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Algebraic dynamics of the lifts of Frobenius

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We study the algebraic dynamics of endomorphisms of projective spaces with coefficients in a p-adic field whose reduction in positive characteristic is the Frobenius. In particular, we prove a version of the dynamical Manin–Mumford conjecture and the dynamical Mordell–Lang conjecture for the coherent backward orbits of such endomorphisms. We also give a new proof of a dynamical version of the Tate–Voloch conjecture in this case. Our method is based on the theory of perfectoid spaces introduced by P. Scholze. In the appendix, we prove that under some technical condition on the field of definition, a dynamical system for a polarized lift of Frobenius on a projective variety can be embedded into a dynamical system for some endomorphism of a projective space.

1. Introduction

In this paper, we write $\mathbb{C}_p$ for the completion of the algebraically closure of $\mathbb{Q}_p$ with the induced norm. Denote by $\mathbb{C}_p^\circ$ its valuation ring and $\mathbb{C}_p^{\circ\circ}$ the maximal ideal of $\mathbb{C}_p^\circ$. Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be an endomorphism taking form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \ldots, x_N) : \cdots : x_N^q + p' P_N(x_0, \ldots, x_N)]$$

where q is a power of p, $p' \in \mathbb{C}_p^{\circ\circ}$, and $P_0, \ldots, P_N$ are homogeneous polynomials of degree q in $\mathbb{C}_p^\circ[x_0, \ldots, x_N]$. We say that F is *a lift of Frobenius on* $\mathbb{P}^N_{\mathbb{C}_p}$.

In this paper we present a new argument for studying the algebraic dynamics for such maps, which is based on the theory of perfectoid spaces introduced by Scholze. In particular, we study some dynamical analogues of diophantine geometry for such maps.

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Dynamical Manin–Mumford conjecture. At first, we recall the dynamical Manin–Mumford conjecture proposed by Zhang [1995].

Dynamical Manin–Mumford Conjecture. Let $F : X_{\mathbb{C}} \to X_{\mathbb{C}}$ be an endomorphism of a quasiprojective variety defined over $\mathbb{C}$. Let V be a subvariety of X. If the Zariski closure of the set of preperiodic points[1] of F contained in V is Zariski dense in V, then V itself is preperiodic, and likewise for periodic points.[2]

This conjecture is a dynamical analogue of the Manin–Mumford conjecture on subvarieties of abelian varieties. More precisely, let V be an irreducible subvariety inside an abelian variety A over $\mathbb{C}$ such that the intersection of the set of torsion points of A and V is Zariski dense in V. Then the Manin–Mumford conjecture asserts that there exists an abelian subvariety V_0 of A and a torsion point $a \in A(\mathbb{C})$ such that $V = V_0 + a$.

The Manin–Mumford conjecture was first proved by Raynaud [1983a; 1983b]. Various versions of this conjecture were proved by Ullmo [1998], Zhang [1998], Buium [1996b], Hrushovski [2001] and Pink and Roessler[2002]. Observe that the dynamical Manin–Mumford conjecture for the map $x \mapsto 2x$ on A implies the classical Manin–Mumford conjecture.

The dynamical Manin–Mumford conjecture does not hold in full generality, as we have some counterexamples [Ghioca et al. 2011; Pazuki 2010; Pazuki 2013]. In particular, Pazuki [2013] shows that counterexamples can come from a lift of Frobenius crossed with a lift of its Verschiebung. This motivated the proposal of several modified versions of the conjecture [Ghioca et al. 2011; Yuan and Zhang 2017].

However, this conjecture is now known to hold in some special cases [Baker and Hsia 2005; Fakhruddin 2014; Medvedev and Scanlon 2014; Ghioca and Tucker 2010; Dujardin and Favre 2017; Ghioca et al. 2011; 2015; 2018]. It seems that the dynamical Manin–Mumford conjecture may be true except a few families of counterexamples.

In this paper, we prove the dynamical Manin–Mumford conjecture for periodic points of lifts of Frobenius on $\mathbb{P}^N$.

Theorem 1.1. *Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. Denote by* Per *the set of periodic closed points in $\mathbb{P}^N_{\mathbb{C}_p}$. Let V be any irreducible subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$ such that $V \cap \mathrm{Per}$ is Zariski dense in V. Then V is periodic i.e., there exists $\ell \geq 1$ such that $F^\ell(V) = V$.*

We note that Medvedev and Scanlon [2014] have proved Theorem 1.1 in the case

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + pP(x_0, x_N) : \cdots : x_{N-1}^q + pP(x_{N-1}, x_N) : x_N^q],$$

where q is a power of p and $P \in \mathbb{Z}_p[x, y]$ is a homogenous polynomial of degree q. Pazuki [2013] studied the lifts of Frobenius on abelian varieties.

We should mention that, recently Scanlon gave a new proof of this theorem without using perfectoid spaces. Since this proof is unpublished and it is completely different from ours, we will discuss it briefly in Section 4 of this paper.

[1] A preperiodic point x is a point satisfying $F^m(x) = F^n(x)$ for some $m > n \geq 0$.

[2] A periodic point x is a point satisfying $F^n(x) = x$ for some $n > 0$.

Dynamical Tate–Voloch conjecture. Let V be an irreducible subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$. There are homogenous polynomials $H_i \in \mathbb{C}_p[x_0, \dots, x_N]$, $i = 1, \dots, m$ satisfying $\|H_i\| = 1$ which define V. For any point $y \in \mathbb{P}^N_{\mathbb{C}_p}(\mathbb{C}_p)$, we may write $y = [y_0 : \cdots : y_N]$, $\max\{|y_i|\}_{0\le i\le N} = 1$. Then we denote by $d(y, V) := \max\{|H_i(y_0, \dots, y_N)|\}_{1\le i\le m}$. Observe that $d(y, V)$ does not depend on the choice of $\{H_i\}_{1\le i\le m}$ or the coordinates $[y_0 : \cdots : y_N]$ of y. It can be viewed as the distance between y and V. Moreover for any quasiprojective variety X and subvariety V of X, by choosing an embedding $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}_p}$, $d(\bullet, \overline{V})$ defines a distance between V and a point in X.

Tate and Voloch [1996] made the following conjecture:

Tate–Voloch Conjecture. Let A be a semiabelian variety over $\mathbb{C}_p$ and V a subvariety of A. Then there exists $c > 0$ such that for any torsion point $x \in A$, we have either $x \in V$ or $d(x, V) > c$.

This conjecture was proved by Scanlon [1999] when A is defined over a finite extension of $\mathbb{Q}_p$. Buium [1996a] proved a dynamical version of this conjecture for periodic points of lifts of Frobenius on any algebraic variety. Here we state it only for the lifts of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$.

Theorem 1.2. *Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. Let V be any irreducible subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$. Then there exists $\delta > 0$ such that for any point $x \in \mathrm{Per}$, either $d(x, V) > \delta$ or $x \in V$.*

In this paper, we give a new proof of this theorem by using the theory of perfectoid spaces.

Dynamical Mordell–Lang conjecture. The Mordell–Lang conjecture on subvarieties of semiabelian varieties (now a theorem of Faltings [1994] and Vojta [1996]) says that if V is a subvariety of a semiabelian variety G defined over $\mathbb{C}$ and Γ is a finitely generated subgroup of $G(\mathbb{C})$, then $V(\mathbb{C}) \bigcap \Gamma$ is a union of at most finitely many translates of subgroups of Γ.

Inspired by this, Ghioca and Tucker proposed the following dynamical analogue of the Mordell–Lang conjecture.

Dynamical Mordell–Lang Conjecture [Ghioca and Tucker 2009]. Let X be a quasiprojective variety defined over $\mathbb{C}$, let $f : X \to X$ be an endomorphism, and V be any subvariety of X. For any point $x \in X(\mathbb{C})$ the set $\{n \in \mathbb{N} \mid f^n(x) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions.[3]

Observe that the dynamical Mordell–Lang conjecture implies the classical Mordell–Lang conjecture in the case $\Gamma \simeq (\mathbb{Z}, +)$.

The dynamical Mordell–Lang conjecture has been proved in many cases. For example, Bell, Ghioca and Tucker [2010] proved this conjecture for étale maps, and the author proved it for endomorphisms of $\mathbb{A}^2_{\overline{\mathbb{Q}}}$ [Xie 2017]. We refer to the book [Bell et al. 2016] for a good survey of this conjecture.

We note that the dynamical Mordell–Lang conjecture is not a full generalization of the Mordell–Lang conjecture. In particular, it considers only the forward orbit but not the backward orbit. In an informal seminar, Zhang asked me the following question:

[3] An arithmetic progression is a set of the form $\{an + b \mid n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$. In particular, when $a = 0$, it contains only one point.

Question 1.3. Let X be a quasiprojective variety over $\mathbb{C}$ and $F : X \to X$ be a finite endomorphism. Let x be a point in $X(\mathbb{C})$. Denote by $O^-(x) := \bigcup_{i=0}^{\infty} F^{-i}(x)$ the backward orbit of x. Let V be a positive dimensional irreducible subvariety of X. If $V \cap O^-(x)$ is Zariski dense in V, what can we say about V?

We note that if V is preperiodic, then $V \cap O^-(x)$ is Zariski dense in V. As with the dynamical Manin–Mumford conjecture, the converse is not true. Indeed, we have the following example. Let $X = \mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ and $f : X \to X$ be the endomorphism defined by $(x, y) \mapsto (x^4, y^6)$. Let V be the diagonal and $x = (1, 1)$. Then $V \cap O^-(x)$ is Zariski dense in V, but V is not preperiodic. We have counterexamples even when F is a polarized endomorphism.[4] The following example is given by Ghioca, which is similar to [Ghioca et al. 2011, Theorem 1.2].

Example 1.4. Let E be the elliptic curve over $\mathbb{C}$ defined by the lattice $\mathbb{Z}[i] \subseteq \mathbb{C}$. Let F_1 be the endomorphism on E defined by the multiplication by 10 and F_2 be the endomorphism on E defined by the multiplication by $6 + 8i$. Set $X := E \times E$, $F := (F_1, F_2)$ on X. Since $|10| = |6 + 8i|$, F is a polarized endomorphism on X. Let V be the diagonal in X and x be the origin. We may check that $V \cap O^-(x)$ is Zariski dense in V, but V is not preperiodic.

As a special case of Question 1.3, we propose the following conjecture.

Conjecture 1.5. Let X be a quasiprojective variety over $\mathbb{C}$ and $F : X \to X$ be a finite endomorphism. Let $\{b_i\}_{i\geq 0}$ be a sequence of points in $X(\mathbb{C})$ satisfying $f(b_i) = b_{i-1}$ for all $i \geq 1$. Let V be a positive dimensional irreducible subvariety of X. If the $\{b_i\}_{i\geq 0} \cap V$ is Zariski dense in V, then V is periodic under F.

Remark 1.6. This conjecture can be viewed as the dynamical Mordell–Lang conjecture for the coherent backward orbits. In fact, it is easy to see that Conjecture 1.5 is equivalent to the following:

Conjecture 1.5*. Let X be a quasiprojective variety over $\mathbb{C}$ and $F : X \to X$ be a finite endomorphism. Let $\{b_i\}_{i\geq 0}$ be a sequence of points in $X(\mathbb{C})$ satisfying $f(b_i) = b_{i-1}$ for all $i \geq 1$. Let V be a subvariety of X. Then the set $\{n \geq 0 \mid b_n \in V\}$ is a union of at most finitely many arithmetic progressions.

Conjecture 1.5 $\Rightarrow$ *Conjecture 1.5**. If $\{b_i\}_{i\geq 0}$ is finite, then the b_i are contained in a periodic circle. Then Conjecture 1.5* trivially holds. Now we assume that $\{b_i\}_{i\geq 0}$ is infinite. Set $W := \bigcap_{n\geq 0} \overline{\{b_i \mid b_i \in V, i \geq n\}}$. Then there exists $N \geq 0$ such that $W = \overline{\{b_i \mid b_i \in V, i \geq N\}}$. We note that $\{n \geq 0 \mid b_n \in V\} \setminus \{n \geq 0 \mid b_n \in W\} \subseteq \{0, \ldots, N\}$ is finite. After replacing b_0 by b_N, we may assume that $N = 0$. If W is empty, then $\{n \geq 0 \mid b_n \in V\} = \{n \geq 0 \mid b_n \in W\} = \varnothing$. If W is not empty, then every irreducible component of W has positive dimension and $\{b_i\}_{i\geq N} \cap W$ is Zariski dense in W. Conjecture 1.5 implies that there exists $r \geq 1$ such that $F^r(W) = W$. If for some index $i \in \{0, \ldots, r-1\}$, there exists $s \geq 0$ such that $b_{i+sr} \notin V$, then $b_{i+nr} \notin V$ for all $n \geq s$. Denote by T_i, $i = 0, \ldots, r-1$, the set of $j \geq 0$ satisfying $b_j \in V$ and

[4] An endomorphism $F : X \to X$ on a projective variety is said to be polarized if there exists an ample line bundle L on X satisfying $F^*L = L^{\otimes d}$, $d \geq 2$.

$j = i \bmod r$. Then T_i is either finite or equal to $\{i + rn \mid n \in \mathbb{N}\}$. It follows that

$$\{n \geq 0 \mid b_n \in V\} = \{n \geq 0 \mid b_n \in W\} = \bigcup_{i=0}^{r-1} T_i$$

is a union of at most finitely many arithmetic progressions. □

Conjecture 1.5 ⇒Conjecture 1.5.* Assume that V is a positive dimensional irreducible subvariety of X such that $\{b_i\}_{i\geq 0} \cap V$ is Zariski dense in V. Then $\{n \geq 0 \mid b_n \in V\}$ is infinite. Conjecture 1.5 shows that $\{n \geq 0 \mid b_n \in V\}$ takes the form $\{n \geq 0 \mid b_n \in V\} = F \cup \left(\bigcup_{j=1}^{s} T_j\right)$ where F is finite and T_j, $j = 1, \ldots, s$, are infinite arithmetic progressions. There exists $j \in \{1, \ldots, s\}$ such that $\{b_i \mid i \in T_j\}$ is Zariski dense in V. Write $T_j = a + r\mathbb{N}$ where $a \geq 0$, $r \geq 1$. Since $F(\{b_i \mid i \in T_j\}) \setminus \{b_i \mid i \in T_j\} = \{a\}$, we have $F^r(V) = V$. □

In this paper, we prove Conjecture 1.5 for the lifts of Frobenius of $\mathbb{P}^N_{\mathbb{C}_p}$.

Theorem 1.7. *Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. Let $\{b_i\}_{i\geq 0}$ be a sequence of points in $\mathbb{P}^N_{\mathbb{C}_p}(\mathbb{C}_p)$ satisfying $f(b_i) = b_{i-1}$ for all $i \geq 1$. Let V be a positive dimensional irreducible subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$. If $\{b_i\}_{i\geq 0} \cap V$ is Zariski dense in V, then V is periodic under F.*

In fact, we prove a stronger statement.

Theorem 1.8. *Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. Let $\{b_i\}_{i\geq 0}$ be a sequence of points in $\mathbb{P}^N_{\mathbb{C}_p}(\mathbb{C}_p)$ satisfying $f(b_i) = b_{i-1}$ for all $i \geq 1$. Let V be a subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$. If there exists a subsequence $\{b_{n_i}\}_{i\geq 0}$ such that $|d(b_{n_i}, V)| \to 0$ when $n \to \infty$, then $b_{n_i} \in V$ for i large enough and there exists $r \geq 0$, such that $\{b_i\}_{i\geq 0} \subseteq \bigcup_{i=0}^{r} F^i(V)$.*

It implies the following Tate–Voloch type statement.

Corollary 1.9. *Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. Let $\{b_i\}_{i\geq 0}$ be a sequence of points in $\mathbb{P}^N_{\mathbb{C}_p}(\mathbb{C}_p)$ satisfying $f(b_i) = b_{i-1}$ for all $i \geq 1$. Let V be a subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$. Then there exists $c > 0$ such that for all $i \geq 0$, either $b_i \in V$ or $d(b_i, V) > c$.*

Overview of the proofs. Let us now see in more detail how our arguments work.

Denote by $K := \mathbb{C}_p$ and $K^\flat := \widehat{\overline{\mathbb{F}_p((t))}}$ the completion of the algebraic closure of $\mathbb{F}_p$. We denote by K° and $K^{\flat\circ}$ the valuation rings of K and $K^\flat$, respectively, and by $K^{\circ\circ}$ and $K^{\flat\circ\circ}$ the maximal ideal of K° and $K^{\flat\circ}$, respectively. Denote by $k := \overline{\mathbb{F}_p}$. We have $k = K^\circ/K^{\circ\circ} = K^{\flat\circ}/K^{\flat\circ\circ}$. Moreover, we have an embedding $k \hookrightarrow K^\flat$.

Let $F : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be an endomorphism taking form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \ldots, x_n) : \cdots : x_N^q + p' P_N(x_0, \ldots, x_N)],$$

where $p' \in K^{\circ\circ}$, q is a power of p, and $P_0, \ldots, P_N$ are homogeneous polynomials of degree q in $K^\circ[x_0, \ldots, x_N]$.

We associate to $\mathbb{P}^N_K$ and $\mathbb{P}^N_{K^\flat}$ nonarchimedean analytic spaces $\mathbb{P}^{N,\mathrm{ad}}_K$ and $\mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$, respectively, with natural embeddings $\mathbb{P}^N_K(K) \subseteq \mathbb{P}^{N,\mathrm{ad}}_K$ and $\mathbb{P}^N_{K^\flat}(K^\flat) \subseteq \mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$. The endomorphism F extends to an endomorphism F^{ad} on $\mathbb{P}^{N,\mathrm{ad}}_K$.

