Density theorems for exceptional eigenvalues
for congruence subgroups

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Using the Kuznetsov formula, we prove several density theorems for exceptional Hecke and Laplacian eigenvalues of Maaß cusp forms of weight 0 or 1 for the congruence subgroups $\Gamma_0(q)$, $\Gamma_1(q)$, and $\Gamma(q)$. These improve and extend upon results of Sarnak and Huxley, who prove similar but slightly weaker results via the Selberg trace formula.

1. Introduction

Let $\kappa \in \{0, 1\}$, let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let χ be a congruence character of Γ satisfying $\chi(-I) = (-1)^\kappa$ should $-I$ be a member of Γ. Denote by $\mathcal{A}_\kappa(\Gamma, \chi)$ the space spanned by Maaß cusp forms of weight κ, level Γ, and nebentypus χ, namely the L^2-closure of the space of smooth functions $f : \mathbb{H} \to \mathbb{C}$ satisfying

- $f(\gamma z) = \chi(\gamma) j_\gamma(z)^\kappa f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$j_\gamma(z) := \frac{cz + d}{|cz + d|},$$

- f is an eigenfunction of the weight κ Laplacian

$$\Delta_\kappa := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\kappa y \frac{\partial}{\partial x},$$

- f is of moderate growth, and

- the constant term is zero in the Fourier expansion of f at every cusp $\mathfrak{a}$ of $\Gamma \backslash \mathbb{H}$ that is singular with respect to χ.

We may choose a basis $\mathcal{B}_\kappa(\Gamma, \chi)$ of the complex vector space $\mathcal{A}_\kappa(\Gamma, \chi)$ consisting of Hecke eigenforms. For $f \in \mathcal{B}_\kappa(\Gamma, \chi)$, we let $\lambda_f = \frac{1}{4} + t_f^2$ denote the eigenvalue of the weight κ Laplacian, where either $t_f \in [0, \infty)$ or $it_f \in \left(0, \frac{1}{2}\right)$. Similarly, we let $\lambda_f(p)$ denote the eigenvalue of the Hecke operator T_p at a prime p, so that $|\lambda_f(p)| < p^{\frac{1}{2}} + p^{-\frac{1}{2}}$. The generalised Ramanujan conjecture states that t_f is real and that $|\lambda_f(p)| \leq 2$ for every prime p. Exceptions to this conjecture are called exceptional eigenvalues. It is known that exceptional Laplacian eigenvalues cannot occur if $\kappa = 1$, while for $\kappa = 0$ there are no exceptional Laplacian eigenvalues for Maaß cusp forms of squarefree conductor less than 857 [Booker

and Strömbergsson 2007, Theorem 1]. The best current bounds towards the generalised Ramanujan conjecture are due to Kim and Sarnak [2003]; they show that

$$\lambda_f \geq \tfrac{1}{4} - \left(\tfrac{7}{64}\right)^2, \quad |\lambda_f(p)| \leq p^{\frac{7}{64}} + p^{-\frac{7}{64}}.$$

Results. In this paper, we use the Kuznetsov formula to prove density results for exceptional eigenvalues for the congruence subgroups

$$\Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \ (\mathrm{mod}\, q) \right\},$$

$$\Gamma_1(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \ (\mathrm{mod}\, q), \ c \equiv 0 \ (\mathrm{mod}\, q) \right\},$$

$$\Gamma(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \ (\mathrm{mod}\, q), \ b, c \equiv 0 \ (\mathrm{mod}\, q) \right\},$$

with χ equal to the trivial character for the latter two congruence subgroups. Recall that

$$\mathrm{vol}(\Gamma \backslash \mathbb{H}) = \frac{\pi}{3}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = \begin{cases} \frac{\pi}{3} q \prod_{p \mid q} \left(1 + \frac{1}{p}\right) & \text{if } \Gamma = \Gamma_0(q), \\ \frac{\pi}{3} q^2 \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) & \text{if } \Gamma = \Gamma_1(q), \\ \frac{\pi}{3} q^3 \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right) & \text{if } \Gamma = \Gamma(q). \end{cases}$$

When χ is the trivial character, we write $\mathcal{B}_\kappa(\Gamma)$ in place of $\mathcal{B}_\kappa(\Gamma, \chi)$, while when $\Gamma = \Gamma_0(q)$, we write this as $\mathcal{B}_\kappa(q, \chi)$. Given positive integers q and q_χ with $q_\chi \mid q$, we factorise $q = \prod_{p^\alpha \| q} p^\alpha$ and $q_\chi = \prod_{p^\gamma \| q_\chi} p^\gamma$, and define

$$\dot{Q} = \dot{Q}(q, q_\chi) = \prod_{\substack{p^\alpha \| q \\ p^\gamma \| q_\chi}} \dot{Q}(p^\alpha, p^\gamma), \quad \ddot{Q} = \ddot{Q}(q, q_\chi) = \prod_{\substack{p^\alpha \| q \\ p^\gamma \| q_\chi}} \ddot{Q}(p^\alpha, p^\gamma),$$

with

$$\dot{Q}(p^\alpha, p^\gamma) := \begin{cases} p^{\lfloor (3\alpha+1)/4 \rfloor - \alpha/2} & \text{if } p \text{ is odd and } \alpha = \gamma \geq 3, \\ 2^{\lfloor (3\alpha+1)/4 \rfloor - \alpha/2} & \text{if } p = 2 \text{ and } \gamma + 1 \geq \alpha \geq 3, \\ 1 & \text{otherwise}, \end{cases}$$

$$\ddot{Q}(p^\alpha, p^\gamma) := \begin{cases} p & \text{if } p \text{ is odd and } \alpha = \gamma \geq 3, \\ 4 & \text{if } p = 2 \text{ and } \alpha = \gamma \geq 3, \\ 2 & \text{if } p = 2 \text{ and } \alpha = \gamma + 1 \geq 3, \\ 1 & \text{otherwise}. \end{cases}$$

Theorem 1.1. *For any fixed finite collection of primes $\mathcal{P}$ not dividing q, any $\alpha_p \in (2, p^{\frac{1}{2}} + p^{-\frac{1}{2}})$ and $0 \leq \mu_p \leq 1$ for all $p \in \mathcal{P}$ with $\sum_{p \in \mathcal{P}} \mu_p = 1$, we have that*

$$\#\left\{ f \in \mathcal{B}_\kappa(\Gamma_1(q)) : t_f \in [0, T], \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \right\}$$
$$\ll_\varepsilon \mathrm{vol}(\Gamma_1(q) \backslash \mathbb{H})^{1 - 3 \sum_{p \in \mathcal{P}} \mu_p (\log \alpha_p / 2) / \log p + \varepsilon} (T^2)^{1 - 4 \sum_{p \in \mathcal{P}} \mu_p (\log \alpha_p / 2) / \log p + \varepsilon}, \quad (1.2)$$

$$\#\big\{ f \in \mathcal{B}_\kappa(\Gamma(q)) : t_f \in [0, T], \ |\lambda_f(p)| \ge \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$
$$\ll_\varepsilon \operatorname{vol}(\Gamma(q)\backslash\mathbb{H})^{1-\frac{8}{3}\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p+\varepsilon}(T^2)^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p+\varepsilon}, \quad (1.3)$$

$$\#\big\{ f \in \mathcal{B}_\kappa(q, \chi) : t_f \in [0, T], \ |\lambda_f(p)| \ge \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$
$$\ll_\varepsilon \operatorname{vol}(\Gamma_0(q)\backslash\mathbb{H})^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p+\varepsilon}(T^2)^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p+\varepsilon}$$
$$\times \min\big\{ \dot{Q}^{4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p}, \ \ddot{Q}^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p} \big\}. \quad (1.4)$$

Theorem 1.1 should be compared to the Weyl law, which states that

$$\#\big\{ f \in \mathcal{B}_\kappa(\Gamma, \chi) : t_f \in [0, T] \big\} \sim \frac{\operatorname{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2.$$

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, so that χ is the trivial character, and $\mathcal{P}$ consisting of a single prime p, Theorem 1.1 is a result of Blomer, Buttcane, and Raulf [Blomer et al. 2014, Proposition 1], improving on a slightly weaker result of Sarnak [1987, Theorem 1.1], who uses the Selberg trace formula in place of the Kuznetsov formula and obtains instead (see [Blomer et al. 2014, Footnote 1])

$$\#\big\{ f \in \mathcal{B}_0(\mathrm{SL}_2(\mathbb{Z})) : t_f \in [0, T], \ |\lambda_f(p)| \ge \alpha \big\} \ll (T^2)^{1-2(\log\alpha/2)/\log p}.$$

Theorem 1.5. *For any fixed finite (possibly empty) collection of primes $\mathcal{P}$ not dividing q, any $\alpha_0 \in \big(0, \frac{1}{2}\big)$, $\alpha_p \in (2, p^{\frac{1}{2}} + p^{-\frac{1}{2}})$, and $0 \le \mu_0, \mu_p \le 1$ for all $p \in \mathcal{P}$ with $\mu_0 + \sum_{p\in\mathcal{P}} \mu_p = 1$, we have that*

$$\#\big\{ f \in \mathcal{B}_0(\Gamma_1(q)) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \ge \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$
$$\ll_\varepsilon \operatorname{vol}(\Gamma_1(q)\backslash\mathbb{H})^{1-3(\mu_0\alpha_0+\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p)+\varepsilon}, \quad (1.6)$$

$$\#\big\{ f \in \mathcal{B}_0(\Gamma(q)) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \ge \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$
$$\ll_\varepsilon \operatorname{vol}(\Gamma(q)\backslash\mathbb{H})^{1-\frac{8}{3}(\mu_0\alpha_0+\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p)+\varepsilon}. \quad (1.7)$$

$$\#\big\{ f \in \mathcal{B}_0(q, \chi) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \ge \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$
$$\ll_\varepsilon \operatorname{vol}(\Gamma_0(q)\backslash\mathbb{H})^{1-4(\mu_0\alpha_0+\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p)+\varepsilon}$$
$$\times \min\big\{ \dot{Q}^{4(\mu_0\alpha_0+\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p)}, \ \ddot{Q}^{1-4(\mu_0\alpha_0+\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p)} \big\}. \quad (1.8)$$

When $\mathcal{P}$ is empty and χ is the trivial congruence character, Theorem 1.5 improves upon a result of Huxley [1986], who uses the Selberg trace formula in place of the Kuznetsov formula and obtains instead this result with the exponent 2 for each of the three congruence subgroups instead of 3, $\frac{8}{3}$, and 4 respectively. When $\mathcal{P}$ is empty and χ is the trivial congruence character, (1.8) is a result of Iwaniec [2002, Theorem 11.7] ; see also [Iwaniec and Kowalski 2004, (16.61)].

Since

$$\left\lfloor \frac{3\alpha+1}{4} \right\rfloor - \frac{\alpha}{2} \le \frac{3\alpha}{10},$$

so that $\dot{Q} \ll \operatorname{vol}(\Gamma_0(q)\backslash\mathbb{H})^{\frac{3}{10}}$, the right-hand side of (1.4) is bounded by

$$\operatorname{vol}(\Gamma_0(q)\backslash\mathbb{H})^{1-\frac{14}{5}\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p+\varepsilon}(T^2)^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p+\varepsilon},$$

while the right-hand side of (1.8) is bounded by

$$\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^{1-\frac{14}{5}\left(\mu_0\alpha_0+\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p\right)+\varepsilon}.$$

On the other hand, taking $\mathcal{P}$ to consist of a single prime in (1.4) recovers the Selberg bound $\lambda_f(p)\ll_\varepsilon p^{\frac{1}{4}+\varepsilon}$ for an individual element $f\in\mathcal{B}_\kappa(q,\chi)$ by taking T sufficiently large, while taking $\mathcal{P}$ to be empty in (1.8) recovers the Selberg bound $\lambda_f\geq\frac{3}{16}$ by embedding f in $\mathcal{B}_\kappa(qQ,\chi)$ and taking Q sufficiently large.

   Finally, we also prove the following improvements of Theorems 1.1 and 1.5 for $\Gamma_1(q)$ with q squarefree via a twisting argument.

Theorem 1.9. *When q is squarefree*, (1.2) *and* (1.6) *hold with the exponent* 3 *replaced by* 4.

Idea of Proof. By Rankin's trick (which is to say Chebyshev's inequality), it suffices to find bounds for

$$\sum_{\substack{f\in\mathcal{B}_\kappa(\Gamma,\chi)\\t_f\in[0,T]}}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p},\qquad\sum_{\substack{f\in\mathcal{B}_0(\Gamma,\chi)\\it_f\in(0,\frac{1}{2})}}X^{2it_f}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p}$$

for nonnegative integers ℓ_p and a positive real number $X\geq1$ to be chosen. To bound these quantities, we begin with the Kuznetsov formula for $\mathcal{B}_\kappa(q,\chi)$; we then use the Atkin–Lehner decomposition to turn this into a Kuznetsov formula for $\mathcal{B}_\kappa(\Gamma,\chi)$. We take a test function in the Kuznetsov formula that localises the spectral sum to cusp forms with $t_f\in[0,T]$ in the case of Theorem 1.1 and to cusp forms with $it_f\in\left(0,\frac{1}{2}\right)$ in the case of Theorem 1.5. We use the Hecke relations to introduce powers of the Hecke eigenvalues into the Kuznetsov formula. By positivity, we discard the contribution of the continuous spectrum, and we are left with bounding the right-hand side of the Kuznetsov formula.

