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# Heights on squares of modular curves

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# Heights on squares of modular curves

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We develop a strategy for bounding from above the height of rational points of modular curves with values in number fields, by functions which are polynomial in the curve's level. Our main technical tools come from effective Arakelov descriptions of modular curves and jacobians. We then fulfill this program in the following particular case:

If $p$ is a not-too-small prime number, let $X_0(p)$ be the classical modular curve of level $p$ over $\mathbb{Q}$. Assume Brumer's conjecture on the dimension of winding quotients of $J_0(p)$. We prove that there is a function $b(p) = O(p^5 \log p)$ (depending only on $p$) such that, for any quadratic number field $K$, the $j$-height of points in $X_0(p)(K)$ which are not lifts of elements of $X_0^+(p)(\mathbb{Q})$ is less or equal to $b(p)$.

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## 1. Introduction

Let $N$ be an integer, $\Gamma_N$ a level-$N$ congruence subgroup of $\mathrm{GL}_2(\mathbb{Z})$, and $X_{\Gamma_N}$ the associated modular curve over some subfield of $\mathbb{Q}(\mu_N)$ which, to simplify the discussion, we assume from now on to be $\mathbb{Q}$. The genus $g_N$ of $X_{\Gamma_N}$ grows roughly as a polynomial function of $N$. So, if $N$ is not too small, $X_{\Gamma_N}$ has only a finite number of rational points with values in any given number field, by Mordell–Faltings. If one is interested in explicitly determining the set of rational points, however, finiteness is of course not sufficient; a much more desirable control would be provided by upper bounds, for some handy height, on those points. Proving such an "effective Mordell" is known to be an extremely hard problem for arbitrary algebraic curves on number fields.

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In the case of modular curves, however, the situation is much better. Indeed, whereas the jacobian of a random algebraic curve should be a somewhat equally random simple abelian variety, it is well-known that the jacobian $J_{\Gamma_N}$ of $X_{\Gamma_N}$ decomposes up to isogeny into a product of quotient abelian varieties defined by Galois orbits of newforms for $\Gamma_N$. Moreover, in many cases, a nontrivial part of those factors happen to have rank zero over $\mathbb{Q}$. Our rustic starting observation is therefore the following: if $J_{\Gamma_N,e}$ is the "winding quotient" of $J_{\Gamma_N}$, that is the largest quotient $J_{\Gamma_N,e}$ with trivial $\mathbb{Q}$-rank, and

$$X_{\Gamma_N} \overset{\iota}{\hookrightarrow} J_{\Gamma_N} \xrightarrow{\pi_e} J_{\Gamma_N,e}$$

is some Albanese map from the curve to its jacobian followed by the projection to $J_{\Gamma_N,e}$, then any rational point on $X_{\Gamma_N}$ has an image which is a torsion point (because rational) on $J_{\Gamma_N,e}$, hence has 0 normalized height. The pull-back of some invertible sheaf defining the (say) theta height on $J_{\Gamma_N,e}$ therefore defines a height on $X_{\Gamma_N}$ which is trivial on rational points. That height in turn necessarily compares to any other natural one, for instance the modular $j$-height. Therefore the $j$-height of any rational point on $X_{\Gamma_N}$ is also zero "up to error terms". Making those error terms explicit would give us the desired upper bound for the height of rational points on $X_{\Gamma_N}$.

That approach can in principle be generalized to degree-$d$ number fields, by considering rational points on symmetric powers $X_{\Gamma_N}^{(d)}$ of $X_{\Gamma_N}$ (at least if $\dim J_{\Gamma_N,e} \geq d$). To be a little bit more precise in the present case of symmetric squares, let us associate to a quadratic point $P$ in $X_0(p)$ the $\mathbb{Q}$-point $Q := (P, {}^{\sigma}P)$ of $X_0(p)^{(2)}$. Its image $\iota(Q)$ via some appropriate Albanese embedding in $J_0(p)$ lies above a torsion point $a$ in $J_e$: assume for simplicity $a = 0$. We therefore know $\iota(Q)$ belongs to the intersection of $\iota(X_0(p)^{(2)})$ with the kernel $\tilde{J}_e^{\perp}$ of the projection

$$\pi_e : J_0(p) \twoheadrightarrow J_e.$$

To improve the situation we can further remark that $\iota(Q)$ actually lies at the intersection of $\iota(X_0(p)^{(2)})$ with the "projection", in some appropriate sense, of the latter surface on $\tilde{J}_e^{\perp}$. Then one can show that this intersection is 0-dimensional (but here we need to assume Brumer's conjecture, see below) so that its theta height is controlled, via some arithmetic Bézout theorem, in terms of the degree and height of the two surfaces we intersect. Using an appropriate version of Mumford's repulsion principle one derives a bound for the height of $\iota(P)$ too (and not only for its sum $\iota(Q)$ with its Galois conjugate). Then one makes the translation again from theta height to $j$-height on $X_0(p)$.

Nontrivial technical work is of course necessary to give sense to the straightforward strategy sketched above. The aim of this article is thus to show the possibility of that approach, by making it work in what we feel to be the simplest nontrivial case: that of quadratic points of the classical modular curve $X_0(p)$ as above (or $X_0(p^2)$, for technical reasons), for $p$ a prime number.[1] In the course of the proof we

[1] Larson and Vaintrob [2014, Corollary 6.5] have proven, under the generalized Riemann hypothesis, the asymptotic triviality of rational points on $X_0(p)$ with values in any given number field which does not contain the Hilbert class field of some quadratic imaginary field. Independently of any conjecture, Momose [1995] had already proven the same result in the case where $K$ is a given quadratic number field. Our method however provides bounds which do not depend on the field, and should generalize to some other congruence subgroups.

are led to assume the already mentioned conjecture of Brumer, which asserts that the winding quotient of $J_0(p) := J_{\Gamma_0(p)}$ has dimension roughly half that of $J_0(p)$. That hypothesis is actually used in only one, technical, but crucial place, where we prove that a morphism between two curves is a generic isomorphism (see last point of Lemma 7.2). Note that a lower bound of $\frac{1}{4}$ (instead of the desired $\frac{1}{2}$) for the asymptotic ratio $\dim J_e / \dim J_0(p)$ has been proven by Iwaniec and Sarnak [2000] and Kowalski, Michel and Vanderkam [Kowalski et al. 2000]. (Actually, $\frac{1}{3}+\varepsilon$ would be sufficient for us; see Lemma 7.2 and the proof of Theorem 7.5 below.) In any case we cannot at the moment get rid of this assumption — note it can in principle be numerically checked in all specific cases. In this setting, our main result is the following (see Theorem 7.5).

**Theorem 1.1.** *For $w_p$ the Fricke involution, set $X_0^+(p) = X_0(p)/w_p$. Assume Brumer's conjecture* (*see Section 2*, (21)).[2] *Then the quadratic points of $X_0(p)$, which are not lifts of elements of $X_0^+(p)(\mathbb{Q})$, have $j$-height bounded from above by $O(p^5 \log p)$.*

*The same holds true for quadratic points of $X_0(p^2)$, without the restriction about $X_0^+(p)$.*

Needless to say, this result cries for both sharpening and generalization. Yet it should be possible to immediately use avatars of Theorem 1.1 to prove that rational points are only cusps and CM points, for some specific modular curves of arithmetic interest. If combined with lower bounds for heights furnished by isogeny theorems as in [Bilu et al. 2013], the above theorem already has consequences on rational points (see Corollary 7.6).

Regarding past works about rational points on modular curves, one can notice that most of them use, at least in parts, some variants of Mazur's method, which can very roughly be divided into two steps: first, map modular curves to winding quotients as described above; then prove some quite delicate properties about completions of that map to $J_e$ (formal immersion criteria). The second step is probably the most difficult to carry over to great generality. Therefore, the method we propose here allows one to use only the first and crucial fact: the mere existence of nontrivial winding quotients. In many cases, the existence of such quotients is known by a deep result of Kolyvagin, Logachev and Kato, à la Birch–Swinnerton-Dyer conjecture, which, again, seems to reflect, from the arithmetic point of view, the special properties of the image locus (in the moduli space of principally polarized abelian varieties) of modular curves, among all algebraic curves, under Torelli's map.

The methods used in this paper are mainly explicit Arakelov techniques for modular curves and abelian varieties. Such techniques and results have been pioneered, as far as we know, by Abbes, Michel and Ullmo at the end of the 1990s (see in particular [Abbes and Ullmo 1995; Michel and Ullmo 1998; Ullmo 2000], whose results we here eagerly use). They have subsequently been revisited and extended in the work developed by Edixhoven and his school, as mainly (but not exhaustively) presented in the orange book [Edixhoven and Couveignes 2011]. That work was motivated by algorithmic Galois-representation issues, but its tools are well suited to our rational points questions, as we wish to show here. We similarly

[2] The weak version of that conjecture we actually need is stated in (22).

hope that the effective Arakelov results about modular curves and jacobians we work out in the present article shall prove useful in other contexts.[3]

The layout of this article is as follows. In Section 2 we start gathering classical instrumental facts on quotients of modular jacobians and regular models of $X_0(p)$ over rings of algebraic integers. In Section 3 we make a precise description of the arithmetic Chow group of $X_0(p)$. Section 4 provides an explicit comparison theorem between $j$-heights and pull-back of normalized theta height on the jacobian. Section 5 computes the degree and Faltings height of the image of symmetric products within modular jacobians. In Section 6 we prove our arithmetic Bézout theorem (in the sense of [Bost et al. 1994]) for cycles in $J_0(p)$, relative to cubist metrics (instead of the more usual Fubini–Study metrics). This seems more natural and has the advantage of being quantitatively more efficient; that constitutes the technical heart of the present paper. Then we apply that arithmetic Bézout to our modular jacobian after technical computations on metric comparisons. Section 7 concludes the computations of the height bounds for quadratic rational points on $X_0(p)$ by making various intersections, projections and manipulations for which to refer to [loc. cit.].

***Convention.*** In order to avoid numerical troubles, we safely assume in all of what follows that primes are by definition strictly larger than 17.

## 2. Curves, jacobians, their quotients and subvarieties

### 2A. *Abelian varieties.*

**2A1.** *Decompositions.* Let $K$ be a field, $J$ an abelian variety of dimension $g$ over $K$ and $\mathcal{L}$ an ample invertible sheaf defining a polarization of $J$. Assume $J$ is $K$-isogenous to a product of two (nonzero) subvarieties, that is, there are abelian subvarieties

$$\iota_A : A \hookrightarrow J, \quad \iota_B : B \hookrightarrow J \tag{1}$$

endowed with polarizations $\iota_A^*(\mathcal{L})$ and $\iota_B^*(\mathcal{L})$, respectively, such that $\iota_A + \iota_B : A \times B \to J$ is an isogeny. (Recall that by convention, all abelian (sub)varieties are assumed to be connected.) Then $\pi_A : J \to A' := J \bmod B$, and similarly $\pi_B : J \to B'$, are called *optimal quotients* of $J$.

To simplify things we also assume from now on that $\mathrm{End}_{\overline{K}}(A, B) = \{0\}$. The product isogeny $\pi := \pi_A \times \pi_B : J \to A' \times B'$ induces isogenies $A \to A'$ and $B \to B'$. We write

$$\Phi : A \times B \to J \to A' \times B'$$

for the obvious composition. Taking for instance dual isogenies of $A \to A'$ and $B \to B'$, we also define an endomorphism

$$\Psi : J \to A' \times B' \to A \times B \to J. \tag{2}$$

[3]For recent investigations related to more general questions of effective bounds of algebraic points on curves, one can check [Checcoli et al. 2016].

When $K = \mathbb{C}$, the above constructions are transparent. There is a $\mathbb{Z}$-lattice $\Lambda$ in $\mathbb{C}^g$, endowed with a symplectic pairing, such that $J(\mathbb{C}) \simeq \mathbb{C}^g/\Lambda$ and one can find a direct sum decomposition $\mathbb{C}^g = \mathbb{C}^{g_A} \oplus \mathbb{C}^{g_B}$ such that if $\Lambda_A = \Lambda \cap \mathbb{C}^{g_A}$ and $\Lambda_B = \Lambda \cap \mathbb{C}^{g_B}$, then

$$A(\mathbb{C}) \simeq \mathbb{C}^{g_A}/\Lambda_A \quad \text{and} \quad B(\mathbb{C}) \simeq \mathbb{C}^{g_B}/\Lambda_B.$$

If $p_A : \mathbb{C}^g \to \mathbb{C}^{g_A}$ and $p_B : \mathbb{C}^g \to \mathbb{C}^{g_B}$ are the $\mathbb{C}$-linear projections relative to that decomposition, the analytic description of $\pi_{A,\mathbb{C}} : J(\mathbb{C}) \to A'(\mathbb{C})$ is then

$$z \bmod \Lambda \mapsto z \bmod (\Lambda + \Lambda_B \otimes \mathbb{R}) = p_A(z) \bmod (p_A(\Lambda)).$$

Summing up, we have lattice inclusions $\Lambda_A \subseteq p_A(\Lambda)$ and $\Lambda_B \subseteq p_B(\Lambda)$, with finite indices, in $\mathbb{C}^g$ such that our isogenies are induced by

$$\Lambda_A \oplus \Lambda_B \subseteq \Lambda \subseteq p_A(\Lambda) \oplus p_B(\Lambda).$$

The isogeny $I'_A : A \to A'$ deduced from the inclusion $\Lambda_A \subseteq p_A(\Lambda)$ has degree $\operatorname{card}(p_A(\Lambda)/\Lambda_A)$. If $N_A$ is a multiple of the exponent of the quotient $p_A(\Lambda)/\Lambda_A$, there is an isogeny $I_{A,N_A} : A' \to A$ such that $I_{A,N_A} \circ I'_A$ and $I'_A \circ I_{A,N_A}$ both are multiplication by $N_A$. The analytic descriptions of the above clearly are:

$$\begin{array}{ccc} A(\mathbb{C}) \simeq \mathbb{C}^{g_A}/\Lambda_A \xrightarrow{I'_A} A'(\mathbb{C}) \simeq \mathbb{C}^{g_A}/p_A(\Lambda) & \text{and} & \mathbb{C}^{g_A}/p_A(\Lambda) \xrightarrow{I_{A,N_A}} \mathbb{C}^{g_A}/\Lambda_A \\ z \longmapsto z & & z \longmapsto N_A z. \end{array} \tag{3}$$

**Remark 2.1.** Instead of considering two immersions as in (1), suppose only $A \hookrightarrow J$ is given, and $K$ is a number field. One might apply [Gaudron and Rémond 2014a, Théorème 1.3] to deduce the existence of an abelian variety $B$ over $K$ such that, with our previous notations, the degree of $A \times B \xrightarrow{+} J$,

$$|A \cap B| = |\Lambda/\Lambda_A \oplus \Lambda_B|,$$

is bounded from above by an explicit function $\kappa(J)$ of the stable Faltings' height $\mathrm{h}_F(J)$,

$$\kappa(J) = ((14g)^{64g^2}[K : \mathbb{Q}] \max(\mathrm{h}_F(J), \log[K : \mathbb{Q}], 1)^2)^{2^{10}g^3},$$

and this does not depend on the choice of the embedding $K \hookrightarrow \mathbb{C}$. Note that when $A$ and $J \bmod A$ are not isogenous (which will be the case for us), then there is actually no choice for that $B \hookrightarrow J$: it has to be the Poincaré complement to $A$. The isogeny $J \to A' \times B'$ given by the two projections has degree $|p_A(\Lambda) \oplus p_B(\Lambda)/\Lambda|$, which also is $|A \cap B| := N$. One can therefore take the $N_A$ appearing in (3) as equal to $N$, and

$$N \leq \kappa(J).$$

Making the same for $B' \to B$, the above morphism $\Psi$ (see (2)) is then simply the multiplication $J \xrightarrow{[N\cdot]} J$ by the integer $N$. Although we will not need numerical estimates for those quantities in what follows, it is straightforward, using [Ullmo 2000], to make them explicit in our setting of modular curves and jacobians.

**2A2.** *Polarizations and heights.* Keeping the above notations and hypothesis, consider in addition now an ample sheaf $\Theta$ on $J$ and let $I_A := I_{A,N} : A' \to A$ (respectively, $I_{B,N}$) be as in (3). We pull-back $\Theta$ along the composed morphism

$$\varphi_A : J \xrightarrow{\pi_A} A' \xrightarrow{I_A} A \xrightarrow{\iota_A} J \tag{4}$$

so that the immersion $\iota_A : A \hookrightarrow J$ defines a polarization $\Theta_A := \iota_A^*(\Theta)$ on $A$, whence a polarization $\Theta_{A'} := {I_A}^*(\Theta_A)$ on $A'$, and finally an invertible sheaf $\Theta_{J,A} := \pi_A^*(\Theta_{A'})$ on $J$. Composing the morphisms

$$J \xrightarrow{\pi_A \times \pi_B} A' \times B' \xrightarrow{I_A \times I_B} A \times B \xrightarrow{\iota_A + \iota_B} J \tag{5}$$

gives the multiplication-by-$N$ map $J \xrightarrow{[\cdot N]} J$. Assuming for simplicity $\Theta$ is symmetric one therefore has

$$[\cdot N]^* \Theta \simeq \Theta^{\cdot \otimes N^2} \simeq \Theta_{J,A} \otimes_{\mathcal{O}_J} \Theta_{J,B}. \tag{6}$$

If $K$ is a number field, the Néron–Tate normalization process associates with $\Theta$ a system of compatible Euclidean norms $\mathrm{h}_\Theta = \|\cdot\|_\Theta^2$ on the finite-dimensional $\mathbb{Q}$-vector spaces $J(F) \otimes_{\mathbb{Z}} \mathbb{Q}$, for $F/K$ running through the number field extensions of $K$, and similarly Euclidean norms

$$\mathrm{h}_{\Theta_A} := \|\cdot\|^2_{\Theta_A^{\cdot\otimes \frac{1}{N^2}}} := \frac{1}{N^2}\|\cdot\|^2_{\Theta_A} \quad \text{and} \quad \mathrm{h}_{\Theta_B} := \frac{1}{N^2}\|\cdot\|^2_{\Theta_B}$$

on $A(F)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $B(F)\otimes_{\mathbb{Z}}\mathbb{Q}$, respectively, such that, under the isomorphisms $J(F)\otimes_{\mathbb{Z}}\mathbb{Q} \simeq (A(F)\otimes_{\mathbb{Z}}\mathbb{Q}) \oplus (B(F)\otimes_{\mathbb{Z}}\mathbb{Q})$, one has

$$\mathrm{h}_\Theta = \mathrm{h}_{\Theta_A} + \mathrm{h}_{\Theta_B}. \tag{7}$$

Recall from (3) the definition of $N_A$, that of the maps $A' \xrightarrow{I_{A,N_A}} A$ and $A \xhookrightarrow{\iota_A} J$. Denote by $[N_A]_A$ the multiplication by $N_A$ restricted to $A$. If $V$ is a closed algebraic subvariety of $J$, define

$$\mathcal{P}_A(V) := (\iota_A [N_A]_A^{-1} I_{A,N_A} \pi_A)(V) \tag{8}$$

as the reduced closed subscheme with relevant support. The map $\mathcal{P}_A$ would simply be the projection of $V$ on $A$ if $J$ were *isomorphic* to the product $A \times B$ of subvarieties and is the best approximation to that projection in our case when $J$ is only isogenous to $A \times B$.

Note that $\mathcal{P}_A(V)$ is a priori highly nonconnected. All its irreducible geometric components are however obtained from each other by translation by an $N_A$-torsion point of $A(\overline{\mathbb{Q}})$. For our later purposes (see the proof of Theorem 7.5), we will have the possibility to replace $\mathcal{P}_A(V)$ by one of its components containing a specific point, say $P_0$: we shall denote that component by $\mathcal{P}_A(V)_{P_0}$ and refer to it as the "pseudoprojection" of $V$ on $A$ containing $P_0$.

Suppose now $J \sim A \times B$ as above is the jacobian of an algebraic curve $X$ on $K$ with positive genus $g$. For $P_0$ a point of $X(K)$ (or more generally a $K$-divisor of degree 1 on $X$) let

$$\iota_{P_0} : X \hookrightarrow J, \quad P \mapsto (P) - (P_0), \tag{9}$$

be the Albanese embedding associated with $P_0$. We define the classical theta divisor $\theta$ on $J$ which is the image of $\iota_{P_0}^{g-1} : X^{g-1} \to J$ and its symmetric version

$$\Theta := (\theta \otimes_{\mathcal{O}_J} [-1]^*\theta)^{\cdot \otimes 1/2} \tag{10}$$

(which is a translate of $\theta$ obtained as $\iota_{\kappa_0}^{g-1}(X^{g-1})$, where $\iota_{\kappa_0} = t_{\kappa_0}^* \iota_{P_0}$ for $t_{\kappa_0}$ the translation by some $\kappa_0$ with $(2g-2)\kappa_0 = \kappa$, the canonical divisor on $X$; of course $\Theta$ does not need to be defined over $K$). Our first aim will be to compare the height functions $\|\iota_{P_0}(\cdot)\|_{\Theta_A{}^{\cdot\otimes 1/N^2}}$ on $X(F)$, when $X$ is a modular curve, with another natural height given by the modular $j$-function.

We will discuss in Section 3 an Arakelov description of Néron–Tate height. We conclude this paragraph by a few remarks as a preparation. Let $B_2 := \{\omega_1, \ldots, \omega_g\}$ be a basis of $H^0(X(\mathbb{C}), \Omega^1_{X/\mathbb{C}}) \simeq H^0(J(\mathbb{C}), \Omega^1_{J/\mathbb{C}})$, which is orthogonal with respect to the norm

$$\|\omega\|^2 = \frac{i}{2} \int_{X(\mathbb{C})} \omega \wedge \bar{\omega}.$$

The transcendent writing-up of the Abel–Jacobi map $\iota_{P_0} : P \mapsto \big(\int_{P_0}^P \omega_i\big)_{1 \le i \le g}$ shows that the pull-back to $X(\mathbb{C})$ of the translation-invariant measure on $J(\mathbb{C})$, normalized to have total mass 1, is

$$\mu_0 = \frac{i}{2g} \sum_{B_2} \frac{\omega \wedge \bar{\omega}}{\|\omega\|^2}. \tag{11}$$

More generally, $\pi_A \circ \iota_{P_0}$ is, over $\mathbb{C}$, the map $P \mapsto \big(\int_{P_0}^P \omega\big)_{\omega \in B_2^A}$, where $B_2^A$ is some orthogonal basis of $H^0(A'(\mathbb{C}), \Omega^1_{A'/\mathbb{C}}) \simeq H^0(J(\mathbb{C}), \pi_A^*(\Omega^1_{A'/\mathbb{C}})) \subseteq H^0(J(\mathbb{C}), \Omega^1_{J/\mathbb{C}})$. Therefore, writing $g_A := \dim(A') = \dim(A)$ (we assume $A \neq 0$), the pull-back to $X(\mathbb{C})$ of the translation-invariant measure on $A'(\mathbb{C})$ (normalized so to have total mass 1 on the curve again) is

$$\mu_A = \frac{i}{2g_A} \sum_{B_2^A} \frac{\omega \wedge \bar{\omega}}{\|\omega\|^2}. \tag{12}$$

**2B.** ***Modular curves.*** Here we recall a few classical facts on the minimal regular model of the modular curve $X_0(p)$, for $p$ a prime number, over a ring of algebraic integers. The first general reference on this topic is [Deligne and Rapoport 1973]; see also [Edixhoven and Couveignes 2011; Menares 2008; 2011].

**2B1.** *The $j$-height.* The quotient of the completed Poincaré upper half-plane $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ by the classical congruence subgroup $\Gamma_0(p)$ defines a Riemann surface $X_0(p)(\mathbb{C})$ which is known to have a geometrically connected smooth and proper model over $\mathbb{Q}$. All through this paper, we denote its genus by $g$.

The first technical theme of this article is the explicit comparison of various heights on $X_0(p)(\overline{\mathbb{Q}})$. When $V$ is an algebraic variety over a number field $K$, any finite $K$-map $\varphi : V \to \mathbb{P}^N_K$ to some projective space defines a naive Weil height on $V(\overline{K})$. This applies in particular when $V$ is a curve and $\varphi$ is the finite morphism defined by an element of the function field of $V$, and in the case of a modular curve $X_\Gamma$ associated with some congruence subgroup $\Gamma$, say, a natural height to choose on $X_\Gamma(\overline{\mathbb{Q}})$ is precisely Weil's height $\mathrm{h}(P) = \mathrm{h}(j(P))$ relative to the classical $j$-function. The degree of the associated map $X_\Gamma \to X(1) \simeq \mathbb{P}^1$

is $[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$, so that number is the class of our Weil height in the Néron–Severi group $\mathrm{NS}(X_\Gamma)$ identified with $\mathbb{Z}$. More explicitly if $X = X_\Gamma$ is defined over the number field $K$, say, the $j$-morphism is

$$X \xrightarrow{\jmath} \mathbb{P}^1_K = \mathrm{Proj}(K[X_0, X_1]) \hookleftarrow \mathbb{A}^1_K = \mathrm{Spec}(K[X_1/X_0])$$
$$P \longmapsto (1, j(P)) = (1/j(P), 1) \leftarrow j(P) = \frac{X_1}{X_0}(P),$$

and the Weil height of a point $P \in X(K)$ is therefore the naive height of its $j$-invariant as an algebraic number

$$\mathrm{h}(P) = \mathrm{h}(j(P)) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log(\max(1, |j(P)|_v))$$

which is also Weil's projective height $\mathrm{h}(\jmath(P))$ with respect to the above basis $(X_0, X_1 = X_0 j)$ of global sections of $\mathcal{O}_{\mathbb{P}^1_K}(1)$. Our Weil height on $X$ is associated with the linear equivalence classes of divisors $D$ corresponding to $\jmath^*(\mathcal{O}_{\mathbb{P}^1_K}(1))$, so that

$$D \sim (\text{poles of } j \text{ on } X)(\sim (\text{zeroes of } j)) \sim \sum_{c \in \{\text{cusps of } X\}} e_c.c$$

where each $e_c$ is the ramification index of $c$ via $\jmath$.

Those considerations lead to explicit comparisons with other heights. Indeed, a more intrinsic way to define heights on algebraic varieties is provided by Arakelov theory. Defining this properly in the case of our modular curves demands a precise description of regular models for them, which we now recall.

**2B2.** *Regular models.* The normalization of the $j$-map $X_0(p) \to X(1)_{/\mathbb{Z}} \simeq \mathbb{P}^1_{/\mathbb{Z}}$ over $\mathbb{Z}$ defines a model for $X_0(p)$ that we call the modular model, it is smooth over $\mathbb{Z}[1/p]$.

We fix a number field $K$, write $\mathcal{O}_K$ for its ring of integers, and deduce by base change a model for $X_0(p)$ over $\mathcal{O}_K$. We know its only singularities are normal crossing, so after a few blow-ups, if necessary, we obtain a regular model of $X_0(p)$ over $\mathcal{O}_K$; see Theorem 1.1.d of the Appendix of [Mazur 1977]. We denote it from now on by $\mathcal{X}_0(p)_{/\mathcal{O}_K}$, or simply $\mathcal{X}_0(p)$ if the context prevents confusion. We stress here that for $F/K$ a field extension, $\mathcal{X}_0(p)_{/\mathcal{O}_F}$ is *not* the base change to $\mathcal{O}_F$ of $\mathcal{X}_0(p)_{/\mathcal{O}_K}$ if $F/K$ ramifies above $p$. Let $v$ be a place of $\mathcal{O}_K$ above $p$, with residue field $k(v)$. The dual graph of $\mathcal{X}_0(p)$ at $v$ is made of two extremal vertices, which we label $C_0$ and $C_\infty$, containing the cusps 0 and $\infty$ respectively (see Figure 1). Those two vertices, which correspond to irreducible components of genus 0, are linked by

$$s := g + 1$$

branches. Each branch corresponds to a singular point $S$ in $\mathcal{X}_0(p)(\mathbb{F}_{p^2})$, which in turn parametrizes an isomorphism class of supersingular elliptic curve $E_S$ in characteristic $p$.

The Fricke involution $w_p$ acts on the dual graph as the continuous isomorphism which exchanges $C_0$ and $C_\infty$ and acts on the branches as a generator of $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$.

