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Cohomologie analytique des arrangements d'hyperplans

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Nous étudions la cohomologie de faisceaux analytiques sur le complémentaire, dans l'espace projectif, d'une collection infinie d'hyperplans bien choisie, comme l'espace symétrique de Drinfeld. En particulier, le faisceau de fonctions inversibles sur ces espaces rigides n'a pas de cohomologie de degré supérieur ou égal à 1. Ceci démontre l'annulation du groupe de Picard, et les méthodes utilisées nous donnent une description pratique des fonctions inversibles globales.

We study the cohomology of some analytic sheaves on the complement in the projective space of a suitable infinite collection of hyperplanes like the Drinfeld symmetric space. In particular, the sheaf of invertible functions on these rigid spaces has no cohomology in degree greater or equal to 1. This proves the vanishing of the Picard group and the methods used give a convenient description of the global invertible functions.

Introduction

Cet article est lié à une série de travaux récents [Colmez et al. 2020a; 2020b; 2021] portant sur la géométrie et la cohomologie p-adique, pour p premier, des espaces symétriques de Drinfeld et de leurs revêtements. Il fait aussi partie d'une autre série d'articles tirés de la thèse de l'auteur [Junger 2022a;

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2022b; 2022c] qui visent à déterminer la partie supercuspidale de la cohomologie de de Rham du premier revêtement de ces espaces (voir le théorème A de [Junger 2022a]). Chacun de ces quatre articles présente des aspects de la géométrie du premier revêtement qui seront utilisés de manière cruciale dans la preuve du résultat principal. Ce travail constitue la première étape de ce programme et se concentre essentiellement sur la géométrie des espaces symétriques eux-mêmes (qui permettra de décrire le premier revêtement dans [Junger 2022b]).

Les espaces symétriques de Drinfeld sont des cas particuliers d'arrangements (infinis) d'hyperplans et l'objet de cet article est de comprendre ce qui se passe pour des arrangements plus généraux. L'étude de leur cohomologie étale p-adique semblant délicate (en effet, les travaux cités utilisent des propriétés spécifiques de l'espace de Drinfeld), nous nous intéresserons plutôt à leur cohomologie analytique à coefficients dans le faisceau $\mathbb{G}_m$ des fonctions inversibles. Notre résultat principal affirme que beaucoup d'arrangements (même infinis) d'hyperplans sont acycliques pour $\mathbb{G}_m$. Par exemple, cela entraîne que le groupe de Picard des espaces de Drinfeld est trivial, ce qui ne semble pas être connu. Il serait intéressant d'avoir des résultats analogues pour la cohomologie étale, mais cela nous semble inaccessible pour le moment. En effet, le calcul de $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ pour l'espace de Drinfeld X de dimension plus grande que 1 semble déjà délicat (la partie de torsion est cependant bien comprise grâce aux résultats de Schneider et Stuhler [1991] et de Colmez, Dospinescu et Nizioł [Colmez et al. 2021]).

Avant de préciser nos résultats principaux, mentionnons-en certaines motivations et applications à l'étude du premier revêtement des espaces de Drinfeld. Soit K une extension finie de $\mathbb{Q}_p$, $\mathcal{O}_K$ son anneau d'entiers, $\mathbb{F} = \mathbb{F}_q$ son corps résiduel et ϖ une uniformisante. Soit aussi C le complété d'une clôture algébrique de K. On note $\mathbb{H}^d_K$ l'espace symétrique de Drinfeld de dimension $d \geq 1$, i.e., l'espace rigide analytique sur K défini par [1]

$$\mathbb{H}^d_K = \mathbb{P}^d_K \setminus \bigcup_{H \in \mathcal{H}} H,$$

avec $\mathcal{H}$ l'ensemble des hyperplans K-rationnels et $\mathbb{P}^d_K$ l'espace projectif rigide analytique de dimension d sur K. L'espace $\mathbb{H}^d_K$ possède un modèle formel semi-stable $\mathbb{H}^d_{\mathcal{O}_K}$, construit par Deligne. Soit D l'algèbre à division sur K d'invariant $1/(d+1)$ et Π_D une uniformisante. Un théorème fondamental de Drinfeld [1976] fournit une interprétation modulaire de l'espace $\mathbb{H}^d_{\mathcal{O}_K}$, et cette description entraîne l'existence d'un $\mathcal{O}_D$-module formel universel $\mathfrak{X}$ sur $\mathbb{H}^d_{\mathcal{O}_K}$. Les points de Π_D-torsion $\mathfrak{X}[\Pi_D]$ forment un schéma formel en $\mathbb{F}_p$-espaces vectoriels de Raynaud. Ces derniers admettent une classification [Raynaud 1974] et sont caractérisés par la donnée des parties isotypiques $(\mathscr{L}_i)_{i \in \mathbb{Z}/(d+1)\mathbb{Z}}$ de $\mathscr{O}(\mathfrak{X}[\Pi_D])$ pour certains caractères de $\mathbb{F}_{q^{d+1}}$, dits fondamentaux. Comprendre les fibrés en droites $(\mathscr{L}_i)_i$ universels sur $\mathbb{H}^d_{\mathcal{O}_K}$ est essentiel pour comprendre la géométrie du premier revêtement Σ^1 de $\mathbb{H}^d_K$. En fibre spéciale, les faisceaux $(\mathscr{L}_i)_i$ sont relativement bien compris et sont étudiés dans [Teitelbaum 1989; 1990; 1993; Grosse-Klönne 2004b]. L'annulation du groupe de Picard de $\mathbb{H}^d_K$, qui découle de nos résultats, fournit donc une description en

1. Il n'est pas immédiat que ce complémentaire d'un nombre infini de parties fermées est bien un espace rigide analytique, mais cela découle de la remarque 2.2.

fibre générique de ces faisceaux localement libres de rang 1 sur $\mathbb{H}^d_{\mathcal{O}_K}$. Dans un travail ultérieur, nous obtiendrons une classification des μ_N-torseurs sur $\mathbb{H}^d_K$ avec $N = q^{d+1}-1$ et nous donnerons une équation explicite du revêtement modéré de l'espace symétrique de Drinfeld.

Passons maintenant à notre résultat principal. Gardons les notations ci-dessus. Soit $\mathcal{A}$ une partie fermée (par exemple une partie finie) de l'espace profini $\mathcal{H}$ et posons

$$\mathrm{Int}(\mathcal{A}) = \mathbb{P}^d_K \setminus \bigcup_{H \in \mathcal{A}} H.$$

Alors $\mathrm{Int}(\mathcal{A})$ possède encore une structure naturelle d'espace rigide analytique sur K. Si L est une extension complète de K et si X est un K-espace analytique, on note $X_L = X \hat{\otimes}_K L$.

Théorème A. *Avec les notations ci-dessus, pour toute partie fermée $\mathcal{A}$ de $\mathcal{H}$ et toute extension complète L de K, on a $\mathrm{H}^i_{\mathrm{an}}(\mathrm{Int}(\mathcal{A})_L, \mathbb{G}_m) = 0$ pour $i \geq 1$.*

Remarque. (1) L'égalité [Berkovich 1993, Proposition 4.1.10]

$$\mathrm{H}^1_{\mathrm{an}}(X, \mathbb{G}_m) = \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) = \mathrm{Pic}(X)$$

est valable pour tout espace analytique X. Ainsi, le groupe de Picard et les fonctions inversibles sur X peuvent être déterminées en calculant sa cohomologie analytique. Ce n'est malheureusement pas le cas des groupes de cohomologie en degrés strictement plus grands que 1.

(2) Nous prouvons aussi une version du théorème dans laquelle le faisceau $\mathbb{G}_m$ est remplacé par le sous-faisceau $\mathcal{O}^{**} = 1 + \mathcal{O}^{++}$ des fonctions $1 + f$ telles que $|f| < 1$ (la norme étant celle spectrale). Si $\mathcal{O}^+$ désigne le faisceau des fonctions f telles que $|f| \leq 1$, il est probable que $\mathrm{H}^i_{\mathrm{an}}(\mathrm{Int}(\mathcal{A})_L, \mathcal{O}^+) = 0$ pour $i \geq 1$, mais nous n'arrivons pas à le démontrer. Le résultat analogue avec $\mathrm{Int}(\mathcal{A})_L$ remplacé par une boule fermée est un théorème de Bartenwerfer [1982] (il est crucial d'utiliser la topologie analytique pour ce genre de résultat, car il est totalement faux pour la topologie étale). Nos méthodes permettent de démontrer que si le résultat de Bartenwerfer est aussi valable pour les polycouronnes, alors $\mathrm{H}^i_{\mathrm{an}}(\mathrm{Int}(\mathcal{A})_L, \mathcal{O}^+) = 0$ pour $i \geq 1$.

Notons $\mathbb{Z}[\![\mathcal{A}]\!]$ le dual du $\mathbb{Z}$-module $\mathcal{L}(\mathcal{A}, \mathbb{Z})$ des fonctions localement constantes sur $\mathcal{A}$ à valeurs dans $\mathbb{Z}$. On voit les éléments de $\mathbb{Z}[\![\mathcal{A}]\!]$ comme des mesures sur $\mathcal{A}$ à valeurs dans $\mathbb{Z}$. On note $\mathbb{Z}[\![\mathcal{A}]\!]^0$ le sous-groupe des mesures de masse totale 0 (l'orthogonal de la fonction constante 1).

Théorème B. *Pour toute partie fermée $\mathcal{A}$ de $\mathcal{H}$ et toute extension complète L de K, il existe un isomorphisme naturel*

$$\mathcal{O}^*(\mathrm{Int}(\mathcal{A})_L)/L^* \simeq \mathbb{Z}[\![\mathcal{A}]\!]^0.$$

Remarque. (1) Ce théorème a été récemment obtenu par Gekeler [2020] pour l'espace symétrique de Drinfeld. Notre méthode est complètement différente.

(2) Si l'on combine le théorème ci-dessus avec la suite exacte de Kummer et l'annulation du groupe de Picard, on obtient une description du groupe $\mathrm{H}^1_{\text{ét}}(\mathrm{Int}(\mathcal{A})_L, \mathbb{Z}/n\mathbb{Z})$ pour tout entier n. Cela semble

suggérer qu'il existe des descriptions explicites de la cohomologie étale en degré cohomologique plus grand. Voir [Colmez et al. 2021] pour le cas de l'espace de Drinfeld.

Nous finissons cette introduction en expliquant les grandes étapes de la preuve de nos résultats principaux. L'ingrédient technique principal est un résultat d'annulation de van der Put [1982], qui affirme que pour tout $r \in p^{\mathbb{Q}}$, le faisceau $\mathscr{O}^{(r)}$ des fonctions de norme spectrale strictement plus petite que r est acyclique sur les boules fermées et les polycouronnes de dimension arbitraire. Pour se ramener à ce type d'espaces, nous utilisons les constructions géométriques de Schneider et Stuhler [1991]. Plus précisément, l'espace $\mathrm{Int}(\mathcal{A})$ possède un recouvrement de type Stein par des affinoïdes $\mathrm{Int}(\mathcal{A}_n)$ obtenus en enlevant de $\mathbb{P}^d_K$ les tubes ouverts d'épaisseur $|\varpi|^n$ autour des hyperplans dans $\mathcal{A}$. Cela nous amène à étudier la géométrie d'un arrangement tubulaire

$$X_I = \mathbb{P}^d_K \setminus \bigcup_{i \in I} H_i(|\varpi|^n),$$

où $H_i(|\varpi|^n)$ est le voisinage tubulaire ouvert d'épaisseur $|\varpi|^n$ de l'hyperplan H_i. Nous allons supposer que ces voisinages tubulaires sont deux à deux distincts. Suivant Schneider et Stuhler, pour comprendre la géométrie de X, il s'agit de comprendre la géométrie des espaces de la forme

$$Y_J = \mathbb{P}^d_K \setminus \bigcap_{j \in J} H_j(|\varpi|^n)$$

avec $J \subset I$. Le point essentiel est que les espaces Y_J sont des fibrations localement triviales en boules fermées au-dessus d'espaces projectifs, dont la dimension dépend de la combinatoire des hyperplans. Cela permet d'utiliser les résultats d'annulation de van der Put et nous ramène à l'étude de certains complexes de Čech relativement explicites. Pour transférer l'étude des faisceaux sur les Y_J à X_I, nous montrons un lemme combinatoire, essentiellement basé sur la suite de Mayer–Vietoris, qui remplace la suite spectrale utilisée par Schneider et Stuhler (et dont l'étude devient assez compliquée dans notre situation). Cela permet de démontrer que les faisceaux $\mathscr{O}^{(r)}$ sont acycliques sur X_I. Un argument basé sur le logarithme tronqué permet d'en déduire l'acyclicité du faisceau $\mathscr{O}^{**} = 1 + \mathscr{O}^{++}$ des fonctions $1 + f$ vérifiant $|f| < 1$ sur les X_I. Enfin, l'étude du quotient $\mathbb{G}_m / \mathscr{O}^{**}$ fait apparaître des complexes de Čech identiques à ceux apparaissant en géométrie algébrique, ce qui permet de passer de $\mathscr{O}^{**}$ à $\mathbb{G}_m$.

Le paragraphe précédent explique la preuve de l'acyclicité de $\mathbb{G}_m$ sur les espaces X_I. Le passage de ces espaces à $\mathrm{Int}(\mathcal{A})$ n'est pas trivial et représente en fait le cœur technique de l'article. Pour expliquer la difficulté, notons que l'on dispose d'un recouvrement Stein $\mathrm{Int}(\mathcal{A}) = \bigcup_{n \geq 1} X_{I_n}$, où les X_{I_n} sont des espaces du même type que ceux introduits ci-dessus, les I_n étant des ensembles finis, de plus en plus grands. On en déduit une suite exacte

$$0 \to \mathrm{R}^1 \varprojlim_n \mathrm{H}^{s-1}_{\mathrm{an}}(X_{I_n}, \mathbb{G}_m) \to \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathbb{G}_m) \to \varprojlim_n \mathrm{H}^s_{\mathrm{an}}(X_{I_n}, \mathbb{G}_m) \to 0.$$

Pour $s > 1$, cela permet de démontrer l'annulation de $\mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathbb{G}_m)$, mais pour $s = 1$, il s'agit de démontrer que

$$R^1 \varprojlim_n \mathscr{O}^*(X_{I_n}) = 0.$$

Pour cela, on se ramène à démontrer le même résultat avec les faisceaux $\mathscr{O}^{**}$ et $\mathscr{O}^{(r)}$ à la place de $\mathbb{G}_m$. Le point crucial à démontrer est alors une version en dimension quelconque du lemme 1.3 de [Colmez et al. 2020a], qui permet de comprendre la flèche de restriction $\mathscr{O}^{**}(X_{I_{n+1}}) \to \mathscr{O}^{**}(X_{I_n})$. Plus précisément, par application du logarithme, nous nous ramenons à montrer, pour r assez petit, qu'il existe une constante c telle que, pour tout n, $\mathscr{O}^{(r)}(X_{I_{n+c}}) \subset \mathcal{O}_L^{(r)} + \varpi \mathscr{O}^{(r)}(X_{I_n})$ avec $\mathcal{O}_L^{(r)}$ le sous-ensemble des éléments de L de norme strictement inférieure ou égale à r. C'est le point le plus délicat de l'article et la preuve en est assez indirecte, car nous n'avons pas de description explicite des groupes $\mathscr{O}^{(r)}(X_{I_n})$.

Notations et conventions. Dans tout l'article, on fixe un nombre premier p et une extension finie K de $\mathbb{Q}_p$. On note $\mathcal{O}_K$ son anneau des entiers, ϖ une uniformisante et $\mathbb{F} = \mathbb{F}_q$ son corps résiduel. On note $C = \hat{\bar{K}}$ la complétion d'une clôture algébrique de K et $\breve{K}$ la complétion de l'extension maximale non ramifiée de K. Soit $L \subset C$ une extension complète de K susceptible de varier (pouvant par exemple être égale à K, $\breve{K}$, ou C), d'anneau des entiers $\mathcal{O}_L$, d'idéal maximal $\mathfrak{m}_L$ et de corps résiduel κ.

Soit S un L-espace rigide analytique.[2] On note respectivement $\mathbb{A}^n_{\mathrm{rig},S}$ et $\mathbb{P}^n_{\mathrm{rig},S}$ les espaces affine et projectif rigides analytiques de dimension relative n sur S. Si $s = (s_i)_{1 \le i \le n}$ est une famille de nombres rationnels, le polydisque rigide fermé sur S de polyrayon $(|\varpi|^{s_i})_i$ sera noté $\mathbb{B}^n_S(|\varpi|^s)$ ou $\mathbb{B}^n_S(s)$ par abus. L'espace $\mathbb{B}^n_S$ sera la boule unité et les boules ouvertes seront notées $\mathring{\mathbb{B}}^n_S$ et $\mathring{\mathbb{B}}^n_S(s)$. Si S est maintenant un schéma, $\mathbb{A}^n_{\mathrm{zar},S}$ sera l'espace affine sur S et $\mathbb{P}^n_{\mathrm{zar},S}$ l'espace projectif.

Si X est un L-espace analytique réduit, on note $\mathscr{O}_X^+$ le faisceau des fonctions à puissances bornées, $\mathscr{O}_X^{++}$ le faisceau des fonctions topologiquement nilpotentes, $\mathscr{O}_X^{(r)}$ le faisceau des fonctions bornées strictement en norme spectrale par r, $\mathscr{O}_X^*$ (ou bien $\mathbb{G}_{m,X}$) le faisceau des fonctions inversibles et $\mathscr{O}_X^{**}$ le faisceau $1 + \mathscr{O}_X^{++}$. Si $X = \mathrm{Sp}(L)$, on écrit $\mathcal{O}_L^{(r)} = \mathscr{O}_X^{(r)}(X)$. Pour tout ouvert affinoïde réduit $U \subset X$, on munit $\mathscr{O}_X^*(U)$ de la topologie induite par le plongement $\mathscr{O}_X^*(U) \to \mathscr{O}_X(U)^2 : f \mapsto (f, f^{-1})$ (muni de la norme spectrale). On notera $K(x)$ le corps valué associé au point fermé $x \in X$.

Si X est un espace analytique sur L (resp. un schéma), la cohomologie d'un faisceau $\mathscr{F}$ sur le site analytique (resp. de Zariski) sera notée $\mathrm{H}^*_{\mathrm{an}}(X, \mathscr{F})$ (resp. $\mathrm{H}^*_{\mathrm{zar}}(X, \mathscr{F})$). Si $\mathcal{U}$ est un recouvrement de X (pour une des topologies précédemment nommées), la cohomologie de Čech de X pour le faisceau $\mathscr{F}$ par rapport au recouvrement $\mathcal{U}$ sera notée $\check{\mathrm{H}}^*(X, \mathscr{F}, \mathcal{U})$ et le complexe de cochaînes sera noté $\check{\mathcal{C}}^*(X, \mathscr{F}, \mathcal{U})$. Pour toutes ces théories cohomologiques, quand $U \subset X$ est un ouvert de X, la cohomologie à support dans le complémentaire de U sera notée $\mathrm{H}^*(X, U)$. Si Λ est un groupe cyclique d'ordre N premier à p et $\overline{X} = X \hat{\otimes} C$, le morphisme de Kummer sera noté $\kappa : \mathscr{O}^*(X) \to \mathrm{H}^1_{\mathrm{\acute{e}t}}(X, \Lambda_X)$ et $\bar{\kappa} : \mathscr{O}^*(X) \to \mathrm{H}^1_{\mathrm{\acute{e}t}}(\overline{X}, \Lambda_{\overline{X}})$ sera la restriction de $\mathscr{O}^*(\overline{X}) \to \mathrm{H}^1_{\mathrm{\acute{e}t}}(\overline{X}, \Lambda_{\overline{X}})$.

Enfin, nous noterons $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$ quand $a, b \in \mathbb{R}$.

2. Dans tout le reste de l'article, les espaces rigides analytiques et les affinoïdes seront supposés « à la Tate ». Ce cadre sera largement suffisant pour nos applications.

1. L'espace des hyperplans K-rationnels

On note $\mathcal{H}$ l'ensemble des hyperplans K-rationnels dans $\mathbb{P}^d$. L'ensemble $\mathcal{H}$ est profini, car il s'identifie à $\mathbb{P}^d(K)$.

Définissons maintenant quelques données relatives à l'ensemble $\mathcal{H}$. Si $a = (a_0, \ldots, a_d) \in C^{d+1} \backslash \{0\}$, alors l_a désignera l'application

$$b = (b_0, \ldots, b_d) \in C^{d+1} \mapsto \langle a, b \rangle := \sum_{0 \leq i \leq d} a_i b_i.$$

Ainsi $\mathcal{H}$ s'identifie à $\{\ker(l_a), a \in K^{d+1} \backslash \{0\}\}$ et à $\mathbb{P}^d(K)$.

Remarque 1.1. L'application précédente permet d'identifier un hyperplan dans $\mathcal{H}$ à sa droite orthogonale par dualité. Nous confondrons alors toujours un élément de $\mathcal{H}$ à sa droite associée. Dans la section suivante, nous attacherons des espaces rigides à certaines parties de $\mathcal{H}$ et nous pourrons décrire explicitement leur géométrie et leur combinatoire uniquement grâce aux relations linéaires sur $\mathcal{O}_K$ entre les générateurs unimodulaires de ces droites.

Le vecteur $a = (a_i)_i \in C^{d+1}$ est dit unimodulaire si $|a|_\infty := \max(|a_i|) = 1$. L'application $a \mapsto H_a := \ker(l_a)$ induit une bijection entre le quotient de l'ensemble des vecteurs unimodulaires $a \in K^{d+1}$ par l'action évidente de $\mathcal{O}_K^*$ et l'ensemble $\mathcal{H}$.

Pour $a \in K^{d+1}$ unimodulaire et $n \geq 1$, on considère l'application $l_a^{(n)}$ définie comme

$$b \in (\mathcal{O}_C/\varpi^n)^{d+1} \mapsto \langle a, b \rangle \in \mathcal{O}_C/\varpi^n$$

et on note

$$\mathcal{H}_n = \{\ker(l_a^{(n)}), a \in K^{d+1} \backslash \{0\} \text{ unimodulaire}\} \simeq \mathbb{P}^d(\mathcal{O}_K/\varpi^n).$$

Alors $\mathcal{H} = \varprojlim_n \mathcal{H}_n$ et chaque $\mathcal{H}_n$ est fini.

Soit $a \in K^{d+1}$ unimodulaire et $z \in \mathbb{P}^d(C)$. La quantité $|l_a(b)|$ ne dépend pas du choix du représentant unimodulaire b de z, et ne dépend que de la classe de a dans $\mathbb{P}^d(K)$. Cela permet de définir les tubes fermés et ouverts de rayon $\varepsilon > 0$ autour de l'hyperplan $H = \ker(l_{a_H}) \in \mathcal{H}$ par

$$\overline{H}(\varepsilon) = \{z \in \mathbb{P}^d(C), |l_{a_H}(z)| \leq \varepsilon\} \quad \text{et} \quad \mathring{H}(\varepsilon) = \{z \in \mathbb{P}^d(C), |l_{a_H}(z)| < \varepsilon\}.$$

Les extensions des scalaires par L seront notées $\overline{H}(\varepsilon)_L$ et $\mathring{H}(\varepsilon)_L$ et les complémentaires dans $\mathbb{P}^d_{\mathrm{rig},L}$ seront $\overline{H}(\varepsilon)_L^c$ et $\mathring{H}(\varepsilon)_L^c$. Il est à noter que $\overline{H}(|\varpi|^n)$ et $\mathring{H}(|\varpi|^n)$ ne dépendent que de la classe de H dans $\mathcal{H}_n$ et $\mathcal{H}_{n+1}$, respectivement. [3]

Remarque 1.2. Dans la définition des tubes ouverts ou fermés, nous procédons à une renormalisation lorsque nous évaluons $|l_a(z)|$ sur un représentant unimodulaire de z. Ce dernier dépend du choix de

3. Cela découle du fait que pour deux vecteurs a_1, a_2, on a l'identité $|l_{a_1}(z) - l_{a_2}(z)| \leq |a_1 - a_2|_\infty |z|_\infty$. En particulier, les inégalités $|l_{a_1}(z)| \leq \varepsilon$ et $|l_{a_2}(z)| \leq \varepsilon$ (resp. $|l_{a_1}(z)| < \varepsilon$ et $|l_{a_2}(z)| < \varepsilon$) sont vérifiées pour les mêmes vecteurs unimodulaires z si $|a_1 - a_2|_\infty \leq \varepsilon$ (resp. $|a_1 - a_2|_\infty < \varepsilon$).

coordonnées initiales. Toutefois, les changements de variables dans $\mathrm{GL}_{d+1}(\mathcal{O}_K)$ permutent les tubes de même rayon. Plus précisément, si $g \in \mathrm{GL}_{d+1}(\mathcal{O}_K)$, alors $g \cdot \overline{H}(\varepsilon) = \overline{(gH)}(\varepsilon)$.

2. Géométrie des arrangements

2A. ***Définitions et exemples.*** Pour toute collection $\mathcal{A}$ de parties de $\mathbb{P}^d_{\mathrm{rig},K}$, on note

$$\mathrm{Int}(\mathcal{A}) = \mathbb{P}^d_{\mathrm{rig},K} \setminus \bigcup_{H \in \mathcal{A}} H \quad \text{et} \quad \mathrm{Uni}(\mathcal{A}) = \mathbb{P}^d_{\mathrm{rig},K} \setminus \bigcap_{H \in \mathcal{A}} H.$$

Dans le cas général, ces constructions n'admettent pas forcément de structure naturelle d'espaces rigides analytiques. Toutefois, c'est le cas lorsque $\mathcal{A}$ est fini et constitué de parties fermées. Nous nous intéresserons dans la suite de l'article aux exemples suivants (où $\mathcal{A}$ est possiblement infini) pour lesquels une telle structure d'espace rigide existe.

Définition 2.1. Une collection $\mathcal{A}$ de parties de $\mathbb{P}^d_{\mathrm{rig},K}$ est appelée

- *arrangement algébrique* ou *algébrique généralisé* (d'hyperplans K-rationnels) si $\mathcal{A}$ est, respectivement, un sous-ensemble fini ou fermé de $\mathcal{H}$;
- *arrangement tubulaire ouvert* ou *fermé d'ordre n* si $\mathcal{A}$ est, respectivement, une famille finie de voisinages tubulaires *fermés* $\overline{H}(|\varpi|^n)$ ou *ouverts* $\mathring{H}(|\varpi|^n)$ avec $H \in \mathcal{H}$.

Remarque 2.2. (1) Pour simplifier l'exposition, nous avons choisi d'étudier les arrangements pour des hyperplans K-rationnels pour K une extension finie de $\mathbb{Q}_p$. Certaines de ces constructions peuvent s'étendre à des corps K beaucoup plus généraux. Par exemple, les arrangements tubulaires (ouverts ou fermés) peuvent être définis sur n'importe quel corps complet. Toutefois, les arguments que nous allons présenter se servent de manière cruciale de l'existence de fibrations dont la construction, que nous rappelons dans la section 2B, nécessite de supposer K de valuation discrète. Pour le cas des arrangements algébriques généralisés, il est nécessaire d'imposer en plus la finitude du corps résiduel de K pour que les $\mathrm{Int}(\mathcal{A})$ soient bien des espaces rigides (voir la note 4 et le troisième point de cette remarque). En caractéristique positive, ces constructions peuvent être transportées mutatis mutandis modulo les hypothèses précédentes sur K. En revanche, certains résultats d'annulation cohomologique que nous allons énoncer ne peuvent être prouvés dans ce cadre grâce aux méthodes de l'article. Nous renvoyons à la remarque 3.4 pour une discussion plus précise.

(2) Se donner un arrangement tubulaire ouvert ou fermé d'ordre n revient à se donner une partie (finie) [4] de $\mathcal{H}_n$ ou de $\mathcal{H}_{n+1}$, respectivement.

(3) Si $m > n$, tout arrangement tubulaire ouvert (ou fermé) d'ordre m induit un arrangement tubulaire ouvert (ou fermé) d'ordre n, appelé sa projection. Plus précisément, la projection d'un arrangement défini par une collection de voisinages tubulaires $(\overline{H}(|\varpi|^m))_{H\in I}$ est l'arrangement défini par la collection de

4. L'hypothèse de finitude est ici redondante sous les conditions que nous avons imposées sur le corps K, car $\mathcal{H}_n$ est fini pour tout n. Cette propriété n'est plus vraie si on raisonne sur un corps plus général dont le corps résiduel peut être infini ou peut ne pas être de valuation discrète.

voisinages tubulaires $(\overline{H}(|\varpi|^n))_{H\in I}$. Cela revient à considérer la projection d'une partie de $\mathcal{H}_m$ ou de $\mathcal{H}_{m+1}$ sur $\mathcal{H}_n$ ou sur $\mathcal{H}_{n+1}$, respectivement. Cette construction s'étend bien sûr au cas d'un arrangement algébrique ou algébrique généralisé. [5]

(4) Une famille d'arrangements tubulaires $(\mathcal{A}_n)_n$ telle que l'ordre de $\mathcal{A}_n$ soit n est dite compatible si, pour tout $m > n$, $\mathcal{A}_n$ est la projection de $\mathcal{A}_m$. Si $\mathcal{A} \subset \mathcal{H}$ est un arrangement algébrique généralisé, on construit par projection deux familles compatibles d'arrangements tubulaires ouverts ou fermés $(\mathcal{A}_n)_n$ par projection. Cette construction a pour intérêt de fournir un recouvrement croissant $\mathrm{Int}(\mathcal{A}) = \bigcup_n \mathrm{Int}(\mathcal{A}_n)$ pour les arrangements algébriques généralisés $\mathcal{A}$.

(5) Les constructions $\mathrm{Int}(\mathcal{A})$ (et $\mathrm{Uni}(\mathcal{A})$ lorsque $\mathcal{A}$ est fini) possèdent des structures naturelles d'espaces rigides analytiques sur K. Le seul cas non trivial est celui d'un arrangement algébrique généralisé, qui découle du point précédent.

(6) Plus précisément, l'espace $\mathrm{Int}(\mathcal{A})$ est un affinoïde ou quasi Stein si $\mathcal{A}$ est un arrangement tubulaire fermé ou ouvert, respectivement.

Exemple 2.3. L'espace symétrique de Drinfeld $\mathbb{H}_K^d$ est l'arrangement d'hyperplans généralisé $\mathrm{Int}(\mathcal{H})$.

Nous allons définir le rang d'un arrangement $\mathcal{A}$, qui permettra de décrire la géométrie de $\mathrm{Uni}(\mathcal{A})$.

Définition 2.4. Nous donnons la notion de rang pour des parties finies de $\mathcal{H}$ et de $\mathcal{H}_n$. D'après la deuxième observation de la remarque 2.2, cela induit une notion de rang pour les arrangements algébriques et tubulaires ouverts ou fermés.

— Si $\mathcal{A} \subset \mathcal{H}$, on se donne, pour tout $H \in \mathcal{A}$, un vecteur a_H unimodulaire tel que $H = \ker(l_{a_H})$. On pose $\mathrm{rg}(\mathcal{A}) = \mathrm{rg}_{\mathcal{O}_K}\big(\sum_{H\in\mathcal{A}} \mathcal{O}_K a_H\big)$.

— Si $\mathcal{A} \subset \mathcal{H}_n$, on se donne, pour tout H dans $\mathcal{A}$, un vecteur a_H unimodulaire dans $\mathcal{O}_K^{d+1}/\varpi^n\mathcal{O}_K^{d+1}$ tel que $H = \ker(l_{a_H})$ et $\tilde{a}_H$ un relevé dans $\mathcal{O}_K^{d+1}$. On écrit [6]

$$\sum_{H\in\mathcal{A}} \mathcal{O}_K \tilde{a}_H = \bigoplus_{i=0}^{d} \varpi^{\alpha_i}\mathcal{O}_K e_i$$

pour (e_i) une base de $\mathcal{O}_K^{d+1}$ bien choisie. On pose alors $\mathrm{rg}(\mathcal{A}) = \mathrm{card}\{i : \alpha_i < n\}$. Cette quantité ne dépend pas des choix des a_H et de leur relevé. Intuitivement, le rang correspond à $\mathrm{rg}(\mathcal{A}) = \mathrm{rg}_{\mathcal{O}_K/\varpi^n\mathcal{O}_K}\big(\sum_{H\in\mathcal{A}}(\mathcal{O}_K/\varpi^n\mathcal{O}_K)a_H\big)$.

2B. ***La suite spectrale associée à un arrangement.*** Dorénavant, pour tout arrangement d'hyperplans $\mathcal{A}$, nous verrons $\mathrm{Int}(\mathcal{A})$ et $\mathrm{Uni}(\mathcal{A})$ comme des L-espaces analytiques par extension des scalaires. Si H désigne la cohomologie de de Rham ou la cohomologie d'un faisceau $\mathscr{F}$ sur le site étale ou analytique, on a

$$E_1^{-r,s} = \bigoplus_{(H_i)_{0\le i\le r}\in\mathcal{A}^{r+1}} \mathrm{H}^s(\mathbb{P}^d_{\mathrm{rig},L}, \mathrm{Uni}(\{H_i\})) \Rightarrow \mathrm{H}^{s-r}(\mathbb{P}^d_{\mathrm{rig},L}, \mathrm{Int}(\mathcal{A})), \tag{1}$$

5. Dans le cas des arrangements algébriques généralisés, il est nécessaire de demander à ce que K soit de valuation discrète et de corps résiduel fini pour que les projections d'ordre $n \in \mathbb{N}$ forment bien une famille finie de voisinages tubulaires.

6. α_i peut être infini et dans ce cas on adopte la convention $\varpi^\infty = 0$.

où $\mathcal{A}$ est un arrangement algébrique tubulaire d'ordre n ouvert ou fermé et $\mathrm{H}(X, Y)$ représente la cohomologie de X à support dans $X \setminus Y$, par un argument général de suites spectrales (voir section 2, proposition 6 et lemme 7 de [Schneider et Stuhler 1991] ainsi que les discussions qui précèdent).

Soit $\mathcal{A}$ un arrangement (algébrique, tubulaire ouvert ou fermé) et $\mathcal{B} \subset \mathcal{A}$ non vide de cardinal $r+1$. Nous allons donc chercher à décrire la géométrie de $\mathrm{Uni}(\mathcal{B})$ suivant si $\mathcal{A}$ est algébrique, tubulaire ouvert ou fermé. Si $r = 0$, $\mathrm{Uni}(\mathcal{B})$ devient un espace affine dans le cas algébrique, une boule ouverte dans le cas tubulaire ouvert et une boule fermée dans le cas tubulaire fermé.

Supposons maintenant $r \neq 0$ et posons $t+1 = \mathrm{rg}(\mathcal{B})$. Par hypothèse, on a $t \neq 0$. Nous allons construire en suivant [Schneider et Stuhler 1991, § 1, Proposition 6] une fibration $f : \mathrm{Uni}(\mathcal{B}) \to \mathbb{P}^t_{\mathrm{rig},L}$. Les fibres seront des espaces affines dans le cas algébrique, des boules ouvertes dans le cas tubulaire ouvert et des boules fermées dans le cas tubulaire fermé. Pour chaque $H_i \in \mathcal{B}$, choisissons un vecteur unimodulaire de K^{d+1} de la même manière que dans la définition 2.4 et écrivons $M := \sum_{0 \leq i \leq r} \mathcal{O}_K a_i \subset M_0 := \sum_{0 \leq i \leq d} \mathcal{O}_K e_i$ où (e_i) est la base canonique de K^{d+1}. Réalisons un changement de base similaire à la définition 2.4 (licite d'après la remarque 1.2) pour obtenir des entiers positifs croissants $(\alpha_i)_{0 \leq i \leq d}$ tels que $\alpha_0 = 0$ et obtenir une décomposition $M = \sum_{0 \leq i \leq d} \varpi^{\alpha_i} \mathcal{O}_K e_i \subset M_0 = \sum_{0 \leq i \leq d} \mathcal{O}_K e_i$. On a alors les descriptions suivantes de $\mathrm{Uni}(\mathcal{B})$, avec la convention que pour la suite on choisit un représentant unimodulaire de chaque point $[b_0, \ldots, b_d]$, i.e., tel que $\max_{0 \leq i \leq d} |b_i| = 1$:

- Dans le cas algébrique,

$$\mathrm{Uni}(\mathcal{B}) = Z^d_t := \{z = [b_0, \ldots, b_d] \in \mathbb{P}^d_{\mathrm{rig},L}, \exists i \leq t, b_i \neq 0\}.$$

- Dans le cas tubulaire fermé, posons $\beta = (\beta_i)_{0 \leq i \leq t} = (n - \alpha_i)_{0 \leq i \leq t}$ et notons

$$\mathrm{Uni}(\mathcal{B}) = X^d_t(\beta) := \{z = [b_0, \ldots, b_d] \in \mathbb{P}^d_{\mathrm{rig},L}, \exists i \leq t, |b_i| \geq |\varpi|^{\beta_i}\}.$$

- Dans le cas tubulaire ouvert, posons $\gamma = (\gamma_i)_{0 \leq i \leq t} = (n + 1 - \alpha_i)_{0 \leq i \leq t}$ et notons

$$\mathrm{Uni}(\mathcal{B}) = Y^d_t(\gamma) := \{z = [b_0, \ldots, b_d] \in \mathbb{P}^d_{\mathrm{rig},L}, \exists i \leq t, |b_i| > |\varpi|^{\gamma_i}\}.$$

La flèche f donnée par $[b_0, \ldots, b_d] \mapsto [b_0, \ldots, b_t]$ induit bien des fibrations[7] $X^d_t(\beta) \to \mathbb{P}^t_{\mathrm{rig},L}$, $Y^d_t(\gamma) \to \mathbb{P}^t_{\mathrm{rig},L}$ ainsi que $Z^d_t \to \mathbb{P}^t_{\mathrm{rig},L}$. Soient $\mathcal{V}(\beta) = \{V(\beta)_i\}$, $\mathring{\mathcal{V}}(\gamma) = \{\mathring{V}(\gamma)_i\}$ et $\mathcal{V} = \{V_i\}$ les recouvrements admissibles de $\mathbb{P}^t_{\mathrm{rig},L}$, où

$$V(\beta)_i = \left\{z = [z_0, \ldots, z_t] \in \mathbb{P}^t_{\mathrm{rig},L}, \forall j \leq t, \left|\frac{z_i}{\varpi^{\beta_i}}\right| \geq \left|\frac{z_j}{\varpi^{\beta_j}}\right|\right\},$$
$$\mathring{V}(\gamma)_i = \left\{z = [z_0, \ldots, z_t] \in \mathbb{P}^t_{\mathrm{rig},L}, \forall j \leq t, \left|\frac{z_i}{\varpi^{\gamma_i}}\right| \geq \left|\frac{z_j}{\varpi^{\gamma_j}}\right|\right\},$$
$$V_i = \{z = [z_0, \ldots, z_t] \in \mathbb{P}^t_{\mathrm{rig},L}, z_i \neq 0\}.$$

7. La flèche f est bien définie sur ces espaces.

Alors, $X_t^d(\beta) \to \mathbb{P}^t_{\mathrm{rig},L}$ se trivialise sur $\mathcal{V}(\beta)$, de même pour $Y_t^d(\gamma) \to \mathbb{P}^t_{\mathrm{rig},L}$ sur $\mathring{\mathcal{V}}(\gamma)$ et $Z_t^d \to \mathbb{P}^t_{\mathrm{rig},L}$ sur $\mathcal{V}$, c'est-à-dire

$$f^{-1}(V(\beta)_i) \cong V(\beta)_i \times \mathbb{B}_L^{d-t}(-\beta_i),$$
$$f^{-1}(\mathring{V}(\gamma)_i) \cong \mathring{V}(\gamma)_i \times \mathring{\mathbb{B}}_L^{d-t}(-\gamma_i),$$
$$f^{-1}(V_i) \cong V_i \times \mathbb{A}^{d-t},$$

par le biais de l'application

$$[z_0, \dots, z_d] \mapsto [z_0, \dots, z_t] \times \left(\frac{z_{t+1}}{z_i}, \dots, \frac{z_d}{z_i}\right).$$

Appelons $\mathcal{U} = \{U_i\}$ le recouvrement adapté (au cas algébrique, tubulaire ouvert ou fermé) et F_i la fibre sur U_i (soit $\mathbb{B}_L^{d-t}(-\beta_i)$ pour les tubulaires fermés, $\mathring{\mathbb{B}}_L^{d-t}(-\gamma_i)$ pour les tubulaires ouverts, $\mathbb{A}^{d-t}$ dans le cas algébrique). La variable sur la base $\mathbb{P}^t_{\mathrm{rig},L}$ sera notée $z = [z_0, \dots, z_t]$, et celle de la fibre, $w = (w_1, \dots, w_{d-t})$. Sur chaque intersection[8] $U_{\{i,j\}}$, l'application de transition rend commutatif le diagramme

$$\begin{array}{ccc} f^{-1}(U_{\{i,j\}}) & \xrightarrow{\sim} & U_{\{i,j\}} \times F_i \\ \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} \times m_{z_i/z_j} \\ f^{-1}(U_{\{i,j\}}) & \xrightarrow{\sim} & U_{\{i,j\}} \times F_j \end{array}$$

où m_{z_i/z_j} est l'homothétie de rapport z_i/z_j. On écrira $f^*(\mathcal{V}(\beta)) = \{f^{-1}(V(\beta)_i)\}$, $f^*(\mathring{\mathcal{V}}(\gamma)) = \{f^{-1}(\mathring{V}(\gamma)_i)\}$, $f^*(\mathcal{V}) = \{f^{-1}(V_i)\}$ les recouvrements de $X_t^d(\beta)$, $Y_t^d(\gamma)$, Z_t^d obtenus.

Dans le cas algébrique, les intersections d'éléments du recouvrement $f^*(\mathcal{V})$ sont des produits de copies de $\mathbb{A}^1$ et de $\mathbb{A}^1 \backslash \{0\}$. Dans le cas tubulaire fermé, les intersections sur $f^*(\mathcal{V}(\beta))$ sont des produits de polycouronnes et de polydisques fermés.

Remarquons que si $t = d$ et $X_d^d(\beta) = \mathbb{P}^d_{\mathrm{rig},L}$, la famille des $(X_t^d(\beta))_{\beta,m,t}$ contient les espaces projectifs. Enfin, il pourra être utile de renormaliser les variables de $\mathbb{P}^t_{\mathrm{rig},L}$ et de les réécrire sous la forme

$$\tilde{z}_i = \frac{z_i}{\varpi^{\beta_i}}.$$

Exemple 2.5. Pour illustrer les constructions précédentes, décrivons ici les affinoïdes $\mathrm{Uni}(\mathcal{A})$ lorsque $\mathcal{A}$ est un arrangement tubulaire fermé d'ordre $n \geq 1$ avec $|\mathcal{A}| = 2$. Choisir $\mathcal{A}$ revient à se donner deux hyperplans H_a et H_b différents dans $\mathcal{H}_{n+1}$, ou par dualité, deux vecteurs unimodulaires a et b qui n'engendrent pas la même droite dans $(\mathcal{O}_K/\varpi^{n+1})^{d+1}$. Quitte à réaliser un changement de variables dans $\mathrm{GL}_{d+1}(\mathcal{O}_K)$ (voir la remarque 1.2), on peut trouver une base (e_i) de $\mathcal{O}_K^{d+1}$ telle que[9]

$$\begin{cases} e_0 = a,\ e_1 = b & \text{si } a \not\equiv b \bmod \varpi, \\ e_0 = a,\ e_0 + \varpi^k e_1 = b & \text{sinon.} \end{cases}$$

8. Pour tout recouvrement $\{U_i\}_{i \in S}$ par des ouverts d'un espace rigide X et toute partie finie $I \subset S$, on pourra noter pour simplifier $U_I := \bigcap_{i \in I} U_i$.

9. Notons que k est le plus grand entier s tel que $a \equiv b \bmod \varpi^s$ et est par conséquent strictement inférieur à $n + 1$.

Nous allons raisonner sur ce système de coordonnées.

Dans le premier cas,

$$\mathring{H}_a(|\varpi^n|)^c = \{z \in \mathbb{P}^d(C) : \forall j \leq d, |z_j| \leq |\varpi^{-n} z_0|\},$$
$$\mathring{H}_b(|\varpi^n|)^c = \{z \in \mathbb{P}^d(C) : \forall j \leq d, |z_j| \leq |\varpi^{-n} z_1|\}.$$

Ainsi $\mathrm{Uni}(\mathcal{A}) = \{z \in \mathbb{P}^d(C) : \exists i \leq 1, \forall j \leq d, |z_i| \geq |\varpi^n z_j|\} = X_1^d(n,n)$. Sous cette présentation, le recouvrement $V(\beta)$ avec $\beta = (n,n)$ correspond au recouvrement $\{\mathring{H}_a(|\varpi^n|)^c, \mathring{H}_b(|\varpi^n|)^c\}$.

Dans le second cas,

$$\begin{aligned}\mathrm{Uni}(\mathcal{A}) &= \left\{z \in \mathbb{P}^d(C) : \forall j \leq d, |z_j| \leq |\varpi^{-n} z_0| \text{ ou } \forall j \leq d, |z_j| \leq |\varpi^{-n}(z_0 + \varpi^k z_1)|\right\} \\ &= \{z \in \mathbb{P}^d(C) : \forall j \leq d, |z_0| \geq |\varpi^n z_j| \text{ ou } \forall j \leq d, |z_1| \geq |\varpi^{n-k} z_j|\} \\ &= X_1^d(n, n-k).\end{aligned}$$

En particulier, $V(\beta)$ avec $\beta = (n, n-k)$ est constitué des éléments

$$\mathring{H}_a(|\varpi^n|)^c = \{z \in \mathbb{P}^d(C) : \forall i \leq d, |z_i| \leq |\varpi^{-n} z_0|\},$$
$$\mathring{H}_{e_1}(|\varpi^{k-n}|)^c = \{z \in \mathbb{P}^d(C) : \forall i \leq d, |z_i| \leq |\varpi^{n-k} z_1|\}.$$

Notons que les vecteurs a et b jouent un rôle symétrique. En les échangeant, on obtient pour l'union la présentation

$$\mathrm{Uni}(\mathcal{A}) = \{z \in \mathbb{P}^d(C) : \forall j \leq d, |z_0 + \varpi^k z_1| \geq |\varpi^n z_j| \text{ ou } \forall j \leq d, |z_1| \geq |\varpi^{n-k} z_j|\} = X_1^d(n, n-k).$$

Sous cette présentation, le recouvrement $V(\beta)$ associé est alors constitué des éléments

$$\{\mathring{H}_b(|\varpi^n|)^c, \mathring{H}_{e_1}(|\varpi^{n-k}|)^c\}.$$

3. Énoncés et stratégies

Dans les énoncés ci-dessous, nous utiliserons systématiquement la topologie analytique. Nous allons prouver (voir le théorème 4.9, le lemme 4.10, ainsi que les théorèmes 5.1, 5.6, et 7.1) :

Théorème 3.1. (1) *Les espaces projectifs, les fibrations Z_t^d, les arrangements tubulaires fermés et les arrangements algébriques généralisés* $\mathrm{Int}(\mathcal{A})$ *sont $\mathscr{O}^{(r)}$-acycliques.*

(2) *Les sections globales de $\mathscr{O}^{(r)}$ sur les arrangements algébriques généralisés* $\mathrm{Int}(\mathcal{A})$ *sont constantes.*

(3) *La cohomologie de $\mathscr{O}^{(r)}$ sur $X_t^d(\beta)$ est concentrée en degrés* 0 *et t. Quand $t \neq 0$, les sections globales sont constantes et la cohomologie en degré t s'identifie au complété p-adique de*

$$\bigoplus_{\substack{\alpha \in \mathbb{N}^{d-t} \\ |\alpha| \geq t+1}} \mathrm{H}^t_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar}, \mathcal{O}_L}, \mathscr{O}(-|\alpha|)) \otimes \mathcal{O}_L^{(r)}.$$

Voir le théorème 4.9, le lemme 4.10, le corollaire 5.14 et le théorème 7.1 pour le résultat suivant :

Théorème 3.2. (1) *Les espaces projectifs, les fibrations* Z_t^d, *les arrangements tubulaires fermés et les arrangements algébriques généralisés* $\mathrm{Int}(\mathcal{A})$ *sont* $\mathscr{O}^{**}$*-acycliques.*

(2) *Les sections globales de* $\mathscr{O}^{**}$ *sur les arrangements algébriques généralisés* $\mathrm{Int}(\mathcal{A})$ *sont constantes.*

(3) *La cohomologie de* $\mathscr{O}^{**}$ *sur* $X_t^d(\beta)$ *est concentrée en degrés* 0 *et* t. *Les sections globales sont constantes quand* $t \neq 0$.

Le résultat suivant est une combinaison des théorèmes 6.1, 6.7, 6.10 et 7.1.

Théorème 3.3. (1) *Les espaces projectifs vérifient*

$$\mathrm{H}^k_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m) = \begin{cases} L^* & \text{si } k = 0, \\ \mathbb{Z} & \text{si } k = 1, \\ 0 & \text{sinon.} \end{cases}$$

(2) *La fibration* $f : X_t^d(\beta) \to \mathbb{P}^t_{\mathrm{rig},L}$ *induit une décomposition en produit direct pour* $s > 0$:

$$\mathrm{H}^*_{\mathrm{an}}(X_t^d(\beta), \mathbb{G}_m) \cong \mathrm{H}^*_{\mathrm{an}}(X_t^d(\beta), \mathscr{O}^{**}) \times \mathrm{H}^*_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m).$$

De plus, les sections globales sont constantes quand $t \neq 0$.

(3) *Les arrangements tubulaires fermés* $\mathrm{Int}(\mathcal{A})$ *sont* $\mathbb{G}_m$*-acycliques et*

$$\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A})) = \mathbb{Z}[\mathcal{A}]^0.$$

(4) *Les arrangements algébriques généralisés* $\mathrm{Int}(\mathcal{A})$ *sont* $\mathbb{G}_m$*-acycliques et*

$$\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/L^* = \mathbb{Z}[\![\mathcal{A}]\!]^0.$$

Pour obtenir ces résultats, il faut d'abord calculer la cohomologie de $\mathscr{O}^{(r)}$ sur $X_t^d(\beta)$ (point 3 du théorème 3.1) grâce aux résultats d'acyclicité de [van der Put 1982] (voir également le théorème 3.5) et au calcul de la cohomologie de Čech sur le recouvrement $f^*(\mathcal{V}(\beta))$ (voir le théorème 5.1). Plus précisément, la fibration f permet de relier le complexe de Čech de $X_t^d(\beta)$ aux complexes de $\mathbb{P}^t_{\mathrm{rig},L}$ pour les faisceaux tordus $\mathscr{O}^{(r)}(k)$ (point 1 du théorème 3.1 et corollaire 5.3 pour un énoncé plus fin). Le résultat se déduit de la cohomologie des faisceaux $\mathscr{O}(k)$ sur les espaces projectifs algébriques sur $\mathcal{O}_L$.

Le résultat d'acyclicité pour les arrangements tubulaires fermés découle de l'annulation de la cohomologie de $X_t^d(\beta)$ à partir du degré $t+1$ et de l'argument combinatoire du lemme 5.7 qui remplace la suite spectrale (1).

Le transfert des énoncés sur $\mathscr{O}^{(r)}$ à $\mathscr{O}^{**}$ résulte de l'argument sur les logarithmes tronqués du lemme 5.13. Pour le faisceau $\mathbb{G}_m$, on calcule encore la cohomologie de Čech des fibrations $X_t^d(\beta)$ sur le recouvrement $f^*(\mathcal{V}(\beta))$. Mais on a pour tout $I \subset [\![0, t]\!]$ une décomposition

$$\mathscr{O}^*\big(f^{-1}(V(\beta)_I)\big) = L^*\mathscr{O}^{**}\big(f^{-1}(V(\beta)_I)\big) \times \left\langle \frac{z_i}{z_j} : i, j \in I \right\rangle_{\mathbb{Z}\text{-Mod}}$$

qui induit les décompositions de la cohomologie du point 2 du théorème 3.3 (voir théorème 6.1) et celle des sections inversibles au théorème 3.3 point 3 (voir corollaire 5.14). Nous notons aussi que le complexe induit par les facteurs directs $\langle z_i/z_j : i, j \in I\rangle_{\mathbb{Z}\text{-Mod}}$ est celui apparaissant en géométrie algébrique, ce qui permet d'établir le point 1 du théorème 3.3 par comparaison. D'après ce qui précède, on sait que la cohomologie de $X_t^d(\beta)$ s'annule à partir du degré $t+1$, ce qui nous donne l'acyclicité des arrangements tubulaires fermés pour $\mathbb{G}_m$, toujours grâce au lemme combinatoire 5.7.

Pour ce qui est des arrangements algébriques généralisés $\mathcal{A}$, ils peuvent être approximés par des arrangements tubulaires fermés compatibles $\mathcal{A}_n$ d'ordre n. On dispose pour tout $s>0$ de la suite exacte

$$0 \to \mathrm{R}^1 \varprojlim_n \mathrm{H}^{s-1}_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}_n), \mathbb{G}_m) \to \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathbb{G}_m) \to \varprojlim_n \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}_n), \mathbb{G}_m) \to 0.$$

Le calcul dans le cas tubulaire fermé induit l'annulation de la cohomologie de $\mathbb{G}_m$ pour $s>1$ et l'égalité $\mathrm{H}^1_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathbb{G}_m) = \mathrm{R}^1 \varprojlim_n \mathcal{O}^*(\mathrm{Int}(\mathcal{A}_n))$. Il s'agit alors de prouver $\mathrm{R}^1 \varprojlim_n \mathcal{O}^*(\mathrm{Int}(\mathcal{A}_n)) = 0$. D'après la décomposition au théorème 3.3 point 3 et la proposition 4.5, il suffit de trouver une constante c indépendante de n pour laquelle on a l'inclusion

$$\mathcal{O}^{(r)}(\mathrm{Int}(\mathcal{A}_n)) \subset \mathcal{O}_L^{(r)} + \varpi \mathcal{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-c})).$$

Pour établir cette identité, on raisonne par récurrence sur le rang de $\mathrm{Int}(\mathcal{A}_n)$ et on se ramène à montrer (voir corollaire 5.10 et lemme 5.11 pour voir que cette condition est bien suffisante) que l'image de la flèche

$$\mathrm{H}^{\mathrm{rg}(\mathcal{A}_n)-1}_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_n), \mathcal{O}^{(r)}) \to \mathrm{H}^{\mathrm{rg}(\mathcal{A}_n)-1}_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_{n-1}), \mathcal{O}^{(r)})$$

est contenue dans $\varpi \mathrm{H}^{\mathrm{rg}(\mathcal{A}_n)-1}_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_{n-1}), \mathcal{O}^{(r)})$. Grâce au théorème 3.1 point 1, on peut voir ces groupes de cohomologie comme des sous-groupes des fonctions bornées de polycouronnes (voir remarque 5.5) dont les flèches de restriction sont explicites et bien comprises d'après le lemme 4.2. Le résultat découle alors de ce cas particulier.

Étudier la cohomologie des arrangements tubulaires fermés via les espaces $X_t^d(\beta)$ est semblable à la stratégie de [Schneider et Stuhler 1991]. Par exemple, le point 3 du théorème 3.1 imite l'axiome d'homotopie de [Schneider et Stuhler 1991, §2]. S'intéresser à $\mathcal{O}^{(r)}$ puis à $\mathcal{O}^{**}$ et enfin à $\mathbb{G}_m$ rappelle la preuve de [van der Put 1982, Theorem 3.25]. L'argument de passage à la limite s'inspire de [Colmez et al. 2020a, section 1.2].

Remarque 3.4. — Tous les calculs sur la cohomologie de Čech qui apparaissent dans la preuve de ces résultats peuvent être réalisés quand K est de caractéristique p. Cette observation suggère que ces résultats peuvent aussi être vérifiés dans ce cadre. Toutefois, nous ne voyons pas comment adapter la preuve du lemme crucial 5.13 dans ce cas. En particulier, nous ne sommes pas en mesure d'établir un analogue du théorème 3.2 ainsi que de sa conséquence le théorème 3.3 en caractéristique p.

— Nous avons choisi de ne pas étudier les arrangements tubulaires ouverts, même si certains raisonnements semblent pouvoir être adaptés. En fait, ces arrangements peuvent s'écrire comme des unions croissantes d'arrangements tubulaires fermés où l'on s'est autorisé des ordres rationnels. Si l'on pouvait établir des résultats similaires pour ces généralisations (la combinatoire des unions est similaire dans ce cadre), on pourrait alors grâce à la suite exacte (8) montrer que la cohomologie des faisceaux étudiés est concentrée en degrés 0 et 1. Un travail futur pourrait chercher à savoir si on a encore l'acyclicité pour les intersections des arrangements tubulaires ouverts lorsque le corps L est sphériquement clos.

Tous nos calculs utilisent de manière cruciale le résultat suivant de van der Put [1982, Theorems 3.10, 3.15, 3.25], décrivant la cohomologie de quelques affinoïdes simples.

Théorème 3.5 (van der Put). *Les produits de polycouronnes et polydisques fermés*[10] *n'ont pas de cohomologie analytique en degré strictement positif pour*

(1) *les faisceaux constants*,

(2) *le faisceau* $\mathscr{O}^{(r)}$,

(3) *le faisceau* $\mathscr{O}^{+}$ *en dimension* 1,

(4) *le faisceau* $\mathbb{G}_m$.

Remarque 3.6. Un théorème de Bartenwerfer [1982] affirme que les boules fermées sont aussi acycliques pour le faisceau $\mathscr{O}^{+}$, en toute dimension. Nous ne savons pas si ce résultat est encore vrai pour les couronnes (sauf en dimension 1, comme indiqué). Si c'était le cas, beaucoup des résultats à suivre pourraient aussi être énoncés pour $\mathscr{O}^{+}$.

4. Cas des arrangements algébriques

Nous traitons d'abord le cas des arrangements algébriques. L'énoncé suivant est un analogue du corollaire 5.12 dans le cas particulier des polycouronnes où le résultat est direct. Pour sa généralisation au corollaire 5.12, nous nous ramenons à ce cas particulier grâce au point technique du corollaire 5.4. Une fois ce résultat établi, les méthodes dans le cas algébrique sont relativement similaires au cas algébrique généralisé.

Lemme 4.1. *On considère le produit de polycouronnes et de polydisques*

$$U = \{x = (x_1, \dots, x_d) \in \mathbb{A}^d_{\mathrm{rig},L} : \forall i, |\varpi|^{-r_i} \geq |x_i| \geq |\varpi|^{s_i}\}$$

où $(r_i)_i$ *et* $(s_i)_i$ *sont des entiers.*[11] *De même, on considère*

$$V = \{x = (x_1, \dots, x_d) \in \mathbb{A}^d_{\mathrm{rig},L} : \forall i, |\varpi|^{-r_i-1} \geq |x_i| \geq |\varpi|^{s_i+1}\}.$$

Alors, on a

$$\mathscr{O}^{+}(V) \subset \mathcal{O}_L + \varpi \mathscr{O}^{+}(U).$$

Démonstration. La description des espaces U et V nous fournit un système de coordonnées commun $(X_i)_i$. La famille de monômes

$$\left(\prod_{i:\nu_i \geq 0} (\varpi^{r_i} X_i)^{\nu_i} \prod_{j:\nu_j < 0} \left(\frac{\varpi^{s_j}}{X_j}\right)^{-\nu_j} \right)_{\nu \in E}$$

10. Plus généralement, les polydisques généralisés au sens de [van der Put 1982, §3.9].

11. On s'autorisera $s_i = \infty$ pour les facteurs isomorphes à une boule fermée.

forme une base de Banach [12] de $\mathscr{O}(U)$ avec

$$E := \{\nu \in \mathbb{Z}^d : \nu_j \geq 0 \text{ si } s_j = \infty\}.$$

Il en est de même pour la famille

$$\left(\prod_{i:\nu_i \geq 0} (\varpi^{r_i+1} X_i)^{\nu_i} \prod_{j:\nu_j \leq 0} \left(\frac{\varpi^{s_j+1}}{X_j} \right)^{\nu_j} \right)_{\nu \in E}$$

sur $\mathscr{O}(V)$. Mais on remarque que pour tout ν $\left(\text{avec } |\nu| = \sum_j \nu_j\right)$,

$$\prod_{i:\nu_i \geq 0} (\varpi^{r_i+1} X_i)^{\nu_i} \prod_{j:\nu_j \leq 0} \left(\frac{\varpi^{s_j+1}}{X_j} \right)^{\nu_j} = \varpi^{|\nu|} \prod_{i:\nu_i \geq 0} (\varpi^{r_i} X_i)^{\nu_i} \prod_{j:\nu_j \leq 0} \left(\frac{\varpi^{s_j}}{X_j} \right)^{\nu_j}.$$

Il est alors aisé de voir que si une section à puissance bornée de V n'a pas de terme constant, sa restriction est dans $\varpi \mathscr{O}^+(U)$. □

Nous avons une version relative de ce résultat :

Lemme 4.2. *Soit Y un affinoïde sur L et soient U, V les affinoïdes définis dans le lemme précédent. On a alors*

$$\mathscr{O}^+(Y \times V) \subset \mathscr{O}^+(Y) + \varpi \mathscr{O}^+(Y \times U).$$

Démonstration. C'est le même argument que pour le lemme 4.1 et cela s'obtient en comparant les développements uniques en série sur les deux espaces U, V :

Lemme 4.3. *Soit $Y = \mathrm{Sp}(A)$ un affinoïde réduit sur L, et soit U comme précédemment. Toute section de $Y \times U$ admet une écriture unique*

$$\sum_\nu f_\nu Z^\nu \quad \textit{avec } f_\nu \in \mathscr{O}(Y) \textit{ et } \|f_\nu\|_Y \|Z^\nu\|_U \to 0$$

où la variable ν parcourt l'ensemble des vecteurs [13] *de $\mathbb{Z}^d$ tels que $\nu_i \geq 0$ quand $s_i = \infty$. De plus, la norme spectrale vérifie l'identité $\left\|\sum_\nu f_\nu Z^\nu\right\|_{Y \times U} = \max_\nu \|f_\nu\|_Y \|Z^\nu\|_U$.*

Démonstration. On commence par établir deux identités classiques sur $\ell^0_\infty(\mathcal{O}_L)$. Donnons-nous une $\mathcal{O}_L$-algèbre plate normée complète, alors les flèches naturelles [14] $\ell^0_\infty(\mathcal{O}_L) \to \ell^0_\infty(B)$ et $\ell^0_\infty(B) \to$

12. Rappelons ici cette notion. Pour cela, considérons A une L-algèbre (ou une $\mathcal{O}_L$-algèbre) normée complète et $\ell^0_\infty(A)$ l'ensemble des suites à valeurs dans A dont le terme général tend vers 0 muni de la norme $\|(a_n)_n\|_\infty := \max_n |a_n|$. Une base de Banach d'un A-module normé complet M est une famille $(e_n)_n \in M^{\mathbb{N}}$ telle que l'application $(a_n)_n \in \ell^0_\infty(A) \mapsto \sum a_n e_n \in M$ est bien définie et réalise une isométrie entre les espaces $\ell^0_\infty(A)$ et M.

13. C'est-à-dire que $\nu \in E$ en reprenant les notations de la preuve du lemme 4.1.

14. Décrivons la topologie sur le module $B[1/\varpi]$. Étant donné une norme sur B définissant la topologie sur cette algèbre, celle-ci peut être prolongée de manière unique en une norme sur $B[1/\varpi]$ qui vérifie la relation $\|b\|_{B[1/\varpi]} = |\varpi^{-k}| \|b\varpi^k\|_B$ pour $b \in \varpi^{-k} B \subset B[1/\varpi]$ (ne dépend pas de k). Cela permet de définir la topologie sur $B[1/\varpi]$ où toute base de voisinage de 0 dans B définit aussi une base de voisinage de 0 dans $B[1/\varpi]$. La complétude de $B[1/\varpi]$ se déduit alors de celle de B.

$\ell^0_\infty(B[1/\varpi])$ induisent des isomorphismes

$$\ell^0_\infty(\mathcal{O}_L)\widehat{\otimes} B \cong \ell^0_\infty(B) \quad \text{et} \quad \ell^0_\infty\Big(B\Big[\frac{1}{\varpi}\Big]\Big) \cong \ell^0_\infty(B)\Big[\frac{1}{\varpi}\Big]. \tag{2}$$

Nous commençons par le deuxième. Vus comme des sous-groupes de $B[1/\varpi]^{\mathbb{N}}$, on a une inclusion évidente entre les modules $\ell^0_\infty(B)[1/\varpi] \subset \ell^0_\infty(B[1/\varpi])$. Prouvons celle en sens opposé. Une suite $(u_n)_n$ dont le terme général tend vers 0 dans $B[1/\varpi]$ est à valeurs dans B à partir d'un certain rang N_0. En particulier, on peut trouver un entier k assez grand tel que $\varpi^k u_n \in B$ pour $n \le N_0$ et $(\varpi^k u_n)_n \in \ell^0_\infty(B)$, ce qui prouve l'inclusion voulue.

Intéressons-nous maintenant à la première identité. On a une flèche naturelle $\ell^0_\infty(\mathcal{O}_L)\otimes B \to \ell^0_\infty(B)$ et nous voulons montrer que c'est un isomorphisme lorsqu'on complète. Les suites à support fini $\ell^c(\mathcal{O}_L) \subset \ell^0_\infty(\mathcal{O}_L)$ et $\ell^c(B) \subset \ell^0_\infty(B)$ forment des sous-groupes denses qui vérifient $\ell^c(B) \cong \ell^c(\mathcal{O}_L)\otimes B$ et cet isomorphisme s'étend par continuité.