   The chief novelty of the proof is the bounds for sums of Kloosterman sums in the Kuznetsov formula for each congruence subgroup. As well as the usual Weil bound, we use character orthogonality for $\Gamma_1(q)$ and $\Gamma(q)$, at which point we only use the trivial bound for the resulting sum of Kloosterman sums. For $\Gamma_0(q)$ and χ the principal character, we may also use the Weil bound, but for χ nonprincipal, additional difficulties arise in bounding the Kloosterman sum, with the bound possibly depending on the conductor of χ; it is for this reason that the bounds (1.4) and (1.8) involve $\dot{Q}$, for $\dot{Q}$ arises when only weaker bounds than the Weil bound are possible for the Kloosterman sums involved.

   We also highlight the key trick to proving Theorem 1.9, namely that the Laplacian eigenvalue and absolute value of a Hecke eigenvalue of a Maaß form remain unchanged under twisting by a Dirichlet character. Twisting may alter the level of a Maaß form, yet Theorem 1.9 involves a favourable situation in which the resulting family of twisted Maaß forms are sufficiently well-behaved that we are able to improve the exponent in the density theorem.

   It is worth mentioning that the results in this paper ought to generalise naturally to cusp forms on GL_2 over arbitrary number fields F. Bruggeman and Miatello [2009] prove a form of the Kuznetsov formula for GL_2 over a totally real field and use this to prove weighted Weyl law for cusp forms. Similarly, Maga [2013] proves a semiadèlic version of the Kuznetsov formula for GL_2 over an arbitrary number field. In the former case, this formula is valid for congruence subgroups of the form $\Gamma_0(\mathfrak{q})$ for a nonzero integral

ideal $\mathfrak{q}$ of the ring of integers $\mathcal{O}_F$ of F and arbitrary congruence characters χ modulo $\mathfrak{q}$, while the latter only treats the case of trivial congruence character but should easily be able to be generalised to arbitrary congruence character; this is precisely what is required for density theorems for the congruence subgroups $\Gamma_0(\mathfrak{q})$, $\Gamma_1(\mathfrak{q})$, and $\Gamma(\mathfrak{q})$.

2. The Kuznetsov formula

The background on automorphic forms and notation in this section largely follows [Duke et al. 2002]; see [Duke et al. 2002, Section 4] for more details. Let $\kappa \in \{0, 1\}$, and let χ be a primitive Dirichlet character modulo q_χ, where q_χ divides q, satisfying $\chi(-1) = (-1)^\kappa$; this defines a congruence character of $\Gamma_0(q)$ via $\chi(\gamma) := \chi(d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(q)$. We denote by $L^2(\Gamma_0(q)\backslash\mathbb{H}, \kappa, \chi)$ the L^2-completion of the space of all smooth functions $f : \mathbb{H} \to \mathbb{C}$ that are of moderate growth and satisfy $f(\gamma z) = \chi(\gamma) j_\gamma(z)^\kappa f(z)$. This space has the spectral decomposition

$$L^2(\Gamma_0(q)\backslash\mathbb{H}, \kappa, \chi) = \mathcal{A}_\kappa(q, \chi) \oplus \mathcal{E}_\kappa(q, \chi)$$

with respect to the weight κ Laplacian, where $\mathcal{A}_\kappa(q, \chi) := \mathcal{A}_\kappa(\Gamma_0(q), \chi)$ is the space spanned by Maaß cusp forms of weight κ, level q, and nebentypus χ, and $\mathcal{E}_\kappa(q, \chi)$ is the space spanned by incomplete Eisenstein series parametrised by the cusps $\mathfrak{a}$ of $\Gamma_0(q)\backslash\mathbb{H}$ that are singular with respect to χ.

We denote by $\mathcal{B}_\kappa(q, \chi)$ an orthonormal basis of Maaß cusp forms $f \in \mathcal{A}_\kappa(q, \chi)$ normalised to have L^2-norm 1:

$$\langle f, f \rangle_q := \int_{\Gamma_0(q)\backslash\mathbb{H}} |f(z)|^2 \, d\mu(z) = 1,$$

where $d\mu(z) = dx \, dy/y^2$ is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}$. Later we will use the Atkin–Lehner decomposition of $\mathcal{A}_\kappa(q, \chi)$ in order to specify that $\mathcal{B}_\kappa(q, \chi)$ can be chosen to consist of linear combinations of Hecke eigenforms. The Fourier expansion of $f \in \mathcal{B}_\kappa(q, \chi)$ is

$$f(z) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \rho_f(n) W_{\mathrm{sgn}(n)\kappa/2, it_f}(4\pi |n| y) e(nx),$$

where $W_{\alpha, \beta}$ is the Whittaker function and

$$\rho_f(n) W_{\mathrm{sgn}(n)\kappa/2, it_f}(4\pi |n| y) = \int_0^1 f(z) e(-nx) \, dx.$$

For a singular cusp $\mathfrak{a}$, we define the Eisenstein series

$$E_\mathfrak{a}(z, s, \chi) := \sum_{\gamma \in \Gamma_\mathfrak{a} \backslash \Gamma_0(q)} \overline{\chi}(\gamma) j_{\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-\kappa} \Im(\sigma_\mathfrak{a}^{-1}\gamma z)^s,$$

which is absolutely convergent for $\Re(s) > 1$ and extends meromorphically to $\mathbb{C}$, with the Fourier expansion

$$\delta_{\mathfrak{a},\infty} y^{\frac{1}{2}+it} + \varphi_{\mathfrak{a},\infty}\left(\tfrac{1}{2} + it, \chi\right) y^{\frac{1}{2}-it} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \rho_\mathfrak{a}(n, t, \chi) W_{\mathrm{sgn}(n)\kappa/2, it}(4\pi |n| y) e(nx)$$

for $s = \frac{1}{2} + it$ with $t \in \mathbb{R} \setminus \{0\}$, where

$$\delta_{\mathfrak{a},\infty} y^{\frac{1}{2}+it} + \varphi_{\mathfrak{a},\infty}\left(\tfrac{1}{2}+it, \chi\right) y^{\frac{1}{2}-it} := \int_0^1 E_{\mathfrak{a}}\left(z, \tfrac{1}{2}+it, \chi\right) dx,$$

$$\rho_{\mathfrak{a}}(n, t, \chi) W_{\mathrm{sgn}(n)\kappa/2, it}(4\pi |n|y) := \int_0^1 E_{\mathfrak{a}}\left(z, \tfrac{1}{2}+it, \chi\right) e(-nx) \, dx.$$

The subspace $\mathcal{E}_\kappa(q, \chi)$ consists of functions $g \in L^2(\Gamma_0(q)\backslash\mathbb{H}, \kappa, \chi)$ that are orthogonal to every Maaß cusp form $f \in \mathcal{A}_\kappa(q, \chi)$; it is the L^2-closure of the space spanned by incomplete Eisenstein series, which are functions of the form

$$E_{\mathfrak{a}}(z, \psi, \chi) := \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} E_{\mathfrak{a}}(z, s, \chi)\widehat{\psi}(s) \, ds \tag{2.1}$$

for some singular cusp $\mathfrak{a}$ and some smooth function of compact support $\psi : \mathbb{R}^+ \to \mathbb{C}$, where $\sigma > 1$ and

$$\widehat{\psi}(s) := \int_0^\infty \psi(x) x^{-s} \frac{dx}{x}.$$

Theorem 2.2 [Duke et al. 2002, Proposition 5.2]. *For $m, n \geq 1$ and $r \in \mathbb{R}$,*

$$\sum_{f \in \mathcal{B}_\kappa(q,\chi)} \frac{4\pi\sqrt{mn}\,\overline{\rho_f}(m)\rho_f(n)}{\cosh\pi(r-t_f)\cosh\pi(r+t_f)} + \sum_{\mathfrak{a}} \int_{-\infty}^\infty \frac{\sqrt{mn}\,\overline{\rho_{\mathfrak{a}}}(m, t, \chi)\rho_{\mathfrak{a}}(n, t, \chi)}{\cosh\pi(r-t)\cosh\pi(r+t)} \, dt$$

$$= \frac{|\Gamma(1-\kappa/2-ir)|^2}{\pi^2}\left(\delta_{m,n} + \sum_{\substack{c=1 \\ c\equiv 0 \,(\mathrm{mod}\, q)}}^\infty \frac{S_\chi(m, n; c)}{c} I_\kappa\left(\frac{4\pi\sqrt{mn}}{c}, r\right)\right),$$

where

$$S_\chi(m, n; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(d)e\left(\frac{md+n\bar{d}}{c}\right),$$

$$I_\kappa(t, r) := -2t \int_{-i}^i (-i\zeta)^{\kappa-1} K_{2ir}(\zeta t) \, d\zeta,$$

with the latter integral being over the semicircle $|z| = 1$, $\Re(z) > 0$.

By the reflection formula for the gamma function, we have that for $r \in \mathbb{R}$,

$$\left|\Gamma\left(1 - \frac{\kappa}{2} - ir\right)\right|^2 = \begin{cases} \pi r/\sinh\pi r & \text{if } \kappa = 0, \\ \pi/\cosh\pi r & \text{if } \kappa = 1. \end{cases}$$

Given a sufficiently well-behaved function h, we may multiply both sides of the pre-Kuznetsov formula for $\kappa = 0$ by

$$\frac{1}{2}\left(h\left(r + \frac{i}{2}\right) + h\left(r - \frac{i}{2}\right)\right)\cosh\pi r$$

and then integrate both sides from $-\infty$ to ∞ with respect to r. This yields the following Kuznetsov formula (see [Blomer et al. 2007, Section 2.1.4; Iwaniec and Kowalski 2004, Theorem 16.3; Knightly and Li 2013, Equation (7.32)]):

Theorem 2.3. *Let $\delta > 0$, and let h be a function that is even, holomorphic in the horizontal strip $|\Im(t)| \leq \frac{1}{2} + \delta$, and satisfies $h(t) \ll (|t| + 1)^{-2-\delta}$. Then*

$$\sum_{f \in \mathcal{B}_0(q,\chi)} 4\pi \sqrt{mn} \, \overline{\rho_f}(m) \rho_f(n) \frac{h(t_f)}{\cosh \pi t_f} + \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \sqrt{mn} \, \overline{\rho_{\mathfrak{a}}}(m, t, \chi) \rho_{\mathfrak{a}}(n, t, \chi) \frac{h(t)}{\cosh \pi t} \, dt$$

$$= \delta_{mn} g_0 + \sum_{\substack{c=1 \\ c \equiv 0 \,(\mathrm{mod}\, q)}}^{\infty} \frac{S_\chi(m, n; c)}{c} g_0 \left(\frac{4\pi \sqrt{mn}}{c} \right),$$

where

$$g_0 := \frac{1}{\pi} \int_{-\infty}^{\infty} r h(r) \tanh \pi r \, dr, \quad g_0(x) := 2i \int_{-\infty}^{\infty} J_{2ir}(x) \frac{r h(r)}{\cosh \pi r} \, dr.$$

The left-hand side of the Kuznetsov formula is called the spectral side; the first term is the contribution from the discrete spectrum, while the second term is the contribution from the continuous spectrum. The right-hand side of the Kuznetsov formula is called the geometric side; the first term is the delta term and the second term is the Kloosterman term.

3. Decomposition of spaces of modular forms

Eisenstein series and Hecke operators. The space $\mathcal{E}_\kappa(q, \chi)$ is spanned by incomplete Eisenstein series of the form (2.1), which are obtained by integrating test functions against Eisenstein series indexed by singular cusps $\mathfrak{a}$; in this sense, the Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$ are a spanning set for $\mathcal{E}_\kappa(q, \chi)$. We may instead choose a different spanning set of Eisenstein series for $\mathcal{E}_\kappa(q, \chi)$; in place of the set of Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$ with $\mathfrak{a}$ a singular cusp, we may instead choose a spanning set of Eisenstein series of the form $E(z, s, f)$ with Fourier expansion

$$c_{1,f}(t) y^{\frac{1}{2}+it} + c_{2,f}(t) y^{\frac{1}{2}-it} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \rho_f(n, t, \chi) W_{\mathrm{sgn}(n)\kappa/2, it}(4\pi |n| y) e(nx)$$

for $s = \frac{1}{2} + it$ with $t \in \mathbb{R} \setminus \{0\}$, where $\mathcal{B}(\chi_1, \chi_2) \ni f$ with $\chi_1 \chi_2 = \chi$ is some finite set depending on χ_1, χ_2 corresponding to an orthonormal basis in the space of the induced representation constructed out of the pair (χ_1, χ_2); see [Blomer et al. 2007, Section 2.1.1] or [Knightly and Li 2013, Chapter 5]. For our purposes, we need not be more specific about $\mathcal{B}(\chi_1, \chi_2)$, other than noting that for each $f \in \mathcal{B}(\chi_1, \chi_2)$, the Eisenstein series $E\left(z, \frac{1}{2} + it, f\right)$ is an eigenfunction of the Hecke operators T_n for $(n, q) = 1$ with Hecke eigenvalues

$$\lambda_f(n, t) = \sum_{ab=n} \chi_1(a) a^{it} \chi_2(b) b^{-it},$$

where for $g : \mathbb{H} \to \mathbb{C}$ a periodic function of period one,

$$(T_n g)(z) := \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \,(\mathrm{mod}\, d)} g\left(\frac{az+b}{d} \right).$$

So for $f \in \mathcal{B}(\chi_1, \chi_2)$,

$$\lambda_f(m, t)\lambda_f(n, t) = \sum_{d|(m,n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}, t\right), \tag{3.1}$$

$$\overline{\lambda_f}(n, t) = \overline{\chi}(n)\lambda_f(n, t), \tag{3.2}$$

$$\rho_f(1, t)\lambda_f(n) = \sqrt{n}\rho_f(n, t), \tag{3.3}$$

whenever $m, n \geq 1$ with $(mn, q) = 1$ and $s = \frac{1}{2} + it$.