We list the supersingular points as $S(1), \ldots, S(s)$ and for each one define

$$w_n := \#\,\mathrm{Aut}(S(n))/\langle \pm 1 \rangle := \#\,\mathrm{Aut}_{\mathbb{F}_{p^2}}(E_{S(n)})/\langle \pm 1 \rangle \qquad (13)$$

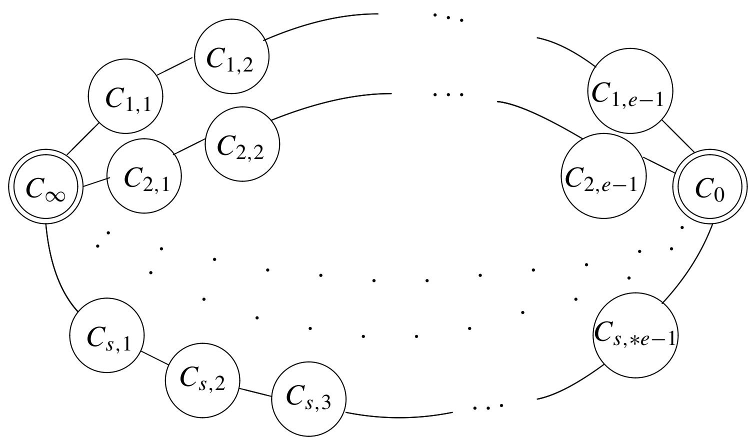


**Figure 1.** Dual graph of $\mathcal{X}_0(p)_{/\mathcal{O}_K}$ at $v$.

which is equal to 1 except in the (at most two) cases when the underlying supersingular elliptic curve has $j$-invariant 1728 or 0, where it is equal to 2 or 3 respectively. Now each path, or branch, on our dual graph at $v$ passes through $(w_n e - 1)$ vertices (for $e$ the ramification index of $K$ at $v$), that is, again, equal to $e-1$ except for at most two branches: one of length $2e-1$ (obtained by blowing-up the supersingular point of moduli $j \equiv 1728 \bmod v$, if it exists) and a path of length $3e-1$ (obtained by blowing-up, if needed, at the supersingular point of moduli $j \equiv 0 \bmod v$). We enumerate the vertices $\{C_{n,m}\}_{1\le m\le w_n e-1}$ in the $n$-th path. We also denote by $w(\mathrm{Eis})$ the familiar quantity $\sum 1/w_n$, the sum being taken over the set of all supersingular points of $\mathcal{X}_0(p)_{/\mathcal{O}_{K,v}}$. The well-known Eichler mass formula says that

$$w(\mathrm{Eis}) = \sum_{1\le n\le s} \frac{1}{w_n} = \frac{p-1}{12} \tag{14}$$

(see for instance [Gross 1987b, p. 117]). Recall that this implies the genus $g$ of $X_0(p)$ is asymptotically equivalent to $p/12$ (the exact formula depending on the residue class of $p$ mod 12) and in any case

$$\frac{p-13}{12} \le g \le \frac{p+1}{12} \tag{15}$$

(see for instance [Gross 1987b, p. 117], again).

Abusing notation a bit, $C_\infty$ will sometimes also be denoted as $C_{n,0}$ and similarly $C_0$ might be written as $C_{n,w_n e}$. We choose as a basis for $\oplus_C \mathbb{Z}\cdot C$ the ordered set

$$\mathcal{B} = (C_\infty, (C_{1,1}, C_{1,2}, \ldots, C_{1,e-1}), (C_{2,1}, \ldots, C_{2,e-1}), \ldots, (C_{s,1}, \ldots, C_{s,w_s e-1}), C_0) \tag{16}$$

(that is, we enumerate the vertices by running through each branch successively, and put the possible branches of length twice or thrice the generic length at the end). At bad places $v$ the intersection matrix restricted to each submodule $\oplus_{m=1}^{w_n e-1} \mathbb{Z}\cdot C_{n,m}$ (for some fixed branch of index $n$) is then $(\log(\#k(v))\cdot \mathcal{M}_0,$

where

$$\mathcal{M}_0 = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \tag{17}$$

whose only dependence on $n$ is that its type is $(w_n e - 1) \times (w_n e - 1)$. That matrix has determinant $(-1)^{w_n e - 1} w_n e$. Define the row vectors

$$L := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad L' := \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(with length implicitly defined by the next lines) and the transpose column vectors

$$V := L^t, \quad V' := L'^t.$$

The intersection matrix on the whole space $\mathbb{Z}^{\mathcal{B}}$ is finally $(\log(\#k(v)) \cdot \mathcal{M})$ for

$$\mathcal{M} = \begin{pmatrix} -s & L & L & \cdots & L & 0 \\ V & \mathcal{M}_0 & 0 & \cdots & 0 & V' \\ V & 0 & \mathcal{M}_0 & \cdots & 0 & V' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V & 0 & 0 & \cdots & \mathcal{M}_0 & V' \\ 0 & L' & L' & \cdots & L' & -s \end{pmatrix}. \tag{18}$$

(This has to be modified in the obvious way when $e_v = 1$.)

**2B3.** *Winding quotients, their dimension.* We denote as usual the jacobian of $X_0(p)_{\mathbb{Q}}$ by $J_0(p)$. As follows from Section 2B2, $\mathcal{X}_0(p)$ is semistable over $\mathbb{Z}$ and the neutral component of the Néron model $\mathcal{J}_0(p)$ of $J_0(p)$ is a semiabelian scheme over $\mathbb{Z}$ (and an abelian scheme over $\mathbb{Z}[1/p]$). Its neutral component represents the neutral component $\mathrm{Pic}^0_{\mathbb{Z}}(\mathcal{X}_0(p))$ of the relative Picard functor of $\mathcal{X}_0(p)$ over $\mathbb{Z}$.

We know from Shimura's theory that the natural decomposition of cotangent spaces into Hecke eigenspaces induces a corresponding decomposition over $\mathbb{Q}$ of abelian varieties up to isogenies:

$$J_0(p) \sim \prod_{f \in B_2/\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} J_f \tag{19}$$

indexed by Galois orbits in some set $B_2$ of newforms. A first useful sorting of this decomposition comes from the sign of the functional equations for the $L$-functions of eigenforms $f$, that is, whether $w_p(f)$ equals $f$ or $-f$. One accordingly writes $J_0(p)^-$ for the optimal quotient abelian variety associated with $\prod_{f, w_p(f)=-f} J_f$ in (19), and similarly $J_0(p)^+$, so that $J_0(p)^- = J_0(p)/(1+w_p)J_0(p)$ and $J_0(p)^+ = J_0(p)/(1-w_p)J_0(p)$. One knows that

$$\dim J_0(p)^- = \left(\tfrac{1}{2} + o(1)\right) \dim J_0(p)$$

(see, e.g., [Royer 2001, Lemme 3.2]).

A more subtle object is the *winding quotient* $J_e$, defined as the optimal quotient of $J_0(p)$ corresponding to $\prod_{f,L(f,1)\neq 0} J_f$ in decomposition (19). One can write

$$J_e = J_0(p)/I_e J_0(p) \tag{20}$$

for some ideal $I_e$ of the Hecke algebra $\mathbb{T}_{\Gamma_0(p)}$. Similarly, $J_e^\perp = J_0(p)/I_e^\perp J_0(p)$ will denote the optimal quotient corresponding to $\prod_{f,L(f,1)=0} J_f$. For obvious reasons regarding signs of functional equations, $J_e$ is contained in $J_0(p)^-$. But more is expected: in line with the principle that "the vanishing order of a (modular) $L$ functions at the critical point should generically be as small as allowed by parity", Brumer [1995] conjectured that, as $p$ tends to infinity,

$$(?) \qquad \dim J_e = (1 - o(1)) \dim J_0(p)^-. \qquad \text{(Brumer)} \tag{21}$$

Equivalently, it is conjectured that $\dim J_e = (\frac{1}{2} + o(1)) \dim J_0(p)$, or that the dimensions of $J_e$ and $J_e^\perp$ should be, asymptotically in $p$, of equal size. Note that (21) above is also implied by the "density conjecture" of [Iwaniec et al. 2000], p. 56 et seq., see also Remark F on p. 65.[4] Actually, what we eventually need in this article (see Section 7) is a weaker form of (21), which is

$$(?) \qquad \dim J_e > \frac{\dim J_0(p)}{3} + \frac{2}{3} \tag{22}$$

for large enough $p$. An important theorem of Iwaniec and Sarnak [2000, Corollary 13] and Kowalski, Michel and Vanderkam [Kowalski et al. 2000] asserts something nearly as good, namely

$$\big(\tfrac{1}{4} - o(1)\big) \dim J_0(p) \le \dim J_e \big(\le \big(\tfrac{1}{2} + o(1)\big) \dim J_0(p)\big) \tag{23}$$

as $p$ goes to infinity $\big($so that $\big(\frac{1}{2} - o(1)\big) \dim J_0(p) \le \dim J_e^\perp \le \big(\frac{3}{4} + o(1)\big) \dim J_0(p)\big)$. Breaking that $\frac{1}{4}$ is known to be closely linked to the Landau–Siegel zero problem. Assuming the generalized Riemann hypothesis for $L$-functions of modular forms, Iwaniec, Luo and Sarnak [2000, Corollary 1.6, (1.54)] prove one can improve $\frac{1}{4}$ to $\frac{9}{32}$. That seems to be all for the moment.

The central object of this paper will eventually be the maps

$$X_0(p)^{(d)} \to J_0(p) \to J_e$$

from symmetric products of $X_0(p)$ (mainly the curve itself and its square) to the winding quotient.

## 3. Arithmetic Chow group of modular curves

We now give a description of the Arakelov geometry of $X_0(p)$, relying on the work of many people; that topic has been pioneered by Abbes and Ullmo [1995], Michel and Ullmo [1998] and Ullmo [2000] and notably developed by Edixhoven and Couveignes [2011] and their coauthors. We shall also use the work

[4]Quoting Olga Balkanova (private communication), *"Theorem 1.1 in [Iwaniec et al. 2000] is proved for the test function $\phi$, whose Fourier transform is supported on the interval* $[-2, 2]$*. The density conjecture claims that the same results are true without restriction on Fourier transform of $\phi$; see formula 1.9 of [loc. cit.]."*

of Bruin [2014], Jorgenson and Kramer [2006] and Menares [2008; 2011] among others. We refer to those articles for general facts on Arakelov theory (see [Chinburg 1986; Edixhoven and de Jong 2011c]).

Let $\mathcal{X}$ be any regular and proper arithmetic surface over the integer ring $\mathcal{O}_K$ of a number field $K$. Fixing in general smooth hermitian metrics $\mu$ on the base changes of $\mathcal{X}$ to $\mathbb{C}$, it follows from the basics of Arakelov theory that for any horizontal divisor $D$ on $\mathcal{X}$ over $\mathcal{O}_K$ there are Green functions $g_{\mu,D}$ on each Archimedean completion $\mathcal{X}(\mathbb{C})$ satisfying the differential equation

$$\Delta g_{\mu,D} = -\delta_D + \deg(D)\mu$$

for $\Delta = 1/(i\pi)\partial\overline{\partial}$ the Laplace operator and $\delta_D$ the Dirac distribution relative to $D_{\mathbb{C}}$ on $\mathcal{X}(\mathbb{C})$. The function $g_{\mu,D}$ is integrable on the compact Riemann surface $\mathcal{X}(\mathbb{C})$ endowed with its measure $\mu$, and uniquely determined up to an additive constant which is often fixed by imposing the normalizing condition that

$$\int_{\mathcal{X}(\mathbb{C})} g_{\mu,D}\mu = 0. \tag{24}$$

When the horizontal divisor $D$ is a section $P_0$ in $\mathcal{X}(\mathcal{O}_K)$, one will sometimes also use the notation $g_\mu(P_0, z)$ for $g_{\mu,P_0}(z)$. The Green functions relative to fixed smooth (1, 1)-forms $\mu$ allows one to define an Arakelov intersection product relative to the $\mu$, which will be denoted by $[\cdot,\cdot]_\mu$ or $[\cdot,\cdot]$ if there is no ambiguity about the implicit form. In particular the index will often be dropped for divisors intersections of which one at least is vertical, where the choice of $\mu$ does not intervene.

We shall denote by $\mu_0$ the canonical Arakelov (1, 1)-form on the Riemann surface $\mathcal{X}(\mathbb{C})$ (assumed to have positive genus), inducing the "flat metric". It corresponds to the pullback, by any Albanese morphism $\mathcal{X}(\mathbb{C}) \to \mathrm{Jac}(\mathcal{X}_K)(\mathbb{C})$, of the "cubist" metric in the sense of Moret-Bailly [1985a] (more about this shortly) on the jacobian $\mathrm{Jac}(\mathcal{X}_K)$, associated with the Néron–Tate normalized height $\mathrm{h}_\Theta$.

We now specialize to the case of $\mathcal{X}_0(p)$ as in Section 2B. If $f$ is a modular form of weight 2 for $\Gamma_0(p)$, let $\|f\|^2$ be its Petersson norm. Because newforms are orthogonal in prime level we have, as in (11),

$$\mu_0 := \frac{i}{2\dim(J_0(p))} \sum_{f\in B_2} \frac{f\frac{dq}{q} \wedge \overline{f\frac{dq}{q}}}{\|f\|^2}. \tag{25}$$

We shall also need to consider Néron–Tate heights $\mathrm{h}_A$ for subabelian varieties $A \hookrightarrow J_0(p)$ as in Section 2A2 (recall $A \neq 0$). The associated (1, 1)-form $\mu_A$ is given by (12). More specifically, we focus on $\mathrm{h}_{\Theta_e}$ on $J_e$ (as in (7) and around, for $A' = J_e$) which induces a height $\mathrm{h}_{\Theta_e} \circ \iota_{e,P_0}$ on $X_0(p)$ via the map $\iota_{e,P_0} : X_0(p) \hookrightarrow J \twoheadrightarrow J_e$. The curvature form of the hermitian sheaf on $X_0(p)$ defining the Arakelov height associated with $\mathrm{h}_{\Theta_e} \circ \iota_{e,P_0}$ is

$$\mu_e := \frac{i}{2\dim(J_e)} \sum_{f\in B_2[I_e]} \frac{f\frac{dq}{q} \wedge \overline{f\frac{dq}{q}}}{\|f\|^2}, \tag{26}$$

where $B_2[I_e]$ stands for the set of newforms killed by the ideal $I_e$ defining $J_e$ as in (20).

**Remark 3.1.** Notice that both $\mu_0$ and $\mu_e$, or any $\mu_A$ above, are invariant by pull-back $w_p^*$ by the Fricke involution. In particular the Arakelov intersection products $[\cdot,\cdot]_{\mu_0}$ and $[\cdot,\cdot]_{\mu_e}$, relative to $\mu_0$ and $\mu_e$ respectively, are $w_p$-invariant. The latter was clear already from the fact that, more generally, $w_p$ is an orthogonal symmetry on $J_0(p)$ endowed with its quadratic form $\mathrm{h}_\Theta$, which respects the orthogonal decomposition $\prod_f J_f$ of (19).

One can now specialize the Hodge index theorem to our modular setting (see [Menares 2011, Theorem 4.16; 2008, Theorem 3.26] or more generally [Moret-Bailly 1985a, p. 85 et seq.]).

**Theorem 3.2.** *Let $K$ be a number field, $\mu$ be a smooth nonzero* (1, 1)*-form on $X_0(p)(\mathbb{C})$ as given in* (12), *and $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu}$ be the arithmetic Chow group with real coefficients up to numerical equivalence of $\mathcal{X}_0(p)$ over $\mathcal{O}_K$, relative to $\mu$. Denote by $\infty$ the horizontal divisor defined by the $\infty$-cusp on $\mathcal{X}_0(p)$ over $\mathbb{Z}$ (which is the Zariski closure of the $\mathbb{Q}$-point $\infty$ in $X_0(p)(\mathbb{Q})$), compactified with the normalizing condition* (24). *Write $\mathbb{R}\cdot X_\infty$ for the line of divisors with real coefficients supported on some fixed full vertical fiber $X_\infty$. Define, for all $v\in\mathrm{Spec}(\mathcal{O}_K)$ above $p$, the $\mathbb{R}$-vector space*

$$G_v := \bigoplus_{C\neq C_\infty} \mathbb{R}\cdot C$$

*where the sum runs through all the irreducible components of $\mathcal{X}_0(p)\times_{\mathcal{O}_K} k(v)$ except $C_\infty$ (the one containing $\infty(k(v))$). Identify finally $J_0(p)(K)/$* torsion *with the subgroup of divisor classes $D_0$ which are compactified under the normalizing condition $g_{D_0}(\infty)=0$ (which is therefore* different *from* (24)). *One has a decomposition*:

$$\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu} = (\mathbb{R}\cdot\infty\oplus\mathbb{R}\cdot X_\infty)\oplus^{\perp}_{v\mid p} G_v \oplus^{\perp} (J_0(p)(K)\otimes\mathbb{R}) \tag{27}$$

*where the "$\oplus^\perp$" means that the direct factors are mutually orthogonal with respect to the Arakelov intersection product. Moreover, the restriction of the self-intersection product to $J_0(p)(K)\otimes\mathbb{R}$ coincides with twice the opposite of the Néron–Tate pairing.*

*Proof.* The proof can be immediately adapted from that of [Menares 2011, Theorem 4.16] for $L^1_2$-admissible measures (a setting allowing to define convenient actions of the Hecke algebra on the Chow group). For further computational use we recall how one decomposes divisors in practice. Take $D$ in $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu}$, with degree $d$ on the generic fiber. There is a vertical divisor $\Phi_D$, with support in fibers above places of bad reduction (that is, of characteristic $p$), such that $(D-d\infty-\Phi_D)$ has a real multiple which belongs to the neutral component $\mathrm{Pic}^0(\mathcal{J}_0(p))_{/\mathcal{O}_K}$. That $\Phi_D$ is well-defined up to multiple of full vertical fibers, so we can assume $\Phi_D$ belongs to $\oplus^\perp G_p$ (and is then unambiguously defined). One associates to $(D-d\infty-\Phi_D)\in\mathbb{R}\cdot\mathcal{J}_0^0(p)(\mathcal{O}_K)$ an element $\delta$ in $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu}$ by imposing a compactification such that $[\infty,\delta]_\mu=0$. The general Hodge index theorem (see for instance [Moret-Bailly 1985a]) then finally asserts that $(D-d\infty-\Phi_D-\delta)$ can be written as an element in $\mathbb{R}\cdot X_\infty$. □

In order to later on interpret the Néron–Tate height (associated with some given (symmetric) invertible sheaf) as an Arakelov height in a suitable sense (see [Abbes 1997] paragraph 3, or [Moret-Bailly 1985b]), we will need to compute explicitly, given $P \in X_0(p)(K)$, the vertical divisor $\Phi_P = \oplus_{v \mid p} \Phi_{P,v}$ such that

$$[C, P - \infty - \Phi_P] = 0 \tag{28}$$

for any irreducible component of any fiber of $\mathcal{X}_0(p) \to \mathrm{Spec}(\mathcal{O}_K)$, as in the proof of Theorem 3.2.

**Lemma 3.3.** *Consider a bad fiber $\mathcal{X}_0(p)_{k(v)}$, with $e_v$ the absolute ramification index of $v$, and write $\#k(v) = p^{f_v}$. Let $P \in X_0(p)(K)$ and let $C_{P,v}$ be the irreducible component of $\mathcal{X}_0(p)_{k(v)}$ which contains $P(k(v))$. As $\mathcal{X}_0(p)$ is assumed to be regular, the section $P$ hits each fiber on its smooth locus, so that the component $P$ belongs to is unambiguously defined in each bad fiber. Write*

$$\Phi_{P,v} = \sum_{n,m} a_{n,m} [C_{n,m}]$$

*with notations as in* (16)*. Recall that, by our convention, $a_{C_\infty} = a_{*,0} = 0$.*

(a) *If $C_{P,v} = C_0$ then for all $n$ and $m$,*

$$a_{n,m} = \frac{-12}{(p-1) \cdot w_n} \cdot m.$$

(*Recall* (*see* (13)) *that* $w_n := \#\,\mathrm{Aut}(S(n))/\langle \pm 1 \rangle \in \{1, 2, 3\}$, *with $S(n)$ the supersingular point corresponding to the branch $\{C_{n,\cdot}\}$.*)

*For further use we henceforth write $\Phi_{C_0}$ for the above vector $\Phi_{P,v} \in \mathbb{Z}^{\mathcal{B}}$.*

(b) *If $C_{P,v} = C_{n_0,m_0} \neq C_0, C_\infty$ then*:

- *For $n = n_0$ and $m \in \{0, m_0\}$, one has*

$$a_{n,m} = \left( \frac{m_0}{w_{n_0} e_v} \left( 1 - \frac{12}{(p-1) w_{n_0}} \right) - 1 \right) \cdot m.$$

- *For $n = n_0$ and $m \in \{m_0, w_{n_0} e_v\}$, one has*

$$a_{n,m} = \left( \frac{m_0}{w_{n_0} e_v} \left( 1 - \frac{12}{(p-1) w_{n_0}} \right) \right) \cdot m - m_0.$$

- *For $n \neq n_0$ and all $m \in \{0, w_n e_v\}$, one has*

$$a_{n,m} = \frac{-12 m_0}{(p-1) w_{n_0} e_v} \cdot \frac{m}{w_n}.$$

(c) *Of course if $C_{P,v} = C_\infty$ then $\Phi_{P,v} = 0$.*

**Remark 3.4.** We have distinguished different cases above because the proof naturally leads to doing so, and it will be of interest below to have the simpler case (a) explicitly displayed. Note however that all outputs are actually covered by the formulae of case (b). Notice also that, in case (a), all coefficients of $\Phi_{P,v}$ satisfy

$$0 \geq a_{n,m} \geq a_0 := a_{C_0} = a_{n, w_n m} = \frac{-12 e_v}{(p-1)}.$$

As for case (b), all coefficients of $\Phi_{P,v}$ satisfy

$$0 \geq a_{n,m} \geq a_{n_0,m_0} = \left(\frac{m_0}{w_{n_0}e_v}\left(1 - \frac{12}{(p-1)w_{n_0}}\right) - 1\right) \cdot m_0$$

(remember $0 \leq m \leq w_n e_v$ for all $m$). Computing the minimum of the above right-hand as a polynomial in $m_0$ gives

$$0 \geq a_{n,m} \geq \frac{-e_v w_{n_0}}{4\left(1 - \frac{12}{(p-1)w_{n_0}}\right)} \geq \frac{-e_v w_{n_0}}{4 - \frac{3}{w_{n_0}}} \geq -3e_v \tag{29}$$

(recalling we always assume $p \geq 17$).

*Proof.* Given the intersection matrix (18) and condition (28), $[C, P - \infty - \Phi_{P,v}] = 0$ for all $C$ in the fiber at $v$, gives the matrix equation

$$\log(\#k(v))\mathcal{M} \cdot \Phi_{P,v} = \log(\#k(v))(-1, 0, \ldots, 1, 0, \ldots, 0)^t \tag{30}$$

where the coefficient 1 (respectively $-1$) in the right-hand column vector is at the place corresponding to $C_{P,v} = C_{n,m}$ (respectively to $C_\infty = C_{n,0}$) in the ordering of our component basis (16). That is however more easily solved by running through the dual graph of $\mathcal{X}_0(p)_{k(v)}$ "branch by branch" as follows. Suppose first that $C_{P,v} = C_0$, and recall $a_{C_\infty} = 0$ by convention. Equation (28) translates into:

- $-1 - \sum_{n=1}^{s} a_{n,1} = 0$, for $C = C_\infty$.
- $1 + sa_0 - \sum_{n=1}^{s} a_{n,w_n e_v - 1} = 0$, for $C = C_0$.
- $a_{n,m-1} - 2a_{n,m} + a_{n,m+1} = 0$, for all others $C = C_{n,m}$.

The equations of the third line in turn define, for each branch (that is, for fixed $n$), a sequence defined by linear double induction with solution $a_{n,m} = m \cdot \alpha_n$ for some $\alpha_n$ which is easily computed to be $-1/(w(\mathrm{Eis}) \cdot w_n) = -12/((p-1)w_n)$ (see (14)). (Note this is true even for $e_v = 1$.)

For case (b), the intersection equations become:

- $-1 - \sum_{n=1}^{s} a_{n,1} = 0$, for $C = C_\infty$.
- $sa_0 - \sum_{n=1}^{s} a_{n,w_n e_v - 1} = 0$, for $C = C_0$.
- $1 - a_{n_0,m_0-1} + 2a_{n_0,m_0} - a_{n_0,m_0+1} = 0$, for $C = C_{P,v} = C_{n_0,m_0}$.
- $a_{n,m-1} - 2a_{n,m} + a_{n,m+1} = 0$, for all others $C = C_{n,m}$.

As above, solving these equations in all branches not containing $C_{P,v}$ gives $a_{n,m} = m\beta_n$ and the same is true in the branch containing $C_{P,v}$ for $m \in \{0, \ldots, m_0\}$. We also see that $a_{n_0,m_0+1} = (m_0+1)\beta_{n_0} + 1$, and then $a_{n_0,m} = m(\beta_{n_0} + 1) - m_0$ for $m \in \{m_0 + 1, w_n e_v\}$. We have $a_0 = w_n e_v \beta_n$ for all $n \neq n_0$, so let $\beta$ be the common value of the $\beta_n$ for $n \neq n_0$ with $w_n = 1$. (There is always such an $n$ as we assumed $p > 13$. Note also those computations still cover the case $e_v = 1$.) From $\beta = a_0/e_v$ and $a_0 = w_{n_0}e_v(\beta_{n_0} + 1) - m_0$ we derive

$$\beta_{n_0} = \frac{a_0 + m_0 - w_{n_0}e_v}{w_{n_0}e_v} = \frac{\beta}{w_{n_0}} + \frac{m_0}{w_{n_0}e_v} - 1.$$

Hence, because of the first equation $\big(-1-\sum_{n=1}^{s} a_{n,1}=0\big)$,

$$0=-1-\beta_{n_0}-\sum_{1\le n\le s, n\neq n_0}\frac{\beta}{w_n}=-\beta w(\mathrm{Eis})-\frac{m_0}{w_{n_0}e_v}$$

so that

$$\beta=\frac{-m_0}{w(\mathrm{Eis})w_{n_0}e_v}=\frac{-12\,m_0}{(p-1)w_{n_0}e_v}.$$ □

**Lemma 3.5.** *Let $\mu$ be some $(1,1)$-form on $X_0(p)(\mathbb{C})$ as in Theorem 3.2.*

(a) *The class in $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu}$ of the cuspidal divisor $(0)-(\infty)$ satisfies*

$$(0)-(\infty)\equiv\Phi^0_{C_0}:=\Phi_{C_0}+\sum_{v|p}\frac{6e_v}{p-1}\Big(\sum_C[C]\Big)=\sum_{v|p}\sum_{n,m}\frac{6}{(p-1)}\Big(e_v-\frac{2m}{w_n}\Big)[C_{n,m}] \tag{31}$$

*with notations as in Lemma 3.3* (a). *This is an eigenvector of the Fricke $\mathbb{Z}$-automorphism $w_p$ with eigenvalue $-1$.*

(b) *One has $[\infty,\infty]_\mu=[0,0]_\mu=[0,\infty]_\mu-6\log p/(p-1)$. If $\mu$ is the Green–Arakelov measure $\mu_0$ then $0\ge[\infty,\infty]_{\mu_0}=O(\log p/p)$ and similarly $[0,\infty]_{\mu_0}=O(\log p/p)$ with $[0,\infty]_{\mu_0}$ nonpositive too, at least for large enough $p$. If $\mu=\mu_e$ (see* (26)*) — or more generally any submeasure of $\mu_0$ — then $[0,\infty]_{\mu_e}=O(p\log p)$.*

*Proof.* By the Manin–Drinfeld theorem, $(0)-(\infty)$ is torsion as a divisor in the generic fiber $\mathcal{X}_0(p)\times_{\mathbb{Z}}\mathbb{Q}$. One therefore has

$$(0)-(\infty)\equiv\Phi+cX_\infty$$

in the decomposition (27) of $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu}$, for $\Phi$ some vertical divisor with support in the fibers above $p$. This divisor is determined by the same equations (28) as $\Phi_{C_0}$ in Lemma 3.3(a). For each $v\mid p$ the full $v$-fiber $\sum_C[C]$ is numerically equivalent to some real multiple of the archimedean fiber $X_\infty$; there is therefore a real number $a$ such that

$$\Phi^0_{C_0}:=\Phi_{C_0}+\sum_{v\,|\,p}\frac{6e_v}{p-1}\Big(\sum_C[C]\Big)\equiv\Phi_{C_0}+aX_\infty.$$

Now $w_p$ switches the cusps 0 and $\infty$ so the divisor $(0)-(\infty)$ is antisymmetric for $w_p$:

$$w_p^*((0)-(\infty))=-((0)-(\infty))$$

and clearly $w_p^*(\Phi^0_{C_0})=-\Phi^0_{C_0}$. The fact that $w_p$ preserves the archimedean fiber concludes the proof of (a).

