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Blocks of the category of smooth ℓ-modular representations of GL(n, F) and its inner forms: reduction to level 0

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Let G be an inner form of a general linear group over a nonarchimedean locally compact field of residue characteristic p, let R be an algebraically closed field of characteristic different from p and let $\mathscr{R}_R(G)$ be the category of smooth representations of G over R. In this paper, we prove that a block (indecomposable summand) of $\mathscr{R}_R(G)$ is equivalent to a level-0 block (a block in which every simple object has nonzero invariant vectors for the pro-p-radical of a maximal compact open subgroup) of $\mathscr{R}_R(G')$, where G' is a direct product of groups of the same type of G.

Introduction

Let F be a nonarchimedean locally compact field of residue characteristic p and let D be a central division algebra of finite dimension over F whose reduced degree is denoted by d. Given $m \in \mathbb{N}^*$, we consider the group $G = \mathrm{GL}_m(D)$ which is an inner form of $\mathrm{GL}_{md}(F)$. Let R be an algebraically closed field of characteristic $\ell \neq p$ and let $\mathscr{R}_R(G)$ be the category of smooth representations of G over R, that are called ℓ-modular when ℓ is positive. In this paper, we are interested in the Bernstein decomposition of $\mathscr{R}_R(G)$ (see [Sécherre and Stevens 2016] or [Vignéras 1998] for $d = 1$) that is its decomposition as a direct sum of full indecomposable subcategories, called *blocks*. Actually a full understanding of blocks of $\mathscr{R}_R(G)$ is equivalent to a full understanding of the whole category.

The main purpose of this paper is to find an equivalence of categories between any block of $\mathscr{R}_R(G)$ and a level-0 block of $\mathscr{R}_R(G')$ where G' is a suitable direct product of inner forms of general linear groups over finite extensions of F. We recall that a level-0 block of $\mathscr{R}_R(G')$ is a block in which every object has nonzero invariant vectors for the pro-p-radical of a maximal compact open subgroup of G'. This result is an important step in the attempt to describe blocks of $\mathscr{R}_R(G)$ because it reduces the problem to the description of level-0 blocks.

In the case of complex representations, Bernstein [1984] found a block decomposition of $\mathscr{R}_{\mathbb{C}}(G)$ indexed by pairs (M, σ) where M is a Levi subgroup of G and σ is an irreducible cuspidal representation

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of M, up to a certain equivalence relation called *inertial equivalence*. In particular an irreducible representation π of G is in the block associated to the inertial class of (M, σ) if its cuspidal support is in this class. Bushnell and Kutzko [1998] introduced a method to describe the blocks of $\mathscr{R}_{\mathbb{C}}(G)$: the *theory of types*. This method consists in associating to every block of $\mathscr{R}_{\mathbb{C}}(G)$ a pair (J, λ), called a type, where J is a compact open subgroup of G and λ is an irreducible representation of J, such that the simple objects of the block are the irreducible subquotients of the compactly induced representation $\operatorname{ind}_J^G(\lambda)$. In this case the block is equivalent to the category of modules over the $\mathbb{C}$-algebra $\mathscr{H}_{\mathbb{C}}(G, \lambda)$ of G-endomorphisms of $\operatorname{ind}_J^G(\lambda)$. Sécherre and Stevens [2012] (see [Bushnell and Kutzko 1999] for $d = 1$) described explicitly this algebra as a tensor product of algebras of type A.

In the case of ℓ-modular representations, Sécherre and Stevens [2016] (see [Vignéras 1998] for $d = 1$) found a block decomposition of $\mathscr{R}_R(G)$ indexed by inertial classes of pairs (M, σ) where M is a Levi subgroup of G and σ is an irreducible supercuspidal representation of M. In particular an irreducible representation π of G is in the block associated to the inertial class of (M, σ) if its supercuspidal support is in this class. We recall that the notions of cuspidal and supercuspidal representations are not equivalent as in complex case; however, Mínguez and Sécherre [2014a] proved the uniqueness of supercuspidal support, up to conjugation, for every irreducible representation of G. We remark that to obtain the block decomposition of $\mathscr{R}_R(G)$, Sécherre and Stevens do not use the same method as Bernstein, but they rely, like us in this paper, on the theory of semisimple types developed in [Sécherre and Stevens 2012] (see [Bushnell and Kutzko 1999] for $d = 1$). Actually, they associate to every block of $\mathscr{R}_R(G)$ a pair $(\boldsymbol{J}, \boldsymbol{\lambda})$, called a *semisimple supertype*. Unfortunately the construction of the equivalence, as in the complex case, between the block and the category of modules over $\mathscr{H}_R(G, \boldsymbol{\lambda})$ does not hold and one of the problems that occurs is that the pro-order of $\boldsymbol{J}$ can be divisible by ℓ. Some partial results on descriptions of algebras which are Morita equivalent to blocks of $\mathscr{R}_R(\mathrm{GL}_n(F))$ are given in [Dat 2012; Helm 2016; Guiraud 2013].

The idea of this paper is the following. We fix a block $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ of $\mathscr{R}_R(G)$ associated to the semisimple supertype $(\boldsymbol{J}, \boldsymbol{\lambda})$ and, as in [Sécherre and Stevens 2016], we can associate to it a compact open subgroup $\boldsymbol{J}_{\max}$ of G, its pro-p-radical $\boldsymbol{J}_{\max}^1$ and an irreducible representation $\boldsymbol{\eta}_{\max}$ of $\boldsymbol{J}_{\max}^1$. We remark that we can extend, not uniquely, $\boldsymbol{\eta}_{\max}$ to an irreducible representation $\boldsymbol{\kappa}_{\max}$ of $\boldsymbol{J}_{\max}$. Thus, we denote $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ the direct sum of blocks of $\mathscr{R}_R(G)$ associated to $(\boldsymbol{J}_{\max}^1, \boldsymbol{\eta}_{\max})$ and we consider the functor

$$\boldsymbol{M}_{\boldsymbol{\eta}_{\max}} = \operatorname{Hom}_G(\operatorname{ind}_{\boldsymbol{J}_{\max}^1}^G \boldsymbol{\eta}_{\max}, -) : \mathscr{R}(G, \boldsymbol{\eta}_{\max}) \longrightarrow \text{Mod-}\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}),$$

where $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}) \cong \operatorname{End}_G(\operatorname{ind}_{\boldsymbol{J}_{\max}^1}^G(\boldsymbol{\eta}_{\max}))$. Using the fact that $\boldsymbol{\eta}_{\max}$ is a projective representation, since $\boldsymbol{J}_{\max}^1$ is a pro-p-group, we prove that $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}$ is an equivalence of categories (Theorem 5.10). This result generalizes Corollary 3.3 of [Chinello 2017] where $\boldsymbol{\eta}_{\max}$ is a trivial character. We can also associate to $(\boldsymbol{J}, \boldsymbol{\lambda})$ a Levi subgroup L of G and a group $B_L^\times$, which is a direct product of inner forms of general linear groups over finite extensions of F and which we have denoted G' above. If K_L is a maximal compact open subgroup of $B_L^\times$ and K_L^1 is its pro-p-radical then $K_L/K_L^1 \cong \boldsymbol{J}_{\max}/\boldsymbol{J}_{\max}^1 = \mathscr{G}$ is a direct product of finite general linear groups. Actually, in [Chinello 2017] it is proved that the K_L^1-invariants functor $\operatorname{inv}_{K_L^1}$ is an equivalence of categories between the level-0 subcategory $\mathscr{R}(B_L^\times, K_L^1)$ of $\mathscr{R}_R(B_L^\times)$, which is the direct sum of its level-0 blocks, and the category of modules over the algebra $\mathscr{H}_R(B_L^\times, K_L^1) \cong \mathrm{End}_{B_L^\times}(\mathrm{ind}_{K_L^1}^{B_L^\times} 1_{K_L^1})$. Now, thanks to the explicit presentation by generators and relations of $\mathscr{H}_R(B_L^\times, K_L^1)$ presented in [Chinello 2017], in this paper we construct a homomorphism $\Theta_{\gamma,\boldsymbol{\kappa}_{\max}} : \mathscr{H}_R(B_L^\times, K_L^1) \longrightarrow \mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ finding elements in $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ satisfying all relations defining $\mathscr{H}_R(B_L^\times, K_L^1)$. This homomorphism depends on the choice of the extension $\boldsymbol{\kappa}_{\max}$ of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$ and on the choice of an intertwining element γ of $\boldsymbol{\eta}_{\max}$. Moreover, using some properties of $\boldsymbol{\eta}_{\max}$, we prove that this homomorphism is actually an isomorphism. We remark that finding this isomorphism is one of the most difficult results obtained in this article and the proof in the case $L = G$ takes about half of the paper (Section 3). In this way we obtain an equivalence of categories $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}} : \mathscr{R}(G, \boldsymbol{\eta}_{\max}) \longrightarrow \mathscr{R}(B_L^\times, K_L^1)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{R}(G, \boldsymbol{\eta}_{\max}) & \xrightarrow{\ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}\ } & \mathscr{R}(B_L^\times, K_L^1) \\ {\scriptstyle \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}}\downarrow \wr & & \wr \downarrow {\scriptstyle \mathrm{inv}_{K_L^1}} \\ \text{Mod-}\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}) & \xrightarrow{\ \Theta^*_{\gamma,\boldsymbol{\kappa}_{\max}}\ } & \text{Mod-}\mathscr{H}_R(B_L^\times, K_L^1). \end{array}$$

Then we obtain

$$\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(\pi, V) = \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\pi, V) \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \mathrm{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1})$$

for every (π, V) in $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$, where the action of $\mathscr{H}_R(B_L^\times, K_L^1)$ on $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\pi, V)$ depends on $\Theta_{\gamma,\boldsymbol{\kappa}_{\max}}$. Hence, $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ induces an equivalence of categories between the block $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ and a level-0 block of $\mathscr{R}_R(B_L^\times)$. To understand this correspondence we need to use the functor

$$\mathsf{K}_{\boldsymbol{\kappa}_{\max}} : \mathscr{R}(G, \boldsymbol{\eta}_{\max}) \longrightarrow \mathscr{R}_R(\boldsymbol{J}_{\max}/\boldsymbol{J}^1_{\max}) = \mathscr{R}_R(\mathscr{G}),$$

where $\boldsymbol{J}_{\max}$ acts on $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\pi) = \mathrm{Hom}_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max}, \pi)$ by $x.\varphi = \pi(x) \circ \varphi \circ \boldsymbol{\kappa}_{\max}(x)^{-1}$ for every representation π of G, $\varphi \in \mathrm{Hom}_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max}, \pi)$ and $x \in \boldsymbol{J}_{\max}$. This functor is strongly used in [Sécherre and Stevens 2016] to define $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ and to prove the Bernstein decomposition of $\mathscr{R}_R(G)$. We also consider the functor $\mathsf{K}_{K_L} : \mathscr{R}(B_L^\times, K_L^1) \to \mathscr{R}_R(K_L/K_L^1) = \mathscr{R}_R(\mathscr{G})$ given by $\mathsf{K}_{K_L}(Z) = Z^{K_L^1}$ for every representation (ϱ, Z) of $B_L^\times$ where $x \in K_L$ acts on $z \in Z^{K_L^1}$ by $x.z = \varrho(x)z$. Then the functors $\mathsf{K}_{K_L} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ and $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}$ are naturally isomorphic (Proposition 5.14) and so $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ is equivalent to the level-0 block $\mathscr{B}$ of $\mathscr{R}_R(B_L^\times)$ such that $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})) = \mathsf{K}_{K_L}(\mathscr{B})$. More precisely, if $\boldsymbol{J}^1$ is the pro-p-radical of $\boldsymbol{J}$, then $\boldsymbol{J}/\boldsymbol{J}^1 = \mathscr{M}$ is a Levi subgroup of $\mathscr{G}$ and the choice of $\boldsymbol{\kappa}_{\max}$ defines a decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$ where $\boldsymbol{\kappa}$ is an irreducible representation of $\boldsymbol{J}$ and $\boldsymbol{\sigma}$ is a cuspidal representation of $\mathscr{M}$ viewed as an irreducible representation of $\boldsymbol{J}$ trivial on $\boldsymbol{J}^1$. If we can consider the pair $(\mathscr{M}, \boldsymbol{\sigma})$ up to the equivalence relation given in Definition 1.14 of [Sécherre and Stevens 2016], then a representation (ϱ, Z) of $B_L^\times$ is in $\mathscr{B}$ if it is generated by the maximal subspace of $Z^{K_L^1}$ such that every irreducible subquotient has supercuspidal support in the class of $(\mathscr{M}, \boldsymbol{\sigma})$.

One question we do not address in this paper is the structure of level-0 blocks of $\mathscr{R}_R(B_L^\times)$ when the characteristic of R is positive. Thanks to results of [Chinello 2017] we know that there is a correspondence between these blocks and the set $\mathscr{E}$ of primitive central idempotents of $\mathscr{H}_R(B_L^\times, K_L^1)$, which are described in Sections 2.5 and 2.6 of [Chinello 2015]. Hence, one possibility for understanding level-0 blocks of

$\mathscr{R}_R(B_L^\times)$ is to describe the algebras $e\mathscr{H}_R(B_L^\times, K_L^1)$ with $e \in \mathscr{E}$. On the other hand, we recall that Dat [2018] proved that every level-0 block of $\mathscr{R}_R(\mathrm{GL}_n(F))$ is equivalent to the unipotent block of $\mathscr{R}_R(G'')$, where G'' is a suitable product of general linear groups over nonarchimedean locally compact fields. Hence, putting together the result of Dat and results of this article, we obtain a method to reduce the description of any block of $\mathscr{R}_R(\mathrm{GL}_n(F))$ to that of a unipotent block. Unfortunately the description of the unipotent block of $\mathscr{R}_R(\mathrm{GL}_n(F))$, or of $\mathscr{R}_R(G)$, is nowadays a hard question and it has no answer yet.

We now summarize the contents of each section of this paper. In Section 1 we present general results on the convolution Hecke algebras $\mathscr{H}_R(\mathtt{G}, \sigma)$ where $\mathtt{G}$ is an arbitrary locally profinite group and σ a representation of an open subgroup $\mathtt{H}$ of $\mathtt{G}$. We see that if σ is finitely generated then $\mathscr{H}_R(\mathtt{G}, \sigma)$ is isomorphic to the endomorphism algebra of $\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}\sigma$. We define two subcategories of $\mathscr{R}_R(\mathtt{G})$ and prove that, when they coincide, they are equivalent to the category of modules over $\mathscr{H}_R(\mathtt{G}, \sigma)$. In Section 2 we introduce the theory of maximal simple types; we consider the Heisenberg representation η associated to a simple character (see Section 2A) and define the groups $B^\times = B_G^\times$ and $K^1 = K_G^1$. In Section 3 we prove that the algebras $\mathscr{H}_R(G, \eta)$ and $\mathscr{H}_R(B^\times, K^1)$ are isomorphic. In Section 4 we introduce the theory of semisimple types, define the representation $\boldsymbol{\eta}_{\max}$ and the group $B_L^\times$, and prove that the algebras $\mathscr{H}_R(B_L^\times, K_L^1)$ and $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ are isomorphic. In Section 5 we prove that $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}$ and $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ are equivalences of categories; we describe the correspondence between blocks of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and of $\mathscr{R}(B_L^\times, K_L^1)$ and investigate the dependence of these results on the choice of the extension of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$.

1. Preliminaries

This section is written in much more generality than the remainder of the paper. We present general results for an arbitrary locally profinite group.

Let $\mathtt{G}$ be a locally profinite group (i.e., a locally compact and totally disconnected topological group) and let R be a unitary commutative ring. We recall that a representation (π, V) of $\mathtt{G}$ over R is smooth if for every $v \in V$ the stabilizer $\{g \in \mathtt{G} \mid \pi(g)v = v\}$ is an open subgroup of $\mathtt{G}$. We denote by $\mathscr{R}_R(\mathtt{G})$ the (abelian) category of smooth representations of $\mathtt{G}$ over R. From now on all representations considered are smooth.

1A. ***Hecke algebras for a locally profinite group.*** In this section we introduce an algebra associated to a representation σ of a subgroup of $\mathtt{G}$ and we prove that it is isomorphic to the endomorphism algebra of the compact induction of σ. This definition generalizes those in Section 1 of [Chinello 2017] that corresponds to the case in which σ is trivial.

Let $\mathtt{H}$ be an open subgroup of $\mathtt{G}$ such that every $\mathtt{H}$-double coset is a finite union of left $\mathtt{H}$-cosets (or equivalently $\mathtt{H} \cap g\mathtt{H}g^{-1}$ is of finite index in $\mathtt{H}$ for every $g \in \mathtt{G}$) and let (σ, V_σ) be a smooth representation of $\mathtt{H}$ over R.

Definition 1.1. Let $\mathscr{H}_R(\mathtt{G}, \sigma)$ be the R-algebra of functions $\Phi : \mathtt{G} \to \mathrm{End}_R(V_\sigma)$ such that $\Phi(hgh') = \sigma(h) \circ \Phi(g) \circ \sigma(h')$ for every $h, h' \in \mathtt{H}$ and $g \in \mathtt{G}$ and whose supports are a finite union of $\mathtt{H}$-double cosets, endowed with convolution product

$$(\Phi_1 * \Phi_2)(g) = \sum_x \Phi_1(x)\Phi_2(x^{-1}g), \tag{1}$$

where x runs over a system of representatives of $\mathtt{G}/\mathtt{H}$ in $\mathtt{G}$. This algebra is unitary and the identity element is σ seen as a function on $\mathtt{G}$ with support equal to $\mathtt{H}$. To simplify the notation, from now on we denote $\Phi_1\Phi_1 = \Phi_1 * \Phi_2$ for all $\Phi_1, \Phi_2 \in \mathscr{H}_R(\mathtt{G}, \sigma)$.

We observe that the sum in (1) is finite since the support of Φ_1 is a finite union of $\mathtt{H}$-double cosets and by hypothesis, every $\mathtt{H}$-double coset is a finite union of left $\mathtt{H}$-cosets. Furthermore, the formula (1) is well defined because for every $h \in \mathtt{H}$ and $x, g \in \mathtt{G}$ we have

$$\Phi_1(xh)\Phi_2((xh)^{-1}g) = \Phi_1(x) \circ \sigma(h) \circ \sigma(h^{-1}) \circ \Phi_2(x^{-1}g) = \Phi_1(x) \circ \Phi_2(x^{-1}g).$$

For every $g \in \mathtt{G}$ we denote by $\mathscr{H}_R(\mathtt{G}, \sigma)_{\mathtt{H}g\mathtt{H}}$ the submodule of $\mathscr{H}_R(\mathtt{G}, \sigma)$ of functions with support in $\mathtt{H}g\mathtt{H}$. If $g_1, g_2 \in \mathtt{G}$, $\Phi_1 \in \mathscr{H}_R(\mathtt{G}, \sigma)_{\mathtt{H}g_1\mathtt{H}}$ and $\Phi_2 \in \mathscr{H}_R(\mathtt{G}, \sigma)_{\mathtt{H}g_2\mathtt{H}}$ then the support of $\Phi_1\Phi_2$ is in $\mathtt{H}g_1\mathtt{H}g_2\mathtt{H}$ and the support of $x \mapsto \Phi_1(x)\Phi_2(x^{-1}g)$ is in $\mathtt{H}g_1\mathtt{H} \cap g\mathtt{H}g_2^{-1}\mathtt{H}$.

Remark 1.2. If g_1 or g_2 normalizes $\mathtt{H}$ then the support of $\Phi_1\Phi_2$ is in $\mathtt{H}g_1g_2\mathtt{H}$ and the support of $x \mapsto \Phi_1(x)\Phi_2(x^{-1}g_1g_2)$ is in $g_1\mathtt{H}$. Hence, we obtain $(\Phi_1\Phi_2)(g_1g_2) = \Phi_1(g_1) \circ \Phi_2(g_2)$.

For every $g \in \mathtt{G}$ we denote by $\mathtt{H}^g = g^{-1}\mathtt{H}g$ and (σ^g, V_σ) the representation of $\mathtt{H}^g$ given by $\sigma^g(x) = \sigma(gxg^{-1})$ for every $x \in \mathtt{H}^g$. We denote by $I_g(\sigma)$ the R-module $\mathrm{Hom}_{\mathtt{H}\cap\mathtt{H}^g}(\sigma, \sigma^g)$ and $I_{\mathtt{G}}(\sigma)$ the set, called the *intertwining* of σ in $\mathtt{G}$, of $g \in \mathtt{G}$ such that $I_g(\sigma) \neq 0$. For every $g \in I_{\mathtt{G}}(\sigma)$ the map $\Phi \mapsto \Phi(g)$ is an isomorphism of R-modules between $\mathscr{H}_R(\mathtt{G}, \sigma)_{\mathtt{H}g\mathtt{H}}$ and $I_g(\sigma)$ and so $g \in \mathtt{G}$ intertwines σ if and only if there exists an element $\Phi \in \mathscr{H}_R(\mathtt{G}, \sigma)$ such that $\Phi(g) \neq 0$.

Let $\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma)$ be the compactly induced representation of σ to $\mathtt{G}$. It is the R-module of functions $f : \mathtt{G} \to V_\sigma$, compactly supported modulo $\mathtt{H}$, such that $f(hg) = \sigma(h)f(g)$ for every $h \in \mathtt{H}$ and $g \in \mathtt{G}$ endowed with the action of $\mathtt{G}$ defined by $x.f : g \mapsto f(gx)$ for every $x, g \in \mathtt{G}$ and $f \in \mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma)$. We remark that, since $\mathtt{H}$ is open, by I.5.2(b) of [Vignéras 1996] it is a smooth representation of $\mathtt{G}$. For every $v \in V_\sigma$ let $i_v \in \mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma)$ be the function with support in $\mathtt{H}$ defined by $i_v(h) = \sigma(h)v$ for every $h \in \mathtt{H}$. Then for every $x \in \mathtt{G}$ the function $x^{-1}.i_v$ has support $\mathtt{H}x$ and takes the value v on x. Hence, for every $f \in \mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma)$ we have

$$f = \sum_{x \in \mathtt{H}\backslash\mathtt{G}} x^{-1}.i_{f(x)} \tag{2}$$

with the sum finite since the support of f is compact modulo H, and so the image i_{V_σ} of $v \mapsto i_v$ generates $\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma)$ as representation of $\mathtt{G}$.

Frobenius reciprocity (I.5.7 of [Vignéras 1996]) states that the map $\mathrm{Hom}_{\mathtt{H}}(\sigma, V) \to \mathrm{Hom}_{\mathtt{G}}(\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma), V)$ given by $\phi \mapsto \psi$ where $\phi(v) = \psi(i_v)$ for every $v \in V_\sigma$ is an isomorphism of R-modules.

Lemma 1.3. *If V_σ is a finitely generated R-module, the map $\xi : \mathscr{H}_R(\mathtt{G}, \sigma) \to \mathrm{End}_{\mathtt{G}}(\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma))$ given by*

$$\xi(\Phi)(f)(g) = (\Phi * f)(g) = \sum_{x \in \mathtt{G}/\mathtt{H}} \Phi(x)f(x^{-1}g)$$

for every $\Phi \in \mathscr{H}_R(\mathtt{G}, \sigma)$, $f \in \mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma)$ and $g \in \mathtt{G}$ is an R-algebra isomorphism whose inverse is given by $\xi^{-1}(\vartheta)(g)(v) = \vartheta(i_v)(g)$ for every $\vartheta \in \mathrm{End}_{\mathtt{G}}(\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\sigma))$, $g \in \mathtt{G}$ and $v \in V_\sigma$.

Proof. See I.8.5–6 of [Vignéras 1996]. □

1B. ***The categories $\mathscr{R}_\sigma(\mathtt{G})$ and $\mathscr{R}(\mathtt{G},\sigma)$.*** In this section we associate to an irreducible projective representation of a compact open subgroup of $\mathtt{G}$ two subcategories of $\mathscr{R}_R(\mathtt{G})$.

Let $\mathtt{K}$ be a compact open subgroup of $\mathtt{G}$ and (σ, V_σ) be an irreducible projective representation of $\mathtt{K}$ such that V_σ is a finitely generated R-module. Then $\rho = \mathrm{ind}_{\mathtt{K}}^{\mathtt{G}}(\sigma)$ is a projective representation of $\mathtt{G}$ by I.5.9(d) of [Vignéras 1996] and so the functor

$$\boldsymbol{M}_\sigma = \mathrm{Hom}_{\mathtt{G}}(\rho, -) : \mathscr{R}_R(\mathtt{G}) \to \text{Mod-}\mathscr{H}_R(\mathtt{G},\sigma)$$

is exact. We remark that for every representation (π, V) of $\mathtt{G}$ the right-action of $\Phi \in \mathscr{H}_R(\mathtt{G},\sigma)$ on $\varphi \in \mathrm{Hom}_{\mathtt{G}}(\rho, V)$ is given by $\varphi.\Phi = \varphi \circ \xi(\Phi)$ where ξ is the isomorphism of Lemma 1.3. Moreover, if V_1 and V_2 are representations of $\mathtt{G}$ and $\epsilon \in \mathrm{Hom}_{\mathtt{G}}(V_1, V_2)$ then $\boldsymbol{M}_\sigma(\phi)$ maps φ to $\phi \circ \varphi$ for every $\varphi \in \mathrm{Hom}_{\mathtt{G}}(\rho, V_1)$.

Definition 1.4. Let $\mathscr{R}_\sigma(\mathtt{G})$ be the full subcategory of $\mathscr{R}_R(\mathtt{G})$ whose objects are representations V such that $\boldsymbol{M}_\sigma(V') \neq 0$ for every irreducible subquotient V' of V.

For every representation V of $\mathtt{G}$ we denote by $V^\sigma = \sum_{\phi \in \mathrm{Hom}_{\mathtt{K}}(\sigma, V)} \phi(\sigma)$ which is a subrepresentation of the restriction of V to $\mathtt{K}$. We denote by $V[\sigma]$ the representation of $\mathtt{G}$ generated by V^σ. If σ is the trivial character of $\mathtt{K}$ then $V^\sigma = V^{\mathtt{K}} = \{v \in V \mid \pi(k)v = v \text{ for all } k \in \mathtt{K}\}$ is the set of $\mathtt{K}$-invariant vectors of V.

Proposition 1.5. *For every representation V of $\mathtt{G}$ we have $V[\sigma] = \sum_{\psi \in \boldsymbol{M}_\sigma(V)} \psi(\rho)$ and so $\boldsymbol{M}_\sigma(V) = \boldsymbol{M}_\sigma(V[\sigma])$. Moreover, if W is a subrepresentation of V then $\boldsymbol{M}_\sigma(W) = \boldsymbol{M}_\sigma(V)$ if and only if $W[\sigma] = V[\sigma]$.*

Proof. By Frobenius reciprocity we have $\mathrm{Hom}_{\mathtt{K}}(\sigma, V) \cong \boldsymbol{M}_\sigma(V)$ and so using (2) we obtain

$$V[\sigma] = \sum_{g \in \mathtt{G}} \pi(g) \sum_{\psi \in \boldsymbol{M}_\sigma(V)} \psi(i_{V_\sigma}) = \sum_{\psi \in \boldsymbol{M}_\sigma(V)} \psi\Big(\sum_{g \in \mathtt{G}} g.i_{V_\sigma}\Big) = \sum_{\psi \in \boldsymbol{M}_\sigma(V)} \psi(\rho),$$

which implies $\boldsymbol{M}_\sigma(V) = \boldsymbol{M}_\sigma(V[\sigma])$. Furthermore, if $W[\sigma] = V[\sigma]$ then $\boldsymbol{M}_\sigma(W) = \boldsymbol{M}_\sigma(V)$ and if $\boldsymbol{M}_\sigma(W) = \boldsymbol{M}_\sigma(V)$ then

$$W[\sigma] = \sum_{\psi \in \boldsymbol{M}_\sigma(W)} \psi(\rho) = \sum_{\psi \in \boldsymbol{M}_\sigma(V)} \psi(\rho) = V[\sigma]. \qquad \square$$

Definition 1.6. Let $\mathscr{R}(\mathtt{G},\sigma)$ be the full subcategory of $\mathscr{R}_R(\mathtt{G})$ whose objects are representations V such that $V = V[\sigma]$. If σ is the trivial character of $\mathtt{K}$ we denote by $\mathscr{R}(\mathtt{G},\mathtt{K})$ the subcategory of representations V generated by $V^{\mathtt{K}}$.

Proposition 1.7. *Let V be a representation of $\mathtt{G}$. The following conditions are equivalent*:

(i) *For every irreducible subquotient U of V we have $\boldsymbol{M}_\sigma(U) \neq 0$.*

(ii) *For every nonzero subquotient W of V we have $\boldsymbol{M}_\sigma(W) \neq 0$.*

(iii) *For every subquotient Z of V we have $Z = Z[\sigma]$.*

(iv) *For every subrepresentation Z of V we have $Z = Z[\sigma]$.*

Proof. (i)→(ii): Let W be a nonzero subquotient of V and $W_1 \subset W_2$ two subrepresentations of W such that $U = W_2/W_1$ is irreducible. By (i) we have $\boldsymbol{M}_\sigma(U) \neq 0$ which implies $\boldsymbol{M}_\sigma(W_2) \neq 0$ and so $\boldsymbol{M}_\sigma(W) \neq 0$.

(ii)→(iii): Let Z be a subquotient of V. By Proposition 1.5 we have $\boldsymbol{M}_\sigma(Z) = \boldsymbol{M}_\sigma(Z[\sigma])$ and so $\boldsymbol{M}_\sigma(Z/Z[\sigma]) = 0$. Hence, by (ii) we obtain $Z = Z[\sigma]$.

(iv)→(i): Let U be an irreducible subquotient of V and $Z_1 \subsetneq Z_2$ be two subrepresentations of V such that $U = Z_2/Z_1$. By (iv) we have $Z_1[\sigma] = Z_1 \neq Z_2 = Z_2[\sigma]$ and by Proposition 1.5 we have $\boldsymbol{M}_\sigma(Z_1) \neq \boldsymbol{M}_\sigma(Z_2)$. Hence, we obtain $\boldsymbol{M}_\sigma(U) \neq 0$. □

Remark 1.8. Proposition 1.7 implies that $\mathscr{R}_\sigma(\mathtt{G})$ is a subcategory of $\mathscr{R}(\mathtt{G}, \sigma)$.

1C. *Equivalence of categories.* In this section we suppose that there exists a compact open subgroup $\mathtt{K}_0$ of $\mathtt{G}$ whose pro-order is invertible in R and we consider the Haar measure $\mathtt{dg}$ on $\mathtt{G}$ with values in R such that $\int_{\mathtt{K}_0} \mathtt{dg} = 1$ (see I.2 of [Vignéras 1996]). We prove that if the two categories introduced in Section 1B are equal then they are equivalent to the category of modules over the algebra introduced in Section 1A.

