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Local root numbers and spectrum of the local descents for orthogonal groups: p-adic case

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We investigate the local descents for special orthogonal groups over p-adic local fields of characteristic zero, and obtain explicit spectral decomposition of the local descents at the first occurrence index in terms of the local Langlands data via the explicit local Langlands correspondence and explicit calculations of relevant local root numbers. The main result can be regarded as a refinement of the local Gan–Gross–Prasad conjecture (2012).

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1. Introduction

Let G be a group and H be a subgroup of G. For any representation π of G, it is a classical problem to look for the spectral decomposition of the restriction of π from G to H. The spectral decomposition problem can also be formulated in a different way. For a given π of G and a subgroup H, which representation σ of H has the property that

$$\mathrm{Hom}_H(\pi, \sigma) \neq 0? \tag{1-1}$$

And what is the dimension of this Hom-space? When π or H is given arbitrarily, it is hard for such a spectral decomposition to be well understood, and those questions may not have reasonable answers.

When G is a Lie group or more generally a locally compact topological group defined by a reductive algebraic group, one may seek geometric conditions on the pair (G, H) such that the multiplicity $m(\pi, \sigma)$, which is the dimension of the Hom-space in (1-1), is bounded and at most one. In such a circumstance, one may seek invariants attached to π and σ that detect the multiplicity $m(\pi, \sigma)$. The local Gan–Gross–Prasad conjecture [2012] for classical groups G defined over a local field F is one of the most successful examples concerning those general questions. When the local field F is a finite extension of the p-adic number field $\mathbb{Q}_p$ for some prime p, the local Gan–Gross–Prasad conjecture for orthogonal groups has been completely resolved by the work of J.-L. Waldspurger [2010; 2012a; 2012b] and of C. Mœglin and Waldspurger [2012].

One of the basic notions in the local Gan–Gross–Prasad conjecture for orthogonal groups G are the so called *Bessel models*. Over a p-adic local field F, Bessel models are defined in terms of a special family of twisted Jacquet functors. It is proved through the work of Aizenbud et al. [2010], Sun and Zhu [2012], Gan et al. [2012], and Jiang et al. [2010b] that the Bessel models over any local fields of characteristic zero are of multiplicity at most one. The local Gan–Gross–Prasad conjecture is to detect the multiplicity (which is either 1 or 0) in terms of the sign of the relevant local ε-factors.

Meanwhile, the Bessel models have been widely used in the theory of the Rankin–Selberg method to study families of automorphic L-functions and to define the corresponding local L-factors and local γ-factors. In terms of representation theory and the local Langlands functoriality, the Bessel models produce the local descent method, which has been successfully used in the explicit construction of certain local Langlands functorial transfers for classical groups [Jiang and Soudry 2003; 2012]. In the spirit of the Bernstein–Zelevinsky derivatives for irreducible admissible representations of general linear groups over p-adic local fields [Bernstein and Zelevinsky 1977], the Bessel models can be regarded as a tool to investigate basic properties of irreducible admissible representations of $G(F)$ in general.

For instance, if G is an odd special orthogonal group SO_{2n+1}, then the local descents constructed via the family of Bessel models may produce representations on the family of even special orthogonal groups SO_{2m}, whose F-ranks should be controlled (up to ± 1) by the F-rank δ of SO_{2n+1}, with $m = n-\delta, n-\delta+1, \dots, n-1, n$. When $m = n$, it is the restriction from SO_{2n+1} to SO_{2n}, which is the case of the classical problem of *symmetric breaking*. Hence the explicit spectral decomposition when a representation π of $\mathrm{SO}_{2n+1}(F)$ descends, via the twisted Jacquet functors of Bessel type, to $\mathrm{SO}_{2m}(F)$ is an interesting and important problem, and may be considered as a refinement of the local Gan–Gross–Prasad conjecture. One of the problems in our mind is to understand the spectral decomposition of the local descent for special orthogonal groups over p-adic local fields in terms of the local Langlands parameters.

We explain our approach below with more details. The method is applicable to other classical groups. In some cases, we have to replace the Bessel models by the Fourier–Jacobi models, following the formulation of the local Gan–Gross–Prasad conjecture in [Gan et al. 2012]. The connection of the results in this paper to automorphic forms is considered in the work of the authors [Jiang and Zhang 2015].

1A. ***Local descents.*** Let F be a nonarchimedean local field of characteristic zero, which is a finite extension of the p-adic number field $\mathbb{Q}_p$ for some prime p. As in [Jiang and Zhang 2015; Arthur 2013, Chapter 9], we use $G_n^* = \mathrm{SO}(V^*, q_{V^*}^*)$ to denote an F-quasisplit special orthogonal group that is defined by a nondegenerate, $\mathfrak{n}$-dimensional quadratic space (V^*, q^*) over F with $n = \left[\frac{\mathfrak{n}}{2}\right]$ and use $G_n = \mathrm{SO}(V, q)$ to denote a pure inner F-form of G_n^*. This means that both quadratic spaces (V^*, q^*) and (V, q) have the same dimension and the same discriminant, as discussed in [Gan et al. 2012], for instance.

Let $\Pi(G_n)$ be the set of equivalence classes of irreducible smooth representations of $G_n(F)$. It is well-known that any $\pi \in \Pi(G_n)$ is also admissible. Let $\mathfrak{r}$ be the F-rank of G_n. Take X^+ to be an $\mathfrak{r}$-dimensional totally isotropic subspace of (V, q), and take X^- to be the dual subspace of X^+. Then one has a polar decomposition of (V, q): $V = X^- \oplus V_0 \oplus X^+$, where (V_0, q_{V_0}) is the F-anisotropic kernel of (V, q). With a suitable choice of the order of the dual bases in X^- and X^+, one must have a minimal parabolic subgroup P_0 of G_n, whose unipotent radical can be realized in the upper triangular matrix form. For any standard F-parabolic subgroup $P = MN$, containing P_0, of G_n, take a character ψ_N of the unipotent radical $N(F)$ of $P(F)$, which is defined through a nontrivial additive character ψ_F of F. One may define the *twisted Jacquet module* for any $\pi \in \Pi(G_n)$ with respect to (N, ψ_N) to be the quotient

$$\mathcal{J}_{N,\psi_N}(V_\pi) := V/V(N, \psi_N),$$

where $V(N, \psi_N)$ is the span of the subset

$$\{\pi(n)v - \psi_N(n)v \mid \forall v \in V_\pi, \forall n \in N(F)\}.$$

Let M_{ψ_N} is the stabilizer of ψ_N in M. Then the twisted Jacquet module $\mathcal{J}_{N,\psi_N}(V_\pi)$ is a smooth representation of $M_{\psi_N}(F)$. In such a generality, one may not have much information about the twisted Jacquet module $\mathcal{J}_{N,\psi_N}(V_\pi)$, as a representation of $M_{\psi_N}(F)$. Following the inspiration of the Bernstein–Zelevinsky theory of derivatives for representations of p-adic GL_n [Bernstein and Zelevinsky 1977], the theory of the local descents is to obtain more explicit information about the twisted Jacquet module $\mathcal{J}_{N,\psi_N}(V_\pi)$ in terms of the given π and its local Langlands parameter, for a family of specially chosen data (N, ψ_N).

To introduce the twisted Jacquet modules of Bessel type, we take a family of partitions of the form:

$$\underline{p}_\ell := [(2\ell+1)1^{\mathfrak{n}-2\ell-1}], \tag{1-2}$$

with $0 \le \ell \le \mathfrak{r}$. Those partitions $\underline{p}_\ell$ are G_n-*relevant* in the sense that they correspond to F-rational unipotent orbits of $G_n(F)$. As in [Jiang and Zhang 2015], the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ corresponding to the partition $\underline{p}_\ell$ defines a unipotent subgroup $V_{\underline{p}_\ell}$ of G_n over F, and each F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ defines a *generic* character $\psi_{\mathcal{O}_\ell}$ of $V_{\underline{p}_\ell}(F)$.

More precisely, let $\{e_{\pm 1}, e_{\pm 2}, \dots, e_{\pm\mathfrak{r}}\}$ be a basis of $X^{\pm}$, respectively such that $q(e_i, e_{-j}) = \delta_{i,j}$ for all $1 \le i, j \le \mathfrak{r}$. Then we may choose the minimal parabolic subgroup P_0 to fix the following totally isotropic flag in (V, q):

$$V_1^+ \subset V_2^+ \subset \cdots \subset V_{\mathfrak{r}}^+ \quad \text{where } V_i^{\pm} = \mathrm{Span}\{e_{\pm 1}, \dots, e_{\pm i}\}. \tag{1-3}$$

For the partition $\underline{p}_\ell$ in (1-2), we consider the standard parabolic subgroup $P_{1^\ell} = M_{1^\ell} N_{1^\ell}$, containing P_0, with the Levi subgroup $M_{1^\ell} \cong \mathrm{GL}_1^{\times \ell} \times G_{n-\ell}$ and $V_{\underline{p}_\ell} = N_{1^\ell}$. Here $V_{\underline{p}_\ell}$ consists of elements of form:

$$V_{\underline{p}_\ell} = \left\{ v = \begin{pmatrix} z & y & x \\ & I_{\mathfrak{n}-2\ell} & y' \\ & & z^* \end{pmatrix} \in G_n : z \in Z_\ell \right\}, \tag{1-4}$$

where Z_ℓ is the standard maximal (upper-triangular) unipotent subgroup of GL_ℓ. Then the F-rational nilpotent orbits $\mathcal{O}_\ell$ in the F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$ correspond to the $\mathrm{GL}_1(F) \times G_{n-\ell}(F)$-orbits of F-anisotropic vectors in $(F^{\mathfrak{n}-2\ell}, q)$. The generic character $\psi_{\mathcal{O}_\ell}$ of $V_{\underline{p}_\ell}(F)$ may be explicitly defined as follows: Fix a nontrivial additive character ψ_F of F. For an anisotropic vector w_0 in $((V_\ell^+ \oplus V_\ell^-)^\perp, q)$ associated to the F-rational orbit $\mathcal{O}_\ell$ in $\mathcal{O}^{\mathrm{st}}_{\underline{p}_\ell}$, define a character ψ_{ℓ, w_0} of $V_{\underline{p}_\ell}(F)$ by

$$\psi_{\mathcal{O}_\ell}(v) = \psi_{\ell, w_0} := \psi\left(\sum_{i=1}^{\ell-1} z_{i,i+1} + q(y_\ell, w_0) \right) \tag{1-5}$$

where $z_{i,j}$ is the entry of the matrix z in the i-th row and j-th column and y_ℓ is the last row of the matrix y in (1-4).

The Levi subgroup M_{1^ℓ} acts on the set of those generic characters ψ_{ℓ, w_0} via the adjoint action on $V_{\underline{p}_\ell}$. We denote by $G_n^{\mathcal{O}_\ell}$ the identity component of the stabilizer in M_{1^ℓ} of the character $\psi_{\mathcal{O}_\ell}$, viewed as a subgroup of $G_{n-\ell}$. By Proposition 2.5 of [Jiang and Zhang 2015], the algebraic group $G_n^{\mathcal{O}_\ell}$ is a special orthogonal group defined over F by a nondegenerate quadratic subspace (W_ℓ, q) of (V, q) with dimension $\mathfrak{n} - 2\ell - 1$. Here if $\psi_{\mathcal{O}_\ell}$ is of form ψ_{ℓ, w_0}, we have that

$$W_\ell = (V_\ell^+ \oplus \mathrm{Span}\{w_0\} \oplus V_\ell^-)^\perp \tag{1-6}$$

and $G_n^{\mathcal{O}_\ell}$ can be identified as the special orthogonal group $\mathrm{SO}(W_\ell, q)$. We refer to [Jiang and Zhang 2015, Proposition 2.5] for more structures of $G_n^{\mathcal{O}_\ell}$.

We define the following subgroup of G_n, which is a semidirect product,

$$R_{\mathcal{O}_\ell} := G_n^{\mathcal{O}_\ell} \ltimes V_{\underline{p}_\ell}. \tag{1-7}$$

For any $\pi \in \Pi(G_n)$, the twisted Jacquet module with respect to the pair $(V_{\underline{p}_\ell}, \psi_{\mathcal{O}_\ell})$ is defined by

$$\mathcal{J}_{\mathcal{O}_\ell}(\pi) = \mathcal{J}_{\mathcal{O}_\ell}(V_\pi) = \mathcal{J}_{V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell}}(V_\pi) := V_\pi / V_\pi(V_{\underline{p}_\ell}, \psi_{\mathcal{O}_\ell}), \tag{1-8}$$

which may also be called the *twisted Jacquet module of Bessel type* of π.

For an irreducible admissible representation σ of $G_n^{\mathcal{O}_\ell}(F)$, the linear functionals that belong to the following Hom-space

$$\operatorname{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}), \tag{1-9}$$

where $\sigma^\vee$ is the admissible dual of σ, are called the *local Bessel functionals* for the pair (π, σ). The uniqueness of local Bessel functionals asserts that

$$\dim \operatorname{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}) \le 1. \tag{1-10}$$

This was proved in [Aizenbud et al. 2010; Sun and Zhu 2012; Gan et al. 2012; Jiang et al. 2010b]. It is clear that

$$\operatorname{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}) \cong \operatorname{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma).$$

It is a natural problem to understand possible irreducible quotients σ of the module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ of $G_n^{\mathcal{O}_\ell}(F)$. The local Gan–Gross–Prasad conjecture determines such a quotient σ by means of the sign of the epsilon factor for the pair (π, σ). One of the main results of this paper is to determine all possible quotients σ for a given π with explicit description of their local Langlands parameters (Theorem 1.7), when the index ℓ is the first occurrence index (given in Definition 1.3).

In order to understand all possible irreducible quotients, we introduce a notion of the ℓ-th *maximal quotient* of π if $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero. Define

$$\mathcal{S}_{\mathcal{O}_\ell}(\pi) := \cap_{\mathfrak{l}_\sigma} \ker(\mathfrak{l}_\sigma), \tag{1-11}$$

where the intersection is taken over all local Bessel functionals $\mathfrak{l}_\sigma$ in $\operatorname{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma)$ for all $\sigma \in \Pi(G_n^{\mathcal{O}_\ell})$. The ℓ-th maximal quotient of π is defined to be

$$\mathcal{Q}_{\mathcal{O}_\ell}(\pi) := \mathcal{J}_{\mathcal{O}_\ell}(\pi)/\mathcal{S}_{\mathcal{O}_\ell}(\pi). \tag{1-12}$$

Of course, if $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is zero, we define $\mathcal{Q}_{\mathcal{O}_\ell}(\pi)$ to be zero. In this paper, we study this quotient for irreducible admissible representations π of $G_n(F)$ with generic local L-parameters, the definition of which will be explicitly given in Section 2A. Since we mainly discuss the local situation, we may call the local L-parameters *the L-parameters* for simplicity.

Proposition 1.1. *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, if the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero, then there exists a $\sigma \in \Pi(G_n^{\mathcal{O}_\ell})$ such that*

$$\operatorname{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma) \neq 0.$$

Namely, the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ of π is nonzero if and only if the ℓ-th maximal quotient $\mathcal{Q}_{\mathcal{O}_\ell}(\pi)$ of π is nonzero.

Proposition 1.1 follows from Lemma 3.1 in Section 3. By Proposition 1.1, if $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero, then the ℓ-th maximal quotient $\mathcal{Q}_{\mathcal{O}_\ell}(\pi)$ is nonzero. In this situation, we set

$$\mathcal{D}_{\mathcal{O}_\ell}(\pi) := \mathcal{Q}_{\mathcal{O}_\ell}(\pi) \tag{1-13}$$

and call $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ the ℓ-th *local descent* of π (with respect to the given F-rational orbit $\mathcal{O}_\ell$). Note that the group $G_n^{\mathcal{O}_\ell}(F)$ and the representation $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ depend on the F-rational structure of orbit $\mathcal{O}_\ell$.

The *theory of the local descents* is to understand the structure of the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ as a representation of $G_n^{\mathcal{O}_\ell}(F)$, in particular, in the situation when ℓ is the first occurrence index. In order to define the notion of the *first occurrence*, we prove the following *stability* of the local descents.

Proposition 1.2. *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, if there exists an ℓ_1 such that the ℓ_1-th local descent $\mathcal{D}_{\mathcal{O}_{\ell_1}}(\pi)$ is nonzero for some F-rational orbit $\mathcal{O}_{\ell_1}$, then the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ is nonzero for every $\ell \le \ell_1$ with a certain compatible $\mathcal{O}_\ell$.*

The details of the compatibility of $\mathcal{O}_{\ell_1}$ and $\mathcal{O}_\ell$ will be given in Proposition 3.3. Proposition 1.2 follows essentially from the relation between multiplicity and parabolic induction as discussed in Section 2D and will be included in Proposition 3.3 on the stability of the local descents in Section 3.

Definition 1.3 (first occurrence index). For $\pi \in \Pi(G_n)$, the first occurrence index $\ell_0 = \ell_0(\pi)$ of π is the integer ℓ_0 in $\{0, 1, \dots, \mathfrak{r}\}$ where $\mathfrak{r}$ is the F-rank of G_n, such that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\ell_0}}(\pi)$ is nonzero for some F-rational orbit $\mathcal{O}_{\ell_0}$, but for any $\ell \in \{0, 1, \dots, \mathfrak{r}\}$ with $\ell > \ell_0$, the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is zero for every F-rational orbit $\mathcal{O}_\ell$ associated to the partition $\underline{p}_\ell$ as defined in (1-2).

It is clear that the definition of the first occurrence index is also applicable to the representations π that may not be irreducible. At the first occurrence index, we define the notion of the local descent of π.

Definition 1.4 (local descent). For any given $\pi \in \Pi(G_n)$, assume that ℓ_0 is the first occurrence index of π. The ℓ_0-th local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ of π is called the first local descent of π (with respect to $\mathcal{O}_{\ell_0}$) or simply the local descent of π, which is the ℓ_0-th nonzero maximal quotient

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) := \mathcal{Q}_{\mathcal{O}_{\ell_0}}(\pi)$$

for the F-rational orbit $\mathcal{O}_{\ell_0}$ associated to the partition $\underline{p}_{\ell_0}$ as defined in (1-2).

It is clear that such an $\mathcal{O}_{\ell_0}$ always exists by the definition of ℓ_0, but may not be unique. Also, when $G_n = G_n^*$ is F-quasisplit and π is generic, i.e., has a nonzero Whittaker model, the first occurrence index is clearly $\ell_0 = n = \left[\frac{\mathfrak{n}}{2}\right]$, where $\mathfrak{n} = \dim V^*$. The discussion related to the first occurrence index in this paper will exclude this trivial case.

1B. ***Main results.*** The main results of the paper are about the spectral properties of the ℓ-th local descents of the irreducible smooth representations of $G_n(F)$ with generic L-parameters. At the first occurrence index, the spectral properties of the local descents are explicit.

Theorem 1.5 (square integrability). *Assume that $\pi \in \Pi(G_n)$ has a generic L-parameter. Then, at the first occurrence index $\ell_0 = \ell_0(\pi)$ of π, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is square-integrable and admissible. Moreover, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a multiplicity-free direct sum of irreducible square-integrable representations, with all its irreducible summands belonging to different Bernstein components.*

The proof of Theorem 1.5 depends on the following weaker result about the ℓ-th local descent for general ℓ. Because it can be deduced from the work on the local Gan–Gross–Prasad conjecture for orthogonal groups of Waldspurger [2010; 2012a; 2012b] for tempered L-parameters, and of Mœglin and Waldspurger [2012] with generic L-parameters, we state it as a corollary and refer to Section 3 for the details.

Proposition 1.6 (irreducible tempered quotient). *For any $\pi \in \Pi(G_n)$ with a generic L-parameter, and for any $\ell \in \{0, 1, \dots, \mathfrak{r}\}$ with $\mathfrak{r}$ being the F-rank of G_n, if the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero, then $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ has a tempered irreducible quotient as a representation of $G_n^{\mathcal{O}_\ell}(F)$, and so does the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$. In other words, if $\mathcal{J}_{\mathcal{O}_\ell}(\pi) \neq 0$, then there exists a $\sigma \in \Pi_{\mathrm{temp}}(G_n^{\mathcal{O}_\ell})$ such that*

$$\mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma) = \mathrm{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{D}_{\mathcal{O}_\ell}(\pi), \sigma) \neq 0.$$

It is worthwhile to mention an analogy of Proposition 1.6 to the *generic summand conjecture* (Conjecture 2.3 in [Jiang and Zhang 2015] and Conjecture 2.4 in [Jiang and Zhang 2017]). In such generality, if the representation π in Proposition 1.6 is unramified, then for any index ℓ, the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ has a tempered, unramified irreducible quotient. While the generic summand conjecture [Jiang and Zhang 2015; 2017] asserts that for any irreducible cuspidal automorphic representation of G_n with a generic global Arthur parameter, its global descent at the first occurrence index is cuspidal and contains at least one irreducible summand that has a generic global Arthur parameter. This assertion serves as a base for the construction of explicit modules for irreducible cuspidal automorphic representations of general classical groups with generic global Arthur parameters as developed in [Jiang and Zhang 2015; 2017]. We will discuss the global impact of the results obtained in this paper to the generic summand conjecture in our future work.

From the setup of the local descents, the theory is closely related to the local Gan–Gross–Prasad conjecture. By applying the theorems of Mœglin and Waldspurger on the local Gan–Gross–Prasad conjecture for generic L-parameters, and by explicit calculations of local L-parameters and the relevant local root numbers via the local Langlands correspondence in the situation considered here, we are able to obtain the following explicit spectral decomposition for the local descent at the first occurrence index. In order to state the result, we briefly explain the notation used in Theorem 1.7 below, and leave the details for Sections 2 and 5.

Write $\mathcal{Z} := F^\times/F^{\times 2}$. After fixing a rationality of the local Langlands correspondence ι_a as in Section 2B, when a $\pi \in \Pi(G_n)$ is determined by the parameter (φ, χ), the abelian group $\mathcal{Z}$ acts on the dual $\hat{\mathcal{S}}_\varphi$ of $\mathcal{S}_\varphi$. Denote by $\mathcal{O}_{\mathcal{Z}}(\pi)$ the $\mathcal{Z}$-orbit in $\hat{\mathcal{S}}_\varphi$ determined by π (see (2-17)). Denote by $[\varphi]_c$ the pair of the local L-parameters which are conjugate to each other via an element c in the complex group $\mathrm{O}(V)(\mathbb{C})$ with $\det(c) = -1$. In Definition 4.1, we introduce the notion of the descent $\mathfrak{D}_\ell(\varphi, \chi)$ and that of the first occurrence index $\ell_0 = \ell_0(\varphi, \chi)$ for the local parameters (φ, χ). The basic structure of the descent $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ at the first occurrence index is given in Theorem 4.6.

Theorem 1.7 (spectral decomposition). *Assume that $\pi \in \Pi(G_n(V_\mathfrak{n}))$ is associated to an equivalence class $[\varphi]_c$ of generic L-parameters.*

(1) *The first occurrence index of π is determined by the first occurrence index of the local parameters via the formula*

$$\ell_0(\pi) = \max_{\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)} \{\ell_0([\varphi]_c, \chi)\}.$$

(2) *For each F-rational orbit $\mathcal{O}_{\ell_0}$, the local descent of π at the first occurrence index $\ell_0 = \ell_0(\pi)$ is a multiplicity-free, direct sum of irreducible, square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$, which can be explicitly given below.*

   (a) *When $\mathfrak{n} = 2n$ is even and for $\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)$*

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \bigoplus_{\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi)} \pi(\phi, \chi_\phi^\star(\varphi, \phi)),$$

   where the local Langlands correspondence ι_a for π is given by $a = \mathrm{disc}(\mathcal{O}_{\ell_0})$ and $\chi_a(\pi) = \chi$, and the quadratic space W defining $G_n^{\mathcal{O}_{\ell_0}}$ is given by $\mathrm{disc}(W) = -\mathrm{disc}(\mathcal{O}_{\ell_0}) \cdot \mathrm{disc}(V_\mathfrak{n})$ *and* (2-25).

