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Big Cohen–Macaulay algebras and the vanishing conjecture for maps of Tor in mixed characteristic

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We prove a version of weakly functorial big Cohen–Macaulay algebras that suffices to establish Hochster and Huneke's vanishing conjecture for maps of Tor in mixed characteristic. As a corollary, we prove an analog of Boutot's theorem that direct summands of regular rings are pseudorational in mixed characteristic. Our proof uses perfectoid spaces and is inspired by the recent breakthroughs on the direct summand conjecture by André and Bhatt.

1. Introduction and preliminaries

In a recent breakthrough, Y. André [2018a] settled Hochster's direct summand conjecture which dates back to 1969.

Theorem 1.1 (André). *Let $A \to R$ be a finite extension of Noetherian rings. If A is regular, then the map is split as a map of A-modules.*

This was previously only known for rings containing a field [Hochster 1975b] and for rings of dimension less than or equal to three [Heitmann 2002]; what is new and striking is the general mixed characteristic case. A simplified and shorter proof of Theorem 1.1 was later found by Bhatt [2018]. But André's argument [2018a] also proved the stronger conjecture that balanced big Cohen–Macaulay algebras exist in mixed characteristic.[1] Recall that B is called a *balanced big Cohen–Macaulay algebra* for the local ring $(R, \mathfrak{m})$ if $\mathfrak{m}B \neq B$ and every system of parameters for R is a regular sequence on B. It is a conjecture of Hochster [1975a; 1975b] that such algebras exist in general and he proved this for rings that contain a field. André's solution in mixed characteristic depends on his deep result in [André 2018b] that gives a generalization of the almost purity theorem: the perfectoid Abhyankar lemma.

The purpose of this paper is to prove that weakly functorial balanced big Cohen–Macaulay algebras exist for *certain* surjective ring homomorphisms in mixed characteristic, a result that has many applications.

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Keywords: big Cohen–Macaulay algebras, direct summand conjecture, perfectoid space, vanishing conjecture for maps of Tor, pseudorational singularities.

[1] André [2018a, Théorème 0.7.1] stated the existence of big Cohen–Macaulay algebras for complete local domains, but the general case follows by killing a minimal prime and taking the completion.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a complete local domain with k algebraically closed, and let $Q \subseteq R$ be a height one prime ideal. Suppose both R and R/Q have mixed characteristic. Then there exists a commutative diagram*:

$$\begin{array}{ccc} R & \longrightarrow & R/Q \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & C \end{array}$$

where B, C are balanced big Cohen–Macaulay algebras for R and R/Q respectively.

Our method of proving *avoids* the perfectoid Abhyankar lemma in [André 2018b] and thus is much shorter than André's argument. More importantly, the weakly functorial property we prove is *new*.[2] We should note that, in equal characteristic, the existence of weakly functorial balanced big Cohen–Macaulay algebras was known in general [Hochster and Huneke 1995]. Nonetheless, the version we prove is strong enough to settle Hochster and Huneke's [1995] vanishing conjecture for maps of Tor in mixed characteristic.

Theorem 4.1. *Let $A \to R \to S$ be maps of Noetherian rings such that $A \to S$ is a local homomorphism of mixed characteristic regular local rings and R is a module-finite torsion-free extension of A. Then for all A-modules M, the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for all $i \geq 1$.*

As a consequence of Theorem 4.1, we prove the following, which is the mixed characteristic analog of Boutot's theorem [1987].[3]

Corollary 4.3. *If $R \to S$ is a ring extension such that S is regular and the map is split as a map of R-modules, then R is pseudorational* (*in particular Cohen–Macaulay*).

It is well known that Theorem 4.1 implies Theorem 1.1 (for example, see [Ranganathan 2000] or [Ma 2018, Remark 4.6]). Recently, Bhatt [2018] gave an alternative and shorter proof of Theorem 1.1: instead of using the perfectoid Abhyankar lemma, Bhatt established a quantitative form of Scholze's Hebbarkeitssatz (the Riemann extension theorem) for perfectoid spaces, and the same idea leads to a proof of a derived variant, i.e., the derived direct summand conjecture. We point out that Theorem 4.1 formally implies such derived variant by [Ma 2018, Remark 5.12] and hence we recover, and in fact generalize, Bhatt's result (see Remark 4.5). Furthermore, although the idea is inspired by [Bhatt 2018], our argument is independent of that work in exposition. We avoid the use of Scholze's Hebbarkeitssatz and the vanishing theorems of perfectoid spaces; instead we study the colon ideals of $A_\infty\langle p^n/g\rangle$ in Lemma 3.4.

Remark 1.2. We should point out that, to the best of our knowledge, Hochster and Huneke's vanishing conjecture for maps of Tor is still open if A and R have mixed characteristic but S has equal characteristic $p > 0$. This case also implies Theorem 1.1 by [Hochster and Huneke 1995, (4.4)]. However, the discussion above shows that the mixed characteristic case we proved (i.e., Theorem 4.1) is enough for almost all applications.

[2] In fact, our version of the existence of weakly functorial big Cohen–Macaulay algebras does not even seem to follow from the perfectoid Abhyankar lemma [André 2018b].

[3] This corollary can be also proved by combining [André 2018a, Remarque 4.2.1] and [Bhatt 2018, Theorem 1.2] (and an extra small argument), see Remark 4.4. However, to the best of our knowledge, the results of [André 2018a; 2018b; Bhatt 2018] are not enough to establish Theorem 4.1.

This paper is organized as follows. In Section 2 we demonstrate a weakly functorial construction of integral perfectoid algebras in Lemma 2.3. Then, in Section 3, we prove Theorem 3.1, and in Section 4, we prove Theorem 4.1 and Corollary 4.3.

Perfectoid algebras. We will freely use the language of perfectoid spaces [Scholze 2012] and almost mathematics [Gabber and Ramero 2003]. In this paper we will always work in the following situation: for a perfect field k of characteristic $p > 0$, we let $W(k)$ be the ring of Witt vectors with coefficients in k. Let K° be the p-adic completion of $W(k)[p^{1/p^\infty}]$ and $K = K^\circ[1/p]$. Then K is a *perfectoid field* in the sense of [Scholze 2012] with $K^\circ \subseteq K$ its ring of integers.

A *perfectoid K-algebra* is a Banach K-algebra R such that the set of power-bounded elements $R^\circ \subseteq R$ is bounded and the Frobenius is surjective on R°/p. A K°-algebra S is called *integral perfectoid* if it is p-adically complete, p-torsion free, satisfies $S = S_*$[4] and the Frobenius induces an isomorphism $S/p^{1/p} \to S/p$. These two categories are equivalent to each other [Scholze 2012, Theorem 5.2] via the functors $R \to R^\circ$ and $S \to S[1/p]$.

Unless otherwise stated, almost mathematics in this paper will always be measured with respect to the ideal (p^{1/p^∞}) in K°.

Partial algebra modifications. We briefly recall Hochster's partial algebra modifications that play a crucial rule in the construction of balanced big Cohen–Macaulay algebras. Our definition and usage of these modifications is basically the same as that in [Hochster 2002, Sections 3 and 4].

Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. We define a *partial algebra modification* of M with respect to a system of parameters $x_1, \dots, x_d$ of R to be a map $M \to M'$ obtained as follows: for some integer $s \geq 0$ and relation $x_{s+1}u_{s+1} = \sum_{j=1}^{s} x_j u_j$, where $u_j \in M$, choose indeterminates $X_1, \dots, X_s$ and an integer $N \geq 1$, let $F = u_{s+1} - \sum_{j=1}^{s} x_j X_j$ and let

$$M' = M[X_1, \dots, X_s]_{\leq N} / F \cdot R[X_1, \dots, X_s]_{\leq N-1},$$

where $M[X_1, \dots, X_s] = M \otimes_R R[X_1, \dots, X_s]$ and thus $M[X_1, \dots, X_s]_{\leq N}$ refers to polynomials of degree at most N (with coefficients in M). The definition of M' makes sense since F has degree one in X_j. It is readily seen that in M', the relation $x_{s+1}u_{s+1} = \sum_{j=1}^{s} x_j u_j$ is trivialized in the sense that u_{s+1} is contained in $(x_1, \dots, x_s)M'$ by construction. We shall refer to the integer N as the *degree bound* of the partial algebra modification. We can then recursively define a sequence of partial algebra modifications of an R-module M.

Now given a local map of local rings $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ we can define a *double sequence of partial algebra modifications* of an R-module M with respect to $R \to S$, a system of parameters $x_1, \dots, x_d$ of R and a system of parameters $y_1, \dots, y_{d'}$ of S as follows: we first form a sequence of partial algebra modifications of M over R with respect to $x_1, \dots, x_d$, say $M = M_0, M_1, \dots, M_r$, and then we form a sequence of partial algebra modifications $N_0 = S \otimes_R M_r, N_1, \dots, N_s$ of N_0 over S with respect to $y_1, \dots, y_{d'}$. When M is an R-algebra, we call this double sequence *bad* if the image of $1 \in M$ in N_s is in $\mathfrak{n}N_s$.

[4] $S_* = \{x \in S[1/p] \mid p^{1/p^k} \cdot x \in S \text{ for all } k\}$. Hence S is almost isomorphic to S_* with respect to (p^{1/p^∞}); thus in practice we will often ignore this distinction since one can always pass to S_* without affecting the issue.

The following was essentially taken from [Hochster 2002, Theorem 4.2], and is one of the main ingredients in our construction.

Theorem 1.3. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings. Then there exists a commutative diagram*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

such that B is a balanced big Cohen–Macaulay algebra for R and C is a balanced big Cohen–Macaulay algebra for S if and only if there is no bad double sequence of partial algebra modifications of R over $R \to S$ with respect to $x_1, \dots, x_d$ of R and $y_1, \dots, y_{d'}$ of S.

This theorem is actually a bit stronger than [Hochster 2002, Theorem 4.2]. Whereas Hochster allows the system of parameters to vary throughout the double sequence, we fix a system of parameters of R and S. But the idea of the proof is the same: one first constructs B' as a direct limit of finite sequences of modifications of R and then constructs C' as a direct limit of finite sequences of modifications of $S \otimes_R B$ over S. It is readily seen that $x_1, \dots, x_d$ and $y_1, \dots, y_{d'}$ are improper-regular sequences on B' and C' respectively. To guarantee that $\mathfrak{m}B' \neq B'$ and $\mathfrak{n}C' \neq C'$ one needs precisely that there is no bad double sequence of partial algebra modifications over $R \to S$. Now B' and C' are not balanced, but that problem is easily remedied. We invoke [Bruns and Herzog 1993, Corollary 8.5.3] to note that $B' \to C'$ induces $B = \widehat{B'}^{\mathfrak{m}} \to C = \widehat{C'}^{\mathfrak{n}}$, a map of balanced big Cohen–Macaulay algebras of $R \to S$.

2. Weakly functorial construction of integral perfectoid algebras

Notation. Throughout this section, $(A, \mathfrak{m}, k)$ will always be a complete and unramified regular local ring of mixed characteristic with k perfect, i.e., $A \cong W(k)[[x_1, \dots, x_{d-1}]]$, where $W(k)$ is the ring of Witt vectors with coefficients in k. Let K° be the p-adic completion of $W(k)[p^{1/p^\infty}]$ and $K = K^\circ[1/p]$. Let $A_{\infty,0}$ be the p-adic completion of $A[p^{1/p^\infty}, x_1^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]$, which is an integral perfectoid K°-algebra.

For any nonzero element $g \in A$, we let $A_{\infty,0} \to A_\infty$ be André's construction of integral perfectoid K°-algebras (for example see [Bhatt 2018, Theorem 2.3]): A_∞ is almost faithfully flat over $A_{\infty,0}$ modulo p such that g admits a compatible system of p^k-th roots in A_∞. We will denote by $A_\infty\langle p^n/g\rangle$ the integral perfectoid K°-algebra which is the ring of bounded functions on the rational subset $\{x \in X \mid |p^n| \leq |g(x)|\}$, where $X = \mathrm{Spa}(A_\infty[1/p], A_\infty)$ is the perfectoid space associated to A_∞. Since g admits a compatible system of p^k-th roots in A_∞, $A_\infty\langle p^n/g\rangle$ can be described almost explicitly as the p-adic completion of $A_\infty[(p^n/g)^{1/p^\infty}]$ [Scholze 2012, Lemma 6.4].

We begin by observing the following:

Lemma 2.1. *Suppose $g \neq 0$ in A/x_1A. Then we have a natural map $A_\infty \to (A/x_1A)_\infty$ sending g^{1/p^k} to $\bar{g}^{1/p^k}$.*

Proof. We first note that there are natural maps

$$A_{\infty,0} \to (A/x_1A)_{\infty,0} \to (A/x_1A)_\infty,$$

where the first map is simply obtained by killing x_1^{1/p^∞}. Thus we have a map

$$A_{\infty,0}\langle T^{1/p^\infty}\rangle \to (A/x_1A)_\infty$$

of integral perfectoid K°-algebras sending T^{1/p^k} to $\bar{g}^{1/p^k}$. Since A_∞ is the ring of functions on the Zariski closed subset of $Y = \mathrm{Spa}(A_{\infty,0}\langle T^{1/p^\infty}\rangle[1/p], A_{\infty,0}\langle T^{1/p^\infty}\rangle)$ defined by $T - g$, the map $A_{\infty,0}\langle T^{1/p^\infty}\rangle \to (A/x_1A)_\infty$ induces a map $A_\infty \to (A/x_1A)_\infty$ sending g^{1/p^k} to $\bar{g}^{1/p^k}$. $\square$

Lemma 2.2. *Let $(R, \mathfrak{m}, k)$ be a complete normal local domain with k perfect, and let $Q \subseteq R$ be a height one prime ideal. Suppose both R and R/Q have mixed characteristic. Then we can find a complete and unramified regular local ring $A \cong W(k)[[x_1, \dots, x_{d-1}]]$ with $A \to R$ a module-finite extension such that*

(1) $Q \cap A = (x_1)$;

(2) $A_{(x_1)} \to R_Q$ *is essentially étale.*

Proof. Let $\{P_i\}$ be all the minimal primes of (p); they all have height one. Since R/Q has mixed characteristic, $p \notin Q$. Thus Q is not contained in any of the P_i. By prime avoidance we can choose $x \in Q$ that is not in $\left(\bigcup_i P_i\right) \cup Q^{(2)}$. Thus the image of x in R_Q generates QR_Q since R is normal, and p, x is part of a system of parameters of R.