Lemma 3.4 (cf. [Conrey et al. 1997, Lemma 3; Hughes and Miller 2007, Lemma 2.8; Petrow and Young 2018, Section 6]). *For any prime $p \nmid q$ and positive integer ℓ, we have that*

$$|\lambda_f(p, t)|^{2\ell} = \sum_{j=0}^{\ell} \alpha_{2j,2\ell}\overline{\chi}(p)^j \lambda_f(p^{2j}, t) \tag{3.5}$$

for any $f \in \mathcal{B}(\chi_1, \chi_2)$ and $s = \frac{1}{2} + it$, where

$$\alpha_{2j,2\ell} = \frac{2j+1}{\ell+j+1}\binom{2\ell}{\ell+j} = \begin{cases} \binom{2\ell}{\ell-j} - \binom{2\ell}{\ell-j-1} & \text{if } 0 \leq j \leq \ell-1, \\ 1 & \text{if } j = \ell, \end{cases} \tag{3.6}$$

so that each $\alpha_{2j,2\ell}$ is positive and satisfies

$$\sum_{j=0}^{\ell} \alpha_{2j,2\ell} = \binom{2\ell}{\ell} \leq 2^{2\ell}. \tag{3.7}$$

Proof. That (3.7) follows from (3.6) is clear. For (3.5), we have that

$$\overline{\chi}(p)^{j/2}\lambda_f(p^j, t) = U_j\left(\frac{\overline{\chi}(p)^{\frac{1}{2}}\lambda_f(p, t)}{2}\right),$$

where U_j is the j-th Chebyshev polynomial of the second kind, because U_j satisfies $U_0(x/2) = 1$, $U_1(x/2) = x$, and the recurrence relation

$$U_{j+1}\left(\frac{x}{2}\right) = xU_j\left(\frac{x}{2}\right) - U_{j-1}\left(\frac{x}{2}\right)$$

for all $j \geq 1$, and $\overline{\chi}(p)^{j/2}\lambda_f(p^j, t)$ satisfies the same recurrence relation from (3.1). Since

$$\frac{2}{\pi} \int_{-1}^{1} U_j(x)U_k(x)\sqrt{1-x^2}\, dx = \delta_{j,k},$$

we have that

$$x^{2\ell} = \sum_{j=0}^{2\ell} \alpha_{j,2\ell}U_j\left(\frac{x}{2}\right),$$

where

$$\alpha_{j,2\ell} = \frac{2^{2\ell+1}}{\pi} \int_{-1}^{1} x^{2\ell}U_j(x)\sqrt{1-x^2}\, dx.$$

This vanishes if j is odd as $U_j(-x) = (-1)^j U_j(x)$, while for j even we have the identity (3.6) from [Gradshteyn and Ryzhik 2007, 7.311.2]. Combined with (3.2), this proves (3.5). □

Atkin–Lehner decomposition for $\Gamma_0(q)$. Similarly, we may choose a basis of $\mathcal{A}_\kappa(q, \chi)$ consisting of linear combinations of Hecke eigenforms. Let $\mathcal{B}_\kappa^*(q, \chi)$ denote the set of newforms of weight κ, level q, and nebentypus χ, and let $\mathcal{A}_\kappa^*(q, \chi)$ denote the subspace of $\mathcal{A}_\kappa(q, \chi)$ spanned by such newforms. Recall that a newform $f \in \mathcal{B}_\kappa^*(q, \chi)$ is an eigenfunction of the weight κ Laplacian Δ_κ with eigenvalue $\frac{1}{4} + t_f^2$ and of every Hecke operator T_n, $n \geq 1$, with eigenvalue $\lambda_f(n)$, as well as the operator $Q_{\frac{1}{2}+it_f,\kappa}$ as defined in [Duke et al. 2002, Section 4], with eigenvalue $\epsilon_f \in \{-1, 1\}$; we say that f is even if $\epsilon_f = 1$ and f is odd if $\epsilon_f = -1$. In particular,

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,q)=1}} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right), \qquad (3.8)$$

$$\rho_f(1)\lambda_f(n) = \sqrt{n}\rho_f(n) \qquad (3.9)$$

whenever $m, n \geq 1$, and

$$\overline{\lambda_f(n)} = \bar{\chi}(n)\lambda_f(n) \qquad (3.10)$$

for $n \geq 1$ with $(n, q) = 1$. Using (3.8) and (3.10), we have the following:

Lemma 3.11. *For any prime $p \nmid q$ and positive integer ℓ, we have that*

$$|\lambda_f(p)|^{2\ell} = \sum_{j=0}^{\ell} \alpha_{2j,2\ell} \bar{\chi}(p)^j \lambda_f(p^{2j}) \qquad (3.12)$$

for any $f \in \mathcal{B}_\kappa^*(q, \chi)$, where once again $\alpha_{2j,2\ell}$ is given by (3.6).

The Atkin–Lehner decomposition states that

$$\mathcal{A}_\kappa(q, \chi) = \bigoplus_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \,(\mathrm{mod}\, q_\chi)}} \bigoplus_{f \in \mathcal{B}_\kappa^*(q_1, \chi)} \bigoplus_{d|q_2} \mathbb{C} \cdot \iota_{d,q_1,q} f,$$

where $\iota_{d,q_1,q} : \mathcal{A}_\kappa(q_1, \chi) \to \mathcal{A}_\kappa(q, \chi)$ is the map $\iota_{d,q_1,q} f(z) = f(dz)$. The map $\iota_{d,q_1,q}$ commutes with the weight k Laplacian Δ_κ and the Hecke operators T_n whenever $n \geq 1$ and $(n, q) = 1$. It follows that if $g = \iota_{d,q_1,q} f$ for some $f \in \mathcal{B}_\kappa^*(q_1, \chi)$, then $t_g = t_f$ and $\lambda_g(n) = \lambda_f(n)$ whenever $n \geq 1$ and $(n, q) = 1$. Note, however, that $\rho_g(1) = 0$ unless $d = 1$, in which case $\rho_g(1) = \rho_f(1)$.

Unfortunately, the inner Atkin–Lehner decomposition

$$\bigoplus_{d|q_2} \mathbb{C} \cdot \iota_{d,q_1,q} f$$

is not an orthogonal decomposition. Nonetheless, one may make use of this decomposition in determining an orthonormal basis of $\mathcal{A}_\kappa(q, \chi)$. For squarefree q and principal nebentypus, this is a result of Iwaniec, Luo, and Sarnak [Iwaniec et al. 2000, Lemma 2.4], while Blomer and Milićević [2015, Lemma 9] have

generalised this to nonsquarefree q. Here we generalise this further to nonprincipal nebentypus; this has also independently been derived by Schulze-Pillot and Yenirce [2018] via a different method.

Lemma 3.13 (cf. [Iwaniec et al. 2000, Lemma 2.4; Blomer and Milićević 2015, Lemma 9]). *Suppose that χ has conductor $q_\chi \mid q$, and suppose that $q_1 q_2 = q$ with $q_1 \equiv 0 \pmod{q_\chi}$. For $f \in \mathcal{B}_\kappa^*(q_1, \chi)$ and $\ell_1, \ell_2 \mid q_2$, we have that*

$$\frac{\langle \iota_{\ell_1, q_1, q} f, \iota_{\ell_2, q_1, q} f \rangle_q}{\langle \iota_{1, q_1, q} f, \iota_{1, q_1, q} f \rangle_q} = A_f\left(\frac{\ell_2}{(\ell_1, \ell_2)} \right) \overline{A_f}\left(\frac{\ell_1}{(\ell_1, \ell_2)} \right),$$

where $A_f(n)$ is the multiplicative function defined on prime powers by

$$A_f(p^t) = \begin{cases} \dfrac{\overline{\lambda_f(p)}}{\sqrt{p}(1 + \chi_{0(q_1)}(p)p^{-1})} & \text{if } t = 1, \\[2ex] \dfrac{\lambda_f(p^t) - \chi_{(q_1)}(p)\lambda_f(p^{t-2})p^{-1}}{p^{t/2}(1 + \chi_{0(q_1)}(p)p^{-1})} & \text{if } t \geq 2, \end{cases}$$

where $\chi_{0(q_1)}$ denotes the principal character modulo q_1 and $\chi_{(q_1)} := \chi \chi_{0(q_1)}$ denotes the Dirichlet character modulo q_1 induced from χ.

Proof. For $\Re(s) > 1$, consider the integral

$$F(s) := \int_{\Gamma_0(q) \backslash \mathbb{H}} f(\ell_1 z) \bar{f}(\ell_2 z) E(z, s) \, d\mu(z), \quad \text{where } E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \Im(\gamma z)^s.$$

Unfolding the integral and using Parseval's identity,

$$F(s) = \int_0^\infty y^{s-1} \sum_{\substack{n_1 = -\infty \\ n_1 \neq 0}}^\infty \sum_{\substack{n_2 = -\infty \\ n_2 \neq 0 \\ \ell_1 n_1 = \ell_2 n_2}}^\infty \rho_f(n_1) \overline{\rho_f}(n_2) W_{\mathrm{sgn}(n_1)\kappa/2, it_f}(4\pi \ell_1 |n_1| y)^2 \, \frac{dy}{y}.$$

From (3.9) and the fact from [Duke et al. 2002, Equation (4.70)] that

$$\rho_f(-n) = \epsilon_f \frac{\Gamma((1+\kappa)/2 + it_f)}{\Gamma((1-\kappa)/2 + it_f)} \rho_f(n)$$

for $n \geq 1$, where $\epsilon_f \in \{-1, 1\}$, we find that

$$F(s) = \frac{|\rho_f(1)|^2}{(4\pi [\ell_1, \ell_2])^{s-1} \sqrt{\ell' \ell''}} \sum_{n=1}^\infty \frac{\lambda_f(\ell'' n) \overline{\lambda_f}(\ell' n)}{n^s}$$

$$\times \int_0^\infty y^{s-1} \left(W_{\kappa/2, it_f}(y)^2 + \left| \frac{\Gamma((1+\kappa)/2 + it_f)}{\Gamma((1-\kappa)/2 + it_f)} \right|^2 W_{-\kappa/2, it_f}(y)^2 \right) \frac{dy}{y},$$

where we have written $n_1 = \ell'' n$, $n_2 = \ell' n$, with $\ell' = \ell_1/(\ell_1, \ell_2)$ and $\ell'' = \ell_2/(\ell_1, \ell_2)$.

Next, by the multiplicativity of the Hecke eigenvalues of f together with the fact that $(\ell', \ell'') = 1$, the sum over n is equal to

$$\sum_{\substack{n=1 \\ (n,\ell'\ell'')=1}}^{\infty} \frac{|\lambda_f(n)|^2}{n^s} \prod_{p^t \| \ell''} \sum_{r=0}^{\infty} \frac{\lambda_f(p^{r+t})\overline{\lambda_f}(p^r)}{p^{rs}} \prod_{p^t \| \ell'} \sum_{r=0}^{\infty} \frac{\lambda_f(p^r)\overline{\lambda_f}(p^{r+t})}{p^{rs}}.$$

From (3.8) and (3.10), we find that

$$\sum_{r=0}^{\infty} \frac{\lambda_f(p^{r+t})\overline{\lambda_f}(p^r)}{p^{rs}} = B_f(p^t; s) \sum_{r=0}^{\infty} \frac{|\lambda_f(p^r)|^2}{p^{rs}},$$

$$\sum_{r=0}^{\infty} \frac{\lambda_f(p^r)\overline{\lambda_f}(p^{r+t})}{p^{rs}} = \overline{B_f}(p^t; \bar{s}) \sum_{r=0}^{\infty} \frac{|\lambda_f(p^r)|^2}{p^{rs}},$$

where $B_f(n; s)$ is defined to be the multiplicative function

$$B_f(p^t; s) = \begin{cases} \dfrac{\lambda_f(p)}{1 + \chi_{0(q_1)}(p)p^{-s}} & \text{if } t = 1, \\ \dfrac{\lambda_f(p^t) - \chi_{(q_1)}(p)\lambda_f(p^{t-2})p^{-s}}{1 + \chi_{0(q_1)}(p)p^{-s}} & \text{if } t \geq 2, \end{cases}$$

so that $A_f(n) = n^{-\frac{1}{2}} B_f(n; 1)$. We surmise that $F(s)$ is equal to

$$\frac{|\rho_f(1)|^2}{(4\pi[\ell_1, \ell_2])^{s-1}\sqrt{\ell'\ell''}} B_f(\ell''; s)\overline{B_f}(\ell'; \bar{s}) \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^s}$$

$$\times \int_0^{\infty} y^{s-1}\left(W_{\kappa/2, it_f}(y)^2 + \left|\frac{\Gamma((1+\kappa)/2+it_f)}{\Gamma((1-\kappa)/2+it_f)}\right|^2 W_{-\kappa/2, it_f}(y)^2\right) \frac{dy}{y}. \quad (3.14)$$