Denote by $\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}}$ the inverse limit of the $\mathbb{P}_K^{N,\mathrm{ad}}$ where the transition maps are F^{ad}. Then we may construct a perfectoid space $\mathbb{P}_K^{N,\mathrm{perf}}$ with an endomorphism F^{perf} for which the topological dynamical system $(\mathbb{P}_K^{N,\mathrm{perf}}, F^{\mathrm{perf}})$ is isomorphic to $(\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}}, T)$ where $T : (x_0, x_1, \ldots) \to (F^{\mathrm{ad}}(x_0), x_0, \ldots)$ is the shift map on $\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}}$. Moreover, we have a natural morphism $\pi : \mathbb{P}_K^{N,\mathrm{perf}} \to \mathbb{P}_K^{N,\mathrm{ad}}$ defined by projection to the first coordinate. This construction has been stated by Scholze [2014, §7].

Similarly, we construct a perfectoid space $\mathbb{P}_{K^\flat}^{N,\mathrm{perf}}$ which is isomorphic to the inverse limit $\varprojlim_{\Phi^s} \mathbb{P}_{K^\flat}^{N,\mathrm{ad}}$ where Φ is the Frobenius endomorphism on $\mathbb{P}_{K^\flat}^{N,\mathrm{ad}}$, and $q = p^s$. Denote by $\pi^\flat : \mathbb{P}_{K^\flat}^{N,\mathrm{perf}} \to \mathbb{P}_{K^\flat}^{N,\mathrm{ad}}$ the morphism defined by the projection to the first coordinate. Since Φ is a homeomorphism on the underlying topological space, $\pi^\flat$ induces an isomorphism from the topological dynamical system $(\mathbb{P}_{K^\flat}^{N,\mathrm{perf}}, \Phi^{s,\mathrm{perf}})$ to $(\mathbb{P}_{K^\flat}^{N,\mathrm{ad}}, \Phi^{s,\mathrm{ad}})$, where Φ^{perf} is the Frobenius on $\mathbb{P}_{K^\flat}^{N,\mathrm{perf}}$.

By the theory of perfectoid spaces, there is a natural homeomorphism of topological space

$$\rho : \mathbb{P}_K^{N,\mathrm{perf}} \to \mathbb{P}_{K^\flat}^{N,\mathrm{perf}}$$

satisfying $\Phi^{s,\mathrm{perf}} \circ \rho = \rho \circ F^{\mathrm{perf}}$.

As an example, we explain the proof of Theorem 1.1. Let V be any subvariety of $\mathbb{P}_{\mathbb{C}_p}^N$ such that $V \cap \mathrm{Per}$ is Zariski dense in V.

It is easy to see that the map $\pi \circ \rho^{-1} \circ (\pi^\flat)^{-1}$ induces a bijection from the set $\mathrm{Per}^\flat$ of periodic points of Φ^s in $\mathbb{P}_{K^\flat}^N(K^\flat)$ to the set Per of periodic points of F in $\mathbb{P}_K^N(K)$. We note that the set of periodic points of Φ^s in $\mathbb{P}_{K^\flat}^N(K^\flat)$ is exactly the set of points defined over k, i.e., the image of $\eta : \mathbb{P}_k^N(k) \hookrightarrow \mathbb{P}_{K^\flat}^N(K^\flat)$. We have a reduction map $\mathrm{red} : \mathbb{P}_K^N(K) \to \mathbb{P}_k^N(k)$. The map $\eta \circ \mathrm{red} : \mathrm{Per} \to \mathrm{Per}^\flat$ is bijective. Moreover, we have that $(\eta \circ \mathrm{red}) \circ (\pi \circ \rho^{-1} \circ (\pi^\flat)^{-1})$ is the identity on $\mathrm{Per}^\flat$.

Denote by $S^\flat$ the Zariski closure of $\eta \circ \mathrm{red}(V \cap \mathrm{Per})$. Since $S^\flat$ is defined over k, it is periodic under Φ^s. The main ingredient of our proof is to show that $S^\flat$ is a subset of $\pi^\flat(\rho(\pi^{-1}(V)))$. If $\pi^\flat(\rho(\pi^{-1}(V)))$ is algebraic, this is obvious. But, a priori, $\pi^\flat(\rho(\pi^{-1}(V)))$ is not algebraic, since the map ρ is very transcendental. Our strategy is to approximate $\pi^\flat(\rho(\pi^{-1}(V)))$ by algebraic subvarieties of $\mathbb{P}_{K^\flat}^N$. For simplicity, assume that V is an hypersurface of $\mathbb{P}_K^N$. Applying the approximation lemma of Scholze [2012, Corollary 6.7], for any $\epsilon > 0$, there exists an algebraic hypersurface H_ϵ of $\mathbb{P}_{K^\flat}^N$ which is ϵ-close to $\pi^\flat(\rho(\pi^{-1}(V)))$. Then $\eta \circ \mathrm{red}(V \cap \mathrm{Per})$ is ϵ-close to H_ϵ. Since $S^\flat$ is the Zariski closure of $\eta \circ \mathrm{red}(V \cap \mathrm{Per})$ in $\mathbb{P}_{K^\flat}^N$, we can show that it is ϵ-close to H_ϵ. Then we can show that $S^\flat$ is contained in $\pi^\flat(\rho(\pi^{-1}(V)))$ by letting ϵ tends to 0. Then we have $S := \pi(\rho^{-1}((\pi^\flat)^{-1}(S^\flat))) \subseteq V$. Since S is periodic and Zariski dense in V, it follows that V is periodic.

In this paper, we mainly consider the lifts of Frobenius on $\mathbb{P}_{\mathbb{C}_p}^N$ for simplicity, since the aim of this paper is to present a new method in dynamics. We suspect that our method can be applied to the more general case where F is a lift of Frobenius on any projective variety over $\mathbb{C}_p$. On the other hand, a lift of Frobenius on a projective variety X can often be extended to some lift of Frobenius on $\mathbb{P}_{\mathbb{C}_p}^N$ for some embedding $\tau : X \hookrightarrow \mathbb{P}_{\mathbb{C}_p}^N$. In the Appendix, we prove the existence of such embedding for *polarized* lifts of Frobenius on some projective varieties under some technical condition on the field of definition. Once this happens, many questions can be reduced to the special case where $X = \mathbb{P}_{\mathbb{C}_p}^N$.

The plan of the paper. The paper is organized as follows. In Section 2, we gather a number of results on the perfectoid spaces in Scholze's papers Scholze 2012; Scholze 2014. In Section 3, we construct the inverse limit and make it a perfectoid space with an automorphism. We also construct its tilt and give the isomorphism between these two topological dynamical systems. In Section 4, we study the periodic points of F. In particular, we prove Theorems 1.1 and 1.2. In Section 5, we study the coherent backward orbits of a point. In particular, we prove Theorems 1.7 and 1.8 and Corollary 1.9. In the Appendix, we study the polarized lift of Frobenius on projective varieties over $\mathbb{C}_p$.

2. Preliminary: perfectoid spaces

In this section, we introduce some necessary background in perfectoid spaces. All the results in this section can be found in Scholze's papers [2012; 2014]. The perfectoid spaces are some nonarchimedean analytic spaces. Following the technique of Scholze [2012], we work with Huber's adic spaces [1993; 1994; 1996].

Adic spaces. In this section, we denote by k a complete nonarchimedean field i.e., a complete topological field whose topology is induced by a nontrivial norm $|\cdot| : k \to [0, \infty)$. Denote by R a topological k-algebra. Moreover we suppose that R is a Tate k-algebra i.e., there exists a subring $R_0 \subseteq R$, such that aR_0, $a \in k^\times$, forms a basis of open neighborhoods of 0.

A subset $M \subseteq R$ is call bounded if $M \subseteq aR_0$, for some $a \in k^\times$. An element $x \in R$ is called power-bounded if $\{x^n \mid n \geq 0\} \subseteq R$ is bounded. Let $R^\circ \subseteq R$ be the subring of power-bounded elements.

Definition 2.1 [Scholze 2012]. An affinoid k-algebra is a pair (R, R^+), where R is a Tate k-algebra and R^+ is an open and integrally closed subring of R°.

A valuation on R is a map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group, such that, $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$ and $|x + y| \leq \max\{|x|, |y|\}$. We say that $|\cdot|$ is continuous, if for all $\gamma \in \Gamma$, the subset $\{x \in R : |x| < \gamma\} \subseteq R$ is open.

To a pair (R, R^+), Huber associates a space $\mathrm{Spa}(R, R^+)$ of equivalence classes of continuous valuations $|\cdot|$ on R such that $|R^+| \leq 1$, and calls it an *affinoid space*.

For a point $x \in \mathrm{Spa}(R, R^+)$, we denote by $f \to |f(x)|$ the associated valuation. It is a fact [Scholze 2012, Proposition 2.12.(iii)] that

$$R^+ = \{f \in R : |f(x)| \leq 1 \text{ for all } x \in \mathrm{Spa}(R, R^+)\}.$$

We equip $\mathrm{Spa}(R, R^+)$ with the topology generated by rational subsets:

$$U(f_1, \ldots, f_n; g) = \{x \in \mathrm{Spa}(R, R^+) : |f_i(x)| \leq |g(x)|\} \subseteq \mathrm{Spa}(R, R^+),$$

where $f_1, \ldots, f_n \in R$ generate R as an ideal and $g \in R$.

The completion $(\hat{R}, \hat{R}^+)$ of an affinoid algebra (R, R^+) is also an affinoid algebra. Then we recall [Huber 1993, Proposition 3.9].

Proposition 2.2. *We have* $\mathrm{Spa}(\hat{R}, \hat{R}^+) \simeq \mathrm{Spa}(R, R^+)$, *identifying rational subsets.*

We say a point $x \in \mathrm{Spa}(R, R^+)$ is a k-point, if the valuation x is induced by a morphism from R to k i.e., there exists a morphism $\phi : R \to k$ such that for any $f \in R$, $|f(x)| = |\phi(f)|$.

Roughly speaking, *adic spaces* over K are the objects obtained by gluing affinoid spaces. The morphisms between the adic spaces are the morphisms glued by the morphisms between affinoid spaces. Because in this paper we only consider some very concrete adic spaces, we give only a very brief definition of the adic spaces. One may find a detailed definition in [Huber 1994].

On an affinoid space $X = \mathrm{Spa}(R, R^+)$, one may define presheaves O_X and O_X^+ on X. Since we do not use these presheaves in this paper, we omit their definition. We do *not* know whether O_X is a sheaf in general. We note that once O_X is a sheaf, O_X^+ is a sheaf also. However, if (R, R^+) is of *topological finite type* then O_X is a sheaf.[5] Assume that O_X is a sheaf on X. For any $x \in X$, the valuation $f \mapsto |f(x)|$ extends to the stalk $O_{X,x}$, and we have $O_{X,x}^+ = \{f \in O_{X,x} : |f(x)| \leq 1\}$. The affinoid spaces X defines a triple $(X, O_X, |\cdot(x)| : x \in X)$.

An adic space over k is a triple $(Y, O_Y, |\cdot(x)| : x \in Y)$, consisting of a locally ringed topological space (Y, O_Y) where O_Y is a sheaf of complete topological k-algebras, and a continuous valuation $|\cdot(x)|$ on $O_{X,x}$ for every $x \in X$, which is locally on Y an affinoid adic space.

Let X be an affinoid space. We say a point $x \in X$ is a k-point if it is a k-point in any (and thus all) affinoid neighborhood of X.

Perfectoid fields. Denote by K a complete nonarchimedean field of residue characteristic $p > 0$ with norm $|\cdot| : K \to \mathbb{R}_{\geq 0}$. Denote by $K^\circ := \{x \in K : |x| \leq 1\}$ its valuation ring.

Definition 2.3. We say K is a *perfectoid field* if $|K| \subseteq \mathbb{R}_{\geq 0}$ is dense in $\mathbb{R}_{\geq 0}$ and the Frobenius map $\Phi : K^\circ/p \to K^\circ/p$ is surjective.

Observe that $\mathbb{C}_p$ and $\widehat{\mathbb{F}_p((t))}$ are perfectoid fields. Set

$$\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup_{i \geq 0} \mathbb{Q}_p(p^{1/p^i}) \quad \text{and} \quad \mathbb{F}_p((t))(t^{1/p^\infty}) := \bigcup_{i \geq 0} \mathbb{F}_p((t))(t^{1/p^i}).$$

Then their completions $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$ are perfectoid fields. Note that $\mathbb{Q}_p$ is not a perfectoid field, since $|\mathbb{Q}_p| = \{0\} \cup \{p^i \mid i \in \mathbb{Z}\} \subseteq \mathbb{R}_{\geq 0}$ is not dense.

For any perfectoid field K, we choose some element $\omega \in K^\times$ such that $|p| \leq |\omega| < 1$. We define

$$K^{\flat\circ} := \varprojlim_{x \mapsto \Phi(x)} K^\circ/\omega.$$

Recall [Scholze 2012, Lemma 3.2].

[5] An affinoid k-algebra (R, R^+) is said to be of topological finite type if R is a quotient of $k\{T_1, \ldots, T_n\}$ for some n, and $R^+ = R^\circ$.

Lemma 2.4. (i) *There exists a multiplicative homeomorphism*

$$\varprojlim_{x\mapsto x^p} K^\circ \xrightarrow{\sim} \varprojlim_{x\mapsto \Phi(x)} K^\circ/\omega = K^{\flat\circ}$$

given by projection. Moreover, we have a map

$$K^{\flat\circ} = \varprojlim_{x\mapsto x^p} K^\circ = \{(x^{(0)}, x^{(1)}, \ldots) \mid x^{(i)} \in K^\circ, (x^{(i+1)})^p = x^i\} \to K^\circ$$

defined by

$$x = (x^{(0)}, x^{(1)}, \ldots) \to x^\# := x^{(0)}.$$

We may define a norm on $K^{\flat\circ}$ by $|x^\#| = |x|$ for all $x \in K^{\flat\circ}$.

(ii) *The addition on*

$$K^{\flat\circ} = \{x := (x^{(0)}, x^{(1)}, \ldots) \mid x^{(i)} \in K^\circ, (x^{(i+1)})^p = x^i\}$$

is given by $(x+y)^i = \lim_{n\to\infty}(x^{(i+n)} + y^{(i+n)})^{p^n}$.

(iii) *There exists an element $\omega^\flat \in \varprojlim_{x\mapsto x^p} K^\circ$, satisfying $(\omega^\flat)^\# = \omega$. Define*

$$K^\flat := K^{\flat\circ}[(\omega^\flat)^{-1}].$$

Then norm $|\cdot|$ on $K^{\flat\circ}$ extends to a norm on $K^\flat$ which makes $K^{\flat\circ}$ the valuation ring of $K^\flat$.

(iv) *There exists a multiplicative homeomorphism*

$$K^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^p} K.$$

Then $K^\flat$ is a perfectoid field of characteristic p. We have $|K^{b\times}| = |K^\times|$, $K^{\flat\circ}/\omega^\flat \simeq K^\circ/\omega$, and $K^{\flat\circ}/\mathfrak{m}^\flat \simeq K^\circ/\mathfrak{m}$, where $\mathfrak{m}$ and $\mathfrak{m}^\flat$ are the maximal ideals of K° and $K^{\flat\circ}$, respectively.

(v) *If K is of characteristic p, then $K^\flat = K$.*

We note that (i) and (ii) of Lemma 2.4 implies that the definition of $K^{\flat\circ}$ is independent of ω.

We call $K^\flat$ the *tilt* of K.

Example 2.5. The tilt of $\mathbb{C}_p$ is $\mathbb{C}_p^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$.

Then we have the following theorem, which was known by the classical work of Fontaine and Wintenberger [1979]

Theorem 2.6. (i) *Let L be a finite extension of K. Then L with its natural topology induced by K is a perfectoid field.*

(ii) *The tilt functor $L \mapsto L^\flat$ induces an equivalence of categories between the category of finite extensions of K and the category of finite extensions of $K^\flat$. This equivalence preserves degrees.*

Almost mathematics. Let K be a perfectoid field and $\mathfrak{m}$ be the maximal ideal of K°.

A K°-module M is said to be *almost zero* if $\mathfrak{m}M = 0$. Define the category of almost K°-modules as

$$K^{\circ a}\text{–mod } K^{\circ a}\text{– mod} := K^\circ\text{–mod } (\mathfrak{m}\text{-torsion}).$$

We have a localization functor $M \mapsto M^a$ from K°-mod to $K^{\circ a}$-mod, whose kernel is the thick subcategory of almost zero modules.

For two $K^{\circ a}$-modules X and Y, we define $\mathrm{alHom}(X, Y) = \mathrm{Hom}(X, Y)^a$.

Proposition 2.7 [Gabber and Ramero 2003]. *The category $K^{\circ a}$–mod is an abelian tensor category, where we define kernels, cokernels and tensor products in the unique way compatible with their definition in K°–mod , that is*

$$M^a \otimes N^a = (M \otimes N)^a$$

for any two K°-modules M and N. For any $L, M, N \in K^{\circ a}$–mod there is a functorial isomorphism

$$\mathrm{Hom}(L, \mathrm{alHom}(M, N)) = \mathrm{Hom}(L \otimes M, N).$$

This means that $K^{\circ a}$-mod has all properties of the category of modules over a ring and thus one can define the notion of $K^{\circ a}$-algebra. Any K°-algebra R defines a $K^{\circ a}$-algebra R^a as the tensor products are compatible. Moreover, localization also gives a functor from R-modules to R^a-modules.

Proposition 2.8 [Gabber and Ramero 2003]. *There exists a right adjoint functor*

$$K^{\circ a}\text{–mod} \to K^\circ\text{–mod} : M \mapsto M_* := \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$$

to the localization functor $M \mapsto M^a$. The adjunction morphism $(M_*)^a \to M$ is an isomorphism. If M is a K°-module, then $(M^a)_* = \mathrm{Hom}(\mathfrak{m}, M)$.

If A is a $K^{\circ a}$-algebra, then A_* has a natural structure as K°-algebra and $(A^a)_* = A$. In particular, any $K^{\circ a}$-algebra comes via localization from a K°-algebra. Furthermore the functor $M \mapsto M_*$ induces a functor from A-modules to A_*-modules, and one can see also that all A-modules come via localization from A_*-modules. The category of A-modules is again an abelian tensor category, and all properties about the category of $K^{\circ a}$-modules stay true for the category of A-modules.

