^{(2)}) \rightarrow \mathbb{H}^2(X, \mathcal{K}_2^\bullet)$ and a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
 H^0(X, \pi_1\tilde{\mathcal{L}}^{(2)}) & \xrightarrow{d} & H^2(X, \pi_2\tilde{\mathcal{L}}^{(2)}) & \longrightarrow & H^0(X, \tilde{\mathcal{L}}^{(2)}) & \longrightarrow & H^1(X, \pi_1\tilde{\mathcal{L}}^{(2)}) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 H^0(X, \mathcal{K}_2/n) & \xrightarrow{d} & H^2(X, {}_n\mathcal{K}_2) & \longrightarrow & \mathbb{H}^2(X, \mathcal{K}_2^\bullet) & \longrightarrow & H^1(X, \mathcal{K}_2/n) & \longrightarrow & 0
 \end{array} \tag{3-6}$$

such that all vertical arrows are isomorphisms.

Proof. The bottom exact sequence follows from the exact triangle

$${}_n\mathcal{K}_2 \rightarrow \mathcal{K}_2^\bullet \rightarrow \mathcal{K}_2/n[-1] \rightarrow {}_n\mathcal{K}_2[1] \tag{3-7}$$

in the derived category of Zariski sheaves of abelian groups on $\mathbf{Sch}_k/\text{zar}$. On the other hand, Lemma 3.1 says that $\tilde{\mathcal{K}}^{(2)}$ is an Eilenberg–Mac Lane complex of the type $(\mathcal{K}_2, 2)$ and there is a homotopy equivalence $\tilde{\mathcal{L}}^{(2)} \rightarrow \mathcal{K}_2^\bullet[2]$. This in particular implies that $\pi_1\tilde{\mathcal{L}}^{(2)} \simeq \mathcal{K}_2/n$, $\pi_2\tilde{\mathcal{L}}^{(2)} \simeq {}_n\mathcal{K}_2$ and $\pi_i\tilde{\mathcal{L}}^{(2)} = 0$ for $i \neq 1, 2$. Applying the Brown–Gersten spectral sequence to $H^0(X, \tilde{\mathcal{L}}^{(2)})$ and using Lemma 3.1, we conclude the proof. The commutativity follows because both rows are obtained by applying the hypercohomology spectral sequence to the map $\tilde{\mathcal{L}}^{(2)} \rightarrow \mathcal{K}_2^\bullet[2]$. □

The key step in extending (3-1) to arbitrary surfaces is the following result.

Lemma 3.3. *The map $H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \rightarrow H^2(X, {}_n\mathcal{K}_2)$ induced by the universal coefficient theorem has a natural factorization*

$$H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) \xrightarrow{c_2} H^2(X, \mathcal{H}^1(\mu_n(2))) \xrightarrow{v} H^2(X, {}_n\mathcal{K}_2)$$

such that the map v is an isomorphism.

Proof. By [Barbieri-Viale et al. 1996, Lemma 6.2, Variant 6.3], there is a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{H}^1(\mu_n(2)) & \xrightarrow{\phi} & \mathcal{K}_3(\mathbb{Z}/n) & \xrightarrow{c_2} & \mathcal{H}^1(\mu_n(2)) \\
 & \searrow & \downarrow \psi & \swarrow & \\
 & & {}_n\mathcal{K}_2 & &
 \end{array} \tag{3-8}$$

such that $c_2 \circ \phi = -1$ and the composite $\psi \circ \phi$ is given by $a \mapsto \{a, \zeta\} \in {}_n\mathcal{K}_2$ (where $\zeta \in k^\times$ is a primitive n -th root of unity). This composite map is surjective and an isomorphism on the regular locus of X by [Barbieri-Viale et al. 1996, Lemma 6.2]. It follows that the induced map $\psi \circ \phi : H^2(X, \mathcal{H}^1(\mu_n(2))) \rightarrow H^2(X, {}_n\mathcal{K}_2)$ is an isomorphism (see [Pedrini and Weibel 1994, Lemma 1.3]). We thus have a diagram

$$\begin{array}{ccccc}
 H^2(X, \mathcal{H}^1(\mu_n(2))) & \xrightarrow{\phi} & H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) & \xrightarrow{c_2} & H^2(X, \mathcal{H}^1(\mu_n(2))) \\
 & \searrow \cong & \downarrow \psi & \swarrow v & \\
 & & H^2(X, {}_n\mathcal{K}_2) & &
 \end{array} \tag{3-9}$$

in which the triangle on the left is commutative. To prove the lemma, it is therefore sufficient to show that $\text{Ker}(c_2) = \text{Ker}(\psi)$. Equivalently, $\text{Ker}(c_2) \subseteq \text{Ker}(\psi)$.

We set $\mathcal{F} = \text{Ker}(c_2)$ so that we have a split exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{K}_3(\mathbb{Z}/n) \xrightarrow{c_2} \mathcal{H}^1(\mu_n(2)) \rightarrow 0. \tag{3-10}$$

By [Pedrini and Weibel 1994, Lemma 1.3], it suffices to show that the composite map $\mathcal{F} \rightarrow \mathcal{K}_3(\mathbb{Z}/n) \xrightarrow{\psi} {}_n\mathcal{K}_2$ of Zariski sheaves is zero on the smooth locus of X . Since this is a local question, it suffices to show that for a regular local ring A which is essentially of finite type over k , the map $\text{Ker}(c_2) \rightarrow {}_n\mathcal{K}_2(A)$ is zero. By the Gersten resolution of $K_2(A)$, Bloch–Ogus resolution for $H_{\acute{e}t}^1(A, \mu_n(2))$ and Gillet’s resolution for $K_3(A, \mathbb{Z}/n)$ (see [Quillen 1973; Bloch and Ogus 1974; Gillet 1986]), we can replace A by its fraction field F .

We now have a commutative diagram

$$\begin{array}{ccccccc}
 K_3^M(F) & \longrightarrow & K_3(F, \mathbb{Z}/n) & \xrightarrow{c_2} & H_{\acute{e}t}^1(F, \mu_n(2)) & \longrightarrow & 0 \\
 \downarrow & & \parallel & & \downarrow & & \\
 K_3(F) & \longrightarrow & K_3(F, \mathbb{Z}/n) & \xrightarrow{\psi} & {}_n\mathcal{K}_2(F) & \longrightarrow & 0.
 \end{array} \tag{3-11}$$

One of the main results of [Levine 1989] (and also [Merkur’ev and Suslin 1990]) shows that the top row in (3-11) is exact. Since the bottom row is clearly exact, we get the desired conclusion. \square

Our first main result of this section is the following extension of (3-1) to surfaces with arbitrary singularities.

Theorem 3.4. *Let X be a reduced quasiprojective surface over k and let $n \geq 1$ be an integer prime to p . Then, there is a short exact sequence*

$$0 \rightarrow H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Z}/n \rightarrow H_{\text{ét}}^3(X, \mu_n(2)) \rightarrow {}_n\text{CH}_0(X) \rightarrow 0.$$

Proof. It follows from Lemma 3.2 and the map between the Brown–Gersten spectral sequences associated to the morphism of simplicial presheaves $\tilde{\mathcal{L}} \rightarrow \tilde{\mathcal{L}}^{(2)}$ that there is a commutative diagram

$$\begin{CD} H^0(X, \mathcal{K}_2/n) @>d>> H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) @>>> H^0(X, \tilde{\mathcal{L}}) @>>> H^1(X, \mathcal{K}_2/n) @>>> 0 \\ @V \simeq VV @VV \psi V @VV V @VV \simeq V \\ H^0(X, \mathcal{K}_2/n) @>d>> H^2(X, {}_n\mathcal{K}_2) @>>> H^0(X, \tilde{\mathcal{L}}^{(2)}) @>>> H^1(X, \mathcal{K}_2/n) @>>> 0. \end{CD} \tag{3-12}$$

Combining this with (3-5) and Lemma 3.2, we obtain a commutative diagram

$$\begin{CD} H^0(X, \mathcal{K}_2/n) @>d>> H^2(X, {}_n\mathcal{K}_2) @>>> \mathbb{H}^2(X, \mathcal{K}_2^\bullet) @>>> H^1(X, \mathcal{K}_2/n) @>>> 0 \\ @V \simeq VV @VV \psi V @VV V @VV \simeq V \\ H^0(X, \mathcal{K}_2/n) @>d>> H^2(X, \mathcal{K}_3(\mathbb{Z}/n)) @>>> H^0(X, \tilde{\mathcal{L}}) @>>> H^1(X, \mathcal{K}_2/n) @>>> 0 \\ @V \simeq c_2 VV @VV c_2 V @VV c_2 V @VV c_2 \simeq V \\ H^0(X, \mathcal{H}^2(\mu_n(2))) @>d>> H^2(X, \mathcal{H}^1(\mu_n(2))) @>>> H_{\text{ét}}^3(X, \mu_n(2)) @>>> H^1(X, \mathcal{H}^2(\mu_n(2))) @>>> 0. \end{CD} \tag{3-13}$$

It follows from Lemma 3.3 that ψ and the corresponding vertical arrow downward are surjective with identical kernels. A simple diagram chase shows that the maps $H^0(X, \tilde{\mathcal{L}}) \rightarrow \mathbb{H}^2(X, \mathcal{K}_2^\bullet)$ and $H^0(X, \tilde{\mathcal{L}}) \rightarrow H_{\text{ét}}^3(X, \mu_n(2))$ are surjective with identical kernels. In particular, there is a natural isomorphism $c_{2,X} : \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\simeq} H_{\text{ét}}^3(X, \mu_n(2))$. Using the exact sequence

$$0 \rightarrow H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Z}/n \rightarrow \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \rightarrow {}_nH^2(X, \mathcal{K}_2) \rightarrow 0$$

and the isomorphisms $H^2(X, \mathcal{K}_2) \xrightarrow{\simeq} \text{CH}_0^{LW}(X) \xrightarrow{\simeq} \text{CH}_0(X)$ (see [Levine 1985, Theorem 7] and [Binda and Krishna 2018, Theorem 3.17]), we now conclude the proof. □

3C. Theorem 1.1 for singular surfaces. As an application of Theorem 3.4, we now prove a version of Theorem 1.1 for singular surfaces. This result was proven for normal projective surfaces in [Barbieri-Viale et al. 1996, Theorem 7.9]. Let $l \neq p$ be a prime number. Let X be a reduced projective surface over k . We shall make no distinction between $\text{CH}_0^{LW}(X)$ and $\text{CH}_0(X)$ in view of [Binda and Krishna 2018, Theorem 3.17].

Lemma 3.5. *$H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ is divisible by l^n for every $n \geq 1$.*

Proof. The exact sequence of Theorem 3.4 is compatible with the maps $\mathbb{Z}/l^n \rightarrow \mathbb{Z}/l^{n+1}$. Taking the direct limit, we obtain a short sequence

$$0 \rightarrow H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \rightarrow H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \xrightarrow{\tau_X} \text{CH}_0(X)\{l\} \rightarrow 0. \tag{3-14}$$

The group on the left is divisible. It is known that $\text{CH}_0(X)_{\text{deg } 0}$ is generated by the images of the maps $\text{Pic}^0(C) \rightarrow \text{CH}_0(X)$, where $C \subsetneq X$ is a reduced Cartier curve. Since $\text{Pic}^0(C)$ is l^n -divisible, it follows that $\text{CH}_0(X)_{\text{deg } 0}$ is l^n -divisible. In particular, $\text{CH}_0(X)\{l\} = \text{CH}_0(X)_{\text{deg } 0}\{l\}$ is also l^n -divisible. The lemma follows. \square

Theorem 3.6. *Given a reduced projective surface X over k and a prime $l \neq p$, the following hold.*

- (1) $H^1(X, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$.
- (2) *The map $\tau_X : H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \rightarrow \text{CH}_0(X)\{l\}$ is an isomorphism.*

Proof. In view of (3-14), the theorem is equivalent to showing that τ_X is injective. To show this, it suffices to prove the stronger assertion that the map $\delta := \rho_X^{\text{semi}} \circ \tau_X : H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \rightarrow \text{CH}_0(X)\{l\} \rightarrow J^2(X)\{l\}$ is an isomorphism.