The result follows by taking the residue at $s = 1$, noting that $E(z, s)$ has residue equal to $1/\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})$ at $s = 1$ independently of $z \in \Gamma_0(q)\backslash\mathbb{H}$, and comparing to the case $\ell_1 = \ell_2 = 1$. $\qquad\square$

Lemma 3.15 (cf. [Blomer and Milićević 2015, Lemma 9]). *An orthonormal basis of $\mathcal{A}_\kappa(q, \chi)$ is given by*

$$\mathcal{B}_\kappa(q, \chi) = \bigsqcup_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \,(\mathrm{mod}\, q_\chi)}} \bigsqcup_{f \in \mathcal{B}_\kappa^*(q_1, \chi)} \bigsqcup_{d|q_2} \left\{ f_d = \sum_{\ell|d} \xi_f(\ell, d)\iota_{\ell, q_1, q} f \right\}, \quad (3.16)$$

where each $f \in \mathcal{B}^*_\kappa(q_1, \chi)$ is normalised such that $\langle \iota_{1,q_1,q} f, \iota_{1,q_1,q} f \rangle_q = 1$, and the function $\xi_f(\ell, d)$ is jointly multiplicative.

$$For\ 0 \le r \le t, \quad \xi_f(p^r, p^t) = \begin{cases} 1 & \text{if } r = t = 0, \\[2mm] -\dfrac{\overline{A_f}(p)}{\sqrt{1 - |A_f(p)|^2}} & \text{if } r = 0 \text{ and } t = 1, \\[4mm] \dfrac{1}{\sqrt{1 - |A_f(p)|^2}} & \text{if } r = t = 1, \\[4mm] \dfrac{\overline{\chi}_{(q_1)}(p)}{p} \dfrac{1}{\sqrt{(1 - \chi_{0(q_1)}(p)p^{-2})(1 - |A_f(p)|^2)}} & \text{if } r = t - 2 \text{ and } t \ge 2, \\[4mm] -\dfrac{\overline{\lambda_f}(p)}{\sqrt{p}} \dfrac{1}{\sqrt{(1 - \chi_{0(q_1)}(p)p^{-2})(1 - |A_f(p)|^2)}} & \text{if } r = t - 1 \text{ and } t \ge 2, \\[4mm] \dfrac{1}{\sqrt{(1 - \chi_{0(q_1)}(p)p^{-2})(1 - |A_f(p)|^2)}} & \text{if } r = t \text{ and } t \ge 2, \\[4mm] 0 & \text{if } 0 \le r \le t - 3 \text{ and } t \ge 3. \end{cases}$$

The key point is that the coefficients $\xi_f(\ell, d)$ are chosen such that the ratio of inner products

$$\delta_f(d_1, d_2) := \frac{\langle f_{d_1}, f_{d_2} \rangle_q}{\langle \iota_{1,q_1,q} f, \iota_{1,q_1,q} f \rangle_q} = \sum_{\ell_1 | d_1} \sum_{\ell_2 | d_2} \xi_f(\ell_1, d_1) \overline{\xi_f}(\ell_2, d_2) \frac{\langle \iota_{\ell_1,q_1,q} f, \iota_{\ell_2,q_1,q} f \rangle_q}{\langle \iota_{1,q_1,q} f, \iota_{1,q_1,q} f \rangle_q}$$

is equal to 1 if $d_1 = d_2$ and 0 otherwise.

Proof. The proof follows the same lines as [Blomer and Milićević 2015, Proof of Lemma 9]; we omit the details. □

Explicit Kuznetsov formula. We may use the explicit basis (3.16) together with (3.10) and (3.9) to rewrite the discrete part of the Kuznetsov formula, noting that for $f \in \mathcal{B}^*_\kappa(q_1, \chi)$, $d \mid q_2$, and $n \ge 1$ coprime to q,

$$\rho_{f_d}(n) = \xi_f(1, d) \rho_f(1) \frac{\lambda_f(n)}{\sqrt{n}}.$$

Similarly, the continuous part can be rewritten in terms of the Eisenstein spanning set $\mathcal{B}(\chi_1, \chi_2)$ with $\chi_1 \chi_2 = \chi$ together with (3.2) and (3.3). This yields the following explicit versions of the pre-Kuznetsov and Kuznetsov formulæ.

Proposition 3.17. *When $m, n \ge 1$ with $(mn, q) = 1$, the pre-Kuznetsov formula has the form*

$$\sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \,(\mathrm{mod}\, q_\chi)}} \sum_{f \in \mathcal{B}^*_\kappa(q_1, \chi)} 4\pi \xi_f |\rho_f(1)|^2 \frac{\overline{\chi}(m) \lambda_f(m) \lambda_f(n)}{\cosh \pi(r - t_f) \cosh \pi(r + t_f)}$$

$$+ \sum_{\substack{\chi_1, \chi_2 \,(\mathrm{mod}\, q) \\ \chi_1 \chi_2 = \chi}} \sum_{f \in \mathcal{B}(\chi_1, \chi_2)} \int_{-\infty}^{\infty} |\rho_f(1, t)|^2 \frac{\overline{\chi}(m) \lambda_f(m, t) \lambda_f(n, t)}{\cosh \pi(r - t) \cosh \pi(r + t)} \, dt$$

$$= \frac{|\Gamma(1 - \kappa/2 - ir)|^2}{\pi^2} \left(\delta_{mn} + \sum_{\substack{c = 1 \\ c \equiv 0 \,(\mathrm{mod}\, q)}}^{\infty} \frac{S_\chi(m, n; c)}{c} I_\kappa \left(\frac{4\pi \sqrt{mn}}{c}, r \right) \right) \quad (3.18)$$

for $\kappa \in \{0, 1\}$, where we define

$$\xi_f := \sum_{d \mid q_2} |\xi_f(1, d)|^2,$$

while the Kuznetsov formula for $\kappa = 0$ has the form

$$\sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \sum_{f \in \mathcal{B}_0^*(q_1, \chi)} \frac{4\pi \xi_f |\rho_f(1)|^2}{\cosh \pi t_f} \overline{\chi}(m) \lambda_f(m) \lambda_f(n) h(t_f)$$
$$+ \sum_{\substack{\chi_1, \chi_2 \pmod{q} \\ \chi_1 \chi_2 = \chi}} \sum_{f \in \mathcal{B}(\chi_1, \chi_2)} \int_{-\infty}^{\infty} \frac{|\rho_f(1, t)|^2}{\cosh \pi t} \overline{\chi}(m) \lambda_f(m, t) \lambda_f(n, t) h(t) \, dt$$
$$= \delta_{mn} g_0 + \sum_{\substack{c=1 \\ c \equiv 0 \pmod{q}}}^{\infty} \frac{S_\chi(m, n; c)}{c} g_0\left(\frac{4\pi \sqrt{mn}}{c}\right). \quad (3.19)$$

In both formulæ, each $f \in \mathcal{B}_\kappa^*(q_1, \chi)$ is normalised such that $\langle \iota_{1, q_1, q} f, \iota_{1, q_1, q} f \rangle_q = 1$.

Atkin–Lehner decomposition for $\Gamma_1(q)$. We recall the decomposition

$$\mathcal{A}_\kappa(\Gamma_1(q)) = \bigoplus_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^\kappa}} \mathcal{A}_\kappa(q, \chi),$$

which follows from the fact that $\Gamma_1(q)$ is a normal subgroup of $\Gamma_0(q)$ with quotient group isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$, noting that $\mathcal{A}_\kappa(q, \chi) = \{0\}$ if $\chi(-1) \neq (-1)^\kappa$. From this, we obtain the natural basis of $\mathcal{A}_\kappa(\Gamma_1(q))$ given by

$$\mathcal{B}_\kappa(\Gamma_1(q)) = \bigsqcup_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^\kappa}} \bigsqcup_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \bigsqcup_{f \in \mathcal{B}_\kappa^*(q_1, \chi)} \bigsqcup_{d \mid q_2} \left\{ f_d = \sum_{\ell \mid d} \xi_f(\ell, d) \iota_{\ell, q_1, q} f \right\}. \quad (3.20)$$

This allows us to use the pre-Kuznetsov and Kuznetsov formulæ (3.18) and (3.19) for $\mathcal{B}_\kappa(\Gamma_1(q))$ and $\mathcal{B}_0(\Gamma_1(q))$, even though ostensibly these two formulæ are only set up for $\mathcal{B}_\kappa(q, \chi)$ and $\mathcal{B}_0(q, \chi)$.

Atkin–Lehner decomposition for $\Gamma(q)$. A similar decomposition also holds for $\mathcal{A}_\kappa(\Gamma(q))$. In this case, the fact that

$$\Gamma_0(q^2) \cap \Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{q}, \ c \equiv 0 \pmod{q^2} \right\}$$
$$= \begin{pmatrix} q^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Gamma(q) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\mathcal{A}_\kappa(\Gamma(q)) = \iota_{q^{-1}} \mathcal{A}_\kappa\left(\Gamma_0(q^2) \cap \Gamma_1(q)\right),$$

where $\iota_{q^{-1}} : \mathcal{A}_\kappa\big(\Gamma_0(q^2) \cap \Gamma_1(q)\big) \to \mathcal{A}_\kappa(\Gamma(q))$ is the map $\iota_{q^{-1}} f(z) = f(q^{-1}z)$. As $\Gamma_0(q^2) \cap \Gamma_1(q)$ is a normal subgroup of $\Gamma_0(q^2)$ with quotient group isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$, we obtain the decomposition

$$\mathcal{A}_\kappa(\Gamma(q)) = \bigoplus_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=(-1)^\kappa}} \iota_{q^{-1}} \mathcal{A}_\kappa(q^2, \chi),$$

thereby allowing us to choose an explicit basis $\mathcal{B}_\kappa(\Gamma(q))$ of $\mathcal{A}_\kappa(\Gamma(q))$ of the form

$$\bigsqcup_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=(-1)^\kappa}} \bigsqcup_{\substack{q_1 q_2 = q^2 \\ q_1 \equiv 0 \ (\mathrm{mod}\ q_\chi)}} \bigsqcup_{f \in \mathcal{B}_\kappa^*(q_1, \chi)} \bigsqcup_{d | q_2} \left\{ \iota_{q^{-1}} f_d = \sum_{\ell | d} \xi_f(\ell, d) \iota_{q^{-1}} \iota_{\ell, q_1, q} f \right\}. \tag{3.21}$$

Once again, this allows us to make use of the pre-Kuznetsov and Kuznetsov formulæ (3.18) and (3.19) for $\mathcal{B}_\kappa(\Gamma(q))$ and $\mathcal{B}_0(\Gamma(q))$.

4. Bounds for Fourier coefficients of newforms

In the Kuznetsov formula (3.19), the Fourier coefficients $|\rho_f(1)|^2$ and the normalisation factor ξ_f both appear naturally. To remove these weights, we obtain lower bounds for $|\rho_f(1)|^2$ and ξ_f. For the former, such bounds are well-known, appearing in some generality in [Duke et al. 2002, Equation (7.16)]; nevertheless, we take this opportunity to correct some of the minor numerical errors in this proof, as well as greatly streamline the proof via the recent work of Li [2010] on obtaining upper bounds for L-functions at the edge of the critical strip.

Lemma 4.1. *For $f \in \mathcal{B}_\kappa^*(q_1, \chi)$, we have that*

$$\xi_f = \sum_{n | q_2^\infty} \frac{|\lambda_f(n)|^2}{n} \prod_{p \| q_2} \left(1 - \frac{\chi_{0(q_1)}(p)}{p^2} \right).$$

In particular, $\xi_f \gg 1$.

Proof. By multiplicativity,

$$\xi_f := \sum_{d | q_2} |\xi_f(1, d)|^2 = \prod_{p^t \| q_2} \sum_{r=0}^{t} |\xi_f(1, p^r)|^2.$$

We have that

$$\sum_{r=0}^{t} |\xi_f(1, p^r)|^2 = \begin{cases} 1 & \text{if } t = 0, \\ \dfrac{1}{1 - |A_f(p)|^2} & \text{if } t = 1, \\ \dfrac{1}{(1 - \chi_{0(q_1)}(p) p^{-2})(1 - |A_f(p)|^2)} & \text{if } t \geq 2. \end{cases}$$

The result then follows from the fact that

$$\frac{1}{1 - |A_f(p)|^2} = \left(1 - \frac{\chi_{0(q_1)}(p)}{p^2} \right) \sum_{k=0}^{\infty} \frac{|\lambda_f(p^k)|^2}{p^k}. \qquad \square$$

For $f \in \mathcal{B}_\kappa(q, \chi)$, we define

$$\nu_f := \Gamma\left(\frac{1+\kappa}{2} + it_f\right)\Gamma\left(\frac{1+\kappa}{2} - it_f\right)|\rho_f(1)|^2.$$

Note that

$$\Gamma\left(\frac{1+\kappa}{2} + it\right)\Gamma\left(\frac{1+\kappa}{2} - it\right) = \begin{cases} \dfrac{\pi}{\cosh \pi t} & \text{if } \kappa = 0, \\[2ex] \dfrac{\pi t}{\sinh \pi t} & \text{if } \kappa = 1. \end{cases}$$

Lemma 4.2. *Suppose that $f \in \mathcal{B}_\kappa^*(q_1, \chi)$ for some $q_1 \mid q$. Then*

$$\frac{\langle \iota_{1,q_1,q} f, \iota_{1,q_1,q} f\rangle_q}{\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})} = \nu_f \operatorname*{Res}_{s=1} \sum_{n=1}^\infty \frac{|\lambda_f(n)|^2}{n^s}.$$