The *global Hecke algebra* $\mathscr{H}_R(\mathtt{G})$ of $\mathtt{G}$ is the R-algebra of locally constant and compactly supported functions $f : \mathtt{G} \to R$ endowed with convolution product given by $(f_1 * f_2)(x) = \int_{\mathtt{G}} f_1(g) f_2(g^{-1}x)\, \mathtt{dg}$ for every $f_1, f_2 \in \mathscr{H}_R(\mathtt{G})$ and $x \in \mathtt{G}$ (see I.3.1 of [Vignéras 1996]). In general $\mathscr{H}_R(\mathtt{G})$ is not unitary but it has enough idempotents by I.3.2 of [loc. cit.]. The categories $\mathscr{R}_R(\mathtt{G})$ and $\mathscr{H}_R(\mathtt{G})$-Mod are equivalent by I.4.4 of [loc. cit.] and we have $\mathrm{ind}_{\mathtt{H}}^{\mathtt{G}}(\tau) = \mathscr{H}_R(\mathtt{G}) \otimes_{\mathscr{H}_R(\mathtt{H})} V_\tau$ for every representation (τ, V_τ) of an open subgroup $\mathtt{H}$ of $\mathtt{G}$ by I.5.2 of [loc. cit.].

Let $\mathtt{K}$ be a compact open subgroup of $\mathtt{G}$, let (σ, V_σ) be an irreducible projective representation of $\mathtt{K}$ as in Section 1B and let $\rho = \mathrm{ind}_{\mathtt{K}}^{\mathtt{G}}(\sigma)$. Since V_σ is a simple projective module over the unitary algebra $\mathscr{H}_R(\mathtt{K})$, it is isomorphic to a direct summand of $\mathscr{H}_R(\mathtt{K})$ itself because any nonzero map $\mathscr{H}_R(\mathtt{K}) \to V_\sigma$ is surjective and splits. Then it is isomorphic to a minimal ideal of $\mathscr{H}_R(\mathtt{K})$ and so there exists an idempotent e of $\mathscr{H}_R(\mathtt{K})$ such that $V_\sigma = \mathscr{H}_R(\mathtt{K})e$. Hence, we obtain $\rho = \mathscr{H}_R(\mathtt{G})e$ because the map $\sum_i (f_i \otimes h_i e) \mapsto \big(\sum_i f_i h_i\big)e$ is an isomorphism of $\mathscr{H}_R(\mathtt{G})$-modules between $\mathscr{H}_R(\mathtt{G}) \otimes_{\mathscr{H}_R(\mathtt{K})} \mathscr{H}_R(\mathtt{K})e$ and $\mathscr{H}_R(\mathtt{G})e$ whose inverse is $fe \mapsto fe \otimes e$.

The algebra $\mathscr{H}_R(\mathtt{G}, \sigma)$ is isomorphic to $\mathrm{End}_{\mathtt{G}}(\rho) \cong \mathrm{End}_{\mathscr{H}_R(\mathtt{G})}(\mathscr{H}_R(\mathtt{G})e)$ by Lemma 1.3 and the map $e\mathscr{H}_R(\mathtt{G})e \to (\mathrm{End}_{\mathscr{H}_R(\mathtt{G})}(\mathscr{H}_R(\mathtt{G})e))^{\mathrm{op}}$ which maps $efe \in e\mathscr{H}_R(\mathtt{G})e$ to the endomorphism $f'e \mapsto f'efe$ of $\mathscr{H}_R(\mathtt{G})e$ is an algebra isomorphism whose inverse is $\varphi \mapsto \varphi(e)$. Then we have $\mathscr{H}_R(\mathtt{G}, \sigma)^{\mathrm{op}} \cong e\mathscr{H}_R(\mathtt{G})e$ and so the categories $e\mathscr{H}_R(\mathtt{G})e$-Mod and Mod-$\mathscr{H}_R(\mathtt{G}, \sigma)$ are equivalent.

Theorem 1.9. *If* $\mathscr{R}_\sigma(\mathtt{G}) = \mathscr{R}(\mathtt{G}, \sigma)$ *then* $V \mapsto \boldsymbol{M}_\sigma(V)$ *is an equivalence of categories between* $\mathscr{R}(\mathtt{G}, \sigma)$ *and* Mod-$\mathscr{H}_R(\mathtt{G}, \sigma)$ *whose quasiinverse is* $W \mapsto W \otimes_{\mathscr{H}_R(\mathtt{G},\sigma)} \rho$.

Proof. We take $A = \mathscr{H}_R(\mathtt{G})$ and $\mathscr{H}_R(\mathtt{G})e = \rho$ as in I.6.6 of [Vignéras 1996]. Since $\mathscr{H}_R(\mathtt{G}, \sigma)^{\mathrm{op}} \cong e\mathscr{H}_R(\mathtt{G})e$, left-actions of $e\mathscr{H}_R(\mathtt{G})e$ become right-actions of $\mathscr{H}_R(\mathtt{G}, \sigma)$. The functor $V \mapsto eV$ of [loc. cit.] from $\mathscr{H}_R(\mathtt{G})$-Mod to $e\mathscr{H}_R(\mathtt{G})e$-Mod becomes the functor $V \mapsto \mathrm{Hom}_{\mathscr{H}_R(\mathtt{G})}(\mathscr{H}_R(\mathtt{G})e, V)$ and so the functor $\boldsymbol{M}_\sigma$. The hypotheses of the theorem "*équivalence de catégories*" in I.6.6 of [Vignéras 1996] are satisfied by the condition $\mathscr{R}_\sigma(\mathtt{G}) = \mathscr{R}(\mathtt{G}, \sigma)$ and so we obtain the result. □

2. Maximal simple types

In this section we introduce the theory of simple types of an inner form of a general linear group over a nonarchimedean locally compact field in the case of modular representations. We refer to Sections 2.1–5 of [Mínguez and Sécherre 2014b] for more details.

Let p be a prime number and let F be a nonarchimedean locally compact field of residue characteristic p. For F' a finite extension of F, or more generally a division algebra over a finite extension of F, we denote by $\mathcal{O}_{F'}$ its ring of integers, by $\varpi_{F'}$ a uniformizer of $\mathcal{O}_{F'}$, by $\wp_{F'}$ the maximal ideal of $\mathcal{O}_{F'}$ and by $\mathfrak{k}_{F'}$ its residue field. Let D be a central division algebra of finite dimension over F whose reduced degree is denoted by d. Given a positive integer m, we consider the ring $A = M_m(D)$ and the group $G = \mathrm{GL}_m(D)$ which is an inner form of $\mathrm{GL}_{md}(F)$. Let R be an algebraically closed field of characteristic different from p.

Let Λ be an $\mathcal{O}_D$-lattice sequence of $V = D^m$. It defines a hereditary $\mathcal{O}_F$-order $\mathfrak{A} = \mathfrak{A}(\Lambda)$ of A whose radical is denoted by $\mathfrak{P}$, a compact open subgroup $U(\Lambda) = U_0(\Lambda) = \mathfrak{A}(\Lambda)^\times$ of G and a filtration $U_k(\Lambda) = 1 + \mathfrak{P}^k$ with $k \geq 1$ of $U(\Lambda)$ (see Section 1 of [Sécherre 2004]). Let $[\Lambda, n, 0, \beta]$ be a simple stratum of A (see for instance Section 1.6 of [Sécherre and Stevens 2008]). Then $\beta \in A$ and the F-subalgebra $F[\beta]$ of A generated by β is a field denoted by E. The centralizer B of E in A is a simple central E-algebra and $\mathfrak{B} = \mathfrak{A} \cap B$ is a hereditary $\mathcal{O}_E$-order of B whose radical is $\mathfrak{Q} = \mathfrak{P} \cap B$.

As in Sections 1.2 and 1.3 of [Sécherre 2005b] we can choose a simple right $E \otimes_F D$-module N such that the functor $V \mapsto \mathrm{Hom}_{E\otimes_F D}(N, V)$ defines a Morita equivalence between the category of modules over $E \otimes_F D$ and the category of vector spaces over $D' = \mathrm{End}_{E\otimes_F D}(N)^{\mathrm{op}}$ which is a central division algebra over E. We set $A(E) = \mathrm{End}_D(N)$ which is a central simple F-algebra. If d' is the reduced degree of D' over E and m' is the dimension of $V' = \mathrm{Hom}_{E\otimes_F D}(N, V)$ over D', then we have $m'd' = md/[E : F]$. Fixing a basis of V' over D' we obtain, via the Morita equivalence above, an isomorphism $N^{m'} \cong V$ of $E \otimes_F D$-modules. If for every $i \in \{1, \dots, m'\}$ we denote by V^i the image of the i-th copy of N by this isomorphism, we obtain a decomposition $V = V^1 \oplus \cdots \oplus V^{m'}$ into simple $E \otimes_F D$-submodules. By Section 1.5 of [Sécherre 2005b] we can choose a basis $\mathscr{B}$ of V' over D' so that Λ decomposes as the direct sum of the $\Lambda^i = \Lambda \cap V^i$ for $i \in \{1, \dots, m'\}$. For every $i \in \{1, \dots, m'\}$, let $\mathrm{e}_i : V \to V^i$ be the projection on V^i with kernel $\bigoplus_{j\neq i} V^j$. In accordance with [Sécherre 2004, 2.3.1] (see also [Bushnell and Henniart 1996]) the family of idempotents $\mathrm{e} = (\mathrm{e}_1, \dots, \mathrm{e}_{m'})$ is a decomposition which conforms to Λ over E.

By 1.4.8 and 1.5.2 of [Sécherre 2005b] there exists a unique hereditary order $\mathfrak{A}(E)$ normalized by $E^\times$ in $A(E)$ whose radical is denoted by $\mathfrak{P}(E)$. For every $i \in \{1, \dots, m'\}$ we have an isomorphism $\mathrm{End}_D(V^i) \cong A(E)$ of F-algebras which induces an isomorphism of $\mathcal{O}_F$-algebras between the hereditary orders $\mathfrak{A}(\Lambda^i)$ and $\mathfrak{A}(E)$. Moreover, to the choice of the basis $\mathscr{B}$ corresponds the isomorphisms $M_{m'}(D') \cong B$ of E-algebras and $M_{m'}(A(E)) \cong A$ of F-algebras.

Remark 2.1. If $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup of $B^\times$, these isomorphisms induce an isomorphism $\mathfrak{B} \cong M_{m'}(\mathcal{O}_{D'})$ of $\mathcal{O}_E$-algebras and, by Lemma 1.6 of [Sécherre 2005a], two isomorphisms $\mathfrak{A} \cong M_{m'}(\mathfrak{A}(E))$ and $\mathfrak{P} \cong M_{m'}(\mathfrak{P}(E))$ of $\mathcal{O}_F$-algebras.

We can associate to $[\Lambda, n, 0, \beta]$ two compact open subgroups $J = J(\beta, \Lambda)$, $H = H(\beta, \Lambda)$ of $U(\Lambda)$ (see 2.4 of [Sécherre and Stevens 2008]). For every integer $k \geq 1$ we set $J^k = J^k(\beta, \Lambda) = J(\beta, \Lambda) \cap U_k(\Lambda)$ and $H^k = H^k(\beta, \Lambda) = H(\beta, \Lambda) \cap U_k(\Lambda)$ which are pro-p-groups. In particular J^1 and H^1 are normal pro-p-subgroups of J and the quotient J^1/H^1 is a finite abelian p-group.

Remark 2.2. We have $J = (U(\Lambda) \cap B^\times)J^1$ and this induce a canonical group isomorphism

$$J/J^1 \cong (U(\Lambda) \cap B^\times)/(U_1(\Lambda) \cap B^\times)$$

(see Section 2.3 of [Mínguez and Sécherre 2014b]). It allows us to associate canonically and bijectively a representation of J trivial on J^1 to a representation of $U(\Lambda) \cap B^\times$ trivial on $U_1(\Lambda) \cap B^\times$.

2A. ***Simple characters, Heisenberg representation and β-extensions.*** Let $[\Lambda, n, 0, \beta]$ be a simple stratum of A. We denote by $\mathscr{C}_R(\Lambda, 0, \beta)$ the set of *simple R-characters* (see Section 2.2 of [Mínguez and Sécherre 2014b] and [Sécherre 2004]) that is a finite set of R-characters of H^1 which depends on the choice of an additive R-character of F which has been fixed once and for all. If $\tilde{m} \in \mathbb{N}^*$ and $[\tilde{\Lambda}, \tilde{n}, 0, \tilde{\beta}]$ is a simple stratum of $M_{\tilde{m}}(D)$ such that there exists an isomorphism of F-algebras $\nu : F[\beta] \to F[\tilde{\beta}]$ with $\nu(\beta) = \tilde{\beta}$, then there exists a bijection $\mathscr{C}_R(\Lambda, 0, \beta) \to \mathscr{C}_R(\tilde{\Lambda}, 0, \tilde{\beta})$ canonically associated to ν, called the *transfer map*. There also exists an equivalence relation, called *endoequivalence*, among simple characters in $\mathscr{C}_R(\Lambda, 0, \beta)$ (see [Broussous et al. 2012]) such that two of them are endoequivalent if they have transfers which intertwine. The equivalence classes of this relation are called *endoclasses*. Let $\theta \in \mathscr{C}_R(\Lambda, 0, \beta)$. By Proposition 2.1 of [Mínguez and Sécherre 2014b] there exists a finite dimensional irreducible representation η of J^1, unique up to isomorphism, whose restriction to H^1 contains θ. It is called the *Heisenberg representation* associated to θ. The intertwining of η is $I_G(\eta) = J^1B^\times J^1 = JB^\times J$ and for every $y \in B^\times$ the R-vector space $I_y(\eta) = \mathrm{Hom}_{J^1 \cap (J^1)^y}(\eta, \eta^y)$ has dimension 1.

A *β-extension* of η (or of θ) is an irreducible representation κ of J extending η such that $I_G(\kappa) = JB^\times J$. By Proposition 2.4 of [Mínguez and Sécherre 2014b], every simple character $\theta \in \mathscr{C}_R(\Lambda, 0, \beta)$ admits a β-extension κ and by formula (2.2) of [Mínguez and Sécherre 2014b] the set of β-extensions of θ is equal to

$$\mathcal{B}(\theta) = \{\kappa \otimes (\chi \circ N_{B/E}) \mid \chi \text{ is a character of } \mathcal{O}_E^\times, \text{ trivial on } 1 + \wp_E\},$$

where $N_{B/E}$ is the reduced norm of B over E and $\chi \circ N_{B/E}$ is seen as a character of J trivial on J^1 thanks to Remark 2.2. We observe that for every $\kappa \in \mathcal{B}(\theta)$ and every $y \in B^\times$, the R-vector space $I_y(\kappa)$ has dimension 1 because it is nonzero and it is contained in $I_y(\eta)$.

2B. ***Maximal simple types.*** Let $[\Lambda, n, 0, \beta]$ be a simple stratum of A such that $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup of $B^\times$. By Remarks 2.1 and 2.2, there exists a group isomorphism $J/J^1 \cong \mathrm{GL}_{m'}(\mathfrak{k}_{D'})$, which depends on the choice of $\mathscr{B}$.

A *maximal simple type* of G associated to $[\Lambda, n, 0, \beta]$ is a pair (J, λ) where λ is an irreducible representation of J of the form $\lambda = \kappa \otimes \sigma$ where $\kappa \in \mathcal{B}(\theta)$ with $\theta \in \mathscr{C}_R(\Lambda, 0, \beta)$ and σ is a cuspidal representation of $\mathrm{GL}_{m'}(\mathfrak{k}_{D'})$ identified with an irreducible representation of J trivial on J^1. If σ is a supercuspidal representation of $\mathrm{GL}_{m'}(\mathfrak{k}_{D'})$ then (J, λ) is called maximal simple *supertype*.

Remark 2.3. The choice of a β-extension $\kappa \in \mathcal{B}(\theta)$ determines the decomposition $\lambda = \kappa \otimes \sigma$. If we choose another β-extension $\kappa' = \kappa \otimes (\chi \circ N_{B/E}) \in \mathcal{B}(\theta)$ we obtain the decomposition $\lambda = \kappa' \otimes \sigma'$ where $\sigma' = \sigma \otimes (\chi^{-1} \circ N_{B/E})$.

2C. *Covers.* Let $\mathcal{M}$ be a Levi subgroup of G, let $\mathcal{P}$ be a parabolic subgroup of G with Levi component $\mathcal{M}$ and unipotent radical $\mathcal{U}$ and let $\mathcal{U}^-$ be the unipotent subgroup opposite to $\mathcal{U}$. We say that a compact open subgroup K of G is *decomposed with respect to* $(\mathcal{M}, \mathcal{P})$ if every element $k \in K$ decomposes uniquely as $k = k_1k_2k_3$ with $k_1 \in K \cap \mathcal{U}^-$, $k_2 \in K \cap \mathcal{M}$ and $k_3 \in K \cap \mathcal{U}$. Furthermore, if π is a representation of K we say that the pair (K, π) is *decomposed with respect to* $(\mathcal{M}, \mathcal{P})$ if K is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and if $K \cap \mathcal{U}$ and $K \cap \mathcal{U}^-$ are in the kernel of π.

Let $\mathcal{M}$ be a Levi subgroup of G. Let K and $K_{\mathcal{M}}$ be two compact open subgroups of G and $\mathcal{M}$ respectively and let ϱ and $\varrho_{\mathcal{M}}$ be two irreducible representations of K and $K_{\mathcal{M}}$ respectively. We say that the pair (K, ϱ) is *decomposed above* $(K_{\mathcal{M}}, \varrho_{\mathcal{M}})$ if (K, ϱ) is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ for every parabolic subgroup $\mathcal{P}$ with Levi component $\mathcal{M}$, if $K \cap \mathcal{M} = K_{\mathcal{M}}$ and if the restriction of ϱ to $K_{\mathcal{M}}$ is equal to $\varrho_{\mathcal{M}}$. For a parabolic subgroup $\mathcal{P}$ of G with Levi component $\mathcal{M}$ and unipotent radical $\mathcal{U}$, let $\varrho_{\mathcal{U}}$ be the Jacquet module of ϱ and $r_{\mathcal{U}}$ be the canonical quotient map $\varrho \to \varrho_{\mathcal{U}}$. A pair (K, ϱ) is a *cover* of $(K_{\mathcal{M}}, \varrho_{\mathcal{M}})$ if it is decomposed above $(K_{\mathcal{M}}, \varrho_{\mathcal{M}})$ and if for every irreducible representations π of G the map $\mathrm{Hom}_K(\varrho, \pi) \to \mathrm{Hom}_{K_{\mathcal{M}}}(\varrho_{\mathcal{M}}, \pi_{\mathcal{U}})$, given by $\varphi \mapsto r_{\mathcal{U}} \circ \varphi$ for every $\varphi \in \mathrm{Hom}_K(\varrho, \pi)$, is injective (see Condition (0.5) of [Blondel 2005]). For more details see [Blondel 2005; Vignéras 1998].

3. The isomorphisms $\mathscr{H}_R(G, \eta) \cong \mathscr{H}_R(B^\times, U_1(\Lambda) \cap B^\times)$

Using the notation of Section 2, let $[\Lambda, n, 0, \beta]$ be a simple stratum of A such that $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup of $B^\times$. Let $\theta \in \mathscr{C}_R(\Lambda, 0, \beta)$ and let η be the Heisenberg representation associated to θ. In this section we want to prove that the algebras $\mathscr{H}_R(G, \eta)$ and $\mathscr{H}_R(B^\times, U_1(\Lambda) \cap B^\times)$ are isomorphic (Theorem 3.43).

Henceforth, for a given $m \in \mathbb{N}$, we denote by $\mathbb{I}_m$ the identity matrix of size m. Thanks to Section 2, from now on we identify A with $M_{m'}(A(E))$, G with $\mathrm{GL}_{m'}(A(E))$, $U(\Lambda)$ with $\mathrm{GL}_{m'}(\mathfrak{A}(E))$, $U_1(\Lambda)$ with $\mathbb{I}_{m'} + M_{m'}(\mathfrak{P}(E))$, $B^\times$ with $\mathrm{GL}_{m'}(D')$, $K_B = U(\Lambda) \cap B^\times$ with $\mathrm{GL}_{m'}(\mathcal{O}_{D'})$ and $K_B^1 = U_1(\Lambda) \cap B^\times$ with $\mathbb{I}_{m'} + M_{m'}(\wp_{D'})$. By Section 2.4 of [Chinello 2017] we know a presentation by generators and relations of the algebra $\mathscr{H}_R(B^\times, K_B^1) \cong \mathscr{H}_{\mathbb{Z}}(B^\times, K_B^1) \otimes_{\mathbb{Z}} R$. Using this presentation we want to find an isomorphism between $\mathscr{H}_R(B^\times, K_B^1)$ and $\mathscr{H}_R(G, \eta)$.

3A. *Root system of* $\mathrm{GL}_{m'}$. In this section we recall some notation and results on the root system of $\mathrm{GL}_{m'}$ contained in Section 2.1 of [Chinello 2017].

We denote by $\mathbf{\Phi} = \{\alpha_{ij} \mid 1 \le i \ne j \le m'\}$ the set of roots of $\mathrm{GL}_{m'}$ relative to the torus of diagonal matrices. Let $\mathbf{\Phi}^+ = \{\alpha_{ij} \mid 1 \le i < j \le m'\}$, $\mathbf{\Phi}^- = -\mathbf{\Phi}^+ = \{\alpha_{ij} \mid 1 \le j < i \le m'\}$ and $\Sigma = \{\alpha_{i,i+1} \mid 1 \le i \le m' - 1\}$

be, respectively, the sets of positive, negative and simple roots relative to the Borel subgroup of upper triangular matrices. For every $\alpha = \alpha_{i,i+1} \in \Sigma$ we write s_α or s_i for the transposition $(i, i+1)$. Let W be the group generated by the s_i which is the group of permutations of m' elements and so the Weyl group of $\mathrm{GL}_{m'}$. Let $\ell : W \to \mathbb{N}$ be the length function of W relative to $s_1, \dots, s_{m'-1}$. The group W acts on $\boldsymbol{\Phi}$ by $w\alpha_{ij} = \alpha_{w(i)w(j)}$ and for every $w \in W$ and $\alpha \in \Sigma$ we have (see (2.2) of [loc. cit.])

$$\ell(ws_\alpha) = \begin{cases} \ell(w)+1 & \text{if } w\alpha \in \boldsymbol{\Phi}^+, \\ \ell(w)-1 & \text{if } w\alpha \in \boldsymbol{\Phi}^-. \end{cases} \tag{3}$$

Remark 3.1. By Proposition 2.2 of [loc. cit.] we have $\ell(w) = |\boldsymbol{\Phi}^+ \cap w\boldsymbol{\Phi}^-| = |\boldsymbol{\Phi}^- \cap w\boldsymbol{\Phi}^+|$.

For every $P \subset \Sigma$ we denote by $\boldsymbol{\Phi}^+_P$ the set of positive roots generated by P, $\boldsymbol{\Phi}^-_P = -\boldsymbol{\Phi}^+_P$, $\Psi^+_P = \boldsymbol{\Phi}^+ \setminus \boldsymbol{\Phi}^+_P$ and $\Psi^-_P = -\Psi^+_P$. We denote by W_P the subgroup of W generated by the s_α with $\alpha \in P$ and by $\hat{P}$ the complement of P in Σ. We abbreviate $\hat{\alpha} = \widehat{\{\alpha\}}$.

Example. If $\alpha = \alpha_{i,i+1}$ then $\hat{\alpha} = \{\alpha_{j,j+1} \in \Sigma \mid j \neq i\}$, $\boldsymbol{\Psi}^+_{\hat{\alpha}} = \{\alpha_{hk} \in \boldsymbol{\Phi}^+ \mid 1 \le h \le i < k \le m'\}$ and $\boldsymbol{\Phi}^+_{\hat{\alpha}} = \{\alpha_{hk} \in \boldsymbol{\Phi}^+ \mid 1 \le h < k \le i \text{ or } i+1 \le h < k \le m'\}$.

Proposition 3.2. *Let $P \subset \Sigma$ and let w be an element of minimal length in $wW_P \in W/W_P$. Then $w\alpha \in \boldsymbol{\Phi}^+$ for every $\alpha \in \boldsymbol{\Phi}^+_P$ and for every $w' \in W_P$ we have $\ell(ww') = \ell(w) + \ell(w')$.*

Proof. Proposition 2.4 and Lemma 2.5 of [Chinello 2017]. □

Proposition 3.2 implies that in each class of W/W_P with $P \subset \Sigma$, there exists a unique element of minimal length and the same holds in each class of $W_P \backslash W$.

If ϖ is a uniformizer of $\mathcal{O}_{D'}$ we identify $\tau_i = \left(\begin{smallmatrix} \mathbb{I}_i & 0 \\ 0 & \varpi \mathbb{I}_{m'-i} \end{smallmatrix}\right)$ with $i \in \{0, \dots, m'\}$, defined in Section 2.2 of [loc. cit.], with elements of $B^\times$ and then of G. For $\alpha = \alpha_{i,i+1} \in \Sigma$ we write $\tau_\alpha = \tau_i$. Let $\boldsymbol{\Delta}$ and $\hat{\Delta}$ be the commutative monoid and group, respectively, generated by τ_α with $\alpha \in \Sigma$. Then we can write every element τ of $\boldsymbol{\Delta}$ uniquely as $\tau = \prod_{\alpha \in \Sigma} \tau_\alpha^{i_\alpha}$ with i_α in $\mathbb{N}$ and uniquely as $\tau = \mathrm{diag}(1, \varpi^{a_1}, \dots, \varpi^{a_{m-1}})$ with $0 \le a_1 \le \dots \le a_{m-1}$. In this case we set $P(\tau) = \{\alpha \in \Sigma \mid i_\alpha = 0\}$ and if $P \subset \{0, \dots, m'\}$ or if $P \subset \Sigma$ we write τ_P in place of $\prod_{x \in P} \tau_x$. We remark that if $P \subset \Sigma$ then $P(\tau_P) = \hat{P}$.

3B. ***The representation $\eta_{\mathcal{P}}$.*** Let $\mathcal{M} = A(E)^\times \times \dots \times A(E)^\times$ (m' copies) which is a Levi subgroup of G and let $\mathcal{P}$ be the parabolic subgroup of G of upper triangular matrices with Levi component $\mathcal{M}$ and unipotent radical $\mathcal{U}$. Let $\mathcal{P}^-$ be the opposite parabolic subgroup of $\mathcal{P}$ and $\mathcal{U}^-$ its unipotent radical.

We write $U = K_B \cap \mathcal{U}$, $M = K_B \cap \mathcal{M}$ and $I_B = K_B^1 MU$. Then U is the group of unipotent upper triangular matrices with coefficients in $\mathcal{O}_{D'}$, M is the group of diagonal matrices with coefficients in $\mathcal{O}_{D'}^\times$ and I_B is the standard Iwahori subgroup of K_B.

We denote by $\tilde{W}$ the group $W \ltimes \hat{\Delta}$ of monomial matrices with coefficients in $\varpi^{\mathbb{Z}}$ which is called the *extended affine Weyl group of $B^\times$*. We recall that $B^\times = I_B \tilde{W} I_B$ and actually it is the disjoint union of $I_B \tilde{w} I_B$ with $\tilde{w} \in \tilde{W}$.

Remark 3.3. By Proposition 2.16 of [Sécherre 2005a], which works for every decomposition that conforms to Λ over E and not necessarily subordinate to $\mathfrak{B}$, the groups J^1 and H^1 are decomposed with respect to $(\mathcal{M}, \mathcal{P})$. Moreover, if $\mathcal{M}' = \prod_{i=1}^{r} \mathrm{GL}_{m'_i}(A(E))$ with $\sum_{i=1}^{r} m'_i = m'$ is a standard Levi subgroup of G containing $\mathcal{M}$ and $\mathcal{P}'$ is the upper standard parabolic subgroup of G with Levi component $\mathcal{M}'$, then J^1 and H^1 are decomposed with respect to $(\mathcal{M}', \mathcal{P}')$.

Let $\mathfrak{J}^1 = \mathfrak{J}^1(\beta, \Lambda)$ and $\mathfrak{H}^1 = \mathfrak{H}^1(\beta, \Lambda)$ be the $\mathcal{O}_F$-lattices of A such that $J^1 = 1 + \mathfrak{J}^1$ and $H^1 = 1 + \mathfrak{H}^1$ (see Section 3.3 of [Sécherre 2004] or Chapter 3 of [Bushnell and Kutzko 1993]). Then they are $(\mathfrak{B}, \mathfrak{B})$-bimodules and we have $\varpi \mathfrak{J}^1 \subset \mathfrak{H}^1 \subset \mathfrak{J}^1 \subset M_{m'}(\mathfrak{P}(E))$.

Since $V^i \cong N$ for every $i \in \{1, \dots, m'\}$, we can identify every Λ^i to a lattice sequence Λ_0 of N with the same period as Λ, every $e^i \beta$ to an element $\beta_0 \in A(E)$ and $\mathfrak{A}(\Lambda_0)$ to $\mathfrak{A}(E)$. By Proposition 2.28 of [Sécherre 2004] the stratum $[\Lambda_0, n, 0, \beta_0]$ of $A(E)$ is simple and the critical exponents $k_0(\beta, \Lambda)$ and $k_0(\beta_0, \Lambda_0)$ are equal (for a definition of the critical exponent see Section 2.1 of [Sécherre 2004]). This implies that β is minimal (i.e., $-k_0(\beta, \Lambda) = n$) if and only if β_0 is minimal. We write $\mathfrak{J}_0^1 = \mathfrak{J}^1(\beta_0, \Lambda_0)$, $\mathfrak{H}_0^1 = \mathfrak{H}^1(\beta_0, \Lambda_0)$, $J_0^1 = J^1(\beta_0, \Lambda_0) = 1 + \mathfrak{J}_0^1$ and $H_0^1 = H^1(\beta_0, \Lambda_0) = 1 + \mathfrak{H}_0^1$.

Proposition 3.4. *We have* $\mathfrak{J}^1 = M_{m'}(\mathfrak{J}_0^1)$ *and* $\mathfrak{H}^1 = M_{m'}(\mathfrak{H}_0^1)$.