Cohen's structure theorem implies the existence of a complete and unramified regular local ring $A \cong W(k)[[x_1, \dots, x_{d-1}]]$ and a module-finite extension $A \to R$ such that the image of x_1 in R is x. It is clear that $Q \cap A = (x_1)$ because $Q \cap A$ is a height one prime of A that contains (x_1), so it must be (x_1). To see $A_{(x_1)} \to R_Q$ is essentially étale, note that the image of x_1, x, generates the maximal ideal QR_Q of R_Q and the extension of residue fields $A_{(x_1)}/(x_1)A_{(x_1)} \to R_Q/QR_Q$ is finite separable since both fields have characteristic 0 (p is inverted when we localize). Thus $A_{(x_1)} \to R_Q$ is unramified. But it is clearly flat because R_Q is x_1-torsion free. Therefore $A_{(x_1)} \to R_Q$ is essentially étale. $\square$

The following is the main result of this section. It is crucial in proving the version of weakly functorial balanced big Cohen–Macaulay algebras that we need.

Lemma 2.3. *Let $(R, \mathfrak{m}, k)$ be a complete normal local domain with k perfect, and let $Q \subseteq R$ be a height one prime ideal. Suppose both R and R/Q have mixed characteristic. We pick $A \cong W(k)[[x_1, \dots, x_{d-1}]]$ such that $A \to R$ is a module-finite extension satisfying the conclusion of Lemma 2.2. Then there exists an element $g \in A$, whose image is nonzero in A/x_1A, such that $A_g \to R_g$ and $(A/x_1A)_g \to (R/Q)_g$ are both finite étale. Furthermore, for every $n > 0$, we have a commutative diagram*:

$$\begin{array}{ccc} R & \longrightarrow & R/Q \\ \downarrow & & \downarrow \\ R_{\infty,n} & \longrightarrow & (R/Q)_{\infty,n} \end{array}$$

where $R_{\infty,n}$ (resp. $(R/Q)_{\infty,n}$) is an integral perfectoid K°-algebra that is almost finite étale over $A_\infty\langle p^n/g\rangle$ (resp. $(A/x_1A)_\infty\langle p^n/g\rangle$).

Proof. Let $g \in A$ be the discriminant of the map $A \to R$; i.e., it defines the locus of $\operatorname{Spec} A$ such that the map $A \to R$ is not essentially étale when localizing. Since $A_{(x_1)} \to R_Q$ is essentially étale, g is nonzero in A/x_1A. Since x_1 generates Q when localizing at Q and we know that $A_g \to R_g$ and hence $(A/x_1A)_g \to (R/x_1R)_g$ are finite étale, replacing g by a multiple we have $A_g \to R_g$ and $(A/x_1A)_g \to (R/Q)_g$ are both finite étale. By Lemma 2.2 we have a commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & R \\ \downarrow & & \downarrow \\ A/x_1A & \longrightarrow & R/Q. \end{array}$$

By Lemma 2.1 we also have a commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & A_\infty & \longrightarrow & A_\infty\langle \frac{p^n}{g}\rangle \\ \downarrow & & \downarrow & & \downarrow \\ A/x_1A & \longrightarrow & (A/x_1A)_\infty & \longrightarrow & (A/x_1A)_\infty\langle \frac{p^n}{g}\rangle. \end{array}$$

Tensoring over A we get a natural commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R \otimes A_\infty\langle \frac{p^n}{g}\rangle \\ \downarrow & & \downarrow \\ R/Q & \longrightarrow & (R/Q) \otimes (A/x_1A)_\infty\langle \frac{p^n}{g}\rangle. \end{array}$$

Since $A_g \to R_g$ and $(A/x_1A)_g \to (R/Q)_g$ are both finite étale and g divides p^n in $A_\infty\langle p^n/g\rangle$ and $(A/x_1A)_\infty\langle p^n/g\rangle$, we know that $(R \otimes A_\infty\langle p^n/g\rangle)[1/p]$ and $((R/Q) \otimes (A/x_1A)_\infty\langle p^n/g\rangle)[1/p]$ are finite étale over $(A_\infty\langle p^n/g\rangle)[1/p]$ and $((A/x_1A)_\infty\langle p^n/g\rangle)[1/p]$ respectively. Therefore

$$\Big(R \otimes A_\infty\Big\langle \frac{p^n}{g}\Big\rangle\Big)\Big[\frac{1}{p}\Big] \to \Big((R/Q) \otimes (A/x_1A)_\infty\Big\langle \frac{p^n}{g}\Big\rangle\Big)\Big[\frac{1}{p}\Big]$$

is a morphism of perfectoid K-algebras; thus it induces a map on the ring of power-bounded elements

$$R_{\infty,n} := \Big(R \otimes A_\infty\Big\langle \frac{p^n}{g}\Big\rangle\Big)\Big[\frac{1}{p}\Big]^\circ \to \Big(R/Q\Big)_{\infty,n} := ((R/Q) \otimes (A/x_1A)_\infty\Big\langle \frac{p^n}{g}\Big\rangle\Big)\Big[\frac{1}{p}\Big]^\circ.$$

The almost purity theorem [Scholze 2012, Theorem 7.9] implies that $R_{\infty,n}$ and $(R/Q)_{\infty,n}$ are integral perfectoid K°-algebras that are almost finite étale over $A_\infty\langle p^n/g\rangle$ and $(A/x_1A)_\infty\langle p^n/g\rangle$ respectively. Therefore we have the desired commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R/Q \\ \downarrow & & \downarrow \\ R_{\infty,n} & \longrightarrow & (R/Q)_{\infty,n} \end{array}$$

□

3. The main result

In this section we continue to use the notation from the beginning of Section 2. The main theorem we want to prove is the following:

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a complete local domain with k algebraically closed, and let $Q \subseteq R$ be a height one prime ideal. Suppose both R and R/Q have mixed characteristic. Then there exists a commutative diagram*:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/Q \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

where B, C are balanced big Cohen–Macaulay algebras for R and R/Q respectively.

To prove this we need several lemmas.

Lemma 3.2. *Let $A \cong W(k)[[x_1, \dots, x_{d-1}]]$ be a complete and unramified regular local ring with k perfect, and let $I = (p, y_1, \dots, y_s)$ be an ideal of A that contains p. Fix a nonzero element $g = p^m g_0 \in A$, where $p \nmid g_0$, and consider the extension $A \to A_\infty \to A_\infty\langle p^n/g\rangle$. Suppose $z \in IA_\infty\langle p^n/g\rangle \cap A_\infty$ for some $n > p^a + m$ (one should think that $n \gg p^a \gg 0$ here). Then we have $(pg)^{1/p^a} z \in IA_\infty$.*

Proof. Using the almost explicit description of $A_\infty\langle p^n/g\rangle$ [Scholze 2012, Lemma 6.4], we have

$$p^{1/p^t} z \in IA_\infty\Big[\widehat{\Big(\frac{p^n}{g}\Big)^{1/p^\infty}}\Big]$$

for some $t > a$. This implies that the image of $p^{1/p^t} z$ in $A_\infty[(p^n/g)^{1/p^\infty}]/p = A_\infty[\widehat{(p^n/g)^{1/p^\infty}}]/p$ is contained in the ideal $(y_1, \dots, y_s)$. Therefore we can write

$$p^{1/p^t} z = pf_0 + y_1 f_1 + \cdots + y_s f_s,$$

where $f_0, f_1, \dots, f_s \in A_\infty[(p^n/g)^{1/p^\infty}]$. Then there exists integers k and h such that $f_0, f_1, \dots, f_s$ are elements in $A_\infty[(p^n/g)^{1/p^k}]$ of degree bounded by $p^k h$. Multiplying by g_0^h to clear all the denominators in f_i, one gets:

$$p^{1/p^t} g_0^h z \in (g_0^{h-(1/p^a)}, p^{(n-m)/p^a}) \cdot (p, y_1, \dots, y_s)A_\infty.$$