Let A be any $K^{\circ a}$-algebra. As in [Scholze 2012], an A-module M is said to be flat if the functor $X \mapsto M \otimes_A X$ on A-modules is exact.

Denote by ω an element in K° satisfying $|p| \le |\omega| < 1$. Let A be a K^a-algebra, we say A is ω-adically complete if $A \simeq \varprojlim A/\omega^n$.

Perfectoid algebras. Fix a perfectoid field K and an element $\omega \in K^\circ$ satisfying $|p| \le |\omega| < 1$.

Definition 2.9. (i) A *perfectoid K-algebra* is a Banach K-algebra R such that the subset $R^\circ \subseteq R$ of powerbounded elements is open and bounded, and the Frobenius morphism $\Phi : R^\circ/\omega \to R^\circ/\omega$ is surjective. Morphisms between perfectoid K-algebras are the continuous morphisms of K-algebras.

(ii) A *perfectoid $K^{\circ a}$-algebra* is a ω-adically complete flat $K^{\circ a}$-algebra A on which Frobenius induces an isomorphism

$$\Phi : A/\omega^{1/p} \simeq A/\omega.$$

Morphisms between perfectoid $K^{\circ a}$-algebras are the morphisms of $K^{\circ a}$-algebras.

(iii) A *perfectoid $K^{\circ a}/\omega$-algebra* is a flat $K^{\circ a}/\omega$-algebra $\bar{A}$ on which Frobenius induces an isomorphism

$$\Phi : \bar{A}/\omega^{1/p} \simeq \bar{A}.$$

Morphisms are the morphisms of $K^{\circ a}/\omega$-algebras.

Let K-Perf denote the category of perfectoid K-algebras and similarly for $K^{\circ a}$-Perf and $K^{\circ a}/\omega$-Perf. Let $K^\flat$ be the tilt of K and $\omega^\flat$ is an element in $K^\flat$ satisfying $(\omega^\flat)^\# = \omega$.

We recall [Scholze 2012, Theorem 5.2].

Theorem 2.10. *We have the following series of equivalences of categories*:

$$K\text{-Perf} \simeq K^{\circ a}\text{-Perf} \simeq (K^{\circ a}/\omega)\text{-Perf} = (K^{\flat a}/\omega^b)\text{-Perf} \simeq K^{\flat a}\text{-Perf} \simeq K^\flat\text{-Perf}.$$

In other words, a perfectoid K-algebra, which is an object over the generic fiber, has a canonical extension to the almost integral level as a perfectoid $K^{\circ a}$-algebra, and perfectoid $K^{\circ a}$-algebras are determined by their reduction modulo ω.

Let R be a perfectoid $K^{\circ a}$-algebra, with $A = R^{\circ a}$. Define

$$A^\flat := \varprojlim_{\Phi} A/\omega,$$

which we regard as a $K^{\flat\circ a}$-algebra via

$$K^{\flat\circ a} = (\varprojlim_{\Phi} K^\circ/\omega)^a = \varprojlim_{\Phi}(K^\circ/\omega)^a = \varprojlim_{\Phi} K^{\circ a}/\omega,$$

and set $R^\flat = A^b_*[(\omega^\flat)^{-1}]$.

Proposition 2.11. *This defines a perfectoid $K^\flat$-algebra $R^\flat$ with corresponding perfectoid $K^{\flat\circ a}$-algebra $A^\flat$, and $R^\flat$ is the tilt of R. Moreover*,

$$R^\flat = \varprojlim_{x\mapsto x^p} R, \quad A^\flat_* = \varprojlim_{x\mapsto x^p} A_*, \quad \text{and} \quad A^\flat_*/\omega^\flat \simeq A_*/\omega.$$

In particular, we have a continuous multiplicative map $R^\flat = \varprojlim_{x\mapsto x^p} R \to R$,

$$x = (x^{(0)}, x^{(1)}, \ldots) \mapsto x^\# := x^{(0)}.$$

Then the equivalence K-Perf $\to K^\flat$-Perf in Theorem 2.10 is given by $R \mapsto R^\flat$.

Proposition 2.12. *Let $R = K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle = K^\circ[\widehat{T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}}][\omega^{-1}]$. Then R is a perfectoid K-algebra, and its tilt $R^\flat$ is given by $K^\flat\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$.*

Perfectoid spaces. Fix a perfectoid field K and an element $\omega \in K^\circ$ satisfying $|p| \leq |\omega| < 1$. Let $K^\flat$ be the tilt of K and $\omega^\flat$ is an element in $K^\flat$ satisfying $(\omega^b)^\# = \omega$.

Definition 2.13. A *perfectoid affinoid K-algebra* is an affinoid K-algebra (R, R^+), where R is a perfectoid K-algebra, and $R^+ \subseteq R^\circ$ is an open and integrally closed subring.

Proposition 2.14. *The association $(R, R^+) \mapsto (R^b, R^{\flat+})$, where $R^{\flat+} = \varprojlim_{(x \to x^p)} R^+$. defines an equivalence between the category of perfectoid affinoid K-algebras and the category of perfectoid affinoid $K^\flat$-algebras.*

Theorem 2.15. *For any $x \in \mathrm{Spa}(R, R^+)$, one may define a point $x^\flat \in \mathrm{Spa}(R^\flat, R^{\flat+})$ by setting $|f(x^\flat)| := |f^\#(x)|$ for $f \in R^\flat$. This defines a homeomorphism $\rho : \mathrm{Spa}(R, R^+) \xrightarrow{\sim} \mathrm{Spa}(R^\flat, R^{\flat+})$ preserving rational subsets.*

Denote by $X := \mathrm{Spa}(R, R^+)$ and $X^\flat := \mathrm{Spa}(R^\flat, R^{\flat+})$. We note that in general the map $R^\flat \to R : f \to f^\#$ is not surjective. For any f in R, $\rho_* f := f \circ \rho^{-1}$ is a continuous function on $X^\flat$ but in general is not contained in $R^\flat$.

We have the following approximation lemma [Scholze 2012, Corollary 6.7].

Lemma 2.16. *For any $f \in R$ and any $c \geq 0$, $\epsilon > 0$, there exists $g_{c,\epsilon} \in R^\flat$ such that for all $x \in X$, we have*

$$|f(x) - g^\#_{c,\epsilon}(x)| \leq |\omega|^{1-\epsilon} \max(|f(x)|, |\omega|^c).$$

Remark 2.17. Note that for $\epsilon < 1$, the given estimate says in particular that for all $x \in X$, we have

$$\max\{|f(x)|, |\omega|^c\} = \max\{|g^\#_{c,\epsilon}(x)|, |\omega|^c\}.$$

Remark 2.18. Let $R := K\langle x_1, \ldots, x_N\rangle$ and $R^+ := R^\circ = K^\circ\langle x_1, \ldots, x_N\rangle$. Then $R^\flat = K^\flat\langle x_1, \ldots, x_N\rangle$ and $R^{\flat+} = K^{\flat\circ}\langle x_1, \ldots, x_N\rangle$.

By Lemma 2.16, for any $c \in \mathbb{Z}^+$, there exists an element $g_c \in K^{\flat\circ}\langle x_1^{1/p^\infty}, \ldots, x_N^{1/p^\infty}\rangle$ such that for all $x \in U_0^{\mathrm{perf}}$, we have

$$|H \circ \pi(x) - g_c^\#(x)| \leq |p|^{1/2} \max(|H \circ \pi(x)|, |p|^c).$$

There exists $\ell \in \mathbb{N}$ and an element $G_c \in K^{\flat\circ}[x_1^{1/p^\ell}, \ldots, x_N^{1/p^\ell}]$ such that $g_c - G_c \in t^{c+1} K^{\flat\circ}\langle x_1^{1/p^\infty}, \ldots, x_N^{1/p^\infty}\rangle$. It follows that for all $x \in U_0^{\mathrm{perf}}$, we have

$$|H \circ \pi(x) - G_c^\#(x)| \leq |p|^{1/2} \max(|H(x)|, |p|^c) = |p|^{1/2} \max(|G_c^\#(x)|, |p|^c),$$

and $G^{p^\ell} \in K^{\flat\circ}[x_1, \ldots, x_N]$.

Moreover, when $K^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$, we may arrange to have $G_c \subseteq E^\circ[x_1^{1/p^\ell}, \ldots, x_N^{1/p^\ell}]$, where E is a finite extension of $\overline{\mathbb{F}_p}((t))$.

We next describe the structure sheaf O_X on $X := \mathrm{Spa}(R, R^+)$. Let $U = U(f_1, \ldots, f_n; g) \subseteq X$ be a rational subset. Equip $R[g^{-1}]$ with the topology making the image of $R^+[f_1/g, \ldots, f_n/g] \to R[g^{-1}]$

open and bounded. Let $R\langle f_1/g, \dots, f_n/g\rangle$ be the completion of $R[g^{-1}]$ with respect to this topology. It is equipped with a subring

$$R\langle f_1/g, \dots, f_n/g\rangle^+ \subseteq R\langle f_1/g, \dots, f_n/g\rangle$$

which is the completion of the integral closure of $R^+[f_1/g, \dots, f_n/g]$. By [Huber 1994, Proposition 1.3], the pair $(O_X(U), O_X^+(U)) := (R\langle f_1/g, \dots, f_n/g\rangle, R\langle f_1/g, \dots, f_n/g\rangle^+)$ depends only on the rational subset $U \subseteq X$ (and not on the choice of $f_1, \dots, f_n, g \in R$). The map

$$\mathrm{Spa}(O_X(U), O_X^+(U)) \to \mathrm{Spa}(R, R^+)$$

is a homeomorphism onto U, preserving rational subsets. Moreover, $(O_X(U), O_X^+(U))$ is initial with respect to this property.

By [Scholze 2012, Theorem 6.3], we have the following:

Theorem 2.19. *For any rational subset $U \subseteq X$, let $U^\flat := \rho(U) \subseteq X^\flat$.*

(i) *The presheaves O_X and $O_{X^\flat}$ are sheaves.*

(ii) *For any rational subset $U \subseteq X$, the pair $(O_X(U), O_X^+(U))$ is a perfectoid affinoid K-algebra, which tilts to $(O_{X^\flat}(U^\flat), O_{X^\flat}^+(U^\flat))$.*

The resulting spaces $\mathrm{Spa}(R, R^+)$, equipped with the two structure sheaves of topological rings O_X and O_X^+, are called *affinoid perfectoid spaces* over K. The morphisms between the affinoid perfectoid spaces over K are the morphisms induced by the morphisms between affinoid perfectoid K-algebras.

One defines *perfectoid spaces* over K to be the objects obtained by gluing affinoid perfectoid spaces. The morphisms between the perfectoid spaces are the morphisms glued by the morphisms between affinoid perfectoid spaces.

We say that a perfectoid space $X^\flat$ over $K^\flat$ is the tilt of a perfectoid space X over K if there exists a functorial isomorphism $\mathrm{Hom}(\mathrm{Spa}(R^\flat, R^{\flat+}), X^\flat) = \mathrm{Hom}(\mathrm{Spa}(R, R^+), X)$ for all perfectoid affinoid K-algebras (R, R^+) with tilts $(R^\flat, R^{\flat+})$.

Theorem 2.20. *Any perfectoid space X over K admits a tilt $X^\flat$, unique up to isomorphism. This induces an equivalence between the category of perfectoid spaces over K and the category of perfectoid spaces over $K^\flat$. The underlying topological spaces of X and $X^\flat$ are naturally identified by ρ. A perfectoid space X is affinoid perfectoid if and only if its tilt $X^\flat$ is affinoid perfectoid. Finally, for any affinoid perfectoid subspace $U \subseteq X$, the pair $(O_X(U), O_X^+(U))$ is a perfectoid affinoid K-algebra with tilt $(O_{X^\flat}(U^\flat), O_{X^\flat}^+(U^\flat))$.*

For any morphism $F : X \to Y$ between perfectoid spaces over K, denote by $F^\flat : X^\flat \to Y^\flat$ the morphism between perfectoid spaces over $K^\flat$ induced by the equivalence of categories.

Points in perfectoid spaces. Fix a perfectoid field K and an element $\omega \in K^\circ$ satisfying $|p| \leq |\omega| < 1$. Let X be a perfectoid space over K.

For any point $x \in X$, let $K(x)$ be the residue field of $O_{X,x}$ and $K(x)^+ \subseteq K(x)$ be the image of $O_{X,x}^+$. By [Scholze 2012, Proposition 2.25], the ω-adic completion of $O_{X,x}^+$ is equal to the ω-adic completion $\widehat{K(x)}^+$ of $\widehat{K(x)}^+$. By [Scholze 2012, Corollary 6.7], $\widehat{K(x)}$ is a perfectoid field.

Definition 2.21. An affinoid perfectoid field is a pair (K, K^+) consisting of a perfectoid field and an open valuation subring $K^+ \subseteq K$.

Then $(\widehat{K(x)}, \widehat{K(x)}^+)$ is an affinoid perfectoid field. Also note that affinoid perfectoid fields (L, L^+) for which $K \subseteq L$ are affinoid K-algebras.

Then we have the following description of points [Scholze 2012, Proposition 2.27].

Proposition 2.22. *The points of X are in bijection with maps $\iota : \mathrm{Spa}(L, L^+) \to X$ to affinoid fields (L, L^+) such that the quotient field of the image of $O_{X,x}$ in L is dense, where x is the image of $\mathrm{Spa}(L, L^+)$ in X.*

Any point $x \in X$ associates to a map $\iota : \mathrm{Spa}(\widehat{K(x)}, \widehat{K(x)}^+) \to X$. By the equivalence of categories, the point $x^\flat \in X^\flat$ associates to a map

$$\iota^\flat : \mathrm{Spa}(\widehat{K(x)}^\flat, \widehat{K(x)}^{\flat+}) \to X.$$

By [Scholze 2012, Lemma 5.21], $\mathrm{Spa}(\widehat{K(x)}^\flat, \widehat{K(x)}^{\flat+})$ is an affinoid perfectoid field. It follows that $\widehat{K^\flat(x^\flat)} = (\widehat{K(x)})^\flat$.

In particular, we have the following:

Lemma 2.23. *For any point $x \in X$, x is a K-point if and only if $x^\flat$ is a $K^\flat$-point in $X^\flat$.*

3. Inverse limit of lifts of Frobenius

In this section, fix a perfectoid field K. Denote by $p > 0$ the characteristic of the residue field $K^\circ/K^{\circ\circ}$ of K.

Let $F : \mathbb{P}_K^N \to \mathbb{P}_K^N$ be a lift of Frobenius i.e., an endomorphism taking the form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \ldots, x_N) : \cdots : x_N^q + p' P_N(x_0, \ldots, x_N)],$$

where $p' \in K^{\circ\circ}$, $q = p^s$ is a power of p, and $P_0, \ldots, P_N$ are homogeneous polynomials of degree q in $K^\circ[x_0, \ldots, x_N]$. Let $\omega \in K^\circ$ be an element satisfying $\max\{|p'|, |p|\} \leq |\omega| < 1$.

Adic projective spaces. At first, we define an adic space $\mathbb{P}_K^{N,\mathrm{ad}}$ which associates to the projective space $\mathbb{P}_K^N$. In fact, by [Scholze 2012, Theorem 2.22], for any projective variety X defined over K with an integral model $\mathfrak{X}$ over K°, we may associate an adic space X^{ad}. But in this paper, we don't need the general theory and we define $\mathbb{P}_K^{N,\mathrm{ad}}$ in the following explicit way:

For any $i \in \{0, \ldots, N\}$, denote by

$$U_i^{\mathrm{ad}} := \mathrm{Spa}\big(K\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle, K\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle^\circ\big)$$

the unit balls. Then we define $\mathbb{P}^{N,\mathrm{ad}}_K$ by gluing the unit balls U_i^{ad} together in the usual way: For any $i \neq j$, $U_i^{\mathrm{ad}} \cap U_j^{\mathrm{ad}} = U(1, z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}; z_{i,j}) \subseteq U_i^{\mathrm{ad}}$. On $U_i^{\mathrm{ad}} \cap U_j^{\mathrm{ad}}$, the transition map $\phi_{i,j}$ is defined by

$$\phi_{i,j}^*(z_{j,k}) = z_{i,k}/z_{i,j} \text{ for } k \neq i, j, \quad \text{and} \quad \phi_{i,j}^*(z_{j,i}) = 1/z_{i,j}.$$

Denote by $R(\mathbb{P}^{N,\mathrm{ad}}_K)$ the set of K-points in $\mathbb{P}^{N,\mathrm{ad}}_K$.