Revenons à la situation de l'énoncé. Par hypothèse de réduction sur Y, A^+ est un anneau de définition [15] de A. Ainsi, on a

$$\mathscr{O}(Y\times U) = (A^+\widehat{\otimes}_{\mathcal{O}_L}\mathscr{O}^+(U))\Big[\frac{1}{p}\Big].$$

Les identités (2) montrent qu'une base de Banach de $\mathscr{O}^+(U)$ définit aussi une base de Banach sur $\mathscr{O}(Y\times U)$. On en déduit l'existence et l'unicité de l'écriture en somme voulue.

Prouvons l'égalité pour la norme spectrale. D'après la discussion précédente, développons une section sous la forme [16] $f = \sum_\nu f_\nu Z^\nu/\|Z^\nu\|$ et appelons $\pi : Y\times U \to Y$ la projection. Pour tout $y \in Y(C)$, la norme spectrale sur $\pi^{-1}(y)$ est donnée par $\max_\nu(|f_\nu(y)|)$; voir le cas d'un corps. La norme spectrale totale vérifie $\|f\|_{Y\times U} = \max_y \|f\|_{\pi^{-1}(y)} = \max_y \max_\nu(|f_\nu(y)|) = \max_\nu(|f_\nu|)$ et on en déduit l'égalité voulue. □

Nous avons aussi un résultat similaire pour les fonctions inversibles des couronnes relatives.

Lemme 4.4. *Soient $I \subset [\![1,n]\!]$, $(s_i)_{i\in I}$ et $(r_i)_{i\in I}$ des nombres rationnels tels que $s_i \ge r_i$ pour tout i, $\mathrm{Sp}(A)$ un L-affinoïde réduit et connexe, et soit D la polycouronne*

$$\{(x_1,\dots,x_n)\in \mathbb{B}^n_L : |\varpi|^{s_i} \le |x_i| \le |\varpi|^{r_i} \text{ si } i \in I\}.$$

Alors

$$\mathscr{O}^*(D\times \mathrm{Sp}(A)) = \mathscr{O}^*(\mathrm{Sp}(A))\mathscr{O}^{**}(D\times\mathrm{Sp}(A)) \times \langle x_i : i\in I\rangle_{\mathbb{Z}\text{-Mod}}.$$

Le résultat reste vrai si la polycouronne D est ouverte.

Démonstration. Supposons vrai le cas d'une couronne fermée et montrons le résultat dans le cas d'une couronne ouverte. On se donne un recouvrement croissant $D = \bigcup_n D_n$ par des couronnes fermées, et on

15. C'est-à-dire que A^+ est une $\mathcal{O}_L$-algèbre topologique plate et p-adiquement complète telle que A est homéomorphe à $A^+[1/\varpi]$ (voir note 14 pour la topologie sur $A^+[1/\varpi]$).

16. On rappelle l'égalité $\|Z^\nu\| = \prod_{i:\nu_i\ge 0}\varpi^{\nu_i r_i}\prod_{j:\nu_j<0}\varpi^{-\nu_j s_j}$ pour $\nu\in E$.

note pour simplifier $D_A = \mathrm{Sp}\, A \times D$ (idem pour $D_{n,A}$). On a alors par hypothèse que

$$\mathscr{O}^*(D_A) = \bigcap_n \mathscr{O}^*(D_{n,A}) = \Big(\bigcap_n A^*\mathscr{O}^{**}(D_{n,A})\Big) \times \langle x_i : i \in I\rangle_{\mathbb{Z}\text{-Mod}}.$$

Il s'agit d'établir $\bigcap_n A^*\mathscr{O}^{**}(D_{n,A}) = A^*\mathscr{O}^{**}(D_A)$.

Prenons u dans cette intersection et écrivons $u = \lambda_n(1+h_n)$ dans chaque $A^*\mathscr{O}^{**}(D_{n,A})$. Fixons $n_0 \in \mathbb{N}$, alors pour tout $n > n_0$, on observe que

$$\frac{\lambda_n}{\lambda_{n_0}} = \frac{1+h_{n_0}}{1+h_n} \in \mathscr{O}^{**}(D_{n_0,A}) \cap A^* = A^{**},$$

donc $u/\lambda_{n_0} = (\lambda_n/\lambda_{n_0})(1+h_n) \in \bigcap_n \mathscr{O}^{**}(D_{n,A}) = \mathscr{O}^{**}(D_A)$ et ainsi $u \in \lambda_{n_0}\mathscr{O}^{**}(D_A)$. L'autre inclusion étant claire, on en déduit le résultat pour les couronnes ouvertes.

On suppose maintenant la couronne D fermée. Par récurrence sur $n = \dim D$, on se ramène au cas $n = 1$ et à la distinction $I = \{1\}$ ou $I = \varnothing$. En effet, en dimension supérieure, D se décompose comme un produit de couronnes fermées $D = D' \times D''$ obtenu en projetant sur la dernière coordonnée pour D'', et sur les autres pour D'. On a, par hypothèse de récurrence sur D'',

$$\mathscr{O}^*(D_A) = \mathscr{O}^*(D'_A)\mathscr{O}^{**}(D_A) \times \langle x_i : i \in I \cap \{n\}\rangle_{\mathbb{Z}\text{-Mod}}.$$

Par hypothèse de récurrence pour D', on a $\mathscr{O}^*(D'_A) = A^*\mathscr{O}^{**}(D'_A) \times \langle x_i : i \in I \text{ et } i < n\rangle_{\mathbb{Z}\text{-Mod}}$. On en déduit le résultat voulu.

Supposons $\dim D = 1$ et notons x la variable sur D. Commençons par le cas où A est une extension complète du corps L. Étant donné une fonction f inversible sur D_A, on peut trouver, d'après [Fresnel et van der Put 2004, Corollary 2.2.4], une fraction rationnelle g n'ayant aucun pôle ni zéro sur $D_A(\hat{\bar{A}}) \subset \hat{\bar{A}}$ telle que $f/g \in \mathscr{O}^{**}(D_A)$. Il est alors suffisant de prouver l'existence d'un entier relatif k et d'une constante $\lambda \in A^*$ telle que $(g/\lambda x^k)\mathscr{O}^{**}(D_A)$. Écrivons g en un produit de monômes $c\prod_{m\in M}(x-m)^{\alpha_m}$ avec $M \subset \hat{\bar{A}}$, et décomposons l'ensemble en $M = M^+ \amalg M^-$ avec

$$M^+ = \{m \in M : |m| > |\varpi|^{r_1}\} \quad \text{et} \quad M^- = \{m \in M : |m| < |\varpi|^{s_1}\}.$$

Comme le groupe de Galois absolu de A agit par isométrie sur $\hat{\bar{A}}$, on a $\prod_{m\in M^+} m^{\alpha_m} \in A$. De plus, on observe les relations

$$(x-m)/(-m) = 1 - x/m \in \mathscr{O}^{**}(D_{\hat{\bar{A}}}) \quad \text{si } m \in M^+,$$
$$(x-m)/x = 1 - m/x \in \mathscr{O}^{**}(D_{\hat{\bar{A}}}) \quad \text{si } m \in M^-.$$

On pose alors $k := \sum_{m\in M^-} \alpha_m$ et $c\prod_{m\in M^+}(-m)^{\alpha_m} \in A$ de telle manière que

$$\frac{g}{\lambda x^k} \in \mathscr{O}^{**}(D_{\hat{\bar{A}}}) \cap \mathscr{O}(D_A) = \mathscr{O}^{**}(D_A),$$

ce qui établit le cas d'un corps.

Revenons au cas général et montrons qu'il découle du cas particulier des corps. Soit u une section inversible de D_A, alors pour tout $z \in \mathrm{Sp}(A)$ on a une décomposition

$$u(z) = \lambda_z(1+h_z)x^{\beta_z} \in \mathscr{O}^*(\mathrm{Sp}(K(z)) \times D) = \mathscr{O}^*(D_z) \tag{3}$$

avec $\lambda_z \in K(z)^*,\ h_z \in \mathscr{O}^{++}(D_z)$ et $\beta_z \in \mathbb{Z}$. Si $I = \varnothing$, on a $\beta_z = 0$ pour tout z. Sinon, nous montrons que la fonction $z \mapsto \beta_z$ est continue sur $\mathrm{Sp}\, A$, d'où localement constante. Soit $z_0 \in \mathrm{Sp}\, A$ fixé, quitte à multiplier u par $x^{-\beta_{z_0}}$, on peut supposer $\beta_{z_0} = 0$. On écrit u comme une somme grâce au lemme 4.3, soit

$$a_0 + \sum_{\nu>0} a_\nu \Big(\frac{x}{\varpi^r}\Big)^\nu + \sum_{\nu>0} a_{-\nu}\Big(\frac{\varpi^s}{x}\Big)^\nu = a_0 + \tilde{u}$$

avec $a_\nu \to 0$ pour le filtre des parties finies. Si $I = \varnothing$, on a $a_\nu = 0$ si $\nu < 0$. Notons que la décomposition $\sum_\nu a_\nu(z_0)x^\nu = \lambda_{z_0}(1+h_{z_0})$ entraîne

$$a_0(z_0) \in \lambda_{z_0} K(z_0)^{**} \quad \text{et} \quad a_\nu(z_0) \in \lambda_{z_0} K(z_0)^{++} \tag{4}$$

pour tout $\nu \neq 0$. On peut trouver un voisinage affinoïde U de z_0 dans $\mathrm{Sp}\, A$ où λ_{z_0} se relève en un élément inversible $\tilde{\lambda} \in \mathscr{O}^*(U)$. Soit $N > 0$ tel que $a_\nu \in \tilde{\lambda}\mathscr{O}^{++}(U)$ pour tout $|\nu| > N$, et on fixe $\varepsilon < 1$ dans $p^{\mathbb{Q}}$ tel que $|a_0(z_0)/\lambda_{z_0} - 1| \le \varepsilon$ et $|a_\nu(z_0)/\lambda_{z_0}| \le \varepsilon$ pour tout $|\nu| \le N$. Considérons l'ouvert affinoïde V de U donné par

$$V := \left\{ z \in U : \left|\frac{a_0(z)}{\tilde{\lambda}} - 1\right| \le \varepsilon \ \text{ et } \ \left|\frac{a_\nu(z)}{\tilde{\lambda}}\right| \le \varepsilon \ \ \forall |\nu| \le N \right\}.$$

Alors $z_0 \in V$ pour ε assez proche de 1 d'après (4). On a dans ce cas $a_0 \in \mathscr{O}^*(V)$ avec $|a_0|_V = |\tilde{\lambda}|$, et $a_\nu/\tilde{\lambda}$ de même que a_ν/a_0 sont dans $\mathscr{O}^{++}(V)$ pour tout $\nu \in \mathbb{Z}$. On en déduit que la restriction de u à la couronne $D_V = V \times D$ s'écrit sous la forme $u = a_0(1 + \tilde{u}/a_0)$ avec $a_0 \in \mathscr{O}^*(V)$ et $\tilde{u}/a_0 \in \mathscr{O}^{++}(D_V)$. Cela montre, par unicité de β_z dans (3), que pour tout $z \in V$ on a $\beta_z = 0$ comme voulu. Ainsi, $z \mapsto \beta_z$ est constante sur les ouverts d'un recouvrement admissible, et donc constante par connexité de $\mathrm{Sp}\, A$.

Supposons maintenant que $\beta_z = 0$ pour tout $z \in \mathrm{Sp}\, A$. L'argument précédent montre que pour tout $z \in \mathrm{Sp}\, A$, on a $a_0(z) \neq 0$ et $\tilde{u}/a_0(z) \in \mathscr{O}^{++}(D_z)$. Donc, on a $a_0 \in A^*$ et $\tilde{u}/a_0 \in \mathscr{O}^{++}(D_A)$, ce qui donne la décomposition voulue,

$$u = a_0\Big(1 + \frac{\tilde{u}}{a_0}\Big) \in A^*\mathscr{O}^{**}(D_A). \qquad \square$$

Le résultat intermédiaire au lemme 4.2 est utile au vu du point technique général suivant.

Proposition 4.5. *Soit $X = \bigcup_n U_n = \bigcup_n \mathrm{Sp}(A_n)$ une réunion croissante de L-affinoïdes. Supposons l'existence d'une constante c indépendante de n telle que*

$$\mathscr{O}^+(U_{n+c}) \subset \mathcal{O}_L + \varpi\mathscr{O}^+(U_n). \tag{5}$$

Alors les sections globales des faisceaux $\mathscr{O}^+$, $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$ et $L^*\mathscr{O}^{**}$ sont constantes et on a

$$\mathrm{R}^1 \varprojlim_n \mathscr{F}(U_n) = 0$$

pour $\mathscr{F} = \mathscr{O}^{+}, \mathscr{O}^{(r)}, \mathscr{O}^{**}, L^{*}\mathscr{O}^{**}$.

Avant de montrer ce résultat clé, commençons par quelques commentaires sur les hypothèses de l'énoncé et sur les foncteurs dérivés de la limite projective. La preuve sera en fait une application du lemme plus général 4.7.

Il est utile d'observer que les conclusions de la proposition sont encore vraies quand on remplace le faisceau $\mathscr{O}^{+}$ par $\mathscr{O}^{++}$ et $\mathcal{O}_L$ par $\mathfrak{m}_L$ dans l'équation (5). En fait, cela découle de l'observation suivante.

Proposition 4.6. *Soit* $X = \bigcup_n U_n = \bigcup_n \mathrm{Sp}(A_n)$ *une réunion croissante de* L*-affinoïdes. Soit* c *un entier et* $0 < r$, *on a alors les implications*

$$\forall n > 0, \mathscr{O}^{+}(U_{n+c}) \subset \mathcal{O}_L + \varpi \mathscr{O}^{+}(U_n) \Rightarrow \forall n > 0, \mathscr{O}^{(r)}(U_{n+2c}) \subset \mathcal{O}_L^{(r)} + \varpi \mathscr{O}^{(r)}(U_n),$$
$$\forall n > 0, \mathscr{O}^{(r)}(U_{n+c}) \subset \mathcal{O}_L^{(r)} + \varpi \mathscr{O}^{(r)}(U_n) \Rightarrow \forall n > 0, \mathscr{O}^{+}(U_{n+2c}) \subset \mathcal{O}_L + \varpi \mathscr{O}^{+}(U_n).$$

Démonstration. Les preuves des deux implications sont quasiment identiques et nous ne traiterons que la première. De plus, quitte à multiplier par une puissance de ϖ, on peut supposer que $|\varpi| < r \leq 1$.

Prenons f dans $\mathscr{O}^{(r)}(U_{n+2c})$ et donc dans $\mathscr{O}^{+}(U_{n+2c})$ par hypothèses sur r. En appliquant deux fois l'hypothèse, on a la chaîne d'inclusions

$$\mathscr{O}^{+}(U_{n+2c}) \subset \mathcal{O}_L + \varpi \mathscr{O}^{+}(U_{n+c}) \subset \mathcal{O}_L + \varpi^2 \mathscr{O}^{+}(U_n) \subset \mathcal{O}_L + \varpi \mathscr{O}^{(r)}(U_n).$$

Ainsi, la fonction f s'écrit $f = \lambda + \varpi \tilde{f}$ avec $\tilde{f} \in \mathscr{O}^{(r)}(U_n)$ et $\lambda \in \mathcal{O}_L$. En particulier, $\lambda = f - \varpi \tilde{f} \in \mathcal{O}_L \cap \mathscr{O}^{(r)}(U_n) = \mathcal{O}_L^{(r)}$ et on en déduit la décomposition voulue. □

Décrivons les quelques propriétés des foncteurs dérivés de la limite projective que nous allons utiliser. Pour A un anneau,[17] I un ensemble ordonné filtré, la catégorie des systèmes projectifs en A-modules indexés par I est abélienne et possède suffisamment d'objets injectifs (voir [Jensen 1972, paragraphe 1]), et le foncteur « limite projective » admet des foncteurs dérivés à droite que l'on notera $\mathrm{R}^i \varprojlim_{j \in I}$.

Dans toute la suite, nous n'étudierons que des systèmes projectifs sur $I = \mathbb{N}$. Un des résultats les plus importants dans ce cas est l'annulation de la plupart des foncteurs dérivés [Jensen 1972, théorème 2.2] :

$$\forall i \geq 2,\ \mathrm{R}^i \varprojlim_n = 0. \tag{6}$$

On peut de plus calculer le premier foncteur dérivé (d'après [Jensen 1972, remarque après le théorème 2.2]) et ce dernier s'inscrit dans une suite exacte

$$0 \to \varprojlim_n M_n \to \prod_n M_n \xrightarrow{\delta} \prod_n M_n \to \mathrm{R}^1 \varprojlim_n M_n \to 0 \tag{7}$$

avec $\delta((m_n)_n) = (m_n - \varphi_{n+1}(m_{n+1}))_n$, où $\varphi_{n+1} : M_{n+1} \to M_n$ est une des fonctions de transition du système projectif. Par abus, les éléments de $\prod_n M_n$ seront appelés cocycles et ceux dans $\mathrm{Im}\,\delta$ seront des cobords.

17. Au vu de la propriété d'invariance décrite dans [Jensen 1972, remarque 1.10], on pourra toujours supposer $A = \mathbb{Z}$.

Précisons la situation dans laquelle nous allons appliquer ces résultats. Prenons X un espace rigide, $\mathcal{U} = \{U_n\}$ un recouvrement admissible croissant par des ouverts affinoïdes et $\mathscr{F}$ un faisceau sur X. En explicitant le complexe de Čech sur ce recouvrement, on obtient grâce à la suite exacte (7) les identifications

$$\check{\mathrm{H}}^0(X,\mathcal{U},\mathscr{F}) \cong \varprojlim_n \mathscr{F}(U_n) \quad \text{et} \quad \check{\mathrm{H}}^1(X,\mathcal{U},\mathscr{F}) \cong \mathrm{R}^1 \varprojlim_n \mathscr{F}(U_n).$$

On peut aussi exprimer la cohomologie de X en fonction de celle des ouverts U_n. Plus précisément, la composition des foncteurs Γ et $\varprojlim_n$ nous fournit une suite spectrale

$$E_2^{i,j} = \mathrm{R}^i \varprojlim_n \mathrm{H}^j_{\mathrm{an}}(U_n, \mathscr{F}_{|U_n}) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{an}}(X,\mathscr{F})$$

qui dégénère d'après le résultat d'annulation (6). On obtient une suite exacte pour tout s (avec pour convention $\mathrm{H}^{-1}_{\mathrm{an}}(U_n, \mathscr{F}_{|U_n}) = 0$) :

$$0 \to \mathrm{R}^1 \varprojlim_n \mathrm{H}^{s-1}_{\mathrm{an}}(U_n, \mathscr{F}_{|U_n}) \to \mathrm{H}^s_{\mathrm{an}}(X,\mathscr{F}) \to \varprojlim_n \mathrm{H}^s_{\mathrm{an}}(U_n, \mathscr{F}_{|U_n}) \to 0. \tag{8}$$

Comme dans le raisonnement précédent, on peut encore interpréter cette suite exacte comme résultant de la dégénérescence de la suite spectrale de Čech sur le recouvrement $\mathcal{U}$ grâce à (6).

Nous allons maintenant prouver l'annulation de $\mathrm{R}^1 \varprojlim_n M_n$ pour des systèmes projectifs $(M_n)_n$ particuliers.

Lemme 4.7. *Soit une suite décroissante de groupes abéliens complets $(G_n)_n$ dont la topologie est induite par des bases de voisinage formées de sous-groupes ouverts $(G_n^{(i)})_i$ avec $G_n = G_n^{(0)}$. Supposons $G_{n+1}^{(i)} \subset G_n^{(i)}$ pour tous i, n (en particulier, les inclusions sont continues).*

S'il existe un sous-groupe $H \subset \bigcap_n G_n$ fermé dans chaque G_n (i.e., $H = \bigcap_i H + G_n^{(i)}$ pour tout n) vérifiant

$$G_{n+c}^{(i)} \subset H + G_n^{(i+1)} \tag{9}$$

pour une constante c indépendante de i et n, alors

$$\bigcap_n G_n = \varprojlim_n G_n = H \quad \textit{et} \quad \mathrm{R}^1 \varprojlim_n G_n = 0.$$

Démonstration. On veut déterminer $\varprojlim_n G_n$ et donc établir l'inclusion $\bigcap_n G_n \subset H$, l'autre étant vérifiée par hypothèse. D'après (9), on vérifie aisément par récurrence l'inclusion $G_{cn} \subset H + G_0^{(n)}$ pour tout n, d'où

$$\bigcap_n G_{cn} \subset \bigcap_n H + G_0^{(n)} = H$$

par hypothèse de fermeture de H.

Calculons maintenant le groupe $\mathrm{R}^1 \varprojlim_n G_n$. Prenons un cocycle $(f_n)_n$ et montrons que c'est un cobord. Toujours d'après (9), on peut trouver par récurrence une suite [18] $(h_n)_n \in H^{\mathbb{N}}$ telle que pour tout n, k et

18. Il suffit de montrer l'inclusion $G_{n+kc+r}^{(i)} \subset G_n^{(i+k)} + H$. Comme on a l'inclusion $G_{n+r}^{(i+k)} \subset G_n^{(i+k)}$, on peut supposer $r = 0$. Quand $k = 1$, le résultat est exactement l'hypothèse (9). Supposons, pour tous i et n, le résultat vrai pour un entier k fixé,

$r < c$, on a

$$f_{n+kc+r} - h_{n+kc+r} \in G_n^{(k)}.$$

Dans ce cas, la somme $\sum_{m\geq n} f_m - h_m$ converge dans G_n pour tout entier n et vérifie

$$\delta\Big(\Big(\sum_{m\geq n} f_m - h_m\Big)_n\Big) = (f_n)_n - (h_n)_n.$$

Donnons-nous $\tilde{h}_0 \in H$ et construisons par récurrence une suite $(\tilde{h}_n)_n$ telle que $\tilde{h}_{n+1} = \tilde{h}_n - h_n$, i.e., $\delta((\tilde{h}_n)_n) = (h_n)_n$. On en déduit que $(f_n)_n$ est en fait le cobord $\delta\big(\big(\sum_{m\geq n} f_m - h_m\big)_n + (\tilde{h}_n)_n\big)$. □

Démonstration de la proposition 4.5. Les constantes $\mathscr{F}(U_n) \cap L$ forment des fermés de $\mathscr{F}(U_n)$ pour $\mathscr{F} = \mathscr{O}^+, \mathscr{O}^{(r)}, \mathscr{O}^{**}, L^*\mathscr{O}^{**}$. Les suites décroissantes $(\mathscr{O}^+(U_n))_n$ et $(\mathscr{O}^{(r)}(U_n)_n)$ de groupes topologiques vérifient clairement l'inclusion (9) et la proposition 4.6 par hypothèse. Montrons que c'est encore le cas pour les suites $(L^*\mathscr{O}^{**}(U_n))_n$ et $(\mathscr{O}^{**}(U_n))_n$. Raisonnons uniquement pour le second, le premier s'en déduira aisément. Soit $1 + \varpi^k f$ avec $f \in \mathscr{O}^{++}(U_n)$, on peut trouver une constante $\lambda \in \mathfrak{m}_L$ telle que $f - \lambda \in \varpi\mathscr{O}^{++}(U_{n-1})$. Alors, on a

$$\frac{1+\varpi^k f}{1+\varpi^k\lambda} = 1 + \varpi^k\frac{f-\lambda}{1+\varpi^k\lambda} \in 1 + \varpi^{k+1}\mathscr{O}^{++}(U_{n-1}).$$

La proposition 4.5 est alors une conséquence directe du lemme précédent. □

La base canonique $(e_i)_{0\leq i\leq d}$ de K^{d+1} définit une collection de $d+1$ hyperplans $V^+(z_i) \subset \mathbb{P}^d_{\mathrm{rig},L}$ et on note $\mathcal{B}$ l'arrangement algébrique $\{V^+(z_i)\}_{0\leq i\leq r}$.

Corollaire 4.8. *L'espace* $\mathrm{Int}(\mathcal{B})$ *défini plus haut est acyclique pour les faisceaux* $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$ *et* $\mathbb{G}_m$. *Les sections globales de* $\mathscr{O}^+$ *et* $\mathscr{O}^{**}$ *sont constantes et*

$$\mathcal{O}^*(\mathrm{Int}(\mathcal{B})) = L^* \times T$$

avec $T = \langle z_i/z_0 : 1 \leq i \leq r\rangle_{\mathbb{Z}\text{-Mod}}$.

Démonstration. On voit cet arrangement d'hyperplans comme le produit $(\mathbb{A}^1_{\mathrm{rig},L}\backslash\{0\})^r \times \mathbb{A}^{d-r}_{\mathrm{rig},L}$. On le recouvre par $(X_n)_n$ où, en posant $x_i = z_i/z_0$,

$$X_n = \big\{x = (x_1,\dots,x_d) \in \mathbb{A}^d_{\mathrm{rig},L} : \forall i \leq r, |\varpi|^{-n} \geq |x_i| \geq |\varpi|^n, \forall j \geq r+1, |\varpi|^{-n} \geq |x_j|\big\}.$$

On a la suite exacte

$$0 \to \mathrm{R}^1 \varprojlim \mathrm{H}^{s-1}_{\mathrm{an}}(X_n, \mathscr{F}) \to \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{B}), \mathscr{F}) \to \varprojlim \mathrm{H}^s_{\mathrm{an}}(X_n, \mathscr{F}) \to 0.$$

---

on a alors une chaîne d'inclusions

$$G^{(i)}_{n+(k+1)c} \subset G^{(i+1)}_{n+kc} + H \subset G^{(i+k+1)}_n + H$$

(par (9) pour la première et par hypothèse de récurrence pour la seconde), ce qui termine l'argument.

Mais $\mathrm{Int}(\mathcal{B}) = \bigcup_n X_n$ est un recouvrement admissible constitué de produits de polycouronnes et polydisques, chacun des termes est acyclique pour les faisceaux $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$ et $\mathbb{G}_m$ d'après le théorème 3.5, d'où, pour $s > 0$,

$$\varprojlim \mathrm{H}^s_{\mathrm{an}}(X_n, \mathscr{F}) = 0 \quad \text{et} \quad \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{B}), \mathscr{F}) = \mathrm{R}^1 \varprojlim \mathrm{H}^{s-1}_{\mathrm{an}}(X_n, \mathscr{F}).$$

Grâce au lemme 4.2, on peut appliquer la proposition 4.5 et on en déduit l'énoncé pour les faisceaux $\mathscr{O}^{(r)}$ et $\mathscr{O}^{**}$. On obtient aussi l'annulation de la cohomologie de $\mathbb{G}_m$ en degré supérieur ou égal à 2.

D'après le lemme 4.4, on a une décomposition en produits directs du système projectif[19] $(\mathscr{O}^*(X_n))_n$, c'est-à-dire

$$(\mathscr{O}^*(X_n))_n = (L^* \mathscr{O}^{**}(X_n))_n \times (T)_n.$$

Ainsi $\mathscr{O}^*(\mathrm{Int}(\mathcal{B})) = \varprojlim_n L^* \mathscr{O}^{**}(X_n) \times \varprojlim_n T = L^* \times T$ (en utilisant la proposition 4.5) et

$$\mathrm{Pic}_L(\mathrm{Int}(\mathcal{B})) = \mathrm{R}^1 \varprojlim_n L^* \mathscr{O}^{**}(X_n) \times \mathrm{R}^1 \varprojlim_n T = 0. \qquad \square$$

Nous pouvons maintenant énoncer le théorème principal de cette section :

Théorème 4.9. *Les arrangements algébriques sont $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$-acycliques et les sections globales sont constantes.*

Lemme 4.10. *Les fibrations Z^d_t sont acycliques pour $\mathscr{O}^{(r)}$ et $\mathscr{O}^{**}$, et les sections globales sont constantes.*

Démonstration. On raisonne sur la suite spectrale de Čech pour le recouvrement $f^*(\mathcal{V})$ de Z^d_t. D'après le corollaire 4.8, chaque intersection est $\mathscr{O}^{(r)}$-acyclique et on se ramène à calculer la cohomologie de Čech sur le recouvrement $f^*(\mathcal{V})$ qui est isomorphe à $\check{\mathcal{C}}^\bullet(Z^d_t, f^*(\mathcal{V}), \mathcal{O}^{(r)}_L)$. Mais le nerf du recouvrement est le simplexe standard Δ^t de dimension t, qui est contractile. Ceci montre l'annulation de la cohomologie en degré supérieur ou égal à 1. On obtient aussi aisément que $\mathscr{O}^{(r)}(Z^d_t) = \mathcal{O}^{(r)}_L$. On raisonne de même pour $\mathscr{O}^{**}$. $\square$

Démonstration du théorème 4.9. D'après le lemme 4.10 et l'identification $\mathbb{P}^d_{\mathrm{rig},L} = Z^d_d$, la flèche d'inclusion $Z^d_t \to \mathbb{P}^d_{\mathrm{rig},L}$ induit alors des isomorphismes

$$\mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathscr{O}^{(r)}) \cong \mathrm{H}^s_{\mathrm{an}}(Z^d_t, \mathscr{O}^{(r)})$$

pour tout s positif. D'où l'annulation de $\mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, Z^d_t, \mathscr{O}^{(r)})$. Alors la suite spectrale (1) dégénère et on obtient $\mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathrm{Int}(\mathcal{A}), \mathscr{O}^{(r)}) = 0$ pour tout s. Ce qui se traduit par

$$\mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathscr{O}^{(r)}) = \begin{cases} \mathcal{O}^{(r)}_L & \text{si } s = 0, \\ 0 & \text{sinon.} \end{cases}$$

On raisonne de même pour $\mathscr{O}^{**}$. $\square$

19. Où T est vu comme un système projectif constant.

5. Cohomologie analytique à coefficients dans $\mathscr{O}^{(r)}$

5A. ***Cohomologie des fibrations $X_t^d(\beta)$.*** Nous allons chercher à déterminer la cohomologie des espaces $X_t^d(\beta)$. Commençons par faire quelques rappels sur les faisceaux localement libres de rang 1 sur $\mathbb{P}^t_{\mathrm{rig},L}$ et $\mathbb{P}^t_{\mathrm{zar},L}$. Dans le cas algébrique, ils sont décrits par les faisceaux tordus $\mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},L}}(k)$ avec k dans $\mathbb{Z}$. Ce faisceau se trivialise sur le recouvrement usuel $\mathcal{V}$ et les fonctions de transition font commuter le diagramme [20] suivant (voir note 8 pour la notation $V_{\{i,j\}}$) :

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},L}}(k)|_{V_{\{i,j\}}} & \xrightarrow{\sim} & \mathscr{O}_{V_i}|_{V_{\{i,j\}}} \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow m_{(\tilde{z}_i/\tilde{z}_j)^{-k}} \\ \mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},L}}(k)|_{V_{\{i,j\}}} & \xrightarrow{\sim} & \mathscr{O}_{V_j}|_{V_{\{i,j\}}} \end{array} \tag{10}$$

En géométrie rigide, on peut encore définir les faisceaux tordus $\mathscr{O}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$, $\mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$ (version à puissance bornée) et $\mathscr{O}^{(r)}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$ grâce aux mêmes morphismes de transition :

$$\begin{array}{ccc} \mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig},L}}(k)|_{V(\beta)_{\{i,j\}}} & \xrightarrow{\sim} & \mathscr{O}^+_{V(\beta)_i}|_{V(\beta)_{\{i,j\}}} \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow m_{(\tilde{z}_i/\tilde{z}_j)^{-k}} \\ \mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig},L}}(k)|_{V(\beta)_{\{i,j\}}} & \xrightarrow{\sim} & \mathscr{O}^+_{V(\beta)_j}|_{V(\beta)_{\{i,j\}}} \end{array}$$

On rappelle que l'on a bien $\tilde{z}_i/\tilde{z}_j \in \mathscr{O}^+(V(\beta)_{\{i,j\}})$. On construit grâce à un diagramme similaire $\mathscr{O}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$ et $\mathscr{O}^{(r)}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$. D'après GAGA (voir aussi théorème 6.1 pour une démonstration plus élémentaire), les $\mathscr{O}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$ sont les seuls faisceaux localement libres de rang 1. Nous pouvons aussi définir ces faisceaux tordus sur les fibrations $X_t^d(\beta)$ en tirant en arrière par f, i.e., $\mathscr{O}_{X_t^d(\beta)}(k) = f^*\mathscr{O}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$, $\mathscr{O}^+_{X_t^d(\beta)}(k) = f^*\mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$ et $\mathscr{O}^{(r)}_{X_t^d(\beta)}(k) = f^*\mathscr{O}^{(r)}_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$. L'un des points techniques de l'argument du lemme 6.5 permettra de montrer que $\mathscr{O}_{X_t^d(\beta)}(k) = \mathscr{O}_{\mathbb{P}^d_{\mathrm{rig},L}}(k)|_{X_t^d(\beta)}$.

Dans le cas algébrique, la cohomologie de Zariski de ces faisceaux tordus est connue et peut être trouvée dans [Hartshorne 1977, Theorem 5.1, Section III] par exemple. Pour tout anneau A et pour k un entier, la cohomologie de $\mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},A}}(k)$ est concentrée en degré 0 si k est positif et en degré t si k est strictement négatif. Plus précisément, on a des isomorphismes

$$\mathrm{H}^0_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},A}, \mathscr{O}(k)) \cong A[T_0,\ldots,T_t]_k \quad \text{si } k \text{ est positif,}$$
$$\mathrm{H}^t_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},A}, \mathscr{O}(k)) \cong \left(\frac{1}{T_0\cdots T_t} A\left[\frac{1}{T_0},\ldots,\frac{1}{T_t}\right]\right)_k \quad \text{si } k \text{ est négatif,}$$

où $A[T_0,\ldots,T_t]_k$ désigne l'ensemble des polynômes homogènes de degré k.

On se propose de calculer la cohomologie de $\mathscr{O}^{(r)}$ des fibrations $X_t^d(\beta)$. Plus précisément, nous souhaitons montrer :

20. Dans ce qui suit, on note aussi $\tilde{z}$ la variable sur $\mathbb{P}^t_{\mathrm{zar},A}$, i.e., $\mathbb{P}^t_{\mathrm{zar},A} = \mathrm{Proj}(A[\tilde{z}_0,\ldots,\tilde{z}_t])$.

Théorème 5.1. — *La cohomologie à coefficients dans $\mathscr{O}^{(r)}(k)$ de l'espace projectif $\mathbb{P}^t_{\mathrm{rig},L}$ est concentrée en degré 0 si k est positif et en degré t si k est strictement négatif. De même, la cohomologie des fibrations $X^d_t(\beta)$ est concentrée en degrés 0 et t.*

— *Plus précisément, on a des isomorphismes*

$$\mathrm{H}^0_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^{(r)}(k)) \cong \mathcal{O}^{(r)}_L \otimes_{\mathcal{O}_L} \mathrm{H}^0_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}, \mathscr{O}(k)) \cong \mathcal{O}^{(r)}_L[T_0, \dots, T_t]_k,$$
$$\mathrm{H}^t_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^{(r)}(k)) \cong \mathcal{O}^{(r)}_L \otimes_{\mathcal{O}_L} \mathrm{H}^0_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}, \mathscr{O}(k)) \cong \Big(\frac{1}{T_0\cdots T_t}\mathcal{O}^{(r)}_L\Big[\frac{1}{T_0}, \dots, \frac{1}{T_t}\Big]\Big)_k.$$

De plus, pour $r \leq r'$, les flèches suivantes sont injectives pour $s \geq 0$:

$$\mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^{(r)}(k)) \to \mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^{(r')}(k)).$$

— *On dispose d'isomorphismes*

$$\mathrm{H}^0_{\mathrm{an}}(X^d_t(\beta), \mathscr{O}^{(r)}(k)) \simeq \bigoplus_{\substack{\alpha\in\mathbb{N}^{d-t}\\ |\alpha|\leq k}} \mathrm{H}^0_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^{(r)}(k-|\alpha|));$$

en particulier, les sections globales des faisceaux $\mathscr{O}^{(r)}(k)$ sont nulles si $k < 0$ et s'identifient à $\mathcal{O}^{(r)}_L$ si $k = 0$. Enfin, $\mathrm{H}^t_{\mathrm{an}}(X^d_t(\beta), \mathscr{O}^{(r)}(k))$ est isomorphe au complété p-adique de

$$\bigoplus_{\substack{\alpha\in\mathbb{N}^{d-t}\\ |\alpha|\geq t+1+k}} \mathrm{H}^t_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^{(r)}(k-|\alpha|)).$$

Démonstration. Les intersections d'éléments des recouvrements $\mathcal{V}(\beta)$ et $f^*(\mathcal{V}(\beta))$ sont des produits de polycouronnes et de polydisques fermés dont les polyrayons sont dans $|L^*|$. Ainsi, on se ramène à calculer la cohomologie de Čech sur les recouvrements $\mathcal{V}(\beta)$ et $f^*(\mathcal{V}(\beta))$ (voir théorème 3.5). De plus, pour toute section non nulle h de $\mathscr{O}^{(r)}(V(\beta)_I)$ ou de $\mathscr{O}^{(r)}\big(f^{-1}(V(\beta)_I)\big)$ (voir note 8 pour la notation), il existe une constante $\lambda \in \mathscr{O}^{(r)}_L$ telle que h/λ soit de norme 1. On en déduit que $\mathscr{O}^{(r)}(V(\beta)_I) = \mathscr{O}^+(V(\beta)_I) \otimes_{\mathcal{O}_L} \mathscr{O}^{(r)}_L$ ou que $\mathscr{O}^{(r)}\big(f^{-1}(V(\beta)_I)\big) = \mathscr{O}^+\big(f^{-1}(V(\beta)_I)\big) \otimes_{\mathcal{O}_L} \mathscr{O}^{(r)}_L$, respectivement. Donc

$$\check{\mathcal{C}}^\bullet\big(\mathbb{P}^t_{\mathrm{rig},L}; \mathscr{O}^{(r)}(k), f^*(\mathcal{V}(\beta))\big) = \check{\mathcal{C}}^\bullet\big(\mathbb{P}^t_{\mathrm{rig},L}; \mathscr{O}^+(k), f^*(\mathcal{V}(\beta))\big) \otimes_{\mathcal{O}_L} \mathscr{O}^{(r)}_L,$$
$$\check{\mathcal{C}}^\bullet(X^d_t(\beta); \mathscr{O}^{(r)}(k), \mathcal{V}(\beta)) = \check{\mathcal{C}}^\bullet(X^d_t(\beta); \mathscr{O}^+(k), \mathcal{V}(\beta)) \otimes_{\mathcal{O}_L} \mathscr{O}^{(r)}_L.$$

Par platitude, on obtient les isomorphismes au niveau des groupes de cohomologie

$$\check{\mathrm{H}}^*\big(\mathbb{P}^t_{\mathrm{rig},L}; \mathscr{O}^{(r)}(k), f^*(\mathcal{V}(\beta))\big) = \check{\mathrm{H}}^*\big(\mathbb{P}^t_{\mathrm{rig},L}; \mathscr{O}^+(k), f^*(\mathcal{V}(\beta))\big) \otimes_{\mathcal{O}_L} \mathscr{O}^{(r)}_L,$$
$$\check{\mathrm{H}}^*(X^d_t(\beta); \mathscr{O}^{(r)}(k), \mathcal{V}(\beta)) = \check{\mathrm{H}}^*(X^d_t(\beta); \mathscr{O}^+(k), \mathcal{V}(\beta)) \otimes_{\mathcal{O}_L} \mathscr{O}^{(r)}_L,$$

et l'injectivité des inclusions quand r varie. Le reste de cette section sera consacré au calcul de ces groupes de cohomologie de Čech sur $\mathscr{O}^+$. Cela repose sur le lemme général suivant :

Lemme 5.2. *Soit $\mathcal{C}^\bullet$ un complexe constitué de $\mathbb{Z}_p$-modules plats tel que $\mathrm{H}^j(\mathcal{C}^\bullet)$ est sans p-torsion pour tout j. On a alors un isomorphisme naturel $\mathrm{H}^j(\widehat{\mathcal{C}^\bullet}) \simeq \widehat{\mathrm{H}^j(\mathcal{C}^\bullet)}$ où les complétions considérées sont réalisées suivant la topologie p-adique.*

Démonstration. Soit $A^j = \mathrm{Im}(d^{j-1} : \mathcal{C}^{j-1} \to \mathcal{C}^j)$ et $B^j = \ker(d^j : \mathcal{C}^j \to \mathcal{C}^{j+1})$. On a une suite exacte $0 \to A^j \to B^j \to \mathrm{H}^j(\mathcal{C}^\bullet) \to 0$, d'où l'exactitude de $0 \to A^j/p^n \to B^j/p^n \to \mathrm{H}^j(\mathcal{C}^\bullet)/p^n \to 0$, car $\mathrm{H}^j(\mathcal{C}^\bullet)$ est sans p-torsion, par hypothèse. Par Mittag-Leffler, on obtient encore une suite exacte $0 \to \widehat{A}^j \to \widehat{B}^j \to \widehat{\mathrm{H}^j(\mathcal{C}^\bullet)} \to 0$. Il suffit donc de montrer que $\widehat{A}^j = \mathrm{Im}(\hat{d}^{j-1} : \widehat{\mathcal{C}}^{j-1} \to \widehat{\mathcal{C}}^j)$ et $\widehat{B}^j = \ker(\hat{d}^j : \widehat{\mathcal{C}}^j \to \widehat{\mathcal{C}}^{j+1})$. Comme $\mathcal{C}^{j+1}$ (et donc A^{j+1}) est sans p-torsion, on a l'exactitude de la suite $0 \to B^j/p^n \to \mathcal{C}^j/p^n \to A^{j+1}/p^n \to 0$, d'où celle de

$$0 \to B^j/p^n \to \mathcal{C}^j/p^n \to \mathcal{C}^{j+1}/p^n,$$

car on a montré que $A^{j+1}/p^n \to B^{j+1}/p^n \to \mathcal{C}^{j+1}/p^n$ est injective. En passant à la limite projective dans les deux suites précédentes, on obtient $\widehat{B}^j = \ker \hat{d}^j$ et $\widehat{A}^{j+1} \cong \widehat{\mathcal{C}}^j/\widehat{B}^j = \widehat{\mathcal{C}}^j/\ker \hat{d}^j \cong \mathrm{Im}\, \hat{d}^j$. □

Corollaire 5.3. *La cohomologie de Čech de $\mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig},L}}(k)$ sur le recouvrement $V(\beta)$ est concentrée en degré* 0 *si k est positif et en degré t si k est strictement négatif. Plus précisément, on a des isomorphismes*

$$\check{\mathrm{H}}^0(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^+(k), V(\beta)) \cong \mathcal{O}_L[T_0, \dots, T_t]_k \quad \text{si } k \text{ est positif},$$
$$\check{\mathrm{H}}^t(\mathbb{P}^t_{\mathrm{rig},L}, \mathscr{O}^+(k), V(\beta)) \cong \Big(\frac{1}{T_0 \cdots T_t}\mathcal{O}_L\Big[\frac{1}{T_0}, \dots, \frac{1}{T_t}\Big]\Big)_k \quad \text{si } k \text{ est négatif.}$$

Démonstration. D'après la description des fonctions analytiques de norme spectrale au plus 1 sur un polydisque ou une polycouronne, le complexe $\check{\mathcal{C}}^\bullet(\mathbb{P}^t_{\mathrm{rig,L}}; \mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig,L}}}(k), \mathcal{V}(\beta))$ est la complétion p-adique du complexe $\check{\mathcal{C}}^\bullet(\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}; \mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}}(k), \mathcal{V})$. Le lemme 5.2 montre alors que les groupes de cohomologie de $\check{\mathcal{C}}^\bullet(\mathbb{P}^t_{\mathrm{rig,L}}; \mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig,L}}}(k), \mathcal{V}(\beta))$ s'identifient aux complétés p-adiques des groupes de cohomologie $\mathrm{H}^*_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}, \mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}}(k))$. Comme ces derniers sont de type fini sur $\mathcal{O}_L$, la complétion est en fait inutile, ce qui permet de conclure. □

Corollaire 5.4. *Soit k un entier. La cohomologie de Čech de $\mathscr{O}^+(k)$ sur $X^d_t(\beta)$ pour le recouvrement $f^*(\mathcal{V}(\beta))$ est concentrée en degrés* 0 *et t. De plus on dispose d'isomorphismes*

$$\check{\mathrm{H}}^0\big(X^d_t(\beta), \mathscr{O}^+(k), f^*(\mathcal{V}(\beta))\big) \simeq \bigoplus_{\substack{\alpha \in \mathbb{N}^{d-t} \\ |\alpha| \le k}} \mathrm{H}^0_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}, \mathscr{O}(k - |\alpha|));$$

en particulier, les sections globales des faisceaux $\mathscr{O}^{(r)}(k)$ sont nulles si $k < 0$ et sont constantes si $k = 0$. Enfin, $\check{\mathrm{H}}^t\big(X^d_t(\beta), \mathscr{O}^+(k), f^*(\mathcal{V}(\beta))\big)$ est isomorphe au complété p-adique de

$$\bigoplus_{\substack{\alpha \in \mathbb{N}^{d-t} \\ |\alpha| \ge t+1+k}} \mathrm{H}^t_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}, \mathscr{O}(k - |\alpha|)).$$

Démonstration. Pour tout $i \in I \subset [\![0, t]\!]$ fixé, on a une trivialisation $f^{-1}(V(\beta)_I) \cong V(\beta)_I \times \mathbb{B}^{d-t}_L(-\beta_i)$ (voir note 8 pour la notation $V(\beta)_I$). D'après le lemme 4.3, tout $\lambda_I \in \mathscr{O}^+(k)(f^{-1}(V(\beta)_I))$ admet une écriture unique qui dépend du choix de l'élément i,

$$\lambda_I = \sum_{\alpha \in \mathbb{N}^{d-t}} \lambda^{(i)}_{I,\alpha}(z)\Big(\frac{w^{(i)}}{\varpi^{-\beta_i}}\Big)^\alpha, \tag{11}$$

où $z=[z_0,\dots,z_t]$ désigne la variable de $V(\beta)_I$ vu comme un ouvert de $\mathbb{P}^t_{\mathrm{rig},L}$, $w^{(i)}=(w^{(i)}_1,\dots,w^{(i)}_{d-t})$ est la variable de $\mathbb{B}^{d-t}(-\beta_i)$, les sections $\lambda^{(i)}_{I,\alpha}$ sont dans $\mathscr{O}^+(k)(V(\beta)_I)$ et tendent vers 0 p-adiquement. On déduit de cette décomposition le fait que le complexe de Čech $\mathscr{O}^+_{X^d_t(\beta)}(k)$ sur le recouvrement $f^*(\mathcal{V}(\beta))$ est la complétion de la somme directe $\bigoplus_{\alpha\in\mathbb{N}^{d-t}}\mathcal{C}^\bullet_\alpha$ où chaque complexe $\mathcal{C}^\bullet_\alpha$ est défini par

$$\mathcal{C}^s_\alpha := \bigoplus_{|I|=s+1}\left(\frac{w^{(i)}}{\varpi^{-\beta_i}}\right)^\alpha \mathscr{O}^+(k)(V(\beta)_I).$$

D'après le lemme 5.2, il suffit de voir que la cohomologie des complexes $\mathcal{C}^\bullet_\alpha$ coïncide avec celle des faisceaux $\mathscr{O}_{\mathbb{P}^t_{\mathrm{zar},\mathcal{O}_L}}(k-|\alpha|)$. Expliquons comment exhiber un tel isomorphisme. La relation $w^{(i)}/\varpi^{-\beta_i} = (\tilde{z}_j/\tilde{z}_i)(w^{(j)}/\varpi^{-\beta_j})$ induit l'identité $\lambda^{(i)}_{I,\alpha}=(\tilde{z}_i/\tilde{z}_j)^{k-|\alpha|}\lambda^{(j)}_{I,\alpha}$ et on peut voir $\lambda_{I,\alpha}:=(\lambda^{(i)}_{I,\alpha})_{i\in I}$ comme un élément de $\mathscr{O}^+(k-|\alpha|)(V(\beta)_I)$. On en déduit alors pour tout α l'isomorphisme

$$\mathcal{C}^\bullet_\alpha \cong \check{\mathcal{C}}^\bullet(\mathbb{P}^t_{\mathrm{rig},L};\mathscr{O}^+_{\mathbb{P}^t_{\mathrm{rig},L}}(k-|\alpha|),\mathcal{V}(\beta)).$$

Le résultat est alors une conséquence du corollaire 5.3. □

Remarque 5.5. En fixant une trivialisation $f^{-1}(V(\beta)_I)\cong V(\beta)_I\times\mathbb{B}^{d-t}_L(-\beta_i)=\mathbb{B}^{d-t}_{V(\beta)_I}(-\beta_i)$ pour $i\in I=[\![0,t]\!]$, on peut considérer le groupe $\check{\mathrm{H}}^t(X^d_t(\beta);\mathscr{O}^+,f^*(\mathcal{V}))$ comme un facteur direct de $\mathscr{O}^+(\mathbb{B}^{d-t}_{V(\beta)_I}(-\beta_i))/\mathscr{O}^+(V(\beta)_I)$.

5B. ***Cohomologie des complémentaires de tubes d'hyperplans.*** Nous pouvons maintenant déterminer la cohomologie de $\mathscr{O}^{(r)}$ d'un arrangement $\mathcal{A}$ tubulaire fermé d'ordre n. Nous souhaitons établir :

Théorème 5.6. *Les arrangements tubulaires fermés* Int($\mathcal{A}$) *sont* $\mathscr{O}^{(r)}$*-acycliques.*

Cela découle du principe général suivant :

Lemme 5.7. *Soit X un L-espace analytique et $\mathcal{U}=\{U_i : i\in I\}$ une famille d'ouverts de X. Soit* H *une théorie cohomologique vérifiant la suite exacte longue de Mayer–Vietoris tel que pour toute famille finie $J\subset I$, les unions $\bigcup_{i\in J}U_i$ n'ont pas de cohomologie en degré supérieur ou égal à $|J|$. Sous ces hypothèses, toutes les intersections finies non vides $\bigcap_{i\in J}U_i$ sont acycliques pour la cohomologie* H.

Il est à noter que d'après le théorème 5.1, les complémentaires des voisinages tubulaires ouverts d'hyperplans vérifient les hypothèses pour H, la cohomologie analytique à coefficients dans $\mathscr{O}^{(r)}$. En effet, pour $\mathcal{A}$ un arrangement tubulaire fermé, la cohomologie d'un espace de la forme Uni($\mathcal{B}$), avec $\mathcal{B}\subset\mathcal{A}$, s'annule en degré supérieur ou égal à rg($\mathcal{B}$) $\leq|\mathcal{B}|$.

Remarque 5.8. Notons que pour appliquer le lemme 5.7, nous avons seulement utilisé le fait que la cohomologie des fibrations $X^d_t(\beta)$ était concentrée entre les degrés 0 et t. Cette propriété se déduit directement du théorème 3.5 par comparaison avec la cohomologie de Čech sur le recouvrement $f^*(\mathcal{V}(\beta))$. Nous pouvons alors nous passer du calcul explicite de ces groupes qui constituent le cœur technique de la preuve du théorème 5.1. Toutefois, la description qui en découle servira de manière cruciale dans la preuve du lemme 5.11.

Démonstration. On peut supposer que $I = [\![1, n]\!]$ et on raisonne par récurrence sur n, le cas $n = 1$ étant évident. Supposons que le résultat est vrai pour $n - 1$. Il suffit de démontrer l'acyclicité de $Y = \bigcap_{i=1}^{n} U_i$ (les autres intersections étant traitées par l'hypothèse de récurrence). Notons $V_i = U_i \cap U_n$ pour $1 \leq i \leq n-1$ et observons que $Y = V_1 \cap \cdots \cap V_{n-1}$. Il suffit donc (grâce à l'hypothèse de récurrence) de montrer que la cohomologie de $\bigcup_{i \in J} V_i$ s'annule en degré supérieur ou égal à $|J|$ quand $J \subset [\![1, n-1]\!]$. Soit donc $k \geq |J|$ et $V^J = \bigcup_{i \in J} V_i = U^J \cap U_n$, où $U^J = \bigcup_{i \in J} U_i$. Une partie de la suite de Mayer–Vietoris s'écrit

$$\mathrm{H}^k(U^J \cup U_n) \to \mathrm{H}^k(U^J) \oplus \mathrm{H}^k(U_n) \to \mathrm{H}^k(V^J) \to \mathrm{H}^{k+1}(U^J \cup U_n).$$

Puisque $k + 1 \geq |J \cup \{n\}|$, le terme $\mathrm{H}^{k+1}(U^J \cup U_n)$ s'annule par hypothèse, et il en est de même de $\mathrm{H}^k(U_n)$ et $\mathrm{H}^k(U^J)$, donc aussi de $\mathrm{H}^k(V^J)$, ce qui permet de conclure. □

Nous pouvons aussi tirer des informations importantes sur les sections globales à puissances bornées des arrangements $\mathrm{Int}(\mathcal{A})$. Nous commencerons par ce lemme général.

Lemme 5.9. *Soit X un espace analytique, $\mathscr{F}$ un faisceau en groupes abéliens et $\mathcal{U} = \{U_i\}$ une famille d'ouverts de X tel que toute intersection finie[21] U_I est $\mathscr{F}$-acyclique. Dans ce cas, on a*

$$\mathscr{F}(U_I) = \sum_{J \in E_I} r_{J,I}(\mathscr{F}(U_J)),$$

où $E_I = \left\{J \subset I : J \neq \varnothing \text{ et } \mathrm{H}_{\mathrm{an}}^{|J|-1}\left(\bigcup_{j \in J} U_j, \mathscr{F}\right) \neq 0\right\}$ et $r_{J,I} : \mathscr{F}(U_J) \to \mathscr{F}(U_I)$ est la flèche de restriction.

Démonstration. On raisonne par récurrence sur le cardinal de I. Le résultat est trivial quand ce dernier vaut 1. Fixons I et supposons le résultat pour toute partie stricte de I. Si $\mathrm{H}_{\mathrm{an}}^{|I|-1}\left(\bigcup_{i \in I} U_i, \mathscr{F}\right) \neq 0$, c'est tautologique, car $I \in E_I$. Sinon, on a par hypothèse

$$\mathrm{H}_{\mathrm{an}}^{|I|-1}\left(\bigcup_{i \in I} U_i, \mathscr{F}\right) = \check{\mathrm{H}}^{|I|-1}\left(\bigcup_{i \in I} U_i, \{U_i : i \in I\}, \mathscr{F}\right) = \mathscr{F}(U_I) / \sum_{i \in I} r_{I \setminus \{i\}, I}(\mathscr{F}(U_{I \setminus \{i\}})) = 0.$$

Mais par hypothèse de récurrence,

$$\sum_{i \in I} r_{I \setminus \{i\}, I}(\mathscr{F}(U_{I \setminus \{i\}})) = \sum_{i \in I} \sum_{J \in E_{I \setminus \{i\}}} r_{J,I}(\mathscr{F}(U_J)) = \sum_{J \in E_I} r_{J,I}(\mathscr{F}(U_J)),$$

car $E_I = \bigcup_{i \in I} E_{I \setminus \{i\}}$. Le résultat s'en déduit. □

Corollaire 5.10 (décomposition en éléments simples). *Soit $\mathcal{A}$ un arrangement tubulaire fermé, on a*

$$\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A})) = \sum_{\substack{\mathcal{B} \subset \mathcal{A} \\ |\mathcal{B}| = \mathrm{rg}(\mathcal{B}) < d+1}} \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{B})).$$

21. Voir note 8 pour la notation U_I.

Démonstration. On reprend les notations du lemme précédent. On remarque l'identité[22] $E_{\mathcal{A}} = \{\mathcal{B} \subset \mathcal{A} : |\mathcal{B}| = \mathrm{rg}(\mathcal{B}) < d+1\}$ d'après le corollaire 5.4 et on conclut. ☐

Lemme 5.11. *Soit $\mathcal{A}_n$ un arrangement tubulaire fermé d'ordre $n > d$ et $\mathcal{A}_{n-d}$ la restriction de $\mathcal{A}_n$ d'ordre $n-d$. On a l'inclusion*

$$\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_n)) \subset \mathcal{O}_L^{(r)} + \varpi \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-d})).$$

Démonstration. D'après le résultat précédent, on peut supposer $|\mathcal{A}_n| = \mathrm{rg}(\mathcal{A}_n) < d+1$. On raisonne par récurrence sur $t = \mathrm{rg}(\mathcal{A}_n) - 1$. Plus précisément, nous montrons que pour tout arrangement tubulaire fermé $\mathcal{B}_m$ d'ordre m quelconque vérifiant $\mathrm{rg}(\mathcal{B}_m) \leq t+1$, on a l'inclusion $\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{B}_m)) \subset \mathcal{O}_L^{(r)} + \varpi \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{B}_{m-(t+1)}))$.

Quand $t = 0$, cela découle du cas de la boule qui a été traité au lemme 4.1. Supposons l'énoncé vrai pour $t-1$ et montrons le résultat pour l'arrangement $\mathcal{A}_n$ de rang $t+1$. Notons $\mathcal{A}_{n-1}$ la projection de $\mathcal{A}_n$ d'ordre $n-1$. On a des recouvrements naturels $\mathrm{Uni}(\mathcal{A}_n) = \bigcup_{H \in \mathcal{A}_n} \mathring{H}(|\varpi|^n)^c$ et $\mathrm{Uni}(\mathcal{A}_{n-1}) = \bigcup_{H \in \mathcal{A}_n} \mathring{H}(|\varpi|^{n-1})$ de cardinal[23] $t+1$ que l'on notera $\mathcal{A}_n^c$ et $\mathcal{A}_{n-1}^c$.

Les intersections d'éléments de $\mathcal{A}_n^c$ ou de $\mathcal{A}_{n-1}^c$ sont $\mathscr{O}^{(r)}$-acycliques d'après le théorème 5.6 et on peut calculer la cohomologie des espaces $\mathrm{Uni}(\mathcal{A}_n)$ et $\mathrm{Uni}(\mathcal{A}_{n-1})$ via les complexes de Čech sur ces recouvrements. Ces derniers sont concentrés entre les degrés 0 et t, on en déduit un isomorphisme[24] $\mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)}) \cong \check{\mathcal{C}}^t(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)}, \mathcal{A}_n^c)/\delta(\check{\mathcal{C}}^{t-1}(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)}, \mathcal{A}_n^c))$ (idem pour $\mathcal{A}_{n-1}$). De plus, ces deux recouvrements sont compatibles avec l'inclusion $\mathrm{Uni}(\mathcal{A}_{n-1}) \subset \mathrm{Uni}(\mathcal{A}_n)$, d'où une flèche entre les complexes de Čech qui induit le morphisme fonctoriel $\mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)}) \xrightarrow{\varphi^{(r)}} \mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_{n-1}), \mathscr{O}^{(r)})$. En explicitant ces complexes, on obtient un diagramme commutatif dont les lignes horizontales sont exactes :

$$\begin{array}{ccccccc} \sum_{a \in \mathcal{A}_n} \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_n \backslash \{a\})) & \longrightarrow & \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_n)) & \longrightarrow & \mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \sum_{a \in \mathcal{A}_{n-1}} \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-1} \backslash \{a\})) & \longrightarrow & \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-1})) & \longrightarrow & \mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_{n-1}), \mathscr{O}^{(r)}) & \longrightarrow & 0 \end{array} \tag{12}$$

On veut montrer

$$\mathrm{Im}(\varphi^{(r)}) \subset \varpi \mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_{n-1}), \mathscr{O}^{(r)}). \tag{13}$$

Quand $\mathrm{rg}(\mathcal{A}_{n-1}) < \mathrm{rg}(\mathcal{A}_n)$, l'inclusion est triviale, car $\mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_{n-1}), \mathscr{O}^{(r)}) = 0$.

Si $\mathrm{rg}(\mathcal{A}_{n-1}) = \mathrm{rg}(\mathcal{A}_n)$, on a des isomorphismes compatibles

$$\mathrm{Uni}(\mathcal{A}_n) \cong X_t^d(\beta) \quad \text{et} \quad \mathrm{Uni}(\mathcal{A}_n) \cong X_t^d(\tilde{\beta})$$

---

22. Notons que lorsque l'on a l'égalité $|\mathcal{B}| = \mathrm{rg}(\mathcal{B}) = d+1$, on a $\mathrm{Uni}(\mathcal{B}) = \mathbb{P}^d_{\mathrm{rig},L}$ qui est $\mathscr{O}^{(r)}$-acyclique d'après le corollaire 5.3.

23. Nous nous autorisons des répétitions dans le deuxième recouvrement.

24. Ici, δ désigne la différentielle du complexe $\check{\mathcal{C}}^\bullet(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)}, \mathcal{A}_n^c)$.

avec $\tilde{\beta} := \beta - (1, \dots, 1)$. D'après la remarque 5.5, $\mathrm{H}^t_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}_n), \mathscr{O}^{(r)})$ est un facteur direct de

$$\mathscr{O}^{(r)}(\mathbb{B}^{d-t}_{V(\beta)_I}(-\beta_i))/\mathscr{O}^{(r)}(V(\beta)_I)$$

pour $I = [\![0, t]\!]$, $i \in I$ fixé (idem pour $\mathrm{Int}(\mathcal{A}_{n-1})$). De plus, la flèche $\varphi^{(r)}$ est induite (notons l'égalité $V(\beta)_I = V(\tilde{\beta})_I$) par la restriction naturelle

$$\mathscr{O}^{(r)}(\mathbb{B}^{d-t}_{V(\beta)_I}(-\beta_i))/\mathscr{O}^{(r)}(V(\beta)_I) \to \mathscr{O}^{(r)}\big(\mathbb{B}^{d-t}_{V(\beta)_I}(-(\beta_i - 1))\big)/\mathscr{O}^{(r)}(V(\beta)_I)$$

et l'image de $\varphi^{(r)}$ est contenue dans $\varpi\mathscr{O}^{(r)}\big(\mathbb{B}^{d-t}_{V(\beta)_I}(-(\beta_i - 1))\big)/\mathscr{O}^{(r)}(V(\beta)_I)$ d'après le lemme 4.2, ce qui entraîne (13).

D'après (12) et (13), on obtient, pour toute fonction $f \in \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_n))$, une décomposition dans $\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-1}))$

$$f = \sum_{a \in \mathcal{A}_{n-1}} f_a + g$$

avec $f_a \in \mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-1} \backslash \{a\}))$ et $g \in \varpi\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-1}))$. Comme $\mathrm{rg}(\mathcal{A}_{n-1} \backslash \{a\}) < \mathrm{rg}(\mathcal{A}_n)$, on a par hypothèse de récurrence

$$f_a \in \mathcal{O}_L^{(r)} + \varpi\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-1-t} \backslash \{a\})) \subset \mathcal{O}_L^{(r)} + \varpi\mathscr{O}^{(r)}(\mathrm{Int}(\mathcal{A}_{n-(t+1)})),$$

ce qui établit le résultat. □

Corollaire 5.12. *Soit $\mathcal{A}_n$ un arrangement tubulaire fermé d'ordre $n > 2d$ et $\mathcal{A}_{n-2d}$ la projection de $\mathcal{A}_n$ d'ordre $n - 2d$. On a l'inclusion*

$$\mathscr{O}^{+}(\mathrm{Int}(\mathcal{A}_n)) \subset \mathcal{O}_L + \varpi\mathscr{O}^{+}(\mathrm{Int}(\mathcal{A}_{n-2d})).$$

Démonstration. Cela découle de la proposition 4.6 et du lemme 5.11. □

5C. ***Cohomologie analytique à coefficients dans* $\mathbf{1}+\mathscr{O}^{++} = \mathscr{O}^{**}$.** Nous allons maintenant nous intéresser aux faisceaux $1 + \mathscr{O}^{++}$ et démontrer un théorème d'acyclicité semblable au théorème 5.6. Le résultat suivant est le point clé de cette section. C'est une application du logarithme tronqué qui permet d'étudier $1 + \mathscr{O}^{++}$ par le biais de $\mathscr{O}^{(r)}$. Ce résultat a été énoncé par Van der Put [1982] (voir § 3.26, remarque en fin de page 195). Nous allons donner les détails de la preuve.