In order to prove this, we first prove a stronger version of its surjectivity assertion, namely, that for every $n \geq 1$, the map $H_{\text{ét}}^3(X, \mu_{l^n}(2)) \rightarrow {}_l^n J^2(X)$ is surjective. In view of Theorem 3.4, it suffices to show that the map ${}_l^n \text{CH}_0(X) \rightarrow {}_l^n J^2(X)$ is surjective. To prove this, we use [Mallick 2009, Theorem 14], which says that we can find a reduced Cartier curve $C \subsetneq X$ (a suitable hypersurface section in a projective embedding) such that $C \cap X_{\text{reg}} \subseteq C_{\text{reg}}$ and the induced map ${}_l^n \text{Pic}^0(C) \rightarrow {}_l^n J^2(X)$ is surjective. The commutative diagram

$$\begin{array}{ccc} {}_l^n \text{Pic}^0(C) & \longrightarrow & {}_l^n \text{CH}_0(X) \\ \simeq \downarrow & & \downarrow \\ {}_l^n J^1(C) & \longrightarrow & {}_l^n J^2(X) \end{array}$$

then proves the desired surjectivity where the left vertical arrow is an isomorphism because the kernel $\text{Ker}(\text{Pic}^0(C) \rightarrow J^1(C))$ is unipotent, and hence uniquely l^n -divisible.

In the rest of the proof, we shall ignore the Tate twist in étale cohomology. Let δ_n denote the composite map $H_{\text{ét}}^3(X, \mu_{l^n}(2)) \rightarrow {}_l^n \text{CH}_0(X) \rightarrow {}_l^n J^2(X)$ so that $\delta = \varinjlim_n \delta_n$. Using the above surjectivity, we get a direct system of short exact sequences

$$0 \rightarrow T_l^n(X) \rightarrow H_{\text{ét}}^3(X, \mu_{l^n}(2)) \rightarrow {}_l^n J^2(X) \rightarrow 0 \tag{3-15}$$

whose direct limit is the short exact sequence

$$0 \rightarrow T_{l^\infty}(X) \rightarrow H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \rightarrow J^2(X)\{l\} \rightarrow 0. \tag{3-16}$$

To prove the theorem, we are only left with showing that $T_{l^\infty}(X) = 0$. Since

$$H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \varinjlim_n H_{\text{ét}}^3(X, \mu_{l^n}(2)) \simeq \varinjlim_n H_{\text{ét}}^3(X, \mathbb{Z}/l^n),$$

it follows that $H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = H_{\text{ét}}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2))\{l\}$. In particular, $T_{l^\infty}(X)$ is an l -primary torsion group.

We next consider a commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_{l^\infty}(X) & \longrightarrow & H_{\acute{e}t}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) & \xrightarrow{\delta} & J^2(X)\{l\} \longrightarrow 0 \\
 & & \downarrow l_1^n & & \downarrow l_2^n & & \downarrow l_3^n \\
 0 & \longrightarrow & T_{l^\infty}(X) & \longrightarrow & H_{\acute{e}t}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) & \xrightarrow{\delta} & J^2(X)\{l\} \longrightarrow 0,
 \end{array}$$

in which the vertical arrows are simply multiplication by l^n . It follows from Lemma 3.5 that the middle vertical arrow is surjective. We have shown above that $H_{\acute{e}t}^3(X, \mu_{l^n}(2)) \rightarrow {}_l J^2(X)$ is surjective. In particular, the map $\ker(l_2^n) \rightarrow \ker(l_3^n)$ is surjective. It follows that $T_{l^\infty} \otimes_{\mathbb{Z}} \mathbb{Z}/l^n = 0$ for every $n \geq 1$. We thus only have to show that T_{l^∞} is finite to finish the proof.

Now, an easy argument that involves dualizing the argument of [Barbieri-Viale and Srinivas 2001, Theorem 2.5.4] (see [Mallick 2009, Proof of Theorem 15, Claim 2]), shows that there is a positive integer N_X , depending only on X and not on the integer n , such that

$$|H_{\acute{e}t}^3(X, \mu_{l^n}(2))| \leq N_X \cdot |{}_l J^2(X)|. \tag{3-17}$$

Combining this with (3-15), it follows that for every $n \geq 1$, either ${}_l J^2(X) = 0$ and hence $T_{l^n}(X) = 0$ or ${}_l J^2(X) \neq 0$ and $|T_{l^n}(X)| \cdot |{}_l J^2(X)| \leq N_X \cdot |{}_l J^2(X)|$. In particular, either $T_{l^n}(X) = 0$ or $|T_{l^n}(X)| \leq N_X$ for every $n \geq 1$. But this implies that $T_{l^\infty}(X)$ is finite. This completes the proof of the theorem. \square

3D. Relation with Bloch’s construction. Bloch [1979, §2] constructed a map $\lambda_X : \text{CH}_0(X)\{l\} \rightarrow H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ for a smooth projective scheme X of dimension $d \geq 1$ over k . We end this section with the following lemma that explains the relation between Bloch’s construction and ours when X is a smooth surface.

Lemma 3.7. *If X is a smooth projective surface over k , then $\tau_X : H_{\acute{e}t}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \rightarrow \text{CH}_0(X)\{l\}$ coincides with the negative of the inverse of Bloch’s map $\lambda_X : \text{CH}_0(X)\{l\} \rightarrow H_{\acute{e}t}^3(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$.*

Proof. To prove the lemma, we have to go back to the construction of $\tau_X : H_{\acute{e}t}^3(X, \mu_n(2)) \rightarrow {}_l \text{CH}_0(X)$ in Section 3B for $n \in k^\times$. Since X is smooth, there is an isomorphism of Zariski sheaves $\mathcal{O}_X^\times/n \xrightarrow{\sim} \mathcal{H}^1(\mu_n(2)) \xrightarrow{\sim} {}_n \mathcal{K}_2$ by [Barbieri-Viale et al. 1996, Lemma 6.2]. Since $H^2(X, \mathcal{O}_X^\times) = 0$, we see that the top cohomology of all these sheaves vanish.

This implies that the map $\mathbb{H}^2(X, \mathcal{K}_2^\bullet) \rightarrow {}_n H^2(X, \mathcal{K}_2)$ is simply the composite

$$\mathbb{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\sim} H^1(X, \mathcal{K}_2/n) \rightarrow H^2(X, {}_n \mathcal{K}_2) \rightarrow {}_n H^2(X, \mathcal{K}_2).$$

Moreover, there is a diagram

$$\begin{array}{ccc}
 \mathbb{H}^2(X, \mathcal{K}_2^\bullet) & \xrightarrow{\sim} & H^1(X, \mathcal{K}_2/n) \\
 \downarrow & & \downarrow \beta_n \simeq \\
 H_{\acute{e}t}^3(X, \mu_n(2)) & \xrightarrow{\simeq} & H^1(X, \mathcal{H}^2(\mu_n(2)))
 \end{array} \tag{3-18}$$

where the right vertical arrow is the Galois symbol map. This induces a unique isomorphism

$$\alpha_n : H_{\acute{e}t}^3(X, \mu_n(2)) \xrightarrow{\sim} \mathbb{H}^2(X, \mathcal{K}_2^\bullet).$$

One checks from (3-13) that τ_X is simply the composite

$$H_{\acute{e}t}^3(X, \mu_n(2)) \xrightarrow{\alpha_n} \mathbb{H}^2(X, \mathcal{K}_2^\bullet) \xrightarrow{\sim} H^1(X, \mathcal{K}_2/n) \xrightarrow{\delta_n} {}_nH^2(X, \mathcal{K}_2). \tag{3-19}$$

We also have a commutative diagram

$$\begin{array}{ccccc} K_2(k(X))/n & \longrightarrow & \coprod_{x \in X^{(1)}} k(x)^\times/n & \longrightarrow & \coprod_{y \in X^{(2)}} \mathbb{Z}/n \\ \downarrow \beta_n & & \downarrow \beta_n & & \downarrow \beta_n \\ H_{\acute{e}t}^2(k(X), \mu_n(2)) & \longrightarrow & \coprod_{x \in X^{(1)}} H_{\acute{e}t}^1(k(x), \mu_n(1)) & \longrightarrow & \coprod_{y \in X^{(2)}} H_{\acute{e}t}^0(k(y), \mathbb{Z}/n) \end{array} \tag{3-20}$$

where all vertical arrows are the Galois symbols and are isomorphisms.

Since the Gersten resolution is universally exact (see [Quillen 1973; Grayson 1985]), the middle cohomology of the top row is $H^1(X, \mathcal{K}_2/n)$ and the Bloch–Ogus resolution (see [Bloch and Ogus 1974]) shows that the middle cohomology of the bottom row is $H^1(X, \mathcal{H}^2(\mu_n(2)))$. The isomorphism β_n in (3-18) is induced by the vertical arrows of (3-20).

For a fixed prime $l \neq p$, the map δ_{lm} in (3-19) becomes an isomorphism on the limit over $m \geq 1$. Assuming this isomorphism, it follows from (3-19) that $\tau_X = \varinjlim_m \alpha_{lm}$. On the other hand, it follows from Bloch’s construction in [Bloch 1979, § 2] that $\lambda_X = \varinjlim_m \beta_{lm}$. Since each α_n (with $n \in k^\times$) is the inverse of β_n by definition, the lemma follows. \square

4. Roitman’s torsion for separably weakly normal surfaces

The Roitman torsion theorem says that for a smooth projective scheme of dimension $d \geq 1$ over k , the Abel–Jacobi map $\rho_X : \text{CH}_0(X)_{\text{deg } 0} \rightarrow J^d(X)$ is an isomorphism on the torsion subgroups. This theorem was extended to normal projective schemes in [Krishna and Srinivas 2002]. However, when X has arbitrary singularities, this theorem is known only for the torsion prime-to- p ; see [Biswas and Srinivas 1999; Mallick 2009]. In this section, we extend the Roitman torsion theorem to separably weakly normal (see below for definition) surfaces. Later in this text, we shall prove a suitable generalization of this theorem in higher dimension. This generalization will be used to prove Theorem 1.2.

4A. Separably weakly normal schemes. The weak normality is a singularity type of schemes, which in characteristic $p > 0$, is closely related to various F -singularities. These F -singularities are naturally encountered while running the minimal model program in positive characteristic. Most of the F -regularity conditions imply weak normality. In characteristic zero, weak normality coincides with the more familiar notion of seminormality. In this text, we shall study the Chow group of 0-cycles on certain singular schemes whose singularities are closely related to weak normality in positive characteristic.

Let A be a reduced commutative Noetherian ring. Let B be the integral closure of A in its total quotient ring. Recall from [Manaresi 1980] that the seminormalization of A is the largest among the subrings A' of B containing A such that

- (1) for all $x \in \text{Spec}(A)$, there exists exactly one $x' \in \text{Spec}(A')$ over x ;
- (2) the canonical homomorphism $k(x) \rightarrow k(x')$ is an isomorphism.

The weak normalization of A is the largest among the subrings A' of B containing A such that

- (1) for all $x \in \text{Spec}(A)$, there exists exactly one $x' \in \text{Spec}(A')$ over x ;
- (2) the field extension $k(x) \rightarrow k(x')$ is finite purely inseparable.

We let $A_s \subset B$ and $A_w \subset B$ denote the seminormalization and weak normalization of A , respectively. One says that A is seminormal (resp. weakly normal) if $A = A_s$ (resp. $A = A_w$). It is clear from the above definition that $A \subset A_s \subset A_w \subset B$. Moreover, $A_s = A_w$ if A is a \mathbb{Q} -algebra. To get a more geometric understanding of these singularities, we make the following definition.

Definition 4.1. Let k be a field and let R be a k -algebra which is finite as a k -vector space. We shall say that R is *weakly separable* over k if it is reduced and either $\text{char}(k) = 0$, or $\text{char}(k) > 0$ and there is no inclusion of rings $k \subsetneq K \subset R$ such that K is a purely inseparable field extension of k .

Recall that a commutative Noetherian ring A is called an S_2 ring if for every prime ideal \mathfrak{p} of A , one has $\text{depth}(A_{\mathfrak{p}}) \geq \min(\text{ht}(\mathfrak{p}), 2)$. It follows easily from this definition that a Cohen–Macaulay ring is S_2 .

Proposition 4.2. *Let A be reduced commutative Noetherian ring. Let B be the integral closure of A in its total quotient ring and let $I \subset B$ be the largest ideal which is contained in A . Assume that B is a finite A -module. Then the following hold.*

- (1) *If A is seminormal, then B/I is reduced. If A is S_2 , the converse also holds.*
- (2) *If A is weakly normal, then B/I is reduced and the inclusion map $A/I \rightarrow B/I$ is generically weakly separable. If A is S_2 , the converse also holds.*

Proof. If A is seminormal, then B/I is reduced by [Traverso 1970, Lemma 1.3]. Suppose now that A is S_2 and B/I is reduced. By [Greco and Traverso 1980, Theorem 2.6], it suffices to show that $A_{\mathfrak{p}}$ is seminormal for every height one prime ideal \mathfrak{p} in A .