To prove (b) we compute

$$0=[0-\infty-\Phi^0_{C_0},\infty]_\mu=[0,\infty]_\mu-[\infty,\infty]_\mu-\frac{6}{p-1}\log p$$

and

$$0=[0-\infty-\Phi^0_{C_0},0]_\mu=[0,0]_\mu-[0,\infty]_\mu+\frac{6}{p-1}\log p$$

so that $[\infty, \infty]_\mu = [0, 0]_\mu = [0, \infty]_\mu - 6\log p/(p-1)$. The cusps 0 and $\infty$ are known not to intersect on $\mathcal{X}_0(p)_{/\mathbb{Z}}$ so that $[0, \infty]_\mu = -g_\mu(0, \infty)$. When $\mu = \mu_0$, this special value of the Arakelov–Green function has been computed by Michel and Ullmo; it satisfies, by [Michel and Ullmo 1998, (12), p. 650],

$$g_{\mu_0}(0, \infty) = \frac{1}{2g}\log p\left(1 + O\left(\frac{\log\log p}{\log p}\right)\right) = O\left(\frac{\log p}{p}\right).$$

Finally, using [Bruin 2014, Theorem 7.1(c) and paragraph 8] and plugging into Bruin's method the estimates of [Michel and Ullmo 1998] regarding the comparison function $F(z) = O((\log p)/p)$ between Green–Arakelov and Poincaré measures, we obtain a bound of shape $O(p\log p)$ for $|g_{\mu_e}(0, \infty)|$ (see also Remark 4.5). This completes the proof of (b). □

Instrumental in the sequel will be the explicit decomposition of the relative dualizing sheaf $\omega$ in the arithmetic Chow group.

**Proposition 3.6.** *The relative dualizing sheaf $\omega$ of the minimal regular model $\mathcal{X}_0(p) \to \mathcal{O}_K$ can be written, in the decomposition* (27) *of $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu_0}$ relative to the canonical Green–Arakelov* (1, 1)*-form $\mu_0$, as*

$$\omega = (2g-2)\infty + \sum_{v\,|\,p} \Phi_{\omega,v} + \omega^0 + [K:\mathbb{Q}]c_\omega X_\infty, \tag{32}$$

*where the above components satisfy the following properties*:

- *The number $c_\omega$ is equal to $\frac{(1-2g)}{[K:\mathbb{Q}]}[\infty, \infty]_{\mu_0}$, so that $0 \le c_\omega \le O(\log p)$.*
- *Set*

$$H_4 := \tfrac{1}{2}\sum_{P\in\mathcal{H}_4}\left(P - \tfrac{1}{2}(0+\infty)\right), \quad H_3 := \tfrac{2}{3}\sum_{p\in\mathcal{H}_3}\left(P - \tfrac{1}{2}(0+\infty)\right)$$

*where the sums run over the sets $\mathcal{H}_4$ and $\mathcal{H}_3$, whose number of elements can be* 0 *or* 2, *of Heegner points of $X_0(p)$ with $j$-invariant* 1728 *and* 0 *respectively. Define*

$$H_4^0 := H_4 + [K:\mathbb{Q}]c_4 X_\infty \quad \text{and} \quad H_3^0 := H_3 + [K:\mathbb{Q}]c_3 X_\infty$$

*for two numbers $c_3$ and $c_4$ with $c_3 = O(\log p)$, and the same for $c_4$.* (*Recall this means the $H_*$ are compactified with the normalizing condition* (24), *whereas the $H_*^0$ are the orthogonal projections on $(J_0(p)(K)\otimes\mathbb{R}) \subseteq \widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu_0}$ of the $H_*$, so that $[\infty, H_*^0]_{\mu_0} = 0$, for $* = 3$ or 4.*) *One sets $\omega^0 := -H_4^0 - H_3^0$, which can be chosen in $J_0(p)^0(\overline{\mathbb{Q}})$.*
- *Finally, the component $\Phi_{\omega,v}$ in each $G_v$ for $v \mid p$ is*

$$\Phi_{\omega,v} = -12\frac{(g-1)}{(p-1)}\sum_{n,m}\frac{m}{w_n}C_{n,m} \tag{33}$$

*with notations as in* (16). *We therefore have $\Phi_{\omega,v} = (g-1)\Phi_{C_0}$ using notations of Lemma 3.3. In particular, recalling $e_v$ is the ramification index of $K/\mathbb{Q}$ at $v$, the coefficients $\omega_{n,m}$ of $\Phi_{\omega,v}$ in* (33) *satisfy*

$$0 \ge \omega_{n,m} \ge -e_v. \tag{34}$$

*Proof.* Many parts of those statements are deduced from [Michel and Ullmo 1998, Section 6] and results of Edixhoven and de Jong [2011b]. See also [Menares 2011, Section 4.4].

We start by estimating $c_\omega$. By Arakelov's adjunction formula,

$$-[\infty,\infty]_{\mu_0}=[\infty,\omega]_{\mu_0}=(2g-2)[\infty,\infty]_{\mu_0}+[K:\mathbb{Q}]c_\omega$$

because of the orthogonality of the decomposition (27). Lemma 3.5 therefore implies

$$0\le c_\omega=\frac{(1-2g)}{[K:\mathbb{Q}]}[\infty,\infty]_{\mu_0}=O(\log p).$$

The computations of the $J_0(p)$-part $\omega^0:=-(H_3^0+H_4^0)$ follows from the Hurwitz formula, as explained in [Michel and Ullmo 1998, paragraph 6, p. 670]. One indeed checks that, on the generic fiber $X_0(p)_{/\mathbb{Q}}=\mathcal{X}_0(p)\times_{\mathbb{Z}}\mathbb{Q}$, the canonical divisor is linearly equivalent to

$$(2g-2)\infty-\Bigl(\tfrac{1}{2}\sum_{j(P)=e^{i\pi/2}}{}'(P-\infty)+\tfrac{2}{3}\sum_{j(P)=e^{2i\pi/3}}{}'(P-\infty)\Bigr)$$

where the sums $\sum'$ are here restricted to points $P$ at which $X_0(p)\to X(1)$ is unramified (these are the Heegner points alluded to in our statement). It follows from the modular interpretation that in each of those sums there are two Heegner points (if any), which are then ordinary at $p$ (recall we assume $p>13>3$). This proves that the $J_0(p)(K)\otimes_{\mathbb{Z}}\mathbb{R}$-part of $\omega$ is indeed $-(H_4^0+H_3^0)$ with $H_4^0=H_4+[K:\mathbb{Q}]c_4X_\infty$ and $H_3^0=H_3+[K:\mathbb{Q}]c_3X_\infty$ for some real numbers $c_3$ and $c_4$. (Note that, as Heegner points are preserved by the Atkin–Lehner involution [Gross 1984, paragraph 5, p. 90] their specializations above $p$ share themselves between the two components $C_0$ and $C_\infty$ of $\mathcal{X}_0(p)_{/\mathbb{F}_p}$, so that $2H_3^0=\sum_{j(P)=e^{i\pi/2}}{}'(P-\infty)$ and $\frac{2}{3}H_4^0=\sum_{j(P)=e^{2i\pi/3}}{}'(P-\infty)$ belong to the neutral component $J_0(p)^0(\mathcal{O}_K)$.) The estimates on $c_3$ and $c_4$ will be justified at the end of the proof.

The bad fibers divisors $\Phi_{\omega,v}:=\sum_{n,m}\omega_{n,m}[C_{n,m}]$ can be computed with the "vertical" adjunction formula [Liu 2002, Chapter 9, Theorem 1.37] as in [Menares 2011, Lemma 4.22]. Indeed, for each irreducible component $C$ in the $v$-fiber having genus 0, one has

$$[C,C+\omega]=-2\log(\#k(v)).$$

If $\mathcal{M}$ is the intersection matrix displayed in (18), and $\delta_{*,*}$ is Kronecker's symbol, we therefore have

$$C\cdot\mathcal{M}\cdot\Phi_{\omega,v}=-2-\frac{1}{\log(\#k(v))}[C,C]-(2g-2)\delta_{C,C_\infty}=\begin{cases}0 & \text{if } C\neq C_\infty, C_0,\\ s-2g & \text{if } C=C_\infty,\\ s-2 & \text{if } C=C_0,\end{cases}\tag{35}$$

that is, as $s=g+1$,

$$\mathcal{M}\cdot\Phi_{\omega,v}=(g-1)(-1,0,\dots,0,1)^t.$$

That equation is (30) (up to a multiplicative scalar), which has been solved in the first case of Lemma 3.3. Therefore

$$\Phi_{\omega,v} = (g-1)\Phi_{C_0}, \quad \text{that is } \omega_{n,m} = \frac{12(1-g)}{(p-1)} \cdot \frac{m}{w_n}. \tag{36}$$

As noted in Remark 3.4 and using (15), this implies the coefficients $\omega_{n,m}$ of $\Phi_{\omega,v}$ satisfy

$$0 \geq \omega_{n,m} \geq \frac{12(1-g)}{p-1} e_v > -e_v.$$

We finally estimate the intersection products

$$c_3 = \frac{-1}{[K:\mathbb{Q}]}[\infty, H_3]_{\mu_0} \quad \text{and} \quad c_4 = \frac{-1}{[K:\mathbb{Q}]}[\infty, H_4]_{\mu_0}.$$

By the adjunction formula and Hriljac–Faltings' theorem [Chinburg 1986, Theorem 5.1(ii)] we compute that for any $P \in X_0(p)(K)$,

$$\begin{aligned} -2[K:\mathbb{Q}]\mathrm{h}_\Theta\Big(P - \frac{1}{2g-2}\omega\Big) &= \Big[P - \frac{1}{2g-2}\omega - \Phi_\omega(P),\, P - \frac{1}{2g-2}\omega - \Phi_\omega(P)\Big]_{\mu_0} \\ &= \frac{1}{(2g-2)^2}[\omega,\omega]_{\mu_0} + \frac{g}{g-1}[P,P]_{\mu_0} - \Phi_\omega(P)^2 \end{aligned}$$

where here $\Phi_\omega(P)$ is a vertical divisor supported at bad fibers such that

$$\Big[C,\, P - \frac{1}{2g-2}\omega - \Phi_\omega(P)\Big] = 0 \tag{37}$$

for any irreducible component $C$ of any bad fiber of $\mathcal{X}_0(p)_{/\mathcal{O}_K}$. Hence

$$\frac{1}{(2g-2)^2}\omega^2 + \frac{g}{g-1}[P,P]_{\mu_0} - \Phi_\omega(P)^2 = -2[K:\mathbb{Q}]\mathrm{h}_\Theta\Big((P-\infty) + \frac{1}{2g-2}(H_3+H_4)\Big). \tag{38}$$

We specialize to the case when $P = P_*^*$ (where the upper star is 1 or 2 and the lower star is 4 or 3) is one of the Heegner points occurring in $H_4$ or $H_3$, respectively. We replace for now the base field $K$ by $F := \mathbb{Q}(P_*^*) = \mathbb{Q}(\sqrt{-1})$ (respectively, $\mathbb{Q}(\sqrt{-3})$). The right-hand of (38), if nonzero, is then

$$-8\log(p)(1+o(1)) \quad \text{or} \quad -12\log(p)(1+o(1)), \quad \text{respectively,} \tag{39}$$

by [Michel and Ullmo 1998, p. 673]. If those Heegner points occur we know that $p$ splits in $F$, so there are two bad primes $v$ and $v'$ on $\mathcal{O}_F$ (therefore two bad fibers on $\mathcal{X}_0(p)_{/\mathcal{O}_F}$ and two $G_v$, $G_{v'}$) to take into account. We compute $\Phi_\omega(P_*^*)$ and $\Phi_\omega(P_*^*)^2$. As mentioned at the beginning of the proof, $P_*^*$ specializes to the component $C_0$ at a place, say $v$, of $F$ above $p$, and to $C_\infty$ at the conjugate place $v'$. Condition (37) therefore gives that, for any irreducible component $C$ of the fiber at $v$,

$$0 = \Big[C,\, P_*^* - \frac{1}{2g-2}\omega - \Phi_\omega(P_*^*)_v\Big] = \Big[C,\, 0 - \infty - \frac{1}{2g-2}\Phi_{\omega,v} - \Phi_\omega(P_*^*)_v\Big]$$

and using Lemma 3.3, 3.5 and (36) one obtains

$$\Phi_\omega(P_*^*)_v = -\frac{1}{2g-2}\Phi_{\omega,v} + \Phi_{C_0,v} = \tfrac{1}{2}\Phi_{C_0,v}$$

whereas, at $v'$

$$\Phi_\omega(P_*^*)_{v'} = -\frac{1}{2g-2}\Phi_{\omega,v'} = -\tfrac{1}{2}\Phi_{C_0,v'}.$$

Using Lemma 3.3 and 3.5 again we therefore have

$$\Phi_\omega(P_*^*)^2 = \sum_{w\mid p} \tfrac{1}{4}\Phi_{C_0,w}^2 = \sum_{w\mid p} \tfrac{1}{4}[\Phi_{C_0,w}, 0-\infty] = \tfrac{1}{2}a_0 \log p = -\frac{6\log(p)}{p-1}. \tag{40}$$

As for the self-intersection of $\omega$ one knows from [Ullmo 2000, Introduction] that

$$\omega^2_{\mathcal{X}_0(p)_{/\mathbb{Z}}} = 3g\log(p)(1+o(1)).$$

As the quantity $\frac{1}{[F:K]}[\omega]^2$ is known to be independent from the number field extension $F/K$, the dualizing sheaf $\omega_{\mathcal{X}_0(p)_{/\mathcal{O}_F}}$ of $\mathcal{X}_0(p)$ over $\mathcal{O}_F$ (instead of $\mathbb{Z}$) satisfies $\omega^2 = 6g\log(p)(1+o(1))$. Summing-up, (38) implies that

$$[P_*^*, P_*^*]_{\mu_0} = O(\log(p)) \tag{41}$$

for each Heegner point $P_*^*$. Now, on the other hand, the vertical divisor $\Phi_{P_*^*}$ in the sense of (28) and Lemma 3.3 is $\Phi_{P_*^*} = \Phi_{C_0,v}$ for the place $v$ of $F$ where $P_*^*$ specializes on $C_0$ and not $C_\infty$. Therefore

$$\begin{aligned} -4\mathrm{h}_\Theta(P_*^*-\infty) &= [P_*^*-\infty-\Phi_{P_*^*}, P_*^*-\infty-\Phi_{P_*^*}]_{\mu_0} \\ &= -2[P_*^*,\infty]_{\mu_0} + [P_*^*, P_*^*]_{\mu_0} + [\infty,\infty]_{\mu_0} - (\Phi_{P_*^*})^2 \end{aligned} \tag{42}$$

whence, using (39), (40), (41) and 3.5(b),

$$[P_*^*,\infty]_{\mu_0} = \tfrac{1}{2}([P_*^*, P_*^*]_{\mu_0} + [\infty,\infty]_{\mu_0} - (\Phi_{C_0,v})^2 + 4\mathrm{h}_\Theta(P_*^*-\infty)) = O(\log p).$$

Putting everything together and using 3.5 once more we conclude that

$$c_4 = -\frac{1}{[K:\mathbb{Q}]}[\infty, H_4]_{\mu_0} = \frac{1}{2[K:\mathbb{Q}]}(-[\infty, P_4^1+P_4^2]_{\mu_0} + [\infty, 0+\infty]_{\mu_0}) = O(\log p) \tag{43}$$

and similarly for $c_3$. (Note that the Arakelov intersection products, in the computations around (42), were performed over $F=\mathbb{Q}(P_*^*)$ and not $K$, although we did not indicate this in the notation in order to keep it from becoming too heavy. We however want quantities over $K$ for the statement of the theorem, so we need considering Arakelov products over $K$ in (43) above.) □

**Remark 3.7.** It may be convenient to write, with notations as in (32), a more symmetric $\omega$ as

$$\omega = (g-1)(\infty+0) + (-H_4^0 - H_3^0) + [K:\mathbb{Q}]c_\omega X_\infty \tag{44}$$

which yields an element with no vertical component at bad fibers.

## 4. $j$-height and $\Theta$-height

In this section we compare two natural heights on $X_0(p)(\overline{\mathbb{Q}})$, namely the $j$-height and the one induced from the Néron–Tate $\Theta$-height on $J_0(p)(\overline{\mathbb{Q}})$. We start with an explicit description of the latter, for which it is actually convenient to use a bit of Zhang's language [1993] about "adelic metrics" which, in our modular setting, has a very concrete form.

Using notations and results from Section 2B2 we therefore consider the limit, as $e_v$ goes to $\infty$, of the dual graph of the special fiber of $\mathcal{X}_0(p)$ at a place $v$ of a $p$-adic local field with ramification index $e_v$ at $p$ (see Figure 1). Here we normalize the length of the $s = g+1$ edges from $C_\infty$ to $C_0$ to be 1, so that the vertex $C_{n,m}$ corresponds to the point of the $n$-th edge with distance $m/(e_v w_n)$ from the origin $C_\infty$. Now associate to any edge $n \in \{1, \ldots, s\}$ the quadratic polynomial function

$$g_n(x) : [0,1] \to \mathbb{R}, \quad x \mapsto \frac{x}{2}\left(\left(w_n - \frac{12}{(p-1)}\right)x - w_n - 12\frac{(g-1)}{(p-1)}\right). \tag{45}$$

For $K$ any number field, $P$ in $X_0(p)(K)$, and $v$ a place of $K$ whose ramification degree and residual degree are still denoted by $e_v$ and $f_v$ respectively, let

$$G(P(K_v)) = e_v f_v \log(p) \cdot g_n(C_{P(k(v))}) \tag{46}$$

where $C_{P(k(v))}$ is the component to which the specialization of $P$ belongs at $v$, identified to a point of the $n$-th edge where it lives.

**Theorem 4.1.** *For any number field $K$, there is an element*

$$\tilde{\omega}_{\Theta,K} = (g \cdot \infty + \Phi_{\Theta,K} + c_{\Theta,K} X_\infty) \tag{47}$$

*of $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu_0}$ such that for any $P \in X_0(p)(K)$ one has, with notations as in Proposition 3.6,*

$$\mathrm{h}_\Theta\big(P - \infty + \tfrac{1}{2}\omega^0\big) = \frac{1}{[K:\mathbb{Q}]}[P, \tilde{\omega}_{\Theta,K}]_{\mu_0} \tag{48}$$

*and the terms of* (47) *satisfy*

$$0 \ge [P, \Phi_{\Theta,K}] \ge -2[K:\mathbb{Q}]\log(p) \quad \text{and} \quad c_{\Theta,K} = [K:\mathbb{Q}]O(\log p). \tag{49}$$

*Passing to the limit on all number fields, the height induced on $X_0(p)(\overline{\mathbb{Q}})$ by pulling-back Néron–Tate's $\Theta$-height on $J_0(p)(\overline{\mathbb{Q}})$ via the embedding $P \mapsto P - \infty + \frac{1}{2}\omega^0$ can be written as*

$$\mathrm{h}_\Theta\big(P - \infty + \tfrac{1}{2}\omega^0\big) = \frac{1}{[K:\mathbb{Q}]}\left(g[P,\infty]_{\mu_0} + \sum_{v \in M_K, v \mid p} G(P(K_v)) + c_{\Theta,K}\right) \tag{50}$$

*where Zhang's Green function $G$ at bad fibers is defined in* (45) *and* (46).

*In any case one has that the height satisfies*

$$\mathrm{h}_\Theta\big(P - \infty + \tfrac{\omega^0}{2}\big) = \frac{1}{[K:\mathbb{Q}]}[P, g \cdot \infty]_{\mu_0} + O(\log p). \tag{51}$$

*Proof.* We prove (48) and (49); from there reformulation (50) and (51) are straightforward.

Recall $\mathcal{X}_0(p)$ denotes the minimal regular model of $X_0(p)$ on $\mathrm{Spec}(\mathcal{O}_K)$, that $\mathcal{J}_0(p)$ is the Néron model of $J_0(p)$ on the same base, and $\mathcal{J}_0(p)^0$ stands for its neutral component. Let $\delta$ be an element of $J_0(p)(K)$, seen as a degree 0 divisor on $X_0(p)$. Up to making a base extension we can assume $\delta$ is linearly equivalent to a sum of points in $X_0(p)(K)$. We shall denote by $\tilde{\delta} = \delta + \Phi_\delta$ (for $\Phi_\delta$ some vertical divisor on $\mathcal{X}_0(p)$, with multiplicity 0 on the component containing $\infty$, following our running conventions) the associated element of the neutral component $\mathcal{J}_0(p)^0(\mathcal{O}_K)$ (that is, the one whose associated divisor has degree zero on each irreducible component, in any fiber, of $\mathcal{X}_0(p)$, and therefore defines a point of $\mathcal{J}_0(p)^0(\mathcal{O}_K)$). For any point $P$ in $X_0(p)(K) \hookrightarrow \mathcal{X}_0(p)(\mathcal{O}_K)$ let similarly $\Phi_P$ be the vertical divisor on $\mathcal{X}_0(p)$, with support on the bad fibers, such that $(P - \infty - \Phi_P)$ has divisor class belonging to the neutral component $\mathcal{J}_0(p)^0(\mathcal{O}_K)$ and, again, $\Phi_P$ has everywhere trivial $\infty$-component, see (28). Recall we can compute $\Phi_P$ explicitly by Lemma 3.3. We write $\Phi_P = \sum_{v \in M_K, v \mid p} \sum_{C_v} a_{C_v}[C_v]$ where the sum is taken on irreducible components $C_v$ of vertical bad fibers of $\mathcal{X}_0(p)$. Using notations of Lemma 3.3 (b) we also define the following new vertical divisor at bad fibers:

$$\Phi_{\vartheta,K} := \sum_{v \in M_K, v \mid p} \sum_{Q_v} a_{C_{Q_v}} C_{Q_v} = \sum_{v \mid p} \sum_{(n_0,m_0)} a^v_{n_0,m_0} C_{n_0,m_0} \tag{52}$$

so that

$$a^v_{n_0,m_0} = \left( \frac{m_0}{w_{n_0} e_v} \left( 1 - \frac{12}{(p-1) w_{n_0}} \right) - 1 \right) \cdot m_0.$$

Our very definitions imply

$$\Phi_P^2 = [P, \Phi_P] = [P, \Phi_{\vartheta,K}] \tag{53}$$

for any $P \in X_0(p)(K)$. Using Faltings' Hodge index theorem we can write the Néron–Tate height $\mathrm{h}_\Theta(P - \infty + \delta)$ as

$$\begin{aligned}
&\mathrm{h}_\Theta(P - \infty + \delta)\\
&\quad = \frac{-1}{2[K:\mathbb{Q}]} [P - \infty + \tilde{\delta} - \Phi_P, P - \infty + \tilde{\delta} - \Phi_P]_{\mu_0}\\
&\quad = \frac{1}{2[K:\mathbb{Q}]} ([P, \omega + 2\infty - 2\tilde{\delta}]_{\mu_0} + 2[P, \Phi_P]_{\mu_0} - [\Phi_P, \Phi_P]_{\mu_0} + [\tilde{\delta}, 2\infty - \tilde{\delta}]_{\mu_0} - [\infty, \infty]_{\mu_0})\\
&\quad = \frac{1}{2[K:\mathbb{Q}]} ([P, \omega + 2\infty - 2\tilde{\delta} + \Phi_{\vartheta,K}]_{\mu_0} + [\tilde{\delta}, 2\infty - \tilde{\delta}]_{\mu_0} - [\infty, \infty]_{\mu_0})\\
&\quad = \frac{1}{[K:\mathbb{Q}]} [P, \tilde{\omega}_\delta]_{\mu_0}
\end{aligned} \tag{54}$$

with

$$\tilde{\omega}_\delta := \left( \tfrac{1}{2}(\omega + \Phi_{\vartheta,K}) + \infty - \tilde{\delta} \right) + c_\delta X_\infty \tag{55}$$

for $X_\infty$ some fixed archimedean fiber of $\mathcal{X}_0(p)$ and $c_\delta$ is the real number

$$c_\delta = \tfrac{1}{2}(-[\infty, \infty]_{\mu_0} + [\tilde{\delta}, 2\infty - \tilde{\delta}]_{\mu_0}). \tag{56}$$

Note that $\tilde{\omega}_\delta$ does not depend on $P$ (as $\Phi_{\vartheta,K}$ was introduced to that aim).

Let us now take $\delta = \frac{1}{2}\omega^0 = -\frac{1}{2}(H_3 + H_4) \in \frac{1}{12}\cdot J_0(p)^0(\mathbb{Q})$, as defined in Proposition 3.6. (This is Riemann's characteristic (the "$\kappa$" of [Hindry and Silverman 2000, p. 138] for instance, that is the generic fiber of the $J_0(p)(\mathbb{Q})\otimes\mathbb{R}$-part of $\omega$ in the decomposition (32).) Set $\Phi_{\Theta,K} := \frac{1}{2}(\Phi_\omega + \Phi_{\vartheta,K})$. Then

$$\tilde{\omega}_\Theta := \tilde{\omega}_\delta = (g\cdot\infty + \Phi_{\Theta,K} + c_{\Theta,K}X_\infty) \tag{57}$$

for $c_{\Theta,K}$ which, still using notations of Proposition 3.6 and its proof, is explicitly given by

$$\begin{aligned}\frac{1}{[K:\mathbb{Q}]}c_{\Theta,K} &= \frac{1}{2}\Big(c_\omega - c_4 - c_3 + \tfrac{1}{2}\mathrm{h}_\Theta(H_3+H_4) - \frac{1}{[K:\mathbb{Q}]}([\infty]^2_{\mu_0} + [\infty, H_3+H_4]_{\mu_0})\Big)\\ &= \frac{1}{2}\Big(c_\omega - \frac{1}{[K:\mathbb{Q}]}[\infty]^2_{\mu_0} + \tfrac{1}{2}\mathrm{h}_\Theta(H_3+H_4)\Big).\end{aligned}$$

As in the proof of Proposition 3.6 we invoke p. 673 of [Michel and Ullmo 1998] to assert $\mathrm{h}_\Theta(H_3+H_4) = O(\log(p))$. We moreover know from the same proposition and from Lemma 3.5 that both $|c_\omega| = O(\log p)$ and $[\infty,\infty]_{\mu_0} = [K:\mathbb{Q}]O(\log p/p)$, so that

$$c_{\Theta,K} = [K:\mathbb{Q}]O(\log p). \tag{58}$$

The contribution of $\Phi_{\Theta,K}$ is controlled by Lemma 3.3 and Remark 3.4. On one hand

$$0 \geq [P, \Phi_{\vartheta,K}] = [P, \Phi_P] = \sum_{v\in M_K, v\,|\,p} a_{C_P,v}\log(\#k_v) \geq \sum_{v\in M_K, v\,|\,p} -3e_v\log(p^{f_v}) \geq -3[K:\mathbb{Q}]\log(p). \tag{59}$$

On the other hand, by (34), the coefficients of the vertical components $\Phi_{\omega,v}$ satisfy $0 \geq \omega_{n,m} \geq -e_v$, so writing $\omega_{n_P,m_P,v}$ for the coefficient in $\Phi_{\omega,v}$ of the component containing $P(k(v))$ we have

$$0 \geq [P, \Phi_\omega] = \sum_{v\,|\,p}\omega_{n_P,m_P,v}\log(\#k(v)) \geq \sum_{v\,|\,p} -e_v\log(p^{f_v}) = -[K:\mathbb{Q}]\log(p). \tag{60}$$

Putting (58), (59) and (60) together completes the proof of (48) and (49) and the proof. □

**Remark 4.2.** Estimates on the Green–Zhang function on $X_0(p)$ as in the above theorem will be extended below to the Néron model over $\overline{\mathbb{Z}}$ of the whole jacobian $J_0(p)$, see Proposition 5.8.