Proof. We prove the result only for $\mathfrak{J}^1$ since the case of $\mathfrak{H}^1$ is similar. We have to prove that for every $i, j \in \{1, \dots, m'\}$ we have $e^i \mathfrak{J}^1 e^j = \mathfrak{J}_0^1$. We need to recall the definition of $\mathfrak{J}(\beta, \Lambda) = \mathfrak{J}^0(\beta, \Lambda)$ and of $\mathfrak{J}^k(\beta, \Lambda)$ with $k \geq 1$. By Proposition 3.42 of [Sécherre 2004] if we set $q = -k_0(\beta, \Lambda)$ and $s = [(q+1)/2]$ (where $[x]$ denotes the integer part of $x \in \mathbb{Q}$) we have $\mathfrak{J}(\beta, \Lambda) = \mathfrak{B} + \mathfrak{P}^s$ if β is minimal and $\mathfrak{J}(\beta, \Lambda) = \mathfrak{B} + \mathfrak{J}^s(\gamma, \Lambda)$ if $[\Lambda, n, q, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, q, \beta]$. Then, if β is minimal, $\mathfrak{J}^k(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap \mathfrak{P}^k$ is equal to $\mathfrak{Q}^k + \mathfrak{P}^s$ if $0 \leq k \leq s-1$ and to $\mathfrak{P}^k$ if $k \geq s$. Otherwise, if $[\Lambda, n, q, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, q, \beta]$, $\mathfrak{J}^k(\beta, \Lambda)$ is equal to $\mathfrak{Q}^k + \mathfrak{J}^s(\gamma, \Lambda)$ if $0 \leq k \leq s-1$ and to $\mathfrak{J}^k(\gamma, \Lambda)$ if $k \geq s$. Similarly we obtain that if β_0 is minimal then $\mathfrak{J}^k(\beta_0, \Lambda_0)$ is equal to $\wp_{D'}^k + \mathfrak{P}(E)^s$ if $0 \leq k \leq s-1$ and to $\mathfrak{P}(E)^k$ if $k \geq s$. Otherwise, if $[\Lambda_0, n, q, \gamma_0]$ is a simple stratum equivalent to $[\Lambda_0, n, q, \beta_0]$, $\mathfrak{J}^k(\beta_0, \Lambda_0)$ is equal to $\wp_{D'}^k + \mathfrak{J}^s(\gamma_0, \Lambda_0)$ if $k \leq s-1$ and to $\mathfrak{J}^k(\gamma_0, \Lambda_0)$ if $k \geq s$. We prove that $e^i \mathfrak{J}^k(\beta, \Lambda) e^j = \mathfrak{J}^k(\beta_0, \Lambda_0)$ for every $k \geq 0$ by induction on q. If $q = n$ and so if β and β_0 are minimal, since $\mathfrak{Q} = M_{m'}(\wp_{D'})$ and $\mathfrak{P} = M_{m'}(\mathfrak{P}(E))$ we have $e^i \mathfrak{Q}^k e^j = \wp_{D'}^k$ and $e^i \mathfrak{P}^k e^j = \mathfrak{P}(E)^k$ for every k and so $e^i \mathfrak{J}^k(\beta, \Lambda) e^j = \mathfrak{J}^k(\beta_0, \Lambda_0)$ for every $k \geq 0$. Now if $q < n$ and so if β and β_0 are not minimal, by Proposition 1.20 of [Sécherre and Stevens 2008] (see also the proof of Theorem 2.2 of [Sécherre 2005b]) we can choose a simple stratum $[\Lambda_0, n, q, \gamma_0]$ equivalent to $[\Lambda_0, n, q, \beta_0]$ such that if γ is the image of γ_0 by the diagonal embedding $A(E) \to A$ then $[\Lambda, n, q, \gamma]$ is a simple stratum equivalent to $[\Lambda, n, q, \beta]$. By the inductive hypothesis we have $e^i \mathfrak{J}^k(\gamma, \Lambda) e^j = \mathfrak{J}^k(\gamma_0, \Lambda_0)$ for every $k \geq 0$ and then we obtain $e^i \mathfrak{J}^k(\beta, \Lambda) e^j = \mathfrak{J}^k(\beta_0, \Lambda_0)$. □

Let θ_0 be the transfer of θ to $\mathscr{C}_R(\Lambda_0, 0, \beta)$. Since H^1 is a pro-p-group, proceeding as in Proposition 2.16 of [Sécherre 2005a], the pair (H^1, θ) is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and the restriction of θ to $H^1 \cap \mathcal{M} = H_0^1 \times \cdots \times H_0^1$ is $\theta_0^{\otimes m'}$. We remark that in general (J^1, η) is not decomposed with respect to $(\mathcal{M}, \mathcal{P})$. We denote by η_0 the Heisenberg representation of θ_0 and we can consider the irreducible representation $\eta_{\mathcal{M}} = \eta_0^{\otimes m'}$ of $J_{\mathcal{M}}^1 = J^1 \cap \mathcal{M} = J_0^1 \times \cdots \times J_0^1$.

We put $J^1_{\mathcal{P}} = (J^1 \cap \mathcal{P})H^1$ and $H^1_{\mathcal{P}} = (J^1 \cap \mathcal{U})H^1$ which are subgroups of J^1. They are normal in J^1 because H^1 contains the derived group of J^1. Moreover, $J \cap \mathcal{P}$ normalizes $J^1_{\mathcal{P}}$ because H^1 is normal in J and $J^1 \cap \mathcal{P}$ is normal in $J \cap \mathcal{P}$. Then $J^1_{\mathcal{P}}$ is normal in $J^1(J \cap \mathcal{P})$.

Remark 3.5. Taking into account Remark 5.7 of [Sécherre and Stevens 2008], Proposition 5.3 of [Sécherre and Stevens 2008] states that $J^1_{\mathcal{P}}$ and $H^1_{\mathcal{P}}$ are decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and so we have $J^1_{\mathcal{P}} = (H^1 \cap \mathcal{U}^-)J^1_{\mathcal{M}}(J^1 \cap \mathcal{U})$ and $H^1_{\mathcal{P}} = (H^1 \cap \mathcal{U}^-)(H^1 \cap \mathcal{M})(J^1 \cap \mathcal{U})$. Moreover, if $\mathcal{M}' = \prod_{i=1}^{r} \mathrm{GL}_{m'_i}(A(E))$ with $\sum_{i=1}^{r} m'_i = m'$ is a standard Levi subgroup of G containing $\mathcal{M}$ and $\mathcal{P}'$ is the upper standard parabolic subgroup of G with Levi component $\mathcal{M}'$, then $J^1_{\mathcal{P}}$ and $H^1_{\mathcal{P}}$ are decomposed with respect to $(\mathcal{M}', \mathcal{P}')$.

Let $\theta_{\mathcal{P}}$ be the character of $H^1_{\mathcal{P}}$ defined by $\theta_{\mathcal{P}}(uh) = \theta(h)$ for every $u \in J^1 \cap \mathcal{U}$ and every $h \in H^1$. Since J^1 is a pro-p-group, proceeding as in Proposition 5.5 of [Sécherre and Stevens 2008] we can construct an irreducible representation $\eta_{\mathcal{P}}$ of $J^1_{\mathcal{P}}$, unique up to isomorphism, whose restriction to $H^1_{\mathcal{P}}$ contains $\theta_{\mathcal{P}}$. Actually it is the natural representation of $J^1_{\mathcal{P}}$ on the $J^1 \cap \mathcal{U}$-invariants of η. Furthermore, $\mathrm{ind}_{J^1_{\mathcal{P}}}^{J^1}(\eta_{\mathcal{P}})$ is isomorphic to η, $I_G(\eta_{\mathcal{P}}) = J^1_{\mathcal{P}} B^\times J^1_{\mathcal{P}}$ and for every $y \in B^\times$ we have $\dim_R(I_y(\eta_{\mathcal{P}})) = 1$. We remark that $(J^1_{\mathcal{P}}, \eta_{\mathcal{P}})$ is decomposed with respect to $(\mathcal{M}, \mathcal{P})$ and the restriction of η_P to $J^1_{\mathcal{M}}$ is $\eta_{\mathcal{M}}$. We denote by $V_{\mathcal{M}}$ the R-vector space of $\eta_{\mathcal{M}}$ and $\eta_{\mathcal{P}}$.

Since $\mathrm{ind}_{J^1_{\mathcal{P}}}^{J^1}(\eta_{\mathcal{P}})$ is isomorphic to η, we can identify the R-vector space V_η of η with the vector space of functions $\varphi : J^1 \to V_{\mathcal{M}}$ such that $\varphi(xj) = \eta_{\mathcal{P}}(x)\varphi(j)$ for every $x \in J^1_{\mathcal{P}}$ and $j \in J^1$. In this case $\eta(j)\varphi : x \mapsto \varphi(xj)$. By the Mackey formula, $V_{\mathcal{M}}$ is a direct summand of V_η and we can identify it with the subspace of functions $\varphi \in V_\eta$ with support in $J^1_{\mathcal{P}}$. This identification is given by $\varphi \mapsto \varphi(1)$ whose inverse is $v \mapsto \varphi_v$ where the support of φ_v is $J^1_{\mathcal{P}}$ and $\varphi_v(1) = v$. Let $\boldsymbol{p} : V_\eta \to V_{\mathcal{M}}$ be the canonical projection, i.e., the restriction of a function in V_η to $J^1_{\mathcal{P}}$, and let $\iota : V_{\mathcal{M}} \to V_\eta$ be the inclusion.

Remark 3.6. In general we cannot define a representation κ_P of $J_P = (J \cap P)H^1$ as in Section 2.3 of [Sécherre 2005a] or in Section 5.5 of [Sécherre and Stevens 2008], because the decomposition e conforms to Λ over E but it is not subordinate to $\mathfrak{B}$. In our case ($\mathfrak{B}$ maximal) the only decomposition which conforms to Λ over E and is subordinate to $\mathfrak{B}$ is the trivial one.

Lemma 3.7. (1) *For every $j \in J^1_{\mathcal{P}}$ we have $\eta(j) \circ \iota = \iota \circ \eta_{\mathcal{P}}(j)$ and $\boldsymbol{p} \circ \eta(j) = \eta_{\mathcal{P}}(j) \circ \boldsymbol{p}$.*

(2) *For every $j \in J^1$ we have*

$$\boldsymbol{p} \circ \eta(j) \circ \iota = \begin{cases} \eta_{\mathcal{P}}(j) & \text{if } j \in J^1_{\mathcal{P}}, \\ 0 & \text{otherwise.} \end{cases}$$

(3) $\sum_{j \in J^1/J^1_{\mathcal{P}}} \eta(j) \circ \iota \circ \boldsymbol{p} \circ \eta(j^{-1})$ *is the identity of* $\mathrm{End}_R(V_{\mathcal{M}})$.

Proof. To prove the first point, let $\varphi_v \in V_{\mathcal{M}}$ and $\varphi \in V_\eta$. Then $\eta(j)(\iota(\varphi_v))(1) = \varphi_v(j) = \eta_P(j)v$ and $\boldsymbol{p}(\eta(j)(\varphi))(1) = \varphi(j) = \eta_{\mathcal{P}}(j)\varphi(1)$. To prove the second point we observe that if $j \in J^1_P$ then $\boldsymbol{p} \circ \eta(j) \circ \iota = \boldsymbol{p} \circ \iota \circ \eta_{\mathcal{P}}(j) = \eta_{\mathcal{P}}(j)$ while if $j \notin J^1_{\mathcal{P}}$ the support of $\eta(j)(\iota(\varphi_v))$ is in $J^1_{\mathcal{P}} j^{-1}$ for every $\varphi_v \in V_{\mathcal{M}}$ and so $\boldsymbol{p} \circ \eta(j) \circ \iota = 0$. Finally, to prove the third point we observe that for every $\varphi \in V_\eta$ the function $\varphi_j = (\eta(j) \circ \iota \circ \boldsymbol{p} \circ \eta(j^{-1}))\varphi$ has support in $J^1_{\mathcal{P}} j^{-1}$ and $\varphi_j(j^{-1}) = \varphi(j^{-1})$. □

We consider the surjective linear map $\mu : \mathrm{End}_R(V_\eta) \to \mathrm{End}_R(V_{\mathcal{M}})$ given by $f \mapsto \boldsymbol{p} \circ f \circ \iota$.

Lemma 3.8. *The map $\zeta : \mathscr{H}_R(G,\eta) \to \mathscr{H}_R(G,\eta_{\mathcal{P}})$ defined by $\Phi \mapsto \mu \circ \Phi$ for every $\Phi \in \mathscr{H}_R(G,\eta)$ is an isomorphism of R-algebras. Moreover, if the support of $\Phi \in \mathscr{H}_R(G,\eta)$ is in $J^1 x J^1$ with $x \in B^\times$ then the support of $\zeta(\Phi)$ is in $J^1_{\mathcal{P}} x J^1_{\mathcal{P}}$.*

Proof. Let $\Phi \in \mathscr{H}_R(G,\eta)$. Then the support of $\mu \circ \Phi$ is contained in the support of Φ which is compact. Furthermore, for every $x_1, x_2 \in J^1_{\mathcal{P}}$ and every $j \in J^1$ we have $\mu(\Phi(x_1 j x_2)) = \boldsymbol{p} \circ \eta(x_1) \circ \Phi(j) \circ \eta(x_2) \circ \iota$ which, by Lemma 3.7, is $\eta_{\mathcal{P}}(x_1) \circ \mu(\Phi(j)) \circ \eta_{\mathcal{P}}(x_2)$. Hence, ζ is well defined and it is R-linear. Let $\Phi_1, \Phi_2 \in \mathscr{H}_R(G,\eta)$. For every $g \in G$ we have

$$\begin{aligned}
((\mu \circ \Phi_1) * (\mu \circ \Phi_2))(g) &= \sum_{x \in G/J^1_{\mathcal{P}}} \boldsymbol{p} \circ \Phi_1(x) \circ \iota \circ \boldsymbol{p} \circ \Phi_2(x^{-1}g) \circ \iota \\
&= \sum_{y \in G/J^1} \sum_{z \in J^1/J^1_{\mathcal{P}}} \boldsymbol{p} \circ \Phi_1(yz) \circ \iota \circ \boldsymbol{p} \circ \Phi_2(z^{-1}y^{-1}g) \circ \iota \\
&= \sum_{y \in G/J^1} \boldsymbol{p} \circ \Phi_1(y) \circ \Bigg(\sum_{z \in J^1/J^1_{\mathcal{P}}} \eta(z) \circ \iota \circ \boldsymbol{p} \circ \eta(z^{-1}) \Bigg) \circ \Phi_2(y^{-1}g) \circ \iota \\
&= \sum_{y \in G/J^1} \boldsymbol{p} \circ \Phi_1(y) \circ \Phi_2(y^{-1}g) \circ \iota \\
\text{(Lemma 3.7)} &= (\mu \circ (\Phi_1 * \Phi_2))(g)
\end{aligned}$$

and so ζ is a homomorphism of R-algebras. Let $\Phi \in \mathscr{H}_R(G,\eta)$ such that $\boldsymbol{p} \circ \Phi(g) \circ \iota = 0$ for every $g \in G$. Then by Lemma 3.7, for every $g' \in G$ we have

$$\begin{aligned}
\Phi(g') &= \sum_{j_1 \in J^1/J^1_{\mathcal{P}}} \eta(j_1) \circ \iota \circ \boldsymbol{p} \circ \eta(j_1^{-1}) \circ \Phi(g') \circ \sum_{j_2 \in J^1/J^1_{\mathcal{P}}} \eta(j_2) \circ \iota \circ \boldsymbol{p} \circ \eta(j_2^{-1}) \\
&= \sum_{j_1, j_2 \in J^1/J^1_{\mathcal{P}}} \eta(j_1) \circ \iota \circ (\boldsymbol{p} \circ \Phi(j_1^{-1} g' j_2) \circ \iota) \circ \boldsymbol{p} \circ \eta(j_2^{-1}) \\
&= 0
\end{aligned}$$

and then ζ is injective. Now, we know that $\mathscr{H}_R(G,\eta) \cong \mathrm{End}_G(\mathrm{ind}^G_{J^1}(\eta))$, $\mathscr{H}_R(G,\eta_{\mathcal{P}}) \cong \mathrm{End}_G(\mathrm{ind}^G_{J^1_{\mathcal{P}}}(\eta_{\mathcal{P}}))$ and $\mathrm{ind}^{J^1}_{J^1_{\mathcal{P}}}(\eta_{\mathcal{P}}) \cong \eta$. Then by transitivity of the induction we have $\mathscr{H}_R(G,\eta) \cong \mathscr{H}_R(G,\eta_{\mathcal{P}})$ and then ζ must be bijective. Furthermore, if $\Phi \in \mathscr{H}_R(G,\eta)$ has support in $J^1 x J^1$ with $x \in B^\times$ then the support of $\zeta(\Phi)$ is in $J^1 x J^1 \cap I_G(\eta_{\mathcal{P}}) = J^1 x J^1 \cap J^1_{\mathcal{P}} B^\times J^1_{\mathcal{P}} = J^1_{\mathcal{P}} x J^1_{\mathcal{P}}$. □

Lemma 3.9. *Let $x_1, x_2 \in B^\times$ and let $\tilde{f}_i \in \mathscr{H}_R(G,\eta)_{J^1 x_i J^1}$ and $\hat{f}_i = \zeta(\tilde{f}_i)$ for $i \in \{1,2\}$.*

(1) *If x_1 or x_2 normalizes $J^1_{\mathcal{P}}$ then the support of $\hat{f}_1 * \hat{f}_2$ is in $J^1_{\mathcal{P}} x_1 x_2 J^1_{\mathcal{P}}$ and*

$$(\hat{f}_1 * \hat{f}_2)(x_1 x_2) = \hat{f}_1(x_1) \circ \hat{f}_2(x_2).$$

(2) *If x_1 or x_2 normalizes J^1 then the support of $\hat{f}_1 * \hat{f}_2$ is in $J^1_{\mathcal{P}} x_1 x_2 J^1_{\mathcal{P}}$ and*

$$(\hat{f}_1 * \hat{f}_2)(x_1 x_2) = \boldsymbol{p} \circ \tilde{f}_1(x_1) \circ \tilde{f}_2(x_2) \circ \iota.$$

Proof. The first point follows from Remark 1.2. If x_1 or x_2 normalizes J^1, by Remark 1.2 the support of $\tilde{f}_1 * \tilde{f}_2$ is in $J^1x_1x_2J^1$ and so the support of $\hat{f}_1 * \hat{f}_2 = \zeta(\tilde{f}_1 * \tilde{f}_2)$ is in $J^1x_1x_2J^1 \cap I_G(\eta_{\mathcal{P}}) = J^1_{\mathcal{P}}x_1x_2J^1_{\mathcal{P}}$ and moreover

$$(\hat{f}_1 * \hat{f}_2)(x_1x_2) = \zeta(\tilde{f}_1 * \tilde{f}_2)(x_1x_2) = \boldsymbol{p} \circ \tilde{f}_1(x_1) \circ \tilde{f}_2(x_2) \circ \iota. \qquad \square$$

Lemma 3.10. *For every $x \in B^\times \cap \mathcal{M}$ and every $y \in I_G(\eta_{\mathcal{P}})$ which normalizes $J^1_{\mathcal{M}}$ we have $I_x(\eta_{\mathcal{P}}) = I_x(\eta_{\mathcal{M}})$ and $I_y(\eta_{\mathcal{P}}) = I_y(\eta_{\mathcal{M}})$. Moreover, every nonzero element in $I_z(\eta_{\mathcal{P}})$, with $z \in I_G(\eta_{\mathcal{P}})$, is invertible.*

Proof. For the first assertion, in both cases the R-vector spaces are 1-dimensional and so it suffices to prove an inclusion. Since $\eta_{\mathcal{M}}$ is the restriction of $\eta_{\mathcal{P}}$ to $J^1_{\mathcal{M}}$, for every $x' \in I_G(\eta_{\mathcal{P}})$ we have $I_{x'}(\eta_{\mathcal{P}}) \subseteq I_{x'}(\eta_{\mathcal{M}})$. For the second assertion, we observe that $I_G(\eta_{\mathcal{P}}) = J^1_{\mathcal{P}}B^\times J^1_{\mathcal{P}} = J^1_{\mathcal{P}}I_B\tilde{W}I_BJ^1_{\mathcal{P}}$. Now I_B normalizes $J^1_{\mathcal{P}}$ since it is contained in $J^1(J \cap \mathcal{P})$ while $\tilde{W}$ normalizes $J^1_{\mathcal{M}}$. Take $z = z_1z_2z_3 \in I_G(\eta_{\mathcal{P}})$ with $z_1 \in J^1_{\mathcal{P}}I_B$, $z_2 \in \tilde{W}$ and $z_3 \in I_BJ^1_{\mathcal{P}}$ and take a nonzero element γ in $I_z(\eta_P)$. Let γ_1 and γ_3 be invertible elements in $I_{z_1^{-1}}(\eta_{\mathcal{P}})$ and in $I_{z_3^{-1}}(\eta_{\mathcal{P}})$ respectively. Then $\gamma_1 \circ \gamma \circ \gamma_3$ is a nonzero element in $I_{z_2}(\eta_{\mathcal{P}}) = I_{z_2}(\eta_{\mathcal{M}})$ and so it is invertible. $\square$

3C. ***The isomorphism $\mathscr{H}_R(J, \eta) \cong \mathscr{H}_R(K_B, K^1_B)$.*** We now prove that the subalgebra $\mathscr{H}_R(K_B, K^1_B)$ of $\mathscr{H}_R(B^\times, K^1_B)$ is isomorphic to the subalgebra $\mathscr{H}_R(J, \eta_{\mathcal{P}})$ of $\mathscr{H}_R(G, \eta_{\mathcal{P}})$ and so to $\mathscr{H}_R(J, \eta)$.

In accordance with Chapter 2 of [Chinello 2017], we denote by $f_x \in \mathscr{H}_R(B^\times, K^1_B)$ the characteristic function of $K^1_BxK^1_B$ for every $x \in B^\times$ and we write $\Phi_1\Phi_2 = \Phi_1 * \Phi_2$ for every Φ_1 and Φ_2 in $\mathscr{H}_R(B^\times, K^1_B)$, in $\mathscr{H}_R(G, \eta)$ or in $\mathscr{H}_R(G, \eta_{\mathcal{P}})$.

We observe that every element in $\mathscr{H}_R(J, \eta_{\mathcal{P}})$ has support in $J \cap J^1_{\mathcal{P}}B^\times J^1_{\mathcal{P}} = J^1_{\mathcal{P}}(J \cap B^\times)J^1_{\mathcal{P}} = J^1_{\mathcal{P}}K_BJ^1_{\mathcal{P}}$ and so its image by ζ^{-1} has support in $J^1K_BJ^1$. This implies that ζ induces an algebra isomorphism from $\mathscr{H}_R(J, \eta)$ to $\mathscr{H}_R(J, \eta_{\mathcal{P}})$. We also remark that $\mathscr{H}_R(K_B, K^1_B)$ is isomorphic to the group algebra $R[K_B/K^1_B] \cong R[J/J^1]$, then we can identify every $\Phi \in \mathscr{H}_R(K_B, K^1_B)$ with a function $\Phi \in \mathscr{H}_R(J, J^1)$.

From now on we fix a β-extension κ of η. We recall that $\mathrm{res}^J_{J^1}\,\kappa = \eta$, $I_G(\eta) = I_G(\kappa) = J^1B^\times J^1$ and for every $y \in B^\times$ we have $I_y(\eta) = I_y(\kappa)$ which is an R-vector space of dimension 1. Then V_η is also the R-vector space of κ and $\kappa(j) \in I_j(\eta)$ for every $j \in J$.

Lemma 3.11. *The map $\Theta' : \mathscr{H}_R(K_B, K^1_B) \to \mathscr{H}_R(J, \eta)$ defined by $\Phi \mapsto \Phi \otimes \kappa$ for every $\Phi \in \mathscr{H}_R(K_B, K^1_B)$ is an algebra isomorphism.*

Proof. The map is well defined since for every $\Phi \in \mathscr{H}_R(K_B, K^1_B)$ we have $\Phi \otimes \kappa : J \to \mathrm{End}_R(V_\eta)$ and $(\Phi \otimes \kappa)(j_1jj'_1) = \Phi(j)\kappa(j_1jj'_1) = \eta(j_1) \circ (\Phi(j)\kappa(j)) \circ \eta(j'_1)$ for every $j \in J$ and $j_1, j'_1 \in J^1$. It is clearly R-linear and

$$\begin{aligned}\Theta'(\Phi_1 * \Phi_2)(j) &= \sum_{x \in J/J^1} \Phi_1(x)\Phi_2(x^{-1}j)\kappa(j) = \sum_{x \in J/J^1} \Phi_1(x)\Phi_2(x^{-1}j)\kappa(x) \circ \kappa(x^{-1}j) \\ &= \sum_{x \in J/J^1} (\Phi_1(x)\kappa(x)) \circ (\Phi_2(x^{-1}j)\kappa(x^{-1}j)) = (\Theta'(\Phi_1) * \Theta'(\Phi_2))(j)\end{aligned}$$

for every $\Phi_1, \Phi_2 \in \mathscr{H}_R(K_B, K_B^1)$ and $j \in J$. Hence, Θ' is an R-algebra homomorphism. It is injective because $\kappa(j) \in \mathrm{GL}(V_\eta)$ for every $j \in J$. Let $\tilde{f} \in \mathscr{H}_R(J, \eta)$ and $j \in J$. Since $\tilde{f}(j) \in I_j(\eta) = \mathrm{Hom}_{J^1}(\eta, \eta^j)$, which is of dimension 1, we have $\tilde{f}(j) \in R\kappa(j)$ and then we can write $\tilde{f}(j) = \Phi(j)\kappa(j)$ with $\Phi : J \to R$. Since $\tilde{f} \in \mathscr{H}_R(J, \eta)$, for every $j_1 \in J^1$ we have

$$\Phi(j_1 j)\kappa(j_1 j) = \tilde{f}(j_1 j) = \eta(j_1)\tilde{f}(j) = \eta(j_1)\Phi(j)\kappa(j) = \Phi(j)\kappa(j_1 j)$$

and so $\Phi \in \mathscr{H}_R(J, J^1)$. We conclude that Θ' is surjective and then it is an algebra isomorphism. □

Composing the restriction of ζ to $\mathscr{H}_R(J, \eta)$ with Θ' we obtain an algebra isomorphism $\mathscr{H}_R(K_B, K_B^1) \to \mathscr{H}_R(J, \eta_{\mathcal{P}})$. For every $x \in K_B$ let $\tilde{f}_x = \Theta'(f_x) \in \mathscr{H}_R(J, \eta)$ which is given by $\tilde{f}_x(y) = \kappa(y)$ for every $y \in J^1 x J^1 = J^1 x$ and let $\hat{f}_x = \zeta(\tilde{f}_x) \in \mathscr{H}_R(J, \eta_{\mathcal{P}})$ which is given by $\hat{f}_x(z) = \boldsymbol{p} \circ \kappa(z) \circ \iota$ for every $z \in J^1_{\mathcal{P}} x J^1_{\mathcal{P}}$.

3D. ***Generators and relations of*** $\boldsymbol{\mathscr{H}_R(B^\times, K_B^1)}$**.** In this section we introduce some notation and recall the presentation by generators and relations of the algebra $\mathscr{H}_R(B^\times, K_B^1)$ presented in [Chinello 2017].

We set $\Omega = K_B \cup \{\tau_0, \tau_0^{-1}\} \cup \{\tau_\alpha \mid \alpha \in \Sigma\}$ and $\boldsymbol{\Omega} = \{f_\omega \mid \omega \in \Omega\}$ which is a finite set. We now define some subgroups of G, through its identification with $\mathrm{GL}_{m'}(A(E))$. For every $\alpha = \alpha_{ij} \in \boldsymbol{\Phi}$ we denote by $\mathcal{U}_\alpha$ the subgroup of matrices $(a_{hk}) \in G$ with $a_{hh} = 1$ for every $h \in \{1, \dots, m'\}$, $a_{ij} \in A(E)$ and $a_{hk} = 0$ if $h \neq k$ and $(h, k) \neq (i, j)$. For every $P \subset \Sigma$ we denote by $\mathcal{M}_P$ the standard Levi subgroup associated to P and by $\mathcal{U}_P^+$ and $\mathcal{U}_P^-$ the unipotent radical of, respectively, upper and lower standard parabolic subgroups with Levi component $\mathcal{M}_P$. We remark that $\mathcal{M} = \mathcal{M}_\varnothing$, $\mathcal{U} = \mathcal{U}_\varnothing$ and $\mathcal{U}^- = \mathcal{U}_\varnothing^-$. Thus, we have $\mathcal{U}_P^+ = \prod_{\alpha \in \Psi_P^+} \mathcal{U}_\alpha$ and $\mathcal{U}_P^- = \prod_{\alpha \in \Psi_P^-} \mathcal{U}_\alpha$. Furthermore, if $P_1 \subset P_2 \subset \Sigma$ then $\mathcal{U}_{P_2}^+$ is a subgroup of $\mathcal{U}_{P_1}^+$ and $\mathcal{U}_{P_2}^-$ a subgroup of $\mathcal{U}_{P_1}^-$.

Remark 3.12. By Proposition 3.4, if we take $\alpha = \alpha_{ij} \in \boldsymbol{\Phi}$ and (a_{hk}) in $\mathcal{U}_\alpha \cap J^1$ or $\mathcal{U}_\alpha \cap H^1$ then a_{ij} is in $\mathfrak{J}_0^1$ or $\mathfrak{H}_0^1$, respectively.

Remark 3.13. In accordance with Section 2.2 of [Chinello 2017] we set $M_P = \mathcal{M}_P \cap K_B$, $U_P^+ = \mathcal{U}_P^+ \cap K_B$ and $U_P^- = \mathcal{U}_P^- \cap K_B$ for every $P \subset \Sigma$ and $U_\alpha = \mathcal{U}_\alpha \cap K_B$ for every $\alpha \in \boldsymbol{\Phi}$.

As in Section 2.3 of [Chinello 2017], for every $\alpha = \alpha_{i,i+1} \in \Sigma$ and $w \in W$ we consider the following sets: $A(w, \alpha) = \{w(j) \mid i+1 \leq j \leq m'\}$, $B(w, \alpha) = \{w(j) - 1 \mid i+1 \leq j \leq m'\}$, $P'(w, \alpha) = A(w, \alpha) \setminus B(w, \alpha)$, $P(w, \alpha) = \{\alpha_{i,i+1} \in \Sigma \mid i \in P'(w, \alpha)\}$ and $Q(w, \alpha) = B(w, \alpha) \setminus A(w, \alpha)$. We remark that $\tau_{P'(w,\alpha)} = \tau_{P(w,\alpha)}$ because $0 \notin P'(w, \alpha)$ and $\tau_{m'} = \mathbb{I}_{m'}$. Moreover, if $\alpha = \alpha_{i,i+1} \in \Sigma$, $w' \in W$ and w is of minimal length in $w' W_{\hat{\alpha}} \in W / W_{\hat{\alpha}}$ then we have

$$w' \tau_i w'^{-1} = w \tau_i w^{-1} = \prod_{h=i+1}^{m'} w \tau_{h-1} \tau_h^{-1} w^{-1} = \prod_{h=i+1}^{m'} \tau_{w(h)-1} \tau_{w(h)}^{-1} = \tau_{P(w,\alpha)}^{-1} \tau_{Q(w,\alpha)}.$$

Lemma 3.14. *The algebra* $\mathscr{H}_R(B^\times, K_B^1)$ *is the* R*-algebra generated by* $\boldsymbol{\Omega}$ *subject to the following relations*:

(1) $f_k = 1$ *for every* $k \in K^1$ *and* $f_{k_1} f_{k_2} = f_{k_1 k_2}$ *for every* $k_1, k_2 \in K$.

(2) $f_{\tau_0} f_{\tau_0^{-1}} = 1$ *and* $f_{\tau_0^{-1}} f_\omega = f_{\tau_0^{-1}\omega\tau_0} f_{\tau_0^{-1}}$ *for every* $\omega \in \Omega$.