Lemma 3.1. *There exists a natural embedding $\tau : \mathbb{P}^N_K(K) \hookrightarrow \mathbb{P}^{N,\mathrm{ad}}_K$. Moreover its image $\tau(\mathbb{P}^N_K(K)) = R(\mathbb{P}^{N,\mathrm{ad}}_K)$.*

Proof of Lemma 3.1. For any point $q \in \mathbb{P}^N_K(K)$, there exists a finite extension L of K, and a point $q' = [x_0 : \cdots : x_N] \in \mathbb{P}^N_L(L)$, such that q is the image of that q' under the natural morphism $\pi_K^L : \mathbb{P}^N_L \to \mathbb{P}^N_K$ induced by the inclusion $K \hookrightarrow L$. Indeed, $(\pi_K^L)^{-1}$ is exactly the Galois orbit of q'. We may suppose that $\max\{|x_0|, \ldots, |x_N|\} = 1$ for all $j = 0, \ldots, N$. Denote by $I_q := \{i : |x_i| = 1\}$. Observe that I_q depends only on q. Pick $i \in I_q$, we define $\tau(q) \in U_i$ to be the point defined by $f \to |f(x_1/x_i, \ldots, x_n/x_i)|$, for all $f \in K\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle$. Here $f(x_1/x_i, \ldots, x_n/x_i) \in L$ depends on the choice of q' in its Galois obit, but the value $|f(x_1/x_i, \ldots, x_n/x_i)|$ depends only on q. Moreover we may check that the definition of $\tau(q)$ does not depend on the choice of $i \in I_q$. Then τ is well defined. Moreover it is easy to check that τ is injective and $\tau(\mathbb{P}^N_K(K)) \subseteq R(\mathbb{P}^{N,\mathrm{ad}}_K)$. By [Bosch et al. 1984, 6.1.2 Corollary 3], the map $\tau : \mathbb{P}^N_K(K) \to R(\mathbb{P}^{N,\mathrm{ad}}_K)$ is surjective. □

Lifts of Frobenius on $\mathbb{P}^{N,\mathrm{ad}}_K$. The endomorphism F induces a natural endomorphism F^{ad} on $\mathbb{P}^{N,\mathrm{ad}}_K$. We define F^{ad} in the following explicit way. For any $i = 0, \ldots, N$, $F^{\mathrm{ad}}|_{U_i^{\mathrm{ad}}} : U_i^{\mathrm{ad}} \to U_i^{\mathrm{ad}}$ is defined to be

$$F^*(z_{i,j}) = \frac{z_{i,j}^q + p' P_j(z_{i,0}, \ldots, z_{i,i-1}, 1, z_{i,i+1}, \ldots, z_{i,N})}{1 + p' P_i(z_{i,0}, \ldots, z_{i,i-1}, 1, z_{i,i+1}, \ldots, z_{i,N})}$$

for all $j \neq i$. We may write

$$\frac{z_{i,j}^q + p' P_j(z_{i,0}, \ldots, z_{i,i-1}, 1, z_{i,i+1}, \ldots, z_{i,N})}{1 + p' P_i(z_{i,0}, \ldots, z_{i,i-1}, 1, z_{i,i+1}, \ldots, z_{i,N})} = z_{i,j}^q + p' Q_{i,j}(z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N})$$

where $Q_{i,j} \in K^\circ\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle$. For any $i \neq j$, we may check that $F_i^{\mathrm{ad}}(U_i^{\mathrm{ad}} \cap U_j^{\mathrm{ad}}) \subseteq U_i^{\mathrm{ad}} \cap U_j^{\mathrm{ad}}$ and

$$F_i^{\mathrm{ad}}|_{U_i^{\mathrm{ad}} \cap U_j^{\mathrm{ad}}} = F_j^{\mathrm{ad}}|_{U_i^{\mathrm{ad}} \cap U_j^{\mathrm{ad}}}.$$

Then we may glue these F_i^{ad} to define $F^{\mathrm{ad}} : \mathbb{P}^{N,\mathrm{ad}}_K \to \mathbb{P}^{N,\mathrm{ad}}_K$. Observe that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^N_K(K) & \xrightarrow{\tau} & \mathbb{P}^{N,\mathrm{ad}}_K \\ {\scriptstyle F|_{\mathbb{P}^N_K(K)}}\downarrow & & \downarrow{\scriptstyle F^{\mathrm{ad}}} \\ \mathbb{P}^N_K(K) & \xrightarrow{\tau^\flat} & \mathbb{P}^{N,\mathrm{ad}}_K \end{array}$$

Now we identify $\mathbb{P}^N_K(K)$ and $\mathbb{P}^N_{K^\flat}(K^\flat)$ with the image of τ and $\tau^\flat$ in $\mathbb{P}^{N,\mathrm{ad}}_K$ $\mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$, respectively.

The inverse limit. The inverse limit $\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}^{N,\mathrm{ad}}_K$ is the topological space $\{(x_0, x_1, \ldots) \in (\mathbb{P}^{N,\mathrm{ad}}_K)^{\mathbb{N}} \mid F^{\mathrm{ad}}(x_i) = x_{i-1} \text{ for all } i \geq 1\}$ with the product topology. There exists a natural automorphism T on $\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}^{N,\mathrm{ad}}_K$ defined by

$$T : (x_0, x_1, \ldots) \to (F^{\mathrm{ad}}(x_0), x_0, x_1, \ldots).$$

The aim of this section is to construct a perfectoid space $(\mathbb{P}^N_K)^{\mathrm{perf}}$ with an automorphism F^{perf} such that the topological dynamical system $((\mathbb{P}^N_K)^{\mathrm{perf}}, F^{\mathrm{perf}})$ is isomorphic to $(\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}^{N,\mathrm{ad}}_K, T)$.

Since $(F^{\mathrm{ad}})^{-1}(U_i^{\mathrm{ad}}) \subseteq U_i^{\mathrm{ad}}$ for all $i = 0, \ldots, N$, we have

$$\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}^{N,\mathrm{ad}}_K = \bigcup_{i=0}^{N} (\varprojlim_{F^{\mathrm{ad}}} U_i^{\mathrm{ad}}).$$

Moreover we have $T(U_i^{\mathrm{ad}}) \subseteq U_i^{\mathrm{ad}}$. It follows that we only need to construct a perfectoid affinoid space U_i^{perf} with an automorphism F_i^{perf} such that the topological dynamical system $(U_i^{\mathrm{perf}}, F_i^{\mathrm{perf}})$ is isomorphic to $(\varprojlim_{F^{\mathrm{ad}}} U_i^{\mathrm{ad}}, T|_{U_i^{\mathrm{ad}}})$ and check that they can be glued together.

Denote $R_i^n := K\langle z_{i,0}^{(n)}, \ldots, z_{i,i-1}^{(n)}, z_{i,i+1}^{(n)}, \ldots, z_{i,N}^{(n)}\rangle$ for all $i = 0, \ldots, N$ and $n \geq 0$. We identify $z_{i,j}^{(0)}$ and $z_{i,j}$ for $i \neq j$. For every $n \geq 0$, we have an embedding $R_i^n \hookrightarrow R_i^{n+1}$ defined by

$$z_{i,j}^{(n)} \mapsto (z_{i,j}^{(n+1)})^q + p' Q_{i,j}(z_{i,0}^{(n+1)}, \ldots, z_{i,i-1}^{(n+1)}, z_{i,i+1}^{(n+1)}, \ldots, z_{i,N}^{(n+1)})$$

where $Q_{i,j}$ is defined in Section 3. Then we denote by

$$R_i := K\langle z_{i,0}^{(\infty)}, \ldots, z_{i,i-1}^{(\infty)}, z_{i,i+1}^{(\infty)}, \ldots, z_{i,N}^{(\infty)}\rangle,$$

the completion of $\bigcup_{n=0}^{\infty} R_i^n$. Denote by $\|\cdot\|$ the norm on R_i induced by the norms on R_i^n, $n \geq 0$.

Lemma 3.2. *For every $i = 1, \ldots, N$, R_i is a perfectoid K-algebra with*

$$R_i^\circ = K^\circ\langle z_{i,0}^{(\infty)}, \ldots, z_{i,i-1}^{(\infty)}, z_{i,i+1}^{(\infty)}, \ldots, z_{i,N}^{(\infty)}\rangle.$$

Its tilt is given by $R_i^\flat = K^\flat\langle z_{i,0}^{1/p^\infty}, \ldots, z_{i,i-1}^{1/p^\infty}, z_{i,i+1}^{1/p^\infty}, \ldots, z_{i,N}^{1/p^\infty}\rangle$.

Proof of Lemma 3.2. Observe that $K^\circ\langle z_{i,0}^{(\infty)}, \ldots, z_{i,i-1}^{(\infty)}, z_{i,i+1}^{(\infty)}, \ldots, z_{i,N}^{(\infty)}\rangle$ is the completion of $\bigcup_{n=0}^{\infty}(R_i^n)^\circ$. It is easy to check that

$$K^\circ\langle z_{i,0}^{(\infty)}, \ldots, z_{i,i-1}^{(\infty)}, z_{i,i+1}^{(\infty)}, \ldots, z_{i,N}^{(\infty)}\rangle \subseteq R_i^\circ.$$

For any $f \in R_i$, there exists a sequence $f_n \in R_i^n$ such that $f_n \to f$ as $n \to \infty$. There exists $M \geq 0$, such that for all $m, n \geq M$, $\|f_n - f_m\| \leq 1$. It follows that $f_n - f_M \in (R_i^n)^\circ$ for all $n \geq M$. Then $\widehat{\bigcup_{n=0}^{\infty}(R_i^n)^\circ}$. If $\|f_M\| \leq 1$, we have $f_M \in (R_i^M)^\circ$ and then $\widehat{\bigcup_{n=0}^{\infty}(R_i^n)^\circ}$. If $\|f_M\| > 1$, we have $\|f^n\| = \|f_M^n\| \to \infty$ as $n \to \infty$. Then f is not power bounded. It follows that

$$R_i^\circ \subseteq K^\circ\langle z_{i,0}^{(\infty)}, \ldots, z_{i,i-1}^{(\infty)}, z_{i,i+1}^{(\infty)}, \ldots, z_{i,N}^{(\infty)}\rangle.$$

It follows that R_i° is open and bounded.

We have $R_i^\circ/\omega = (K^\circ/\omega)\langle z_{i,0}^{(\infty)}, \dots, z_{i,i-1}^{(\infty)}, z_{i,i+1}^{(\infty)}, \dots, z_{i,N}^{(\infty)}\rangle$ is the completion of $\bigcup_{n=0}^{\infty} R_i^n/\omega$. The embedding $R_i^n/\omega \to R_i^n/\omega$ is given by

$$z_{i,j}^{(n)} \bmod \omega \mapsto (z_{i,j}^{(n+1)})^q + p' Q_{i,j}(z_{i,0}^{(n+1)}, \dots, z_{i,i-1}^{(n+1)}, z_{i,i+1}^{(n+1)}, \dots, z_{i,N}^{(n+1)}) \bmod \omega = (z_{i,j}^{(n+1)})^q \bmod \omega.$$

It follows that

$$R_i^\circ/\omega = (K^\circ/\omega)\langle z_{i,0}^{1/p^\infty}, \dots, z_{i,i-1}^{1/p^\infty}, z_{i,i+1}^{1/p^\infty}, \dots, z_{i,N}^{1/p^\infty}\rangle.$$

Then the Frobenius morphism $\Phi : R_i^\circ/\omega \to R_i^\circ/\omega$ is surjective. It follows that R_i is a perfectoid K-algebra.

By Proposition 2.12 and the categorical equivalence in Theorem 2.10, we have

$$R_i^\flat = K^\flat\langle z_{i,0}^{1/p^\infty}, \dots, z_{i,i-1}^{1/p^\infty}, z_{i,i+1}^{1/p^\infty}, \dots, z_{i,N}^{1/p^\infty}\rangle. \qquad \square$$

We define $U_i^{\mathrm{perf}} := \mathrm{Spa}(R_i, R_i^\circ)$ and $F_i^{\mathrm{perf}} : U_i^{\mathrm{perf}} \to U_i^{\mathrm{perf}}$ the map induced by the morphism $R_i \to R_i$ defined by

$$z_{i,j}^{(n)} \to z_{i,j}^{(n-1)} \text{ for all } n \geq 1 \quad \text{and} \quad z_{i,j}^{(0)} \to (z_{i,j}^{(0)})^q + p' Q_{i,j}(z_{i,0}^{(0)}, \dots, z_{i,i-1}^{(0)}, z_{i,i+1}^{(0)}, \dots, z_{i,N}^{(0)}).$$

Then we define $(\mathbb{P}_K^N)^{\mathrm{perf}}$ by gluing U_i^{perf} together in the usual way: For any $i \neq j$, $U_i^{\mathrm{perf}} \cap U_j^{\mathrm{perf}} = U(1, z_{i,0}^{(0)}, \dots, z_{i,i-1}^{(0)}, z_{i,i+1}^{(0)}, \dots, z_{i,N}^{(0)}; z_{i,j}^{(0)}) \subseteq U_i^{\mathrm{perf}}$. On $U_i^{\mathrm{perf}} \cap U_j^{\mathrm{perf}}$, the transition map $\phi_{i,j}$ is defined to be

$$(\phi_{i,j}^{\mathrm{perf}})^*(z_{j,k}^{(n)}) = z_{i,k}^{(n)}/z_{i,j}^{(n)} \text{ for } k \neq i, j \quad \text{and} \quad (\phi_{i,j}^{\mathrm{perf}})^*(z_{j,i}^{(n)}) = 1/z_{i,j}^{(n)}.$$

It is easy to check that for all $i \neq j$,

$$F_i^{\mathrm{perf}}(U_i^{\mathrm{perf}} \cap U_j^{\mathrm{perf}}) \subseteq U_i^{\mathrm{perf}} \cap U_j^{\mathrm{perf}}$$

and $F_i^{\mathrm{perf}} = F_j^{\mathrm{perf}}$ on $U_i^{\mathrm{perf}} \cap U_j^{\mathrm{perf}}$. Then we define F^{perf} by gluing F_i^{perf} for $i = 0, \dots, N$.

Then we have the following:

Theorem 3.3. *There exists a natural homeomorphism* $\psi : (\mathbb{P}_K^N)^{\mathrm{perf}} \to \varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}}$ *which makes the following diagram commutative*:

$$\begin{array}{ccc} \mathbb{P}_K^{N,\mathrm{perf}} & \xrightarrow{\psi} & \varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}} \\ {\scriptstyle F^{\mathrm{perf}}}\downarrow & & \downarrow{\scriptstyle T} \\ \mathbb{P}_K^{N,\mathrm{perf}} & \xrightarrow{\psi} & \varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}} \end{array}$$

In other words, the topological dynamical systems $(\mathbb{P}_K^{N,\mathrm{perf}}, F^{\mathrm{perf}})$ *and* $(\varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}}, T)$ *are isomorphic by* ψ.

Moreover a point $x \in \mathbb{P}_K^{N,\mathrm{perf}}$, *with image* $\psi(x) = (x_0, x_1, \dots)$, *is a* K*-point if and only if* x_n *is a* K*-point for every* $n \geq 0$.

Proof of Theorem 3.3. Denote by $B_i := \bigcup_{n=0}^{\infty} R_i^n$. We have $B_i^\circ = \bigcup_{n=0}^{\infty} R_i^{n\circ}$. Then $\mathrm{Spa}(B, B^\circ)$ is an affinoid space and we have $R_i = \widehat{B_i}$ and $R_i^\circ = \widehat{B_i^\circ}$. By Proposition 2.2, the natural morphism $\mu_i : \mathrm{Spa}(R_i, R_i^\circ) \to \mathrm{Spa}(B_i, B_i^\circ)$ is a homeomorphism.

Denote by $\psi_i^n : U_i^{\mathrm{perf}} \to U_i^{\mathrm{ad}}$ the map induced by the morphism

$$K\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle \to R_i^n \subseteq B_i \subseteq R_i$$

by sending $z_{i,j} \to z_{i,j}^{(n)}$. It is easy to check that ψ_i^n could be glued to a map $\psi^n : \mathbb{P}_K^{N,\mathrm{perf}} \to \mathbb{P}_K^{N,\mathrm{ad}}$.

Since $F^{\mathrm{ad}} \circ \psi^{n+1} = \psi^n$ for all $n \geq 0$, it induces a map

$$\psi := \varprojlim_n \psi^n : \mathbb{P}_K^{N,\mathrm{perf}} \to \varprojlim_{F^{\mathrm{ad}}} \mathbb{P}_K^{N,\mathrm{ad}}.$$

By checking in the affinoid spaces U_i^{perf}, it is easy to check that $T \circ \psi = \psi \circ F^{\mathrm{perf}}$.

So we only need to show that ψ is a homeomorphism. We only need to show it in U_i. Denote by

$$\psi_i := \psi|_{U_i^{\mathrm{perf}}} = \varprojlim_n \psi_i^n.$$

Now we define a morphism $\theta_i : \varprojlim_{F^{\mathrm{ad}}} U_i^{\mathrm{ad}} \to \mathrm{Spa}(B_i, B_i^\circ)$ as the following: Let $(x_0, x_1, \ldots)$ be a point in $\varprojlim_{F^{\mathrm{ad}}} U_i^{\mathrm{ad}}$. For every $n \geq 0$, we identify $U_i \to \mathrm{Spa}(R_i^n, R_i^{n\circ})$ by $z_{i,j} \to z_{i,j}^{(n)}$. Then x_n defines a valuation on R_i^n with valuation group $\Gamma_n := \{|f(x_n)| : f \in R_i^n\}$. Moreover, for any $\ell \geq n$, and $f \in R_i^n$, we have $|f(x_l)| = |f(x_n)|$. Then we define $\theta_i((x_0, x_1, \ldots))$ to be the natural valuation $B_i = \bigcup_{n=0}^{\infty} R_i^n \to \bigcup_{n=0}^{\infty} \Gamma_n$ by gluing all the valuations x_n on R_i^n. Since all the rational subset of $\mathrm{Spa}(B_i, B_i^\circ)$ are defined over some R_i^n, it is easy to check that θ is continuous. It is easy to check that $\psi_i \circ (\mu_i^{-1} \circ \theta_i) = \mathrm{id}$ and $(\mu_i^{-1} \circ \theta_i) \circ \psi_i = \mathrm{id}$. It follows that ψ_i is a homeomorphism.

Let x be a K-point in U_i^{perf} and $\psi_i(x) = (x_0, x_1, \ldots)$. For any $n \geq 0$, we have

$$K \subseteq R_i^n/\{|f(x_n)| = 0 : f \in R_i^n\} \subseteq R_i/\{|f(x)| = 0 : f \in R_i\} = K.$$

It follows that x_n is a K-point.

Let x be a point in U_i^{perf} and set $\psi_i(x) = (x_0, x_1, \ldots)$. We suppose that all x_n are K-points. Then $m_i^n := \{|f(x_n)| = 0 : f \in R_i^n\}$ is a maximal ideal in R_i^n and $R_i^n/m_i^n = K$. The valuation $R_i^n/m_i^n \to \mathbb{R}$ induced by x_i is the norm on K.