**Remark 4.3.** As already noticed, the involution $w_p$ acts as an isometry (actually, an orthogonal symmetry) with respect to the quadratic form $\mathrm{h}_\Theta$ on $J_0(p)(K)\otimes_{\mathbb{Z}}\mathbb{R}$. Indeed $w_p$ acts as multiplication by $\pm1$ on each factor of Shimura's decomposition up to isogeny

$$J_0(p) \sim \prod_{f\in G_{\mathbb{Q}}\cdot S_2(\Gamma_0(p))^{\mathrm{new}}} J_f$$

whose factors are $\mathrm{h}_\Theta$-orthogonal subspaces. (See also [Menares 2008, Corollaire 4.3] or [Menares 2011, Theorem 4.5(3)].) As $w_p(\omega^0) = \omega^0$ (see the proof of Proposition 3.6) this implies

$$\mathrm{h}_\Theta\big(P - \infty + \tfrac{1}{2}\omega^0\big) = \mathrm{h}_\Theta\big(w_p\big(P - \infty + \tfrac{1}{2}\omega^0\big)\big) = \mathrm{h}_\Theta\big(w_p(P) - 0 + \tfrac{1}{2}\omega^0\big) = \mathrm{h}_\Theta\big(w_p(P) - \infty + \tfrac{1}{2}\omega^0\big)$$

using once more that $(0)-(\infty)$ is torsion, so that

$$[P, \tilde{\omega}_\Theta]_{\mu_0} = [w_p(P), \tilde{\omega}_\Theta]_{\mu_0} = [P, w_p^*(\tilde{\omega}_\Theta)]_{w_p^*(\mu_0)} = [P, w_p^*(\tilde{\omega}_\Theta)]_{\mu_0} \tag{61}$$

(see Remark 3.1). This suggests it could sometimes be convenient to write $\tilde{\omega}_\Theta$ in a $w_p$-eigenbasis of $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu}$ instead of that of Theorem 3.2, for instance

$$\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu_0} = \mathbb{R}\cdot\tfrac{1}{2}(0+\infty)\oplus\mathbb{R}\cdot X_\infty \oplus_{v\mid p}\Gamma_v \oplus (J_0(p)(K)\otimes\mathbb{R}) \tag{62}$$

where now the $\Gamma_v$ decompose as the direct sum of eigenspaces $\Gamma_v^{w_p=-1}$ and $\Gamma_v^{w_p=+1}$, with bases

$$\{C^-_{n,m} := C_{n,m} - w_p(C_{n,m})\}_{\substack{1\le n\le s\\ 0\le m\le ew_n/2}} \quad\text{and}\quad \{C^+_{n,m} := C_{n,m} + w_p(C_{n,m}) - C_0 - C_\infty\}_{\substack{1\le n\le s\\ 1\le m\le ew_n/2}} \tag{63}$$

respectively. Using 3.5 and Proposition 3.6, a lengthy but easy computation allows one to check that

$$\tilde{\omega}_\Theta = g\cdot\tfrac{1}{2}(0+\infty) + \Phi^+_\Theta + \gamma_\Theta X_\infty$$

where $\Phi^+_\Theta$ is an explicit vertical divisor above $p$ with $w_p^*(\Phi^+_\Theta) = \Phi^+_\Theta$, so that indeed

$$w_p^*(\tilde{\omega}_\Theta) = \tilde{\omega}_\Theta$$

thus recovering (61).

Consider for instance the case of $\mathcal{X}_0(p)$ over $\mathbb{Z}$, for $p\equiv 1 \bmod 12$ (that is, $\mathcal{X}_0(p)_{/\mathbb{Z}}$ is regular, so that there is no need to blow-up singular points of width larger than 1). Here $\Gamma_v = \Gamma_v^- = \mathbb{R}\cdot C_0^- = \mathbb{R}\cdot([C_\infty]-[C_0])$) and one readily checks that

$$\tilde{\omega}_\Theta = \frac{g}{2}(0+\infty) + \gamma_\Theta X_\infty \tag{64}$$

that is, there is no $\Gamma_v$-component at all in that case. Evaluating $\mathrm{h}_\Theta\big(\tfrac{1}{2}\omega^0\big)$ as in the proof of Proposition 3.6 and using 3.5,

$$\gamma_\Theta = -\frac{g}{2}[\infty, 0+\infty]_{\mu_0} + \mathrm{h}_\Theta\big(\tfrac{1}{2}\omega^0\big) = g\,O(\log p/p) + O(\log p) = O(\log p).$$

We then turn to the $j$-height, first making a comparison of $\mathrm{h}_j$ with the "degree component" (in the sense of Theorem 3.2) of the hermitian sheaf $\omega$.

**Proposition 4.4.** *Let* $\mathrm{h}_j$ *be Weil's* $j$*-height on* $X_0(p)$ *as defined in Section 2B, and let* $\mu_0$ *and* $\mu_e$ *be the* $(1,1)$*-forms defined in* (25) *and* (26). *Recall* $\sup_{X_0(p)(\mathbb{C})} g_\mu$ *stands for the upper bound for all Green functions* $g_{\mu,a}$ *relative to some point* $a$ *of* $X_0(p)(\mathbb{C})$ *and to the measure* $\mu$.

*If* $p$ *is a prime number*, $K$ *is a number field*, *and* $P$ *belongs to* $X_0(p)(K)$, *then*

$$\mathrm{h}_j(P) \le (p+1)\left(\frac{1}{[K:\mathbb{Q}]}[P,\infty]_{\mu_0} + \sup_{X_0(p)(\mathbb{C})} g_{\mu_0} + O(1)\right) \le \frac{(p+1)}{[K:\mathbb{Q}]}[P,\infty]_{\mu_0} + O(p^2\log p) \tag{65}$$

*and similarly*

$$\mathrm{h}_j(P) \le (p+1)\left(\frac{1}{[K:\mathbb{Q}]}[P,\infty]_{\mu_e} + \sup_{X_0(p)(\mathbb{C})} g_{\mu_e} + O(1)\right) \le \frac{(p+1)}{[K:\mathbb{Q}]}[P,\infty]_{\mu_e} + O(p^3). \tag{66}$$

**Remark 4.5.** As explained in the proof below, the function $O(p^2 \log p)$ of (65) comes from [Wilms 2017, Corollary 1.5] together with [Ullmo 2000, Corollaire 1.3] for the estimate of Faltings' $\delta$ invariant for $X_0(p)$, which imply the suprema of our functions verify

$$\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} \leq O(p \log p). \tag{67}$$

The function $O(p^3)$ of (66) in turns follows from the main result of [Bruin 2014]. Indeed this states explicitly that $\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} \leq 0.088 \cdot p^2 + 7.7 \cdot p + 1.6 \cdot 10^4$ [loc. cit., Theorem 1.2]. It follows from measures comparison (see (74) below) and the method of P. Bruin that this holds for $\sup_{X_0(p)(\mathbb{C})} g_{\mu_e}$ too, so that

$$\sup_{X_0(p)(\mathbb{C})} g_{\mu_e} \leq O(p^2). \tag{68}$$

It seems that, at least in the case of $X_0(p)$, if we plug into Bruin's method the estimates of [Michel and Ullmo 1998] regarding the comparison function $F(z)$ between Green–Arakelov and Poincaré measures, we recover bounds of shape $O(p \log p)$ instead of $O(p^2)$ (see [Bruin 2014], p. 263 and §8 (Theorem 7.1 in particular)), and the same again holds true for the Green function $g_{\mu_e}$. One should therefore be able to obtain the same error term $O(p^2 \log p)$ for (66) as for (65).

Note that the main theorems of [Jorgenson and Kramer 2006; Aryasomayajula 2013] might even yield that the above functions $O(p^2)$ or $O(p \log p)$ could be replaced by a uniform bound $O(1)$.

*Proof.* This is essentially a question of measure comparisons on $X_0(p)(\mathbb{C})$ between $j^*(\mu_{FS})$ on one hand (where $\mu_{FS}$ is the Fubini–Study (1, 1)-form on $X(1)(\mathbb{C}) \simeq \mathbb{P}^1(\mathbb{C})$) and the Green–Arakelov form $\mu_0$ (respectively, $\mu_e$) on the other hand. We adapt the main result of [Edixhoven and de Jong 2011a].

We define first a somewhat canonical Arakelov intersection product $[\cdot,\cdot]_{\mu_{FS}}$ on the projective line using $\mu_{FS}$. Write $\mathbb{P}^1_{/\mathcal{O}_K} = \mathrm{Proj}(\mathcal{O}_K[x_0, x_1]) = \overline{\mathrm{Spec}}^{\mathrm{Zar}}(\mathcal{O}_K[j])$ (with $j = x_1/x_0$), so that the horizontal divisor $\infty(\mathcal{O}_K)$ is $V(x_0)$ and, for any $P = [x_0 : x_1]$, let the associated Green function be

$$g_{\mu_{FS},\infty}(P) = g_{\mu_{FS},\infty}(j(P)) = \tfrac{1}{2} \log\left(\frac{|x_0|^2}{|x_0|^2 + |x_1|^2}\right) = -\tfrac{1}{2}\log(1 + |j(P)|^2)$$

at any point different from $\infty = [0 : 1]$. (We note in passing this ad hoc Green function does not need to fulfill the normalization condition (24).) Then for any $P$ in $X(1)(K)$ one easily checks that

$$\left| \mathrm{h}_j(P) - \frac{1}{[K : \mathbb{Q}]}[j(P), \infty]_{\mu_{FS}} \right| \leq \tfrac{1}{2}\log(2). \tag{69}$$

Applying [Edixhoven and de Jong 2011a], Theorem 9.1.3 and its proof to the setting described above gives, for any $P$ in $X_0(p)(K)$,

$$[j(P), \infty]_{\mu_{FS}} \leq [P, j^*(\infty)]_{\mu_0} + (p+1)\sum_\sigma \sup_{X_0(p)_\sigma} g_{\mu_0} + \tfrac{1}{2}\sum_\sigma \int_{X_0(p)_\sigma} \log(|j|^2 + 1)\mu_0 \tag{70}$$

where $\sigma$ runs through the infinite places of $K$ and $X_0(p)_\sigma := X_0(p) \times_{\mathcal{O}_K,\sigma} \mathbb{C}$.

We estimate the right-hand terms of (70). As for the last integrals we recall that, on the union of disks of ray $|q| < r$ around the cusps (that is, on the image in $X_0(p)(\mathbb{C})$ of the open subset $D_r := \{z \in \mathcal{H} : \Im(z) > -(\log r)/2\pi\}$ in Poincaré upper half-plane $\mathcal{H}$) for some fixed $r$ in $]0, 1[$, one has

$$\left|\frac{f(q)}{q}\right| \leq \frac{2}{(1-r)^2}$$

for any newform $f$ in $S_2(\Gamma_0(p))$. (See for instance [Edixhoven and de Jong 2011b], Lemma 11.3.7 and its proof.) We also know that the Petersson norm of such an $f$ satisfies $\|f\|^2 \geq \pi e^{-4\pi}$ [Edixhoven and de Jong 2011b, Lemma 11.1.2]. Choose $r = \frac{1}{2}$ to fix ideas. On $D_{1/2}$, we have (see (25)):

$$\mu_0 = \frac{i}{2\dim(J)} \sum_{f \in B_2} \frac{f\frac{dq}{q} \wedge \overline{f\frac{dq}{q}}}{\|f\|^2} \leq \frac{64e^{4\pi}}{\pi} \frac{i}{2} dq \wedge \overline{dq}.$$

(Sharper bounds should be achievable, but the one above is good enough for our present purpose.) It follows that there exists some real $A$ such that, in the decomposition

$$\int_{X_0(p)(\mathbb{C})} \log(|j|^2+1)\mu_0 = \int_{X_0(p)(\mathbb{C}) \cap D_{1/2}} \log(|j|^2+1)\mu_0 + \int_{X_0(p)(\mathbb{C}) \setminus D_{1/2}} \log(|j|^2+1)\mu_0 \tag{71}$$

the first term of the right-hand side satisfies

$$\int_{X_0(p)(\mathbb{C}) \cap D_{1/2}} \log(|j|^2+1)\mu_0 \leq \frac{64e^{4\pi}}{\pi}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)] \int_{X(1)(\mathbb{C}) \cap D_{1/2}} \log(|j|^2+1)\tfrac{i}{2}\, dq \wedge \overline{dq} \leq (p+1)A.$$

As for the second term, remembering that $\mu_0$ has total mass 1 on $X_0(p)(\mathbb{C})$ we check that

$$\int_{X_0(p)(\mathbb{C}) \setminus D_{1/2}} \log(|j|^2+1)\mu_0 \leq M_{1/2} := \max_{X(1)(\mathbb{C}) \setminus D_{1/2}} (\log(|j|^2+1))$$

whence the existence of some absolute real number $A_0$ such that

$$\int_{X_0(p)(\mathbb{C})} \log(|j|^2+1)\mu_0 \leq (p+1)A_0. \tag{72}$$

Putting this together with (70) we obtain a constant $C$ for which (69) reads

$$\mathrm{h}_j(P) \leq \frac{1}{[K:\mathbb{Q}]}[P, j^*(\infty)]_{\mu_0} + (p+1)(\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} + A_0).$$

With notations of 3.5, one further has

$$j^*(\infty) = p(0) + (\infty) \equiv (p+1)\infty + p \cdot \Phi^0_{C_0} \tag{73}$$

as elements of $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu_0}$. Using 3.5(a) we get

$$|[P, \Phi^0_{C_0}]| \leq [K:\mathbb{Q}]\frac{6\log p}{p-1}$$

so that, with (67),

$$\begin{aligned} \mathrm{h}_j(P) &\le \frac{1}{[K:\mathbb{Q}]}[P,(p+1)\infty]_{\mu_0} + (p+1)\big(\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} + A_0\big) + O(\log p) \\ &\le \frac{1}{[K:\mathbb{Q}]}[P,(p+1)\infty]_{\mu_0} + C_0\cdot p^2\log p, \end{aligned}$$

which is (65).

The proof of (66) proceeds along the same lines, with one more ingredient. Applying Theorem 9.1.3 of [Edixhoven and de Jong 2011a] with the measure $\mu_e$ instead of $\mu_0$ gives the corresponding version of (70). To obtain an upper bound for $\sup_{X_0(p)(\mathbb{C})} g_{\mu_e}$ we recall that the theorem of Kowalski, Michel and Vanderkam asserts that $\dim(J_e)\ge \dim(J_0(p))/5$ for large enough $p$; see (23). Our measure $\mu_e := \frac{1}{\dim(J_e)}\sum_{S_e}\frac{i}{2}\big(f\frac{dq}{q}\wedge\overline{f\frac{dq}{q}}\big)/\|f\|^2$ (see (26)) therefore satisfies

$$0\le \mu_e \le \frac{g}{\dim(J_e)}\mu_0 \le 5\mu_0. \tag{74}$$

This shows that, as in (68), Bruin's theorem [2014, Theorem 7.1] provides a universal $c_e$ such that

$$\sup_{X_0(p)(\mathbb{C})} g_{\mu_e} \le c_e\, p^2. \tag{75}$$

Using (72) we obtain

$$\int_{X_0(p)(\mathbb{C})} \log(|j|^2+1)\mu_e \le (p+1)A_e. \tag{76}$$

Finally, equivalence (73) remains naturally true in the Chow group $\widehat{\mathrm{CH}}(p)^{\mathrm{num}}_{\mathbb{R},\mu_e}$ relative to the measure $\mu_e$ instead of $\mu_0$, as remarked in 3.5(a). This completes the proof of (66). □

We can finally relate $\mathrm{h}_j$ and the Néron–Tate height $\mathrm{h}_\Theta$ relative to the $\Theta$-divisor (see (10)).

**Theorem 4.6.** *There are real numbers $\gamma$ and $\gamma_1$ such that the following holds. Let $K$ a number field and $p$ a prime number. Let $\omega^0 := -(H_4+H_3)$ be the 0-component of the canonical sheaf $\omega$ on $X_0(p)$ over $K$ (as in Proposition 3.6 and Theorem 4.1). If $P$ is a point of $X_0(p)(K)$ then*

$$\mathrm{h}_j(P) \le (12+o(1))\cdot \mathrm{h}_\Theta\big(P-\infty+\tfrac{1}{2}\omega^0\big) + \gamma\cdot p^2\log p, \tag{77}$$

$$\mathrm{h}_j(P) \le (24+o(1))\cdot \mathrm{h}_\Theta(P-\infty) + \gamma_1\cdot p^2\log p. \tag{78}$$

**Remark 4.7.** Theorem 4.6 offers only one direction of inequality between $j$-height and $\Theta$-height; with our method of proof, it is harder to give an effective form to the reverse inequality, because of the metrics comparisons we use (see below).

Notice also that going through the above proofs using the estimate $\sup_{X_0(p)(\mathbb{C})} g_{\mu_0} = O(1)$ of [Jorgenson and Kramer 2006] and [Aryasomayajula 2013] (see Remark 4.5) would even give an error term of shape $O(p)$ instead of $O(p^2\log p)$ in (78).

Those results are in some sense (hopefully sharp) special cases of the main results of [Pazuki 2012], after rewriting the $j$-function in terms of classical $\Theta$.

*Proof.* Using Theorem 4.1, (51), Proposition 4.4 and (15) we obtain

$$\mathrm{h}_j(P) \leq 12\frac{p+1}{p-13}\mathrm{h}_\Theta\big(P - \infty + \tfrac{1}{2}\omega^0\big) + O(p^2 \log p).$$

The last estimate (78) of the theorem comes from the fact that $\mathrm{h}_\Theta$ is a quadratic form and that

$$\mathrm{h}_\Theta(\omega^0) = O(\log p) \tag{79}$$

by the results of [Michel and Ullmo 1998] now many times mentioned. □

## 5. Height of modular curves and the various $W_d$

We prove in this section a certain number of technical results about heights of cycles in the modular jacobian, which will be useful in the sequel. For applications of the explicit arithmetic Bézout theorem displayed in next section (Proposition 6.1), we indeed first need estimates for the degree and height of the image of $X_0(p)$, together with its various $d$-th symmetric-products (usually called "$W_d$"), within either $J_0(p)$ or its quotient $J_e$, relative to the $\Theta$-polarization. (For more general considerations on this topic, we also refer to [de Jong 2018].) We estimate those heights both in the normalized Néron–Tate sense and for some good ("Moret-Bailly") projective models, to be defined shortly.

Let us first define the height of cycles relative to some hermitian bundle. For further details on this we refer to [Zhang 1995], or to [Abbes 1997, Section 2] for a more informal introduction.

**Definition 5.1.** Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathcal{X}$ be an arithmetic scheme over $\mathcal{O}_K$, that is an integral scheme which is projective and flat over $\mathcal{O}_K$, having smooth generic fiber $X$ over $K$. Let $\mathcal{F}$ be a generically ample and relatively semiample hermitian sheaf with smooth metric, see [Zhang 1995, Section 5]. We denote by $\hat{c}_1(\mathcal{F})$ the first arithmetic Chern class of $\mathcal{F}$, and similarly by $c_1(F)$ the first Chern class of $F$.

Such a pair $(\mathcal{X}, \mathcal{F})$ will be called a model, in the sense of Zhang, of its pull-back $(X, F) = (\mathcal{X}_K, \mathcal{F}_K)$ to the generic fiber.

Consider a model $(\mathcal{X}, \mathcal{F})$ as in Definition 5.1, and let $Y$ be a $d$-dimensional subvariety of $X$. The degree of $Y$ with respect to $F$ is as usual the nonnegative integer given by the $d$-th power self-intersection of $c_1(F)$ with $Y$, that is

$$\deg_F(Y) = (c_1(F)^d | Y).$$

We shall sometimes also write that quantity as $\deg_{\mathcal{F}}(Y)$.

Now let $\mathcal{Y} \to \mathcal{X}$ be some "generic resolution of singularities" of $Y$ (that is, some good integral model for some desingularization of $Y$, see Section 1 of [Zhang 1995]). The height of $Y$ with respect to $\mathcal{F}$ will similarly be the real number obtained by taking the $(\dim \mathcal{Y})$-th power self-intersection of $\hat{c}_1(\mathcal{F})$ with $\mathcal{Y}$, divided by the degree of $Y$ and normalized so that

$$\mathrm{h}_{\mathcal{F}}(Y) = \frac{(\hat{c}_1(\mathcal{F})^{d+1} | \mathcal{Y})}{[K:\mathbb{Q}](d+1)\deg_F(Y)}. \tag{80}$$

One can check that definition[5] does not depend on the desingularization $\mathcal{Y} \to \mathcal{X}$.

Instrumental to us will here be Zhang's control of heights in terms of essential minima. Recall that the (first) essential minimum $\mu^{\mathrm{ess}}_{\mathcal{F}}(Y)$ of $Y$ is the minimum of the set of real numbers $\mu$ such that there is a sequence of points $(x_n)$ in $Y(\overline{\mathbb{Q}})$ which is Zariski dense in $Y$ and $\mathrm{h}_{\mathcal{F}}(x_n) \leq \mu$ for all $n$. Zhang's theorem [1995, (5.2)] then asserts that

$$\mathrm{h}_{\mathcal{F}}(Y) \leq \mu^{\mathrm{ess}}_{\mathcal{F}}(Y). \tag{81}$$

Note that if $\mathrm{h}_{\mathcal{F}} \geq 0$ on $Y(\overline{\mathbb{Q}})$ one also knows from [Zhang 1995, Theorem 5.2] the reverse inequality

$$\mathrm{h}_{\mathcal{F}}(Y) \geq \frac{\mu^{\mathrm{ess}}_{\mathcal{F}}(Y)}{d+1}. \tag{82}$$

If $(\mathcal{X}, \mathcal{F})$ is a model over $\mathcal{O}_K$, in the sense of Definition 5.1, of a polarized abelian variety $(X, F)$ over $K = \mathrm{Frac}(\mathcal{O}_K)$, and $Y$ again is a $d$-dimensional subvariety of the generic fiber $X$, we still define its normalized Néron–Tate height relative to $F$ as the limit

$$\mathrm{h}_F(Y) := \lim_{n\to\infty} \frac{1}{N^{2n}} \mathrm{h}_{\mathcal{F}}([N^n]Y)$$

where $N$ is any fixed integer larger than 1 and $[N^n]Y$ is the image of $Y$ under multiplication by $N^n$ in $X$. This normalized height, which is a direct generalization of the classical notion of Néron–Tate height for points, is known not to depend neither on the model $\mathcal{X}$ of $X$, nor the extension $\mathcal{F}$ of $F$, nor its hermitian structure (and not on $N$), so that the notation $\mathrm{h}_F(\cdot)$ is finally unambiguous. We refer to [Abbes 1997], Proposition-Définition 3.2 of Section 3 for more details. We will actually use the extension of the two inequalities (81) and (82) to the case where the heights and essential minima are those given by the limit process defining Néron–Tate height (which is known to be nonnegative on points) that is, with obvious notations

$$\frac{\mu^{\mathrm{ess}}_{F}(Y)}{d+1} \leq \mathrm{h}_F(Y) \leq \mu^{\mathrm{ess}}_{F}(Y), \tag{83}$$

see Théorème 3.4 of [Abbes 1997]. As we will see in Section 5C and below, Moret-Bailly theory allows, under certain conditions, to interpret Néron–Tate heights as Arakelov projective heights (that is, without going through limit process).

**5A.** ***Néron–Tate heights.*** We shall apply the above to cycles in modular abelian varieties endowed with their symmetric theta divisor: the notation $\mathrm{h}_\Theta$ will always stand for normalized Néron–Tate height of cycles.

**Proposition 5.2.** *Let $X$ be the image via $\pi_A \circ \iota_\infty : X_0(p) \to A$ of the modular curve $X_0(p)$ mapped to a nonzero quotient $\pi_A : J_0(p) \to A$ of its jacobian, endowed with the polarization $\Theta_A$ induced by the*

[5] It could have been simpler to systematically use the definition of height of [Bost et al. 1994, Section 3.1] which does not demand desingularization, as we do in the proof of Proposition 6.1 at the end of Section 6. We could not find references however for Zhang's inequality (see (81)) in that setting, so we stick to the above definitions.

*$\Theta$-divisor* (*see* (4), (9) *and around*). *The degree and normalized Néron–Tate height of $X$ satisfy*

$$\deg_{\Theta_A}(X)=\dim(A)=O(p) \quad \textit{and} \quad \mathrm{h}_{\Theta_A}(X)=O(\log p).$$

*Proof.* If $(A,\Theta_A)=(\mathrm{Jac}(X_0(p)),\Theta)$, it is well-known that the $\Theta$-degree of $X_0(p)$ (or in fact any curve) embedded in its jacobian via some Albanese embedding, equals its genus. That can be seen in many ways, among which one can invoke Wirtinger's theorem [Griffiths and Harris 1978, p. 171], which yields in fact the desired result for any quotient $(A,\Theta_A)$; using the notation before (12) we have

$$\deg_{\Theta_A}(X)=\int_{X_0(p)}\sum_{f\in B_2^A}\frac{i}{2}\frac{f\frac{dq}{q}\wedge\overline{f\frac{dq}{q}}}{\|f\|^2}=\dim A\le g(X_0(p)).$$

We then apply once more the fact (15) that the genus $g(X_0(p))$ is roughly $p/12$. (We could also more simply say that the degree is decreasing by projection, as in the argument below.)

As for the height, the main result of [Michel and Ullmo 1998] gives that the essential minimum of the normalized Néron–Tate height $\mu_{\Theta}^{\mathrm{ess}}(X_0(p))$ is $O(\log p)$. As the height of points decreases by projection (see Section 2A2 and in particular (7)) the same is true for $\mu_{\Theta_A}^{\mathrm{ess}}(X)$ and we conclude with Zhang's (83). □

Now for the Néron–Tate normalized height of symmetric squares and variants.

**Proposition 5.3.** *Assume $X:=X_0(p)$ has gonality strictly larger than* 2 (*which is true as soon as $p>71$, see [Ogg 1974]*). *Let $\iota:=\iota_\infty : X_0(p)\hookrightarrow J_0(p)$ be the Albanese embedding as in Proposition 5.2. Let $X^{(2)}$ be the symmetric square $X_0(p)^{(2)}$ embedded in $J_0(p)$ via $(P_1,P_2)\mapsto\iota(P_1)+\iota(P_2)$, and similarly let $X^{(2),-}$ be the image of $(P_1,P_2)\mapsto\iota(P_1)-\iota(P_2)$. Let $X^{(2)}_{e^\perp}$ and $X^{(2),-}_{e^\perp}$ be the projections of $X^{(2)}$ and $X^{(2),-}$, respectively, to $J_e^\perp$ (the "orthogonal complement" to the winding quotient $J_e$; see Section 2B3). Then with notations as in Proposition 5.2 taking $A=J_0(p)$ and $A=J_e^\perp$ respectively one has*

$$\deg_\Theta(X^{(2)})=O(p^2)=\deg_\Theta(X^{(2),-}),\quad \mathrm{h}_\Theta(X^{(2)})=O(\log p)=\mathrm{h}_\Theta(X^{(2),-})$$

*and the same holds for the quotient objects*

$$\deg_{\Theta_e^\perp}(X^{(2)}_{e^\perp})=O(p^2)=\deg_{\Theta_e^\perp}(X^{(2),-}_{e^\perp}),\quad \mathrm{h}_{\Theta_e^\perp}(X^{(2)}_{e^\perp})=O(\log p)=\mathrm{h}_{\Theta_e^\perp}(X^{(2),-}_{e^\perp}).$$

*Proof.* Denoting by $p_1$ and $p_2$ the obvious projections below we factor in the common way (see [Mumford 1966], paragraph 3, Proposition 1 on p. 320) our maps over $\mathbb{Q}$ as follows:

$$\begin{array}{ccccccc} & & & & & \nearrow^{p_2} & A \\ X_0(p)\times X_0(p) & \xrightarrow{\pi_A\iota\times\pi_A\iota} & A\times A & \xrightarrow{M} & A\times A & & \\ & & (x,y) & \longmapsto & (x+y,\,x-y) & \searrow_{p_1} & A \end{array} \tag{84}$$

so $X^{(2)}=p_1\circ M\circ(\pi_A\iota\times\pi_A\iota)(X_0(p)\times X_0(p))$ and $X^{(2),-}=p_2\circ M\circ(\pi_A\iota\times\pi_A\iota)(X_0(p)\times X_0(p))$ when $A=J_0(p)$, and the same with $X^{(2)}_{e^\perp}$ and $X^{(2),-}_{e^\perp}$ with $A=J_e^\perp$. We endow $A\times A$ with the hermitian sheaf

${\Theta_A}^{\boxtimes 2} := p_1^*\Theta_A \otimes p_2^*\Theta_A$. Then $M^*({\Theta_A}^{\boxtimes 2}) \simeq ({\Theta_A}^{\boxtimes 2})^{\otimes 2}$ [Mumford 1966, p. 320]. Therefore, writing $X$ for $\pi_A \iota(X_0(p))$ in short and using Proposition 5.2,

$$\deg_{{\Theta_A}^{\boxtimes 2}}(M(X \times X)) = 4 \deg_{{\Theta_A}^{\boxtimes 2}}(X \times X) = 8(\deg_{\Theta_A}(X))^2 = O(g^2).$$

As degree decreases by our projections and $O(g^2) = O(p^2)$, $\deg_{\Theta_A}(X^{(2)})$ and $\deg_{\Theta_A}(X^{(2),-})$ are $O(p^2)$.