(3) $f_{\tau_\alpha} f_x = f_{\tau_\alpha x \tau_\alpha^{-1}} f_{\tau_\alpha}$ *for every* $\alpha \in \Sigma$ *and* $x \in M_{\hat{\alpha}}$.

(4) $f_u f_{\tau_\alpha} = f_{\tau_\alpha}$ *if* $u \in U_{\alpha'}$ *with* $\alpha' \in \Psi^+_{\hat{\alpha}}$, *for every* $\alpha \in \Sigma$.

(5) $f_{\tau_\alpha} f_u = f_{\tau_\alpha}$ *if* $u \in U_{\alpha'}$ *with* $\alpha' \in \Psi^-_{\hat{\alpha}}$, *for every* $\alpha \in \Sigma$.

(6) $f_{\tau_\alpha} f_{\tau_{\alpha'}} = f_{\tau_{\alpha'}} f_{\tau_\alpha}$ *for every* $\alpha, \alpha' \in \Sigma$.

(7) $\big(\prod_{\alpha' \in P(w,\alpha)} f_{\tau_{\alpha'}}\big) f_w f_{\tau_\alpha} f_{w^{-1}} = q^{\ell(w)} \big(\prod_{\alpha'' \in Q(w,\alpha)} f_{\tau_{\alpha''}}\big)\big(\sum_u f_u\big)$ *for every* $\alpha \in \Sigma$ *and* w *of minimal length in* $wW_{\hat{\alpha}} \in W/W_{\hat{\alpha}}$ *and where* u *runs over a system of representatives of* $(U \cap wU^- w^{-1})K_B^1/K_B^1$ *in* $U \cap wU^- w^{-1}$.

Proof. The only difference between this presentation and that in [Chinello 2017] is relation 3 which is equivalent to relations 3, 4 and 7 of Definition 2.21 of [Chinello 2017] because $\mathcal{M} \cap K_B$, $U_{\alpha'}$ with $\alpha' \in \mathbf{\Phi}_{\hat{\alpha}}$ and $W_{\hat{\alpha}}$ generate $M_{\hat{\alpha}}$. □

Hence, to define an algebra homomorphism from $\mathscr{H}_R(B^\times, K_B^1)$ to $\mathscr{H}_R(G, \eta_{\mathcal{P}})$, it is sufficient to choose elements $\hat{f}_\omega \in \mathscr{H}_R(G, \eta_{\mathcal{P}})$ for every $\omega \in \Omega$ such that the $\hat{f}_\omega$ respect the relations of Lemma 3.14. We remark that we can take $\hat{f}_\omega \in \mathscr{H}_R(G, \eta_{\mathcal{P}})_{J^1_{\mathcal{P}}\omega J^1_{\mathcal{P}}}$ for every $\omega \in \Omega$ and we recall that in Section 3C we have defined $\hat{f}_k$ for every $k \in K_B$ as the image of f_k by $\zeta \circ \Theta'$.

3E. ***Some decompositions of $J^1_{\mathcal{P}}$-double cosets.*** In this section we introduce some notation and some tools that we will use to construct elements in $\mathscr{H}_R(G, \eta_{\mathcal{P}})_{J^1_{\mathcal{P}}\tau_i J^1_{\mathcal{P}}}$ with $i \in \{0, \dots, m'-1\}$.

Lemma 3.15. *Let* $\tau \in \mathbf{\Delta}$ *and* $P = P(\tau)$.

(1) *We have* $J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap \mathcal{U}^-_P)(J^1_{\mathcal{P}} \cap \mathcal{M}_P)(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P) = (J^1_{\mathcal{P}} \cap \mathcal{U}^+_P)(J^1_{\mathcal{P}} \cap \mathcal{M}_P)(J^1_{\mathcal{P}} \cap \mathcal{U}^-_P)$.

(2) *We have* $(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P)^\tau \subset H^1 \cap \mathcal{U}^+_P \subset J^1_{\mathcal{P}} \cap \mathcal{U}^+_P$, $(J^1_{\mathcal{P}} \cap \mathcal{U}^-_P)^{\tau^{-1}} \subset (J^1 \cap \mathcal{U}^-_P)^{\tau^{-1}} \subset H^1 \cap \mathcal{U}^-_P = J^1_{\mathcal{P}} \cap \mathcal{U}^-_P$ *and* $(J^1_{\mathcal{P}} \cap \mathcal{M}_P)^\tau = J^1_{\mathcal{P}} \cap \mathcal{M}_P$.

(3) *We have* $(J^1_{\mathcal{P}} \cap \mathcal{U})^\tau \subset J^1_{\mathcal{P}} \cap \mathcal{U}$, $(J^1_{\mathcal{P}} \cap \mathcal{U}^-)^{\tau^{-1}} \subset J^1_{\mathcal{P}} \cap \mathcal{U}^-$ *and* $(J^1_{\mathcal{M}})^\tau = J^1_{\mathcal{M}}$.

Proof. The first point follows from Remark 3.5. To prove the second point we observe that Remark 3.12 implies that $(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P)^\tau = (J^1 \cap \prod_{\alpha \in \mathbf{\Psi}^+_P} \mathcal{U}_\alpha)^\tau$ is contained in $(\mathbb{I}_{m'} + \varpi \mathfrak{J}^1) \cap \mathcal{U}^+_P$ which is in $H^1 \cap \mathcal{U}^+_P \subset J^1_{\mathcal{P}} \cap \mathcal{U}^+_P$. Similarly we prove $(J^1 \cap \mathcal{U}^-_P)^{\tau^{-1}} \subset H^1 \cap \mathcal{U}^-_P$. Moreover, since $\varpi^{-1}\mathfrak{J}^1_0\varpi = \mathfrak{J}^1_0$ and $\varpi^{-1}\mathfrak{H}^1_0\varpi = \mathfrak{H}^1_0$, we have $(J^1_{\mathcal{P}} \cap \mathcal{M}_P)^\tau = J^1_{\mathcal{P}} \cap \mathcal{M}_P$. To prove the third point, we observe that $(J^1_{\mathcal{P}} \cap \mathcal{U})^\tau \subset ((J^1_{\mathcal{P}} \cap \mathcal{M}_P)(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P))^\tau \cap \mathcal{U}$ which is in $(J^1_{\mathcal{P}} \cap \mathcal{M}_P)(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P) \cap \mathcal{U} = J^1_{\mathcal{P}} \cap \mathcal{U}$. Similarly we prove $(J^1_{\mathcal{P}} \cap \mathcal{U}^-)^{\tau^{-1}} \subset J^1_{\mathcal{P}} \cap \mathcal{U}^-$. Finally, since $\varpi^{-1}\mathfrak{J}^1_0\varpi = \mathfrak{J}^1_0$ we obtain $(J^1_{\mathcal{M}})^\tau = J^1_{\mathcal{M}}$. □

Lemma 3.16. *Let* $\tau, \tau' \in \mathbf{\Delta}$ *and* $w \in W$.

(1) *We have* $J^1_{\mathcal{P}}\tau J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap \mathcal{U}^-_{P(\tau)})\tau J^1_{\mathcal{P}} = J^1_{\mathcal{P}}\tau(J^1_{\mathcal{P}} \cap \mathcal{U}^+_{P(\tau)})$ *and* $J^1_{\mathcal{P}}\tau^{-1}J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap \mathcal{U}^+_{P(\tau)})\tau^{-1}J^1_{\mathcal{P}} = J^1_{\mathcal{P}}\tau^{-1}(J^1_{\mathcal{P}} \cap \mathcal{U}^-_{P(\tau)})$.

(2) *We have* $(J^1_{\mathcal{P}})^w J^1_{\mathcal{P}} = (J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)J^1_{\mathcal{P}}$.

(3) *We have* $J^1_{\mathcal{P}}\mathcal{U}^- J^1_{\mathcal{P}} \cap \mathcal{U} = J^1_{\mathcal{P}} \cap \mathcal{U}$ *and* $J^1_{\mathcal{P}}\mathcal{U} J^1_{\mathcal{P}} \cap \mathcal{U}^- = J^1_{\mathcal{P}} \cap \mathcal{U}^-$.

(4) *We have* $J^1_{\mathcal{P}}\tau J^1_{\mathcal{P}}\tau' J^1_{\mathcal{P}} = J^1_{\mathcal{P}}\tau\tau' J^1_{\mathcal{P}}$ *and* $(J^1_{\mathcal{P}})^{\tau} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{\tau'^{-1}} J^1_{\mathcal{P}} = J^1_{\mathcal{P}}$.

Proof. Let $P = P(\tau)$.

(1) By Lemma 3.15 we have $J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap \mathcal{U}^-_P)(J^1_{\mathcal{P}} \cap \mathcal{M}_P)(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P)$ and so we obtain $J^1_{\mathcal{P}}\tau J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap \mathcal{U}^-_P)\tau(J^1_{\mathcal{P}} \cap \mathcal{M}_P)^{\tau}(J^1_{\mathcal{P}} \cap \mathcal{U}^+_P)^{\tau} J^1_{\mathcal{P}}$ which is equal to $(J^1_{\mathcal{P}} \cap \mathcal{U}^-_P)\tau J^1_{\mathcal{P}}$ by Lemma 3.15. We prove the other equalities similarly.

(2) Since $(H^1 \cap \mathcal{U}^-)^w \subset J^1_{\mathcal{P}}$ and $(J^1_{\mathcal{M}})^w = J^1_{\mathcal{M}}$ we obtain $(J^1_{\mathcal{P}})^w J^1_{\mathcal{P}} = (J^1 \cap \mathcal{U})^w J^1_{\mathcal{P}}$. Moreover, we have $(J^1 \cap \mathcal{U})^w \cap \mathcal{U} \subset J^1_{\mathcal{P}}$ and so $(J^1_{\mathcal{P}})^w J^1_{\mathcal{P}} = (J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)J^1_{\mathcal{P}}$.

(3) We have $J^1_{\mathcal{P}}\mathcal{U}^- J^1_{\mathcal{P}} \cap \mathcal{U} = (J^1_{\mathcal{P}} \cap \mathcal{U})\big((J^1_{\mathcal{P}} \cap \mathcal{M})\mathcal{U}^-(J^1_{\mathcal{P}} \cap \mathcal{M}) \cap \mathcal{U}\big)(J^1_{\mathcal{P}} \cap \mathcal{U})$ which is contained in $(J^1_{\mathcal{P}} \cap \mathcal{U})(\mathcal{P}^- \cap \mathcal{U})(J^1_{\mathcal{P}} \cap \mathcal{U}) = J^1_{\mathcal{P}} \cap \mathcal{U}$. We prove the second statement similarly.

(4) By point 1, we have $J^1_{\mathcal{P}}\tau J^1_{\mathcal{P}}\tau' J^1_{\mathcal{P}} = J^1_{\mathcal{P}}\tau(J^1_{\mathcal{P}} \cap \mathcal{U}^+_{P(\tau)})\tau' J^1_{\mathcal{P}}$ which is equal to $J^1_{\mathcal{P}}\tau\tau'(J^1_{\mathcal{P}} \cap \mathcal{U}^+_{P(\tau)})^{\tau'} J^1_{\mathcal{P}}$. By Lemma 3.15 it is in $J^1_{\mathcal{P}}\tau\tau'(J^1_{\mathcal{P}} \cap \mathcal{U})^{\tau'} J^1_{\mathcal{P}} \subset J^1_{\mathcal{P}}\tau\tau' J^1_{\mathcal{P}}$ and so we have $J^1_{\mathcal{P}}\tau J^1_{\mathcal{P}}\tau' J^1_{\mathcal{P}} = J^1_{\mathcal{P}}\tau\tau' J^1_{\mathcal{P}}$. By point 1, $(J^1_{\mathcal{P}})^{\tau} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{\tau'^{-1}} J^1_{\mathcal{P}}$ is contained in $(J^1_{\mathcal{P}} \cap \mathcal{U}^-)^{\tau} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}} \cap \mathcal{U})^{\tau'^{-1}} J^1_{\mathcal{P}} = \big((J^1_{\mathcal{P}} \cap \mathcal{U}^-)^{\tau} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}} \cap \mathcal{U})^{\tau'^{-1}}\big) J^1_{\mathcal{P}}$ which is contained in $(\mathcal{U}^- J^1_{\mathcal{P}} \cap \mathcal{U})J^1_{\mathcal{P}}$ and so it is equal to $J^1_{\mathcal{P}}$ by point 3. □

Remark 3.17. We can prove results similar to Lemmas 3.15 and 3.16 with J^1 in place of $J^1_{\mathcal{P}}$.

Lemma 3.18. *Let* $\alpha = \alpha_{i,i+1} \in \Sigma$, $w \in W$ *and* $P = P(w, \alpha)$. *Then* $\Psi^+_{\hat{P}} \cap w\Psi^-_{\hat{\alpha}} = \mathbf{\Phi}^+ \cap w\Psi^-_{\hat{\alpha}}$ *and* $\Psi^-_{\hat{P}} \cap w\Psi^+_{\hat{\alpha}} = \mathbf{\Phi}^- \cap w\Psi^+_{\hat{\alpha}}$. *If in addition* w *is of minimal length in* $wW_{\hat{\alpha}} \in W/W_{\hat{\alpha}}$ *then* $\mathbf{\Phi}^+ \cap w\Psi^-_{\hat{\alpha}} = \mathbf{\Phi}^+ \cap w\mathbf{\Phi}^-$ *and* $\mathbf{\Phi}^- \cap w\Psi^+_{\hat{\alpha}} = \mathbf{\Phi}^- \cap w\mathbf{\Phi}^+$.

Proof. This follows from Lemma 2.19 of [Chinello 2017]. □

From now on, we set $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0) = [\mathfrak{J}^1_0 : \mathfrak{H}^1_0]$ and $\delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0) = [\mathfrak{H}^1_0 : \varpi\mathfrak{H}^1_0]$.

Remark 3.19. By Remark 3.12, for every $\alpha \in \mathbf{\Phi}$, $\alpha' \in \mathbf{\Phi}^+$ and $\alpha'' \in \mathbf{\Phi}^-$ we have $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0) = [J^1 \cap \mathcal{U}_\alpha : H^1 \cap \mathcal{U}_\alpha]$ and $\delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0) = [H^1 \cap \mathcal{U}_{\alpha'} : (H^1 \cap \mathcal{U}_{\alpha'})^{\tau_{\alpha'}}] = [H^1 \cap \mathcal{U}_{\alpha''} : (H^1 \cap \mathcal{U}_{\alpha''})^{\tau^{-1}_{\alpha''}}]$. In particular $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)$ and $\delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0)$ are powers of p and so they are invertible in R.

From now on we fix $1 \leq i \leq m' - 1$ and we consider $\alpha = \alpha_{i,i+1}$, w of minimal length in $wW_{\hat{\alpha}}$, $P = P(w, \alpha)$ and $Q = Q(w, \alpha)$.

Remark 3.20. Lemma 3.18 implies that $w\mathcal{U}^-_{\hat{\alpha}}w^{-1} \cap \mathcal{U}^+_{\hat{P}} = w\mathcal{U}^-w^{-1} \cap \mathcal{U}^+$ and $w\mathcal{U}^+_{\hat{\alpha}}w^{-1} \cap \mathcal{U}^-_{\hat{P}} = w\mathcal{U}w^{-1} \cap \mathcal{U}^-$. Moreover, we have $\ell(w) = |\Psi^+_{\hat{P}} \cap w\Psi^-_{\hat{\alpha}}| = |\Psi^-_{\hat{P}} \cap w\Psi^+_{\hat{\alpha}}|$ by Remark 3.1.

We define

$$\mathcal{V}(w, \alpha) = (J^1_{\mathcal{P}} \cap w\mathcal{U}^+_{\hat{\alpha}}w^{-1} \cap \mathcal{U}^-_{\hat{P}})^{w\tau^{-1}_\alpha w^{-1}} \tag{4}$$

which is a pro-p-group. We remark that it is equal to $(J^1_{\mathcal{P}} \cap w\mathcal{U}w^{-1} \cap \mathcal{U}^-)^{w\tau^{-1}_\alpha w^{-1}}$ by Remark 3.20 and to $(H^1 \cap w\mathcal{U}^+_{\hat{\alpha}}w^{-1} \cap \mathcal{U}^-_{\hat{P}})^{w\tau^{-1}_\alpha w^{-1}}$ since $J^1_{\mathcal{P}} \cap \mathcal{U}^-_{\hat{P}} = H^1 \cap \mathcal{U}^-_{\hat{P}}$. Then $\mathcal{V}(w, \alpha)$ is equal to

$$\prod_{\alpha' \in w\mathbf{\Psi}^+_{\hat{\alpha}} \cap \mathbf{\Psi}^-_{\hat{P}}} (H^1 \cap \mathcal{U}_{\alpha'})^{w\tau^{-1}_\alpha w^{-1}} = \prod_{\alpha'' \in \mathbf{\Psi}^+_{\hat{\alpha}} \cap w^{-1}\mathbf{\Psi}^-_{\hat{P}}} (H^1 \cap \mathcal{U}_{\alpha''})^{\tau^{-1}_\alpha w^{-1}} = \prod_{\alpha' \in w\mathbf{\Psi}^+_{\hat{\alpha}} \cap \mathbf{\Psi}^-_{\hat{P}}} (\mathbb{I}_{m'} + \varpi^{-1}\mathfrak{H}^1) \cap \mathcal{U}_{\alpha'}$$

which is $(\mathbb{I}_{m'} + \varpi^{-1}\mathfrak{H}^1) \cap w\mathcal{U}_{\hat{\alpha}}^+ w^{-1} \cap \mathcal{U}_{\hat{P}}^-$. We remark that $\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1 = J_{\mathcal{P}}^1 \cap w\mathcal{U}_{\hat{\alpha}}^+ w^{-1} \cap \mathcal{U}_{\hat{P}}^-$ which is equal to $H^1 \cap w\mathcal{U}_{\hat{\alpha}}^+ w^{-1} \cap \mathcal{U}_{\hat{P}}^-$ since $J_{\mathcal{P}}^1 \cap \mathcal{U}^- = H^1 \cap \mathcal{U}^-$.

Lemma 3.21. *The group $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^-$ is in $\mathcal{V}(w,\alpha)$, it normalizes $\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1$ and*

$$(wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^-) \cap (\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1) = wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- \cap K_B^1.$$

Proof. We recall that by Remark 3.13 we have $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- = w\mathcal{U}_{\hat{\alpha}}^+ w^{-1} \cap \mathcal{U}_{\hat{P}}^- \cap K_B$. Since $U_{\alpha'} = \tau_\alpha (K_B^1 \cap U_{\alpha'}) \tau_\alpha^{-1}$ for every $\alpha' \in \Psi_{\hat{\alpha}}^+$ (see Lemma 2.9 of [Chinello 2017]), then we have $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- = (K_B^1 \cap wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^-)^{w\tau_\alpha^{-1}w^{-1}}$ which is contained in $\mathcal{V}(w,\alpha)$. Moreover, the group $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^-$ normalizes $\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1 = \mathcal{V}(w,\alpha) \cap H^1$ because we have $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- \subset K_B$ and K_B normalizes H^1. Finally, since $\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1 = H^1 \cap w\mathcal{U}_{\hat{\alpha}}^+ w^{-1} \cap \mathcal{U}_{\hat{P}}^-$, we have $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- \cap \mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1 = wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- \cap H^1$ and, since $K_B \cap H^1 = K_B^1$, it is equal to $wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- \cap K_B \cap H^1 = wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^- \cap K_B^1$. ☐

By Lemma 3.21 the group $\mathcal{V}' = (wU_{\hat{\alpha}}^+ w^{-1} \cap U_{\hat{P}}^-)(\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1)$ is a subgroup of $\mathcal{V}(w,\alpha)$. We set

$$d(w,\alpha) = [\mathcal{V}(w,\alpha) : \mathcal{V}'] \in R$$

which is nonzero because it is a power of p.

Remark 3.22. We have $\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1 = H^1 \cap w\mathcal{U}_{\hat{\alpha}}^+ w^{-1} \cap \mathcal{U}_{\hat{P}}^- = \prod_{\alpha' \in w\Psi_{\hat{\alpha}}^+ \cap \Psi_{\hat{P}}^-} H^1 \cap \mathcal{U}_{\alpha'}$. Hence, by Remarks 3.19 and 3.20 we have

$$[\mathcal{V}(w,\alpha) : \mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1] = [\varpi^{-1}\mathfrak{H}_0^1 : \mathfrak{H}_0^1]^{\ell(w)} = \delta(\mathfrak{H}_0^1, \varpi\mathfrak{H}_0^1)^{\ell(w)}.$$

On the other hand we have $[\mathcal{V}(w,\alpha) : \mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1] = d(w,\alpha)[\mathcal{V}' : \mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1]$ which is equal to $d(w,\alpha)[(wU^+ w^{-1} \cap U^-)(\mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1) : \mathcal{V}(w,\alpha) \cap J_{\mathcal{P}}^1]$ and by Remark 3.20 to $d(w,\alpha)[wUw^{-1} \cap U^- : wUw^{-1} \cap U^- \cap K_B^1] = d(w,\alpha) q^{\ell(w)}$ where q is the cardinality of $\mathfrak{k}_{D'}$. So, if we denote $\partial = \delta(\mathfrak{H}_0^1, \varpi\mathfrak{H}_0^1)/q \in R^\times$ then $d(w,\alpha) = \partial^{\ell(w)}$.

Lemma 3.23. *We have $(J_{\mathcal{P}}^1)^{\tau_P} J_{\mathcal{P}}^1 \cap (J_{\mathcal{P}}^1)^{w\tau_\alpha^{-1}w^{-1}} J_{\mathcal{P}}^1 = \mathcal{V}(w,\alpha) J_{\mathcal{P}}^1$.*

Proof. We have $(J_{\mathcal{P}}^1)^{w\tau_\alpha^{-1}w^{-1}} = (H^1 \cap w^{-1}\mathcal{U}^- w)^{\tau_\alpha^{-1}w^{-1}} (J_{\mathcal{M}}^1)^{w\tau_\alpha^{-1}w^{-1}} (J^1 \cap w^{-1}\mathcal{U}w)^{\tau_\alpha^{-1}w^{-1}}$. Now we consider the decompositions $H^1 \cap w^{-1}\mathcal{U}^- w = (H^1 \cap w^{-1}\mathcal{U}^- w \cap \mathcal{U})(H^1 \cap w^{-1}\mathcal{U}^- w \cap \mathcal{U}^-)$ and $J^1 \cap w^{-1}\mathcal{U}w = (J^1 \cap w^{-1}\mathcal{U}w \cap \mathcal{U}^-)(J^1 \cap w^{-1}\mathcal{U}w \cap \mathcal{U})$. By Lemma 3.18 we have $J^1 \cap w^{-1}\mathcal{U}w \cap \mathcal{U}^- = J^1 \cap w^{-1}\mathcal{U}w \cap \mathcal{U}_{\hat{\alpha}}^-$ and so $(J^1 \cap w^{-1}\mathcal{U}w \cap \mathcal{U}^-)^{\tau_\alpha^{-1}w^{-1}}$ is contained in $(J^1 \cap \mathcal{U}_{\hat{\alpha}}^-)^{\tau_\alpha^{-1}w^{-1}} \subset (H^1 \cap \mathcal{U}_{\hat{\alpha}}^-)^{w^{-1}} \subset J_{\mathcal{P}}^1$ and, by Lemma 3.15, $(H^1 \cap w^{-1}\mathcal{U}^- w \cap \mathcal{U}^-)^{\tau_\alpha^{-1}w^{-1}}$ is contained in $(H^1 \cap \mathcal{U}^-)^{\tau_\alpha^{-1}w^{-1}} \subset (H^1 \cap \mathcal{U}^-)^{w^{-1}} \subset J_{\mathcal{P}}^1$. Then, since $(J_{\mathcal{M}}^1)^{w\tau_\alpha^{-1}w^{-1}} = J_{\mathcal{M}}^1$ by Lemma 3.15 and since $(H^1 \cap \mathcal{U}^- \cap w\mathcal{U}w^{-1})^{w\tau_\alpha^{-1}w^{-1}} = \mathcal{V}(w,\alpha)$, we obtain $(J_{\mathcal{P}}^1)^{w\tau_\alpha^{-1}w^{-1}} \subset \mathcal{V}(w,\alpha) J_{\mathcal{P}}^1 (J^1 \cap \mathcal{U} \cap w\mathcal{U}w^{-1})^{w\tau_\alpha^{-1}w^{-1}}$. By Lemma 3.16 and by previous calculations we have

$$(J_{\mathcal{P}}^1)^{\tau_P} J_{\mathcal{P}}^1 \cap (J_{\mathcal{P}}^1)^{w\tau_\alpha^{-1}w^{-1}} J_{\mathcal{P}}^1 = \big((J_{\mathcal{P}}^1 \cap \mathcal{U}_{\hat{P}}^-)^{\tau_P} \cap \mathcal{V}(w,\alpha) J_{\mathcal{P}}^1 (J^1 \cap \mathcal{U} \cap w\mathcal{U}w^{-1})^{w\tau_\alpha^{-1}w^{-1}} J_{\mathcal{P}}^1\big) J_{\mathcal{P}}^1.$$

Now, since $w\tau_\alpha^{-1}w^{-1} = \tau_Q^{-1}\tau_P$, the group $\mathcal{V}(w,\alpha)$ is contained both in $(\mathcal{U}_{\hat{P}}^-)^{\tau_Q^{-1}\tau_P} = (\mathcal{U}_{\hat{P}}^-)^{\tau_P}$ and in $(J_{\mathcal{P}}^1 \cap \mathcal{U}^-)^{\tau_Q^{-1}\tau_P} \subset (J_{\mathcal{P}}^1 \cap \mathcal{U}^-)^{\tau_P} \subset (J_{\mathcal{P}}^1)^{\tau_P}$ by Lemma 3.15. This implies $\mathcal{V}(w,\alpha) \subset (J_{\mathcal{P}}^1 \cap \mathcal{U}_{\hat{P}}^-)^{\tau_P}$ and so

$(J^1_{\mathcal{P}})^{\tau_P} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{w\tau_\alpha^{-1}w^{-1}} J^1_{\mathcal{P}} = \mathcal{V}(w,\alpha)\big((J^1_{\mathcal{P}} \cap \mathcal{U}^-_{\hat{P}})^{\tau_P} \cap J^1_{\mathcal{P}}(J^1 \cap \mathcal{U} \cap w\mathcal{U}w^{-1})^{w\tau_\alpha^{-1}w^{-1}} J^1_{\mathcal{P}}\big) J^1_{\mathcal{P}}$. Now we have $(J^1_{\mathcal{P}} \cap \mathcal{U}^-_{\hat{P}})^{\tau_P} \cap J^1_{\mathcal{P}}(J^1 \cap \mathcal{U} \cap w\mathcal{U}w^{-1})^{w\tau_\alpha^{-1}w^{-1}} J^1_{\mathcal{P}} \subset \mathcal{U}^- \cap J^1_{\mathcal{P}}\mathcal{U}J^1_{\mathcal{P}}$ that is in $J^1_{\mathcal{P}}$ by point 3 of Lemma 3.16. □

3F. *The group $\tilde{W}$.* In this section we use a presentation by generators and relations of $\tilde{W}$ to find a subgroup of $\mathrm{Aut}_R(V_{\mathcal{M}})$ isomorphic to a quotient of $\tilde{W}$.

Remark 3.24. We know that the Iwahori–Hecke algebra (see I.3.14 of [Vignéras 1996]) is a deformation of the R-algebra $R[\tilde{W}]$ and so it is not difficult to show that $\tilde{W}$ is the group generated by $s_1, \dots, s_{m'-1}$ and $\tau_{m'-1}$ subject to relations $s_is_j = s_js_i$ for every i and j such that $|i-j|>1$, $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for every $i \neq m'-1$, $s_i^2 = 1$ for every i, $\tau_{m'-1}s_i = s_i\tau_{m'-1}$ for every $i \neq m'-1$ and $\tau_{m'-1}s_{m'-1}\tau_{m'-1}s_{m'-1} = s_{m'-1}\tau_{m'-1}s_{m'-1}\tau_{m'-1}$.