Let $\mathfrak{p} \subset A$ be a prime ideal of height one. Let $\{x_1, \dots, x_r\} \subset B$ be a subset whose image in B/A generates it as an A -module. We now note that $(A_{\mathfrak{p}} : B_{\mathfrak{p}}) = (A : B)_{\mathfrak{p}}$, where

$$I := (A : B) = \{a \in A \mid aB \subset A\} = \text{ann}(B/A).$$

The first equality uses the fact that B/A is a finite A -module so that $I = \bigcap_{i=1}^r (A : x_i)$ and $(J \cap J')_{\mathfrak{p}} = J_{\mathfrak{p}} \cap J'_{\mathfrak{p}}$; see [Matsumura 1986, Example 4.8]. Since $B_{\mathfrak{p}}/I_{\mathfrak{p}} = (B/I)_{\mathfrak{p}}$ is reduced (by our assumption) and $A_{\mathfrak{p}}$ is 1-dimensional, it follows from [Bass and Murthy 1967, Proposition 7.2] and [Traverso 1970, Theorem 3.6] that $A_{\mathfrak{p}}$ is seminormal. This proves (1).

Suppose now that A is weakly normal. Then it is clearly seminormal. In particular, B/I is reduced by (1). We now show that $A/I \hookrightarrow B/I$ is generically weakly separable. Note that as A is reduced and generically weakly normal, I must have height at least one in A . Let \mathfrak{p} be a minimal prime of I in A and let $k(\mathfrak{p})$ be its residue field. Then $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ is the Jacobson radical of $B_{\mathfrak{p}}$ and hence $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is a product of finite field extensions of $k(\mathfrak{p})$.

We consider the commutative diagram

$$\begin{array}{ccc}
 A_{\mathfrak{p}} & \longrightarrow & B_{\mathfrak{p}} \\
 \downarrow & & \downarrow \\
 k(\mathfrak{p}) & \longrightarrow & B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \xrightarrow{\simeq} \prod_{i=1}^s k(\mathfrak{q}_i),
 \end{array} \tag{4-1}$$

in which the square is Cartesian, the horizontal arrows are injective finite morphisms and the vertical arrows are surjective. Since $A_{\mathfrak{p}}$ is weakly normal (see [Manaresi 1980, Theorem IV.3]) and $B_{\mathfrak{p}}$ is its integral closure, it means that $A_{\mathfrak{p}}$ is weakly normal in $B_{\mathfrak{p}}$. We conclude from [Yanagihara 1983, Proposition 3] that $k(\mathfrak{p})$ is weakly normal in $\prod_{i=1}^s k(\mathfrak{q}_i)$. We now apply Lemma 2 there to deduce that the lower horizontal arrow in (4-1) is weakly separable, as desired.

Conversely, suppose that A is S_2 and the inclusion $A/I \hookrightarrow B/I$ is generically weakly separable map of reduced schemes. By [Manaresi 1980, Corollary IV. 4], it suffices to show that $A_{\mathfrak{p}}$ is weakly normal for every height one prime ideal \mathfrak{p} in A .

Let $\mathfrak{p} \subset A$ be a prime ideal of height one. Since $I_{\mathfrak{p}} = (A_{\mathfrak{p}} : B_{\mathfrak{p}})$ as shown above, it follows that $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ (hence $A_{\mathfrak{p}}$ is weakly normal) if $I \not\subset \mathfrak{p}$. We can therefore assume that $I \subset \mathfrak{p}$. Since A is generically normal (and hence weakly normal), I must have height at least one in A . It follows that \mathfrak{p} must be a minimal prime ideal of I .

It follows from (1) that $A_{\mathfrak{p}}$ is seminormal. In particular, $I_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$. Furthermore, $\mathfrak{p}B_{\mathfrak{p}}$ is the Jacobson radical of $B_{\mathfrak{p}}$ such that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is a finite product of finite field extensions of $k(\mathfrak{p})$. This gives rise to a Cartesian square of rings as in (4-1). Our assumption says that the lower horizontal arrow in (4-1) is weakly separable. We conclude again from [Yanagihara 1983, Lemma 2, Proposition 3] that $A_{\mathfrak{p}}$ is weakly normal. This proves (2). □

Example 4.3. Using Proposition 4.2, we can construct many examples of seminormal rings which are not weakly normal. Let k be perfect field of characteristic $p > 0$ and consider the Cartesian square of rings

$$\begin{array}{ccc}
 A & \xrightarrow{\psi} & k[x, t] \\
 \downarrow & & \downarrow \\
 k[x] & \xrightarrow{\phi} & \frac{k[x, t]}{(t^p - x)}
 \end{array} \tag{4-2}$$

in which the right vertical arrow is the canonical surjection and ϕ is the canonical inclusion. In particular, one has $A = \{(f(x), g(x, t)) \mid f(t^p) = g(t^p, t)\}$. Since ϕ is a finite purely inseparable map of reduced

rings which is not an isomorphism, it follows that ψ is not an isomorphism. It follows from [Proposition 4.2](#) that A is seminormal but not weakly normal.

Example 4.4. We now provide an example of a weakly normal S_2 ring A with normalization B and reduced conductor I such that the map $A/I \rightarrow B/I$ is not generically separable. Let k be a perfect field of characteristic $p > 0$ and consider the Cartesian square

$$\begin{array}{ccc}
 A & \xrightarrow{\psi} & k[x, t] \\
 \downarrow & & \downarrow \\
 k[x] & \xrightarrow{\phi} & \frac{k[x, t]}{t(t^p - x)}
 \end{array} \tag{4-3}$$

where the right vertical arrow is the canonical quotient map and the lower horizontal arrow is given by $\phi(x) = x$. One can easily check (e.g., use Eisenstein’s criterion) that $t^p - x$ is an irreducible polynomial in $k[x, t]$. If we let $\mathfrak{q}_1 = (t)$ and $\mathfrak{q}_2 = (t^p - x)$, then we see that $k(x) \rightarrow k(\mathfrak{q}_1)$ is an isomorphism and $k(x) \rightarrow k(\mathfrak{q}_2) \xrightarrow{\sim} k(x^{1/p})$ is purely inseparable. In particular, ϕ induces a weakly separable map between the function fields. It follows from [\[Yanagihara 1983, Proposition 3\]](#) that A is weakly normal. We just saw however that ϕ is not generically separable.

Note also that the kernels of the vertical arrows in (4-3) are isomorphic and the kernel on the right is a principal ideal generated by $f(x, t) = t(t^p - x)$. Since $f(x, t)$ a nonzero-divisor in $k[x, t]$ and lies in A , it must be a nonzero-divisor in A . Since $k[x]$ is Cohen–Macaulay, it follows that A is Cohen–Macaulay too; see [\[Matsumura 1986, Theorem 17.3\]](#). In particular, it is S_2 .

Motivated by [Example 4.4](#), we make the following definition.

Definition 4.5. Let A be a commutative reduced Noetherian ring with finite normalization B . Let $I \subset B$ be the largest ideal lying in A . We shall say that A is *separably weakly normal* if B/I is reduced and the induced map $A/I \rightarrow B/I$ is generically separable. We shall say that a Noetherian scheme X is (separably) weakly normal (resp. seminormal) if the coordinate ring of every affine open in X is (separably) weakly normal (resp. seminormal).

It follows from [Proposition 4.2](#) that if A is S_2 and separably weakly normal, then it is weakly normal. On the other hand, [Example 4.4](#) shows that a weakly normal S_2 ring may not be separably weakly normal. For \mathbb{Q} -algebras, seminormality implies separably weak normality by [Proposition 4.2](#). The three singularity types coincide for \mathbb{Q} -algebras which are S_2 . Our goal in the rest of this text is to study the torsion in the Chow group of 0-cycles on schemes which are separably weakly normal.

Remark 4.6. Since we only deal with schemes which are separably weakly normal in the rest of this text, the reader can, in principle, only read [Definition 4.5](#) in [Section 4A](#) and move to the next subsection. Our discussion of seminormality and weak normality is primarily meant to motivate the reader to [Definition 4.5](#), and to give a comparison of various nonnormal singularity types which are closely related yet different.

We end this subsection by recalling the notion of conducting ideals and conducting subschemes. Let A be a reduced commutative Noetherian ring and let B be a subring of the integral closure of A in its total quotient ring such that $A \subset B$. Assume that B is a finite A -module. Recall that an ideal $I \subset A$ is called a conducting ideal for the inclusion $A \subset B$ if $I = IB$. It is clear from this definition that $I \subset (A : B)$. Furthermore, one knows that $(A : B)$ is the largest conducting ideal for $A \subset B$; see [Huneke and Swanson 2006, Example 2.11]. If X is a reduced Noetherian scheme and $f : X' \rightarrow X$ is a finite birational map, then a closed subscheme $Y \subset X$ is called a conducting subscheme if $\mathcal{I}_Y \subset \mathcal{O}_X$ is a sheaf of conducting ideals for the inclusion of sheaves of rings $\mathcal{O}_X \subset f_*(\mathcal{O}_{X'})$.

4B. The torsion theorem for separably weakly normal surfaces. For the remaining part of this section, we shall identify the two Chow groups $\text{CH}_0^{LW}(X)$ and $\text{CH}_0(X)$ for curves and surfaces using [Binda and Krishna 2018, Theorem 3.17]. To prove the Roitman torsion theorem for a separably weakly normal projective surface, we need the following excision result for our Chow group. We fix an algebraically closed field k .

Lemma 4.7. *Let X be a reduced quasiprojective separably weakly normal surface over k and let $f : \bar{X} \rightarrow X$ denote the normalization map. Let $Y \subset X$ denote the smallest conducting closed subscheme with $\bar{Y} = Y \times_X \bar{X}$. Then there is an exact sequence*

$$SK_1(\bar{X}) \oplus SK_1(Y) \rightarrow SK_1(\bar{Y}) \rightarrow \text{CH}_0(X) \rightarrow \text{CH}_0(\bar{X}) \rightarrow 0. \tag{4-4}$$

Proof. We have a commutative diagram of relative and birelative K -theory exact sequences:

$$\begin{array}{ccccccccccc}
 & & & & K_0(X, \bar{X}, Y) & & & & K_{-1}(X, \bar{X}, Y) & & (4-5) \\
 & & & & \downarrow & & & & \downarrow & & \\
 K_1(X) & \rightarrow & K_1(Y) & \longrightarrow & K_0(X, Y) & \longrightarrow & K_0(X) & \rightarrow & K_0(Y) & \longrightarrow & K_{-1}(X, Y) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 K_1(\bar{X}) & \rightarrow & K_1(\bar{Y}) & \longrightarrow & K_0(\bar{X}, \bar{Y}) & \longrightarrow & K_0(\bar{X}) & \rightarrow & K_0(\bar{Y}) & \longrightarrow & K_{-1}(\bar{X}, \bar{Y}) \\
 & & & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & K_{-1}(X, \bar{X}, Y) & & & & K_{-2}(X, \bar{X}, Y) & &
 \end{array}$$

Using the Thomason–Trobaugh spectral sequence and the results of Bass [1968] that the sheaves $\mathcal{K}_{i,(X,\bar{X},Y)}$ on Y vanish for $i \leq 0$, we see that $K_i(X, \bar{X}, Y) = 0$ for $i \leq -1$. It also follows from the spectral sequence that $K_0(X, \bar{X}, Y) \simeq H^1(Y, \mathcal{K}_{1,(X,\bar{X},Y)})$. It follows from [Geller and Weibel 1983, Theorem 0.2] that $H^1(Y, \mathcal{K}_{1,(X,\bar{X},Y)}) \simeq H^1(X, \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathcal{O}_Y} \Omega_{\bar{Y}/Y}^1)$. Since X is separably weakly normal, it follows from Definition 4.5 that the map $\bar{Y} \rightarrow Y$ is a finite generically étale map of reduced schemes. Hence $\Omega_{\bar{Y}/Y}^1$ has 0-dimensional support and we conclude that $H^1(X, \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathcal{O}_Y} \Omega_{\bar{Y}/Y}^1) = 0$. In particular, $K_0(X, \bar{X}, Y) = 0$.