   (b) *When $\mathfrak{n} = 2n + 1$ is odd,*

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \bigoplus_{\substack{\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi) \\ \det(\phi) = \mathrm{disc}(\mathcal{O}_{\ell_0}) \cdot \mathrm{disc}(V_\mathfrak{n})}} \pi_{-\mathrm{disc}(\mathcal{O}_{\ell_0})}(\phi, \chi_\phi^\star(\varphi, \phi)),$$

   where the quadratic space defining $G_n^{\mathcal{O}_{\ell_0}}$ is given by (2-24) *and* (2-26).

Theorems 1.5 and 1.7 will be proved in Section 5, not only using the result of Proposition 1.6, but also using the proof of Proposition 1.6 in Section 3, with refinement. Moreover, in order to keep tracking the behavior of the local L-parameters and the sign of the local root numbers with the local descents, we need explicit information about the descent of local L-parameters $\mathfrak{D}_{\ell_0}(\varphi, \chi)$, given in Theorem 4.6.

We note that it is possible that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = 0$ for some F-rational orbit $\mathcal{O}_{\ell_0}$. But there exists at least one F-rational orbit $\mathcal{O}_{\ell_0}$ such that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) \neq 0$. Also an explicit formula for the character $\chi_\phi^\star(\varphi, \phi)$ can be found in Corollary 5.4.

Some more comments on Theorem 1.7 are in order.

First of all, in terms of the local Gan–Gross–Prasad conjecture, the spectral decomposition as given in Theorem 1.7 can be interpreted as follows: For any $\pi \in \Pi(G_n)$ with a generic L-parameter, at the first occurrence index, the spectral decomposition explicitly determines in terms of the local Langlands data of all possible irreducible representations σ of $G_n^{\mathcal{O}_{\ell_0}}(F)$ that form the distinguished pair with the given π as required by the local Gan–Gross–Prasad conjecture. Meanwhile, this spectral decomposition indicates that for such a given π, if a $\sigma \in \Pi(G_n^{\mathcal{O}_{\ell_0}})$ can be paired with π as in the local Gan–Gross–Prasad conjecture, then σ must be square integrable. Hence Theorem 1.7 and Corollary 5.4 can be regarded as a refinement of the local Gan–Gross–Prasad conjecture.

Secondly, it is interesting to compare briefly Theorem 1.7 with the local descent of the first named author and Soudry [Jiang and Soudry 2003; 2012], see also [Jiang et al. 2010a]. For instance, one takes G_n^* to be the F-split SO_{2n+1}. Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(F)$, which is of symplectic type, i.e., the local exterior square L-function of τ has a pole at $s=0$. The local descent in [Jiang and Soudry 2003; Jiang et al. 2010a] is to take $\pi=\pi(\tau,2)$ to be the unique Langlands quotient of the induced representation of the F-split $\mathrm{SO}_{4n}(F)$ with supercuspidal support (GL_{2n},τ). According to the endoscopic classification of Arthur [2013], $\pi=\pi(\tau,2)$ has a nongeneric (nontempered) local Arthur parameter $(\tau,1,2)$, and has the local L-parameter $\phi_\tau|\cdot|^{\frac{1}{2}}\oplus\phi_\tau|\cdot|^{-\frac{1}{2}}$, where ϕ_τ is the local L-parameter of τ. Now the local descent in [Jiang and Soudry 2003; Jiang et al. 2010a] shows that $\mathcal{D}_{n-1}(\pi)$ with $\ell_0=n-1$ being the first occurrence index is an irreducible generic supercuspidal representation of $\mathrm{SO}_{2n+1}(F)$ with the generic local Arthur parameter $(\tau,1,1)$ or the local L-parameter ϕ_τ. In this case, τ is the image of $\mathcal{D}_{n-1}(\pi)$ under the local Langlands functorial transfer from SO_{2n+1} to GL_{2n}. The result is even simpler than Theorem 1.7, as expected. However, from the point of view of the local Gan–Gross–Prasad conjecture, the result in [Jiang and Soudry 2003; Jiang et al. 2010a] can be viewed as a case of the local Gan–Gross–Prasad conjecture for nontempered local Arthur parameters. Hence the work to extend Theorem 1.7 to the representations with general local Arthur parameters is closely related to the local Gan–Gross–Prasad conjecture for nontempered local Arthur parameters. This is definitely a very interesting topic, but we will not discuss it with any more details in this paper.

Finally, we would like to elaborate an application of Theorem 1.7. For a generic local L-parameter ϕ of an F-quasisplit G_n^*, the local L-packet $\Pi_\phi(G_n^*)$ as defined in [Mœglin and Waldspurger 2012] contains a generic member, i.e., a member with a nonzero Whittaker model. From the relation between unipotent orbits of $G_n^*(F)$ and the twisted Jacquet modules for $G_n^*(F)$. The Whittaker model corresponds to the twisted Jacquet module associated to the regular unipotent orbit of G_n^*. In general, other members in the local L-packet $\Pi_\phi(G_n^*)$ may not have a nonzero twisted Jacquet module associated to the regular unipotent orbit. Hence it is desirable to know that for any member π in the local L-packet $\Pi_\phi(G_n^*)$, what kind twisted Jacquet modules does π have?

Let $\mathcal{P}(G_n^*)$ be the set of orthogonal partitions $\underline{p}=[p_1p_2\cdots p_r]$ associated to the F-stable unipotent orbits of $G_n^*(F)$. As defined in [Jiang 2014, Section 4] and similar to the definition of the twisted Jacquet modules of Bessel type, we may construct a twisted Jacquet module associated to any F-rational unipotent orbit $\mathcal{O}_{\underline{p}}$ in the corresponding F-stable orbit $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$. More precisely, we may construct, for any $\mathcal{O}_{\underline{p}}$ in $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$, a unipotent subgroup $V_{\mathcal{O}_{\underline{p}}}$ of G_n^* and a character $\psi_{\mathcal{O}_{\underline{p}}}$, and define the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\underline{p}}}(\pi)$ for any irreducible smooth representation π of $G_n^*(F)$. Now, we define $\mathfrak{p}(\pi)$ to be the subset of $\mathcal{P}(G_n^*)$, consisting of partitions $\underline{p}$ with the property that there exists an F-rational $\mathcal{O}_{\underline{p}}$ in the F-stable $\mathcal{O}_{\underline{p}}^{\mathrm{st}}$ such that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\underline{p}}}(\pi)$ is nonzero. Let $\mathfrak{p}^m(\pi)$ be the subset of $\mathfrak{p}(\pi)$ consisting of all maximal members in $\mathfrak{p}(\pi)$. Following [Kawanaka 1987; Mœglin 1996], one may take $\mathfrak{p}^m(\pi)$ to be the algebraic version of the *wave-front set* of π. It is generally believed [Jiang and Liu 2016, Conjecture 3.1] that if π is tempered, then the set $\mathfrak{p}^m(\pi)$ contains only one partition. One expects that this property holds for general π. Assume that $\mathfrak{p}^m(\pi) = \{\underline{p} = [p_1 p_2 \cdots p_r]\}$ with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$. In order to determine the algebraic wave-front set $\mathfrak{p}^m(\pi)$, it is an important step to understand how the largest part p_1 in the partition $\underline{p} \in \mathfrak{p}^m(\pi)$ is determined by the local Langlands data associated to π, via the local Langlands correspondence for $G_n^*(F)$. Here is our conjecture.

Conjecture 1.8. *Assume that $\pi \in \Pi_\phi(G_n^*)$ has a generic local L-parameter ϕ. Then the largest part p_1 in the partition $\underline{p} = [p_1 p_2 \cdots p_r]$ that belongs to $\mathfrak{p}^m(\pi)$ is equal to $2\ell_0 + 1$, where $\ell_0 = \ell_0(\pi)$ is the first occurrence index in the local descents.*

Assume that Conjecture 1.8 holds. Then part (1) of Theorem 1.7 asserts that the largest part p_1 of the partition $\underline{p}$ in the algebraic wave-front set $\mathfrak{p}^m(\pi)$ of π is completely determined by the property of the local Langlands data associated to π, if π has a generic local L-parameter. From this point of view, Theorem 1.7 serves as a base of an induction argument to determine the remaining parts $p_2, \ldots, p_r$. In Section 6, we will discuss Conjecture 1.8 via some examples. However, we expect that the induction argument is long and complicate, and hence leave it to our future work. Waldspruger [2018] confirmed this conjecture for a special family of generic local L-parameters of special odd orthogonal groups.

1C. ***The structure of the proofs and the paper.*** The local Gan–Gross–Prasad conjecture as proved in [Waldspurger 2010; 2012a; 2012b; Mœglin and Waldspurger 2012] is the starting point and the technical backbone of this paper; we state it in Section 2. In order to understand the F-rationality of the local L-parameters and the F-rationality of the local descents, we reformulate the local Langlands correspondence and the local Gan–Gross–Prasad conjecture in terms of the basic rationality data given by the underlying quadratic forms and quadratic spaces. This is discussed in Section 2B. It is clear that one might formulate such rationality in terms of rigid inner forms as discussed by T. Kaletha [2016]. Due to the nature of the current paper, the authors thought it more convenient and direct to use the formulation in Section 2B. In addition to the local Langlands correspondence as proved by Arthur [2013], we need the result for even special orthogonal groups as discussed by H. Atobe and W. T. Gan [2017].

We start to prove Proposition 1.6 in Section 3. First we show (Lemma 3.1) that for any $\pi \in \Pi(G_n)$, if the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ of Bessel type is nonzero, then it has an irreducible quotient. Then by applying the relation of multiplicity with parabolic induction (Proposition 2.6 as proved by Mœglin and Waldspurger [2012]), we obtain (Corollary 3.2) the relation of the first occurrence index with parabolic induction and the result that for any $\pi \in \Pi(G_n)$ with generic L-parameters, every irreducible quotient of the local descent at the first occurrence index is square-integrable. It is clear that Corollary 3.2 is one step towards Theorem 1.5. Finally, Proposition 1.6 follows from the stability of local descents (Proposition 3.3) and its proof.

In order to prove Theorems 1.5 and 1.7, we have to work on the local L-parameters. We define in Section 4 the *descent of local L-parameters*. In order to explicitly determine the structure of the descent of local L-parameters, we first calculate explicitly the local epsilon factors associated to a local L-parameter or a pair of local L-parameters, and keep tracking the local Langlands data through the process of local descents. With help from the theorem of Mœglin and Waldspurger [2012] on the local Gan–Gross–Prasad conjecture for orthogonal groups with generic L-parameters and the explicit local Langlands correspondence, we undertake a long and tedious calculation of the characters that parametrize the distinguished pair of representations as given by the local Gan–Gross–Prasad conjecture and determine the descent of the local L-parameters. Theorem 4.6 describes explicitly all the local L-parameters occurring in the descent of the local L-parameter associated to the initially given representation π. The point is that all the local L-parameters occurring in the descent of local L-parameters are discrete local L-parameters.

With Theorem 4.6 in hand, we are able to determine in Section 5 the local L-parameters for all the irreducible quotients of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$. We then use the structure of the Bernstein components of the local descent to prove Theorem 5.2; the irreducible quotients of local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ belong to different Bernstein components and are square integrable. Hence the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a multiplicity free direct sum of irreducible square-integrable representations and hence is square-integrable, which is Theorem 1.5. Theorem 1.7, which gives an explicit spectral decomposition of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$, can now be deduced from Theorems 1.5 and 4.6.

In Section 6 we provide even more explicit results for two special families of generic local L-parameters: the local cuspidal L-parameters as discussed by A.-M. Aubert, A. Moussaoui, and M. Solleveld [2015] (where they are called the local cuspidal Langlands parameters) and the local discrete unipotent L-parameters that correspond to the discrete unipotent representations in the sense of G. Lusztig [1995]. Moreover, we discuss Conjecture 1.8 via examples.

2. On the local Gan–Gross–Prasad conjecture

2A. ***Generic local L-parameters and local Vogan packets.*** We recall from [Mœglin and Waldspurger 2012] the notion of the generic local L-parameters and their structure. Without the assumption of the generalized Ramanujan conjecture, the localization of the generic global Arthur parameters [2013] will be examples of the generic local L-parameters defined and discussed in [Mœglin and Waldspurger 2012] for a p-adic local field F of characteristic zero. Following [loc. cit.], we denote by $\Phi_{\mathrm{gen}}(G_n^*)$ the set of conjugacy classes of the generic local L-parameters of G_n^*. We may simply call $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$ the *generic L-parameters* since we only consider the local situation in this paper.

Now we recall from [loc. cit.] the definition of generic local L-parameters for orthogonal groups. We denote by $\mathcal{W}_F$ the local Weil group of F. The local Langlands group of F, which is denoted by $\mathcal{L}_F$, is equal to the local Weil–Deligne group $\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C})$. The local L-parameters for $G_n^*(F)$ are of the form

$$\phi : \mathcal{L}_F \to {}^L G_n^* \tag{2-1}$$

with the property that the restriction of ϕ to the local Weil group $\mathcal{W}_F$ is Frobenius semisimple and trivial on an open subgroup of the inertia group $\mathcal{I}_F$ of F, the restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic, and ϕ is compatible with the projections of $\mathcal{L}_F$ and ${}^L G_n^*$ to the Weil group $\mathcal{W}_F$ in the definition. Then, for each given local L-parameter ϕ, there exists a datum $(L^*, \phi^{L^*}, \underline{\beta})$ such that:

(1) L^* is a Levi subgroup of $G_n^*(F)$ of the form

$$L^* = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_t} \times G_{n_0}^*.$$

(2) ϕ^{L^*} is a local L-parameter of L^* given by

$$\phi^{L^*} := \phi_1 \oplus \cdots \oplus \phi_t \oplus \phi_0 : \mathcal{L}_F \to {}^L L^*,$$

where ϕ_j is a local tempered L-parameter of GL_{n_j} for $j = 1, 2, \ldots, t$, and ϕ_0 is a local tempered L-parameter of $G_{n_0}^*$.

(3) $\underline{\beta} := (\beta_1, \ldots, \beta_t) \in \mathbb{R}^t$, such that $\beta_1 > \beta_2 > \cdots > \beta_t > 0$.

(4) The parameter ϕ can be expressed as

$$\phi = (\phi_1 \otimes |\cdot|^{\beta_1} \oplus \phi_1^\vee \otimes |\cdot|^{-\beta_1}) \oplus \cdots \oplus (\phi_t \otimes |\cdot|^{\beta_t} \oplus \phi_t^\vee \otimes |\cdot|^{-\beta_t}) \oplus \phi_0.$$

Following [Arthur 2013; Mœglin and Waldspurger 2012], the local L-packets can be formed for all L-parameters ϕ as displayed above, and are denoted by $\Pi_\phi(G_n^*)$. Now a local L-parameter ϕ is called *generic* if the associated local L-packet $\Pi_\phi(G_n^*)$ contains a generic member, i.e., a member with a nonzero Whittaker model with respect to a certain Whittaker data for G_n^*. The set of all generic local L-parameters of G_n^* is denoted by $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$. It was proved in [Mœglin and Waldspurger 2012] that all the members in such local L-packets are given by irreducible standard modules. Note that the situation here is more general than that considered in [Arthur 2013] and hence the members in a generic local L-packet may not be unitary. By [Mœglin and Waldspurger 2012], the local Gan–Gross–Prasad conjecture, which we call the local GGP conjecture for short, holds for all generic L-parameters $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$.

Recall that an F-quasisplit special orthogonal group $G_n^* = \mathrm{SO}(V^*, q^*)$ and its pure inner F-forms $G_n = \mathrm{SO}(V, q)$ share the same L-group ${}^L G_n^*$. As explained in [Gan et al. 2012, §7], if the dimension $\mathfrak{n} = \dim V = \dim V^*$ is odd, one may take $\mathrm{Sp}_{\mathfrak{n}-1}(\mathbb{C})$ to be the L-group ${}^L G_n^*$, and if the dimension $\mathfrak{n} = \dim V = \dim V^*$ is even, one may take $\mathrm{O}_{\mathfrak{n}}(\mathbb{C})$ to be ${}^L G_n^*$ when $\mathrm{disc}(V^*)$ is not a square in $F^\times$ and take $\mathrm{SO}_{\mathfrak{n}}(\mathbb{C})$ to be ${}^L G_n^*$ when $\mathrm{disc}(V^*)$ is a square in $F^\times$. Let S_ϕ be the centralizer of the image of ϕ in $\mathrm{SO}_{\mathfrak{n}}(\mathbb{C})$ or $\mathrm{Sp}_{\mathfrak{n}-1}(\mathbb{C})$, and S_ϕ° be its identity connected component group. Define the component group $\mathcal{S}_\phi := S_\phi / S_\phi^\circ$, which is an abelian 2-group.

By Theorem 1.5.1 of [Arthur 2013], and its extension to the generic L-parameters in $\Phi_{\mathrm{gen}}(G_n^*)$ in [Mœglin and Waldspurger 2012], the local L-packets $\Pi_\phi(G_n^*)$ are of multiplicity free and there exists a bijection

$$\pi \mapsto <\cdot, \pi> = \chi_\pi(\cdot) = \chi(\cdot) \tag{2-2}$$

between the finite set $\Pi_\phi(G_n^*)$ and the dual of $\mathcal{S}_\phi / Z({}^L G_n^*)$ of the finite abelian 2-group $\mathcal{S}_\phi$ associated to ϕ. Following the Whittaker normalization of Arthur, the trivial character χ corresponds to a generic member in the local L-packet $\Pi_\phi(G_n^*)$ with a chosen Whittaker character. There is an F-rationality issue on which the bijection may depend. We will discuss this issue explicitly in Section 2B. Under the bijection in (2-2), we may write

$$\pi = \pi(\phi, \chi) \tag{2-3}$$

in a unique way for each member $\pi \in \Pi_\phi(G_n^*)$ with $\chi = \chi_\pi$ as in (2-2). For any pure inner F-forms G_n of G_n^*, the same formulation works [Arthur 2013; Kaletha 2016]. The local Vogan packet associated to any generic L-parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$ is defined to be

$$\Pi_\phi[G_n^*] := \bigcup_{G_n} \Pi_\phi(G_n), \tag{2-4}$$

where G_n runs over all pure inner F-forms of the given F-quasisplit G_n^*. The L-packet $\Pi_\phi(G_n)$ is defined to be *empty* if the parameter ϕ is not G_n-relevant.

According to the structure of the generic L-parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$, one may easily figure out the structure of the abelian 2-group $\mathcal{S}_\phi$. Write

$$\phi = \bigoplus_{i \in \mathrm{I}} m_i \phi_i, \tag{2-5}$$

which is the decomposition of ϕ into simple and generic ones. The simple, generic local L-parameter ϕ_i can be written as $\rho_i \boxtimes \mu_{b_i}$, where ρ_i is an a_i-dimensional irreducible representation of $\mathcal{W}_F$ and μ_{b_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension b_i. We denote by $\Phi_{\mathrm{sg}}(G_n^*)$ the set of all simple, generic local L-parameters of G_n^*. In the decomposition (2-5), ϕ_i is called *of good parity* if $\phi_i \in \Phi_{\mathrm{sg}}(G_{n_i}^*)$ with $G_{n_i}^*$ being the same type as G_n^* where $n_i := \left[\frac{a_i b_i}{2}\right]$. We denote by I_{gp} the subset of I consisting of indices i such that ϕ_i is of good parity and by I_{bp} the subset of I consisting of indices i such that ϕ_i is self-dual, but not of good parity. We set $\mathrm{I}_{\mathrm{nsd}} := \mathrm{I} - (\mathrm{I}_{\mathrm{gp}} \cup \mathrm{I}_{\mathrm{bp}})$ for the indices of non-self-dual ϕ_i. Hence we may write $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$ in the following more explicit way:

$$\phi = \left(\bigoplus_{i \in \mathrm{I}_{\mathrm{gp}}} m_i \phi_i\right) \oplus \left(\bigoplus_{j \in \mathrm{I}_{\mathrm{bp}}} 2m'_j \phi_j\right) \oplus \left(\bigoplus_{k \in \mathrm{I}_{\mathrm{nsd}}} m_k (\phi_k \oplus \phi_k^\vee)\right), \tag{2-6}$$

where $2m'_j = m_j$ in (2-5) for $j \in \mathrm{I}_{\mathrm{bp}}$. According to this explicit decomposition, it is easy to know that

$$\mathcal{S}_\phi \cong \mathbb{Z}_2^{\#\mathrm{I}_{\mathrm{gp}}} \quad \text{or} \quad \mathbb{Z}_2^{\#\mathrm{I}_{\mathrm{gp}}-1}. \tag{2-7}$$

The latter case occurs if G_n is even orthogonal, and some orthogonal summand ϕ_i for $i \in \mathrm{I}_{\mathrm{gp}}$ has odd dimension.

In all cases, when G_n is even or odd orthogonal, for any $\phi \in \Phi_{\mathrm{gen}}(G_n^*)$, we write elements of $\mathcal{S}_\phi$ in the following form

$$(e_i)_{i \in \mathrm{I}_{\mathrm{gp}}} \in \mathbb{Z}_2^{\#\mathrm{I}_{\mathrm{gp}}}, \quad \text{(or simply denoted by } (e_i)) \tag{2-8}$$

where each e_i corresponds to ϕ_i-component in the decomposition (2-6) for $i \in \mathrm{I}_{\mathrm{gp}}$. The component group $\mathrm{Cent}_{\mathrm{O_n}(\mathbb{C})}(\phi)/\mathrm{Cent}_{\mathrm{O_n}(\mathbb{C})}(\phi)^\circ$ is denoted by A_ϕ. Then $\mathcal{S}_\phi$ consists of elements in A_ϕ with determinant 1, which is a subgroup of index 1 or 2. Also write elements in A_ϕ of form (e_i) where $e_i \in \{0, 1\}$ corresponds to the ϕ_i-component in the decomposition (2-6) for $i \in \mathrm{I}_{\mathrm{gp}}$. When G_n is even orthogonal and some ϕ_i for $i \in \mathrm{I}_{\mathrm{gp}}$ has odd dimension, then $(e_i)_{i\in\mathrm{I}_{\mathrm{gp}}}$ is in $\mathcal{S}_\phi$ if and only if $\sum_{i\in\mathrm{I}_{\mathrm{gp}}} e_i \dim\phi_i$ is even.

An L-parameter ϕ is of orthogonal type or symplectic type if its image $\mathrm{Im}(\phi)$ lies in $\mathrm{O}_{\mathfrak{n}}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$, respectively. In this paper, a self-dual L-parameter refers to be of either orthogonal type or symplectic type. Let ρ be an irreducible smooth representation of $\mathcal{W}_F$, which is Frobenius semisimple and trivial on an open subgroup of the inertia group $\mathcal{I}_F$ of F. Similarly, ρ is of orthogonal type or symplectic type if $\rho\boxtimes 1$ is of orthogonal type or symplectic type, respectively. In this case, $\rho\boxtimes 1$ is a discrete L-parameter.

2B. ***Rationality and the local Langlands correspondence.*** As explained in Section 1B, one of our motivations is to study the algebraic version of the wave-front set $\mathfrak{p}^m(\pi)$ of π via this local descent method. Our main approach is to perform an induction argument on the parts of partitions $\underline{p}$ in $\mathfrak{p}^m(\pi)$. In this inductive argument, the representations are descended to the ones of special orthogonal groups with different parity alternatively. We need to keep tracking the F-rational nilpotent orbits $\mathcal{O}_\ell$, which give the nonzero local descents. Meanwhile, $\mathcal{O}_\ell$ also determines the quadratic forms of the descendant special orthogonal groups. On the other hand, one needs to fix a normalization of the Whittaker datum in order to fix the local Langlands correspondence. Such a normalization depends also on the quadratic forms of the special orthogonal groups. Hence, we will fix the normalization for the parent representations and track the F-rational forms $\mathcal{O}_\ell$ for their descendants, then the normalization for their descendants will be determined. In this sense, we refer those normalizations as the F-rationality of the local Langlands correspondence for $G_n = \mathrm{SO}(V_{\mathfrak{n}})$ in terms of the F-rationality of the underlying quadratic space $(V_{\mathfrak{n}}, q_{\mathfrak{n}})$. When more quadratic spaces get involved in the discussion, we denote by $q_{\mathfrak{n}}$ the quadratic form of $V_{\mathfrak{n}}$, which was simply denoted by q before.