Lemme 5.13. *Soit X un espace rigide quasi compact.*

(1) *Si X est $\mathscr{O}^{(r)}$-acyclique pour tout $0 < r \leq 1$, alors X est $1 + \mathscr{O}^{++}$-acyclique.*

(2) *Si la cohomologie de X à coefficients dans $\mathscr{O}^{(r)}$ est concentrée en degrés 0 et t pour tout $0 < r \leq 1$, et la flèche naturelle $\mathrm{H}^t_{\mathrm{an}}(X, \mathscr{O}^{(r)}) \to \mathrm{H}^t_{\mathrm{an}}(X, \mathscr{O}^{(r')})$ est injective pour $r' \geq r$, alors la cohomologie de X à coefficients dans $1 + \mathscr{O}^{++}$ est concentrée en degrés 0 et t.*

Démonstration. On suppose que la cohomologie de X à coefficients dans $\mathscr{O}^{(r)}$ est concentrée en degrés 0 et t pour tout $r > 0$, et la flèche naturelle $\mathrm{H}^t_{\mathrm{an}}(X, \mathscr{O}^{(r)}) \to \mathrm{H}^t_{\mathrm{an}}(X, \mathscr{O}^{(r')})$ est injective pour $r' \geq r$. Le premier point s'en déduit quand $t = 0$. Soit $s \notin \{0, t\}$, on veut l'annulation de la cohomologie de $1 + \mathscr{O}^{++}$ en degré s. Remarquons que $1 + \mathscr{O}^{++} = \varinjlim_{r \to 1^-} 1 + \mathscr{O}^{(r)}$, donc (par quasi-compacité),

$$\mathrm{H}^s_{\mathrm{an}}(X; 1 + \mathscr{O}^{++}) = \varinjlim_{r \to 1^-} \mathrm{H}^s_{\mathrm{an}}(X; 1 + \mathscr{O}^{(r)}).$$

On fixe $r < 1$. On a la suite exacte

$$0 \to 1 + \mathscr{O}^{(r^2)} \to 1 + \mathscr{O}^{(r)} \to \mathscr{O}^{(r)}/\mathscr{O}^{(r^2)} \to 0$$

où la surjection est donnée par $(1 + x) \mapsto x$. Par hypothèse, $\mathscr{O}^{(r)}/\mathscr{O}^{(r^2)}$ a une cohomologie analytique concentrée en degrés 0 et t d'après la suite exacte [25]

$$0 \to \mathscr{O}^{(r^2)} \to \mathscr{O}^{(r)} \to \mathscr{O}^{(r)}/\mathscr{O}^{(r^2)} \to 0.$$

On a donc une surjection

$$\mathrm{H}^s_{\mathrm{an}}(X; 1 + \mathscr{O}^{(r^2)}) \to \mathrm{H}^s_{\mathrm{an}}(X; 1 + \mathscr{O}^{(r)}). \tag{14}$$

Il suffit de prouver que $\mathrm{H}^s_{\mathrm{an}}(X; 1 + \mathscr{O}^{(r)}) = 0$ pour r petit. Si $r < |p|$ et $\|x\| < r$, alors pour tout n, on a [26]

$$\left\| \frac{x^n}{n!} \right\| < \|x\| \quad \text{et} \quad \left\| \frac{x^n}{n} \right\| < \|x\|.$$

Les séries usuelles du logarithme et de l'exponentielle sont bien définies et vérifient

$$\|\exp(x) - 1\| = \|\log(1 + x)\| = \|x\|$$

et elles induisent des morphismes inverses l'un de l'autre,

$$\exp : \mathscr{O}^{(r)} \xrightarrow{\sim} 1 + \mathscr{O}^{(r)} \quad \text{et} \quad \log : 1 + \mathscr{O}^{(r)} \xrightarrow{\sim} \mathscr{O}^{(r)},$$

d'où l'annulation de $\mathrm{H}^s_{\mathrm{an}}(X; 1 + \mathscr{O}^{(r)})$ par hypothèse. □

Les théorèmes 5.1 et 5.6 et les corollaires 5.3 et 5.4 de la section 5 permettent d'appliquer directement ce résultat et d'obtenir (on utilise partout la topologie analytique) :

Corollaire 5.14. *Les propriétés suivantes sont vérifiées* :

(1) *Les arrangements tubulaires fermés* $\mathrm{Int}(\mathcal{A})$ *sont* $\mathscr{O}^{**}$*-acycliques.*

(2) *Les espaces projectifs* $\mathbb{P}^d_{\mathrm{rig},L}$ *sont* $\mathscr{O}^{**}$*-acycliques.*

(3) *La cohomologie des fibrations* $X^d_t(\beta)$ *pour le faisceau* $\mathscr{O}^{**}$ *est concentrée en degrés* 0 *et* t.

Le résultat précédent nous permet d'appliquer le lemme 5.9 pour obtenir une version multiplicative de la décomposition en éléments simples du corollaire 5.10.

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25. Il est à noter que nous avons utilisé l'hypothèse d'injectivité de $\mathrm{H}^t_{\mathrm{an}}(X, \mathscr{O}^{(r)}) \to \mathrm{H}^t_{\mathrm{an}}(X, \mathscr{O}^{(r')})$ pour montrer l'annulation de $\mathrm{H}^{t-1}_{\mathrm{an}}(X; 1 + \mathscr{O}^{(r)})$.

26. Justifions la première inégalité. Par hypothèse, on a supposé que $p^{1+\varepsilon}$ divisait x pour ε un rationnel assez petit et on rappelle l'identité $v_p(n!) = (n - s_p(n))/(p-1) \leq n - 1$ avec $s_p(n)$ la somme des chiffres dans l'écriture de n en base p, d'où $\|x^{n-1}/n!\| < 1$. On en déduit alors $\|x^n/n!\| < \|x\|$.

Corollaire 5.15 (décomposition en éléments simples). *Soit $\mathcal{A}$ un arrangement tubulaire fermé, on a*

$$\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A})) = \sum_{\substack{\mathcal{B}\subset\mathcal{A} \\ |\mathcal{B}|=\mathrm{rg}(\mathcal{B})<d+1}} \mathscr{O}^{**}(\mathrm{Int}(\mathcal{B})).$$

6. Cohomologie analytique à coefficients dans $\mathbb{G}_m$

6A. ***Cohomologie des fibrations $X_t^d(\beta)$.*** Nous souhaitons montrer le théorème suivant :

Théorème 6.1. *Soient $s \geq 1$ et $t \geq 1$. Les fonctions inversibles de $X_t^d(\beta)$ sont constantes et l'application f^* en cohomologie donnée par la fibration $f : X_t^d(\beta) \to \mathbb{P}^t_{\mathrm{rig},L}$ induit une décomposition*

$$\mathrm{H}^s_{\mathrm{an}}(X_t^d(\beta), \mathbb{G}_m) \cong \mathrm{H}^s_{\mathrm{an}}(X_t^d(\beta), \mathscr{O}^{**}) \times \mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m). \tag{15}$$

De plus, l'inclusion $\iota : X_t^d(\beta) \to \mathbb{P}^d_{\mathrm{rig},L}$ induit une bijection entre $\mathrm{H}^*_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathbb{G}_m)$ et le facteur direct $\mathrm{H}^*_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m)$. Enfin, pour tout corps F, on a une identification

$$\mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m) = \mathrm{H}^s_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F}, \mathbb{G}_m) = \begin{cases} \mathbb{Z} & si\ s = 1, \\ \{0\} & si\ s > 1. \end{cases}$$

Corollaire 6.2. *Soit $\mathcal{A}$ un arrangement tubulaire fermé. La cohomologie analytique à coefficients dans $\mathbb{G}_m$ de* $\mathrm{Uni}(\mathcal{A})$ *s'annule en degré supérieur ou égal à* $\mathrm{rg}(\mathcal{A})$.

Démonstration du théorème 6.1. D'après le théorème 3.5 point 4, les recouvrements $\mathcal{V}(\beta)$ et $f^*(\mathcal{V}(\beta))$ ont des intersections $\mathbb{G}_m$-acycliques et on se ramène à calculer la cohomologie de Čech sur ces recouvrements. Pour toute intersection $V(\beta)_I$, on fixe une trivialisation $f^{-1}(V(\beta)_I) \cong V(\beta)_I \times \mathbb{B}_L^{d-t}(-\beta_{i_0})$ pour $i_0 \in I$ et on a, d'après le lemme 4.4,

$$\mathscr{O}^*(V(\beta)_I) = L^*\mathscr{O}^{**}(V(\beta)_I) \times T_I^{(t)}$$

et

$$\mathscr{O}^*(f^{-1}(V(\beta)_I)) = L^*\mathscr{O}^{**}(f^{-1}(V(\beta)_I)) \times T_I^{(t)},$$

où $T_I^{(t)} = \langle \tilde{z}_i/\tilde{z}_j : i, j \in I\rangle_{\mathbb{Z}\text{-Mod}} \subset \langle \tilde{z}_i/\tilde{z}_j : i, j \in [\![0, t]\!]\rangle_{\mathbb{Z}\text{-Mod}}$.

Introduisons le complexe $(\mathcal{C}^i(T_\bullet^{(t)}))_{0\leq i\leq t} := \big(\bigoplus_{I\subset[\![0,t]\!]:|I|=i+1} T_I^{(t)}\big)_i$ avec pour différentielles les sommes alternées des inclusions. C'est un facteur direct du complexe de Čech $\check{\mathcal{C}}^\bullet\big(X_t^d(\beta); \mathbb{G}_m, f^*(\mathcal{V}(\beta))\big)$ et on en déduit un isomorphisme

$$\check{\mathrm{H}}^s\big(X_t^d(\beta); \mathbb{G}_m, f^*(\mathcal{V}(\beta))\big) \cong \check{\mathrm{H}}^s\big(X_t^d(\beta); L^*\mathscr{O}^{**}, f^*(\mathcal{V}(\beta))\big) \times \mathrm{H}^s(\mathcal{C}^*(T_\bullet^{(t)})). \tag{16}$$

Nous allons, dans la suite de l'argument, calculer les deux termes apparaissant dans ce produit direct. Commençons par la cohomologie du complexe $\mathcal{C}^*(T_\bullet^{(t)})$.

Proposition 6.3. *On a pour tout corps F*

$$\mathrm{H}^s_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F}; \mathbb{G}_m) \cong \mathrm{H}^s(\mathcal{C}^*(T_\bullet^{(t)})) = \begin{cases} \mathbb{Z} & si\ s = 1, \\ \{0\} & si\ s > 1. \end{cases}$$

Démonstration. Nous allons procéder en calculant $\mathrm{H}^s_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F};\mathbb{G}_m)$ de deux manières différentes. Dans un premier temps, nous utiliserons la suite exacte (17) (voir plus bas) pour donner une expression explicite à ces groupes de cohomologie. Puis, nous étudierons la cohomologie de Čech de $\mathbb{G}_m$ sur le recouvrement standard $\mathcal{V} = \{V_i\}$ de l'espace projectif pour relier $\mathrm{H}^s_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F};\mathbb{G}_m)$ à la cohomologie du complexe $\mathcal{C}^*(T^{(t)}_\bullet)$. (Cet argument pourrait être vu comme un analogue algébrique de la décomposition (16).)

Pour réaliser ces deux étapes, nous allons rappeler quelques propriétés du foncteur des diviseurs de Cartier. Nous renvoyons à [Hartshorne 1977, Section II.6; Görtz et Wedhorn 2010, Chapter 11, Sections 11.9–11.14] pour leur étude sur un schéma localement factoriel. Nous aurons seulement besoin ici du cas où l'espace considéré $X = \mathrm{Spec}(A)$ est affine de sections globales factorielles où la situation est beaucoup plus simple. Appelons S l'ensemble des éléments irréductibles dans A à un inversible près et, pour tout ouvert U de X, appelons $S(U)$ les éléments f de S tels que $V(f)$ ne rencontre pas U. Intéressons-nous (la propriété de recollement étant claire) au faisceau

$$\mathrm{Div} : U \subset X \mapsto \mathbb{Z}[S(U)].$$

Les ouverts standards $D(f) = \mathrm{Spec}(A[1/f])$ de X sont encore affines de sections globales factorielles et l'ensemble des éléments irréductibles dans A à un inversible près est S privé des facteurs irréductibles de f, à savoir $S(D(f))$. Si on note $\mathcal{K}_X = \mathrm{Frac}(A)$ les fractions rationnelles sur X, on en déduit l'identité

$$\mathrm{Div}(D(f)) = \mathcal{K}_X/\mathscr{O}^*(D(f))$$

par factorialité de $A[1/f]$. Comme les ouverts standards forment une base de voisinage de X, on a la suite exacte de faisceaux

$$0 \to \mathbb{G}_m \to \mathcal{K}_X \to \mathrm{Div} \to 0. \tag{17}$$

De plus, Div est flasque par construction et le faisceau constant $\mathcal{K}_X$ l'est aussi par irréductibilité de X.

Revenons à notre cas d'étude. Le recouvrement standard $\mathcal{V}$ de l'espace projectif est constitué d'ouverts affines dont les sections globales sont factorielles. On peut construire comme dans la discussion précédente le faisceau Div sur chacun de ces ouverts $V_i \in \mathcal{V}$, et ces derniers se recollent sur l'espace projectif tout entier, car chaque $V_i \cap V_j$ est encore affine de sections factorielles. Par recollement, on obtient encore une suite exacte

$$0 \to \mathbb{G}_m \to \mathcal{K}_{\mathbb{P}^t_{\mathrm{zar},F}} \xrightarrow{\pi} \mathrm{Div} \to 0$$

où les faisceaux $\mathcal{K}_{\mathbb{P}^t_{\mathrm{zar},F}}$ et Div sont encore flasques;[27] ils n'ont pas de cohomologie en degré strictement positif. Par suite exacte longue, on en déduit les égalités

$$\mathrm{H}^s_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F};\mathbb{G}_m) = 0 \quad \text{si } s \geq 2,$$
$$\mathrm{H}^1_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F};\mathbb{G}_m) = \mathrm{Div}(\mathbb{P}^t_{\mathrm{zar},F})/\pi(\mathcal{K}_{\mathbb{P}^t_{\mathrm{zar},F}}) \cong \mathbb{Z}.$$

27. C'est une notion locale.

Nous laissons au lecteur le soin de vérifier que les sections globales de Div sur l'espace projectif s'identifient au module libre sur $\mathbb{Z}$ engendré par les éléments irréductibles de $K[z_0, \ldots, z_d]$ à une unité près et que $\pi(\mathcal{K}_{\mathbb{P}^t_{\mathrm{zar},F}})$ s'identifie au sous-ensemble des éléments de masse totale nulle. L'isomorphisme ci-dessus est alors induit par le degré.

On relie maintenant $\mathrm{H}^*_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F}; \mathbb{G}_m)$ à $\mathrm{H}^*(\mathcal{C}^*(T^{(t)}_\bullet))$. On voit chaque intersection $V_I := \bigcap_{i\in I} V_i$ comme le spectre d'un anneau factoriel et on déduit de (17), soit

$$0 \to \mathbb{G}_m \to \mathcal{K}_{V_I} \to \mathrm{Div} \to 0,$$

que chaque intersection V_I est $\mathbb{G}_m$-acyclique.[28] En particulier, on peut calculer $\mathrm{H}^*_{\mathrm{zar}}(\mathbb{P}^t_{\mathrm{zar},F}; \mathbb{G}_m)$ via la cohomologie de Čech sur le recouvrement $\mathcal{V}$ qui admet la décomposition

$$\check{\mathrm{H}}^s(\mathbb{P}^t_{\mathrm{zar},F}; \mathbb{G}_m, \mathcal{V}) = \check{\mathrm{H}}^s(\mathbb{P}^t_{\mathrm{zar},F}; F^*, \mathcal{V}) \times \mathrm{H}^s(\mathcal{C}^*(T^{(t)}_\bullet)) = \mathrm{H}^s(\mathcal{C}^*(T^{(t)}_\bullet)).$$

La dernière égalité s'obtient par contractibilité du nerf de $\mathcal{V}$, ce qui termine l'argument. □

Étudions maintenant le terme $\check{\mathrm{H}}^s(X^d_t(\beta); L^*\mathscr{O}^{**}, f^*(\mathcal{V}(\beta))$ et décrivons un peu plus précisément le complexe de Čech associé. Pour toute partie I, on a une suite exacte de groupes

$$1 \to 1 + \mathfrak{m}_L \to L^* \times \mathscr{O}^{**}(f^{-1}(V(\beta)_I)) \to L^*\mathscr{O}^{**}(f^{-1}(V(\beta)_I)) \to 1,$$

d'où une suite exacte au niveau des complexes de Čech sur le recouvrement $f^*(\mathcal{V}(\beta))$ et une suite exacte longue entre les cohomologies associées. Comme le nerf du recouvrement $f^*(\mathcal{V}(\beta))$ est contractile, on a,[29] pour $s \geq 1$,

$$\check{\mathrm{H}}^s\big(X^d_t(\beta); 1 + \mathfrak{m}_L, f^*(\mathcal{V}(\beta))\big) = \check{\mathrm{H}}^s\big(X^d_t(\beta); L^*, f^*(\mathcal{V}(\beta))\big) = 0.$$

On en déduit une suite d'isomorphismes

$$\check{\mathrm{H}}^s\big(X^d_t(\beta); L^*\mathscr{O}^{**}, f^*(\mathcal{V}(\beta))\big) \cong \check{\mathrm{H}}^s\big(X^d_t(\beta); \mathscr{O}^{**}, f^*(\mathcal{V}(\beta))\big) \cong \mathrm{H}^s_{\mathrm{an}}(X^d_t(\beta), \mathscr{O}^{**}),$$

où le dernier isomorphisme s'obtient par acyclicité des polycouronnes (théorème 3.5 point 2 et lemme 5.13).

Revenons à la cohomologie des fibrations. Dans le cas particulier où $t = d$ et $f = \mathrm{Id}$, on traite alors le cas de l'espace projectif. D'après (16), le corollaire 5.14 point 2 et la proposition 6.3, on a pour $s \geq 1$ des isomorphismes

$$\mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}; \mathbb{G}_m) \cong \mathrm{H}^s(\mathcal{C}^*(T^{(d)}_\bullet)) = \begin{cases} L^* & \text{si } s = 0, \\ \mathbb{Z} & \text{si } s = 1, \\ \{0\} & \text{si } s > 1. \end{cases}$$

Remarque 6.4. L'identification ci-dessus entre les groupes de degré 1 est explicite en termes de fibrés en droite. On voit que la famille de fonctions inversibles $(\tilde{z}^k_i/\tilde{z}^k_j)_{0\leq i,j\leq d}$ apparaissant dans le diagramme

28. En degré 1, nous avons utilisé l'égalité $\mathrm{Div}(V_I) = \mathcal{K}_X/\mathscr{O}^*(V_I)$ qui découle de la construction de Div.

29. Pour $s \geq 1$, cet argument prouve aussi l'annulation de $\check{\mathrm{H}}^s\big(X^d_t(\beta); A, f^*(\mathcal{V}(\beta))\big)$ pour tout faisceau constant A, ce qui établit $\mathrm{H}^s_{\mathrm{an}}(X^d_t(\beta); A) = 0$ par le théorème 3.5. Enfin, grâce au lemme 5.7, nous obtenons pour tout arrangement tubulaire $\mathcal{A}$ l'annulation $\mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), A)$.

(10) forme un cocycle dans $\mathcal{C}^*(T_\bullet^{(d)})$. Cela prouve que l'isomorphisme $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^d_{\mathrm{rig},L})$ est donné par $k \mapsto \mathscr{O}_{\mathbb{P}^d_{\mathrm{rig},L}}(k)$.

Dans le cas général, on voit que la fibration f identifie les facteurs directs isomorphes à $\mathcal{C}^*(T_\bullet^{(d)})$ dans les complexes $\check{\mathcal{C}}^\bullet\big(X_t^d(\beta); \mathbb{G}_m, f^*(\mathcal{V}(\beta))\big)$ et $\check{\mathcal{C}}^\bullet(\mathbb{P}^t_{\mathrm{rig},L}; \mathbb{G}_m, \mathcal{V}(\beta))$ La décomposition (16) devient

$$\mathrm{H}^s_{\mathrm{an}}(X_t^d(\beta), \mathbb{G}_m) \cong \mathrm{H}^s_{\mathrm{an}}(X_t^d(\beta), \mathscr{O}^{**}) \times \mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m)$$

pour $s \geq 1$. En degré 0, on a

$$\mathscr{O}^*(X_t^d(\beta)) = \check{\mathrm{H}}^0\big(X_t^d(\beta); L^*\mathscr{O}^{**}, f^*(\mathcal{V}(\beta))\big) = L^*\mathscr{O}^{**}(X_t^d(\beta)) = L^*,$$

car $\mathrm{H}^0(\mathcal{C}^*(T_\bullet^{(t)})) = 0$ (voir proposition 6.3) et $\mathscr{O}^{**}(X_t^d(\beta)) = 1 + \mathfrak{m}_L$ (voir corollaire 5.4).

Nous terminons l'argument en caractérisant l'image de $\mathrm{H}^*_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathbb{G}_m)$ dans $\mathrm{H}^*_{\mathrm{an}}(X_t^d(\beta), \mathbb{G}_m)$ dans la décomposition (16).

Lemme 6.5. *L'inclusion* $\iota : X_t^d(\beta) \to \mathbb{P}^d_{\mathrm{rig},L}$ *induit un isomorphisme de* $\mathrm{H}^*_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathbb{G}_m)$ *sur le facteur direct* $\mathrm{H}^*_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m)$ *de* $\mathrm{H}^*_{\mathrm{an}}(X_t^d(\beta), \mathbb{G}_m)$.

Démonstration. Il suffit de montrer l'isomorphisme en degré 1, car les groupes en degrés supérieurs sont nuls et les sections constantes sont identifiées en degré 0. On rappelle que l'espace

$$X_t^d(\beta) = \{z = [z_0, \ldots, z_d] \in \mathbb{P}^d_{\mathrm{rig},L}, \exists i \leq t, \forall j \leq d, |z_i| \geq |\varpi|^{\beta_i}|z_j|\}$$

admet un recouvrement

$$X_t^d(\beta) = \bigcup_{i \leq t}\{z = [z_0, \ldots, z_d] \in \mathbb{P}^d_{\mathrm{rig},L}, \tilde{z}_i \neq 0 \text{ et } \forall j \leq t, |\tilde{z}_i| \geq |\tilde{z}_j|\} = \bigcup_{i \leq t} f^*(V(\beta))_i.$$

De plus, on a pour $0 \leq i \leq d$ la famille d'ouverts

$$V_i := \{z = [z_0, \ldots, z_d] \in \mathbb{P}^d_{\mathrm{rig},L}, \tilde{z}_i \neq 0\}$$

qui recouvre l'espace projectif tout entier et qui vérifie $f^*(V(\beta))_i \subset V_i$. La famille de fonctions inversibles

$$\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)_{0 \leq i,j \leq d} \in \prod_{i,j} \mathscr{O}^*(V_i \cap V_j) = \check{\mathcal{C}}^1(\mathbb{P}^d_{\mathrm{rig},L}, \mathbb{G}_m, \mathcal{V})$$

définit un cocycle et correspond donc à un fibré en droite sur $\mathbb{P}^d_{\mathrm{rig},L}$. D'après la remarque 6.4, la classe de ce fibré engendre $\mathrm{H}^1_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathbb{G}_m)$. Par compatibilité des recouvrements $f^*(\mathcal{V}(\beta))$ et $\mathcal{V}$, la restriction de cette classe à $X_t^d(\beta)$ est donnée par le cocycle

$$\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)_{0 \leq i,j \leq t} \in \prod_{i,j} \mathscr{O}^*\big(f^*(V(\beta))_i \cap f^{-1}(V(\beta))_j\big) = \check{\mathcal{C}}^1\big(X_t^d(\beta), \mathbb{G}_m, f^*(\mathcal{V}(\beta))\big)$$

qui, toujours par la remarque 6.4, engendre le facteur direct $\mathrm{H}^1_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m)$ de $\mathrm{H}^1_{\mathrm{an}}(X_t^d(\beta), \mathbb{G}_m)$. Ceci conclut l'argument. □

Remarque 6.6. On a en fait montré un résultat plus fort; on a un diagramme commutatif

$$\begin{array}{ccc} \mathrm{H}^s_{\mathrm{an}}(X^d_t(\beta), \mathbb{G}_m) & \xleftarrow{\iota^*} & \mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^d_{\mathrm{rig},L}, \mathbb{G}_m) \\ {\scriptstyle f^*}\uparrow & \swarrow{\scriptstyle \varphi^*}\ \sim & \\ \mathrm{H}^s_{\mathrm{an}}(\mathbb{P}^t_{\mathrm{rig},L}, \mathbb{G}_m) & & \end{array}$$

où φ est le morphisme de $\mathbb{P}^t_{\mathrm{rig},L}$ dans $\mathbb{P}^d_{\mathrm{rig},L}$ donné par $[z_0, \dots, z_t] \mapsto [z_0, \dots, z_t, 0, \dots, 0]$. Par contre, les morphismes au niveau des espaces ne commutent pas.

Ainsi tous les points ont été démontrés. □

6B. *Cohomologie des arrangements tubulaires fermés.* Nous sommes maintenant en mesure de déterminer la cohomologie à coefficients dans $\mathbb{G}_m$ pour les arrangements tubulaires fermés et ainsi donner l'un des résultats principaux de cet article.

Théorème 6.7. *Pour tout arrangement tubulaire fermé $\mathcal{A}$, les intersections* $\mathrm{Int}(\mathcal{A})$ *sont $\mathbb{G}_m$-acycliques.*

Démonstration. On peut appliquer le lemme 5.7, car la cohomologie des unions $\mathrm{Uni}(\mathcal{A})$ à coefficients dans $\mathbb{G}_m$ s'annule en degré supérieur ou égal à $\mathrm{rg}(\mathcal{A})$ d'après le corollaire 6.2 où $\mathrm{rg}(\mathcal{A}) \leq |\mathcal{A}|$. □

Nous avons aussi un résultat de structure pour les fonctions inversibles d'un arrangement tubulaire fermé. Nous aurons besoin de quelques notations.

Définition 6.8. Si S est un ensemble fini et A est un anneau, on note le sous-ensemble $A[S]^0 \subset A[S]$ du module libre sur A engendré par S constitué des éléments de masse totale nulle. [30]

Remarque 6.9. Si $\mathcal{A}$ est un arrangement tubulaire fermé, nous faisons le choix d'un système de représentants des éléments de $\mathcal{A}$ par des éléments de $\mathcal{H}$ puis par des vecteurs unimodulaires que l'on voit comme des formes linéaires $(l_a)_{a \in \mathcal{A}}$. Cela permet d'identifier $\mathbb{Z}[\mathcal{A}]^0$ au sous-groupe de $\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))$

$$\left\langle \frac{l_a(z)}{l_b(z)} : a, b \in \mathcal{A} \right\rangle_{\mathbb{Z}\text{-Mod}}.$$

Théorème 6.10. *Soit $\mathcal{A}$ un arrangement tubulaire fermé. On a un isomorphisme*

$$\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A})) \simeq \mathbb{Z}[\mathcal{A}]^0.$$

Remarque 6.11. Le théorème précédent montre plus précisément que la composée

$$\mathbb{Z}[\mathcal{A}]^0 \subset \mathscr{O}^*(\mathrm{Int}(\mathcal{A})) \twoheadrightarrow \mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}))$$

ne dépend pas du choix du système de représentants et est un isomorphisme.

30. C'est-à-dire les éléments $\sum_{s \in S} a_s \delta_s$ tels que $\sum_{s \in S} a_s = 0$.

Démonstration. Comme dans la remarque 6.9, pour tout voisinage tubulaire $a \in \mathcal{A}$, on fixe une forme linéaire l_a représentée par un vecteur unimodulaire encore noté a telle que $a = \mathring{\ker}(l_a)(|\varpi^n|)$.

On introduit un faisceau T grâce aux suites exactes

$$0 \to \mathcal{O}^{**} \to \mathbb{G}_m \to \mathbb{G}_m/\mathcal{O}^{**} \to 0,$$
$$0 \to L^*/(1+\mathfrak{m}_L) \to \mathbb{G}_m/\mathcal{O}^{**} \to T \to 0.$$

D'après le théorème 6.1, les flèches naturelles $\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathcal{O}^{**}) \to \mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m)$ sont injectives et les espaces $\mathrm{Uni}(\mathcal{A})$ sont acycliques pour les faisceaux constants (note 29). On en déduit des isomorphismes

$$\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m/\mathcal{O}^{**}) = \frac{\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m)}{\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathcal{O}^{**})} = \begin{cases} L^*/(1+\mathfrak{m}_L) & \text{si } s = 0, \\ \mathbb{Z} & \text{si } s = 1 \text{ et } |\mathcal{A}| \neq 1, \\ 0 & \text{sinon}, \end{cases}$$
$$\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), T) = \frac{\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m/\mathcal{O}^{**})}{\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), L^*/(1+\mathfrak{m}_L))} = \begin{cases} \mathbb{Z} & \text{si } s = 1 \text{ et } |\mathcal{A}| \neq 1, \\ 0 & \text{sinon}, \end{cases}$$

d'où l'acyclicité des espaces $\mathrm{Int}(\mathcal{A})$ pour les faisceaux $\mathbb{G}_m/\mathcal{O}^{**}$ et T par le lemme 5.7. En particulier, la cohomologie à coefficients dans T de $\mathrm{Uni}(\mathcal{A})$ peut se calculer grâce au complexe de Čech sur le recouvrement $\mathcal{A}^c$ constitué des complémentaires des voisinages tubulaires $a \in \mathcal{A}$. On sait aussi que $\mathcal{O}^{**}$ et les faisceaux constants (voir note 29) n'ont pas de cohomologie en degré supérieur ou égal à 1 sur $\mathrm{Int}(\mathcal{A})$, d'où les égalités

$$(\mathbb{G}_m/\mathcal{O}^{**})(\mathrm{Int}(\mathcal{A})) = \mathcal{O}^*(\mathrm{Int}(\mathcal{A}))/\mathcal{O}^{**}(\mathrm{Int}(\mathcal{A})) \quad \text{et} \quad T(\mathrm{Int}(\mathcal{A})) = \mathcal{O}^*(\mathrm{Int}(\mathcal{A}))/L^*\mathcal{O}^{**}(\mathrm{Int}(\mathcal{A})).$$

Nous chercherons à décrire les sections globales de T sur $\mathrm{Int}(\mathcal{A})$. Montrons par récurrence sur $|\mathcal{A}|$ que la flèche décrite dans la remarque 6.11 est un isomorphisme

$$T(\mathrm{Int}(\mathcal{A})) \cong \mathbb{Z}[\mathcal{A}]^0.$$

Le reste de l'argument consiste à relier les sections de T sur $\mathrm{Int}(\mathcal{A})$ aux groupes $\mathrm{H}^s_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), T)$.

Quand $|\mathcal{A}| = 1$, l'espace $\mathrm{Int}(\mathcal{A})$ est une boule et on a directement $\mathcal{O}^*(\mathrm{Int}(\mathcal{A})) = L^*\mathcal{O}^{**}(\mathrm{Int}(\mathcal{A}))$.

Pour $|\mathcal{A}| = 2$, on note l_a, l_b les deux formes linéaires associées. La suite exacte de Mayer–Vietoris établit un isomorphisme $T(\mathrm{Int}(\mathcal{A})) \cong \mathrm{H}^1_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), T) \cong \mathbb{Z}$. De plus, d'après la discussion précédente, la flèche surjective $\mathbb{G}_m \to T$ induit un diagramme commutatif

$$\begin{array}{ccccc} \mathcal{O}^*(\mathrm{Int}(\mathcal{A})) & = & \check{\mathcal{C}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{A}^c, \mathbb{G}_m) & \twoheadrightarrow & \mathrm{H}^1_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m) \\ \downarrow\!\!\!\downarrow & & \downarrow & & \downarrow \wr \\ \mathcal{O}^*(\mathrm{Int}(\mathcal{A}))/L^*\mathcal{O}^{**}(\mathrm{Int}(\mathcal{A})) & = & \check{\mathcal{C}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{A}^c, T) & = & \mathrm{H}^1_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), T) \end{array}$$

Il suffit de prouver que l_a/l_b engendre $\mathrm{Pic}(\mathrm{Uni}(\mathcal{A})) = \mathrm{H}^1_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m)$. En suivant l'exemple 2.5, on peut trouver un changement de variables tel que

$$\begin{cases} a = e_0,\ b = e_1 \text{ ou} \\ a = e_0,\ b = e_0 + \varpi^k e_1 \quad \text{avec } 0 < k < n. \end{cases}$$

Dans le premier cas, on a $\mathrm{Uni}(\mathcal{A}) \cong X_1^d(n,n)$ et les recouvrements $\mathcal{V}(n,n)$ et $\mathcal{A}^c$ coïncident. Mais d'après la remarque 6.4, on a directement [31] $\check{\mathrm{H}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{V}(n,n), \mathbb{G}_m) = (l_a/l_b)^{\mathbb{Z}}$. Dans le second cas, on introduit l'ouvert $U \subset \mathrm{Uni}(\mathcal{A})$ (noté $\mathring{H}_{e_1}(|\varpi^{n-k}|)^c$ dans l'exemple 2.5) défini par

$$U = \{z \in \mathbb{P}^d(C) : |z_i| \le |\varpi^{n-k} z_1|\}.$$

Il suit que si on échange a et b, on a deux isomorphismes de $\mathrm{Uni}(\mathcal{A})$ vers $X_1^d(n, n-k)$ et donc deux recouvrements $\mathcal{V}(n,n-k)^{(1)} = \{a^c, U\}$ et $\mathcal{V}(n,n-k)^{(2)} = \{b^c, U\}$. Les cohomologies de Čech sur ces deux recouvrements sont bien comprises grâce à la proposition 6.3 et à la remarque 6.4, et on a

$$\check{\mathrm{H}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{V}(n,n-k)^{(1)}, \mathbb{G}_m) = \left(\frac{l_a}{l_{e_1}}\right)^{\mathbb{Z}} \quad \text{et} \quad \check{\mathrm{H}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{V}(n,n-k)^{(2)}, \mathbb{G}_m) = \left(\frac{l_b}{l_{e_1}}\right)^{\mathbb{Z}}.$$

Le triplet $(l_a/l_b, l_b/l_{e_1}, l_{e_1}/l_a)$ définit un 1-cocycle sur le recouvrement $\{a^c, b^c, U\}$ dont l'image engendre $\check{\mathrm{H}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{V}(n,n-k)^{(1)}, \mathbb{G}_m)$ d'après l'équation qui précède. Sa projection $(l_a/l_b, l_b/l_{e_1}, l_{e_1}/l_a) \mapsto l_a/l_b$ sur la cohomologie sur le recouvrement $\mathcal{A}^c$ engendre ainsi encore

$$\check{\mathrm{H}}^1(\mathrm{Uni}(\mathcal{A}), \mathcal{A}^c, \mathbb{G}_m) = \mathrm{H}^1_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), \mathbb{G}_m),$$

ce qui conclut l'argument.

Si $|\mathcal{A}| = 3$, on se donne encore l_a, l_b, l_c des formes linéaires associées aux voisinages tubulaires. Étudions le complexe $\check{\mathcal{C}}^\bullet(\mathrm{Uni}(\mathcal{A}), \mathcal{A}^c, T) = \check{\mathcal{C}}^\bullet$ qui calcule les groupes $\mathrm{H}^*_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), T)$. Quand $s \neq 1$, tous ces groupes ainsi que $\check{\mathcal{C}}^3$ sont nuls et on obtient l'exactitude de la suite

$$0 \to \mathrm{H}^1_{\mathrm{an}}(\mathrm{Uni}(\mathcal{A}), T) \to \check{\mathcal{C}}^1/\delta(\check{\mathcal{C}}^0) \to \check{\mathcal{C}}^2 \to 0. \tag{18}$$

On a $\check{\mathcal{C}}^2 = T(\mathrm{Int}(\mathcal{A}))$ mais aussi, d'après les cas de cardinal 1 et 2, les identités

$$\check{\mathcal{C}}^0 = T(\mathrm{Int}(\{a\})) \times T(\mathrm{Int}(\{b\})) \times T(\mathrm{Int}(\{c\})) = 0,$$
$$\check{\mathcal{C}}^1 = T(\mathrm{Int}(\{b,c\})) \times T(\mathrm{Int}(\{c,a\})) \times T(\mathrm{Int}(\{a,b\})) = \left(\frac{l_b}{l_c}\right)^{\mathbb{Z}} \times \left(\frac{l_c}{l_a}\right)^{\mathbb{Z}} \times \left(\frac{l_a}{l_b}\right)^{\mathbb{Z}}.$$

En remplaçant les termes précédents dans la suite exacte (18), on obtient

$$0 \to \mathbb{Z} \xrightarrow{\alpha} \left(\frac{l_b}{l_c}\right)^{\mathbb{Z}} \times \left(\frac{l_c}{l_a}\right)^{\mathbb{Z}} \times \left(\frac{l_a}{l_b}\right)^{\mathbb{Z}} \xrightarrow{\beta} T(\mathrm{Int}(\mathcal{A})) \to 0,$$

avec β le produit des trois termes. Il suffit maintenant de prouver que $\mathrm{Im}\,\alpha = \ker\beta$ coïncide avec le sous-groupe G engendré par le triplet $(l_b/l_c, l_c/l_a, l_a/l_b)$. On a clairement l'inclusion $G \subset \ker\beta$. Le

31. On peut aussi utiliser le fait que $\mathrm{Int}(\mathcal{A})$ est ici une polycouronne et déduire le résultat du lemme 4.4.

quotient $\ker\beta/G$ est de torsion [32] et s'injecte dans le groupe $(l_b/l_c)^{\mathbb{Z}} \times (l_c/l_a)^{\mathbb{Z}} \times (l_a/l_b)^{\mathbb{Z}}/G$ qui est sans torsion. On en déduit l'annulation de $\ker\beta/G = 0$ ainsi que les isomorphismes $T(\mathrm{Int}(\mathcal{A})) \cong (l_b/l_c)^{\mathbb{Z}} \times (l_c/l_a)^{\mathbb{Z}} \times (l_a/l_b)^{\mathbb{Z}}/G \cong \mathbb{Z}[\mathcal{A}]^0$.

Maintenant $|\mathcal{A}| \geq 4$, et supposons le résultat pour tout arrangement tubulaire $\mathcal{B}$ tel que $|\mathcal{B}| < |\mathcal{A}|$. On note encore $\check{\mathcal{C}}^\bullet$ le complexe $\check{\mathcal{C}}^\bullet(\mathrm{Uni}(\mathcal{A}), \mathcal{A}^c, T)$. On connaît l'annulation des groupes de cohomologie $\mathrm{H}_{\mathrm{an}}^{|\mathcal{A}|-1}(\mathrm{Uni}(\mathcal{A}), T) = \mathrm{H}_{\mathrm{an}}^{|\mathcal{A}|-2}(\mathrm{Uni}(\mathcal{A}), T) = 0$, d'où une suite exacte

$$\check{\mathcal{C}}^{|\mathcal{A}|-3} \to \check{\mathcal{C}}^{|\mathcal{A}|-2} \to \check{\mathcal{C}}^{|\mathcal{A}|-1} \to 0. \tag{19}$$

Mais par hypothèse de récurrence, on a

$$\check{\mathcal{C}}^{|\mathcal{A}|-3} = \bigoplus_{c,d\in\mathcal{A}} T(\mathrm{Int}(\mathcal{A}\backslash\{c,d\})) = \bigoplus_{c,d\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c,d\}]^0,$$
$$\check{\mathcal{C}}^{|\mathcal{A}|-2} = \bigoplus_{c\in\mathcal{A}} T(\mathrm{Int}(\mathcal{A}\backslash\{c\})) = \bigoplus_{c\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c\}]^0.$$

En remplaçant ces deux termes et en observant que $\check{\mathcal{C}}^{|\mathcal{A}|-1} = T(\mathrm{Int}(\mathcal{A}))$, la suite exacte (19) devient

$$\bigoplus_{c,d\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c,d\}]^0 \xrightarrow{\varphi} \bigoplus_{c\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c\}]^0 \to T(\mathrm{Int}(\mathcal{A})) \to 0.$$

Il reste à établir l'isomorphisme $\mathbb{Z}[\mathcal{A}]^0 \cong \mathrm{Coker}(\varphi)$. Chaque fraction l_a/l_b peut se voir dans $\mathbb{Z}[\mathcal{A}\backslash\{c,d\}]^0$ ou $\mathbb{Z}[\mathcal{A}\backslash\{c\}]^0$ pour $a,b,c,d \in \mathcal{A}$ distincts. Pour les distinguer, nous introduisons la notation

$$\left(\frac{l_a}{l_b}\right)^{(c,d)} \in \mathbb{Z}[\mathcal{A}\backslash\{c,d\}]^0 \quad \text{et} \quad \left(\frac{l_a}{l_b}\right)^{(c)} \in \mathbb{Z}[\mathcal{A}\backslash\{c\}]^0.$$

Chacune des familles $((l_a/l_b)^{(c,d)})_{a,b,c,d\in\mathcal{A}}$, $((l_a/l_b)^{(c)})_{a,b,c\in\mathcal{A}}$ engendre $\bigoplus_{c,d\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c,d\}]^0$ et $\bigoplus_{c\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c\}]^0$, respectivement. Le groupe $\mathrm{Im}(\varphi)$ est engendré par les éléments $\varphi((l_a/l_b)^{(c,d)}) = (l_a/l_b)^{(c)}(l_b/l_a)^{(d)}$. Ainsi, la flèche

$$\bigoplus_{c\in\mathcal{A}} \mathbb{Z}[\mathcal{A}\backslash\{c\}]^0 \to \mathbb{Z}[\mathcal{A}]^0, \quad \left(\frac{l_a}{l_b}\right)^{(c)} \mapsto \frac{l_a}{l_b},$$

induit l'isomorphisme $\mathrm{Coker}(\varphi) \cong \mathbb{Z}[\mathcal{A}]^0$ voulu. □

7. Étude des arrangements algébriques généralisés

Théorème 7.1. *Si $\mathcal{A}$ est un arrangement algébrique généralisé, alors* $\mathrm{Int}(\mathcal{A})$ *est acyclique pour les faisceaux $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$ et $\mathbb{G}_m$ en topologie analytique. Les sections sur* $\mathrm{Int}(\mathcal{A})$ *de $\mathscr{O}^+$, $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$ sont constantes et on a une suite exacte*

$$0 \to L^* \to \mathscr{O}^*(\mathrm{Int}(\mathcal{A})) \to \mathbb{Z}[\![\mathcal{A}]\!]^0 \to 0.$$

32. C'est un quotient de $\mathbb{Z}$ par un sous-groupe non trivial.

Démonstration. Considérons la famille $(\mathcal{A}_n)_n$ d'arrangements tubulaires fermés compatible définie dans la remarque 2.2. On obtient alors un recouvrement croissant de $\mathrm{Int}(\mathcal{A}) = \bigcup \mathrm{Int}(\mathcal{A}_n)$ qui en fait un espace analytique quasi Stein. Si $\mathscr{F}$ est l'un des faisceaux $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$, $\mathbb{G}_m$, on a la suite exacte

$$0 \to R^1 \varprojlim_n \mathrm{H}^{s-1}_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}_n), \mathscr{F}) \to \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathscr{F}) \to \varprojlim_n \mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}_n), \mathscr{F}) \to 0.$$

Par acyclicité des arrangements tubulaires d'hyperplans (voir le théorème 5.6, le corollaire 5.14 et le théorème 6.10), on a

$$\mathrm{H}^s_{\mathrm{an}}(\mathrm{Int}(\mathcal{A}), \mathscr{F}) = \begin{cases} \varprojlim_n \mathscr{F}(\mathrm{Int}(\mathcal{A}_n)) & \text{si } s = 0, \\ R^1 \varprojlim_n \mathscr{F}(\mathrm{Int}(\mathcal{A}_n)) & \text{si } s = 1, \\ 0 & \text{si } s \geq 2. \end{cases}$$

On peut appliquer la proposition 4.5 grâce au point technique du corollaire 5.12 pour obtenir l'acyclicité de $\mathrm{Int}(\mathcal{A})$ pour $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$. On en déduit aussi la description des sections globales de $\mathscr{O}^{+}$, $\mathscr{O}^{(r)}$, $\mathscr{O}^{**}$, ce qui donne en particulier une autre démonstration de [Berkovich 1995a, Lemma 3].

Pour $\mathbb{G}_m$, on a (théorème 6.10) une suite exacte de systèmes projectifs

$$0 \to \big(L^* \mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n))\big)_n \to \big(\mathscr{O}^{*}(\mathrm{Int}(\mathcal{A}_n))\big)_n \to (\mathbb{Z}[\mathcal{A}_n]^0)_n \to 0.$$

En appliquant le foncteur $\varprojlim_n$, on obtient une suite exacte longue

$$0 \to L^* \to \mathscr{O}^*(\mathrm{Int}(\mathcal{A})) \to \mathbb{Z}[\![\mathcal{A}]\!]^0 \to R^1 \varprojlim_n L^* \mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n)) \to R^1 \varprojlim_n \mathscr{O}^*(\mathrm{Int}(\mathcal{A}_n)) \to R^1 \varprojlim_n \mathbb{Z}[\mathcal{A}_n]^0.$$

On a $R^1 \varprojlim_n L^* \mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n)) = R^1 \varprojlim_n \mathbb{Z}[\mathcal{A}_n]^0 = 0$ d'après la surjectivité de $\mathbb{Z}[\mathcal{A}_{n+1}]^0 \to \mathbb{Z}[\mathcal{A}_n]^0$ et la proposition 4.5. Donc

$$\mathrm{Pic}_L(\mathrm{Int}(\mathcal{A})) = R^1 \varprojlim_n \mathscr{O}^*(\mathrm{Int}(\mathcal{A}_n)) = 0$$

et la suite suivante est exacte :

$$0 \to L^* \to \mathscr{O}^*(\mathrm{Int}(\mathcal{A})) \to \mathbb{Z}[\![\mathcal{A}]\!]^0 \to 0. \qquad \square$$

8. Quelques commentaires sur la cohomologie étale et de de Rham des arrangements d'hyperplans

8A. ***Cohomologie étale l-adique et de de Rham.*** En appliquant la suite exacte de Kummer, on obtient d'après les théorèmes 6.7, 6.10 et 7.1 :

Corollaire 8.1. *Soit m un entier. On a les diagrammes suivants.*

(1) *Si $\mathcal{A}_n$ est un arrangement tubulaire fermé et m est premier à p :*

$$\begin{array}{ccc} \mathscr{O}^*(\mathrm{Int}(\mathcal{A}_n))/L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n)) & \xrightarrow{\ \kappa\ } & \mathrm{H}^1_{\text{ét}}(\mathrm{Int}(\mathcal{A}_n), \mu_m)/\kappa(L^*) \\ \uparrow & & \wr\uparrow \\ \mathbb{Z}[\mathcal{A}_n]^0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}[\mathcal{A}_n]^0 \end{array}$$

(2) *Si $\mathcal{A}$ est un arrangement algébrique généralisé* :

$$\begin{array}{ccc}
\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/L^* & \xrightarrow{\kappa} & \mathrm{H}^1_{\text{ét}}(\mathrm{Int}(\mathcal{A}), \mu_m)/\kappa(L^*) \\
\uparrow & & \wr\uparrow \\
\mathbb{Z}[\![\mathcal{A}]\!]^0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z}[\![\mathcal{A}]\!]^0
\end{array}$$

Démonstration. Dans les deux cas, le groupe de Picard de $\mathrm{Int}(\mathcal{A})$ est trivial, d'où on obtient par la suite de Kummer l'isomorphisme

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Int}(\mathcal{A}), \mathbb{Z}/m\mathbb{Z}) \cong \mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/\big(\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))\big)^m.$$

Dans le second cas, la suite exacte du théorème 7.1 devient

$$0 \to L^*/(L^*)^m \to \mathscr{O}^*(\mathrm{Int}(\mathcal{A}))/\big(\mathscr{O}^*(\mathrm{Int}(\mathcal{A}))\big)^m \to \mathbb{Z}/m\mathbb{Z}[\![\mathcal{A}]\!]^0 \to 0,$$

car $\mathbb{Z}[\![\mathcal{A}]\!]^0$ est sans m-torsion. L'argument se termine en identifiant $L^*/(L^*)^m$ et $\kappa(L^*)$.

On raisonne de manière similaire dans le premier cas en étudiant la suite exacte

$$0 \to L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n))/\big(L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n))\big)^m \to \mathscr{O}^*(\mathrm{Int}(\mathcal{A}_n))/\big(\mathscr{O}^*(\mathrm{Int}(\mathcal{A}_n))\big)^m \to \mathbb{Z}/m\mathbb{Z}[\mathcal{A}]^0 \to 0.$$

Mais $\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n))$ est m-divisible quand m est premier à p, car la série formelle $(X-1)^{1/m}$ converge sur $\mathscr{O}^{++}(\mathrm{Int}(\mathcal{A}_n))$. On obtient la suite d'identifications qui conclut la preuve :

$$L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n))/\big(L^*\mathscr{O}^{**}(\mathrm{Int}(\mathcal{A}_n))\big)^m \cong L^*/(L^*)^m \cong \kappa(L^*). \qquad \square$$

Proposition 8.2. *Soit n un entier, $\mathcal{A}$ un arrangement tubulaire ouvert d'hyperplans d'ordre n et $\widetilde{\mathcal{A}}$ sa projection fermée d'ordre $n-1$. Alors l'inclusion* $\mathrm{Int}(\widetilde{\mathcal{A}}) \to \mathrm{Int}(\mathcal{A})$ *induit un isomorphisme au niveau des groupes de cohomologie de de Rham,*[33] *et de même pour la cohomologie étale l-adique pour $L = C = \hat{\bar{K}}$.*

Démonstration. Écrivons H l'une des deux cohomologies considérées (avec $L = C$ pour la cohomologie étale l-adique). La suite spectrale (1) calculant H pour l'arrangement $\mathcal{A}$ ou $\widetilde{\mathcal{A}}$ sera notée $E_j^{-r,s}(\mathcal{A})$ ou $E_j^{-r,s}(\widetilde{\mathcal{A}})$, respectivement. Nous allons les comparer pour établir le résultat.

Considérons $\mathcal{B}$ une partie de $\mathcal{A}$ et $\widetilde{\mathcal{B}}$ sa projection dans $\widetilde{\mathcal{A}}$. On a $\mathrm{rg}(\mathcal{B}) = \mathrm{rg}(\widetilde{\mathcal{B}}) = t+1$. Alors il existe $\beta \in \mathbb{N}^{t+1}$ tel que $\mathrm{Uni}(\mathcal{B}) \cong Y_t^d(\beta)$ et $\mathrm{Uni}(\widetilde{\mathcal{B}}) \cong X_t^d(\beta)$. Les deux cohomologies H vérifient l'axiome d'homotopie, i.e., pour tout espace analytique X, on a des isomorphismes induits par les projections naturelles (voir l'axiome d'homotopie à la propriété I, §2, page 55 de [Schneider et Stuhler 1991]) pour la boule ouverte. Pour la boule fermée, on obtient

$$\mathrm{H}^*(X \times \mathbb{B}) \cong \mathrm{H}^*(X) \cong \mathrm{H}^*(X \times \mathring{\mathbb{B}}),$$

voir la formule Künneth [Grosse-Klönne 2002, Lemma 3, page 74] ou [Grosse-Klönne 2004a, Proposition 3.3] en de Rham et [Berkovich 1996, Lemma 3.3] en l-adique. Ainsi, les fibrations induisent des

33. Tous les groupes de cohomologie de de Rham sont calculés sur le site surconvergent. Notons que cette notion coïncide avec la cohomologie usuelle dans le cas tubulaire ouvert où les espaces sont partiellement propres.

isomorphismes entre la cohomologie de $\mathbb{P}^t_{\mathrm{rig},L}$ et celles de $Y^d_t(\beta)$, $X^d_t(\beta)$ compatibles par commutativité du diagramme suivant, où la flèche horizontale est l'inclusion naturelle :

$$\begin{array}{ccc} X^d_t(\beta) & \longrightarrow & Y^d_t(\beta) \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^t_{\mathrm{rig},L} & \end{array}$$

Par somme directe, on obtient un isomorphisme entre les suites spectrales, d'où le résultat. □

8B. ***Cohomologie étale p-adique des arrangements algébriques d'hyperplans.*** Ici, $L = C$ et on verra $\mathrm{Int}(\mathcal{A})$ comme un C-espace analytique par extension des scalaires pour $\mathcal{A}$ un arrangement d'hyperplans K-rationnels.

Proposition 8.3. *Soit $\mathcal{A}$ un arrangement algébrique K-rationnel, on a un isomorphisme canonique*

$$\mathrm{H}^*_{\text{ét}}(\mathrm{Int}(\mathcal{A}), \mathbb{Q}_p) \otimes C \cong \mathrm{H}^*_{\mathrm{dR}}(\mathrm{Int}(\mathcal{A})).$$

Remarque 8.4. Le résultat récent [Colmez et al. 2021, Theorem 5.1] semble suggérer que l'on a encore le résultat pour les arrangements algébriques généralisés.

Démonstration. Appelons $E_j^{-r,s}(\text{ét})$ et $E_j^{-r,s}(\mathrm{dR})$ les suites spectrales calculant respectivement la cohomologie étale p-adique et la cohomologie de de Rham. Nous allons exhiber un isomorphisme canonique $E_j^{-r,s}(\text{ét}) \otimes C \to E_j^{-r,s}(\mathrm{dR})$. Considérons alors une union $\mathrm{Uni}(\mathcal{B})$ et écrivons-la Z^d_t. Nous allons montrer

$$\mathrm{H}^*_{\text{ét}}(Z^d_t, \mathbb{Q}_p) \otimes C \cong \mathrm{H}^*_{\mathrm{dR}}(Z^d_t).$$

Appelons Λ le faisceau constant $\mathbb{Z}/p^n\mathbb{Z}$. D'après un résultat de Berkovich [1995b, Lemma 2.2], pour tout espace analytique S, tout entier m et $\phi : \mathbb{A}^m_{\mathrm{rig},S} \to S$, on a $R^i\phi_*\Lambda_{\mathbb{A}^m_{\mathrm{rig},S}} = 0$ pour $i \geq 1$. On a alors, par la suite spectrale de Leray, pour toute intersection $f^{-1}(V_I)$ de $f^*(\mathcal{V})$, $R\psi_*\Lambda_{f^{-1}(V_I)} = R\psi_*\Lambda_{V_I}$, où $\psi : X \to \mathrm{Sp}(C)$ pour tout C-espace analytique X. Par Čech, on obtient que $R\psi_*\Lambda_{Z^d_t} = R\psi_*\Lambda_{\mathbb{P}^t_{\mathrm{rig},C}}$, d'où un isomorphisme

$$\mathrm{H}^i_{\text{ét}}(Z^d_t, \mathbb{Q}_p) \cong \mathrm{H}^i_{\text{ét}}(\mathbb{P}^t_{\mathrm{rig},C}, \mathbb{Q}_p).$$

De plus, d'après [de Jong et van der Put 1996, Theorem 7.3.2], on a un isomorphisme canonique $\mathrm{H}^i_{\text{ét}}(\mathbb{P}^t_{\mathrm{rig},C}, \mathbb{Q}_p) \cong \mathrm{H}^i_{\text{ét}}(\mathbb{P}^t_{\mathrm{zar},C}, \mathbb{Q}_p)$. Par étude du cas algébrique, on en déduit que $\mathrm{H}^*_{\text{ét}}(\mathbb{P}^t_{\mathrm{rig},C}, \mathbb{Q}_p) \otimes C$ est engendré en tant que C-algèbre graduée par l'image du faisceau tordu $\mathcal{O}(1)$ par l'application de Kummer $\mathrm{Pic}(\mathbb{P}^t_{\mathrm{rig},C}) \to \mathrm{H}^2_{\text{ét}}(\mathbb{P}^t_{\mathrm{rig},C}, \mathbb{Q}_p)$. On construit alors un isomorphisme en identifiant les classes logarithmiques. Ces morphismes commutent bien aux différentielles de la suite spectrale. On en déduit le résultat à la convergence. □

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Distinction inside L-packets of SL(n)

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If E/F is a quadratic extension p-adic fields, we first prove that the $\mathrm{SL}_n(F)$-distinguished representations inside a distinguished unitary L-packet of $\mathrm{SL}_n(E)$ are precisely those admitting a degenerate Whittaker model with respect to a degenerate character of $N(E)/N(F)$. Then we establish a global analogue of this result. For this, let E/F be a quadratic extension of number fields, and let π be an $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$. Let (σ, d) be the unique pair associated to π, where σ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_E)$ with $n = dr$. Using an unfolding argument, we prove that an element of the L-packet of π is distinguished with respect to $\mathrm{SL}_n(\mathbb{A}_F)$ if and only if it has a degenerate Whittaker model for a degenerate character ψ of type $r^d := (r, \dots, r)$ of $N_n(\mathbb{A}_E)$ which is trivial on $N_n(E + \mathbb{A}_F)$, where N_n is the group of unipotent upper triangular matrices of SL_n. As a first application, under the assumptions that E/F splits at infinity and r is odd, we establish a local–global principle for $\mathrm{SL}_n(\mathbb{A}_F)$-distinction inside the L-packet of π. As a second application we construct examples of distinguished cuspidal automorphic representations π of $\mathrm{SL}_n(\mathbb{A}_E)$ such that the period integral vanishes on some canonical realization of π, and of everywhere locally distinguished representations of $\mathrm{SL}_n(\mathbb{A}_E)$ such that their L-packets do not contain any distinguished representation.

1. Introduction

The present work fits in the study of local distinction and periods of automorphic forms, with respect to Galois pairs of reductive groups. It is motivated by earlier works, namely, [Anandavardhanan and Prasad 2003; 2018] in the local context and [Anandavardhanan and Prasad 2006; 2013] in the global context, which investigated distinction in the presence of L-packets.

In probing distinction inside an L-packet for SL(2), the key finding of [Anandavardhanan and Prasad 2003; 2006] was that distinction inside an L-packet that contains at least one distinguished representation can be characterized in terms of Whittaker models; i.e., distinguished representations in such "distinguished" L-packets are precisely the ones which admit a Whittaker model with respect to a nontrivial character of E/F (resp. $\mathbb{A}_E/(E + \mathbb{A}_F)$) in the local (resp. global) case. A crucial role in the global papers on SL(2) [Anandavardhanan and Prasad 2006; 2013] is played by "multiplicity one for SL(2)"; i.e., a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_L)$ appears exactly once in the space of cusp forms on $\mathrm{SL}_2(\mathbb{A}_L)$ [Ramakrishnan 2000].

More recently, the results of [AP 2018] generalized [AP 2003] from $n = 2$ to any n. Thus, in [AP 2018], it is proved, amongst many other results, that if π is a generic $\mathrm{SL}_n(F)$-distinguished representation

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of $\mathrm{SL}_n(E)$, then the distinguished members of the L-packet of π are the representations which are ψ-generic with respect to some nondegenerate character ψ satisfying $\psi^\theta = \psi^{-1}$, where θ denotes the Galois involution. Such a relationship between distinction and genericity is expected more generally [Prasad 2015]; indeed, if ψ is a nondegenerate character such that $\psi^\theta = \psi^{-1}$, then according to [Prasad 2015, Conjecture 13.3, (3)], for any quasisplit Galois pair, ψ-generic members of a distinguished L-packet are distinguished.

Somewhat surprisingly, even the finite field analogue of this characterization of distinction in a generic L-packet turned out to be nontrivial and was settled only fairly recently [Anandavardhanan and Matringe 2020, Theorem 5.1].

In this paper, we first prove a generalization of the above-mentioned local result of [AP 2003; 2018] for unitary L-packets of $\mathrm{SL}_n(E)$ and degenerate Whittaker models (see Theorem 3.9).

Theorem 1.1. *If $\tilde{\pi}$ is an irreducible unitary representation of $\mathrm{GL}_n(E)$ of type $(n_1, \dots, n_d)$, where $(n_1, \dots, n_d)$ is the partition of n defined in Section 3A, and if the L-packet associated to $\tilde{\pi}$ contains a representation distinguished by $\mathrm{SL}_n(F)$, then its distinguished members are those which admit a ψ-degenerate Whittaker model for ψ of type $(n_1, \dots, n_d)$ satisfying $\psi^\theta = \psi^{-1}$.*

Our proof of Theorem 1.1 builds on the work of Matringe [2014], which classified unitary representations of $\mathrm{GL}_n(E)$ which are distinguished with respect to $\mathrm{GL}_n(F)$, making use of which we can adapt the techniques of [AP 2003] and [AP 2018] to the unitary context. Such a result hints at the possibility of a generalization of the prediction of Dipendra Prasad [2015] relating distinction for Galois pairs inside distinguished generic L-packets to distinguished Whittaker models, to nongeneric L-packets. We feel that, thanks in particular to the work [Kemarsky 2015], the same result could be obtained in the Archimedean setting and we leave this question to experts. This would allow the removal of the assumption that the number field is split at infinity in some of our global results.

Now we come to the global results of this paper. The study of global representations of $\mathrm{SL}(n)$, already quite involved for $n = 2$ as can be seen from [AP 2006; 2013], is considerably more difficult for several reasons, one of which is that "multiplicity one" is not true for $\mathrm{SL}(n)$ for $n \geq 3$, as was first shown in the famous work of D. Blasius [Blasius 1994; Lapid 1999].

In this paper, we prove the most basic result about characterizing distinction inside a distinguished L-packet in terms of Whittaker models, thus generalizing [AP 2006, Theorem 4.2] from $n = 2$ to any n, and we cover not just cuspidal representations but the full residual spectrum (see Theorem 6.10). We emphasize that the L-packets that we consider in this work are defined by restriction of cusp forms, except in the abstract, where the results are formulated in terms of "the" L-packet consisting of automorphic members of the representation-theoretic L-packet.

Theorem 1.2. *Suppose $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ is an irreducible square-integrable automorphic representation of $\mathrm{GL}_{dr}(\mathbb{A}_E)$, where σ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_E)$. Assume that the L-packet determined by $\tilde{\pi}$ contains an $\mathrm{SL}_{dr}(\mathbb{A}_F)$-distinguished representation. Then an irreducible square-integrable automorphic representation π of this L-packet is $\mathrm{SL}_{dr}(\mathbb{A}_F)$-distinguished if and only if there exists a degenerate* *character ψ of type $r^d := (r, \dots, r)$* (see Section 5B) *of $N_n(\mathbb{A}_E)$, trivial on $N_n(E + \mathbb{A}_F)$, such that π has a degenerate ψ-Whittaker model.*

There are two main ideas in proving Theorem 1.2. First we settle the cuspidal case by creating an inductive setup based on an unfolding method, and make use of the base case for $n = 2$, which is known by [AP 2006, Theorem 4.2]. We mention here that the method that we follow to create this inductive setup is very parallel to that employed in [Dijols and Prasad 2019, Section 5] (see Remark 6.4). Having established the cuspidal case for all r, we do one more induction, this time in d, where $n = dr$, the case $d = 1$ being the cuspidal case. In order to work this out, the key ingredient is the work of Yamana [2015], which is the global counterpart of [Matringe 2014], and we need to do one more unfolding argument as well.

As an application of Theorem 1.2, we establish a local–global principle for square-integrable representations for $(\mathrm{SL}_n(\mathbb{A}_E), \mathrm{SL}_n(\mathbb{A}_F))$ (see Theorem 8.4).

Theorem 1.3. *Let E/F be a quadratic extension of number fields split at the Archimedean places. Suppose $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ is a square-integrable automorphic representation of $\mathrm{GL}_{dr}(\mathbb{A}_E)$, where σ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_E)$, where we assume that r is odd. Suppose that the L-packet determined by $\tilde{\pi}$ contains an $\mathrm{SL}_{dr}(\mathbb{A}_F)$-distinguished representation. Let π be an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ which belongs to this L-packet. Write $\pi = \bigotimes_v' \pi_v$, for v varying through the places of F. Then π is distinguished with respect to $\mathrm{SL}_n(\mathbb{A}_F)$ if and only if each π_v is $\mathrm{SL}_n(F_v)$-distinguished.*

Remark 1.4. Such a local–global principle was proved in [AP 2006] for cuspidal representations of $\mathrm{SL}_2(\mathbb{A}_E)$ by quite involved arguments. In contrast, our proof is reasonably elementary, making use of the assumption that r is odd.

Another important objective of the present paper is to analyze distinction vis-à-vis the phenomenon of higher multiplicity for SL(n). As mentioned earlier, unlike in the case of SL(2), a cuspidal representation may appear in the space of cusp forms with multiplicity more than 1 for SL(n) for $n \geq 3$ [Blasius 1994; Lapid 1999].

In our first set of examples, we give a precise answer regarding the nonvanishing of the period integral on the canonical realizations of a cuspidal representation inside the L-packets obtained from restricting the cusp forms on $\mathrm{GL}_n(\mathbb{A}_E)$. We exhibit two types of examples of cuspidal representations of $\mathrm{SL}_n(\mathbb{A}_E)$ of multiplicity $m(\pi)$ more than 1 in the space of cusp forms which are $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished (see Sections 10B and 10C). In one set of examples, F is any number field and E/F is chosen so that the period integral vanishes on some of the $m(\pi)$ many canonical realizations but not on all the canonical realizations. In the second set of examples, F is any number field and E/F is chosen so that the period integral does not vanish in any of the $m(\pi)$ many canonical realizations inside the L-packets.

Then we tweak the method employed to construct the above examples to also show that the local–global principle fails at the level of nondistinguished L-packets for SL(n) (see Section 10D). Namely, we give examples of cuspidal representations π of $\mathrm{SL}_n(\mathbb{A}_E)$ which are distinguished at every place, but such that the L-packet of π contains no distinguished representation. Such a phenomenon was observed for SL(2) as well by an explicit construction in [AP 2006, Theorem 8.2]. The construction in [AP 2006] is somewhat involved, whereas our analogous examples in Section 10D are conceptually simpler; however, the methods here are tailor-made for n odd.

All our examples in Section 10 of cuspidal representations of $\mathrm{SL}_n(\mathbb{A}_E)$ of high multiplicity that are $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished, which highlight a variety of different phenomena, owe a lot to the examples of Blasius [1994] of high cuspidal multiplicity. Blasius makes use of the representation theory of the Heisenberg group H and, in particular, the fact that different Heisenberg representations are such that their value at any element of the group are conjugate in $\mathrm{PGL}_n(\mathbb{C})$, but they are projectively inequivalent [Blasius 1994, Section 1.1]. To give a rough idea, Blasius [Blasius 1994] produces high multiplicity examples on $\mathrm{SL}_n(\mathbb{A}_E)$ by transferring this representation-theoretic information about Heisenberg groups to Galois groups of L/E for suitable number fields, via Shafarevich's theorem, and then to the automorphic side via the strong Artin conjecture, which is a theorem in the situation at hand, because $\mathrm{Gal}(L/E) \simeq H$ is nilpotent, by Arthur and Clozel [1989, Theorem 7.1]. For our examples, we start with an involution on H and consider the corresponding semidirect product $H \rtimes \mathbb{Z}/2$, which cuts out extensions $L \supset E \supset F$, and play with these involutions to construct a variety of examples answering several natural questions about distinction for the pair $(\mathrm{SL}_n(\mathbb{A}_E), \mathrm{SL}_n(\mathbb{A}_F))$.