A diagram chase in (4-5) shows that there is an exact sequence

$$K_1(\bar{X}) \oplus K_1(Y) \rightarrow K_1(\bar{Y}) \rightarrow \tilde{K}_0(X) \rightarrow \tilde{K}_0(\bar{X}) \oplus \tilde{K}_0(Y) \rightarrow \tilde{K}_0(\bar{Y}),$$

where $\tilde{K}_0(-) = \text{Ker}(K_0(-) \rightarrow H^0(-, \mathbb{Z}))$. Using the map of sheaves $\mathcal{O}_X^\times \rightarrow f_*(\mathcal{O}_{\bar{X}}^\times)$, this exact sequence maps to a similar unit-Pic exact sequence

$$U(\bar{X}) \oplus U(Y) \rightarrow U(\bar{Y}) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(\bar{X}) \oplus \text{Pic}(Y) \rightarrow \text{Pic}(\bar{Y}).$$

Taking the kernels and using the Levine’s formula $SK_0(Z) := \text{Ker}(\tilde{K}_0(Z) \rightarrow \text{Pic}(Z)) \simeq H^2(Z, \mathcal{K}_2) \simeq \text{CH}_0(Z)$ for a reduced surface Z , we get (4-4). □

Lemma 4.8. *Let Y be a reduced curve over k and let $r \geq 0$ denote the number of irreducible components of Y which are projective over k . If Y is affine, then $SK_1(Y)$ is uniquely divisible. If Y is projective, then $SK_1(Y)$ is divisible, $SK_1(Y)\{l\} \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^r$ for a prime $l \neq p$ and $SK_1(Y)\{p\} = 0$.*

Proof. It follows from [Barbieri-Viale et al. 1996, Theorem 5.3] that $SK_1(Y) \simeq (k^\times)^r \oplus V$, where V is uniquely divisible. The lemma follows directly from this isomorphism. □

Theorem 4.9. *Let X be a reduced projective separably weakly normal surface over an algebraically closed field k of exponential characteristic p . Then, $A^2(X)$ is a semiabelian variety and the Abel–Jacobi map $\rho_X : \text{CH}_0(X)_{\text{deg } 0} \rightarrow A^2(X)$ is an isomorphism on the torsion subgroups.*

Proof. We can find a finite collection of reduced complete intersection Cartier curves $\{C_1, \dots, C_r\}$ on X such that the induced map of algebraic groups $\prod_{i=1}^r \text{Pic}^0(C_i) \rightarrow A^2(X)$ is surjective (see, for instance, [Esnault et al. 1999, (7.1), p. 657]). In characteristic zero, we can further assume by [Cumino et al. 1983, Corollary 2.5] that each of these curves is weakly normal. Since a surjective morphism of smooth connected algebraic groups restricts to a surjective map on their unipotent and semiabelian parts and since $\text{Pic}^0(C)$ is semiabelian if C is a weakly normal curve (easy to check), it follows that $A^2(X)$ is a semiabelian variety in characteristic zero.

In characteristic $p \geq 2$, the surjections $\prod_{i=1}^r \text{Pic}^0(C_i) \twoheadrightarrow A^2(X) \twoheadrightarrow J^2(X)$ and [Krishna 2015a, Lemma 2.7] together imply that the induced maps $(\text{CH}^2(X)_{\text{deg } 0})_{\text{tors}} \rightarrow A^2(X)_{\text{tors}} \rightarrow J^2(X)_{\text{tors}}$ are also surjective. Since $A^2_{\text{unip}}(X)$ is a p -primary torsion group of bounded exponent, the theorem will follow if we prove, in any characteristic, that the composite map $\rho_X^{\text{semi}} : \text{CH}_0(X)_{\text{deg } 0} \rightarrow A^2(X) \rightarrow J^2(X)$ is injective on the torsion subgroup.

It follows from [Mallick 2009, Theorem 15] that the map $\text{CH}_0(X)_{\text{deg } 0}\{l\} \rightarrow J^2(X)\{l\}$ is an isomorphism for every prime $l \neq p$. This also follows immediately from Theorem 3.4 and the proof of Theorem 3.6. We can thus assume that $p \geq 2$ and $l = p$.

Let $f : \bar{X} \rightarrow X$ be the normalization and consider the commutative diagram

$$\begin{CD}
 \mathrm{CH}_0(X)_{\mathrm{deg} 0}\{p\} @>f^*>> \mathrm{CH}_0(\bar{X})_{\mathrm{deg} 0}\{p\} \\
 @VVV @VVV \\
 J^2(X)\{p\} @>>> J^2(\bar{X})\{p\}.
 \end{CD}
 \tag{4-6}$$

The right vertical arrow is an isomorphism by [Krishna and Srinivas 2002, Theorem 1.6]. Thus, it suffices to prove the stronger assertion that the map $f^* : \mathrm{CH}_0(X)\{p\} \rightarrow \mathrm{CH}_0(\bar{X})\{p\}$ is an isomorphism. It is enough to show that $A := \mathrm{Ker}(\mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(\bar{X}))$ is uniquely p -divisible.

It follows from Lemma 4.7 that there is an exact sequence

$$SK_1(\bar{X}) \oplus SK_1(Y) \rightarrow SK_1(\bar{Y}) \rightarrow A \rightarrow 0.$$

Following the notations of Lemma 4.7, it follows from Lemma 4.8 that $SK_1(Y)$ and $SK_1(\bar{Y})$ are uniquely p -divisible. Furthermore, it follows from [Krishna 2015a, Theorem 5.6] that $SK_1(\bar{X}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$. An elementary argument now shows that A must be uniquely p -divisible. This finishes the proof. \square

Corollary 4.10. *If X is a reduced projective separably weakly normal surface over $\bar{\mathbb{F}}_p$, then $\mathrm{CH}_0(X)_{\mathrm{deg} 0}$ is finite-dimensional. That is, the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow A^2(X)$ is an isomorphism.*

Proof. In view of Theorem 4.9, it suffices to show that for any reduced projective scheme X of dimension $d \geq 1$ over $\bar{\mathbb{F}}_p$, the group $\mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$ is a torsion abelian group.

Given $\alpha \in \mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$, we can find a reduced Cartier curve $C \subseteq X$ such that α lies in the image of the pushforward map $\mathrm{Pic}^0(C) \rightarrow \mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$. It is therefore enough to show that $\mathrm{Pic}^0(C)$ is torsion. But this is a special case of the more general fact that $G(\bar{\mathbb{F}}_p)$ is torsion whenever G is a smooth commutative algebraic group over $\bar{\mathbb{F}}_p$. \square

5. Bloch’s torsion theorem for 0-cycles with modulus

We continue with our assumption that k is algebraically closed with exponential characteristic p . Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor on X . We shall prove Theorem 1.1 in this section.

5A. Relative étale cohomology. Given an étale sheaf \mathcal{F} on \mathbf{Sch}_k and a finite map $f : Y \rightarrow X$ in \mathbf{Sch}_k , let $\mathcal{F}_{(X,Y)} := \mathrm{Cone}(\mathcal{F}|_X \rightarrow f_*(\mathcal{F}|_Y))[-1]$ be the chain complex of étale sheaves on X . The exactness of f_* on étale sheaves implies that there is a long exact sequence of étale (hyper)cohomology groups

$$0 \rightarrow H_{\acute{e}t}^0(X, \mathcal{F}_{(X,Y)}) \rightarrow H_{\acute{e}t}^0(X, \mathcal{F}) \rightarrow H_{\acute{e}t}^0(Y, \mathcal{F}) \rightarrow H_{\acute{e}t}^1(X, \mathcal{F}_{(X,Y)}) \rightarrow \dots \tag{5-1}$$

If $Y \hookrightarrow X$ is a closed immersion with complement $j : U \hookrightarrow X$, then one checks immediately from the above definition that $H_{\acute{e}t}^*(X, \mathcal{F}_{(X,Y)})$ is canonically isomorphic to $H_{\acute{e}t}^*(X, j_!(\mathcal{F}|_U))$. We conclude that if X is projective over k and $Y \hookrightarrow X$ is closed, then $H_{\acute{e}t}^*(X, \mathcal{F}_{(X,Y)})$ is same as the étale cohomology with compact support $H_{c,\acute{e}t}^*(X \setminus Y, \mathcal{F})$ of $X \setminus Y$.

Let us now consider an abstract blow-up diagram in \mathbf{Sch}_k :

$$\begin{array}{ccc}
 Y' & \xrightarrow{f'} & X' \\
 \pi' \downarrow & & \downarrow \pi \\
 Y & \xrightarrow{f} & X.
 \end{array} \tag{5-2}$$

This is a Cartesian square in which the horizontal arrows are closed immersions, π is a proper morphism such that $X' \setminus Y' \xrightarrow{\sim} X \setminus Y$. The proper base change theorem for the étale cohomology implies that for a torsion constructible étale sheaf \mathcal{F} on \mathbf{Sch}_k , the cohomology groups $H_{\acute{e}t}^*(-, \mathcal{F})$ satisfy the *cdh*-descent. In particular, the canonical map

$$\pi^* : H_{\acute{e}t}^i(X, \mathcal{F}_{(X,Y)}) \rightarrow H_{\acute{e}t}^i(X', \mathcal{F}_{(X',Y')}) \tag{5-3}$$

is an isomorphism for every $i \geq 0$. We shall write the *relative étale cohomology groups* $H_{\acute{e}t}^*(X, \mathcal{F}_{(X,Y)})$ for a closed immersion $Y \hookrightarrow X$ as $H_{\acute{e}t}^*(X|Y, \mathcal{F})$.

5B. A weak Lefschetz-type theorem for the double. We now come back to our situation of X being a smooth projective scheme of dimension $d \geq 1$ over k and $D \subset X$ an effective Cartier divisor. Let $\{E_1, \dots, E_r\}$ be the set of irreducible components of D_{red} . If $d \geq 3$, we choose, as in Section 2E, a closed embedding $X \hookrightarrow \mathbb{P}_k^N$ and a smooth hypersurface section $\tau : Y = X \cap H_1 \hookrightarrow X$ such that Y is not contained in D and $Y \cap E_i$ is integral for every $1 \leq i \leq r$. We set $F = D \cap Y$. We shall use these notations throughout the rest of this section.

Given a prime-to- p integer n , it is clear from the definition of the relative étale cohomology and its *cdh*-descent (see Section 5A) associated to the Cartesian square (2-1) (see Lemma 2.2) that the pullback maps via the closed immersions $\iota_{\pm} : X \hookrightarrow S_X$ induce, for each $i \geq 0$, a split exact sequence of étale cohomology

$$0 \rightarrow H_{\acute{e}t}^i(X|D, \mu_n(j)) \xrightarrow{p_{\pm,*}} H_{\acute{e}t}^i(S_X, \mu_n(j)) \xrightarrow{\iota_{\pm}^*} H_{\acute{e}t}^i(X, \mu_n(j)) \rightarrow 0. \tag{5-4}$$

Here, the splitting of ι_{\pm}^* is given by the pullback $\Delta^* : H_{\acute{e}t}^i(X, \mu_n(j)) \rightarrow H_{\acute{e}t}^i(S_X, \mu_n(j))$.

Let us next recall from Gabber’s construction [Fujiwara 2002] of the Gysin morphism for étale cohomology (see, also [Navarro 2018, Definition 2.1]) that the regular closed embeddings $\tau : Y \hookrightarrow X$ and $\tau_1 : S_Y \hookrightarrow S_X$ induce Gysin morphisms $\tau_* : H_{\acute{e}t}^i(Y, \mu_n(j)) \rightarrow H_{\acute{e}t}^{i+2}(X, \mu_n(j+1))$ and $\tau_{1,*} : H_{\acute{e}t}^i(S_Y, \mu_n(j)) \rightarrow H_{\acute{e}t}^{i+2}(S_X, \mu_n(j+1))$ for $i \geq 0$. Furthermore, it follows from the Cartesian square (2-8) and [Navarro 2018, Corollary 2.12] (see also [Fujiwara 2002, Proposition 1.1.3]) that the pullback via the closed immersions $\iota_{\pm} : X \hookrightarrow S_X$ induces a commutative diagram

$$\begin{array}{ccc}
 H_{\acute{e}t}^i(S_Y, \mu_n(j)) & \xrightarrow{\iota_{\pm}^*} & H_{\acute{e}t}^i(Y, \mu_n(j)) \\
 \tau_{1,*} \downarrow & & \downarrow \tau_* \\
 H_{\acute{e}t}^{i+2}(S_X, \mu_n(j+1)) & \xrightarrow{\iota_{\pm}^*} & H_{\acute{e}t}^{i+2}(X, \mu_n(j+1)).
 \end{array} \tag{5-5}$$

Lemma 5.1. *If $d \geq 3$, then the Gysin maps $\tau_* : H_{\acute{e}t}^{2d-3}(Y, \mu_n(d-1)) \rightarrow H_{\acute{e}t}^{2d-1}(X, \mu_n(d))$ and $\tau_{1,*} : H_{\acute{e}t}^{2d-3}(S_Y, \mu_n(d-1)) \rightarrow H_{\acute{e}t}^{2d-1}(S_X, \mu_n(d))$ are isomorphisms.*

Proof. The first isomorphism is a well known consequence of the weak Lefschetz theorem for étale cohomology. The main problem is to prove the second isomorphism. Since k is algebraically closed, we shall replace μ_n by the constant sheaf $\Lambda = \mathbb{Z}/n$.