Define the discriminant of the quadratic space $V_{\mathfrak{n}}$ by

$$\mathrm{disc}(V_{\mathfrak{n}}) = (-1)^{\mathfrak{n}(\mathfrak{n}-1)/2}\det(V_{\mathfrak{n}}) \in F^\times/F^{\times 2}$$

and, similar to [O'Meara 2000, p. 167], define the Hasse invariant of $V_{\mathfrak{n}}$ by

$$\mathrm{Hss}(V_{\mathfrak{n}}) = \prod_{1\le i\le j\le \mathfrak{n}} (\alpha_i, \alpha_j),$$

where $V_{\mathfrak{n}}$ is decomposed orthogonally as $Fv_1 \oplus Fv_2 \oplus \cdots \oplus Fv_{\mathfrak{n}}$ with $q_{\mathfrak{n}}(v_i, v_i) = \alpha_i \in F^\times$. According to this definition, if $V_{\mathfrak{n}}$ is decomposed orthogonally as $V_{\mathfrak{n}} = W \oplus U$, we have, by [O'Meara 2000, Remark 58:3], the following formulas:

$$\mathrm{disc}(V_{\mathfrak{n}}) = (-1)^{ab}\,\mathrm{disc}(W)\,\mathrm{disc}(U) \tag{2-9}$$

$$\mathrm{Hss}(V_{\mathfrak{n}}) = \mathrm{Hss}(W)\,\mathrm{Hss}(U)((-1)^{a(a-1)/2}\,\mathrm{disc}(W), (-1)^{b(b-1)/2}\,\mathrm{disc}(U)) \tag{2-10}$$

where $a = \dim W$ and $b = \dim U$.

Recall from Section 1A that for each F-rational orbit $\mathcal{O}_\ell$ the nondegenerate character $\psi_{\mathcal{O}_\ell}$ is given by ψ_{ℓ,w_0} defined in (1-5) where w_0 is an anisotropic vector w_0. Define

$$\operatorname{disc}(\mathcal{O}_\ell) := \operatorname{disc}(V_\ell^+ \oplus Fw_0 \oplus V_\ell^-) \quad \text{and} \quad \operatorname{Hss}(\mathcal{O}_\ell) := \operatorname{Hss}(V_\ell^+ \oplus Fw_0 \oplus V_\ell^-),$$

where $V_\ell^\pm$ is defined in (1-3). Since the quadratic space $V_\ell^+ \oplus Fw_0 \oplus V_\ell^-$ is split, one has

$$\operatorname{disc}(\mathcal{O}_\ell) = q_{\mathfrak{n}}(w_0, w_0) \quad \text{and} \quad \operatorname{Hss}(\mathcal{O}_\ell) = (-1,-1)^{\ell(\ell+1)/2}((-1)^{\ell+1}, \operatorname{disc}(\mathcal{O}_\ell)).$$

Let W_ℓ be defined in (1-6). We have the decomposition

$$V_{\mathfrak{n}} = (V_\ell^+ \oplus Fw_0 \oplus V_\ell^-) \oplus W_\ell. \tag{2-11}$$

Then one has $\operatorname{disc}(W_\ell) = (-1)^{\mathfrak{n}-1} \operatorname{disc}(V_{\mathfrak{n}}) \operatorname{disc}(\mathcal{O}_\ell)$, and it is easy to check that

$$\operatorname{Hss}(W_\ell) = (-1,-1)^{\ell(\ell+1)/2} \operatorname{Hss}(V_{\mathfrak{n}})((-1)^\ell \operatorname{disc}(\mathcal{O}_\ell), (-1)^{\mathfrak{n}(\mathfrak{n}-1)/2+\ell} \operatorname{disc}(V_{\mathfrak{n}})). \tag{2-12}$$

In order to fix the F-rationality of the local Langlands correspondence, we adopt the (QD) *condition* of Waldspurger [2012a, p. 119] for the even special orthogonal group $(V_{\mathfrak{n}}, q_{\mathfrak{n}})$:

(QD) *The special orthogonal group of* $(V_{\mathfrak{n}}, q_{\mathfrak{n}}) \oplus (F, q_0)$ *is split, where* (F, q_0) *is the one-dimensional quadratic space with* $q_0(x,y) = -a \cdot xy$.

Here $a \in \mathcal{Z}$ will be specified later. In [Waldspurger 2012a], a is denoted $2\nu_0$.

Lemma 2.1. *Assume that* $\mathfrak{n}$ *is even. Then* $(V_{\mathfrak{n}}, q_{\mathfrak{n}})$ *satisfies* (QD) *if and only if*

$$\operatorname{Hss}(V_{\mathfrak{n}}) = (-1,-1)^{n(n+1)/2}((-1)^n a, \operatorname{disc}(V_{\mathfrak{n}})), \tag{2-13}$$

where $n = \frac{\mathfrak{n}}{2}$.

Proof. Write $V' = (V_{\mathfrak{n}}, q_{\mathfrak{n}}) \oplus (F, q_0)$. Since $\operatorname{disc}(V') = -a \cdot \operatorname{disc}(V_{\mathfrak{n}})$, we deduce

$$\operatorname{Hss}(V') = \operatorname{Hss}(V_{\mathfrak{n}})((-1)^n \operatorname{disc}(V_{\mathfrak{n}}), -a)(-a,-a) = \operatorname{Hss}(V_{\mathfrak{n}})((-1)^{n+1} \operatorname{disc}(V_{\mathfrak{n}}), -a). \tag{2-14}$$

Note that $\mathrm{SO}(V')$ is split if and only if V' is isometric to the quadratic space $\mathbb{H}^n \oplus Fv_0$ for some v_0, where $\mathbb{H}$ is a hyperbolic plane, equivalently $q_{\mathfrak{n}}(v_0, v_0) = \operatorname{disc}(V')$ and $\operatorname{Hss}(V') = \operatorname{Hss}(\mathbb{H}^n \oplus Fv_0)$. Under the assumption that $q_{\mathfrak{n}}(v_0, v_0) = \operatorname{disc}(V')$, by (2-10), we have

$$\begin{aligned} \operatorname{Hss}(\mathbb{H}^n \oplus Fv_0) &= \operatorname{Hss}(\mathbb{H}^n)((-1)^n, \operatorname{disc}(V'))(\operatorname{disc}(V'), \operatorname{disc}(V')) \\ &= (-1,-1)^{n(n+1)/2}(-1, \operatorname{disc}(V'))^{n+1}. \end{aligned} \tag{2-15}$$

$\mathrm{SO}(V')$ is split if and only if (2-14) equals (2-15). By using the relation that $\operatorname{disc}(V') = -a \cdot \operatorname{disc}(V_{\mathfrak{n}})$ and by simplifying the equality, we obtain this lemma. □

For example, suppose $\dim V_0^* = 2$, that is, $\mathrm{SO}(V_{\mathfrak{n}}^*)$ is a quasisplit and nonsplit even special orthogonal group, whose pure inner form $\mathrm{SO}(V_{\mathfrak{n}})$ satisfies $\operatorname{disc}(V_{\mathfrak{n}}) = \operatorname{disc}(V_{\mathfrak{n}}^*)$ and $\operatorname{Hss}(V_{\mathfrak{n}}) = -\operatorname{Hss}(V_{\mathfrak{n}}^*)$. Note that $\mathrm{SO}(V_{\mathfrak{n}})$ and $\mathrm{SO}(V_{\mathfrak{n}}^*)$ are F-isomorphic. Recall that $\Pi_\phi[G_n^*]$ is a generic Vogan packet of $\mathrm{SO}(V_{\mathfrak{n}}^*)$. The issue is which orthogonal group shall be assigned to $\chi \in \hat{\mathcal{S}}_\phi$ with $\chi((1)) = -1$ [Gan et al. 2012, §10]. We fix the F-rationality of the local Langlands correspondence following [Waldspurger 2012a, §4.6]. This means that for $\pi(\phi,\chi)\in\Pi_\phi[G_n^*]$ of $\mathrm{SO}(V')$ where $V'\in\{V_{\mathfrak{n}},V_{\mathfrak{n}}^*\}$, the quadratic space V' is determined by (2-13).

For the even special orthogonal groups, the local Langlands correspondence needs more explanation. Define c to be an element in $\mathrm{O}(V_{\mathfrak{n}})\setminus\mathrm{SO}(V_{\mathfrak{n}})$ with $\det(c)=-1$. For instance, when $\mathfrak{n}$ is odd, we can take $c=-I_{\mathfrak{n}}$. Consider the conjugate action of c on $\Pi(G_n)$, from which arises an equivalence relation $\sim_c$ on $\Pi(G_n)$. Obviously, when $\mathfrak{n}$ is odd, the c-conjugation is trivial. We only discuss the even special orthogonal case here. Denote by $\Pi(G_n)/\sim_c$ the set of equivalence classes. For $\sigma\in\Pi(G_n)$, let $[\sigma]_c$ denote the equivalence class of σ. Similarly, one has an analogous equivalence relation on the set $\Phi(G_n)$ of all L-parameters of G_n, which is also denoted by $\sim_c$.

Let us recall the desiderata of the weak local Langlands correspondence for even special orthogonal groups $\mathrm{SO}(V_{\mathfrak{n}})$ from [Atobe and Gan 2017, Desideratum 3.2]. For the needs of this paper, we only recall some partial facts from their desiderata, which has been verified in [loc. cit.], in order to fix the rationality of the local Langlands correspondence.

A Weak Local Langlands Correspondence for $G_n=\mathrm{SO}(V_{2n})$:

(1) There exists a canonical surjection

$$\bigsqcup_{G_n}\Pi(G_n)/\sim_c\to\Phi(G_n^*)/\sim_c,$$

where G_n runs over all pure inner forms of G_n^*. Note that the preimage of ϕ under the above map is the Vogan packet $\Pi_\phi[G_n^*]$ associated to ϕ.

(2) Let $\Phi^c(G_n^*)/\sim_c$ be the subset of $\Phi(G_n^*)/\sim_c$ consisting of the ϕ which contain an irreducible orthogonal subrepresentation of $\mathcal{L}_F$ with odd dimension. The following are equivalent:
   - $\phi\in\Phi^c(G_n^*)/\sim_c$.
   - Some $[\sigma]_c\in\Pi_\phi[G_n^*]$ satisfies $\sigma\circ\mathrm{Ad}(c)\cong\sigma$.
   - All $[\sigma]_c\in\Pi_\phi[G_n^*]$ satisfy $\sigma\circ\mathrm{Ad}(c)\cong\sigma$.

(3) For each $a\in\mathcal{Z}$, there exists a bijection

$$\iota_a\colon\Pi_\phi[G_n^*]\longrightarrow\hat{\mathcal{S}}_\phi,$$

which satisfies the endoscopic and twisted endoscopic character identities (refer to [Arthur 2013; Kaletha 2016] for instance).

(4) For $[\sigma]_c\in\Pi_\phi[G_n^*]$ and $a\in\mathcal{Z}$, the following are equivalent:
   - $\sigma\in\Pi(\mathrm{SO}(V_{2n}))$.
   - $\iota_a([\sigma]_c)((1))$ and $\mathrm{Hss}(V_{2n})$ satisfy the following equation

$$\mathrm{Hss}(V_{2n})=\iota_a([\sigma]_c)((1))(-1,-1)^{n(n+1)/2}((-1)^na,\mathrm{disc}(V_{2n})).\tag{2-16}$$

Note that the subscript a in ι_a is used to indicate the F-rationality of the Whittaker datum. The details can be found in [Atobe and Gan 2017, §3].

By the above weak local Langlands correspondence for $\mathrm{SO}(V_{2n})$ and the local Langlands correspondence for $\mathrm{SO}(V_{2n+1})$, each irreducible admissible representation $\pi \in \Pi(G_n)$ is associated to an equivalence class $[\phi]_c$ of L-parameters under c-conjugation. Following [Gan et al. 2012, §9 and §10] and [Atobe and Gan 2017, Proposition 3.5], define the action of $\mathcal{Z}$ on $\hat{\mathcal{S}}_\phi$ via an one-dimensional twist given by the local Langlands correspondence ι_a, which is

$$a \cdot \chi \to \chi \otimes \eta_a$$

where $\eta_a((e_i)_{i \in \mathrm{I_{gp}}}) = (\det \phi_i, a)_F^{e_i}$ and $(\cdot, \cdot)_F$ is the Hilbert symbol defined over F. Denote by $\mathcal{O}_{\mathcal{Z}}(\pi)$ the orbit in $\hat{\mathcal{S}}_\phi$ corresponding to π. More precisely, if $\pi = \pi(\phi, \chi)$ under the local Langlands correspondence ι_a for some a, one has

$$\mathcal{O}_{\mathcal{Z}}(\pi) = \{\chi \otimes \eta_\alpha \in \hat{\mathcal{S}}_\phi : \alpha \in \mathcal{Z}\}. \tag{2-17}$$

Note that the set $\mathcal{O}_{\mathcal{Z}}(\pi)$ is uniquely determined by π and independent of the choice of the local Langlands correspondence ι_a.

In the rest of this paper, the local Langlands correspondence refers to the weak local Langlands correspondence for even special orthogonal groups $\mathrm{SO}(V_{2n})$*, and the local Langlands correspondence for odd special orthogonal groups* $\mathrm{SO}(V_{2n+1})$.

Under the local Langlands correspondence ι_a of $G_n(F)$ with some $a \in \mathcal{Z}$, for an L-parameter ϕ and a character $\chi \in \hat{\mathcal{S}}_\phi$, denote by $\pi_a(\phi, \chi)$ the corresponding irreducible admissible representation of $G_n(F)$. Conversely, given an irreducible admissible representation π of $G_n(F)$, denote by $\phi_a(\pi)$ and $\chi_a(\pi)$ the associated L-parameter and its corresponding character in $\hat{\mathcal{S}}_\phi$, respectively.

When $G_n = \mathrm{SO}(V_{2n+1})$, the local Langlands correspondence is unique and independent of the choice of a. We denote the trivial action by $\mathcal{Z}$, then $\mathcal{O}_{\mathcal{Z}}(\pi)$ contains only π, and we simply write $\pi(\phi, \chi)$ and $(\phi(\pi), \chi(\pi))$, respectively.

Remark 2.2. Let ϕ be an L-parameter of $\mathrm{SO}(V_{2n})$. Suppose that all irreducible orthogonal summands of ϕ are even dimensional. Then the c-conjugate L-parameter ϕ^c is different from ϕ because ϕ^c is not $G_n^\vee(\mathbb{C}) = \mathrm{SO}_{2n}(\mathbb{C})$-conjugate to ϕ. It follows that $\Pi_\phi[G_n^*]$ and $\Pi_{\phi^c}[G_n^*]$ are two different Vogan packets. However, the conjugation $\mathrm{Ad}(c) \colon \pi_a(\phi, \chi) \mapsto \pi_a(\phi^c, \chi)$ gives a bijection between $\Pi_\phi[G_n^*]$ and $\Pi_{\phi^c}[G_n^*]$. According to [Atobe and Gan 2017, §3], the c-conjugation stabilizes the Whittaker datum associated to ι_a. Thus, the corresponding characters of $\mathcal{S}_\phi$ associated to π and $\pi \circ \mathrm{Ad}(c)$ under the same local Langlands correspondence are identical.

2C. ***On the local GGP conjecture: multiplicity one.*** The local GGP conjecture was explicitly formulated in [Gan et al. 2012] for general classical groups. We recall the case of orthogonal groups here.

Let $\mathfrak{n}$ and $\mathfrak{m}$ be two positive integers with different parity. For a relevant pair $G_n = \mathrm{SO}(V, q_V)$ and $H_m = \mathrm{SO}(W, q_W)$, and an F-quasisplit relevant pair $G_n^* = \mathrm{SO}(V^*, q_V^*)$ and $H_m^* = \mathrm{SO}(W^*, q_W^*)$ in the sense of [Gan et al. 2012], where $m = \left[\frac{\mathfrak{m}}{2}\right]$ with $\mathfrak{m} = \dim W = \dim W^*$, we are going to discuss the local

L-parameters for the group $G_n^* \times H_m^*$ and its relevant pure inner F-form $G_n \times H_m$. Consider admissible group homomorphism:

$$\phi : \mathcal{L}_F = \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_n^* \times {}^L H_m^*, \tag{2-18}$$

with the properties described for the local L-parameters in (2-1). We consider those L-parameters analogous to $\Phi_{\mathrm{gen}}(G_n^*)$, and denote the set of those L-parameters by $\Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$. To each parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$, one defines the associated local L-packet $\Pi_\phi(G_n^* \times H_m^*)$, as in [Mœglin and Waldspurger 2012]. For any relevant pure inner F-form $G_n \times H_m$, if a parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$ is $G_n \times H_m$-relevant, it defines a local L-packet $\Pi_\phi(G_n \times H_m)$, following [Arthur 2013; Mœglin and Waldspurger 2012]. If a parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$ is not $G_n \times H_m$-relevant, the corresponding local L-packet $\Pi_\phi(G_n \times H_m)$ is defined to be the empty set. The local Vogan packet associated to a parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$ is defined to be the union of the local L-packets $\Pi_\phi(G_n \times H_m)$ over all relevant pure inner F-forms $G_n \times H_m$ of the F-quasisplit group $G_n^* \times H_m^*$, which is denoted by

$$\Pi_\phi[G_n^* \times H_m^*].$$

The local GGP conjecture is formulated in terms of the local Bessel functionals as introduced in Section 1A. For a given relevant pair (G_n, H_m), assuming that $\mathfrak{n} \geq \mathfrak{m}$, take a partition of the form:

$$\underline{p}_\ell := [(2\ell+1)1^{\mathfrak{n}-2\ell-1}],$$

where $2\ell+1 = \dim W^\perp = \mathfrak{n}-\mathfrak{m}$. As in Section 1A, the F-stable nilpotent orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$ corresponding to the partition $\underline{p}_\ell$ defines a unipotent subgroup $V_{\underline{p}_\ell}$ and a generic character $\psi_{\mathcal{O}_\ell}$ associated to any F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$. Following [Jiang and Zhang 2015], there is an F-rational orbit $\mathcal{O}_\ell$ in the F-stable orbit $\mathcal{O}_{\underline{p}_\ell}^{\mathrm{st}}$ such that the subgroup $H_m = G_n^{\mathcal{O}_\ell}$ normalizes the unipotent subgroup $V_{\underline{p}_\ell}$ and stabilizes the character $\psi_{\mathcal{O}_\ell}$. As in Section 1A again, the uniqueness of local Bessel functionals asserts that

$$\dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma^\vee, \psi_{\mathcal{O}_\ell}) \leq 1,$$

as proved in [Aizenbud et al. 2010; Sun and Zhu 2012; Gan et al. 2012; Jiang et al. 2010b]. The stronger version in terms of Vogan packets is formulated as follows.

Theorem 2.3 (Mœglin–Waldspurger). *Let G_n^* and H_m^* be a relevant pair as given above. For any local L-parameter $\phi \in \Phi_{\mathrm{gen}}(G_n^* \times H_m^*)$, the following identity holds*:

$$\sum_{\pi\otimes\sigma\in\Pi_\phi[G_n^*\times H_m^*]} \dim \mathrm{Hom}_{R_{\mathcal{O}_\ell}(F)}(\pi \otimes \sigma, \psi_{\mathcal{O}_\ell}) = 1. \tag{2-19}$$

Theorem 2.3 is the orthogonal group case of the general local GGP conjecture over p-adic local fields. It was proved by Waldspurger [2010; 2012a; 2012b] for tempered local L-parameters, and by Mœglin and Waldspurger [2012] for generic local L-parameters.

2D. ***On multiplicity and parabolic induction.*** We now discuss the relation between multiplicity in the local GGP conjecture and parabolic induction as given in the work of Mœglin and Waldspurger in a series of papers [Waldspurger 2010; 2012a; 2012b; Mœglin and Waldspurger 2012] for orthogonal groups.

Let ϕ and φ be generic L-parameters of different type and of even dimension. Denote by $\pi^{\mathrm{GL}}(\phi)$ and $\pi^{\mathrm{GL}}(\varphi)$ the two irreducible representations of general linear groups via the local Langlands functoriality, which is independent with the choice of φ in $[\varphi]_c$. Define

$$\mathcal{E}(\varphi,\phi)=\det(\varphi)(-1)^n\cdot\varepsilon\bigl(\tfrac{1}{2},\pi^{\mathrm{GL}}(\varphi)\times\pi^{\mathrm{GL}}(\phi),\psi_F\bigr), \tag{2-20}$$

where $\varepsilon(s,\cdot,\psi_F)$ is the ε-factor defined by H. Jacquet, I. Piatetski-Shapiro and J. Shalika [1983]. Recall that $\det(\varphi)(-1)=(\det(\varphi),-1)_F$ and $(\cdot,\cdot)_F$ is the Hilbert symbol defined over F. Decompose φ and ϕ as in (2-6), their index sets are denoted by $\mathrm{I}^{\phi}_{\mathrm{gp}}$ and $\mathrm{I}^{\varphi}_{\mathrm{gp}}$ respectively, and define the character $\chi^\star(\phi,\varphi)$ to be the pair $(\chi^\star_\phi(\phi,\varphi),\chi^\star_\varphi(\phi,\varphi))$ (or simply $(\chi^\star_\phi,\chi^\star_\varphi)$), where

$$\chi^\star_\phi((e_{i'})_{i'\in\mathrm{I}^{\phi}_{\mathrm{gp}}})=\prod_{i'\in\mathrm{I}^{\phi}_{\mathrm{gp}}}\mathcal{E}(\varphi,\phi_{i'})^{e_{i'}} \tag{2-21}$$

and

$$\chi^\star_\varphi((e_{i})_{i\in\mathrm{I}^{\varphi}_{\mathrm{gp}}})=\prod_{i\in\mathrm{I}^{\varphi}_{\mathrm{gp}}}\mathcal{E}(\varphi_{i},\phi)^{e_{i}}. \tag{2-22}$$

By convention, if φ or ϕ equals 0, then $\chi^\star(\phi,\varphi)=(1,1)$.

Note that $\chi^\star_\phi$ and $\chi^\star_\varphi$ belong to $\hat{\mathcal{S}}_\phi$ and $\hat{\mathcal{S}}_\varphi$, respectively, and are independent of the choice of ψ_F defining local root numbers (see [Gan et al. 2012, §6]). It is easy to see that

$$\chi^\star_\phi((1))=\chi^\star_\varphi((1))=\mathcal{E}(\varphi,\phi).$$

By [Gan et al. 2012, §6 and §18], the character $\chi^\star(\phi,\varphi)$ of $\mathcal{S}_\phi\times\mathcal{S}_\varphi$ only depends on ϕ and $[\varphi]_c$.

For $\pi\in\Pi(G_n)$ with $G_n=\mathrm{SO}(V_{2n})$ and $\sigma\in\Pi(H_m)$ with $H_m=\mathrm{SO}(V_{2m+1})$, define the multiplicity for the pair (π,σ) by

$$m(\pi,\sigma)=\begin{cases}\dim\mathrm{Hom}_{R_{\mathcal{O}_\ell}}(\pi,\sigma\otimes\psi_{\mathcal{O}_\ell}) & \text{if } n>m,\\ \dim\mathrm{Hom}_{R_{\mathcal{O}_\ell}}(\sigma,\pi\otimes\psi_{\mathcal{O}_\ell}) & \text{if } n\le m.\end{cases} \tag{2-23}$$

Theorem 2.4 [Waldspurger 2012a, Theorem in §4.9]. *Assume that the twisted endoscopic identities and twisted endoscopic character identities as described in* [Waldspurger 2012a, §4.2 and §4.3] *hold. Suppose that φ and ϕ are generic L-parameters of $G_n=\mathrm{SO}(V_{2n})$ and $H_m=\mathrm{SO}(W_{2m+1})$. There exists an isometry class of quadratic spaces $V\times W$, unique up to a scalar multiplication, that satisfies the following conditions*:

$$\mathrm{disc}(V)=\det(\varphi). \tag{2-24}$$

$$\mathrm{Hss}(W)=(-1,-1)^{m(m+1)/2}((-1)^{m+1},\mathrm{disc}(W))\mathcal{E}(\varphi,\phi). \tag{2-25}$$

$$\mathrm{Hss}(V)=(-1,-1)^{n(n+1)/2}((-1)^n.\,\mathrm{disc}(W),\mathrm{disc}(V))\mathcal{E}(\varphi,\phi). \tag{2-26}$$

Moreover, under the local Langlands correspondence ι_a

$$m(\pi_a(\varphi, \chi_\varphi^\star), \pi_a(\phi, \chi_\phi^\star)) = 1,$$

where $a = -\operatorname{disc}(W)\operatorname{disc}(V)$.