There exists a continuous morphism $\bigcup_{n=0}^{\infty} R_i^n \to K$ obtained by gluing the morphisms $R_i^n \to R_i^n/m_i^n \hookrightarrow K$. We can extend this morphism to a continuous morphism $g : R_i^n = \widehat{\bigcup_{n=0}^{\infty} R_i^n} \to K$. The valuation $f \to |g(f)|$ defines a point $y \in U_i^{\mathrm{perf}}$, which is a K-point. Observe that for all $f \in R_i^n$, $|f(y)| = |f(x_i)|$. Then we have $\psi(y) = (x_0, x_1, \ldots) = \psi(x)$. Then $y = x$ and so, x is a K-point. □

For every $i = 0, \ldots, N$, the embedding $K\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle \subseteq R_i$ induces a map $U_i^{\mathrm{perf}} \to U_i^{\mathrm{ad}}$. We define $\pi : \mathbb{P}_K^{N,\mathrm{perf}} \to \mathbb{P}_K^{N,\mathrm{ad}}$ by gluing these maps. It is easy to check that

$$F^{\mathrm{ad}} \circ \pi^\flat = \pi^\flat \circ F^{\mathrm{perf}}.$$

For any point $x \in \mathbb{P}_K^{N,\mathrm{perf}}$ with $\psi(x) = (x_0, x_1, \ldots)$, we have $\pi(x) = x_0$.

Passing to the tilt. Denote by $U_i^{\flat,\mathrm{perf}} := \mathrm{Spa}(R_i^{\flat}, R_i^{b\circ})$ and $\Phi_i^{s,\mathrm{perf}} : U_i^{\flat,\mathrm{perf}} \to U_i^{\flat,\mathrm{perf}}$ the s-th power of the Frobenius i.e., the map induced by the morphism

$$R_i^{\flat} \to R_i^{\flat} : f \to f^q.$$

We define $(\mathbb{P}^N_{K^\flat})^{\mathrm{perf}}$ by gluing $U_i^{\flat,\mathrm{perf}}$ together in the usual way: For any $i \neq j$, $U_i^{\flat,\mathrm{perf}} \cap U_j^{\flat,\mathrm{perf}} = U(1, z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}; z_{i,j}) \subseteq U_i^{\flat,\mathrm{perf}}$.

On $U_i^{\flat,\mathrm{perf}} \cap U_j^{\flat,\mathrm{perf}}$, the transition map $\phi_{i,j}^{\flat}$ is defined to be

$$(\phi_{i,j}^{\flat,\mathrm{perf}})^*(z_{j,k}^{1/p^n}) = z_{i,k}^{1/p^n}/z_{i,j}^{1/p^n} \text{ for } k \neq i, j \quad \text{and} \quad (\phi_{i,j}^{\flat,\mathrm{perf}})^*(z_{j,i}^{1/p^n}) = 1/z_{i,j}^{1/p^n}.$$

By reducing modulo ω and the categorical equivalence in Theorem 2.10, we see that $\phi_{i,j}^{\flat,\mathrm{perf}} = (\phi_{i,j}^{\mathrm{perf}})^{\flat}$ and $\Phi_i^{s,\mathrm{perf}} = (F_i^{\mathrm{perf}})^{\flat}$. It follows that $(\mathbb{P}^N_{K^\flat})^{\mathrm{perf}} = ((\mathbb{P}^N_K)^{\mathrm{perf}})^{\flat}$ and we can define $\Phi^{s,\mathrm{perf}}$ by gluing $\Phi_i^{s,\mathrm{perf}}$ together. Moreover, we have $\Phi^{s,\mathrm{perf}} = (F^{\mathrm{perf}})^{\flat}$. Then we have the following:

Theorem 3.4. *The following diagram is commutative*:

$$\begin{array}{ccc} \mathbb{P}^{N,\mathrm{perf}}_K & \xrightarrow{\rho} & \mathbb{P}^{N,\mathrm{perf}}_{K^\flat} \\ {\scriptstyle F^{\mathrm{perf}}}\downarrow & & \downarrow{\scriptstyle \Phi^{s,\mathrm{perf}}} \\ \mathbb{P}^{N,\mathrm{perf}}_K & \xrightarrow{\rho} & \mathbb{P}^{N,\mathrm{perf}}_{K^\flat} \end{array}$$

In other words, the topological dynamical systems $(\mathbb{P}^{N,\mathrm{perf}}_K, F^{\mathrm{perf}})$ *and* $(\mathbb{P}^{N,\mathrm{perf}}_{K^\flat}, \Phi^s)$ *are isomorphic by* ρ.

For any $i \in \{0, \ldots, N\}$, denote by

$$U_i^{\flat,\mathrm{ad}} := \mathrm{Spa}\big(K^\flat\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle, K^\flat\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle^\circ\big).$$

As in Section 3, we define $\mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$ by gluing $U_i^{\flat,\mathrm{ad}}$, $i = 0, \ldots, N$. Denote by $\phi_i^{s,\mathrm{ad}}$ the s-th power of the Frobenius on U_i i.e., the map induced by the morphism $f \to f^q$ on $K^\flat\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle$.

By Lemma 3.1, we have a natural embedding $\tau^\flat : \mathbb{P}^N_{K^\flat}(K^\flat) \hookrightarrow \mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$. Then $\tau(\mathbb{P}^N_K(K)) = R(\mathbb{P}^{N,\mathrm{ad}}_K)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^N_{K^\flat}(K^\flat) & \xrightarrow{\tau} & \mathbb{P}^{N,\mathrm{ad}}_{K^\flat} \\ {\scriptstyle \Phi^s|_{\mathbb{P}^N_K(K)}}\downarrow & & \downarrow{\scriptstyle \Phi^{s,\mathrm{ad}}} \\ \mathbb{P}^N_{K^\flat}(K^\flat) & \xrightarrow{\tau} & \mathbb{P}^{N,\mathrm{ad}}_{K^\flat} \end{array}$$

where Φ^s is the s-th power of the Frobenius on $P^N_{K^\flat}$.

For every $i = 0, \ldots, N$, the embedding $K^\flat\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle \subseteq R_i^{\flat}$ induces a map $U_i^{\flat,\mathrm{perf}} \to U_i^{\flat,\mathrm{ad}}$. We define $\pi^\flat : \mathbb{P}^{N,\mathrm{perf}}_{K^\flat} \to \mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$ by gluing these maps. It is easy to check that

$$\Phi^{s,\mathrm{ad}} \circ \pi^\flat = \pi^\flat \circ \Phi^{s,\mathrm{perf}}. \tag{1}$$

By [Scholze 2012, Theorem 8.5] $\pi^\flat$ is a homeomorphism. Moreover, we have the following:

Lemma 3.5. *The map $\pi^\flat$ induces a bijection between $R(\mathbb{P}^{N,\mathrm{perf}}_{K^\flat})$ and $R(\mathbb{P}^{N,\mathrm{ad}}_{K^\flat})$.*

Proof of Lemma 3.5. It is clear that if x is a $K^\flat$-point then $\pi^\flat(x)$ is a $K^\flat$-point.

Now we suppose that $\pi^\flat(x)$ is a $K^\flat$-point. We suppose that x is contained in $U_i^{\flat,\mathrm{perf}}$ and then $x_0 := \pi^\flat(x) \in U_i^{\flat,\mathrm{ad}}$. Since x_0 is a $K^\flat$-point, it defines a morphism

$$g_0 : R_i^{\flat,0} := K^\flat\langle z_{i,0}, \ldots, z_{i,i-1}, z_{i,i+1}, \ldots, z_{i,N}\rangle \to K^\flat.$$

It follows that the Frobenius map $f \to f^p$ on $K^\flat$ is a field automorphism. For any $f \in R_i^{\flat,n} := K^\flat\langle z_{i,0}^{1/p^n}, \ldots, z_{i,i-1}^{1/p^n}, z_{i,i+1}^{1/p^n}, \ldots, z_{i,N}^{1/p^n}\rangle$, we have $f^{p^n} \in R_i^{\flat,0}$. Then the morphism g_0 extends to a morphism $g_n : R_i^{\flat,n} \to K^\flat$ by sending f to $(g_0(f^{p^n}))^{1/p^n}$. We glue g_n to define a continuous morphism $\bigcup_{n=0}^\infty R_i^{\flat,n} \to K^\flat$ and then extend it to a continuous morphism

$$g : R_i^\flat = \widehat{\left(\bigcup_{n=0}^{\infty} R_i^{\flat,n}\right)} \to K^\flat.$$

Then g induces a $K^\flat$-point $y \in U_i^{\flat,\mathrm{perf}}$. Since $\pi^\flat(y) = x_0 = \pi^\flat(x)$, we have $y = x$. Then x is a $K^\flat$-point. □

4. Periodic points

In this section, we denote by $K = \mathbb{C}_p$. Then K is a perfectoid field and $K^\flat$ is the completion of the algebraical closure of $\mathbb{F}_p((t))$. We may suppose that $|p| = |t| = p^{-1}$.

Let $F : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be an endomorphism taking form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \ldots, x_N) : \cdots : x_N^q + p' P_N(x_0, \ldots, x_N)],$$

where $p' \in K^{\circ\circ}$, q is a power of p, and $P_0, \ldots, P_N$ are homogeneous polynomials of degree q in $K^\circ[x_0, \ldots, x_N]$. The aim of this section is to study the periodic points of F. In particular, we prove Theorem 1.1 and 1.2.

Recall that Per is the set of periodic closed points in $\mathbb{P}^N_K$.

Let V be any irreducible subvariety of $\mathbb{P}^N_K$. Suppose that V is defined by the equations $H_j(x_0, \ldots, x_N) = 0$, $j = 1, \ldots, m$, where H_j are homogenous polynomials. We may suppose that $\|H_j\| = 1$ for all $j = 1, \ldots, m$. For any $i = 0, \ldots, N$, denote by

$$V_i^{\mathrm{ad}} := \{x \in U_i^{\mathrm{ad}} : |H_{i,j}(x)| = 0,\ j = 1, \ldots, m\},$$

where $H_{i,j} := H(z_{i,0}, \ldots, z_{i,i-1}, 1, z_{i,i+1}, \ldots, z_{i,N})$. Observe that $\|H_{i,j}\| = 1$.

Set $R(V_i^{\mathrm{ad}}) := R(\mathbb{P}^{N,\mathrm{ad}}_K) \cap V_i^{\mathrm{ad}}$, $V^{\mathrm{ad}} := \bigcup_{i=0}^N V_i^{\mathrm{ad}}$ and $R(V^{\mathrm{ad}}) := R(\mathbb{P}^{N,\mathrm{ad}}_K) \cap V^{\mathrm{ad}}$. Then we have $\tau(R(V)) = R(V^{\mathrm{ad}})$.

Observe that for all points $x \in R(U_i^{\mathrm{ad}})$, we have $d(x, V) = \max\{|H_{i,j}(x)|\}$.

Passing to the reduction. Since K is algebraically closed, we have $\mathbb{P}^N_K(K) = \mathbb{P}^N_K(K)$. Denote by $k = \overline{\mathbb{F}_p}$, we have $k = K^\circ/K^{\circ\circ}$. At first, there exists a reduction map

$$\mathrm{red} : \mathbb{P}^N_K(K) \to \mathbb{P}^N_k(k)$$

defined by the following: For any point $x \in \mathbb{P}^N_K(K)$, we may write it as $x = [x_0 : \cdots : x_N]$ where $x_i \in K^\circ$, $i = 0, \ldots, N$ and $\max\{|x_i|, i = 0, \ldots, N\} = 1$. Then we define $\mathrm{red}(x) = [\overline{x_1} : \cdots : \overline{x_N}]$ where $\overline{x_i}$ is the image of x_i in $k = K^\circ/K^{\circ\circ}$. Observe that $\mathrm{red} \circ F = \overline{\Phi}^s \circ \mathrm{red}$, where $\overline{\phi}$ is the Frobenius on $\mathbb{P}^N_k$. For every point $y \in \mathbb{P}^N_k(k)$, there exists $m > 0$ such that $\overline{\Phi}^{sm}(y) = y$. Then we have $D_y := \mathrm{red}^{-1}(y) \simeq (K^{\circ\circ})^N$ is a polydisc fixed by F. Since $F^m|_{D_y}$ is attracting, $D_y \cap \mathrm{Per}$ has exactly one point. It follows that red induces a bijection between Per and $\mathbb{P}^N_k(k)$.

Similarly, we can define the reduction map $\mathrm{red}^\flat : \mathbb{P}^N_{K^\flat}(K^\flat) \to \mathbb{P}^N_k(k)$. This map induces a bijection between $\mathrm{Per}^\flat$ and $\mathbb{P}^N_k(k)$ where $\mathrm{Per}^\flat$ is the set of Φ^s-periodic closed points of $\mathbb{P}^N_{K^\flat}$.

Since k is a subfield of $K^\flat$, there exists an embedding $\eta : \mathbb{P}^N_k(k) \hookrightarrow \mathbb{P}^N_{K^\flat}(K^\flat)$. Observe that the image $\eta(\mathbb{P}^N_k(k))$ is exactly $\mathrm{Per}^\flat$. Moreover we have $\mathrm{red}^\flat \circ \eta = \mathrm{id}$. We may check that the map

$$\phi := \eta \circ \mathrm{red} : \mathrm{Per} \to \mathrm{Per}^\flat$$

is a bijection satisfying $\Phi^s \circ \phi = \phi \circ F$.

Passing to the tilt. Denote by $\mathrm{Per}^{\mathrm{ad}} = \tau(\mathrm{Per})$. It is exactly the set of periodic K-points in $\mathbb{P}^{N,\mathrm{ad}}_K$. For any point $x \in \mathrm{Per}^{\mathrm{ad}}$, denote by $n > 0$ a period of x under F^{ad}. We define a map $\chi : \mathrm{Per}^{\mathrm{ad}} \to \varprojlim_{F^{\mathrm{ad}}} \mathbb{P}^{N,\mathrm{ad}}_K$ by sending x to $(x_0, x_1, \ldots)$ where $x_i = (F^{\mathrm{an}})^{kn-i}(x)$ where $kn \geq i$. We note that $\chi(x)$ does not depend on the choice of n and k. Since $\pi \circ \chi = \mathrm{id}$, χ is injective. We have that $\chi(\mathrm{Per}^{\mathrm{ad}})$ is exactly the set of Per_T, where Per_T is the set of points $(x_0, x_1, \ldots) \in \varprojlim_{F^{\mathrm{ad}}} \mathbb{P}^{N,\mathrm{ad}}_K$ which is periodic under T such that every x_n is a K-point.

Denote by $\mathrm{Per}^{\flat,\mathrm{ad}}$ the set of $K^\flat$-points in $\mathbb{P}^{N,\mathrm{ad}}_{K^\flat}$ which are periodic under $\Phi^{s,\mathrm{ad}}$. By applying Lemma 3.1 over $K^\flat$, there exists a bijection $\tau^\flat : \mathbb{P}^N_{K^\flat}(K^\flat) \to R(\mathbb{P}^{N,\mathrm{ad}}_{K^\flat})$ and we have

$$\tau^\flat \circ \Phi^s = \Phi^{s,\mathrm{ad}} \circ \tau^\flat.$$

It follows that $\tau^\flat$ induces a bijection between $\mathrm{Per}^{\flat,\mathrm{ad}}$ and the set $\mathrm{Per}^\flat$ of Φ^s-periodic points in $R(\mathbb{P}^N_{K^\flat}) = \mathbb{P}^N_{K^\flat}(K^\flat)$.

By Theorems 3.3, 3.4, (1) and Lemma 3.5, the map

$$\iota := \pi^\flat \circ \rho \circ \psi^{-1} \circ \chi : \mathrm{Per}^{\mathrm{ad}} \to \mathrm{Per}^{\flat,\mathrm{ad}}$$

is bijective.

Denote by $\mathrm{Per}^{\mathrm{ad}}_i := \mathrm{Per}^{\mathrm{ad}} \cap U^{\mathrm{ad}}_i$ and $\mathrm{Per}^{\flat,\mathrm{ad}}_i := \mathrm{Per}^{\flat,\mathrm{ad}} \cap U^{\flat,\mathrm{ad}}_i$ for every $i = 0, \ldots, N$. Then we have $\iota(\mathrm{Per}^{\mathrm{ad}}_i) = \mathrm{Per}^{\flat,\mathrm{ad}}_i$.

Observe that for every point $x \in \mathrm{Per}$, we have $\mathrm{red}(x) = \mathrm{red}^\flat \circ \iota \circ \tau(x)$. Then on Per we have

$$\phi = \eta \circ \mathrm{red} = \eta \circ \mathrm{red}^\flat \circ \iota \circ \tau = \iota \circ \tau.$$

Proof of Theorem 1.2. We only need to show this theorem for the periodic points in U_i^{ad} for all $i = 0, \ldots, N$. Without the loss of generality, we only need to show that there exists $\delta > 0$ such that for all $x \in \mathrm{Per} \cap U_0^{\mathrm{ad}}$, either $d(x, V) > \delta$ or $x \in V$.

At first, we prove our theorem for hypersurfaces.

Lemma 4.1. *Let $H \in K[x_1, \ldots, x_N]$ be a polynomial. Then there exists $\varepsilon > 0$, such that for all $x \in \mathrm{Per} \cap U_0^{\mathrm{ad}}$, either $|H(x)| > \varepsilon$ or $H(x) = 0$.*

By this lemma, for any $H_{0,j}$, $j = 1, \ldots, m$, we have $\varepsilon_j > 0$ such that for all $x \in \mathrm{Per} \cap U_0^{\mathrm{ad}}$, either $|H_{0,j}(x)| > \varepsilon_j$ or $H_{0,j}(x) = 0$. Set $\delta := \min_{1 \le j \le m}\{\varepsilon_j\}$. Let x be a point in $\mathrm{Per} \cap U_0^{\mathrm{ad}}$ satisfying $d(x, V) \le \delta$. Then for all $j = 1, \ldots, m$, we have $H_{0,j}(x) = 0$. It follows that $x \in V$.

We only need to prove Lemma 4.1. To do this, we need the following lemma:

Lemma 4.2. *Let $E/\overline{\mathbb{F}_p}((t))$ be a finite extension. Then, for some $u \in E$ satisfying $|u| = |t|^{1/[E:\overline{\mathbb{F}_p}((t))]}$, $E = \overline{\mathbb{F}_p}((u))$.*

Proof of Lemma 4.2. Observe that E is a discrete valuation field.