By definition of essential minima,

$$\mu^{\mathrm{ess}}_{{\Theta_A}^{\boxtimes 2}}(X \times X) \leq 2\mu^{\mathrm{ess}}_{\Theta_A}(X).$$

This implies that $\mu^{\mathrm{ess}}_{{\Theta_A}^{\boxtimes 2}}(M(X \times X)) \leq 4\mu^{\mathrm{ess}}_{\Theta_A}(X)$. Invoking (83) again and Proposition 5.2 together with the fact that the height of points also decreases by projection,

$$\mu^{\mathrm{ess}}_{\Theta_A}(X^{(2)}) \leq \mu^{\mathrm{ess}}_{{\Theta_A}^{\boxtimes 2}}(M(X \times X)) \leq 4\mu^{\mathrm{ess}}_{\Theta_A}(X) \leq 8\mathrm{h}_{\Theta_A}(X) \leq O(\log p).$$

Therefore

$$\mathrm{h}_{\Theta_A}(X^{(2)}) = O(\log p). \qquad \square$$

Note that this proof applies more generally to any subquotient of $J_0(p)$.

**5B.** ***Moret-Bailly models and associated projective heights.*** To build-up the projective models of the jacobian (over $\mathbb{Z}$, or finite extensions), and associated heights, that we shall need for our arithmetic Bézout, we use Moret-Bailly theory, in the sense of [Moret-Bailly 1985b], as follows. For more about similar constructions in the general setting of abelian varieties we refer to [Bost 1996, 2.4 and 4.3]; see also [Pazuki 2012].

Let therefore $(J, L(\Theta))$ stand for the principally polarized abelian variety $J_0(p)$ endowed with the invertible sheaf associated with its symmetric theta divisor, defined over some small extension of $\mathbb{Q}$ (see (89) below and around for more details). Endow the complex base-changes of the associated invertible sheaf $L(\Theta)$ with its cubist hermitian metric. If $\mathcal{N}_{J,\mathcal{O}_K}$ is the Néron model of $J$ over the ring of integers $\mathcal{O}_K$ of a number field $K$, we know it is a semistable scheme over $\mathcal{O}_K$, whose only nonproper fibers are above primes $\mathfrak{P}$ of characteristic $p$, where it then is purely toric. At any such $\mathfrak{P}$, with ramification index $e_{\mathfrak{P}}$, the group scheme $\mathcal{N}_{J,\mathcal{O}_K}$ has components group

$$\Phi_{\mathfrak{P}} \simeq (\mathbb{Z}/N_0 e_{\mathfrak{P}}\mathbb{Z}) \times (\mathbb{Z}/e_{\mathfrak{P}}\mathbb{Z})^{g-1} \tag{85}$$

for $g := \dim J$ and $N_0 := \mathrm{num}((p-1)/12)$ (see, e.g., [Le Fourn 2016, Proposition 2.11]).

We choose and fix an integer $N > 0$ and a number field $K \supseteq \mathbb{Q}(J[2N])$, for all this paragraph, so that all the $2N$-torsion points in $J$ have values in $K$. One then observes from (85) that $2N$ divides all the ramification indices $e_{\mathfrak{P}}$, and Proposition II.1.2.2 on p. 45 of [Moret-Bailly 1985b] asserts that $L(\Theta)$ has a cubist extension, let us denote it by $\mathcal{L}(\Theta)$, to the open subgroup scheme $\mathcal{N}_{J,N}$ of the Néron model $\mathcal{N}_{J,\mathcal{O}_K}$ over $\mathcal{O}_K$ whose fibers have component group killed by $N$.

Such an extension $\mathcal{L}(\Theta)$ is actually symmetric [Moret-Bailly 1985b, Remarque II.1.2.6.2] and unique (see Théorème II.1.1.i on p. 40 of [loc. cit.]). Moreover $\mathcal{L}(\Theta)$ is ample on $\mathcal{N}_{J,N}$ [loc. cit., Proposition VI.2.1

on p. 134]. Its powers $\mathcal{L}(\Theta)^{\otimes r}$ are even very ample on $\mathcal{N}_{J,N} \times_{\mathcal{O}_K} \mathcal{O}_K[1/2p]$ as soon as $r \geq 3$, as follows from the general theory of theta functions. Provided $N > 1$, the sheaf $\mathcal{L}(\Theta)^{\otimes N}$ is spanned by its global sections on the whole of $\mathcal{N}_{J,N}$ [loc. cit., Proposition VI.2.2], although we shall not use that last fact as such.

Picking-up a basis of *generic* global sections in $H^0(J_0(p)_K, L(\Theta)^{\otimes N})$, with $N \geq 3$, we thus defines a map $J_0(p)_K \xrightarrow{\jmath_N} \mathbb{P}^n_K$, for $n = N^g - 1$. Assume our generic global sections extend to a set $\mathcal{S}$ in $H^0(\mathcal{N}_{J,N}, \mathcal{L}(\Theta)^{\otimes N})$. Let $\mathcal{J} \overset{\jmath}{\hookrightarrow} \mathbb{P}^n_{\mathcal{O}_K}$ be the schematic closure in $\mathbb{P}^n_{\mathcal{O}_K}$ of the generic fiber $(\mathcal{N}_{J,N})_K = J_K$ via the associated composed embedding $J_K \hookrightarrow \mathbb{P}^n_K \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$. Define $\mathcal{M} = \jmath^* \mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)$ on $\mathcal{J}$. Let on the other hand $\mathcal{M}_{\mathcal{N}_{J,N}} := \left(\sum_{s \in \mathcal{S}} \mathcal{O}_K \cdot s\right)$ be the subsheaf of $\mathcal{L}(\Theta)^{\otimes N}$ on $\mathcal{N}_{J,N}$ spanned by $\mathcal{S}$. Write $\nu : \widetilde{\mathcal{N}}_{J,N} \to \mathcal{N}_{J,N}$ for the blowup at base points for $\mathcal{M}_{\mathcal{N}_{J,N}}$ on $\mathcal{N}_{J,N}$, that is, the blowup along the closed subscheme of $\mathcal{N}_{J,N}$ defined by the sheaf $\mathcal{L}(\Theta)^{\otimes N}/\mathcal{M}_{\mathcal{N}_{J,N}}$. We have a commutative diagram

$$\begin{array}{ccccc} & & \widetilde{\mathcal{N}}_{J,N} & & \\ & \nearrow & \downarrow{\scriptstyle \iota_{\mathcal{N}}} & \searrow{\scriptstyle \jmath_{\mathcal{N}}} & \\ J_K & \hookrightarrow & \mathcal{J} & \overset{\jmath}{\hookrightarrow} & \mathbb{P}^n_{\mathcal{O}_K} \end{array} \tag{86}$$

where the only nontrivial map $\jmath_{\mathcal{N}}$ (whence $\iota_{\mathcal{N}}$) is deduced from the fundamental properties of blowups. Considering the complex base-changes of the generic fiber we note that $\mathcal{M}$ is automatically endowed with a cubist hermitian structure induced by that of $L(\Theta)_{\mathbb{C}}$ (see [Bost 1996], (4.3.3) and following lines).

**Definition 5.4.** Given an integer $N \geq 3$, and a number field $K$ containing $\mathbb{Q}(J_0(p)[2N])$, we define the "good model" for $(J_0(p), L(\Theta)^{\otimes N})$ relative to some finite set $\mathcal{S}$ in $H^0(\mathcal{N}_{J,N}, \mathcal{L}(\Theta)^{\otimes N})$, which spans $H^0(J_0(p), L(\Theta)^{\otimes N})$, as the projective scheme $\mathcal{J}$ over $\mathrm{Spec}(\mathcal{O}_K)$ enhanced with the hermitian sheaf $\mathcal{M}$ constructed above, and $\mathrm{h}_{\mathcal{M}}$ the associated height.

Outside base points for $\mathcal{M}_{\mathcal{N}_{J,N}}$ on $\mathcal{N}_{J,N}$ the blowup $\nu : \widetilde{\mathcal{N}}_{J,N} \to \mathcal{N}_{J,N}$ is an isomorphism and on that open locus we have

$$\mathcal{L}(\Theta)^{\otimes N} \simeq \mathcal{M}_{\mathcal{N}_{J,N}} \simeq \iota_{\mathcal{N}}^* \mathcal{M} = \jmath_{\mathcal{N}}^* \mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1) \tag{87}$$

so we dwell on the fact that the height $\mathrm{h}_{\mathcal{M}}$ of our "good models" for $(J_0(p), L(\Theta)^{\otimes N})$ will indeed compute ($N$ times) the Néron–Tate height of *certain* $\overline{\mathbb{Q}}$-points (those whose closure factorizes through $\mathcal{N}_{J,N}$ deprived from the base points for $\mathcal{S}$), but definitely *not all*. For arbitrary points, still, one can deduce from the work of Bost ([Bost 1996], 4.3) the following inequality.

**Proposition 5.5.** *For any point $P$ in $J_0(p)(\overline{\mathbb{Q}})$, the height $\mathrm{h}_{\mathcal{M}}(P)$ of Definition 5.4 satisfies*

$$\mathrm{h}_{\mathcal{M}}(P) \leq N \mathrm{h}_{\Theta}(P).$$

*Proof.* We briefly adapt [Bost 1996, 2.4 and 4.3] using our above notations. Of course this statement has nothing to see with modular jacobians, and holds for any abelian variety over a number field. Let $N'$ be some integer such that $P$ defines a section of $\mathcal{N}_{J,N'}(\mathcal{O}_F)$ for some ring of integers $\mathcal{O}_F$. Up to replacing

$\mathcal{O}_F$ by a sufficiently ramified finite extension, we can assume $L(\Theta)^{\otimes N}$ has a cubist extension $\mathcal{L}(\Theta)^{\otimes N}$ to all of $\mathcal{N}_{J,N'}$ over $\mathcal{O}_F$ [Moret-Bailly 1985b, Proposition II.1.2.2]. One has

$$\mathrm{h}_{\Theta}(P) = \frac{1}{N}\frac{1}{[F:\mathbb{Q}]}\widehat{\deg}(P^*(\mathcal{L}(\Theta)^{\otimes N})).$$

As in (86) however we see that there is no well-defined map from $\mathcal{N}_{J,N'}$ to $\mathbb{P}^n_{\mathcal{O}_F}$ because $\mathcal{L}(\Theta)^{\otimes N}$ needs not be spanned by elements of $\mathcal{S}$ on all of $\mathcal{N}_{J,N'}$ (even though it is, by hypothesis, on the generic fiber). To remedy this we adapt the construction (86).

If $\pi' : \mathcal{N}_{J,N'} \to \operatorname{Spec}(\mathcal{O}_F)$ is the structural morphism, we define now $\mathcal{M}'_{\mathcal{N}} := \left(\sum_{s\in\mathcal{S}} \mathcal{O}_F \cdot s\right)$ as the subsheaf of $\mathcal{L}(\Theta)^{\otimes N}$ on $\mathcal{N}_{J,N'}$ spanned by $\mathcal{S}$, still endowed with the metric induced by that of $\mathcal{L}(\Theta)^{\otimes N}$. One checks (see [Bost 1996, (4.3.8)]) that the projective model $\mathcal{J}_{\mathcal{O}_F}$ of $(\mathcal{N}_{J,N'})_F \simeq J_F$ in $\mathbb{P}^n_{\mathcal{O}_F}$ defined as in (86) yields a sheaf $\mathcal{M}'$ on $\mathcal{J}_{\mathcal{O}_F}$, whence a height $\mathrm{h}_{\mathcal{M}'}$, which coincides with the height $\mathrm{h}_{\mathcal{M}}$ on the base change of the good model $\mathcal{J}_{\mathcal{O}_K}$.

Replacing $\mathcal{N}_{J,N'}$ by its blowup $\nu' : \widetilde{\mathcal{N}}_{J,N'} \to \mathcal{N}_{J,N'}$ at base points for $\mathcal{M}'_{\mathcal{N}}$ in $\mathcal{L}(\Theta)^{\otimes N}$ on $\mathcal{N}_{J,N'}$, we keep on following construction (86) to obtain maps $\iota'_{\mathcal{N}} : \widetilde{\mathcal{N}}_{J,N'} \to \mathcal{J}_{\mathcal{O}_F}$ and $\jmath'_{\mathcal{N}} : \widetilde{\mathcal{N}}_{J,N'} \to \mathbb{P}^n_{\mathcal{O}_F}$ such that the Zariski closure of $\jmath'_{\mathcal{N}}(\widetilde{\mathcal{N}}_{J,N'})$ identifies with $\mathcal{J}_{\mathcal{O}_F}$. We moreover have

$$\iota'^*_{\mathcal{N}}(\mathcal{M}') = \nu'^*(\mathcal{L}(\Theta)^{\otimes N}) \otimes \mathcal{O}(-E)$$

where $E$ is the exceptional divisor of the blowup which is by definition effective. The section $P$ of $\mathcal{N}_{J,N'}(\mathcal{O}_F)$ lifts to some $\tilde{P}$ of $\widetilde{\mathcal{N}}_{J,N'}(\mathcal{O}_F)$. Let $\varepsilon_P$ be the section of $\mathcal{J}(\mathcal{O}_F)$ defined by the Zariski closure of $P(F)$ in $\mathcal{J}$. One can finally compute

$$\begin{aligned}\mathrm{h}_{\mathcal{M}}(P) = \mathrm{h}_{\mathcal{M}'}(P) &= \frac{1}{[F:\mathbb{Q}]}\widehat{\deg}(\varepsilon_P^*(\mathcal{M}')) = \frac{1}{[F:\mathbb{Q}]}\widehat{\deg}(\tilde{P}^*(\iota'^*_{\mathcal{N}}(\mathcal{M}'))) \\ &\le \frac{1}{[F:\mathbb{Q}]}\widehat{\deg}(\tilde{P}^*(\nu'^*(\mathcal{L}(\Theta)^{\otimes N}))) = \frac{1}{[F:\mathbb{Q}]}\widehat{\deg}(P^*(\mathcal{L}(\Theta)^{\otimes N})) = N\,\mathrm{h}_{\Theta}(P). \quad \square\end{aligned}$$

The following straightforward generalization to higher dimension will be useful in next section.

**Corollary 5.6.** *If $Y$ is a $d$-dimensional irreducible subvariety of $J_0(p)$ then*

$$\mathrm{h}_{\mathcal{M}}(Y) \le (d+1)\,N\,\mathrm{h}_{\Theta}(Y).$$

*Proof.* Combine Zhang's formulas (81) and (83) with Proposition 5.5. □

Recall from (8) that one can define the "pseudoprojection" $\mathcal{P}_{\tilde{J}_{e^\perp}}(\iota_\infty(X_0(p)))$ of the image of $X_0(p) \xhookrightarrow{\iota_\infty} J_0(p)$ on the subabelian variety $\tilde{J}_{e^\perp} \subseteq J_0(p)$. Let $X_{e^\perp}$ be any of its irreducible components. Define similarly $X^{(2)}$, $X^{(2),-}$, $X^{(2)}_{e^\perp}$ and $X^{(2),-}_{e^\perp}$ as in Proposition 5.3. Note that, by construction, the degree and normalized Néron–Tate height of $X_{e^\perp}$ (and other similar pseudoprojections: $X^{(2)}_{e^\perp}$ etc.), as an irreducible subvariety of $J_0(p)$ endowed with $\mathrm{h}_\Theta$, are those of $\pi_{J_e^\perp}(X_0(p)) = X^{(2),-}_{e^\perp}$ relative to the only natural hermitian sheaf of $J_e^\perp$, that is, the $\Theta_e^\perp = \Theta_{J_e^\perp}$ described in paragraph 2A2 and estimated in Proposition 5.2.

**Corollary 5.7.** *For any fixed integer $N \ge 3$, and any number field $K$ containing $\mathbb{Q}(J_0(p)[2N])$, let $(\mathcal{J}, \mathcal{M})$ be the good model for $(J_0(p), L(\Theta)^{\otimes N})$, and $\mathrm{h}_{\mathcal{M}}$ the associated projective height, given in*

*Definition 5.4. Let $X$ be the image of $X_0(p) \stackrel{\iota_\infty}{\hookrightarrow} J_0(p)$, and more generally $X^{(2)}$, $X^{(2),-}$, $X^{(2)}_{e^\perp}$ and $X^{(2),-}_{e^\perp}$ be the objects $X^{(2)}, \dots$ defined in Proposition 5.3 (or their pseudoprojections). Then their $\mathcal{M}^{\otimes \frac{1}{N}}$-heights are bounded from above by similar functions as their Néron–Tate height (Proposition 5.3). Explicitly, $\mathrm{h}_{\mathcal{M}^{\otimes \frac{1}{N}}}(X_0(p))$ is less than $O(\log p)$, and $\mathrm{h}_{\mathcal{M}^{\otimes \frac{1}{N}}} X^{(2)}$, etc., are all less than $O(\log p)$. Similarly the $\mathcal{M}^{\otimes \frac{1}{N}}$-degree of $X_0(p)$ is $O(p)$, and the $\mathcal{M}^{\otimes \frac{1}{N}}$-degrees of $X^{(2)}$, etc., are all $O(p^2)$.*

*Proof.* Combine Zhang's formulas (81) and (83) with Propositions 5.2, 5.3 and 5.5. □

**5C. *Estimates on Green–Zhang functions for $J_0(p)$.*** We shall later on need some control on the $p$-adic Néron–Tate metric of $\Theta$ as alluded to in Remark 4.3. (Those statements can probably be best formulated in the setting of Berkovich theory, for which one might check in particular [Ducros 2007, Proposition 2.12] and [Thuillier 2005]. A useful point of view is also proposed by that of "tropical jacobians", see [Mikhalkin and Zharkov 2008; de Jong and Shokrieh 2018]. We will content ourselves here with our down-to-earth point of view). We therefore define

$$\hat{\Phi}_p := \varinjlim_{K_{\mathfrak{P}} \supseteq \mathbb{Q}_p} \Phi_{\mathfrak{P}}$$

as the direct limit, on a tower of totally ramified extensions $K_{\mathfrak{P}}/\mathbb{Q}_p$, of the component groups $\Phi_{\mathfrak{P}}$ of the Néron models of $J_0(p)$ at $\mathfrak{P}$, see (85). The compatible embeddings

$$Z := \langle C_0 - C_\infty \rangle \simeq \langle (0) - (\infty) \rangle \simeq \mathbb{Z}/N_0\mathbb{Z} \hookrightarrow \Phi_{\mathfrak{P}}$$

for each $\mathfrak{P}$ induce an exact sequence $0 \to Z \to \hat{\Phi}_p \to \varinjlim_{e_{\mathfrak{P}}} (\mathbb{Z}/e_{\mathfrak{P}}\mathbb{Z})^g \simeq (\mathbb{Q}/\mathbb{Z})^g \to 0$. Passing to the real completion yields a presentation:

$$0 \to Z \simeq \mathbb{Z}/N_0\mathbb{Z} \to \hat{\Phi}_{p,\mathbb{R}} \to (\mathbb{R}/\mathbb{Z})^g \to 0 \tag{88}$$

(where $\hat{\Phi}_{p,\mathbb{R}}$ must be the "skeleton", in the sense of Berkovich, of the Néron model over $\overline{\mathbb{Z}}_p$ of $J_0(p)$, and the tropical jacobian, see [de Jong and Shokrieh 2018], of the curve $X_0(p)$ above $p$). The right-hand side of (88) is more canonically written $(\mathbb{R}/\mathbb{Z})^g \simeq (\mathbb{R}/\mathbb{Z})^s/\Delta(\mathbb{R})$, for $\Delta$ the almost diagonal map [Le Fourn 2016, Proposition 2.11.(c)]

$$\Delta(z) \mapsto \left(\tfrac{1}{w_i} z\right)_{1 \le i \le g+1}.$$

We then sum up useful properties about theta divisors and theta functions "over $\overline{\mathbb{Z}}$".

As $J_0(p)$ is principally polarized over $\mathbb{Q}$, the complex extension of scalars $J_0(p)(\mathbb{C})$ can be given a classical complex uniformization $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ for some $\tau$ in Siegel's upper half-plane. The associated Riemann theta function

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp(i\pi {}^t m \cdot \tau \cdot m + 2i\pi {}^t m \cdot z) \tag{89}$$

defines the tautological global section 1 of a trivialization of $\mathcal{O}_{J_0(p)}(\Theta_{\mathbb{C}})(= \mathcal{M}_{\mathbb{C}}^{\otimes 1/N})$ for $\Theta_{\mathbb{C}}$ the image $W_{g-1}$ of some $(g-1)$-st power of $X_0(p)$ in $J_0(p)$. More precisely, Riemann's classical results (e.g., [Griffiths and Harris 1978], Theorem on p. 338) assert that $\mathrm{div}(\theta(z)) = \Theta_{\mathbb{C}}$ is the divisor with support

$\{\kappa_{P_0}+\sum_{i=1}^{g-1}\iota_{P_0}(P_i),\ P_i\in X_0(p)(\mathbb{C})\}$, where for any $P_0\in X_0(p)(\mathbb{C})$ we write $\iota_{P_0}:X_0(p)\hookrightarrow J_0(p)$ for the Albanese morphism with base point $P_0$, and $\kappa=\kappa_{P_0}=$ "$\frac{1}{2}(\iota_{P_0}(K_{X_0(p)}))$" for the image of Riemann's characteristic, which is some preimage under duplication in $J_0(p)$ of the image of some canonical divisor: $\omega^0=\iota_{P_0}(K_{X_0(p)})$ (see Theorem 4.6 above).

Among the translates $\Theta_D=t_D^*\Theta$, for $D\in J_0(p)(\mathbb{C})$, of the above symmetric $\Theta$, the divisor $\Theta_\kappa=t_\kappa^*\Theta=\sum_{i=1}^{g-1}\iota_\infty(X_0(p)_{\mathbb{Q}})$ defines an invertible sheaf $L(\Theta_\kappa)$ on $J_0(p)$ over $\mathbb{Q}$. If $\mathcal{N}_{J,1}$ denotes the neutral component of the Néron model of $J$ over $\mathbb{Z}$ and $\mathcal{L}(\Theta_\kappa)$ is the cubist extension of $L(\Theta_\kappa)$ to $\mathcal{N}_{J,1}$ (compare [Moret-Bailly 1985b, Proposition II.1.2.2], as in Section 5B above), we know that $H^0(\mathcal{N}_{J,1},\mathcal{L}(\Theta_\kappa))$ is a (locally) free $\mathbb{Z}$-module of rank 1, so that the complex base-change $H^0(J_0(p)(\mathbb{C}),L(\Theta_{\kappa,\mathbb{C}}))$ is similarly a complex line. This means that if $s_\theta$ is a generator of the former space, whose image in the later we denote by $s_{\theta,\mathbb{C}}$, there is a nonzero complex number $C_\vartheta$ such that

$$s_{\theta,\mathbb{C}}(z)=C_\vartheta\cdot\theta(z+\kappa). \tag{90}$$

Up to making some base-change from $\mathbb{Z}$ to some $\mathcal{O}_K$ we can now forget about $\kappa$ and come back to the symmetric $\Theta$; we define a global section

$$s_{\mathcal{J}^0}:=(t_{-\kappa}^*)s_\theta\in H^0(\mathcal{N}_{J,1},\mathcal{L}(\Theta)_{\mathcal{O}_K})\quad\text{so that}\quad s_{\mathcal{J}^0,\mathbb{C}}(z)=C_\vartheta\cdot\theta(z). \tag{91}$$

If one replaces $\mathcal{N}_{J,1}$ by the Néron model, say $\mathcal{N}_{\mathcal{O}_{K_1}}$, of $J_0(p)$ over any extension $K_1$ of $K$, then [Moret-Bailly 1985b, Proposition II.1.2.2] insures that up to making some further field extension $K_2/K_1$ the sheaf $L(\Theta)_{K_2}$ has a cubist extension $\mathcal{L}(\Theta)_{\mathcal{O}_{K_2}}$ to $\mathcal{N}_{\mathcal{O}_{K_1}}\times_{\mathcal{O}_{K_1}}\mathcal{O}_{K_2}$. Therefore $s_{\mathcal{J}^0}$ extends to a *rational* section (we shall sometimes write *meromorphic* section) of $\mathcal{L}(\Theta)_{\mathcal{O}_{K_2}}$ on $\mathcal{N}_{\mathcal{O}_{K_1}}\times_{\mathcal{O}_{K_1}}\mathcal{O}_{K_2}$. Abusing notations we still denote that extended section by $s_{\mathcal{J}^0}$, and write accordingly $\Theta$ for its divisor $\operatorname{div}(s_{\mathcal{J}^0})$ on $\mathcal{N}_{\mathcal{O}_{K_1}}\times_{\mathcal{O}_{K_1}}\mathcal{O}_{K_2}$. Because $s_{\mathcal{J}^0}$ is well defined (and nonzero) on the neutral component of the Néron model, its poles on $\mathcal{N}_{\mathcal{O}_{K_1}}\times_{\mathcal{O}_{K_1}}\mathcal{O}_{K_2}$ can only show-up at places of bad reduction.

**Proposition 5.8.** *The multiplicity of the $\Theta$-divisor at any component of the Néron model of $J_0(p)$ over $\overline{\mathbb{Z}}$, normalized to be* 0 *along the neutral component*, *is $O(p)$.*

*Proof.* We start by the following observations. Let us write $s_{\mathcal{J}^0,\mathbb{C}}(z)=C_\vartheta\cdot\theta(z)$ as in (91). Take $D$ in $J_0(p)(\mathbb{C})$ which can written as the linear equivalence class of some divisor

$$D=\sum_{i=1}^{g}-(Q_i-\infty)$$

for points $Q_i$ in $X_0(p)(\mathbb{C})$. We associate to $D$ the embedding

$$\iota_{\kappa+D}:X_0(p)\hookrightarrow J_0(p),\qquad P\mapsto\operatorname{cl}(P-\infty+\kappa+D)$$

where $\kappa$ is Riemann's characteristic (see just before (57)). For such a $D$ whose $Q_i$ are assumed to belong to $X_0(p)(\overline{\mathbb{Q}})$, we know from the proof of Theorem 4.1 (see (54)) that

$$\mathrm{h}_\Theta(\iota_{\kappa+D}(P))=\frac{1}{[K(P,D):\mathbb{Q}]}[P,\tilde{\omega}_D]_{\mu_0} \tag{92}$$

with

$$\tilde{\omega}_D = \sum_i Q_i + \Phi_D + c_D X_\infty \tag{93}$$

and $\Phi_D$ is the explicit vertical divisor

$$\Phi_D = \tfrac{1}{2}(\Phi_\omega + \Phi_\vartheta) - \sum_{i=1}^{g} \Phi_{Q_i} \tag{94}$$

at each bad place, with notations as those of the proof of Theorem 4.1; see (55).