It must be mentioned that the assumption on the t.e. identities and t.e. character identities for the quasisplit groups in Theorem 2.4 is removed by the work of Mœglin and Waldspurger [2016] on the stabilization of the twisted trace formula.

We note that Atobe and Gan [2017, Theorem 5.2] give a version of local GGP conjecture in terms of the weak local Langlands correspondence.

Remark 2.5. Suppose that $V \times W$ satisfies the conditions in Theorem 2.4, so $\lambda V \times \lambda W$ for any $\lambda \in \mathcal{Z}$. More precisely, after choosing $\operatorname{disc}(W) \in \mathcal{Z}$, one can choose an F-rational orbit $\mathcal{O}_\ell$ with $\operatorname{disc}(\mathcal{O}_\ell) = (-1)^{\min\{2m+1,2n\}} \operatorname{disc}(V)\operatorname{disc}(W)$. Then there is a unique isometry class $V \times W$ satisfying (2-24), (2-25) and (2-26), which is associated with $\mathcal{O}_\ell$.

The relation between the multiplicity and the parabolic induction is given by the following proposition of Mœglin and Waldspurger.

Proposition 2.6 [Mœglin and Waldspurger 2012, Proposition 1.3]. *Assume that* π *is the induced representation*

$$\operatorname{Ind}_P^{G_n} \tau_1|\det|^{\alpha_1} \otimes \cdots \otimes \tau_r|\det|^{\alpha_r} \otimes \pi_0$$

and σ *is the induced representation*

$$\operatorname{Ind}_Q^{H_m} \tau_1'|\det|^{\beta_1} \otimes \cdots \otimes \tau_t'|\det|^{\beta_t} \otimes \sigma_0$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_t \geq 0$, *all* τ_i *and* τ_j' *are unitary tempered irreducible representations of general linear groups, and* π_0 *and* σ_0 *are tempered irreducible representations of classical groups of smaller rank. Then the following equation of multiplicities holds*:

$$m(\pi, \sigma^\vee) = m(\pi_0, \sigma_0).$$

3. Proof of Proposition 1.6

The proof of Proposition 1.6 takes a few steps. We first prove the following lemma, which implies Proposition 1.1.

Lemma 3.1. *For any* $\pi \in \Pi(G_n)$, *the following hold*:

(1) *Each Bernstein component of the* ℓ*-th twisted Jacquet module* $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ *is finitely generated.*

(2) *If* $\mathcal{J}_{\mathcal{O}_\ell}(\pi) \neq 0$, *then there exists an irreducible representation* $\sigma \in \Pi(G_n^{\mathcal{O}_\ell})$ *such that*

$$\operatorname{Hom}_{G_n^{\mathcal{O}_\ell}(F)}(\mathcal{J}_{\mathcal{O}_\ell}(\pi), \sigma) \neq 0.$$

(3) *If* $\mathcal{J}_{\mathcal{O}_\ell}(\pi) \neq 0$, *then the* ℓ*-th local descent* $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ *of* π *is not zero.*

Proof. It is clear that (3) follows from (2). Assume that (1) holds. Then (2) follows, since any smooth representation of finite length has an irreducible quotient. Hence we only need to show that (1) holds.

For any $\pi \in \Pi(G_n)$, let $\mathrm{End}(\pi)$ be the space of continuous endomorphisms of the space of π. We have the following $G_n(F) \times G_n(F)$-equivariant homomorphism:

$$f \in \mathcal{S}(G_n(F)) := C_c^\infty(G_n(F)) \mapsto \pi(f) \in \mathrm{End}(\pi) \simeq \pi \otimes \pi^\vee,$$

where $\pi(f)(v) := \int_{G_n(F)} f(g)\pi(g)v\, \mathrm{d}g$ is the convolution operator induced by π for all Bruhat–Schwartz functions $f \in \mathcal{S}(G_n(F))$. It is not hard to check that this homomorphism is surjective. By the separation of variables, the homomorphism induces a surjective homomorphism

$$\mathcal{S}(G_n(F) \times G_n^{\mathcal{O}_\ell}(F)) = \mathcal{S}(G_n(F)) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)) \twoheadrightarrow \pi \otimes \pi^\vee \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)).$$

By taking the $(V_{\underline{p}_\ell}, \psi_{\mathcal{O}_\ell})$-coinvariant for the action by the left translation of $\mathcal{S}(G_n(F))$, one has the projection

$${}_{(V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell})}(\mathcal{S}(G_n)(F)) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)) \twoheadrightarrow \mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \pi^\vee \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)).$$

Then taking $G_n^{\mathcal{O}_\ell}$-coinvariant for the action by the left translation, one obtains a surjective homomorphism

$${}_{G_n^{\mathcal{O}_\ell}}[{}_{(V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell})}(\mathcal{S}(G_n(F))) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))] \twoheadrightarrow \pi^\vee \otimes {}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))].$$

Note that $G_n^{\mathcal{O}_\ell}(F)$ acts on ${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))]$ via the left translation on the second variable. As smooth representations of $G_n^{\mathcal{O}_\ell}(F)$, one has a surjection

$${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))] \twoheadrightarrow \mathcal{J}_{\mathcal{O}_\ell}(\pi).$$

We need to show that the map

$$\mathfrak{p}\colon \mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F)) \to \mathcal{J}_{\mathcal{O}_\ell}(\pi)$$

defined by $v \otimes f \mapsto \int_{G_n^{\mathcal{O}_\ell}(F)} f(h^{-1})\mathcal{J}_{\mathcal{O}_\ell}(\pi)(h)v\, \mathrm{d}h$ factors through the quotient ${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))]$ and is surjective. First, for any smooth vector $v \in \mathcal{J}_{\mathcal{O}_\ell}(\pi)$, suppose that v is fixed by a compact open subgroup K_0 of $G_n^{\mathcal{O}_\ell}(F)$. Let 1_{K_0} be a characteristic function of K_0 in $\mathcal{S}(G_n^{\mathcal{O}_\ell}(F))$. By choosing a suitable nonzero constant c_0, we have $v \otimes c_0 \cdot 1_{K_0} \mapsto v$. It follows that this map is surjective. Then, if $v \otimes f$ is of form $\sum(\pi(h_i)v_i \otimes L(h_i)f_i - v_i \otimes f_i)$ for some h_i, v_i and f_i, one must have that $\mathfrak{p}(v \otimes f) = 0$. Thus, the map $\mathfrak{p}$ factors through the quotient ${}_{G_n^{\mathcal{O}_\ell}}[\mathcal{J}_{\mathcal{O}_\ell}(\pi) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell}(F))]$.

Because

$$\mathcal{S}((G_n^{\mathcal{O}_\ell} \ltimes V_{\underline{p}_\ell})\backslash G_n \times G_n^{\mathcal{O}_\ell}, \psi_{\mathcal{O}_\ell}) = {}_{G_n^{\mathcal{O}_\ell}}({}_{(V_{\underline{p}_\ell},\psi_{\mathcal{O}_\ell})}(\mathcal{S}(G_n)) \otimes \mathcal{S}(G_n^{\mathcal{O}_\ell})),$$

we obtain a projection

$$\mathcal{S}((G_n^{\mathcal{O}_\ell}(F) \ltimes V_{\underline{p}_\ell}(F))\backslash G_n(F) \times G_n^{\mathcal{O}_\ell}(F), \psi_{\mathcal{O}_\ell}) \twoheadrightarrow \pi^\vee \otimes \mathcal{J}_{\mathcal{O}_\ell}(\pi).$$

By Theorem A and the subsequent remark in [Aizenbud et al. 2012], as a representation of $G_n(F)\times G_n^{\mathcal{O}_\ell}(F)$, each Bernstein component of

$$\mathcal{S}((G_n^{\mathcal{O}_\ell}(F)\ltimes V_{\underline{p}_\ell}(F))\backslash G_n(F)\times G_n^{\mathcal{O}_\ell}(F),\psi_{\mathcal{O}_\ell})$$

is finitely generated, and so is each Bernstein component of $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$. This finishes the proof of (1). □

Assume that $\pi\in\Pi(G_n)$ has a generic L-parameter. By the corollary in [Mœglin and Waldspurger 2012, §2.14], π can be written as the irreducible induced representation (standard module)

$$\operatorname{Ind}_{P(F)}^{G_n(F)}\tau_1|\det|^{\alpha_1}\otimes\cdots\otimes\tau_t|\det|^{\alpha_t}\otimes\pi_0, \tag{3-1}$$

where $\alpha_1>\alpha_2>\cdots>\alpha_t>0$, and all τ_i and π_0 are irreducible unitary tempered representations. One may write the Levi subgroup of P as $\mathrm{GL}_{a_1}\times\cdots\times\mathrm{GL}_{a_t}\times G_{n_0}$ and has that $\pi_0\in\Pi_{\mathrm{temp}}(G_{n_0})$.

As a corollary to Proposition 2.6, one has:

Corollary 3.2. *For any $\pi\in\Pi(G_n)$ with a generic L-parameter and written as in (3-1), the following hold*:

(1) *The first occurrence indices $\ell_0(\pi)$ and $\ell_0(\pi_0)$, with π_0 being as in (3-1), enjoy the relation*:

$$\ell_0(\pi)=n-n_0+\ell_0(\pi_0).$$

(2) *Every irreducible quotient of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi)$ of π is square-integrable.*

Finally we prove the following proposition, which proves Proposition 1.2 and finishes the proof of Proposition 1.6.

Let $\pi\in\Pi(G_n)$ of a generic L-parameter. Assume that the character $\psi_{\mathcal{O}_{\ell_0}}$ is given by ψ_{ℓ_0,w_0} (defined in (1-5)). Then the quadratic space W_{ℓ_0} defining $G^{\mathcal{O}_{\ell_0}}$ is of form (1-6). For each $\ell\le\ell_0$, we may choose the *compatible* F-rational orbit $\mathcal{O}_\ell$ such that its corresponding character is ψ_{ℓ,w_0}. Then its stabilizer $G^{\mathcal{O}_\ell}=\mathrm{SO}(W_\ell,q)$, where $W_\ell=X^+_{\ell_0-\ell}\oplus W_{\ell_0}\oplus X^-_{\ell_0-\ell}$ and $X^\pm_{\ell_0-\ell}=\mathrm{Span}\{e^\pm_{\ell+1},e^\pm_{\ell+2},\dots,e^\pm_{\ell_0}\}$.

Proposition 3.3 (stability of local descent). *For any $\pi\in\Pi(G_n)$ with a generic L-parameter, then the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ is nonzero for $\ell\le\ell_0$ and the above compatible $\mathcal{O}_\ell$.*

Proof. By Corollary 3.2, there exists an F-rational orbit $\mathcal{O}_{\ell_0}$ associated to the partition $\underline{p}_{\ell_0}$ such that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_{\ell_0}}(\pi)$ has an irreducible quotient σ belonging to $\Pi_{\mathrm{temp}}(G_n^{\mathcal{O}_{\ell_0}})$ and hence we have that $m(\pi,\sigma)\neq 0$.

For any occurrence index $\ell<\ell_0(\pi)$, we want to show that the twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ is nonzero for the F-rational orbit $\mathcal{O}_\ell$, which is compatible with $\mathcal{O}_{\ell_0}$. Note that the quadratic spaces defining both orthogonal groups $G_n^{\mathcal{O}_\ell}$ and $G_n^{\mathcal{O}_{\ell_0}}$ have the same anisotropic kernel if $\mathcal{O}_\ell$ is compatible with $\mathcal{O}_{\ell_0}$. Let τ be a unitary non-self-dual irreducible supercuspidal representation of $\mathrm{GL}_{\ell_0-\ell}(F)$. Define

$$\sigma_1=\operatorname{Ind}_{P_{\ell_0-\ell}(F)}^{G_n^{\mathcal{O}_\ell}(F)}\tau\otimes\sigma,$$

where $P_{\ell_0-\ell}$ is a parabolic subgroup of $G_n^{\mathcal{O}_\ell}$, whose Levi subgroup is isomorphic to $\mathrm{GL}_{\ell_0-\ell}\times G_n^{\mathcal{O}_{\ell_0}}$. It is clear that σ_1 is an irreducible tempered representation of $G_n^{\mathcal{O}_\ell}(F)$. By applying Proposition 2.6, we obtain an identity: $m(\pi, \sigma_1^\vee) = m(\pi, \sigma) \neq 0$. Hence the ℓ-th twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$ has a quotient $\sigma_1^\vee$, as representations of $G_n^{\mathcal{O}_\ell}(F)$. In particular, the ℓ-local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ is nonzero, and has the irreducible tempered representation $\sigma_1^\vee$ as a quotient. This finishes the proof. □

4. Descent of local L-parameters

We introduce a notion of the *descent of the local L-parameters* and determine the structure of the descent of local L-parameters, which forms one of the technical cores of the proofs of the main results in the paper. To do so, we have to calculate explicitly the relevant local root numbers.

Let $\varphi \in \Phi_{\text{gen}}(G_n^*)$, which is a $2n$-dimensional, self-dual local L-parameter, either of orthogonal type or of symplectic type. For $\chi \in \hat{\mathcal{S}}_\varphi$, define the *$\ell$-th descent* of (φ, χ) for any $\ell \in \{0, 1, \dots, n\}$, which is denoted by $\mathfrak{D}_\ell(\varphi, \chi)$, to be the set of generic self-dual local L-parameters ϕ, satisfying the following conditions:

(1) ϕ are local L-parameters of dimension $2(n-\ell)$ for $G_n^{\mathcal{O}_\ell}(F)$, which have the type different from that of φ.

(2) The equation $\chi_\varphi^\star(\varphi, \phi) = \chi$ holds.

Definition 4.1 (descent for L-parameters). For a parameter φ in $\Phi_{\text{gen}}(G_n)$ and $\chi \in \hat{\mathcal{S}}_\varphi$, the first occurrence index $\ell_0 := \ell_0(\varphi, \chi)$ of (φ, χ) is the integer ℓ_0 in $\{0, 1, \dots, n\}$, such that $\mathfrak{D}_{\ell_0}(\varphi, \chi) \neq \varnothing$ but for any $\ell \in \{0, 1, \dots, n\}$ with $\ell > \ell_0$, $\mathfrak{D}_\ell(\varphi, \chi) = \varnothing$. The ℓ_0-th descent $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ of (φ, χ) is called the first descent of (φ, χ) or simply the descent of (φ, χ).

Note that $\ell_0(\varphi, \chi) = n$ if and only if $\chi = 1$ by convention. As in the local descent on the representation side, this case will be excluded from the discussion at the first occurrence index, because in this case, the Bessel model becomes the Whittaker model.

Note that by definition $\mathfrak{D}_\ell(\varphi, \chi) = \mathfrak{D}_\ell(\varphi^c, \chi)$, and hence $\mathfrak{D}_\ell(\varphi, \chi)$ is stable under c-conjugate. It follows that $\mathfrak{D}_\ell([\varphi]_c, \chi)$ is well defined, and that $\phi^c \in \mathfrak{D}_\ell(\varphi, \chi)$ if and only if $\phi \in \mathfrak{D}_\ell(\varphi, \chi)$. In fact, the c-conjugation preserves the Bessel models, and hence the local descent of representations is stable under c-conjugate and any two c-conjugate L-parameters have the same local descent.

4A. *Local root number.* Let ϕ be a local L-parameter of G_n and ψ_F be a nontrivial additive character of F, as before. The *local epsilon factor* defined by P. Deligne and J. Tate [1979] is given by

$$\varepsilon(s, \phi, \psi_F) = \varepsilon\left(\tfrac{1}{2}, \phi, \psi_F\right) q^{a(\phi,\psi)(1/2-s)}, \tag{4-1}$$

where $a(\phi, \psi)$ is the Artin conductor of ϕ and $\varepsilon\left(\tfrac{1}{2}, \phi, \psi_F\right)$ is the *local root number*. By [Gross and Reeder 2006, Proposition 15.1] and [Tate 1979], one also has the following properties of the local epsilon factors:

- $\varepsilon(s, \phi, \psi_a) = \det\phi(a)|a|^{-\dim\phi(1/2-s)}\varepsilon(s, \phi, \psi_F)$, where ψ_a is defined by $\psi_a(x) = \psi_F(ax)$.
- $\varepsilon(s, \phi \otimes |\cdot|^{s_0}, \psi_F) = q_F^{-s_0 a(\phi,\psi)}\varepsilon(s, \phi, \psi_F)$.
- $\varepsilon(s, \phi, \psi_F)\varepsilon(1-s, \phi^\vee, \psi_F) = \det(\phi)(-1)$.

- $\varepsilon(s, \phi \oplus \phi', \psi_F) = \varepsilon(s, \phi, \psi_F)\varepsilon(s, \phi', \psi_F)$.

Denote by μ_n the algebraic n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ in this paper. Let us decompose ϕ as

$$\phi = \oplus_{n\geq 0}\rho_n \otimes \mu_{n+1},$$

where (ρ_n, V_{ρ_n}) is a semisimple complex representation of $\mathcal{W}_F$, which may possibly be zero. By the work of B. Gross and M. Reeder [2010] for instance, one has the local epsilon factor

$$\varepsilon(s, \phi, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big) q^{a(\phi)(1/2-s)},$$

where $a(\phi) = \sum_{n\geq 0}(n+1)a(\rho_n) + \sum_{n\geq 1} n \cdot \dim V_{\rho_n}^{\mathcal{I}}$, and

$$\varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big) = \prod_{n\geq 0} \varepsilon\big(\tfrac{1}{2}, \rho_n, \psi_F\big)^{n+1} \prod_{n\geq 1} \det(-\rho_n(\mathrm{Fr})|V_{\rho_n}^{\mathcal{I}})^n, \tag{4-2}$$

with $\mathcal{I} = \mathcal{I}_F$ being the inertia group and V_{ρ_n} being the vector space defining ρ_n, and with $a(\rho_n)$ being the Artin conductor of ρ_n. More details on the normalization of ψ and the Haar measure defining the Artin conductor can be found in [Gross and Reeder 2010, §2].

By [Gan et al. 2012, Propositions 5.1 and 5.2], if ϕ is a self-dual L-parameter and $\det(\phi) = 1$, then $\varepsilon\big(\tfrac{1}{2}, \phi, \psi_F\big)$ is independent of the choice of ψ_F and is simply denoted by $\varepsilon(\phi)$. Moreover, $\varepsilon(\phi) = \pm 1$.

For example, if τ is a character of $F^\times$, we may rewrite τ as $|\cdot|^{s_0}\omega$, where ω is a unitary character of $F^\times$. Then

$$\varepsilon(s, \tau, \psi_F) = q^{n(1/2-s-s_0)} \frac{g(\bar{\omega}, \psi_{\varpi^{-n}})}{|g(\bar{\omega}, \psi_{\varpi^{-n}})|},$$

where n is the conductor of ω (i.e., ω is trivial on $1 + \varpi^n \mathfrak{o}_F$ but not trivial on $1 + \varpi^{n-1}\mathfrak{o}_F$) and the Gauss sum is defined by

$$g(\omega, \psi_a) = \int_{\mathfrak{o}_F^\times} \omega(u)\psi_F(au)\, du.$$

In particular, if τ is unramified, then $\varepsilon(s, \tau, \psi_F) = 1$. If τ is a ramified quadratic character, then it is of conductor 1. Thus $\varepsilon(s, \tau, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \tau, \psi_F\big)q^{1/2-s}$ and $\varepsilon^2\big(\tfrac{1}{2}, \tau, \psi_F\big) = \tau(-1)$.

More generally, let $\pi = [\tau|\cdot|^{1-r/2}, \tau|\cdot|^{r-1/2}]$ be the square-integrable representation of a general linear group determined by the line segment of Bernstein and Zelevinsky [1977], with the local L-parameter $\varphi_\tau \boxtimes \mu_r$. Here φ_τ is the associated irreducible representation of $\mathcal{W}_F$ via the local Langlands correspondence for general linear groups. Denote by ω_π the central character of π. If τ is a quadratic character of $\mathrm{GL}_1(F)$ and $\omega_\pi = 1$ (that is, if τ is nontrivial, then r is even), then

$$\varepsilon(s, \varphi_\tau \boxtimes \mu_r, \psi_F) = \begin{cases} q^{(r-1)(1/2-s)} & \text{if } \tau = 1 \text{ and } r \text{ is odd},\\ -\tau(\varpi)q^{(r-1)(1/2-s)} & \text{if } \tau \text{ is unramified and } r \text{ is even},\\ \tau(-1)^{r/2}q^{r(1/2-s)} & \text{if } \tau \text{ is ramified.}\end{cases} \tag{4-3}$$

We remark that under the assumption $\varphi_\tau \boxtimes \mu_r$ is self-dual and $\det(\varphi_\tau \boxtimes \mu_r) = 1$. If τ is a self-dual supercuspidal representation of $\mathrm{GL}_a(F)$ with $a > 1$, then

$$\varepsilon(s, \varphi_\tau \boxtimes \mu_r, \psi_F) = \varepsilon\big(\tfrac{1}{2}, \tau, \psi_F\big)^r q^{ra(\tau)(1/2-s)}. \tag{4-4}$$

Lemma 4.2. *Let $\varphi_1 = \varphi_\pi \boxtimes \mu_n$ and $\varphi_2 = \varphi_\tau \boxtimes \mu_m$ be two irreducible local L-parameters, with φ_π and φ_τ being self-dual and of dimensions a and b, respectively.*

(1) *If m and n are even, then $\varepsilon(\varphi_1 \otimes \varphi_2) = 1$.*

(2) *If $\varphi_\pi \otimes \varphi_\tau$ is of symplectic type, then*

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \begin{cases} 1 & \text{when } m+n \text{ is odd,} \\ \varepsilon(\pi \times \tau) & \text{when } mn \text{ is odd.} \end{cases}$$

(3) *If $\varphi_\pi \otimes \varphi_\tau$ is of orthogonal type and $\pi \ncong \tau$, then*

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2},$$

when $m+n$ is odd, and $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$, when mn is odd.

(4) *If $\varphi_\pi \otimes \varphi_\tau$ is of orthogonal type and $\pi \cong \tau$, then*

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2}(-1)^{\min\{m,n\}} = (-1)^{\min\{m,n\}},$$

when $m+n$ is odd, and $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$, when mn is odd.

Remark 4.3. If mn is odd, and φ_π and φ_τ are of orthogonal type, then $\det(\varphi_\pi \otimes \varphi_\tau) = \pm 1$ and $\varepsilon(\varphi_\pi \otimes \varphi_\tau)$ depends possibly on the additive character ψ_F. Thus, we add ψ_F in the ε for this case. In this case the local root number is calculated in [Gan et al. 2012, Theorem 6.2(2)]. However, after the normalization in (2-20), the local root number is independent on ψ_F (see [Gan et al. 2012, Theorem 6.2(1)]).