Lemma 3.25. *Let $i \in \{1,\dots,m'-1\}$, $\alpha = \alpha_{i,i+1}$, $w \in W$ be of minimal length in $wW_{\hat{\alpha}}$ and $\Phi \in \mathscr{H}_R(G,\eta_{\mathcal{P}})_{J^1_{\mathcal{P}}\tau_i J^1_{\mathcal{P}}}$. Then the support of $\hat{f}_w\Phi\hat{f}_{w^{-1}}$ is in $J^1_{\mathcal{P}}w\tau_iw^{-1}J^1_{\mathcal{P}}$ and*

$$(\hat{f}_w\Phi\hat{f}_{w^{-1}})(w\tau_iw^{-1}) = \delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{\ell(w)}\hat{f}_w(w)\circ\Phi(\tau_i)\circ\hat{f}_{w^{-1}}(w^{-1}).$$

Proof. Since w and w^{-1} normalize J^1, by Lemma 3.9 the support of $\hat{f}_w\Phi\hat{f}_{w^{-1}}$ is in $J^1_{\mathcal{P}}w\tau_iw^{-1}J^1_{\mathcal{P}}$. We recall that

$$(\hat{f}_w\Phi\hat{f}_{w^{-1}})(w\tau_iw^{-1}) = \sum_{x\in G/J^1_{\mathcal{P}}}(\hat{f}_w\Phi)(w\tau_ix)\hat{f}_{w^{-1}}(x^{-1}w^{-1}).$$

By point 2 of Lemma 3.16, the support of the function $x \mapsto (\hat{f}_w\Phi)(w\tau_ix)\hat{f}_{w^{-1}}(x^{-1}w^{-1})$ is contained in $(J^1_{\mathcal{P}})^{w\tau_i}J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^wJ^1_{\mathcal{P}} = (J^1_{\mathcal{P}})^{w\tau_i}J^1_{\mathcal{P}} \cap (J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)J^1_{\mathcal{P}}$. Since w is of minimal length in $wW_{\hat{\alpha}}$, by Lemma 3.18 we have $J^1 \cap \mathcal{U}^w \cap \mathcal{U}^- = J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-_{\hat{\alpha}}$ which is included in $(J^1_{\mathcal{P}})^{w\tau_i}$ because $(J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-_{\hat{\alpha}})^{\tau_i^{-1}w^{-1}} = ((J^1 \cap \mathcal{U}^-_{\hat{\alpha}})^{\tau_i^{-1}} \cap \mathcal{U}^w)^{w^{-1}}$ that by Lemma 3.15 is included in $(H^1 \cap \mathcal{U}^-_{\hat{\alpha}})^{w^{-1}} \cap \mathcal{U}$ and so in $J^1_{\mathcal{P}}$. Hence, we obtain $(J^1_{\mathcal{P}})^{w\tau_i}J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^wJ^1_{\mathcal{P}} = (J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)J^1_{\mathcal{P}}$. Now, since $(J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)^{w^{-1}}$ and $(J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)^{\tau_i^{-1}w^{-1}}$ are contained in $J^1 \cap \mathcal{U}$ and so in the kernel of $\eta_{\mathcal{P}}$ and since we have $[(J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)J^1_{\mathcal{P}} : J^1_{\mathcal{P}}] = [J^1 \cap \mathcal{U}^w \cap \mathcal{U}^- : H^1 \cap \mathcal{U}^w \cap \mathcal{U}^-] = \delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{\ell(w)}$ we obtain $(\hat{f}_w\Phi\hat{f}_{w^{-1}})(w\tau_iw^{-1}) = \delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{\ell(w)}(\hat{f}_w\Phi)(w\tau_i)\circ\hat{f}_{w^{-1}}(w^{-1})$. To conclude we observe that by Lemma 3.9 the support of $\hat{f}_w\Phi$ is contained in $J^1_{\mathcal{P}}w\tau_iJ^1_{\mathcal{P}}$ and by points 1 and 2 of Lemma 3.16 the support of $x \mapsto (\hat{f}_w)(wx)\Phi(x^{-1}\tau_i)$ is in $(J^1_{\mathcal{P}})^wJ^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{\tau_i^{-1}}J^1_{\mathcal{P}} = (J^1 \cap \mathcal{U}^w \cap \mathcal{U}^-)J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}} \cap \mathcal{U}^+_{P(\tau_i)})^{\tau_i^{-1}}J^1_{\mathcal{P}}$, which is contained in $(\mathcal{U}J^1_{\mathcal{P}} \cap \mathcal{U}^-)J^1_{\mathcal{P}} = J^1_{\mathcal{P}}$ by point 3 of Lemma 3.16. Hence, $(\hat{f}_w\Phi)(w\tau_i) = \hat{f}_w(w)\circ\Phi(\tau_i)$. □

Lemma 3.26. *Let $w \in W$ and $\alpha \in \Sigma$. Then*

$$\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\boldsymbol{p}\circ\kappa(s_\alpha)\circ\iota = \begin{cases}\boldsymbol{p}\circ\kappa(ws_\alpha)\circ\iota & \text{if } w\alpha>0,\\ \delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{-1}\boldsymbol{p}\circ\kappa(ws_\alpha)\circ\iota & \text{if } w\alpha<0.\end{cases}$$

Proof. By Lemma 3.11 we have $\hat{f}_w\hat{f}_{s_\alpha} = \hat{f}_{ws_\alpha}$ and then $(\hat{f}_w\hat{f}_{s_\alpha})(ws_\alpha) = \boldsymbol{p}\circ\kappa(ws_\alpha)\circ\iota$. On the other hand we have

$$(\hat{f}_w\hat{f}_{s_\alpha})(ws_\alpha) = \sum_{x\in G/J^1_{\mathcal{P}}}(\hat{f}_w)(wx)\hat{f}_{s_\alpha}(x^{-1}s_\alpha).$$

Moreover, by point 2 of Lemma 3.16, the support of the function $x \mapsto \hat{f}_w(wx)\hat{f}_{s_\alpha}(x^{-1}s_\alpha)$ is contained in $(J^1_{\mathcal{P}})^w J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{s_\alpha} J^1_{\mathcal{P}} = (J^1_{\mathcal{P}})^w J^1_{\mathcal{P}} \cap (J^1 \cap \mathcal{U}^{s_\alpha} \cap \mathcal{U}^{-1}) J^1_{\mathcal{P}} = ((J^1_{\mathcal{P}})^w J^1_{\mathcal{P}} \cap J^1 \cap \mathcal{U}_{-\alpha}) J^1_{\mathcal{P}}$ which is equal to $J^1_{\mathcal{P}}$ if $w(-\alpha) < 0$ and to $(J^1 \cap \mathcal{U}_{-\alpha}) J^1_{\mathcal{P}}$ if $w(-\alpha) > 0$. Hence, if $w\alpha > 0$ we obtain $(\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = \boldsymbol{p} \circ \kappa(w) \circ \iota \circ \boldsymbol{p} \circ \kappa(s_\alpha) \circ \iota$ while if $w\alpha < 0$, since $(J^1 \cap \mathcal{U}_{-\alpha})^{w^{-1}}$ and $(J^1 \cap \mathcal{U}_{-\alpha})^{s_\alpha}$ are contained in $J^1 \cap \mathcal{U}$ and so in the kernel of $\eta_{\mathcal{P}}$ and since we have $[(J^1 \cap \mathcal{U}_{-\alpha}) J^1_{\mathcal{P}} : J^1_{\mathcal{P}}] = [J^1 \cap \mathcal{U}_{-\alpha} : H^1 \cap \mathcal{U}_{-\alpha}] = \delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)$, we obtain $(\hat{f}_w \hat{f}_{s_\alpha})(ws_\alpha) = \delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0) \boldsymbol{p} \circ \kappa(w) \circ \iota \circ \boldsymbol{p} \circ \kappa(s_\alpha) \circ \iota$. ☐

From now on we fix a nonzero element $\gamma \in I_{\tau_{m'-1}}(\eta_{\mathcal{P}})$, which is invertible by Lemma 3.10, and a square root $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{1/2}$ of $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)$ in R. We consider the function $\hat{f}_{\tau_{m'-1}} \in \mathscr{H}_R(G, \eta_{\mathcal{P}})_{J^1_{\mathcal{P}} \tau_{m'-1} J^1_{\mathcal{P}}}$ defined by $\hat{f}_{\tau_{m'-1}}(j_1 \tau_{m'-1} j_2) = \eta_{\mathcal{P}}(j_1) \circ \gamma \circ \eta_{\mathcal{P}}(j_2)$ for every $j_1, j_2 \in J^1_{\mathcal{P}}$ and the subgroup $\widetilde{\mathcal{W}}$ of $\mathrm{Aut}_R(V_{\mathcal{M}})$ generated by γ and by $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{1/2} \boldsymbol{p} \circ \kappa(s_i) \circ \iota$ with $i \in \{1, \ldots, m'-1\}$.

Lemma 3.27. *The function that maps s_i to $\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{1/2} \boldsymbol{p} \circ \kappa(s_i) \circ \iota$ for every $i \in \{1, \ldots, m'-1\}$ and $\tau_{m'-1}$ to γ extends to a surjective group homomorphism $\varepsilon : \tilde{W} \to \widetilde{\mathcal{W}}$.*

Proof. Let $\delta = \delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)$. To prove that ε is a group homomorphism we use the presentation of $\tilde{W}$ given in Remark 3.24. For every $i, j \in \{1, \ldots, m'-1\}$ such that $|i-j| > 1$ we have $\varepsilon(s_i)\varepsilon(s_j) = \delta \boldsymbol{p} \circ \kappa(s_i) \circ \iota \circ \boldsymbol{p} \circ \kappa(s_j) \circ \iota$ which, by Lemma 3.26, is equal to $\delta \boldsymbol{p} \circ \kappa(s_i s_j) \circ \iota = \delta \boldsymbol{p} \circ \kappa(s_j s_i) \circ \iota = \varepsilon(s_j)\varepsilon(s_i)$. For every $i \neq m'-1$ we have $\varepsilon(s_i)\varepsilon(s_{i+1})\varepsilon(s_i) = \delta^{3/2} \boldsymbol{p} \circ \kappa(s_i) \circ \iota \circ \boldsymbol{p} \circ \kappa(s_{i+1}) \circ \iota \circ \boldsymbol{p} \circ \kappa(s_i) \circ \iota$ which, by Lemma 3.26, is equal to $\delta^{3/2} \boldsymbol{p} \circ \kappa(s_i s_{i+1} s_i) \circ \iota = \delta^{3/2} \boldsymbol{p} \circ \kappa(s_{i+1} s_i s_{i+1}) \circ \iota = \varepsilon(s_{i+1})\varepsilon(s_i)\varepsilon(s_{i+1})$. For every i we have $\varepsilon(s_i)^2 = \delta \boldsymbol{p} \circ \kappa(s_i) \circ \iota \circ \boldsymbol{p} \circ \kappa(s_i) \circ \iota$ which, by Lemma 3.26, is equal to $\boldsymbol{p} \circ \kappa(s_i s_i) \circ \iota$ which is the identity of $\mathrm{Aut}_R(V_{\mathcal{M}})$. Let $\tau = \tau_{m'-1}$ and $\hat{f}_\tau = \hat{f}_{\tau_{m'-1}}$. For every $i \neq m'-1$ we have $\varepsilon(\tau)\varepsilon(s_i) = \delta^{1/2} \gamma \circ \boldsymbol{p} \circ \kappa(s_i) \circ \iota$ which is equal to $\delta^{1/2}(\hat{f}_\tau \hat{f}_{s_i})(\tau s_i)$ since the support of $x \mapsto \hat{f}_\tau(\tau x)\hat{f}_{s_i}(x^{-1}s_i)$ is contained in $(J^1_{\mathcal{P}})^\tau J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{s_i} J^1_{\mathcal{P}} = ((J^1_{\mathcal{P}} \cap \mathcal{U}^-_{P(\tau)})^\tau J^1_{\mathcal{P}} \cap J^1_{\mathcal{P}} \cap \mathcal{U}_{\alpha_{i+1,i}}) J^1_{\mathcal{P}} = J^1_{\mathcal{P}}$. Hence, by Lemma 3.9 we have $\varepsilon(\tau)\varepsilon(s_i) = \delta^{1/2} \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_\tau)(\tau) \circ \kappa(s_i) \circ \iota$. Since $\zeta^{-1}(\hat{f}_\tau)(\tau) \in I_\tau(\eta) = I_\tau(\kappa)$ and $s_i \in J \cap J^\tau$ we obtain $\varepsilon(\tau)\varepsilon(s_i) = \delta^{1/2} \boldsymbol{p} \circ \kappa(s_i) \circ \zeta^{-1}(\hat{f}_\tau)(\tau) \circ \iota = \delta^{1/2}(\hat{f}_{s_i} \hat{f}_\tau)(s_i \tau)$, which is equal to $\delta^{1/2} \boldsymbol{p} \circ \kappa(s_i) \circ \iota \circ \gamma = \varepsilon(s_i)\varepsilon(\tau)$ since the support of $x \mapsto \hat{f}_{s_i}(s_i x)\hat{f}_\tau(x^{-1}\tau)$ is contained in $(J^1_{\mathcal{P}})^{s_i} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{\tau^{-1}} J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap \mathcal{U}_{\alpha_{i+1,i}} \cap (J^1_{\mathcal{P}} \cap \mathcal{U}^+_{P(\tau)})^{\tau^{-1}} J^1_{\mathcal{P}}) J^1_{\mathcal{P}} = J^1_{\mathcal{P}}$. It remains to prove the last relation. Let $s = s_{m'-1}$ and $\tau = \tau_{m'-1}$. Then $\tau s \tau s = \tau_{m'-2} = s \tau s \tau$ and by Lemma 3.9 we have $(\hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(\tau s \tau s) = \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_\tau \hat{f}_s \hat{f}_\tau)(\tau s \tau) \circ \kappa(s) \circ \iota$. Now, since $\zeta^{-1}(\hat{f}_\tau \hat{f}_s \hat{f}_\tau)(\tau s \tau) \in I_{\tau s \tau}(\kappa)$ and $s = s^{\tau s \tau} \in J \cap J^{\tau s \tau}$, we obtain $(\hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(\tau_{m'-2}) = \boldsymbol{p} \circ \kappa(s) \circ \zeta^{-1}(\hat{f}_\tau \hat{f}_s \hat{f}_\tau)(\tau s \tau) \circ \iota = (\hat{f}_s \hat{f}_\tau \hat{f}_s \hat{f}_\tau)(\tau_{m'-2})$. On the other hand we have

$$(\hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(\tau_{m'-2}) = (\hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(\tau s \tau s) = \sum_{x \in G/J^1_{\mathcal{P}}} \hat{f}_\tau(\tau x)(\hat{f}_s \hat{f}_\tau \hat{f}_s)(x^{-1} s \tau s).$$

The support of $x \mapsto \hat{f}_\tau(\tau x)(\hat{f}_s \hat{f}_\tau \hat{f}_s)(x^{-1} s \tau s)$ is in $(H^1 \cap \mathcal{U}_{\alpha'})^\tau J^1_{\mathcal{P}}$ with $\alpha' = \alpha_{m',m'-1}$ by Lemma 3.23. For every $x \in (H^1 \cap \mathcal{U}_{\alpha'})^\tau$ the elements $x^{\tau^{-1}}$ and $(x^{-1})^{s \tau s}$ are in $H^1 \cap \mathcal{U}$ and so in the kernel of $\eta_{\mathcal{P}}$. Then $(\hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(\tau_{m'-2}) = (\hat{f}_s \hat{f}_\tau \hat{f}_s \hat{f}_\tau)(\tau_{m'-2})$ is equal to $\delta(\mathfrak{H}^1_0, \varpi \mathfrak{H}^1_0) \gamma \circ (\hat{f}_s \hat{f}_\tau \hat{f}_s)(s \tau s)$ and by Lemma 3.25 it is also equal to $\delta(\mathfrak{H}^1_0, \varpi \mathfrak{H}^1_0) \varepsilon(\tau)\varepsilon(s)\varepsilon(\tau)\varepsilon(s)$. Now, if $\alpha'' = \alpha_{m'-2,m'-1}$ then $\alpha' \notin \boldsymbol{\Psi}^+_{\widehat{\alpha''}} \cup \boldsymbol{\Psi}^-_{\widehat{\alpha''}}$ and so we have $(J^1_{\mathcal{P}})^s J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{\tau^{-1}_{m'-2}} J^1_{\mathcal{P}} = J^1_{\mathcal{P}} = (J^1_{\mathcal{P}})^{\tau_{m'-2}} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^s J^1_{\mathcal{P}}$. Hence, $(\hat{f}_s \hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(s\tau s\tau s)$ is equal both to

$$\hat{f}_s(s) \circ (\hat{f}_\tau \hat{f}_s \hat{f}_\tau \hat{f}_s)(\tau_{m'-2}) = \delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0)\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{-1/2}\varepsilon(s)\varepsilon(\tau)\varepsilon(s)\varepsilon(\tau)\varepsilon(s)$$

and also to

$$\begin{aligned}(\hat{f}_s \hat{f}_\tau \hat{f}_s \hat{f}_\tau)(\tau_{m'-2}) \circ \hat{f}_s(s) &= \delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0)\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{-1/2}\varepsilon(\tau)\varepsilon(s)\varepsilon(\tau)\varepsilon(s)^2 \\ &= \delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0)\delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{-1/2}\varepsilon(\tau)\varepsilon(s)\varepsilon(\tau).\end{aligned}$$

This implies $\varepsilon(\tau)\varepsilon(s)\varepsilon(\tau)\varepsilon(s) = \varepsilon(s)\varepsilon(\tau)\varepsilon(s)\varepsilon(\tau)$ since both $\delta(\mathfrak{H}^1_0, \varpi\mathfrak{H}^1_0)$ and $\delta^{-1/2}$ are invertible in R. We conclude that ε is a group homomorphism and it is clearly surjective. □

Remark 3.28. For every $w \in W$ we have $\varepsilon(w) = \delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{\ell(w)/2} \boldsymbol{p} \circ \kappa(w) \circ \iota$.

Lemma 3.29. *For every $\tilde{w} \in \tilde{W}$ we have $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_{\mathcal{P}})$.*

Proof. Since $\eta_{\mathcal{M}}$ is the restriction of $\eta_{\mathcal{P}}$ to the group $J^1_{\mathcal{M}}$, we have $\varepsilon(w) = \delta(\mathfrak{J}^1_0, \mathfrak{H}^1_0)^{\ell(w)/2} \hat{f}_w(w) \in I_w(\eta_{\mathcal{M}})$ for every $w \in W$ and $\gamma \in I_{\tau_{m'-1}}(\eta_{\mathcal{M}})$. Then, since every $w \in W$ and $\tau_{m'-1}$ normalize $J^1_{\mathcal{M}}$, we have $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_{\mathcal{M}})$ for every $\tilde{w} \in \tilde{W}$ and so $\varepsilon(\tilde{w}) \in I_{\tilde{w}}(\eta_{\mathcal{P}})$ by Lemma 3.10. □

Lemma 3.30. *For every $\tau', \tau'' \in \boldsymbol{\Delta}$, $\gamma' \in I_{\tau'}(\eta_{\mathcal{P}})$ and $\gamma'' \in I_{\tau''}(\eta_{\mathcal{P}})$ we have $\gamma' \circ \gamma'' = \gamma'' \circ \gamma'$.*

Proof. We recall that for every $\tau \in \boldsymbol{\Delta}$ the vector space $I_\tau(\eta_{\mathcal{P}})$ is 1-dimensional and so there exist elements $c', c'' \in R$ such that $\gamma' = c'\varepsilon(\tau')$ and $\gamma'' = c''\varepsilon(\tau'')$. We obtain $\gamma' \circ \gamma'' = c'c''\varepsilon(\tau') \circ \varepsilon(\tau'') = c'c''\varepsilon(\tau'\tau'') = c'c''\varepsilon(\tau''\tau') = \gamma'' \circ \gamma'$. □

3G. *The isomorphisms $\mathscr{H}_R(G, \eta_P) \cong \mathscr{H}_R(B^\times, K^1_B)$.* In this section we define the elements $\hat{f}_{\tau_i} \in \mathscr{H}_R(G, \eta_{\mathcal{P}})_{J^1_{\mathcal{P}}\tau_i J^1_{\mathcal{P}}}$ for every $i \in \{0, \dots, m'-1\}$ and we prove that $\hat{f}_\omega$ with $\omega \in \Omega$ respect the relations of Lemma 3.14 obtaining an algebra homomorphism from $\mathscr{H}_R(B^\times, K^1_B)$ to $\mathscr{H}_R(G, \eta_{\mathcal{P}})$.

For every $i \in \{0, \dots, m'-1\}$ we put $\gamma_i = \partial^{(m'-i)(m'-i-1)/2}\varepsilon(\tau_i)$ where ∂ is the power of p defined in Remark 3.22. Then γ_i is an invertible element in $I_{\tau_i}(\eta_{\mathcal{P}})$ and $\gamma_{m'-1} = \gamma$.

Lemma 3.31. *We have, for every $i \in \{1, \dots, m'-1\}$,*

$$\gamma_{i-1} \circ \gamma_i^{-1} = \partial^{m'-i}\varepsilon(\tau_{i-1}\tau_i^{-1}) \quad \textit{and} \quad \gamma_i = \prod_{h=i+1}^{m'} \partial^{m'-h}\varepsilon(\tau_i).$$

Proof. Since $((m'-(i-1))(m'-(i-1)-1)-(m'-i)(m'-i-1))/2 = m'-i$ we have that $\gamma_{i-1} \circ \gamma_i^{-1} = \partial^{m'-i}\varepsilon(\tau_{i-1})\varepsilon(\tau_i)^{-1} = \partial^{m'-i}\varepsilon(\tau_{i-1}\tau_i^{-1})$. The second statement is true because

$$\sum_{h=i+1}^{m'} m'-h = \sum_{j=0}^{m'-i-1} j = \frac{(m'-i)(m'-i-1)}{2}.$$ □

For every $i \in \{0, \dots, m'-1\}$ we consider the function $\hat{f}_{\tau_i} \in \mathscr{H}_R(G, \eta_{\mathcal{P}})_{J^1_{\mathcal{P}}\tau_i J^1_{\mathcal{P}}}$ defined by $\hat{f}_{\tau_i}(j_1\tau_i j_2) = \eta_{\mathcal{P}}(j_1) \circ \gamma_i \circ \eta_{\mathcal{P}}(j_2)$ for every $j_1, j_2 \in J^1_{\mathcal{P}}$. We remark that in general $\hat{f}_{\tau_i}$ is not invertible but since τ_0 normalizes $J^1_{\mathcal{P}}$ the function $\hat{f}_{\tau_0}$ is invertible in $\mathscr{H}_R(G, \eta_{\mathcal{P}})$ with inverse $\hat{f}_{\tau_0^{-1}} : \tau_0^{-1}J^1_{\mathcal{P}} \to \mathrm{End}_R(V_{\mathcal{M}})$ defined by $\hat{f}_{\tau_0^{-1}}(\tau_0^{-1}j) = \gamma_0^{-1} \circ \eta_{\mathcal{P}}(j)$ for every $j \in J^1_{\mathcal{P}}$.

Lemma 3.32. *The map $\Theta'' : \mathbf{\Omega} \to \mathscr{H}_R(G, \eta_P)$ given by $f_\omega \mapsto \hat{f}_\omega$ for every $f_\omega \in \mathbf{\Omega}$ is well defined.*

Proof. The map is well defined on f_k with $k \in K_B$ because Θ' is a homomorphism and it is well defined on τ_i with $i \in \{0, \dots, m'-1\}$ because $K_B^1 \tau_i K_B^1 = K_B^1 \tau_j K_B^1$ implies $i = j$. □

Lemma 3.33. *The function $\hat{f}_{\tau_i} \hat{f}_{\tau_j}$ is in $\mathscr{H}_R(G, \eta_{\mathcal{P}})_{J^1_{\mathcal{P}} \tau_i \tau_j J^1_{\mathcal{P}}}$ and $(\hat{f}_{\tau_i} \hat{f}_{\tau_j})(\tau_i \tau_j) = \gamma_i \circ \gamma_j$, for every $i, j \in \{0, \dots, m'-1\}$*

Proof. If i or j is 0 then the result follows from Lemma 3.9 since τ_0 normalizes $J^1_{\mathcal{P}}$. Otherwise, by point 4 of Lemma 3.16 the support of $\hat{f}_{\tau_i} \hat{f}_{\tau_j}$ is contained in $J^1_{\mathcal{P}} \tau_i J^1_{\mathcal{P}} \tau_j J^1_{\mathcal{P}} = J^1_{\mathcal{P}} \tau_i \tau_j J^1_{\mathcal{P}}$ and the support of $x \mapsto \hat{f}_{\tau_i}(\tau_i x) \hat{f}_{\tau_j}(x^{-1} \tau_j)$ is contained in $(J^1_{\mathcal{P}})^{\tau_i} J^1_{\mathcal{P}} \cap (J^1_{\mathcal{P}})^{\tau_j^{-1}} J^1_{\mathcal{P}} = J^1_{\mathcal{P}}$. Hence, we obtain $(\hat{f}_{\tau_i} \hat{f}_{\tau_j})(\tau_i \tau_j) = \sum_{x \in G/J^1_{\mathcal{P}}} \hat{f}_{\tau_i}(\tau_i x) \hat{f}_{\tau_j}(x^{-1} \tau_j) = \hat{f}_{\tau_i}(\tau_i) \circ \hat{f}_{\tau_j}(\tau_j) = \gamma_i \circ \gamma_j$. □

By Lemmas 3.33 and 3.30 we obtain $\hat{f}_{\tau_i} \hat{f}_{\tau_j} = \hat{f}_{\tau_j} \hat{f}_{\tau_i}$ for every $i, j \in \{0, \dots, m'-1\}$. So, if $P \subset \{0, \dots, m'-1\}$ we denote by γ_P the composition of γ_i with $i \in P$, which is well defined by Lemma 3.30, and by $\hat{f}_{\tau_P}$ the product of $\hat{f}_{\tau_i}$ with $i \in P$, which is well defined because the $\hat{f}_{\tau_i}$ commute. Furthermore, by point 4 of Lemma 3.16 we obtain that the support of $\hat{f}_{\tau_P}$ is $J^1_{\mathcal{P}} \tau_P J^1_{\mathcal{P}}$ and by Lemma 3.33 we have $\hat{f}_{\tau_P}(\tau_P) = \gamma_P$.

Lemma 3.34. *We have $\hat{f}_{\tau_i} \hat{f}_x = \hat{f}_{\tau_i x \tau_i^{-1}} \hat{f}_{\tau_i}$ for every $i \in \{0, \dots, m'-1\}$ and every $x \in M_{\widehat{\alpha_{i,i+1}}} = K_B \cap \mathcal{M}_{\widehat{\alpha_{i,i+1}}}$ if $i \neq 0$ or $x \in K_B$ if $i = 0$.*

Proof. Since x normalizes J^1, by Lemma 3.9 the supports of $\hat{f}_{\tau_i} \hat{f}_x$ and of $\hat{f}_{\tau_i x \tau_i^{-1}} \hat{f}_{\tau_i}$ are contained in $J^1_{\mathcal{P}} \tau_i x J^1_{\mathcal{P}}$ and $(\hat{f}_{\tau_i} \hat{f}_x)(\tau_i x) = \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \kappa(x) \circ \iota$, which is equal to $\boldsymbol{p} \circ \kappa(\tau_i x \tau_i^{-1}) \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \iota = (\hat{f}_{\tau_i x \tau_i^{-1}} \hat{f}_{\tau_i})(\tau_i x)$ because $\zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \in I_{\tau_i}(\kappa)$ and $x \in J \cap J^{\tau_i}$. □

Lemma 3.35. *Let $i \in \{1, \dots, m'-1\}$ and $\alpha \in \Psi^+_{\widehat{\alpha_{i,i+1}}}$. Then for every $u \in U_\alpha$ and $u' \in U_{-\alpha}$ we have $\hat{f}_u \hat{f}_{\tau_i} = \hat{f}_{\tau_i}$ and $\hat{f}_{\tau_i} \hat{f}_{u'} = \hat{f}_{\tau_i}$.*

Proof. The elements $\tau_i^{-1} u \tau_i$ and $\tau_i u' \tau_i^{-1}$ are in $K^1_B \subset J^1_{\mathcal{P}}$ and so, since u and u' normalize J^1, by Lemma 3.9 the supports of $\hat{f}_u \hat{f}_{\tau_i}$ and of $\hat{f}_{\tau_i} \hat{f}_{u'}$ are in $J^1_{\mathcal{P}} u \tau_i J^1_{\mathcal{P}} = J^1_{\mathcal{P}} \tau_i J^1_{\mathcal{P}} = J^1_{\mathcal{P}} \tau_i u' J^1_{\mathcal{P}}$. Now since $\zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \in I_{\tau_i}(\eta) = I_{\tau_i}(\kappa)$ and $u \in J \cap J^{\tau_i^{-1}}$, by Lemma 3.9 we have $(\hat{f}_u \hat{f}_{\tau_i})(u \tau_i) = \boldsymbol{p} \circ \kappa(u) \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \iota = \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \eta(\tau_i^{-1} u \tau_i) \circ \iota$. By Lemma 3.7 we obtain $(\hat{f}_u \hat{f}_{\tau_i})(u \tau_i) = \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \iota \circ \eta_{\mathcal{P}}(\tau_i^{-1} u \tau_i) = \hat{f}_{\tau_i}(\tau_i) \circ \eta_{\mathcal{P}}(\tau_i^{-1} u \tau_i) = \hat{f}_{\tau_i}(u \tau_i)$. Similarly we have $\hat{f}_{\tau_i}(\tau_i u') = \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \kappa(u') \circ \iota = \boldsymbol{p} \circ \eta(\tau_i u' \tau_i^{-1}) \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \iota$ which is equal to $\eta_{\mathcal{P}}(\tau_i u' \tau_i^{-1}) \circ \boldsymbol{p} \circ \zeta^{-1}(\hat{f}_{\tau_i})(\tau_i) \circ \iota = \eta_{\mathcal{P}}(\tau_i u' \tau_i^{-1}) \circ \hat{f}_{\tau_i}(\tau_i) = \hat{f}_{\tau_i}(\tau_i u')$. □

We introduce some subgroups of G, through its identification with $\mathrm{GL}_{m'}(A(E))$, in order to find the support of $\hat{f}_{\tau_P} \hat{f}_w \hat{f}_{\tau_\alpha} \hat{f}_{w^{-1}}$. We recall that $\mathfrak{A}(E)$ is the unique hereditary order normalized by $E^\times$ in $A(E)$ and $\mathfrak{P}(E)$ is its radical.

- Let $\mathcal{Z}$ be the set of matrices (z_{ij}) such that $z_{ii} = 1$, $z_{ij} \in \varpi^{-1} \mathfrak{P}(E)$ if $i < j$ and $z_{ij} = 0$ if $i > j$.
- Let $\mathcal{V}$ be the group $(J^1 \cap w \mathcal{U}^-_{\hat{\alpha}} w^{-1} \cap \mathcal{U}^+_{\hat{P}})^{w \tau_\alpha w^{-1}} = \prod_{\alpha' \in w \Psi^-_{\hat{\alpha}} \cap \Psi^+_{\hat{P}}} (\mathbb{I}_{m'} + \varpi^{-1} \mathfrak{J}^1) \cap \mathcal{U}_{\alpha'} \subset \mathcal{Z}$. We remark that it is different from $\mathcal{V}(w, \alpha)$ defined by (4).
- Let $\tilde{I}^1$ be the group of matrices (m_{ij}) such that $m_{ii} \in 1 + \mathfrak{P}(E)$, $m_{ij} \in \mathfrak{A}(E)$ if $i < j$ and $m_{ij} \in \mathfrak{P}(E)$ if $i > j$.

- Let $\boldsymbol{W} = W \ltimes M$ be the subgroup of $B^\times$ of monomial matrices with coefficients in $\mathcal{O}_{D'}^\times$. Then $B^\times$ is the disjoint union of $I_B(1)wI_B(1)$ with $w \in \boldsymbol{W}$, where $I_B(1) = K^1U$ is the standard *pro-p-Iwahori subgroup of* K_B, i.e., the pro-p-radical of I_B.

Lemma 3.36. *We have* $J^1_{\mathcal{P}}\tau_P J^1_{\mathcal{P}} w\tau_\alpha w^{-1} J^1_{\mathcal{P}} = J^1_{\mathcal{P}}\tau_Q \mathcal{V} J^1_{\mathcal{P}}$.

Proof. We proceed in a similar way to the beginning of the proof of Lemma 3.23: we can prove that $J^1_{\mathcal{P}} w\tau_\alpha w^{-1} J^1_{\mathcal{P}} = (J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1}) w\tau_\alpha w^{-1} J^1_{\mathcal{P}}$. Now we consider the decomposition of the group $(J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1})$ into the product $(J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1} \cap \mathcal{U}^-)(J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1} \cap \mathcal{U})$. By Lemma 3.15 we have $(J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1} \cap \mathcal{U}^-)^{\tau_P^{-1}} \subset J^1_{\mathcal{P}}$ and by Lemma 3.18 we have $J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1} \cap \mathcal{U} = J^1_{\mathcal{P}} \cap w\mathcal{U}^-_{\hat{\alpha}} w^{-1} \cap \mathcal{U}^+_{\hat{P}}$. □

Lemma 3.37. *Let* $\tau \in \boldsymbol{\Delta}$. *If* $z \in \mathcal{Z}$ *is such that* $\tilde{I}^1\tau z\tilde{I}^1 \cap \boldsymbol{W} \neq \varnothing$ *then* $\tilde{I}^1\tau z\tilde{I}^1 \cap \boldsymbol{W} = \{\tau\}$.