Recall from [SGA 4]₂ 1977, §2] that the line bundle $\mathcal{O}_X(Y)$ (which we shall write in short as $\mathcal{O}(Y)$) on X has a canonical class $[\mathcal{O}(Y)] \in H_{Y,\acute{e}t}^1(X, \mathbb{G}_m)$ and its image via the boundary map $H_{Y,\acute{e}t}^1(X, \mathbb{G}_m) \rightarrow H_{Y,\acute{e}t}^2(X, \Lambda)$ is Deligne’s localized Chern class $c_1(Y)$. Here, $H_{Y,\acute{e}t}^*(X, -)$ denotes the étale cohomology with support in Y . On the level of the derived category $D^+(Y, \Lambda)$, this Chern class is given in terms of the map $c_1(Y) : \Lambda \rightarrow \tau^1\Lambda(1)[2]$. Using this Chern class, Gabber’s Gysin morphism $\tau_* : H_{\acute{e}t}^*(Y, \Lambda(j)) \rightarrow H_{\acute{e}t}^{*+2}(X, \Lambda(j+1))$ is the one induced on the cohomology by the composite map $\tau_* : \tau_*(\Lambda_Y) \rightarrow \tau_*\tau^1(\Lambda_Y(1)[2]) \rightarrow \Lambda_X(1)[2]$ in $D^+(X, \Lambda)$.

Corresponding to the Cartesian square (2-8), we have $\pi^*(\mathcal{O}(S_Y)) = \mathcal{O}(Y \amalg Y)$ and hence we have a commutative diagram

$$\begin{CD} H_{S_Y,\acute{e}t}^1(S_X, \mathbb{G}_m) @>\partial>> H_{S_Y,\acute{e}t}^2(S_X, \Lambda) \\ @V\pi^*VV @VV\pi^*V \\ H_{Y \amalg Y,\acute{e}t}^1(X \amalg X, \mathbb{G}_m) @>\partial>> H_{Y \amalg Y,\acute{e}t}^2(X \amalg X, \Lambda). \end{CD} \tag{5-6}$$

We thus have commutative diagrams

$$\begin{array}{ccc} \tau_{1,*}(\Lambda_{S_Y}) & \xrightarrow{\tau_{1,*}} & \Lambda_{S_X}(1)[2] \\ \pi^* \downarrow & & \downarrow \pi^* \\ \pi_*\psi_*(\Lambda_{Y \amalg Y}) & \xrightarrow{\pi_*\psi_*} & \pi_*\Lambda_{X \amalg X}(1)[2] \end{array} \qquad \begin{array}{ccc} \tau_*(\Lambda_Y) & \xrightarrow{\tau_*} & \Lambda_X(1)[2] \\ \iota_F^* \downarrow & & \downarrow \iota_D^* \\ \tau_*\iota_{F,*}(\Lambda_F) & \xrightarrow{\tau_*} & \iota_*(\Lambda_D(1)[2]) \end{array} \tag{5-7}$$

in $D^+(S_X, \Lambda)$ and $D^+(X, \Lambda)$.

We next observe that the canonical map $\Lambda_{X \amalg X} \rightarrow \Lambda_X \oplus \Lambda_X$ induced by the inclusions of the two components is an isomorphism. We thus have a sequence of maps

$$\Lambda_{S_X} \xrightarrow{\pi^*} \pi_*(\Lambda_{X \amalg X}) \simeq \Lambda_{X_+} \oplus \Lambda_{X_-} \longrightarrow \Lambda_D,$$

where the last map is the difference of two restrictions $\Lambda_{X_{\pm}} \rightarrow \iota_*(\Lambda_D)$. Furthermore, it is easy to check that the sequence

$$0 \rightarrow \Lambda_{S_X}(j) \xrightarrow{\pi^*} \pi_*(\Lambda_{X \amalg X}(j)) \rightarrow \Lambda_D(j) \rightarrow 0$$

is exact. A combination of this with (5-7) yields a commutative diagram of exact triangles in $D^+(S_X, \Lambda)$:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tau_{1,*}(\Lambda_{S_Y}) & \xrightarrow{\pi^*} & \pi_*\psi_*(\Lambda_{Y \sqcup Y}) & \longrightarrow & \tau_{1,*}(\Lambda_F) \longrightarrow 0 \\
 & & \tau_{1,*} \downarrow & & \downarrow \pi_*\psi_* & & \downarrow \tau_{1,*} \\
 0 & \longrightarrow & \Lambda_{S_X}(1)[2] & \xrightarrow{\pi^*} & \pi_*(\Lambda_{X \sqcup X}(1)[2]) & \longrightarrow & \Lambda_D(1)[2] \longrightarrow 0.
 \end{array} \tag{5-8}$$

Since all the underlying maps in (2-8) are finite, we obtain a commutative diagram of long exact sequence of cohomology groups

$$\begin{array}{cccccccc}
 H_{\acute{e}t}^{2d-4}(Y \sqcup Y, \Lambda(d-1)) & \rightarrow & H_{\acute{e}t}^{2d-4}(F, \Lambda(d-1)) & \rightarrow & H_{\acute{e}t}^{2d-3}(S_Y, \Lambda(d-1)) & \rightarrow & H_{\acute{e}t}^{2d-3}(Y \sqcup Y, \Lambda(d-1)) & \rightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow \tau_{1,*} & & \downarrow & \\
 H_{\acute{e}t}^{2d-2}(X \sqcup X, \Lambda(d)) & \longrightarrow & H_{\acute{e}t}^{2d-2}(D, \Lambda(d)) & \longrightarrow & H_{\acute{e}t}^{2d-1}(S_X, \Lambda(d)) & \longrightarrow & H_{\acute{e}t}^{2d-1}(X \sqcup X, \Lambda(d)) & \rightarrow 0
 \end{array} \tag{5-9}$$

Since $d \geq 3$, it follows from the weak Lefschetz theorem for étale cohomology (see [Milne 1980, Theorem VI.7.1]) that the vertical arrow on the left end of (5-9) is surjective and the one on the right end is an isomorphism. We next note that the inclusion $F = D \cap H_1 \hookrightarrow D$ induces a bijection between the irreducible components of F and D by our choice of the hypersurface H_1 (see Section 2E). It follows from this, together with the isomorphism $H_{\acute{e}t}^{2d-2}(D, \Lambda) \xrightarrow{\sim} H_{\acute{e}t}^{2d-2}(D^N, \Lambda)$ and [Milne 1980, Lemma VI.11.3], that the second vertical arrow from the left in (5-9) is an isomorphism. A diagram chase now shows that $\tau_{1,*}$ is an isomorphism. This proves the lemma. \square

Remark 5.2. We should warn the reader here that Lemma 5.1 proves an analogue of the weak Lefschetz theorem only for a specific étale cohomology group. We do not expect this to be true for other cohomology groups in general and it will depend critically on D .

5C. The torsion in Chow group with modulus and relative étale cohomology. Let $D \subset X$ be an effective Cartier divisor on a smooth scheme X as above. Recall from [Binda and Krishna 2018, § 4, 5] that there are maps $p_{\pm,*} : \text{CH}_0(X|D) \rightarrow \text{CH}_0(S_X)$ and $l_{\pm}^* : \text{CH}_0(S_X) \rightarrow \text{CH}_0(X)$, where $p_{\pm,*}([x]) = l_{\pm,*}([x])$ and l_{\pm}^* is induced by the projection map $\mathcal{Z}_0(S_X \setminus D) = \mathcal{Z}_0(X_+ \setminus D) \oplus \mathcal{Z}_0(X_- \setminus D) \rightarrow \mathcal{Z}_0(X_{\pm} \setminus D)$. It follows at once from this description of the projection maps l_{\pm}^* that there is a commutative diagram

$$\begin{array}{ccc}
 \text{CH}_0(S_Y)_{\text{deg } 0} & \xrightarrow{l_{\pm}^*} & \text{CH}_0(Y)_{\text{deg } 0} \\
 \tau_{1,*} \downarrow & & \downarrow \tau_* \\
 \text{CH}_0(S_X)_{\text{deg } 0} & \xrightarrow{l_{\pm}^*} & \text{CH}_0(X)_{\text{deg } 0}.
 \end{array} \tag{5-10}$$

We shall use the following decomposition theorem from [Binda and Krishna 2018, Theorem 7.1].

Theorem 5.3. *The projection map $\Delta : S_X \rightarrow X$ induces a flat pullback $\Delta^* : \text{CH}_0(X) \rightarrow \text{CH}_0(S_X)$ such that $l_{\pm}^* \circ \Delta^* = \text{Id}_{\text{CH}_0(X)}$. Moreover, there is a split exact sequence*

$$0 \rightarrow \text{CH}_0(X|D) \xrightarrow{p_{+,*}} \text{CH}_0(S_X) \xrightarrow{l_{+}^*} \text{CH}_0(X) \rightarrow 0. \tag{5-11}$$

Theorem 5.4. *Let X be a smooth projective scheme of dimension $d \geq 1$ over an algebraically closed field k of exponential characteristic p . Then for any prime $l \neq p$, there is an isomorphism*

$$\lambda_{X|D} : \text{CH}_0(X|D)\{l\} \xrightarrow{\sim} H_{\acute{e}t}^{2d-1}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)).$$

Proof. If $D = \emptyset$, we take $\lambda_{X|D}$ to be the isomorphism $\lambda_X : \text{CH}_0(X)\{l\} \xrightarrow{\sim} H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d))$ given by Bloch [1979, §2].

We now assume $D \neq \emptyset$ and let S_X be the double of X along D . We shall first prove by induction on d that there exists an isomorphism

$$\lambda_{S_X} : \text{CH}_0(S_X)\{l\} \xrightarrow{\sim} H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \tag{5-12}$$

such that $\iota_{\pm}^* \circ \lambda_{S_X} = \lambda_X \circ \iota_{\pm}^*$.

When $d = 1$, it follows easily from the Kummer sequence and [Levine and Weibel 1985, Proposition 1.4] that there is a natural isomorphism $H_{\acute{e}t}^1(C, \mu_n(1)) \xrightarrow{\sim} {}_n\text{CH}_0(C)$ for any reduced curve C over k and any integer $n \geq 1$ prime to p . The naturality of this isomorphism proves our assertion. Note that this isomorphism coincides with that of Bloch when C is smooth, as one directly checks (or see [Bloch 1979, Proposition 3.6]).

We next assume $d = 2$. In this case, we have shown in Theorem 3.4 that there is a homomorphism $\tau_Y : H_{\acute{e}t}^3(Y, \mu_n(2)) \rightarrow {}_n\text{CH}_0(Y)$ for any reduced quasiprojective surface Y over k and any integer n prime to p . We claim that the following diagram commutes:

$$\begin{array}{ccc} H_{\acute{e}t}^3(S_X, \mu_n(2)) & \xrightarrow{\tau_{S_X}} & {}_n\text{CH}_0(S_X) \\ \iota_-^* \downarrow & & \downarrow \iota_-^* \\ H_{\acute{e}t}^3(X, \mu_n(2)) & \xrightarrow{\tau_X} & {}_n\text{CH}_0(X) \end{array} \tag{5-13}$$

To prove this, recall from the construction of τ_Y in Section 3 that there are isomorphisms

$$H_{\acute{e}t}^3(Y, \mu_n(2)) \xrightarrow{\sim} \mathbb{H}^2(Y, \mathcal{K}_2^{\bullet}) \quad \text{and} \quad \text{CH}_0(Y) \xrightarrow{\sim} \text{CH}_0^{LW}(X) \xrightarrow{\sim} H^2(Y, \mathcal{K}_2)$$

(see [Binda and Krishna 2018, Theorem 3.17]) which are clearly functorial for the normalization map $Y^N \rightarrow Y$. Since X_{\pm} are two disjoint components of the normalization S_X^N , we see that these isomorphisms are functorial for the inclusions $\iota_{\pm} : X_{\pm} \hookrightarrow S_X$.