From this we know:

$$p^{1/p^t} g_0^h z = g_0^{h-(1/p^a)}(ph_0 + y_1 h_1 + \cdots + y_s h_s) \quad \text{in } A_\infty/p^{(n-m)/p^a},$$

where $h_0, h_1, \dots, h_s \in A_\infty$. Rewriting this we have

$$g_0^{h-(1/p^a)}(p^{1/p^t} g_0^{1/p^a} z - ph_0 - y_1 h_1 - \cdots - y_s h_s) = 0 \quad \text{in } A_\infty/p^{(n-m)/p^a}.$$

Since $p \nmid g_0$, g_0 is a nonzero divisor on A/p. This implies $g_0^{h-(1/p^a)}$ is an almost nonzero divisor on $A_\infty/p^{(n-m)/p^a}$ since $A \to A_{\infty,0}$ is faithfully flat and $A_{\infty,0} \to A_\infty$ is almost faithfully flat modulo p.

Hence $p^{1/p^t} g_0^{1/p^a} z - ph_0 - y_1h_1 - \cdots - y_sh_s$ is killed by (p^{1/p^∞}). In particular, since $t > a$, we know

$$(pg_0)^{1/p^a} z \in (p, y_1, \ldots, y_s) \quad \text{in } A_\infty/p^{(n-m)/p^a}.$$

Finally, since $n > p^a + m$ and g is a multiple of g_0, we have

$$(pg)^{1/p^a} z \in (p, y_1, \ldots, y_s)A_\infty.$$

This finishes the proof. □

Lemma 3.3. *Let $A \cong W(k)[[x_1, \ldots, x_{d-1}]]$ be a complete and unramified regular local ring with k perfect. Fix a nonzero element $g = p^m g_0 \in A$ where $p \nmid g_0$ and $n > m$. Suppose $z \in A_\infty[(p^n/g)^{1/p^\infty}]$ and $p^D z \in A_\infty$ for some $D > 0$. Then $p^D z \in p^{D-(1/p^t)} A_\infty$ for all t.*

Proof. There exist $k \gg 0$ such that $z \in A_\infty[(p^n/g)^{1/p^k}]$. Choosing a high enough power of g_0 to clear denominators, we get $g_0^h z \in A_\infty$. So $g_0^h(p^D z) \in p^D A_\infty$. Since g_0 is a nonzerodivisor on A/p^D and A_∞/p^D is almost faithfully flat over A/p^D, $p^{1/p^t} p^D z \in p^D A_\infty$ for all t. Since A_∞ is p-torsion free, $p^D z \in p^{D-(1/p^t)} A_\infty$ for all t. □

Lemma 3.4. *Let $A \cong W(k)[[x_1, \ldots, x_{d-1}]]$ be a complete and unramified regular local ring with k perfect. Fix a nonzero element $g = p^m g_0 \in A$ where $p \nmid g_0$, and consider the extension $A \to A_\infty \to A_\infty\langle p^n/g\rangle$ for every n. Suppose S is an almost finite projective $A_\infty\langle p^n/g\rangle$-algebra. If $p^a + m < n$, then we have $(p^{1/p^\infty})(pg)^{1/p^a}$ annihilates $(p, x_1, \ldots, x_s)S : x_{s+1}/(p, x_1, \ldots, x_s)S$ for all $s < d-1$.*

Proof. Suppose $y \in (p, x_1, \ldots, x_s)A_\infty\langle p^n/g\rangle : x_{s+1}$. Since y is an element of $A_\infty\langle p^n/g\rangle$, for every $t > 0$, $p^{1/p^t} y \in A_\infty[\widehat{(p^n/g)}^{1/p^\infty}]$ and $x_{s+1}p^{1/p^t} y \in (p, x_1, \ldots, x_s)A_\infty[\widehat{(p^n/g)}^{1/p^\infty}]$ by [Scholze 2012, Lemma 6.4]. Thus modulo p, $p^{1/p^t} y$ gives an element in $A_\infty[\widehat{(p^n/g)}^{1/p^\infty}]/p = A_\infty[(p^n/g)^{1/p^\infty}]/p$. We pick $z \in A_\infty[(p^n/g)^{1/p^\infty}]$ such that $z \equiv p^{1/p^t} y$ modulo $pA_\infty[\widehat{(p^n/g)}^{1/p^\infty}]$.

Now the image of $x_{s+1}z \in A_\infty[(p^n/g)^{1/p^\infty}]$ in $A_\infty[(p^n/g)^{1/p^\infty}]/p = A_\infty[\widehat{(p^n/g)}^{1/p^\infty}]/p$ is contained in the ideal $(x_1, \ldots, x_s)(A_\infty[\widehat{(p^n/g)}^{1/p^\infty}]/p)$. Therefore, we know

$$x_{s+1}z \in (p, x_1, \ldots, x_s)A_\infty\Big[\Big(\frac{p^n}{g}\Big)^{1/p^\infty}\Big]$$

and thus

$$z \in (p, x_1, \ldots, x_s)A_\infty\Big[\Big(\frac{p^n}{g}\Big)^{1/p^\infty}\Big] : x_{s+1}.$$

Next we write $z = u + (p^n/g)^{1/p^a} u'$ such that $g_0^{1/p^a} u \in A_\infty$, $u' \in A_\infty[(p^n/g)^{1/p^\infty}]$, and we also write $x_{s+1}z = v + (p^n/g)^{1/p^a} v'$ such that $g_0^{1/p^a} v \in (p, x_1, \ldots, x_s)A_\infty$, $v' \in (p, x_1, \ldots, x_s)A_\infty[(p^n/g)^{1/p^\infty}]$. We consider two expressions of $x_{s+1}g_0^{1/p^a} z$:

$$x_{s+1}g_0^{1/p^a} u + p^{(n-m)/p^a} x_{s+1}u' = x_{s+1}g_0^{1/p^a} z = g_0^{1/p^a} v + p^{(n-m)/p^a} v'.$$

From this we know that

$$x_{s+1}(g_0^{1/p^a} u) = g_0^{1/p^a} v + p^{(n-m)/p^a}(v' - x_{s+1}u'). \tag{3.4.1}$$

It follows from (3.4.1) that $p^{(n-m)/p^a}(v'-x_{s+1}u') \in A_\infty$ (since the other two terms are in A_∞). Thus by Lemma 3.3, $p^{(n-m)/p^a}(v'-x_{s+1}u') \in pA_\infty$ since $n > p^a+m$. But now (3.4.1) tells us that

$$x_{s+1}(g_0^{1/p^a}u) \in (p, x_1, \dots, x_s)A_\infty + pA_\infty = (p, x_1, \dots, x_s)A_\infty.$$

Since $g_0^{1/p^a}u \in A_\infty$ and $p, x_1, \dots, x_{s+1}$ is an almost regular sequence on A_∞,

$$(pg_0)^{1/p^a}u \in (p, x_1, \dots, x_s)A_\infty.$$

But now

$$(pg_0)^{1/p^a}z = (pg_0)^{1/p^a}u + p^{1/p^a}p^{(n-m)/p^a}u'.$$

Therefore

$$(pg_0)^{1/p^a}z \in (p, x_1, \dots, x_s)A_\infty + pA_\infty\Big[\Big(\frac{p^n}{g}\Big)^{1/p^\infty}\Big] \subseteq (p, x_1, \dots, x_s)A_\infty\Big[\Big(\frac{p^n}{g}\Big)^{1/p^\infty}\Big].$$