Since $\overline{\mathbb{F}_p}$ is algebraically closed, the extension $E/\overline{\mathbb{F}_p}((t))$ is totally ramified. It follows that $E = \overline{\mathbb{F}_p}((t))(u)$ where the minimal polynomial of u over $\overline{\mathbb{F}_p}((t))$ is an Eisenstein polynomial. It follows that $|u| = |t|^{1/[E:\overline{\mathbb{F}_p}((t))]}$ and uE° is the maximal ideal of E°. For every $f \in E^\circ$, f can be written as $\sum_{i \ge 0} a_i u^i$ where $a_i \in \overline{\mathbb{F}_p}$ for all $i \ge 0$. This concludes our proof. □

Lemma 4.3. *For any polynomial $G \in K^{\flat\circ}[x_1, \ldots, x_N]$ and $\varepsilon > 0$, there exists a polynomial $G_\varepsilon \in K^{\flat\circ}[x_1, \ldots, x_N]$ satisfying $\deg G_\varepsilon \le \deg G$, $\|G - G_\varepsilon\| < \varepsilon$ (resp. $\le \varepsilon$) and G_ε has the form $G_\varepsilon = \sum_{i \ge 0}^m u^i g_i$ where $g_i \in \overline{\mathbb{F}_p}[x_1, \ldots, x_N]$, $u \in \overline{\mathbb{F}_p((t))}^\circ$ with norm $|u| = |t|^{1/[\overline{\mathbb{F}_p}((t))(u):\overline{\mathbb{F}_p}((t))]}$ and $|u|^m \ge \varepsilon$ (resp. $> \varepsilon$).*

Proof of Lemma 4.3. By Lemma 4.2, there exists $u \in \overline{\mathbb{F}_p((t))}^\circ$ with norm $|u| = |t|^{1/[\overline{\mathbb{F}_p}((t))(u):\overline{\mathbb{F}_p}((t))]}$ and $H \in \overline{\mathbb{F}_p}((t))(u)[x_1, \ldots, x_N]$ such that $\deg H \le \deg G$, $\|G - H\| < \varepsilon$ and H takes form $H = \sum_{i \ge 0}^\infty u^i g_i$ where $g_i \in \overline{\mathbb{F}_p}[x_1, \ldots, x_N]$. Let m be the largest integer such that $|u|^m \ge \varepsilon$ (resp. $> \varepsilon$). Set $G_\varepsilon = \sum_{i \ge 0}^m u^i g_i$ then we conclude our proof. □

Proof of Lemma 4.1. We may suppose that $H \ne 0$ and $\|H\| = 1$.

By Remark 2.18, for any $c \in \mathbb{Z}^+$, there exists $\ell \in \mathbb{N}$ and an element $G_c \in K^{\flat\circ}[x_1^{1/p^\ell}, \ldots, x_N^{1/p^\ell}]$ such that for all $x \in U_0^{\mathrm{perf}}$, we have

$$|H \circ \pi(x) - G_c^{\#}(x)| \le |p|^{1/2} \max(|H(x)|, |p|^c) = |p|^{1/2} \max(|G_c^{\#}(x)|, |p|^c),$$

and $G^{p^\ell} \in K^{\flat\circ}[x_1, \ldots, x_N]$. By Lemma 4.3, we may suppose that $G_c^{p^\ell} = \sum_{i \ge 0}^m u^i g_{c,i}$ where $g_{c,j} \in \overline{\mathbb{F}_p}[x_1, \ldots, x_N]$, $u \in \overline{\mathbb{F}_p((t))}^\circ$ with norm $|u| = |t|^{1/[\overline{\mathbb{F}_p}((t))(u):\overline{\mathbb{F}_p}((t))]}$ and $|u|^m > |t|^{(c+1/2)p^\ell}$.

Denote by I_c the ideal of $K^\flat[x_1, \ldots, x_N]$ generated by all $g_{c,i}$.

If $x \in R(U_0^\flat)$ is a point such that for all $g \in I_c$, $g(x) = 0$, then we have

$$\begin{aligned}\left|H(\pi(\rho^{-1}((\pi^b)^{-1}(x))))\right| &= \left|H(\pi(\rho^{-1}((\pi^b)^{-1}(x)))) - G_c^{\#}(\rho^{-1}((\pi^\flat)^{-1}(x)))\right| \\ &\leq \max\{|p|^{1/2}|G_c^{\#}(\rho^{-1}((\pi^\flat)^{-1}(x))|, |p|^{c+1/2}\} \\ &= |p|^{c+1/2}.\end{aligned}$$

On the other hand we have the following lemma.

Lemma 4.4. *Let x be a point in $\mathrm{Per} \cap U_0^{\mathrm{ad}}$ satisfying $|H(x)| \leq |p|^{c+1/2}$. Then for all $g \in I_c$, we have $g(\phi(x)) = 0$.*

Proof of Lemma 4.4. Observe that $|(H - G_c^{\#})(\chi(x))| \leq |p|^{1/2}\max(|H(x)|, |p|^c)$ and $|H(x)| \leq |p|^{c+1/2}$. We have $|G_c(\phi(x))| = |G_c^{\#}(\chi(x))| \leq |p|^{c+1/2}$. For all $j \geq 0$, we have $g_{c,j}(\phi(x)) \in k$. It follows that either $|g_{c,j}(\phi(x))| = 1$ or $g_{c,j}(\phi(x)) = 0$ for all $j \geq 0$. If $g_{c,j}(\phi(x)) = 0$ for all $j \geq 0$, then for all $g \in I_c$ we have $g(\phi(x)) = 0$. Otherwise, let j_0 be the smallest j satisfying $|g_{c,j}(\phi(x))| = 1$. It follows that $|G_c(\phi(x))| = |u|^{j_0/p^\ell}$. Since $|G_c(\phi(x))| = |G_c^{\#}(\chi(x))| \leq |p|^{c+1/2}$, we get a contradiction. □

Set $I := \sum_{c\geq 1} I_c$. Since $K^\flat[x_1, \ldots, x_N]$ is Noetherian, there exists $M \in \mathbb{Z}^+$, such that $I = \sum_{c=1}^{M} I_c$. Set $\varepsilon := |p|^{M+1/2}$. Let x be a point in $\mathrm{Per} \cap U_0^{\mathrm{ad}}$ satisfying $|H(x)| \leq \varepsilon = |p|^{M+1/2}$. By Lemma 4.4, for all $g \in I = \sum_{c=1}^{M} I_c$, we have $|g(\phi(x))| = 0$. It follows that for all $c \geq 1$ and $g \in I_c$, we have $|g(\phi(x))| = 0$. Then we have $|H(x)| \leq |p|^{c+1/2}$ for all $c \geq 0$. Let c tend to infinity, we have $H(x) = 0$. We conclude our proof of Lemma 4.1. □

Proof of Theorem 1.1. Suppose that $V \cap \mathrm{Per}$ is Zariski dense in V. We claim the following:

Lemma 4.5. *There exists a Zariski dense subset $S \subseteq V$ with the property that $F^\ell(S) = S$ for some positive integer ℓ.*

Since S is Zariski dense in V and $S = F^\ell(S)$ is Zariski dense in $F^\ell(V)$. It follows that $V = F^\ell(V)$. Then Lemma 4.5 implies Theorem 1.1.

Proof of Lemma 4.5. Since $\bigcup_{i=0}^{N} \tau^{-1}(\mathrm{Per}_i^{\mathrm{ad}} \cap V_i^{\mathrm{ad}}) = \mathrm{Per}$ is Zariski dense in V, there exists $i = 0, \ldots, N$, such that $\tau^{-1}(\mathrm{Per}_i^{\mathrm{ad}} \cap V_i^{\mathrm{ad}})$ is Zariski dense in V. We may suppose that $i = 0$.

Let Z be the Zariski closure of $\phi(\tau^{-1}(\mathrm{Per}_0^{\mathrm{ad}} \cap V_0^{\mathrm{ad}})) \subseteq \mathbb{P}^N_{K^\flat}$. Since $\phi(\tau^{-1}(\mathrm{Per}_0^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}))$ is defined over k and it is Zariski dense in Z, Z is defined over $k = \overline{\mathbb{F}_p}$. Then Z is defined over a finite extension of $\mathbb{F}_p$. It follows that there exists $\ell \geq 1$, such that $\Phi^{sl}(Z) = Z$.

Set $S^{\flat,\mathrm{ad}} := \tau^\flat(Z(K^\flat)) \cap U_0^\flat$. We have $\iota(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}) \subseteq S^{\flat,\mathrm{ad}} \cap \pi^\flat(\rho(\pi^{-1}(V^{\mathrm{ad}})))$.

We claim the following:

Lemma 4.6. *We have $S^{\flat,\mathrm{ad}} \subseteq \pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$.*

Remark 4.7. We note that if $\pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$ is algebraic, our lemma is easy. Since $\phi(\tau^{-1}(\mathrm{Per}_0^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}))$ is Zariski dense in Z, and $\pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$ is algebraic, we have $S^{\flat,\mathrm{ad}} \subseteq \pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$.

But in general $\pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$ is not algebraic since the map ρ is not algebraic. Our proof of Lemma 4.6 is based on Lemma 2.16, which allows us to approximate $\pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$ by algebraic subvarieties.

By assuming Lemma 4.6, we have $\pi(\rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}}))) \subseteq V_0^{\mathrm{ad}}$. Set $S = \tau^{-1}(\pi(\rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}}))))$. We have $S \subseteq V$ is a Zariski dense subset of V. Moreover, we have $F^\ell(S) = S$. This concludes the proof of Lemma 4.5. □

Now we only need to prove Lemma 4.6. First, we need the following:

Lemma 4.8. *Let $H \in K^b[z_{0,1}, \ldots, z_{0,N}]$ be a polynomial with norm 1. Suppose that for every point $x \in \iota(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}})$, we have $|H(x)| \leq 1/p^s$, where $s \in \mathbb{Z}^+$. Then for every point $y \in S^{\flat,\mathrm{ad}}$, we have $|H(y)| \leq 1/p^s$.*

Proof of Lemma 4.8. Observe that we have a map

$$R(U_0^{\mathrm{ad}}) = (K^{\flat\circ})^N \to (K^{\flat\circ}/(t^s))^N = \mathbb{A}^N_{K^{\flat\circ}/(t^s)}(K^{\flat\circ}/(t^s))$$

defined by $(x_1, \ldots, x_n) \mapsto (\overline{x_1}, \ldots, \overline{x_N})$ where $\overline{x_i} = x_i \mod t^s$. Denote by

$$\overline{H} := H \mod t^s.$$

For every point $x \in \iota(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}})$, we have $\overline{H}(\bar{x}) = 0$. Observe that

$$\overline{\iota(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}})} = (\phi(\tau^{-1}(\mathrm{Per}_0^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}))) \times_{\mathrm{Spec}\, k} \mathrm{Spec}(K^{\flat,\circ}/(t^s))$$

is Zariski dense in $Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}(K^{\flat,\circ}/(t^s))$. It follows that

$$Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}(K^{\flat,\circ}/(t^s)) \subseteq \{\overline{H} = 0\}.$$

Then we have

$$\overline{S^{\flat,\mathrm{ad}}} = Z(k) \times_{\mathrm{Spec}\, k} \mathrm{Spec}(K^{\flat,\circ}/(t^s)) \subseteq Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}(K^{\flat,\circ}/(t^s)) \subseteq \{\overline{H} = 0\}.$$

It follows that for every $x \in S^{\flat,\mathrm{ad}}$, we have $H(x) = 0 \mod t^s$. Then we have $|H(x)| \leq 1/p^s$, for all $x \in S^{\flat,\mathrm{ad}}$. □

Proof of Lemma 4.6. Now we apply Lemma 2.16 to $H_{0,j} \in K\langle z_{0,1}, \ldots, z_{0,N}\rangle \subseteq R_0^{\mathrm{perf}}$ for every $j = 1, \ldots, m$. For any $s \geq 2$ there exists $h_s \in R_0^{\flat,\mathrm{perf}}$ such that for all $x \in U_0^{\mathrm{perf}}$, we have

$$|H_{0,j}(x) - h_s^\#(x)| \leq |t|^{1/2} \max(|H_{0,j}(x)|, |t|^s) = |t|^{1/2} \max\{|h_s^\#(x)|, |t|^s\} < 1. \tag{2}$$

It follows that $\|h_s\| = \|H_{0,j}\| = 1$.

For every point $x^\flat \in (\pi^\flat)^{-1}(\iota(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}))$, we have

$$x := \rho^{-1}(x^\flat) \in \pi^{-1}(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}).$$

Then we have $H_{0,j}(x) = 0$. By (2) we have

$$|h_s(x^\flat)| \leq |t|^{s+1/2} = |t|^{1/2} \max\{|h_s(x^\flat)|, |t|^s\} = 1/p^{s+1/2}.$$

Since $h_s \in R_0^{\flat,\mathrm{perf}} = K^\flat\langle z_{0,1}^{1/p^\infty}, \dots, z_{0,N}^{1/p^\infty}\rangle$, there are $r \geq 0$ and a function

$$g_s \in K^\flat[z_{0,1}^{1/p^r}, \dots, z_{0,N}^{1/p^r}]$$

such that $\|h_s - g_s\| < 1/p^s$. It follows that $g_s^{p^r} \in K^\flat[z_{0,1}, \dots, z_{0,N}]$ and

$$\|h_s^{p^r} - g_s^{p^r}\| \leq |p|^{sp^r}.$$

Then for every point $x^\flat \in (\pi^\flat)^{-1}(\iota(\mathrm{Per}^{\mathrm{ad}} \cap V_0^{\mathrm{ad}}))$, we have

$$|g_s^{p^r}(\pi^\flat(x^\flat))| = |g_s^{p^r}(x^\flat)| = |h_s^{p^r}(x^\flat) + (g_s^{p^r}(x^\flat) - h_s^{p^r}(x^\flat))| \leq |p|^{sp^r}.$$

By Lemma 4.8, for all $y \in S^{\flat,\mathrm{ad}}$, we have $|g_s^{p^r}(y)| \leq |p|^{sp^r}$. Then we have $|h_s((\pi^\flat)^{-1}(y)| \leq 1/p^s$ and

$$|h_s^\#(\rho^{-1}((\pi^\flat)^{-1}(y))| = |h_s((\pi^\flat)^{-1}(y)| \leq 1/p^s$$

for all $y \in S^{\flat,\mathrm{ad}}$.

By (2), we have

$$|H_{0,j}(x) - h_s^\#(x)| \leq |t|^{1/2} \max\{|h_s^\#(x)|, |t|^s\} = 1/p^{s+1/2}$$

for all $x \in \rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}}))$. It follows that for all $x \in \rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}}))$, we have $|H_{0,j}(x)| \leq 1/p^s$. Let $s \to \infty$, we have $|H_{0,j}(x)| = 0$ for all $x \in \rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}}))$. Since $|H_{0,j}(x)| = |H_{0,j}(\pi(x))|$, we have $|H_{0,j}(y)| = 0$ for all $j = 1, \dots, m$ and $y \in \pi(\rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}})))$. It follows that $\pi(\rho^{-1}((\pi^\flat)^{-1}(S^{\flat,\mathrm{ad}}))) \subseteq V_0^{\mathrm{ad}}$. Then we have $S^{\flat,\mathrm{ad}} \subseteq \pi^\flat(\rho(\pi^{-1}(V_0^{\mathrm{ad}})))$. □

Scanlon's proof of Theorem 1.1. In this section, we discuss Scanlon's proof of Theorem 1.1. In this proof, we don't need the perfectoid spaces.

Let V be a subvariety of $\mathbb{P}^N$ such that $\mathrm{Per} \cap V$ is Zariski dense in V. We want to show that V is periodic.

We first treat the case where F is defined over $\overline{\mathbb{Q}_p}^\circ$. Since all points in Per are defined over $\overline{\mathbb{Q}_p}$ and $\mathrm{Per} \cap V$ is Zariski dense in V, V is defined over $\overline{\mathbb{Q}_p}$. There exists a finite extension K_p of $\mathbb{Q}_p$ such that F is defined over K_p i.e., F takes form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \dots, x_n) : \cdots : x_N^q + p' P_N(x_0, \dots, x_N)],$$

where $p' \in K_p^{\circ\circ}$, q is a power of p, $P_0, \dots, P_N$ are homogeneous polynomials of degree $q = p^s$ in $K_p^\circ[x_0, \dots, x_N]$. After replacing F by a suitable iterate, we may assume that the residue field $\tilde{K} := K^\circ/K^{\circ\circ}$ is fixed by the q-power Frobenius.

By the structure of the absolute Galois group of K_p, there exists an element $\sigma \in \mathrm{Gal}(\overline{K_p}/K_p)$ which lifts the q-power Frobenius. Then we have the following lemma:

Lemma 4.9 [Medvedev and Scanlon 2014]. *We have* $\mathrm{Per} = \{x \in \mathbb{P}^N(\overline{\mathbb{Q}_p}) : F(x) = \sigma(x)\}$.

Proof of Lemma 4.9. Recall that the reduction map

$$\mathrm{red} : \mathbb{P}^N(\overline{\mathbb{Q}_p}) \to \mathbb{P}^N(\overline{\mathbb{F}_p})$$

gives a bijection between Per and $\mathbb{P}^N(\overline{\mathbb{F}_p})$.

Let x be any point in Per. We have that $F(x) \in \mathrm{Per}$ and $\mathrm{red}(F(x)) = \mathrm{red}(x)^q$. On the other hand, we have that $\sigma(x) \in \mathrm{Per}$ and $\mathrm{red}(\sigma(x)) = \mathrm{red}(x)^q$. Then we have $F(x) = \sigma(x)$.