Proof. We let $\ell_1 = \ell_2 = 1$ in (3.14) and take the residue at $s = 1$, yielding

$$\frac{\langle \iota_{1,q_1,q} f, \iota_{1,q_1,q} f\rangle_q}{\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})}$$
$$= |\rho_f(1)|^2 \operatorname*{Res}_{s=1} \sum_{n=1}^\infty \frac{|\lambda_f(n)|^2}{n^s} \int_0^\infty \left(W_{\kappa/2, it_f}(y)^2 + \left|\frac{\Gamma((1+\kappa)/2 + it_f)}{\Gamma((1-\kappa)/2 + it_f)}\right|^2 W_{-\kappa/2, it_f}(y)^2\right) \frac{dy}{y},$$

since the residue of $E(z, s)$ at $s = 1$ is $1/\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})$. We have by [Gradshteyn and Ryzhik 2007, 7.611.4] that for $\kappa \in \mathbb{C}$ and $-\frac{1}{2} < \Re(it) < \frac{1}{2}$,

$$\int_0^\infty W_{\kappa/2, it}(y)^2 \frac{dy}{y} = \frac{\pi}{\sin 2\pi it} \frac{\psi((1-\kappa)/2 + it) - \psi((1-\kappa)/2 - it)}{\Gamma((1-\kappa)/2 + it)\Gamma((1-\kappa)/2 - it)},$$

where ψ is the digamma function; note that a slightly erroneous version of this appears in [Duke et al. 2002, Equation (19.6)]. By the gamma and digamma reflection formulæ, we find that

$$\int_0^\infty \left(W_{\kappa/2, it_f}(y)^2 + \left|\frac{\Gamma((1+\kappa)/2 + it_f)}{\Gamma((1-\kappa)/2 + it_f)}\right|^2 W_{-\kappa/2, it_f}(y)^2\right) \frac{dy}{y} = \Gamma\left(\frac{1+\kappa}{2} + it_f\right)\Gamma\left(\frac{1+\kappa}{2} - it_f\right) \quad (4.3)$$

assuming that $t_f \in [0, \infty)$ if $\kappa = 1$ and $t_f \in [0, \infty)$ or $it_f \in \left(0, \frac{1}{2}\right)$ if $\kappa = 0$. $\square$

Corollary 4.4. *Suppose that $f \in \mathcal{B}_\kappa^*(q_1, \chi)$ for some $q_1 \mid q$. Then*

$$\nu_f \gg_\varepsilon \frac{\langle \iota_{1,q_1,q} f, \iota_{1,q_1,q} f\rangle_q}{\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})} (q(3 + t_f^2))^{-\varepsilon}. \quad (4.5)$$

Proof. It is known that

$$\sum_{n=1}^\infty \frac{|\lambda_f(n)|^2}{n^s} = \frac{\zeta(s)L(s, \mathrm{ad}\, f)}{\zeta(2s)} \prod_{p \mid q} P_{f,p}(p^{-s}),$$

where for each prime p dividing q, $P_{f,p}(z)$ is a rational function satisfying $p^{-\varepsilon} \ll_\varepsilon P_{f,p}(p^{-1}) \leq 1$. The work of Li [Li 2010, Theorem 2] then shows that

$$L(1, \mathrm{ad}\, f) \ll \exp\left(C \frac{\log(q(3 + t_f^2))}{\log\log(q(3 + t_f^2))}\right)$$

for some absolute constant $C > 0$, thereby yielding the result. $\square$

5. Bounds for sums of Kloosterman sums

We denote by

$$S(m, n; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^{\times}} e\left(\frac{md + n\bar{d}}{c}\right)$$

the usual Kloosterman sum with trivial character, for which the Weil bound holds:

$$|S(m, n; c)| \leq \tau(c)\sqrt{(m, n, c)c}. \tag{5.1}$$

We also require bounds for Kloosterman sums with nontrivial character. For $c \equiv 0 \pmod{q}$, $m, n \geq 1$, and $(a, q) = 1$, we have that

$$\sum_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^{\kappa}}} \bar{\chi}(a) S_{\chi}(m, n; c) = \frac{1}{2} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^{\times}} \sum_{\chi \pmod{q}} \bar{\chi}(a)\big(\chi(d) + (-1)^{\kappa} \chi(-d)\big) e\left(\frac{md + n\bar{d}}{c}\right).$$

We break this up into two sums. In the second sum, we can replace d with $-d$ and χ with $\bar{\chi}$ and use character orthogonality to see that

$$\sum_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^{\kappa}}} \bar{\chi}(a) S_{\chi}(m, n; c) = \begin{cases} \varphi(q)\Re\big(S_{a(q)}(m, n; c)\big) & \text{if } \kappa = 0, \\ i\varphi(q)\Im\big(S_{a(q)}(m, n; c)\big) & \text{if } \kappa = 1, \end{cases} \tag{5.2}$$

where we set

$$S_{a(q)}(m, n; c) := \sum_{\substack{d \in (\mathbb{Z}/c\mathbb{Z})^{\times} \\ d \equiv a \pmod{q}}} e\left(\frac{md + n\bar{d}}{c}\right).$$

If $c = c_1 c_2$ with $(c_1, c_2) = 1$ and $c_1 c_2 \equiv 0 \pmod{q}$, then we let $d = c_2 \bar{c}_2 d_1 + c_1 \bar{c}_1 d_2$, where $d_1 \in (\mathbb{Z}/c_1\mathbb{Z})^{\times}$, $d_2 \in (\mathbb{Z}/c_2\mathbb{Z})^{\times}$, and $c_2 \bar{c}_2 \equiv 1 \pmod{c_1}$, $c_1 \bar{c}_1 \equiv 1 \pmod{c_2}$. By the Chinese remainder theorem,

$$S_{a(q)}(m, n; c) = S_{a((q, c_1))}(m\bar{c}_2, n\bar{c}_2; c_1) S_{a((q, c_2))}(m\bar{c}_1, n\bar{c}_1; c_2).$$

To bound $S_{a(q)}(m, n; c)$, it therefore suffices to find bounds for $S_{a(p^\alpha)}(m, n; p^\beta)$ for any prime p and any $\beta \geq \alpha \geq 1$. The trivial bound is merely

$$|S_{a(p^\alpha)}(m, n; p^\beta)| \leq p^{\beta - \alpha}. \tag{5.3}$$

Somewhat surprisingly, this is sufficient for our needs. Indeed, we cannot do better than this when $\beta = \alpha$, and in our applications, this will be the dominant contribution.

We also require bounds for $S_{\chi}(m, n; c)$. Unfortunately, it is not necessarily the case that this is bounded by $\tau(c)\sqrt{(m, n, c)c}$, which can be observed numerically at [LMFDB 2013]; see also [Knightly and Li 2013, Example 9.9].

Lemma 5.4. *Let p be an odd prime, let χ_{p^γ} be a Dirichlet character of conductor p^γ, and suppose that $(mn, p) = 1$. Then for $\beta \geq \gamma \geq 0$, we have that*

$$|S_{\chi_{p^\gamma}}(m, n; p^\beta)| \leq 2p^{\beta/2}$$

unless $\beta = \gamma \geq 3$, in which case we only have that

$$|S_{\chi_{p^\gamma}}(m, n; p^\beta)| \leq 2p^{\lfloor (3\beta+1)/4 \rfloor}.$$

   Similarly, let χ_{2^γ} be a Dirichlet character of conductor 2^γ, and suppose that $(mn, 2) = 1$. Then for $\beta \geq \gamma \geq 0$, we have that

$$|S_{\chi_{2^\gamma}}(m, n; 2^\beta)| \leq 8 \cdot 2^{\beta/2}$$

unless $\gamma + 1 \geq \beta \geq 3$, in which case we only have that

$$|S_{\chi_{2^\gamma}}(m, n; 2^\beta)| \leq 4 \cdot 2^{\lfloor (3\beta+1)/4 \rfloor}.$$

Proof. This follows from [Knightly and Li 2013, Propositions 9.4, 9.7, 9.8, and Lemmata 9.6]. □

Lemma 5.5. *When $(m, n) = 1$, we have that*

$$\sum_{\substack{c \leq 4\pi\sqrt{mn} \\ c \equiv 0 \,(\mathrm{mod}\, q)}} \frac{|S_{a(q)}(m, n; c)|}{c^{\frac{3}{2}}} \ll \frac{(\log(mn + 1))^2}{q^{\frac{3}{2}}} \prod_{p | q} \frac{1}{1 - p^{-\frac{1}{2}}}, \tag{5.6}$$

$$\sum_{\substack{c \leq 4\pi\sqrt{mn} \\ c \equiv 0 \,(\mathrm{mod}\, q^2)}} \frac{|S_{a(q)}(m, n; c)|}{c^{\frac{3}{2}}} \ll \frac{(\log(mn + 1))^2}{q^2} \prod_{p | q} \frac{1}{1 - p^{-\frac{1}{2}}}. \tag{5.7}$$

If we additionally assume that $(mn, q) = 1$, then given a Dirichlet character χ modulo q, we have that

$$\sum_{\substack{c \leq 4\pi\sqrt{mn} \\ c \equiv 0 \,(\mathrm{mod}\, q)}} \frac{|S_\chi(m, n; c)|}{c^{\frac{3}{2}}} \ll (\log(mn + 1))^2 \frac{2^{\omega(q)} \dot{Q}}{\varphi(q)}. \tag{5.8}$$

Proof. We write $q = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$, so that the left-hand side of (5.6) is

$$\sum_{\beta_1 = \alpha_1}^{\infty} \cdots \sum_{\beta_\ell = \alpha_\ell}^{\infty} \frac{1}{(p_1^{\beta_1} \cdots p_\ell^{\beta_\ell})^{\frac{3}{2}}} \sum_{\substack{c \leq 4\pi\sqrt{mn}\, p_1^{-\beta_1} \cdots p_\ell^{-\beta_\ell} \\ (c, q) = 1}} \frac{1}{c^{\frac{3}{2}}}$$

$$\times |S(m\overline{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}}, n\overline{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}}; c)| |S_{a(q)}(m\bar{c}, n\bar{c}; p_1^{\beta_1} \cdots p_\ell^{\beta_\ell})|.$$

Using the Weil bound (5.1) for the first Kloosterman sum and the trivial bound (5.3) for the second, we find that this is bounded by

$$\frac{1}{q} \sum_{\beta_1 = \alpha_1}^{\infty} \cdots \sum_{\beta_\ell = \alpha_\ell}^{\infty} \frac{1}{\sqrt{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}}} \sum_{\substack{c \leq 4\pi\sqrt{mn} \\ (c, q) = 1}} \frac{\tau(c)\sqrt{(m, n, c)}}{c}.$$

If $(m, n) = 1$, the inner sum is bounded by a constant multiple of $(\log(mn + 1))^2$, and so the sum is bounded by a constant multiple of

$$\frac{(\log(mn + 1))^2}{q} \sum_{\beta_1=\alpha_1}^{\infty} \cdots \sum_{\beta_\ell=\alpha_\ell}^{\infty} \frac{1}{\sqrt{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}}},$$

which yields (5.6) upon evaluating these geometric series. (5.7) follows similarly. Finally, (5.8) follows via the same method but using Lemma 5.4 to bound the Kloosterman sums, yielding the bound

$$8 \cdot 2^{\omega(q)} \dot{Q} \sum_{\beta_1=\alpha_1}^{\infty} \cdots \sum_{\beta_\ell=\alpha_\ell}^{\infty} \frac{1}{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}} \sum_{\substack{c \leq 4\pi\sqrt{mn} \\ (c,q)=1}} \frac{\tau(c)}{c}$$

for the left-hand side of (5.8), from which the result easily follows. $\qquad \square$

Lemma 5.9. *When $(m, n) = 1$, we have that*

$$\sum_{\substack{c > 4\pi\sqrt{mn} \\ c \equiv 0 \pmod q}} \frac{|S_{a(q)}(m, n; c)|}{c^2} \left(1 + \log \frac{c}{4\pi\sqrt{mn}}\right) \ll \frac{(\log(mn + 1))^2}{(mn)^{\frac{1}{4}}} \frac{1}{q^{\frac{3}{2}}} \prod_{p \mid q} \frac{1}{1 - p^{-\frac{1}{2}}}, \qquad (5.10)$$

$$\sum_{\substack{c > 4\pi\sqrt{mn} \\ c \equiv 0 \pmod{q^2}}} \frac{|S_{a(q)}(m, n; c)|}{c^2} \left(1 + \log \frac{c}{4\pi\sqrt{mn}}\right) \ll \frac{(\log(mn + 1))^2}{(mn)^{\frac{1}{4}}} \frac{1}{q^2} \prod_{p \mid q} \frac{1}{1 - p^{-\frac{1}{2}}}. \qquad (5.11)$$

If we additionally assume that $(mn, q) = 1$, then given a Dirichlet character χ modulo q, we have that

$$\sum_{\substack{c > 4\pi\sqrt{mn} \\ c \equiv 0 \pmod q}} \frac{|S_\chi(m, n; c)|}{c^2} \left(1 + \log \frac{c}{4\pi\sqrt{mn}}\right) \ll \frac{(\log(mn + 1))^2}{(mn)^{\frac{1}{4}}} \frac{2^{\omega(q)} \dot{Q}}{\varphi(q)}. \qquad (5.12)$$