Because $z \equiv p^{1/p^t}y$ modulo $pA_\infty\widehat{[(p^n/g)^{1/p^\infty}]}$ and g is a multiple of g_0, we have

$$p^{1/p^t}(pg)^{1/p^a}y \in (p, x_1, \dots, x_s)A_\infty\widehat{\Big[\Big(\frac{p^n}{g}\Big)^{1/p^\infty}\Big]} \subseteq (p, x_1, \dots, x_s)A_\infty\Big\langle\frac{p^n}{g}\Big\rangle.$$

Since this is true for all $t > 0$, we have $(p^{1/p^\infty})(pg)^{1/p^a}$ annihilates

$$\frac{(p, x_1, \dots, x_s)A_\infty\langle\frac{p^n}{g}\rangle : x_{s+1}}{(p, x_1, \dots, x_s)A_\infty\langle\frac{p^n}{g}\rangle}.$$

Finally, since S is an almost finite projective $A_\infty\langle p^n/g\rangle$-algebra, by [Gabber and Ramero 2003, Lemma 2.4.31],

$$\frac{(p, x_1, \dots, x_s)S : x_{s+1}}{(p, x_1, \dots, x_s)S} = \mathrm{Hom}_S(S/x_{s+1}, S/(p, x_1, \dots, x_s))$$

is almost isomorphic to

$$S \otimes \mathrm{Hom}_{A_\infty\langle p^n/g\rangle}\Big(A_\infty\Big\langle\frac{p^n}{g}\Big\rangle/x_{s+1}, A_\infty\Big\langle\frac{p^n}{g}\Big\rangle/(p, x_1, \dots, x_s)\Big) = S \otimes \frac{(p, x_1, \dots, x_s)A_\infty\langle\frac{p^n}{g}\rangle : x_{s+1}}{(p, x_1, \dots, x_s)A_\infty\langle\frac{p^n}{g}\rangle}.$$

Therefore $(p^{1/p^\infty})(pg)^{1/p^a}$ annihilates $(p, x_1, \dots, x_s)S : x_{s+1}/(p, x_1, \dots, x_s)S$ as well. □

We need the following lemma:

Lemma 3.5 [Hochster 2002, Lemma 5.1]. *Let M be an R-module and let T be an R-algebra with a map $\alpha : M \to T[1/c]$. Let $M \to M'$ be a partial algebra modification of M with respect to part of a system of parameters $p, x_1, \dots, x_s, x_{s+1}$ with degree bound D. Suppose $x_{s+1}t_{s+1} = pt_0 + x_1t_1 + \dots + x_st_s$ with $t_j \in T$ implies $ct_{s+1} \in (p, x_1, \dots, x_s)T$ and $\alpha(M) \subseteq c^{-N}T$. Then there is an R-linear map $\beta : M' \to T[1/c]$ extending α with image contained in $c^{-N'}T$ where $N' = ND + N + D$ depends only on N and D.*

Proof of Theorem 3.1. Let R' be the normalization of R and let Q' be a height one prime of R' that lies over Q. Note that the residue field of R' is still k since we assumed k is algebraically closed. If we can construct weakly functorial big Cohen–Macaulay algebras for $R' \to R'/Q'$ then the same follows for $R \to R/Q$. Thus without loss of generality we can assume $(R, \mathfrak{m}, k)$ is normal. Let

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/Q \arrow[d] \\ R_{\infty,n} \arrow[r] & (R/Q)_{\infty,n} \end{tikzcd}$$

be the commutative diagram constructed in Lemma 2.3. Moreover, abusing notation slightly, suppose $g = p^{m_1} g_0$ in A and $\bar{g} = p^{m_2} \bar{g}_0$ in A/x_1A such that $p \nmid g_0$ and $p \nmid \bar{g}_0$.

Now $R_{\infty,n}$ and $(R/Q)_{\infty,n}$ are almost finite étale over $A_\infty\langle p^n/g\rangle$ and $(A/x_1A)_\infty\langle p^n/g\rangle$ respectively, in particular they are almost finite projective over $A_\infty\langle p^n/g\rangle$ and $(A/x_1A)_\infty\langle p^n/g\rangle$ respectively (see [Scholze 2012, Definition 4.3 and Proposition 4.10]). Lemma 3.4 shows that, for every n and p^a such that $n > p^a + m_1 + m_2$, with $c = (pg)^{2/p^a}$, if $x_{s+1}t_{s+1} = pt_0 + x_1t_1 + \cdots + x_st_s$ with $t_j \in R_{\infty,n}$ (resp. $(R/Q)_{\infty,n}$), we have that $ct_{s+1} \in (p, x_1, \ldots, x_s)R_{\infty,n}$ (resp. $(R/Q)_{\infty,n}$).

By Theorem 1.3, it suffices to show that there is no bad double sequence of partial algebra modifications of R. Suppose there is one:

$$R \to M_1 \to \cdots \to M_r \to (R/Q) \otimes M_r \to N_1 \to \cdots \to N_s.$$

We claim that there exists a commutative diagram:

$$\begin{tikzcd}[column sep=small] R \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & M_r \arrow[r] \arrow[d, "\alpha"] & (R/Q) \otimes M_r \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & N_s \arrow[d, "\beta"] \\ R_{\infty,n}\left[\frac{1}{c}\right] \arrow[r, "\equiv"] & R_{\infty,n}\left[\frac{1}{c}\right] \arrow[r, "\equiv"] & \cdots \arrow[r, "\equiv"] & R_{\infty,n}\left[\frac{1}{c}\right] \arrow[r] & (R/Q)_{\infty,n}\left[\frac{1}{c}\right] \arrow[r, "\equiv"] & (R/Q)_{\infty,n}\left[\frac{1}{c}\right] \arrow[r, "\equiv"] & \cdots \arrow[r, "\equiv"] & (R/Q)_{\infty,n}\left[\frac{1}{c}\right] \end{tikzcd}$$

The leftmost vertical map is the natural one; the first half of the diagram exists by Lemma 3.5; the middle commutative diagram exists because the composite map $M_r \to R_{\infty,n}[1/c] \to (R/Q)_{\infty,n}[1/c]$ induces a map $(R/Q) \otimes M_r \to (R/Q)_{\infty,n}[1/c]$ since $(R/Q)_{\infty,n}[1/c]$ is an R/Q-algebra; the second half of the diagram exists by Lemma 3.5 again.

Let $D > 0$ be an integer larger than the degree bounds for all the partial algebra modifications in this sequence. Applying Lemma 3.5 repeatedly to the first half of the diagram, we know there is an integer M depending only on D, but not on n and p^a, such that the image of α is contained in $c^{-M}R_{\infty,n}$. The image of the map $(R/Q) \otimes M_r \to (R/Q)_{\infty,n}[1/c]$ is contained in $c^{-M}(R/Q)_{\infty,n}$ because $R_{\infty,n}[1/c] \to (R/Q)_{\infty,n}[1/c]$ is induced by $R_{\infty,n} \to (R/Q)_{\infty,n}$. But then applying Lemma 3.5 repeatedly to the second half of the diagram, we know that there exists an integer N depending on M and D (and hence only on D), but not on n and p^a, such that the image of β is contained in $c^{-N}(R/Q)_{\infty,n}$.