Last we mention that we give proofs of some elementary, and probably standard, facts on Archimedean and global L-packets of SL_n for which we could not find accessible sources in the literature. They follow from [Aizenbud et al. 2015] in the Archimedean setting, and from [Jiang and Liu 2013] in the global setting.

2. Notation

We denote by δ_G the character of a locally compact group G such that $\delta_G\lambda$ is a right-invariant Haar measure on G if λ is a left-invariant Haar measure on G. We denote by $\mathcal{M}_{a,b}$ the algebraic group of $a \times b$ matrices. We denote by G_n the algebraic group GL_n, by T_n its diagonal torus and by N_n the group of upper triangular matrices in G_n. We set

$$U_n = \left\{ u_n(x) = \begin{pmatrix} I_{n-1} & x \\ \cdot & 1 \end{pmatrix} : x \in (\mathbb{A}^1)^{n-1} \right\} \subset N_n,$$

where $\mathbb{A}^1$ denotes the affine line. For $k \le n$, we embed G_k inside G_n via $g \mapsto \mathrm{diag}(g, I_{n-k})$ and set $P_n = G_{n-1}U_n$, the mirabolic subgroup of G_n. We denote by $N_{n,r}$ the group of matrices

$$k(a, x, u) = \begin{pmatrix} a & x \\ \cdot & u \end{pmatrix}$$

with $a \in G_{n-r}$, $x \in \mathcal{M}_{n-r,r}$ and $u \in N_r$. We denote by $U_{n,r}$ the unipotent radical of $N_{n,r}$, which consists of the matrices $k(I_{n-r}, x, u)$. Note that $N_{n,n} = N_n$ and

$$U_{n,r} = U_n \cdots U_{r+1}.$$

For a subgroup H of G_n, we denote by H° the intersection of H with SL_n.

3. Non-Archimedean theory

Let E/F be a quadratic extension of p-adic fields with Galois involution θ. We denote by $|\cdot|_E$ and $|\cdot|_F$ the respective normalized absolute values. In this section, by abuse of notation, we set $G = G(E)$ for any algebraic group defined over E. We denote by ν_E (or ν), the character $|\cdot|_E \circ \det$ of G_n. We fix a nontrivial character ψ_0 of E which is trivial on F.

3A. ***The type of an irreducible* GL*-representation via derivatives.*** If ψ is a nondegenerate (smooth complex) character of N_n, we denote by ψ^k its restriction to U_k for $k \le n$. We denote by Rep($\bullet$) the category of smooth complex representations of $\bullet$. Bernstein and Zelevinsky [1976; 1977] introduced the functors

$$\Phi^-_{\psi^n} : \mathrm{Rep}(P_n) \to \mathrm{Rep}(P_{n-1}) \quad \text{and} \quad \Psi^- : \mathrm{Rep}(P_n) \to \mathrm{Rep}(G_{n-1}).$$

For $(\tau, V) \in \mathrm{Rep}(P_n)$, one has

$$\Phi^-_{\psi^n}(V) = V/V(U_n, \psi^n),$$

where $V(U_n, \psi^n)$ is the space spanned by the differences $\tau(u)v - \psi^n(u)v$ for $u \in U_n$ and $v \in V$, but the action of P_{n-1} on $\Phi^-_{\psi^n}(V)$ is normalized by twisting by $\delta_{P_n}^{-1/2}$. Similarly

$$\Psi^-(V) = V/V(U_n, 1),$$

where the action of G_{n-1} on $\Psi^-(V)$ is normalized by twisting by $\delta_{P_n}^{-1/2}$ again.

The functor $\Phi^-_{\psi^n}$ does not in fact depend on ψ in the sense that for $\tau \in \mathrm{Rep}(P_n)$ one has $\Phi^-_{\psi^n}(\tau) \simeq \Phi^-_{\psi'^n}(\tau)$ whenever ψ and ψ' are nondegenerate characters of N_n. Hence we simply write $\Phi^-(\tau)$ for it. For $\tau \in \mathrm{Rep}(P_n)$, we set

$$\tau_{(k)} = (\Phi^-)^{k-1}(\tau) \in \mathrm{Rep}(P_{n+1-k}),$$

and

$$\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau) \in \mathrm{Rep}(G_{n-k}),$$

which is called the k-th derivative of τ. The k-th shifted derivative of τ is given by

$$\tau^{[k]} = \nu^{1/2}\tau^{(k)}.$$

Note that these definitions apply when τ is a representation of G_n which we consider as a representation of P_n by restriction.

Let $\tilde{\pi}$ be an irreducible smooth representation of G_n. We denote by $\tilde{\pi}^{[n_1]}$ its highest (nonzero) shifted derivative, by $\tilde{\pi}^{[n_1,n_2]} := (\tilde{\pi}^{[n_1]})^{[n_2]}$ the highest shifted derivative of $\tilde{\pi}^{[n_1]}$, and so on. All the representations $\tilde{\pi}^{[n_1,n_2,\ldots,n_i]}$ are irreducible thanks to [Zelevinsky 1980, Theorem 8.1]. This defines a finite sequence of positive integers $(n_1, \ldots, n_d)$ such than $n_1 + \cdots + n_d = n$. In fact, [Zelevinsky 1980, Theorem 8.1] implies that this sequence is a partition of n, i.e., $n_1 \ge n_2 \ge \cdots \ge n_d$. We call $(n_1, \ldots, n_d)$ the *partition associated to* $\tilde{\pi}$. We will also say that $\tilde{\pi}$ is *of type* $(n_1, \ldots, n_d)$. Note that by [Bernstein 1984, Section 7.4], if $\tilde{\pi}$ is unitary, then all the representations $\tilde{\pi}^{[n_1,n_2,\ldots,n_i]}$ are unitary as well.

Example 3.1. Using the product notation for normalized parabolic induction, if δ is an essentially square-integrable representation of G_r we set

$$\mathrm{Sp}(d,\delta)=\mathrm{LQ}(|\cdot|_E^{(d-1)/2}\delta\times\cdots\times|\cdot|_E^{(1-d)/2}\delta)$$

to be the Langlands quotient of the parabolically induced representation

$$|\cdot|_E^{(d-1)/2}\delta\times\cdots\times|\cdot|_E^{(1-d)/2}\delta.$$

More generally, if $\tau=\delta_1\times\cdots\times\delta_l$ is a generic unitary representation of G_r written as a commutative product of essentially square-integrable representations [Zelevinsky 1980, Theorem 9.7], we set

$$\mathrm{Sp}(d,\tau)=\mathrm{Sp}(d,\delta_1)\times\cdots\times\mathrm{Sp}(d,\delta_l),$$

which is a commutative product by the results of Tadić [1986, Theorem D]. In this situation, [Bernstein and Zelevinsky 1977, 4.5, Lemma], together with the computation of the highest derivative of Speh representations [Offen and Sayag 2008, § 3.5 (3.3); Tadić 1987, § 6.1], implies that the partition of $n=rd$ associated to $\mathrm{Sp}(d,\tau)$ is $r^d:=(r,\dots,r)$. Conversely one can check using the same results that an irreducible unitary representation of G_n of type r^d is of the form $\mathrm{Sp}(d,\tau)$ for a unitary generic representation τ of G_r. We refer to Section 4B for the details in the Archimedean setting, which are the same as in the non-Archimedean setting.

3B. *Degenerate Whittaker models and* L*-packets.* Let ψ_{n_i} be a nondegenerate character of the group N_{n_i}. By [Zelevinsky 1980, Section 8], if the representation $\tilde{\pi}$ is of type $(n_1,\dots,n_d)$, then it has a unique degenerate Whittaker model with respect to

$$(\psi_{n_1}\otimes\cdots\otimes\psi_{n_d})\begin{pmatrix}u_d & \cdots & \cdot\\ & \ddots & \vdots\\ & & u_1\end{pmatrix}=\psi_{n_1}(u_1)\dots\psi_{n_d}(u_d)$$

for $u_i\in N_{n_i}$. We often use the notation

$$\psi_{n_1,\dots,n_d}:=\psi_{n_1}\otimes\cdots\otimes\psi_{n_d},$$

which has the advantage of being short but could mislead the reader, so we insist on the fact that $\psi_{n_1,\dots,n_d}$ depends on the characters ψ_{n_i} and not only on the positive integers n_i. We will say that $\psi_{n_1,\dots,n_d}$ is *of type* $(n_1,\dots,n_d)$. If all the n_i are equal then we set

$$\psi_{1,\dots,d}:=\psi_{n_1,\dots,n_d}.$$

The L-packet associated to $\tilde{\pi}$ is the finite set of irreducible representations of $G_n^\circ=\mathrm{SL}_n(E)$ appearing in the restriction of $\tilde{\pi}$, and is denoted by $\mathrm{L}(\tilde{\pi})$. We refer to [Hiraga and Saito 2012, Section 2] for its basic properties, which we now state (see also [Gelbart and Knapp 1982] or [Tadić 1992]). Any irreducible representation π of G_n° arises in the restriction of an irreducible representation of G_n and two irreducible representations of G_n containing π are twists of each other by a character. Hence it makes sense to set $\mathrm{L}(\pi)=\mathrm{L}(\tilde{\pi})$, and call this finite set the L-packet of π (or the L-packet determined by $\tilde{\pi}$). *We define* *the type of* π (*or the type of* $\mathrm{L}(\pi)$) *to be that of* $\tilde{\pi}$. Of course two irreducible representations of G_n determining the same L-packet have the same type.

Clearly the group $\mathrm{diag}(E^\times, I_{n-1})$ acts transitively on $\mathrm{L}(\tilde{\pi})$ and the existence of a degenerate Whittaker model for irreducible representations of G_n then has the following immediate consequence.

Lemma 3.2. *Suppose that* $\tilde{\pi}$ *is an irreducible representation of* G_n *of type* $(n_1, \dots, n_d)$. *Then the group* $\mathrm{diag}(E^\times, I_{n-1})$ *acts transitively on* $\mathrm{L}(\tilde{\pi})$ *and every member of* $\mathrm{L}(\tilde{\pi})$ *has a* (*necessarily unique*) *degenerate* ψ*-Whittaker model for some* ψ *of type* $(n_1, \dots, n_d)$.

Uniqueness of degenerate Whittaker models for $\tilde{\pi}$, together with Lemma 3.2, then has the following well-known consequence.

Proposition 3.3. *If* $\tilde{\pi}$ *is an irreducible representation of* G_n *then the representations in* $\mathrm{L}(\tilde{\pi})$ *appear with multiplicity one in the restriction of* $\tilde{\pi}$ *to* G_n°.

In fact we can be more precise. The following lemma follows from the fact that if $\tilde{\pi}$ is of type $(n_1, \dots, n_d)$ then $\tilde{\pi}^{[n_1,\dots,n_{k-1}]}$ is of type $(n_k, \dots, n_d)$ (see Section 3A).

Lemma 3.4. *If* $\tilde{\pi}$ *is an irreducible representation of* G_n *of type* $(n_1, \dots, n_d)$, *then* $\mathrm{L}(\tilde{\pi}^{[n_1,\dots,n_{k-1}]})$ *contains a unique irreducible representation of* $G^\circ_{n_k+\cdots+n_d}$ *with a degenerate Whittaker model with respect to* $\psi_{n_k,\dots,n_d}$.

Again $\mathrm{L}(\tilde{\pi}^{[n_1,\dots,n_{k-1}]})$ only depends on $\mathrm{L}(\tilde{\pi}) = \mathrm{L}(\pi)$ (because derivatives commute with character twists), and we set

$$\mathrm{L}(\pi)^{[n_1,\dots,n_{k-1}]} := \mathrm{L}(\tilde{\pi}^{[n_1,\dots,n_{k-1}]})$$

for any irreducible representation $\tilde{\pi}$ of G_n such that $\pi \in \mathrm{L}(\tilde{\pi})$.

Definition 3.5. Let π be an irreducible representation of G_n°. Let $\pi^{[n_1,\dots,n_{k-1}]}(\psi_{n_k,\dots,n_d})$ denote the irreducible representation of $G^\circ_{n_k+\cdots+n_d}$ isolated in Lemma 3.4, i.e., the unique representation in $\mathrm{L}(\pi)^{[n_1,\dots,n_{k-1}]}$ with a degenerate Whittaker model with respect to $\psi_{n_k,\dots,n_d}$. In particular, $\pi(\psi_{n_1,\dots,n_d})$ denotes the unique irreducible representation of G_n° in $\mathrm{L}(\pi)$ with a degenerate Whittaker model with respect to $\psi_{n_1,\dots,n_d}$.

Remark 3.6. We do not claim that if $\pi(\psi) = \pi(\psi')$, then ψ and ψ' are in the same T_n°-conjugacy class.

3C. ***Distinguished representations inside a distinguished* L*-packet.*** Let $\tilde{\pi}$ be an irreducible representation of G_n of type $(n_1, \dots, n_d)$. We start by making explicit the relation between the degenerate Whittaker models $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ and $\mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$.

Lemma 3.7. *The map*

$$W \mapsto W|_{G_{n-n_1}}$$

is surjective from $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ *to* $\mathcal{W}(\nu_E^{(n_1-1)/2}\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$.

Proof. By the same proof as in [Cogdell and Piatetski-Shapiro 2017, Proposition 1.2], the map

$$W \mapsto W|_{P_{n-n_1+1}}$$

is a surjection from $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ to $\mathcal{W}(\nu_E^{(n-n_1+1)/2}\tilde{\pi}_{(n_1-1)}, \psi_{n_2,\dots,n_d})$. But then, because $W(gu) = W(g)$ for $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$, $g \in G_{n-n_1}$ and $u \in U_{n-n_1+1}$, we deduce that $W|_{G_{n-n_1}} \in \mathcal{W}(\nu_E^{n_1/2}\tilde{\pi}^{(n_1)}, \psi_{n_2,\dots,n_d})$ and that

$$W \mapsto W|_{G_{n-n_1}}$$

is surjective from $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ to $\mathcal{W}(\nu_E^{n_1/2}\tilde{\pi}^{(n_1)}, \psi_{n_2,\dots,n_d})$. The result follows. □

We denote by $\mathcal{K}(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$ the generalized Kirillov model of $\tilde{\pi}$ (see [Zelevinsky 1980, Section 5]) with respect to $\tilde{\pi}^{[n_1]}$ and ψ_{n_1}. It is, by definition, the image of the unique embedding of $\tilde{\pi}|_{P_n}$ into the space of functions $K : P_n \to \pi^{[n_1]}$ which satisfy

$$K(k(a, x, u_1)p) = \nu(a)^{(n_1-1)/2}\psi_{n_1}(u_1)\pi^{[n_1]}(a)K(p)$$

for $k(a, x, u_1) \in N_{n,n_1}$.

Let $\tilde{\pi}$ be an irreducible representation of G_n with degenerate Whittaker model $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$. Then, by Lemma 3.7, for any $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ and $g \in G_n$, the map

$$g_1 \mapsto \nu_E^{(n_1-1)/2}W(g_1 g)$$

belongs to $\mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$. We set

$$I(W) : G_n \to \mathcal{W}(\nu_E^{(n_1-1)/2}\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$$

to be the map defined by

$$I(W)(g) : g_1 \in G_{n-n_1} \mapsto W(g_1 g).$$

Hence I realizes $\mathcal{W}(\tilde{\pi}, \psi_{n_1,n_2,\dots,n_d})$ inside the induced representation

$$\mathrm{Ind}_{N_{n,n_1}}^{G_n}(\mathcal{W}(\nu_E^{(n_1-1)/2}\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d}) \otimes \psi_{n_1}).$$

Then the map $W \mapsto I(W)|_{P_n}$ is a bijection

$$\mathcal{W}(\tilde{\pi}, \psi_{n_1,n_2,\dots,n_d}) \to \mathcal{K}(\tilde{\pi}, \mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d}), \psi_{n_1})).$$

The following is now a consequence of the results of [Matringe 2014].

Proposition 3.8. *Let $\tilde{\pi}$ be an irreducible unitary representation of G_n of type $(n_1, \dots, n_d)$ which is distinguished with respect to G_n^θ, with degenerate Whittaker model $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$, and suppose that $\psi_{n_1,\dots,n_d}$ is trivial on $N_n(F)$. Then the invariant linear form on $\tilde{\pi}$ is expressed as a local period on $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ by*

$$\lambda(W) = \int_{N_{n,n_1}^\theta \backslash P_n^\theta} \int_{N_{n-n_1,n_2}^\theta \backslash P_{n-n_1}^\theta} \cdots \int_{N^\theta_{n-\sum_{i=1}^{d-1} n_i, n_d} \backslash P^\theta_{n-\sum_{i=1}^{d-1} n_i}} W(p_d \cdots p_2 p_1)\, dp_d \cdots dp_2\, dp_1.$$

For all $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$, the integral above is well-defined inductively in the sense that

$$x \mapsto \int_{N^\theta_{n-\sum_{i=1}^{c} n_i, n_{c+1}} \backslash P^\theta_{n-\sum_{i=1}^{c} n_i}} \cdots \int_{N^\theta_{n-\sum_{i=1}^{d-1} n_i, n_d} \backslash P^\theta_{n-\sum_{i=1}^{d-1} n_i}} W(p_d \cdots p_{c+1} x)\, dp_d \cdots dp_{c+1}$$

defines an absolutely convergent function on

$$N^{\theta}_{n-\sum_{i=1}^{c-1} n_i, n_c} \setminus P^{\theta}_{n-\sum_{i=1}^{c-1} n_i}$$

for c descending from d to 1 (for $c = d$ the first integral above is just W by convention).

Proof. The proof is by induction on d. For $d = 1$, the representation is unitary generic, and the fact that

$$W \mapsto \int_{N_n^{\theta} \setminus P_n^{\theta}} W(p)\, dp$$

is well defined is due to Flicker [1988, Section 4], and that it is G_n^{θ}-invariant is a result due to Youngbin Ok (see [Matringe 2014, Proposition 2.5] for a more general statement in the unitary context). Then, for a general d, by [Matringe 2014, Proposition 2.4], if $\tilde{\pi}$ is distinguished, so is $\tilde{\pi}^{[n_1]}$, and we take $L \in \mathrm{Hom}_{G^{\theta}_{n-n_1}}(\tilde{\pi}^{[n_1]}, \mathbb{C}) \setminus \{0\}$. By [Matringe 2014, Propositions 2.2 and 2.5], the linear form

$$\lambda_K : K \mapsto \int_{N^{\theta}_{n,n_1} \setminus P_n^{\theta}} L(K(p_1))\, dp_1 \tag{1}$$

is, up to scaling, the unique G_n^{θ}-invariant linear form on $\mathcal{K}(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$ and it is given by an absolutely convergent integral for all $K \in \mathcal{K}(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$. We realize $\tilde{\pi}$ as $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ and $\tilde{\pi}^{[n_1]}$ as $\mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$. Then by induction for all $W' \in \mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$ we have

$$L(W') = \int_{N^{\theta}_{n-n_1,n_2} \setminus P^{\theta}_{n-n_1}} \cdots \int_{N^{\theta}_{n-\sum_{i=1}^{d-1} n_i, n_d} \setminus P^{\theta}_{n-\sum_{i=1}^{d-1} n_i}} W'(p_r \cdots p_2)\, dp_r \cdots dp_2,$$

which is well defined in the sense of the statement of the proposition because $\tilde{\pi}^{[n_1]}$ is unitary. Applying it to $W' = K(p_1) = I(W_K)(p_1)$ for the unique $W_K \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ such that the previous equality holds, the result follows in view of the discussion preceding the proposition. ☐

Theorem 3.9. *Let π be an irreducible unitary representation of $\mathrm{SL}_n(E)$ of type $(n_1, \dots, n_d)$ which is $\mathrm{SL}_n(F)$-distinguished. Then the $\mathrm{SL}_n(F)$-distinguished representations in $\mathrm{L}(\pi)$ are precisely the representations $\pi(\psi)$ for a character ψ of N_n of type $(n_1, \dots, n_d)$ such that $\psi|_{N_n^{\theta}} \equiv \mathbf{1}$.*

Proof. The proof follows exactly along the same lines of the generic case, as in [AP 2003, Section 3] and [AP 2018, Section 4], making use of Proposition 3.8 in lieu of Flicker's invariant linear form mentioned above. ☐

Theorem 3.9 has the following consequences.

Proposition 3.10. *Let π be an irreducible unitary representation of G_n° of type $(n_1, \dots, n_d)$, and fix $\psi_{n_1,\dots,n_d}$, a character of N_n of this type trivial on N_n^{θ}. If $\pi(\psi_{n_1,\dots,n_d})$ is $\mathrm{SL}_n(F)$-distinguished, then the representation $\pi^{[n_1,\dots,n_{k-1}]}(\psi_{n_k,\dots,n_d})$ is $\mathrm{SL}_{\sum_{i=k}^{d} n_i}(F)$-distinguished for all $k = 1, \dots, d$.*

Proof. According to [AP 2018, Lemma 3.2], up to twisting $\tilde{\pi}$ by an appropriate character, we can suppose that it is $\mathrm{GL}_n(F)$-distinguished. Then $\tilde{\pi}^{[n_1,\dots,n_{k-1}]}$ is distinguished as we already saw (see proof of

Proposition 3.8). Now $\pi^{[n_1,\dots,n_{k-1}]}(\psi_{n_k,\dots,n_d})$ belongs to $\mathrm{L}(\pi^{[n_1,\dots,n_{k-1}]})$ and it has a degenerate Whittaker model with respect to the distinguished character $\psi_{n_k,\dots,n_d}$, so the result follows from Theorem 3.9. □

Proposition 3.10 can be strengthened for Speh representations.

Theorem 3.11. *Let τ be a generic representation of G_r and let ψ_i be a nondegenerate character of N_r trivial on N_r^θ for $i = 1, \dots, d$. Fix $1 \le k \le d$, and then $\pi(\psi_{1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(d, \tau))$ is $\mathrm{SL}_n(F)$-distinguished if and only if $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(k, \tau))$ is $\mathrm{SL}_{kr}(F)$-distinguished.*

Proof. One direction follows from Proposition 3.10. Conversely suppose that

$$\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(k, \tau))$$

is $\mathrm{SL}_{kr}(F)$-distinguished. Then, thanks to [Matringe 2014, Theorem 2.13], up to a twist, $\mathrm{Sp}(k, \tau)$ is distinguished, so τ is, and hence $\mathrm{Sp}(d, \tau)$ is. But then because $\pi(\psi_{1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(d, \tau))$ has a $\psi_{1,\dots,d}$-degenerate Whittaker model and $\psi_{1,\dots,d}$ is trivial on N_n^θ, we deduce that $\pi(\psi_{1,\dots,d})$ is $\mathrm{SL}_n(F)$-distinguished, thanks to Theorem 3.9. □

We will give the global analogue of this result in Theorem 6.16.

4. Archimedean prerequisites for the global theory

Here $E = \mathbb{C}$ or $\mathbb{R}$, and by abuse of notation we write $G = G(E)$ for any algebraic group defined over E. We set $|a + ib|_{\mathbb{C}} = a^2 + b^2$ and denote by $|\cdot|_{\mathbb{R}}$ the usual absolute value on $\mathbb{R}$. We then denote by ν_E the character of G_n obtained by composing $|\cdot|_E$ with det. For G a reductive subgroup of G_n we write $\mathcal{SAF}(G)$ for the category of smooth admissible Fréchet representations of G of moderate growth as in [Aizenbud et al. 2015], in which we work. We use the same product notation for parabolic induction in $\mathcal{SAF}(G_n)$ as in [Aizenbud et al. 2015].

We only consider unitary characters of N_n. The nondegenerate characters of N_n are of the form

$$\psi_\lambda : \begin{pmatrix} 1 & z_1 & \cdots & \cdots & \cdot \\ & 1 & z_2 & \cdots & \cdot \\ & & \ddots & \ddots & \cdot \\ & & & 1 & z_{n-1} \\ & & & & 1 \end{pmatrix} \mapsto \exp\left(i \sum_{i=1}^{n-1} \Re(\lambda_i z_i) \right)$$

with $\lambda_i \in E^*$. Then for a partition $(n_1, \dots, n_r)$ of n and nondegenerate characters ψ_{n_i} of N_{n_i} we define the degenerate character $\psi_{n_1,\dots,n_d}$ of N_n as in Section 3B and we also write $\psi_{1,\dots,d} := \psi_{n_1,\dots,n_d}$ when all the n_i's are equal. We again say $\psi_{n_1,\dots,n_d}$ is of type $(n_1, \dots, n_d)$, so that the set of characters of a given type forms a single T_n-conjugacy class. We call a member of this conjugacy class a degenerate character of type $(n_1, \dots, n_d)$. For a degenerate character ψ of N_n and an irreducible representation $\tilde{\pi}$ of G_n, by a ψ-Whittaker functional, we mean a nonzero continuous linear form L from $\tilde{\pi}$ to $\mathbb{C}$ satisfying

$$L(\tilde{\pi}(n)v) = \psi(n)L(v)$$

for $n \in N_n$ and $v \in \tilde{\pi}$. We will say that $\tilde{\pi}$ has a unique ψ-Whittaker model if the space of ψ-Whittaker functionals on the space of $\tilde{\pi}$ is one-dimensional.

4A. ***The Tadić classification of the unitary dual of G_n.*** We recall that irreducible square-integrable representations of G_n for $n \geq 1$ exist only when $n = 1$ if $E = \mathbb{C}$ and when $n = 1$ or 2 if $E = \mathbb{R}$. When $n = 1$ these are just the unitary characters of $E^\times$. For $d \in \mathbb{N}$ and an irreducible square-integrable representation δ of G_n ($n = 1$ or $n \in \{1, 2\}$ depending on whether E is $\mathbb{C}$ or $\mathbb{R}$) we denote by

$$\mathrm{Sp}(d, \delta) = \mathrm{LQ}(\nu_E^{(d-1)/2}\delta \times \cdots \times \nu_E^{(1-d)/2}\delta)$$

the Langlands quotient of $\nu_E^{(d-1)/2}\delta \times \cdots \times \nu_E^{(1-d)/2}\delta$. In particular, $\mathrm{Sp}(d, \chi) = \chi \circ \det$ when χ is a unitary character of G_1. By [Tadić 2009], the representations

$$\pi(\mathrm{Sp}(d, \delta), \alpha) := \nu^\alpha \mathrm{Sp}(d, \delta) \times \nu^{-\alpha}\mathrm{Sp}(d, \delta)$$

are irreducible unitary when $\alpha \in \left(0, \frac{1}{2}\right)$, and any irreducible representation π of G_n can be written in a unique manner as a commutative product

$$\tilde{\pi} = \prod_{i=1}^{r} \mathrm{Sp}(d_i, \delta_i) \prod_{j=r+1}^{s} \pi(\mathrm{Sp}(d_j, \delta_j), \alpha_j).$$

When all the d_i and d_j are equal to one, the representation

$$\tau = \prod_{i=1}^{r} \delta_i \prod_{j=r+1}^{s} \pi(\delta_j, \alpha_j)$$

is generic unitary (it has a unique ψ-Whittaker model for any nondegenerate character ψ of N_n), according to [Jacquet 2009, p. 4], and we set

$$\mathrm{Sp}(d, \tau) = \prod_{i=1}^{r} \mathrm{Sp}(d, \delta_i) \prod_{j=r+1}^{s} \pi(\mathrm{Sp}(d, \delta_j), \alpha_j),$$

which is thus an irreducible unitary representation.

We note that according to the proof of [Gourevitch and Sahi 2013, 4.1.1], which refers to [Vogan 1986] and [Sahi and Stein 1990], a Speh representation $\mathrm{Sp}(d, \delta)$ for δ an irreducible square-integrable representation of G_2 is the same thing as the Speh representations of Vogan's classification as presented in [Aizenbud et al. 2015, 4.1.2(c)]. Hence the Vogan classification as stated in [Aizenbud et al. 2015, 4.1.2] is immediately related to that of Tadić:

- The unitary characters of [Aizenbud et al. 2015, 4.1.2(a)] are the representations of the form $\mathrm{Sp}(d, \chi)$ for χ a unitary character of G_1.
- The Stein complementary series of [Aizenbud et al. 2015, 4.1.2(b)] are the representations of the form $\pi(\mathrm{Sp}(d, \chi), \alpha)$ for χ a unitary character of G_1.

- The Speh representations of [Aizenbud et al. 2015, 4.1.2(c)] are the representations of the form $\mathrm{Sp}(d,\delta)$ for δ an irreducible square-integrable representation of G_2.
- The Speh complementary series of [Aizenbud et al. 2015, 4.1.2(d)] are the representations of the form $\pi(\mathrm{Sp}(d,\delta),\alpha)$ for δ an irreducible square-integrable representation of G_2.

The third and fourth cases occur only when $E=\mathbb{R}$.

4B. ***Degenerate Whittaker models of irreducible unitary representations.*** In this section we recall the results of Aizenbud, Gourevitch, and Sahi on degenerate Whittaker models for $\mathrm{GL}_n(E)$ for $E=\mathbb{C}$ or $\mathbb{R}$. We believe that with the material developed by these authors, together with the real analogue of Ok's result due to Kemarsky [2015], the results obtained in [Matringe 2014] and Section 3 are in reach. However, being inexperienced in such matters, we leave this for experts, and simply recall immediate implications of the results in [Aizenbud et al. 2015] that we will need for our global applications.

To any irreducible representation $\tilde{\pi}$ of G_n, Sahi [1989] attached an irreducible representation $A(\tilde{\pi})$ of G_{n-n_1} for some $0<n_1\leq n$, the *adduced representation of* $\tilde{\pi}$, and proved that it satisfied

$$A\biggl(\prod_{i=1}^{r}\mathrm{Sp}(d_i,\delta_i)\prod_{j=r+1}^{s}\pi(\mathrm{Sp}(d_j,\delta_j),\alpha_j)\biggr)=\prod_{i=1}^{r}A(\mathrm{Sp}(d_i,\delta_i))\prod_{j=r+1}^{s}A\bigl(\pi(\mathrm{Sp}(d_j,\delta_j),\alpha_j)\bigr)$$

with respect to the Tadić classification. The adduced representation is the Archimedean highest shifted derivative, and from [Sahi 1990; Gourevitch and Sahi 2013; Aizenbud et al. 2015] (see [Aizenbud et al. 2015, Section 4]) one has

$$A\biggl(\prod_{i=1}^{r}\mathrm{Sp}(d_i,\delta_i)\prod_{j=r+1}^{s}\pi(\mathrm{Sp}(d_j,\delta_j),\alpha_j)\biggr)=\prod_{i=1}^{r}\mathrm{Sp}(d_i-1,\delta_i)\prod_{j=r+1}^{s}\pi(\mathrm{Sp}(d_j-1,\delta_j),\alpha_j). \tag{2}$$

One can then take the adduced of the adduced representation of the irreducible unitary representation $\tilde{\pi}$ and so on, and obtain the "depth sequence" $\bar{n}:=(n_1,\dots,n_d)$ attached to $\tilde{\pi}$, which forms a partition of n. We call this depth sequence the *type of* $\tilde{\pi}$. The combination of [Gourevitch and Sahi 2013, Theorem A] and [Aizenbud et al. 2015, Theorem 4.2.3] says:

Theorem 4.1. *Let $\tilde{\pi}$ be an irreducible unitary representation of G_n of type $(n_1,\dots,n_d)$, and ψ be any character of N_n of type $(n_1,\dots,n_d)$. Then $\tilde{\pi}$ has a unique degenerate ψ-Whittaker model.*

For an irreducible representation π of $\mathrm{SL}_n(E)$, the notion of a degenerate Whittaker model is defined similarly. This notion depends on the T_n°-conjugacy class of the degenerate character ψ and not just its type. The L-packet of π is defined as in the p-adic case, and we refer to [Hiraga and Saito 2012, end of Section 2]. Note that [Hiraga and Saito 2012] deals with Harish-Chandra modules but their results remain valid in the context of $\mathcal{SAF}(G_n)$, thanks to the Casselman–Wallach equivalence of categories (see [Wallach 1988, Chapter 11]). If $\tilde{\pi}$ is an irreducible unitary representation of G_n, it follows from [Gourevitch and Sahi 2013, Theorem A] that the type of $\tilde{\pi}$ depends only on $\mathrm{L}(\tilde{\pi})$, and *we define the type of an irreducible unitary representation π of G_n° to be that of any irreducible representation $\tilde{\pi}$ of G_n such that $\pi\in\mathrm{L}(\tilde{\pi})$.*

Remark 4.2. If $\tilde{\pi}$ is an irreducible representation of G_n°, then $\tilde{\pi}|_{G_n^\circ}$ contains an irreducible unitary representation if and only if it is unitary up to a character twist.

As in the p-adic case, Theorem 4.1 has the following consequence.

Corollary 4.3. *Let $\tilde{\pi} \in \mathcal{SAF}(G_n)$ be an irreducible unitary representation of type $(n_1, \dots, n_d)$. Then the group $\mathrm{diag}(E^\times, I_{n-1})$ acts transitively on $\mathrm{L}(\tilde{\pi})$ and every $\pi \in \mathrm{L}(\tilde{\pi})$ has a (necessarily unique) degenerate ψ-Whittaker model for some character ψ of N_n of type $(n_1, \dots, n_d)$. Moreover, $\tilde{\pi}|_{G_n^\circ}$ is multiplicity-free.*

We note that the computation of the adduced representation given in (2) implies:

Theorem 4.4 (Aizenbud, Gourevitch, and Sahi). *Let τ be an irreducible generic representation of G_r. The Speh representation $\mathrm{Sp}(d, \tau)$ has type r^d, and conversely an irreducible unitary representation of G_n of type r^d is of the form $\mathrm{Sp}(d, \tau)$ for some unitary generic representation τ of G_r.*

We end by giving the Archimedean analogue of Definition 3.5 for Speh representations.

Definition 4.5. Let π be an irreducible unitary representation of G_n° of type r^d, and let τ be an irreducible unitary generic representation of G_r such that $\pi \in \mathrm{L}(\mathrm{Sp}(d, \tau))$. For $\psi_{1,\dots,d}$ a character of N_n of type r^d, we denote by $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ the unique representation in $\mathrm{L}(\mathrm{Sp}(k, \tau))$ with a $\psi_{d-k+1,\dots,d}$-degenerate Whittaker model.

Remark 4.6. The representation $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ above depends only on $\mathrm{L}(\pi)$.

5. The global setting

In this section, E/F is a quadratic extension of number fields with associated Galois involution θ. We denote by $\mathbb{A}_E$ and $\mathbb{A}_F$ the rings of adeles of E and F respectively. We denote by $\mathrm{GL}_n(\mathbb{A}_E)^1$ the elements of $\mathrm{GL}_n(\mathbb{A}_E)$ which have determinant of adelic norm equal to 1, and for any subgroup H of $\mathrm{GL}_n(\mathbb{A}_E)$, by H^1 we denote the intersection of H with $\mathrm{GL}_n(\mathbb{A}_E)^1$. We recall that $\mathbb{A}_F^\times = \mathbb{A}_F^1 \times (\mathbb{A}_F)_{>0}$, where $(\mathbb{A}_F)_{>0}$ is $\mathbb{R}_{>0} \otimes_{\mathbb{Q}} 1 \subset \mathbb{R} \otimes_{\mathbb{Q}} F$ sitting inside $\mathbb{A}_F$. In particular, passing to the groups of unitary characters, we have $\widehat{\mathbb{A}_F^\times} = \widehat{\mathbb{A}_F^1} \times \widehat{(\mathbb{A}_F)_{>0}}$, and for $\lambda \in \mathbb{R}$ we denote by α_λ the unitary character of $\mathbb{A}_F^\times$ corresponding to $(\alpha, (|\cdot|_{\mathbb{A}_F}^{i\lambda})|_{\widehat{(\mathbb{A}_F)_{>0}}}) \in \widehat{\mathbb{A}_F^1} \times \widehat{(\mathbb{A}_F)_{>0}}$. Namely, extending α_0 is the extension of α which is trivial on $\widehat{(\mathbb{A}_F)_{>0}}$ and $\alpha_\lambda = \alpha_0 |\cdot|_{\mathbb{A}_F}^{i\lambda}$. In particular α_λ is automorphic if and only if $\alpha \in \widehat{F^\times \backslash \mathbb{A}_F^1}$.

5A. *Square-integrable automorphic representations and their* L*-packets.*

For $\omega \in \widehat{E^\times \backslash \mathbb{A}_E^\times}$, we denote by

$$L^2(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

the space of smooth L^2-automorphic forms on which the center $\mathbb{A}_E^\times$ of $\mathrm{GL}_n(\mathbb{A}_E)$ acts by ω, and by

$$L_d^2(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

its discrete part. We then denote by $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ the dense $\mathrm{GL}_n(\mathbb{A}_E)$-submodule of $L_d^2(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ consisting of smooth automorphic forms (see [Cogdell 2004, Lecture 2]). We say that $\tilde{\pi}$ is a *square-integrable automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ if it is a closed (for the

Fréchet topology) irreducible $\mathrm{GL}_n(\mathbb{A}_E)$-submodule of $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ for some Hecke character ω. The space $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ contains the space of smooth cusp forms

$$\mathcal{A}_0^\infty(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

as a $\mathrm{GL}_n(\mathbb{A}_E)$-invariant subspace. A *cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ is a closed irreducible $\mathrm{GL}_n(\mathbb{A}_E)$-submodule of $\mathcal{A}_0^\infty(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$, for some Hecke character ω.

Let σ be a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$, and

$$\tilde{\pi} = \mathrm{Sp}(d, \sigma) = \bigotimes_v{}' \, \mathrm{Sp}(d, \sigma_v)$$

be the restricted tensor product of the representations $\mathrm{Sp}(d, \sigma_v)$ for v varying through the places of E. By [Jacquet 1984], this is a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, where $n = dr$. By [Mœglin and Waldspurger 1989], any irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ is of this form for a unique pair (σ, d), and moreover $\mathrm{Sp}(d, \sigma)$ appears with multiplicity one in $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ (this of course was already known for $d = 1$ by the pioneering independent results of Piatetski-Shapiro and Shalika).

We define the spaces $L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ and $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ in the same way that we defined their GL-analogues. Also, similarly, the notions of square-integrable and cuspidal automorphic representations of $\mathrm{SL}_n(\mathbb{A}_E)$ are defined. We set

$$L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c := \bigoplus_{\omega \in \widehat{E^\times \backslash \mathbb{A}_E^\times}} L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega),$$

which is well known to be multiplicity-free.

Notation 5.1. We denote by

$$\mathrm{Res} : L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c \to L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$$

the restriction of functions from $\mathrm{GL}_n(\mathbb{A}_E)$ to $\mathrm{SL}_n(\mathbb{A}_E)$.

We recall from [Hiraga and Saito 2012, Chapter 4] (see in particular [Hiraga and Saito 2012, Remark 4.23] for square-integrable representations) the following facts. If $\tilde{\pi} \subset L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c$ is an irreducible submodule, then by Corollary 5.5 of the next section the representation $\mathrm{Res}(\tilde{\pi})$ is multiplicity-free, and we denote by $\mathrm{L}(\tilde{\pi})$ the set of irreducible submodules of $\mathrm{Res}(\tilde{\pi})$, and call it the L-packet attached to $\tilde{\pi}$. Moreover if $\tilde{\pi}' \subset L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c$ is also an irreducible submodule, then $\mathrm{Res}(\tilde{\pi})$ and $\mathrm{Res}(\tilde{\pi}')$ are either in direct sum or equal, and they are equal if and only if $\tilde{\pi}$ and $\tilde{\pi}'$ are twists of each other by an automorphic character of $\mathbb{A}_E^\times$; i.e., $\mathrm{L}(\tilde{\pi}) \cap \mathrm{L}(\tilde{\pi}') \neq \varnothing$ if and only if they are equal if and only if $\tilde{\pi}$ and $\tilde{\pi}'$ are twists of each other by an automorphic character of $\mathbb{A}_E^\times$. For π an irreducible submodule of $L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ we set

$$m(\pi) = \dim \mathrm{Hom}_{\mathrm{SL}_n(\mathbb{A}_E)}\big(\pi, L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))\big),$$

and call it the multiplicity of π in $L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$. This is known to be finite.

If π is a square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$, then there are exactly $m(\pi)$ L-packets containing a representation isomorphic to π, and if π_0 is a representation isomorphic to π contained in an L-packet, we call π_0 a *canonical realization* of π. In particular, if π is such a canonical realization, the L-packet $\mathrm{L}(\pi)$ of π is well defined (it is by definition equal to $\mathrm{L}(\tilde{\pi})$ for $\pi \subset \mathrm{Res}(\tilde{\pi})$).

5B. ***Degenerate Whittaker models and square-integrable* L*-packets.*** Let $n = dr$. Let σ be a smooth unitary cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ and let $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ be the associated square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. We set U_{r^d} to be the unipotent radical of the parabolic subgroup of type r^d of $\mathrm{GL}(n)$, denoted by P_{r^d}. Let

$$\psi_{1,\dots,d}(\mathrm{diag}(n_1, \dots, n_d)u) = \prod_{i=1}^{d} \psi_i(n_i),$$

where ψ_i is a nondegenerate character of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$ and $u \in U_{r^d}(\mathbb{A}_E)$. For $\varphi \in \pi$, we set

$$p_{\psi_{1,\dots,d}}(\varphi) = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi(n)\psi_{1,\dots,d}^{-1}(n)\,dn.$$

By [Jiang and Liu 2013, Corollary 3.4], there exists $\varphi \in \mathrm{Sp}(d, \sigma)$ such that $p_{\psi_{1,\dots,d}}(\varphi) \neq 0$: we will say that φ *has a nonzero Fourier coefficient of type* r^d or *a degenerate Whittaker model of type* r^d. Of course when $d = 1$ this result is due to the pioneering works of Piatetski-Shapiro and Shalika.

Remark 5.2. The result [Jiang and Liu 2013, Corollary 3.4] could also be deduced by the techniques used in Section 6, using the $E = F \times F$-analogue of Yamana's formula [2015, Theorem 1.1] (see Theorem 6.7). Also following Section 6 in the case where E is split, one would conclude that any square-integrable representation of $\mathrm{SL}_n(\mathbb{A}_E)$ in the L-packet determined by $\mathrm{Sp}(d, \sigma)$ has a degenerate Whittaker model of type r^d. However for the sake of variety we offer a different proof of this fact here, using the results of [Jiang and Liu 2013] rather than those of [Yamana 2015] (or rather its split analogue).

Definition. We say that a square-integrable representation π of $\mathrm{SL}_n(\mathbb{A}_E)$ is *of type* r^d if it belongs to $\mathrm{L}(\mathrm{Sp}(d, \sigma))$ for an irreducible (unitary) cuspidal automorphic representation σ of $G_r(\mathbb{A}_E)$.

We say that $\tilde{\pi}$ (resp. π) *has a degenerate Whittaker model of type* r^d if there is $\varphi \in \tilde{\pi}$ (resp. $\varphi \in \pi$) with a nonzero Fourier coefficient of type r^d. In particular $\mathrm{Sp}(d, \sigma)$ has a degenerate Whittaker model of type r^d.

We denote by ψ a nondegenerate character of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$. We set

$$(\mathbf{1} \otimes \psi)\begin{pmatrix} I_{n-r} & x \\ \cdot & u_1 \end{pmatrix} = \psi(u_1) \quad \text{for} \quad \begin{pmatrix} I_{n-r} & x \\ \cdot & u_1 \end{pmatrix} \in U_{n,r}(\mathbb{A}_E).$$

For $\varphi \in \tilde{\pi}$, we set

$$\varphi_{U_{n,r},\psi}(g) = \int_{U_{n,r}(E)\backslash U_{n,r}(\mathbb{A}_E)} \varphi(u\,\mathrm{diag}(g, I_r))(\mathbf{1} \otimes \psi^{-1})(u)\,du$$

for $g \in \mathrm{GL}_{n-r}(\mathbb{A}_E)$.

Remark 5.3. Note that the function $\varphi_{U_{n,r},\psi}$ is nothing but the integral of the constant term of φ along the $(n-r,r)$ parabolic against ψ^{-1} on $N_r(E)\backslash N_r(\mathbb{A}_E)$. By [Yamana 2015, Lemma 6.1], there is a positive character δ of $\mathrm{GL}_{n-r}(\mathbb{A}_E)$ such that the function $\delta\otimes\varphi_{U_{n,r},\psi}$ belongs to $\mathrm{Sp}(d-1,\sigma)$; in particular, $(\varphi_{U_{n,r},\psi})|_H$ belongs to $\mathrm{Res}_H(\mathrm{Sp}(d-1,\sigma))$ (restriction of cusp forms) for any subgroup H of $\mathrm{GL}_n(\mathbb{A}_E)^1$, for example $H=\mathrm{SL}_n(\mathbb{A}_E)$.

Proposition 5.4. *A square-integrable automorphic representation π of $\mathrm{SL}_n(\mathbb{A}_E)$ of type r^d has a degenerate Whittaker model of type r^d.*

Proof. We will prove the stronger claim: for any $\varphi\in\tilde\pi$ such that $\varphi|_{\mathrm{SL}_n(\mathbb{A}_E)}\neq 0$, there is $h_0\in\mathrm{SL}_r(\mathbb{A}_E)$ (embedded in $\mathrm{SL}_n(\mathbb{A}_E)$ in the upper left block) such that $\rho(h_0)\varphi$ has a nonzero Fourier coefficient of type r^d. If $d=1$, we are in the cuspidal (and hence generic) case and the result follows from the same inductive procedure of Lemma 6.1 and Proposition 6.3, but applied to E diagonally embedded inside $E\times E$ (instead of $F\subset E$ considered there). If $d\geq 2$, by [Jiang and Liu 2013, Proposition 3.1(1)] applied to φ, there is a nondegenerate character ψ of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$ such that $\varphi_{U_{n,r},\psi}$ is nonzero on $\mathrm{SL}_{n-r}(\mathbb{A}_E)$ (because $N_{n,r}(E)\backslash P_n(E)=N^\circ_{n,r}(E)\backslash P^\circ_n(E)$ since $d\geq 2$). We conclude by induction, thanks to Remark 5.3. ☐

Corollary 5.5. *If $\tilde\pi$ is an irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ of type r^d, then $\mathrm{Res}(\tilde\pi)$ is multiplicity-free. Moreover, for any automorphic character ψ of $N_n(\mathbb{A}_E)$ of type r^d, the L-packet $\mathrm{L}(\tilde\pi)$ contains a unique member $\pi(\psi)$ with a ψ-Whittaker model, and the group $\mathrm{diag}(E^\times, I_{n-1})$ acts transitively on $\mathrm{L}(\tilde\pi)$.*

Proof. Thanks to multiplicity one inside local L-packets (see Proposition 3.3 and Corollary 4.3), it follows that the representations in $\mathrm{L}(\tilde\pi)$ appear with multiplicity one in $\mathrm{Res}(\tilde\pi)$. Moreover, we deduce that $T_n(E)$ acts transitively on $\mathrm{L}(\tilde\pi)$: by Proposition 5.4 any representation in $\mathrm{L}(\tilde\pi)$ has a degenerate Whittaker model of type r^d. Note that two automorphic characters of type r^d of $N_n(\mathbb{A}_E)$ are conjugate to each other by $T_n(E)$ and this implies that for each automorphic character ψ of type r^d of $N_n(\mathbb{A}_E)$ there is a representation $\pi(\psi)$ in $\mathrm{L}(\tilde\pi)$ with a ψ-Whittaker model. Moreover $\mathrm{L}(\tilde\pi)$ has at most one representation with a ψ-Whittaker model by local multiplicity one of degenerate Whittaker models and this implies the uniqueness of $\pi(\psi)$ in the statement. Finally for $t\in T_n(E)$ and $t'=\mathrm{diag}(\det(t), I_{n-1})$, the representations π^t and $\pi^{t'}$ in $\mathrm{L}(\tilde\pi)$ are isomorphic, hence equal by multiplicity one inside $\mathrm{L}(\tilde\pi)$. ☐

5C. ***Distinguished representations and distinguished* L*-packets.*** Take $\chi\in\widehat{F^\times\backslash\mathbb{A}_F^\times}$, and choose $\omega\in\widehat{E^\times\backslash\mathbb{A}_E^\times}$, a Hecke character such that $\omega|_{\mathbb{A}_F^\times}=\chi^n$. We denote by $\tilde p_{n,\chi}$ the linear form called the χ-period integral on $L_d^{2,\infty}(\mathbb{A}_E^\times\mathrm{GL}_n(E)\backslash\mathrm{GL}_n(\mathbb{A}_E),\omega)$, given by

$$\tilde p_{n,\chi}(\phi)=\int_{\mathbb{A}_F^\times\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)}\phi(h)\chi^{-1}(\det(h))\,dh.$$

It is well defined on $\mathcal{A}_0^\infty(\mathbb{A}_E^\times\mathrm{GL}_n(E)\backslash\mathrm{GL}_n(\mathbb{A}_E),\omega)$ by [Ash et al. 1993, Proposition 1] and in general by [Yamana 2015, Lemma 3.1]. Indeed up to a positive constant, $\tilde p_{n,\chi}(\phi)$ is equal to

$$\int_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1}\phi(h)\chi^{-1}(\det(h))\,dh.$$

Definition. We say that a square-integrable automorphic representation

$$\tilde{\pi} \subset L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

is *χ-distinguished* (or simply *distinguished* when $\chi \equiv 1$) if $\tilde{p}_{n,\chi}$ is nonvanishing on $\tilde{\pi}$.

We denote by p_n the period integral on $L_d^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ given by

$$p_n(\phi) = \int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \phi(h)\, dh.$$

It is again well defined on $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ thanks to [Ash et al. 1993, Proposition 1] and on the space $L_d^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ by the arguments in [Yamana 2015, Lemma 3.1].

Definition. We say that a square-integrable representation

$$\pi \subset L_d^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$$

is *distinguished* if p_n does not vanish on π. We give another useful formula for the SL_n-period integral following [AP 2006, Proposition 3.2].

Proposition 5.6. *Let $\tilde{\pi}$ be a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. The period integral*

$$\varphi \mapsto \int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h)\, dh$$

is given by an absolutely convergent integral on $\mathrm{Res}(\tilde{\pi})$. Moreover, for any $\varphi \in \tilde{\pi}$, we have

$$\int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h)\, dh = \sum_\alpha \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\alpha(\det(h))\, dh,$$

where the sum is over all characters α of the compact abelian group $F^\times\backslash \mathbb{A}_F^1$.

Proof. For the absolute convergence of the integrals, the arguments of [Yamana 2015, Lemma 3.1] adapt in a straightforward manner and we do not repeat them. The proof of the second point is now essentially that of [AP 2006, Proposition 3.2]. Indeed,

$$\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\, dh = \int_{F^\times\backslash \mathbb{A}_F^1} \left(\int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h \operatorname{diag}(x, I_{n-1}))\, dh \right) dx,$$

and one applies Fourier inversion on the compact abelian group $F^\times\backslash \mathbb{A}_F^1$. □

Remark 5.7. The sum of the $(\mathrm{GL}(n, \mathbb{A}_F)^1, \alpha)$-periods over all characters α of the group $F^\times\backslash \mathbb{A}_F^1$ is in fact a finite sum. We denote by $\omega_{\tilde{\pi}}$ the central character of $\tilde{\pi}$. The first observation is that we may assume that $\tilde{\pi}$ is distinguished with respect to $\mathrm{GL}_n(\mathbb{A}_F)$. Indeed if $\tilde{\pi}$ is $(\mathrm{GL}_n(\mathbb{A}_F)^1, \alpha)$-distinguished, then it is $(\mathrm{GL}_n(\mathbb{A}_F), \alpha')$-distinguished for α' the unique character of $\mathbb{A}_F^\times$ extending α and equal to $\omega_{\tilde{\pi}}^n$ on $(\mathbb{A}_F)_{>0}$, but then we take $\alpha'' \in \widehat{E^\times\backslash \mathbb{A}_E^\times}$ with $\alpha''|_{\mathbb{A}_F^\times} = \alpha'$ and replace $\tilde{\pi}$ by $\tilde{\pi} \otimes \alpha''^{-1}$. With this assumption $\tilde{\pi}$ is Galois conjugate self-dual by strong multiplicity one for the residual spectrum [Mœglin and Waldspurger 1989] and the fact that $\mathrm{Sp}(d, \sigma_v)$ is distinguished and hence Galois conjugate self-dual for any finite place v [Flicker 1991]. Now, if the $(\mathrm{GL}(n,\mathbb{A}_F)^1,\alpha)$-period is also nonzero then we have $\tilde{\pi}\cong\tilde{\pi}\otimes\alpha'\circ N_{E/F}$ for the unique character $\alpha'\in\widehat{F^\times\backslash\mathbb{A}_F^\times}$ extending α and equal to $\omega_{\tilde{\pi}}^n$ on $(\mathbb{A}_F)_{>0}$, and writing $\tilde{\pi}=\mathrm{Sp}(d,\sigma)$, we see that $\sigma\cong\sigma\otimes\alpha'\circ N_{E/F}$. As σ is a cuspidal representation and because $N_{E/F}(\mathbb{A}_E^\times)$ has finite index in $\mathbb{A}_F^\times$, the set of such characters α' (hence of that of the characters α) follows from [Ramakrishnan 2000, Lemma 3.6.2] (which is [Hiraga and Saito 2012, Lemma 4.11]).

Definition 5.8. We say that the L-packet determined by a square-integrable representation of $\mathrm{GL}_n(\mathbb{A}_E)$ is distinguished if it contains a distinguished representation of $\mathrm{SL}_n(\mathbb{A}_E)$.

6. Distinction inside global L-packets

The aim of this section is to establish our main result, Theorem 6.10, which asserts that distinguished representations inside distinguished L-packets are those with a degenerate ψ-Whittaker model for some distinguished ψ, and to give a first application of it (Theorem 6.16). The proof is an induction based on the unfolding method, and has two steps, the first one being the cuspidal step (corresponding to $d=1$).

6A. ***The cuspidal case.*** Here we characterize members of distinguished L-packets of $\mathrm{SL}_n(\mathbb{A}_E)$ with nonvanishing $\mathrm{SL}_n(\mathbb{A}_F)$-period in terms of Whittaker periods. The following lemma is a generalization of [AP 2006, Lemma 4.3], but the proof there does not generalize to this case. We denote by Q_n the proper parabolic subgroup of SL_n containing $P_n^\circ=\mathrm{SL}_{n-1}.U_n$. For $n\geq 3$, we set

$$R_n=\{\mathrm{diag}(x,I_{n-2},x^{-1}):x\in\mathbb{G}_m\},$$

so Q_n is the semidirect product $P_n^1.R_n$.

Lemma 6.1. *Take $n\geq 3$. Let φ be a cusp form on $\mathrm{SL}_n(\mathbb{A}_E)$ such that*

$$\int_{\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A}_F)}\varphi(h)\,dh\neq 0.$$

Then there is $h_0\in\mathrm{SL}_n(\mathbb{A}_F)$ (and in fact in $R_n(\mathbb{A}_F)$) such that

$$\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)}\varphi(hh_0)\,dh\neq 0,$$

where this integral is absolutely convergent.

Proof. By [Sakellaridis and Venkatesh 2017, Section 18.2], there is $s\in\mathbb{C}$ such that for $\Re(s)$ large enough, the integral $\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)}\varphi(p)\delta_{Q_n}^s(p)\,dp$ is absolutely convergent. Moreover, it has meromorphic continuation, and there is a meromorphic function $r(s)$ with $r(0)=0$ such that $r(s)\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)}\varphi(h)\delta_{Q_n}^s(h)\,dh$ tends to $\int_{\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A}_F)}\varphi(h)\,dh\neq 0$ when $s\to 0$. In particular there is an $s\in\mathbb{R}$ large enough in the realm of absolute convergence that

$$0\neq\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)}\varphi(p)\delta_{Q_n}^s(p)\,dp=\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)}\int_{R_n(F)\backslash R_n(\mathbb{A}_F)}\varphi(pa)\delta_{Q_n}^s(a)\,dp\,da,$$

and hence there is an $a\in R_n(\mathbb{A}_F)$ such that $\delta_{Q_n}^s(a)\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)}\varphi(pa)\,dp\neq 0$ and the result follows. □

Remark 6.2. A result similar to Lemma 6.1 is [Dijols and Prasad 2019, Proposition 8], which is proved via unfolding an Eisenstein series $E(h, s)$ on $\mathrm{SL}_n(\mathbb{A}_F)$ and using that

$$\mathrm{Res}_{s=1}\left(\int_{\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A}_F)} \varphi(h)E(h,s)\,dh\right) = \mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}(\varphi),$$

a trick that [DP 2019] attributes to [Ash et al. 1993]. A straightforward adaptation of the proof of [DP 2019, Proposition 8] can also be used to prove Lemma 6.1. Though our proof here looks much shorter where we appeal to [Sakellaridis and Venkatesh 2017, Section 18.2], the core of [SV 2017, Proposition 18.2.1] is, however, the equality (18.6) and what follows in [loc. cit.], and it relies on the exact same considerations on Eisenstein series as in [DP 2019, Proposition 8]. Hence the proof above is in fact essentially the same as that of [DP 2019, Proposition 8] but the main part of the argument is contained in the statement of [SV 2017, Section 18.2]. Note that [SV 2017, Section 18.2] is done in general for any semisimple group.

We recall that $U_{n,k} = U_n \cdots U_{k+1} < N_n = U_{n,1}$. For $\psi_{n,k}$ a character of $U_{n,k}(\mathbb{A}_E)$ and φ a cusp form on $\mathrm{SL}_n(\mathbb{A}_E)$, we set

$$\varphi_{\psi_{n,k}}(x) = \int_{U_{n,k}(E)\backslash U_{n,k}(\mathbb{A}_E)} \varphi(nx)\psi_{n,k}^{-1}(n)\,dn$$

for $x \in \mathrm{SL}_n(\mathbb{A}_E)$. When $k = 1$ and $\psi := \psi_{n,1}$ is nondegenerate, we write $\varphi_\psi = W_{\varphi,\psi}$. Note that the integrals defining $\varphi_{\psi_{n,k}}$ and $W_{\varphi,\psi}$ make sense for any smooth cuspidal function on $P_n^\circ(\mathbb{A}_F)$ and define smooth functions on $P_n^\circ(\mathbb{A}_E)$ which restrict to $P_{n-1}^\circ(\mathbb{A}_E)$ as smooth cuspidal functions again. This defines an appropriate setting for inductive proofs. The reader familiar with it will recognize what is often called the unfolding method in the following proof (see [Jacquet and Shalika 1990, Section 6] for a famous and difficult instance of this technique).

Proposition 6.3. *Let φ be a smooth cuspidal function on $P_n^\circ(\mathbb{A}_E)$ such that*

$$\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)} \varphi(h)\,dh \neq 0.$$

Then there is a nondegenerate character ψ of $N_n(\mathbb{A}_E)/N_n(E+\mathbb{A}_F)$ such that $W_{\varphi,\psi}$ does not vanish on $\mathrm{SL}_{n-1}(\mathbb{A}_F)$. In particular, thanks to Lemma 6.1, if π is an $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished cuspidal automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$, then it is ψ-generic for a nondegenerate character ψ of $N_n(\mathbb{A}_E)/N_n(E+\mathbb{A}_F)$.

Proof. We induct on n, and observe that the $n = 2$ case is part of the proof of [AP 2006, Theorem 4.2]. Supposing that $n \geq 3$, we have, by hypothesis,

$$\int_{\mathrm{SL}_{n-1}(F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \varphi(uh)\,du\,dh \neq 0.$$

Set

$$\varphi^{U_n,F}(x) = \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \varphi(ux)\,du$$

for $x \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$. By the Poisson formula for $(F\backslash\mathbb{A}_F)^{n-1} \subset (E\backslash\mathbb{A}_E)^{n-1}$, we have

$$\varphi^{U_n,F}(x) = \sum_{\psi_{n,n-1}\in\widehat{U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)}} \varphi_{\psi_{n,n-1}}(x),$$

which is in turn equal to

$$\sum_{\psi_{n,n-1}\in\widehat{U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)}\backslash\{1\}} \varphi_{\psi_{n,n-1}}(x)$$

by cuspidality of φ, where the convergence of the series is absolute. For fixed nondegenerate $\psi^0_{n,n-1}$ of $U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)$, one has

$$\varphi^{U_n,F}(x) = \sum_{\psi_{n,n-1}\in\widehat{U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)}} \varphi_{\psi_{n,n-1}}(x) = \sum_{\gamma\in P^\circ_{n-1}(F)\backslash \mathrm{SL}_{n-1}(F)} \varphi_{\psi^0_{n,n-1}}(\gamma x)$$

because, for $n \geq 3$, the group $\mathrm{SL}_{n-1}(F)$ acts transitively on the set of nontrivial characters of $U_n(\mathbb{A}_E)$ trivial on $U_n(E+\mathbb{A}_F)$, and the stabilizer of $\psi^0_{n,n-1}$ is $P^\circ_{n-1}(F)$. Hence

$$0 \neq \int_{\mathrm{SL}_{n-1}(F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \varphi(uh)\, du\, dh = \int_{P^\circ_{n-1}(F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(h)\, dh,$$

where the right-hand side is absolutely convergent (by Fubini). Now

$$\int_{P^\circ_{n-1}(F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(h)\, dh = \int_{P^\circ_{n-1}(\mathbb{A}_F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{P^\circ_{n-1}(F)\backslash P^\circ_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(hx)\, dh\, dx,$$

and this implies that

$$\int_{P^\circ_{n-1}(F)\backslash P^\circ_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(hh_0)\, dh \neq 0$$

for some $h_0 \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$. The function $\varphi_0 = (\rho(h_0)\varphi)_{\psi^0_{n,n-1}} = \rho(h_0)\varphi_{\psi^0_{n,n-1}}$ restricts to a smooth cuspidal function on $P^\circ_{n-1}(\mathbb{A}_E)$, and we can apply our induction hypothesis to it, to conclude that $W_{\varphi_0,\psi'}$ is nonzero on $\mathrm{SL}_{n-1}(\mathbb{A}_F)$ for some nondegenerate character ψ' of $N_{n-1}(\mathbb{A}_E)$ trivial on $N_{n-1}(\mathbb{A}_F+E)$. Setting

$$\psi := \psi' \otimes \psi^0_{n,n-1} : n'.u \mapsto \psi'(n')\psi^0_{n,n-1}(u),$$

one checks that, by definition,

$$W_{\varphi_0,\psi'}(x) = W_{\rho(h_0)\varphi,\psi}(x) = W_{\varphi,\psi}(xh_0)$$

for $x \in \mathrm{SL}_{n-1}(\mathbb{A}_E)$. The result follows. □

Remark 6.4. As mentioned in Section 1 our strategy in proving Proposition 6.3 is to have an inductive setup to reduce the proof to the case of $n = 2$. In the finite field cuspidal case as well as in the p-adic field tempered case, such an inductive machinery can be set up via Clifford theory [DP 2019, Proposition 1], and this is carried out in [AP 2018, Proposition 4.2 and Remark 4]. A similar approach in the number field case can be carried out as well by making use of the global analogue of [DP 2019, Proposition 1], which is [DP 2019, Proposition 6]. This was brought to our attention by Prasad. In fact, [DP 2019, Proposition 6] is stated more generally and our inductive setup would follow by taking $H = \mathrm{SL}_{n-1}(\mathbb{A}_F)$ and $A = U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)$, in the notation of [DP 2019, Proposition 6].

Remark 6.5. Though not relevant to this paper, we remark here that the inductive strategy in the finite cuspidal and p-adic tempered cases mentioned in Remark 6.4 do not seem to generalize to cover all the generic representations. However, the final result, that distinction is characterized by genericity for a nondegenerate character of $N(E)/N(F)$, is established via other methods. In the p-adic case, this is done in [AP 2018], and this we have further generalized in Theorem 1.1 of the present paper. In the finite field case, the general result is established in [Anandavardhanan and Matringe 2020].

Remark 6.6. We seize the occasion to fill a small gap in the literature, using the ideas of this paper: namely, the unfolding of the Asai L-function. The proofs given in [Flicker 1988, p. 303] and [Zhang 2014, p. 558] are a bit quick. Here we add the details to the proof of [Flicker 1988, 2 Proposition, p. 303]. The transition between the second and third lines of the equality there relies on the following step: for φ a cusp form on $\mathrm{GL}_n(\mathbb{A}_E)$,

$$\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \varphi(n)\,dn = \sum_{\gamma\in N_n(F)\backslash P_n(F)} W_{\varphi,\psi}(\gamma),$$

where both the "integrals" are absolutely convergent and ψ is a nondegenerate character of $N_n(\mathbb{A}_E)$ trivial on $N_n(\mathbb{A}_F+E)$. We use the same notation as in Proposition 6.3, and denote by $\psi^0_{n,n-1}$ the restriction of ψ to $U_n(\mathbb{A}_E)$.