For a surface Y , the map τ_Y is then the natural map $\mathbb{H}^2(Y, \mathcal{K}_2^{\bullet}) \rightarrow {}_nH^2(Y, \mathcal{K}_2)$ obtained via the exact sequence $\mathbb{H}^2(Y, \mathcal{K}_2^{\bullet}) \rightarrow H^2(Y, \mathcal{K}_2) \xrightarrow{n} H^2(Y, \mathcal{K}_2)$. The commutativity of (5-13) then follows immediately from the naturality of the complex of Zariski sheaves on $\mathcal{K}_2[-1] \rightarrow \mathcal{K}_2^{\bullet} \rightarrow \mathcal{K}_2$ on Sch_k . This proves the claim.

Theorem 3.6 says that $\tau_Y : H_{\acute{e}t}^3(Y, \mathbb{Q}_l/\mathbb{Z}_l(2)) \xrightarrow{\sim} \text{CH}_0(Y)\{l\}$ is an isomorphism on the limit for every prime $l \neq p$. We let $\lambda_Y : \text{CH}_0(Y)\{l\} \rightarrow H_{\acute{e}t}^3(Y, \mathbb{Q}_l/\mathbb{Z}_l(2))$ be the negative of the inverse of this isomorphism. When Y is smooth over k , it follows from Lemma 3.7 that λ_Y agrees with Bloch’s isomorphism. It follows

then from (5-13) that

$$\iota_{\pm}^* \circ \lambda_{S_X} = \lambda_X \circ \iota_{\pm}^*. \tag{5-14}$$

We now assume $\dim(X) \geq 3$. We choose a closed embedding $X \hookrightarrow \mathbb{P}_k^N$ and hypersurfaces H_1, \dots, H_{d-2} as on page 1437. We continue to use the notations of Proposition 2.3. We set $Y = X_1 := X \cap H_1$ and assume by induction that there is an isomorphism $\lambda_{S_Y} : \text{CH}_0(S_Y)\{l\} \xrightarrow{\sim} H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1))$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{CH}_0(S_Y)\{l\} & \xrightarrow{\lambda_{S_Y}} & H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \\ \iota_{\pm}^* \downarrow & & \downarrow \iota_{\pm}^* \\ \text{CH}_0(Y)\{l\} & \xrightarrow{\lambda_Y} & H_{\acute{e}t}^{2d-3}(Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \end{array} \tag{5-15}$$

We now consider the diagram

$$\begin{array}{ccc} \text{CH}_0(S_Y)\{l\} & \xrightarrow{\lambda_{S_Y}} & H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \\ \tau_{1,*} \downarrow & & \downarrow \tau_{1,*} \\ \text{CH}_0(S_X)\{l\} & \xrightarrow{\lambda_{S_X}} & H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \end{array} \tag{5-16}$$

It follows from Proposition 2.3 that the left vertical arrow is an isomorphism and Lemma 5.1 says that the right vertical arrow is an isomorphism. Since λ_{S_Y} is an isomorphism too, it follows that there is a unique isomorphism $\lambda_{S_X} : \text{CH}_0(S_X)\{l\} \xrightarrow{\sim} H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d))$ such that (5-16) commutes.

We have to check that $\iota_{\pm}^* \circ \lambda_{S_X} = \lambda_X \circ \iota_{\pm}^*$. For this, we consider the diagram

$$\begin{array}{ccccc} & & \iota_{\pm}^* & & \\ & & \longrightarrow & & \\ \text{CH}_0(S_Y)\{l\} & \xrightarrow{\quad} & \text{CH}_0(Y)\{l\} & \xrightarrow{\quad} & \text{CH}_0(X)\{l\} \\ & \searrow \tau_{1,*} & \downarrow & \searrow \tau_* & \\ & & \text{CH}_0(S_X)\{l\} & \xrightarrow{\quad} & \text{CH}_0(X)\{l\} \\ & & \downarrow & & \downarrow \lambda_X \\ & & \text{CH}_0(S_X)\{l\} & \xrightarrow{\quad} & \text{CH}_0(X)\{l\} \\ & & \downarrow \lambda_{S_X} & & \downarrow \lambda_X \\ H_{\acute{e}t}^{2d-3}(S_Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) & \xrightarrow{\quad} & H_{\acute{e}t}^{2d-3}(Y, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) & \xrightarrow{\quad} & H_{\acute{e}t}^{2d-3}(X, \mathbb{Q}_l/\mathbb{Z}_l(d-1)) \\ & \searrow \tau_{1,*} & \downarrow \iota_{\pm}^* & \searrow \tau_* & \\ & & H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) & \xrightarrow{\quad} & H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \\ & & \downarrow \iota_{\pm}^* & & \\ & & H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) & \xrightarrow{\quad} & H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \end{array}$$

We need to show that the front face of this cube commutes. Since $\tau_{1,*} : \text{CH}_0(S_Y)\{l\} \rightarrow \text{CH}_0(S_X)\{l\}$ is an isomorphism, it suffices to show that all other faces of the cube commute. The top face commutes

by (5-10) and the bottom face commutes by (5-5). The left face commutes by (5-16) and the right face commutes by [Bloch 1979, Proposition 3.3]. Finally, the back face commutes by (5-15) and we are done.

To finish the proof of the theorem, we use (5-4) and Theorem 5.3 and consider the diagram of split exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathrm{CH}_0(X|D)\{l\} & \xrightarrow{p_{+,*}} & \mathrm{CH}_0(S_X)\{l\} & \xrightarrow{\iota_-^*} & \mathrm{CH}_0(X)\{l\} \longrightarrow 0 \\
 & & \downarrow & & \downarrow \lambda_{S_X} & & \downarrow \lambda_X \\
 0 & \longrightarrow & H_{\acute{e}t}^{2d-1}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)) & \xrightarrow{p_{+,*}} & H_{\acute{e}t}^{2d-1}(S_X, \mathbb{Q}_l/\mathbb{Z}_l(d)) & \xrightarrow{\iota_-^*} & H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(d)) \longrightarrow 0.
 \end{array} \tag{5-17}$$

It follows from (5-12) that the square on the right is commutative. Moreover, the maps λ_{S_X} and λ_X are both isomorphisms. We conclude that there is an isomorphism

$$\lambda_{X|D} : \mathrm{CH}_0(X|D)\{l\} \xrightarrow{\sim} H_{\acute{e}t}^{2d-1}(X|D, \mathbb{Q}_l/\mathbb{Z}_l(d)). \quad \square$$

5D. Applications. We now deduce two applications of Theorem 5.4. Since the étale cohomology of $\mu_n(j)$ is nilinvariant whenever $(n, p) = 1$, it follows from (5-1) that $H_{\acute{e}t}^*(X|D, \mu_n(j)) \xrightarrow{\sim} H_{\acute{e}t}^*(X|D_{\mathrm{red}}, \mu_n(j))$. Using Theorem 5.4, we therefore obtain the following result about the prime-to- p torsion in the Chow group with modulus.

Theorem 5.5. *Let X be a smooth projective scheme of dimension $d \geq 1$ over k let $D \subset X$ be an effective Cartier divisor. Then, the restriction map ${}_n\mathrm{CH}_0(X|D) \rightarrow {}_n\mathrm{CH}_0(X|D_{\mathrm{red}})$ is an isomorphism for every integer n prime to p .*

Our second application is the following extension of Theorem 3.6 to the cohomology of relative K_2 -sheaf on a smooth surface. Recall that for a closed immersion $Y \hookrightarrow X$ in \mathbf{Sch}_k , the relative K -theory sheaf $\mathcal{K}_{i,(X,Y)}$ is the Zariski sheaf on X associated to the presheaf $U \mapsto K_i(U, Y \cap U)$.

Theorem 5.6. *Let X be a smooth projective surface over an algebraically closed field k of exponential characteristic p and let $l \neq p$ be a prime. Let $D \subset X$ be an effective Cartier divisor. Then, the following hold.*

- (1) $H^1(X, \mathcal{K}_{2,(X,D)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$.
- (2) $H^2(X, \mathcal{K}_{2,(X,D)})\{l\} \simeq H_{\acute{e}t}^3(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2))$.

Proof. It follows from Theorem 5.4 that $H_{\acute{e}t}^3(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \mathrm{CH}_0(X|D)\{l\}$. On the other hand, a combination of [Binda and Krishna 2018, Theorem 1.7] and [Krishna 2015b, Lemma 2.1] shows that there is a canonical isomorphism $\mathrm{CH}_0(X|D) \xrightarrow{\sim} H^2(X, \mathcal{K}_{2,(X,D)})$. This proves (2).

To prove (1), we consider the following commutative diagram for any integer $n \in k^\times$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & (5-18) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & H^1(S_X, \mathcal{K}_{2,(S_X, X_-)}) \otimes_{\mathbb{Z}} \mathbb{Z}/n & \rightarrow & H_{\acute{e}t}^3(S_X|X_-, \mu_n(2)) & \rightarrow & {}_n\text{CH}_0(X|D) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & H^1(S_X, \mathcal{K}_{2,S_X}) \otimes_{\mathbb{Z}} \mathbb{Z}/n & \longrightarrow & H_{\acute{e}t}^3(S_X, \mu_n(2)) & \longrightarrow & {}_n\text{CH}_0(S_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & H^1(X, \mathcal{K}_{2,X}) \otimes_{\mathbb{Z}} \mathbb{Z}/n & \longrightarrow & H_{\acute{e}t}^3(X, \mu_n(2)) & \longrightarrow & {}_n\text{CH}_0(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

The columns on the left and in the middle are exact by the splitting $\Delta \circ \iota_- = \text{Id}_X$ in (2-1). The column on the right is exact by Theorem 5.3. The middle and the bottom rows are exact by Theorem 3.4. It follows that the top row is also exact. Using the isomorphism $\iota_+^* : H_{\acute{e}t}^3(S_X|X_-, \mu_n(2)) \xrightarrow{\sim} H_{\acute{e}t}^3(X|D, \mu_n(2))$ (see (5-3)) and taking the direct limit of the terms in (5-18) with respect to the direct system $\{\mathbb{Z}/l^n\}_n$, we obtain a short exact sequence

$$0 \rightarrow H^1(S_X, \mathcal{K}_{2,(S_X, X_-)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \rightarrow H_{\acute{e}t}^3(X|D, \mathbb{Q}_l/\mathbb{Z}_l(2)) \rightarrow \text{CH}_0(X|D)\{l\} \rightarrow 0. \tag{5-19}$$

Using (5-19) Theorem 5.4, we conclude that $H^1(S_X, \mathcal{K}_{2,(S_X, X_-)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l = 0$ for every prime $l \neq p$. To finish the proof, it suffices now to show that the pullback map $\iota_+^* : H^1(S_X, \mathcal{K}_{2,(S_X, X_-)}) \rightarrow H^1(X, \mathcal{K}_{2,(X, D)})$ is surjective.

Given an open subset $W \subset D$, let $U = S_X \setminus (D \setminus W)$ be the open subset of S_X . Let $\mathcal{K}_{i,(S_X, X_-, D)}$ be the Zariski sheaf on D associated to the presheaf

$$W \mapsto K_i(U, X_+ \cap U, X_- \cap U) = \text{hofib}((K(U, X_- \cap U) \xrightarrow{\iota_+^*} K(X_+ \cap U, D \cap U));$$

see [Pedrini and Weibel 1994, Proposition A.5]. There is an exact sequence of K -theory sheaves

$$\nu_*(\mathcal{K}_{2,(S_X, X_-, D)}) \rightarrow \mathcal{K}_{2,(S_X, X_-)} \rightarrow \mathcal{K}_{2,(X_+, D)} \rightarrow \nu_*(\mathcal{K}_{1,(S_X, X_-, D)}),$$

where $\nu : D \hookrightarrow S_X$ is the inclusion. We have $\mathcal{K}_{1,(S_X, X_-, D)} = \mathcal{I}_D/\mathcal{I}_D^2 \otimes_D \Omega_{D/X}^1$ by [Geller and Weibel 1983, Theorem 1.1] and the latter term is zero. We thus get an exact sequence

$$\nu_*(\mathcal{K}_{2,(S_X, X_-, D)}) \rightarrow \mathcal{K}_{2,(S_X, X_-)} \rightarrow \mathcal{K}_{2,(X_+, D)} \rightarrow 0.$$

Since $H^2(S_X, \iota_*(\mathcal{K}_{2,(S_X, X_-, D)}) = H^2(D, \mathcal{K}_{2,(S_X, X_-, D)}) = 0$, we get

$$H^1(X, \mathcal{K}_{2,(S_X, X_-)}) \rightarrow H^1(X, \mathcal{K}_{2,(X_+, D)}) \simeq H^1(X, \mathcal{K}_{2,(X, D)}). \quad \square$$

6. Roitman's torsion theorem for 0-cycles with modulus

In this section, we prove the Roitman torsion theorem ([Theorem 1.2](#)) for the Chow group of 0-cycles with modulus. We shall deduce this result by proving a more general Roitman torsion theorem for singular varieties which are obtained by joining two smooth schemes along a common reduced Cartier divisor. As before, we assume the base field k to be algebraically closed of exponential characteristic p .