Proof. For every $r \in \{1, \dots, m'\}$ we denote by $\boldsymbol{\Delta}_{(r)}$, $\mathcal{Z}_{(r)}$, $\tilde{I}^1_{(r)}$ and $\boldsymbol{W}_{(r)}$ the subsets of $\mathrm{GL}_r(A(E))$ similar to those defined for $\mathrm{GL}_{m'}(A(E))$. We prove the statement of the lemma by induction on r. If $r = 1$ we have $\boldsymbol{\Delta}_{(1)} = \varpi^{\mathbb{Z}}$, $\mathcal{Z}_{(1)} = \{1\}$, $\tilde{I}^1_{(1)} = 1 + \mathfrak{P}(E)$ and $\boldsymbol{W}_{(1)} = \varpi^{\mathbb{Z}}$ and we have $(1+\mathfrak{P}(E))\varpi^a(1+\mathfrak{P}(E)) \cap \varpi^{\mathbb{Z}} = \varpi^a(1+\mathfrak{P}(E)) \cap \varpi^{\mathbb{Z}} = \{\varpi^a\}$ for every $a \in \mathbb{Z}$. Now we suppose the statement true for every $r < m'$. Let $x, y \in \tilde{I}^1$ such that $x\tau z y \in \boldsymbol{W}$. We proceed by steps.

First step: We consider the decomposition $\tilde{I}^1 = (\tilde{I}^1 \cap \mathcal{U}^-)(\tilde{I}^1 \cap \mathcal{U})(\tilde{I}^1 \cap \mathcal{M})$ and we write $x = x_1x_2x_3$ with $x_1 \in \tilde{I}^1 \cap \mathcal{U}^-$, $x_2 \in \tilde{I}^1 \cap \mathcal{U}$ and $x_3 \in \tilde{I}^1 \cap \mathcal{M}$. Then we have

$$x\tau z y = x_1\tau((\tau^{-1}x_2\tau)(\tau^{-1}x_3\tau)z(\tau^{-1}x_3^{-1}\tau))(\tau^{-1}x_3\tau)y.$$

We observe that $\tau^{-1}x_3\tau$ is a diagonal matrix with coefficients in $1 + \mathfrak{P}(E)$ and the conjugate of z by this element is in $\mathcal{Z}$. Moreover, $\tau^{-1}x_2\tau$ is in $\tilde{I}^1 \cap \mathcal{U}$ and if we multiply it by an element of $\mathcal{Z}$ we obtain another element of $\mathcal{Z}$. If we set $z_1 = \tau^{-1}x_2x_3\tau z\tau^{-1}x_3^{-1}\tau \in \mathcal{Z}$ then $\tilde{I}^1\tau z\tilde{I}^1 = \tilde{I}^1\tau z_1\tilde{I}^1$ and $(\tilde{I}^1 \cap \mathcal{U}^-)\tau z_1\tilde{I}^1 \cap \boldsymbol{W} \neq \varnothing$. Hence, we can suppose $x \in \tilde{I}^1 \cap \mathcal{U}^-$.

Second step: Let $a_1 \leq \cdots \leq a_{m'} \in \mathbb{N}$ such that $\tau = \mathrm{diag}(\varpi^{a_i})$ and let $s \in \mathbb{N}^*$ such that $a_1 = \cdots = a_s$ and $a_1 < a_{s+1}$. We want to prove $z_{ij} \in \mathfrak{A}(E)$ for every $i \in \{1, \dots, s\}$ so we assume the opposite and we look for a contradiction. Let v be the valuation on $A(E)$ associated to $\mathfrak{P}(E)$ and let

$$b = \min\{\mathrm{v}(\varpi^{a_1}z_{ij}) \mid 1 \leq i \leq s, 1 \leq j \leq m'\},$$
$$k = \min\{1 \leq j \leq m' \mid \text{there exists } z_{ij} \text{ with } 1 \leq i \leq s \text{ such that } \mathrm{v}(\varpi^{a_1}z_{ij}) = b\}.$$

Let $1 \leq h \leq s$ be such that $\mathrm{v}(\varpi^{a_1}z_{hk}) = b$. By hypothesis the element z_{hk} is not in $\mathfrak{A}(E)$ and so $h < k$ and

$$(a_1 - 1)\mathrm{v}(\varpi) < b < a_1\mathrm{v}(\varpi). \tag{5}$$

We observe that for every $i \in \{1, \dots, m'\}$ and $j > i$ we have $\mathrm{v}(\varpi^{a_i}z_{ij}) \geq b$: if $i \leq s$ by definition of b and if $i > s$ because $\mathrm{v}(\varpi^{a_i}z_{ij}) = a_i\mathrm{v}(\varpi) + \mathrm{v}(z_{ij}) > (a_i - 1)\mathrm{v}(\varpi) \geq a_1\mathrm{v}(\varpi) > b$. We consider the coefficient at position (h, k) of $x\tau zy$ which is equal to

$$\sum_{e=1}^{m'}\sum_{f=1}^{m'} x_{he}\varpi^{a_e} z_{ef} y_{fk} = \sum_{e=1}^{h}\sum_{f=e}^{m'} x_{he}\varpi^{a_1} z_{ef} y_{fk},$$

since $x_{he}=0$ if $e>h$ and $z_{ef}=0$ if $f<e$. Now,

- if $e=h$ and $f=k$ then $\mathrm{v}(x_{hh}\varpi^{a_1}z_{hk}y_{kk})=b$ because $x_{hh}=1$, and $y_{kk}\in 1+\mathfrak{P}(E)$;
- if $e=h$ and $f<k$ then $\mathrm{v}(x_{hh}\varpi^{a_1}z_{hf}y_{fk})>b$ by definition of k;
- if $e=h$ and $f>k$ then $\mathrm{v}(x_{hh}\varpi^{a_1}z_{hf}y_{fk})>b$ because $y_{fk}\in\mathfrak{P}(E)$;
- if $e<h$ then $\mathrm{v}(x_{he}\varpi^{a_1}z_{ef}y_{fk})>b$ because $x_{he}\in\mathfrak{P}(E)$.

We obtain an element of valuation b. Then b must be a multiple of $\mathrm{v}(\varpi)$ because $x\tau zy\in \boldsymbol{W}$ but this in contradiction with (5). Hence, $z_{ij}\in\mathfrak{A}(E)$ for every $i\in\{1,\dots,s\}$. Now, we can write $z=z'z''$ with $z'_{ii}=1$ for all i, $z'_{ij}=z_{ij}$ if $i\in\{s+1,\dots,m'\}$ and $j>i$ and $z'_{ij}=0$ otherwise and $z''_{ii}=1$ for all i, $z''_{ij}=z_{ij}$ if $i\in\{1,\dots,s\}$ and $j>i$ and $z''_{ij}=0$ otherwise. Then $z''\in\tilde{I}^1$ and so $\tilde{I}^1\tau z\tilde{I}^1=\tilde{I}^1\tau z'\tilde{I}^1$ and $(\tilde{I}^1\cap\mathcal{U}^-)\tau z'\tilde{I}^1\cap\boldsymbol{W}\neq\varnothing$. Then we can suppose z of the form $\left(\begin{smallmatrix}\mathbb{I}_s & 0\\ 0 & \hat{z}\end{smallmatrix}\right)$ with $\hat{z}\in\mathcal{Z}_{(m'-s)}$.

Third step: We write $x=x'x''$ with $x'_{ii}=1$ for all i, $x'_{ij}=x_{ij}$ if $i\in\{s+1,\dots,m'\}$ and $j<i$ and $x'_{ij}=0$ otherwise and $x''_{ii}=1$ for all i, $x''_{ij}=x_{ij}$ if $i\in\{1,\dots,s\}$ and $j<i$ and $x''_{ij}=0$ otherwise. Then $\tau^{-1}x''\tau\in\tilde{I}^1$ and it commutes with z. Then we can suppose x is of the form $\left(\begin{smallmatrix}\mathbb{I}_s & 0\\ x''' & \hat{x}\end{smallmatrix}\right)$ with $x'''\in M_{(m'-s)\times s}(\mathfrak{P}(E))$ and $\hat{x}\in\tilde{I}^1_{(m'-s)}$.

Fourth step: Let $\tau=\left(\begin{smallmatrix}\varpi^{a_1}\mathbb{I}_s & 0\\ 0 & \hat{\tau}\end{smallmatrix}\right)$ with $\hat{\tau}\in\boldsymbol{\Delta}_{(m'-s)}$ and $y=\left(\begin{smallmatrix}y_1 & y_2\\ y_3 & \hat{y}\end{smallmatrix}\right)$ with $y_1\in\tilde{I}^1_{(s)}$, $y_2\in M_{s\times(m'-s)}(\mathfrak{A}(E))$, $y_3\in M_{(m'-s)\times s}(\mathfrak{P}(E))$ and $\hat{y}\in\tilde{I}^1_{(m'-s)}$. Then the product $x\tau zy$ is

$$\begin{pmatrix}\mathbb{I}_s & 0\\ x''' & \hat{x}\end{pmatrix}\begin{pmatrix}\varpi^{a_1}\mathbb{I}_s & 0\\ 0 & \hat{\tau}\end{pmatrix}\begin{pmatrix}\mathbb{I}_s & 0\\ 0 & \hat{z}\end{pmatrix}\begin{pmatrix}y_1 & y_2\\ y_3 & \hat{y}\end{pmatrix}=\begin{pmatrix}\varpi^{a_1}y_1 & \varpi^{a_1}y_2\\ t & x'''\varpi^{a_1}y_2+\hat{x}\hat{\tau}\hat{z}\hat{y}\end{pmatrix}$$

where $t=x'''\varpi^{a_1}y_1+\hat{x}\hat{\tau}\hat{z}y_3$. Since $x\tau zy$ is in $\boldsymbol{W}$ and since $y_1\in\tilde{I}^1_{(s)}$ is invertible then $\varpi^{a_1}y_1$ must be in $\boldsymbol{W}_{(s)}$ and so $y_1=\mathbb{I}_s$. This also implies $\varpi^{a_1}y_2=t=0$ since $x\tau zy$ is a monomial matrix and so $x\tau zy=\left(\begin{smallmatrix}\varpi^{a_1}\mathbb{I}_s & 0\\ 0 & \hat{x}\hat{\tau}\hat{z}\hat{y}\end{smallmatrix}\right)$ with $\hat{x}\hat{\tau}\hat{z}\hat{y}\in\boldsymbol{W}_{(m'-s)}$. Now, since $\tilde{I}^1_{(m'-s)}\hat{\tau}\hat{z}\tilde{I}^1_{(m'-s)}\cap\boldsymbol{W}_{(m'-s)}\neq\varnothing$, by the inductive hypothesis we have $\hat{x}\hat{\tau}\hat{z}\hat{y}=\hat{\tau}$ and so $x\tau zy=\tau$. ☐

Lemma 3.38. *We have* $J^1_{\mathcal{P}}\tau_P J^1_{\mathcal{P}}w\tau_\alpha w^{-1}J^1_{\mathcal{P}}\cap J^1_{\mathcal{P}}B^\times J^1_{\mathcal{P}}=J^1_{\mathcal{P}}\tau_Q(U\cap wU^-w^{-1})J^1_{\mathcal{P}}$.

Proof. By Lemma 3.36 we have $J^1_{\mathcal{P}}\tau_P J^1_{\mathcal{P}}w\tau_\alpha w^{-1}J^1_{\mathcal{P}}=J^1_{\mathcal{P}}\tau_Q\mathcal{V}J^1_{\mathcal{P}}$. Now, since $\mathfrak{J}^1\subset M_{m'}(\mathfrak{P}(E))$ we have $\mathcal{V}\subset\mathcal{Z}$ and $J^1_{\mathcal{P}}\subset\tilde{I}^1$ and so we obtain

$$\begin{aligned}J^1_{\mathcal{P}}\tau_P J^1_{\mathcal{P}}w\tau_\alpha w^{-1}J^1_{\mathcal{P}}\cap B^\times &\subset \tilde{I}^1\tau_Q\mathcal{Z}\tilde{I}^1\cap K^1_BU\boldsymbol{W}UK^1_B=K^1_BU(\tilde{I}^1\tau_Q\mathcal{Z}\tilde{I}^1\cap\boldsymbol{W})UK^1_B\\ &\text{(Lemma 3.37)}=K^1_BU\tau_QUK^1_B=K^1_B\tau_QUK^1_B.\end{aligned}$$

This implies $J^1_{\mathcal{P}}\tau_P J^1_{\mathcal{P}}w\tau_\alpha w^{-1}J^1_{\mathcal{P}}\cap B^\times=J^1_{\mathcal{P}}\tau_Q\mathcal{V}J^1_{\mathcal{P}}\cap K^1_B\tau_QUK^1_B$. Let now $v\in\mathcal{V}$ be such that $J^1_{\mathcal{P}}\tau_Q vJ^1_{\mathcal{P}}\cap K^1_B\tau_QUK^1_B\neq\varnothing$. Then $v\in\tau_Q^{-1}J^1_{\mathcal{P}}K^1_B\tau_QUK^1_BJ^1_{\mathcal{P}}\cap\mathcal{V}\subset\tau_Q^{-1}J^1_{\mathcal{P}}\tau_QUJ^1_{\mathcal{P}}\cap\mathcal{U}$. Now $U=K_B\cap\mathcal{U}\subset J\cap\mathcal{P}$ normalizes $J^1_{\mathcal{P}}$ and so $v\in\tau_Q^{-1}J^1_{\mathcal{P}}\tau_QJ^1_{\mathcal{P}}U\cap\mathcal{U}$ which is in $(\tau_Q^{-1}(J^1_{\mathcal{P}}\cap\mathcal{U}^-_{\hat{Q}})\tau_QJ^1_{\mathcal{P}}\cap\mathcal{U})U$ by point 1 of Lemma 3.16. Hence, by point 3 of Lemma 3.16 we obtain $v\in UJ^1_{\mathcal{P}}\cap\mathcal{V}\subset UJ^1\cap\mathcal{V}$. By Lemma 3.18 we have $U\cap wU^-w^{-1}=U^+_{\hat P}\cap wU^-_{\hat\alpha}w^{-1}$ and proceeding in a way similar to the proof of Lemma 3.21 we can prove $U^+_{\hat P}\cap wU^-_{\hat\alpha}w^{-1}\subset\mathcal{V}$. We obtain

$$\begin{aligned}UJ^1\cap\mathcal{V}&=(U\cap wU^-w^{-1})(U\cap wUw^{-1})J^1\cap\mathcal{V}\\&=(U\cap wU^-w^{-1})(J^1(U\cap wUw^{-1})\cap\mathcal{V})\\&=(U\cap wU^-w^{-1})(J^1(w^{-1}Uw\cap U)\cap\mathcal{V}^w)^{w^{-1}}.\end{aligned}$$

By the definition of $\mathcal{V}$ we have $\mathcal{V}^w=(J^1_{\mathcal{P}}\cap w\mathcal{U}^-_{\hat\alpha}w^{-1}\cap\mathcal{U}^+_{\hat P})^{w\tau_\alpha}\subset(\mathcal{U}^-_{\hat\alpha})^{\tau_\alpha}\subset\mathcal{U}^-$ and then $UJ^1\cap\mathcal{V}\subset(U\cap wU^-w^{-1})(J^1\mathcal{U}\cap\mathcal{U}^-)^{w^{-1}}$ which, by Remark 3.17, is equal to $(U\cap wU^-w^{-1})J^1$. Hence v is in $(U\cap wU^-w^{-1})J^1\cap UJ^1_{\mathcal{P}}=(U\cap wU^-w^{-1})(J^1\cap U)J^1_{\mathcal{P}}$ which is contained in $(U\cap wU^-w^{-1})K^1_BJ^1_{\mathcal{P}}=(U\cap wU^-w^{-1})J^1_{\mathcal{P}}$ and so $J^1\tau_PJ^1_{\mathcal{P}}w\tau_\alpha w^{-1}J^1_{\mathcal{P}}\cap J^1_{\mathcal{P}}B^\times J^1_{\mathcal{P}}=J^1_{\mathcal{P}}\tau_Q(U\cap wU^-w^{-1})J^1_{\mathcal{P}}$. □

Lemma 3.39. *For every $u\in U\cap wU^-w^{-1}$ we have*

$$(\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(\tau_Qu)=q^{\ell(w)}d(w,\alpha)\delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{\ell(w)}\gamma_P\circ\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\gamma_i\circ\boldsymbol{p}\circ\kappa(w^{-1})\circ\iota\circ\boldsymbol{p}\circ\kappa(u)\circ\iota.$$

Proof. By Lemma 3.38 the support of $\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}}$ is contained in $J^1_{\mathcal{P}}\tau_Q(U\cap wU^-w^{-1})J^1_{\mathcal{P}}$. Let $u\in U\cap wU^-w^{-1}$. By Lemma 3.18 we have $U\cap wU^-w^{-1}=U^+_{\hat P}\cap wU^-_{\hat\alpha}w^{-1}$, by Lemma 3.35 we have $\hat f_{\tau_\alpha}=\hat f_{\tau_\alpha}\hat f_{w^{-1}uw}$ and by Lemma 3.11 we have $\hat f_{w^{-1}uw}\hat f_{w^{-1}}=\hat f_{w^{-1}}\hat f_u$. Since u is in $U=K_B\cap\mathcal{U}\subset J\cap\mathcal{P}$, it normalizes $J^1_{\mathcal{P}}$ and then by Lemma 3.9 we obtain $(\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(\tau_Qu)=(\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}}\hat f_u)(\tau_Qu)=(\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(\tau_Q)\circ\boldsymbol{p}\circ\kappa(u)\circ\iota$. It remains to calculate

$$(\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(\tau_Q)=\sum_{x\in G/J^1_{\mathcal{P}}}\hat f_{\tau_P}(\tau_Px)(\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(x^{-1}w\tau_\alpha w^{-1}).$$

By Lemma 3.23 the support of the function $x\mapsto\hat f_{\tau_P}(\tau_Px)(\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(x^{-1}w\tau_\alpha w^{-1})$ is in $\mathcal{V}(w,\alpha)J^1_{\mathcal{P}}$. Now, since for every $x\in\mathcal{V}(w,\alpha)=(J^1_{\mathcal{P}}\cap w\mathcal{U}^+_{\hat\alpha}w^{-1}\cap\mathcal{U}^-_{\hat P})^{w\tau_\alpha^{-1}w^{-1}}$ we have $(x^{-1})^{w\tau_\alpha w^{-1}}\in J^1_{\mathcal{P}}\cap\mathcal{U}^-$ and $x^{\tau_{\mathcal{P}}^{-1}}\in(J^1_{\mathcal{P}}\cap w\mathcal{U}^+_{\hat\alpha}w^{-1}\cap\mathcal{U}^-_{\hat P})^{\tau_{\mathcal{Q}}^{-1}}\subset(J^1_{\mathcal{P}}\cap\mathcal{U}^-)^{\tau_{\mathcal{Q}}^{-1}}$ which is in $J^1_{\mathcal{P}}\cap\mathcal{U}^-$ by Lemma 3.15, then $(x^{-1})^{w\tau_\alpha w^{-1}}$ and $x^{\tau_{\mathcal{P}}^{-1}}$ are in the kernel of $\eta_{\mathcal{P}}$. We obtain

$$\begin{aligned}(\hat f_{\tau_P}\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(\tau_Q)&=[\mathcal{V}(w,\alpha):\mathcal{V}(w,\alpha)\cap H^1]\hat f_{\tau_P}(\tau_P)\circ(\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(w\tau_\alpha w^{-1})\\ \text{(Remark 3.22)}&=d(w,\alpha)q^{\ell(w)}\hat f_{\tau_P}(\tau_P)\circ(\hat f_w\hat f_{\tau_\alpha}\hat f_{w^{-1}})(w\tau_\alpha w^{-1})\\ \text{(Lemma 3.25)}&=d(w,\alpha)q^{\ell(w)}\delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{\ell(w)}\gamma_P\circ\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\gamma_i\circ\boldsymbol{p}\circ\kappa(w^{-1})\circ\iota.\end{aligned}$$

The result follows. □

Lemma 3.40. *We have $\gamma_Q=d(w,\alpha)\delta(\mathfrak{J}^1_0,\mathfrak{H}^1_0)^{\ell(w)}\gamma_P\circ\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\gamma_i\circ\boldsymbol{p}\circ\kappa(w^{-1})\circ\iota$.*

Proof. By the definition of $P=P(w,\alpha)$ and $Q=Q(w,\alpha)$ (see Section 3D) we have

$$\tau_P^{-1}\tau_Q=w\tau_iw^{-1}=\prod_{h=i+1}^{m'}\tau^{-1}_{w(h)}\tau_{w(h)-1}$$

and so

$$
\begin{aligned}
\gamma_P^{-1}\gamma_Q &= \prod_{h=i+1}^{m'} \gamma_{w(h)}^{-1}\gamma_{w(h)-1} \\
\text{(Lemma 3.31)} &= \prod_{h=i+1}^{m'} \partial^{m'-w(h)}\varepsilon(\tau_{w(h)}^{-1}\tau_{w(h)-1}) \\
&= \Bigl(\prod_{h=i+1}^{m'} \partial^{m'-w(h)}\Bigr)\varepsilon(w\tau_i w^{-1}) \\
\text{(Lemma 3.31)} &= \Bigl(\prod_{h=i+1}^{m'} \partial^{m'-w(h)}\Bigr)\Bigl(\prod_{h=i+1}^{m'} \partial^{h-m'}\Bigr)\varepsilon(w)\circ\gamma_i\circ\varepsilon(w^{-1}) \\
\text{(Remark 3.28)} &= \Bigl(\prod_{h=i+1}^{m'} \partial^{m'-w(h)}\Bigr)\Bigl(\prod_{h=i+1}^{m'} \partial^{h-m'}\Bigr)\delta(\mathfrak{J}_0^1,\mathfrak{H}_0^1)^{\ell(w)}\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\gamma_i\circ\boldsymbol{p}\circ\kappa(w^{-1})\circ\iota \\
&= \Bigl(\prod_{h=i+1}^{m'} \partial^{h-w(h)}\Bigr)\delta(\mathfrak{J}_0^1,\mathfrak{H}_0^1)^{\ell(w)}\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\gamma_i\circ\boldsymbol{p}\circ\kappa(w^{-1})\circ\iota.
\end{aligned}
$$

It remains to prove that $d(w,\alpha)=\prod_{h=i+1}^{m'}\partial^{h-w(h)}$. Since by Remark 3.22 we have $d(w,\alpha)=\partial^{\ell(w)}$, it is sufficient to prove $\sum_{h=i+1}^{m'} h-w(h)=\ell(w)$. We prove this statement by induction on $\ell(w)$. If $\ell(w)=1$, since w is of minimal length in $wW_{\hat{\alpha}}$, we have $w=s_\alpha=(i,i+1)$ and

$$\sum_{h=i+1}^{m'} h-w(h)=i+1-w(i+1)+\sum_{h=i+2}^{m'} h-w(h)=i+1-i+0=1.$$

Let now w be of length $\ell(w)=n>1$. By Lemma 2.12 of [Chinello 2017] there exists $\alpha_{j,j+1}\in P$ and $w'\in W$ of length $n-1$ such that $w=s_jw'$. Then w' is of minimal length in $w'W_{\hat{\alpha}}$ and so we can use the inductive hypothesis. Moreover, by definition of P, there exist $\hat{h}\in\{i+1,\dots,m'\}$ such that $j=w(\hat{h})$ and $j+1\neq w(h)$ for every $h\in\{i+1,\dots,m'\}$ and then $w(h)=w'(h)$ for every $h\in\{i+1,\dots,m'\}$ different from $\hat{h}$. We obtain $\sum_{h=i+1}^{m'} h-w(h)=\sum_{h\neq\hat{h}}(h-w(h))+\hat{h}-w'(\hat{h})+w'(\hat{h})-w(\hat{h})$ which is equal to

$$\sum_{h\neq\hat{h}}(h-w'(h))+\hat{h}-w'(\hat{h})+(s_j(j))-j=\sum_{h=i+1}^{m'} h-w'(h)+j+1-j=\ell(w')+1=\ell(w). \quad \square$$

Lemma 3.41. *We have $\hat{f}_{\tau_P}\hat{f}_w\hat{f}_{\tau_\alpha}\hat{f}_{w^{-1}}=q^{\ell(w)}\hat{f}_{\tau_Q}\sum_u\hat{f}_u$ where u runs over a system of representatives of $(U\cap wU^-w^{-1})K^1/K^1$ in $U\cap wU^-w^{-1}$.*

Proof. By Lemma 3.38 the support of $\hat{f}_{\tau_P}\hat{f}_w\hat{f}_{\tau_\alpha}\hat{f}_{w^{-1}}$ is contained in $J_{\mathcal{P}}^1\tau_Q(U\cap wU^-w^{-1})J_{\mathcal{P}}^1$. For every $u'\in U\cap wU^-w^{-1}$, by Lemmas 3.39 and 3.40, $(\hat{f}_{\tau_P}\hat{f}_w\hat{f}_{\tau_\alpha}\hat{f}_{w^{-1}})(\tau_Q u')$ is equal to

$$q^{\ell(w)}d(w,\alpha)\delta(\mathfrak{J}_0^1,\mathfrak{H}_0^1)^{\ell(w)}\gamma_P\circ\boldsymbol{p}\circ\kappa(w)\circ\iota\circ\gamma_i\circ\boldsymbol{p}\circ\kappa(w^{-1})\circ\iota\circ\boldsymbol{p}\circ\kappa(u')\circ\iota=q^{\ell(w)}\gamma_Q\circ\boldsymbol{p}\circ\kappa(u')\circ\iota.$$

To conclude we observe that $\big(\hat{f}_{\tau_Q}\sum_u \hat{f}_u\big)(\tau_Q u') = (\hat{f}_{\tau_Q}\hat{f}_{u'})(\tau_Q u') = \gamma_Q \circ \boldsymbol{p} \circ \kappa(u') \circ \iota$ □

Proposition 3.42. *The map Θ'' of Lemma 3.32 respect the relations of Lemma 3.14.*

Proof. By Lemma 3.11 the map Θ'' respects relation 1. By Lemma 3.34 it respects relation 3 and $\hat{f}_{\tau_0^{-1}}\hat{f}_k = \hat{f}_{\tau_0^{-1}k\tau_0}\hat{f}_{\tau_0^{-1}}$ for every $k \in K_B$ and by Lemmas 3.33 and 3.30 it respects relations 2 and 6. Moreover, it respects relations 4 and 5 by Lemma 3.35 and relation 7 by Lemma 3.41. □

Theorem 3.43. *For every nonzero $\gamma \in I_{\tau_{m'-1}}(\eta)$ and every β-extension κ of η there exists an algebra isomorphism $\Theta_{\gamma,\kappa} : \mathscr{H}_R(B^\times, K_B^1) \to \mathscr{H}_R(G,\eta)$.*

Proof. By Proposition 3.42 and by Lemma 3.8 there exists an algebra homomorphism from $\mathscr{H}_R(B^\times, K_B^1)$ to $\mathscr{H}_R(G,\eta)$ which depends on the choice of a β-extension of η and of an element in $I_{\tau_{m'-1}}(\eta_{\mathcal{P}})$, which is isomorphic to $I_{\tau_{m'-1}}(\eta)$ by Lemma 3.8. Let Ξ be a set of representatives of K_B^1-double cosets of $B^\times$. Then $\{f_x \mid x \in \Xi\}$ is a basis of $\mathscr{H}_R(B^\times, K_B^1)$ as an R-vector space and, since $I_G(\eta) = J^1 B^\times J^1$ and $\dim_R(I_y(\eta)) = 1$ for every $y \in I_G(\eta)$, the set $\{\Theta_{\gamma,\kappa}(f_x) \mid x \in \Xi\}$ is a set of generators of $\mathscr{H}_R(G,\eta)$ as an R-vector space and so $\Theta_{\gamma,\kappa}$ is surjective. Moreover, the set $\{\Theta_{\gamma,\kappa}(f_x) \mid x \in \Xi\}$ is linearly independent and so $\Theta_{\gamma,\kappa}$ is also injective. □

Remark 3.44. Let κ and κ' be two β-extensions of η. By Section 2A there exists a character χ of $\mathcal{O}_E^\times$ trivial on $1+\wp_E$ such that $\kappa' = \kappa \otimes (\chi \circ N_{B/E})$. If we consider χ trivial on ϖ_E and we write $\tilde{\chi} = \chi \circ N_{B/E}$, which is a character of $B^\times$, then $\Theta_{\gamma,\kappa}^{-1} \circ \Theta_{\gamma,\kappa'}$ maps f_x to $\tilde{\chi} f_x = \tilde{\chi}(x) f_x$ for every $x \in B^\times$.

4. Semisimple types

Using the notation of Section 2, in this section we present the construction of semisimple types of G with coefficients in R. We refer to Sections 2.8–9 of [Mínguez and Sécherre 2014b] for more details.

Let $r \in \mathbb{N}^*$ and let $(m_1, \dots, m_r)$ be a family of strictly positive integers such that $\sum_{i=1}^r m_i = m$. For every $i \in \{1, \dots, r\}$ we fix a maximal simple type (J_i, λ_i) of $\mathrm{GL}_{m_i}(D)$ and a simple stratum $[\Lambda_i, n_i, 0, \beta_i]$ of $A_i = M_{m_i}(D)$ such that $J_i = J(\beta_i, \Lambda_i)$. Then, the centralizer B_i of $E_i = F[\beta_i]$ in A_i is isomorphic to $M_{m_i'}(D_i')$ for a suitable E_i-division algebra D_i' of reduced degree d_i' and a suitable $m_i' \in \mathbb{N}^*$. Moreover, $U(\Lambda_i) \cap B_i^\times$ is a maximal compact open subgroup of $B_i^\times$ which we identify with $\mathrm{GL}_{m_i'}(\mathcal{O}_{D_i'})$.

Let M be the standard Levi subgroup of G of block diagonal matrices of sizes $m_1, \dots, m_r$. The pair (J_M, λ_M) with $J_M = \prod_{i=1}^r J_i$ and $\lambda_M = \bigotimes_{i=1}^r \lambda_i$ is called a *maximal simple type* of M.

For every $i \in \{1, \dots, r\}$ we fix a simple character $\theta_i \in \mathscr{C}_R(\Lambda_i, 0, \beta_i)$ contained in λ_i and we observe that this choice does not depend on the choices of the β-extensions implicit in λ_i. Grouping θ_i according their endoclasses, we obtain a partition $\{1, \dots, r\} = \bigsqcup_{j=1}^l I_j$ with $l \in \mathbb{N}^*$. Up to renumbering the (J_i, λ_i) we can suppose that there exist integers $0 = a_0 < a_1 < \cdots < a_l = r$ such that we have $I_j = \{i \in \mathbb{N} \mid a_{j-1} < i \leq a_j\}$. For every $j \in \{1, \dots, l\}$ we denote $m^j = \sum_{i \in I_j} m_i$ and $m'^j = \sum_{i \in I_j} m_i'$ and we consider the standard Levi subgroup L of G containing M of block diagonal matrices of sizes $m^1, \dots, m^l$.