Moreover, it is well known that there is a subset of $J_0(p)(\mathbb{C})$ which is open for the complex topology, and even the Zariski topology, in which all points $D = \sum_1^g -(Q_i - \infty)$ as above are such that

$$\dim_{\mathbb{C}} H^0(X_0(p)(\mathbb{C}), L(-D + g \cdot \infty)_{\mathbb{C}}) = \dim_{\mathbb{C}} H^0(X_0(p)(\mathbb{C}), \iota_{\kappa+D}^* L(\Theta_{\mathbb{C}})) = 1 \tag{95}$$

so that $\iota_{\kappa+D}^*(\Theta_{\mathbb{C}}) = \sum_i Q_{i,\mathbb{C}}$, the latter being an equality between effective divisors, not just a linear equivalence [Griffiths and Harris 1978, pp. 336–340]. As the height $\mathrm{h}_\Theta$, in the Néron model of $J_0(p)$, can be understood as the Arakelov intersection with $\Theta = \mathrm{div}(s_{\mathcal{J}^0})$ it follows that, on the curve $X_0(p)$, $\mathrm{div}(s_{\mathcal{J}^0,\mathbb{C}}) \cap \iota_{\kappa+D}(X_0(p))(\mathbb{C}) = \bigcup_i \iota_{\kappa+D}(Q_{i,\mathbb{C}})$ or $\mathrm{div}(\iota_{\kappa+D}^*(s_{\mathcal{J}^0,\mathbb{C}})) = \sum_i Q_i$ over $\mathbb{C}$. More precisely, extending base to some ring of integers $\mathcal{O}_K$ so that the $Q_i$ define sections of the minimal regular model $\mathcal{X}_0(p)_{\mathcal{O}_K}$ of $X_0(p)$ over $\mathcal{O}_K$, and making if necessary a further base extension such that $\mathcal{L}(\Theta)$ has a cubist extension on the whole Néron model of $J_0(p)$ over $\mathcal{O}_K$ (as after (91)), one sees that $s_{\mathcal{J}^0}$ defines a meromorphic section of $\mathcal{L}(\Theta)_{\mathcal{O}_K}$ and the restriction to the generic fiber $X_0(p)_K$ of $\mathrm{div}(\iota_{\kappa+D}^*(s_{\mathcal{J}^0}))$ has to be equal (and not merely linearly equivalent) to $\sum_i Q_i$. Now in such a situation, the multiplicity of $\mathrm{div}(s_{\mathcal{J}^0})$ on a component of the Néron model to which $\mathcal{X}_0(p)_{\mathcal{O}_K}^{\mathrm{smooth}}$ is mapped via $\iota_{\kappa+D}$, can be read on the multiplicity of $\iota_{\kappa+D}^*(s_{\mathcal{J}^0})$ along that component of $\mathcal{X}_0(p)_{\mathcal{O}_K}^{\mathrm{smooth}}$. In turn, because of decompositions of the arithmetic Chow group similar to that of Theorem 3.2, multiplicities of $\mathrm{div}(s_{\mathcal{J}^0})$ are determined by the $\Phi_D$ of (93), up to constant addition of vertical fibers. The property that $\mathrm{div}(s_{\mathcal{J}^0})$ has multiplicity 0 along the neutral component of the Néron model (see (91)) fixes that last indetermination. Now if $\mathfrak{P}$ is a place of bad reduction for $\mathcal{X}_0(p)_{\mathcal{O}_K}$, and if the $Q_i$ move slightly in the $\mathfrak{P}$-adic topology (without modifying their specialization component at $\mathfrak{P}$), the vertical divisor $\Phi_D$ does not change either at $\mathfrak{P}$, and the above reasoning regarding the components values of $\Theta$ is actually independent from the fact that condition (95) holds true or not (provided, we insist, that the specialization components of the $Q_i$ at $\mathfrak{P}$ do not vary).

We shall gain some flexibility with a last preliminary remark. If $k$ is any integer between 0 and $N_0 - 1$ (recall $N_0$ is the order of the Eisenstein element $(0 - \infty)$), the divisor $\tilde{\omega}_D$ of (93) can still be written as

$$\tilde{\omega}_D = \big(k \cdot 0 + (g-k) \cdot \infty - k\Phi_{C_0} + \tfrac{1}{2}(\Phi_\omega + \Phi_\vartheta) - \tilde{D}\big) + c_D X_\infty$$

so that if

$$D = \Big(\sum_{i=1}^{g} -(Q_i - \infty)\Big) + k(0 - \infty) = \sum_{i=1}^{k} -(Q_i - 0) + \sum_{i=k+1}^{g} -(Q_i - \infty)$$

then $\tilde{\omega}_D = \sum_{i=1}^{g} Q_i + \Phi_D + c_D X_\infty$ where $\Phi_D$ is still

$$\Phi_D = \tfrac{1}{2}(\Phi_\omega + \Phi_\vartheta) - \sum_{i=1}^{g} \Phi_{Q_i}. \tag{96}$$

Coming back to the proof of the present Proposition 5.8, and assuming first $D = 0$, it follows from what we have just discussed that the multiplicity of the $\Theta$-divisor on the components of the jacobian to which the components of $\mathcal{X}_0(p)^{\mathrm{smooth}}_{\mathcal{O}_K}$ map under $\iota_\kappa$ is given by the functions $g_n$ and $G$ of (45) and (46); see Theorem 4.1. To obtain the multiplicity of the $\Theta$-divisor on *all* components of the jacobian we shall shift our Albanese embeddings $\iota_{\kappa+D}$ in order to explore all of $J_0(p)/J_0(p)^0$ with successive translations of $\mathcal{X}_0(p)^{\mathrm{smooth}}_{\mathcal{O}_K}$ inside $J_0(p)$.

To be more explicit, let $\mathfrak{C}$ be an element of the component group $J_0(p)/J_0(p)^0$ at $\mathfrak{P}$, and $D = \sum_{i=1}^{g}(P_i - \infty)$ be a divisor, with all $P_i$ in $X_0(p)(K)$, which reduces to $\mathfrak{C}$ at $\mathfrak{P}$. For all $r$ in $\{1, \dots, g\}$, set $D_r = \sum_{i=1}^{r}(P_i - \infty)$ and let also $k_r$ in $\{1, \dots, N_0 - 1\}$ and $Q_{i,r}$ be $g$ associated points on the curve such that one can write both

$$D_r = \sum_{i=1}^{r}(P_i - \infty) \quad \text{and} \quad D_r = \sum_{i=1}^{g} -(Q_{i,r} - \infty) + k_r(0 - \infty).$$

As always in this proof, up to making a finite base-field extension one can assume all points have values in $K$. Recall also from the discussion above that one can move slightly the $Q_i$ in the $\mathfrak{P}$-adic topology, as all that interests us here is the component $\mathfrak{C}_r$, $1 \le r \le g$, of $(J_0(p)/J_0(p)^0)_{\mathfrak{P}}$ to which $D_r$ maps. One can therefore assume if one wishes that $\iota^*_{\kappa+D_r}(\Theta_{\mathbb{C}}) = \sum_i Q_{i,\mathbb{C}}$ (equality, not just linear equivalence). The presentation of $\Phi_{\mathfrak{P}}$ given in (88) and above also shows one can assume that the specialization components at $\mathfrak{P}$ of the $Q_{i,r}$, in $\mathcal{X}_0(p)^{\mathrm{smooth}}_{\mathcal{O}_K}$, which are not $C_\infty$, are all different (see Figure 1).

Taking first $D = 0$, that is, using the map $\iota_\kappa$, we already remarked that (94) implies the value $V_1$ of $\mathrm{div}(s_{\mathcal{J}^0})$ on $\mathfrak{C}_1$ is $V_1 = \left[\tfrac{1}{2}(\Phi_\vartheta + \Phi_\omega), P_1\right] = \tfrac{1}{2}([\Phi_\omega, P_1] + [\Phi_{P_1}]^2)$ (see (53)). By Remark 3.4 and (34), $|V_1| \le 2$.

Going one step further we reach $\mathfrak{C}_2$ by considering the Albanese image $\iota_{\kappa+D_1}(\mathcal{X}_0(p)^{\mathrm{smooth}}_{\mathcal{O}_K})$ and looking at the image of $P_2$. Here we need not to forget that the $\infty$-cusp in $X_0(p)$ now maps to $\mathfrak{C}_1$, so the normalization of components-divisor on the *curve* $\mathcal{X}_0(p)^{\mathrm{smooth}}_{\mathcal{O}_K}$ at $\mathfrak{P}$ cannot be fixed to be 0 along the $\infty$-component any longer; it needs to take the value $V_1$ found above, in order to match with the normalization of the theta divisor on the jacobian. Applying the same reasoning as before with formula (96) gives that the value of $\Theta$ on $\mathfrak{C}_2$ is

$$V_2 = \left[P_2, \tfrac{1}{2}(\Phi_\omega + \Phi_\vartheta) - \sum_{i=1}^{g} \Phi_{Q_{i,1}} + V_1\right] = \tfrac{1}{2}([\Phi_\omega, P_2] + [\Phi_{P_2}]^2) - \sum_{i=1}^{g}[\Phi_{Q_{i,1}}, P_2] + V_1$$

so that $|V_2| \le 9$ invoking Remark 3.4 again, and recalling the $Q_{i,1}$ specialize to different branches of Figure 1.

From there the inductive process is clear which yields that the value of $\Theta$ on $\mathfrak{C}_r$ has absolute value less or equal to $7r$, whence the proof of Proposition 5.8. □

**5D. *Explicit modular version of Mumford's repulsion principle.*** We conclude this section by writing-down, for later use, an explicit version of Mumford's well-known "repulsion principle" for points, in the case of modular curves.

**Proposition 5.9.** *For $P$ and $Q$ two different points of $X_0(p)(\overline{\mathbb{Q}})$ one has*

$$\mathrm{h}_\Theta(P-Q) \geq \frac{g-2}{4g}(\mathrm{h}_\Theta(P-\infty)+\mathrm{h}_\Theta(Q-\infty)) - O(p\log p). \tag{97}$$

*Proof.* Let $K$ be a number field such that both $P$ and $Q$ have values in $K$. Using notations of Section 3, the adjunction formula and Hodge index theorem give

$$\begin{aligned} 2[K:\mathbb{Q}]\mathrm{h}_\Theta(P-Q) &= -[P-Q-\Phi_P+\Phi_Q, P-Q-\Phi_P+\Phi_Q]_{\mu_0} \\ &= [P+Q,\omega]_{\mu_0} + 2[P,Q]_{\mu_0} + [\Phi_P-\Phi_Q]^2 \\ &\geq [P+Q,\omega]_{\mu_0} - 2[K:\mathbb{Q}]\sup g_{\mu_0} + [\Phi_P-\Phi_Q]^2. \end{aligned}$$

In the same way,

$$\begin{aligned} [P,\omega]_{\mu_0} &= 2[K:\mathbb{Q}]\mathrm{h}_\Theta(P-\infty) - 2[P,\infty]_{\mu_0} + [\infty]^2_{\mu_0} - [\Phi_P]^2 \\ &\geq [K:\mathbb{Q}]\mathrm{h}_\Theta\big(P-\infty+\tfrac{1}{2}\omega^0\big) - 2[P,\infty]_{\mu_0} + [\infty]^2_{\mu_0} - [\Phi_P]^2 \end{aligned}$$

where the last inequality comes from the quadratic nature of $\mathrm{h}_\Theta$, plus the fact that the error term of (97) allows us to assume $\mathrm{h}_\Theta(P-\infty) \geq 1/(12-8\sqrt{2})\mathrm{h}_\Theta(\omega^0) = O(\log p)$ (see (79) and the end of proof of Theorem 4.6). Now by (51),

$$\mathrm{h}_\Theta\big(P-\infty+\tfrac{1}{2}\omega^0\big) = \frac{1}{[K:\mathbb{Q}]}[P, g\cdot\infty]_{\mu_0} + O(\log p)$$

and using Remark 3.4 and 3.5 gives

$$[P,\omega]_{\mu_0} \geq \frac{g-2}{g}[K:\mathbb{Q}]\mathrm{h}_\Theta\big(P-\infty+\tfrac{1}{2}\omega^0\big) + [K:\mathbb{Q}]O(\log p).$$

As $[\Phi_P,\Phi_Q] = [P,\Phi_Q] = [Q,\Phi_P]$, we have $|[\Phi_P,\Phi_Q]| \leq 3[K:\mathbb{Q}]\log p$ using Remark 3.4 again. Putting everything together with Remark 4.5 about $\sup g_{\mu_0}$ we obtain

$$\mathrm{h}_\Theta(P-Q) \geq \frac{g-2}{2g}\big(\mathrm{h}_\Theta\big(P-\infty+\tfrac{1}{2}\omega^0\big) + \mathrm{h}_\Theta\big(Q-\infty+\tfrac{1}{2}\omega^0\big)\big) - O(p\log p)$$

which, by our previous remarks, can again be written as

$$\mathrm{h}_\Theta(P-Q) \geq \frac{g-2}{4g}(\mathrm{h}_\Theta(P-\infty)+\mathrm{h}_\Theta(Q-\infty)) - O(p\log p).$$ □

(For large $p$, the angle between two points of equal large enough height is here therefore at least $\arccos\frac{3}{4}-\varepsilon > \frac{\pi}{6}$. Of course the natural value is $\frac{\pi}{2}$, to which one tends when sharpening the computations.)

## 6. Arithmetic Bézout theorem with cubist metric

We display in this section an explicit version of Bézout arithmetic theorem, in the sense of Philippon [1994] or Bost, Gillet and Soulé [1994], for intersections of cycles in our modular abelian varieties over number fields, with the following variants: we use Arakelov heights (as in Section 5 above, see (80)) on higher-dimensional cycles and we endow the implicit hermitian sheaf for this height with its cubist metric (instead of Fubini–Study).

It indeed seems that one generally uses Fubini–Study metrics for arithmetic Bézout because they are the only natural explicit ones available on a general projective space (a necessary frame for the approach we follow for Bézout-like statements). They moreover have the pleasant feature that the relevant projective embeddings have tautological basis of global sections with sup-norm less than 1 which, for instance, allows for proving that the induced Faltings height is nonnegative on effective cycles [Faltings 1991, Proposition 2.6]. For our present purposes however, we need bounds for the Néron–Tate heights of points, that is, Arakelov heights induced by cubist metrics. One could in principle have tried working with Fubini–Study metrics as in [Bost et al. 1994] and then directly compare with Néron–Tate heights, but comparison terms tend to be huge. In the case of rational points, for instance (that is, horizontal cycles of relative dimension 0), within jacobians, those error terms are bounded by Manin and Zarhin [1972] linearly in the ambient projective dimension, that is exponential in the dimension of the abelian variety. In other words, for our modular curves, the error terms would be exponential in the level $p$. It is therefore much preferable to stick to cubist metrics. This implies we avoid the use of joins as in [Bost et al. 1994], as those need a sheaf metrization on the whole of the ambient projective spaces, and we instead use plain Segre embeddings. The extra numerical cost essentially consists of the appearance of modest binomial coefficients, which do not significantly alter the quantitative bounds we eventually obtain.

We also need to work with projective models which are "almost" compactifications of relevant Néron models of our jacobians. This we do with the help of Moret-Bailly theory as introduced in Section 5.

Let us recall that there still is another approach for such arithmetic Bézout theorems which uses Chow forms [Philippon 1994; Rémond 2000]. That is however known to amount to working again with Faltings' height relative to the Fubini–Study metrics [Philippon 1994; Soulé 1991] that we said we cannot afford.

Finally, regarding generality, it would of course be desirable to have a proof available for arbitrary abelian varieties. Many of the present arguments are however quite particular to our application to $J_0(p)$. We therefore prefer working in our concrete setting from the beginning, instead of considering a somewhat artificial generality.

**Proposition 6.1** (arithmetic Bézout theorem for $J_0(p)$). *Let $(J_0(p), \Theta)$ be defined over some number field $K$, endowed with the principal and symmetric polarization $\Theta$. Let $V$ and $W$ be two irreducible $K$-subvarieties of $J_0(p)$, of dimension $d_V := \dim_K V$ and $d_W := \dim_K W$, respectively, such that*

$$d_V + d_W \leq g = \dim J_0(p)$$

*and assume $V \cap W$ has dimension* 0.

*If $P$ is an element of $(V \cap W)(K)$ then its Néron–Tate $\Theta$-height satisfies*

$$\mathrm{h}_\Theta(P) \leq \frac{4^{d_V+d_W}}{2}\,\frac{(d_V+d_W+1)!}{d_V!\,d_W!}\deg_\Theta(V)\deg_\Theta(W)[(d_W+1)\mathrm{h}_\Theta(W)+(d_V+1)\mathrm{h}_\Theta(V)+O(p\log p)]. \tag{98}$$

**Remark 6.2.** The general aspect of the above release of arithmetic Bézout might look a bit different from the original ones, as can be found in [Bost et al. 1994]; this is due to the fact that our definition of the height of some cycle $Y$ (see Section 5, (80)) amounts to dividing its height in the sense of [loc. cit.] by the product of the degree and absolute dimension of $Y$.

Let us first sketch the strategy of proof, which occupies the rest of this Section 6. We henceforth fix a prime number $p$ and some perfect square integer $N := r^2$. (We shall eventually take $r = 2$.) We write $(\mathcal{J}, \mathcal{M})$ for the Moret-Bailly projective model of $(J_0(p), L(\Theta)^{\otimes N})$ given by Definition 5.4, relative to some given set of global sections $\mathcal{S}$ in $H^0(\mathcal{N}_{J,N}, \mathcal{L}(\Theta)^{\otimes N})$, of size $N^g$, to be described later (Lemma 6.5). That model is defined over some ring of integers $\mathcal{O}_K$. Consider the morphisms

$$\begin{array}{ccccc} \mathcal{J} & \xrightarrow{\Delta} & \mathcal{J}\times\mathcal{J} & & \\ & & \mathcal{P}\big\downarrow & \searrow^{\iota} & \\ & & \mathbb{P}^n_{\mathcal{O}_K}\times\mathbb{P}^n_{\mathcal{O}_K} & \xrightarrow{S} & \mathbb{P}^{n^2+2n}_{\mathcal{O}_K} \end{array} \tag{99}$$

where $\Delta$ is the diagonal map, $n = N^g - 1$, $\mathcal{P}$ is the product of two $\mathcal{S}$-embeddings $\mathcal{J} \xhookrightarrow{J} \mathbb{P}^n = \mathbb{P}^n_{\mathcal{O}_K}$ and the application $\iota : \mathcal{J}\times\mathcal{J} \to \mathbb{P}^{n^2+2n}$ is the composition of the Segre embedding $S$ with $\mathcal{P}$. As sheaves,

$$S^*(\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)) = \mathcal{O}_{\mathbb{P}^n}(1)\otimes_{\mathcal{O}_K}\mathcal{O}_{\mathbb{P}^n}(1) \quad\text{and}\quad \mathcal{P}^*(\mathcal{O}_{\mathbb{P}^n}(1)\otimes_{\mathcal{O}_K}\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{M}\otimes_{\mathcal{O}_K}\mathcal{M} =: \mathcal{M}^{\boxtimes 2}$$

so that

$$\iota^*(\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)) = \mathcal{M}^{\boxtimes 2}$$

and

$$\Delta^*\iota^*\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1) = \mathcal{M}\otimes_{\mathcal{O}_\mathcal{J}}\mathcal{M} = \mathcal{M}^{\otimes 2}. \tag{100}$$

We naturally endow the sheaves $\mathcal{M}^{\boxtimes 2}$, $\mathcal{M}^{\otimes 2}$, and so on with the hermitian structures induced by the cubist metric on the various $\mathcal{M}_\sigma$ for $\sigma : K \hookrightarrow \mathbb{C}$, denoted by $\|\cdot\|_{\mathrm{cub}}$.

We then pick two copies $(x_i)_{0\leq i\leq n}$ and $(y_j)_{0\leq j\leq n}$ of the canonical basis of global sections for each $\mathcal{O}_{\mathbb{P}^n}(1)$ on the two factors of $\mathbb{P}^n_{\mathcal{O}_K}\times\mathbb{P}^n_{\mathcal{O}_K}$ of (99), which give our basis $\mathcal{S}$ by restriction to $\mathcal{J}$. Then we provide the sheaf $\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)$ on $\mathbb{P}^{n^2+2n}_{\mathcal{O}_K}$ with the basis of global sections $(z_{i,j})_{0\leq i,j\leq n}$, each of which is mapped to $x_i\otimes_{\mathcal{O}_K} y_j$ under $S^*$. Define $\mathcal{D}$ as the diagonal linear subspace of $\mathbb{P}^{n^2+2n}_{\mathcal{O}_K}$ defined by the *linear* equations $z_{i,j} = z_{j,i}$ for all $i$ and $j$.

Let $V,\ W \subseteq J = \mathcal{J}_K$ be two closed subvarieties over $K$. The support of $V\cap W$ is the same as that of $(\iota\circ\Delta)^{-1}(\mathcal{D}\cap\iota(V\times W))$. To bound from above the height of points in $V\cap W$ it is therefore sufficient to estimate Faltings' height of $\mathcal{D}\cap\iota(V\times W)$, relative to the hermitian line bundle $\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)_{|\iota(J\times J)}$ endowed with the cubist metric. As $\mathcal{D}$ is a linear subspace that height is essentially the same as that of $(V\times W)$, up to an explicit error term which depends on the degree. In turn this error term is a priori linear in the number of (relevant) equations for $\mathcal{D}$, and this is way too high. But if one knows $V \cap W$ has dimension 0, it is enough to choose $(\dim V + \dim W)$ equations (up to perhaps increasing a bit the size of the set whose height we estimate), which makes the error term much smaller.

That is the basic strategy of proof for Proposition 6.1. To make it effective however we must control the "error terms" alluded to in the preceding lines, and those crucially depend on the supremum, on the set $\mathcal{S}$, of values for the cubist metric of global sections defining the projective embedding $\mathcal{J} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$. We shall build that $\mathcal{S}$ using theta functions as follows.

Recall Riemann's theta function on $J_0(p)$ introduced in Section 5C; see (89). Its usual analytic norm is

$$\|\theta(z)\|_{\mathrm{an}} := \det(\Im(\tau))^{1/4} \exp(-\pi y \Im(\tau)^{-1} y)|\theta(z)| \tag{101}$$

for $z = x + iy \in \mathbb{C}^g$ (see [Moret-Bailly 1990, (3.2.2)]). That analytic metric will have to be compared to the cubist one, about which we recall the following basic facts.

Let $A$ be an abelian variety over a number field $K$, which extends to a semiabelian scheme $\mathcal{A}$ over the ring of integers $\mathcal{O}_K$. We endow $\mathcal{A}$ with a symmetric ample invertible sheaf $\mathcal{L}$. Define, for $I \subseteq \{1, 2, 3\}$, the projection $p_I : \mathcal{A}^3 \to \mathcal{A}$, $p_I(x_1, x_2, x_3) = \sum_{i \in I} x_i$. It is known to follow from the theorem of the cube [Moret-Bailly 1985b] that the sheaf $\mathcal{D}_3(\mathcal{L}) := \bigotimes_{I \subseteq \{1,2,3\}} p_I^* \mathcal{L}^{\otimes(-1)^{|I|}}$ is trivial on $\mathcal{A}^3$. Let us therefore fix an isomorphism $\phi : \mathcal{O}_{\mathcal{A}^3} \to \mathcal{D}_3(\mathcal{L})$. For every complex place $\sigma$ of $\mathcal{O}_K$ one can endow $\mathcal{L}_\sigma$ with some cubist metric $\|\cdot\|_\sigma$ such that one obtains through $\phi$ the trivial metric on $\mathcal{O}_{\mathcal{A}^3}$. Each cubist metric $\|\cdot\|_\sigma$ is determined only up to multiplication by some constant factor so we perform the following rigidification to remove that ambiguity. If $0_{\mathcal{A}} : \mathrm{Spec}(\mathcal{O}_K) \to \mathcal{A}$ denotes the zero section, we replace $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_K} (\pi^* 0_{\mathcal{A}}^* \mathcal{L}^{\otimes -1})$ on $\mathcal{A}$. Then

$$0_{\mathcal{A}}^*(\mathcal{L}) \simeq \mathcal{O}_K$$

and we demand that the $\|\cdot\|_\sigma$ be adjusted so that the above sheaf isomorphism is an isometry at each $\sigma$, where $\mathcal{O}_K$ is endowed with the trivial metric so that $\|1\| = 1$. This uniquely determines our cubist metrics $\|\cdot\|_\sigma$. Now by construction the hermitian sheaf $\mathcal{L}$ on $\mathcal{A}$ defines a height $\mathrm{h}$ verifying the expected normalization condition $\mathrm{h}(0) = 0$.

Having the same curvature form, the analytic and cubist metrics are known to differ by constant factors, at each complex place, on the theta sheaf, as we shall use in the proof of Lemma 6.4 below.

Recall we also defined in (91) a "meromorphic theta function $s_{\mathcal{J}^0}$ over $\overline{\mathbb{Z}}$", which can be generalized; we have $[r]^* \mathcal{L}(\Theta)_{|\mathcal{N}_{J,1}} \simeq \mathcal{L}(\Theta)^{\otimes r^2}$ on $\mathcal{N}_{J,r}$ [Pazuki 2012, Proposition 5.1], so we define a global section

$$s_{\mathcal{M}} := ([r]^* t_{-\kappa}^*) s_{\mathcal{J}^0} \in H^0(\mathcal{N}_{J,r}, [r]^* \mathcal{L}(\Theta)_{\mathcal{O}_K}). \tag{102}$$

We will shortly show how to control the supremum of $\|s_{\mathcal{J}^0}\|_{\mathrm{cub}}$, therefore of $\|s_{\mathcal{M}}\|_{\mathrm{cub}}$, on $J_0(p)(\mathbb{C})$ (see Lemma 6.4). Writing $N = r^2$, we shall moreover fix the morphism $\jmath_{\mathcal{M}} : \widetilde{\mathcal{N}}_{J,N} \to \mathcal{J} \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$ of (86) by mapping the canonical coordinates $(x_i)_{0 \le i \le n}$ to sections $(s_i)$ which will be translates by $r$-torsion points of a multiple of the above $s_{\mathcal{M}}$ by some constant, as explained in Lemma 6.5 and its proof.

This will allow us to control as well the supremum of those $s_i$, relative to the cubist metrics, on the complex base change of our abelian varieties, as is required by the proof of arithmetic Bézout theorems.

We now start the technical preparation for the proof of Proposition 6.1, for which we need some lemmas on the behavior of heights and degree under Segre maps, comparison between cubist and analytic metrics on theta functions, and estimates for all.

**Lemma 6.3.** *There is an infinite sequence $(P_i)_{i\in\mathbb{N}}$ of points in $X_0(p)(\overline{\mathbb{Q}})$ which are ordinary at all places dividing $p$ and have everywhere integral $j$-invariant. Moreover their normalized theta height satisfies $\mathrm{h}_\Theta\big(P_i-\infty+\frac{1}{2}\omega^0\big)=O(p^3)$, with notations of Theorem 4.1.*

*Proof.* Let $(\zeta_i)_{\mathbb{N}}$ be a infinite sequence of roots of unity. One can assume none are congruent to some supersingular $j$-invariant in characteristic $p$, modulo any place of $\overline{\mathbb{Q}}$ above $p$. (Indeed, as the supersingular $j$-invariants are quadratic over $\mathbb{F}_p$, it is enough for instance to choose for the $\zeta_i$ some primitive $\ell_i$-roots of unity, with $\ell_i$ running through the set of primes larger than $p^2-1$.) Lift each $j$-invariant equal to $\zeta_i$ to some point $P_i$ in $X_0(p)(\overline{\mathbb{Q}})$. By construction, this makes a sequence of points with $j$-height $\mathrm{h}_j(P_i)$ equal to 0. As for their (normalized) theta height one sees from Theorem 4.1 that

$$\mathrm{h}_\Theta\big(P_i-\infty+\tfrac{1}{2}\omega^0\big)=\frac{1}{[K(P_i):\mathbb{Q}]}[P_i,\tilde{\omega}_\Theta]_{\mu_0}=\frac{-1}{[K(P_i):\mathbb{Q}]}\sum_{\sigma:K(P_i)\hookrightarrow\mathbb{C}} g\cdot g_{\mu_0}(\infty,\sigma(P_i))+O(\log p)$$

as the contribution at finite places of $[P_i,\infty]$ is 0. It is therefore enough to bound the $|g_{\mu_0}(\infty,\sigma(P_i))|$.