Proof. As before, with $q = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$, the left-hand side of (5.10) is bounded by

$$\frac{1}{q} \sum_{\beta_1=\alpha_1}^{\infty} \cdots \sum_{\beta_\ell=\alpha_\ell}^{\infty} \frac{1}{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}} \sum_{\substack{c > 4\pi\sqrt{mn} \, p_1^{-\beta_1} \cdots p_\ell^{-\beta_\ell} \\ (c,q)=1}} \frac{\tau(c)\sqrt{(m, n, c)} \log c}{c^{\frac{3}{2}}}.$$

If $(m, n) = 1$, then the inner sum is bounded by a constant multiple of

$$\frac{(\log(mn + 1))^2}{(mn)^{\frac{1}{4}}} \sqrt{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}}.$$

It follows that the sum is bounded by a constant multiple of

$$\frac{(\log(mn + 1))^2}{(mn)^{\frac{1}{4}}} \frac{1}{q} \sum_{\beta_1=\alpha_1}^{\infty} \cdots \sum_{\beta_\ell=\alpha_\ell}^{\infty} \frac{1}{\sqrt{p_1^{\beta_1} \cdots p_\ell^{\beta_\ell}}},$$

which gives (5.10). The proof of (5.11) is analogous, while (5.12) again follows upon using Lemma 5.4 to bound the Kloosterman sums. $\qquad \square$

Lemma 5.13 (cf. [Iwaniec and Kowalski 2004, Equation (16.50)]). *For all $\frac{1}{2} < \sigma < 1$,*

$$\sum_{\substack{c=1 \\ c \equiv 0 \,(\mathrm{mod}\, q)}}^{\infty} \frac{|S_{a(q)}(m,n;c)|}{c^{1+\sigma}} \leq \frac{18\tau((m,n))}{(2\sigma-1)^2} \frac{1}{q^{1+\sigma}} \prod_{p|q} \frac{1}{1-p^{-\sigma}}, \tag{5.14}$$

$$\sum_{\substack{c=1 \\ c \equiv 0 \,(\mathrm{mod}\, q^2)}}^{\infty} \frac{|S_{a(q)}(m,n;c)|}{c^{1+\sigma}} \leq \frac{18\tau((m,n))}{(2\sigma-1)^2} \frac{1}{q^{1+2\sigma}} \prod_{p|q} \frac{1}{1-p^{-\sigma}}. \tag{5.15}$$

If we additionally assume that $(m,n) = (mn, q) = 1$, then given a Dirichlet character χ modulo q, we have that

$$\sum_{\substack{c=1 \\ c \equiv 0 \,(\mathrm{mod}\, q)}}^{\infty} \frac{|S_\chi(m,n;c)|}{c^{1+\sigma}} \leq \frac{72}{(2\sigma-1)^2} \frac{2^{\omega(q)} \dot{Q}}{\varphi(q) q^{\sigma-\frac{1}{2}}}. \tag{5.16}$$

Proof. Once again writing $q = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$ and bounding the Kloosterman sums, we have that

$$\sum_{\substack{c=1 \\ c \equiv 0 \,(\mathrm{mod}\, q)}}^{\infty} \frac{|S_{a(q)}(m,n;c)|}{c^{1+\sigma}} \leq \sum_{\substack{c=1 \\ (c,q)=1}}^{\infty} \frac{\tau(c)\sqrt{(m,n,c)}}{c^{\frac{1}{2}+\sigma}} \frac{1}{q} \sum_{\beta_1=\alpha_1}^{\infty} \cdots \sum_{\beta_\ell=\alpha_\ell}^{\infty} \frac{1}{(p_1^{\beta_1} \cdots p_\ell^{\beta_\ell})^\sigma}$$

$$= \sum_{\substack{c=1 \\ (c,q)=1}}^{\infty} \frac{\tau(c)\sqrt{(m,n,c)}}{c^{\frac{1}{2}+\sigma}} \frac{1}{q^{1+\sigma}} \prod_{p|q} \frac{1}{1-p^{-\sigma}}$$

$$\leq \zeta\left(\sigma+\tfrac{1}{2}\right)^2 \sum_{d|(m,n)} \frac{\tau(d)}{d^\sigma} \frac{1}{q^{1+\sigma}} \prod_{p|q} \frac{1}{1-p^{-\sigma}}$$

$$\leq \frac{18\tau((m,n))}{(2\sigma-1)^2} \frac{1}{q^{1+\sigma}} \prod_{p|q} \frac{1}{1-p^{-\sigma}}.$$

This proves (5.14). The inequality (5.15) follows by a similar argument, as does (5.16) once the Kloosterman sums are bounded via Lemma 5.4. $\qquad\square$

6. Bounds for test functions

We require bounds for the test function that we will obtain by multiplying the pre-Kuznetsov formula (3.18) by a function dependent on r and then integrating both sides over $r \in [0, T]$.

Lemma 6.1. *For $T \geq 1$, let*

$$h_{\kappa,T}(t) := \frac{\pi^2}{\Gamma((1+\kappa)/2+it)\Gamma((1+\kappa)/2-it)} \int_0^T \frac{r|\Gamma(1-\kappa/2+ir)|^{-2}}{\cosh \pi(r-t) \cosh \pi(r+t)} \, dr$$

$$= \begin{cases} \cosh \pi t \displaystyle\int_0^T \frac{\sinh \pi r}{\cosh \pi(r-t) \cosh \pi(r+t)} \, dr & \text{if } \kappa = 0, \\[2ex] \dfrac{\sinh \pi t}{t} \displaystyle\int_0^T \frac{r \cosh \pi r}{\cosh \pi(r-t) \cosh \pi(r+t)} \, dr & \text{if } \kappa = 1. \end{cases}$$

Then $h_{\kappa,T}(t)$ is positive for all $t \in \mathbb{R}$ and additionally, should κ be equal to 0, for $it \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Furthermore, $h_{\kappa,T}(t) \gg 1$ for $t \in [0, T]$.

Proof. Using the fact that

$$\cosh \pi(r - t) \cosh \pi(r + t) = \cosh^2 \pi t + \sinh^2 \pi r = \sinh^2 \pi t + \cosh^2 \pi r,$$

it is clear that $h_{\kappa,T}(t)$ is positive for all $t \in \mathbb{R}$ and additionally, should κ be equal to 0, if $it \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

For $\kappa = 0$, we have that

$$\begin{aligned}
h_{0,T}(t) &= \frac{\cosh \pi t}{\pi} \int_1^{\cosh \pi T} \frac{1}{x^2 + \sinh^2 \pi t}\, dx \\
&= \frac{\coth \pi t}{\pi} \arctan \frac{\sinh \pi t (\cosh \pi T - 1)}{\sinh^2 \pi t + \cosh \pi T},
\end{aligned}$$

where the second line follows from the arctangent subtraction formula. The first expression shows that $h_{0,T}(t) \gg 1$ when t is small, while when t is large, the argument of arctan is essentially

$$\frac{e^{\pi(T+t)} - e^{\pi t}}{e^{2\pi t} + e^{\pi T}},$$

and this is bounded from below provided that $t \leq T$, so that again $h_{0,T}(t) \gg 1$.

For $\kappa = 1$, we can similarly show via integration by parts that

$$\begin{aligned}
h_{1,T}(t) &= \frac{\sinh \pi t}{\pi^2 t} \int_0^{\sinh \pi T} \frac{\operatorname{arsinh} x}{x^2 + \cosh^2 \pi t}\, dx \\
&= \frac{\tanh \pi t}{\pi^2 t} \int_0^{\sinh \pi T} \frac{\arctan(\sinh \pi T / \cosh \pi t) - \arctan(x / \cosh \pi t)}{\sqrt{x^2 + 1}}\, dx.
\end{aligned}$$

The first expression shows that $h_{1,T}(t) \gg 1$ when t is small, while when t is large, we break up the second expression into two integrals: one from 0 to $\sinh \frac{\pi t}{2}$ and one from $\sinh \frac{\pi t}{2}$ to $\sinh \pi T$. Trivially bounding the numerator in each integral, we find that

$$\begin{aligned}
h_{1,T}(t) &\geq \frac{\tanh \pi t}{2\pi} \left(\arctan(\sinh \pi T / \cosh \pi t) - \arctan(\sinh(\pi t/2) / \cosh \pi t)\right) \\
&= \frac{\tanh \pi t}{2\pi} \arctan \frac{\cosh \pi t \left(\sinh \pi T - \sinh(\pi t/2)\right)}{\cosh^2 \pi t + \sinh \pi T \sinh(\pi t/2)}.
\end{aligned}$$

The argument of arctan is essentially

$$\frac{e^{\pi(T+t)} - e^{3\pi t/2}}{e^{2\pi t} + e^{\pi(T+t/2)}},$$

and this is bounded from below provided that $t \leq T$, while $\tanh \pi t$ is bounded from below provided that t is larger than some fixed constant. It follows again that $h_{1,T}(t) \gg 1$.   □

We also require the following bound, which arises from the Kloosterman term in the pre-Kuznetsov formula (3.18).

Lemma 6.2. *For $\kappa \in \{0, 1\}$ and $T > 0$, we have the bound*

$$\int_0^T r I_\kappa(a, r)\, dr \ll \begin{cases} \sqrt{a} & \text{if } a \geq 1, \\ a(1 + \log(1/a)) & \text{if } 0 < a < 1 \end{cases} \tag{6.3}$$

uniformly in T.

Proof. From [Kuznetsov 1980, Equation (5.13)], we have that

$$\int_0^T r I_0(a, r)\, dr = a \int_0^\infty \frac{\tanh \xi}{\xi}(1 - \cos 2T\xi) \sin(a \cosh \xi)\, d\xi.$$

Similarly, using the fact that

$$K_{2ir}(\zeta) = \int_0^\infty e^{-\zeta \cosh \xi} \cos 2r\xi\, d\xi$$

for $r \in \mathbb{R}$ and $\Re(\zeta) > 0$ from [Gradshteyn and Ryzhik 2007, 8.432.1], we have that

$$\int_0^T r I_1(a, r)\, dr = -2a \int_0^\infty \int_0^T r \cos 2r\xi\, dr \int_{-i}^i e^{-\zeta a \cosh \xi}\, d\zeta\, d\xi.$$

Evaluating each of the inner integrals and then integrating by parts, we find that

$$\begin{aligned} \int_0^T & r I_1(a, r)\, dr \\ & = ia \int_0^\infty \frac{\tanh \xi}{\xi}(1 - \cos 2T\xi) \cos(a \cosh \xi)\, d\xi - i \int_0^\infty \frac{\tanh \xi}{\xi}(1 - \cos 2T\xi)\frac{\sin(a \cosh \xi)}{\cosh \xi}\, d\xi. \end{aligned}$$

From here, one can show via stationary phase on subintervals of $(0, \infty)$ that $\int_0^T r I_0(a, r)\, dr$ and the first term in the above expression for $\int_0^T r I_1(a, r)\, dr$ both are bounded by a constant multiple of

$$\begin{cases} \sqrt{a} & \text{if } a \geq 1, \\ a(1 + \log(1/a)) & \text{if } 0 < a < 1; \end{cases}$$

see [Kuznetsov 1980, Equation (5.14)]. The second term in the expression for $\int_0^T r I_1(a, r)\, dr$ is uniformly bounded for $a \geq 1$, so we need only consider when $0 < a < 1$. In this case, the fact that $|\sin x| \leq \min\{1, |x|\}$ for $x \in \mathbb{R}$ implies that this is bounded by

$$2a \int_0^{\log(1/a)} \frac{\tanh \xi}{\xi}\, d\xi + 2 \int_{\log(1/a)}^\infty \frac{\tanh \xi}{\xi}\frac{1}{\cosh \xi}\, d\xi \ll a(1 + \log(1/a)). \qquad \square$$

7. Sarnak's density theorem for exceptional Hecke eigenvalues

We are now in a position to prove Theorem 1.1.

Proof of (1.2). By Rankin's trick,

$$\#\big\{f \in \mathcal{B}_\kappa(\Gamma_1(q)) : t_f \in [0, T],\ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\} \leq \prod_{p \in \mathcal{P}} \alpha_p^{-2\ell_p} \sum_{\substack{f \in \mathcal{B}_\kappa(\Gamma_1(q)) \\ t_f \in [0, T]}} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}$$

for any nonnegative integers ℓ_p to be chosen. Using the explicit basis (3.20) of $\mathcal{A}_\kappa(\Gamma_1(q))$ together with the lower bound (4.5) for ν_f,

$$
\sum_{\substack{f \in \mathcal{B}_\kappa(\Gamma_1(q)) \\ t_f \in [0,T]}} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p} = \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q_1,\chi) \\ t_f \in [0,T]}} \tau(q_2) \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}
$$

$$
\ll_\varepsilon q^{1+\varepsilon} T^\varepsilon \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q_1,\chi) \\ t_f \in [0,T]}} \xi_f \nu_f \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}.
$$

We take $m = 1$ and $n = \prod_{p \in \mathcal{P}} p^{2j_p}$ in the pre-Kuznetsov formula (3.18), multiply both sides by $\prod_{p \in \mathcal{P}} \alpha_{2j_p,2\ell_p} \bar{\chi}(p)^{j_p}$, and sum over all $0 \leq j_p \leq \ell_p$, over all $p \in \mathcal{P}$, and over all Dirichlet characters χ modulo q satisfying $\chi(-1) = (-1)^\kappa$. We then multiply both sides by $\pi^2 r |\Gamma(1 - \kappa/2 + ir)|^{-2}$ and integrate both sides with respect to r from 0 to T.