Now we chase the above diagram and we see that on the one hand, the element $1 \in R$ maps to $1 \in (R/Q)_{\infty,n}[1/c]$. But on the other hand, since the sequence is bad, the image of $1 \in R$ in N_s is in $\mathfrak{m}N_s$ and hence the image of $1 \in R$ is contained in $\mathfrak{m}c^{-N}(R/Q)_{\infty,n}$ in $(R/Q)_{\infty,n}[1/c]$. Therefore we have $1 \in \mathfrak{m}((pg)^{2/p^a})^{-N}(R/Q)_{\infty,n}$, that is,

$$(pg)^{2N/p^a} \in \mathfrak{m}(R/Q)_{\infty,n}.$$

Because $\mathfrak{m}$ is the maximal ideal of R and $A = W(k)[\![x_1, \dots, x_{d-1}]\!] \to R$ is module-finite, $\mathfrak{m}^{N'} \subseteq (p, x_1, \dots, x_{d-1})R$ for some fixed N'. We thus have:

$$(pg)^{2NN'/p^a} \in (p, x_2, \dots, x_{d-1})(R/Q)_{\infty,n}.$$

Since $(R/Q)_{\infty,n}$ is almost finite étale over $(A/x_1A)_\infty\langle p^n/g\rangle$, we know that

$$(pg)^{(2NN'+1)/p^a} \in (p, x_2, \dots, x_{d-1})(A/x_1A)_\infty\Big\langle\frac{p^n}{g}\Big\rangle \cap (A/x_1A)_\infty.$$

But now Lemma 3.2 implies $(pg)^{(2NN'+2)/p^a} \in (p, x_2, \dots, x_{d-1})(A/x_1A)_\infty$ for all p^a. Because N, N' do not depend on p^a, we know that $pg \in (p, x_2, \dots, x_{d-1})^m(A/x_1A)_\infty$ for all $m > 0$. Since $(A/x_1A)_\infty$ is almost faithfully flat over $(A/x_1A)_{\infty,0}$ mod p^m, we know that

$$p^2g \in (p, x_2, \dots, x_{d-1})^m(A/x_1A)_{\infty,0} \cap (A/x_1A) = (p, x_2, \dots, x_{d-1})^m(A/x_1A)$$

for all $m > 0$ by faithful flatness of $(A/x_1A)_{\infty,0}$ over A/x_1A. But then

$$p^2g \in \cap_m(p, x_2, \dots, x_{d-1})^m(A/x_1A) = 0,$$

which is a contradiction. □

Remark 3.6. We point out that the quantitative form of Scholze's Hebbarkeitssatz [Bhatt 2018, Theorem 4.2] implies Lemma 3.2 and the following weaker form of Lemma 3.4: if $\{S_n\}_n$ is a pro-system such that S_n is an almost finite projective $A_\infty\langle p^n/g\rangle$-algebra, then for every $k \geq 1$ and $n \geq p^a + m$, $(p^{1/p^\infty})(pg)^{1/p^a}$ annihilates the image of $(p,x_1,\dots,x_s)S_{k+n} : x_{s+1}/(p,x_1,\dots,x_s)S_{k+n}$ in $(p,x_1,\dots,x_s)S_k : x_{s+1}/(p,x_1,\dots,x_s)S_k$. This weaker form is enough to establish Theorem 3.1, but one needs to modify the proof of Lemma 3.5 and Theorem 3.1: to extend each partial algebra modification to $R_{\infty,n}[1/c]$ one needs to decrease n roughly by p^a in order to trivialize bad relations (and keep control on the denominators). We leave it to the interested reader to carry out the details.

4. Applications

The results obtained in the preceding section are strong enough to establish the mixed-characteristic case of Hochster and Huneke's vanishing conjecture for maps of Tor [1995].

Theorem 4.1. *Let $A \to R \to S$ be maps of Noetherian rings such that $A \to S$ is a local homomorphism of mixed characteristic regular local rings and R is a module-finite torsion-free extension of A. Then for all A-modules M, the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for all $i \geq 1$.*

We need the following important reduction. This reduction is known to experts and is proved implicitly in [Ranganathan 2000, Chapter 5.2] and [Hochster 2017, Section 13]. We will give a sketch of the proof.

Lemma 4.2. *To prove Theorem 4.1, we can assume $(A, \mathfrak{m})$ is complete, R is a complete local domain, and $S = A/xA$ where $x \in \mathfrak{m} - \mathfrak{m}^2$.*

Sketch of proof. We can assume M is finitely generated. Replacing M by its first module of syzygies over A repeatedly, we only need to prove the case $i = 1$. We may further assume $M = A/I$ by [Ranganathan 2000, Lemma 5.2.1] or [Hochster 2017, Page 15].[5] Next by [Hochster and Huneke 1995, (4.5)(a)], we can assume A and S are both complete, R is a complete local domain, and $A \to S$ is surjective; i.e., $S = A/P$ where P is generated by part of a regular system of parameters of A (note that $p \notin P$ since S has mixed characteristic). It follows that $S = R/Q$ for some prime ideal Q of R lying over P. After all these reductions, we note that by [Hochster 2017, Lemma 13.6], $\mathrm{Tor}_1^A(A/I, R) \to \mathrm{Tor}_1^A(A/I, S)$ vanishes if and only if $IQ \cap P = IP$.

We next want to reduce to the case that P is generated by one element. The trick is to replace A by its extended Rees ring $\widetilde{A} = A[Pt, t^{-1}]$, R by $\widetilde{R} = R[Pt, t^{-1}]$ and S by $\widetilde{S} = \widetilde{A}/t^{-1}\widetilde{A}$. Since P is generated by part of a regular system of parameters, $\widetilde{A}$ and $\widetilde{S}$ are still regular. The point is that there is a homogeneous prime ideal $\widetilde{Q} \subseteq \widetilde{R}$ that contains Q and contracts to $t^{-1}\widetilde{A} \subseteq \widetilde{A}$ (see [Ranganathan 2000, Proof of Theorem 5.2.6] or [Hochster 2017, Page 16]), thus we have $\widetilde{A} \to \widetilde{R} \to \widetilde{S}$. Therefore if we can prove Theorem 4.1 for $\widetilde{A} \to \widetilde{R} \to \widetilde{S}$ and $M = \widetilde{A}/I\widetilde{A}$, then [Hochster 2017, Lemma 13.6] implies that $I\widetilde{Q} \cap t^{-1}\widetilde{A} = It^{-1}\widetilde{A}$. Comparing the degree 0 part, we see that $IQ \cap P = IP$.

Finally, we can localize $\widetilde{A}$ and $\widetilde{S}$ and complete, and reduce to the case $\widetilde{R}$ is a complete local domain as in [Hochster and Huneke 1995, (4.5)(a)]. Note that $\widetilde{S}$ is obtained from $\widetilde{A}$ by killing one element (and we may assume $\widetilde{S}$ still has mixed characteristic after localization). We thus obtain all the desired reductions. $\square$

Proof of Theorem 4.1. By Lemma 4.2, we may assume R is a complete local domain and $S = A/xA$. It follows that $S = R/Q$ for a height one prime Q of R. Since $A \to S$ and $R \to S$ are both surjective, A, R, S have the same residue field k. We fix a coefficient ring $W(k)$ of A, then the images of $W(k)$ in R and S are also coefficient rings of R and S. Replacing A, R, S by their faithfully flat extensions $A\widehat{\otimes}_{W(k)}W(\bar{k})$, $R\widehat{\otimes}_{W(k)}W(\bar{k})$, $S\widehat{\otimes}_{W(k)}W(\bar{k})$ does not affect whether the map on Tor vanishes or not. Thus without loss of generality we may assume k is algebraically closed.