Proof. Since $\mu_n \otimes \mu_m = \bigoplus_{i=1}^{\min\{m,n\}} \mu_{n+m+1-2i}$, as representations of $\mathrm{SL}_2(\mathbb{C})$, one has

$$\varphi_1 \otimes \varphi_2 = \bigoplus_{i=1}^{\min\{m,n\}} (\varphi_\pi \otimes \varphi_\tau) \boxtimes \mu_{n+m+1-2i}.$$

If $\pi \ncong \chi\tau$ for any unramified character χ, there is no $\varphi_\pi \otimes \varphi_\tau(\mathcal{I})$-invariant vector. Following (4-4), one has

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \prod_{i=1}^{\min\{m,n\}} \varepsilon(\pi \times \tau, \psi_F)^{n+m+1-2i} = \varepsilon(\pi \times \tau, \psi_F)^{mn}. \tag{4-5}$$

Suppose that $\pi \cong \chi\tau$ for some unramified character χ. Following (4-2) and (6.2.5) in [Bushnell and Kutzko 1993], one has

$$\varepsilon((\varphi_\pi \otimes \varphi_\tau) \boxtimes \mu_r) = \varepsilon(\pi \times \tau, \psi_F)^r (-\chi(\varpi)^{d(\pi)})^{r-1},$$

where $d(\pi)$ is the number of all unramified characters χ such that $\pi \cong \chi\pi$. And an unramified character χ' satisfies $\pi \cong \chi'\pi$ if and only if the order of χ' divides $d(\pi)$. It follows that

$$\begin{aligned}\varepsilon(\varphi_1 \otimes \varphi_2) &= \prod_{i=1}^{\min\{m,n\}} \varepsilon(\pi \times \tau, \psi_F)^{n+m+1-2i}(-\chi(\varpi)^{d(\pi)})^{n+m-2i} \\ &= \varepsilon(\pi \times \tau, \psi_F)^{mn}(-\chi(\varpi)^{d(\pi)})^{mn-\min\{m,n\}}.\end{aligned} \tag{4-6}$$

Since π and τ are self-dual, the unramified character χ has the property that $\pi \cong \chi^2\pi$. Hence, the order of χ^2 divides $d(\pi)$, equivalently $\chi(\varpi)^{d(\pi)} = \pm 1$. If $\chi(\varpi)^{d(\pi)} = -1$, then the order of χ equals $2d(\pi)$, which implies $\pi \ncong \chi\pi$ and then $\pi \ncong \tau$. In this case, $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)^{mn}$.

If $\chi(\varpi)^{d(\pi)} = 1$, then the order χ divides of $d(\pi)$, which implies $\pi \cong \chi\tau \cong \tau$ and

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)^{mn}(-1)^{mn-\min\{m,n\}}. \tag{4-7}$$

By (4-6), one obtains (4-7).

Combining with the case $\pi \ncong \chi\tau$ for any unramified χ, we summarize

$$\varepsilon(\varphi_1 \otimes \varphi_2) = \begin{cases} \varepsilon(\pi \times \tau, \psi_F)^{mn} & \text{if } \pi \ncong \tau, \\ \varepsilon(\pi \times \tau, \psi_F)^{mn}(-1)^{mn-\min\{m,n\}} & \text{if } \pi \cong \tau. \end{cases}$$

Recall that $\varepsilon(\varphi)^2 = (\det\varphi)(-1) = \pm 1$ if φ is self-dual. Since $\varphi_\pi \otimes \varphi_\tau$ is self-dual, we have

$$\varepsilon(\varphi_\pi \otimes \varphi_\tau)^2 = \det(\varphi_\pi)(-1)^b \det(\varphi_\tau)(-1)^a. \tag{4-8}$$

If m and n are even, then 4 divides mn and $\min\{m,n\}$ is even. By (4-5), (4-6) and (4-8), we have $\varepsilon(\varphi_1 \otimes \varphi_2) = 1$.

Suppose that $\varphi_\pi \otimes \varphi_\tau$ is of symplectic type. Then $\det(\varphi_\pi \otimes \varphi_\tau) = 1$ and φ_π and φ_τ are of different type. If $m+n$ is odd, then mn is even and $\varepsilon(\pi \times \tau, \psi_F)^{mn} = 1$. As φ_π and φ_τ are irreducible and of different type, there is no unramified character χ satisfying $\pi \cong \chi\tau$. Therefore, $\varepsilon(\varphi_1 \otimes \varphi_2) = 1$ if $m+n$ is odd, and $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau)$ if mn is odd.

Suppose that $\varphi_\pi \otimes \varphi_\tau$ is of orthogonal type. If $\pi \ncong \tau$, then, when mn is even,

$$\varepsilon(\varphi_1 \otimes \varphi_2) = (\varepsilon(\pi \times \tau, \psi_F)^2)^{mn/2} = \det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2},$$

which is independent of the choice of ψ_F; and when mn is odd, $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$, which depends on the choice of ψ_F.

If $\pi \cong \tau$, then, when mn is even, $\varepsilon(\varphi_1 \otimes \varphi_2)$ equals

$$\det(\varphi_\pi)(-1)^{bmn/2} \det(\varphi_\tau)(-1)^{amn/2}(-1)^{\min\{m,n\}} = (-1)^{\min\{m,n\}}$$

and when mn is odd, $\varepsilon(\varphi_1 \otimes \varphi_2) = \varepsilon(\pi \times \tau, \psi_F)$ as $mn - \min\{m,n\}$ is even. □

In some special cases, the result is simple. The following example is about the quadratic unipotent L-parameters, which will be more explicitly discussed in Section 6.

Example 4.4. Let $\varphi_\pi = \chi \boxtimes \mu_n$ and $\varphi_\tau = \xi \boxtimes \mu_m$, where χ and ξ are quadratic characters. If $m+n$ is odd, then

$$\varepsilon(\pi \times \tau) = \begin{cases} (-\chi\xi(\varpi))^{\min\{m,n\}} & \text{if } \chi\xi \text{ is unramified,} \\ \chi\xi(-1)^{mn/2} & \text{if } \chi\xi \text{ is ramified.} \end{cases}$$

Remark 4.5. Let φ and ϕ be discrete L-parameters of different type. Then $\chi^\star(\varphi, \phi) = (\chi^\star_\varphi, \chi^\star_\phi)$, with $\chi^\star_\varphi$ and $\chi^\star_\phi$ as defined in (2-21) and (2-22), yields a pair of characters on A_φ and A_ϕ, which are independent of ψ_F. In fact, one may decompose φ and ϕ as $\varphi = \boxplus_{i=1}^r \varphi_i$ and $\phi = \boxplus_{j=1}^s \phi_j$, where φ_i and ϕ_j are irreducible and of different type for all i and j. Since $\varphi_i \otimes \phi_j$ is of symplectic type, $\mathcal{E}(\varphi_i, \phi)$ and $\mathcal{E}(\varphi, \phi_j)$ are ± 1 and independent of ψ_F.

In the remark, we extend the character $\chi^\star(\varphi, \phi)$ to be a character of $A_\varphi \times A_\phi$, which is well-defined and independent of ψ_F. And it is allowed that the orthogonal parameter φ or ϕ is of odd dimension. Then the character $\chi^\star(\varphi, \phi)$ is still well defined.

4B. *Descent of local L-parameters.* The main result of this subsection is Theorem 4.6, which explicitly determines the descent of local L-parameters.

Let φ be a local L-parameter of G_n for a square-integrable representation of $G_n(F)$. It can be decomposed as

$$\varphi = \boxplus_{i=1}^r \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2\alpha_{i,j}} \boxplus \boxplus_{i=1}^s \boxplus_{j=1}^{s_i} \varrho_i \boxtimes \mu_{2\beta_{i,j}+1}, \tag{4-9}$$

where all ρ_i and ϱ_i are irreducible self-dual distinct representations of $\mathcal{W}_F$ of dimension a_i and b_i, respectively, and $1 \le \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,r_i}$ and $0 \le \beta_{i,1} < \beta_{i,2} < \cdots < \beta_{i,s_i}$ for all i are integers. For such a local L-parameter φ as in (4-9), we define the even and odd parts according to the dimension of the μ_k:

$$\varphi_e := \boxplus_{i=1}^r \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2\alpha_{i,j}} \tag{4-10}$$

$$\varphi_o := \boxplus_{i=1}^s \boxplus_{j=1}^{s_i} \varrho_i \boxtimes \mu_{2\beta_{i,j}+1}. \tag{4-11}$$

For an irreducible representation ρ of $\mathcal{W}_F$, denote by $\varphi(\rho)$ the ρ-isotypic component of φ when restricted to $\mathcal{W}_F$. It is clear that $\varphi(\rho)$ is still a local L-parameter. For instance, $\varphi(\rho_i) = \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2\alpha_{i,j}}$. Hence $\varphi(\rho) = 0$ if ρ is not isomorphic to any of the ρ_i or ϱ_i for all i.

For a local L-parameter φ of G_n for a square-integrable representation of $G_n(F)$, we decompose it as $\varphi = \boxplus_{i \in \mathrm{I}} \varphi_i$. Let φ' be a subrepresentation of φ, i.e., $\varphi' = \boxplus_{i \in \mathrm{I}'} \varphi_i$ where $\mathrm{I}' \subseteq \mathrm{I}$. For an element $(e_i) \in \mathcal{S}_\varphi$, denote by $(e_i)|_{\varphi'}$ the elements in $\mathcal{S}_\varphi$ such that $e_i = 0$ for $i \in \mathrm{I}'$. For example, the element $1_{\varphi'}$ is given by $e_i = 1$ for $i \in \mathrm{I}'$ and $e_i = 0$ for $i \notin \mathrm{I}'$.

Define the *sign alternative index set*, $\operatorname{sgn}_{*,\rho}(\chi)$, as follows: when $\rho \cong \rho_i$ for some i,

$$\operatorname{sgn}_{o,\rho_i}(\chi) := \{j : 0 \le j < r_i,\ \chi(1_{\rho_i \boxtimes \mu_{2\alpha_{i,j}} \boxplus \rho_i \boxtimes \mu_{2\alpha_{i,j+1}}}) = -1\} \tag{4-12}$$

and when $\rho \cong \varrho_i$ for some i,

$$\operatorname{sgn}_{e,\varrho_i}(\chi) = \{j : 1 \le j < s_i,\ \chi(1_{\varrho_i \boxtimes \mu_{2\beta_{i,j}+1} \boxplus \varrho_i \boxtimes \mu_{2\beta_{i,j+1}+1}}) = -1\}. \tag{4-13}$$

By convention, $\alpha_{i,0}=0$. For each (φ,χ), define

$$\varphi_{\mathrm{sgn}}=\boxplus_{i=1}^{r}\boxplus_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\rho_i\boxtimes\mu_{2\alpha_{i,j}+1}\boxplus\boxplus_{i=1}^{s}\boxplus_{j\in\mathrm{sgn}_{e,\varrho_i}(\chi)}\varrho_i\boxtimes\mu_{2\beta_{i,j}+2}. \tag{4-14}$$

Note that φ_{sgn} is uniquely determined by the given local Langlands data (φ,χ), but it may possibly be zero. If φ_{sgn} is not zero, then φ_{sgn} and φ are of different type. It is worthwhile to mention that φ_{sgn} is a common component of each elements in the local descent $\mathfrak{D}_{\ell}(\varphi,\chi)$ (see Definition 4.1), whose dimension gives an upper bound for the index of the first occurrence, i.e., $2\ell_0\le\dim\varphi-\dim\varphi_{\mathrm{sgn}}$. According to Section 6A, such types of L-parameters are closely related to the cuspidal local L-parameters, as discussed in [Aubert et al. 2015], which, by definition, have the property that their L-packets contain at least one irreducible supercuspidal representation.

From φ, we define two new parameters $\lceil\varphi_o\rceil$ and $\varphi^{\dagger}$ by

$$\lceil\varphi_o\rceil=\boxplus_{i=1}^{s}\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}\quad\text{and}\quad\varphi^{\dagger}=\boxplus_{i=1}^{s}\varrho_i\boxtimes 1. \tag{4-15}$$

Note that for each i, the piece $\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}$ is the one with maximal dimension among the summands $\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}$'s for $j=1,2,\dots,s_i$. Both $\lceil\varphi_o\rceil$ and $\varphi^{\dagger}$ are of the same type as φ, and are possibly of orthogonal type and have odd dimension. Define $\chi_{\lceil\varphi_o\rceil}=\chi|_{\lceil\varphi_o\rceil}$, the restriction of χ on the elements $((e_i)|_{\lceil\varphi_o\rceil})$, to be a character of $A_{\lceil\varphi_o\rceil}$. Also $\chi_{\lceil\varphi_o\rceil}$ is considered as a character of $\mathcal{S}_{\lceil\varphi_o\rceil}$ by restriction.

Note that there is an isomorphism from $A_{\lceil\varphi_o\rceil}$ to $A_{\varphi^{\dagger}}$ given by $(e_i)\in A_{\lceil\varphi_o\rceil}\mapsto(e_i')\in A_{\varphi^{\dagger}}$ and $e_i=e_i'$, where e_i and e_i' correspond to the component $\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}$ in $A_{\lceil\varphi_o\rceil}$ and $\varrho_i\boxtimes 1$ in $A_{\varphi^{\dagger}}$, respectively. Hence we have that $\mathcal{S}_{\lceil\varphi_o\rceil}\cong\mathcal{S}_{\varphi^{\dagger}}$.

For a local generic L-parameter φ with the decomposition (2-6), define its discrete part by

$$\varphi_{\square}=\boxplus_{i\in\mathrm{I}_{\mathrm{gp}}}\varphi_i, \tag{4-16}$$

which is a discrete L-parameter.

Theorem 4.6. *Let φ be a generic L-parameter of even dimension with the discrete part $\varphi_{\square}$ of the decomposition (4-9). Then for $\chi\in\hat{\mathcal{S}}_{\varphi}$, the descent of the parameter (φ,χ) at the first occurrence index ℓ_0 can be completely determined by the following*

$$\mathfrak{D}_{\ell_0}(\varphi,\chi)=\bigcup_{\psi}[\psi\boxplus\varphi_{\mathrm{sgn}}]_c,$$

where ψ runs over all discrete L-parameters satisfying the following conditions with minimal dimension:

(1) $\dim\psi\equiv\sum_{i=1}^{r}\#\,\mathrm{sgn}_{o,\rho_i}(\chi)\cdot\dim\rho_i \pmod 2$.

(2) $\psi=(\boxplus_{i=1}^{k}\delta_i\boxtimes 1)\boxplus(\boxplus_{i=1}^{r}m_i\rho_i\boxtimes\mu_{2\alpha_{i,r_i}+1})\boxplus(\boxplus_{j=1}^{s}n_j\varrho_j\boxtimes\mu_{2\beta_{j,s_j}+2})$ *with multiplicity $m_i,n_j\in\{0,1\}$, which are multiplicity-free.*

(3) *All δ_i and ρ_i are of the same type as ψ and*

$$\{\delta_i:1\le i\le k\}\cap\{\rho_i:1\le i\le r\}=\varnothing.$$

(4) $\chi_{\lceil\varphi_o\rceil}|_{\mathcal{S}_{\lceil\varphi_o\rceil}}=\chi^{\star}_{\varphi^{\dagger}}|_{\mathcal{S}_{\varphi^{\dagger}}}$ *via $\mathcal{S}_{\lceil\varphi_o\rceil}\cong\mathcal{S}_{\varphi^{\dagger}}$, where $\chi^{\star}_{\varphi^{\dagger}}$ is the character in $\chi^{\star}(\varphi^{\dagger},\psi\boxplus\varphi_{\mathrm{sgn},o})$.*

Note that φ_{sgn}, $\varphi^{\dagger}$ and $\lceil\varphi_o\rceil$ are associated to $\varphi_{\square}$ and $\varphi_{\mathrm{sgn},o}=\boxplus_{i=1}^{r}\boxplus_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\rho_i\boxtimes\mu_{2\alpha_{i,j}+1}$. Recall that $[\phi]_c=\{\phi,\phi^c\}$ is the c-conjugacy class of ϕ, and $\chi^{\star}(\varphi^{\dagger},\psi)$ is well defined even though $\varphi^{\dagger}$ or ψ is of orthogonal type of odd dimension, following from Remark 4.5. In addition, $\psi=0$ is allowed. We will see some examples in Section 6.

Now, let us sketch the proof of Theorem 4.6. First, we only need to calculate the character $\chi^{\star}(\varphi,\phi)$ defined in (2-21) and (2-22) for general discrete parameters φ and ϕ by Corollary 3.2. Following from Lemma 4.2 and their decompositions of form (4-9), we may reduce $\chi^{\star}(\varphi,\phi)$ to the symplectic root numbers of type $\mathcal{E}(\varrho_i,\varrho'_{\ell})$, where ϱ_i and ϱ'_{ℓ} are irreducible self-dual distinct representations of $\mathcal{W}_F$ and are of different type. The formula is stated in Lemma 4.7 below.

Next, at the first occurrence ℓ_0, we have the property that the descent of the parameter (φ,χ) has the minimal dimension such that $\mathfrak{D}_{\ell_0}(\varphi,\chi)$ is not empty. Following this property, we apply Lemma 4.7 repeatedly on the descent parameters in $\mathfrak{D}_{\ell_0}(\varphi,\chi)$ and then we give a refined description of those descent parameters in Theorem 4.6. Its proof will be given in Section 4C.

The rest of Section 4B is devoted to calculating the character $\chi^{\star}_{\varphi}(\varphi,\phi)$ for two discrete parameters φ and ϕ. Due to symmetry, the formula for $\chi^{\star}_{\phi}(\varphi,\phi)$ is similar. In order to state Lemma 4.7, we introduce more notation first.

For an L-parameter φ with the decomposition (4-9), define

$$\varphi_*^{\diamond m}(\rho):=\begin{cases}\boxplus_{\{j:\,1\le j\le r_i,\ \alpha_{i,j}\diamond m\}}\rho_i\boxtimes\mu_{2\alpha_{i,j}} & \text{if } \rho\cong\rho_i \text{ for some } i,\\ \boxplus_{\{j:\,1\le j\le s_i,\ \beta_{i,j}\diamond m\}}\varrho_i\boxtimes\mu_{2\beta_{i,j}+1} & \text{if } \rho\cong\varrho_i \text{ for some } i,\\ 0 & \text{otherwise,}\end{cases}\tag{4-17}$$

where $\diamond\in\{>,<,\ge,\le\}$, $*=e$ if φ and ρ are of the different type and $*=o$, otherwise. Also define

$$\varphi_*^{-}(\rho)=\varphi_*\boxminus\varphi_*(\rho),\tag{4-18}$$

which is the remaining part of the parameter φ_* without the ρ-isotypic component $\varphi_*(\rho)$. For a subrepresentation φ' of φ, write $\#\varphi'$ for the number of the irreducible summands in φ'.

We decompose ϕ as in (4-9), i.e.,

$$\phi=\boxplus_{i=1}^{r'}\boxplus_{j=1}^{r'_i}\rho'_i\boxtimes\mu_{2\alpha'_{i,j}}\boxplus\boxplus_{i=1}^{s'}\boxplus_{j=1}^{s'_i}\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1},\tag{4-19}$$

where ρ'_i and ϱ'_i are of dimension a'_i and b'_i, respectively.

Lemma 4.7. *Let φ and ϕ be discrete L-parameters decomposed as in (4-9) and (4-19) respectively, and be of different type. Then as the character of $\mathcal{S}_{\varphi}$,*

$$\chi^{\star}_{\varphi}((e_{i,j})|_{\varphi_e})=\prod_{i=1}^{r}\prod_{j=1}^{r_i}(\det(\rho_i)(-1)^{\alpha_{i,j}\dim\phi}(-1)^{\#\phi_o^{<\alpha_{i,j}}(\rho_i)})^{e_{i,j}}\tag{4-20}$$

$$\chi^{\star}_{\varphi}((e_{i,j})|_{\varphi_o})=\prod_{i=1}^{s}\prod_{j=1}^{s_i}\left(\left(\prod_{l=1}^{s'}\mathcal{E}(\varrho_i,\varrho'_l)^{s'_l}\right)(-1)^{\#\phi_e^{>\beta_{i,j}}(\varrho_i)}\right)^{e_{i,j}},\tag{4-21}$$

where $\mathcal{E}(\cdot,\cdot)$ is defined in (2-20).

It is clear that $A_\varphi \cong A_{\varphi_e} \times A_{\varphi_o}$ and $\mathcal{S}_\varphi \cong A_{\varphi_e} \times \mathcal{S}_{\varphi_o}$. The isomorphism is given by $(e_i) \mapsto ((e_i)|_{\varphi_e}, (e_i)|_{\varphi_o})$. Equations (4-20) and (4-21) give a formula for all values of $\chi^\star_\varphi$ on $\mathcal{S}_\varphi$. Over A_φ, the formula (4-21) will be slightly different. Note that ρ_i in (4-9) and ρ'_i in (4-19) are of different types, so are ϱ_i and ϱ'_i.

Proof. To prove this lemma, we evaluate $\chi^\star_\varphi$ at three types of elements in $\mathcal{S}_\varphi$:

Type (1): $1_{\rho\boxtimes\mu_{2\alpha}}$.

Type (2): $1_{\varrho\boxtimes\mu_{2\beta+1}}$ where $\dim\varrho$ is even.

Type (3): $1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}}$ where $\dim\varrho$ and $\dim\varrho_0$ are odd.

For each type, the evaluation is reduced to the calculation of the symplectic roots of form $\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi)$, where $\varsigma\boxtimes\mu_\kappa$ is the summand occurring in the subscript of above types. From (4-10), (4-11), and (4-18), we may write that $\phi = \phi_e \boxplus \phi_o = \phi_e \boxplus \phi_o(\varsigma) \boxplus \phi_o^-(\varsigma)$. Then

$$\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi) = \varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi_e)\cdot\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi_o(\varsigma))\cdot\varepsilon((\varsigma\boxtimes\mu_\kappa)\otimes\phi_o^-(\varsigma)). \tag{4-22}$$

Finally, we apply Lemma 4.2 to calculate each factor on the right hand side of (4-22).

Type (1): We verify (4-20). Consider a summand of form $\rho\boxtimes\mu_{2\alpha}$ in φ. Write $a = \dim\rho$. Since the dimension of $\rho\boxtimes\mu_{2\alpha}$ is even, $\langle (e_i)|_{\rho\boxtimes\mu_{2\alpha}}\rangle \cong \mathbb{Z}_2$ is a subgroup of $\mathcal{S}_\varphi$ in both symplectic and orthogonal types. For all types, it is sufficient to show

$$\chi^\star_\varphi(1_{\rho\boxtimes\mu_{2\alpha}}) = \det(\rho)(-1)^{\alpha\dim\phi}(-1)^{\#\phi_o^{<\alpha}(\rho)}. \tag{4-23}$$

Here $1_{\rho\boxtimes\mu_{2\alpha}}$ corresponds to the nontrivial element in $\langle (e_i)|_{\rho\boxtimes\mu_{2\alpha}}\rangle$.

By definition, if φ is of symplectic type, then

$$\chi^\star_\varphi(1_{\rho\boxtimes\mu_{2\alpha}}) = \det(\phi)(-1)^{a\alpha}\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi) \tag{4-24}$$

and if φ is of orthogonal type, then $\chi^\star_\varphi(1_{\rho\boxtimes\mu_{2\alpha}}) = \varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi)$, as $\det(\rho)=1$. Similar to (4-22), we have $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi)$ equal to

$$\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_e)\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho))\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^-(\rho)).$$

As discussed in Remark 4.3, the local root number in this case is independent of the choice of the additive character ψ_F. We omit ψ_F in the calculation. By Lemma 4.2, $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes(\rho'_i\boxtimes\mu_{2\alpha'_{i,j}})) = 1$. It follows that $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_e)=1$.

Next, we calculate $\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho))$. As defined in (4-17), we may write $\phi_o(\rho) = \phi_o^{<\alpha}(\rho)\boxplus\phi_o^{\geq\alpha}(\rho)$. It follows that

$$\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho)) = \varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{<\alpha}(\rho))\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{\geq\alpha}(\rho)).$$

Since $\phi_o^{<\alpha}(\rho)=\boxplus_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}<\alpha\}}\varrho'_{i_0}\boxtimes\mu_{2\beta'_{i_0,j}+1}$, we have, by part (4) of Lemma 4.2, that

$$\begin{aligned}\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{<\alpha}(\rho))&=\prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}<\alpha\}}\det(\rho)(-1)^{b'_{i_0}\alpha(2\beta'_{i_0,j}+1)}\det(\varrho'_{i_0})(-1)^{a\alpha(2\beta'_{i_0,j}+1)}(-1)^{2\beta'_{i_0,j}+1}\\&=(-1)^{\#\phi_o^{<\alpha}(\rho)}\times\prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}<\alpha\}}\det(\rho)(-1)^{b'_{i_0}\alpha}\det(\varrho'_{i_0})(-1)^{a\alpha},\end{aligned}\tag{4-25}$$

where i_0 is the index of ϱ'_{i_0} with $\varrho'_{i_0}=\rho$ and $\#\phi_o^{<\alpha}(\rho)$ is the number of irreducibles in $\phi_o^{<\alpha}(\rho)$, i.e., the cardinality of $\{1\le j\le s'_{i_0}:\beta'_{i_0,j}<\alpha_{i,j}\}$.