Let us write

$$\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \varphi(n)\,dn = \int_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A}_F)} \varphi^{U_n,F}(n)\,dn.$$

By induction applied to the cusp form $\varphi^{U_n,F}$ on $\mathrm{GL}_{n-1}(\mathbb{A}_E)$, we have

$$\int_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A}_F)} \varphi^{U_n,F}(n)\,dn = \sum_{\gamma'\in N_{n-1}(F)\backslash P_{n-1}(F)} \varphi^{U_n,F}(\gamma').$$

Now replace $\varphi^{U_n,F}(\gamma')$ by $\sum_{\gamma\in P_{n-1}(F)U_n(F)\backslash P_n(F)} \varphi_{\psi^0_{n,n-1}}(\gamma\gamma')$ this time (still by the Poisson formula and because $P_n(F)$ also acts transitively on the set of nontrivial characters of $U_n(\mathbb{A}_E)$ trivial on $U_n(E+\mathbb{A}_F)$, the stabilizer of $\psi^0_{n,n-1}$ being $P_{n-1}(F)U_n(F)$). We get

$$\begin{aligned}\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \varphi(n)\,dn &= \sum_{\gamma'\in N_{n-1}(F)\backslash P_{n-1}(F)}\ \sum_{\gamma\in P_{n-1}(F)U_n(F)\backslash P_n(F)} W_{\varphi_{\psi^0_{n,n-1}},\psi|_{N_{n-1}(\mathbb{A}_E)}}(\gamma\gamma')\,dn \\ &= \sum_{\gamma'\in N_{n-1}(F)\backslash P_{n-1}(F)}\ \sum_{\gamma\in P_{n-1}(F)U_n(F)\backslash P_n(F)} W_{\varphi,\psi}(\gamma\gamma') \\ &= \sum_{\gamma\in N_{n-1}(F)U_n(F)\backslash P_n(F)} W_{\varphi,\psi}(\gamma),\end{aligned}$$

which is what we wanted.

6B. ***The square-integrable case.*** Our aim in this section is to show that if π is distinguished then π has a nonvanishing Fourier coefficient with respect to a character of type r^d of $N_n(\mathbb{A}_E)$ which is trivial on $N_n(E+\mathbb{A}_F)$ (see Proposition 6.11). The key ingredient in achieving this is Proposition 6.8 below.

The following result is [Yamana 2015, Theorem 1.1] slightly reformulated for our purposes.

Theorem 6.7. *Let $n = rd$ with $r \geq 2$ and $d \geq 2$, and let ψ be a nondegenerate unitary character of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$. Fix a character α of $F^\times \backslash \mathbb{A}_F^1$. Then, for $\varphi \in \tilde{\pi} = \mathrm{Sp}(d, \sigma)$, we have*

$$\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\alpha(\det h)\, dh = \int_{N^\circ_{n-1,r-1}(\mathbb{A}_F)\backslash \mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{\mathrm{GL}_{n-r}(F)\backslash \mathrm{GL}_{n-r}(\mathbb{A}_F)^1} (\alpha\varphi)_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)\,\mathrm{diag}(h, 1))\, dm\, dh.$$

Proof. We denote by ω_σ the central character of σ. We extend α as α_0 to $\mathbb{A}_F^\times$. We then extend α_0 to an automorphic character of β of $\mathbb{A}_E^\times$. Then we claim that the following equality holds:

$$\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\alpha_0(\det h)\, dh = \int_{N_{n-1,r-1}(\mathbb{A}_F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \int_{\mathrm{GL}_{n-r}(F)\backslash \mathrm{GL}_{n-r}(\mathbb{A}_F)^1} (\alpha_0\varphi)_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)\,\mathrm{diag}(h, 1))\, dm\, dh.$$

Indeed, if $\alpha_0^r \cdot \omega_\sigma|_{\mathbb{A}_F^\times}$ is trivial, then this follows from the second part of [Yamana 2015, Theorem 1.1] applied to $\beta \otimes \pi$. If $\alpha_0^r \cdot \omega_\sigma|_{\mathbb{A}_F^\times} \not\equiv 1$, then it follows from the first part of [Yamana 2015, Theorem 1.1] applied to $\beta \otimes \pi$, with the extra observation that the right-hand side of the equality also vanishes, thanks to Remark 5.3 and the first part of [Yamana 2015, Theorem 1.1] again if $d \geq 3$, and for central character reasons when $d = 2$. We can now replace the quotient $N_{n-1,r-1}(\mathbb{A}_F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)$ by $N^\circ_{n-1,r-1}(\mathbb{A}_F)\backslash \mathrm{SL}_{n-1}(\mathbb{A}_F)$ and the statement follows. □

From Theorem 6.7, we deduce its $\mathrm{SL}(n)$ version by making use of Proposition 5.6.

Proposition 6.8. *With notation and assumptions ($r, d \geq 2$) as in Theorem 6.7, for $\varphi \in \mathrm{Res}(\tilde{\pi})$ we have*

$$p_n(\varphi) = \int_{N^\circ_{n-1,r-1}(\mathbb{A}_F)\backslash \mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{\mathrm{SL}_{n-r}(F)\backslash \mathrm{SL}_{n-r}(\mathbb{A}_F)} \varphi_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)\,\mathrm{diag}(h, 1))\, dm\, dh.$$

Proof. We relate the $\mathrm{SL}(n, \mathbb{A}_F)$-period p_n to the $(\mathrm{GL}(n, \mathbb{A}_F)^1, \alpha)$-periods via Proposition 5.6. Applying Theorem 6.7 to each summand of the sum over characters α of $F^\times \backslash \mathbb{A}_F^\times$ just selected, we once again apply Proposition 5.6 to the right-hand side sum to conclude the proof. □

Setting

$$(\rho(g)\varphi)_{n-r,\psi} := m \in \mathrm{GL}_{n-r}(\mathbb{A}_E) \mapsto \varphi_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)g),$$

Proposition 6.8 implies the following observation, which we state as a lemma.

Lemma 6.9. *With notation and assumptions ($r, d \geq 2$) as in Theorem 6.7, suppose that $\varphi \in \mathrm{Res}(\tilde{\pi})$ is such that $p_n(\varphi) \neq 0$. Then there is $h \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$ such that*

$$p_{n-r}\big((\rho(\mathrm{diag}(h, 1))\varphi)_{n-r,\psi}\big) \neq 0.$$

We now state the main theorem of this section, which holds without the previous assumptions on r and d, as do all the results that we state from now on.

Theorem 6.10. *Let* $\mathrm{L}(\tilde{\pi})$ *be a distinguished square-integrable* L*-packet of* $\mathrm{SL}_n(\mathbb{A}_E)$ *of type* r^d*. Then the period integral* p_n *does not vanish on* $\pi \in \mathrm{L}(\tilde{\pi})$ *if and only if there exists a degenerate character* $\psi_{1,\dots,d}$ *of type* r^d *of* $N_n(\mathbb{A}_E)$ *trivial on* $N_n(E+\mathbb{A}_F)$ *such that* $p_{\psi_{1,\dots,d}}$ *does not vanish on* π.

The key direction of Theorem 6.10 is Proposition 6.11, which follows from Lemma 6.9 by an inductive argument (see also the proof of Proposition 6.3).

Proposition 6.11. *Let* π *be an irreducible square-integrable automorphic representation of* $\mathrm{SL}_n(\mathbb{A}_E)$ *of type* r^d *which is distinguished with respect to* $\mathrm{SL}_n(\mathbb{A}_F)$*, so that there exists* $\varphi \in \pi$ *such that* $p_n(\varphi) \neq 0$*. Then there exist d nondegenerate characters* ψ_i *of* $N_r(\mathbb{A}_E)$ *trivial on* $N_r(E+\mathbb{A}_F)$ *and* $\varphi' \in \pi$ *such that*

$$p_{\psi_{1,\dots,d}}(\varphi') = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi'(n)\psi_{1,\dots,d}^{-1}(n)\,dn \neq 0.$$

Moreover, φ' *can be chosen to be a right* $\mathrm{SL}_{n-1}(\mathbb{A}_F)$*-translate of* φ.

Proof. The theorem is immediate from Lemma 6.9 by an inductive argument, but we have to treat the case $r=1$ separately. If $r=1$ then π is the trivial character of $\mathrm{SL}_n(\mathbb{A}_E)$ and the claim is obvious. So we suppose that $r \geq 2$. If $d=1$ the result is proved in Proposition 6.3, so we assume $d \geq 2$. Since $\varphi \in \pi$ is such that $p_n(\varphi) \neq 0$, by Lemma 6.9, we get $h \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$ such that $p_{n-r}((\rho(h)\varphi)_{n-r,\psi}) \neq 0$. Therefore, by induction and thanks to Remark 5.3, we get $d-1$ nondegenerate characters ψ_i for $i=2,\dots,d$ of $N_r(\mathbb{A}_E)$, trivial on $N_r(E+\mathbb{A}_F)$, such that

$$p_{\psi_{2,\dots,d}}[\rho(x)(\rho(h)\varphi)_{n-r,\psi}] = \int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} (\rho(h)\varphi)_{n-r,\psi}(nx)\psi_{2,\dots,d}^{-1}(n)\,dn \neq 0,$$

for some $x = \mathrm{diag}(y,1)$ for $y \in \mathrm{SL}_{n-r-1}(\mathbb{A}_F)$. But setting $\psi_1 := \psi$,

$$\begin{aligned}
&\int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} (\rho(h)\varphi)_{n-r,\psi_1}(nx)\psi_{2,\dots,d}^{-1}(n)\,dn \\
&\qquad= \int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} \varphi_{U_{n,r},\psi}(\mathrm{diag}(nx, I_r)h)\psi_{1,\dots,d-1}^{-1}(n)\,dn \\
&\qquad= \int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} \int_{U_{n,r}(E)\backslash U_{n,r}(\mathbb{A}_E)} \varphi(u\,\mathrm{diag}(nx, I_r)h)\psi_{1,\dots,d-1}^{-1}(n)(\mathbf{1}\otimes\psi^{-1})(u)\,dn\,du \\
&\qquad= \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi(n\,\mathrm{diag}(x, I_r)h)\psi_{1,\dots,d}^{-1}(n)\,dn,
\end{aligned}$$

and the result follows. □

To end the proof of Theorem 6.10, it now suffices to prove the following implication, which is part of the proof of [AP 2006, Theorem 4.2], which we repeat.

Lemma 6.12. *Let* $\mathrm{L}(\tilde{\pi})$ *be a distinguished* L*-packet of* $\mathrm{SL}_n(\mathbb{A}_E)$ *of type* r^d*. If* $\pi \in \mathrm{L}(\tilde{\pi})$ *is* ψ*-generic with respect to a degenerate character* ψ *of type* r^d *of* $N_n(\mathbb{A}_E)$ *trivial on* $N_n(E+\mathbb{A}_F)$*, then* p_n *does not vanish on* π.

Proof. By definition there is $\pi' \in \mathrm{L}(\tilde{\pi})$ such that p_n does not vanish on it. By Proposition 6.11, the representation π' is ψ'-generic for a degenerate character ψ' of type r^d of $N_n(\mathbb{A}_E)$ trivial on $N_n(E+\mathbb{A}_F)$.

Now there is $t \in T_n(F)$ such that $\psi = \psi'^t$ where $\psi'^t(n) = \psi'(t^{-1}nt)$. And then the representation π'^t given by $\pi'^t(g) = \pi'(t^{-1}gt)$ appears in $\mathrm{L}(\tilde{\pi})$ and is ψ-generic. We deduce that $\pi = \pi'^t$, by the local uniqueness of degenerate Whittaker models, and the result follows since $t \in \mathrm{GL}_n(F)$. □

Let us now state a simple but very useful consequence of Theorem 6.10, whose proof idea we have already employed in the proof of Lemma 6.12. We formulate this with an application in Section 9 in mind.

Corollary 6.13. *Let π be a square-integrable automorphic $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished representation of $\mathrm{SL}_n(\mathbb{A}_E)$, and let $\mathrm{L}(\tilde{\pi}')$ be a distinguished* L*-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ containing an isomorphic copy of π. Then the period p_n does not vanish on the unique representation in $\mathrm{L}(\tilde{\pi}')$ isomorphic to π.*

Proof. Call π' the isomorphic copy of π in $\mathrm{L}(\tilde{\pi}')$. Thanks to Theorem 6.10, π is ψ-generic for ψ a distinguished degenerate character of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$ of the correct type, and therefore π' has a locally ψ_v-degenerate Whittaker model for every place v of F. By Theorem 6.10 again, the ψ-generic representation π'' in $\mathrm{L}(\tilde{\pi}')$ is also $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished. But thanks to multiplicity one of local degenerate Whittaker models, two locally ψ-generic automorphic representations in the same L-packet are equal, so $\pi' = \pi''$, and we deduce that p_n does not vanish on π'. □

As a corollary to Theorem 6.10, we state and prove one more variation of the above theme. This is applied in Section 8.

Proposition 6.14. *Let π be a canonical realization of an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$. The group $\mathrm{diag}(F^\times, I_{n-1})$ acts transitively on the set of distinguished members of $\mathrm{L}(\pi)$.*

Proof. From Theorem 6.10 and the local uniqueness of degenerate Whittaker models, we easily deduce that $T_n(F)$ acts transitively on the set of distinguished members of $\mathrm{L}(\pi)$, and that the representations in $\mathrm{L}(\pi)$ appear with multiplicity one. However, for $t \in T_n(F)$ and $t' = \mathrm{diag}(\det(t), I_{n-1})$, the representations π^t and $\pi^{t'}$ in $\mathrm{L}(\pi)$ are isomorphic, hence equal by multiplicity one inside $\mathrm{L}(\pi)$. □

6C. *Automorphy and distinction of the highest derivative for* $\mathbf{SL}_n(\mathbb{A}_E)$. As a first application of Theorem 6.10, we end this section with an analogue of [Yamana 2015, Theorem 1.2] in the context of $\mathrm{SL}_n(\mathbb{A}_E)$.

Lemma 6.15. *Let π be a canonical realization of an irreducible square-integrable representation of $\mathrm{SL}_n(\mathbb{A}_E)$ of type r^d, and write*

$$\pi \simeq \bigotimes_v{}' \pi_v.$$

Then, for any $k \in [1, d]$, the representation

$$\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d}) := \bigotimes_v{}' \pi_v^{[r^{d-k}]}(\psi_{d-k+1,\dots,d,v})$$

(see Definitions 3.5 and 4.5) *is automorphic. If σ is a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ such that a canonical realization of π belongs to $\mathrm{L}(\mathrm{Sp}(d, \sigma))$, then $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ is in fact isomorphic to the unique element of $\mathrm{L}(\mathrm{Sp}(k, \sigma))$ with a $\psi_{d-k+1,\dots,d}$-Whittaker model.*

Proof. Let μ be the member of $\mathrm{L}(\mathrm{Sp}(k,\sigma))$ with a $\psi_{d-k+1,\dots,d}$-Whittaker model. Then for all places v, the representation μ_v is the member of $\mathrm{L}(\mathrm{Sp}(k,\sigma_v))$ with a ψ_v-Whittaker model, and therefore it must be $\pi_v^{[r^{d-d}]}(\psi_{d-k+1,\dots,d,v})$ and the result follows. □

Here is our SL-analogue of [Yamana 2015, Theorem 1.2].

Theorem 6.16. *Suppose that $\psi_{1,\dots,d}$ is a character of $N_n(\mathbb{A}_E)$ of type r^d trivial on $N_n(E+\mathbb{A}_F)$. Let π be a canonical realization of an irreducible square-integrable representation of $\mathrm{SL}_n(\mathbb{A}_E)$ of type r^d and fix $k\in[1,d]$. Then $\pi(\psi_{1,\dots,d})$ is $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished if and only if $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ is $\mathrm{SL}_{kr}(\mathbb{A}_F)$-distinguished.*

Proof. The proof is the same as that of Theorem 3.11, using [Yamana 2015, Theorem 1.2] in lieu of [Matringe 2014, Theorem 2.13]. □

7. Characterization of distinguished square-integrable global L-packets

Here we generalize the characterization of distinguished L-packets given in [AP 2006], which turns out to be convenient in the proof of our main applications, namely, the local–global principle inside distinguished L-packets of Section 8 and the study of the behavior of distinction with respect to higher multiplicity in Section 10. The proof is based on the following well-known theorem, which is a consequence of the work of Jacquet and Shalika [1981] on the one hand and Flicker and Zinoviev [1988; 1995] on the other.

Theorem 7.1. *Denote by $\omega_{E/F}$ the quadratic character attached to E/F by global class field theory, and let $\tilde{\pi}$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then $\tilde{\pi}$ is conjugate self-dual, i.e., $\tilde{\pi}^\vee\simeq\tilde{\pi}^\theta$ if and only if π is either distinguished or $\omega_{E/F}$-distinguished (and in fact not both together).*

Proof. Let π_1, π_2 and π_3 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$. By the aforementioned references, the partial Rankin–Selberg $L^S(s,\pi_1,\pi_2)$ has a pole at $s=1$, which is necessarily simple, if and only if $\pi_2\simeq\pi_1^\vee$, whereas the partial Asai L-function $L^S_{\mathrm{As}}(s,\pi_3)$ has a pole (necessarily simple) at $s=1$ if and only if π_3 is $\mathrm{GL}_n(\mathbb{A}_F)$-distinguished. The result now follows from the equality

$$L^S(s,\pi_1,\pi_1^\theta)=L^S_{\mathrm{As}}(s,\pi_1)L^S_{\mathrm{As}}(s,\omega\otimes\pi_1),$$

where ω is any Hecke character of $\mathbb{A}_E^\times$ extending $\omega_{E/F}$. □

First it implies the following lemma.

Lemma 7.2. *Let α be a character of $F^\times\backslash\mathbb{A}_F^1$ and σ be a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ with central character ω. The restriction of ω to $(\mathbb{A}_F)_{>0}$ coincides with the restriction of $|\cdot|_{\mathbb{A}_F}^{ir\lambda}$ for some $\lambda\in\mathbb{R}$, and we extend α to $\mathbb{A}_F^\times$ as the automorphic character $\alpha_{-\lambda}$. Suppose that the period integral*

$$\tilde{p}^1_{r,\alpha^{-1}}:\phi\mapsto\int_{\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F)^1}\phi(h)\alpha(\det(h))\,dh$$

is nonzero on σ. Then α^r and ω^{-1} coincide on $\mathbb{A}_F^1$; i.e., $(\alpha_{-\lambda}\circ\det)^{-1}$ restricts as ω to $\mathbb{A}_F^\times$, and σ is $\alpha_{-\lambda}^{-1}$-distinguished, and thus $\sigma^\vee\simeq(\alpha_{-\lambda}\circ N_{E/F})\otimes\sigma^\theta$.

Proof. The fact that α^r and ω^{-1} must coincide on $\mathbb{A}_F^1$ if $\tilde{p}^1_{r,\alpha^{-1}}$ does not vanish on $\tilde{\pi}$ follows from central character considerations and the fact that $\tilde{p}^1_{r,\alpha^{-1}}$ is α^{-1}-equivariant under $\mathrm{GL}_r(\mathbb{A}_F)^1$. But then for $\phi \in \tilde{\pi}$ the function $\alpha_{-\lambda} \otimes \phi : g \mapsto \alpha_{-\lambda}(\det(g))\phi(g)$ is $\mathbb{A}_F^\times$-invariant and we conclude that $\tilde{p}_{r,\alpha_{-\lambda}^{-1}}$ and $\tilde{p}^1_{r,\alpha^{-1}}$ agree up to a positive constant; in particular, σ is $\alpha_{-\lambda}^{-1}$-distinguished. Therefore, for β an automorphic character extending $\alpha_{-\lambda}$ to $\mathbb{A}_E^\times$, the representation $\beta \otimes \sigma$ is distinguished and we conclude that $\sigma^\vee \simeq (\alpha_{-\lambda} \circ N_{E/F}) \otimes \sigma^\theta$, thanks to Theorem 7.1. □

Now the characterization of square-integrable distinguished L-packets follows.

Proposition 7.3. *Let $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ an irreducible square-integrable representation of $\mathrm{GL}_n(\mathbb{A}_E)$, with σ a unitary cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$. Then $\mathrm{L}(\tilde{\pi})$ is distinguished if and only if there is an automorphic character $\alpha \in \widehat{F^\times\backslash\mathbb{A}_F^\times}$ such that $\tilde{\pi}^\vee \simeq (\alpha \circ N_{E/F}) \otimes \tilde{\pi}^\theta$ or, equivalently, $\sigma^\vee \simeq (\alpha \circ N_{E/F}) \otimes \sigma^\theta$.*

Proof. If $\tilde{\pi}^\vee \simeq (\alpha \circ N_{E/F}) \otimes \tilde{\pi}^\theta$, which is equivalent to $\sigma^\vee \simeq (\alpha \circ N_{E/F}) \otimes \sigma^\theta$, then $\alpha \otimes \sigma$ is conjugate self-dual hence an automorphic twist of σ distinguished by $\mathrm{GL}_r(\mathbb{A}_F)$ thanks to Theorem 7.1. Hence by [Yamana 2015, Theorem 1.2] an automorphic twist of $\tilde{\pi}$ is distinguished by $\mathrm{GL}_n(\mathbb{A}_F)$, and $\mathrm{L}(\tilde{\pi})$ is distinguished thanks to Proposition 5.6 by a straightforward generalization of the second part of the proof of [AP 2006, Proposition 3.2]. Conversely if $\mathrm{L}(\tilde{\pi})$ is distinguished, then by Proposition 5.6 and Lemma 7.2, an automorphic twist of $\tilde{\pi}$ is distinguished and the result follows from Theorem 7.1. □

8. Local global principle for distinguished L-packets when r is odd

This section establishes a local–global principle for distinction inside a square-integrable L-packet of type r^d of $\mathrm{SL}_n(\mathbb{A}_E)$, when r is odd.

Our proof makes use of the setup of [AP 2013, Section 7], where such a result is proved for a cuspidal L-packet of $\mathrm{SL}_2(\mathbb{A}_E)$. The proof there is somewhat intricate and relied crucially on an analysis of the fibers of the Asai lift (see [AP 2013, Remark in Section 7]). Here our arguments are more elementary due to the fact that r is odd. This is consistent with the earlier works [Anandavardhanan 2005; AP 2018].

For the moment, however, r is general. Let π be a canonical realization of an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ and denote by $\tilde{\pi}$ a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that π is realized in $\mathrm{Res}(\tilde{\pi})$.

We borrow the notation of [AP 2013, Section 7]. We consider $\mathbb{A}_E^\times$ as a subgroup of $\mathrm{GL}_n(\mathbb{A}_E)$ via the mapping $x \mapsto \mathrm{diag}(x, I_{n-1})$. This group acts by conjugation on isomorphism classes of an irreducible representation π of $\mathrm{SL}_n(\mathbb{A}_E)$. The orbit of π under this action is the representation-theoretic L-packet of π, say $\mathrm{L}'(\pi)$. Let $G_\pi < \mathbb{A}_E^\times$ be the stabilizer of π. Then (see [Hiraga and Saito 2012, p. 23])

$$G_\pi = \bigcap_{\chi \in X(\tilde{\pi})} \operatorname{Ker} \chi,$$

where

$$X(\tilde{\pi}) = \{\chi \in \widehat{E^\times\backslash\mathbb{A}_E^\times} \mid \tilde{\pi} \otimes \chi \cong \tilde{\pi}\},$$

which is a finite abelian group (see Remark 5.7).

Remark 8.1. Note that $\mathrm{L}(\pi)$ identifies with the automorphic members of $\mathrm{L}'(\pi)$. Indeed $\mathrm{L}(\pi)$ clearly identifies with a subset of $\mathrm{L}'(\pi)$. On the other hand, if π' is an automorphic member of $\mathrm{L}'(\pi)$, then any of its canonical realizations has a degenerate ψ-Whittaker model of type r^d thanks to Proposition 5.4. However $\mathrm{L}(\pi)$ also contains a member π'' with a degenerate ψ-Whittaker model according to Corollary 5.5. We conclude that $\pi' \simeq \pi''$ by local uniqueness of degenerate Whittaker models.

We start with an elementary observation.

Proposition 8.2. *Suppose that $\tilde{\pi}$ is a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ which is Galois conjugate self-dual, i.e., $\tilde{\pi}^\vee \cong \tilde{\pi}^\theta$, and that $\pi \in \mathrm{L}(\tilde{\pi})$. Then G_π is stable under the action of θ.*

Proof. As $\tilde{\pi}$ is Galois conjugate self-dual, it follows that the finite abelian group $X(\tilde{\pi})$ is stable under the Galois action, and thus G_π is Galois stable. Alternatively, note that if π_1 and π_2 are in the same L-packet then $G_{\pi_1} = G_{\pi_2}$. Indeed, $\pi_2 = \pi_1^y$, for some $y \in \mathbb{A}_E^\times$, and by definition, $G_{\pi_2} = y^{-1} G_{\pi_1} y = G_{\pi_1}$ as the groups are abelian. In particular, $G_{\pi^\theta} = G_{\pi^\vee}$ as $\tilde{\pi}^\vee \cong \tilde{\pi}^\theta$. Observe also that $G_{\pi^\vee} = G_\pi$. Thus, if $x \in G_\pi$ then $x^\theta \in G_{\pi^\theta} = G_{\pi^\vee} = G_\pi$. □

Assumption. From now on, we assume that E is split at the Archimedean places, so that the Archimedean analogue of Theorem 3.9 obviously holds.

As in [AP 2013, Section 7], we define the groups

$$H_0 = \mathbb{A}_E^\times, \quad H_1 = \mathbb{A}_F^\times G_\pi, \quad H_2 = E^\times G_\pi, \quad H_3 = F^\times G_\pi,$$

and we observe that:

(1) The set $H_0 \cdot \pi$ is the L-packet of representations of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by π (see, for instance, [Hiraga and Saito 2012, Corollary 2.8]).

(2) The set $H_1 \cdot \pi$ is the set of locally distinguished representations in the L-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by π (by Theorem 3.9 and its Archimedean analogue).

(3) The set $H_2 \cdot \pi$ is the set of automorphic representations in the L-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by π (by Corollary 5.5).

(4) The set $H_3 \cdot \pi$ is the set of globally distinguished representations in the L-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by π (by Proposition 6.14).

We also record the following observation as a lemma.

Lemma 8.3. *Let π as above be of type r^d. Then, for an $x \in \mathbb{A}_E^\times$, we have $x^r \in G_\pi$.*

Proof. If π has a $\psi_{1,\dots,d}$-Whittaker model with respect to the automorphic character $\psi_{1,\dots,d}$, then

$$\pi^{\mathrm{diag}(xI_r, I_{n-r})} \in \mathrm{L}'(\pi).$$

In particular, for finite places v, the local representation $\pi_v^{\mathrm{diag}(x_v I_r, I_{n-r})}$ has a $\psi_{1,\dots,d,v}$-Whittaker model because $\mathrm{diag}(x_v I_r, I_{n-r})$ fixes $\psi_{1,\dots,d,v}$ by conjugation, and hence both π_v and $\pi_v^{\mathrm{diag}(xI_r, I_{n-r})}$ have a $\psi_{1,\dots,d,v}$-Whittaker model inside $\mathrm{L}(\pi_v)$, so they are equal, and the lemma follows. □

Next we state the local–global principle for $(\mathrm{SL}_n(\mathbb{A}_E), \mathrm{SL}_n(\mathbb{A}_F))$ for square-integrable automorphic representations (for r odd).

Theorem 8.4. *Let π be a canonical realization of an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ such that $\mathrm{L}(\pi)$ is distinguished. Assume that r is odd and write $\pi = \bigotimes_v' \pi_v$, but this time for v varying through the places of F (hence here π_v is π_w for w the place in E lying over v if v does not split in E, and $\pi_v = \pi_{w_1} \otimes \pi_{w_2}$ if v splits into (w_1, w_2)). Then π is distinguished with respect to $\mathrm{SL}_n(\mathbb{A}_F)$ if and only if each π_v is $\mathrm{SL}_n(F_v)$-distinguished.*

Proof. One direction is obvious, so we suppose that π is locally distinguished. We can always suppose that $\tilde{\pi}$ is conjugate self-dual by Proposition 7.3.

The group G_π is Galois stable by Proposition 8.2. As in [AP 2013, Theorem 7.1], we need to prove that the group

$$(H_1 \cap H_2)/H_3$$

is trivial. In order to show that $H_1 \cap H_2 \subseteq H_3$, we claim that $H_2 \cap \mathbb{A}_F^\times \subseteq H_3$.

So let $x \in E^\times G_\pi \cap \mathbb{A}_F^\times$. Note that $x^2 = xx^\theta$, as $x \in \mathbb{A}_F^\times$. Since G_π is Galois stable, we see that $x^2 \in F^\times G_\pi = H_3$. Indeed, writing $x = hk$ for $h \in E^\times$ and $k \in G_\pi$, we get

$$x^2 = xx^\theta = hkh^\theta k^\theta = hh^\theta kk^\theta \in F^\times G_\pi.$$

Also $x^r \in G_\pi$ by Lemma 8.3. We have thus shown that both x^2 and x^r are in H_3. It follows that $x \in H_3$, as r is odd. □

Remark 8.5. The simplifying role played by the fact that r is odd in the proof of Theorem 8.4 is quite analogous to its role in the proof of local multiplicity one, when n is odd, for the pair $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ (see [Anandavardhanan 2005, p. 183] or [AP 2018, p. 1703]).

9. Higher multiplicity for SL_n

We now suppose $n \geq 3$ and recall consequences of the works of Blasius [1994], Lapid [1998; 1999], and Hiraga and Saito [2005; 2012]. This section contains no original result.

9A. ***Different notions of multiplicity.*** Let π be a cuspidal automorphic representation π of $\mathrm{SL}_n(\mathbb{A}_E)$. There are several other notions of multiplicity for π, both on the automorphic side and on the Galois parameter side of the putative global Langlands correspondence. We shall need to pass from one to another, and we explain the process in this paragraph. We follow Lapid [1998, p. 293; 1999, p. 162]. First we consider the automorphic side. Thus, let $\tilde{\pi}$ and $\tilde{\pi}'$ be two cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_E)$. We write

(i) $\tilde{\pi} \sim_s \tilde{\pi}'$ if $\tilde{\pi} \simeq \tilde{\pi}' \otimes \eta$ for a Hecke character η of $\mathbb{A}_E^\times$,

(ii) $\tilde{\pi} \sim_{ew} \tilde{\pi}'$ if $\tilde{\pi}_v \simeq \tilde{\pi}'_v \otimes \eta_v$ for a character η_v of $E_v^\times$ at each place v of E,

(iii) $\tilde{\pi} \sim_w \tilde{\pi}'$ if $\tilde{\pi}_v \simeq \tilde{\pi}'_v \otimes \eta_v$ for a character η_v of $E_v^\times$ for almost places v of E.

One denotes by $M(\mathrm{L}(\tilde{\pi}))$ the number of $\sim_s$ equivalence classes in the $\sim_{ew}$ equivalence class of $\tilde{\pi}$, and by $\mathcal{M}(\mathrm{L}(\tilde{\pi}))$ the number of $\sim_s$ equivalence classes in the $\sim_w$ equivalence class of $\tilde{\pi}$. It was expected by Labesse and Langlands [1979] that if π is a cuspidal automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ contained in $\mathrm{L}(\tilde{\pi})$, then its multiplicity $m(\pi)$ inside the cuspidal automorphic spectrum is equal to $M(\mathrm{L}(\tilde{\pi}))$, so that in particular $M(\mathrm{L}(\tilde{\pi}))$ is finite. This was proved for $\mathrm{SL}_2(\mathbb{A}_E)$ in [Labesse and Langlands 1979] and in general for $\mathrm{SL}_n(\mathbb{A}_E)$ by Hiraga and Saito [2012, Theorem 1.6].

On the other hand, the multiplicity $\mathcal{M}(\mathrm{L}(\tilde{\pi}))$, which is conjectured to be finite and bounded by a function of n in [Lapid 1999, Conjecture 1], is certainly at least equal to $M(\mathrm{L}(\tilde{\pi}))$ by definition, and related to a similar multiplicity on the "Galois parameter side". To this end we introduce equivalence relations $\sim_s$ and $\sim_w$ on the set of representations of a group G. Letting ϕ and ϕ' be two morphisms from G to $\mathrm{GL}_n(\mathbb{C})$, we write

(i) $\phi \sim_s \phi'$ if there is $x \in \mathrm{PGL}_n(\mathbb{C})$ such that $\overline{\phi'(g)} = x^{-1}\overline{\phi(g)}x \in \mathrm{PGL}_n(\mathbb{C})$ for all $g \in G$, in which case we say that ϕ and ϕ' are strongly equivalent;

(ii) $\phi \sim_w \phi'$ if for all $g \in G$, there is $x_g \in \mathrm{PGL}_n(\mathbb{C})$ such that $\overline{\phi'(g)} = x_g^{-1}\overline{\phi(g)}x_g \in \mathrm{PGL}_n(\mathbb{C})$, in which case we say that ϕ and ϕ' are weakly equivalent.

We denote by $\mathcal{M}(\phi)$ the number of $\sim_s$ equivalence classes in the $\sim_w$ equivalence class of ϕ. One of the main achievements of [Lapid 1998; 1999] is the following result (see [Lapid 1998, Theorems 6 and 2]).

Theorem 9.1. *Let L be a Galois extension of E with respective Weil groups W_L and W_E such that* $\mathrm{Gal}(L/E)$ *is nilpotent, and let χ be a Hecke character of $\mathbb{A}_E^\times$ such that $\phi = \mathrm{Ind}_{W_L}^{W_E}(\chi)$ is irreducible. Denote by $\tilde{\pi} = \tilde{\pi}(\phi)$ the cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ *associated to* $\mathrm{Ind}_{W_E}^{W_F}(\chi)$ *by* [Arthur and Clozel 1989]. *Then $\mathcal{M}(\phi) = \mathcal{M}(\mathrm{L}(\tilde{\pi}))$.*

Remark 9.2. In the proof of this result Lapid invokes the Chebotarev density theorem to argue that for such representations, the relations $\sim_s$ and $\sim_w$ are compatible on the Galois parameter side and the automorphic side, and shows that if $\tilde{\pi}' \sim_w \tilde{\pi}$ (i.e., $\tilde{\pi}'$ is almost everywhere a twist of $\tilde{\pi}$) for $\tilde{\pi}$ as in the statement of Theorem 9.1, then $\tilde{\pi}'$ is of Galois type, i.e., there exists a Galois representation ϕ', necessarily unique, of W_E with Satake parameters equal to those of $\tilde{\pi}'$ at almost every place of E. We shall use these facts as well in what follows.

Remark 9.3. In particular suppose that $\tilde{\pi}$ and $\mathcal{M}(\phi)$ are as in the statement of Theorem 9.1, and suppose moreover that the weak equivalence class of $\tilde{\pi}$ (its $\sim_w$ class) is the same as its $\sim_{ew}$ class. Then, for any $\pi \in \mathrm{L}(\tilde{\pi})$, we have

$$m(\pi) = M(\mathrm{L}(\tilde{\pi})) = \mathcal{M}(\mathrm{L}(\tilde{\pi})) = \mathcal{M}(\phi).$$

Note that the middle equality can in general be a strict inequality; see for example [Blasius 1994, Proposition 2.5].

9B. ***Examples of higher cuspidal multiplicity due to Blasius.*** In this section we recall the first fundamental construction, due to Blasius [1994], of representations appearing with a multiplicity greater than one in the cuspidal spectrum of $\mathrm{SL}_n(\mathbb{A}_E)$. In view of the more recent results of Lapid and of Hiraga and Saito recalled in Section 9A, we give a slightly more modern treatment of the construction of Blasius, however following its exact same lines. For p a fixed prime number, we denote by H_p the Heisenberg subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ of upper triangular unipotent matrices with order p^3. Blasius considers finite products of Heisenberg groups

$$H_{p_i} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z}/p_i \right\},$$

where for our purpose we restrict a finite number of odd primes p_i possibly equal for $i \neq j$. For each index i, we denote by Z_i the center of H_{p_i}, and by $\mathcal{L}_i$ the Lagrangian subgroup of H_{p_i} given by $a = 0$. We then set $H = \prod_i H_{p_i}$, $\mathcal{L} = \prod_i \mathcal{L}_i$ and $Z = \prod_i Z_i$.

Now let E be our number field. Since H is a product of p-groups it is solvable, and therefore by the well-known result of Shafarevich in inverse Galois theory, there is a Galois extension L/E such that $\mathrm{Gal}(L/E) = H$. Now take for each i a nontrivial character χ_i of Z_i and extend χ_i to a character $\tilde{\chi}_i$ of $\mathcal{L}_i$ by

$$\tilde{\chi}_i \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \chi_i \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now set $\chi = \bigotimes_i \chi_i$ to be the corresponding character of Z, and call it a *regular* character of Z (meaning all the χ_i are nontrivial) and $\tilde{\chi} = \bigotimes_i \tilde{\chi}_i$ to be the corresponding character of $\mathcal{L} = \mathrm{Gal}(L/L_{\mathcal{L}})$ (for $L_{\mathcal{L}}$ an extension of E). This character can be seen as a Hecke character of the Weil group $W_{L_{\mathcal{L}}}$ (which is trivial on W_L). The induced representation $I_\chi = \mathrm{Ind}_{W_{L_{\mathcal{L}}}}^{W_E}(\tilde{\chi})$ is an irreducible representation of H of dimension $n = \prod_i p_i$, and when χ varies, the representations I_χ are nonisomorphic and describe all the irreducible representations of H, their number being equal to

$$m(n) = \prod_i (p_i - 1).$$

We then set $\tilde{\pi}_\chi$ to be the cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ attached to I_χ in [Arthur and Clozel 1989]. By Theorem 9.1 we obtain the following result from Section 1.1 of [Blasius 1994].

Proposition 9.4. *In the situation above, let* $\pi \subset \mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash\mathrm{SL}_n(\mathbb{A}_E))$ *be an irreducible summand of* $\tilde{\pi}_\chi$. *Then* $\mathcal{M}(\mathrm{L}(\tilde{\pi}_\chi)) = m(n)$.

Proof. According to Theorem 9.1, it is sufficient to check that the conjugacy class of $I_\chi(w)$ in $\mathrm{PGL}_n(\mathbb{C})$ is independent of χ for any $w \in W_E$ but that the I_χ's are inequivalent projective representations. This is done in [Blasius 1994, Section 1.1]. □

We are, however, looking for information on $m(\pi)$ rather than $\mathcal{M}(\mathrm{L}(\tilde{\pi}_\chi))$. Therefore we follow Blasius again to put us in a situation where $\mathcal{M}(\mathrm{L}(\tilde{\pi}_\chi)) = M(\mathrm{L}(\tilde{\pi}_\chi))$ in order to apply Remark 9.3. To this end we select L as in the proof of [Blasius 1994, Proposition 2.1], such that at all the places in L lying above p for each p dividing $|H|$, L is unramified.

Then in such a situation, by [Blasius 1994, Proposition 2.1(2)], we deduce that two representations $\tilde{\pi}_\chi$ and $\tilde{\pi}_{\chi'}$, for regular characters χ and χ' of Z, are not only weakly equivalent (which we already know from [Blasius 1994, Section 1.1] and Section 9A), but they are in fact in the same $\sim_{ew}$-class, i.e., they are twists of each other at every place of E. Finally, by Remark 9.2, if $\tilde{\pi}$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ weakly equivalent to $\tilde{\pi}_\chi$, it is of Galois type with Galois parameter, say, ϕ. Because for every $w \in W_E$, the conjugacy class of $I_\chi(w)$ in $\mathrm{GL}_n(\mathbb{C})$ is equal to that of $\phi(w)$, we deduce that I_χ and ϕ have the same kernel, and are thus in fact both irreducible representations of H. This implies that ϕ is itself of the form $I_{\chi'}$ for a regular character χ' of Z; in particular, the $\sim_w$ class of π is equal to its $\sim_{ew}$ class. In view of Remark 9.3, the outcome of this discussion is the following result, which also follows from the proof of [Blasius 1994, Proposition 3.3].

Proposition 9.5. *Let E be a number field and let L be an extension of E such that $\mathrm{Gal}(L/E) \simeq H$ and such that L is unramified at every place of L lying over a prime divisor of the cardinality $n = |H|$. Let χ be a regular character of Z and let $\pi \in \mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ be an irreducible summand of $\tilde{\pi}_\chi$. Then $m(\pi) = m(n)$, and the L-packets containing a copy of π are those of the form $\mathrm{L}(\tilde{\pi}_{\chi'})$ for a regular character χ' of Z, and they are all different.*

Remark 9.6. Such extensions L of E exist in abundance by Shafarevich's theorem in inverse Galois theory.

Remark 9.7. Blasius [1994] had conjectured that two L-packets, say $\mathrm{L}(\tilde{\pi})$ and $\mathrm{L}(\tilde{\pi}')$, would be isomorphic if $\tilde{\pi}$ and $\tilde{\pi}'$ are locally isomorphic at every place up to a character twist [Blasius 1994, Conjecture on p. 239]. This conjecture was later proved by Hiraga and Saito [2005]. Lacking the truth of the conjecture at that point in time, [Blasius 1994] resorted to a trick using complex conjugation. Note that reading out the precise multiplicity $m(\pi)$ is an immediate consequence of this result.

10. Two questions

In this section we attempt to answer two natural and important questions. We thank Raphaël Beuzart-Plessis and Prasad for posing the first of these questions to us in the context of this paper. We then consider one more question, which in the case of SL(2) was answered by an explicit construction in [AP 2006, Theorem 8.2]. The key ingredient in all our constructions is the explicit nature of the examples of cuspidal representations of high multiplicity in [Blasius 1994; Lapid 1998; 1999]. In these examples, we also need to make a crucial use of the main result of this paper (see Theorem 6.10).

10A. *Questions.* We formulate two natural questions, for each of which we provide answers in the later subsections.

Question 10.1. Consider the natural decomposition of $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ into L-packets. Let π_1 and π_2 be two canonical realizations of an irreducible submodule of $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ such that $\pi_1 \simeq \pi_2$ but which belong to two different L-packets $\mathrm{L}(\tilde{\pi}_1) \neq \mathrm{L}(\tilde{\pi}_2)$. If p_n does not vanish on π_1, then is it true that it does not vanish on π_2?

Remark 10.2. We shall see in Section 10C that the answer is no in general. Thus, for $n \geq 3$, there are cuspidal automorphic representations of $\mathrm{SL}_n(\mathbb{A}_E)$ which are locally distinguished, but with at least one canonical realization in the space of smooth cusp forms on which p_n vanishes.

The following question arises immediately after the above remark.

Question 10.3. For $n \geq 3$, are there cuspidal automorphic representations of $\mathrm{SL}_n(\mathbb{A}_E)$ which are locally distinguished at every place of F, but not globally? In fact is it even possible to construct such a representation with no canonical realization belonging to a distinguished L-packet?

We shall see in Section 10D that such representations do exist. Note that though Question 10.1 is not meaningful for $\mathrm{SL}_2(\mathbb{A}_E)$ according to Ramakrishnan's multiplicity one result [Ramakrishnan 2000], the issues addressed by Remark 10.2, as well as Question 10.3, make sense for $n = 2$. In this case both questions are answered in [AP 2006]. In fact it is sufficient to answer Question 10.3 for $n = 2$, and this is done by [AP 2006, Theorem 8.2], the proof of which is quite involved: there are indeed cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ which are locally distinguished at every place of F but not globally. We shall provide easier examples of this type in Section 10C for $n \geq 3$.

10B. ***Distinguished cuspidal representations of higher multiplicity.*** Now we need to construct cuspidal representations π of $\mathrm{SL}_n(\mathbb{A}_E)$ which are $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished with $m(\pi) \geq 2$ for odd n.

Let us explain our general recipe for this, using the examples of Blasius in Section 9B. We take $n \geq 3$ odd and write it as $n = \prod_i p_i$. We set $H = \prod_i H_{p_i}$ as before and take an involution θ of the group H. Associated to this involution is the semidirect product

$$G = H \rtimes \mathbb{Z}/2,$$

where $\mathbb{Z}/2$ acts on H via θ. Now let F be any number field and let L be an extension of F such that $\mathrm{Gal}(L/F) \simeq G$. In fact we choose L in such a way that L/F is unramified at each place of F lying above any p dividing n. Note that all these can be done by Shafarevich's theorem since G is solvable. Let E be the fixed field of H so that

$$\mathrm{Gal}(L/E) \simeq H \quad \text{and} \quad \mathrm{Gal}(E/F) = \langle \theta \rangle.$$

Take an irreducible representation ρ of H. It identifies with I_{χ_ρ} for χ_ρ a regular character of Z and we set $\tilde{\pi}(\rho) = \tilde{\pi}_{\chi_\rho}$ (see Section 9B). In particular, because L/E is unramified at places of E lying above the prime divisors of n, if π belongs to $\mathrm{L}(\tilde{\pi}(\rho))$, we obtain $m(\pi) = m(n)$ thanks to Proposition 9.5. In this situation, we have the following very useful result due to the rigidity of the representation theory of Heisenberg groups, which we will apply in order to produce examples answering Question 10.1.

Proposition 10.4. *In the situation described above, take an irreducible representation ρ of H and denote by c_ρ its central character. The* L*-packet* $\mathrm{L}(\tilde{\pi}(\rho))$ *is distinguished if and only if* $c_\rho(z^\theta) = c_\rho(z^{-1})$ *for all* $z \in Z$.

Proof. By Proposition 7.3, $\mathrm{L}(\tilde{\pi}(\rho))$ is distinguished if and only if $(\tilde{\pi}(\rho)^{\vee})^{\theta} \simeq \mu \otimes \tilde{\pi}(\rho)$ for a Hecke character μ factoring through $N_{E/F}$. This is equivalent to $\tilde{\pi}((\rho^{\vee})^{\theta}) \simeq \mu \otimes \tilde{\pi}(\rho)$. However as the L-packets determined by different irreducible representations are different thanks to Proposition 9.5, we easily deduce that $\mathrm{L}(\tilde{\pi}(\rho))$ is distinguished if and only if ρ is conjugate self-dual, i.e., $\rho^{\vee} \simeq \rho^{\theta}$. The result now follows from the fact that ρ is determined by its central character. □

In view of Corollary 6.13, a consequence of Proposition 10.4 is the following.

Corollary 10.5. *In the situation of Proposition 10.4, let ρ be an irreducible representation of H such that $c_{\rho}^{\theta} = c_{\rho}^{-1}$, and $\pi \in \mathrm{L}(\tilde{\pi}(\rho))$ such that $\mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}$ does not vanish on π. Then the canonical copies of π on which $\mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}$ does not vanish are those contained in the* L*-packets of the form* $\mathrm{L}(\tilde{\pi}(\rho'))$ *with ρ' an irreducible representation of H such that $c_{\rho'}^{\theta} = c_{\rho'}^{-1}$.*

10C. ***Examples for Question 10.1.*** We first give two examples for which we answer Question 10.1. In the first one, all the canonical copies of the considered distinguished representation have a nonvanishing period, whereas in the second example only some of the canonical copies of the considered distinguished representation have a nonvanishing period and some others do not have a nonvanishing period.

For the first set of examples, the group H is as in Section 10B and the involution that we consider on it, for a, b and c in $\prod_i \mathbb{Z}/p_i$, is given by

$$\theta : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & -c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case because the associated involution acts as the inversion on Z, Proposition 10.4 tells us that all L-packets $\mathrm{L}(\tilde{\pi}(\rho))$ are distinguished when ρ varies in the set of irreducible classes of representations of H, and that if one fixes a representation π in one L-packet on which $\mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}$ does not vanish, then it does not vanish on any of the $m(n)$ canonical copies of π.

For the second set of examples, we consider H as above (of odd cardinality n) and $H' = H \times H$ (which is in fact a special type of H) endowed with the switching involution

$$\theta : (x, y) \mapsto (y, x).$$

In this case Proposition 10.4 tells us that the distinguished L-packets of $\mathrm{SL}_{n^2}(\mathbb{A}_E)$ of the form $\mathrm{L}(\tilde{\pi}(\rho'))$ are the $m(n)$ ones such that $\chi_{\rho'}$ is of the form $\chi \otimes \chi^{-1}$ with χ regular, whereas the others are not. Then again by Corollary 10.5 we conclude that if π is a fixed distinguished representation of $\mathrm{SL}_{n^2}(\mathbb{A}_E)$ appearing in one of the $m(n)^2$ many L-packets above, then the period $\mathcal{P}_{\mathrm{SL}_{n^2}(\mathbb{A}_F)}$ does not vanish on the $m(n)$ canonical copies inside the distinguished $m(n)$ many distinguished L-packets, and does vanish on the $m(n)^2 - m(n)$ remaining ones.

10D. ***Examples for Question 10.3.*** Now we give a set of examples answering Question 10.3, using again Proposition 10.4.

For simplicity we take $H = H_p$ for p an odd prime (i.e., $n = p$), and we also take L/F, hence in particular E/F, split at Archimedean places (however we explain in Remark 10.6 how to get rid of this assumption). Let θ be an involution of H such that $z^\theta = z$ for all $z \in Z$. Thus, we may take the trivial involution or the involution of H given by

$$\theta : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -a & c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $z^\theta = z$ for all $z \in Z$, Proposition 10.4 implies that no L-packet of the form $\tilde{\pi}(\rho)$ for ρ an irreducible representation of H is distinguished because, as $|Z|$ is odd, the only character of Z of order ≤ 2 is trivial.

It remains to prove that if we fix ρ as above, and set $\tilde{\pi} = \tilde{\pi}(\rho)$, then $\mathrm{L}(\tilde{\pi})$ contains an automorphic representation π such that π_v is $\mathrm{SL}_p(F_v)$-distinguished for every place v of F. This is equivalent to showing that $\tilde{\pi}_v$ is $(\mathrm{GL}_p(F_v), \gamma_v)$-distinguished for some character γ_v of $F_v^\times$, which is what we do. Recall that by [Blasius 1994, Proposition 2.1],

$$\tilde{\pi}_v^\theta \simeq \tilde{\pi}_v^\vee \otimes \eta_v$$

at each place v for a character η_v of $E_v^\times$.

If a place v of F splits in E as (v_1, v_2) then the above condition implies $\tilde{\pi}_v$ is of the form $(\tau, \tau^\vee \otimes \nu)$, which is distinguished for the character ν of $\mathbb{F}_v^\times$.

Now let v be such that it does not split in E; in particular, v is finite. We set $B_p(E_v)$ the upper triangular Borel subgroup of $\mathrm{GL}_p(E_v)$.

We write as before $\tilde{\pi} = \tilde{\pi}(\rho)$ for ρ an irreducible representation of H. We denote by $\mathcal{L}$ and $\mathcal{L}'$ the first and the second Lagrangian subgroups of H given by $a = 0$ and $b = 0$ respectively (see Section 9B). By the proof of [Blasius 1994, Proposition 2.1] the local Galois group of H_v is an abelian subgroup of H, hence either trivial or equal to Z, $\mathcal{L}$ or $\mathcal{L}'$. We recall that $\rho = \mathrm{Ind}_{\mathcal{L}}^H(\tilde{\chi})$, where

$$\tilde{\chi}\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \chi(c)$$

for χ a nontrivial character of $\mathbb{Z}/p$. We fix μ a nontrivial character $\mathbb{Z}/p$ and set

$$\tilde{\mu}\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \mu(b).$$

Similarly we set

$$\tilde{\chi}'\begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \chi(c) \quad \text{and} \quad \tilde{\mu}'\begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mu(a).$$

Clearly if H_v is trivial or equal to Z, then $\rho|_{H_v}$ is a sum of copies of the same character, and hence $\tilde{\pi}_v$ is of the form

$$\mathrm{Ps}(\alpha, \dots, \alpha) = \mathrm{Ind}_{B_p(E_v)}^{\mathrm{GL}_p(E_v)}(\alpha \otimes \cdots \otimes \alpha),$$

where induction is normalized, hence $\alpha|_{F_v^\times}$-distinguished by [Matringe 2011, Theorem 5.2]. Now we consider the case $H_v = \mathcal{L}$. Then, by Mackey theory,

$$\rho|_{\mathcal{L}} = \tilde{\chi}.\Big(\bigoplus_{k=0}^{p-1} \tilde{\mu}^k\Big).$$

Thus the corresponding principal series is of the form

$$\tilde{\pi}_v = \mathrm{Ps}(\alpha, \alpha\beta, \alpha\beta^{-1}, \dots, \alpha\beta^{(p-1)/2}, \alpha\beta^{-(p-1)/2}). \tag{3}$$

If θ is the trivial involution we trivially have $\beta = \beta^\theta$ so (3) takes the form

$$\tilde{\pi}_v = \alpha \otimes \mathrm{Ps}(1, \beta, \beta^{-\theta}, \dots, \beta^{(p-1)/2}, (\beta^{(p-1)/2})^{-\theta}),$$

which is distinguished by [Matringe 2011, Theorem 5.2].

If θ is the nontrivial involution such that $z^\theta = z$ for $z \in Z$ then note that θ fixes $\tilde{\chi}$ whereas it sends $\tilde{\mu}$ to its inverse. We set $\mu_k = \alpha\beta^k$ for $k = 1, \dots, \frac{1}{2}(p-1)$, so that (3) takes the form

$$\tilde{\pi}_v = \mathrm{Ps}(\alpha, \mu_1, \mu_1^\theta, \dots, \mu_{(p-1)/2}, \mu_{(p-1)/2}^\theta).$$

Now because for $k = 1, \dots, \frac{1}{2}(p-1)$, one has $\alpha^2 = \mu_k\mu_k^\theta$ and hence $\alpha|^2_{F_v^\times} = \mu_k|^2_{F_v^\times}$, but as both characters in this equality have odd order p we deduce that $\alpha|_{F_v^\times} = \mu_k|_{F_v^\times}$. So

$$\tilde{\pi}_v = \alpha \otimes \mathrm{Ps}(1, \alpha^{-1}\mu_1, \alpha^{-1}\mu_1^\theta, \dots, \alpha^{-1}\mu_{(p-1)/2}, \alpha^{-1}\mu_{(p-1)/2}^\theta),$$

and all the characters appearing in the principal series have trivial restriction to $F_v^\times$, and thus we deduce again from [Matringe 2011, Theorem 5.2] that $\tilde{\pi}_v$ is $\alpha|_{F_v^\times}$-distinguished.

Finally, when $H_v = \mathcal{L}'$,

$$\rho|_{\mathcal{L}'} = \tilde{\chi}'.\Big(\bigoplus_{k=0}^{p-1} \tilde{\mu}'^k\Big),$$

and an analogous argument proves that $\tilde{\pi}_v$ is distinguished by a character.

Hence $\mathrm{L}(\tilde{\pi})$ does not contain any distinguished representation but it contains cuspidal representations which are everywhere locally distinguished.

Remark 10.6. In constructing examples in this section, we chose L/F such that the Archimedean places split in order to have E/F split at the Archimedean places. This assumption can be removed because the characterization of a generic distinguished principal series, as in [Matringe 2011, Theorem 5.2], is true also for $(\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{R}))$. Namely, a generic principal series $\mathrm{Ps}(\chi_1, \dots, \chi_n)$ of $\mathrm{GL}_n(\mathbb{C})$ is $\mathrm{GL}_n(\mathbb{R})$-distinguished if and only if there is an involution ϵ of in the symmetric group S_n such that $\chi_{\epsilon(i)} = \chi_i^{-\theta}$ for any $i = 1, \dots, n$, and moreover, $(\chi_i)|_{\mathbb{R}^\times} = 1$ if $\epsilon(i) = i$. The direct implication is a special case of [Kemarsky 2015, Theorem 1.2], whereas the other implication can be obtained as follows. First up to reordering (which is possible as the principal series is generic by assumption) we can suppose that there is $1 \le s \le \lfloor \frac{1}{2}n \rfloor$ such that $\chi_{2i} = \chi_{2i-1}^{-\theta}$ for $i = 1, \dots, s$, and that $(\chi_i)|_{\mathbb{R}^\times} = 1$ for $i = 2s+1, \dots, n$. Now a principal series $\mathrm{Ps}(\chi, \chi^{-\theta})$ of $\mathrm{GL}_2(\mathbb{C})$ is $\mathrm{GL}_2(\mathbb{R})$-distinguished. Indeed by [Carmona and Delorme

1994, Théorème 3], for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough, there is a $\mathrm{GL}_2(\mathbb{R})$-invariant continuous linear form L_s on $\mathrm{Ps}(\chi|\cdot|_{\mathbb{R}}^s, \chi^{-\theta}|\cdot|_{\mathbb{R}}^{-s})$, and a nonzero holomorphic function h on $\mathbb{C}$ such that $h(s)L_s(f_s)$ extends to a holomorphic function on $\mathbb{C}$ for any flat section f_s of $\mathrm{Ps}(\chi|\cdot|_{\mathbb{R}}^s, \chi^{-\theta}|\cdot|_{\mathbb{R}}^{-s})$. Moreover by [Carmona and Delorme 1994, Théorème 3] the meromorphic function $s \mapsto L_s(f_s)$ is nonzero for some choice of f_s, which by density we can suppose to be $U(2, \mathbb{C}/\mathbb{R})$-finite because L_s is continuous for $\mathrm{Re}(s)$ large enough. A standard leading-term argument then allows to regularize L_s at $s = 0$ to define a nonzero $\mathrm{GL}_2(\mathbb{R})$-invariant linear form L on the dense subspace of $U(2, \mathbb{C}/\mathbb{R})$-finite vectors in $\mathrm{Ps}(\chi, \chi^{-\theta})$. Finally one extends L to a necessarily nonzero element of $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{R})}(\mathrm{Ps}(\chi, \chi^{-\theta}), \mathbb{C})$ by [Brylinski and Delorme 1992, Théorème 1]. Once we have this result, the transitivity of parabolic induction together with a closed-orbit-contribution argument allows to define a nonzero $\mathrm{GL}_n(\mathbb{R})$-invariant linear form on $\mathrm{Ps}(\chi_1, \dots, \chi_n)$.

Remark 10.7. It is not hard to extend the examples obtained in this section in the cuspidal case, to the square-integrable case, using the results of this paper.

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Multiplicities of jumping numbers

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We study multiplicities of jumping numbers of multiplier ideals in a smooth variety of arbitrary dimension. We prove that the multiplicity function is a quasipolynomial, hence proving that the Poincaré series is a rational function. We further study when the various components of the quasipolynomial have the highest possible degree and relate it to jumping numbers contributed by Rees valuations. Finally, we study the special case of monomial ideals.

1. Introduction

Let X be a smooth variety over an algebraically closed field of characteristic 0. Associated to each closed subscheme Z of X is a family of ideals, called the *multiplier ideals* (Definition 2.2), that quantify the singularities of Z. The multiplier ideals are indexed by a positive real parameter c (and are denoted by $\mathscr{J}(c \cdot Z)$), and form a decreasing family of ideals of $\mathcal{O}_X$.

As c varies over the positive real numbers, the stalks $\mathscr{J}(c \cdot Z)_x$ of these ideals at any point x change exactly at a discrete set of rational numbers c_i called the *jumping numbers* of Z at x (Definition 2.3). So we get a descending chain of ideals

$$\mathcal{O}_{X,x} \supsetneq \mathscr{J}(c_1 \cdot Z)_x \supsetneq \mathscr{J}(c_2 \cdot Z)_x \supsetneq \cdots. \tag{1-1}$$

The jumping numbers, first defined and studied in [Ein et al. 2004], are interesting invariants of the singularity of Z at x. For example, the smallest jumping number is the well-known log-canonical threshold (lct) of the subscheme. The log-canonical threshold, and more generally any jumping number in the interval [lct, lct + 1), is a root of $b_Z(-s)$, where $b_Z(s)$ is the famous *Bernstein–Sato polynomial* of Z; see [Kollár 1997; Ein et al. 2004; Budur et al. 2006]. Jumping numbers were connected to the Hodge spectrum of a hypersurface by Budur [2003].

Jumping numbers have been studied extensively in the case when the dimension of X is two, starting with [Smith and Thompson 2007]. For example, explicit formulas for the jumping numbers have been calculated in [Kuwata 1999; Galindo et al. 2016; Järvilehto 2011; Naie 2009; Hyry and Järvilehto 2018]. More algorithms for computing jumping numbers can be found in [Tucker 2010; Alberich-Carramiñana et al. 2016; 2017].

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The purpose of this paper is to study jumping numbers in higher dimensions. We do this by studying a natural refinement of the jumping numbers, namely *multiplicities of jumping numbers*, first introduced in [Ein et al. 2004]. More precisely, fix a closed subscheme Z of a smooth variety X, and an irreducible component Z_1 of Z. All the (stalks of) multiplier ideals of Z will have finite colength in the local ring at Z_1. So any jumping number c at Z_1 has a natural *multiplicity*, $m(c)$, that measures the change in the multiplier ideal at c, namely

$$m(c) := \lambda(\mathscr{J}(\mathfrak{a}^{c-\varepsilon})_x / \mathscr{J}(\mathfrak{a}^c)_x) \quad \text{for } 0 < \varepsilon \ll 1. \tag{1-2}$$

Here, $\mathscr{J}(\mathfrak{a}^c)_x$ denotes the stalk of the multiplier ideal of Z at x, the generic point of Z_1 in X, and λ denotes the length as an $\mathcal{O}_{X,x}$ module. If the real number c is not a jumping number, we define its multiplicity to be zero, compatibly with (1-2).

For any jumping number c, we obtain another jumping number by adding any positive integer, so it is natural to consider the sequence of multiplicities of the jumping numbers $c+n$, as n ranges through the natural numbers. Our first main theorem is:

Theorem 3.3. *Let Z be a closed subscheme of X and Z_1 an irreducible component of Z. For any positive real number c and natural number n, let $m(c+n)$ denote the multiplicity of $c+n$ of Z along Z_1. Then the sequence of multiplicities*

$$(m(c+n))_{n\in\mathbb{N}}$$

is a polynomial function of n of degree less than the codimension of Z_1 in X.

Theorem 3.3 allows us to interpret the multiplicity function $m(c)$ as a quasipolynomial in c; see Corollary 3.4.

Theorem 3.3 generalizes the work of Alberich-Carramiñana, Àlvarez Montaner, Dachs-Cadefau and González-Alonso when X is a surface [Alberich-Carramiñana et al. 2017]. They compute the multiplicities explicitly in terms of the intersection matrix of the exceptional divisors in a log resolution. In higher dimension, we instead use the numerical intersection theory of divisors as developed by Kleiman [1966].

The polynomial of Theorem 3.3 encodes interesting information about the divisors relevant for computing jumping numbers as developed in [Smith and Thompson 2007]. For example, its degree tells us precisely whether (possibly some translate of) the jumping number is contributed by a Rees valuation:

Theorem 4.6. *For a closed subscheme Z of X and any positive real number c, consider the multiplicity polynomial $m(c+n)$ along an irreducible component Z_1 of codimension h in X. Then the degree of this polynomial is equal to $h-1$ if and only if $c+h-1$ is a jumping number contributed — in the sense of* [Smith and Thompson 2007] *— by some Rees valuation of Z centered at Z_1.*

Theorem 4.6 motivates the term *Rees coefficient* of the jumping number c for the coefficient of n^{h-1} in the polynomial $m(c+n)$ from Theorem 3.3. The Rees coefficient is not necessarily the "leading coefficient" of the polynomial $m(c+n)$ because it can be zero; see Example 4.9.

We prove several applications of Theorem 4.6. For example, we show that certain jumping numbers of Z can be computed directly from the normalized blowup of Z without finding a full log resolution; see

Corollary 4.11 for a precise statement. As another consequence, we answer a question posed by Joaquín Moraga: we prove that every divisorial valuation over X contributes a jumping number for some divisor; see Theorem 4.15.

The Rees coefficient of a jumping number c is the same as the Rees coefficient of any translate of c by an integer. So we can consider the Rees coefficient of a class of jumping numbers modulo $\mathbb{Z}$; there are finitely many such classes, by discreteness of jumping numbers. We prove in Theorem 4.17 that the sum of the Rees coefficients for the distinct classes of jumping numbers modulo $\mathbb{Z}$ is equal to the *Hilbert–Samuel multiplicity* of Z at Z_1 (scaled by $1/(h-1)!$). So the Rees coefficients can be thought of as refinements of the Hilbert–Samuel multiplicity.

In Section 5, we study the special case of point schemes defined by *monomial ideals*. We prove formulas for the multiplicities and for the Rees coefficients of each jumping number in this case (Theorem 5.3). In particular, we see that the Rees coefficient of every jumping number of a monomial scheme is positive (Corollary 5.5). Thus, Theorem 4.6 implies that, for monomial ideals, all jumping numbers (after translation by some integer) are contributed by Rees valuations.

Finally, in Section 6, we examine a generating function for multiplicities of a jumping number. For an irreducible component Z_1 of a fixed subscheme Z, we define a Poincaré series from the multiplicities $m(c)$, and prove that it is a rational function in a suitable sense; see Theorem 6.1. This generalizes the previous results from [Galindo and Monserrat 2010] and [Alberich-Carramiñana et al. 2017], valid for point schemes in dimension two. Theorem 6.1, which is valid in any dimension, was independently proved by Àlvarez Montaner and Núñez-Betancourt [2022] using different methods.

2. Review of multiplier ideals and intersection theory

Throughout this paper, we work over an algebraically closed field k of characteristic zero.

2A. ***Multiplier ideals and jumping numbers.*** Let X be a smooth variety over k. Fix a coherent ideal sheaf $\mathfrak{a}$ of $\mathcal{O}_X$ and let Z be the subscheme defined by $\mathfrak{a}$. We will now recall the definition of the multiplier ideals $\mathscr{J}(c \cdot Z)$, interchangeably denoted by $\mathscr{J}(\mathfrak{a}^c)$, referring the reader to [Lazarsfeld 2004b] for details.

Definition 2.1. A *log resolution* of the ideal $\mathfrak{a}$ is a projective, birational map $\mu : Y \to X$ such that

(a) Y is smooth,

(b) $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, where F is an effective divisor, and

(c) $F + K_{Y|X}$ has simple normal crossing support, where $K_{Y|X}$ denotes the relative canonical divisor of μ.

Such log resolutions exist by Hironaka's theorem on resolution of singularities.