On the spectral side, (3.1), (3.5), and Lemma 6.1 allow us to use positivity to discard the contribution from the continuous spectrum, while we may discard the contribution of the discrete spectrum with $t \notin [0, T]$ via (3.8), (3.12), and Lemma 6.1, so that the spectral side is bounded from below by a constant multiple of

$$
\sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q_1,\chi) \\ t_f \in [0,T]}} \xi_f \nu_f \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}.
$$

On the geometric side, we only pick up the delta term when $j_p = 0$ for all $p \in \mathcal{P}$, in which case the term is bounded by a constant multiple of $qT^2 \prod_{p \in \mathcal{P}} \alpha_{0,2\ell_p}$. For $\kappa = 0$, we use (5.2) to write the Kloosterman term in the form

$$
\frac{\varphi(q)}{\pi} \sum_{\substack{j_p=0 \\ p \in \mathcal{P}}}^{\ell_p} \prod_{p \in \mathcal{P}} \alpha_{2j_p,2\ell_p} \sum_{\substack{c=1 \\ c \equiv 0 \pmod q}}^{\infty} \frac{\Re\left(S_{\prod_{p \in \mathcal{P}} p^{j_p}(q)}\left(1, \prod_{p \in \mathcal{P}} p^{2j_p}; c\right)\right)}{c} \int_0^T r I_0\left(\frac{4\pi \prod_{p \in \mathcal{P}} p^{j_p}}{c}, r\right) dr.
$$

For $\kappa = 1$, the Kloosterman term is the same except with $i\Im$ in place of $\Re$ and I_1 in place of I_0. In either case, we bound the integral via (6.3), which allows us to use (5.6) and (5.10) to bound the summation over c, so that the Kloosterman term is bounded by a constant multiple of

$$
\frac{1}{\sqrt{q}} \prod_{p'|q} \frac{1}{1 - p'^{-\frac{1}{2}}} \sum_{\substack{j_p=0 \\ p \in \mathcal{P}}}^{\ell_p} \prod_{p \in \mathcal{P}} \alpha_{2j_p,2\ell_p} p^{j_p/2} \left(\log\left(\prod_{p \in \mathcal{P}} p^{2j_p} + 1\right)\right)^2.
$$

We bound the summation over j_p and over $p \in \mathcal{P}$ via (3.7), thereby obtaining

$$
\#\{f \in \mathcal{B}_\kappa(\Gamma_1(q)) : t_f \in [0,T], \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\}
$$

$$
\ll_\varepsilon q^{1+\varepsilon} T^\varepsilon \prod_{p \in \mathcal{P}} \left(\frac{\alpha_p}{2}\right)^{-2\ell_p} \left(qT^2 + \frac{\prod_{p \in \mathcal{P}} p^{\ell_p/2}\left(\log \prod_{p \in \mathcal{P}} p^{\ell_p/2}\right)^2}{\sqrt{q}} \prod_{p'|q} \frac{1}{1 - p'^{-\frac{1}{2}}}\right).
$$

It remains to take

$$\ell_p = \left\lfloor \frac{\mu_p \log(\mathrm{vol}(\Gamma_1(q)\backslash\mathbb{H})^{\frac{3}{2}} T^4)}{\log p} \right\rfloor. \qquad \square$$

Proof of (1.3). We use (3.21), (5.7), and (5.11) in place of (3.20), (5.6), and (5.10), thereby finding that

$$\#\big\{ f \in \mathcal{B}_k(\Gamma(q)) : t_f \in [0, T],\ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \big\} \leq \prod_{p \in \mathcal{P}} \alpha_p^{-2\ell_p} \sum_{\substack{f \in \mathcal{B}_k(\Gamma(q)) \\ t_f \in [0,T]}} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p},$$

with

$$\sum_{\substack{f \in \mathcal{B}_k(\Gamma(q)) \\ t_f \in [0,T]}} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p} = \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q^2 \\ q_1 \equiv 0\ (\mathrm{mod}\ q_\chi)}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q_1, \chi) \\ t_f \in [0,T]}} \tau(q_2) \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}$$

$$\ll_\varepsilon q^{2+\varepsilon} T^\varepsilon \prod_{p \in \mathcal{P}} 2^{2\ell_p} \left(q T^2 + \frac{\prod_{p \in \mathcal{P}} p^{\ell_p/2} \big(\log \prod_{p \in \mathcal{P}} p^{\ell_p/2}\big)^2}{q} \prod_{p' | q} \frac{1}{1 - p'^{-\frac{1}{2}}} \right).$$

Taking

$$\ell_p = \left\lfloor \frac{\mu_p \log(\mathrm{vol}(\Gamma(q)\backslash\mathbb{H})^{\frac{4}{3}} T^4)}{\log p} \right\rfloor$$

completes the proof. $\qquad \square$

Proof of (1.4). Using (3.16), (5.8), and (5.12) in place of (3.20), (5.6), and (5.10),

$$\#\big\{ f \in \mathcal{B}_k(q, \chi) : t_f \in [0, T],\ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$

$$\ll_\varepsilon q^{1+\varepsilon} T^\varepsilon \prod_{p \in \mathcal{P}} \left(\frac{\alpha_p}{2} \right)^{-2\ell_p} \left(T^2 + \prod_{p \in \mathcal{P}} p^{\ell_p/2} \Big(\log \prod_{p \in \mathcal{P}} p^{\ell_p/2} \Big)^2 \frac{2^{\omega(q)} \dot{Q}}{\varphi(q)} \right).$$

Upon taking

$$\ell_p = \left\lfloor \frac{\mu_p \log(\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^2 T^4 \dot{Q}^{-2})}{\log p} \right\rfloor,$$

we conclude that

$$\#\big\{ f \in \mathcal{B}_k(q, \chi) : t_f \in [0, T],\ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$

$$\ll_\varepsilon \big(\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H}) T^2\big)^{1 - 4 \sum_{p \in \mathcal{P}} \mu_p (\log \alpha_p/2)/\log p + \varepsilon} \dot{Q}^{4 \sum_{p \in \mathcal{P}} \mu_p (\log \alpha_p/2)/\log p}. \quad (7.1)$$

On the other hand, by the inclusion $\mathcal{A}_\kappa(q, \chi) \subset \mathcal{A}_\kappa(q\ddot{Q}, \chi)$,

$$\#\big\{ f \in \mathcal{B}_k(q, \chi) : t_f \in [0, T],\ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \big\}$$

$$\leq \#\big\{ f \in \mathcal{B}_k(q\ddot{Q}, \chi) : t_f \in [0, T],\ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \big\}.$$

Since $q_{\chi\psi^2} \mid q_\chi$, we have that $\dot{Q}(q\ddot{Q}, q_{\chi\psi^2}) = 1$. Consequently, (7.1) yields the bound

$$\#\{f \in \mathcal{B}_\kappa(q, \chi) : t_f \in [0, T], \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\}$$
$$\ll_\varepsilon \left(\mathrm{vol}(\Gamma_0(q\ddot{Q})\backslash\mathbb{H})T^2\right)^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p + \varepsilon}$$
$$\ll_\varepsilon \left(\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})T^2\right)^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p + \varepsilon} \ddot{Q}^{1-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p}. \quad \square$$

Remark 7.2. Should we wish to improve (1.4) to be uniform in $\mathcal{P}$, then one needs to take into account the fact that

$$\prod_{p\in\mathcal{P}}\left(\frac{\alpha_p}{2}\right)^{-2\ell_p} = \left(\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})T^2\right)^{-4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p + \varepsilon}\dot{Q}^{4\sum_{p\in\mathcal{P}}\mu_p(\log\alpha_p/2)/\log p}$$
$$\times \prod_{p\in\mathcal{P}}\left(\frac{\alpha_p}{2}\right)^{2\left(\mu_p\log(\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^2T^4\dot{Q}^{-2})\right)/\log p},$$

where $\{x\}$ denotes the fractional part of x, and the last term need not necessarily be $\ll_\varepsilon (\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})T^2)^\varepsilon$. For this reason, [Blomer et al. 2014, Proposition 1] is not correct in the generality in which it is stated, namely the claim that the result is uniform for $T > p$. Instead, one requires that $p \ll_\varepsilon T^\varepsilon$.

8. Huxley's density theorem for exceptional laplacian eigenvalues

Theorem 1.5 is proved similarly to Theorem 1.1, though we use the Kuznetsov formula (3.19) with a carefully chosen test function in place of the pre-Kuznetsov formula (3.18), and we require different methods to bound the Kloosterman term.

Proof of (1.6). We again use Rankin's trick with nonnegative integers ℓ_p and a positive real number $X \geq 1$ to be chosen:

$$\#\{f \in \mathcal{B}_0(\Gamma_1(q)) : it_f \in (\alpha_0, \tfrac{1}{2}), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\}$$
$$\leq X^{-2\alpha_0}\prod_{p\in\mathcal{P}}\alpha_p^{-2\ell_p}\sum_{\substack{f\in\mathcal{B}_0(\Gamma_1(q))\\ it_f\in(0,\frac{1}{2})}}X^{2it_f}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p}.$$

Again using (3.20) and (4.5),

$$\sum_{\substack{f\in\mathcal{B}_0(\Gamma_1(q))\\ it_f\in(0,\frac{1}{2})}}X^{2it_f}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p} = \sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi(-1)=1}}\sum_{\substack{q_1q_2=q\\ q_1\equiv 0\ (\mathrm{mod}\ q_\chi)}}\sum_{\substack{f\in\mathcal{B}_0^*(q_1,\chi)\\ it_f\in(0,\frac{1}{2})}}\tau(q_2)X^{2it_f}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p}$$
$$\ll_\varepsilon q^{1+\varepsilon}\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi(-1)=1}}\sum_{\substack{q_1q_2=q\\ q_1\equiv 0\ (\mathrm{mod}\ q_\chi)}}\sum_{\substack{f\in\mathcal{B}_0^*(q_1,\chi)\\ it_f\in(0,\frac{1}{2})}}\xi_f\nu_f X^{2it_f}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p}.$$

We take $m = 1$, $n = \prod_{p\in\mathcal{P}}p^{2j_p}$, and

$$h(t) = h_X(t) = \left(\frac{X^{it} + X^{-it}}{t^2 + 1}\right)^2$$

in the Kuznetsov formula (3.19), multiply both sides by $\prod_{p\in\mathcal{P}}\alpha_{2j_p,2\ell_p}\overline{\chi}(p)^{j_p}$, and sum over all $0\leq j_p\leq\ell_p$, over all $p\in\mathcal{P}$, and over all even Dirichlet characters modulo q. On the spectral side, we discard all but the discrete spectrum for which $it_f\in(0,\frac{1}{2})$ via positivity, so that the spectral side is bounded from below by a constant multiple of

$$\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi(-1)=1}}\sum_{\substack{q_1q_2=q\\ q_1\equiv 0\ (\mathrm{mod}\ q_\chi)}}\sum_{\substack{f\in\mathcal{B}_0^*(q_1,\chi)\\ it_f\in(0,\frac{1}{2})}}\xi_f\nu_f X^{2it_f}\prod_{p\in\mathcal{P}}|\lambda_f(p)|^{2\ell_p}.$$

We only pick up the delta term on the geometric side when $j_p=0$ for all $p\in\mathcal{P}$, in which case the term is bounded by a constant multiple of $q\prod_{p\in\mathcal{P}}2^{2\ell_p}$. We write the Kloosterman term in the form

$$\frac{\varphi(q)}{2\pi i}\sum_{\substack{j_p=0\\ p\in\mathcal{P}}}^{\ell_p}\prod_{p\in\mathcal{P}}\alpha_{2j_p,2\ell_p}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{\substack{c=1\\ c\equiv 0\ (\mathrm{mod}\ q)}}^{\infty}\frac{\Re\big(S_{\prod_{p\in\mathcal{P}}p^{j_p}(q)}\big(1,\prod_{p\in\mathcal{P}}p^{2j_p};c\big)\big)}{c}J_s\left(\frac{4\pi\prod_{p\in\mathcal{P}}p^{j_p}}{c}\right)\frac{sh_X(is/2)}{\cos(\pi s/2)}ds$$

for any $\frac{1}{2}<\sigma<1$. We have, via [Gradshteyn and Ryzhik 2007, 8.411.4], the bound

$$J_s(x)\ll\frac{x^\sigma}{|\Gamma(s+\frac{1}{2})|}\ll e^{\pi|s|/2}\left(\frac{x}{|s|}\right)^\sigma,$$

and so the integral in the Kloosterman term is bounded by a constant multiple of

$$\prod_{p\in\mathcal{P}}p^{j_p\sigma}\sum_{\substack{c=1\\ c\equiv 0\ (\mathrm{mod}\ q)}}^{\infty}\frac{\big|S_{\prod_{p\in\mathcal{P}}p^{j_p}(q)}\big(1,\prod_{p\in\mathcal{P}}p^{2j_p};c\big)\big|}{c^{1+\sigma}}\int_{\sigma/2-i\infty}^{\sigma/2+i\infty}|r^{\frac{3}{4}}h_X(ir)|\,dr.$$