6A. Join of two smooth schemes along a common divisor. Let X_+ and X_- be two smooth connected quasiprojective schemes of dimension $d \geq 1$ over k and let $X_+ \xleftarrow{i_+} D \xrightarrow{i_-} X_-$ be two closed embeddings such that D is a reduced effective Cartier divisor on each X_{\pm} via these embeddings. Let X be the quasiprojective scheme over k such that the square

$$\begin{array}{ccc} D & \xrightarrow{i_+} & X_+ \\ i_- \downarrow & & \downarrow \iota_+ \\ X_- & \xrightarrow{\iota_-} & X \end{array} \quad (6-1)$$

is co-Cartesian in \mathbf{Sch}_k .

It is easy to check that all arrows in (6-1) are closed immersions and X is a reduced scheme with irreducible components $\{X_+, X_-\}$ with $X_{\text{sing}} = D$. In particular, the canonical map $\pi = \iota_+ \amalg \iota_- : X_+ \amalg X_- \rightarrow X$ is the normalization map. Let $U = X_{\text{reg}} = (X_+ \setminus D) \amalg (X_- \setminus D)$. We shall use the following important further properties of X .

Lemma 6.1. *The scheme X satisfies the following properties.*

- (1) *It is Cohen–Macaulay.*
- (2) *It is separably weakly normal.*
- (3) *It is weakly normal.*
- (4) *The map $D \rightarrow X_+ \times_X X_-$ is an isomorphism.*

Proof. Because X_+ and X_- are smooth of dimension d , they are Cohen–Macaulay. Since D is an effective Cartier divisor on a smooth scheme of dimension d , it is Cohen–Macaulay of dimension $d - 1$. The statement (1) now follows from [[Ananthnarayan et al. 2012](#), Lemma 1.2, Lemma 1.5, (1.5.3)]. Second statement follows because $D \subset X$ is the smallest conducting closed subscheme for the normalization $\pi : X^N \rightarrow X$ which is reduced. Furthermore, the map $D \times_X X^N \simeq D \amalg D \xrightarrow{\pi} D$ is just the collapse map and hence generically separable. The statement (3) follows from the previous two and (4) follows from [Lemma 2.2](#). \square

6B. The main result. We shall use the setup of [Section 6A](#) throughout this section. We assume from now on that $d = \dim(X) \geq 3$. Let $C \hookrightarrow X$ be a reduced Cartier curve (see [Section 2A](#)) such that each irreducible component of $C \setminus D$ is smooth and $C \setminus D$ has only double point singularities.

Lemma 6.2. *There exists a locally closed embedding $X \hookrightarrow \mathbb{P}_k^n$ such that for a general set of distinct hypersurfaces $H_1, \dots, H_{d-2} \subset \mathbb{P}_k^n$ of large degree containing C , the intersection $L = H_1 \cap \dots \cap H_{d-2}$ satisfies the following.*

- (1) $X \cap L$ is reduced.
- (2) $D \cap L$ is reduced.
- (3) $X_{\pm} \cap L$ is an integral normal surface whose singular locus is contained in $C_{\text{sing}} \cap D$.
- (4) The inclusion $C \subset (X \cap L)$ is a local complete intersection along D .

Proof. Since X is reduced of dimension $d \geq 3$, and since $C \subset X$ is a local complete intersection along $D = X_{\text{sing}}$, it follows from [Levine 1987, Lemmas 1.3, 1.4] (see also [Biswas and Srinivas 1999, Sublemma 1]) that for all $m \gg 1$, there is an open dense subset $\mathcal{U}_n(C, m)$ of the scheme $\mathcal{H}_n(C, m)$ of hypersurfaces in \mathbb{P}_k^n of degree m containing C such that the following hold.

- (a) For general distinct $H_1, \dots, H_{d-2} \in \mathcal{U}_n(C, m)$, the scheme-theoretic intersection $L = H_1 \cap \dots \cap H_{d-2}$ has the property that $X \cap L$ is reduced away from C .
- (b) $D \cap L$ is reduced away from C .
- (c) $X_{\pm} \cap L$ is integral away from C .
- (d) $C \subset (X \cap L)$ is a local complete intersection along D .

Note that as X_{\pm} is smooth and $X_{\pm} \cap L$ is a complete intersection in X_{\pm} , it follows that $X_{\pm} \cap L$ is a Cohen–Macaulay surface. In particular, it has no embedded component. Since C is nowhere dense in $X_{\pm} \cap L$, it follows from (c) that $X_{\pm} \cap L$ must be irreducible. Since C does not contain the generic point of $X_{\pm} \cap L$, it follows from (b) and Lemma 6.3 that $X_{\pm} \cap L$ is integral.

Setting $W := C \setminus (C_{\text{sing}} \cap D)$ and following [Kleiman and Altman 1979, § 5], let $W(\Omega_C^1, e)$ denote the locally closed subset of points in W where the embedding dimension of W is e . It follows from our assumption on the singularities of $C \setminus D$ that $\max_e \{\dim(W(\Omega_C^1, e)) + e\} \leq 2$. We conclude from their Theorem 7 that for all $m \gg 1$, there is an open dense subset $\mathcal{W}_n(W, m)$ of the scheme $\mathcal{H}_n(W, m)$ of hypersurfaces of \mathbb{P}_k^n of degree m containing W such that for general distinct $H_1, \dots, H_{d-2} \in \mathcal{W}_n(W, m)$, the scheme-theoretic intersection $L = H_1 \cap \dots \cap H_{d-2}$ has the following properties:

- (a') $X \cap L$ is a complete intersection in X of dimension two.
- (b') $(X \cap L) \setminus D$ is smooth.
- (c') $X_{\pm} \cap L$ is smooth away from $C_{\text{sing}} \cap D$.

It follows from (c') that $X_{\pm} \cap L$ is a Cohen–Macaulay surface whose singular locus is contained in $C_{\text{sing}} \cap D$. It follows from Serre's normality condition that $X_{\pm} \cap L$ is normal. Since C is the closure of W in X , we must have $\mathcal{H}_n(W, m) = \mathcal{H}_n(C, m)$ and in particular, $D \subset X \cap L$. Combining (a) – (d) and (a') – (c') together, we conclude that there is a closed immersion $Y \subset X$ with the following properties:

- (1) $\dim(Y) = 2$.
- (2) The inclusions $Y \subset X$, $(X_{\pm} \cap Y) \subset X_{\pm}$ and $(D \cap Y) \subset D$ are all complete intersections.
- (3) Y is reduced away from $C \cap D$.
- (4) $(D \cap Y) \subset Y$ is a Cartier divisor which is reduced away from C .
- (5) $X_{\pm} \cap Y$ is an integral normal surface which is smooth away from $C_{\text{sing}} \cap D$.
- (6) $C \subset Y$ is a local complete intersection along D .
- (7) $Y \setminus D$ is smooth.

If $Y \subset X$ is as above, then (6) says that the local rings of Y at $C \cap D$ contain regular elements. In particular, $C \cap D$ can contain no embedded point of Y . We conclude from (3) and [Lemma 6.3](#) that a surface Y as above must be reduced. Since $D \subset X_{\pm}$ is a Cartier divisor, it is Cohen–Macaulay and hence (2) shows that $D \cap Y$ is a complete intersection Cohen–Macaulay curve inside D . Since $C \cap D$ can contain no generic point of $D \cap Y$, it follows from (4) and [Lemma 6.3](#) that $D \cap Y$ is a reduced curve. This finishes the proof of the lemma. \square

The following is a straightforward application of the prime avoidance theorem in commutative algebra and its proof is left to the reader.

Lemma 6.3. *Let R be commutative Noetherian ring such that $R_{\mathfrak{p}}$ is reduced for every associated prime ideal \mathfrak{p} of R . Then R must be reduced.*

Lemma 6.4. *Let $X = X_+ \amalg_D X_-$ be as in (6-1) and let $Y = X \cap L$ be the complete intersection surface as obtained in [Lemma 6.2](#). Assume that X is projective. Then, $A^2(Y)$ is a semiabelian variety and the Roitman torsion theorem holds for Y . That is, the Abel–Jacobi map $\rho_Y : \text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow A^2(Y)$ is an isomorphism on the torsion subgroups.*

Proof. We let $Y_{\pm} = X_{\pm} \cap Y$ and $E = D \cap Y$. Let $\iota'_{\pm} : Y_{\pm} \hookrightarrow Y$ be the inclusion maps. It follows from the construction of $Y \subset X$ and an easy variant of [\[Binda and Krishna 2018, Lemma 2.2\]](#) that the canonical map $Y_+ \amalg_E Y_- \rightarrow Y$ is an isomorphism. It also follows from [Lemma 6.2](#) that $\iota'_+ \amalg \iota'_- : Y_+ \amalg Y_- \rightarrow Y$ is the normalization map. Since Y is the join of normal surfaces Y_+ and Y_- along the common closed subscheme E , it follows that the nonnormal locus of Y is the support of E . Let us denote the map $\iota'_+ \amalg \iota'_-$ by π' .

Let $\mathcal{I}_{\pm} \subset \mathcal{O}_{Y_{\pm}}$ denote the defining ideal sheaf for the inclusion $E \subset Y_{\pm}$. It is then immediate from (6-1) that $\pi'^*(\mathcal{I}_E) = \mathcal{I}_+ \times \mathcal{I}_-$ which is actually in \mathcal{O}_Y under the inclusion $\pi'^* : \mathcal{O}_Y \hookrightarrow \pi'_*(\mathcal{O}_{Y_+ \amalg Y_-})$. In other words, $E \hookrightarrow Y$ is a conducting subscheme for π' . Since $\pi'^*(E) = E \amalg E$ is reduced by [Lemma 6.2](#) and since the support of E is the nonnormal locus of Y , we see that π' is the normalization map for which E is the smallest conducting subscheme and is reduced. Since $\pi' : E \amalg E \rightarrow E$ is just the collapse map, it is clearly generically separable. We conclude (see [Definition 4.5](#)) that Y is a reduced projective surface which is separably weakly normal. The lemma now follows from [Theorem 4.9](#). \square

Theorem 6.5. *Let X_+ and X_- be two smooth projective schemes of dimension $d \geq 1$ over k and let $X = X_+ \amalg_D X_-$ be as in (6-1). Then, the Albanese variety $A^d(X)$ is a semiabelian variety and the Abel–Jacobi map $\rho_X : \text{CH}_0^{LW}(X)_{\text{deg}0} \rightarrow A^d(X)$ is isomorphism on the torsion subgroups.*

Proof. It follows from Lemma 6.1 that X is separably weakly normal and weakly normal. Hence, in characteristic zero, a general reduced Cartier curve on X containing a 0-cycle is weakly normal by [Cumino et al. 1983, Corollary 2.5]. We can thus repeat the argument of the proof of Theorem 4.9, which shows that ρ_X is surjective and reduces the remaining proof to showing only that the restriction of ρ_X to the p -primary torsion subgroup of $\text{CH}_0^{LW}(X)_{\text{deg}0}$ is injective when $p \geq 2$.

When $d = 1$, the scheme X is a weakly normal curve, and it is well known in this case that $\text{Pic}^0(X) \simeq A^1(X) \simeq J^1(X)$. The case $d = 2$ is Lemma 6.4. So we assume $d \geq 3$.

Let $\alpha \in \mathcal{Z}_0(X)$ be such that $n\alpha = 0$ in $\text{CH}_0^{LW}(X)$ for some integer $n = p^m$. Equivalently, $n\alpha \in \mathcal{R}_0^{LW}(X)$. Let us assume further that $\rho_X^{\text{semi}}(\alpha) = 0$.

We can now use [Biswas and Srinivas 1999, Lemma 2.1] to find a reduced Cartier curve C on X and a function $f \in \mathcal{O}_{C,S}^\times$ such that $n\alpha = \text{div}(f)$, where $S = C \cap X_{\text{sing}} = C \cap D$. Since part of $n\alpha$ supported on any connected component of C is also of the form $n\alpha' = \text{div}_{C'}(f')$ for some Cartier curve C' and some f' , we can assume that C is connected. Let $\{C_1, \dots, C_r\}$ denote the set of irreducible components of C .