Let $j \in \{1, \dots, l\}$. We choose a simple stratum $[\Lambda^j, n^j, 0, \beta^j]$ of $M_{m^j}(D)$ as in Section 2.8 of [Mínguez and Sécherre 2014b] (see also Section 6.2 of [Sécherre and Stevens 2016]); in particular we can assume that for every $i\in I_j$ there exist an embedding $\iota_i : F[\beta^j]\to A_i$ such that $\beta_i=\iota_i(\beta^j)$ and that the characters θ_i with $i\in I_j$ are related by the transfer maps. If we denote by B^j the centralizer of $E^j=F[\beta^j]$ in $M_{m^j}(D)$, there exist an E^j-division algebra D'^j and an isomorphism that identifies B^j to $M_{m'^j}(D'^j)$ and $U(\Lambda^j)\cap B^{j\times}$ to the standard parabolic subgroup of $\mathrm{GL}_{m'^j}(\mathcal{O}_{D'^j})$ associated to m'_i with $i\in I_j$. We denote by θ^j the transfer of θ_i with $i\in I_j$ to $\mathscr{C}_R(\Lambda^j,0,\beta^j)$, which does not depend on i, and we fix a β-extension κ^j of θ^j. In Section 2.8 of [Mínguez and Sécherre 2014b] the authors define two compact open subgroups $\boldsymbol{J}_j\subset J(\beta^j,\Lambda^j)$ and $\boldsymbol{J}_j^1\subset J^1(\beta^j,\Lambda^j)$ of G such that $\boldsymbol{J}_j/\boldsymbol{J}_j^1\cong\prod_{i\in I_j}J_i/J_i^1$, and representations $\boldsymbol{\kappa}_j$ of $\boldsymbol{J}_j$ and $\boldsymbol{\eta}_j$ of $\boldsymbol{J}_j^1$ such that

$$\mathrm{ind}_{\boldsymbol{J}_j^1}^{J^1(\beta^j,\Lambda^j)}\boldsymbol{\eta}_j\cong\mathrm{res}^{J(\beta^j,\Lambda^j)}_{J^1(\beta^j,\Lambda^j)}\kappa^j,\quad \mathrm{ind}_{\boldsymbol{J}_j}^{J(\beta^j,\Lambda^j)}\boldsymbol{\kappa}_j\cong\kappa^j,\quad \boldsymbol{J}_j\cap M=\prod_{i\in I_j}J_i,\quad \mathrm{res}^{\boldsymbol{J}_j}_{\boldsymbol{J}_j\cap M}\boldsymbol{\kappa}_j=\bigotimes_{i\in I_j}\kappa_i,$$

where $\kappa_i\in\mathcal{B}(\theta_i)$ for every $i\in I_j$. We denote by η_i the restriction of κ_i to $J^1(\beta_i,\Lambda_i)$ for every $i\in I_j$. We obtain a decomposition $\lambda_i=\kappa_i\otimes\sigma_i$ for every $i\in I_j$ where σ_i is a representation of J_i trivial on J_i^1. We denote by $\boldsymbol{\sigma}_j$ the representation $\bigotimes_{i\in I_j}\sigma_i$ viewed as a representation of $\boldsymbol{J}_j$ trivial on $\boldsymbol{J}_j^1$ and we set $\boldsymbol{\lambda}_j=\boldsymbol{\kappa}_j\otimes\boldsymbol{\sigma}_j$. Then $(\boldsymbol{J}_j,\boldsymbol{\lambda}_j)$ is a cover of $\big(\prod_{i\in I_j}J_i,\bigotimes_{i\in I_j}\lambda_i\big)$ by Proposition 2.26 of [Mínguez and Sécherre 2014b], $(\boldsymbol{J}_j,\boldsymbol{\kappa}_j)$ is decomposed above $\big(\prod_{i\in I_j}J_i,\bigotimes_{i\in I_j}\kappa_i\big)$ and $(\boldsymbol{J}_j^1,\boldsymbol{\eta}_j)$ is a cover of $\big(\prod_{i\in I_j}J_i^1,\bigotimes_{i\in I_j}\eta_i\big)$ by Proposition 2.27 of the same reference.

We set

$$J_M^1=\prod_{i=1}^r J_i^1,\quad \kappa_M=\bigotimes_{i=1}^r\kappa_i,\quad \eta_M=\bigotimes_{i=1}^r\eta_i,\quad \boldsymbol{J}_L=\prod_{j=1}^l\boldsymbol{J}_j,\quad \boldsymbol{J}_L^1=\prod_{j=1}^l\boldsymbol{J}_j^1,$$

$$\boldsymbol{\lambda}_L=\bigotimes_{j=1}^l\boldsymbol{\lambda}_j,\quad \boldsymbol{\kappa}_L=\bigotimes_{j=1}^l\boldsymbol{\kappa}_j,\quad \boldsymbol{\eta}_L=\bigotimes_{j=1}^l\boldsymbol{\eta}_j,\quad \boldsymbol{\sigma}_L=\bigotimes_{j=1}^l\boldsymbol{\sigma}_j.$$

By construction $(\boldsymbol{J}_L,\boldsymbol{\lambda}_L)$ and $(\boldsymbol{J}_L^1,\boldsymbol{\eta}_L)$ are covers of (J_M,λ_M) and (J_M^1,η_M) respectively and $(\boldsymbol{J}_L,\boldsymbol{\kappa}_L)$ is decomposed above (J_M,κ_M).

Proposition 2.28 of [loc. cit.] defines a cover $(\boldsymbol{J},\boldsymbol{\lambda})$ of $(\boldsymbol{J}_L,\boldsymbol{\lambda}_L)$ and so of (J_M,λ_M), that we call a *semisimple type* of G. If the (J_i,λ_i) are maximal simple supertypes, we call $(\boldsymbol{J},\boldsymbol{\lambda})$ a *semisimple supertype* of G. The semisimple type $(\boldsymbol{J},\boldsymbol{\lambda})$ is associated to a stratum $[\boldsymbol{\Lambda},\boldsymbol{n},0,\boldsymbol{\beta}]$ of A, which is not necessarily simple (Section 2.9 of [loc. cit.]). We denote by B the centralizer of $\boldsymbol{\beta}$ in A, $B_L^\times=B^\times\cap L=\prod_{j=1}^l B^{j\times}$ and $\boldsymbol{J}^1=\boldsymbol{J}\cap U_1(\boldsymbol{\Lambda})$. By Propositions 2.30 and 2.31 of [loc. cit.] there exists a unique pair $(\boldsymbol{J}^1,\boldsymbol{\eta})$ decomposed above $(\boldsymbol{J}_L^1,\boldsymbol{\eta}_L)$ and so above (J_M^1,η_M). Its intertwining set is $I_G(\boldsymbol{\eta})=\boldsymbol{J}B_L^\times\boldsymbol{J}$ and for every $y\in B_L^\times$ the R-vector space $I_y(\boldsymbol{\eta})$ is 1-dimensional. We also have the isomorphisms

$$\boldsymbol{J}/\boldsymbol{J}^1\cong\boldsymbol{J}_L/\boldsymbol{J}_L^1\cong\prod_{i=1}^r J_i/J_i^1\cong\prod_{i=1}^r\mathrm{GL}_{m_i'}(\mathfrak{k}_{D_i'}).$$

We can identify $\boldsymbol{\sigma}_L$ with an irreducible representation $\boldsymbol{\sigma}$ of $\boldsymbol{J}$ trivial on $\boldsymbol{J}^1$. By Proposition 2.33 of [loc. cit.] there exists a unique pair $(\boldsymbol{J},\boldsymbol{\kappa})$ decomposed above $(\boldsymbol{J}_L,\boldsymbol{\kappa}_L)$ and so above (J_M,κ_M). Moreover, we have

$\boldsymbol{\eta}=\operatorname{res}^{\boldsymbol{J}}_{\boldsymbol{J}^1}\boldsymbol{\kappa}$, $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$ and $I_G(\boldsymbol{\kappa})=\boldsymbol{J}B_L^{\times}\boldsymbol{J}$. We denote by $\mathscr{M}$ the finite group $\prod_{i=1}^r \mathrm{GL}_{m'_i}(\mathfrak{k}_{D'_i})$. Then we can identify $\boldsymbol{\sigma}$ to a cuspidal (supercuspidal if $(\boldsymbol{J},\boldsymbol{\lambda})$ is a semisimple supertype) representation of $\mathscr{M}$.

Remark 4.1. The choice of β-extensions $\kappa^j\in\mathcal{B}(\theta^j)$ for every $j\in\{1,\dots,l\}$ determines $\kappa_i\in\mathcal{B}(\theta_i)$ for every $i\in\{1,\dots,r\}$, $\boldsymbol{\kappa}^j$ for every $j\in\{1,\dots,l\}$, $\boldsymbol{\kappa}_L$ and $\boldsymbol{\kappa}$ and so the decompositions $\lambda_i=\kappa_i\otimes\sigma_i$, $\boldsymbol{\lambda}_j=\boldsymbol{\kappa}_j\otimes\boldsymbol{\sigma}_j$ and $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$.

4A. ***The representation $\boldsymbol{\eta}_{\mathbf{max}}$.*** In this section we associate to every semisimple supertype $(\boldsymbol{J},\boldsymbol{\lambda})$ of G an irreducible projective representation $\boldsymbol{\eta}_{\max}$ of a compact open subgroup of G and we prove that the algebra $\mathscr{H}_R(G,\boldsymbol{\eta}_{\max})$ is isomorphic to $\mathscr{H}_R(B_L^{\times},K_L^1)$ where K_L^1 is the pro-p-radical of the maximal compact open subgroup of $B_L^{\times}$.

For every $j\in\{1,\dots,l\}$ we choose a simple stratum $[\Lambda_{\max,j},n_{\max,j},0,\beta^j]$ of $M_{m^j}(D)$ such that $U(\Lambda_{\max,j})\cap B^{j\times}$ is a maximal compact open subgroup of $B^{j\times}$ containing $U(\Lambda^j)\cap B^{j\times}$ as in Section 6.2 of [Sécherre and Stevens 2016]. Then we can identify $U(\Lambda_{\max,j})\cap B^{j\times}$ to $\mathrm{GL}_{m'^j}(\mathcal{O}_{D'^j})$. Let $J_{\max,j}=J(\beta^j,\Lambda_{\max,j})$ and $J^1_{\max,j}=J^1(\beta^j,\Lambda_{\max,j})$. We can also choose $\theta_{\max,j}\in\mathscr{C}_R(\Lambda_{\max,j},0,\beta^j)$ such that its transfer to $\mathscr{C}_R(\Lambda^j,0,\beta^j)$ is θ^j. We fix a β-extension $\kappa_{\max,j}$ of $\theta_{\max,j}$ and we denote by $\eta_{\max,j}$ its restriction to $J^1_{\max,j}$. By (5.2) of [Sécherre and Stevens 2016], there exists a unique $\kappa^j\in\mathcal{B}(\theta^j)$ such that

$$\operatorname{ind}_{J(\beta^j,\Lambda^j)}^{(U(\Lambda_j)\cap B^{j\times})U_1(\Lambda^j)}\kappa^j\cong\operatorname{ind}_{(U(\Lambda_j)\cap B^{j\times})J^1_{\max,j}}^{(U(\Lambda^j)\cap B^{j\times})U_1(\Lambda^j)}\kappa_{\max,j}\tag{6}$$

and so by Remark 4.1 the choice of $\kappa_{\max,j}$ determines $\boldsymbol{\kappa}_j$. We set

$$J_{\max}=\prod_{j=1}^{l}J_{\max,j},\quad J^1_{\max}=\prod_{j=1}^{l}J^1_{\max,j},\qquad \kappa_{\max}=\bigotimes_{j=1}^{l}\kappa_{\max,j},$$
$$\eta_{\max}=\bigotimes_{j=1}^{l}\eta_{\max,j},\quad K_L=\prod_{j=1}^{l}U(\Lambda_{\max,j})\cap B^{j\times},\ K_L^1=\prod_{j=1}^{l}U_1(\Lambda_{\max,j})\cap B^{j\times}.$$

If we denote by $\mathscr{G}$ the finite group $\prod_{j=1}^{l}\mathrm{GL}_{m'^j}(\mathfrak{k}_{D'^j})$, we obtain $J_{\max}/J^1_{\max}\cong K_L/K_L^1\cong\mathscr{G}$ and $(\mathscr{M},\boldsymbol{\sigma})$ is a supercuspidal pair of $\mathscr{G}$.

As before in this section, by Propositions 2.30, 2.31 and 2.33 of [Mínguez and Sécherre 2014b] we can define two compact open subgroups $\boldsymbol{J}_{\max}$ and $\boldsymbol{J}^1_{\max}$ of G such that $\boldsymbol{J}_{\max}/\boldsymbol{J}^1_{\max}\cong J_{\max}/J^1_{\max}\cong\mathscr{G}$ and pairs $(\boldsymbol{J}_{\max},\boldsymbol{\kappa}_{\max})$ and $(\boldsymbol{J}^1_{\max},\boldsymbol{\eta}_{\max})$ decomposed above $(J_{\max},\kappa_{\max})$ and $(J^1_{\max},\eta_{\max})$ respectively. Then we have $I_G(\boldsymbol{\kappa}_{\max})=I_G(\boldsymbol{\eta}_{\max})=\boldsymbol{J}_{\max}B_L^{\times}\boldsymbol{J}_{\max}$ and the R-vector spaces $I_y(\boldsymbol{\eta}_{\max})$ and $I_y(\boldsymbol{\kappa}_{\max})$ have dimension 1 for every $y\in B_L^{\times}$.

Remark 4.2. Since for every $j\in\{1,\dots,l\}$ the choice of $\kappa_{\max,j}\in\mathcal{B}(\theta_{\max,j})$ determines $\boldsymbol{\kappa}_j$, the choice of $\kappa_{\max}$ determines $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_{\max}$ and so the decomposition $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$. On the other hand $\boldsymbol{\eta}_{\max}$, the group $\mathscr{G}$ and the conjugacy class of $\mathscr{M}$ are uniquely determined by the semisimple supertype $(\boldsymbol{J},\boldsymbol{\lambda})$, independently by the choice of $\kappa_{\max}$ or of $\boldsymbol{\kappa}$.

Proposition 4.3. *The algebras $\mathscr{H}_R(G,\boldsymbol{\eta}_{\max})$ and $\bigotimes_{j=1}^{l}\mathscr{H}_R(\mathrm{GL}_{m^j}(D),\eta_{\max,j})$ are isomorphic.*

Proof. By Lemma 1.3 and by Lemma 2.4 and Proposition 2.5 of [Guiraud 2013] there exists an algebra isomorphism $\bigotimes_{j=1}^{l} \mathscr{H}_R(\mathrm{GL}_{m^j}(D), \eta_{\max,j}) \to \mathscr{H}_R(L, \eta_{\max})$. Now, since $I_G(\boldsymbol{\eta}_{\max}) \subset \boldsymbol{J}_{\max} L \boldsymbol{J}_{\max}$ the subalgebra $\mathscr{H}_R(\boldsymbol{J}_{\max} L \boldsymbol{J}_{\max}, \boldsymbol{\eta}_{\max})$ of $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ of functions with support in $\boldsymbol{J}_{\max} L \boldsymbol{J}_{\max}$ is equal to $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ and so by Sections II.6–8 of [Vignéras 1998] there exists an algebra isomorphism $\mathscr{H}_R(L, \eta_{\max}) \to \mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ which preserves the support. □

Corollary 4.4. *The R-algebras $\mathscr{H}_R(B_L^\times, K_L^1)$ and $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ are isomorphic.*

Proof. By Remark 1.5 of [Chinello 2017] (see also Theorem 6.3 of [Krieg 1990]) the algebra $\mathscr{H}_R(B_L^\times, K_L^1)$ is isomorphic to $\bigotimes_{j=1}^{l} \mathscr{H}_R(B^{j\times}, U_1(\Lambda_{\max,j}) \cap B^{j\times})$ and then by Theorem 3.43 we obtain, for every $j \in \{1, \ldots, l\}$,

$$\mathscr{H}_R(B^{j\times}, U_1(\Lambda_{\max,j}) \cap B^{j\times}) \cong \mathscr{H}_R(\mathrm{GL}_{m^j}(D), \eta_{\max,j}). \qquad \square$$

Remark 4.5. By Theorem 3.43 the isomorphism of Corollary 4.4 depends on the choice of a β-extension $\kappa_{\max,j}$ of $\eta_{\max,j}$ and of an intertwining element of $\eta_{\max,j}$ for every $j \in \{1, \ldots, l\}$. Using Proposition 4.3, the tensor product of these intertwining elements becomes an intertwining element of $\boldsymbol{\eta}_{\max}$.

Remark 4.6. The procedure that associates $\boldsymbol{\eta}_{\max}$ to $(\boldsymbol{J}, \boldsymbol{\lambda})$ depends on several noncanonical choices, for example the choice of the isomorphism $B_L^\times \to \prod \mathrm{GL}_{m'^j}(D'^j)$. To obtain a canonical correspondence, we denote by $\boldsymbol{\Theta}_i$ the endoclass of θ_i with $i \in \{1, \ldots, r\}$ and we canonically associate to $(\boldsymbol{J}, \boldsymbol{\lambda})$ the formal sum

$$\boldsymbol{\Theta}(\boldsymbol{J}, \boldsymbol{\lambda}) = \boldsymbol{\Theta} = \sum_{i=1}^{r} \frac{m_i d}{[E_i : F]} \boldsymbol{\Theta}_i.$$

Furthermore, the group $\mathscr{G}$ and the $\mathscr{G}$-conjugacy class of $\mathscr{M}$ depend only on $(\boldsymbol{J}, \boldsymbol{\lambda})$ and actually the group $\mathscr{G}$ depends only on $\boldsymbol{\Theta}$ because $m'^j[\mathfrak{k}_{D'^j} : \mathfrak{k}_{E^j}] = m^j d/[E^j : F] = \sum_{i \in I_j} m_i d/[E_i : F]$ which is the coefficient of $\boldsymbol{\Theta}_i$ in $\boldsymbol{\Theta}$. We refer to Section 6.3 of [Sécherre and Stevens 2016] for more details.

5. The category equivalence $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) \simeq \mathscr{R}(B_L^\times, K_L^1)$

Using the notation of Section 4, in this section we prove that there exists an equivalence of categories between $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and $\mathscr{R}(B_L^\times, K_L^1)$. This allows to reduce the description of a positive-level block of $\mathscr{R}_R(G)$ to the description of a level-0 block of $\mathscr{R}_R(B_L^\times)$.

5A. ***The category $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$.*** In this section we associate to a semisimple supertype $(\boldsymbol{J}, \boldsymbol{\lambda})$ of G a subcategory of $\mathscr{R}_R(G)$. We refer to [Sécherre and Stevens 2016] for more details.

From now on we fix an extension $\boldsymbol{\kappa}_{\max}$ of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$, as in Section 4A. This uniquely determines a decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$ where $\boldsymbol{\kappa}$ is an irreducible representation of $\boldsymbol{J}$ and $\boldsymbol{\sigma}$ is a supercuspidal representation of $\mathscr{M}$ viewed as an irreducible representation of $\boldsymbol{J}$ trivial on $\boldsymbol{J}^1$. We consider the functor $\mathsf{K}_{\boldsymbol{\kappa}_{\max}} : \mathscr{R}_R(G) \to \mathscr{R}(\boldsymbol{J}_{\max}/\boldsymbol{J}_{\max}^1) = \mathscr{R}_R(\mathscr{G})$ given by $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\pi) = \mathrm{Hom}_{\boldsymbol{J}_{\max}^1}(\boldsymbol{\eta}_{\max}, \pi)$ for every representation π of G, with $\boldsymbol{J}_{\max}$ acting on $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\pi)$ by

$$x.\varphi = \pi(x) \circ \varphi \circ \boldsymbol{\kappa}_{\max}(x)^{-1} \tag{7}$$

for every $x \in \boldsymbol{J}_{\max}$. We denote by $\pi(\boldsymbol{\kappa}_{\max})$ this representation of $\mathscr{G}$. We remark that if V_1 and V_2 are representations of G and $\phi \in \mathrm{Hom}_G(V_1, V_2)$ then $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\phi)$ maps φ to $\phi\circ\varphi$ for every $\varphi \in \mathrm{Hom}_G(\rho, V_1)$. For more details on this functor see Section 5 of [Mínguez and Sécherre 2014b] and [Sécherre and Stevens 2016].

We recall that we have $\boldsymbol{\sigma} = \bigotimes_{i=1}^{r} \sigma_i$ where σ_i is a supercuspidal representation of $\mathrm{GL}_{m'_i}(\mathfrak{k}_{D'_i})$. We put $\Gamma_{\mathscr{M}} = \prod_{j=1}^{l} \mathrm{Gal}(\mathfrak{k}_{D'^j}/\mathfrak{k}_{E^j})^{|I_j|}$. The equivalence class of $(\mathscr{M}, \boldsymbol{\sigma})$ (see Definition 1.14 of [Sécherre and Stevens 2016]) is the set, denoted by $[\mathscr{M}, \boldsymbol{\sigma}]$, of supercuspidal pairs $(\mathscr{M}', \boldsymbol{\sigma}')$ of $\mathscr{G}$ such that there exists $\epsilon \in \Gamma_{\mathscr{M}}$ such that $(\mathscr{M}', \boldsymbol{\sigma}')$ is $\mathscr{G}$-conjugate to $(\mathscr{M}, \boldsymbol{\sigma}^{\epsilon})$.

Let $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\boldsymbol{J}, \boldsymbol{\lambda})$. For every representation V of G let $V[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$ be the subrepresentation of V generated by the maximal subspace of $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(V)$ such that every irreducible subquotient has supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$ and let $V[\boldsymbol{\Theta}]$ be the subrepresentation of V generated by $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(V)$ (see Section 9.1 of [Sécherre and Stevens 2016]).

Definition 5.1. Let $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ be the full subcategory of $\mathscr{R}_R(G)$ of representations V such that $V = V[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$. This does not depend on the choice of $\boldsymbol{\kappa}_{\max}$ (see Section 10.1 of [loc. cit.]).

Remark 5.2. For every representation V of G we have $V[\boldsymbol{\Theta}, \boldsymbol{\sigma}][\boldsymbol{\Theta}, \boldsymbol{\sigma}] = V[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$ (see Lemma 9.2 of [loc. cit.]) and so $V[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$ is an object of $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$.

We define the *equivalence class of* $(\boldsymbol{J}, \boldsymbol{\lambda})$ to be the set $[\boldsymbol{J}, \boldsymbol{\lambda}]$ of semisimple supertypes $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{\lambda}})$ of G such that $\mathrm{ind}_{\tilde{\boldsymbol{J}}}^{G}(\tilde{\boldsymbol{\lambda}}) \cong \mathrm{ind}_{\boldsymbol{J}}^{G}(\boldsymbol{\lambda})$.

Theorem 5.3. *The category $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ depends only on the class $[\boldsymbol{J}, \boldsymbol{\lambda}]$ and it is a block of $\mathscr{R}_R(G)$.*

Proof. This follows from Propositions 10.2 and 10.5 and Theorem 10.4 of [Sécherre and Stevens 2016]. □

Remark 5.4. The proof in [loc. cit.] of Theorem 5.3 uses the notions of inertial class of a supercuspidal pair of G and of supercuspidal support (see 1.1.3, 2.1.2 and 2.1.3 of [Mínguez and Sécherre 2014a]). These notions are very important in the study of representations of $\mathrm{GL}_m(D)$ but in this article they are not used explicitly.

5B. ***The category equivalence.*** Let $(\boldsymbol{J}, \boldsymbol{\lambda})$ be a semisimple supertype of G and let $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\boldsymbol{J}, \boldsymbol{\lambda})$ be the formal sum of endoclasses associated to it. In general there exist several semisimple supertypes of G associated to $\boldsymbol{\Theta}$. We put $\boldsymbol{X} = \boldsymbol{X}_{\boldsymbol{\Theta}} = \{[\boldsymbol{J}', \boldsymbol{\lambda}'] \mid \boldsymbol{\Theta}(\boldsymbol{J}', \boldsymbol{\lambda}') = \boldsymbol{\Theta}\}$. In this section we prove that the sum $\bigoplus_{[\boldsymbol{J}', \boldsymbol{\lambda}'] \in \boldsymbol{X}} \mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$ is equivalent to the level-0 subcategory of $\mathscr{R}_R(B_L^{\times})$.

Let $\boldsymbol{Y} = \boldsymbol{Y}_{\boldsymbol{\Theta}}$ be the set of equivalence classes of supercuspidal pairs of $\mathscr{G}$, that is uniquely determined by $\boldsymbol{\Theta}$ by Remark 4.6. Let $\boldsymbol{\kappa}_{\max}$ be a fixed extension of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$ as in Section 4A and let $\mathsf{K} = \mathsf{K}_{\boldsymbol{\kappa}_{\max}}$. By Proposition 10.7 of [Sécherre and Stevens 2016] there exists a bijection

$$\phi_{\boldsymbol{\kappa}_{\max}} : \boldsymbol{X} \to \boldsymbol{Y} \tag{8}$$

given by $\phi_{\boldsymbol{\kappa}_{\max}}([\boldsymbol{J}', \boldsymbol{\lambda}']) = [\mathscr{M}, \boldsymbol{\sigma}]$ if the supercuspidal supports of irreducible subquotients of $\mathsf{K}(V)$ are in $[\mathscr{M}, \boldsymbol{\sigma}]$ for every (or equivalently for one) object V of $\mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$. This is equivalent to saying that there exists $\boldsymbol{\kappa}$ as in Section 4 (which depends on $\boldsymbol{\kappa}_{\max}$) such that $\boldsymbol{\lambda}' = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}'$ with $(\mathscr{M}, \boldsymbol{\sigma}') \in [\mathscr{M}, \boldsymbol{\sigma}]$.

Proposition 5.5 [Sécherre and Stevens 2016, Corollary 9.4]. *For every representation V of G we have*

$$V[\boldsymbol{\Theta}] = \bigoplus_{[\mathscr{M}', \boldsymbol{\sigma}'] \in \boldsymbol{Y}} V[\boldsymbol{\Theta}, \boldsymbol{\sigma}']. \tag{9}$$

Proposition 5.6 [loc. cit., Lemma 10.3]. *If $[\boldsymbol{J}', \boldsymbol{\lambda}'] \in \boldsymbol{X}$ and W is a simple object of $\mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$ then $\mathsf{K}(W) \neq 0$.*

Since $\boldsymbol{J}^1_{\max}$ has invertible pro-order in R, the representation $\boldsymbol{\eta}_{\max}$ is projective and so we can use the notation and results of Section 1B. We have defined the functor

$$\boldsymbol{M}_{\boldsymbol{\eta}_{\max}} : \mathscr{R}_R(G) \to \text{Mod-}\,\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$$

by $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V) = \mathrm{Hom}_G(\mathrm{ind}^G_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max}), V)$ and $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\phi) : \varphi \mapsto \varphi \circ \phi$ for all representations V and V_1 of G, $\phi \in \mathrm{Hom}_G(V, V_1)$ and $\varphi \in \mathrm{Hom}_G(\mathrm{ind}^G_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max}), V)$.

Remark 5.7. Frobenius reciprocity induces a natural isomorphism between the functor $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}$ composed with the forgetful functor $\text{Mod-}\,\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}) \to \text{Mod-}\,R$ and the functor $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}$ composed with the forgetful functor $\mathscr{R}_R(\mathscr{G}) \to \text{Mod-}\,R$. This implies that for every representation V of G the subrepresentation $V[\boldsymbol{\Theta}]$ of V is the subrepresentation $V[\boldsymbol{\eta}_{\max}]$ defined in Section 1B.

We have also defined the full subcategories $\mathscr{R}_{\boldsymbol{\eta}_{\max}}(G)$ and $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ of $\mathscr{R}_R(G)$. We recall that $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ is the category of V such that $V = V[\boldsymbol{\Theta}]$ and $\mathscr{R}_{\boldsymbol{\eta}_{\max}}(G)$ is the category of V such that $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V') \neq 0$ for every irreducible subquotient V' of V.

Lemma 5.8. *We have $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) = \mathscr{R}_{\boldsymbol{\eta}_{\max}}(G)$.*

Proof. Thanks to Remark 1.8 it is sufficient to prove $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) \subset \mathscr{R}_{\boldsymbol{\eta}_{\max}}(G)$. Let V be a representation in $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$. By Proposition 5.5 we have $V = \bigoplus_{\boldsymbol{Y}} V[\boldsymbol{\Theta}, \boldsymbol{\sigma}']$ and by Remark 5.2 the representation $V[\boldsymbol{\Theta}, \boldsymbol{\sigma}']$ is an object of $\mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$ where $[\boldsymbol{J}', \boldsymbol{\lambda}'] = \phi^{-1}_{\boldsymbol{\kappa}_{\max}}([\mathscr{M}, \boldsymbol{\sigma}']) \in \boldsymbol{X}$. Hence, we obtain the inclusion $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) \subset \bigoplus_{\boldsymbol{X}} \mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$. Let now W be an object of $\bigoplus_{\boldsymbol{X}} \mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$ and W' an irreducible subquotient of W. Then W' is an irreducible object of $\mathscr{R}(\boldsymbol{J}', \boldsymbol{\lambda}')$ for a $[\boldsymbol{J}', \boldsymbol{\lambda}'] \in \boldsymbol{X}$ and so by Proposition 5.6 we have $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(W) \neq 0$. Therefore, by Remark 5.7 we have $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(W') \neq 0$ which implies $\bigoplus_{\boldsymbol{X}} \mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda}') \subset \mathscr{R}_{\boldsymbol{\eta}_{\max}}(G)$. □

Remark 5.9. We have proved that $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) = \mathscr{R}_{\boldsymbol{\eta}_{\max}}(G) = \bigoplus_{[\boldsymbol{J}, \boldsymbol{\lambda}] \in \boldsymbol{X}} \mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$. Moreover, by Proposition 1.7, a representation V of G is in this category if and only if it satisfies one of the following equivalent conditions:

- $V = V[\boldsymbol{\Theta}]$.
- For every subquotient Z of V we have $Z = Z[\boldsymbol{\Theta}]$.
- For every irreducible subquotient U of V we have $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(U) \neq 0$.
- For every nonzero subquotient W of V we have $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(W) \neq 0$.