We take

$$\sigma=\frac{1}{2}+\frac{1}{\log\big(X\prod_{p\in\mathcal{P}}p^{\ell_p}\big)},$$

so that the integral is bounded by a constant multiple of $\sqrt{X}$, and use (5.14) to bound the summation over c and (3.7) to bound the summation over j_p and $p\in\mathcal{P}$ in order to find that

$$\#\big\{f\in\mathcal{B}_0(\Gamma_1(q)):it_f\in\big(\alpha_0,\tfrac{1}{2}\big),\ |\lambda_f(p)|\geq\alpha_p\text{ for all }p\in\mathcal{P}\big\}$$

$$\ll_\varepsilon q^{1+\varepsilon}X^{-2\alpha_0}\prod_{p\in\mathcal{P}}\left(\frac{\alpha_p}{2}\right)^{-2\ell_p}\left(q+\sqrt{X}\prod_{p\in\mathcal{P}}p^{\ell_p/2}\left(\log\left(X\prod_{p\in\mathcal{P}}p^{\ell_p}\right)\right)^2\frac{1}{\sqrt{q}}\prod_{p'|q}\frac{1}{1-p'^{-\frac{1}{2}}}\right).$$

The result follows upon taking

$$X=\mathrm{vol}(\Gamma_1(q)\backslash\mathbb{H})^{3\mu_0/2},\quad \ell_p=\left\lfloor\frac{\mu_p\log\mathrm{vol}(\Gamma_1(q)\backslash\mathbb{H})^{\frac{3}{2}}}{\log p}\right\rfloor.\qquad\square$$

Proof of (1.7). By using (3.21) and (5.15) in place of (3.20) and (5.14), we obtain

$$\#\big\{f \in \mathcal{B}_0(\Gamma(q)) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\}$$

$$\ll_\varepsilon q^{2+\varepsilon} X^{-2\alpha_0} \prod_{p \in \mathcal{P}} \Big(\frac{\alpha_p}{2}\Big)^{-2\ell_p} \Big(q + \sqrt{X} \prod_{p \in \mathcal{P}} p^{\ell_p/2} \Big(\log\Big(X \prod_{p \in \mathcal{P}} p^{\ell_p}\Big)\Big)^2 \frac{1}{q} \prod_{p'|q} \frac{1}{1 - p'^{-\frac{1}{2}}}\Big),$$

and it remains to take

$$X = \mathrm{vol}(\Gamma(q)\backslash\mathbb{H})^{4\mu_0/3}, \quad \ell_p = \Big\lfloor \frac{\mu_p \log \mathrm{vol}(\Gamma(q)\backslash\mathbb{H})^{\frac{4}{3}}}{\log p} \Big\rfloor. \qquad \square$$

Proof of (1.8). We use (3.16) and (5.16) in place of (3.20) and (5.14), so that

$$\#\big\{f \in \mathcal{B}_0(q, \chi) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\}$$

$$\ll_\varepsilon q^{1+\varepsilon} X^{-2\alpha_0} \prod_{p \in \mathcal{P}} \Big(\frac{\alpha_p}{2}\Big)^{-2\ell_p} \Big(1 + \sqrt{X} \prod_{p \in \mathcal{P}} p^{\ell_p/2} \Big(\log\Big(X \prod_{p \in \mathcal{P}} p^{\ell_p}\Big)\Big)^2 \frac{2^{\omega(q)}\dot{Q}}{\varphi(q)}\Big).$$

We find that

$$\#\big\{f \in \mathcal{B}_0(q, \chi) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\}$$

$$\ll_\varepsilon \mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^{1 - 4\big(\mu_0\alpha_0 + \sum_{p \in \mathcal{P}} \mu_p(\log \alpha_p/2)/\log p\big) + \varepsilon} \dot{Q}^{4\big(\mu_0\alpha_0 + \sum_{p \in \mathcal{P}} \mu_p(\log \alpha_p/2)/\log p\big)}.$$

by taking

$$X = \mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^{2\mu_0} \dot{Q}^{-2\mu_0}, \quad \ell_p = \Big\lfloor \frac{\mu_p \log(\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^2 \dot{Q}^{-2})}{\log p} \Big\rfloor.$$

Again, we also have that

$$\#\big\{f \in \mathcal{B}_0(q, \chi) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\}$$

$$\leq \#\big\{f \in \mathcal{B}_0(q\ddot{Q}, \chi\psi^2) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\}$$

for any primitive character ψ modulo $\ddot{Q}$, which implies that

$$\#\big\{f \in \mathcal{B}_0(q, \chi) : it_f \in \big(\alpha_0, \tfrac{1}{2}\big) \in [0, T], \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\big\}$$

$$\ll_\varepsilon \mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})^{1 - 4\big(\mu_0\alpha_0 + \sum_{p \in \mathcal{P}} \mu_p(\log \alpha_p/2)/\log p\big) + \varepsilon} \ddot{Q}^{1 - 4\big(\mu_0\alpha_0 + \sum_{p \in \mathcal{P}} \mu_p(\log \alpha_p/2)/\log p\big)}. \qquad \square$$

9. Improving Theorems 1.1 and 1.5 for $\Gamma_1(q)$ via twisting

In this section, we prove Theorem 1.9 Let $f \in \mathcal{B}_\kappa^*(q, \chi)$ be a newform, and for a primitive character ψ modulo q_ψ with $q_\psi \mid q$, we let $f \otimes \psi$ denote the twist of f by ψ; this is the *newform* whose Hecke eigenvalues $\lambda_{f \otimes \psi}(n)$ are equal to $\lambda_f(n)\psi(n)$ whenever $(n, q) = 1$. By [Atkin and Li 1978, Proposition 3.1], the weight of $f \otimes \psi$ is κ, the level of $f \otimes \psi$ divides q^2, and the nebentypus is the primitive character that induces $\chi\psi^2$. We make crucial use of the fact that twisting by a Dirichlet character preserves the Laplacian eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$ and the absolute value $|\lambda_f(n)|$ of the Hecke eigenvalues of f for all

$(n, q) = 1$. Moreover, if $f_1 \in \mathcal{B}_\kappa^*(q_1, \chi_1)$, $f_2 \in \mathcal{B}_\kappa^*(q_2, \chi_2)$ are such that there exist primitive Dirichlet characters ψ_1 modulo q_{ψ_1} and ψ_2 modulo q_{ψ_2} with $q_{\psi_1}, q_{\psi_2} \mid q$ such that

$$f_1 \otimes \psi_1 = f_2 \otimes \psi_2,$$

then $f_2 = f_1 \otimes \psi_1 \overline{\psi_2}$.

Lemma 9.1. *If q is squarefree, ψ is a primitive Dirichlet modulo q_ψ, where $q_\psi \mid q$, and $f \in \mathcal{B}_\kappa^*(q, \chi)$, then the level of $f \otimes \psi$ divides q if and only if $\overline{\psi}$ divides χ, in the sense that $\psi \chi$ has conductor dividing q_χ.*

Proof. This follows via the methods of [Humphries 2017]. For $p \mid q$, let π_p be the local component of the cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to the newform f, so that the central character ω_p of π_p is the local component of the Hecke character ω that is the idèlic lift of χ. As q is squarefree, π_p is either a principal series representation or a special representation.

In the former case, $\pi_p = \omega_{p,1} \boxplus \omega_{p,2}$ with central character $\omega_p = \omega_{p,1} \omega_{p,2}$, where $\omega_{p,1}, \omega_{p,2}$ are characters of $\mathbb{Q}_p^\times$ with conductor exponents $c(\omega_{p,1}), c(\omega_{p,2}) \in \{0, 1\}$ such that the conductor exponent $c(\pi_p)$ of π_p is $c(\omega_{p,1}) + c(\omega_{p,2}) = 1$. The twist $\pi_p \otimes \omega_p'$ of π_p by a character ω_p' of $\mathbb{Q}_p^\times$ of conductor exponent $c(\omega_p') \in \{0, 1\}$ is $\omega_{p,1} \omega_p' \boxplus \omega_{p,2} \omega_p'$ with corresponding conductor exponent $c(\pi_p \otimes \omega_p') = c(\omega_{p,1} \omega_p') + c(\omega_{p,2} \omega_p')$. For this to be at most 1, either ω_p' is unramified, or one of $c(\omega_{p,1} \omega_p')$, $c(\omega_{p,2} \omega_p')$ must be equal to 0, so that $\overline{\omega_p'}$ is equal to $\omega_{p,1}$ or $\omega_{p,2}$ up to multiplication by an unramified character.

In the latter case, $\pi_p = \omega_{p,1} \mathrm{St}$ with central character $\omega_p = \omega_{p,1}^2$ such that $c(\omega_{p,1}) = 0$, so that $c(\pi_p) = 1$. The twist of π_p by ω_p' is $\omega_{p,1} \omega_p' \mathrm{St}$, with corresponding conductor exponent $c(\pi_p \otimes \omega_p') = \max\{1, 2c(\omega_{p,1} \omega_p')\}$. For this to be at most 1, ω_p' must be unramified.

It follows that if the Hecke character ω' is the idèlic lift of ψ, then the conductor of $\pi \otimes \omega'$ divides q if and only if the conductor of $\omega' \omega$ divides the conductor of ω. $\qquad\square$

From this, we have the following.

Corollary 9.2. *Let q be squarefree. Given a newform g of level dividing q^2, there exist at most $\tau(q)$ newforms f of level dividing q that can be twisted by a Dirichlet character of conductor dividing q to give g.*

Proof. Suppose that $f_1 \in \mathcal{B}_\kappa^*(q_1, \chi_1)$ and $f_2 \in \mathcal{B}_\kappa^*(q_2, \chi_2)$ with q_1 and q_2 dividing q are such that there exist Dirichlet characters ψ_1 and ψ_2 of conductors dividing q for which $f_1 \otimes \psi_1 = f_2 \otimes \psi_2 = g$. Then $f_2 = f_1 \otimes \psi_1 \overline{\psi_2}$, and Lemma 9.1 implies that $\overline{\psi_1} \psi_2$ divides χ_1. Since the conductor of χ_1 divides q_1, the level of f_1, the proof is complete by noting that the number of Dirichlet characters ψ_2 modulo q for which this may occur is bounded by the number of divisors of q. $\qquad\square$

Lemma 9.3. *Let q be squarefree, let $\mathcal{P}$ be a finite collection of primes not dividing q, let E_0 be a measurable subset of $[0, \infty) \cup i\left(0, \frac{1}{2}\right)$, and let E_p be a measurable subset of $[0, \infty)$ for each $p \in \mathcal{P}$. Then*

$$\#\left\{f \in \mathcal{B}_\kappa(\Gamma_1(q)) : t_f \in E_0, \ |\lambda_f(p)| \in E_p \text{ for all } p \in \mathcal{P}\right\}$$

$$\leq \frac{\tau(q)^2}{\varphi(q)} \#\left\{f \in \mathcal{B}_\kappa(\Gamma(q)) : t_f \in E_0, \ |\lambda_f(p)| \in E_p \text{ for all } p \in \mathcal{P}\right\}.$$

Proof. From (3.20),

$$\#\big\{f \in \mathcal{B}_\kappa(\Gamma_1(q)) : t_f \in E_0, \ |\lambda_f(p)| \in E_p \text{ for all } p \in \mathcal{P}\big\}$$

is equal to

$$\sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \tau(q_2) \#\big\{f \in \mathcal{B}_\kappa^*(q_1, \chi) : t_f \in E_0, \ |\lambda_f(p)| \in E_p \text{ for all } p \in \mathcal{P}\big\},$$

which, in turn, is equal to

$$\frac{1}{\varphi(q)} \sum_{\psi \pmod q} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q \\ q_1 \equiv 0 \pmod{q_\chi}}} \tau(q_2)$$

$$\times \#\big\{f \otimes \psi : f \in \mathcal{B}_\kappa^*(q_1, \chi), \ t_f \in E_0, \ |\lambda_f(p)| \in E_p \text{ for all } p \in \mathcal{P}\big\},$$

as twisting preserves Laplacian eigenvalues and the absolute value of Hecke eigenvalues. Each twist $g = f \otimes \psi$ of some $f \in \mathcal{B}_\kappa^*(q_1, \chi)$ is a newform of weight κ, level dividing q^2, and nebentypus of conductor dividing q, and Corollary 9.2 implies that there are at most $\tau(q)$ newforms of level dividing q that can be twisted by a Dirichlet character of conductor dividing q to yield g. Since $\tau(q_2) \leq \tau(q)$, the above quantity is bounded by

$$\frac{\tau(q)^2}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q^2 \\ q_1 \equiv 0 \pmod{q_\chi}}} \#\big\{g \in \mathcal{B}_\kappa^*(q_1, \chi) : t_g \in E_0, \ |\lambda_g(p)| \in E_p \text{ for all } p \in \mathcal{P}\big\},$$

while the explicit basis (3.21) of $\mathcal{B}_\kappa(\Gamma(q))$ implies that

$$\#\{g \in \mathcal{B}_\kappa(\Gamma(q)) : t_g \in E_0, \ |\lambda_g(p)| \in E_p \text{ for all } p \in \mathcal{P}\}$$

is equal to

$$\sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^\kappa}} \sum_{\substack{q_1 q_2 = q^2 \\ q_1 \equiv 0 \pmod{q_\chi}}} \tau(q_2) \#\{g \in \mathcal{B}_\kappa^*(q_1, \chi) : t_g \in E_0, \ |\lambda_g(p)| \in E_p \text{ for all } p \in \mathcal{P}\}.$$

This yields the result. $\qquad\square$

Combining this with the fact that $\mathrm{vol}(\Gamma(q) \backslash \mathbb{H}) = q \, \mathrm{vol}(\Gamma_1(q) \backslash \mathbb{H})$, we deduce Theorem 1.9. It is likely that a more careful analysis could obtain this same result even when q is not squarefree via the methods in [Humphries 2017].

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Volume 12   No. 7   2018