Since $\phi_o^{\ge\alpha}(\rho)=\boxplus_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}\ge\alpha\}}\varrho'_{i_0}\boxtimes\mu_{2\beta'_{i_0,j}+1}$, we have, by part (4) of Lemma 4.2 again, that

$$\begin{aligned}\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{\ge\alpha}(\rho))&=\prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}\ge\alpha\}}\det(\rho)(-1)^{b'_{i_0}\alpha(2\beta'_{i_0,j}+1)}\det(\varrho'_{i_0})(-1)^{a\alpha(2\beta'_{i_0,j}+1)}(-1)^{2\alpha}\\&=\prod_{\{1\le j\le s'_{i_0}\,:\,\beta'_{i_0,j}\ge\alpha\}}\det(\rho)(-1)^{b'_{i_0}\alpha}\det(\varrho'_{i_0})(-1)^{a\alpha}.\end{aligned}\tag{4-26}$$

On the other hand, because $\phi_o^{-}(\rho)=\boxplus_{\substack{i=1\\i\ne i_0}}^{s'}\boxplus_{j=1}^{s'_i}\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1}$, we have, by part (3) of Lemma 4.2, that

$$\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{-}(\rho))=\prod_{\substack{i=1\\i\ne i_0}}^{s'}\prod_{j=1}^{s'_i}\det(\rho)(-1)^{b'_i\alpha(2\beta'_{i,j}+1)}\det(\varrho'_i)(-1)^{a\alpha(2\beta'_{i,j}+1)}.\tag{4-27}$$

Finally, by taking the product of (4-25), (4-26), and (4-27), we obtain that

$$\begin{aligned}&\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o(\rho))\cdot\varepsilon((\rho\boxtimes\mu_{2\alpha})\otimes\phi_o^{-}(\rho))\\&\qquad=\left(\prod_{i=1}^{s'}\prod_{j=1}^{s'_i}\det(\rho)(-1)^{b'_i\alpha}\det(\varrho'_i)(-1)^{a\alpha}\right)\times(-1)^{\#\phi_o^{<\alpha}(\rho)}\\&\qquad=\det(\rho)(-1)^{(\sum_{i=1}^{s'}b'_is'_i)\alpha}\times\left(\prod_{i=1}^{s'}\det(\varrho'_i)(-1)^{s'_i}\right)^{a\alpha}\times(-1)^{\#\phi_o^{<\alpha}(\rho)}\\&\qquad=\det(\rho)(-1)^{\alpha\dim\phi}\times\det(\phi)(-1)^{a\alpha}\times(-1)^{\#\phi_o^{<\alpha}(\rho)}.\end{aligned}$$

Indeed, since ϕ_e is of even dimension, $\sum_{i=1}^{s'}b'_is'_i$ and $\dim\phi$ are of the same parity.

If φ is of symplectic type, continuing with (4-24), we obtain (4-23). If φ is of orthogonal type, then ϕ is of symplectic type, which implies that $\det(\phi)=1$. One also has (4-23).

Next, we show (4-21) by considering summands of form $\varrho\boxtimes\mu_{2\beta+1}$ in φ. Write $b=\dim\varrho$.

Type (2): Assume that ϱ is of even dimension (that is, b is even). In this case, since $\varrho\boxtimes\mu_{2\beta+1}$ has even dimension, $\langle(e_i)|_{\varrho\boxtimes\mu_{2\beta+1}}\rangle\cong\mathbb{Z}_2$ is a subgroup of $\mathcal{S}_\varphi$ regardless to the type of φ. Similarly, denote by

$1_{\varrho\boxtimes\mu_{2\beta+1}}$ the nontrivial element in $\langle (e_i)|_{\varrho\boxtimes\mu_{2\beta+1}}\rangle$. By definition, $\chi^\star_\varphi(1_{\varrho\boxtimes\mu_{2\beta+1}})$ equals

$$\begin{cases} \det(\phi)(-1)^{b(2\beta+1)/2}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi) & \text{if } \varphi \text{ is of symplectic type,} \\ \det(\varrho)(-1)^{m}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi) & \text{if } \varphi \text{ is of orthogonal type,} \end{cases} \tag{4-28}$$

where $m=\frac{\dim\phi}{2}$. Under the assumption, we need to show that

$$\chi^\star_\varphi(1_{\varrho\boxtimes\mu_{2\beta+1}}) = \prod_{i=1}^{s'} \mathcal{E}(\varrho,\varrho'_i)^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho)}. \tag{4-29}$$

Recall from (2-20) that

$$\mathcal{E}(\varrho,\varrho'_i) = \begin{cases} \det(\varrho'_i)(-1)^{b/2}\varepsilon(\varrho\otimes\varrho'_i) & \text{if } \varphi \text{ is of symplectic type,} \\ \det(\varrho)(-1)^{b'_i/2}\varepsilon(\varrho\otimes\varrho'_i) & \text{if } \varphi \text{ is of orthogonal type.} \end{cases}$$

We may write that $\phi=\phi_e\boxplus\phi_o=\phi_e(\varrho)\boxplus\phi_e^-(\varrho)\boxplus\phi_o$, following from (4-10), (4-11), and (4-18) again. It follows that

$$\begin{aligned} \varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi) &= \varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e)\cdot\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_o,\psi_F), \\ \varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e) &= \varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e(\varrho))\cdot\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho)). \end{aligned}$$

Here only the term $\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_o,\psi_F)$ is possibly dependent on ψ_F. By part (2) of Lemma 4.2, when $\varrho\otimes\varrho'_i$ is of symplectic type, we have that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes(\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1}),\psi_F) = \varepsilon(\varrho\otimes\varrho'_i),$$

which is independent of ψ_F. As $\phi_o=\boxplus_{i=1}^{s'}\boxplus_{j=1}^{s'_i}\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1}$, we have that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_o,\psi_F) = \prod_{i=1}^{s'}\prod_{j=1}^{s'_i}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes(\varrho'_i\boxtimes\mu_{2\beta'_{i,j}+1}),\psi_F) = \prod_{i=1}^{s'}\varepsilon(\varrho\otimes\varrho'_i)^{s'_i},$$

which is independent of ψ_F.

Now, let us calculate the first two terms: $\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e(\varrho))$ and $\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho))$. Since $\phi_e^-(\varrho)=\boxplus_{\substack{i=1\\ i\neq i_0}}^{r'}\boxplus_{j=1}^{r'_i}\rho'_i\boxtimes\mu_{2\alpha'_{i,j}}$, by part (3) of Lemma 4.2, we have that

$$\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho)) = \prod_{\substack{i=1\\ i\neq i_0}}^{r'}\prod_{j=1}^{r'_i}\det(\varrho)(-1)^{a'_i\alpha'_{i,j}(2\beta+1)}\det(\rho'_i)(-1)^{b\alpha'_{i,j}(2\beta+1)}, \tag{4-30}$$

where i_0 is the index of ρ'_{i_0} with $\rho'_{i_0}=\varrho$.

Following (4-25) and (4-26), we write $\phi_e(\varrho)=\phi_e^{\leq\beta}(\varrho)\boxplus\phi_e^{>\beta}(\varrho)$. By part (4) of Lemma 4.2, since $\phi_e^{\leq\beta}(\varrho)=\boxplus_{\{1\leq j\leq r'_{i_0}\,:\,\alpha'_{i_0,j}\leq\beta\}}\rho'_{i_0}\boxtimes\mu_{2\alpha'_{i_0,j}}$, we have that

$$\begin{aligned}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^{\leq\beta}(\varrho))&=\prod_{\{1\leq j\leq r'_{i_0}\,:\,\alpha'_{i_0,j}\leq\beta\}}\det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}(2\beta+1)}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}(2\beta+1)}\cdot(-1)^{2\alpha'_{i_0,j}}\\&=\prod_{\{1\leq j\leq r'_{i_0}\,:\,\alpha'_{i_0,j}\leq\beta\}}\det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}}.\end{aligned}\tag{4-31}$$

Since $\phi_e^{>\beta}(\varrho)=\boxplus_{\{1\leq j\leq r'_{i_0}\,:\,\alpha'_{i_0,j}>\beta\}}\rho'_{i_0}\boxtimes\mu_{2\alpha'_{i_0,j}}$, we have, by part (4) of Lemma 4.2, that

$$\begin{aligned}\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^{>\beta}(\varrho))&=\prod_{\{1\leq j\leq r'_{i_0}\,:\,\alpha'_{i_0,j}>\beta\}}\det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}(2\beta+1)}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}(2\beta+1)}\cdot(-1)^{2\beta+1}\\&=(-1)^{\#\phi_e^{>\beta}(\varrho)}\times\prod_{\{1\leq j\leq r'_{i_0}\,:\,\alpha'_{i_0,j}>\beta\}}\det(\varrho)(-1)^{a'_{i_0}\alpha'_{i_0,j}}\det(\rho'_{i_0})(-1)^{b\alpha'_{i_0,j}},\end{aligned}\tag{4-32}$$

where $\#\phi_e^{>\beta}(\varrho)$ is the number of irreducibles in $\phi_e^{>\beta}(\varrho)$.

Finally, by taking the product of (4-30), (4-31), and (4-32), we obtain that

$$\begin{aligned}&\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e(\varrho))\cdot\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi_e^-(\varrho))\\&\qquad=\left(\prod_{i=1}^{r'}\prod_{j=1}^{r'_i}\det(\varrho)(-1)^{a'_i\alpha'_{i,j}}\det(\rho'_i)(-1)^{b\alpha'_{i,j}}\right)\times(-1)^{\#\phi_e^{>\beta}(\varrho)}\\&\qquad=\det(\varrho)(-1)^{\sum_{i=1}^{r'}\sum_{j=1}^{r'_i}a'_i\alpha'_{i,j}}\cdot(-1)^{\#\phi_e^{>\beta}(\varrho)}.\end{aligned}$$

Recall that b is even and $\det\rho'_i(-1)^b=1$.

If φ is of symplectic type, then ϱ is of symplectic type, which implies that $\det(\varrho)(-1)=1$. Continuing with (4-28) and by

$$\det(\phi)(-1)=\det(\phi_o)(-1)=\prod_{i=1}^{s'}\det(\varrho_i)(-1)^{s'_i},$$

one has (4-29).

If φ is of orthogonal type and b is even, then ϱ and ρ'_i are of orthogonal type and ϱ'_i is of symplectic type. Because

$$m\equiv\sum_{i=1}^{r'}\sum_{j=1}^{r'_i}a'_i\alpha'_{i,j}+\sum_{i=1}^{s'}s'_i\tfrac{b'_i}{2}\mod 2,$$

continuing with (4-28), one obtains (4-29).

Type (3): Assume that $b=\dim\varrho$ is odd, which implies that φ is of orthogonal type. Let $\varrho_0\boxtimes\mu_{2\beta_0+1}$ be any different summand in φ such that $b_0:=\dim\varrho_0$ is odd. Consider the following subgroup of $\mathcal{S}_\varphi$

$$\langle(e_i)|_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}}:(e_i)\in\mathcal{S}_\varphi\rangle\cong\mathbb{Z}_2.$$

Denote by $1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}}$ the nontrivial element in the above subgroup. If $\varrho_0\boxtimes\mu_{2\beta_0+1}$ does not exist, we do not need to consider this case as $\chi_\varphi^\star$ is a character of $\mathcal{S}_\varphi$. Then we have

$$\chi_\varphi^\star(1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}})=\det(\varrho)(-1)^m\det(\varrho_0)(-1)^m\varepsilon((\varrho\boxtimes\mu_{2\beta+1})\otimes\phi)\varepsilon((\varrho_0\boxtimes\mu_{2\beta_0+1})\otimes\phi).$$

Following the above calculation, one obtains that

$$\begin{aligned}\chi_\varphi^\star&(1_{\varrho\boxtimes\mu_{2\beta+1}\boxplus\varrho_0\boxtimes\mu_{2\beta_0+1}})\\ &=\det(\varrho)(-1)^m\cdot\det(\varrho_0)(-1)^m\times\prod_{i=1}^{r'}\prod_{j=1}^{r'_i}(\det(\varrho)(-1)\det(\varrho_0)(-1))^{a'_i\alpha'_{i,j}}\prod_{i=1}^{r'}\prod_{j=1}^{r'_i}\det(\rho'_i)(-1)^{(b+b_0)\alpha'_{i,j}}\\ &\qquad\times\prod_{i=1}^{s'}\varepsilon(\varrho\otimes\varrho'_i)^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho)}\prod_{i=1}^{s'}\varepsilon(\varrho_0\otimes\varrho'_i)^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho_0)}\\ &=\prod_{i=1}^{s'}(\det(\varrho)(-1)^{b'_i/2}\varepsilon(\varrho\otimes\varrho'_i))^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho)}\times\prod_{i=1}^{s'}(\det(\varrho_0)(-1)^{b'_i/2}\varepsilon(\varrho_0\otimes\varrho'_i))^{s'_i}(-1)^{\#\phi_e^{>\beta}(\varrho_0)}.\end{aligned}$$

Finally, by putting all the calculations above together, we obtain (4-21). □

At the end of this section, we present an example of Lemma 4.7, which also will be used in the proof of Theorem 4.6. Let $\varphi^\dagger$ be a discrete L-parameter of the decomposition defined in (4-15).

Example 4.8. Let ψ be a discrete L-parameter of the different type from $\varphi^\dagger$ and ψ be given in item (2) of Theorem 4.6. Applying Lemma 4.7 to both $\varphi^\dagger$ and $\psi\boxplus\varphi_{\mathrm{sgn},o}$, one has

$$\chi_{\varphi^\dagger}^\star((e_i))=\prod_{i=1}^{s}\left(\left(\prod_{l=1}^{k}\mathcal{E}(\varrho_i,\delta_l)\right)\cdot\left(\prod_{j=1}^{r}\mathcal{E}(\varrho_i,\rho_j)^{m_j+\#\mathrm{sgn}_{o,\rho_j}(\chi)}\right)\cdot(-1)^{n_i}\right)^{e_i},\tag{4-33}$$

where $\mathrm{sgn}_{o,\rho_i}(\chi)$ is defined in (4-12).

Note that from this example the explicit description of $\mathfrak{D}_{\ell_0}(\varphi,\chi)$ is reduced to finding a ψ of minimal dimension satisfying (4-33).

4C. *Proof of Theorem 4.6.* Following Corollary 3.2, we have

$$\mathfrak{D}_{\ell_0}(\varphi,\chi)=\mathfrak{D}_{\ell_0}(\varphi_\square,\chi),$$

and all local L-parameters ϕ in $\mathfrak{D}_{\ell_0}(\varphi_\square,\chi)$ are discrete. The proof is reduced to the case where φ is discrete. It is enough to show that items (1), (2), (3), and (4) are *necessary* and *sufficient* to characterize the set $\mathfrak{D}_{\ell_0}(\varphi,\chi)$. First, assume that $\chi_\varphi^\star=\chi$. Applying Lemma 4.7, we conclude that items (2) and (3) are the necessary conditions for the descent parameters in $\mathfrak{D}_{\ell_0}(\varphi,\chi)$. Note that item (1) holds by the definition. Then assume that ϕ is of the decomposition $\psi\boxplus\varphi_{\mathrm{sgn}}$, where ψ is given in items (2) and satisfies (3). We show that for such ϕ, $\chi_\varphi^\star(\varphi,\phi)=\chi$ is equivalent to $\chi_{\lceil\varphi_o\rceil}=\chi_{\varphi^\dagger}^\star$, which is item (4). The calculation of $\chi_{\varphi^\dagger}^\star$ is given in Example 4.8. Finally, by the minimality condition on $\dim\phi$, items (1), (2), (3), and (4) are sufficient.

Necessity. Take ϕ in $\mathfrak{D}_{\ell_0}(\varphi,\chi)$. Then ϕ is of the minimal even dimension such that $\chi_\varphi^\star=\chi$ for the pair (φ,ϕ). By Corollary 3.2, ϕ is discrete.

First, we consider the subrepresentation $\phi_o(\rho_i)$. By (4-20) in Lemma 4.7, the parity of $\#\phi_o^{<\alpha_{i,j}}(\rho_i)$ is determined by $\chi(1_{\rho_i\boxtimes\mu_{\alpha_{i,j}}})$. By (4-21), the value $\chi_\varphi^\star((e_{i,j})|_{\varphi_o})$ is partially affected by $\varepsilon(\varrho\otimes\varrho'_{i'})^{s'_{i'}}$ with $\varrho'_{i'}\cong\rho_i$ where $s'_{i'}=\#\phi_o(\rho_i)$, and more precisely by the parity of $s'_{i'}$. By the minimality of $\dim\phi$, one has that

$$\phi_o^{<\alpha_{i,r_i}}(\rho_i)=\boxplus_{i=1}^{r}\boxplus_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\rho_i\boxtimes\mu_{2\alpha_{i,j}+1},$$

where $\mathrm{sgn}_{o,\rho_i}(\chi)$ is defined in (4-12).

Next, we consider the component $\phi_e(\varrho_i)$. For $1\le j_1<j_2\le s_i$, by Lemma 4.7, we have that

$$\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j_1}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j_2}+1}})=(-1)^{\#\phi_e^{>\beta_{i,j_1}}(\varrho_i)+\#\phi_e^{>\beta_{i,j_2}}(\varrho_i)}. \tag{4-34}$$

The parity of $\#\phi_e^{>\beta_{i,j_1}}(\varrho_i)\pm\#\phi_e^{>\beta_{i,j_2}}(\varrho_i)$ is uniquely determined by $\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j_1}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j_2}+1}})$. In addition, $\chi_\varphi^\star((e_i)|_{\varphi^-(\varrho)})$ is independent of $\phi_e(\varrho_i)$. The only requirement on $\phi_e^{<\beta_{i,s_i}}(\varrho_i)$ is that (4-34) holds for all $1\le j_1<j_2\le s_i$. Then by the minimality of $\dim\phi$ one has, for $1\le j<r_i$, that

$$\phi_e^{>\beta_{i,j}}(\varrho_i)\boxminus\phi_e^{>\beta_{i,j+1}}(\varrho_i)=\begin{cases}\varrho_i\boxtimes\mu_{2(\beta_{i,j}+1)} & \text{if } \chi(\cdot)=-1,\\ 0 & \text{if } \chi(\cdot)=1.\end{cases}$$

Here $\chi(\cdot)=\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j+1}+1}})$ for simplicity. Thus, we obtain that

$$\phi_e^{<\beta_{i,s_i}}(\varrho_i)=\boxplus_{i=1}^{s}\boxplus_{j\in\mathrm{sgn}_{e,\varrho_i}(\chi)}\varrho_i\boxtimes\mu_{2\beta_{i,j}+2},$$

where $\mathrm{sgn}_{e,\varrho_i}(\chi)$ is defined in (4-13).

Now we rewrite $\phi=\psi\boxplus\varphi_{\mathrm{sgn}}$, where ψ and φ_{sgn} (see (4-14) for definition) have no common irreducibles. By the above discussion, Example 4.8 and Lemma 4.7, we may assume that $\psi_o^{\le\alpha_{i,r_i}}(\rho_i)$ and $\psi_e^{\le\beta_{i,s_i}}(\varrho_i)$ are zero for all ρ_i and ϱ_i. In the rest of the proof, we will repeatedly apply Lemma 4.7 and the minimality of $\dim\phi$ to obtain the requirements on ψ. First, no matter what $\psi_e(\rho)$ for $\rho\notin\{\varrho_1,\dots,\varrho_s\}$ are, $\chi_\varphi^\star$ does not change in (4-20) and (4-21). It follows that $\psi_e(\rho)=0$ for $\rho\notin\{\varrho_1,\dots,\varrho_s\}$. Next, note that for all ϱ_i only the parity of $\#\psi_e^{>\beta_{i,s_i}}(\varrho_i)$ has nontrivial contribution in (4-20) and (4-21), which implies, for all $1\le j\le s$, that

$$\psi_e(\varrho_i)=n_j\varrho_j\boxtimes\mu_{2\beta_{j,s_j}+2}, \tag{4-35}$$

where $n_j\in\{0,1\}$. Finally, let us consider $\psi_o(\rho)$ in two cases: $\rho\notin\{\rho_1,\dots,\rho_r\}$ or $\rho\cong\rho_i$ for some i. If $\rho\notin\{\rho_1,\dots,\rho_r\}$, only the parity of $\#\psi_o(\rho)$ is involved in (4-21). When $\psi_o(\rho)\ne0$, one has that

$$\psi_o(\rho)=\rho\boxtimes 1. \tag{4-36}$$

If $\rho\cong\rho_i$ for some i, only the parity of $\#\,\mathrm{sgn}_{o,\rho_i}(\chi)-\#\psi_o^{\ge\alpha_{i,r_i}}(\rho_i)$ possibly changes the value of $\chi_\varphi^\star$ in (4-21). Thus, we obtain that

$$\psi_o(\rho_i)=m_i\rho_i\boxtimes\mu_{2\alpha_{i,r_i}+1}, \tag{4-37}$$

where $m_i \in \{0, 1\}$. Combining (4-35), (4-36) and (4-37), we obtain items (2) and (3), which are necessary conditions on ψ.

Sufficiency. Assume that $\phi = \psi \boxplus \varphi_{\mathrm{sgn}}$ and ψ is of the form in item (2) satisfying item (3). By the above discussion and (4-20), $\chi|_{\varphi_e} = \chi_\varphi^\star|_{\varphi_e}$. In order to complete the proof, it is sufficient to show that $\chi|_{\varphi_o} = \chi_\varphi^\star|_{\varphi_o}$ if and only if $\chi_{\lceil\varphi_o\rceil} = \chi_{\varphi^\dagger}^\star$. In the rest of the proof, all the characters are on the corresponding subgroups $\mathcal{S}_{\varphi_o}$, $\mathcal{S}_{\lceil\varphi_o\rceil}$ and $\mathcal{S}_{\varphi^\dagger}$.

By applying (4-21) in Lemma 4.7 to φ and $\varphi^\dagger$, and by (4-33) in Example 4.8, we have

$$\chi_\varphi^\star((e_{i,j})|_{\varphi_o}) = \chi_{\varphi^\dagger}^\star\left(\left(\sum_{k=1}^{s_i} e_{i,k}\right)\right) \cdot \prod_{i=1}^{s}\prod_{j=1}^{s_i}((-1)^{\#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j}}(\varrho_i)})^{e_{i,j}}. \tag{4-38}$$

Define $f\colon (e_{i,j})|_{\varphi_o} \in \mathcal{S}_{\varphi_o} \mapsto \left(\sum_{j=1}^{s_i} e_{i,j}\right) \in A_{\varphi^\dagger}$, where $\mathcal{S}_{\varphi_o}$ is considered as a subgroup of $\mathcal{S}_\varphi$. It is easy to check that f is surjective on $\mathcal{S}_{\varphi^\dagger}$. (In general it is not surjective on $A_{\varphi^\dagger}$.) Denote $f|_{\lceil\varphi_o\rceil}$ to be the restriction map into $\mathcal{S}_{\lceil\varphi_o\rceil}$. Then $f|_{\lceil\varphi_o\rceil}$ is an isomorphism. Following (4-38) and by $\#\varphi_{\mathrm{sgn},e}^{>\beta_{i,s_i}}(\varrho_i) = 0$ for all $1 \le i \le s$, one has that

$$\chi_\varphi^\star((e_{i,j})|_{\lceil\varphi_o\rceil}) = \chi_{\varphi^\dagger}^\star((e_{i,s_i})) = \chi_{\varphi^\dagger}^\star(f((e_{i,j})|_{\lceil\varphi_o\rceil})). \tag{4-39}$$

By (4-39), if $\chi|_{\varphi_o} = \chi_\varphi^\star|_{\varphi_o}$ then $\chi_{\lceil\varphi_o\rceil} = \chi_{\varphi^\dagger}^\star$ as $f|_{\lceil\varphi_o\rceil}$ is an isomorphism.