Theorem 5.10. *The functor $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}$ is an equivalence of categories between*

$$\mathscr{R}(G, \boldsymbol{\eta}_{\max}) \quad \textit{and} \quad \text{Mod-}\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}).$$

Proof. We apply Theorem 1.9 with $\mathtt{G} = G$ and $\sigma = \boldsymbol{\eta}_{\max}$. □

Remark 5.11. We recall that a level-0 representation of $B_L^\times$ is a representation generated by its K_L^1-invariant vectors. It is equivalent to say that all irreducible subquotients have nonzero K_L^1-invariant vectors (see Section 3 of [Chinello 2017]). The category $\mathscr{R}(B_L^\times, K_L^1)$ is called the *level-*0 *subcategory* of $\mathscr{R}_R(B_L^\times)$. By Section 3 of [Chinello 2017] and Theorem 1.9, the K_L^1-invariant functor $\mathrm{inv}_{K_L^1}$ induces an equivalence of categories between $\mathscr{R}(B_L^\times, K_L^1)$ and Mod-$\mathscr{H}_R(B_L^\times, K_L^1)$ whose quasiinverse is

$$W \mapsto W \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \mathrm{ind}_{K_L^1}^{B_L^\times}(1).$$

We recall that if (ϱ, Z) is a representation of $B_L^\times$ then the action of $\Phi \in \mathscr{H}_R(B_L^\times, K_L^1)$ on $z \in Z^{K_L^1}$ is given by $z.\Phi = \sum_{x \in K_L^1 \backslash B_L^\times} \Phi(x)\varrho(x^{-1})z$.

Corollary 5.12. *There exists an equivalence of categories between $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and $\mathscr{R}(B_L^\times, K_L^1)$.*

Proof. By Corollary 4.4 the algebras $\mathscr{H}_R(B_L^\times, K_L^1)$ and $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ are isomorphic. We obtain an equivalence of categories between Mod-$\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ and Mod-$\mathscr{H}_R(B_L^\times, K_L^1)$ and so between $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and $\mathscr{R}(B_L^\times, K_L^1)$ by Theorem 5.10 and Remark 5.11. □

Now we want to describe the functor that induces this equivalence of categories. We recall that we have fixed an isomorphism $B_L^\times \cong \prod \mathrm{GL}_{m'^j}(D'^j)$ and an extension $\boldsymbol{\kappa}_{\max}$ of $\boldsymbol{\eta}_{\max}$. We also fix a nonzero intertwining element γ of $\boldsymbol{\eta}_{\max}$ as in Remark 4.5. By Corollary 4.4 we have an isomorphism $\Theta_{\gamma,\boldsymbol{\kappa}_{\max}}$: $\mathscr{H}_R(B_L^\times, K_L^1) \to \mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ which induces an equivalence of categories $\Theta^*_{\gamma,\boldsymbol{\kappa}_{\max}}$: Mod-$\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}) \to$ Mod-$\mathscr{H}_R(B_L^\times, K_L^1)$. We obtain the diagram

$$\begin{array}{ccc} \mathscr{R}(G, \boldsymbol{\eta}_{\max}) & \xrightarrow{\text{Corollary 5.12}} & \mathscr{R}(B_L^\times, K_L^1) \\ \Big\downarrow {\scriptstyle \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}} & & \Big\uparrow {\scriptstyle \text{Remark 5.11}} \\ \text{Mod-}\mathscr{H}_R(G, \boldsymbol{\eta}_{\max}) & \xrightarrow{\Theta^*_{\gamma,\boldsymbol{\kappa}_{\max}}} & \text{Mod-}\mathscr{H}_R(B_L^\times, K_L^1). \end{array} \tag{10}$$

The functor $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}} : \mathscr{R}(G, \boldsymbol{\eta}_{\max}) \to$ Mod-$\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ is an equivalence of categories by Theorem 5.10. By Lemma 1.3 the right action of $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ on $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V)$ is given by $(m.\Psi)(f) = m(\Psi * f)$ for every $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V)$, $\Psi \in \mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ and $f \in \mathrm{ind}_{\boldsymbol{J}^1_{\max}}^G(\boldsymbol{\eta}_{\max})$. The right-action of $\Phi \in \mathscr{H}_R(B_L^\times, K_L^1)$ on a $\mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$-module N is given by $N.\Phi = N.\Theta_{\gamma,\boldsymbol{\kappa}_{\max}}(\Phi)$. By Remark 5.11 the functor $W \mapsto W \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \mathrm{ind}_{K_L^1}^{B_L^\times}(1)$ is an equivalence of categories between Mod-$\mathscr{H}_R(B_L^\times, K_L^1)$ and $\mathscr{R}(B_L^\times, K_L^1)$ where, by Lemma 1.3, the left-action of $\Phi \in \mathscr{H}_R(B_L^\times, K_L^1)$ on $f \in \mathrm{ind}_{K_L^1}^{B_L^\times}(1)$ is given by $\Phi.f = \Phi * f$. Moreover, the left-action of $x \in B_L^\times$ on $w \otimes f \in W \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \mathrm{ind}_{K_L^1}^{B_L^\times}(1)$ is given by $x.(w \otimes f) = w \otimes (x.f)$.

Composing these three functors we obtain the equivalence of categories of Corollary 5.12 which we denote by $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ and is given by

$$\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(\pi, V) = \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\pi, V) \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \operatorname{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1}) \tag{11}$$

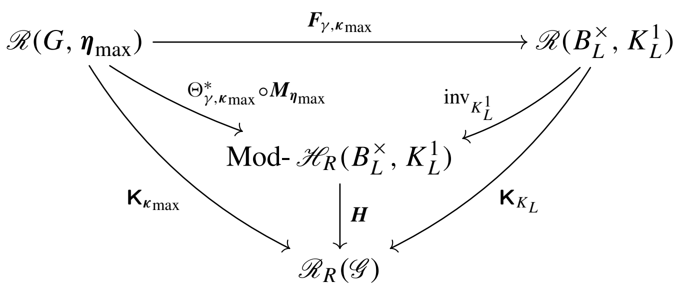
for every (π, V) in $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$, where the right-action of $\Phi \in \mathscr{H}_R(B_L^\times, K_L^1)$ on $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\pi, V)$ is given by $(m.\Phi)(f) = m(\Theta_{\gamma,\boldsymbol{\kappa}_{\max}}(\Phi) * f)$ for every $f \in \operatorname{ind}_{\boldsymbol{J}_{\max}^1}^G(\boldsymbol{\eta}_{\max})$. We remark that if V_1 and V_2 are in $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and $\phi \in \operatorname{Hom}_G(V_1, V_2)$ then $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(\phi)$ maps $m \otimes f$ to $(\phi \circ m) \otimes f$ for every $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V_1)$ and $f \in \operatorname{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1})$.

5C. *Correspondence between blocks.* In this section we discuss the correspondence among blocks of $\mathscr{R}(B_L^\times, K_L^1)$ and those of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ induced by the equivalence of categories $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ defined in (11).

We consider the functor $\mathsf{K}_{K_L} : \mathscr{R}(B_L^\times, K_L^1) \to \mathscr{R}_R(K_L/K_L^1) = \mathscr{R}_R(\mathscr{G})$ given by $\mathsf{K}_{K_L}(Z) = Z^{K_L^1}$ and $\mathsf{K}_{K_L}(\phi) = \phi_{|Z^{K_L^1}}$ for all representations (ϱ, Z) and (ϱ_1, Z_1) of $B_L^\times$ and every $\phi \in \operatorname{Hom}_{B_L^\times}(Z, Z_1)$, where $x \in K_L$ acts on $z \in Z^{K_L^1}$ by $x.z = \varrho(x)z$. It is the functor presented in Section 5A when we replace G by $B_L^\times$ and $\boldsymbol{\kappa}_{\max}$ by the trivial representation of K_L. We also consider the functor $\boldsymbol{H} : \text{Mod-}\mathscr{H}_R(B_L^\times, K_L^1) \to \mathscr{R}_R(K_L/K_L^1)$ given by $\boldsymbol{H}(W) = (\varrho', W)$ and $\boldsymbol{H}(\phi) = \phi$ for all $\mathscr{H}_R(B_L^\times, K_L^1)$-modules W and W_1 and every $\phi \in \operatorname{Hom}_{\mathscr{H}_R(B_L^\times, K_L^1)}(W, W_1)$, where $\varrho'(k)w = w.f_{k^{-1}}$ for every $k \in K_L$ and $w \in W$.

Remark 5.13. The functor K_{K_L} is the composition of $\operatorname{inv}_{K_L^1}$ (see Remark 5.11) and the functor $\boldsymbol{H}$. Actually if (ϱ, Z) is an object of $\mathscr{R}(B_L^\times, K_L^1)$ then $\boldsymbol{H}(\operatorname{inv}_{K_L^1}(Z)) = \boldsymbol{H}(Z^{K_L^1}) = (\varrho', Z^{K_L^1})$ where $\varrho'(k)z = z.f_{k^{-1}} = \sum_{x \in K_L^1 \backslash B_L^\times} f_{k^{-1}}(x)\varrho(x^{-1})z = \varrho(k)z$ for every $z \in Z^{K_L^1}$ and $k \in K_L$.

We obtain the diagram



(12)

Proposition 5.14. *There exists a natural isomorphism between $\mathsf{K}_{K_L} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ and $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}$.*

Proof. By Remark 5.13 we have $\mathsf{K}_{K_L} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}} = \boldsymbol{H} \circ \operatorname{inv}_{K_L^1} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ and by (10) we have a natural isomorphism between $\operatorname{inv}_{K_L^1} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ and $\Theta^*_{\gamma,\boldsymbol{\kappa}_{\max}} \circ \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}$ so it is sufficient to find a natural isomorphism $\mathfrak{Z} : \boldsymbol{H} \circ \Theta^*_{\gamma,\boldsymbol{\kappa}_{\max}} \circ \boldsymbol{M}_{\boldsymbol{\eta}_{\max}} \to \mathsf{K}_{\boldsymbol{\kappa}_{\max}}$. For every object (π, V) of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$, let $\mathfrak{Z}_V : \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V) \to \mathsf{K}_{\boldsymbol{\kappa}_{\max}}(V)$ be the isomorphism of R-modules given by Remark 5.7. The action of $x \in K_L/K_L^1 \cong \mathscr{G}$ on $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\pi, V)$ is given by $x.m = m.\Theta_{\gamma,\boldsymbol{\kappa}_{\max}}(f_{x^{-1}}) = m.\tilde{f}_{x^{-1}}$ where $\tilde{f}_{x^{-1}} \in \mathscr{H}_R(G, \boldsymbol{\eta}_{\max})$ has support $x^{-1}\boldsymbol{J}_{\max}^1$ and $\tilde{f}_{x^{-1}}(x^{-1}) = \boldsymbol{\kappa}_{\max}(x^{-1})$ while the action of $x \in \boldsymbol{J}_{\max}/\boldsymbol{J}_{\max}^1 \cong \mathscr{G}$ on $\varphi \in \mathsf{K}_{\boldsymbol{\kappa}_{\max}}(V)$ is given by (7). We have to prove that $\mathfrak{Z}_V(x.m) = x.\mathfrak{Z}_V(m)$ for every $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\pi, V)$ and $x \in \mathscr{G}$. We recall that in Section 1A we defined elements $i_v : \boldsymbol{J}^1_{\max} \to V_{\boldsymbol{\eta}_{\max}}$ with $v \in V_{\boldsymbol{\eta}_{\max}}$ such that $m(i_v) = \mathfrak{Z}_V(m)(v)$, which generate $\operatorname{ind}^G_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max})$ as a representation of G. Then for every $v \in V_{\boldsymbol{\eta}_{\max}}$ we have

$$\mathfrak{Z}_V(x.m)(v) = (x.m)(i_v) = (m.\tilde{f}_{x^{-1}})(i_v) = m(\tilde{f}_{x^{-1}} * i_v).$$

The support of $\tilde{f}_{x^{-1}} * i_v$ is $\boldsymbol{J}^1_{\max}x^{-1}$ and $(\tilde{f}_{x^{-1}} * i_v)(x^{-1}) = \tilde{f}_{x^{-1}}(x^{-1})v = \boldsymbol{\kappa}_{\max}(x^{-1})v$. We obtain $\mathfrak{Z}_V(x.m)(v) = m(x.i_{\boldsymbol{\kappa}_{\max}(x^{-1})v}) = \pi(x)(m(i_{\boldsymbol{\kappa}_{\max}(x^{-1})v})) = \pi(x)(\mathfrak{Z}_V(m)(\boldsymbol{\kappa}_{\max}(x^{-1})v)) = (x.\mathfrak{Z}_V(m))(v)$. Now, let V_1 and V_2 be two objects of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and let $\phi \in \operatorname{Hom}_G(V_1, V_2)$. Then for every $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V_1)$ and every $v \in V_{\boldsymbol{\eta}_{\max}}$ we have $\mathfrak{Z}_{V_2}(\boldsymbol{H}(\Theta^*_{\gamma,\boldsymbol{\kappa}_{\max}}(\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(\phi)))(m))(v) = \mathfrak{Z}_{V_2}(\phi \circ m)(v)$ which is equal to $(\phi \circ m)(i_v)$. On the other hand we have $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\phi)(\mathfrak{Z}_{V_1}(m))(v) = \phi(\mathfrak{Z}_{V_1}(m)(v))$ which is equal to $\phi(m(i_v))$. This shows that $\mathfrak{Z}$ is a natural isomorphism. □

Now we look for a block decomposition of $\mathscr{R}(B_L^\times, K_L^1)$. Let $[\mathscr{M}, \boldsymbol{\sigma}] \in \boldsymbol{Y}$. Then $\mathscr{M} = \prod_{j=1}^l \mathscr{M}_j$ and $\boldsymbol{\sigma} = \bigotimes_{j=1}^l \boldsymbol{\sigma}_j$ where $\mathscr{M}_j \cong \boldsymbol{J}_j/\boldsymbol{J}_j^1$ and $[\mathscr{M}_j, \boldsymbol{\sigma}_j]$ is a class of supercuspidal pairs of $\mathrm{GL}_{m'^j}(\mathfrak{k}_{D'^j})$. For every $j \in \{1, \dots, l\}$, replacing G by $B^{j\times}$ and $\boldsymbol{\kappa}_{\max}$ by the trivial character of $U(\Lambda_{\max,j}) \cap B^{j\times}$ in Definition 5.1, we obtain an abelian full subcategory $\mathscr{R}(U(\Lambda_{\max,j}) \cap B^{j\times}, \boldsymbol{\sigma}_j)$ of $\mathscr{R}_R(B^{j\times})$ whose objects are representations V_j of $B^{j\times}$ generated by the maximal subspace of ${V_j}^{U_1(\Lambda_{\max,j})\cap B^{j\times}}$ for which every irreducible subquotient has supercuspidal support in $[\mathscr{M}_j, \boldsymbol{\sigma}_j]$. We obtain a full subcategory $\mathscr{R}(K_L, \boldsymbol{\sigma})$ of $\mathscr{R}_R(B_L^\times)$ (and of $\mathscr{R}(B_L^\times, K_L^1)$) whose objects are representations V of $B_L^\times$ generated by the maximal subspace of $V^{K_L^1}$ such that every irreducible subquotient has supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$. Theorem 5.3 and Remark 5.9 give a block decomposition of $\mathscr{R}(B^{j\times}, U_1(\Lambda_{\max,j}) \cap B^{j\times})$ for every $j \in \{1, \dots, l\}$ and so we obtain a block decomposition

$$\mathscr{R}(B_L^\times, K_L^1) = \bigoplus_{[\mathscr{M},\boldsymbol{\sigma}]\in \boldsymbol{Y}} \mathscr{R}(K_L, \boldsymbol{\sigma}).$$

We recall that we have a block decomposition $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) = \bigoplus_{[\boldsymbol{J},\boldsymbol{\lambda}]\in \boldsymbol{X}} \mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ by Remark 5.9 and a bijection $\phi_{\boldsymbol{\kappa}_{\max}} : \boldsymbol{X} \to \boldsymbol{Y}$ defined in (8) which depends on the choice of $\boldsymbol{\kappa}_{\max}$.

Theorem 5.15. *Let $[\boldsymbol{J}, \boldsymbol{\lambda}] \in \boldsymbol{X}$ and $[\mathscr{M}, \boldsymbol{\sigma}] = \phi_{\boldsymbol{\kappa}_{\max}}([\boldsymbol{J}, \boldsymbol{\lambda}]) \in \boldsymbol{Y}$. Then $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ induces an equivalence of categories between the block $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ of $\mathscr{R}_R(G)$ and the block $\mathscr{R}(K_L, \boldsymbol{\sigma})$ of $\mathscr{R}_R(B_L^\times)$.*

Proof. If V is an object of $\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$, by Proposition 5.14 there exists an isomorphism of representations of $\mathscr{G}$ between $\mathsf{K}_{K_L}(\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(V))$ and $\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(V)$. Then irreducible subquotients of $(\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(V))^{K_L^1}$ have supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$ and so $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(V)$ is in $\mathscr{R}(K_L, \boldsymbol{\sigma})$. □

We remark that the matching of the blocks of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and of $\mathscr{R}(B_L^\times, K_L^1)$ does not depend on the choice of the intertwining element γ of $\boldsymbol{\eta}_{\max}$ while the equivalence of categories between these blocks, induced by $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(V)$, depends on this choice.

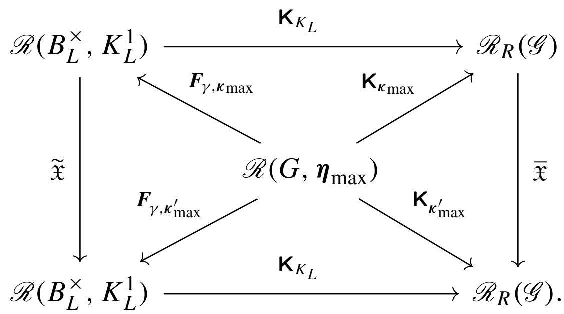
5D. ***Dependence on the choice of $\boldsymbol{\kappa}_{\max}$.*** In this section we discuss the dependence of results of Sections 5A, 5B and 5C on the choice of the extension of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$.

Let $(\boldsymbol{J}, \boldsymbol{\lambda})$ be a semisimple supertype of G. We have just seen in Remark 4.6 that the group $\mathscr{G}$ depends only on $\boldsymbol{\Theta}(\boldsymbol{J}, \boldsymbol{\lambda})$ and by Remark 4.6 and Theorem 5.3 the $\mathscr{G}$-conjugacy class of $\mathscr{M}$ and the category

$\mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ do not depend on the choice of the extension of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$. Moreover, the sum (9) does not depend on this choice because a different one permutes the terms $V[\boldsymbol{\Theta}, \boldsymbol{\sigma}']$ in $V[\boldsymbol{\Theta}]$. Then $V[\boldsymbol{\Theta}]$, the equalities $\mathscr{R}(G, \boldsymbol{\eta}_{\max}) = \mathscr{R}_{\boldsymbol{\eta}_{\max}}(G) = \bigoplus_{[\boldsymbol{J},\boldsymbol{\lambda}]\in X} \mathscr{R}(\boldsymbol{J}, \boldsymbol{\lambda})$ and the equivalence of Theorem 5.10 do not depend on the choice of the extension of $\boldsymbol{\eta}_{\max}$.

Let γ be a fixed nonzero intertwining element of $\boldsymbol{\eta}_{\max}$ as in Remark 4.5. Using notation of Section 4A let $\boldsymbol{\kappa}_{\max}$ and $\boldsymbol{\kappa}'_{\max}$ be two extensions of $\boldsymbol{\eta}_{\max}$ to $\boldsymbol{J}_{\max}$ and let $\kappa_{\max} = \bigotimes_{j=1}^{l} \kappa_{\max,j}$ and $\kappa'_{\max} = \bigotimes_{j=1}^{l} \kappa'_{\max,j}$ be the restrictions to $J_{\max}$ of $\boldsymbol{\kappa}_{\max}$ and $\boldsymbol{\kappa}'_{\max}$ respectively. Then, for every $j \in \{1, \dots, l\}$, $\kappa_{\max,j}$ and $\kappa'_{\max,j}$ are β-extensions of $\theta_{\max,j}$ and so by Section 2A there exists a character χ_j of $\mathcal{O}_{E^j}^{\times}$ trivial on $1+\wp_{E^j}$ such that $\kappa'_{\max,j} = \kappa_{\max,j} \otimes (\chi_j \circ N_{B^j/E^j})$. Let χ and $\overline{\chi}$ be the character $\bigotimes_{j=1}^{l}(\chi_j \circ N_{B^j/E^j})$ viewed as characters of $\boldsymbol{J}_{\max}$ trivial on $\boldsymbol{J}^1_{\max}$ and of $\mathscr{G}$ respectively and, if we consider χ_j trivial on ϖ_{E^j} for every $j \in \{1, \dots, l\}$, let $\tilde{\chi} = \bigotimes_{j=1}^{l}(\chi_j \circ N_{B^j/E^j})$ viewed as a character of $B_L^{\times}$.

We consider the functors $\widetilde{\mathfrak{X}} : \mathscr{R}(B_L^{\times}, K_L^1) \to \mathscr{R}(B_L^{\times}, K_L^1)$ and $\overline{\mathfrak{X}} : \mathscr{R}_R(\mathscr{G}) \to \mathscr{R}_R(\mathscr{G})$ given by $\widetilde{\mathfrak{X}}(\varrho) = \varrho \otimes \tilde{\chi}^{-1}$, $\widetilde{\mathfrak{X}}(\tilde{\phi}) = \tilde{\phi}$, $\overline{\mathfrak{X}}(\tau) = \tau \otimes \overline{\chi}^{-1}$ and $\overline{\mathfrak{X}}(\bar{\phi}) = \bar{\phi}$ for every ϱ, ϱ_1 in $\mathscr{R}(B_L^{\times}, K_L^1)$, every $\tilde{\phi} \in \mathrm{Hom}_{B_L^{\times}}(\varrho, \varrho_1)$, all representations τ and τ_1 of $\mathscr{G}$ and every $\bar{\phi} \in \mathrm{Hom}_{\mathscr{G}}(\tau, \tau_1)$. We consider the following diagram.



(13)

Lemma 5.16. *We have $\mathsf{K}_{\boldsymbol{\kappa}'_{\max}} = \overline{\mathfrak{X}} \circ \mathsf{K}_{\boldsymbol{\kappa}_{\max}}$ and so for every representation (π, V) in $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ we have $\pi(\boldsymbol{\kappa}'_{\max}) = \pi(\boldsymbol{\kappa}_{\max}) \otimes \overline{\chi}^{-1}$.*

Proof. The space of $\mathsf{K}_{\boldsymbol{\kappa}'_{\max}}(V)$ and $\overline{\mathfrak{X}}(\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(V))$ is $\mathrm{Hom}_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max}, V)$. Let φ be in this space and $x \in \boldsymbol{J}_{\max}$. Let Q be the standard parabolic subgroup of G with Levi component L, let N be the unipotent radical of Q such that $Q = LN$ and let N^- be the unipotent radical opposite to N. We choose $x_1 \in \boldsymbol{J}_{\max} \cap N^-$, $x_2 \in J_{\max}$ and $x_3 \in \boldsymbol{J}_{\max} \cap N$ such that $x = x_1x_2x_3$. Since $(\boldsymbol{\kappa}_{\max}, \boldsymbol{J}_{\max})$ and $(\boldsymbol{\kappa}'_{\max}, \boldsymbol{J}_{\max})$ are decomposed above $(\kappa_{\max}, J_{\max})$ and $(\kappa'_{\max}, J_{\max})$ respectively, we obtain $\pi(\boldsymbol{\kappa}'_{\max})(x)(\varphi) = \pi(x) \circ \varphi \circ \boldsymbol{\kappa}'_{\max}(x^{-1})$ which is equal to $\pi(x) \circ \varphi \circ \kappa'_{\max}(x_2^{-1}) = \pi(x) \circ \varphi \circ \kappa_{\max}(x_2^{-1})\chi(x_2^{-1}) = \pi(\boldsymbol{\kappa}_{\max})(x)(\varphi)\chi(x_2)^{-1}$. Since $\boldsymbol{J}_{\max} \cap N = \boldsymbol{J}^1_{\max} \cap N$ and $\boldsymbol{J}_{\max} \cap N^- = \boldsymbol{J}^1_{\max} \cap N^-$ we obtain $\chi(x_2)^{-1} = \chi(x)^{-1}$. Now, let V_1 and V_2 be two objects of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and let $\phi \in \mathrm{Hom}_G(V_1, V_2)$. Then for every $\varphi \in \mathrm{Hom}_{\boldsymbol{J}^1_{\max}}(\boldsymbol{\eta}_{\max}, V_1)$ we have $\mathsf{K}_{\boldsymbol{\kappa}'_{\max}}(\phi)(\varphi) = \phi \circ \varphi = \overline{\mathfrak{X}}(\mathsf{K}_{\boldsymbol{\kappa}_{\max}}(\phi))(\varphi)$. □

Lemma 5.17. *We have $\mathsf{K}_{K_L} \circ \widetilde{\mathfrak{X}} = \overline{\mathfrak{X}} \circ \mathsf{K}_{K_L}$.*

Proof. Let (ϱ, Z) be in $\mathscr{R}(B_L^{\times}, K_L^1)$. The space of $\mathsf{K}_{K_L}(\widetilde{\mathfrak{X}}(Z))$ and $\overline{\mathfrak{X}}(\mathsf{K}_{K_L}(Z))$ is $Z^{K_L^1}$. Let $x \in K_L$ and let $\bar{x}$ be the projection of x in $K_L/K_L^1 \cong \mathscr{G}$. For every $z \in Z^{K_L^1}$ we have $\mathsf{K}_{K_L}(\widetilde{\mathfrak{X}}(\varrho))(\bar{x})(z) = \tilde{\chi}(x^{-1})\varrho(x)v$

while $\overline{\mathfrak{X}}(\mathsf{K}_{K_L}(\varrho))(\bar{x})(z) = \bar{\chi}(\bar{x}^{-1})\varrho(x)v$. Now, let Z_1 and Z_2 be two objects of $\mathscr{R}(B_L^\times, K_L^1)$ and let $\phi \in \mathrm{Hom}_{B_L^\times}(Z_1, Z_2)$. Then we have $\mathsf{K}_{K_L}(\widetilde{\mathfrak{X}}(\phi)) = \phi_{|Z_1^{K_L^1}} = \overline{\mathfrak{X}}(\mathsf{K}_{K_L}(\phi))$. □

We remark that by Proposition 5.14 and Lemmas 5.16 and 5.17, the functor $\mathsf{K}_{K_L} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}'_{\max}}$ is naturally isomorphic to $\mathsf{K}_{\boldsymbol{\kappa}'_{\max}}$ which is equal to $\overline{\mathfrak{X}} \circ \mathsf{K}_{\boldsymbol{\kappa}_{\max}}$ which is naturally isomorphic to $\overline{\mathfrak{X}} \circ \mathsf{K}_{K_L} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$ which is equal to $\mathsf{K}_{K_L} \circ \widetilde{\mathfrak{X}} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$.

Proposition 5.18. *There exists a natural isomorphism between* $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}'_{\max}}$ *and* $\widetilde{\mathfrak{X}} \circ \boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}$.

Proof. For every object (π, V) in $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$, the space of $\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}'_{\max}}(V)$ and $\widetilde{\mathfrak{X}}(\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(V))$ is

$$\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V) \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \mathrm{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1}).$$

If $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V)$ and $f \in \mathrm{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1})$, in the first case the right-action of $\Phi \in \mathscr{H}_R(B_L^\times, K_L^1)$ on m and the left-action of $x \in B_L^\times$ on $m \otimes f$ are given by $m \star' \Phi = m.\Theta_{\gamma,\boldsymbol{\kappa}'_{\max}}(\Phi)$ and $x \diamond' (m \otimes f) = m \otimes x.f$ while in the second case they are given by $m \star \Phi = m.\Theta_{\gamma,\boldsymbol{\kappa}_{\max}}(\Phi)$ and $x \diamond (m \otimes f) = \tilde{\chi}(x^{-1})m \otimes x.f$. Let $\mathfrak{z}_V$ be the automorphism of $\boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V) \otimes_{\mathscr{H}_R(B_L^\times, K_L^1)} \mathrm{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1})$ that maps $m \otimes f$ to $m \otimes \tilde{\chi} f$ for every $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V)$ and $f \in \mathrm{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1})$. By Remark 3.44 we have $m \star' \Phi = m \star \tilde{\chi}\Phi$ and then

$$\begin{aligned}\mathfrak{z}_V(m \star' \Phi \otimes f) &= (m \star' \Phi) \otimes (\tilde{\chi} f)\\ &= (m \star \tilde{\chi}\Phi) \otimes (\tilde{\chi} f)\\ &= m \otimes ((\tilde{\chi}\Phi) * (\tilde{\chi} f))\\ &= m \otimes \tilde{\chi}(\Phi * f)\\ &= \mathfrak{z}_V(m \otimes (\Phi * f)).\end{aligned}$$

This implies that $\mathfrak{z}_V$ is well defined as an R-linear automorphism. Moreover, for every $x \in B_L^\times$ we have $\mathfrak{z}_V(x \diamond' (m \otimes f)) = m \otimes \tilde{\chi}(x.f) = \tilde{\chi}(x^{-1})m \otimes x.(\tilde{\chi} f) = x \diamond \mathfrak{z}_V(m \otimes f)$ and so $\mathfrak{z}_V$ is an isomorphism of representations of $B_L^\times$. Now, let V_1 and V_2 be two objects of $\mathscr{R}(G, \boldsymbol{\eta}_{\max})$ and let $\phi \in \mathrm{Hom}_G(V_1, V_2)$. Then for every $m \in \boldsymbol{M}_{\boldsymbol{\eta}_{\max}}(V_1)$ and $f \in \mathrm{ind}_{K_L^1}^{B_L^\times}(1_{K_L^1})$ we have $\mathfrak{z}_{V_2}(\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}'_{\max}}(\phi)(m \otimes f)) = \mathfrak{z}_{V_2}((\phi \circ m) \otimes f) = (\phi \circ m) \otimes \tilde{\chi} f$ which is equal to $\widetilde{\mathfrak{X}}(\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(\phi))(m \otimes \tilde{\chi} f) = \widetilde{\mathfrak{X}}(\boldsymbol{F}_{\gamma,\boldsymbol{\kappa}_{\max}}(\phi))(\mathfrak{z}_{V_1}(m \otimes f))$. □

By Remark 4.2, the representations $\boldsymbol{\kappa}_{\max}$ and $\boldsymbol{\kappa}'_{\max}$ determine two decompositions $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$ and $\boldsymbol{\lambda} = \boldsymbol{\kappa}' \otimes \boldsymbol{\sigma}'$ where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are supercuspidal representations of $\mathscr{M}$ viewed as irreducible representations of $\boldsymbol{J}_L$ trivial on $\boldsymbol{J}_L^1$. Hence, the bijection $\phi_{\boldsymbol{\kappa}'_{\max}} \circ \phi_{\boldsymbol{\kappa}_{\max}}^{-1}$ permutes the elements of $\boldsymbol{Y}$ and it maps $[\mathscr{M}, \boldsymbol{\sigma}]$ to $[\mathscr{M}, \boldsymbol{\sigma}']$. Let $\boldsymbol{\kappa}_L$ and $\boldsymbol{\kappa}'_L$ be the restrictions to $\boldsymbol{J}_L$ of $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ respectively. By (6) and by (2.20) of [Mínguez and Sécherre 2014b] we have $\boldsymbol{\kappa}'_L = \boldsymbol{\kappa}_L \otimes \chi$ and so $\boldsymbol{\sigma}' = \boldsymbol{\sigma} \otimes \bar{\chi}^{-1}$.

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