Let x be any point in $\mathbb{P}^N(\overline{\mathbb{Q}_p})$ satisfying $F(x) = \sigma(x)$. Since x is defined over a finite extension of K_p, there exists $n \geq 1$ such that $\sigma^n(x) = x$. It follows that

$$F^n(x) = F^{n-1}(\sigma(x)) = \sigma(F^{n-1}(x)) = \cdots = \sigma^n(x) = x.$$

Then x is periodic. □

Observe that $\sigma(V)$ is a subvariety of $\mathbb{P}^N$. Then we have

$$\sigma(V \cap \mathrm{Per}) = F(V \cap \mathrm{Per}) \subseteq \sigma(V) \cap F(V).$$

Since $V \cap \mathrm{Per}$ is Zariski dense in V, we have $\sigma(V) = F(V)$. Since V is defined over a finite extension of $\mathbb{Q}_p$, there exists $n \geq 1$ such that $\sigma^n(V) = V$. It follows that

$$F^n(V) = F^{n-1}(\sigma(V)) = \sigma(F^{n-1}(V)) = \cdots = \sigma^n(V) = V.$$

Then V is periodic.

Now we treat the general case.

There exists a subring $R \subseteq \mathbb{C}_p^\circ$ which is finitely generated over $\mathbb{Z}$ such that F is defined over R. Let $m := R \cap \mathbb{C}_p^{\circ\circ}$ be a maximal ideal of R. By Lemma A.3, there exists $\sigma \in \mathrm{Gal}(\mathbb{C}_p/\mathbb{Q})$ such that $\sigma(R) \subseteq \overline{\mathbb{Q}_p}^\circ \subseteq \mathbb{C}_p^\circ$ and $\sigma(m) = \overline{\mathbb{Q}_p}^{\circ\circ} \cap R$.

Denote by F^σ the Galois conjugate of F by σ i.e., F^σ is obtained by changing every coefficient of F by its image under σ. Since $F^\sigma \bmod \mathbb{C}_p^{\circ\circ} = F \bmod \mathbb{C}_p^{\circ\circ}$, F^σ is a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. Moreover it is defined over $\overline{\mathbb{Q}_p}^\circ$.

Since $V \cap \mathrm{Per}$ is Zariski dense in V, $\sigma(V) \cap \sigma(\mathrm{Per})$ is Zariski dense in $\sigma(V)$. Moreover $\sigma(\mathrm{Per})$ is exactly the set of periodic points of F^σ. Then the previous argument shows that $\sigma(V)$ is periodic under F^σ. It follows that V is periodic under F.

5. Coherent backward orbits

In this section, we let $K = \mathbb{C}_p$. Then K is a perfectoid field and $K^\flat$ is the completion of the algebraic closure of $\mathbb{F}_p((t))$. We may suppose that $|p| = |t| = p^{-1}$. Let $k = \overline{\mathbb{F}_p}$ which is a subfield of $K^\flat$.

Let $F : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be an endomorphism taking form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \ldots, x_N) : \cdots : x_N^q + p' P_N(x_0, \ldots, x_N)],$$

where $p' \in K^{\circ\circ}$, q is a power of p, and $P_0, \ldots, P_N$ are homogeneous polynomials of degree q in $K^\circ[x_0, \ldots, x_N]$.

The aim of this section is to prove Theorems 1.7 and 1.8.

Without loss of generality, we may suppose that $b_0 \in R(U_0^{\mathrm{ad}})$. It follows that $b_i \in R(U_0^{\mathrm{ad}})$ for all $i \geq 0$. Set $w := \pi^\flat \circ \rho \circ \psi^{-1}((b_0, b_1, \ldots)) \in \mathbb{P}^N_{K^\flat}(K_b)$. Then $w \in R(U_0^{\flat,\mathrm{ad}}) := \{[1 : x_1 : \cdots : x_N] : |x_i| \leq 1\} \subseteq \mathbb{P}^N_{K^\flat}(K_b)$. It follows that $w^{1/q^n} \subseteq R(U_0^{\flat,\mathrm{ad}})$ for all $n \geq 0$.

If $\{b_i\}_{i\geq 0}$ is infinite, we may suppose that $b_1 \neq b_0$ and then the b_i, for $i \geq 0$, are all different. Let Z be the reduced subvariety of $U_0^\flat := \operatorname{Spec} K^\flat[x_1, \ldots, x_N]$, whose support is the union of all positive dimensional irreducible components of the Zariski closure of $\{w^{1/q^{n_i}}\}_{i\geq 0}$.

There exists $A \geq 0$, such that Z is the Zariski closure of $\{w^{1/q^{n_i}}\}_{i\geq A}$ in $U_0^\flat$. Moreover, for all $n \geq A$, Z is the Zariski closure of $\{w^{1/q^{n_i}}\}_{i\geq n}$ in $U_0^\flat$.

Denote by $I(Z)$ the ideal in $K^\flat[x_1, \ldots, x_N]$ which defines Z.

For every polynomial $f = \sum_I a_I x^I \in K^\flat[x_1, \ldots .x_N]$ and $i \in \mathbb{Z}$, we denote by $f^{\sigma^i} := \sum_I a_I^{q^i} x^I$. Observe that $f(y^{1/q^i}) = (f^{\sigma^i}(y))^{1/q^i}$ for all $i \geq 0$ and $y \in R(U_0^{\flat,\mathrm{ad}})$.

Then we have the following lemma:

Lemma 5.1. *Let $f \in k[x_1, \ldots, x_N]$ be a polynomial defined over k. If there exists $c \in (0, 1)$ and $B \geq A$, such that for all $i \geq B$, $|f(w^{1/q^{n_i}})| \leq c$, then $f \in I(Z)$.*

Proof of Lemma 5.1. There exists $L \geq 1$ such that f is defined over $\mathbb{F}_{q^L}$. Then we have $f^{\sigma^{nL}} = f$ for all $n \geq 0$. For $t = 0, \ldots, L-1$, set $T_t := \{i \geq B \mid n_i = t \bmod L\}$.

For all $t = 0, \ldots, L-1$ satisfying $\#T_t = \infty$, we have

$$|f(w^{1/q^t})|^{1/q^{n_i-t}} = |f^{\sigma^{n_i-t}}(w^{1/q^t})|^{1/q^{n_i-t}} = |f(w^{1/q^{n_i}})| \leq c,$$

for all $i \in T_t$. It follows that $|f(w^{1/q^t})| \leq c^{q^{n_i-t}}$ for all $i \in T_t$. Since T_t is infinite, n_i can be arbitrary large. Then we have $|f(w^{1/q^t})| = 0$ for all $i \in T_t$. It follows that

$$|f(w^{1/q^{n_i}})| = |f^{\sigma^{n_i-t}}(w^{1/q^t})|^{1/q^{n_i-t}} = |f(w^{1/q^t})|^{1/q^{n_i-t}} = 0$$

for all $i \in T_t$. Set

$$T' := \bigsqcup_{\substack{0\leq t\leq L-1\\ \#T_t=\infty}} T_t.$$

It follows that $f(w^{1/q^{n_i}}) = 0$ for all $i \in T'$. Since $\{i \geq A\} \setminus T'$ is finite, $\{w^{1/q^{n_i}}\}_{i\in T'}$ is Zariski dense in Z. Then $f \in I(Z)$. □

Lemma 5.2. *We have that Z is defined over k. In particular, there exists $r \geq 1$ such that $\Phi^{sr}(Z) = Z$ and $\{w^{1/q^i}\}_{i\in\mathbb{Z}} \subseteq \bigcup_{i=0}^{r-1} \Phi^{si}(Z)$.*

Proof of Lemma 5.2. We only need to show that $I(Z)$ is generated by finitely many polynomials in $k[x_1, \ldots, x_N] \subseteq K^\flat[x_1, \ldots, x_N]$. In fact, if $I(Z) = (g_1, \ldots, g_l)$ and $g_i \in k[x_1, \ldots, x_N]$ for all $i = 1, \ldots, \ell$, then there exists $r \geq 1$ such that all the coefficients of $g_i, i = 1, \ldots, \ell$, are defined over $\mathbb{F}_{q^r}$. Then we have $\Phi^{sr}(Z) = Z$. Moreover, there exists $j \geq 0$, such that $w^{1/p^j} \in Z$. It follows that $\{w^{1/q^i}\}_{i\in\mathbb{Z}} \subseteq \bigcup_{i\in\mathbb{Z}} \Phi^{si}(Z) = \bigcup_{i=0}^{r-1} \Phi^{si}(Z)$.

Write $I(Z) = (f_1, \ldots, f_m)$ where $m \geq 1$ and $f_i \in K^\flat[x_1, \ldots, x_N]$ for all $i = 1, \ldots, m$. Denote by $d := \max_{0\leq i\leq m}\{\deg(f_i)\}$.

By Lemma 4.3, for all $i = 1, \ldots, m$, there exists a sequence of polynomial $\{f_{i,n}\}_{n\geq 1}$ such that $\|f_i - f_{i,n}\| \leq |t^n|$ and taking form $f_{i,n} = \sum_{j=0}^{m_{i,n}} u_{i,n}^j f_{i,n,j}$ where $f_{i,n,j} \in \overline{\mathbb{F}_p}[x_1, \ldots, x_N]$ of degree at most d, $u_{i,n} \in \overline{\mathbb{F}_p((t))}^\circ$ with norm $|u_{i,n}| = |t|^{1/[\overline{\mathbb{F}_p}((t))(u_{i,n}):\overline{\mathbb{F}_p}((t))]}$ and $|u_{i,n}|^{m_{i,n}} > |t|^n$.

We claim that $f_{i,j,n} \in I(Z)$ for all $j = 0, \ldots, m_{i,n}$.

We prove that claim by induction on j. For $j = 0$, we have

$$|f_{i,0,n}(w^{1/q^{n_l}})| = \left|f_{i,n}(w^{1/q^{n_l}}) - \sum_{j\geq 1} u_{i,n}^j f_{i,j,n}(w^{1/q^{n_l}})\right| \leq \max\{|t^n|, |u_{i,n}|\} < 1$$

for all $\ell \geq A$. By Lemma 5.2, we have $f_{i,0} \in I(Z)$.

If $j \geq 1$ and $f_{i,0,n}, \ldots, f_{i,j-1,n} \in I(Z)$, then

$$\begin{aligned}
|f_{i,j,n}(w^{1/q^{n_l}})| &= \left|u_{i,n}^{-j}(f_{i,n}(w^{1/q^{n_l}}) - \sum_{0\leq t'\leq j-1} u_{i,n}^{t'} f_{i,t',n}(w^{1/q^{n_l}})) - \sum_{t'\geq j+1} u_{i,n}^{t'-j} f_{i,j,n}(w^{1/q^{n_l}})\right| \\
&\leq \max\left\{\frac{|t|^n}{|u_{i,n}|^j}, \left|\sum_{t'\geq j+1} u_{i,n}^{t'-j} f_{i,j,n}(w^{1/q^{n_l}})\right|\right\} \\
&\leq \max\left\{\frac{|t|^n}{|u_{i,n}|^j}, |u_{i,n}|\right\} \\
&< 1
\end{aligned}$$

for all $\ell \geq A$. By Lemma 5.2, we have $f_{i,j,n} \in I(Z)$. This concludes the proof of the claim. It follows that $f_{i,n} \in I$.

Set $I_d := \{f \in I \mid \deg(f) \leq d\}$. Then I_d is a finite-dimensional $K^\flat$-vector space. For all $n \geq 0$ and $j = 0, \ldots, m_{i,n}$, denote by $I_{i,j,n}$ the K_b-vector space spanned by $f_{i,0,0} \ldots, f_{i,j,0}, \ldots, f_{i,0,n} \ldots, f_{i,j,n}$. Then $\bigcup_{n\geq 0, j=0,\ldots,m_{i,n}} I_{i,j,n}$ is a subspace of I_d. Since $\dim I_d$ is finite, $\bigcup_{n\geq 0, j=0,\ldots,m_{i,n}} I_{i,j,n}$ is closed. Observe that f_i is contained in the closure of $\bigcup_{n\geq 0, j=0,\ldots,m_{i,n}} I_{i,j,n}$, we have $f_i \in \bigcup_{n\geq 0, j=0,\ldots,m_{i,n}} I_{i,j,n}$. There exists $l_i \geq 0$, such that $f_i \in I_{i,m_{i,l_i},l_i}$. It follows that $I = (f_1, \ldots, f_m) \subseteq \sum_{1\leq i\leq m}(I_{i,m_{i,l_i},l_i}) \subseteq I$. Then we have $I = (f_{i,j,n})_{1\leq i\leq m, 0\leq n\leq l_i, 0\leq j\leq m_{l_i}}$ and $f_{i,j} \in k[x_1, \ldots, x_N]$ for all i, j. □

Proof of Theorem 1.8. Let V be a subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$ such that there exists a subsequence $\{b_{n_i}\}_{i\geq 0}$ such that $|d(b_{n_i}, V)| \to 0$ when $i \to \infty$. We need to show that $b_{n_i} \in V$ for i large enough and there exists $r \geq 0$, such that $\{b_i\}_{i\geq 0} \subseteq \bigcup_{i=0}^{r-1} F^i(V)$.

If $\{b_i\}_{i\geq 0}$ is finite, Theorem 1.8 is trivial. So we suppose that $\{b_i\}_{i\geq 0}$ is infinite.

Let $I(V)$ denote the ideal in $K[x_1, \ldots, x_N]$ which defines $V \cap U_1$. Then for any point in $R(U_0^{\mathrm{ad}})$, we have $d(y, V) = \max\{|H(y)| : H \in I(V) \text{ and } \|H\| = 1\}$.

Let Z denote the union of all positive dimensional irreducible components of the Zariski closure of $\{w^{1/q^{n_i}}\}_{i\geq 0}$. There exists an $A \geq 0$, such that Z is that Zariski closure of $\{w^{1/q^{n_i}}\}_{i\geq A}$ in $U_0^\flat := \mathrm{Spec}\, K^\flat[x_1, \ldots, x_N]$. Moreover, for all $n \geq A$, Z is the Zariski closure of $\{w^{1/q^{n_i}}\}_{i\geq n}$ in $U_0^\flat$.

Let $I(Z)$ denote the ideal in $K^\flat[x_1, \ldots, x_N]$ which defines Z. Let H be a polynomial in $I(V)$.

Lemma 5.3. *For any point $x \in Z \cap R(U_0^{\flat,\mathrm{ad}})$, we have $H(\pi(\rho^{-1}((\pi^\flat)^{-1}(x)))) = 0$.*

Proof of Lemma 5.3. By Remark 2.18, for any $c \in \mathbb{Z}^+$, there exists $\ell \in \mathbb{N}$ and an element $G_c \in K^{\flat\circ}[x_1^{1/p^\ell}, \ldots, x_N^{1/p^\ell}]$ such that for all $x \in U_0^{\mathrm{perf}}$, we have

$$|H \circ \pi(x) - G_c^{\#}(x)| \le |p|^{1/2} \max(|H(x)|, |p|^c) = |p|^{1/2} \max(|G_c^{\#}(x)|, |p|^c),$$

and $G_c^{p^\ell} \in K^{\flat\circ}[x_1, \ldots, x_N]$. By Lemma 4.3, we may suppose that $G_c^{p^\ell} = \sum_{i\ge 0}^{m} u^i g_i$ where $g_i \in \overline{\mathbb{F}_p}[x_1, \ldots, x_N]$, $u \in \overline{\mathbb{F}_p((t))}^\circ$ with norm $|u| = |t|^{1/[\overline{\mathbb{F}_p}((t))(u):\overline{\mathbb{F}_p}((t))]}$ and $|u|^m > |t|^{(c+1/2)p^\ell}$.

There exists $A_1 \ge 0$, such that $|H(b_{n_i})| \le |p|^{c+1}$, for all $i \ge A_1$.

For all $i \ge A_1$, we have

$$|G_c(w^{1/q^{n_i}})| \le \max\{|H(b_{n_i})|, |H(b_{n_i}) - G_c^{\#}(\rho^{-1}(w^{1/q^{n_i}}))|\} \le |p|^{c+1/2}.$$

Then we have

$$|G_c(w^{1/q^{n_i}})^{p^\ell}| \le |t|^{p^\ell(c+1/2)}.$$

We claim that for all $j = 0, \ldots, m$, we have $g_j \in I(Z)$.

We prove this claim by induction on j. Suppose that for all $0 \le t' < j \le m$, we have $g_{t'} \in I(Z)$. For all $i \ge \max\{A, A_1\}$, we have

$$\left| u^j g_j(w^{1/q^{n_i}}) + \sum_{t'\ge j+1} u^{t'} g_{t'}(w^{1/q^{n_i}}) \right| = |G_c(w^{1/q^{n_i}})^{p^\ell}| \le |t|^{p^\ell(c+1/2)}.$$

It follows that $|g_j(w^{1/q^{n_i}})| \le \max\{|t|^{p^\ell(c+1/2)}/|u|^j, |u|\} < 1$ for all $i \ge \max\{A, A_1\}$. Then Lemma 5.1 implies that $g_j \in I(Z)$ for $j = 0, \ldots, m$. This proves the claim.

Then for any $x \in Z \cap R(U_0^{\flat,\mathrm{ad}})$, we have

$$\begin{aligned} |H(\pi(\rho^{-1}((\pi^b)^{-1}(x))))| &= |H(\pi(\rho^{-1}((\pi^b)^{-1}(x)))) - G_c^{\#}(\rho^{-1}((\pi^\flat)^{-1}(x)))| \\ &\le \max\{|p|^{1/2}|G_c^{\#}(\rho^{-1}((\pi^\flat)^{-1}(x))), |p|^{c+1/2}\} \\ &= |p|^{c+1/2}. \end{aligned}$$

Let c tend to infinity, then we have $|H(\pi(\rho^{-1}((\pi^b)^{-1}(x))))| = 0$. We complete the proof of our lemma. □

This lemma shows that $S := \pi(\rho^{-1}((\pi^b)^{-1}(Z \cap R(U_0^{\flat,\mathrm{ad}})))) \subseteq V$. Then $b_{n_i} \in V$ for $i \ge A$. By Lemma 5.2, there exists $r \ge 1$ such that $\Phi^{sr}(Z) = Z$ and $\{w^{1/p^i}\}_{i\in\mathbb{Z}} \subseteq \bigcup_{i=0}^{r} \Phi^{si}(Z)$. It follows that $\{b_i\}_{i\ge 0} \subseteq \bigcup_{i=0}^{r-1} F^i(S) \subseteq \bigcup_{i=0}^{r-1} F^i(V)$.