Now $|j(P_i)|_\sigma=1$ for all $\sigma:K(P_i)\hookrightarrow\mathbb{C}$, so the corresponding elements $\tau$ in the usual fundamental domain in Poincaré upper half-plane for $X_0(p)$ or $X(p)$ are absolutely bounded, and the same for the absolute values of $q_\tau=e^{2i\pi\tau}$. (For a useless explicit estimate of this bound one can check Corollary 2.2 of [Bilu and Parent 2011] which proposes $|q_\tau|\geq e^{-2500}$.) From this, running through the proof of Theorem 11.3.1 of [Edixhoven and de Jong 2011b], and adapting it to the case of $X_0(p)$ instead of $X_1(pl)$, we deduce that the $\sigma(P_i)$ do not belong to the open neighborhood, in the atlas of [loc. cit.], of the cusp $\infty$ in $X_0(p)(\mathbb{C})$. Therefore Proposition 10.13 of [Merkl 2011] applies and gives, with notations of that work,

$$|g_{\mu_0}(\infty,\sigma(P_i))|=|g_{\mu_0}(\infty,\sigma(P_i))-h_\infty(\sigma(P_i))|=O(p^2) \tag{103}$$

(see Theorem 11.3.1 of [Edixhoven and de Jong 2011b] and its proof). □

**Lemma 6.4.** *Let $s_\theta$ be the "theta function over $\mathbb{Z}$", that is, the global section introduced just before* (90). *One has*

$$\sup_{J_0(p)(\mathbb{C})}(\log\|s_\theta\|_{\mathrm{cub}})\leq O(p\log p). \tag{104}$$

*Proof.* Writing $s_{\theta,\mathbb{C}}(z)=C_\vartheta\cdot\theta(z+\kappa)$ as in (90), we shall bound from above both $|C_\vartheta|$ and the contribution of the difference between cubist and analytic metrics. Then we will use upper bounds for the analytic norm of the theta function due to P. Autissier and proven in the Appendix of the present paper.

We invoke again some key arguments of the proof of Proposition 5.8. For $D$ in $J_0(p)(\mathbb{C})$, written as the linear equivalence class of some divisor $\sum_{i=1}^{g}(P_i-\infty)$ on $X_0(p)(\mathbb{C})$, we indeed once more consider the embedding

$$\iota_{\kappa-D} : X_0(p) \hookrightarrow J_0(p), \quad P \mapsto \mathrm{cl}(P - \infty + \kappa - D)$$

as in Proposition 5.8. For such a $D$ whose $P_i$ are assumed to belong to $X_0(p)(\overline{\mathbb{Q}})$, we recall (92) that

$$\mathrm{h}_\Theta(\iota_{\kappa-D}(P)) = \frac{1}{[K(P,D):\mathbb{Q}]}\Big[P, \sum_i P_i + \Phi_D + c_D X_\infty\Big]_{\mu_0}.$$

If the $P_i$ all have everywhere ordinary reduction, as will be the case in (105) below, the vertical divisor $\Phi_D$ will contribute at most $O(\log p)$ to the height of points (see Remark 3.4).

Note that we can fulfill condition (95) considering only points $P_i$ of same type as occurring in Lemma 6.3 (which, in particular, are ordinary and have integral $j$-invariants), because those $P_i$ make a Zariski-dense subset of $X_0(p)(\overline{\mathbb{Q}})$ (and the onto-ness of the map $X_0(p)^{(g)} \xrightarrow{\iota_\infty^g} J_0(p)$). We therefore conclude as in the proof of Proposition 5.8 that $\mathrm{div}(\iota_{\kappa-D}^*(s_\theta))$ has indeed to be $\left(\sum_i P_i + \Phi_D\right)$ on $X_0(p)^{\mathrm{smooth}}_{\mathcal{O}_K}$.[6]

On the other hand, for some of those choices of $(P_i)_{1\le i\le g}$, our $\mathbb{Z}$-theta function $s_\theta$ does not vanish at $\iota_{\kappa-D}(\infty)(\mathbb{C})$, so $\mathrm{h}_\Theta(\iota_{\kappa-D}(\infty))$ can also be computed as the Arakelov degree:

$$\mathrm{h}_\Theta(\iota_{\kappa-D}(\infty)) = \widehat{\deg}(\infty^*\iota_{\kappa-D}^*(\mathcal{L}(\Theta))).$$

Integrality of the $P_i$ shows the intersection numbers $[\infty, P_i]$ have trivial nonarchimedean contribution. The only finite contribution to our Arakelov degree therefore comes from intersection with vertical components, that is, if $K_D$ is a sufficiently large field over which $D$ is defined, then for a set of elements $(z_\sigma)_{\sigma:K_D\hookrightarrow\overline{\mathbb{Q}}}$ which lift $\sigma(-D)$ in the complex tangent space of $J_0(p)$ to 0 one has

$$\begin{aligned}\mathrm{h}_\Theta(\iota_{\kappa-D}(\infty)) &= \widehat{\deg}(0^*_{\mathcal{J}_0(p)}(t^*_{\kappa-D}\mathcal{L}(\Theta))) = \widehat{\deg}(0^*_{\mathcal{J}_0(p)}(t^*_{-D}\mathcal{L}(\Theta_\kappa)))\\ &= -\frac{1}{[K_D:\mathbb{Q}]}\sum_{K_D\overset{\sigma}{\hookrightarrow}\mathbb{C}} \log\|s_\theta(z_\sigma)\|_{\mathrm{cub}} + O(\log p),\end{aligned}$$

whence, as $s_{\theta,\mathbb{C}}(z) = C_\vartheta\cdot\theta(z+\kappa)$,

$$\log|C_\vartheta| = -\mathrm{h}_\Theta(\iota_{\kappa-D}(\infty)) - \frac{1}{[K_D(\kappa):\mathbb{Q}]}\sum_{K_D(\kappa)\overset{\sigma}{\hookrightarrow}\mathbb{C}} \log\|\theta((z+\kappa)_\sigma)\|_{\mathrm{cub}} + O(\log p). \tag{105}$$

Following [Gaudron and Rémond 2014b, paragraph 8] we now write $J_0(p)(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ for $\tau$ in Siegel's fundamental domain, write $z\in\mathbb{C}^g$ as $z = \tau\cdot p + q$ for $p, q\in\mathbb{R}^g$, and introduce the function $F:\mathbb{C}^g\to\mathbb{C}$ defined as

$$F(z) = \det(2\Im(z))^{1/4}\sum_{n\in\mathbb{Z}^g}\exp(i\pi\,{}^t(n+p)\tau(n+p) + 2i\pi\,{}^t nq).$$

One then has $|F(z)| = 2^{g/4}\|\theta(z)\|_{\mathrm{an}}$. Indeed there is a constant $A\in\mathbb{R}_+^*$ such that $|F(z)| = A\cdot\|\theta(z)\|_{\mathrm{an}}$ (see the end of proof of Lemma 8.3 of [loc. cit.]), $\int_{J_0(p)(\mathbb{C})}|F|^2\,d\nu = 1$ (where $d\nu$ is the probability

[6] Although we shall not use this, one can check that $\mathrm{h}_\Theta(\iota_{\kappa-D}(\infty)) = \left\|-\left(\sum_i P_i - \infty\right) + \frac{1}{2}\omega^0\right\|_\Theta^2 = O(p^5)$ by Lemma 6.3 and (79).

Haar measure on $J_0(p)(\mathbb{C})$, see [loc. cit., Lemma 8.2(1)]), and $\int_{J_0(p)(\mathbb{C})} \|\theta(z)\|_{\mathrm{an}}^2\, d\nu = 2^{-g/2}$ (see, e.g., [Moret-Bailly 1990, (3.2.1) and (3.2.2)]). Therefore Lemme 8.3 of [Gaudron and Rémond 2014b] gives, using definitions of [loc. cit., Théorème 8.1],

$$-\frac{1}{[K_D(\kappa):\mathbb{Q}]} \sum_{K_D(\kappa) \overset{\sigma}{\hookrightarrow} \mathbb{C}} \big(\log\|\theta((z+\kappa)_\sigma)\|_{\mathrm{an}} + \tfrac{g}{4}\log 2\big) \leq \mathrm{h}_\Theta(\iota_{\kappa-D}(\infty)) + \tfrac{1}{2}\mathrm{h}_F(J_0(p)) + \tfrac{g}{4}\log 2\pi.$$

Remember Faltings' height of $J_0(p)$ is known to satisfy $\mathrm{h}_F(J_0(p)) = O(p\log p)$ by [Ullmo 2000, Théorème 1.2]. (We remark that Ullmo's normalization of Faltings' height differs from that of Gaudron and Rémond, but the difference term is linear in $g = O(p)$ so the bound $O(p\log p)$ remains valid for the above $\mathrm{h}_F(J_0(p))$). Writing $\|\cdot\|_{\mathrm{cub}} = e^{\varphi}\|\cdot\|_{\mathrm{an}}$ we therefore see that (105) implies

$$\log|C_\vartheta| + \varphi \leq \tfrac{1}{2}\mathrm{h}_F(J_0(p)) + O(p) \leq O(p\log p).$$

Given this upper bound for $e^{\varphi}|C_\vartheta|$ we can now go the other way round to derive an upper bound for $\|s_\theta\|_{\mathrm{cub}} = C_\vartheta \cdot \|\theta(z+\kappa)\|_{\mathrm{cub}}$, by using estimates for analytic theta functions. For any principally polarized complex abelian variety whose complex invariant $\tau$ is chosen within Siegel's fundamental domain $F_g$, Autissier's result in the Appendix (Proposition A.1 below) indeed gives, with notations as in (101), that

$$\frac{1}{\det(\Im(\tau))^{1/4}}\|\theta(z)\|_{\mathrm{an}} = \exp(-\pi y \Im(\tau)^{-1} y)|\theta(z)| \leq g^{g/2}. \tag{106}$$

We refer to the Appendix for a bound which is slightly sharper.[7]

As for the factor $\det(\Im(\tau))^{1/4}$, Lemma 11.2.2 of [Edixhoven and de Jong 2011b] gives the general result

$$\det(\Im(z))^{1/2} \leq \frac{(2g)!V_{2g}}{2^g V_g} \prod_{g+1\leq i\leq 2g} \lambda_i,$$

where for any $k$ we write $V_k$ for the volume of the unit ball in $\mathbb{R}^k$ endowed with its standard Euclidean structure, and the $\lambda_r$ are the successive minima, relative to the Riemann form, of the lattice $\Lambda = \mathbb{Z}^g + \tau\cdot\mathbb{Z}^g$. To bound the $\lambda_i$ we need to invoke an avatar of [loc. cit., Lemma 11.2.3]. But the very same proof shows that for any integer $N$, the group $\Gamma_0(N)$ has a set of generators having entries of absolute value less or equal to the very same bound $N^6/4$. (That term could be improved, but this would have an invisible impact on the final bounds so we here content ourselves with it.) We can therefore rewrite the proof of Lemma 11.2.4 verbatim. This gives that $\Lambda$ is generated by elements having naive hermitian norm $\|x\|_E^2$ less or equal to $gp^{46}$. Finally, in our case the Gram matrix is diagonal (no $2\times 2$-blocks, at the difference of Lemma 11.1.4 of [loc. cit.]) so Lemma 11.2.5 a fortiori holds: if $\|\cdot\|_P$ denotes the hermitian product on $\mathbb{C}^g$ induced by the polarization, $\|\cdot\|_P^2 \leq e^{4\pi}/\pi\|\cdot\|_E^2$. This allows to conclude as in p. 228 of [loc. cit.]:

$$\Big(\prod_{i=g+1}^{2g} \lambda_i\Big)^2 \leq \Big(\frac{e^{4\pi}}{\pi} gp^{46}\Big)^g$$

[7] Works of Igusa and Edixhoven and de Jong [2011b, pp. 231–232] give $1/\det(\Im(\tau))^{1/4}\|\theta(z)\|_{\mathrm{an}} \leq 2^{3g^3+5g}$.

so that

$$\log(\det(\Im(\tau))) \leq O(p \log p)$$

and combining with (106),

$$\log\|\theta(z)\|_{\mathrm{an}} \leq O(p \log p).$$

Putting everything together finally yields

$$\begin{aligned}\sup_{z\in J_0(p)(\mathbb{C})} \log\|s_{\theta,\mathbb{C}}(z)\|_{\mathrm{cub}} &= \sup_{z\in J_0(p)(\mathbb{C})} \log\|C_\vartheta \cdot \theta(z+\kappa)\|_{\mathrm{cub}} \\ &= (\log|C_\vartheta| + \varphi) + \sup_{z\in J_0(p)(\mathbb{C})} \log\|\theta(z+\kappa)\|_{\mathrm{an}} \\ &\leq O(p \log p).\end{aligned}$$

□

**Lemma 6.5.** *Assume the same hypothesis and notations as in Definition 5.4. After possibly making some finite base extension one can pick a set $\mathcal{S}$ in $H^0(\mathcal{N}_{J,4}, \mathcal{L}(\Theta)^{\otimes 4})$ of $4^g$ global sections $(s_i)_{1\leq i\leq 4^g}$, which span $\mathcal{L}(\Theta)^{\otimes 4}$ on $\mathcal{N}_{J,4}[1/2p]$, and verify*

$$\sup_{J_0(p)} (\log\|s_i\|_{\mathrm{cub}}) \leq O(p \log p). \tag{107}$$

*Proof.* We fix $N = r^2 = 4$ for the construction of a good model as in Definition 5.4. Up to making a base extension, we can assume $L(\Theta)^{\otimes 4}$ and $[2]^*L(\Theta)$ have cubist extensions $\mathcal{L}(\Theta)^{\otimes 4}$ and $[2]^*\mathcal{L}(\Theta))$ on $\mathcal{N}_{J,4}$, respectively. As $\Theta$ is symmetric one knows there is an isomorphism $[2]^*\mathcal{L}(\Theta) \to \mathcal{L}(\Theta)^{\otimes 4}$ which actually is an isometry [Pazuki 2012, Proposition 5.1], by which we identify those two objects from now on. On the other hand, every element $x$ of $J_0(p)[4](\overline{\mathbb{Q}}) = J_0(p)[4](K)$ defines a section $\tilde{x}$ in $\mathcal{N}_{J,4}(\mathrm{Spec}(\mathcal{O}_K))$. Letting $t_{\tilde{x}}$ denote the translation by $\tilde{x}$ on $\mathcal{N}_{J,4}$ we have

$$t_{\tilde{x}}^*\mathcal{L}(\Theta)^{\otimes 4} \simeq \mathcal{L}(\Theta)^{\otimes 4}. \tag{108}$$

(This is indeed true over $\mathbb{C}$ by Lemma 2.4.7.c of [Birkenhake and Lange 2004], hence over $K$, then over $\mathrm{Spec}(\mathcal{O}_K)$ by uniqueness of cubist extensions.) The interpretation as Néron–Tate heights shows that as $\mathcal{L}(\Theta)$ is endowed with its cubist metric, this isomorphism even is an isometry. Recall the section $s_{\mathcal{M}}$ defined in (102), belonging to $H^0(\mathcal{N}_{J,2}, [2]^*\mathcal{L}(\Theta))$. Up to making an extension to some larger base ring of integer, we may assume $s_{\mathcal{M}}$ extends as a meromorphic section on $\mathcal{N}_{J,4}$ and Proposition 5.8, which gives estimates on the poles of $s_{\mathcal{J}^0}$ at bad components, implies that $s_{\mathcal{M}}$ is actually holomorphic (has no pole on the new components) after multiplication by some power $C_1$ of $p$ with $\log C_1 = O(p \log p)$. We can therefore define a set $(s_i)_{1\leq i\leq 4^g}$ in $H^0(\mathcal{N}_{J,4}, [2]^*\mathcal{L}(\Theta))$ made of $4^g$ elements of shape

$$s_i := t_{\tilde{x}_i}^* C_1 \cdot s_{\mathcal{M}} \tag{109}$$

for $\tilde{x}_i$ running through a set of representatives, in $J_0(p)[4](K)$, of $J_0(p)[4]/J_0(p)[2]$. Note that one can explicitly lift $s_{\mathcal{M}}$ on the complex tangent space at 0 of $J_0(p)(\mathbb{C})$ as

$$s_{\mathcal{M},\mathbb{C}}(z) = C_\vartheta \cdot \theta(2 \cdot z) \tag{110}$$

where $C_\vartheta$ is defined in the proof of Lemma 6.4 and the $s_{i,\mathbb{C}}$ are constant multiple of the basis denoted by $\mathrm{h}_{\vec{a},\vec{b}}(\vec{z})$ in [Mumford 1984], Proposition II.1.3.iii on p. 124.[8] From here, Lemma 6.4 and Proposition 5.8 give (107).

By the theory of theta functions [Pazuki 2012, Proposition 2.5 and its proof; Mumford 1966; Moret-Bailly 1985b, Chapitre VI] the $s_i$ make a generic basis of global sections, which span $\mathcal{L}(\Theta)^{\otimes 4}$ over $\mathrm{Spec}(\mathcal{O}_K[1/2p])$. □

**Lemma 6.6.** *Let $V$ and $W$ be two closed $K$-subvarieties, with dimension $d_V$ and $d_W$ respectively, of a smooth projective variety $A$ over a number field $K$, endowed with an ample sheaf $M$. Assume the flat projective scheme $(\mathcal{A}, \mathcal{M})$ over $\mathrm{Spec}(\mathcal{O}_K)$, with $\mathcal{M}$ an hermitian sheaf on $\mathcal{A}$, is a model for $(A, M)$. Let $\mathcal{V}$ and $\mathcal{W}$ be the Zariski closure in $\mathcal{A}$ of $V$ and $W$ respectively. Then, with definitions as in [Bost et al. 1994, §3.1],*

$$(c_1(M^{\boxtimes 2})^{d_V+d_W} \mid (V \times W)) = \binom{d_V+d_W}{d_V}(c_1(M)^{d_V} \mid V)(c_1(M)^{d_W} \mid W) \tag{111}$$

*and*

$$\begin{aligned}(\hat{c}_1(\mathcal{M}^{\boxtimes 2})^{d_V+d_W+1} \mid \mathcal{V}\times\mathcal{W}) \\ = \binom{d_V+d_W+1}{d_V}(c_1(M)^{d_V} \mid V)(\hat{c}_1(\mathcal{M})^{d_W+1} \mid \mathcal{W}) + \binom{d_V+d_W+1}{d_W}(\hat{c}_1(\mathcal{M})^{d_V+1} \mid \mathcal{V})(c_1(M)^{d_W} \mid W).\end{aligned} \tag{112}$$

**Remark 6.7.** Equation (111) can be read as

$$\deg_{M^{\boxtimes 2}}(V \times W) = \binom{d_V+d_W}{d_V} \deg_M(V) \deg_M(W).$$

Equation (112) in turn fits with Zhang's interpretation (83) in terms of essential minima, compare the proof of Proposition 6.1 below.

*Proof of Lemma 6.6.* For (111), one can realize it is elementary, or refer to Lemme 2.2 of [Rémond 2010], or proceed as follows. Using (2.3.18), (2.3.19), and Proposition 3.2.1(iii) of [Bost et al. 1994], and noticing

$$c_1(M^{\boxtimes 2}) = c_1(M) \times \mathbf{1} + \mathbf{1} \times c_1(M)$$

(and same with $\hat{c}_1(\mathcal{M})$ and $\hat{c}_1(\mathcal{M}^{\boxtimes 2})$ instead) one computes

[8] where it seems by the way that the expression "$\mathrm{h}_{\vec{a},\vec{b}}(\vec{z}) = \vartheta\begin{bmatrix}\vec{a}/k \\ \vec{b}/k\end{bmatrix}(\ell\cdot\vec{z}, \Omega)$" should read "$\cdots = \vartheta\begin{bmatrix}\vec{a}/k \\ \vec{b}/k\end{bmatrix}(k\cdot\vec{z}, \Omega)$" (notations of [loc. cit.]).

$$
\begin{aligned}
(c_1(M^{\boxtimes 2})^{d_V+d_W} \mid (V\times W)) &= \Big(\sum_{k=0}^{d_V+d_W}\binom{d_V+d_W}{k} c_1(M)^k \times c_1(M)^{d_V+d_W-k} \mid V\times W\Big)\\
&= \sum_{k=0}^{d_V+d_W}\binom{d_V+d_W}{k}(c_1(M)^k \times c_1(M)^{d_V+d_W-k} \mid V\times W)\\
&= \sum_{k=0}^{d_V+d_W}\binom{d_V+d_W}{k}(c_1(M)^k \mid V)(c_1(M)^{d_V+d_W-k} \mid W)\\
&= \binom{d_V+d_W}{k}(c_1(M)^{d_V} \mid V)(c_1(M)^{d_W} \mid W),
\end{aligned}
$$

where the last equality comes from the fact that the only nonzero term in the line before occurs for $k = d_V$.

An analogous computation, using [Bost et al. 1994, (2.3.19)], can be used for the arithmetic degree:

$$
\begin{aligned}
&(\hat{c}_1(\mathcal{M}^{\boxtimes 2})^{d_V+d_W+1} \mid \mathcal{V}\times\mathcal{W})\\
&= \sum_{k=0}^{d_V+d_W+1}\binom{d_V+d_W+1}{k}(\hat{c}_1(\mathcal{M})^k \times \hat{c}_1(\mathcal{M})^{d_V+d_W+1-k} \mid \mathcal{V}\times\mathcal{W})\\
&= \binom{d_V+d_W+1}{d_V}(c_1(M)^{d_V} \mid V)(\hat{c}_1(\mathcal{M})^{d_W+1} \mid \mathcal{W})\binom{d_V+d_W+1}{d_W}(\hat{c}_1(\mathcal{M})^{d_V+1} \mid \mathcal{V})(c_1(M)^{d_W} \mid W). \quad \square
\end{aligned}
$$

For the rest of this Section we fix the model $(\mathcal{J}, \mathcal{M})$ for $(J_0(p), \Theta)$ (see (99)) as the one built with the set $\mathcal{S}$ of $N^g = 4^g$ sections provided by Lemma 6.5. Before settling the proof of the arithmetic Bézout theorem, we need a last lemma on the comparison between the projective height on $(\mathcal{J}, \mathcal{M})$ and its Néron–Tate avatar.

**Lemma 6.8.** *Up to translation by torsion points, the projective height* $\mathrm{h}_{\mathcal{M}}$ *on points in* $J_0(p)(\overline{\mathbb{Q}})$ *(associated with the good model* $(\mathcal{J}, \mathcal{M})$*) differs from the Néron–Tate theta-height* $4\mathrm{h}_\Theta$ *by an error term of shape* $O(p\log p)$.

*Proof.* Lemma 6.5 implies that the elements of $\mathcal{S}$ extend as holomorphic sections to *any* component of the Néron model $\overline{\mathcal{N}}$ of $J_0(p)$ over $\overline{\mathbb{Z}}$ (see (109)). As remarked in the proof of Lemma 6.5, Mumford's algebraic theory of theta-functions implies that the sections in $\mathcal{S}$ do define a projective embedding of $\overline{\mathcal{N}}$ over $\overline{\mathbb{Z}}[1/2p]$; the only fibers of $\overline{\mathcal{N}}$ over $\overline{\mathbb{Z}}$ where base points for $\mathcal{S}$ can show up are above 2 and $p$. If one seeks to approximate the Néron–Tate height of a given point $P$ in $J_0(p)(\overline{\mathbb{Q}})$ by the projective height of our good model $(\mathcal{J}, \mathcal{M})$, one needs the section of the Néron model $\overline{\mathcal{N}}$ defined by $P$ to avoid those base points, or at least control their length.

Given $P$ in $J_0(p)(\overline{\mathbb{Q}})$, we claim one can translate $P$ by some torsion point in $J_0(p)(\overline{\mathbb{Q}})$ so that the translated new point $P+t$ does avoid base points in characteristic 2. Indeed, choose a Galois extension $F/\mathbb{Q}$ such that the base locus is defined over $\mathrm{Spec}(\mathcal{O}_F \otimes \mathbb{F}_2)$. Summing-up, as divisors, all the Galois conjugates of that base locus in each fiber of characteristic 2, one obtains a constant cycle $C_\kappa$, in each fiber at $\kappa$, which is defined over $\mathbb{F}_2$. (In our case one actually could have taken $F = \mathbb{Q}$.) Density of torsion points then shows that one can replace our point $P$ by $P+t$, for some torsion point $t$, such that $P+t$

does not belong to $C_{\kappa_0}$ for some $\kappa_0$, then for all $\kappa$ of characteristic 2 because $C_\kappa$ is constant. This proves our claim. Now in characteristic $p$, we know from Proposition 5.8 again that possible base points have length at most $O(p)$, which gives an estimate of size $O(p \log p)$ for the difference error term between projective height on $\mathcal{J}$ and Néron–Tate height [Pazuki 2012, Proposition 4.1]. ☐

*Proof of Proposition 6.1.* Before proceeding we will allow ourselves for this proof only, in order to not overcomplicate the computations, to work with heights defined as in [Bost et al. 1994, §3.1]. Namely, for $\mathcal{Y}$ a cycle of dimension $(d+1)$ in a regular arithmetic variety endowed with a hermitian sheaf $\mathcal{F}$, we multiply our definition (80) of its height by degree and absolute dimension and we set

$$\mathrm{h}'_{\mathcal{F}}(\mathcal{Y}) = \frac{(\hat{c}_1(\mathcal{F})^{d+1} \mid \mathcal{Y})}{[K:\mathbb{Q}]}.$$

Note that h and h′ coincide on $K$-rational points, in which case we might use either notation.

Construction (99) gives a $\mathbb{Q}$-embedding $V \times W \overset{\iota}{\hookrightarrow} \mathbb{P}^{n^2+2n}$ via a Segre map. We set

$$s_{\underline{i},\underline{j}} := \iota^*(z_{\underline{i},\underline{j}} - z_{\underline{j},\underline{i}})$$

for all $(\underline{i},\underline{j})$, and denote by $\mathcal{O}_N$ the ambiant line bundle $\iota^*(\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)) = \mathcal{M}^{\boxtimes 2}$ as before (100). (Recall we will eventually specialize to $N=4$.) Set also $O_N := \mathcal{O}_N \otimes \mathbb{Q}$. We intersect $\iota(V \times W)$ with one of the $\mathrm{div}(z_{\underline{i_0},\underline{j_0}} - z_{\underline{j_0},\underline{i_0}})_{\mathbb{Q}}$ such that the two cycles meet properly; define

$$J_1 = \mathrm{div}(s_{\underline{i_0},\underline{j_0}\,\mathbb{Q}}) \cap (V \times W)$$

in the generic fiber $(J_0(p) \times J_0(p))_{\mathbb{Q}}$. As $\mathrm{div}(z_{\underline{i_0},\underline{j_0}} - z_{\underline{j_0},\underline{i_0}})$ is a projective hyperplane we have by definition

$$\deg_{O_N}(J_1) = \deg_{O_N}(V \times W).$$

For the same linearity reason, a similar statement is true for heights. Indeed, let $\mathcal{V}$ and $\mathcal{W}$ denote the schematic closure in $\mathcal{J}$ of $V$ and $W$ respectively, and $\mathcal{J}_1$ the schematic closure of $J_1$ in $\mathcal{J} \times \mathcal{J}$, which satisfies

$$\mathrm{h}'_{\mathcal{O}_N}(\mathcal{J}_1) \le \mathrm{h}'_{\mathcal{O}_N}(\mathrm{div}(s_{\underline{i_0},\underline{j_0}}) \cap (\mathcal{V} \times \mathcal{W}))$$

(as there might be vertical components in the intersection of the right-hand side which do not intervene in the left, and contribute positively to the height).