Suppose that $\chi_{\lceil\varphi_o\rceil} = \chi_{\varphi^\dagger}^\star$. Let $\varrho_i \boxtimes \mu_{2\beta_{i,j}+1}$ be an irreducible subrepresentation of φ_o. We consider two cases: $\dim\varrho_i$ is even and $\dim\varrho_i$ is odd, respectively, as $\mathcal{S}_{\varphi_o}$ is generated by the elements of two types $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}$ and $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+1}}$ with $i \ne i'$, or $i = i'$ and $j \ne j'$.

Assume that $\dim\varrho_i$ is even. In this case, $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}$ is in $\mathcal{S}_{\varphi_o}$. Set

$$j_0 = \#\{j \le l < s_i : l \in \mathrm{sgn}_{e,\varrho_i}(\chi)\} = \#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j}}(\varrho_i).$$

By the definition of $\mathrm{sgn}_{e,\varrho_i}(\chi)$, one has that

$$\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}) = (-1)^{j_0}\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}}) = (-1)^{j_0}\chi_{\lceil\varphi_o\rceil}(1_{\varrho_i\boxtimes\mu_{2\beta_{i,s_i}+1}}).$$

By (4-38), we have that $\chi_\varphi^\star(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}) = \chi_{\varphi^\dagger}^\star(1_{\varrho_i\boxtimes 1})(-1)^{\#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j}}(\varrho_i)}$. Thus, we obtain that $\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}}) = \chi_\varphi^\star(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}})$.

Assume that $\dim\varrho_i$ is odd. Consider the elements in $\mathcal{S}_{\varphi_o}$ of form $1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+1}}$ with $i \ne i'$, or $i = i'$ and $j \ne j'$. Suppose that $i = i'$ and $j < j'$. Set

$$j_0 = \#\{j \le l < j' : l \in \mathrm{sgn}_{e,\varrho_i}(\chi)\} = \#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j}}(\varrho_i) - \#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j'}}(\varrho_i).$$

One has that

$$\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}}) = (-1)^{j_0} = (-1)^{\#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j}}(\varrho_i) - \#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j'}}(\varrho_i)}.$$

Since $f(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}}) = 0 \in \mathcal{S}_{\varphi^\dagger}$, one has, by (4-38), that

$$\chi_\varphi^\star(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}}) = \chi_{\varphi^\dagger}^\star(0)(-1)^{\#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j}}(\varrho_i) + \#\varphi_{\mathrm{sgn},e}^{>\beta_{i,j'}}(\varrho_i)},$$

which is equal to $\chi(1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_i\boxtimes\mu_{2\beta_{i,j'}+1}})$.

Suppose that $i\neq i'$. Denote $1_{(i,j),(l,k)}=1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_l\boxtimes\mu_{2\beta_{l,k}+1}}$. When $i=l$ and $j=k$, define that $1_{(i,j),(l,k)}=0$. Then $1_{(i,j),(l,k)}$ is in $\mathcal{S}_{\varphi_o}$. We rewrite that

$$1_{\varrho_i\boxtimes\mu_{2\beta_{i,j}+1}\boxplus\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+1}}=1_{(i,j),(i,s_i)}+1_{(i,s_i),(i',s_{i'})}+1_{(i',s_{i'}),(i',j')}.$$

In the above case, we proved that $\chi(1_{(i,j),(i,s_i)})=\chi^\star_\varphi(1_{(i,j),(i,s_i)})$ and $\chi(1_{(i',s_{i'}),(i',j')})=\chi^\star_\varphi(1_{(i',s_{i'}),(i',j')})$.

It remains to show that $\chi(1_{(i,s_i),(i',s_{i'})})=\chi^\star_\varphi(1_{(i,s_i),(i',s_{i'})})$ with $i\neq i'$. By definition, one has that $\chi(1_{(i,s_i),(i',s_{i'})})=\chi_{\lceil\varphi_o\rceil}(1_{(i,s_i),(i',s_{i'})})$. Note that $\#\varphi^{>\beta_{i,s_i}}_{\mathrm{sgn},e}(\varrho_i)=\#\varphi^{>\beta_{i',s_{i'}}}_{\mathrm{sgn},e}(\varrho_{i'})=0$. It then follows from (4-38) that $\chi^\star_\varphi(1_{(i,s_i),(i',s_{i'})})=\chi^\star_{\varphi^\dagger}(1_{\varrho_i\boxtimes 1\boxplus\varrho_{i'}\boxtimes 1})$, which implies that $\chi(1_{(i,s_i),(i',s_{i'})})=\chi^\star_\varphi(1_{(i,s_i),(i',s_{i'})})$. Therefore, we complete the proof of Theorem 4.6.

5. Proof of Theorems 1.5 and 1.7

In this section, we will apply our main results from Theorem 4.6 to prove Theorems 1.5 and 1.7. First, we use the descents of discrete parameters studied in Section 4 to recover the local descents of representations via the local Langlands corresponding.

Proposition 5.1 (part (1) of Theorem 1.7). *For a $\pi\in\Pi(G_n)$ with a generic L-parameter φ, the first occurrence index of π can be calculated by*

$$\ell_0(\pi)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}=n-\tfrac{1}{2}\dim(\varphi_\square)+\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\},$$

where $\mathcal{O}_{\mathcal{Z}}(\pi)$ is defined in (2-17) and $\varphi_\square$ is defined in (4-16). Moreover, suppose that χ_0 is a character such that

$$\ell_0(\varphi_\square,\chi_0)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\}$$

holds, and that $\pi=\pi_a(\varphi,\chi_0)$ under the local Langlands correspondence ι_a. Define $\pi_\square:=\pi_a(\varphi_\square,\chi_0)\in\Pi(G_{n_0})$ where $n_0=\frac{1}{2}\dim\varphi_\square$. If $\mathrm{disc}(\mathcal{O}_{\ell_0(\pi)})=\mathrm{disc}(\mathcal{O}_{\ell_0(\pi_\square)})$, then an irreducible square integrable quotient occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi)$ if and only if it occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi_\square)}}(\pi_\square)$.

Proof. For each $\chi\in\mathcal{S}_\varphi\cong\mathcal{S}_{\varphi_\square}$ and a generic L-parameter ϕ of type different from that of φ, since $\chi^\star(\varphi,\phi)=\chi^\star(\varphi_\square,\phi)$, we have $\mathfrak{D}_\ell(\varphi,\chi)=\mathfrak{D}_\ell(\varphi_\square,\chi)$. Hence $\ell_0(\varphi,\chi)=n-\frac{1}{2}\dim(\varphi_\square)+\ell_0(\varphi_\square,\chi)$. It suffices to show that $\ell_0(\pi)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}$.

By Corollary 3.2, $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is nonzero for some rational orbit $\mathcal{O}_{\ell_0}$ if and only if there exists a nonzero irreducible square-integrable representation σ of $G_n^{\mathcal{O}_{\ell_0}}$ such that $m(\pi,\sigma)\neq 0$. If $m(\pi,\sigma)\neq 0$, assume that $\pi=\pi_a(\varphi,\chi_\pi)$ and $\sigma=\pi_a(\phi,\chi_\sigma)$. By Theorem 2.4, $(\chi_\pi,\chi_\sigma)=\chi^\star(\varphi,\phi)$ and then $[\phi]_c$ is in $\mathfrak{D}_{\ell_0(\pi)}(\varphi,\chi_\pi)$. It follows that

$$\ell_0(\pi)\leq\ell_0(\varphi,\chi_\pi)\leq\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}.$$

On the other hand, let χ_0 be in $\mathcal{O}_{\mathcal{Z}}(\pi)$ such that

$$\ell_0(\varphi,\chi_0)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi,\chi)\}=n-\tfrac{1}{2}\dim(\varphi_\square)+\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\}.$$

Take an equivalence class $[\phi]_c$ of the local L-parameters in $\mathfrak{D}_{\ell_0^{\max}}(\varphi,\chi_0)$ at the first occurrence index $\ell_0^{\max}=\ell_0(\varphi,\chi_0)$. Choose the local Langlands correspondence ι_a such that $\pi=\pi_a(\varphi,\chi_0)$. By Definition 4.1, $\chi_0=\chi_\varphi^\star(\varphi,\phi)$. Denote $\chi_\phi=\chi_\phi^\star(\varphi,\phi)$ and $\sigma=\pi_a(\phi,\chi_\phi)$ under the same Langlands correspondence ι_a. By Theorem 2.4, we have $m(\pi,\sigma)=1$ and σ is an irreducible quotient in $\mathcal{D}_{\mathcal{O}_{\ell_0^{\max}}}(\pi)$. Then $\mathcal{D}_{\mathcal{O}_{\ell_0^{\max}}}(\pi)\neq 0$ and $\ell_0(\pi)\geq\ell_0^{\max}$. Hence, we have that $\ell_0(\pi)=\ell_0^{\max}$.

Consider the local descents given by the rational orbits $\mathcal{O}_{\ell_0(\pi)}$ and $\mathcal{O}_{\ell_0(\pi_\square)}$ with $\mathrm{disc}(\mathcal{O}_{\ell_0(\pi)})=\mathrm{disc}(\mathcal{O}_{\ell_0(\pi_\square)})$. Referring to Theorem 2.4 and Remark 2.5, the local Langlands correspondence ι_b for π is determined by $\mathrm{disc}(\mathcal{O}_{\ell_0(\pi)})$, so is the same choice for $\pi_\square$. Recall that the corresponding character of $\mathcal{S}_\varphi\cong\mathcal{S}_{\varphi_\square}$ is denoted by $\chi_b:=\chi_b(\pi)=\chi_b(\pi_\square)$.

By the definition in (2-17), $\mathcal{O}_{\mathcal{Z}}(\pi)=\mathcal{O}_{\mathcal{Z}}(\pi_\square)$ and then $\ell_0(\pi_\square)=\max_{\chi\in\mathcal{O}_{\mathcal{Z}}(\pi)}\{\ell_0(\varphi_\square,\chi)\}$. By the above proof, $\ell_0(\pi)=n-\frac{1}{2}\dim\varphi_\square+\ell_0(\pi_\square)$. If $\mathrm{disc}(\mathcal{O}_{\ell_0(\pi)})=\mathrm{disc}(\mathcal{O}_{\ell_0(\pi_\square)})$, then $G_n^{\mathcal{O}_{\ell_0(\pi)}}\cong G_{n_0}^{\mathcal{O}_{\ell_0(\pi_\square)}}$.

If $\ell_0(\varphi,\chi_b(\pi))<\ell_0(\pi)$ (equivalently, if $\ell_0(\varphi_\square,\chi_b(\pi_\square))<\ell_0(\pi_\square)$), then both $\mathfrak{D}_{\ell_0(\pi)}(\varphi,\chi_b)$ and $\mathfrak{D}_{\ell_0(\pi_\square)}(\varphi_\square,\chi_b)$ are empty. By Theorem 2.4, $\mathcal{D}_{\ell_0(\pi)}(\pi)=\mathcal{D}_{\ell_0(\pi_\square)}(\pi_\square)=0$.

Assume that $\ell_0(\varphi,\chi_a(\pi))=\ell_0(\pi)$ for some ι_a as denoted in the proposition. With the above notation, an irreducible square-integrable representation σ of $G_n^{\mathcal{O}_{\ell_0}}$ occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi)$ if and only if $(\chi_a(\pi),\chi_\sigma)=\chi^\star(\varphi,\phi)$ by Theorem 2.4. Recall that by the definition of $\pi_\square$, $\chi_a(\pi)=\chi_a(\pi_\square)$ under the same local Langlands correspondence. By $\chi^\star(\varphi,\phi)=\chi^\star(\varphi_\square,\phi)$, we have $\chi^\star(\varphi_\square,\phi)=(\chi_a(\pi_\square),\chi_\sigma)$, which is equivalent that σ is also an irreducible quotient of $\mathcal{D}_{\mathcal{O}_{\ell_0(\pi)}}(\pi_\square)$. □

In order to prove Theorem 1.5, it remains to show that the irreducible quotients of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ (at the first occurrence index $\ell_0=\ell_0(\pi)$) belong to different Bernstein components of $G_n^{\mathcal{O}_{\ell_0}}(F)$.

Theorem 5.2 (Theorem 1.5). *For any $\pi\in\Pi(G_n)$ with a generic L-parameter, irreducible quotients of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ at the first occurrence index $\ell_0=\ell_0(\pi)$ of π belong to different Bernstein components. Moreover, $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ can be written as a multiplicity-free direct sum of irreducible square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$, and hence is square-integrable and admissible.*

Remark 5.3. In general, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ at the first occurrence index could be a direct sum of infinitely many irreducible nonsupercuspidal square-integrable representations. When $\ell<\ell_0$, the descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ may not be completely reducible.

Proof. Assume that $\pi=\pi_a(\varphi,\chi)$ under local Langlands correspondence ι_a for both even and odd special orthogonal groups discussed in Section 2B. The local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a smooth representation of $G_n^{\mathcal{O}_{\ell_0}}(F)$. By the Bernstein decomposition, $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a direct sum of its Bernstein components. Thus, it is sufficient to show that if σ_1 and σ_2 are nonisomorphic irreducible quotients of $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$, i.e.,

$$m(\pi_a(\varphi,\chi),\sigma_1)=m(\pi_a(\varphi,\chi),\sigma_2)=1,$$

then σ_1 and σ_2 have different cuspidal supports. Corollary 3.2 asserts that σ_1 and σ_2 are square-integrable.

By Proposition 5.1, we may assume, without loss of generality, that φ is discrete. Then $\sigma_1=\pi_a(\phi_1,\xi_1)$ and $\sigma_2=\pi_a(\phi_2,\xi_2)$, where ϕ_1 and ϕ_2 are of the form in Theorem 4.6. We have decompositions

$\phi_i = \psi_i \boxplus \varphi_{\mathrm{sgn}}$ for $i = 1, 2$. Let λ_{ϕ_i} be the infinitesimal characters of ϕ_i (as explained in [Arthur 2013, p. 69], for instance). That is,

$$\lambda_{\phi_i}(w) = \phi_i\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

Referring to [Aubert et al. 2015], if $\lambda_{\phi_1} \neq \lambda_{\phi_2}$, then $\Pi_{\phi_1}(G_n^{\mathcal{O}_{\ell_0}})$ and $\Pi_{\phi_2}(G_n^{\mathcal{O}_{\ell_0}})$ belong to different Bernstein components. Because $\phi_1 \neq \phi_2$, we have that $\psi_1 \neq \psi_2$, which implies that $\lambda_{\psi_1} \neq \lambda_{\psi_2}$ by the definition of ψ_i. Then $\lambda_{\phi_1} \neq \lambda_{\phi_2}$. Hence each Bernstein component of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ has a unique irreducible quotient, which is square-integrable by Corollary 3.2. By the definition of the ℓ-th local descent $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$, which is the ℓ-th maximal quotient of the ℓ-th twisted Jacquet module $\mathcal{J}_{\mathcal{O}_\ell}(\pi)$, it follows that each Bernstein component of the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is an irreducible square-integrable representation of $G_n^{\mathcal{O}_{\ell_0}}(F)$. Therefore, the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is a multiplicity-free direct sum of irreducible square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$, and hence is itself square-integrable. Moreover, for any compact open subgroup K of $G_n^{\mathcal{O}_{\ell_0}}(F)$, there are only finitely many square-integrable representations of $G_n^{\mathcal{O}_{\ell_0}}(F)$ with K-fixed vectors. Thus, the K-invariant subspace of $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is finite and hence the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is admissible. $\square$

Now, let us finish the proof of part (2) of Theorem 1.7. Following Theorem 5.2, for each F-rational orbit $\mathcal{O}_{\ell_0}$, $\mathcal{D}_{\ell_0}(\pi)$ is either zero or a direct sum of square-integrable representations. Recall that an irreducible square-integrable representation σ is a subrepresentation of $\mathcal{D}_{\ell_0}(\pi)$ if and only if the data associated to σ belong to $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ for some $\chi \in \mathcal{O}_{\mathcal{Z}}(\pi)$ by Theorem 2.4. To prove this theorem, we will characterize σ by the descents of L-parameters in Theorem 4.6.

Fix a choice of F-rational orbit $\mathcal{O}_{\ell_0}$. Then the quadratic space W defining $G_n^{\mathcal{O}_{\ell_0}}$ is determined by $\mathrm{disc}(W) = (-1)^{\mathrm{n}-1} \mathrm{disc}(\mathcal{O}_{\ell_0}) \mathrm{disc}(V_{\mathrm{n}})$ (refer to Remark 2.5). Following Theorem 2.4, we choose the local Langlands correspondence ι_a where $a = (-1)^{\mathrm{n}} \mathrm{disc}(\mathcal{O}_\ell)$ for the even special orthogonal group. For the odd special orthogonal group, the normalization is unique.

Assume that G_n is a even special orthogonal group. Let $\pi = \pi_a(\varphi, \chi_a)$ where $\chi_a = \chi_a(\pi)$ via ι_a. If $\ell_0(\varphi, \chi_a) < \ell_0(\pi)$, then $\mathfrak{D}_{\ell_0}(\varphi, \chi_a)$ is empty and the local descent $\mathcal{D}_{\ell_0}(\pi)$ is zero for the F-rational orbit $\mathcal{O}_{\ell_0}$. If $\ell_0(\varphi, \chi_a) = \ell_0(\pi)$, then $\mathfrak{D}_{\ell_0}(\varphi, \chi_a)$ is not empty. By the definition of $\mathfrak{D}_{\ell_0}(\varphi, \chi_a)$, for $\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi_a)$, $\chi_a = \chi^\star(\varphi, \phi)$ and denote χ_ϕ to $\chi^\star(\varphi, \phi)$. Under ι_a, by (2-25), the corresponding representation $\pi_a(\phi, \chi_\phi)$ of $G_n^{\mathcal{O}_{\ell_0}}$ occurs in $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$. This gives the decomposition (a) in Theorem 1.7.

Assume that G_n is an odd special orthogonal group. In this case, the set $\mathcal{O}_{\mathcal{Z}}(\pi)$ is a singleton and $\pi = \pi(\varphi, \chi)$. Thus $\ell_0(\varphi, \chi_\pi) = \ell_0(\pi)$ and $\ell_0(\pi) = \ell_0(\varphi, \chi)$. Let W be the quadratic space defining $G_n^{\mathcal{O}_{\ell_0}}$. By (2-24), only the L-parameters ϕ satisfying $\det \phi = \mathrm{disc}(\mathcal{O}_{\ell_0}) \mathrm{disc}(V_{\mathrm{n}})$ correspond representations of $G_n^{\mathcal{O}_{\ell_0}}$. Thus if $\det \phi = \mathrm{disc}(\mathcal{O}_{\ell_0}) \mathrm{disc}(V_{\mathrm{n}})$, by the definition of $\mathfrak{D}_{\ell_0}(\varphi, \chi)$, W satisfies (2-24) and (2-26). Then $\pi_a(\phi, \chi_\phi^\star(\varphi, \phi))$ is a representation of $G_n^{\mathcal{O}_{\ell_0}}$, where $a = -\mathrm{disc}(\mathcal{O}_{\ell_0})$. This gives the decomposition (b) in Theorem 1.7.

With Proposition 5.1 together, we finally complete the proof of Theorem 1.7

Furthermore, by following Theorem 4.6 and Lemma 4.7, we will give a formula for $\chi^\star_\phi(\varphi,\phi)$ (simply written as $\chi^\star_\phi$) in Theorem 1.7. For L-parameters $\phi\in\mathfrak{D}_{\ell_0}(\varphi,\chi)$, $(\phi,\chi^\star_\phi)$ determine the irreducible square-integrable representations $\sigma=\pi_a(\phi,\chi^\star_\phi(\varphi,\phi))$ of $G_n^{\mathcal{O}_{\ell_0}}(F)$, via the given local Langlands correspondence ι_a. These σ occur as irreducible summands in the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi))$.

Assume that $\phi\in\mathfrak{D}_{\ell_0}(\varphi,\chi)$ is as given in Theorem 4.6, which can be written as

$$\phi=(\boxplus_{l=1}^k\delta_l\boxtimes 1)\boxplus(\boxplus_{\mathrm{i}=1}^r m_{\mathrm{i}}\rho_{\mathrm{i}}\boxtimes\mu_{2\alpha_{\mathrm{i},r_{\mathrm{i}}}+1})\boxplus(\boxplus_{\mathrm{i}'=1}^s n_{\mathrm{i}'}\varrho_{\mathrm{i}'}\boxtimes\mu_{2\beta_{\mathrm{i}',s_{\mathrm{i}'}}+2})\boxplus\varphi_{\mathrm{sgn}}.$$

Here the notation follows from Theorem 4.6. We may use more self-explanatory notation for the elements in A_ϕ to make the formula clearer. A general element of A_ϕ is written as

$$((e_{\delta,l}),(e_{\rho,\mathrm{i}}),(e_{\varrho,\mathrm{i}'}),(e_{i,j}),(\mathfrak{e}_{i',j'})).$$

Those components correspond to the components determined by the summands: $\delta_{\mathrm{i}}\boxtimes 1$, $m_{\mathrm{i}}\rho_{\mathrm{i}}\boxtimes\mu_{2\alpha_{\mathrm{i},r_{\mathrm{i}}}+1}$, $n_{\mathrm{i}'}\varrho_{\mathrm{i}'}\boxtimes\mu_{2\beta_{\mathrm{i}',s_{\mathrm{i}'}}+2}$, $\rho_i\boxtimes\mu_{2\alpha_{i,j}+1}$, and $\varrho_{i'}\boxtimes\mu_{2\beta_{i',j'}+2}$, respectively. Note that $(e_{i,j})$ and $(\mathfrak{e}_{i',j'})$ are indexed by $1\le i\le r$ and $j\in\mathrm{sgn}_{o,\rho_i}(\chi)$, and by $1\le i'\le s$ and $j'\in\mathrm{sgn}_{e,\varrho_{i'}}(\chi)$, respectively.

Corollary 5.4. *With the notation as in Theorem 1.7 and Theorem 4.6, for $\phi\in\mathfrak{D}_{\ell_0}(\varphi,\chi)$, the character $\chi^\star_\phi((e_{\delta,l}),(e_{\rho,\mathrm{i}}),(e_{\varrho,\mathrm{i}'}),(e_{i,j}),(\mathfrak{e}_{i',j'}))$ can be explicitly written as the following product*:

$$\prod_{l=1}^k\Bigl(\prod_{l'=1}^s\mathcal{E}(\delta_l,\varrho_{l'})^{s_{l'}}\Bigr)^{e_{\delta,l}}\times\prod_{\mathrm{i}=1}^r\Bigl(\prod_{l'=1}^s\mathcal{E}(\rho_{\mathrm{i}},\varrho_{l'})^{s_{l'}}\Bigr)^{m_{\mathrm{i}}e_{\rho,\mathrm{i}}}\times\prod_{i=1}^r\prod_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}\Bigl(\prod_{l'=1}^s\mathcal{E}(\rho_i,\varrho_{l'})^{s_{l'}}\cdot(-1)^{r_i-j}\Bigr)^{e_{i,j}}$$
$$\times\prod_{\mathrm{i}'=1}^s(-1)^{n_{\mathrm{i}'}s_{\mathrm{i}'}e_{\mathrm{i}'}}\times\prod_{i'=1}^s\prod_{j'\in\mathrm{sgn}_{e,\varrho_{i'}}(\chi)}(-1)^{j'\mathfrak{e}_{i',j'}},$$

which can also be written as the following product:

$$\prod_{l=1}^k\Bigl(\prod_{l'=1}^s\mathcal{E}(\delta_l,\varrho_{l'})^{s_{l'}}\Bigr)^{e_{\delta,l}}\times\Bigl(\prod_{l'=1}^s\mathcal{E}(\rho_{\mathrm{i}},\varrho_{l'})^{s_{l'}}\Bigr)^{\sum_{\mathrm{i}=1}^r m_{\mathrm{i}}e_{\rho,\mathrm{i}}+\sum_{i=1}^r\sum_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}e_{i,j}}$$
$$\times\prod_{i=1}^r\prod_{j\in\mathrm{sgn}_{o,\rho_i}(\chi)}(-1)^{(r_i-j)e_{i,j}}\times\prod_{\mathrm{i}'=1}^s(-1)^{n_{\mathrm{i}'}s_{\mathrm{i}'}e_{\mathrm{i}'}}\times\prod_{i'=1}^s\prod_{j'\in\mathrm{sgn}_{e,\varrho_{i'}}(\chi)}(-1)^{j'\mathfrak{e}_{i',j'}}.$$

6. Examples

We will consider the descents for two special families of the discrete local L-parameters. The spectral decomposition of the local descents in these cases can be even more explicitly described. Also, we will discuss Conjecture 1.8 via some examples.