Definition 2.2. For any positive real number c, the *multiplier ideal* of $\mathfrak{a}$ at c is defined as

$$\mathscr{J}(\mathfrak{a}^c) = \mu_*(\mathcal{O}_Y(K_{Y|X} - \lfloor cF \rfloor)),$$

where $\mu : Y \to X$ is any log resolution of $\mathfrak{a}$ and F is an effective divisor with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. The multiplier ideal is independent of the choice of the log resolution.

As the positive real parameter c increases, the stalks of the multiplier ideal $\mathscr{J}(\mathfrak{a}^c)_x$ at any point x decrease. The numbers c at which the stalks change are called the *jumping numbers* of the ideal $\mathfrak{a}$ (or Z) at x. More precisely:

Definition 2.3 [Ein et al. 2004]. A positive real number c is a *jumping number* of the subscheme Z at a point $x \in X$ if

$$\mathscr{J}(\mathfrak{a}^c)_x \subsetneq \mathscr{J}(\mathfrak{a}^{c-\varepsilon})_x \quad \text{for all } \varepsilon > 0.$$

It follows from the definition of the multiplier ideal in terms of a log resolution that the jumping numbers of any subscheme Z are discrete and rational (see Section 2A1 below).

2A1. *Candidate jumping numbers.* Let $\mu : Y \to X$ be any log resolution of $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for an effective divisor F. Suppose $F = \sum_i a_i D_i$ for prime divisors D_i. Then we call the numbers of the form n/a_i for natural numbers n the *candidate jumping numbers* of $\mathfrak{a}$. These are precisely the values of c where the divisor $K_{Y|X} - \lfloor cF \rfloor$ changes, and hence the set of jumping numbers at any point is certainly contained in the set of candidate jumping numbers. The candidate jumping numbers make it clear that the set of jumping numbers is rational and discrete. Moreover, there is a uniform ε such that $c_{i+1} - c_i > \varepsilon$ for any two consecutive jumping numbers c_i and c_{i+1}.

Note that this definition is slightly different from the candidate jumping numbers as defined in [Smith and Thompson 2007], where they were defined to be the set of rational numbers where the divisor $K_{Y|X} - \lfloor cF \rfloor$ changes and is not effective. The main reason for this deviation is that now for every candidate jumping number c, its fractional part $\{c\}$ is also a candidate jumping number. Hence, every jumping number c can be written as $\{c\} + \lfloor c \rfloor$, i.e., an integer translate of a candidate jumping number that lies in the interval (0, 1].

We now recall two of the main results from the theory of multiplier ideals in the form that we will use them. The reference for these results is [Lazarsfeld 2004b, Chapter 9].

Theorem 2.4 (local vanishing theorem). *Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal, and $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$ and let $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for an effective divisor F. Then for any $c > 0$, the following higher direct images vanish*:

$$R^j \mu_* \mathcal{O}_Y(K_{Y|X} - \lfloor cF \rfloor) = 0 \quad \textit{for } j > 0.$$

Theorem 2.5 (Skoda's theorem). *Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal, $c > 0$ be any real number and $m \geq d := \dim(X)$ be an integer. Then we have*

$$\mathscr{J}(\mathfrak{a}^{c+m}) = \mathfrak{a} \cdot \mathscr{J}(\mathfrak{a}^{c+m-1}).$$

2B. *Numerical intersection theory.* Now we review Kleiman's numerical intersection theory of divisors as developed in [Kleiman 1966] and also explained in [Debarre 2001]. We only state the main facts that we need here.

Let Y be a proper scheme of dimension d over a field L. Let $\mathscr{L}_1, \ldots, \mathscr{L}_r$ be line bundles on Y and $\mathscr{F}$ a coherent sheaf on Y. Then we have the following theorem that Kleiman [1966] attributes to Snapper:

Theorem 2.6. *Consider the function* $f(m_1, \dots, m_r) = \chi(\mathscr{F} \otimes \mathscr{L}_1^{m_1} \otimes \cdots \otimes \mathscr{L}_r^{m_r})$ *for* $m_1, \dots, m_r \in \mathbb{Z}$, *where* χ *denotes the Euler characteristic* (*over* L). *Then there is a polynomial* $P(x_1, \dots, x_r)$ *with coefficients in* $\mathbb{Q}$ *and of total degree* $\leq \dim \operatorname{Supp}(\mathscr{F})$ *such that* $P(m_1, \dots, m_r) = f(m_1, \dots, m_r)$ *for all* $m_1, \dots, m_r \in \mathbb{Z}$.

Definition 2.7 [Kleiman 1966, Section 2, Definition 1]. Suppose that $\dim \operatorname{Supp}(\mathscr{F}) \leq r$, and then define the intersection number $(\mathscr{F}; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r)$ to be the coefficient of the term $x_1 \cdots x_r$ in the polynomial $P(x_1, \dots, x_r)$ as in Theorem 2.6. In particular, if $\dim \operatorname{Supp}(\mathscr{F}) < r$, then $(\mathscr{F}; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r) = 0$, since the degree of $P(x_1, \dots, x_r)$ is strictly less than r by Theorem 2.6.

For any coherent sheaf $\mathscr{F}$ with $\dim(\operatorname{Supp}(\mathscr{F})) \leq r$, this defines an integer-valued multilinear form on $\operatorname{Pic}(Y)^r$. If a line bundle $\mathscr{L}$ is defined by a Cartier divisor D, then we sometimes write D instead of $\mathscr{L}$ in the intersection form. We will use the following properties of the intersection numbers:

Proposition 2.8. *Let* Y *be a proper scheme over a field* L *of dimension* d. *Let* $\mathscr{L}_1, \dots, \mathscr{L}_r$ *be line bundles on* Y, D *an effective Cartier divisor and* $\mathscr{F}$ *a coherent sheaf on* Y. *Suppose* $\dim \operatorname{Supp}(\mathscr{F}) = r$. *Then*:

(1) *If* $\mathscr{F}$ *is a locally free sheaf* (in this case $r = d$), *then*

$$(\mathscr{F}; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_{d-1} \cdot \mathcal{O}_Y(D)) = (\mathscr{F}|_D; \mathscr{L}_1|_D \cdot \ldots \cdot \mathscr{L}_{d-1}|_D).$$

(2) *If* $\mathscr{F}'$ *and* $\mathscr{F}''$ *are two other coherent sheaves and there is an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

then $(\mathscr{F}; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r) = (\mathscr{F}'; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r) + (\mathscr{F}''; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r)$.

(3) *If* $\mathscr{L}$ *is any line bundle on* Y *and* $P(x)$ *is the polynomial such that* $P(n) = \chi(\mathscr{F} \otimes \mathscr{L}^n)$ *for* $n \in \mathbb{Z}$ (which exists by Theorem 2.6 above), *then*

$$(\mathscr{F}; \underbrace{\mathscr{L} \cdot \ldots \cdot \mathscr{L}}_{r \text{ times}}) = \alpha r!,$$

where α *is the coefficient of* x^r *in* $P(x)$.

(4) *Let* $V = \operatorname{Supp}(\mathscr{F})$, *and let* $V_1, \dots, V_s$ *be its irreducible components. Let* $l_i = \operatorname{length}(\mathscr{F} \otimes \mathcal{O}_{V_i})$, *where* $\mathcal{O}_{V_i}$ *is the stalk of* $\mathcal{O}_Y$ *at the generic point of* V_i. *Then, assuming* $r \geq \dim(V)$, *we have*

$$(\mathscr{F}; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_r) = \sum_{i=1}^{s} l_i (\mathscr{L}_1|_{V_i} \cdot \ldots \cdot \mathscr{L}_r|_{V_i}).$$

In particular, if $\mathscr{F}$ *is an invertible sheaf, then* $(\mathscr{F}; \mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_d) = (\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_d)$.

(5) *Let* $\pi : Y' \to Y$ *be a map of finite type of integral projective varieties of dimension* d. *Then* $(\pi^*\mathscr{L}_1 \cdot \ldots \cdot \pi^*\mathscr{L}_d)_{Y'} = \deg(\pi)(\mathscr{L}_1 \cdot \ldots \cdot \mathscr{L}_d)_Y$.

Proof. Parts (1), (2), (4) and (5) are proved in [Kleiman 1966, Section 2] as Propositions 4, 3, 5 and 6, respectively. Note that even though it is assumed that the ground field L is algebraically closed in [Kleiman 1966], the hypothesis is not required in Section 2 there. For instance, see [Debarre 2001].

(3) If $Q(x_1, \dots, x_r)$ is the polynomial such that $Q(m_1, \dots, m_r) = \chi(\mathscr{F} \otimes \mathscr{L}^{m_1} \otimes \cdots \otimes \mathscr{L}^{m_r})$, then $Q(x_1, \dots, x_r) = P(x_1 + \cdots + x_r)$. Since $P(x)$ has a degree at most r, the coefficient of $x_1 \cdots x_r$ in $P(x_1 + \cdots + x_r)$ is $\alpha r!$, which by definition is the required intersection number. □

3. The polynomial nature of multiplicities

In this section, we prove Theorem 3.3 on the polynomial behavior of multiplicities of jumping numbers. We begin with some preliminary definitions.

Notation 3.1. We fix the following notation throughout this section: Let X be a smooth variety of dimension d over k. We fix a closed subscheme Z of X and an irreducible component Z_1 of Z. Let x be the generic point of Z_1 in X and $(A, \mathfrak{m}, L)$ the local ring at x in X. The Krull dimension of A — or, equivalently, the codimension of Z_1 in X — will be denoted by h. Let $\mathfrak{a} \subset \mathcal{O}_X$ denote the ideal of Z, and observe that the stalk of $\mathfrak{a}$ at x is an $\mathfrak{m}$-primary ideal of A. Abusing notation, we often write $\mathfrak{a}$ and $\mathscr{J}(\mathfrak{a}^c)$ for the stalks of the ideal sheaf $\mathfrak{a}$ and the multiplier ideal at x, respectively, and think of these as ideals in A.

Definition 3.2 [Ein et al. 2004]. For any real number $c > 0$ of $\mathfrak{a}$ at x, the *multiplicity of c at x* is defined to be

$$m(c) := \lambda(\mathscr{J}(\mathfrak{a}^{c-\varepsilon})_x / \mathscr{J}(\mathfrak{a}^c)_x), \tag{3-1}$$

for sufficiently small positive ε. Here, λ denotes the length as an A-module.

The multiplicity is well defined, because $\mathscr{J}(\mathfrak{a}^c)$ is either A or an $\mathfrak{m}$-primary ideal for all c since $\mathfrak{a}$ is $\mathfrak{m}$-primary and $\mathscr{J}(\mathfrak{a}^c) \supset \mathscr{J}(\mathfrak{a}^{\lceil c \rceil}) \supset \mathfrak{a}^{\lceil c \rceil}$. Note that even though we define $m(c)$ for any positive real number c by expression (3-1), it will be nonzero if and only if c is a jumping number, by the definition of jumping number.

Our focus in this section is on the function $m(c+n)$, where $c > 0$ is any fixed real number and n varies over the natural numbers $\mathbb{Z}_{\geq 0}$. So we define

$$f_c(n) := m(c+n).$$

The main theorem about $f_c(n)$ is the following:

Theorem 3.3. *Let Z be a closed subscheme of a smooth variety X and Z_1 be an irreducible component of Z. Then for each $c > 0$, the function $f_c(n)$ is a polynomial function of n of degree less than the codimension h of Z_1 in X.*

Recall that a function $g : \mathbb{R}_{>0} \to \mathbb{R}$ is called a *quasipolynomial* if g can be written as

$$g(x) = a_r(x)x^r + \cdots + a_0(x),$$

where each $a_i(x)$ is a periodic function of x with integral period.

Corollary 3.4. *Let Z be a closed subscheme of a smooth variety X. The multiplicity function $m(c)$ of Z along one of its components Z_1 is a quasipolynomial in c.*

Proof. By Theorem 3.3, for each c in the interval $(0, 1]$, the multiplicity $m(c+n)$ can be written as a polynomial $P_c(c+n)$ of degree less than h, where h is the codimension of Z_1 in X. So we can write $m(c)$ as

$$m(x) = a_{h-1}(x)x^{h-1} + \cdots + a_0(x),$$

where $a_i(x)$ is the coefficient of x^i in the polynomial P_c, for c the fractional part $c = x - \lfloor x \rfloor$ of x. Note that each a_i is a periodic function with period 1. □

Before we prove the theorem, it will be convenient to introduce some notation and the notion of *jumping divisor* of a candidate jumping number:

Definition 3.5. Let c be positive real number c and $\mathfrak{a}$ an ideal sheaf on the smooth variety X. Fix a log resolution $\mu : Y \to X$ of $\mathfrak{a}$, and let F be the unique effective exceptional divisor such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. For $\varepsilon > 0$ small enough, the divisor $\lfloor cF \rfloor - \lfloor (c-\varepsilon)F \rfloor$ is reduced and does not change as $\varepsilon \to 0$. The *jumping divisor* E^c of c is

$$E^c = \lfloor cF \rfloor - \lfloor (c-\varepsilon)F \rfloor, \tag{3-2}$$

where ε is a sufficiently small positive number.

The jumping divisor E^c is a reduced nonzero divisor whenever c is a candidate jumping number of $\mathfrak{a}$, by definition of candidate jumping number (see Section 2A1). Otherwise, E^c is zero. Further note that $E^c = E^{c+n}$ for any natural number n, as follows from formula (3-2) by properties of rounding down.

Remark 3.6. The notion of a jumping divisor was introduced as the *maximal jumping divisor* in [Alberich-Carramiñana et al. 2016] along with its *minimal* variant in the two-dimensional case. For simplicity, we drop the adjective "maximal" from the name and do not discuss the minimal jumping divisor here.

Remark 3.7. Let $\mu : Y \to X$ be any log resolution of the closed subscheme Z defined by $\mathfrak{a}$. Let $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Since we do not assume X to be a projective variety (in fact, we will assume it to be affine), the divisor F or its components are not necessarily projective varieties. However, we can realize each prime component of the divisor F as a projective variety over a suitable field, after a suitable base change, as guaranteed by the next lemma.

Lemma 3.8. *With notation as in Remark 3.7*:

(1) *The pushforward $\mu_*\mathcal{O}_Y(-F)$ is the integral closure of the ideal $\mathfrak{a}$.*

(2) *If $E \subset Y$ is an irreducible component of F, then $\mu(E)$ is contained in Z. Conversely, all prime divisors that are mapped inside Z by μ occur as irreducible components of F.*

(3) *If Z is supported at a closed point x, then each component of F is a smooth projective variety over k and is contracted by μ to x.*

(4) *More generally, if $E \subset Y$ is a prime component of F and $p \in Z$ is the center of E on X* (i.e., generic point of $\mu(E)$), *then*

$$E' = E \times_X \textbf{\textit{Spec}}(\mathcal{O}_{X,p})$$

is a smooth projective variety over the residue field $\kappa(p)$ of $p \in X$.

(5) *The dimension of E' is $h-1$, where h is the dimension of $\mathcal{O}_{X,p}$.*

Proof. (1) See [Lazarsfeld 2004b, Remark 9.6.4].

(2) The forward implication follows from part (1) and the converse from the fact that $\mu^{-1}(\mathfrak{a}) = \mathcal{O}_Y(-F)$.

(3) This is a special case of part (4).

(4) Since we are assuming our log resolutions are projective maps, E' is projective over $\boldsymbol{Spec}(\mathcal{O}_{X,p})$. Further, the fiber over p is a projective variety over $\kappa(p)$. Hence, so is every closed subvariety of the fiber. Since E' is contracted to $x \in \boldsymbol{Spec}(\mathcal{O}_{X,p})$ by μ, E' is a projective variety over $\kappa(p)$. Smoothness follows from the fact that E is smooth over k.

(5) Let $\kappa(E) = \kappa(E')$ denote the function field (over k) of the variety E and $\kappa(p)$ denote the residue field of X at p. Then

$$\dim_{\kappa(p)} E' = \text{tr. deg.}_{\kappa(p)}\kappa(E') = \text{tr. deg.}_{k}\kappa(E') - \text{tr. deg.}_{k}\kappa(p) = d - 1 - \dim_k(Z_1) = h - 1,$$

where Z_1 denotes the closure of p in X and tr. deg.$_\kappa$ denotes the transcendence degree of a field over κ. □

Notation 3.9. For any effective divisor $D \subset F$ and a point p in X, we denote by D_p the scheme $D \times_X \boldsymbol{Spec}(\mathcal{O}_{X,p})$ over $\boldsymbol{Spec}(\mathcal{O}_{X,p})$. In particular, E^c_x denotes the base change of the jumping divisor E^c to $\boldsymbol{Spec}(\mathcal{O}_{X,x})$. Note E^c_x can be empty. But in that case, it follows from (3-4) below that the multiplicity of c at x is zero, and hence c can not be a jumping number.

We can now prove Theorem 3.3.

Proof of Theorem 3.3. Since the multiplicities $m(c)$ depend only on the stalk of the multiplier ideal at x, we may assume X is the local affine scheme $\boldsymbol{Spec}(\mathcal{O}_{X,x})$.

Let $\mu : Y \to X$ be any log resolution of the closed subscheme Z defined by the ideal $\mathfrak{a}$. Let $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. We now prove that for any $c > 0$, the multiplicity $m(c)$ can be calculated as an Euler characteristic on Y. To see this, we start with the exact sequence

$$0 \to \mathcal{O}_Y(-E^c) \to \mathcal{O}_Y \to \mathcal{O}_{E^c} \to 0.$$

Let $\mathscr{L}$ be the line bundle on Y corresponding to the divisor $K_{Y|X} + E^c - \lfloor cF \rfloor = K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor$. Tensoring with the invertible sheaf associated to $\mathscr{L}$, we get

$$0 \to \mathcal{O}_Y(\mathscr{L}) \otimes \mathcal{O}_Y(-E^c) \to \mathcal{O}_Y(\mathscr{L}) \to \mathcal{O}_{E^c}(\mathscr{L}|_{E^c}) \to 0. \tag{3-3}$$

Observing that $\mathcal{O}_Y(\mathscr{L}) \otimes \mathcal{O}_Y(-E^c) \cong \mathcal{O}_Y(K_{Y|X} - \lfloor cF \rfloor)$, the local vanishing theorem (Theorem 2.4) tells us that the sequence remains exact after applying μ_*. Thus we have the exact sequence

$$0 \to \mathscr{J}(\mathfrak{a}^c) \to \mathscr{J}(\mathfrak{a}^{c-\varepsilon}) \to H^0(E^c, \mathscr{L}|_{E^c}) \to 0, \tag{3-4}$$

where the first map is just the inclusion of the ideal $\mathscr{J}(\mathfrak{a}^c)$ inside $\mathscr{J}(\mathfrak{a}^{c-\varepsilon})$.

Since X is affine, applying the local vanishing theorem (Theorem 2.4) again, we have that the first two sheaves in the short exact sequence (3-3) have vanishing higher cohomology. Using the long exact sequence for (3-3) of sheaf cohomology groups, we get

$$H^p(E^c, \mathscr{L}|_{E^c}) = 0 \quad \text{for all } \ p > 0. \tag{3-5}$$

Since E^c is reduced, by Lemma 3.8, it is a projective variety over L (the residue field at x). Putting this together with (3-3) and (3-5), we get

$$m(c) = \lambda(\mathscr{J}(\mathfrak{a}^{c-\varepsilon})/\mathscr{J}(\mathfrak{a}^c)) = \lambda(H^0(E^c, \mathscr{L}|_{E^c})) = \dim_L(H^0(E^c, \mathscr{L}|_{E^c})) = \chi_L(\mathscr{L}|_{E^c}), \tag{3-6}$$

where the third equality holds because the $\mathcal{O}_{X,x}$-module $H^0(E^c, \mathscr{L}|_{E^c})$ is already a vector space over L, since E^c is a projective variety over L.

By (3-6), we have

$$f_c(n) = \chi_L\big(\mathcal{O}_Y(K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor)|_{E^c} \otimes \mathcal{O}_Y(-nF)|_{E^c}\big) \tag{3-7}$$

for all $n \geq 0$. Since we are fixing c and varying n, we may apply Theorem 2.6 on the complete scheme E^c over L with $r = 1$, $\mathscr{F} = \mathcal{O}_Y(K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor)|_{E^c}$ and $\mathscr{L}_1 = \mathcal{O}_Y(-F)|_{E^c}$ to get a polynomial $Q_c(x)$ of degree at most the dimension of E^c such that $Q_c(n) = \chi(\mathscr{F} \otimes \mathscr{L}_1^n) = f_c(n)$ for all $n \geq 0$. The proof is now complete by noting that the dimension of E^c is equal to $h-1$ by Lemma 3.8, where h is the codimension of Z_1. □

We can also explicitly determine the coefficient of n^{h-1} in the polynomial $m(c+n)$:

Theorem 3.10. *Fix $c > 0$. With notation as before, the coefficient ρ_c of the term n^{h-1} in the polynomial $f_c(n)$ describing the multiplicities $m(c+n)$ can be computed on the log resolution Y using the formula*

$$\rho_c = \frac{(-1)^{h-1}}{(h-1)!}(\underbrace{F|_{E_x^c} \cdot \ldots \cdot F|_{E_x^c}}_{h-1 \text{ times}}), \tag{3-8}$$

where $E_x^c = E^c \times_X \mathbf{Spec}(\mathcal{O}_{X,x})$ is a projective variety over L. When the subscheme Z is supported only at a closed point x in X, the formula for ρ_c may be written as follows (where d is the dimension of X):

$$\rho_c = \frac{(-1)^{d-1}}{(d-1)!}(\underbrace{F \cdot \ldots \cdot F}_{d-1 \text{ times}} \cdot E^c). \tag{3-9}$$

Proof. Let $\mathscr{F}_c = \mathcal{O}_Y(K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor)$ and $\mathscr{L} = \mathcal{O}_Y(-F)$. Then $f_c(n) = \chi(\mathscr{F}|_{E_x^c} \otimes \mathscr{L}|^n_{E_x^c})$ by (3-7). Using part (3) of Proposition 2.8, we have $\rho_c = (\mathscr{F}_c|_{E_x^c}; \mathscr{L}|_{E_x^c} \cdot \ldots \cdot \mathscr{L}|_{E_x^c})/(h-1)!$. Since $\mathscr{F}_c$ is a line bundle on Y and by definition $\mathscr{L} = \mathcal{O}_Y(-F)$, using part (4) of Proposition 2.8, we have

$$\rho_c = (-1)^{h-1}\frac{(F|_{E_x^c} \cdot \ldots \cdot F|_{E_x^c})}{(h-1)!}.$$

If Z is supported only at a closed point x, then $E_x^c = E^c$ and is projective over k. By part (1) of Proposition 2.8, we can compute ρ_c on any projective closure of Y as

$$\frac{(\mathscr{L}|_{E^c} \cdot \ldots \cdot \mathscr{L}|_{E^c})}{(d-1)!} = \frac{(\mathscr{L} \cdot \ldots \cdot \mathscr{L} \cdot E^c)}{(d-1)!} = (-1)^{d-1}\frac{(F \cdot \ldots \cdot F \cdot E^c)}{(d-1)!}.$$ □

Definition 3.11. Given Z and Z_1, we define the *Rees coefficient* of $c > 0$ (denoted by ρ_c) to be the coefficient of the term n^{h-1} in the polynomial $f_c(n)$. Equivalently, in light of Theorem 3.3, ρ_c is the unique number such that

$$m(c+n) = \rho_c n^{h-1} + o(n^{h-2}) \quad \text{as } n \to \infty,$$

where h is the codimension of Z_1 in X.

The Rees coefficient of a jumping number will be studied in more detail in the next section.

Remark 3.12. When $\dim(X) = 2$ and Z is a point scheme, the polynomial $f_c(n)$ has degree at most 1. In this case, we have

$$m(c+n) = m(c) + \rho_c n, \quad \text{where } \rho_c = -F \cdot E^c.$$

This recovers the *linear* formula for $m(c+n)$ proved in [Alberich-Carramiñana et al. 2017].

Remark 3.13. In higher dimensions and when Z is still a point scheme, the polynomials $f_c(n)$ have other coefficients that can be computed as follows:

If $f_c(n) = m(c) + \alpha_1^c n + \cdots + \alpha_{d-1}^c n^{d-1}$, then

$$\alpha_j^c = \frac{1}{j!} \int_{E^c} c_1(\mathscr{L}|_{E^c})^j \cap \tau_{E^c, j}(\mathscr{F}_c|_{E^c}), \tag{3-10}$$

where $\mathscr{F}_c = \mathcal{O}_Y(K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor)$ and $\mathscr{L} = \mathcal{O}_Y(-F)$, c_1 denotes the first Chern class of a line bundle and $\tau_{E^c, j}$ denotes the degree j component of the Todd class of a sheaf. This formula comes from (3-7) and the Riemann–Roch theorem for singular varieties. See [Fulton 1998, Example 18.3.6] for the details.

The interpretation of the multiplicity $m(c)$ as the dimension of global sections as in (3-6) also implies that the function under consideration, $f_c(n)$, is a nondecreasing function, which we now note:

Proposition 3.14. *Let Z be a closed subscheme of a smooth variety X and Z_1 an irreducible component of Z with generic point x. Fix any $c > 0$. Then $m(c+1) \geq m(c)$.*

Proof. As before, we assume X is the local scheme $\mathbf{Spec}(\mathcal{O}_{X,x})$. It is sufficient to deal with the case when c is a jumping number since otherwise $m(c) = 0$ and $m(c+1) \geq 0$ by definition. So we assume that c is a jumping number, in which case $m(c) > 0$. By (3-6), we need to show that

$$\dim_L H^0\big(E^c, \mathcal{O}_Y(K_{Y|X} - \lfloor (c+1-\varepsilon)F \rfloor)|_{E^c}\big) \geq \dim_L H^0\big(E^c, \mathcal{O}_Y(K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor)|_{E^c}\big).$$

Denoting by $\mathscr{F}$ and $\mathscr{L}$ the invertible sheaves $\mathcal{O}_Y(K_{Y|X} - \lfloor (c-\varepsilon)F \rfloor)$ and $\mathcal{O}_Y(-F)$ on Y, respectively, we need to prove that

$$\dim_L H^0(E^c, \mathscr{F}|_{E^c} \otimes \mathscr{L}|_{E^c}) \geq \dim_L H^0(E^c, \mathscr{F}|_{E^c}).$$

But, since $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, $\mathscr{L}$ is generated by its global sections (by the generators of $\mathfrak{a}$) on Y, the same is true for $\mathscr{L}|_{E^c}$ on E^c. In particular, $\mathscr{L}|_{E^c}$ has a nonzero global section. Hence, choosing a nonzero global section of $\mathscr{L}|_{E^c}$, we have an injective map

$$\mathcal{O}_{E^c} \hookrightarrow \mathcal{O}_{E^c}(\mathscr{L}).$$

Tensoring with $\mathscr{F}|_{E^c}$ preserves injectivity since $\mathscr{F}$ was invertible. Hence, we have

$$\mathcal{O}_{E^c}(\mathscr{F}) \hookrightarrow \mathcal{O}_{E^c}(\mathscr{L} \otimes \mathscr{F}).$$

Since taking global sections also preserves injectivity, our claim is proved. □

Remark 3.15. This proposition implies that if c is a jumping number of Z at Z_1, then so is $c+1$. This also follows directly from the definition of multiplier ideals and jumping numbers. Though this proposition says that if we fix a "c" then $m(c+n)$ is nondecreasing with n, this does not imply that the multiplicity $m(c)$ increases with c (i.e., as c varies over the jumping numbers). In fact, this is not true. As we will see in the next section, $m(c+n)$ increases at different rates for different jumping numbers c.

Theorem 3.3 can be thought of as a finiteness statement for multiplier ideals of a subscheme at its irreducible components. For example, we can use Theorem 3.3 to deduce a well-known periodicity result for jumping numbers:

Proposition 3.16. *Let c be a positive real number, and let Z_1 be an h-dimensional component of a subscheme Z of a smooth variety X. If $c > h-1$, then c is a jumping number of Z at Z_1 if and only if $c+1$ is a jumping number of Z at Z_1.*

Proof. By Proposition 3.14, if c is a jumping number, then $c+1$ is also a jumping number (without any assumptions on c). So it is enough to prove that if $c+1$ is a jumping number of Z at Z_1 then so is c. If c is not a jumping number, then by Proposition 3.14, $m(c-n) = 0$ for all n such that $0 \leq n \leq h-1$. By Theorem 3.3, $m(c+n)$ is a polynomial of degree at most $h-1$. Since $m(c+n)$ has h zeroes, this implies that $m(c+n)$ is identically zero. So $c+1$ can not be a jumping number either. □

4. Properties of the Rees coefficient

In this section, we investigate some properties of the *Rees coefficient* of a jumping number (Definition 3.11). In Section 4A, we prove some criteria for the positivity of the Rees coefficient. Next, we relate the Rees coefficients to the Hilbert–Samuel multiplicity (Section 4B). We begin with some preliminary observations about the Rees coefficients.

Recall that the Rees coefficient ρ_c of c is defined to be the coefficient of n^{h-1} in the polynomial $m(c+n)$, where $m(c)$ denotes the multiplicity of c of a closed subscheme Z along one of its irreducible components Z_1 and h denotes the codimension of Z_1 in X.

Proposition 4.1. *Let X be a smooth variety and fix a closed subscheme Z of X and an irreducible component Z_1 of Z. Fix a positive real number c. Then the Rees coefficient satisfies the following*:

(1) $\rho_c \geq 0$ *for all* $c > 0$.

(2) $\rho_c \in (1/(h-1)!)\mathbb{N}_{\geq 0}$.

(3) $\rho_c = \rho_{c+n}$ *for any positive integer* n.

(4) $\rho_c = 0$ *if c is not a candidate jumping number of* $\mathfrak{a}$ (see Section 2A1).

(5) ρ_c *is positive implies that $c+n$ is a jumping number at Z_1 for some n.*

Proof. By Theorem 3.3, $m(c+n)$ is a polynomial in n of degree at most $h-1$. Since ρ_c is the coefficient of n^{h-1}, parts (1), (2) and (3) follow from the fact that $m(c+n)$ is a nonnegative integer for all n. Parts (4) and (5) follow from the fact that $m(c+n)$ is positive exactly when $c+n$ is a jumping number. In particular, if $m(c+n) > 0$ for some n, then c (equivalently $c+n$) must be a candidate jumping number. □

Since ρ_c is the same as ρ_{c+n} for any natural number n, we think of ρ_c as a $\mathbb{Q}$-valued function on $\mathbb{Q}/\mathbb{Z}$. This function is an invariant of the subscheme Z along a component Z_1 and describes interesting properties of jumping numbers at Z_1 (or rather their classes in $\mathbb{Q}/\mathbb{Z}$). We turn to explaining these properties.

4A. ***Criteria for positivity.*** We first discuss the issue of when the Rees coefficient ρ_c is positive. More precisely, for any closed subscheme Z and an irreducible component Z_1, we ask for which classes in $\mathbb{Q}/\mathbb{Z}$ of jumping numbers c is the Rees coefficient ρ_c positive? Equivalently, for which jumping numbers c at Z_1 does the multiplicity $m(c+n)$ along Z_1 grow like n^{h-1} as $n \to \infty$ (where $h = \operatorname{codim}_X Z_1$)? In this subsection, we prove several criteria for when the Rees coefficient is positive. Theorem 4.2 is a summary of the results of this section:

Theorem 4.2. *Let Z be a closed subscheme of X* (*with ideal* $\mathfrak{a}$) *and Z_1 be an irreducible component of Z with generic point x in X. Let h be the codimension of Z_1 in X and fix any $c > 0$. Then the following are equivalent*:

(i) *The Rees coefficient ρ_c is positive.*

(ii) *The polynomial $m(c+n)$ has degree $h-1$ in n.*

(iii) *The jumping divisor E^c* (Definition 3.5) *contains a* ***Rees valuation*** *of* $\mathfrak{a}$ *centered at x on some* (equivalently any) *log resolution of* $\mathfrak{a}$.

(iv) $c+h-1$ *is a jumping number* ***contributed by*** *some* ***Rees valuation*** *of* $\mathfrak{a}$ *centered at x.*

(v) *The class of c in $\mathbb{Q}/\mathbb{Z}$ is* ***contributed by*** *some* ***Rees valuation*** *of* $\mathfrak{a}$ *centered at x.*

We explain the statement of the theorem and the various terms appearing in it before giving the proof.

Jumping numbers contributed by divisors. The notion of a jumping number *contributed by a divisor* on a log resolution was defined by Smith and Thompson [2007]. This notion was studied further in [Tucker 2010] and [Baumers et al. 2018]. This definition naturally generalizes to classes of jumping numbers in $\mathbb{Q}/\mathbb{Z}$. In this terminology, Theorem 4.2 characterizes the classes *contributed by* the Rees valuations on any log resolution as exactly the classes whose Rees coefficient is positive (hence the choice of name). We now review the definition of contribution by a divisor here. Recall that E^c_x is the base change of the jumping divisor E^c (Definition 3.5), defined as $E^c_x = (\lfloor cF \rfloor - \lfloor (c-\varepsilon)F \rfloor) \times_X \boldsymbol{Spec}(\mathcal{O}_{X,x})$.

Definition 4.3. Let $\mu : Y \to X$ be a log resolution of an ideal $\mathfrak{a} \subset \mathcal{O}_X$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, and let $E \subset F$ be a prime divisor. Then, for any point x of X, a jumping number c of $\mathfrak{a}$ at x is said to be *contributed by E* if

(1) E_x is nonempty,

(2) $E_x \subset E_x^c$ and

(3) $\mathscr{J}(\mathfrak{a}^c)_x = \mu_*(\mathcal{O}_Y(K_{Y|X} - \lfloor cF \rfloor))_x \subsetneq \mu_*(\mathcal{O}_Y(K_{Y|X} - \lfloor cF \rfloor + E))_x$.

We say that a class $[c] \in \mathbb{Q}/\mathbb{Z}$ is *contributed by* E if $c+n$ is contributed by E for some natural number n.

The condition that E_x is nonempty is saying that E is centered on $\boldsymbol{Spec}(\mathcal{O}_{X,x})$ and $E_x \subset E_x^c$ is just saying that c is a candidate jumping number for E. We also note that contribution by a prime divisor depends only on the valuation defined by E and not on the chosen log resolution. So we say that a jumping number is "contributed by a valuation ν" if it is contributed by a divisor on a log resolution whose associated valuation is ν.

Rees valuations. Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal. If $\nu : \widetilde{X} \to X$ denotes the normalization of the blowup of X along $\mathfrak{a}$, then $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$, for an effective divisor E on $\widetilde{X}$. Then the valuations corresponding to the prime components of E are called the *Rees valuations* of $\mathfrak{a}$.

Let $(A, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ be an ideal. The set of Rees valuations of $\mathfrak{a}$ is a minimal set of discrete, rank-one valuations $\nu_1, \dots, \nu_r$ of A satisfying the property that:

$$\text{For all } n \in \mathbb{N}, \text{ the integral closure } \overline{\mathfrak{a}^n} \text{ equals } \bigcap_{i=1}^{r} (\mathfrak{a}^n V_i \cap A), \text{ where } V_i \text{ is the valuation ring of } \nu_i.$$

In other words, the Rees valuations are a minimal set of valuations that determine the integral closure of all the powers of the ideal $\mathfrak{a}$. We refer to [Huneke and Swanson 2006, Chapter 10] for details about Rees valuations. This is also explained in [Lazarsfeld 2004b, Section 9.6].

Given the normalized blowup $\widetilde{X}$, or more generally any normal variety Y mapping properly and birationally to X, we abuse terminology by saying "E is a Rees valuation of $\mathfrak{a}$" to refer to a prime divisor E on Y corresponding to a Rees valuation of $\mathfrak{a}$.

We now turn to the proof of Theorem 4.2.

Proof of Theorem 4.2. We first note the following easy implications in the theorem: The equivalence between (i) and (ii) is clear from Definition 3.11 of the Rees coefficient. That (iv) implies (iii) follows from Definition 4.3 of jumping numbers contributed by divisors (so does (v) implies (iii)). We have that (iv) implies (v) immediately from Definition 4.3 of classes of jumping numbers contributed by a divisor. So to complete the proof, we will prove that (i) is equivalent to (iii) and (iii) implies (iv). These will be proved as the two main theorems of this subsection.

We first prove part (i) is equivalent to part (iii) of Theorem 4.2:

Theorem 4.4. *Let Z be a closed subscheme of X and Z_1 an irreducible component of Z with generic point x in X. Fix any $c > 0$. Then, the Rees coefficient ρ_c at x is positive if and only if on some* (equivalently every) *log resolution, the jumping divisor E^c* (Definition 3.5) *contains a Rees valuation of Z centered at x.*

Let $\mu : Y \to X$ be a log resolution of the ideal $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. When X is a surface and $\mathfrak{a}$ is supported at a closed point, it follows from the results of Lipman [1969] that, for any irreducible exceptional curve $E \subset Y$, $(-F \cdot E)$ is nonnegative and is positive exactly when E corresponds to a Rees valuation of $\mathfrak{a}$. It follows from the formula (3-8) that the Rees coefficient ρ_c is positive if and only if a

Rees valuation appears in the jumping divisor E^c. Theorem 4.4 can be thought of as a generalization of Lipman's result in this context to higher dimensions. To prove this theorem, we need a key lemma which is a generalization to higher dimensions of a version of [Lipman 1969, Lemma 21.2].

Lemma 4.5. *If $E \subset Y$ is an irreducible component of F (and x is the generic point of Z_1, an irreducible component of Z), then*

$$(\underbrace{-F|_{E_x} \cdot \dots \cdot -F|_{E_x}}_{h-1 \text{ times}}) \geq 0,$$

where $E_x = E \times_X \boldsymbol{Spec}(\mathcal{O}_{X,x})$ and h is the codimension of Z_1 in X. Moreover, the intersection number $(-F|_{E_x} \cdot \dots \cdot -F|_{E_x})$ is positive if and only if the divisor E is a Rees valuation of $\mathfrak{a}$ centered at x.

Proof. Since the relevant intersection numbers depend only on the local ring at x in X, we assume X is the local scheme $\boldsymbol{Spec}(\mathcal{O}_{X,x})$. Recall that by Lemma 3.8, E_x (henceforth denoted just by E) is a smooth projective variety over L (the residue field of X at x) of dimension $h-1$. So the intersection numbers make sense.

The setup of the proof is as follows: Let $\tilde{\mu} : \widetilde{X} \to X$ be the normalized blowup of $\mathfrak{a}$, the ideal of Z in X. Since Y is normal (it is smooth) and $\mathfrak{a} \cdot \mathcal{O}_Y$ is locally principal, the universal property of normalization and of blowing-up provides a factorization

$$\mu : Y \xrightarrow{\pi} \widetilde{X} \xrightarrow{\tilde{\mu}} X,$$

i.e., $\mu = \tilde{\mu} \circ \pi$. Let $\mathscr{L}$ denote the invertible sheaf $\mathcal{O}_Y(-F) = \mathfrak{a} \cdot \mathcal{O}_Y$ on Y and $\widetilde{\mathscr{L}}$ be the invertible sheaf $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}}$ on $\widetilde{X}$, and we have $\mathscr{L} = \pi^*(\widetilde{\mathscr{L}})$. Then, since $\widetilde{\mathscr{L}}$ is relatively ample for $\tilde{\mu}$, if $D \subset \widetilde{X}$ is any effective divisor, then $\widetilde{\mathscr{L}}|_D$ is ample.

Let $B = \{G_j\}$ denote the finite set of prime divisors on $\widetilde{X}$ in the support of $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}}$ and $C = \{E_i\}$ denote the finite set of irreducible components of F on Y. Then, a divisor E_i from the set C corresponds to a Rees valuation of $\mathfrak{a}$ if and only if π maps E_i birationally onto some divisor G_j from B. And if E_i does not correspond to a Rees valuation, then π maps E_i onto a proper subset of G_j from some j. In any case, for each E_i in B, we have a G_j in C such that π restricts to a map

$$\begin{array}{ccc} E_i & \xrightarrow{\pi|_{E_i}} & G_j \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\pi} & \widetilde{X} \end{array}$$

with π birational exactly when E_i corresponds to a Rees valuation of $\mathfrak{a}$. Now, we want to understand the intersection number $(\mathscr{L}|_{E_i} \cdot \dots \cdot \mathscr{L}|_{E_i})$. By Proposition 2.8(5), we have

$$(\underbrace{\mathscr{L}|_{E_i} \cdot \dots \cdot \mathscr{L}|_{E_i}}_{h-1 \text{ times}}) = \deg(\pi|_{E_i}) \times (\underbrace{\widetilde{\mathscr{L}}|_{G_j} \cdot \dots \cdot \widetilde{\mathscr{L}}|_{G_j}}_{h-1 \text{ times}}). \tag{4-1}$$

If E_i is not centered at x then E_i does not appear on Y since we are over $\boldsymbol{Spec}(\mathcal{O}_{X,x})$, and if E_i does not correspond to a Rees valuation then $\deg(\pi|_{E_i})$ is zero. Therefore, in both cases, the intersection number $(\mathscr{L}|_{E_i}\cdot\ldots\cdot\mathscr{L}|_{E_i})$ is zero. The only case when $(\mathscr{L}|_{E_i}\cdot\ldots\cdot\mathscr{L}|_{E_i})$ can be nonzero is when E_i is a Rees valuation centered at x. But in that case, $\pi|_{E_i}$ is birational onto G_j and $\widetilde{\mathscr{L}}$ is ample on G_j. Hence, by (4-1), we get that $(\mathscr{L}|_{E_i}\cdot\ldots\cdot\mathscr{L}|_{E_i})$ is positive. This completes the proof of the lemma. □

Proof of Theorem 4.4. The theorem now follows from Lemma 4.5 and Theorem 3.10:

$$\rho_c = (-1)^{h-1}\frac{(F|_{E^c_x}\cdot\ldots\cdot F|_{E^c_x})}{(h-1)!} = \sum_{E\subset E^c}(-1)^{h-1}\frac{(F|_{E_x}\cdot\ldots\cdot F|_{E_x})}{(h-1)!},$$

where the second equality holds because of Proposition 2.8(4). Lemma 4.5 tells us that each intersection number on the right-hand side is nonnegative and is positive exactly when E_x corresponds to a Rees valuation of $\mathfrak{a}$ centered at x. □

Finally, we conclude the proof of Theorem 4.2, by proving that part (iii) implies (iv).

Theorem 4.6. *Let Z be a closed subscheme of X (with ideal $\mathfrak{a}$) and Z_1 an irreducible component of Z with generic point x in X. Fix a positive real number c. Suppose on some log resolution $\mu : Y \to X$ of $\mathfrak{a}$, the jumping divisor E^c* (Definition 3.5) *contains a Rees valuation E of $\mathfrak{a}$ centered at x. Then $c+h-1$ is a jumping number at x contributed by E (where $h = \operatorname{codim}_X Z_1$).*

The proof of this theorem has two main steps:

- First, we reinterpret the notion of a jumping number *contributed by a divisor D* in terms of nonvanishing of the space of global sections of a certain sheaf on the divisor D.
- Next, we prove the nonvanishing of the required space of global sections by pushing the sheaf forward to the normalized blowup of $\mathfrak{a}$ and using a theorem of Mumford.

Both steps crucially rely on vanishing theorems, namely the Kawamata–Viehweg vanishing theorem and the local vanishing theorem (Theorem 2.4). We now recall the main theorems we need for the proof in the form that we will use them.

Theorem 4.7 (Kawamata–Viehweg vanishing theorem [Lazarsfeld 2004b, Theorem 9.1.18]). *Let X be a smooth projective variety of dimension n and let N be an integral divisor on X. Assume that*

$$N \equiv_{\text{num}} B + \Delta,$$

where B is a big and nef $\mathbb{Q}$-divisor and $\Delta = \sum a_i \Delta_i$ is a $\mathbb{Q}$-divisor with simple normal crossings support and such that $0 \le a_i < 1$ for each i. Then

$$H^i(X, \mathscr{O}_X(K_X + N)) = 0 \quad \textit{for all } i > 0.$$

Theorem 4.8 (Mumford's theorem [Lazarsfeld 2004a, Theorem 1.8.5]). *Let X be a projective variety and $\mathcal{L}$ a globally generated ample line bundle on X. Suppose a coherent sheaf $\mathcal{F}$ on X is m-regular with respect to $\mathcal{L}$, i.e.,*

$$H^i(X, \mathcal{F}\otimes\mathcal{L}^{m-i}) = 0 \quad \textit{for } i > 0\,.$$

Then, $\mathcal{F}\otimes\mathcal{L}^m$ is generated by its global sections.

Proof of Theorem 4.6. Contribution by divisors at x depends only on the local ring at x (see Definition 4.3).

So we may replace X by the affine local scheme $\boldsymbol{Spec}(\mathcal{O}_{X,x})$. Further, since by assumption $E \subset E^c$ and E is centered at x, E^c_x contains E_x (henceforth denoted by E), so E^c_x is nonempty and c is a candidate jumping number (defined in Section 2A1).

Step 1: Let $q > 0$ be a candidate jumping number and G be a prime divisor on Y with $G \subset E^q$. Then we claim that q is a jumping number contributed by G if and only if $H^0(G, \mathcal{O}_G(K_G - \lfloor qF \rfloor|_G)) \neq 0$. To see this, consider the following exact sequence on Y:

$$0 \to \mathcal{O}_Y(-G) \to \mathcal{O}_Y \to \mathcal{O}_G \to 0.$$

Tensoring with $\mathcal{O}_Y(K_{Y|X} - \lfloor qF \rfloor + G)$, we get

$$0 \to \mathcal{O}_Y(K_{Y|X} - \lfloor qF \rfloor) \to \mathcal{O}_Y(K_{Y|X} - \lfloor qF \rfloor + G) \to \mathcal{O}_G((K_{Y|X} - \lfloor qF \rfloor + G)|_G) \to 0.$$

Applying μ_*, using the local vanishing theorem (Theorem 2.4) and the fact that X is affine by assumption, we see that

$$\mu_*(\mathcal{O}_Y(K_{Y|X} - \lfloor qF \rfloor + G))/\mu_*(\mathcal{O}_Y(K_{Y|X} - \lfloor qF \rfloor)) \cong H^0(G, \mathcal{O}_G(K_{Y|X} - \lfloor qF \rfloor + G)|_G). \quad (4\text{-}2)$$

Further, since G is a smooth divisor, we have

$$\mathcal{O}_G(K_{Y|X} - \lfloor qF \rfloor + G)|_G \cong \mathcal{O}_G(K_G - \mu^* K_X|_G - \lfloor qF \rfloor|_G), \quad (4\text{-}3)$$

where we have used the adjunction formula for $G \subset Y$. Using (4-2) and (4-3), we obtain that q is a jumping number at x contributed by G (Definition 4.3) if and only if

$$H^0(G, \mathcal{O}_G(K_G - \lfloor qF \rfloor|_G)) \neq 0. \quad (4\text{-}4)$$

Here we are using the following observation: since G maps to x (see Lemma 3.8), the canonical bundle ω_X pulls back to the trivial bundle on G. Note also that since G is a smooth projective variety over L (the residue field of x), K_G denotes the canonical divisor of G over L. This is justified by the following argument:

Let φ denote the structure map $G \to \boldsymbol{Spec}(L)$. We have the following exact sequence coming from the first exact sequence for differentials [Hartshorne 1977, Chapter II, 8.11]:

$$0 \to \varphi^* \Omega_{K|_k} \to \Omega_{G|_k} \to \Omega_{G|_L} \to 0.$$

This sequence is exact on the left because $\varphi^* \Omega_{L|_k}$ is a free sheaf on G of rank equal to the transcendence degree of L over k $(= d - h)$ and the sequence is exact at the generic point of G. Since G is smooth over L, the other two sheaves are locally free of ranks $d - 1$ and $h - 1$. Taking top exterior powers and using the freeness of the first sheaf, we get the required isomorphism:

$$\omega_{G|_k} \cong \omega_{G|_L}.$$

This completes Step 1 of the proof, where we have proved that the statement that $q + n$ is a jumping number contributed by G for a natural number n is equivalent to the statement

$$\text{the invertible sheaf } \mathcal{O}_G(K_G - \lfloor qF \rfloor|_G) \otimes \mathcal{O}_G(-nF|_G) \text{ has a nonzero global section.} \quad (4\text{-}5)$$

Step 2: Now, we use the hypothesis that the divisor $E \subset E^c$ is a Rees valuation centered at x to conclude that the sheaf $\mathcal{O}_E(K_E - \lfloor cF \rfloor|_E) \otimes \mathcal{O}_E(-(h-1)F|_E)$ has a nonzero global section.

To do this, like in the proof of Lemma 4.5, we consider the normalized blowup $\widetilde{X}$ of the ideal $\mathfrak{a}$ and by the universal property of blowing-up and normalization, the map μ factors through $\widetilde{X}$. The Rees valuation E on Y is the strict transform of a prime divisor $\widetilde{E}$ on $\widetilde{X}$ and E maps birationally onto $\widetilde{E}$ (since $\widetilde{X}$ is normal and $\widetilde{E}$ is a divisor). Note that $\widetilde{E}$ is also a projective variety over L. This is summarized in this picture:

$$\begin{array}{ccccc} E & \xrightarrow{f|_E} & \widetilde{E} & \longrightarrow & \boldsymbol{Spec}(L) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{f} & \widetilde{X} & \longrightarrow & X \end{array}$$

Let $\mathscr{L} = \mathcal{O}_Y(-F) = \mathfrak{a} \cdot \mathcal{O}_Y$ and $\widetilde{\mathscr{L}} = \mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}}$. Then $\widetilde{\mathscr{L}}$ is very ample on $\widetilde{X}$ and $\mathscr{L} = f^*(\widetilde{\mathscr{L}})$. So, using the projection formula along $f|_E$, we have

$$H^0(E, \mathcal{O}_E(K_E - \lfloor cF \rfloor|_E) \otimes \mathscr{L}^n) \cong H^0\big(\widetilde{E}, f|_{E*}(\mathcal{O}_E(K_E - \lfloor cF \rfloor|_E)) \otimes \widetilde{\mathscr{L}}^n\big).$$

So it is enough to show that $H^0\big(\widetilde{E}, f|_{E*}(\mathcal{O}_E(K_E - \lfloor cF \rfloor|_E)) \otimes \widetilde{\mathscr{L}}^{h-1}\big)$ is nonzero. We do this by using Mumford's theorem (Theorem 4.8). And to check the required cohomology vanishing conditions, we use the Kawamata–Viehweg vanishing theorem (Theorem 4.7).

Note that $\mathscr{L}$ is a globally generated line bundle on Y. Hence, by Bertini's theorem [Hartshorne 1977, Chapter III, Corollary 10.9], we may choose a smooth divisor (possibly disconnected) D linearly equivalent to $-F$. Further, we may also assume that D intersects E transversally (in particular, no component of D is E). Having chosen such a D, we note that $-\lfloor cF \rfloor = \lceil -cF \rceil \sim_{\mathbb{Q}} cD + \sum a_i E_i$ as $\mathbb{Q}$-divisors, with $0 \le a_i < 1$ and where the E_i are the irreducible components of F in some order. Let $\Delta = \sum a_i E_i$. Then, since $E \subset E^c$, the coefficient of E in cF is already an integer, and hence the coefficient of E in Δ is 0. So we have $-\lfloor cF \rfloor|_E \sim_{\mathbb{Q}} cD|_E + \Delta|_E$. Since the support of Δ is the union of divisors in the support of F and does not contain E, it follows that $\Delta|_E$ has simple normal crossings support (since F was snc). Further, all the coefficients of $\Delta|_E$ are in the interval [0, 1). Next, $\mathscr{L}|_E$ is the pullback of $\widetilde{\mathscr{L}}|_{\widetilde{E}}$ along $f|_E$; since $f|_E : E \to \widetilde{E}$ is a projective birational map, and $\widetilde{\mathscr{L}}$ is very ample on $\widetilde{E}$, $\mathscr{L}|_E$ is big and nef on E. So $cD|_E$ is a big and nef $\mathbb{Q}$-divisor. Since E is smooth and projective over L and all the relevant hypotheses remain true after base-changing to an algebraic closure of L, we can apply the Kawamata–Viehweg vanishing theorem (Theorem 4.7) on E with $N = -\lfloor cF \rfloor|_E \equiv_{\text{num}} cD|_E + \Delta|_E$ to conclude that

$$H^i(E, \mathcal{O}_E(K_E - \lfloor cF \rfloor|_E)) = 0 \quad \text{for all } i > 0.$$

By the same argument, for all natural numbers $n \in \mathbb{N}_{\ge 0}$, we have

$$H^i\big(E, \mathcal{O}_E(K_E - \lfloor (c+n)F \rfloor|_E)\big) = 0 \quad \text{for all } i > 0 \text{ and all } n \ge 0. \tag{4-6}$$

Now, [Lazarsfeld 2004a, Lemma 4.3.10] implies that $R^j f|_{E*}\mathcal{O}_E(K_E - \lfloor (c+n)F \rfloor|_E) = 0$ for all $j > 0$ and all $n \ge 0$. By the Leray spectral sequence, we then have isomorphisms

$$H^i\big(E, \mathcal{O}_E(K_E - \lfloor (c+n)F \rfloor|_E)\big) \cong H^i\big(\widetilde{E}, f|_{E*}\mathcal{O}_E(K_E - \lfloor (c+n)F \rfloor|_E)\big) \tag{4-7}$$

for all $i \geq 0$ and all $n \geq 0$. So if $\mathscr{F}$ denotes the (coherent, since $f|_E$ is proper) sheaf $f|_{E*}\mathcal{O}_E(K_E - \lfloor cF \rfloor|_E)$ on $\widetilde{E}$, then, putting together (4-6) and (4-7), we get

$$H^i(\widetilde{E}, \mathscr{F} \otimes \widetilde{\mathscr{L}}^n|_{\widetilde{E}}) = 0 \quad \text{for all } i > 0 \text{ and } n \geq 0. \tag{4-8}$$

Finally, (4-8) implies that $\mathscr{F} \otimes \widetilde{\mathscr{L}}^{h-1}|_{\widetilde{E}}$ is 0-regular with respect to the very ample line bundle $\widetilde{\mathscr{L}}|_{\widetilde{E}}$, i.e.,

$$H^i(\widetilde{E}, \mathscr{F} \otimes \widetilde{\mathscr{L}}^{h-1}|_{\widetilde{E}} \otimes \widetilde{\mathscr{L}}^{-i}|_{\widetilde{E}}) = 0 \quad \text{for all } i > 0.$$

This is because since the dimension of $\widetilde{E}$ is $h-1$, we only need to check the vanishing of cohomology groups till $i = h-1$. But if $i \leq h-1$, then (4-8) gives the required vanishing. So, by Mumford's theorem (Theorem 4.8), we get that $\mathscr{F} \otimes \widetilde{\mathscr{L}}^{h-1}|_{\widetilde{E}}$ is globally generated and, in particular,

$$H^0(\widetilde{E}, \mathscr{F} \otimes \widetilde{\mathscr{L}}^{h-1}|_E) \neq 0.$$

So, using (4-5), we are done. This completes the proof of Theorem 4.6 and hence of Theorem 4.2. □

Example 4.9. The Rees coefficient ρ_c (Definition 3.11) of a jumping number c is not always positive. Indeed, consider the ideal $\mathfrak{a} = (x^5 + y^3, y^4)$ in the polynomial ring $k[x, y]$ defining a point scheme supported at the origin in $X = \mathbb{A}^2$. This ideal has Rees coefficient zero for all of its jumping numbers less than one.

To check this, observe that since dimension of X is two, ρ_c can be computed on any log resolution $\mu : Y \to X$ of $\mathfrak{a}$, with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, as

$$\rho_c = -F \cdot E^c,$$

where E^c is the jumping divisor of c as defined in Definition 3.5. This follows by specializing Theorem 3.10 to the surface case, although it is also proved in [Alberich-Carramiñana et al. 2017, Theorem 4.1].

A log resolution Y of $\mathfrak{a}$ can be computed by hand and requires 9 successive blowups. Let $E_1, \ldots, E_9$ denote the exceptional divisors obtained in the order they are labeled. Then, the intersection matrix for the resolution is

$$\begin{pmatrix} -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The relative canonical divisor is

$$K_{Y|X} = E_1 + 2E_2 + 4E_3 + 7E_4 + 8E_5 + 9E_6 + 10E_7 + 11E_8 + 12E_9.$$

If $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, then

$$F = 3E_1 + 5E_2 + 9E_3 + 15E_4 + 16E_5 + 17E_6 + 18E_7 + 19E_8 + 20E_9.$$

Using this information, we compute that the jumping numbers of $\mathfrak{a}$ less than one are $\frac{8}{15}$, $\frac{11}{15}$, $\frac{13}{15}$ and $\frac{14}{15}$, for example, by hand using the formula from [Alberich-Carramiñana et al. 2017] or by using the Macaulay2 package [MultiplierIdealsDim2]. We can then compute that the jumping divisors for each of these jumping numbers turn out to be the same, namely E_4. Therefore, using the intersection matrix above, we can compute that $-F \cdot E_4$ is zero, and conclude that the Rees coefficient for each of the jumping numbers $\frac{8}{15}$, $\frac{11}{15}$, $\frac{13}{15}$ and $\frac{14}{15}$ is zero in this case. □

Theorem 4.2 and Example 4.9 suggest that the classes of jumping numbers in $\mathbb{Q}/\mathbb{Z}$ naturally come in various types depending on the type of growth of the multiplicities of its translates. The type of jumping numbers for which the multiplicity grows fastest (i.e., for which the Rees coefficient is positive) is described by Theorem 4.2. The next corollary states that the class of *integer* jumping numbers always has maximal growth of its multiplicities — that is, $\rho_1 > 0$:

Corollary 4.10. *Let Z be a closed subscheme of the smooth variety X, with irreducible component Z_1 of codimension h. Then we have*:

(1) *The Rees coefficient ρ_1 of the real number* 1 *along Z_1 is always positive.*

(2) *There are at most $(h-1)!\,\rho_1$ Rees valuations of Z centered at Z_1.*

(3) *The codimension h is always a jumping number of $\mathfrak{a}$ contributed by each of the Rees valuations of $\mathfrak{a}$ centered at the generic point of Z_1.*

Part (3) of Corollary 4.10 recovers the fact that, for a regular local ring $(A, \mathfrak{m})$ of dimension h essentially of finite type over k, h is always a jumping number of each $\mathfrak{m}$-primary ideal $\mathfrak{a}$.

Proof. Consider a log resolution $\mu : Y \to X$ of Z with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$.

(1) Since the jumping divisor E^1 (Definition 3.5) is the same as F_{red}, it contains all the exceptional divisors corresponding to the Rees valuations. So E^1_x contains all the Rees valuations of $\mathfrak{a}$ centered at x. So ρ_1 must be positive by Theorem 4.4.

(2) By Theorem 3.10, we have

$$\rho_c = (-1)^{h-1} \frac{(F|_{E^c_x} \cdot \ldots \cdot F|_{E^c_x})}{(h-1)!} = \sum_{E \subset E^c} (-1)^{h-1} \frac{(F|_{E_x} \cdot \ldots \cdot F|_{E_x})}{(h-1)!},$$

where the second equality holds by Proposition 2.8(4). Since the intersection numbers $(-F|_{E_x} \cdot \ldots \cdot -F|_{E_x})$ are nonnegative integers (by Lemma 4.5) and are positive exactly for the Rees valuations at x, each such valuation adds at least $1/(h-1)!$ to ρ_1. Hence the claim.

(3) The last statement follows immediately from Theorem 4.6 since E^1 contains each of the Rees valuations at x. □

For a closed subscheme Z of X with an irreducible component Z_1, Theorem 4.2 and the polynomial nature of the multiplicities $m(c+n)$ guarantee more jumping numbers of Z along Z_1 in the interval $(h-1, h]$ (where h is the dimension of the local ring at Z_1). The following observations generalize similar ones made in [Alberich-Carramiñana et al. 2017] when $\dim X = 2$.

Corollary 4.11. *Fix an irreducible component Z_1 of a closed subscheme Z of the smooth variety X. Let $\nu_1, \dots, \nu_r$ be all the Rees valuations of Z* (see page 95) *centered at the generic point x of Z_1. Set a_i to be the order of vanishing of Z along the valuation ν_i, that is, set*

$$a_i = \min\{\nu_i(f) | f \in \mathfrak{a}\},$$

where $\mathfrak{a}$ is the ideal of Z. Then we have:

(1) *Each rational number ℓ/a_i (where $\ell \in \mathbb{Z}$) in the interval $(h-1, h]$ is a jumping number at x contributed by ν_i* (in the sense of Definition 4.3).

(2) *If c is a jumping number at x in the interval $(h-1, h]$ such that c is not an integer translate of a smaller jumping number at x, then c is a rational number of the form in part* (1).

(3) *More generally, if $c < h$ is a jumping number such that c is not an integer translate of a smaller jumping number at x, then $m(c+n)$ is polynomial* (*in n*) *of degree at least $\lfloor c \rfloor$.*

Proof. Clearly (3) implies (2). So we just prove (1) and (3):

(1) Let $\mu : Y \to X$ be a log resolution of Z and let $E_1, \dots, E_r$ denote the exceptional divisors on Y corresponding to $\nu_1, \dots, \nu_r$ respectively. We see that for any number of the form ℓ/a_i, E_i is contained in the corresponding jumping divisor (Definition 3.5) and hence the Rees coefficient ρ_{ℓ/a_i} is positive by Theorem 4.4. The claim now follows immediately from Theorem 4.6.

(3) If c is a jumping number such that $c - n$ is not a jumping number for any n, then $m(\{c\} + n)$ is a polynomial (Theorem 3.3) which is zero for $0 \le n \le \lfloor c \rfloor - 1$. Here $\{c\}$ $(= c - \lfloor c \rfloor)$ denotes the fractional part of the real number c. This means the degree must be at least $\lfloor c \rfloor$. □

Remark 4.12. Corollary 4.11 is interesting because it ensures that many jumping numbers in the interval $(h-1, h]$ can be computed easily from the normalized blowup of a closed subscheme Z (without computing a full log resolution of Z). Those that cannot are integer translates of a smaller jumping number.

In this context, we recall the result of Budur [2003] mentioned in the introduction which implies that when Z is a point scheme, the jumping numbers (along with their multiplicity) of Z in the interval (0, 1) can be interpreted as coming from the cohomology of the Milnor fiber of a general element of $\mathfrak{a}$ (the ideal of Z). Putting this result together with Corollary 4.11 gives us an interpretation of essentially all jumping numbers of a point scheme in a smooth surface. In higher dimension, this only gives us an understanding of jumping numbers in the intervals (0, 1) and $(d-1, d]$. We raise the following two questions towards understanding all the other jumping numbers.

Question 4.13. For each integer j in the interval $[0, h-2]$, how do we characterize the classes of jumping numbers $[c]$ such that $m(c+n)$ grows like n^j? In particular, can we characterize such classes of jumping numbers in terms of contribution by divisors?

Question 4.14. Given a class of jumping numbers $[c] \in \mathbb{Q}/\mathbb{Z}$ contributed by a Rees valuation, how do we find the smallest jumping number in $[c]$? More generally, how do we find the smallest jumping number in any given class of jumping numbers in $\mathbb{Q}/\mathbb{Z}$? Equivalently, how do we characterize the jumping numbers $c \leq h-1$ of $\mathfrak{a}$ such that c is not a translate of a smaller jumping number of $\mathfrak{a}$?

On the valuations contributing jumping numbers. As another consequence of Theorem 4.6, we prove the following theorem:

Theorem 4.15. *Let X be a smooth variety over k. Let ν be a divisorial valuation over X, i.e., a divisorial valuation centered at a (not necessarily closed) point x in X. Then there is an effective integral divisor D on X and a jumping number c of D at x such that c is* ***contributed by*** *ν.*

Proof. Let $(A, \mathfrak{m})$ denote the local ring at x, and let h be the Krull dimension of A. First, suppose $h = 1$, i.e., ν comes from a divisor E on X. Then taking $D = E$ and $c = 1$ works. So we assume $h > 1$.

Next, we claim that there is an ideal $\mathfrak{a}$ in A such that $\mathfrak{a}$ is $\mathfrak{m}$-primary and ν is a Rees valuation of $\mathfrak{a}$ (defined on page 95). This follows immediately from [Huneke and Swanson 2006, Proposition 10.4.4]. So we may choose an ideal $\mathfrak{a} \subset \mathcal{O}_X$ such that x corresponds to a minimal component of $\mathfrak{a}$ and the stalk of $\mathfrak{a}$ at x has ν as a Rees valuation. Then, by Corollary 4.10, we know that h is a jumping number of $\mathfrak{a}$ at x contributed by ν. Now, choose a log resolution $\mu : Y \to X$ of $\mathfrak{a}$ and let $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, where $F = \sum a_i E_i$ for prime components E_i. We may assume ν corresponds to E_1 on Y. Now, the fact that h is contributed by ν means (by definition) that

$$\mathscr{J}(\mathfrak{a}^h)_x = \mu_*(\mathcal{O}_Y(K_{Y|X} - hF))_x \subsetneq \mu_*(\mathcal{O}_Y(K_{Y|X} - hF + E_1))_x. \tag{4-9}$$

Now, if we choose D to be a general member of the ideal $\mathfrak{a}^n$ (i.e., a general k-linear combination of its generators) for any $n > h$, then a general enough D will be reduced (since $h > 1$), and further, it will satisfy $\mu^* D = \widetilde{D} + nF$, where $\widetilde{D}$ is the strict transform of D. Now, we claim that h/n is a jumping number of D at x contributed by E_1. To verify this, we need to check that

$$\mathscr{J}\Big(\frac{h}{n}D\Big)_x = \mu_*\Big(\mathcal{O}_Y\Big(K_{Y|X} - \Big\lfloor \frac{h}{n}(nF + \widetilde{D})\Big\rfloor\Big)\Big)_x \subsetneq \mu_*\Big(\mathcal{O}_Y\Big(K_{Y|X} - \Big\lfloor \frac{h}{n}(nF + \widetilde{D})\Big\rfloor + E_1\Big)\Big)_x,$$

which holds if and only if

$$\mu_*(\mathcal{O}_Y(K_{Y|X} - hF))_x \subsetneq \mu_*(\mathcal{O}_Y(K_{Y|X} - hF + E_1))_x,$$

which holds because h/n is less than one and $\widetilde{D}$ is reduced. So we are done by (4-9). □

Remark 4.16. Theorem 4.15 shows that the set of valuations that contribute some jumping number of some divisor in X includes all divisorial valuations over X, in contrast to the valuations computing only the log-canonical threshold of divisors, which are known to satisfy many special properties (see [Blum 2021]). This negatively answers a question raised by Joaquín Moraga, asking whether any valuation contributing a jumping number also computes a log-canonical threshold. We thank him for asking this question and useful related conversations.