By Theorem 3.1, we have a commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S = R/Q \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

[5] In this process we may lose A and S being local, but we can always localize A and S again to assume they are local (and have mixed characteristic, since otherwise Theorem 4.1 is known).

where B and C are balanced big Cohen–Macaulay algebras for R and S respectively. This induces a commutative diagram:

$$\begin{array}{ccc} \operatorname{Tor}_i^A(M, R) & \longrightarrow & \operatorname{Tor}_i^A(M, S) \\ \downarrow & & \downarrow \\ \operatorname{Tor}_i^A(M, B) & \longrightarrow & \operatorname{Tor}_i^A(M, C) \end{array}$$

Since B is a balanced big Cohen–Macaulay algebra over R (and hence also over A), it is faithfully flat over A so $\operatorname{Tor}_i^A(M, B) = 0$ for all $i \geq 1$. Moreover, C is faithfully flat over S since it is a balanced big Cohen–Macaulay algebra over S and S is regular, thus $\operatorname{Tor}_i^A(M, S) \to \operatorname{Tor}_i^A(M, C)$ is injective. Chasing the diagram above we know that the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for all $i \geq 1$. □

A local ring $(R, \mathfrak{m})$ of dimension d is called *pseudorational* if it is normal, Cohen–Macaulay, analytically unramified (i.e., the completion $\widehat{R}$ is reduced), and if for every projective and birational map $\pi\colon W \to \operatorname{Spec} R$, the canonical map $H^d_{\mathfrak{m}}(R) \to H^d_E(W, \mathcal{O}_W)$ is injective where $E = \pi^{-1}(\mathfrak{m})$ denotes the closed fiber. In characteristic 0, pseudorational singularities are the same as rational singularities. Very recently, Kovács [2017] has proved a remarkable result that, in *all* characteristics, if $\pi\colon X \to \operatorname{Spec} R$ is projective and birational, where X is Cohen–Macaulay and R is pseudorational, then $\boldsymbol{R}\pi_*\mathcal{O}_X = R$.

In equal characteristic, direct summands of regular rings are pseudorational [Boutot 1987; Hochster and Huneke 1990]. This is usually called Boutot's theorem. It is well known that the vanishing conjecture for maps of Tor in a given characteristic implies that direct summands of regular rings are Cohen–Macaulay [Hochster and Huneke 1995, (4.3)]. What we want to prove next is the analog of Boutot's theorem that direct summands of regular rings are pseudorational in mixed characteristic. This in fact also follows formally from the vanishing conjecture for maps of Tor [Ma 2018]. Since the full details were not written down explicitly there, we give a complete argument here. We first recall the following Sancho de Salas exact sequence [1987].

Let $T = R[Jt] = R \oplus Jt \oplus J^2t^2 \oplus \cdots$ and let $W = \operatorname{Proj} T \to \operatorname{Spec} R$ be the blow up with $E = \pi^{-1}(\mathfrak{m})$. Pick $f_1, \dots, f_n \in Jt = [T]_1$ such that $U = \{U_i = \operatorname{Spec}[T_{f_i}]_0\}$ is an affine open cover of W. We have an exact sequence of chain complexes:

$$0 \to \check{C}^\bullet(U, \mathcal{O}_W)[-1] \to [C^\bullet(f_1, \dots, f_n, T)]_0 \to R \to 0.$$

Since $\check{C}^\bullet(U, \mathcal{O}_W) \cong \boldsymbol{R}\pi_*\mathcal{O}_W$ and $C^\bullet(f_1, \dots, f_n, T) = [\boldsymbol{R}\Gamma_{T_{>0}}T]_0$, the above sequence gives us (after rotating) an exact triangle:

$$[\boldsymbol{R}\Gamma_{T_{>0}}T]_0 \to R \to \boldsymbol{R}\pi_*\mathcal{O}_W \xrightarrow{+1}$$

Applying $\boldsymbol{R}\Gamma_{\mathfrak{m}}$, we get:

$$[\boldsymbol{R}\Gamma_{\mathfrak{m}+T_{>0}}T]_0 \to \boldsymbol{R}\Gamma_{\mathfrak{m}}R \to \boldsymbol{R}\Gamma_{\mathfrak{m}}\boldsymbol{R}\pi_*\mathcal{O}_W \xrightarrow{+1} .$$

Taking cohomology we get the Sancho de Salas exact sequence:

$$\begin{tikzcd} \arrow[r] & h^d([\boldsymbol{R}\Gamma_{\mathfrak{m}+T_{>0}}T]_0) \arrow[r] \arrow[d, "="] & h^d(\boldsymbol{R}\Gamma_{\mathfrak{m}}R) \arrow[r] \arrow[d, "="] & h^d(\boldsymbol{R}\Gamma_{\mathfrak{m}}\boldsymbol{R}\pi_* O_W) \arrow[r] \arrow[d, "="] & {} \\ \arrow[r] & {[H^d_{\mathfrak{m}+T_{>0}}(T)]_0} \arrow[r] & H^d_{\mathfrak{m}}(R) \arrow[r] & H^d_E(W, O_W) \arrow[r] & {} \end{tikzcd} \tag{4.2.1}$$

We also recall that $R \to S$ is *pure* if $R \otimes M \to S \otimes M$ is injective for every R-module M. This is slightly weaker than saying that $R \to S$ splits as a map of R-modules. If R is an A-algebra and $R \to S$ is pure, then $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ is injective for every i [Hochster and Huneke 1995, (2.1)(h)], in particular, $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(S)$ is injective for every i.

We are ready to prove the following corollary. We state the result in the local setting, but the general case reduces immediately to the local case.

Corollary 4.3. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a pure map of local rings such that $(S, \mathfrak{n})$ is regular of mixed characteristic. Then R is pseudorational. In particular, direct summands of regular rings are pseudorational.*

Proof. We can complete R and S at $\mathfrak{m}$ and $\mathfrak{n}$ respectively to assume both R and S are complete; R is normal since pure subrings of normal domains are normal. By Cohen's structure theorem, we have a module-finite extension $A \to R$ such that A is a complete regular local ring. Let $x_1, \dots, x_d$ be a regular system of parameters of A. We apply Theorem 4.1 to $M = A/(x_1, \dots, x_d)$. We have

$$\operatorname{Tor}_i^A(A/(x_1, \dots, x_d), R) \to \operatorname{Tor}_i^A(A/(x_1, \dots, x_d), S)$$

vanishes for all $i \geq 1$. However, we also know that this map is injective because $R \to S$ is pure. Thus we have $\operatorname{Tor}_i^A(A/(x_1, \dots, x_d), R) = H_i(x_1, \dots, x_d, R) = 0$ for all $i \geq 1$. This implies $x_1, \dots, x_d$ is a regular sequence on R and hence R is Cohen–Macaulay. Obviously, the complete local domain R is analytically unramified.