6A. ***Cuspidal Local L-parameters.*** In order to understand the summand φ_{sgn} defined in (4-14) occurring in the local descent $\mathfrak{D}_{\ell_0}(\varphi,\chi)$, we give an example on such summands for cuspidal local L-parameters, which is called *cuspidal Langlands parameters* in [Aubert et al. 2015, Definition 6.8], and their descents.

Aubert, Moussaoui, and Solleveld conjectured [2015, Conjecture 7.5] that an L-packet $\Pi_\varphi(G_n)$ of a reductive group contains a supercuspidal representation if and only if φ is a cuspidal L-parameter. For the split orthogonal groups and symplectic group cases, Moussaoui [2017] verified this conjecture and gave a description of cuspidal L-parameters. Take

$$\varphi = \boxplus_{i=1}^{r} \boxplus_{j=1}^{r_i} \rho_i \boxtimes \mu_{2j} \boxplus \boxplus_{i=1}^{s} \boxplus_{j=1}^{s_i} \varrho_i \boxtimes \mu_{2j-1},$$

where $\sum_{i=1}^{s} s_i b_i$ (here $b_i = \dim \varrho_i$) is even. By [Moussaoui 2017, Proposition 3.7], φ is a cuspidal L-parameter for split even orthogonal group G_n when φ is of orthogonal type and $\prod_{i=1}^{s} \det \varrho_i = 1$.

Following (2-8), we write the elements in $\mathcal{S}_\varphi$ in the form $((e_{i,j}), (\mathfrak{e}_{i,j}))$, indexed by the set

$$\{e_{i,j} : 1 \le i \le r,\ 1 \le j \le r_i\} \cup \{\mathfrak{e}_{i,j} : 1 \le i \le s,\ 1 \le j \le s_i\}.$$

The index sets correspond to the summands $\rho_i \boxtimes \mu_{2j}$ and $\varrho_i \boxtimes \mu_{2j-1}$ respectively. If ϱ_i has even dimension for all $1 \le i \le s$, then

$$\mathcal{S}_\varphi = \{((e_{i,j}), (\mathfrak{e}_{i,j})) \in \mathbb{Z}_2^r \times \mathbb{Z}_2^s : e_{i,j}, \mathfrak{e}_{i,j} \in \{0, 1\}\}.$$

Otherwise, $\mathcal{S}_\varphi$ is the subgroup of $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ consisting of elements with the condition that $\sum_{i,j} \mathfrak{e}_{i,j} \dim \varrho_i$ is even. Define the character χ in $\hat{\mathcal{S}}_\varphi$ by

$$\chi((e_{i,j}), (\mathfrak{e}_{i,j})) = \prod_{i=1}^{r} \prod_{j=1}^{r_i} (-1)^{(j+r_i)e_{i,j}} \cdot \prod_{i=1}^{s} \prod_{j=1}^{s_i} (-1)^{(j+s_i)\mathfrak{e}_{i,j}}.$$

Remark 6.1. In some cases, the associated representation $\pi_a(\varphi, \chi)$ is supercuspidal. For instance, take $\varphi = \boxplus_{i=1}^{r} \boxplus_{j=1}^{2r_i} \rho_i \boxtimes \mu_{2j}$. By definition, $\chi(1_{\rho_i \boxtimes \mu_2}) = -1$ for all $1 \le i \le r$ and $\chi((1)) = 1$ where (1) is the element with $e_{i,j} = 1$ for all i and j. Thus, $\pi_a(\varphi, \chi)$ is a representation of a symplectic group or a split even special orthogonal group, because $\det(\varphi) = 1$. By [Moussaoui 2017], $\pi_a(\varphi, \chi)$ is supercuspidal.

Under the above assumption, by (4-14), we have that

$$\varphi_{\mathrm{sgn}} = \boxplus_{i=1}^{r} \boxplus_{j=1}^{2[r_i/2]} \rho_i \boxtimes \mu_{2(r_i-j)+1} \boxplus \boxplus_{i=1}^{s} \boxplus_{j=1}^{s_i-1} \varrho_i \boxtimes \mu_{2j}.$$

By convention, the summand $\varrho_i \boxtimes \mu_{2j}$ is empty if $s_i = 1$. As $\sum_{j=1}^{s} s_j b_j$ is even and

$$\dim \varphi - \dim \varphi_{\mathrm{sgn}} = \sum_{i=1}^{r} 2a_i \left[\tfrac{1}{2}(r_i + 1)\right] + \sum_{j=1}^{s} s_j b_j,$$

we deduce that $\dim \varphi_{\mathrm{sgn}}$ is even. By the definition in (4-15), one has that

$$\lceil \varphi_o \rceil = \boxplus_{i=1}^{s} \varrho_i \boxtimes \mu_{2s_i-1} \text{ and } \chi_{\lceil \varphi_o \rceil} = 1.$$

According to Theorem 4.6, if $\phi \in \mathfrak{D}_{\ell_0}(\varphi, \chi)$, we may decompose ϕ as $\phi = \psi \boxplus \varphi_{\mathrm{sgn}}$, where ψ satisfies the conditions in Theorem 4.6. Then $\psi = 0$ is the unique choice. In fact, $\dim \psi = 0$ is even and of the minimal dimension. By convention, $\chi^{\star}_{\varphi^\dagger}(\varphi^\dagger, \psi) = 1$. Thus,

$$\mathfrak{D}_{\ell_0}(\varphi, \chi) = [\varphi_{\mathrm{sgn}}]_c = \{\varphi_{\mathrm{sgn}}, \varphi_{\mathrm{sgn}}^c\} \quad \text{and} \quad \ell_0 = \sum_{i=1}^{r} a_i \left[\tfrac{1}{2}(r_i + 1)\right] + \sum_{j=1}^{s} \tfrac{1}{2} s_j b_j. \tag{6-1}$$

Similarly, assume that the elements in $\mathcal{S}_\phi$ are of form $((e'_{i,j}),(\mathfrak{e}'_{i,j}))$, where $e'_{i,j}$ and $\mathfrak{e}'_{i,j}$ correspond to a $\rho_i \boxtimes \mu_{2(r_i-j)+1}$-component and $\varrho_i \boxtimes \mu_{2j}$-component respectively. Define the character $\zeta \in \hat{\mathcal{S}}_\phi$ with the following conditions for all i:

- $\zeta(1_{\rho_i\boxtimes\mu_{2(r_i-j)+1}}) = 1$ when $j = 2\left[\frac{r_i}{2}\right]$ and $\zeta(1_{\varrho_i\boxtimes\mu_2}) = -1$.
- $\zeta(1_{\rho_i\boxtimes\mu_{2(r_i-j)-1}\boxplus\rho_i\boxtimes\mu_{2(r_i-j)+1}}) = -1$ for $1 \le j < 2\left[\frac{r_i}{2}\right]$.
- $\zeta(1_{\varrho_i\boxtimes\mu_{2j}\boxplus\varrho_i\boxtimes\mu_{2(j+1)}}) = -1$ for $1 \le j < s_i - 1$.

By Lemma 4.7, one has that $(\chi,\zeta) = \chi^\star(\varphi,\phi)$. By choosing the quadratic spaces and the Langlands correspondence as in Theorem 2.4, we have that $m(\pi_a(\varphi,\chi),\pi_a(\phi,\zeta)) = 1$.

For example, when $G_n = \mathrm{SO}(V_{2n+1})$ is an odd special orthogonal group, take $\mathcal{O}_{\ell_0}$ with $\mathrm{disc}(\mathcal{O}_{\ell_0}) = \mathrm{disc}(V_{2n+1})$ as $\det(\varphi_{\mathrm{sgn}}) = 1$, which is the unique F-rational orbit such that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi)) \ne 0$ and

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi)) = \begin{cases} \pi_a(\varphi_{\mathrm{sgn}},\zeta) & \text{if some } \dim \rho_i \text{ is odd,} \\ \pi_a(\varphi_{\mathrm{sgn}},\zeta) \oplus \pi_a(\varphi^c_{\mathrm{sgn}},\zeta) & \text{otherwise,} \end{cases}$$

where $a = -\,\mathrm{disc}(V_{2n+1})$. Note that $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi(\varphi,\chi))$ is a representation of a pure inner form of split even orthogonal group as $\det(\varphi_{\mathrm{sgn}}) = 1$.

6B. *Discrete unipotent representations.* We follow the definition of unipotent representations given by Lusztig [1995]: $\pi(\phi,\chi)$ is a unipotent representation if and only if ϕ is trivial on the inertia subgroup $\mathcal{I} = \mathcal{I}_F$ of the local Weil group $\mathcal{W}_F$, and such a ϕ is called a unipotent local L-parameter. We apply the local descent method to give an explicit description on the descent of discrete unipotent representations.

Denote by $\xi_{\mathrm{un}} = (\,\cdot\,,\epsilon)_F$ the nontrivial unramified quadratic character of $F^\times$, which is also regarded as a character $\mathcal{W}_F$ via the local class field theory. Here ϵ is a nonsquare element in F with absolute value 1. Let φ be a discrete unipotent L-parameter. Then it can be written as

$$\varphi = \begin{cases} \boxplus_{i=1}^r 1 \boxtimes \mu_{2a_i} \boxplus \boxplus_{j=1}^s \xi_{\mathrm{un}} \boxtimes \mu_{2b_j} & \text{if } G_n = \mathrm{SO}_{2n+1}, \\ \boxplus_{i=1}^r 1 \boxtimes \mu_{2a_i+1} \boxplus \boxplus_{j=1}^s \xi_{\mathrm{un}} \boxtimes \mu_{2b_j+1} & \text{if } G_n = \mathrm{SO}_{2n}. \end{cases} \tag{6-2}$$

Recall the calculation on the local root number from Example 4.4. We obtain the following.

Corollary 6.2. *Let $\pi(\varphi,\eta)$ be a discrete unipotent representation of $G_n = \mathrm{SO}(V_{2n+1})$. Then $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is irreducible and*

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \begin{cases} \pi_a(\varphi_{\mathrm{sgn}},\zeta) & \textit{if } \mathrm{disc}(\mathcal{O}_{\ell_0}) = \det(\varphi_{\mathrm{sgn}})\,\mathrm{disc}(V_{2n+1}), \\ 0 & \textit{otherwise,} \end{cases}$$

where $\zeta = \chi^\star_{\varphi_{\mathrm{sgn}}}(\varphi,\varphi_{\mathrm{sgn}})$ and $a = -\det(\varphi_{\mathrm{sgn}})\,\mathrm{disc}(V_{2n+1})$.

Note that when $G_n = \mathrm{SO}(V_{2n+1})$, the descent φ_{sgn} is invariant under c-conjugate. And the local descent $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is an irreducible unipotent discrete representation.

Now, let us consider the case $G_n = \mathrm{SO}(V_{2n})$. Suppose that an irreducible unipotent discrete representation π in the L-packet $\Pi_\varphi(G_n)$. One can choose a local Langlands correspondence ι_a such that

$\pi = \pi_a(\varphi, \eta)$ with the condition that $\eta(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$. To prove this, let us consider the action of $\mathcal{Z}$ on $\hat{\mathcal{S}}_\varphi$. In this case, as $\det\varphi = \epsilon^s$, the character η_ϖ (defined in Section 2B) is equal to

$$\eta_\varpi(((e_i), (\mathfrak{e}_j))) = \prod_{j=1}^{s} (-1)^{\mathfrak{e}_j},$$

where ϖ is a uniformizer in F. Then $\chi(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$ or $\chi \otimes \eta_\varpi(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$ for any $\chi \in \hat{\mathcal{S}}_\varphi$. Hence, by the definition of $\mathcal{O}_{\mathcal{Z}}(\pi)$ in (2-17), there exists a character χ in $\mathcal{O}_{\mathcal{Z}}(\pi)$ such that $\chi(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$. Then we can choose ι_a such that $\chi_a(\pi) = \chi$, which is the desired normalization for the local Langlands correspondence.

Corollary 6.3. *Let π be a discrete unipotent representation of $G_n = \mathrm{SO}(V_{2n})$ in $\Pi_\varphi[G_n^*]$. Choose the local Langlands correspondence ι_a such that $\pi = \pi_a(\pi, \eta)$ with $\eta(1_{1\boxtimes\mu_{2a_r+1}\boxplus\xi_{\mathrm{un}}\boxtimes\mu_{2b_s+1}}) = 1$. Then $\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi)$ is an irreducible representation of $\mathrm{SO}(V_{2m+1})$ with $m = \dim\varphi_{\mathrm{sgn}}/2$ and*

$$\mathcal{D}_{\mathcal{O}_{\ell_0}}(\pi) = \begin{cases} \pi(\varphi_{\mathrm{sgn}}, \zeta) & \text{if } \mathrm{disc}(\mathcal{O}_{\ell_0}) = a, \\ 0 & \text{otherwise}, \end{cases}$$

where $\zeta = \chi^\star_{\varphi_{\mathrm{sgn}}}(\varphi, \varphi_{\mathrm{sgn}})$, $\mathrm{disc}(V_{2m+1}) = -a\epsilon^s$ and

$$\mathrm{Hss}(V_{2m+1}) = (-1, -1)^{m(m+1)/2}((-1)^{m+1}, \mathrm{disc}(V_{2m+1}))\eta((1)).$$

6C. ***On Conjecture 1.8.*** We will show that Conjecture 1.8 holds for certain representations of SO_7^*.

Referring to [Collingwood and McGovern 1993], for SO_7^*, all stable unipotent orbits are parametrized by the following partitions $\underline{p}$, respectively

$$[7], \quad [5, 1^2], \quad [3^2, 1], \quad [3, 2^2], \quad [3, 1^4], \quad [2^2, 1^3], \quad [1^7], \tag{6-3}$$

where the powers indicate the multiplicities in the partitions, and the corresponding unipotent orbits are listed following the topological order. In particular, [7] is for the regular unipotent orbit and $[1^7]$ is for the trivial orbit. In this case, this topological order is a total order. Hence, for an irreducible smooth representation π, the set $\mathfrak{p}^m(\pi)$ is a singleton. We may assume that

$$\mathfrak{p}^m(\pi) = \{\underline{p} = [p_1 p_2 \cdots p_r]\}, \quad \text{where } p_1 \geq p_2 \geq \cdots \geq p_r > 0. \tag{6-4}$$

Let π be an irreducible square-integrable representation of $\mathrm{SO}(V_7, F)$ and φ be its L-parameter. We are going to apply our main results to the following two types of φ

(1) $\varphi = \chi_1 \boxtimes \mu_4 \boxplus \chi_2 \boxtimes \mu_2$ or

(2) $\varphi = \chi_1 \boxtimes \mu_2 \boxplus \chi_2 \boxtimes \mu_2 \boxplus \chi_3 \boxtimes \mu_2$,

where χ_i are quadratic characters, and to verify that Conjecture 1.8 holds for the representations in the corresponding Vogan packets. For simplicity, we assume that -1 is a square in $F^\times$. Then $(\det\varphi, -1)_F = 1$ in (2-20) for all L-parameters φ. We take V_7 satisfying $\mathrm{disc}(V_7) = -1 = 1 \bmod F^{\times 2}$. For the odd special orthogonal group, the local Langland correspondence is unique and denote χ_π to be the corresponding character in $\hat{\mathcal{S}}_\varphi$.

To verify Conjecture 1.8, we will construct an L-parameter ϕ in $\mathfrak{D}_\ell(\varphi, \chi_\pi)$ for some $\ell \in \{1, 2\}$, which implies the descent $\mathfrak{D}_\ell(\varphi, \chi_\pi)$ is not empty. By Theorem 1.7, for such ϕ the F-rational orbit $\mathcal{O}_\ell$ and the quadratic space W defining $G_n^{\mathcal{O}_\ell}$ are determined by $\mathrm{disc}(\mathcal{O}_\ell) = \mathrm{disc}(W) = \det \phi \mod F^{\times 2}$. Following (2-25) and (2-26), after simplification, $\mathrm{Hss}(V_7) = \mathrm{Hss}(W) = \chi_\pi((1))$. The local Langlands correspondence ι_a for the even special orthogonal group $\mathrm{SO}(W)$ is normalized by $a = \det \phi \mod F^{\times 2}$. Hence we obtain that the following irreducible square-integrable representation of $\mathrm{SO}(W, F)$

$$\sigma = \pi_{\det \phi}(\phi, \chi^\star(\varphi, \phi))$$

occurs in $\mathcal{D}_{\mathcal{O}_\ell}(\pi)$ for the above chosen F-rational orbit, where

$$\mathrm{disc}(W) = \det \phi \text{ and } \mathrm{Hss}(W) = \mathrm{Hss}(V_7).$$

It follows that p_1 in (6-4) is greater than or equal to $2\ell + 1$. Because $\ell \in \{1, 2\}$, we have a lower bound $p_1 \geq 3$ by the total order in (6-3).

Now, if $p_1 = 7$, π is generic and then $\ell_0 = 3$. Conjecture 1.8 holds. If $p_1 = 5$, then the nonvanishing of the twisted Jacquet module associated $\underline{p} = [51^2]$ is equivalent to the nonvanishing of the local descent $\mathcal{D}_{\mathcal{O}_2}(\pi)$ by Lemma 3.1. By definition, the first occurrence index ℓ_0 equals 2. If $p_1 = 3$, by the above lower bound $p_1 \geq 3$ we have $p_1 = 3$. Then $\ell_0 = 1$ in this case, which implies the conjecture.

Furthermore, for these two types of parameters, our results may explicitly determine $\mathfrak{p}^m(\pi)$ in terms of (φ, χ_π) associated to π. The detailed calculation will be given in the remainder of this paper.

Type (1): Assume that $\varphi = \chi_1 \boxtimes \mu_4 \boxplus \chi_2 \boxtimes \mu_2$. Then $\mathcal{S}_\varphi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Denote by ζ_+ and ζ_- the trivial and nontrivial characters of $\mathbb{Z}_2$, respectively. Then we may write the characters of $\mathcal{S}_\varphi$ as $\zeta_\pm \otimes \zeta_\pm$. Its Vogan packet $\Pi_\varphi[\mathrm{SO}_7^*]$ contains four representations $\pi(\varphi, \zeta_\pm \otimes \zeta_\pm)$.

If $\pi = \pi(\varphi, \zeta_+ \otimes \zeta_+)$, then π is the unique generic representation in $\Pi_\varphi[\mathrm{SO}_7^*]$. We have $\ell_0 = 3$ and $\underline{p} = [7]$.

For $\pi = \pi(\varphi, \zeta_- \otimes \zeta_-)$, choose

$$\phi = \begin{cases} \chi_1 \boxplus \chi_2 & \text{if } \chi_1 \neq \chi_2, \\ \chi_1 \boxplus \chi' & \text{if } \chi_1 = \chi_2, \end{cases}$$

where χ' is a quadratic character not isomorphic to χ_1 or χ_2. Since π is nongeneric, we have $\ell_0(\pi) \leq 2$. By Lemma 4.7, $\phi \in \mathfrak{D}_2(\varphi, \zeta_- \otimes \zeta_-)$ and then $\ell_0(\pi) \geq 2$. It follows that $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

When $\pi = \pi(\varphi, \zeta_+ \otimes \zeta_-)$ or $\pi(\varphi, \zeta_- \otimes \zeta_+)$, $\mathrm{Hss}(V_7) = -1$ (i.e., SO_7 is nonsplit) and $\ell_0 \leq 2$. Similarly, we have $\chi_2 \boxplus \chi' \in \mathfrak{D}_2(\varphi, \zeta_+ \otimes \zeta_-)$ and $\phi = \chi_1 \boxplus \chi' \in \mathfrak{D}_2(\varphi, \zeta_- \otimes \zeta_+)$. In both cases, $\ell_0 \geq 2$ and then $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

Type (2): Assume that $\varphi = \chi_1 \boxtimes \mu_2 \boxplus \chi_2 \boxtimes \mu_2 \boxplus \chi_3 \boxtimes \mu_2$. Since π is square-integrable, χ_1, χ_2 and χ_3 are distinct. Then $\mathcal{S}_\varphi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the character of $\mathcal{S}_\varphi$ is of form $\zeta_1 \otimes \zeta_2 \otimes \zeta_3$, where ζ_i for $1 \leq i \leq 3$ are characters of $\mathbb{Z}_2$.

Let $\pi = \pi(\varphi, \zeta_1 \otimes \zeta_2 \otimes \zeta_3)$. If $\zeta_i = \zeta_+$ for all $1 \le i \le 3$, then π is generic and $\ell_0(\pi) = 3$. All the other representations are nongeneric. For the remaining cases, we always have $\ell_0(\pi) \le 2$, i.e., $\underline{p} \ne [7]$.

As $F^\times/(F^\times)^2$ contains at least 4 elements, there exists a character χ' of $F^\times$ such that $\chi'^2 = 1$ and χ' is not isomorphic to any χ_i for $1 \le i \le 3$. When $\zeta_{i_0} = \zeta_-$ for some i_0 and $\zeta_i = \zeta_+$ for all $i \ne i_0$, we may take $\phi = \chi_{i_0} \boxplus \chi'$. In this case, SO_7 is nonsplit. It follows that $\phi \in \mathfrak{D}_2(\varphi, \otimes_{i=1}^3 \zeta_i)$ and then $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

If $\zeta_{i_0} = \zeta_+$ for some i_0 and $\zeta_i = \zeta_-$ for all $i \ne i_0$, we may take $\phi = \chi_i \boxplus \chi_j$ where $\{i, j\} = \{1, 2, 3\} \smallsetminus \{i_0\}$. One has that $\phi \in \mathfrak{D}_2(\varphi, \otimes_{i=1}^3 \zeta_i)$ and hence $\ell_0(\pi) = 2$ and $\mathfrak{p}^m(\pi) = [5, 1^2]$.

If $\zeta_i = \zeta_-$ for all $1 \le i \le 3$, we may take $\phi = \boxplus_{i=1}^3 \chi_i \boxplus \chi'$, which is in $\mathfrak{D}_1(\varphi, \otimes_{i=1}^3 \zeta_i)$. We have the lower bound $\ell_0(\pi) \ge 1$. By the above discussion, this implies Conjecture 1.8 for this representation. In this case, one may also explicitly determine $\mathfrak{p}^m(\pi)$ by calculating the symplectic root number $\varepsilon(\tau \times \chi_i)$ where τ is a supercuspidal representation of $\mathrm{GL}_2(F)$ with the central character $\omega_\tau = 1$ (i.e., of symplectic type). We omit the details here.

Remark 6.4. Beside the above *Type* (1) and *Type* (2), for any irreducible smooth representation π of $\mathrm{SO}(V_7, F)$ in a generic L-packet, we may obtain the lower bound $\ell_0(\pi) \ge 1$ by using an alternative global argument. Then Conjecture 1.8 holds for all pure inner forms of SO_7^*. However, such global arguments only work for the special orthogonal groups of lower rank.

Remark 6.5 (counter example). We give an example to show that Conjecture 1.8 may not be true for nontempered representations. Let $G_n^* = \mathrm{SO}_4^*$ be the split even orthogonal group. Note that SO_4^* only has 4 stable unipotent orbits, whose corresponding partitions are

$$[3, 1], [2^2]^I, [2^2]^{II}, [1^4].$$

Here $[2^2]^I$ and $[2^2]^{II}$ are the same partitions of 4 but give two different unipotent orbits. We take π to be the irreducible nongeneric nontempered infinite dimensional representation of SO_4^*. Then it is not generic and has a nonzero twisted Jacquet model associated to $[2^2]^I$ or $[2^2]^{II}$. In this case, the largest part p_1 is even and not equal to $2\ell + 1$ for all ℓ. In general, one can find a family of nontempered representations π of SO_{2n}^*, whose largest part p_1 in the partitions of $\mathfrak{p}^m(\pi)$ are even. Hence Conjecture 1.8 fails for those nontempered representations.

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