4B. ***Relation to Hilbert–Samuel multiplicity.*** In this section, we relate the Hilbert–Samuel multiplicity of the subscheme Z at an irreducible component Z_1 to the Rees coefficients ρ_c associated to the jumping numbers of $\mathfrak{a}$. We first recall the relevant definitions.

Let $(A, \mathfrak{m})$ be the local ring of X at the generic point of Z_1 and $\mathfrak{a}$ be the ideal of Z in A. Let h denote the Krull dimension of A. The Hilbert–Samuel multiplicity of $(A, \mathfrak{m})$ with respect to the $\mathfrak{m}$-primary ideal $\mathfrak{a}$, denoted by $e_{\mathfrak{a}}(A)$, is defined to be the number

$$e_{\mathfrak{a}}(A) = \lim_{n\to\infty} \frac{h!\,\lambda(A/\mathfrak{a}^n)}{n^h},$$

where λ denotes the length as an A-module. The number $e_{\mathfrak{a}}(A)$ is always a positive integer.

Recall that for any real number $c > 0$, the Rees coefficient ρ_c of Z along Z_1 was defined in Definition 3.11 so that $m(c+n) = \rho_c n^{h-1} + o(n^{h-2})$ as $n \to \infty$. Here $m(c)$ denotes the multiplicity of the number c (Definition 3.2).

Theorem 4.17. *We have*

$$\sum_{c\in(0,1]} \rho_c = \frac{e_{\mathfrak{a}}(A)}{(h-1)!}.$$

The main idea relating these two numbers is that the Hilbert–Samuel multiplicity can be computed using multiplier ideals, which is a consequence of Skoda's theorem:

Lemma 4.18. *We have*

$$e_{\mathfrak{a}}(A) = \lim_{n\to\infty} \frac{h!\,\lambda(A/\mathscr{J}(\mathfrak{a}^n))}{n^h}.$$

Proof. If d is the dimension of the ambient variety X, for $n \geq d$, we have

$$\mathfrak{a}^n \subset \mathscr{J}(\mathfrak{a}^n) \subset \mathfrak{a}^{n-d+1},$$

where the second containment holds because of Skoda's theorem (Theorem 2.5). This gives us

$$\lambda(A/\mathfrak{a}^{n-d+1}) \leq \lambda\big(A/(\mathscr{J}(\mathfrak{a}^n))\big) \leq \lambda(A/\mathfrak{a}^n).$$

When we divide by n^h and take limit as $n \to \infty$, since both the first and third terms approach $e_{\mathfrak{a}}(A)/h!$, so does the middle term. □

Proof of Theorem 4.17. We first note that ρ_c can be nonzero only for the candidate jumping numbers (defined in Section 2A1). Hence, the sum is finite. Now to prove the proposition, we use Lemma 4.18 to compute $e_{\mathfrak{a}}(A)$ as follows:

$$\frac{e_{\mathfrak{a}}(A)}{h!} = \lim_{n\to\infty} \frac{\lambda(A/\mathscr{J}(\mathfrak{a}^n))}{n^h} = \lim_{n\to\infty} \frac{\sum_{c\leq n} m(c)}{n^h} = \lim_{n\to\infty} \frac{\sum_{c\in(0,1]} \sum_{j=0}^{n-1} m(c+j)}{n^h}.$$

Since $m(c+j) = \rho_c j^{h-1} + o(j^{h-1})$ as $j \to \infty$, by using the fact that $\sum_{j=0}^{n} j^i = n^{i+1}/(i+1) + o(n^i)$ as $n \to \infty$, we have $\sum_{j=0}^{n-1} m(c+j) = (\rho_c/h)n^h + o(n^{h-1})$ as $n \to \infty$. Then we have

$$\frac{e_{\mathfrak{a}}(A)}{h!} = \lim_{n\to\infty} \frac{\sum_{c\in(0,1]} (\rho_c/h)n^h + o(n^{h-1})}{n^h} = \frac{\sum_{c\in(0,1]} \rho_c}{h},$$

which concludes the proof. □

5. Special case: monomial ideals

In this section, we derive an explicit formula for the multiplicities of jumping numbers and Rees coefficients (Theorem 5.3) of an arbitrary cofinite monomial ideal (see Definitions 3.2 and 3.11). In particular, we see that the Rees coefficient is positive for the class of any jumping number in this case (Corollary 5.5).

We introduce some notation. Let $X = \mathbb{A}^d$ and $R = k[x_1, \ldots, x_d]$, and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary monomial ideal, where $\mathfrak{m} = (x_1, \ldots, x_d)$. Fix the lattice $M = \mathbb{N}^d$ inside $M_{\mathbb{R}} = \mathbb{R}^d$. For any $v = (v_1, \ldots, v_d) \in M$, we write $x^v = x_1^{v_1} \cdots x_d^{v_d}$ for the corresponding monomial. For any ideal generated by monomials $\mathfrak{a}$ in R, the *Newton polyhedron* $P(\mathfrak{a}) \subset \mathbb{R}^d$ of $\mathfrak{a}$ is the convex hull of all vectors $v \in M$ such that x^v belongs to $\mathfrak{a}$. Note that $P(\mathfrak{a})$ is an unbounded polyhedral (i.e., bounded by polygonal faces) region in the first orthant of $\mathbb{R}^d$. For any positive real number c, let $P(c \cdot \mathfrak{a})$ denote the polyhedron obtained by scaling the vectors in $P(\mathfrak{a})$ by a factor of c. A theorem of Howald [2001] provides a formula for the multiplier ideals of $\mathfrak{a}$:

Theorem 5.1 [Lazarsfeld 2004b, Theorem 9.3.27]. *Fix a monomial ideal $\mathfrak{a}$ in R and a positive real number c. Then the multiplier ideal $\mathscr{J}(\mathfrak{a}^c)$ is the monomial ideal generated by all monomials x^v such that*

$$v + \mathbf{1} \in \operatorname{Int}(P(c \cdot \mathfrak{a})),$$

where $\mathbf{1}$ is the vector $(1, \ldots, 1)$ *and* $\operatorname{Int}(P(c \cdot \mathfrak{a}))$ *is the interior of the scaled Newton polyhedron $P(c \cdot \mathfrak{a})$.*

We can now state our formula for the multiplicity and Rees coefficient of a jumping number.

Notation 5.2. We will use the following notation throughout this section.

- For any set $S \subset \mathbb{R}^d$, $\#(S)$ denotes the number of lattice points in S.
- If P is a face of a d-dimensional polyhedron in $\mathbb{R}^d$, then $\operatorname{vol}(P)$ denotes the $(d-1)$-volume of P.

Theorem 5.3. *Fix an $\mathfrak{m}$-primary monomial ideal $\mathfrak{a}$ and a positive real number c. Then the multiplicity of c is given by*

$$m(c) = \#\left\{ \bigcup_i c P_i^\circ \cap M \right\}, \tag{5-1}$$

where the P_i are all the bounded faces of the Newton polyhedron $P(\mathfrak{a})$ and

$$c P_i^\circ := c P_i - \left(\bigcup_{1 \le j \le d} j\text{-th coordinate hyperplane} \right) = c P_i \cap (\mathbf{1} + M).$$

Moreover, the Rees coefficient ρ_c is given by

$$\rho_c = \sum_{c\mathcal{H}_i \cap M \neq \varnothing} \operatorname{vol}(P_i),$$

where $\mathcal{H}_i$ is the unique hyperplane containing the face P_i.

To prove Theorem 5.3, we need the following lemma:

Lemma 5.4. *For each jumping number c of the monomial ideal $\mathfrak{a}$, the k-vector space $\mathscr{J}(\mathfrak{a}^{c-\epsilon})/\mathscr{J}(\mathfrak{a}^c)$ is isomorphic to the vector space generated by the monomials in $\partial P(c \cdot \mathfrak{a}) \cap (\mathbf{1} + M)$, where $\partial P(c \cdot \mathfrak{a})$ is the boundary of the dilated* (*by c*) *Newton polyhedron $P(c \cdot \mathfrak{a})$.*

Proof. Howald's theorem implies that a real number $c > 0$ is a jumping number of $\mathfrak{a}$ if and only if $cP_i^\circ \cap M$ is nonempty for some i. Moreover, the monomials that are contained in $\mathscr{J}(\mathfrak{a}^{c-\epsilon})$ but not in $\mathscr{J}(\mathfrak{a}^c)$ are exactly the monomials x^v such that $v + \mathbf{1}$ is on the boundary of $P(c \cdot \mathfrak{a})$. Therefore, the k-vector space $\mathscr{J}(\mathfrak{a}^{c-\epsilon})/\mathscr{J}(\mathfrak{a}^c)$ is isomorphic to the vector space generated by the monomials in $\partial P(c \cdot \mathfrak{a}) \cap (\mathbf{1} + M)$. □

Proof of Theorem 5.3. The condition that $\mathfrak{a}$ is an $\mathfrak{m}$-primary monomial ideal is equivalent to the condition that the region $P(\mathfrak{a})$ intersects each coordinate axis. In this case, the boundary of $P(\mathfrak{a})$ is the union of the bounded faces $P_1, \dots, P_r$ and the unbounded faces exactly along the coordinate hyperplanes. The faces $P_1, \dots, P_r$ are polytopes of dimension $d - 1$ with vertices in M. Each polytope P_i is contained in a unique hyperplane $\mathcal{H}_i \subset \mathbb{R}^d$.

For the first claim, we note that addition by $\mathbf{1}$ gives a one-to-one correspondence between vectors in M and the vectors in M that do not lie on any of the coordinate hyperplanes. Hence, the vectors v in M such that $v + \mathbf{1} \in \partial P(c \cdot \mathfrak{a})$ are in one-to-one correspondence with the vectors in $\partial P(c \cdot \mathfrak{a})$ that do not lie on any of the coordinate hyperplanes. But, since the components of the boundary that do not lie on the coordinate hyperplanes are exactly the bounded components $P_1, \dots, P_r$, this proves the first claim. So the formula for the multiplicity follows from this and Lemma 5.4 above.

For $c > 0$, let cP_i denote the polytope obtained by scaling each vector in P_i by c and let $c\mathcal{H}_i$ be the corresponding scaled hyperplane.

For the last claim, we use the following fact: Let Q be a convex polyhedron of dimension ℓ in $\mathbb{R}^\ell$. Then

$$\mathrm{vol}(Q) = \lim_{t \to \infty} \frac{\#(tQ \cap \mathbb{Z}^\ell)}{t^\ell}.$$

We actually need a slightly more general version of this formula: Suppose $\{Q_n\}_{n \ge 1}$ is a sequence of polyhedra of dimension ℓ in $\mathbb{R}^\ell$ such that Q_n is some (possibly noninteger) translate of $c_n Q_1$, where $\{c_n\}$ is a sequence of real numbers such that $c_n \to \infty$ as $n \to \infty$. Then

$$\mathrm{vol}(Q_1) = \lim_{n \to \infty} \frac{\#(Q_n \cap M)}{c_n^\ell}.$$

This statement actually follows immediately from the previous statement with the additional observation that when we translate a polyhedron by any vector, the difference in the number of lattice points is bounded above by a fixed multiple of volume of the boundary of the polyhedron. Hence, the difference does not really matter for the limit.

Suppose $c\mathcal{H}_i \cap M \neq \varnothing$. Then the number points of $(c+n)\mathcal{H}_i \cap M$ in $P((c+n) \cdot \mathfrak{a})$ is the same as the number of lattice points in a translate of $((c+n)/c)(cP_i)$ in $c\mathcal{H}_i$. This is because any lattice point on P_i gives us a natural bijection between the lattice points in $c\mathcal{H}_i$ and $(c+n)\mathcal{H}_i$ (by translating). Under this translation, we can identify $(c+n)P_i$ in $(c+n)\mathcal{H}_i$ with a translate of $((c+n)/c)(cP_i)$ in $c\mathcal{H}_i$. Hence, using the formula above with $\ell = d - 1$, $\mathbb{R}^{d-1} = c\mathcal{H}_i$, $Q_n = (c+n)P_i$ and $c_n = (c+n)/c$, we have

$$\mathrm{vol}(P_i) = \frac{1}{c^{d-1}} \mathrm{vol}(cP_i) = \lim_{n \to \infty} \frac{\#\big(((c+n)/c)P_i \cap M\big)}{((c+n)/c)^{d-1}}.$$

Since the number of lattice points in the intersection of $(c+n)P_i$ and the coordinate hyperplanes or the intersection of $(c+n)P_i$ and $(c+n)P_j$ for $j \neq i$ grows at a rate of at most n^{d-2}, the formula for ρ_c follows. □

Corollary 5.5. *If c is a jumping number of an $\mathfrak{m}$-primary monomial ideal $\mathfrak{a} \subset k[x_1, \ldots, x_d]$, then the Rees coefficient ρ_c is positive.*

6. Poincaré series

Let X be a smooth variety. For any closed subscheme Z of X and an irreducible component Z_1 of Z, we can assemble the multiplicities of jumping numbers (Definition 3.2) into a generating function called the *Poincaré series* of Z at Z_1. More precisely, this is the generating function defined by

$$\phi_{Z,Z_1}(T) = \sum_{c\in(0,\infty)} m(c)T^c = \sum_{c\in(0,1]} \sum_{n=0}^{\infty} m(c+n)T^{c+n}.$$

This series is clearly only a countable sum, because the jumping numbers of Z are a discrete subset of rational numbers. Moreover, there is an $\ell \in \mathbb{N}$ such that every jumping number has a denominator a factor of ℓ, i.e., every jumping number is of the form n/ℓ for $n \in \mathbb{N}$. Then, having chosen such an ℓ and setting $z = T^{1/\ell}$, we see that $\phi_{Z,Z_1}(T)$ is actually a power series in z. We use the polynomial nature of $m(c+n)$ proved in Theorem 3.3 to prove that this power series is actually a rational function:

Theorem 6.1. *$\phi_{Z,Z_1}(z)$ is a rational function of z, where $z = T^{1/\ell}$ and ℓ is as above. In fact, we have the formula*

$$\phi_{Z,Z_1}(T) = \sum_{c\in(0,1]} \left(\frac{m(c)}{(1-T)} + \frac{\gamma_{c,1}T}{(1-T)^2} + \cdots + \frac{\gamma_{c,h-2}T}{(1-T)^{h-1}} + \frac{\rho_c(h-1)!T}{(1-T)^h} \right) T^c \tag{6-1}$$

for some rational numbers $\gamma_{c,i}$, where h denotes the codimension of Z_1 in X.

Proof. For any $c \in (0, 1]$, by Theorem 3.3 we have

$$m(c+n) = \alpha_{0,c} + \alpha_{1,c}n + \cdots + \alpha_{h-1,c}n^{h-1}.$$

(Of course, $\alpha_{h-1,c} = \rho_c$ and $\alpha_{0,c} = m(c)$.) Now, the list of polynomials

$$p_0(n) = 1, \quad p_1(n) = n, \quad p_2(n) = \binom{n+1}{2}, \quad \ldots, \quad p_{h-1}(n) = \binom{n+h-2}{h-1}$$

has exactly one polynomial of each degree between 0 and $h-1$. So these polynomials form a basis over $\mathbb{Q}$ of the space of rational polynomials of degree less than h. Thus, we can write the polynomial $m(c+n)$ in n in terms of the $p(i)$'s as follows:

$$m(c+n) = \sum_{i=0}^{h-1} \gamma_{c,i}\, p_i(n) \quad \text{for some rational numbers } \gamma_{c,i}. \tag{6-2}$$

Inspecting the constant and the leading coefficient, we get $\gamma_{c,0} = m(c)$ and $\gamma_{c,h-1} = (h-1)! \times \rho_c$. Thus,

$$\begin{aligned}\phi_{Z,Z_1}(T) &= \sum_{c\in(0,\infty)} m(c)T^c = \sum_{c\in(0,1]} \sum_{n=0}^{\infty} m(c+n)T^{c+n} \\ &= \sum_{c\in(0,1]} T^c \sum_{n=0}^{\infty} (\gamma_{c,0}p_o(n) + \gamma_{c,1}p_1(n) + \cdots + \gamma_{c,h-1}p_{h-1}(n))T^n \\ &= \sum_{c\in(0,1]} T^c \biggl(\gamma_{c,0} \sum_{n=0}^{\infty} p_o(n)T^n + \gamma_{c,1} \sum_{n=0}^{\infty} p_1(n)T^n + \cdots + \gamma_{c,h-1} \sum_{n=0}^{\infty} p_{h-1}(n)T^n \biggr).\end{aligned}$$

Now the theorem follows from the following elementary observation:

$$\sum_{n=0}^{\infty} p_i(n)T^n = \begin{cases} 1/(1-T) & \text{if } i = 0, \\ T/(1-T)^{i+1} & \text{if } i \geq 1. \end{cases}$$

When $i = 0$ the formula is clear. For $i \geq 1$, the formula is equivalent to

$$\frac{1}{(1-T)^{i+1}} = \sum_{j=0}^{\infty} \binom{j+i}{i} T^j,$$

which is a well-known combinatorial identity counting degree j monomials in $i+1$ variables. □

Remark 6.2. Theorem 6.1 was proved independently by Àlvarez Montaner and Núñez-Betancourt [2022] using different methods. They also prove an analogous theorem for test ideals, which are positive-characteristic counterparts of multiplier ideals. We are grateful to them for conveying the explicit form for the Poincaré series as in (6-1).

Remark 6.3. The case of Theorem 6.1 when Z is a point scheme and X is a surface was proved by Alberich-Carramiñana et al. [2017], generalizing the work of Galindo and Monserrat [2010] in the case of simple complete ideals.

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A classification of the weak Lefschetz property for almost complete intersections generated by uniform powers of general linear forms

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We use Macaulay's inverse system to study the Hilbert series for almost complete intersections generated by uniform powers of general linear forms. This allows us to give a classification of the weak Lefschetz property for these algebras, settling a conjecture by Migliore, Miró-Roig, and Nagel.

1. Introduction

Let $\Bbbk$ be a field of characteristic zero and let $\ell_1, \ell_2, \dots, \ell_r \in \Bbbk[x_1, x_2, \dots, x_n]$ be general linear forms. For positive integers $d_1, d_2, \dots, d_r$, consider the quotient $\Bbbk[x_1, x_2, \dots, x_n]/\langle \ell_1^{d_1}, \ell_2^{d_2}, \dots, \ell_r^{d_r}\rangle$. This algebra ties together several areas of contemporary mathematics.

From the algebraic point of view, it is in a natural way linked to the long-standing conjectures by Fröberg [1985] and Iarrobino [1997] on the Hilbert series of generic forms.

Powers of general linear forms are also tightly connected to the study of fat-points schemes via Macaulay's inverse system, as was noticed by Emsalem and Iarrobino [1995]. This bridge to geometry relates the study of powers of general linear forms to the Alexander–Hirschowitz theorem [Alexander and Hirschowitz 1995; Chandler 2002] and the Segre–Gimigliano–Harbourne–Hirschowitz (SGHH) conjecture [Ciliberto 2001].

In this paper we consider the uniform almost complete intersection case, that is, algebras of the form $\Bbbk[x_1, x_2, \dots, x_n]/(\ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d)$, from the perspective of the weak Lefschetz property.

Recall that a graded algebra A satisfies the weak Lefschetz property (WLP) if there exists a linear form ℓ such that the multiplication map $\times\ell : A_i \to A_{i+1}$ has maximal rank for all degrees i, while A satisfies the strong Lefschetz property (SLP) if the multiplication map $\times\ell^j : A_i \to A_{i+j}$ has maximal rank for all i and all j. For an introduction to the Lefschetz properties, see, e.g., [Harima et al. 2013; Migliore and Nagel 2013].

The WLP for the class of algebras that we consider holds for $n = 1$ and $n = 2$, since all graded artinian quotients in one or two variables have the SLP, the argument being trivial for the univariate case, while the case of two variables, which requires characteristic zero, is attributed to Harima, Migliore, Nagel,

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and Watanabe [Harima et al. 2003]. For $n = 3$, Schenck and Seceleanu [2010] showed that the WLP holds for *any* quotient by an artinian ideal generated by powers of linear forms. Migliore, Miró-Roig, and Nagel [Migliore et al. 2012] showed that for even $n \geq 4$, the WLP fails for almost complete intersections generated by uniform powers, except in the case $(n, d) = (4, 2)$. They also gave results in the odd uniform case and in the mixed-degree case, and provided a conjecture for the unproven part of the odd uniform case.

To simplify the statement of the conjecture and the presentation in this paper in general, we let $R_{n,m,d}$ denote the ring $\Bbbk[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_m^d \rangle$, where $\ell_1, \ell_2, \dots, \ell_m$ are general linear forms and $\Bbbk$ is a field of characteristic zero.

Conjecture 1 [Migliore et al. 2012, Conjecture 6.6]. *Let $n \geq 9$ be an odd integer. Then $R_{n,n+1,d} = \Bbbk[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$ fails the WLP if and only if $d > 1$. Furthermore, if $n = 7$, then $R_{n,n+1,d}$ fails the WLP when $d = 3$.*

Independently, Harbourne, Schenck, and Seceleanu [Harbourne et al. 2011] gave a more general but less precise conjecture.

Conjecture 2 [Harbourne et al. 2011, Conjecture 1.5]. *Let $r + 1 \geq n \geq 5$. Then the algebra $R_{n,r,d} = \Bbbk[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_r^d \rangle$ fails the WLP if $d \gg 0$.*

Since then, the main focus have been on Conjecture 1. Miró-Roig [2016] has shown the failure of the WLP when $d = 2$, Nagel and Trok [2019] have shown the failure when both n and d are large enough, and also when $n \geq 9$, $d - 2 \gg 0$, and $d - 2$ is divisible by n. Ilardi and Vallès [2019] have settled the case $(n, d) = (7, 3)$, while Miró-Roig and Tran [2020] have shown the failure in the cases $9 \leq n = 2m+1 \leq 17$ and $d \geq 4$, and in the cases $d = 2r$, $1 \leq r \leq 8$, and $9 \leq n \leq 4r(r+2) - 1$.

We cover all the remaining cases, settling Conjecture 1, and provide the following classification.

Theorem 1.1. *Let $d, n \geq 1$. Then $R_{n,n+1,d} = \Bbbk[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$ fails the WLP except when $n \leq 3$, $d = 1$ or $(n, d) \in \{(4, 2), (5, 2), (5, 3), (7, 2)\}$, and in these cases, the WLP holds.*

In Section 2, we introduce some notation. In Section 3, we use the theory for inverse systems to determine the degree of the Hilbert series for $R_{n,n+2,d}$. In Section 4, we give an upper bound for the degree of the Hilbert series for $R_{n,n+2,d}$ under the assumption that $R_{n+1,n+2,d}$ has the WLP. By comparing this upper bound with the actual degree, we can draw the conclusion that $R_{n+1,n+2,d}$ fails the WLP in all but a finite number of cases. The remaining cases are then dealt with separately in Section 5.

2. Preliminaries

We begin by giving some background on Hilbert series, Fröberg's conjecture, and inverse systems.

The Hilbert series for a standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ is the power series $\sum \dim_{\Bbbk} A_i t^i$ and is denoted by $\mathrm{HS}(A, t)$. The Hilbert function of A is the function $i \mapsto \dim_{\Bbbk} A_i$.

If A is an artinian graded algebra and f is a form in A of degree d such that the map $\times f : A_i \to A_{i+d}$ has maximal rank for all i, then it is an easy exercise to check that the Hilbert series for $A/(f)$ is equal to $[\mathrm{HS}(A, t) \cdot (1 - t^d)]$. The bracket notation means that we truncate the series before the first nonpositive term.

Fröberg [1985] has conjectured that if A is the polynomial ring modulo an ideal generated by general forms, then the map induced by multiplication by a general form of degree d has maximal rank. Normally, this conjecture is expressed equivalently as follows: if $f_1, \dots, f_r$ are general forms in $\Bbbk[x_1, x_2, \dots, x_n]$ of degrees $d_1, d_2, \dots, d_r$, then the Hilbert series for $\Bbbk[x_1, x_2, \dots, x_n]/\langle f_1, f_2, \dots, f_r\rangle$ equals $\bigl[\prod(1-t^{d_i})/(1-t)^n\bigr]$.

Fröberg [1985] also proves that $\bigl[\prod(1-t^{d_i})/(1-t)^n\bigr]$ is a lower bound for possible Hilbert series among forms of degrees $d_1, d_2, \dots, d_r$ in the lexicographic sense, so for a fixed signature $(n, d_1, d_2, \dots, d_r)$, the conjecture can be verified with an example.

The conjecture is, except for a few cases, open for $r-1 > n \geq 4$. For some recent results, see [Nenashev 2017]. The case $r = n+1$ is due to Stanley [1980] and is of particular importance for this paper.

Let A be a monomial complete intersection, i.e., $A = \Bbbk[x_1, \dots, x_n]/\langle x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n}\rangle$. Stanley showed that the multiplication map $\times(x_1 + x_2 + \cdots + x_n)^d : A_i \to A_{i+d}$ has full rank for every i and d, not only settling the $n+1$ case of the Fröberg conjecture, but also opening up the area of the Lefschetz properties for graded algebras. If we perform a linear change of coordinates, Stanley's result is equivalent to the fact that complete intersections generated by powers of general linear forms have the SLP.

When restricted to the equigenerated case $d = d_1 = \cdots = d_{n+1}$, this implies that

$$\mathrm{HS}(R_{n,n+1,d}, t) = \left[\frac{(1-t^d)^{n+1}}{(1-t)^n}\right].$$

Suppose now that $R_{n,n+1,d}$ satisfies the WLP. Then the map induced by multiplication by a general linear form ℓ has maximal rank in every degree, so the Hilbert series for $R_{n,n+1,d}/(\ell)$ equals

$$\left[(1-t)\left[\frac{(1-t^d)^{n+1}}{(1-t)^n}\right]\right] = \left[(1-t)\frac{(1-t^d)^{n+1}}{(1-t)^n}\right] = \left[\frac{(1-t^d)^{n+1}}{(1-t)^{n-1}}\right],$$

where the first equality follows from [Fröberg 1985, Lemma 4].

Since $R_{n,n+1,d}/(\ell)$ is isomorphic to $R_{n-1,n+1,d}$, this gives that $R_{n,n+1,d}$ has the WLP if and only if the Hilbert series of $R_{n-1,n+1,d}$ is the one expected by Fröberg's conjecture, that is,

$$R_{n,n+1,d} \text{ has the WLP} \quad \text{if and only if} \quad \mathrm{HS}(R_{n-1,n+1,d}, t) = \left[\frac{(1-t^d)^{n+1}}{(1-t)^{n-1}}\right]. \tag{1}$$

For an ideal I in $\Bbbk[x_1, x_2, \dots, x_n]$, we consider the dual polynomial ring $\Bbbk[X_1, X_2, \dots, X_n]$, where x_i acts like $\partial/\partial X_i$ for $i = 1, 2, \dots, n$, and the inverse system of I, denoted by I^{-1}, is the submodule annihilated by I under this action. We use the notation $f \circ F$ for the action of the form $f \in \Bbbk[x_1, x_2, \dots, x_n]$ on the form $F \in \Bbbk[X_1, X_2, \dots, X_n]$. By duality,

$$\dim_{\Bbbk}[I^{-1}]_d = \dim_{\Bbbk}[\Bbbk[x_1, x_2, \dots, x_n]/I]_d, \tag{2}$$

and this will enable us to use the inverse system in order to obtain lower bounds for the Hilbert series.

Finally, the action of the d-th power of a general linear form ℓ on a form F in $\Bbbk[X_1, X_2, \dots, X_n]$ will be of particular importance to us, and we therefore recall the general Leibniz rule,

$$\ell^d \circ F^n = \sum_{d_1+d_2+\cdots+d_n=d} \frac{d!}{d_1!d_2!\cdots d_n!}(\ell^{d_1} \circ F)(\ell^{d_2} \circ F)\cdots(\ell^{d_n} \circ F).$$

3. The degree of the Hilbert series for $R_{n,n+2,d}$

By definition, the degree of the Hilbert series for an artinian graded algebra A is equal to $\max\{j \mid A_j \neq 0\}$. Since A is artinian and nonzero, this number also agrees with the Castelnuovo–Mumford regularity of A; see [Eisenbud 2005].

Let

$$s(n,d) = \begin{cases} \frac{1}{2}(n+1)(d-1) & \text{if } n \text{ is odd,} \\ \lfloor \frac{1}{2}n(n+2)(d-1)/(n+1) \rfloor & \text{if } n \text{ is even.} \end{cases}$$

We will show that $\deg(\mathrm{HS}(R_{n,n+2,d}, t)) = s(n,d)$ for all $n, d \geq 1$.

Sturmfels and Xu [2010] have shown that $\deg(\mathrm{HS}(R_{n,n+2,2,}, t)) = s(n,2)$, and that the dimension of $R_{n,n+2,2}$ in degree $s(n,2)$ is equal to $2^{n/2}$ if n is even, and equal to 1 if n is odd.

Nagel and Trok [2019] proved that $\deg(\mathrm{HS}(R_{n,n+2,d}, t)) \leq s(n,d)$. They also proved that equality holds when n is odd, in which case the dimension of $R_{n,n+2,d}$ in degree $s(n,d)$ is equal to 1, and when n is even and $n+1$ divides $d-1$ or $d \geq n^2+n+2$, in which case the dimension of $R_{n,n+2,d}$ in degree $s(n,d)$ is equal to a binomial coefficient.

By (2), we have

$$\dim_{\Bbbk}[\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+2}^d \rangle^{-1}]_s \neq 0 \implies \deg(\mathrm{HS}(R_{n,n+2,d}, t)) \geq s,$$

so in order to show that $s(n,d)$ is a lower bound, it is enough to show that the inverse system is nonzero in degree $s(n,d)$.

Although it is sufficient to show that $s(n,d)$ is a lower bound for $\deg(\mathrm{HS}(R_{n,n+2,d}, t))$ in the unproven part of the case n even, we show, for completeness, that $s(n,d)$ is a lower bound for all n.

We begin with an alternative proof of the case n odd. The argument is short and also gives the main idea behind the more involved proof for the even case.

Proposition 3.1. *Let $n \geq 1$ be odd, and let $d \geq 1$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is nonzero in degree $s(n,d)$.*

Proof. When $d = 1$, we have $s(n,1) = 0$, and the value of the Hilbert function in degree 0 is equal to 1. The case $d = 2$ follows from the result by Sturmfels and Xu; in particular, there is a form F of degree $s(n,d)$ such that $\ell_i^2 \circ F = 0$ for $i = 1, \dots, n+2$. For the case $d > 2$, it follows from the pigeonhole principle in conjunction with the general Leibniz rule that $\ell_i^d \circ F^{d-1} = 0$ for $i = 1, \dots, n+2$. Finally, the degree of F^{d-1} equals

$$(d-1)s(n,2) = (d-1)\frac{n-1}{2} = s(n,d).$$

□

We now turn to the even case. Also here we are able to reduce the argument to the result by Sturmfels and Xu in degree 2.

Lemma 3.2. *Let n be even and let $1 \le d \le 3$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is nonzero in degree $s(n,d)$.*

Proof. The cases $d=1$ and $d=2$ are dealt with similarly as in Proposition 3.1.

We now consider the case $d=3$. Let F be a form such that $\ell_i^2 \circ F = 0$ for all i. Since F is in the inverse system of the ideal generated by $n+2$ squares of general linear forms, we can choose F of degree $s(n,2)$.

By the pigeonhole principle and the general Leibniz rule, we have $\ell_i^3 \circ F^2 = 0$. The degree of F^2 is $2s(n,2)$. Since

$$s(n,2) = \left\lfloor \frac{n(n+2)}{2(n+1)} \right\rfloor = \left\lfloor \frac{n(n+1)}{2(n+1)} + \frac{n}{2(n+1)} \right\rfloor = \left\lfloor \frac{n}{2} + \frac{n}{2(n+1)} \right\rfloor = \frac{n}{2}$$

and

$$s(n,3) = \left\lfloor \frac{n(n+2)}{n+1} \right\rfloor = \left\lfloor n + \frac{n+1}{n+2} \right\rfloor = n,$$

we get that the degree of F^2 equals $s(n,3)$, which shows that the value of the Hilbert function is nonzero in degree $s(n,3)$. □

Lemma 3.3. *Let n be even and let $4 \le d \le n+1$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is nonzero in degree $s(n,d)$.*

Proof. Let F_i be such that $\ell_i \circ F_i = 0$ and $\ell_j^2 \circ F_i = 0$ for all j. Since F_i is in the inverse system of the ideal generated by one general linear form and $n+1$ squares of general linear forms, we can choose F_i to be of degree $s(n-1,2)$.

Next, let G be such that $\ell_i \circ G = 0$ for $i \ge d$, and $\ell_i^2 \circ G = 0$ for all i. Now G is in the inverse system of the ideal generated by $n+2-d+1$ general linear forms and $d-1$ squares of general linear forms, so we can choose G of degree $s(n-(n+2-d+1),2) = s(d-3,2)$.

It follows by the pigeonhole principle and the general Leibniz rule that $\ell_i^d \circ GF_1 \cdots F_{d-1} = 0$ for $i = 1, \ldots, n+2$, and we are done if we can show that the degree of the form $GF_1 \cdots F_{d-1}$ is equal to $s(n,d)$, that is, that $s(d-3,2) + (d-1)s(n-1,2) = s(n,d)$.

Suppose first that d is odd and write $d-1 = 2c$. We get

$$s(d-3,2) = \left\lfloor \frac{(d-1)(d-3)}{2(d-2)} \right\rfloor = \left\lfloor \frac{c(2c-2)}{2c-1} \right\rfloor = \left\lfloor c - \frac{c}{2c-1} \right\rfloor = c-1,$$

$$(d-1) \cdot s(n-1,2) = cn,$$

$$s(n,d) = \left\lfloor \frac{(n+2)nc}{n+1} \right\rfloor = \left\lfloor nc + \frac{nc}{n+1} \right\rfloor = nc + \left\lfloor c - \frac{c}{n+1} \right\rfloor = nc + c - 1,$$

where we in the last step have used that $c < n+1$. This proves the case d odd.

Suppose now that d is even. We get

$$\begin{aligned} s(d-3,2) &= \frac{d-2}{2}, \\ (d-1)\cdot s(n-1,2) &= \frac{n(d-1)}{2}, \\ s(n,d) &= \left\lfloor \frac{(n+2)n(d-1)}{2(n+1)} \right\rfloor = \left\lfloor \frac{n(d-1)}{2} + \frac{n(d-1)}{2(n+1)} \right\rfloor = \frac{n(d-1)}{2} + \left\lfloor \frac{n(d-1)}{2(n+1)} \right\rfloor \\ &= \frac{n(d-1)}{2} + \left\lfloor \frac{d-1}{2} - \frac{d-1}{2(n+1)} \right\rfloor = \frac{n(d-1)}{2} + \left\lfloor \frac{d}{2} - \frac{1}{2} - \frac{d-1}{2(n+1} \right\rfloor \\ &= \frac{n(d-1)}{2} + \frac{d}{2} - 1, \end{aligned}$$

where we in the last step have used that $d-1 < n+1$. This finishes the proof. □

Theorem 3.4. *The degree of the Hilbert series for* $R_{n,n+2,d}$ *equals* $s(n,d)$.

Proof. The case n odd was established by Nagel and Trok, so we only need to consider the case n even. Moreover, by [Nagel and Trok 2019, Theorem 4.4], the degree of the Hilbert series of $R_{n,n+2,d}$ is less than or equal to $s(n,d)$, so it is sufficient to prove that the $R_{n,n+2,d}$ is nonzero in degree $s(n,d)$.

Write $d = c + a(n+1)$, where $1 \le c \le n+1$. By Lemmas 3.2 and 3.3, there is a form F_c of degree $s(n,c)$ such that $\ell_i^c \circ F_c = 0$ for $i = 1, \dots, n+2$. Let $F = F_1 \cdots F_{n+2}$, where F_i is such that $\ell_i \circ F_i = 0$ and $\ell_j^2 \circ F_i = 0$ for all j. Then, by the pigeonhole principle and the general Leibniz rule, we have that $\ell^{(n+1)+1} \circ F = 0$, or more generalized, that $\ell^{a(n+1)+1} \circ F^a = 0$.

It follows that $\ell_i^{c+a(n+1)} \circ F_c F^a = 0$. Thus we are done if we can show that the degree of $F_c F^a$ is equal to $s(n,d)$, that is, that $s(n,c) + (n+2)a \cdot s(n-1,2) = s(n,d)$, which we verify by the calculation

$$\begin{aligned} s(n,d) &= \left\lfloor \frac{(n+2)n(c+a(n+1)-1)}{2(n+1)} \right\rfloor = \left\lfloor \frac{(n+2)n(a(n+1))}{2(n+1)} + \frac{(n+2)n(c-1)}{2(n+1)} \right\rfloor \\ &= \frac{(n+2)na}{2} + \left\lfloor \frac{(n+2)n(c-1)}{2(n+1)} \right\rfloor = (n+2)a \cdot s(n-1,2) + s(n,c). \end{aligned}$$ □

4. An upper bound for the smallest inflection point of the Hilbert function of a complete intersection

In order to use the results from the previous section to draw conclusions about the WLP, we need an upper bound for the degree of the expected Hilbert series

$$\left[\frac{(1-t^d)^{n+2}}{(1-t)^n} \right] \tag{3}$$

for $R_{n,n+2,d}$ given by Fröberg's conjecture. Since

$$\frac{(1-t^d)^{n+2}}{(1-t)^n} = (1-t)^2(1+t+\cdots+t^{d-1})^{n+2},$$

we are interested in the lowest degree where the coefficients of the polynomial $(1-t)^2(1+t+\cdots+t^{d-1})^{n+2}$

are nonpositive. We will provide the necessary bounds by induction on n. The induction step is Lemma 4.1 below and the base of the induction is given in Lemma 4.2.

For the statements of these lemmas, we introduce the following notation. For a sequence $a_0, a_1, \dots, a_n$ of integers, let $\Delta(a)$ be the sequence of differences $\Delta(a) = a_0, a_1 - a_0, \dots, a_n - a_{n-1}, -a_n$. For simplicity we will assume that all sequences are zero outside the range of indices for which they are defined. The generating series of this sequence is $\sum_{i=0}^{n+1} \Delta(a)_i t^i = (1-t)\sum_{i=0}^{n} a_i t^i$. Instead of looking at the range of indices for which the coefficients of the polynomial in (3) are nonpositive, we will look at where the first difference of the coefficients of $(1+t+\cdots+t^{d-1})^{n+2}$ are decreasing.

Lemma 4.1. *Let $n \geq 4$, let $d \geq 1$, and let*

$$\sum_{i=0}^{n(d-1)} a_i t^i = (1+t+\cdots+t^{d-1})^n \quad \textit{and} \quad \sum_{i=0}^{(n+1)(d-1)} b_i t^i = (1+t+\cdots+t^{d-1})^{n+1}.$$

Suppose that for some $s \in \frac{1}{2}\mathbb{N}$ with $\frac{1}{2}n(d-1) - s \geq \frac{1}{2}(d-1)$ we have that

$$\Delta(a)_i \geq \Delta(a)_{i+1} \quad \textit{for } s \leq i \leq (d-1)n - s \tag{4}$$

and

$$\Delta(a)_{s-j} \geq \Delta(a)_{s+j+1} \quad \textit{for } 0 \leq j \leq s. \tag{5}$$

Then

$$\Delta(b)_i \geq \Delta(b)_{i+1} \quad \textit{for } s + \tfrac{1}{2}(d-1) \leq i \leq (d-1)(n+1) - \big(s + \tfrac{1}{2}(d-1)\big) \tag{6}$$

and

$$\Delta(b)_{s+(d-1)/2-j} \geq \Delta(b)_{s+(d-1)/2+j+1} \quad \textit{for } 0 \leq j \leq s + \tfrac{1}{2}(d-1). \tag{7}$$

In all cases, the index $j \in \frac{1}{2}\mathbb{N}$ takes only values that make the indices integers.

Proof. For simplicity, we will denote $\frac{1}{2}(d-1)$ by m, which is an integer when d is odd and a half integer for even d. Observe that the sequences $\Delta(a)_i$ and $\Delta(b)_i$ are antisymmetric around $mn + \frac{1}{2}$ and $m(n+1) + \frac{1}{2}$ respectively and that they are positive in the first half and negative in the second half. In particular, this gives that it is sufficient to prove (6) for $i < m(n+1)$.

We have that $(1-t)(1+t+\cdots+t^{d-1})^{n+1} = (1-t^d)(1+t+\cdots+t^{d-1})^n$, which shows that $\Delta(b)_i = a_i - a_{i-d}$ and

$$\Delta(b)_i - \Delta(b)_{i+1} = a_i - a_{i-d} - a_{i+1} + a_{i+1-d} = \Delta(a)_{i+1-d} - \Delta(a)_{i+1} = \Delta(a)_{i-2m} - \Delta(a)_{i+1}. \tag{8}$$

Hence we can use (4) to prove (6) when $i - 2m \geq s$ and $i \leq 2mn - s$, i.e., in the range $s + 2m \leq i \leq 2mn - s$.

Since, by the antisymmetry of $\Delta(b)_i$, we do not need to look at $i \geq m(n+1)$ and since we have that $2mn - s \geq m(n+1)$ by the assumption that $mn - s \geq m$, it only remains to prove (6) for i in the range $s + m \leq i < s + 2m$. In order to do this, we use (5), and for $0 \leq j \leq m$, we have that

$$\Delta(a)_{s-m+j} \geq \Delta(a)_{s+m-j+1}.$$

Moreover, by (4), we have

$$\Delta(a)_{s+m-j+1} \geq \Delta(a)_{s+m+j+1},$$

which with $i = s+m+j$ in (8) now gives

$$\Delta(b)_{s+m+j} \geq \Delta(b)_{s+m+j+1} \quad \text{for } 0 \leq j \leq m,$$

where we only consider the j that make $j+m$ an integer. This finishes the proof of (6).

For (7) we have two cases, $j \geq m$ and $j \leq m$. In the first case we write

$$\Delta(b)_{s+m-j} \geq \Delta(b)_{s+m+j+1} \iff a_{s+m-j} - a_{s-m-j-1} \geq a_{s+m+j+1} - a_{s-m+j},$$

and the latter can be written

$$\Delta(a)_{s+m-j} + \cdots + \Delta(a)_{s-m-j} \geq \Delta(a)_{s+m+j+1} + \cdots + \Delta(a)_{s-m+j+1}.$$

This holds termwise because of (5) if $j \geq m$.

For the second case, we write

$$\Delta(b)_{s+m-j} \geq \Delta(b)_{s+m+j+1} \iff a_{s-m+j} - a_{s-m-j-1} \geq a_{s+m+j+1} - a_{s+m-j},$$

and the latter can be written as

$$\Delta(a)_{s-m-j} + \cdots + \Delta(a)_{s-m+j} \geq \Delta(a)_{s+m+j+1} + \cdots + \Delta(a)_{s+m-j+1},$$

and again this holds termwise because of (5) when $j \leq m$. This finishes the proof of (7). □

Even though we do have the WLP for $R_{3,4,d}$, we start the induction with this as the base case. The following lemma gives us the value of s that can be used in the induction step of Lemma 4.1.

Lemma 4.2. *For $d > 1$, let $a_0 + a_1 t + \cdots + a_{4(d-1)} t^{4(d-1)} = (1+t+\cdots+t^{d-1})^4$. Then for $s = \lfloor \frac{4}{3}(d-1) \rfloor$,*

$$\Delta(a)_j \geq \Delta(a)_{j+1} \quad \textit{for } s \leq j \leq 4(d-1)-s \qquad \textit{and} \qquad \Delta(a)_{s-j} \geq \Delta(a)_{s+j+1} \quad \textit{for } 0 \leq j \leq s.$$

Proof. The coefficients of the polynomial $(1-t^d)^3/(1-t)^3 = (1+t+\cdots+t^{d-1})^3$ are unimodal and symmetric around degree $\frac{3}{2}(d-1)$. Hence the coefficients of the polynomial

$$(1-t^d)^4/(1-t)^3 = \Delta(a)_0 + \Delta(a)_1 t + \cdots + \Delta(a)_{4d-3}$$

are antisymmetric around $\frac{1}{2}(4(d-2)+1) = 2d-2$, having positive coefficients up to degree $2d-2$ and thereafter negative coefficients satisfying $\Delta(a)_{2d-2-i} = -\Delta(a)_{2d-1+i}$ for $0 \leq i \leq 2d-2$.

We can write down an explicit formula for the positive coefficients as

$$\Delta(a)_j = \begin{cases} \binom{j+2}{2} & \text{for } 0 \leq j \leq d-1, \\ \binom{j+2}{2} - 4\binom{j+2-d}{2} & \text{for } d \leq j \leq 2d-2. \end{cases}$$

The second expression can be written as a quadratic polynomial in j as

$$g(j) = \binom{j+2}{2} - 4\binom{j+2-d}{2} = -\frac{3}{2}\left(j^2 - \left(\frac{8d}{3} - 3\right)j + \frac{4d^2}{3} - 4d + 2\right),$$

which is symmetric around $j = \frac{4}{3}d - \frac{3}{2}$. Thus we have that $\Delta(a)_j \geq \Delta(a)_{j+1}$ when

$$\tfrac{1}{2}(j + (j+1)) \geq \tfrac{4}{3}d - \tfrac{3}{2} \iff j \geq \tfrac{4}{3}d - 2 \iff j \geq s$$

for the positive coefficients. For the negative coefficients, we use the antisymmetry to conclude that $\Delta(a)_j \geq \Delta(a)_{j+1}$ when $j \leq s + 2 \cdot (2d - 2 - s) = 4(d-1) + s$.

Moreover, for $0 \leq j \leq s$, we have $\Delta(a)_{s-j} \geq g(s-j)$ and $\Delta(a)_{s+j+1} = g(s+j+1)$ when $\Delta(a)_{s+j+1}$ is positive. Thus it is enough to verify that $g(s-j) \geq g(s+j+1)$. Indeed,

$$\frac{(s-j)+(s+j+1)}{2} \geq \frac{4d}{3} - \frac{3}{2} \iff s \geq \frac{4d}{3} - 2 = \left\lfloor \frac{4(d-1)}{3} \right\rfloor. \qquad \square$$

We now use Lemmas 4.1 and 4.2 to give an upper bound on the degree of the expected Hilbert series for $n+2$ general forms of degree d.

Proposition 4.3. *An upper bound for the smallest inflection point of the Hilbert function of a complete intersection generated by $n+2$ forms of degree d in $n+2 \geq 4$ variables is given by $\lfloor \frac{4}{3}(d-1) \rfloor + \frac{1}{2}(n-2)(d-1)$; that is, for $n \geq 2$, we have*

$$\deg\left(\left[\frac{(1-t^d)^{n+2}}{(1-t)^n}\right]\right) \leq \left\lfloor \frac{4(d-1)}{3} \right\rfloor + \frac{(n-2)(d-1)}{2}.$$

Proof. Induction on n with Lemma 4.2 as the base case and with Lemma 4.1 as the induction step proves the statement about the inflection point. As seen before, this inflection point corresponds to the degree of the expected Hilbert series, since

$$\left[\frac{(1-t^d)^{n+2}}{(1-t)^n}\right] = \left[(1-t)^2 \frac{(1-t^d)^{n+2}}{(1-t)^{n+2}}\right]. \qquad \square$$

The upper bound in Proposition 4.3 is far from being sharp, and in the proof of Theorem 4.5 we will need a better bound in some cases. For this purpose we will in Lemma 4.4 below give a more general version of Proposition 4.3 that could be used with a different base case than Lemma 4.2. Lemma 4.4 also gives the connection to the WLP that will be used in Theorem 4.5.

Lemma 4.4. *Let $n \geq 4$ and $d \geq 1$, and suppose that $\tilde{s}$ is an integer such that the assumptions in Lemma 4.1 are satisfied. If $\tilde{s} < s(n-2, d)$, then*

$$\deg\left(\left[\frac{(1-t^d)^{n+k}}{(1-t)^{n-2+k}}\right]\right) < s(n-2+k, d) \quad \textit{for all even integers } k \geq 0,$$

and in particular, $R_{n-1+k,n+k,d}$ fails the WLP for all even integers $k \geq 0$.

Proof. If n is even, a computation reveals that $s(n+k, d) - s(n-2+k, d) \geq d-1$, and if n is odd, then $s(n+k, d) - s(n-2+k, d) = d-1$. From this we conclude that $s(n-2, d) + \frac{1}{2}(k+2)(d-1) \leq s(n+k, d)$.

On the other hand, repeated use of Lemma 4.1 gives $\deg([(1-t^d)^{n+k}/(1-t)^{n-2+k}]) \leq \tilde{s} + k \cdot \frac{1}{2}(d-1)$. Thus we have

$$\deg\left(\left[\frac{(1-t^d)^{n+k}}{(1-t)^{n-2+k}}\right]\right) \leq \tilde{s} + k \cdot \frac{d-1}{2} < s(n-2, d) + ((k-2)+2) \cdot \frac{d-1}{2} \leq s(n+k-2, d).$$

The second part of the proposition follows since by (1), the WLP for $R_{m+1,m+2,d}$ fails if the Hilbert series for $R_{m,m+2,d}$ does not equal $[(1-t^d)^{m+2}/(1-t)^m]$. $\square$

Theorem 4.5. *Let $n \geq 4$ and $d \geq 2$. Then $R_{n,n+1,d}$ fails the WLP except possibly for*

$$(n, d) \in \{(4, 2),\ (5, 2),\ (5, 3),\ (5, 5),\ (7, 2),\ (7, 3),\ (9, 2),\ (9, 3),\ (11, 2),\ (11, 3)\}.$$

Proof. According to Proposition 4.3, we have

$$\deg\left(\left[\frac{(1-t^d)^{n+2}}{(1-t)^n}\right]\right) \leq \left\lfloor \frac{4(d-1)}{3} \right\rfloor + (n-2)\frac{d-1}{2},$$

and by Lemma 4.4, the WLP fails for $R_{n+1,n+2,d}$ if

$$s(n, d) > \left\lfloor \frac{4(d-1)}{3} \right\rfloor + (n-2)\frac{d-1}{2}.$$

Thus for even n the WLP fails for $R_{n+1,n+2,d}$ if

$$\left\lfloor \frac{n(n+2)(d-1)}{2(n+1)} \right\rfloor - \left\lfloor \frac{4(d-1)}{3} \right\rfloor \geq (n-2)\frac{d-1}{2} + 1.$$

In order to show this, it is sufficient to show that

$$\frac{n(n+2)(d-1)}{2(n+1)} - \frac{4(d-1)}{3} \geq (n-2)\frac{d-1}{2} + 2,$$

which can be written as

$$d \geq 2 + 12 \cdot \frac{n+1}{n-2},$$

which for $n = 4$ gives $d \geq 32$.

In the same way, for odd $n > 2$, we want to show that

$$\frac{(n+1)(d-1)}{2} - \left\lfloor \frac{4(d-1)}{3} \right\rfloor \geq (n-2)\frac{d-1}{2} + 1,$$

and here it is sufficient to show that

$$\frac{(n+1)(d-1)}{2} - \frac{4(d-1)}{3} \geq (n-2)\frac{d-1}{2} + 1,$$

which is equivalent to $d \geq 7$.

Thus to this point we have by Lemma 4.4 that $R_{n+1,n+2,d}$ fails for even $n \geq 4$ and $d \geq 32$, and for odd $n \geq 3$ and $d \geq 7$.

For the remaining cases we introduce the notation

$$\tilde{s}(n, d) = \min\{s : s \text{ satisfies the hypotheses in Lemma 4.1}\}.$$

For $n = 3$, we check with [Macaulay2] that, in the range $2 \leq d < 7$ except for $d = 2$, we have $s(3, d) > \tilde{s}(3, d)$, so by Lemma 4.4, the WLP for $R_{n+1,n+2,d}$ fails for all odd $n \geq 3$ and $d > 2$. Since $s(5, 2) = 3 > \tilde{s}(5, 2) = 2$, we also get that the WLP for $R_{n+1,n+2,d}$ fails for all odd $n \geq 5$ and $d = 2$.

For $n = 4$, we check that $s(4, d) > \tilde{s}(4, d)$ in the range $2 \leq d < 32$ except for $d \in \{2, 3, 5\}$. Since $s(6, 5) = 13 > \tilde{s}(6, 5) = 12$, we have that the WLP for $R_{n+1,n+2,d}$ fails for all even $n \geq 6$ and $d > 3$ according to Lemma 4.4. For $d \in \{2, 3\}$ and even n, we need to go to $n = 12$ to get $s(12, 2) = 6 > \tilde{s}(12, 2) = 5$ and $s(12, 3) = 12 > \tilde{s}(12, 3) = 11$.

Thus we have shown that the WLP for $R_{n,n+1,d}$ fails for all $n \geq 4$ and $d \geq 2$ except possibly for

$$(n, d) \in \{(4, 2),\ (5, 2),\ (5, 3),\ (5, 5),\ (7, 2),\ (7, 3),\ (9, 2),\ (9, 3),\ (11, 2),\ (11, 3)\}. \qquad \square$$

5. Explicit formulas, the remaining cases, and the proof of Theorem 1.1

As we saw from the previous section there are ten cases to consider in order to finish the proof of our main theorem. Four of them do satisfy the WLP, and now we have to deal with the remaining cases, which are

$$(n, d) \in \{(5, 5),\ (7, 3),\ (9, 2),\ (9, 3),\ (11, 2),\ (11, 3)\}.$$

The case (5, 5) was handled by Migliore, Miró-Roig and Nagel in [Migliore et al. 2012], the case (7, 3) by Ilardi and Vallès [2019] and the cases (9, 2) and (11, 2) by Sturmfels and Xu [2010]. Thus there are two remaining cases: $(n, d) = (9, 3)$ and $(n, d) = (11, 3)$. We will now deal with these two cases, but we will also give new arguments for the other four since the method we use is the same.

We will for each case provide a set of elements of generators for the inverse system in the top degree that shows that the Hilbert function of $R_{n-1,n+1,d}$ is not the one expected from the Fröberg conjecture.

We start by establishing an explicit formula for the form of degree $\frac{1}{2}(n+1)$ in $\Bbbk[X_1, X_2, \ldots, X_n]$ that is annihilated by the squares of $n+2$ general linear forms when n is odd. We will do this in two different ways with different sets of parameters. The two versions are useful in different situations. In the first version, we observe that for $n+2$ general forms we can, by a change of variables, assume that n are the variables and one is the sum of the variables. The last form will have general coefficients.

Observe that the references to the result by Sturmfels and Xu [2010] in Section 3 can be replaced by the use of Theorem 5.1 to get a completely self-contained proof of our main result.

We will use $V(y_1, y_2, \ldots, y_m)$ to denote the Vandermonde determinant in variables $y_1, y_2, \ldots, y_m$.

Theorem 5.1. *Let $n = 2k - 1$ for a positive integer k. The form*

$$F = \det \begin{bmatrix} X_1 & a_1X_1 & a_1^2X_1 & \cdots & a_1^{k-1}X_1 & a_1 & a_1^2 & \cdots & a_1^{k-1} \\ X_2 & a_2X_2 & a_2^2X_2 & \cdots & a_2^{k-1}X_2 & a_2 & a_2^2 & \cdots & a_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_n & a_nX_n & a_n^2X_n & \cdots & a_n^{k-1}X_n & a_n & a_n^2 & \cdots & a_n^{k-1} \end{bmatrix}$$
$$= \frac{1}{k!(k-1)!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) V(a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_k}) V(a_{\sigma_{k+1}}, a_{\sigma_{k+2}}, \ldots, a_{\sigma_n}) \prod_{j=k+1}^{n} a_{\sigma_j} \prod_{j=1}^{k} X_{\sigma_j}$$

is the unique form of degree k in $\Bbbk[X_1, X_2, \ldots, X_n]$ that is annihilated by the squares of the linear forms $x_1, x_2, \ldots, x_n,\ x_1 + x_2 + \cdots + x_n$, and $a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

Proof. By work of Nagel and Trok [2019], there is a unique such form and it is sufficient for us to prove that this particular form is annihilated by the squares of the linear forms. The equality between the two formulas follows from the generalized Laplace expansion over the first k columns. Since the form is square-free, it is annihilated by the squares of the variables and it remains for us to check that it is annihilated by the squares of the last two linear forms, $\ell_{n+1} = x_1 + x_2 + \cdots + x_n$ and $\ell_{n+2} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

We can start by computing $\ell_{n+1}^2 \circ F = (x_1 + x_2 + \cdots + x_n)^2 \circ F$, where we by the Leibniz rule for determinants get a sum over terms where we substitute $X_i = 1$ for $i = 1, 2, \ldots, n$ in two of the first k columns. In all these terms, there will be a repeated column so all terms are zero.

In the same way we get that $\ell_{n+1}^2 \circ F = (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 \circ F = 0$. This time we substitute $X_i = a_i$ for $i = 1, 2, \ldots, n$ in two of the first k columns, which again results in repeated columns. □

For the second version of this formula, we observe that $n+2$ general points in $\mathbb{P}^{n-1}$ are on a rational normal curve and we can, by a change of coordinates, assume that this curve is the moment curve with parametrization $(1 : t : t^2 : \cdots : t^{n-1})$. Thus we can assume that the $n+2$ linear forms are given by $\ell_i = \sum_{j=1}^n \alpha_i^{j-1} x_j$ for $i = 1, 2, \ldots, n+2$, where $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ are general elements of the field $\Bbbk$.

Theorem 5.2. *For $n = 2k-1$, the form of degree $k = \frac{1}{2}(n+1)$ that is annihilated by the squares of the linear forms $\ell_i = \sum_{j=1}^n \alpha_i^{j-1} x_j$ for $i = 1, 2, \ldots, n+2$ is given by*

$$F = \det \begin{bmatrix} x_1 & x_2 & \cdots & x_k \\ x_2 & x_3 & \cdots & x_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k+1} & \cdots & x_n \end{bmatrix}.$$

Proof. Again we use that Nagel and Trok [2019] have shown that there is a unique such form and it is sufficient for us to prove that this determinant is annihilated by the squares of the linear forms.

We observe that $\ell_i^2 \circ F$ is given by a sum with signs over all ways of substituting x_j with α_i^{j-1} in two of the rows of the matrix. These two rows become linearly dependent and thus $\ell_i^2 \circ F = 0$ for $i = 1, 2, \ldots, n+2$. □

The advantage of this second version is that the formula does not depend on the parameters. Moreover, we can also use it to find formulas for the unique forms that are annihilated by some of the linear forms and the squares of the remaining linear forms.

Theorem 5.3. *For $0 < k \le \frac{1}{2}(n+1)$, the unique form of degree k that is annihilated by the linear forms $\ell_i = \sum_{j=1}^n \alpha_i^{j-1} x_j$ for $i = 1, 2, \ldots, n-2k+1$ and by the squares of the remaining $2k+1$ linear forms is given by*

$$F = \det \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-k} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{n-2k+1} & \alpha_{n-2k+1}^2 & \cdots & \alpha_{n-2k+1}^{n-k} \\ x_1 & x_2 & x_3 & \cdots & x_{n+1-k} \\ x_2 & x_3 & x_4 & \cdots & x_{n+2-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k+1} & x_{k+2} & \cdots & x_n \end{bmatrix}.$$

Proof. The uniqueness is given by Nagel and Trok [2019] and it is enough for us to show the vanishing.

We have that $\ell^2 \circ F = 0$ for any linear form $\ell = \sum_{j=1}^{n} \alpha^{j-1} x_j$ for the same reason as in the previous theorem. Applying ℓ_i, where $i = 1, 2, \dots, n-2k+1$, to F gives a sum over the k determinants we get by replacing x_j with α_i^{j-1} in each of the k lowest rows. Hence $\ell_i \circ F = 0$ for $i = 1, 2, \dots, n-2k+1$. □

We can now treat the sporadic cases not covered by Theorem 4.5 where the WLP fails.

Theorem 5.4. *The WLP fails for* $R_{n,n+1,d} = \Bbbk[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$ *in the cases* $(n, d) \in \{(5, 5), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}$. *In particular, the Hilbert series of* $R_{4,6,5}$, $R_{6,8,3}$, $R_{8,10,2}$, $R_{8,10,3}$, $R_{10,12,2}$, *and* $R_{10,12,3}$ *are*:

ring	*Hilbert series*
$R_{4,6,5}$	$1+4t+10t^2+20t^3+35t^4+50t^5+60t^6+60t^7+45t^8+14t^9$
$R_{8,10,2}$	$1+6t+21t^2+48t^3+78t^4+84t^5+43t^6$
$R_{8,10,2}$	$1+8t+26t^2+40t^3+16t^4$
$R_{8,10,3}$	$1+8t+36t^2+110t^3+250t^4+432t^5+561t^6+492t^7+171t^8$
$R_{10,12,2}$	$1+10t+43t^2+100t^3+121t^4+32t^5$
$R_{10,12,3}$	$1+10t+55t^2+208t^3+595t^4+1342t^5+2431t^6+3520t^7+3916t^8+2860t^9+683t^{10}$

These differ in the leading term from, respectively, the $10t^9$, $42t^6$, $15t^4$, $135t^8$, $22t^5$ *and* $88t^{10}$ *that are expected by the Fröberg conjecture.*

Proof. We consider the ring $R_{n,n+2,d}$ and denote our set of $n+2$ general linear forms by

$$\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_{n+2}\} = \left\{ \sum_{i=1}^{n} a_1^{i-1} x_i, \sum_{i=1}^{n} a_2^{i-1} x_i, \dots, \sum_{i=1}^{n} a_{n+2}^{i-1} x_i \right\},$$

where $a_1, a_2, \dots, a_{n+2}$ are general elements of $\Bbbk$.

Using Theorem 5.3, we can find a formula for the unique form of degree k that is annihilated by $n-2k+1$ of the $n+2$ linear forms in $\mathcal{L}$ and by the squares of the remaining $2k+1$ linear forms in $\mathcal{L}$. For $\mathcal{S} \subseteq \mathcal{L}$ of size $n-2k+1$, we denote this unique form by $F_{\mathcal{S}}$. Observe that the coefficients of $F_{\mathcal{S}}$ are polynomials in the parameters $a_1, a_2, \dots, a_{n+2}$. In each of the six cases of the theorem, we produce a set of forms in the inverse system of $R_{n,n+2,d}$ in the top degree such that the dimension of the subspace they span agrees with the stated value of the Hilbert function in the socle degree. It will be enough to verify this for a specialization of the parameters, since a specialization can only lower the dimension. Thus this gives a lower bound for the Hilbert function in the socle degree. On the other hand, the computation of the Hilbert function of $R_{n,n+2,d}$ for a specialization provides an upper bound. Since they agree we can make the desired conclusion.

In all the cases, we use the specialization $a_i = i-1$ for $i = 1, 2, \dots, n+2$ to verify the dimension using Macaulay2.

For $R_{4,6,5}$ we need 14 linearly independent forms of degree 9. We use forms that can be written as $F = F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_3} F_{\mathcal{S}_4} F_{\mathcal{S}_5} F_{\mathcal{S}_6}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 3$ and $|\mathcal{S}_4| = |\mathcal{S}_5| = |\mathcal{S}_6| = 1$, such that each linear form in $\mathcal{L}$ is contained in exactly two of the six subsets. Observe that the three first factors are linear and the three last are quadratic. For each $\ell \in \mathcal{L}$ we get $\ell^5 \circ F = 0$ by the pigeonhole principle and the general Leibniz rule since ℓ annihilates two of the factors and ℓ^2 annihilates the remaining four factors.

For $R_{6,8,3}$ we need 43 linearly independent forms of degree 6. We use forms that can be written as $F_{\mathcal{S}_1}F_{\mathcal{S}_2}F_{\mathcal{S}_3}F_{\mathcal{S}_4}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = 5$ and $|\mathcal{S}_3| = |\mathcal{S}_4| = 3$, and each linear form in $\mathcal{L}$ is contained in exactly two of the subsets. These forms are annihilated by the squares of all linear forms in $\mathcal{L}$ since each linear form annihilates two of the factors and the square of the linear form annihilates the remaining two factors.

For $R_{8,10,2}$ we need 16 linearly independent forms of degree 4. These can be obtained as $F = F_{\mathcal{S}}F_{\mathcal{L}\setminus\mathcal{S}}$ for subsets $\mathcal{S}$ of size five. These are products of two quadrics and they are annihilated by the squares of the linear forms in $\mathcal{L}$ since for each ℓ in $\mathcal{L}$ we have that ℓ annihilates one of the factors and ℓ^2 annihilates the other.