Proof of Corollary 1.9. Let V be a subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$ of positive dimension. We need to show there exists $c > 0$ such that for all $i \ge 0$ either $b_i \in V$ or $d(b_i, V) > c$.

Otherwise, there exists a subsequence $\{b_{n_i}\}_{i\ge 0} \subseteq \{b_i\}_{i\ge 0} \setminus V$ such that $d(b_{n_i}, V)$ tends to 0. By Theorem 1.8, we have $b_{n_i} \in V$ for sufficiently large i, which is a contradiction. □

Proof of Theorem 1.7. Let V be a positive subvariety of $\mathbb{P}^N_{\mathbb{C}_p}$ such that $\{b_i\}_{i\geq 0}\cap V$ is Zariski dense in V. Let $\{n_1 < n_2 < \cdots\}$ be the set of $n \geq 0$ such that $b_n \in V$. We need to show that V is periodic under F.

If $\{b_i\}_{i\geq 0}$ is finite, then all points in $\{b_i\}_{i\geq 0}$ are periodic. Moreover V is a union of finitely many periodic points. So V is periodic.

Now we may suppose that $\{b_i\}_{i\geq 0}$ is infinite. Denote by $I(V)$ the ideal in $K[x_1,\ldots,x_N]$ which defines $V\cap U_1$. Let H be a polynomial $I(V)$. By Lemma 5.3, for any point $x\in Z\cap R(U_0^{\flat,\mathrm{ad}})$, we have $H(\pi(\rho^{-1}((\pi^b)^{-1}(x))))=0$.

It follows that $S:=\pi(\rho^{-1}((\pi^b)^{-1}(Z\cap R(U_0^{\flat,\mathrm{ad}}))))\subseteq V$. Since $b_{n_i}\in S$ for all $i\geq A$, S is Zariski dense in V. Since $\Phi^{rs}(Z\cap R(U_0^{\flat,\mathrm{ad}}))=Z\cap R(U_0^{\flat,\mathrm{ad}})$, we have $F^r(S)=S$. It follows that $F^r(V)=V$. This concludes the proof. □

Appendix

Let X be any projective variety over $\mathbb{C}_p$ and $F: X\to X$ be an endomorphism. Let $\mathfrak{X}\to \operatorname{Spec}\mathbb{C}_p^\circ$ be a finitely presented projective scheme which is flat over $\operatorname{Spec}\mathbb{C}_p^\circ$ whose generic fiber is X and $\mathcal{L}$ an ample line bundle on $\mathfrak{X}$. If there exists an endomorphism $\tilde{F}$ of $\mathfrak{X}$ over $\mathbb{C}_p^\circ$ such that $\tilde{F}^*\mathcal{L}=\mathcal{L}^{\otimes q}$ where $q=p^s$, $s\geq 1$, the restriction of $\tilde{F}$ on the generic fiber is F and the restriction $\bar{F}$ of $\tilde{F}$ on the special fiber $\bar{X}$ is a power of the Frobenius, then we say that *F is a polarized lift of Frobenius on X with respect to* $(\mathfrak{X},\tilde{F},\mathcal{L})$. In particular, a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$ in the previous sections is a lift of Frobenius on X with respect to a pair $(\mathbb{P}^N_{\mathbb{C}_p^\circ},\tilde{F},O_{\mathbb{P}^N_{\mathbb{C}_p^\circ}}(1))$.

Now assume that F is a polarized lift of Frobenius on X with respect to the pair $(\mathfrak{X},\tilde{F},\mathcal{L})$ and we identify X with the generic fiber of $\mathfrak{X}$.

In this appendix, we show that under a technical condition, the dynamical system (X,F) can be embedded in a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$ (with respect to some $(\mathbb{P}^N_{\mathbb{C}_p^\circ},\tilde{F},O_{\mathbb{P}^N_{\mathbb{C}_p^\circ}}(1))$).

Theorem A.1. *Assume that $\mathfrak{X}$ and $\tilde{F}$ are defined over $\overline{\mathbb{Q}_p}^\circ\subseteq\mathbb{C}_p^\circ$. Then there exists $N\geq 1$, a lift of Frobenius G on $\mathbb{P}^N_{\mathbb{C}_p}$ and an embedding $\tau: X\hookrightarrow\mathbb{P}^N_{\mathbb{C}_p}$ such that $\tau\circ F^l=G\circ\tau$ for some $l\geq 1$.*

This theorem can be viewed as a version of [Fakhruddin 2003, Proposition 2.1] for the lifts of Frobenius.

As an application, it implies the dynamical Manin–Mumford Conjecture and Conjecture 1.5, for any polarized lift of Frobenius on X with respect to some $(\mathfrak{X},\tilde{F},\mathcal{L})$.

Corollary A.2. *Let V be any positive dimensional irreducible subvariety of X. Denote by Per_F the set of periodic closed points in X. Let $\{b_i\}_{i\geq 0}$ be a sequence of closed points in X satisfying $f(b_i)=b_{i-1}$ for all $i\geq 1$. Then we have that*:

(i) *If $V\cap\mathrm{Per}_F$ is Zariski dense in V, then V is periodic.*

(ii) *If the $\{b_i\}_{i\geq 0}\cap V$ is Zariski dense in V, then V is periodic under F.*

Proof of Corollary A.2. There exists subring $R \subseteq \mathbb{C}_p^\circ$ which is finitely generated over $\mathbb{Z}$ such that $\mathfrak{X}$, $\tilde{F}$ and $\mathcal{L}$ are defined over R, i.e., there exists a projective scheme $\mathfrak{X}_R$ over Spec R with an endomorphism $\tilde{F}_R$ and an ample line bundle $\mathcal{L}_R$ such that $\mathfrak{X} = \mathfrak{X}_R \otimes_R \mathbb{C}_p^\circ$, $\mathcal{L} = \mathcal{L}_R \otimes_R \mathbb{C}_p^\circ$, $\tilde{F} = \tilde{F}_R \otimes_R \mathbb{C}_p^\circ$ and $\tilde{F}_R^* \mathcal{L}_R = \mathcal{L}_R^{\otimes q}$.

If $R \subseteq \overline{\mathbb{Q}_p}^\circ$, Theorem A.1 reduces it to the case where $X = \mathbb{P}^N_{\mathbb{C}_p}$ and F is a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. We conclude the proof by applying Theorems 1.1 and 1.8.

Now we assume that $R \not\subseteq \overline{\mathbb{Q}_p}^\circ$. Set $m := R \cap \mathbb{C}_p^{\circ\circ}$. It is a maximal ideal of R.

Lemma A.3. *Let R be a subring of $\mathbb{C}_p^\circ$ which is finitely generated over $\mathbb{Z}$. Let $m := R \cap \mathbb{C}_p^{\circ\circ}$ be a maximal ideal of R. Then there exists $\sigma \in \mathrm{Gal}(\mathbb{C}_p/\mathbb{Q})$ such that $\sigma(R) \subseteq \overline{\mathbb{Q}_p}^\circ \subseteq \mathbb{C}_p^\circ$ and $\sigma(m) = \overline{\mathbb{Q}_p}^{\circ\circ} \cap R$.*

Now consider $\mathfrak{X}^\sigma := \mathfrak{X}_R \otimes_R^\sigma \mathbb{C}_p^\circ$, $\mathcal{L}^\sigma := \mathcal{L}_R \otimes_R^\sigma \mathbb{C}_p^\circ$, $\tilde{F}^\sigma := \tilde{F} \otimes_R^\sigma \mathbb{C}_p^\circ$, $X^\sigma := X_R \otimes_R^\sigma \mathbb{C}_p$ and $F^\sigma := X_R \otimes_R^\sigma \mathbb{C}_p$. In the tensor product $\bullet \otimes_R^\sigma \mathbb{C}_p$, we use the embedding $\sigma|_R$. We note that if we view $\mathbb{C}_p$ as an abstract field, (X, F) and (X^σ, F^σ) are Galois conjugate. Since the statements of (i) and (ii) are purely algebraic, we only need to show it for (X^σ, F^σ). Observe that the special fiber $\overline{X}^\sigma$ of $\mathfrak{X}^\sigma$ is

$$\overline{X}^\sigma = \mathfrak{X}_R \otimes_R^\sigma (\mathbb{C}_p^\circ/\mathbb{C}_p^{\circ\circ}) = \mathfrak{X}_R \otimes_R (R/m) \otimes_{R/m}^{\bar{\sigma}} (\mathbb{C}_p^\circ/\mathbb{C}_p^{\circ\circ}) = \mathfrak{X}_R \otimes_R^{\bar{\sigma}} (\mathbb{C}_p^\circ/\mathbb{C}_p^{\circ\circ}) \simeq \overline{X}.$$

Moreover the restriction of F^σ on $\overline{X}^\sigma$ is exactly $\overline{F}$ under this identification. So F^σ is some power of Frobenius and F is a lift of the Frobenius with respect to $(\mathfrak{X}^\sigma, \tilde{F}^\sigma, \mathcal{L}^\sigma)$. Since $(\mathfrak{X}^\sigma, \tilde{F}^\sigma, \mathcal{L}^\sigma)$ is defined over $\sigma(R) \subseteq \overline{\mathbb{Q}_p}^\circ$, Theorem A.1 reduces it to the case where $X = \mathbb{P}^N_{\mathbb{C}_p}$ and F is a lift of Frobenius on $\mathbb{P}^N_{\mathbb{C}_p}$. We conclude the proof by applying Theorems 1.1 and 1.8. □

Proof of Lemma A.3. Since $\mathbb{C}_p$ is algebraically closed, any embedding $R \hookrightarrow \mathbb{C}_p$ extends to an automorphism in $\mathrm{Gal}(\mathbb{C}_p/\mathbb{Q})$. We only need to find an embedding $\sigma : R \hookrightarrow \mathbb{C}_p^\circ \subseteq \mathbb{C}_p$ satisfying $\sigma(m) \subseteq \mathbb{C}_p^{\circ\circ}$. Indeed since $\sigma^{-1}(\mathbb{C}_p^{\circ\circ} \cap \sigma(R))$ is a maximal ideal of R which contains m, we have $\sigma(m) = \sigma(R) \cap \mathbb{C}_p^{\circ\circ}$.

Let $t_1, \ldots, t_l \in R$ be a set of generators of R over $\mathbb{Z}$. Let $u_1, \ldots, u_s$ be a set of generators of m. Set $Y := \mathrm{Spec}\, R$ and $Y_{\mathbb{C}_p} := \mathrm{Spec}\, R \otimes_{\mathbb{Z}} \mathbb{C}_p$. We endow $Y_{\mathbb{C}_p}(\mathbb{C}_p)$ with the p-adic topology induced by the topology on $\mathbb{C}_p$. An element $f \in R$ can be viewed as an analytic function on $Y_{\mathbb{C}_p}(\mathbb{C}_p)$.

Denote by $i : R \hookrightarrow \mathbb{C}_p$ the inclusion. It defines a point $o \in Y_{\mathbb{C}_p}(\mathbb{C}_p)$. Set $U := \{x \in Y_{\mathbb{C}_p}(\mathbb{C}_p) : |t_i| \leq 1, i = 1, \ldots, l, \text{ and } |u_i| < 1, i = 1, \ldots, s\}$. Then U is an open neighborhood of o.

For any nonzero element P of R, denote by V_P the subscheme of $Y_{\mathbb{C}_p}$ defined by $\{P = 0\}$. Since the set of nonzero prime ideals is countable, and $Y_{\mathbb{C}_p}(\mathbb{C}_p)$ has a complete metric, $Y_{\mathbb{C}_p}(\mathbb{C}_p) \setminus \left(\bigcup_{R\setminus\{0\}} V_P\right)$ is dense in $Y_{\mathbb{C}_p}(\mathbb{C}_p)$. Then there exists a point $y \in U \setminus \left(\bigcup_{R\setminus\{0\}} V_P\right)$. It defines a morphism $\sigma : R \to \mathbb{C}_p$ by $f \mapsto f(y)$. Because $y \in U$, we have $\sigma(t_i) \in \mathbb{C}^\circ$, $i = 1, \ldots, l$, and $\sigma(u_i) \in \mathbb{C}_p^{\circ\circ}$, $i = 1, \ldots, s$. It follows that $\sigma(R) \subseteq \mathbb{C}_p^\circ$ and $\sigma(m) \subseteq \mathbb{C}_p^{\circ\circ}$. Since $y \notin \left(\bigcup_{R\setminus\{0\}} V_P\right)$, $\sigma : R \to \mathbb{C}_p$ is an embedding. This concludes the proof. □

Proof of Theorem A.1. In this section, we assume that $\mathfrak{X}$, $\tilde{F}$ and $\mathcal{L}$ are defined over $\overline{\mathbb{Q}_p}^\circ \subseteq \mathbb{C}_p^\circ$. Since $\mathfrak{X}$ is finitely presented, there exists a finite extension K of $\mathbb{Q}_p$ such that $\mathfrak{X}$, $\tilde{F}$ and $\mathcal{L}$ are defined over K°. We note that $R := K^\circ$ is a discrete valuation ring. Set $m := K^{\circ\circ}$ the maximal ideal of R and π a generator of m.

There exists a flat and geometrically irreducible projective scheme $\mathfrak{X}_R$ over Spec R an ample line bundle $\mathcal{L}_R$ and an endomorphism $\tilde{F}_R$ such that $\mathfrak{X} = \mathfrak{X}_R \otimes_R \mathbb{C}_p^\circ$ $\mathcal{L} = \mathcal{L}_R \otimes_R \mathbb{C}_p^\circ$ and $\tilde{F} = \tilde{F}_R \otimes_R \mathbb{C}_p^\circ$. We

may assume that $\tilde{F}_R^*\mathcal{L}_R = \mathcal{L}_R^{\otimes q}$. We denote by X_s the special fiber of $\mathfrak{X}_R$ and F_s the restriction of $\tilde{F}_R$ on X_s. Let X_K be the generic fiber of $\mathfrak{X}_R$ and F_K the restriction of $\tilde{F}_R$ on X_K. Write $L_K := \mathcal{L}_R|_{X_K}$ and $L_s := \mathcal{L}_R|_{X_s}$. Since $\mathcal{L}_R$ is ample, after replacing $\mathcal{L}_R$ by a suitable power, we may assume that $\mathcal{L}$ is very ample and the morphisms

$$\Psi_R := H^0(X_R, \mathcal{L}_R)^{\otimes q} \to H^0(X_R, \mathcal{L}_R^{\otimes q}) \quad \text{and} \quad \Psi_s := H^0(X_s, \mathcal{L}_s)^{\otimes q} \to H^0(X_s, \mathcal{L}_R^{\otimes q})$$

are surjective. Moreover, we may assume that

$$H^i(X_R, \mathcal{L}_R) = H^i(X_R, \mathcal{L}_R^{\otimes q}) = 0$$

for all $i \geq 1$. It follows that the natural morphisms

$$r_1 : H^0(X_R, \mathcal{L}_R) \otimes_R R/m \to H^0(X_s, \mathcal{L}_s) \quad \text{and} \quad r_q : H^0(X_R, \mathcal{L}_R^{\otimes q}) \otimes_R R/m \to H^0(X_s, \mathcal{L}_s^{\otimes q})$$

are isomorphisms.

Since $\tilde{F}_R^*\mathcal{L}_R = \mathcal{L}_R^{\otimes q}$, it induces morphisms

$$\tilde{F}_R^* : H^0(X_R, \mathcal{L}_R) \to H^0(X_R, \mathcal{L}_R^{\otimes q}) \quad \text{and} \quad \tilde{F}_s^* : H^0(X_s, \mathcal{L}_s) \to H^0(X_s, \mathcal{L}_s^{\otimes q}).$$

We have $r_q \circ \tilde{F}_R^* = F_s^* \circ r_1$.

Since R is a discrete valuation ring, and $H^0(X_R, \mathcal{L}_R)$ has no torsions, $H^0(X_R, \mathcal{L}_R)$ is a free R-module. Let $s_0, \dots s_N$ a basis of $H^0(X_R, \mathcal{L}_R)$. We note that

$$r_q(\tilde{F}^*(s_i)) = r_1(s_i)^q$$

for $i = 0, \dots, N$. It follows that

$$\tilde{F}^*(s_i) - \Psi_R(s_i^q) \in mH^0(X_R, \mathcal{L}_R^{\otimes q}) = \pi H^0(X_R, \mathcal{L}_R^{\otimes q})$$

for $i = 0, \dots, N$. In other words, there exists $g_i \in H^0(X_R, \mathcal{L}_R^{\otimes q})$ such that

$$\tilde{F}^*(s_i) = s_i^q + \pi g_i, i = 1, \dots, N.$$

Since Ψ_R is surjective, there exists $G_i \in R[x_0, \dots, x_N]$ homogenous of degree q such that $g_i = G_i(s_0, \dots, s_N), i = 1, \dots, N$. It follows that

$$\tilde{F}^*(s_i) = s_i^q + \pi G_i(s_0, \dots, s_N), \quad i = 1, \dots, N.$$

Let $G_K : \mathbb{P}_K^N \to \mathbb{P}_K^N$ be the morphism

$$[x_0, \dots, x_N] \mapsto [x_0^q + \pi G_0(x_0, \dots, x_N) : \cdots : x_N^q + \pi G_N(x_0, \dots, x_N)].$$

Set $G := G_K \otimes_K \mathbb{C}_p : \mathbb{P}_{\mathbb{C}_p}^N \to \mathbb{P}_{\mathbb{C}_p}^N$. It is a lift of Frobenius on $\mathbb{P}_{\mathbb{C}_p}^N$. Let $\tau_K : X \to \mathbb{P}_K^N$ be the morphism

$$x \mapsto [s_0(x) : \cdots : s_N(x)].$$

Since L_K is very ample, τ is an embedding. We may check that $G_K \circ \tau_K = \tau_K \circ F_K$. We conclude the proof by setting $\tau := \tau_K \otimes_K \mathbb{C}_p$. □

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