Proposition 3.2.1(iv) of [Bost et al. 1994] gives, with notations of [loc. cit.], that

$$\mathrm{h}'_{\mathcal{O}_N}(\mathrm{div}(s_{\underline{i_0},\underline{j_0}}) \cap (\mathcal{V} \times \mathcal{W})) = \mathrm{h}'_{\mathcal{O}_N}(\mathcal{V} \times \mathcal{W}) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma: K \hookrightarrow \mathbb{C}} \int_{(V \times W)_\sigma(\mathbb{C})} \log \| s_{\underline{i_0},\underline{j_0}\,\mathbb{C}} \| c_1(\mathcal{O}_N)^{d_V + d_W} \quad (113)$$

where $\|\cdot\| = \|\cdot\|_{\mathrm{cub}}$ shall denote the cubist metric, or the metric induced by the cubist metric on products or powers of relevant sheaves. To estimate the last integral we note that at any point of $(V \times W)_\sigma(\mathbb{C})$ and for any $(\underline{i}, \underline{j})$,

$$\begin{aligned}\|s_{\underline{i},\underline{j}}\| = \|z_{\underline{i},\underline{j}} - z_{\underline{j},\underline{i}}\|_{\mathcal{M}^{\boxtimes 2}} &\le \|z_{\underline{i},\underline{j}}\|_{\mathcal{M}^{\boxtimes 2}} + \|z_{\underline{j},\underline{i}}\|_{\mathcal{M}^{\boxtimes 2}} \\ &\le \|x_{\underline{i}}\|_{\mathcal{M}}\|y_{\underline{j}}\|_{\mathcal{M}} + \|x_{\underline{j}}\|_{\mathcal{M}}\|y_{\underline{i}}\|_{\mathcal{M}} \le 2(\sup_{\underline{i}}\|x_{\underline{i}}\|_{\mathcal{M}})^2 \\ &\le \exp(2\log(\sup\|s_i\|_{\text{cub}}) + \log 2)\end{aligned}$$

with notations of Lemma 6.5. Setting $M_{\mathcal{J},\mathcal{M}} = \log(\sup\|s_i\|_{\text{cub}})$ we obtain

$$\mathrm{h}'_{\mathcal{O}_N}(\mathcal{J}_1) \le \mathrm{h}'_{\mathcal{O}_{N_0}}(\mathcal{V}\times\mathcal{W}) + (2M_{\mathcal{J},\mathcal{M}} + \log 2)\deg_{\mathcal{O}_N}(V\times W).$$

Call $I_1$ one of the reduced irreducible components of $J_1$ containing the point $\iota(\Delta(P))$ of $V\cap W$ considered in the statement of Proposition 6.1 and let $\mathcal{I}_1$ denote its Zariski closure in $\mathcal{J}$. It has $\mathcal{O}_N$-height (and degree) less than or equal to those of $\mathcal{J}_1$, so that again

$$\mathrm{h}'_{\mathcal{O}_N}(\mathcal{I}_1) \le \mathrm{h}'_{\mathcal{O}_N}(\mathcal{V}\times\mathcal{W}) + (2M_{\mathcal{J},\mathcal{M}} + \log 2)\deg_{\mathcal{O}_N}(V\times W)$$

and we can iterate the process with $I_1$ in place of $V\times W$: we obtain some $J_2$, $\mathcal{J}_2$, $I_2$, $\mathcal{I}_2$ such that

$$\begin{aligned}\mathrm{h}'_{\mathcal{O}_N}(\mathcal{I}_2) &\le \mathrm{h}'_{\mathcal{O}_N}(\mathcal{I}_1) + (2M_{\mathcal{J},\mathcal{M}} + \log 2)\deg_{\mathcal{O}_N}(I_1) \\ &\le \mathrm{h}'_{\mathcal{O}_N}(\mathcal{V}\times\mathcal{W}) + 2(2M_{\mathcal{J},\mathcal{M}} + \log 2)\deg_{\mathcal{O}_N}(V\times W).\end{aligned}$$

(The only obstruction to this step is if all the $s_{\underline{k},\underline{l}}$ vanish on $I_1$, which implies it is contained in the diagonal of $J_0(p)\times J_0(p)$ - so that $I_1 = \iota(\Delta(P))$ by construction and that means we are already done.) Processing, one builds a sequence $(\mathcal{I}_k)$ of integral closed subschemes of $\mathcal{J}\times\mathcal{J}$, with decreasing dimension, such that the last step gives

$$\mathrm{h}'_{\mathcal{O}_N}(\mathcal{I}_{d_V+d_W}) \le \mathrm{h}'_{\mathcal{O}_N}(\mathcal{V}\times\mathcal{W}) + (d_V + d_W)(2M_{\mathcal{J},\mathcal{M}} + \log 2)\deg_{O_N}(V\times W).$$

Now

$$\mathrm{h}'_{\mathcal{O}_N}(\mathcal{I}_{d_V+d_W}) \ge \mathrm{h}'_{\mathcal{O}_N}(\Delta(P,P)) = \mathrm{h}'_{\mathcal{M}^{\otimes 2}}(P) = \mathrm{h}_{\mathcal{L}^{\otimes 2N}}(P) = 2N\,\mathrm{h}_\Theta(P) + O(p\log p),$$

for $\mathrm{h}_\Theta(P)$ the Néron–Tate normalized theta height. Indeed the statement of the present Proposition 6.1 is invariant by translation of every object by some fixed torsion point, so that one can apply Lemma 6.8.

Using Lemma 6.6 and Corollary 5.6 and writing $\mathrm{h}'_\Theta(Y) = (\dim(Y)+1)\deg_\Theta(Y)\mathrm{h}_\Theta(Y)$ we therefore obtain

$$\begin{aligned}&2N\mathrm{h}_\Theta(P) \\ &\le N^{d_v+d_W+1}\left[(d_W+1)\binom{d_V+d_W+1}{d_V}\mathrm{h}'_\Theta(W)\deg_\Theta(V) + (d_V+1)\binom{d_V+d_W+1}{d_W}\mathrm{h}'_\Theta(V)\deg_\Theta(W)\right] \\ &\qquad + N^{d_V+d_W}(d_V+d_W)(2M_{\mathcal{J},\mathcal{M}} + \log 2)\binom{d_V+d_W}{d_V}\deg_\Theta(V)\deg_\Theta(W) + O(p\log p).\end{aligned}$$

From here, fixing $N = 4$, the bound $M_{\mathcal{J},\mathcal{M}} \le O(p\log p)$ (Lemma 6.5) concludes the proof, after expressing quantities $\mathrm{h}'_\Theta$ back into $\mathrm{h}_\Theta$. □

That arithmetic Bézout theorem will be our principal tool in the sequel.

## 7. Height bounds for quadratic points on $X_0(p)$

**Proposition 7.1.** *Let $\iota : X \hookrightarrow J$ be some Albanese map from a curve (of positive genus) over some field $K$ to its jacobian $J$. Let $\pi : J \to A$ be some quotient of $J$, with $\dim(A) > 1$, and $X'$ be the normalization of the image $\pi \circ \iota(X)$ of $X$ in $A$. Then the map $\pi' : X \to X'$ induced by $\pi \circ \iota$ verifies*

$$\deg(\pi') \leq \frac{\dim(J) - 1}{\dim(A) - 1}.$$

*Proof.* The map $\pi \circ \iota$ induces an inclusion of function fields which defines the map $\pi' : X \to X'$. If $J'$ is the jacobian of $X'$, Albanese functoriality says that $\pi$ factorizes through surjective morphisms $J \to J' \to A$. Hurwitz formula writes

$$\deg(\pi') = \frac{\dim(J) - 1 - \frac{1}{2}\deg R}{\dim(J') - 1}$$

for $R$ the ramification divisor of $\pi'$, whence the result. □

**Lemma 7.2.** *For all large enough prime $p$, let $X := X_0(p)$ and $\pi_e : J_0(p) \twoheadrightarrow J_e$ be the projection. Let*

$$\iota_{P_0} : X_0(p) \hookrightarrow J_0(p), \quad P \mapsto \mathrm{cl}(P - P_0)$$

*for some $P_0$ in $X_0(p)(\overline{\mathbb{Q}})$ such that $w_p(P_0) = P_0$ (there are roughly $\sqrt{p}$ such points, [Gross 1987a, Proposition 3.1]) and set $\varphi_e := \pi_e \circ \iota_{P_0}$. Then:*

- *If $a \in J_e(\mathbb{Q})$ is some (necessarily torsion) point, the equality $\varphi_e(X_0(p)) = a - \varphi_e(X_0(p))$ implies*
$$\varphi_e(X_0(p)) = a + \varphi_e(X_0(p)) \tag{114}$$
*and $a = 0$.*
- *If $d$ is the degree of the map $X_0(p) \to \widetilde{\varphi_e(X_0(p))}$ to the normalization of $\varphi_e(X_0(p))$, then $d$ is either* 1, 3 *or* 4.
- *Assuming moreover Brumer's conjecture (see* (21) *and* (22)*) equality* (114) *implies $d = 1$ for large enough $p$.*

*Proof.* Notice first that, by our choice of $P_0$ (whence $\iota$), and because $J_e$ belongs to the $w_p$-minus part of $J_0(p)$, one has

$$\varphi_e(w_p(P)) = w_p(\varphi_e(P)) = -\varphi_e(P),$$

for all $P \in X_0(p)(\mathbb{C})$, whence equality (114). So let $n$ be the order of $a$, which also is that of the automorphism "translation by $a$ restricted to $\varphi_e(X_0(p))$". We remark that the degree $d$ cannot equal 2, as otherwise the extension of fraction fields $K(X_0(p))/K(\varphi_e(X_0(p)))$ would be Galois and $X_0(p)$ would possess an involution different from $w_p$, which it does not by Ogg's theorem [1977] (or even [Kenku and Momose 1988]). If $d = 1$, the same reason that $\mathrm{Aut}(X_0(p)) = \langle w_p \rangle$ implies that $n = 1$. Let now $X'$ be the normalization of the quotient of $\varphi_e(X_0(p))$ by the automorphism $P \mapsto P + a$, that is, the image of $\varphi_e(X_0(p))$ by the quotient morphism $J_e \twoheadrightarrow J_e/\langle a \rangle$. Let $\pi$ be the composed map $J_0(p) \xrightarrow{\varphi_e} J_e \to J_e/\langle a \rangle$.

The degree of $X_0(p) \to X'$ is $d \cdot n$ and Proposition 7.1 together with the left part of inequalities (23) implies

$$d \cdot n \le \frac{g-1}{\left(\frac{1}{4} - o(1)\right)g - 1} \le 4 + o(1)$$

for large enough $p$. This shows that if $d = 3$ or 4 one still has $a = 0$, whence the proposition's first two statements. Assuming (22) we have $d \cdot n < 3$, so that $d = 1$ and $a = 0$ by previous arguments. ☐

**Remark 7.3.** Replace, in Lemma 7.2, the map $X_0(p) \to J_e$ by $X_0(p) \xrightarrow{\varphi} J_0(p)^-$ (by which the former factorizes, by the way). The above proof shows that the map $X_0(p) \to \varphi(X_0(p))$ is of generic degree 1 (independently on any conjecture), but of course it need not be injective on points: a finite number of points can be mapped together to singular points on $\varphi(X_0(p))$. In our case one checks those are among the Heegner points $P$ such that $P = w_p(P)$ (for which we again refer to [Gross 1987a, Proposition 3.1]). Indeed, the endomorphism of $J_0(p)$ defined by multiplication by $(1 - w_p)$ factorizes through $\varphi$ and $\cdot(1 - w_p)$ is the map considered in (4) and what follows, inducing multiplication by 2 on tangent spaces. Therefore, if $P$ maps to a multiple point of $\varphi(X_0(p))$, it also maps to a multiple point of $(1-w_p)\circ\iota(X_0(p))$. Now assuming $X_0(p)$ has gonality larger than 2 (which is true as soon as $p > 71$ [Ogg 1974, Theorem 2]) the equality $\mathrm{cl}((1 - w_p)P) = \mathrm{cl}((1 - w_p)P')$ in $J_0(p)$, for some $P'$ on $X_0(p)$ different from $P$, implies $P = w_pP$ and $P' = w_pP'$. That is, $P$ and $P'$ are Heegner points.

**Lemma 7.4.** *Suppose $P$ belongs to $X_0(p^2)(K)$ for some quadratic number field $K$, and $P$ is not a complex multiplication point. Then for one of the two natural degeneracy morphisms $\pi$ from $X_0(p^2)$ to $X_0(p)$, the point $Q := \pi(P)$ in $X_0(p)(K)$ does not define a $\mathbb{Q}$-valued point of the quotient curve $X_0^+(p) := X_0(p)/w_p$.*

*Proof.* Using the modular interpretation, we write $P = (E, C_{p^2})$ for $E$ an elliptic curve over $K$ and $C_{p^2}$ a cyclic $K$-isogeny of degree $p^2$, from which we obtain the two points $Q_1 := (E, p \cdot C_{p^2})$ and $Q_2 := (E/p \cdot C_{p^2}, C_{p^2} \bmod p \cdot C_{p^2})$ in $X_0(p)(K)$. Assume both $Q_1$ and $Q_2$ do define elements of $X_0^+(p)(\mathbb{Q})$. If $\sigma$ denotes a generator of $\mathrm{Gal}(K/\mathbb{Q})$ we then have

$$w_p(Q_1) = (E/p \cdot C_{p^2}, E[p] \bmod p \cdot C_{p^2}) \simeq \sigma(Q_1)$$

and

$$w_p(Q_2) = (E/C_{p^2}, E[p] + C_{p^2} \bmod C_{p^2}) \simeq \sigma(Q_2).$$

Therefore $E \simeq {}^{\sigma}(E/p \cdot C_{p^2}) \simeq E/C_{p^2}$, which means $E$ has complex multiplication. ☐

We can now conclude with the main result of this paper.

**Theorem 7.5.** *There is an integer $C$ such that the following holds. If $p$ is a prime number such that* (22), *the weak form of Brumer's conjecture, holds and $P$ is a quadratic point of $X_0(p)$ (that is, $P$ is an element of $X_0(p)(K)$ for some quadratic number field $K$) which does not come from $X_0(p)^+(\mathbb{Q})$, then its $j$-height satisfies*

$$\mathrm{h}_j(P) < C \cdot p^5 \log p. \tag{115}$$

*If $P$ is a quadratic point of $X_0(p^2)$ then the same conclusion holds without further assumption apart from* (22).

*Proof.* In the case $P$ is a quadratic point of $X_0(p^2)$, by Lemma 7.4, one can deduce from $P$ a point $P'$ in $X_0(p)(K)$ which does not induce an element of $X_0^+(p)(\mathbb{Q})$ and whose $j$-height, say, is equal to $\mathrm{h}_j(P)+O(\log p)$ for an explicit function $O(\log p)$ (see, e.g., [Pellarin 2001, inequality (51) on p. 240; Bilu et al. 2013, Proposition 4.4(i)]). Replace $P$ by $P'$ if necessary. By Theorem 4.6 it is now sufficient to prove that $\mathrm{h}_\Theta(P-\infty)=O(p^5\log p)$.

Keep the notation of Lemma 7.2. By construction, the point

$$a := \varphi_e(P)+\varphi_e({}^\sigma P)=\varphi_e(P)-\varphi_e(w_p({}^\sigma P))=\varphi_e(P-w_p({}^\sigma P))$$

is torsion. First assume $a=0$. Set $X^{(2),-} := \{\iota_\infty(x)-\iota_\infty(y),\,(x,y)\in X_0(p)^2\}$ as in Proposition 5.3. Recall from Section 2 that $\tilde{I}_{J_e^\perp,N_e^\perp}: J_e^\perp \to \tilde{J}_e^\perp$ is the map defined as in (3), that $\iota_{\tilde{J}_e^\perp,N_e^\perp}$ is the embedding $\tilde{J}_e^\perp \hookrightarrow J_0(p)$, and denote by $[N_{\tilde{J}_e^\perp}]_{\tilde{J}_e^\perp}$ the multiplication by $N_{\tilde{J}_e^\perp}$ restricted to $\tilde{J}_e^\perp$. As in (8) and before Corollary 5.7 we use our pseudoprojections and define

$$\tilde{X}^{(2),-} := \iota_{\tilde{J}_e^\perp,N_e^\perp}[N_{\tilde{J}_e^\perp}]_{\tilde{J}_e^\perp}^{-1}\tilde{I}_{J_e^\perp,N_e^\perp}\pi_{J_e^\perp}(X^{(2),-}).$$

Then $P-w_p({}^\sigma P)$ belongs to $X^{(2),-}\cap \tilde{J}_e^\perp$, and even to the intersection of surfaces (in the generic fiber)

$$X^{(2),-}\cap \tilde{X}^{(2),-}.$$

Recall (see (8)) that $\tilde{X}^{(2),-}$ is a priori highly nonconnected, being the inverse image of multiplication by $N_{\tilde{J}_e^\perp}$ in $\tilde{J}_e^\perp$ of the (irreducible) surface $\tilde{I}_{J_e^\perp,N_e^\perp}\pi_{J_e^\perp}(X^{(2),-})$. However, in what follows we can replace $\tilde{X}^{(2),-}$ by one of its connected components containing $P-w_p({}^\sigma P)$. Denote that component by $\tilde{X}_P^{(2),-}$.

By construction, the theta degree and height of $\tilde{X}_P^{(2),-}$, as an irreducible subvariety of $J_0(p)$ endowed with $\Theta$, are those of $\pi_{J_e^\perp}(X^{(2),-})=X_{e^\perp}^{(2),-}$ relative to the only natural hermitian sheaf of $J_e^\perp$, that is, the $\Theta_e^\perp=\Theta_{J_e^\perp}$ described in paragraph 2A2. One can therefore apply Proposition 5.3 to obtain that all theta degrees are $O(p^2)$, all Néron–Tate theta heights are $O(\log p)$. We claim the dimension of $(X^{(2),-}\cap \tilde{X}_P^{(2),-})$ is zero. That intersection indeed corresponds to pairs of distinct points on $X_0(p)$ having same image (0) under $\varphi_e$. On the other hand, Brumer's conjecture implies $X_0(p)\to\varphi_e(X_0(p))$ has generic degree one (see Lemma 7.2), so our intersection points correspond to singular points on $\varphi_e(X_0(p))$, which of course make a finite set.

We therefore are in position to apply our arithmetic Bézout theorem (Proposition 6.1), which yields $\mathrm{h}_\Theta(P-w_p({}^\sigma P))\le O(p^5\log p)$. The two points $(P-\infty)$ and $(w_p({}^\sigma P)-\infty)$ have same $\Theta$-height (recall $w_p$ is an isometry on $J_0(p)$ for $\mathrm{h}_\Theta$, compare the end of Remark 4.3), and are by hypothesis different, so one can apply them Mumford's repulsion principle (Proposition 5.9) to obtain

$$\mathrm{h}_\Theta(P-\infty)\le O(p^5\log p). \tag{116}$$

Let us finally deal with the case when the torsion point $a=\varphi_e(P)+\varphi_e({}^\sigma P)$ is nonzero. We adapt the previous argument: pick a lift $\tilde{a}\in J_0(p)(\overline{\mathbb{Q}})$ of $a$ by $\pi_e^\perp$ which also is torsion, and let $t_{\tilde{a}}$ be the translation by $\tilde{a}$ in $J_0(p)$. Replace $(P-w_p({}^\sigma P))$ by $t_{\tilde{a}}^*(P-w_p({}^\sigma P))$, $X^{(2),-}$ by $t_{\tilde{a}}^*X^{(2),-}$ and $\tilde{X}^{(2),-}$ by

$$\widetilde{t_{\tilde{a}}^*X}^{(2),-}=\iota_{\tilde{J}_e^\perp,N_e^\perp}[N_{\tilde{J}_e^\perp}]_{\tilde{J}_e^\perp}^{-1}\tilde{I}_{J_e^\perp,N_e^\perp}\pi_{J_e^\perp}(t_{\tilde{a}}^*X^{(2),-}).$$

Now $t_{\tilde{a}}^*(P-w_p({}^\sigma P))$ belongs to $(t_{\tilde{a}}^*X^{(2),-}\cap\widetilde{t_{\tilde{a}}^*X}^{(2),-})$. The theta degree and height of $t_{\tilde{a}}^*X^{(2),-}$ and $\widetilde{t_{\tilde{a}}^*X}^{(2),-}$ (or rather, as above, some connected component $\widetilde{t_{\tilde{a}}^*X}_P^{(2),-}$ of it containing $t_{\tilde{a}}^*(P-w_p({}^\sigma P))$) are the same as for the former objects in the case $a=0$. The fact that the intersection

$$t_{\tilde{a}}^*X^{(2),-}\cap\widetilde{t_{\tilde{a}}^*X}_P^{(2),-}$$

is zero-dimensional comes from the fact that otherwise, we would have $\varphi_e(X_0(p))=a-\varphi_e(X_0(p))$, a contradiction with our present hypothesis $a\neq 0$ by Lemma 7.2. The height bound for $P$ is therefore the same as (116). □

**Corollary 7.6.** *Under the assumptions of Theorem 7.5, if $p$ is a large enough prime number and $P$ is a quadratic point of $X_0(p^\gamma)$ for some integer $\gamma$, such that $P$ is not a cusp nor a complex multiplication point, then $\gamma\leq 10$.*

*Proof.* Let $P$ be a point in $X_0(p^\gamma)(K)$, which is not a cusp nor a CM point, for some quadratic number field $K$. Then the isogeny bounds of [Gaudron and Rémond 2014b, Theorem 1.4] imply there is some real $\kappa$ with

$$p^\gamma<\kappa(\mathrm{h}_j(P))^2.$$

Now Theorem 7.5 gives that there is some absolute real constant $B$ such that, if $p\geq B$ then $\gamma\leq 10$. □

**Remark 7.7.** A similar (but technically simpler) approach for the morphism $X_0(p)\to J_e$ over $\mathbb{Q}$ should give (independently of any conjecture) a bound of shape $O(p^3\log p)$ for the $j$-height of $\mathbb{Q}$-rational (noncuspidal) points of $X_0(p)$ (which are known not to exist for $p>163$ by Mazur's theorem). The same should apply for $\mathbb{Q}$-points of $X_{\mathrm{split}}(p)$ (and here again, we obtain a weak version of known results).

Actually, sharpening results directly coming from Section 4 (that is, avoiding the use of Bézout) might even yield the full strength of the above results about $X_0(p)(\mathbb{Q})$ and $X_{\mathrm{split}}(p)(\mathbb{Q})$, with more straightforward (unconditional) proofs.

## Appendix: An upper bound for the theta function

by Pascal Autissier

In this appendix, I give a new upper bound for the norm of the classical theta function on any complex abelian variety. This result, apart from its role in the present paper (see Section 6), has been used by Wilms [2017] to bound the Green–Arakelov function on curves.

***Result.*** Let $g$ be a positive integer. Write $\mathbb{H}_g$ for the Siegel space of symmetric matrices $Z \in \mathrm{M}_g(\mathbb{C})$ such that $\operatorname{Im} Z$ is positive definite. To every $Z \in \mathbb{H}_g$ is associated the theta function defined by,

$$\theta_Z(z) = \sum_{m\in\mathbb{Z}^g} \exp(i\pi\,{}^{\mathrm{t}}mZm + 2i\pi\,{}^{\mathrm{t}}mz), \quad \forall z \in \mathbb{C}^g,$$

and its norm defined by,

$$\|\theta_Z(z)\| = \sqrt[4]{\det Y}\exp(-\pi\,{}^{\mathrm{t}}yY^{-1}y)|\theta_Z(z)|, \quad \forall z = x+iy \in \mathbb{C}^g,$$

where $Y = \operatorname{Im} Z$.

My contribution here is the following:

**Proposition A.1.** *Let $Z \in \mathbb{H}_g$ and assume that $Z$ is Siegel-reduced. Put $c_g = (g+2)/2$ if $g \le 3$ and $c_g = \big((g+2)/(\pi\sqrt{3})\big)^{g/2}(g+2)/2$ if $g \ge 4$. The upper bound $\|\theta_Z(z)\| \le c_g(\det \operatorname{Im} Z)^{1/4}$ holds for every $z \in \mathbb{C}^g$.*

Let us remark that $c_g \le g^{g/2}$ for every $g \ge 2$. In comparison, Edixhoven and de Jong [2011b, p. 231] obtained the statement of Proposition A.1 with $c_g$ replaced by $2^{3g^3+5g}$.

***Proof.*** Fix a positive integer $g$. Denote by $\mathbb{S}_g$ the set of symmetric matrices $Y \in \mathrm{M}_g(\mathbb{R})$ that are positive definite. Let us recall a special case of the functional equation for the theta function (see [Mumford 1983, (5.6), p. 195]: for every $Y \in \mathbb{S}_g$ and every $z \in \mathbb{C}^g$, one has

$$\theta_{iY^{-1}}(-iY^{-1}z) = \sqrt{\det Y}\exp(\pi\,{}^{\mathrm{t}}zY^{-1}z)\theta_{iY}(z). \tag{117}$$

**Lemma A.2.** *Let $Z \in \mathbb{H}_g$ and $z \in \mathbb{C}^g$. Putting $Y = \operatorname{Im} Z$, one has the inequality*

$$\|\theta_Z(z)\| \le \|\theta_{iY}(0)\| = \theta_{iY}(0)\sqrt[4]{\det Y}.$$

*Proof.* Put $y = \operatorname{Im} z$. One has

$$|\theta_Z(z)| = \left|\sum_{m\in\mathbb{Z}^g} \exp(i\pi\,{}^{\mathrm{t}}mZm + 2i\pi\,{}^{\mathrm{t}}mz)\right| \le \sum_{m\in\mathbb{Z}^g} |\exp(i\pi\,{}^{\mathrm{t}}mZm + 2i\pi\,{}^{\mathrm{t}}mz)| = \theta_{iY}(iy),$$

that is, $\|\theta_Z(z)\| \le \|\theta_{iY}(iy)\|$. The functional equation (117) gives $\|\theta_{iY^{-1}}(Y^{-1}y)\| = \|\theta_{iY}(iy)\|$, and one deduces

$$\|\theta_Z(z)\| \le \|\theta_{iY^{-1}}(Y^{-1}y)\|. \tag{118}$$

Applying again (118) with $Z$ replaced by $iY^{-1}$ and $z$ by $Y^{-1}y$, one gets

$$\|\theta_{iY^{-1}}(Y^{-1}y)\| \le \|\theta_{iY}(0)\|.$$

Whence the result. □

Let $Y \in \mathbb{S}_g$. Define $\lambda(Y) = \min_{m\in\mathbb{Z}^g-\{0\}} {}^{\mathrm{t}}mYm$. For every $t \in \mathbb{R}_+^*$, put

$$f_Y(t) = \theta_{itY}(0) = \sum_{m\in\mathbb{Z}^g} \exp(-\pi t\,{}^{\mathrm{t}}mYm).$$

**Lemma A.3.** *Let $Y \in \mathbb{S}_g$ and put $\lambda = \lambda(Y)$. The following properties hold*:

(a) *The function $\mathbb{R}_+^* \to \mathbb{R}$ that maps $t$ to $t^{g/2} f_Y(t)$ is increasing.*

(b) *One has the estimate $f_Y((g+2)/(2\pi\lambda)) \leq (g+2)/2$.*

*Proof.*

(a) The functional equation (117) implies $\sqrt{\det Y}\, t^{g/2} f_Y(t) = f_{Y^{-1}}(1/t)$ for every $t \in \mathbb{R}_+^*$; conclude by remarking that $f_{Y^{-1}}$ is decreasing.

(b) Part $(a)$ gives $\frac{\mathrm{d}}{\mathrm{d}t}[t^{g/2} f_Y(t)] \geq 0$, that is, $\frac{g}{2t} f_Y(t) \geq -f_Y'(t)$ for every $t > 0$. On the other hand,

$$-\tfrac{1}{\pi} f_Y'(t) = \sum_{m \in \mathbb{Z}^g} {}^{\mathrm{t}}mYm \exp(-\pi t\, {}^{\mathrm{t}}mYm) \geq \sum_{m \in \mathbb{Z}^g - \{0\}} \lambda \exp(-\pi t\, {}^{\mathrm{t}}mYm) = \lambda[f_Y(t) - 1].$$

One infers $\frac{g}{2t} f_Y(t) \geq \pi\lambda[f_Y(t) - 1]$. Choosing $t = (g+2)/(2\pi\lambda)$, one obtains the result. □

**Proposition A.4.** *Let $Y \in \mathbb{S}_g$. Putting $\lambda = \lambda(Y)$, one has the upper bound*

$$\theta_{iY}(0) \leq \frac{g+2}{2} \max\left[\left(\frac{g+2}{2\pi\lambda}\right)^{g/2}, 1\right].$$

*Proof.* Put $t = (g+2)/(2\pi\lambda)$. If $t \geq 1$, then Lemma A.3(a) implies the inequality $f_Y(1) \leq t^{g/2} f_Y(t)$. If $t \leq 1$, then $f_Y(1) \leq f_Y(t)$ since $f_Y$ is decreasing. In any case, one obtains

$$\theta_{iY}(0) = f_Y(1) \leq \max(t^{g/2}, 1) f_Y(t).$$

Conclude by applying Lemma A.3(b). □

Now, to prove Proposition A.1 from Lemma A.2 and Proposition A.4, it suffices to observe that if $Z \in \mathbb{H}_g$ is Siegel-reduced, then $\lambda(\operatorname{Im} Z) \geq \frac{\sqrt{3}}{2}$ (see [Igusa 1972, Lemma 15, p. 195]).

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As already stressed, the influence of the orange book [Edixhoven and Couveignes 2011] should be obvious all over this text. We have used many results of the deep effective Arakelov study of modular

[9] Although, as goes without saying, he bears no responsibility for the mistakes which remain.

[10] They have already been used by R. Wilms [2017]; see the introduction to the Appendix.

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