We now check the last condition of pseudorationality. Let $W \to \operatorname{Spec} R$ be a projective birational map, thus $W \cong \operatorname{Proj} T = \operatorname{Proj} R \oplus Jt \oplus J^2t^2 \oplus \cdots$ for some ideal $J \subseteq R$. We now apply the Sancho de Salas exact sequence (4.2.1) to get:

$$\begin{tikzcd} {[H^d_{\mathfrak{m}+T_{>0}}(T)]_0} \arrow[r] \arrow[d, hook] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "="] & H^d_E(W, O_W) \\ H^d_{\mathfrak{m}+T_{>0}}(T) \arrow[r] & H^d_{\mathfrak{m}}(R). \end{tikzcd}$$

Thus in order to show $H^d_{\mathfrak{m}}(R) \to H^d_E(W, O_W)$ is injective, it suffices to show $H^d_{\mathfrak{m}+T_{>0}}(T) \to H^d_{\mathfrak{m}}(R)$ vanishes. We can localize T at the maximal ideal $\mathfrak{m} + T_{>0}$, complete, and kill a minimal prime without affecting whether the map vanishes or not. Hence it is enough to show that if $(T, \mathfrak{m}) \twoheadrightarrow (R, \mathfrak{m})$ is a surjection such that T is a complete local domain of dimension $d + 1$, then $H^d_{\mathfrak{m}}(T) \to H^d_{\mathfrak{m}}(R)$ vanishes. By Cohen's structure theorem there exists $(A, \mathfrak{m}_0) \to (T, \mathfrak{m})$ a module-finite extension such that A is a complete regular local ring. We consider the chain of maps

$$A \to T \to R \to S$$

Applying Theorem 4.1 to $A \to T \to S$ and $M = H^{d+1}_{\mathfrak{m}_0}(A)$, we know that the composite map

$$\mathrm{Tor}_1^A(H^{d+1}_{\mathfrak{m}_0}(A), T) \to \mathrm{Tor}_1^A(H^{d+1}_{\mathfrak{m}_0}(A), R) \to \mathrm{Tor}_1^A(H^{d+1}_{\mathfrak{m}_0}(A), S)$$

vanishes. Since the Čech complex on a regular system of parameters gives a flat resolution of $H^{d+1}_{\mathfrak{m}_0}(A)$ over A, we know that $\mathrm{Tor}_1^A(H^{d+1}_{\mathfrak{m}_0}(A), N) \cong H^d_{\mathfrak{m}_0}(N)$ for every A-module N. Thus the composite map

$$H^d_{\mathfrak{m}}(T) \to H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(S)$$

vanishes. But then $H^d_{\mathfrak{m}}(T) \to H^d_{\mathfrak{m}}(R)$ vanishes because $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(S)$ is injective since $R \to S$ is pure. □

Remark 4.4. Corollary 4.3 can be also obtained by combining the main results of [André 2018a; Bhatt 2018] and using the following argument: the existence of weakly functorial big Cohen–Macaulay algebras for injective ring homomorphisms [André 2018a, Remarque 4.2.1] implies that direct summands of regular rings are Cohen–Macaulay, but we also know they are derived splinters (because this is true for regular rings by [Bhatt 2018, Theorem 1.2] and it is easy to see that direct summand of derived splinters are still derived splinters). Now the argument of [Kovács 2017, Lemma 7.5] implies that Cohen–Macaulay derived splinters are pseudorational.

Remark 4.5. Last we point out that by [Ma 2018, Remark 5.12], Theorem 4.1 gives a new proof of the derived direct summand conjecture [Bhatt 2018, Theorem 6.1], that is, if R is a complete regular local ring of mixed characteristic and $\pi\colon X \to \operatorname{Spec} R$ is a proper surjective map, then $R \to \boldsymbol{R}\pi_* O_X$ splits in the derived category of R-modules. Our proof is different from Bhatt's in that it does not use Scholze's vanishing theorem [2012, Proposition 6.14]. In fact, tracing the arguments of [Ma 2018, Theorem 5.11 and Remark 5.13], one can show that our Theorem 3.1 leads to a stronger result that complete local rings that are pure inside all their big Cohen–Macaulay algebras (e.g., complete regular local rings) are derived splinters.

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References

[André 2018a] Y. André, "La conjecture du facteur direct", *Publ. Math. Inst. Hautes Études Sci.* **127** (2018), 71–93. MR

[André 2018b] Y. André, "Le lemme d'Abhyankar perfectoïde", *Publ. Math. Inst. Hautes Études Sci.* **127** (2018), 1–70. MR

[Bhatt 2018] B. Bhatt, "On the direct summand conjecture and its derived variant", *Invent. Math.* **212**:2 (2018), 297–317. MR

[Boutot 1987] J.-F. Boutot, "Singularités rationnelles et quotients par les groupes réductifs", *Invent. Math.* **88**:1 (1987), 65–68. MR Zbl

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Adv. Math. **39**, Cambridge Univ. Press, 1993. MR Zbl

[Gabber and Ramero 2003] O. Gabber and L. Ramero, *Almost ring theory*, Lect. Notes in Math. **1800**, Springer, 2003. MR Zbl

[Heitmann 2002] R. C. Heitmann, "The direct summand conjecture in dimension three", *Ann. of Math.* (2) **156**:2 (2002), 695–712. MR Zbl

[Hochster 1975a] M. Hochster, "Big Cohen–Macaulay modules and algebras and embeddability in rings of Witt vectors", pp. 106–195 in *Conference on commutative algebra*, edited by A. V. Geramita, Queen's Papers on Pure and Appl. Math. **42**, Queen's Univ., Kingston, ON, 1975. MR Zbl

[Hochster 1975b] M. Hochster, *Topics in the homological theory of modules over commutative rings* (Lincoln, NE, 1974), CBMS Regional Conference Series in Math. **24**, Amer. Math. Soc., Providence, RI, 1975. MR Zbl

[Hochster 2002] M. Hochster, "Big Cohen–Macaulay algebras in dimension three via Heitmann's theorem", *J. Algebra* **254**:2 (2002), 395–408. MR Zbl

[Hochster 2017] M. Hochster, "Homological conjectures and lim Cohen–Macaulay sequences", pp. 173–197 in *Homological and computational methods in commutative algebra* (Cortona, Italy, 2016), edited by A. Conca et al., Springer INdAM Ser. **20**, Springer, 2017. MR

[Hochster and Huneke 1990] M. Hochster and C. Huneke, "Tight closure, invariant theory, and the Briançon–Skoda theorem", *J. Amer. Math. Soc.* **3**:1 (1990), 31–116. MR Zbl

[Hochster and Huneke 1995] M. Hochster and C. Huneke, "Applications of the existence of big Cohen–Macaulay algebras", *Adv. Math.* **113**:1 (1995), 45–117. MR Zbl

[Kovács 2017] S. J. Kovács, "Rational singularities", preprint, 2017. arXiv

[Ma 2018] L. Ma, "The vanishing conjecture for maps of Tor and derived splinters", *J. Eur. Math. Soc.* **20**:2 (2018), 315–338. MR Zbl

[Ranganathan 2000] N. Ranganathan, *Splitting in module-finite extension rings and the vanishing conjecture for maps of Tor*, Ph.D. thesis, University of Michigan, 2000, Available at https://search.proquest.com/docview/304609163.

[Sancho de Salas 1987] J. B. Sancho de Salas, "Blowing-up morphisms with Cohen–Macaulay associated graded rings", pp. 201–209 in *Géométrie algébrique et applications, I* (La Rábida, Spain, 1984), edited by J.-M. Aroca et al., Travaux en Cours **22**, Hermann, Paris, 1987. MR Zbl

[Scholze 2012] P. Scholze, "Perfectoid spaces", *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 245–313. MR Zbl

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