We set $U = X \setminus D = (X_+ \setminus D) \amalg (X_- \setminus D)$. Let $\phi : X' \rightarrow X$ be a successive blow-up at smooth points such that the following hold.

- (1) The strict transform D_i of each C_i is smooth along $\phi^{-1}(U)$.
- (2) $D_i \cap D_j \cap \phi^{-1}(U) = \emptyset$ for $i \neq j$.
- (3) Each D_i intersects the exceptional divisor E (which is reduced) transversely at smooth points.

It is clear that there exists a finite set of blown-up closed points $T = \{x_1, \dots, x_n\} \subset U$ such that $\phi : \phi^{-1}(X \setminus T) \rightarrow X \setminus T$ is an isomorphism. In particular, $\phi : X'_{\text{sing}} = \phi^{-1}(D) \rightarrow D$ is an isomorphism. If we identify $\phi^{-1}(D)$ with D and let $X'_\pm = \text{Bl}_{T \cap X_\pm}(X_\pm)$, it becomes clear that $X' = X'_+ \amalg_D X'_-$. We set $U' = X' \setminus D = \phi^{-1}(U)$. Let C' denote the strict transform of C with components $\{C'_1, \dots, C'_r\}$.

Since ϕ is an isomorphism over an open neighborhood of D , it follows that $\phi^{-1}(S) \simeq S$ and the map $C' \rightarrow C$ is an isomorphism along D . In particular, we have $f \in \mathcal{O}_{C',S}^\times$ and $\phi_*(\text{div}_{C'}(f)) = \text{div}_C(f) = n\alpha$. Since $\text{Supp}(\alpha) \subset C'$, we can find $\alpha' \in \mathcal{Z}_0(X')$ supported on C' such that $\phi_*(\alpha') = \alpha$. This implies that $\phi_*(n\alpha' - \text{div}_{C'}(f)) = 0$. Setting $\beta = n\alpha' - \text{div}_{C'}(f)$, it follows that β must be a 0-cycle on the exceptional divisor such that $\phi_*(\beta) = 0$.

We can now write $\beta = \sum_{i=0}^n \beta_i$, where β_i is a 0-cycle on X' supported on $\phi^{-1}(x_i)$ for $1 \leq i \leq n$ and β_0 is supported on the complement E . We must then have $\phi_*(\beta_i) = 0$ for all $0 \leq i \leq n$. Since ϕ is an isomorphism away from $T = \{x_1, \dots, x_n\}$, we must have $\beta_0 = 0$. We can therefore assume that β is a 0-cycle on E .

Next, we note that each $\phi^{-1}(\{x_i\})$ is a $(d - 1)$ -dimensional projective variety whose irreducible components are point blow-ups of \mathbb{P}_k^{d-1} , intersecting transversely in X'_{reg} . Moreover, we have $\phi_*(\beta_i) = 0$

for the pushforward map $\phi_* : \mathcal{Z}_0(\pi^{-1}(\{x_i\})) \rightarrow \mathbb{Z}$, induced by $\phi : \phi^{-1}(\{x_i\}) \rightarrow \text{Spec}(k(x_i)) = k$. But this means that $\deg(\beta) = \sum_{i=0}^n \deg(\beta_i) = 0$. In particular, there are finitely many smooth projective rational curves $L_j \subset E$ and rational functions $g_j \in k(L_j)$ such that $\beta = \sum_{j=1}^s (g_j)_{L_j}$.

Using the argument of [Bloch 1980, Lemma 5.2], we can further choose L_j 's so that $C'' := C' \cup (\bigcup_j L_j)$ is a connected reduced curve with following properties.

- (1) Each component of C'' is smooth along U' .
- (2) $C'' \cap U'$ has only ordinary double point singularities.

In particular, the embedding dimension of C'' at each of its singular points lying over U is two. Furthermore, $C'' \cap D = (C'' \setminus (\bigcup_j L_j)) \cap D = C' \cap D$. This implies that C'' is a Cartier curve on X' .

We now fix a closed embedding $X' = X'_+ \amalg_D X'_- \hookrightarrow \mathbb{P}_k^N$ and choose a complete intersection surface $j' : Y' \subset X'$ as in Lemma 6.2. Let $\pi' : X'_+ \amalg X'_- \rightarrow X'$ denote the normalization map with $\pi'^{-1}(Y') = Y'_+ \amalg Y'_-$. Note that $E' = Y' \cap D$ is a reduced curve and $Y'_{\text{sing}} = E'$. Since $C'' \cap E' = C' \cap E'$ and since C'' is Cartier on Y' , it follows that C' and L_j 's are also Cartier curves on Y' . Furthermore, α' is an element of $\mathcal{Z}_0(Y', E')$ such that $n\alpha' = \text{div}_{C'}(f) + \sum_j \text{div}_{L_j}(g_j)$. In particular, $\alpha' \in \text{CH}_0^{LW}(Y')$ and $n\alpha' = 0$ in $\text{CH}_0^{LW}(Y')$. Note that this also implies that $\alpha' \in \text{CH}_0^{LW}(Y')_{\text{deg } 0}$.

It follows from [Mallick 2009, Theorem 14] that there exists a commutative diagram

$$\begin{CD}
 \text{CH}_0^{LW}(Y')_{\text{deg } 0} @>\rho_{Y'}^{\text{semi}}>> J^2(Y') @>\pi'^*>> J^2(Y'_+) \times J^2(Y'_-) \\
 @V j'_* VV @VV j'_* V @VV j'_* V \\
 \text{CH}_0^{LW}(X')_{\text{deg } 0} @>\rho_{X'}^{\text{semi}}>> J^d(X') @>\pi'^*>> J^d(X'_+) \times J^d(X'_-)
 \end{CD} \tag{6-2}$$

Since $\phi : X' \rightarrow X$ is a blow-up at smooth closed points and since the Albanese variety of a smooth projective scheme is a birational invariant, it follows from the construction of $J^d(X)$ in Section 2D and [Esnault et al. 1999, Lemma 2.5] that the canonical pushforward map $\phi_* : \mathcal{Z}_0(X') \rightarrow \mathcal{Z}_0(X)$ gives rise to a commutative diagram

$$\begin{CD}
 \text{CH}_0^{LW}(X')_{\text{deg } 0} @>\rho_{X'}^{\text{semi}}>> J^d(X') \\
 @V \phi_* VV @VV \phi_* V \\
 \text{CH}_0^{LW}(X)_{\text{deg } 0} @>\rho_X^{\text{semi}}>> J^d(X),
 \end{CD} \tag{6-3}$$

in which the two vertical arrows are isomorphisms. It follows that $\alpha' \in \text{CH}_0^{LW}(Y')_{\text{deg } 0}$ is a 0-cycle such that $\rho_{X'}^{\text{semi}} \circ j'_*(\alpha') = 0$ in $J^d(X')$. Using (6-2), we conclude that

$$j'_* \circ \pi'^* \circ \rho_{Y'}^{\text{semi}}(\alpha') = \pi'^* \circ \rho_{X'}^{\text{semi}} \circ j'_*(\alpha') = 0. \tag{6-4}$$

Since $Y'_+ \amalg Y'_- \rightarrow Y'$ is the normalization map (see Lemma 6.4), we know that $J^2(Y'_+) \times J^2(Y'_-)$ is the universal abelian variety quotient of $J^2(Y')$. In particular, the map $J^2(Y')\{p\} \rightarrow J^2(Y'_+)\{p\} \times J^2(Y'_-)\{p\}$ is an isomorphism. On the other hand, $Y'_\pm \subset X'_\pm$ are the iterated hypersurface sections of normal projective schemes and hence it follows from [Lang 1959, Chapter 8, § 2, Theorem 5] that the right vertical arrow in (6-2) is an isomorphism of abelian varieties. Note here that X' or Y' need not be smooth for this isomorphism. It follows therefore from (6-4) that $\rho_{Y'}^{\text{semi}}(\alpha') = 0$. Lemma 6.4 now implies that $\alpha' = 0$ and we finally get $\alpha = \phi_*(\alpha') = 0$. □

6C. Applications of Theorem 6.5. We now obtain some applications of Theorem 6.5. Our first result is the following comparison theorem.

Theorem 6.6. *Let X_+ and X_- be two smooth projective schemes of dimension $d \geq 1$ over k and let $X = X_+ \amalg_D X_-$ be as in (6-1). Then, the canonical map $\text{CH}_0^{LW}(X) \rightarrow \text{CH}_0(X)$ is an isomorphism.*

Proof. Let L denote the kernel of the surjection $\text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0(X)$. Using the factorization $\text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0(X) \rightarrow K_0(X)$ (see [Binda and Krishna 2018, Lemma 3.13] and [Levine 1987, Corollary 2.7]), we know that L is a torsion subgroup of $\text{CH}_0^{LW}(X)_{\text{deg } 0}$ of bounded exponent. On the other hand, we also have a factorization $\text{CH}_0^{LW}(X)_{\text{deg } 0} \twoheadrightarrow \text{CH}_0(X)_{\text{deg } 0} \twoheadrightarrow J^d(X)$ by (2-4). Since $J^d(X)$ is the universal semiabelian variety quotient of $A^d(X)$, we can use Theorem 6.5 to replace $J^d(X)$ by $A^d(X)$. We therefore have a factorization $\text{CH}_0^{LW}(X)_{\text{deg } 0} \twoheadrightarrow \text{CH}_0(X)_{\text{deg } 0} \twoheadrightarrow A^d(X)$ of ρ_X^{semi} . Another application of Theorem 6.5 now shows that L is torsion-free. Hence, it must be zero. □

As second application of Theorem 6.5, we now prove the Roitman torsion theorem for the Chow group of 0-cycles with modulus when the underlying divisor is reduced.

Theorem 6.7. *Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Assume that D is reduced. Then, the Albanese variety with modulus $A^d(X|D)$ is a semiabelian variety and the Abel–Jacobi map $\rho_{X|D} : \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow A^d(X|D)$ is isomorphism on the torsion subgroups.*

Proof. By Theorem 6.6 and [Binda and Krishna 2018, (11.2)], there is a commutative diagram of split exact sequences:

$$\begin{array}{ccccccc}
 0 \rightarrow \text{CH}_0(X|D)_{\text{deg } 0} & \xrightarrow{p_{+,*}} & \text{CH}_0^{LW}(S_X)_{\text{deg } 0} & \xrightarrow{i_-^*} & \text{CH}_0(X)_{\text{deg } 0} & \rightarrow & 0 \\
 & & \downarrow \rho_{S_X} & & \downarrow \rho_X & & \\
 0 \rightarrow A^d(X|D) & \xrightarrow{p_{+,*}} & A^d(S_X) & \xrightarrow{i_-^*} & A^d(X) & \rightarrow & 0.
 \end{array}$$

We note here that the constructions of § 11 there are based on the assumption that Theorem 6.6 holds. Since X is smooth over k , we know that $A^d(X)$ is an abelian variety. Moreover, the right vertical arrow is isomorphism on the torsion subgroups by Roitman [1980] and Milne [1982]. We conclude that the theorem is equivalent to showing that $A^d(S_X)$ is a semiabelian variety and ρ_{S_X} is isomorphism on the torsion subgroups. But this follows from Theorem 6.5. □

Corollary 6.8. *Let X be a smooth projective scheme of dimension $d \geq 1$ over $\bar{\mathbb{F}}_p$ and let $D \subset X$ be a reduced effective Cartier divisor. Then, $\mathrm{CH}_0(X|D)_{\mathrm{deg} 0}$ is finite-dimensional. That is, the Abel–Jacobi map $\rho_X : \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ is an isomorphism.*

Proof. In view of [Theorem 6.7](#), we only need to show that $\mathrm{CH}_0(S_X)$ is a torsion abelian group. But this is already shown in the proof of [Corollary 4.10](#). \square

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Generalized Fourier coefficients of multiplicative functions	1311
LILIAN MATHIESEN	
A blowup algebra for hyperplane arrangements	1401
MEHDI GARROUSIAN, ARON SIMIS and ȘTEFAN O. TOHĂNEANU	
Torsion in the 0-cycle group with modulus	1431
AMALENDU KRISHNA	
Continuity of the Green function in meromorphic families of polynomials	1471
CHARLES FAVRE and THOMAS GAUTHIER	
Local root numbers and spectrum of the local descents for orthogonal groups: p-adic case	1489
DIHUA JIANG and LEI ZHANG	
Bases for quasisimple linear groups	1537
MELISSA LEE and MARTIN W. LIEBECK	