For $R_{8,10,3}$ we will produce a set of 171 linearly independent forms of degree 8 that are annihilated by the cubes of the linear forms. These forms are obtained as $F = F_{\mathcal{S}_1}F_{\mathcal{S}_2}F_{\mathcal{S}_2}F_{\mathcal{S}_2}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = |\mathcal{S}_4| = 5$ and each linear form in $\mathcal{L}$ is contained in two of the subsets. Thus we get that $\ell^3 \circ (F_{\mathcal{S}_1}F_{\mathcal{S}_2}F_{\mathcal{S}_2}F_{\mathcal{S}_2}) = 0$ for all $\ell \in \mathcal{L}$ since ℓ annihilates two of the factors and ℓ^2 annihilates the remaining two.

For $R_{10,12,2}$ we provide a set of 32 linearly independent forms of degree 5 that are annihilated by the squares of the linear forms in $\mathcal{L}$. We do this by forms $F = F_{\mathcal{S}}F_{\mathcal{L}\setminus\mathcal{S}}$ for subsets $\mathcal{S}$ of size five. Each linear form ℓ in $\mathcal{L}$ annihilates one of the factors and ℓ^2 the other. Hence $\ell^2 \circ F_{\mathcal{S}}F_{\mathcal{L}\setminus\mathcal{S}} = 0$.

For $R_{10,12,3}$ we provide a set of 683 linearly independent forms of degree 10 that are annihilated by the cubes of the linear forms in $\mathcal{L}$. We do this by forms $F = F_{\mathcal{S}_1}F_{\mathcal{S}_2}F_{\mathcal{S}_3}F_{\mathcal{S}_4}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = 5$, $|\mathcal{S}_3| = |\mathcal{S}_4| = 7$, and every $\ell \in \mathcal{L}$ is contained in two of the subsets. Now $\ell^3 \circ (F_{\mathcal{S}_1}F_{\mathcal{S}_2}F_{\mathcal{S}_3}F_{\mathcal{S}_4}) = 0$ for all $\ell \in \mathcal{L}$ since ℓ annihilates two of the factors and ℓ^2 annihilates the remaining two. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 4.5 we have that the WLP for $R_{n,n+1,d}$ fails when $n \geq 4$ and $d \geq 2$ expect possibly for the cases

$$(n, d) \in \{(4, 2),\ (5, 2),\ (5, 3),\ (5, 5),\ (7, 2),\ (7, 3),\ (9, 2),\ (9, 3),\ (11, 2),\ (11, 3)\}.$$

In the cases (4, 2), (5, 2), (5, 3) and (7, 2), we can verify by one example that they do satisfy the WLP and for the remaining cases Theorem 5.4 shows that they fail to satisfy the WLP.

Finally, we refer to the introduction for references to the cases $n = 2$ and $n = 3$, and for the cases $n = 1$ and $d = 1$, the WLP is trivially satisfied. □

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A classification of modular compactifications of the space of pointed elliptic curves by Gorenstein curves

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We classify the Deligne–Mumford stacks $\mathcal{M}$ compactifying the moduli space $\mathcal{M}_{1,n}$ of smooth n-pointed curves of genus one under the condition that the points of $\mathcal{M}$ represent Gorenstein curves with distinct smooth markings. This classification uncovers new moduli spaces $\overline{\mathcal{M}}_{1,n}(Q)$, which we may think of as coming from an enrichment of the notion of level used to define Smyth's m-stable spaces. Finally, we construct a cube complex of Artin stacks interpolating between the $\overline{\mathcal{M}}_{1,n}(Q)$'s, a multidimensional analogue of the wall-and-chamber structure seen in the log minimal model program for $\overline{\mathcal{M}}_g$.

1. Introduction

The moduli stack $\mathcal{M}_{g,n}$ of smooth genus g algebraic curves with n marked points is not proper, so one searches for compactifications, that is, proper Deligne–Mumford stacks $\mathcal{M}$ such that $\mathcal{M}_{g,n}$ embeds as a dense open substack of $\mathcal{M}$. In this paper we construct a new family of modular compactifications of $\mathcal{M}_{1,n}$. We then show that these moduli spaces exhaust the semistable modular compactifications of $\mathcal{M}_{1,n}$ with Gorenstein singularities and distinct markings.

Let us now set up some notation in order to give the definition of these new moduli spaces.

Definition 1.1. Given a positive integer n, let $\operatorname{Part}(n)$ be the set of partitions of $\{1, \dots, n\}$. Give $\operatorname{Part}(n)$ a partial order by $P_1 \preceq P_2$ if the partition P_1 is refined by the partition P_2.

Denote by $\mathfrak{Q}_n$ the collection of subsets $Q \subseteq \operatorname{Part}(n)$ such that

(i) Q is downward closed;

(ii) Q does not contain the discrete partition $\{\{1\}, \dots, \{n\}\}$.

Definition 1.2. Let p be a closed point of an algebraic curve C over an algebraically closed field k, and let $\nu : \widetilde{C} \to C$ be the normalization. The *number of branches at* p is

$$m(p) = |\nu^{-1}(p)|.$$

The *delta invariant of* p is

$$\delta(p) = \dim_k(\nu_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C).$$

The *genus of* p is

$$g(p) = \delta(p) - m(p) - 1.$$

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The genus of a singularity captures its contribution to the genus of C. In particular, if C is connected and proper, $p_1, \dots, p_e$ are the singularities of C, and $\widetilde{C}_1, \dots, \widetilde{C}_v$ are the irreducible components of $\widetilde{C}$, then the arithmetic genus of C is

$$g(C) = \sum_{i=1}^{e} g(p_i) + \sum_{j=1}^{v} g(\widetilde{C}_j) + b_1(\Delta_C),$$

where $b_1(\Delta_C)$ is the first Betti number of the simplicial complex Δ_C with vertices $\widetilde{C}_1, \dots, \widetilde{C}_v$ and, for each p_i, an $(m(p_i)-1)$-simplex whose vertices are glued to the components meeting $\nu^{-1}(p_i)$.

Definition 1.3. A closed point p of an algebraic curve C over an algebraically closed field is an *elliptic Gorenstein singularity* if $\mathscr{O}_{C,p}$ is Gorenstein and $g(p) = 1$.

It is shown in [Smyth 2011a] that the elliptic Gorenstein singularities are classified by their number of branches, m. If $m = 1$, p is a cusp; for $m = 2$, p is a tacnode; for $m \geq 3$, p is the union of the coordinate axes of $\mathbb{A}^{m-1}$ with one more line transverse to each of the coordinate hyperplanes of $\mathbb{A}^{m-1}$. Given such a singularity, we will call the irreducible components to which p belongs the *branches* of p.

Definition 1.4. A *subcurve* Z of a proper algebraic curve C over an algebraically closed field is a connected reduced closed subscheme of C.

Let $(C, p_1, \dots, p_n)$ be a curve of arithmetic genus one together with n marked closed points over an algebraically closed field. Let Z be a subcurve of C of genus one and let Σ be the divisor of markings. We define the *level of* Z, $\operatorname{lev}(Z)$, to be the partition of $\{1, \dots, n\}$ where $a, b \in \{1, \dots, n\}$ lie in the same subset if and only if the markings p_a and p_b lie in the same connected component of $(C - Z) \cup \Sigma$.

If $q \in C$ is an elliptic Gorenstein singularity, we say the *level of* q, $\operatorname{lev}(q)$ is the partition of $\{1, \dots, n\}$ where $a, b \in \{1, \dots, n\}$ lie in the same subset if and only if the markings p_a and p_b lie in the same connected component of the normalization of C at q (i.e., if the rational tails containing p_a and p_b are connected via a nodal path to the same branch of the singularity).

Remark 1.5. If Z_1 and Z_2 are two genus one subcurves of C and $Z_1 \subseteq Z_2$, then $\operatorname{lev}(Z_1) \preceq \operatorname{lev}(Z_2)$.

Remark 1.6. The level of C considered in [Smyth 2011a] is the cardinality $|\operatorname{lev}(Z)|$, where Z is the minimal subcurve of C of genus one.

The level of an elliptic Gorenstein singularity q defined here was called the "combinatorial type of C" in [Smyth 2011b, Definition 2.15].

Definition 1.7. Let $Q \in \mathfrak{Q}_n$. A Q-stable curve over a scheme S consists of

(i) $\pi : C \to S$, a flat and proper morphism of schemes, and

(ii) $\sigma_1, \dots, \sigma_n : S \to C$, sections of π with disjoint images,

such that, for each geometric fiber $(C_s, p_1, \dots, p_n)$,

(i) C_s is a connected reduced Gorenstein scheme of dimension 1 with arithmetic genus one;

(ii) (level condition on subcurves) if $Z \subseteq C_s$ is a subcurve of genus one, then $\operatorname{lev}(Z) \notin Q$;

(iii) (level condition on singularities) if $q \in Z$ is a genus one singularity, then $\operatorname{lev}(q) \in Q$;

(iv) $H^0(C, \Omega_C^\vee(-\Sigma)) = 0$.

We define the *moduli space* $\overline{\mathcal{M}}_{1,n}(Q)$ *of Q-stable n-marked curves of genus one* to be the stack over $\mathbb{Z}\left[\frac{1}{6}\right]$ whose S-points are the Q-stable curves over S.

Our first main result is that this defines a modular compactification of $\mathcal{M}_{1,n}$.

Theorem 5.1. *For each $Q \in \mathfrak{Q}_n$, $\overline{\mathcal{M}}_{1,n}(Q)$ is a proper irreducible Deligne–Mumford stack over $\mathbb{Z}\left[\frac{1}{6}\right]$ containing $\mathcal{M}_{1,n}$.*

When $Q = \{S \in \operatorname{Part}(n) : |S| \leq m\}$ for some m, we recover the m-stable compactification $\overline{\mathcal{M}}_{1,n}(m)$ of Smyth [2011a]. We may regard the spaces $\overline{\mathcal{M}}_{1,n}(Q)$ as "combinatorial remixes" of the m-stable spaces, since each of the curves of $\overline{\mathcal{M}}_{1,n}(Q)$ for some Q belong to some $\overline{\mathcal{M}}_{1,n}(m)$ for various m. Despite this, the Q-stable spaces are surprisingly plentiful: for $n = 5$, there are only 5 m-stable spaces, but 79,814,831 Q-stable spaces.

All of the Q-stable spaces arise from compatible choices of how to contract the universal curve of the moduli space of radially aligned log curves (defined in [Ranganathan et al. 2019] and [Santos-Parker 2017]), analogously to the "extremal assignments" of [Smyth 2013]. It was systematic enumeration of such contractions using the log-geometric techniques of [Bozlee 2020] that led to the discovery of the Q-stable spaces. This leads to a resolution of the rational map between the Deligne–Mumford–Knudsen space and each Q-stable space.

Theorem 4.13. *For each $Q \in \mathfrak{Q}_n$, there is a diagram of stacks*

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \\ \swarrow & & \searrow \\ \overline{\mathcal{M}}_{1,n} & & \overline{\mathcal{M}}_{1,n}(Q) \end{array}$$

such that both arrows are proper and restrict to an isomorphism on $\mathcal{M}_{1,n}$.

We will also find the construction of contractions of families of curves to be helpful sporadically throughout the paper.

Our next main theorem is that the Q-stable spaces account for all sufficiently nice modular compactifications in genus one, taking us one step further in the classification of modular compactifications of the moduli space of pointed algebraic curves. To that end, we introduce some definitions.

Definition 1.8. Let $\mathcal{U}_{1,n}$ be the stack of Gorenstein, connected, reduced curves of genus one with n distinct smooth marked points and no infinitesimal automorphisms. For the purposes of this paper, a *modular compactification* is an open Deligne–Mumford substack $\mathcal{M}$ of $\mathcal{U}_{1,n}$, proper over $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right]$.

In the language of [Smyth 2013], a modular compactification in our sense is a semistable modular compactification whose curves are Gorenstein with distinct smooth markings, except that the base is chosen as $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right]$ instead of $\operatorname{Spec} \mathbb{Z}$.

Theorem 1.9. *If $\mathcal{M}$ is a modular compactification of $\mathcal{M}_{1,n}$, then $\mathcal{M} = \overline{\mathcal{M}}_{1,n}(\mathcal{Q})$ for some $\mathcal{Q}$.*

We prove this classification theorem over the course of Section 6.

Finally, in Section 7, we construct a cube complex of mildly nonseparated Artin stacks interpolating between the $\overline{\mathcal{M}}_{1,n}(\mathcal{Q})$'s. This complex yields a multidimensional analogue of the wall-and-chamber structure seen in the log minimal model program for $\overline{\mathcal{M}}_g$.

This paper gives the first general classification of Gorenstein modular compactifications of $\mathcal{M}_{g,n}$ in genus greater than 0. In future work we hope to use similar ideas to construct and classify modular compactifications of $\mathcal{M}_{g,n}$. For instance, Battistella [2022] has constructed a sequence of modular compactifications of $\mathcal{M}_{2,n}$ parametrized by a level analogous to that of [Smyth 2011a], and our more flexible notion of level should also yield combinatorial variations of Battistella's moduli spaces.

It would also be natural to search for similar results on modular compactifications in which the marked points are permitted to come together, as in the spaces of weighted stable curves of [Hassett 2003]. The thesis of Andy Fry [2021] suggests that it is necessary to consider more general collisions of markings than those permitted by weights. This will be pursued in future work with Vance Blankers.

2. Examples

In this section we give some examples to illustrate the nature and variety of $\mathcal{Q}$-stable spaces. We start by describing how to count $\mathcal{Q}$-stability conditions.

Definition 2.1. An *antichain* in a partially ordered set P is a subset $A \subseteq P$ such that no distinct elements of A are comparable.

Proposition 2.2. *Let P be a finite partially ordered set. There is a bijection*

$$\{\mathcal{Q} \subsetneq P : \mathcal{Q} \text{ downward closed}\} \leftrightarrow \{A \subseteq P : A \text{ is a nonempty antichain}\}$$

given left-to-right by taking $\mathcal{Q}$ to the set of minimal elements of $P - \mathcal{Q}$, and right-to-left by taking A to the complement of the upward closure of A.

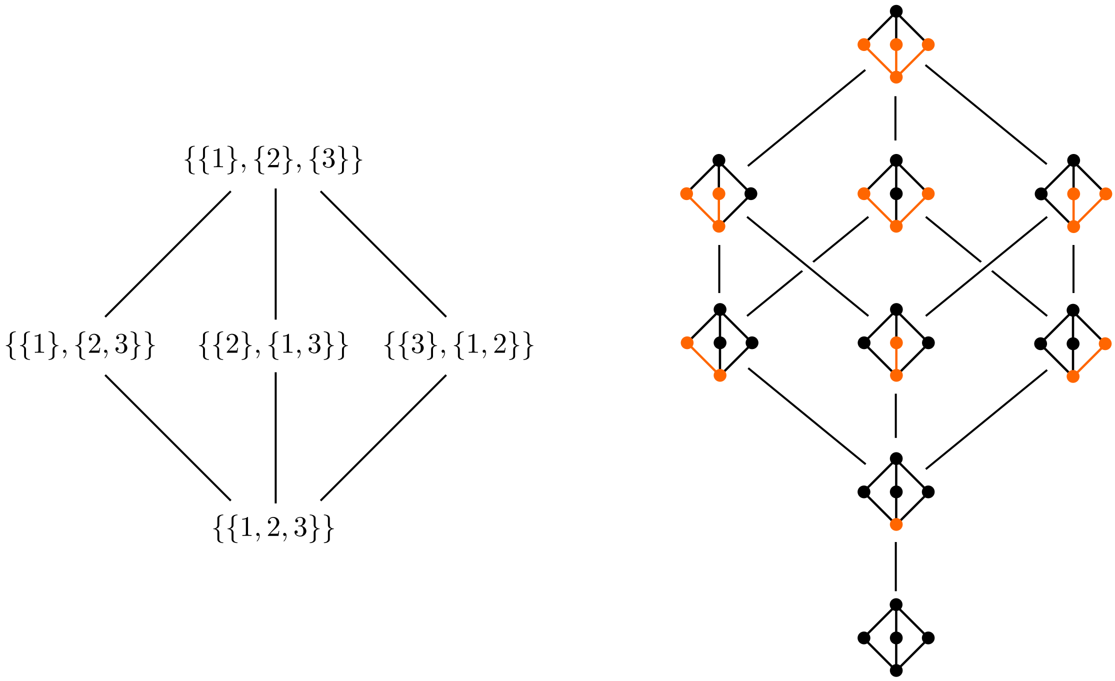
Proof. Omitted. □

The number of nonempty antichains of the lattice of partitions of n elements are counted in OEIS sequence A302251 [Machacek 2018]. We learn that there are

- 9 $\mathcal{Q}$-stable compactifications of $\mathcal{M}_{1,3}$,
- 346 $\mathcal{Q}$-stable compactifications of $\mathcal{M}_{1,4}$,
- 79,814,831 $\mathcal{Q}$-stable compactifications of $\mathcal{M}_{1,5}$.

By contrast, for a given n, there are only n compactifications of $\mathcal{M}_{1,n}$ by m-stable spaces.

Since the properties of being downward closed and of being a proper subset are preserved by finite union and intersection, the set $\mathfrak{Q}_n$ forms a lattice under union and intersection.



Part(3) downward closed proper subsets of Part(3)

Figure 1. The partially ordered set of partitions of $\{1, 2, 3\}$ and the lattice of Q-stability conditions for $n = 3$. An orange dot indicates that the corresponding partition on the left is included in Q.

Example 2.3. Consider the case $n = 3$. The Hasse diagram of Part(3) and the corresponding lattice of Q-stability conditions for $n = 3$ are displayed in Figure 1. Visually, $\mathfrak{Q}_3$ consists of a cube and a whisker: we will show later that the lattice is always a "union of cubes" and consider a way to fill in the interior of the cube.

We see that there are 9 Q-stable spaces for $n = 3$, in agreement with the count just above. Three of those are m-stable spaces: $\overline{\mathcal{M}}_{1,3}$ corresponds to the subset at the bottom of the diagram, $\overline{\mathcal{M}}_{1,3}(1)$ to the subset just above, and $\overline{\mathcal{M}}_{1,3}(2)$ to the subset at the top of whole diagram.

In Figure 2 we give some examples of 3-pointed curves and the Q's for which they are considered stable.

Example 2.4. For $n = 3$ none of the new stability conditions — that is, the Q's such that $\overline{\mathcal{M}}_{1,n}(Q)$ is not an m-stable space — are symmetric with respect to the markings. This is a coincidence for low n.

Say that a proper downward closed subset Q of Part(n) is *symmetric* if Q is fixed by the natural S_n action. The orbits of partitions of $\{1, \ldots, n\}$ are in bijection with the integer partitions of n, so we may equivalently think of a symmetric Q as a proper downward closed subset of the partially ordered set of integer partitions of n ordered by refinement. Since the property of being symmetric is preserved under intersection and union, the set of symmetric Q-stable conditions for n also form a lattice under union and intersection.

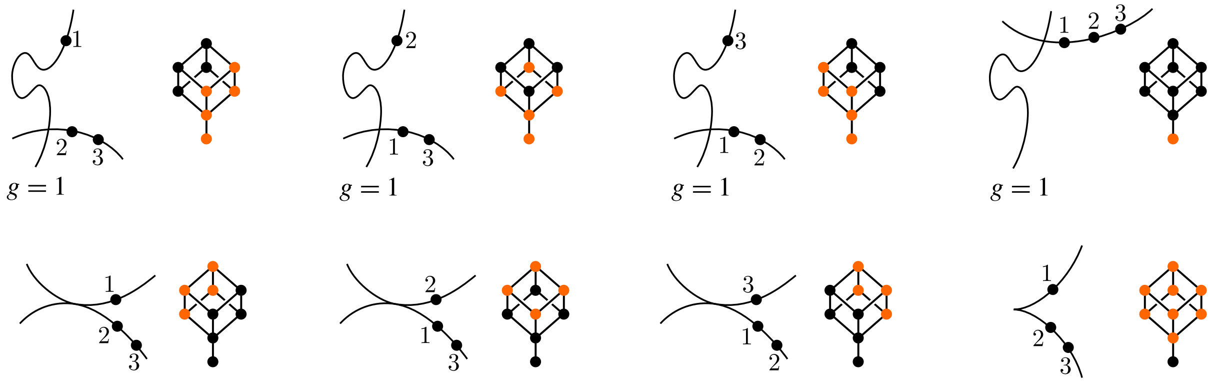


Figure 2. Some Gorenstein 3-pointed curves of genus one. Next to each curve we indicate for which choices of Q the curve is Q-stable: an orange dot means that the curve is stable with respect to the corresponding point in the diagram of downward closed subsets in Figure 1.

Consider the set of symmetric Q-stability conditions when $n = 5$. The Hasse diagram of the integer partitions of 5 and the lattice of symmetric Q-stable conditions, colored by the corresponding subset of the integer partitions of 5, is shown in Figure 3. The 5 m-stable spaces are given by the subsets down the middle of the diagram on the right; the 4 remaining subsets on the sides of that diagram yield new moduli spaces.

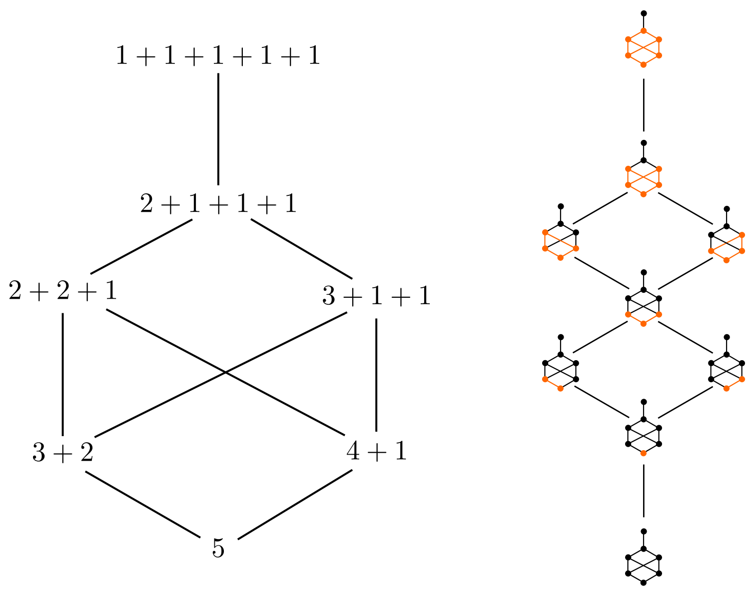


Figure 3. On the left, the integer partitions of 5. On the right, the lattice of symmetric Q-stability conditions when $n = 5$, thought of as subsets of the set of integer partitions. An orange dot indicates that the corresponding integer partition in the diagram on the left is included in Q.

3. Preliminaries

Tropical curves. Our main tool is the log-geometric approach to tropical geometry. We will use the framework of [Cavalieri et al. 2020]. All of our monoids will be commutative and we take $\mathbb{N}$ to include zero. We will prefer additive notation for the operation of P.

Recall that a monoid P is

(i) *sharp* if its only invertible element is the identity,

(ii) *integral* if $a+b=a+c$ implies $b=c$ for all $a, b, c \in P$,

(iii) *finitely generated* if there is a surjective monoid homomorphism $\mathbb{N}^r \to P$ for some integer r,

(iv) *saturated* if P is integral and for any $a \in P^{gp}$ and $n \in \mathbb{Z}_{>0}$, $n \cdot a \in P$ implies $a \in P$,

(v) *fs* if P is finitely generated, integral, and saturated.

We begin by recalling the definition of tropical curve, which is essentially a graph whose edges are labeled with "lengths" from an fs sharp monoid.

Definition 3.1. An n-marked *tropical curve* Γ *with edge lengths in an fs sharp monoid* P consists of:

(i) A finite set $X(\Gamma) = V(\Gamma) \sqcup F(\Gamma)$. The elements of $V(\Gamma)$ are called the *vertices* of Γ and the elements of $F(\Gamma)$ are called the *flags* of Γ.

(ii) A *root map* $r_\Gamma : X(\Gamma) \to X(\Gamma)$ which is idempotent with image $V(\Gamma)$.

(iii) An involution $\iota_\Gamma : X(\Gamma) \to X(\Gamma)$ that fixes $V(\Gamma)$. The subsets $\{f, \iota_\Gamma(f)\}$ of $F(\Gamma)$ of size two are called *edges*, and the set of all edges is denoted by $E(\Gamma)$. The subsets $\{f, \iota_\Gamma(f)\}$ of $F(\Gamma)$ of size one are called *legs*, and the set of all legs is denoted by $L(\Gamma)$.

(iv) A bijection $l : \{1, \dots, n\} \to L(\Gamma)$.

(v) A function $g : V(\Gamma) \to \mathbb{N}$. Given a vertex v, $g(v)$ is called the *genus of* v.

(vi) A function $\delta : E(\Gamma) \to P$. Given an edge e, $\delta(e)$ is called the *length of* e.

We imagine that each flag f is half of an edge starting at the vertex $r_\Gamma(f)$. Given an edge $e = \{f, \iota_\Gamma(f)\}$, we say the vertices $r_\Gamma(f)$ and $r_\Gamma(\iota_\Gamma(f))$ are *incident* to e.

Definition 3.2. The *genus* of a tropical curve Γ is

$$g(\Gamma) = b_1(\Gamma) + \sum_{v \in V(\Gamma)} g(v),$$

where $b_1(\Gamma)$ is the first Betti number of Γ, that is, $|E(\Gamma)| - |V(\Gamma)| + n$, where n is the number of connected components of Γ.

Definition 3.3. A tropical curve is *stable* if it is connected and not an isolated vertex of genus one, and the valence of each of its vertices of genus 0 is at least 3.

Definition 3.4. A *piecewise linear function* f on a tropical curve Γ with edge lengths in P consists of

(i) a value $f(v) \in P$ for each vertex $v \in V(\Gamma)$,

(ii) a slope $m(l) \in \mathbb{N}$ for each leg $l \in L(\Gamma)$

such that whenever e is an edge with ends v and w, $f(v) - f(w)$ is an integer multiple of $\delta(e)$.

The set of all piecewise linear functions on Γ is denoted by $\mathrm{PL}(\Gamma)$.

Given a tropical curve Γ with edge lengths in P and a morphism of fs sharp monoids $\pi^\sharp : P \to P'$, we may apply $\pi^\sharp$ to the edge lengths of Γ and contract edges of length zero to arrive at a new tropical curve. Composing with an isomorphism gives us the notion of a weighted edge contraction, which we define below.

Definition 3.5. Let Γ and Γ' be tropical curves with edge lengths in P and P', respectively. A *weighted edge contraction* $\pi : \Gamma' \to \Gamma$ (note the variance!) consists of

(i) a function $\pi : X(\Gamma) \to X(\Gamma')$,

(ii) a morphism of monoids $\pi^\sharp : P \to P'$

such that

(i) π preserves ends of flags, that is, $\pi \circ r_\Gamma = r_{\Gamma'} \circ \pi$;

(ii) π preserves edges, that is, $\pi \circ \iota_\Gamma = \iota_{\Gamma'} \circ \pi$;

(iii) π sends legs of Γ bijectively to legs of Γ' and preserves their markings;

(iv) for each flag $f \in F(\Gamma')$, the preimage $\pi^{-1}(f)$ has exactly one element (automatically a flag);

(v) for each vertex $v \in V(\Gamma')$, the preimage $\pi^{-1}(v)$ is a connected weighted graph of genus $g(v)$;

(vi) the flags of an edge $e \in E(\Gamma)$ are sent by π to a vertex of Γ' if and only if $\pi^\sharp(\delta(e)) = 0$;

(vii) for each edge $e \in E(\Gamma)$ with $\pi^\sharp(\delta(e)) \neq 0$, the image of e is an edge e' of length $\delta(e') = \pi^\sharp(\delta(e))$.

We will call a weighted edge contraction a *face contraction* if there is a subset $S \subseteq P$ such that the map $\pi^\sharp$ is of the form

$$P \to S^{-1}P \to S^{-1}P/(S^{-1}P)^* \xrightarrow{\sim} P',$$

where the first arrow is localization, the second is the quotient by the submonoid of invertible elements, and the third is an isomorphism. (These are the edge contractions associated to face inclusions in the category of rational polyhedral cones [Cavalieri et al. 2020, Definition 2.25].) In the case that P is a finite free monoid $\mathbb{N}^r$, the face contractions are those induced by the projections of $\mathbb{N}^r$ onto subsets of its coordinates.

Given a weighted edge contraction $\pi : \Gamma' \to \Gamma$ there is an induced map

$$\pi^* : \mathrm{PL}(\Gamma) \to \mathrm{PL}(\Gamma')$$

given by taking f with values $f(v)$ and slopes $m(l)$ to the piecewise linear function $\pi^* f$ with values $(\pi^* f)(v) = \pi^\sharp(f(v))$ for $v \in V(\Gamma)$ and the same slopes.

We take weighted edge contractions to be the morphisms in the category of tropical curves. In particular, an isomorphism of tropical curves is an invertible weighted edge contraction.

Log curves and their tropicalizations. The natural notion of family of curves in logarithmic geometry admits both an underlying family of pointed nodal curves and a tropicalization, connecting the tropical and algebrogeometric worlds. F. Kato [2000] introduced the notion of a family of log curves.

Definition 3.6. (cf. [Kato 2000, Definition 1.2]) Let S be an fs log scheme. A *log curve* over S is a log smooth and integral morphism $\pi : C \to S$ of fs log schemes such that every geometric fiber of π is a reduced and connected curve.

Kato [2000, Theorem 1.3] has shown that the underlying morphism of schemes of a log curve is a family of nodal curves, and the data in the log structure records some marked points. We borrow this statement of Kato's local structure theorem from [Ranganathan et al. 2019].

Theorem 3.7. *Let $\pi : C \to S$ be a family of proper log curves. If $x \in C$ is a geometric point with image $s \in S$, then there are étale neighborhoods V of x and U of s such that $V \to U$ has a strict morphism to an étale-local model $V' \to U'$, where $V' \to U'$ is one of the following*:

(i) *The smooth germ*: $V' = \mathbb{A}^1_{U'} \to U'$ *and the log structure on V' is pulled back from the base.*

(ii) *The germ of a marked point*: $V' = \mathbb{A}^1_{U'} \to U'$ *with the log structure pulled back from the toric log structure on* $\mathbb{A}^1$.

(iii) *The node*: $V' = \operatorname{Spec} \mathscr{O}_{U'}[x, y]/(xy - t)$ *for* $t \in \mathscr{O}_{U'}$. *The log structure on V is pulled back from the multiplication map $\mathbb{A}^2 \to \mathbb{A}^1$ of toric varieties along the morphism $U' \to \mathbb{A}^1$ of logarithmic schemes induced by t.*

The tropicalization of a log curve is the dual graph of its underlying nodal curve, enriched with the data of the smoothing parameters from its log structure.

Definition 3.8. Given an n-marked log curve $\pi : C \to S$, where S is a log point, the *tropicalization* $\operatorname{trop}(C)$ of C is the n-marked tropical curve with edge lengths in $\Gamma(S, \overline{M}_S)$ which has

(i) a vertex for each component of C;

(ii) an edge for each node of C, incident to the components of C which form the branches of the node;

(iii) a leg for each marked point of C, rooted at the component of C to which the marked point belongs;

and

(a) for each vertex v, the genus $g(v)$ is the genus of the normalization of the corresponding component of C;

(b) for each edge e, the length $\delta(e) \in \Gamma(S, \overline{M}_S)$ is the smoothing parameter of the node e.

See Figure 4 for an example of tropicalization. Note that the tropicalization may contain loops: consider the nodal cubic.

Sections of the characteristic sheaf of C are interpreted tropically as piecewise linear functions.

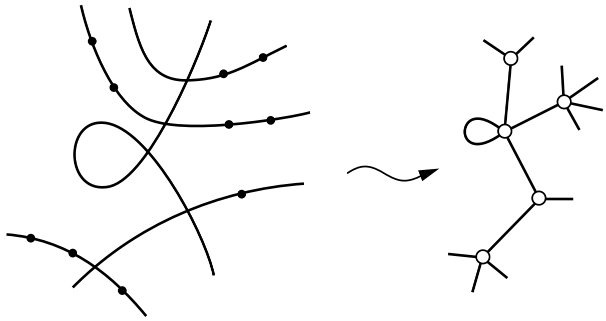


Figure 4. A typical log curve and its tropicalization.

Theorem 3.9. *Let $\pi : C \to S$ be a log curve over the spectrum of an algebraically closed field. Then there is a bijection*

$$\mathrm{PL} : \Gamma(C, \overline{M}_C) \xrightarrow{\sim} \mathrm{PL}(\mathrm{trop}(C)), \quad \sigma \mapsto \mathrm{PL}(\sigma),$$

where

(i) *the value of* $\mathrm{PL}(\sigma)$ *at a vertex v of* $\Gamma(C)$ *is the stalk of σ at the generic point of the corresponding component of C*;

(ii) *the slope of* $\mathrm{PL}(\sigma)$ *at a leg l of* $\Gamma(C)$ *is the image of σ in* $(\overline{M}_C/\pi^{-1}\overline{M}_S)_p \cong \mathbb{N}$, *where p is the marked point corresponding to l.*

Proof. See, for example, [Cavalieri et al. 2020, Remark 7.3]. □

For a general log curve, this interpretation extends nicely over an étale neighborhood of each point.

Theorem 3.10. *Let $\pi : C \to S$ be a log curve and let s be a geometric point of S. Then there is an étale neighborhood U of s in S such that*

(i) $\Gamma(U, \overline{M}_S) \to \overline{M}_{S,s}$ *and* $\Gamma(C|_U, \overline{M}_C) \to \Gamma(C|_s, \overline{M}_{C|_s})$ *are isomorphisms*;

(ii) *for each geometric point t of U, there is a canonical face contraction*

$$\mathrm{trop}(C_s) \to \mathrm{trop}(C_t)$$

induced by

$$\overline{M}_{S,s} \xleftarrow{\sim} \Gamma(U, \overline{M}_S) \to \overline{M}_{S,t}.$$

Moreover, this face contraction respects associated piecewise linear functions in the sense that

$$\begin{array}{ccccc} \Gamma(C|_s, \overline{M}_{C|_s}) & \xleftarrow{\sim} & \Gamma(C|_U, \overline{M}_C) & \longrightarrow & \Gamma(C|_t, \overline{M}_{C|_t}) \\ \downarrow{\scriptstyle \mathrm{PL}} & & & & \downarrow{\scriptstyle \mathrm{PL}} \\ \mathrm{PL}(\mathrm{trop}(C|_s)) & & \longrightarrow & & \mathrm{PL}(\mathrm{trop}(C|_t)) \end{array}$$

commutes.

Proof. This follows, for example, from the existence of "uniform sets of charts", constructed in [Bozlee 2020, Proposition 2.3.13]. □

It follows that to define a section of the characteristic sheaf of C it is equivalent to specify a piecewise linear function on each geometric fiber of C so that the resulting piecewise linear functions are compatible with generization.

Definition 3.11. An n-marked log curve is a log curve $\pi : C \to S$ equipped with disjoint sections $\sigma_1, \dots, \sigma_n : S \to C$ with image the marked points of C. An n-marked log curve is *stable* if its underlying family of marked nodal curves is Deligne–Mumford stable.

Theorem 3.12 [Kato 2000, Theorem 4.5]. *There is a log structure on $\overline{\mathcal{M}}_{g,n}$, called the* ***basic log structure****, such that $\overline{\mathcal{M}}_{g,n}$ represents the stack of stable n-marked log curves of genus g over the category of fs log schemes.*

We will generally regard $\overline{\mathcal{M}}_{g,n}$ as a stack with the basic log structure and will freely confuse $\overline{\mathcal{M}}_{g,n}$ with its underlying algebraic stack. A stable log curve $\pi : C \to S$ is said to have the *basic log structure* if its log structure is pulled back from that of the universal stable log curve of $\overline{\mathcal{M}}_{g,n}$. In the case that S is a geometric point, π has the basic log structure if and only if the characteristic monoid $\overline{M}_S$ is freely generated by the edge lengths of $\operatorname{trop}(C)$.

Radially aligned curves. We now build up the terminology to work with radially aligned curves. These were introduced by Santos-Parker [2017] under the name of ordered log curves and then popularized in [Ranganathan et al. 2019].

Definition 3.13. Let Γ be a tropical curve. A *path* W in Γ is a sequence $v_0 e_1 v_1 e_2 \cdots e_k v_k$ of vertices and edges in Γ such that the vertices v_i are distinct and v_{i-1} and v_i are the ends of the edge e_i for all i. Given subsets A and B of $V(\Gamma)$, we say that W is a path from A to B if $v_0 \in A$, $v_k \in B$, and $v_i \notin A \cup B$ for $i \neq 0, k$.

Definition 3.14. Given a proper curve C over the spectrum of an algebraically closed field, a *subcurve* of C is a union of irreducible components of C, possibly empty.

The *core* of C is the minimal connected subcurve of C with the same genus as C. Analogously the core of a tropical curve Γ is the minimal connected vertex-induced subgraph of the same genus as Γ.

Definition 3.15. Given a tropical curve Γ of genus one, we define a piecewise linear function λ on Γ measuring "distance from the core" as follows. If v is a vertex in the core of Γ, we set

$$\lambda(v) = 0.$$

If v is a vertex outside of the core of Γ, we let $W = v_0 e_1 v_1 e_2 \cdots e_k v_k$ be the unique path from the core of Γ to v and set

$$\lambda(v) = \sum_{i=1}^{k} \delta_{e_i}.$$

Finally, we set the slope of λ to be 1 at all marked points.

This is compatible with generization, so for any stable log curve $(\pi : C \to S; \sigma_1, \dots, \sigma_n)$ of genus one, we let $\lambda \in \Gamma(S, \overline{M}_S)$ be the unique section of the characteristic bundle whose restriction to geometric fibers has corresponding piecewise linear function as in the last paragraph.

Definition 3.16. If P is any fs sharp monoid, we give the elements of P a partial order by the rule $p \le q$ if and only if there exists $r \in P$ with $q = p + r$.

Definition 3.17. A stable n-marked tropical curve of genus one with edge lengths in P is *radially aligned* if, for each pair of vertices v, w of Γ, $\lambda(v)$ is comparable to $\lambda(w)$ in P.

Given such a radially aligned curve, let

$$0 < \rho_1 < \cdots < \rho_k$$

be the distinct values of $\lambda(v)$ as v varies over the components of C, and let $\delta_1, \dots, \delta_l$ be the lengths of the edges of $\operatorname{trop}(C)$ internal to the core of Γ. Let $e_1 = \rho_1$, $e_2 = \rho_2 - \rho_1, \dots, e_k = \rho_k - \rho_{k-1}$. If P is freely generated by

$$\{e_1, \dots, e_k\} \cup \{\delta_1, \dots, \delta_l\},$$

then we say that Γ is a *basic radially aligned tropical curve*. An element of P is said to have *no contribution from the core* if it lies in the submonoid generated by $e_1, \dots, e_k$.

A stable log curve $(\pi : C \to S; \sigma_1, \dots, \sigma_n)$ of genus one with n markings is *radially aligned* or has a *basic radially aligned log structure* if the tropicalizations of its geometric fibers with their pulled back log structure are respectively radially aligned or basic radially aligned. An element $\rho \in \Gamma(S, \overline{M}_S)$ has *no contribution from the core* if the same holds of its stalks at the geometric points of S.

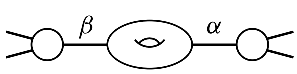
There is a moduli stack with log structure parametrizing radially aligned log curves.

Theorem 3.18 [Ranganathan et al. 2019, Proposition 3.3.4]. (i) *There is a Deligne–Mumford stack with locally free log structure $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ whose S-points for S an fs log scheme are the n-marked radially aligned curves $\pi : C \to S$ over S. We say its log structure is* **the** *basic radially aligned log structure.*

(ii) *There is a natural map $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}$ induced by a logarithmic blowup and it restricts to an isomorphism on $\mathcal{M}_{1,n}$.*

A stable log curve $(\pi : C \to S; \sigma_1, \dots, \sigma_n)$ has a basic radially aligned log structure precisely when the log structure on $\pi : C \to S$ is that pulled back from the universal curve $\mathcal{C}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ along the map $S \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. We remark that a fixed family of nodal curves may be enhanced to a family of basic radially aligned log curves in more than one way, which we illustrate with an example.

Example 3.19. Let $\pi : C \to S = \operatorname{Spec} k$ be a stable curve with the basic log structure over the spectrum of an algebraically closed field, and suppose that its tropicalization is



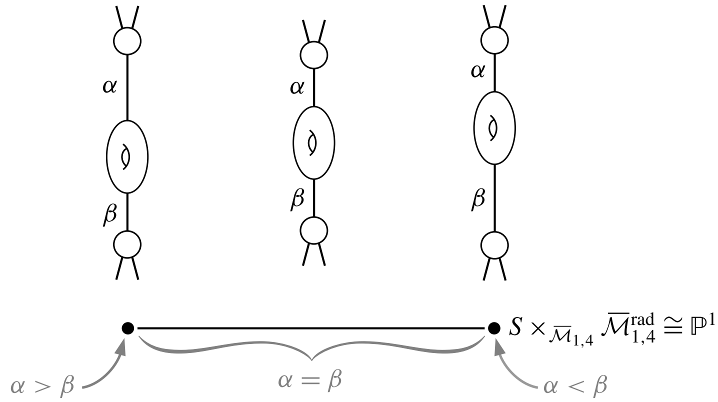


Figure 5. The $\mathbb{P}^1$ of radial alignments of a basic stable log curve with two edges.

There is an associated map $S \to \overline{\mathcal{M}}_{1,4}$. The basic log structure on S comes from the chart

$$\mathbb{N}\tilde{\alpha} \oplus \mathbb{N}\tilde{\beta} \to k$$

sending $\tilde{\alpha}, \tilde{\beta} \mapsto 0$. The edge lengths α and β are the respective images of $\tilde{\alpha}$ and $\tilde{\beta}$ in the characteristic sheaf. Locally in $\overline{\mathcal{M}}_{1,4}$ near the image of S, the map $\overline{\mathcal{M}}_{1,4}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,4}$ is given by the log blowup of the log ideal generated by α and β, as these are the distances that we wish to make comparable. We refer the interested reader to [Ogus 2018, Chapter III, Section 2.6] for details on log blowups.

We may compute all of the basic radially aligned log structures on C by computing the restriction of this blowup to S. We construct the blowup by first freely adjoining the element $\tilde{\alpha} - \tilde{\beta}$ to the log structure of S, adjoining an element to k for $\tilde{\alpha} - \tilde{\beta}$ to map to, doing likewise for $\tilde{\beta} - \tilde{\alpha}$, and finally gluing over the overlap. That is, $S \times_{\overline{\mathcal{M}}_{1,4}} \overline{\mathcal{M}}_{1,4}^{\mathrm{rad}}$ possesses a cover by two open sets $U = \operatorname{Spec} k[t]$ (where $\alpha \geq \beta$) and $V = \operatorname{Spec} k[t^{-1}]$ (where $\beta \geq \alpha$) with log structure on U induced by

$$\mathbb{N}(\tilde{\alpha} - \tilde{\beta}) \oplus \mathbb{N}\tilde{\beta} \to k[t], \qquad \tilde{\alpha} - \tilde{\beta} \mapsto t \quad \text{and} \quad \tilde{\beta} \mapsto 0,$$

and log structure on V induced by

$$\mathbb{N}\tilde{\alpha} \oplus \mathbb{N}(\tilde{\beta} - \tilde{\alpha}) \to k[t^{-1}], \qquad \tilde{\alpha} \mapsto 0 \quad \text{and} \quad \tilde{\beta} - \tilde{\alpha} \mapsto t^{-1}.$$

The two charts are glued in the obvious way. Notice that on the intersection $U \cap V = \operatorname{Spec} k[t, t^{-1}]$, the sections $\tilde{\alpha} - \tilde{\beta}$ and $\tilde{\beta} - \tilde{\alpha}$ of the log structure restrict to units, so that their images in the characteristic sheaf are 0. It follows that α and β are equal over a $\mathbb{G}_m$'s worth of possible basic radially aligned enhancements of C. See Figure 5.

Definition 3.20. Let Γ be a radially aligned tropical curve with ordered radii $0 < \rho_1 < \cdots < \rho_k$.

Given a radius ρ, we may form a tropical curve $\widetilde{\Gamma}(\rho)$ by subdividing the edges and legs of Γ where $\lambda = \rho$, then deleting the locus where $\lambda < \rho$. We define the *partition associated to the radius* ρ to be the partition of $\{1, \ldots, n\}$ induced by the components of $\widetilde{\Gamma}(\rho)$, and we denote it by $\operatorname{part}(\rho)$.

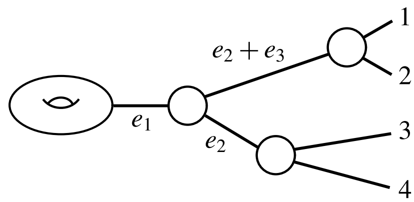


Figure 6. A basic radially aligned curve with partition type $\{\{1, 2, 3, 4\}\} \prec \{\{1, 2\}, \{3, 4\}\} \prec \{\{1, 2\}, \{3\}, \{4\}\}$. We draw a torus to indicate a vertex of genus one.

We say that the resulting strict chain of partitions

$$\operatorname{part}(\rho_1) \prec \operatorname{part}(\rho_2) \prec \cdots \prec \operatorname{part}(\rho_k)$$

is the *partition type* of Γ. See Figure 6 for an example.

It would be natural to include the partition part(0) in the partition type as well, but we choose not to for a few reasons. The first is that we always have $\operatorname{part}(0) = \{\{1, 2, \dots, n\}\}$, since part(0) is the partition of the markings induced by deleting no components. So including part(0) in the list does not convey more information. For another, unlike the other comparisons, the comparison $\operatorname{part}(0) \preceq \operatorname{part}(\rho_1)$ need not be strict: it may be that both are the indiscrete partition. For example, see Figure 6.

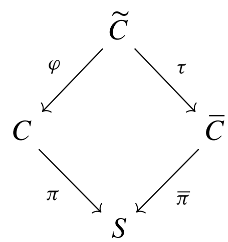
4. Contractions of the universal radially aligned curve

Part of the utility of families of radially aligned curves is that they are easy to contract to families of curves with Gorenstein singularities, even at the level of a universal curve. By exploring the possible contractions of the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ we find regular birational maps $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}(Q)$ for each Q. It was this computation that identified the Q-stable moduli spaces.

The following theorem says that in order to contract a family of radially aligned curves, all we need is the data of a tropical radius for each curve in the family. This idea was key for the results of [Santos-Parker 2017] and [Ranganathan et al. 2019] and can be done using their language; see [Santos-Parker 2017, Section 5] and [Ranganathan et al. 2019, Section 3.7]. We give a proof using the language of [Bozlee 2020] for its convenience and generality.

Theorem 4.1. *Let $\pi : C \to S$ be a family of n-marked radially aligned log curves. Let $\rho \in \Gamma(S, \overline{M}_S)$ be a section of the characteristic monoid such that "ρ is a radius at all geometric points of s"; that is, for each geometric point s of S, there is a vertex v of $\operatorname{trop}(C|_s)$ such that $\lambda(v) = \rho|_s$.*

Then there is a diagram



where

(i) *φ is a log blowup inducing the subdivision at the locus where $\lambda = \rho$ on tropicalizations*;

(ii) *$\overline{\pi} : \overline{C} \to S$ is a flat and proper family of Gorenstein curves of genus one*;

(iii) *τ is a surjective map whose restriction to geometric fibers contracts the locus* (*if nonempty*) *where $\lambda < \rho$ to an elliptic singularity of level* $\mathrm{part}(\rho|_s)$ *and restricts further to an isomorphism in the complement of this locus.*

Moreover, formation of the diagram commutes with base change in S.

Proof. The construction of φ is standard. Using the language of [Bozlee 2020], we then define a mesa $\bar{\lambda} \in \Gamma(\widetilde{C}, \overline{M}_{\widetilde{C}})$ on the resulting family of log curves $\widetilde{C} \to S$ with the formula

$$\bar{\lambda} = \max\{\rho - \lambda, 0\}.$$

It is easy to check that $\bar{\lambda}$ defines a steep mesa with support on the locus where $\lambda < \rho$, so the main theorem of [Bozlee 2020] yields the claimed diagram with the required properties. To see that the elliptic singularities of $\overline{C}$ have the claimed level, we note that the branches of the singularity will be the images of the connected components of the locus in $\widetilde{C}$ where $\lambda \geq \rho$. These are precisely the connected components considered in the definition of $\mathrm{part}(\rho)$. □

We want to apply this theorem to the universal curve of $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$. Our next task is to reduce the problem of enumerating the possible radii $\rho \in \Gamma(\overline{\mathcal{M}}_{1,n}, \overline{M}_{\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}})$ to something manageable.

Lemma 4.2. *Let Γ be a basic radially aligned curve. Then*:

(i) *If the core of Γ consists of a vertex with a self-loop*, *then the only nontrivial automorphism of Γ is the automorphism reversing the loop*, *but the identity on everything else.*

(ii) *If the core of Γ consists of a pair of vertices with two edges*, *the only nontrivial automorphism of Γ is the automorphism exchanging the edges of the core*, *and the identity on everything else.*

(iii) *Otherwise*, *Γ has no nonidentity automorphisms.*

Proof. Suppose Γ is a basic radially aligned tropical curve. Let $\varphi : \Gamma \to \Gamma$ be an invertible weighted edge contraction.

We argue that φ is the identity on the vertices of Γ. Let v be a vertex of Γ. Notice that the complement of the core of Γ consists of a forest of trees, each of which we can root at the vertex that attaches to the core. Furthermore, due to stability,

(i) if v is a vertex outside of the core of Γ, then v is uniquely identified by the markings that lie on v and the descendants of v;

(ii) if v is a vertex inside the core of Γ, there is at least one tree attached to v, and those trees are uniquely identified by their markings.

An automorphism of Γ must in particular preserve the markings. Therefore, φ fixes all of the vertices of Γ. This implies that φ fixes all edges except possibly those who share incident vertices. This implies the result. $\square$

Remark 4.3. For a fixed n, there are only finitely many isomorphism classes of n-marked tropical curves with the basic radially aligned log structure.

Lemma 4.4. *Let Γ be an n-marked basic radially aligned tropical curve with edge lengths in P. Write $\mathcal{M}_\Gamma^{\mathrm{rad}}$ for the substack of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ of curves whose tropicalizations are isomorphic to Γ. Then $\mathcal{M}_\Gamma^{\mathrm{rad}}$ is an irreducible locally closed substack of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$.*

Proof. Write $\mathcal{M}_\Gamma$ for the locally closed substack of stable n-marked curves of genus one whose dual graph is the underlying graph of Γ. Recall that $\mathcal{M}_\Gamma \cong \prod_{v\in V(\Gamma)} \mathcal{M}_{g(v),\mathrm{val}(v)}$, where the valence $\mathrm{val}(v)$ is the number of flags incident to v. Since the $\mathcal{M}_{g(v),\mathrm{val}(v)}$'s are geometrically irreducible, so is $\mathcal{M}_\Gamma$. There is a forgetful map $\mathcal{M}_\Gamma^{\mathrm{rad}} \to \mathcal{M}_\Gamma$ given by forgetting the log structure.

We recall from [Ranganathan et al. 2019, Proposition 3.3.4] that the map $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}$ is locally given as follows. Suppose given a map $S \to \overline{\mathcal{M}}_{1,n}$ such that S admits a global chart by a monoid Q. This induces a map $S \to V = \operatorname{Spec} \mathbb{Z}[Q]$. Let σ be the rational polyhedral cone dual to Q. Let Σ be the fan obtained by subdividing σ along the hyperplanes where $\lambda(v) = \lambda(w)$ as v and w range among the vertices of $\mathrm{trop}(C_s)$, and let W be the toric variety associated to Σ. Then

$$S \times_{\overline{\mathcal{M}}_{1,n}} \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \cong S \times_V W.$$

Suppose that $S \to \mathcal{M}_{1,n}^{\mathrm{rad}}$ factors through $\mathcal{M}_\Gamma^{\mathrm{rad}}$. Then Q is the free monoid on the edges of Γ. By construction, there is a cone of Σ associated to each possible choice of ordering of the distances $\lambda(v)$ as v varies over $V(\Gamma)$. Since Γ is basic radially aligned, these distances are ordered, and their ordering determines a cone τ of Σ. Let $W_\tau \subseteq W$ be the torus orbit associated to τ. One may check that the locus in $S \times_V W$ in which the tropicalization is isomorphic to Γ is precisely the locus $S \times_V W_\tau$: this is the locus in which the stalks of the characteristic sheaf agree with P. Letting S vary over a smooth cover of $\mathcal{M}_\Gamma^{\mathrm{rad}}$, we see that $\mathcal{M}_\Gamma^{\mathrm{rad}} \to \mathcal{M}_\Gamma$ is smooth-locally a W_τ-bundle. Since the target is irreducible and W_τ is irreducible, $\mathcal{M}_\Gamma^{\mathrm{rad}}$ is irreducible. Moreover it is a locally closed substack of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ as this is true of W_τ inside W. $\square$

Lemma 4.5. *Let I be the set of isomorphism classes of n-marked basic radially aligned tropical curves. Fix a representative Γ with edge lengths in P_Γ for each isomorphism class.*

Giving a section ρ of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ with no contribution from the core is equivalent to specifying for each isomorphism class of basic radially aligned tropical curves an element $\rho_\Gamma \in P_\Gamma$ with no contribution from the core such that whenever $\Gamma, \Gamma' \in I$ and $\pi : \Gamma \to \Gamma'$ is a face contraction with $\pi^\sharp : P_\Gamma \to P_{\Gamma'}$ a quotient map, then $\pi^* \rho_\Gamma = \rho_{\Gamma'}$

Proof. For brevity, write $\overline{M}$ for the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Since $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is a Deligne–Mumford stack, we may choose an étale cover $U \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ by a scheme U. For each point x of U, choose an algebraic closure $\bar{k}(x)$ of the residue field $k(x)$, and write $\bar{x}^U : \operatorname{Spec} \bar{k}(x) \to U$ for the natural map to U and $\bar{x}$ for its composite with $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Write $\Gamma_{\bar{x}}$ for the tropicalization of the basic radially aligned log curve associated to $\bar{x}$ and write $P_{\bar{x}}$ for its associated monoid, i.e., $\overline{M}_{\bar{x}}$.

We may apply Theorem 3.10 to find an étale neighborhood $U_{\bar{x}}$ of $\bar{x}^U$ in U over which the pullback of the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ has the properties of the theorem. Since U is an étale cover of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ and the log structure on $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is divisorial, the maps $P_{\bar{x}} \xleftarrow{\sim} \Gamma(U_{\bar{x}}, \overline{M}) \to \overline{M}_t$ vary through *all* of the quotients of $P_{\bar{x}}$ by its generators as t varies through the geometric points of $U_{\bar{x}}$. Then, since the $U_{\bar{x}}$'s form a cover, we may identify the global sections of $\overline{M}$ with elements $(\rho_{\bar{x}})$ of $\prod_{\bar{x}} \Gamma(U_{\bar{x}}, \overline{M}) \cong \prod_{\bar{x}} P_{\bar{x}}$, suitably compatible on overlaps.

Now, for a pair of points $\bar{x}$ and $\bar{y}$, sections $\rho_{\bar{x}} \in \Gamma(U_{\bar{x}}, \overline{M})$ and $\rho_{\bar{y}} \in \Gamma(U_{\bar{y}}, \overline{M})$ agree on $U_{\bar{x},\bar{y}} := U_{\bar{x}} \times_{\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}} U_{\bar{y}}$ if and only if their stalks at geometric points $\bar{z}$ of $U_{\bar{x},\bar{y}}$ agree. This translates to the statement that whenever $\Gamma_{\bar{z}}$ is a face contraction of both $\Gamma_{\bar{x}}$ and $\Gamma_{\bar{y}}$, and $\varphi : \Gamma_{\bar{z}} \to \Gamma_{\bar{z}}$ is an automorphism, the stalk of $\rho_{\bar{x}}$ at $\bar{z}$ is $\varphi^\sharp$ applied to the stalk of $\rho_{\bar{y}}$ at $\bar{z}$.

Suppose $\bar{x}$ and $\bar{y}$ are points, $\Gamma \in I$ and $\Gamma_{\bar{x}} \cong \Gamma \cong \Gamma_{\bar{y}}$. Then, in the notation of the previous lemma, since $\mathcal{M}_{\Gamma}^{\mathrm{rad}}$ is irreducible, $U_{\bar{x},\bar{y}}$ must contain a point $\bar{z}$ that also maps into $\mathcal{M}_{\Gamma}^{\mathrm{rad}}$. Then the elements corresponding to $\rho_{\bar{x}}$ and $\rho_{\bar{y}}$ on P_Γ must differ by at most an automorphism of Γ. By Lemma 4.2, they are actually equal, so we have a well-defined element ρ_Γ of P_Γ. The agreement of stalks at other points implies that the ρ_Γ's are compatible with edge contraction. We obtain the converse by reversing this construction. □

In view of Lemma 4.5, we introduce the notion of *universal radius*.

Definition 4.6. An n-marked *universal radius* consists of the data of an element $\rho_\Gamma \in P_\Gamma$ for each n-marked basic radially aligned curve Γ so that

(i) for each Γ, $\rho_\Gamma = \lambda(v)$ for some vertex v of Γ;

(ii) if Γ and Γ' are two n-marked radially aligned curves and $\pi : \Gamma \to \Gamma'$ is a face contraction, then $\pi^* \rho_\Gamma = \rho_{\Gamma'}$.

We use the shorthand notation (ρ_Γ) for the tuple of radii making up a universal radius, and denote by $\mathfrak{R}_n^{\mathrm{uni}}$ the set of n-marked universal radii.

Remark 4.7. Condition (i) implies each ρ_Γ has no contribution from the core. Condition (ii) implies that we only need to keep track of the *finite* data of a choice of radius ρ_Γ for each isomorphism class of n-marked basic radially aligned curves. The maps $P_\Gamma \to P_{\Gamma'}$ induced by face contractions are just the coordinate projections. Therefore all we have to worry about to satisfy condition (ii) is what happens when we set various subsets of the generators of P_Γ (that is, $\delta_1, \dots, \delta_l$ and $e_1, \dots, e_k$ in the notation of Definition 3.17) to zero.

We have therefore reduced the problem of finding a section of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to giving the finite collection of tropical data that make up a universal radius. This is still a fair amount of data: see Figure 7 for an example when $n = 3$. We will see that we can reduce the data of a universal

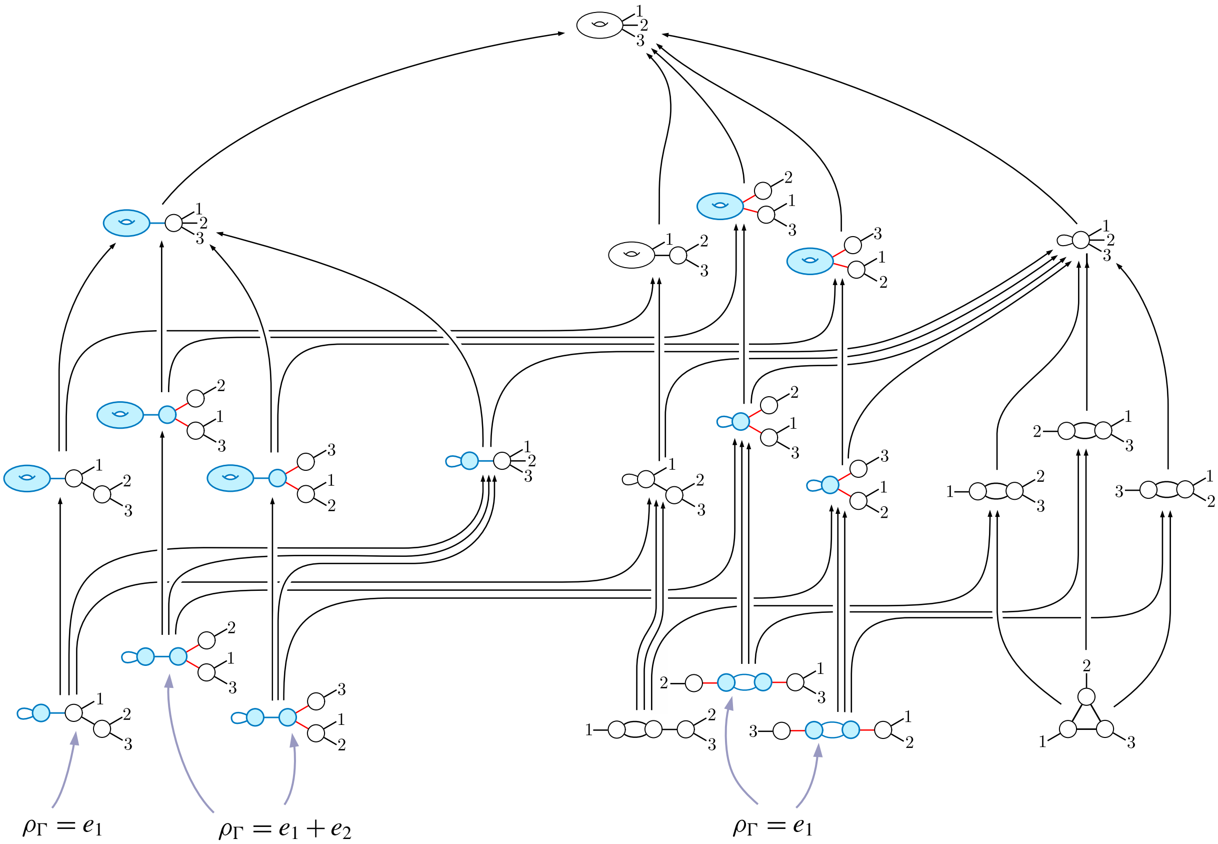


Figure 7. An example of the subdivided universal curve $\widetilde{C}$ on $\overline{\mathcal{M}}_{1,3}^{\mathrm{rad}}$ associated to a universal radius. A torus indicates a vertex of genus one. We have labeled the nonzero ρ_Γ's on the most degenerate tropical curves using the notation of Definition 3.17. The radii on other tropical curves can be deduced by following the indicated face contractions. The red edges of a particular curve have equal length: they come about by subdividing at the radius ρ_Γ. The blue components are those to be contracted.

radius further to that of a downward closed subset Q of partitions on $\{1, \dots, n\}$. In Figure 7, for example, the corresponding downward closed subset will be

$$Q = \big\{\{\{1, 2, 3\}\},\ \{\{2\}, \{1, 3\}\},\ \{\{3\}, \{1, 2\}\}\big\}.$$

This can be read off from the second row of the figure.

Definition 4.8. We say that Γ is a *k-layer tree* if Γ is an n-marked basic radially aligned tropical curve with k nonzero radii $0 < \rho_1 < \cdots < \rho_k$ and smooth core.

Lemma 4.9. *Let Γ be a basic radially aligned tropical curve with k nonzero radii $0 < \rho_1 < \cdots < \rho_k$. Then, for each index $i < k$, there is a strict refinement* $\mathrm{part}(\rho_i) \prec \mathrm{part}(\rho_{i+1})$.

Proof. Notice that if Γ is replaced by the tropical curve in which all edges of the core are contracted, the sequence of partitions remains the same. Therefore we may assume that Γ is a k-layer tree.

Orient the tree Γ by taking its core as a root. Resuming the notation of Definition 3.20, let V_i be the set of roots of the forest $\widetilde{\Gamma}(\rho_i)$ for each i. Observe that for each i, there are bijections between V_i, the connected components of $\widetilde{\Gamma}(\rho_i)$, and the parts of the partition $\mathrm{part}(\rho_i)$. Notice that the connected components of $\widetilde{\Gamma}(\rho_{i+1})$ factor through the connected components of $\widetilde{\Gamma}(\rho_i)$, so the partition of $\{1, \dots, n\}$ induced by the connected components of $\widetilde{\Gamma}(\rho_{i+1})$ refines that induced by the components of $\widetilde{\Gamma}(\rho_i)$.

To see that the refinement is strict, let v be a vertex of Γ such that $\lambda(v) = \rho_i$. By stability, there must be at least two flags leaving v in the direction of increasing λ. Then there are at least two vertices of V_{i+1} that belong to the component of $\Gamma(\rho_i)$ containing v. It follows that the refinement is strict. $\square$

Proposition 4.10. *1-layer trees are in bijection with the nondiscrete partitions of* $\{1, \dots, n\}$.

Proof. Given a nondiscrete partition p of $\{1, \dots, n\}$, we can construct a 1-layer tree $\Gamma(p)$ as follows. Let $p = \{p_1, \dots, p_r\}$. Start with a genus 1 vertex v and then for each $1 \le i \le r$ attach a vertex v_i to v such that

(i) v_i is distance ρ_1 from v, and

(ii) the elements of p_i are precisely the legs attached to v_i.

After stabilizing, we obtain $\Gamma(p)$.

If Γ is a 1-layer tree, then $\Gamma \mapsto \mathrm{part}(\rho_1)$ gives a map from 1-layer trees to nondiscrete partitions. The maps $p \mapsto \Gamma(p)$ and $\Gamma \mapsto \mathrm{part}(\rho_1)$ are inverses. $\square$

Definition 4.11. Let Γ be a radially aligned tropical curve with monoid

$$\mathbb{N}e_1 \oplus \cdots \oplus \mathbb{N}e_k \oplus \mathbb{N}\delta_1 \oplus \cdots \oplus \mathbb{N}\delta_l,$$

and let Γ_{e_i} denote the tropical curve corresponding the monoid map $P \to \mathbb{N}$ taking $e_i \mapsto 1$ and $e_j \mapsto 0$ for all $j \neq i$, and $\delta_j \mapsto 0$ for all j. We define Γ_{δ_i} similarly; it is the tropical curve corresponding to the monoid map $P \to \mathbb{N}$ taking $\delta_i \mapsto 1$ and all other generators to 0.

Consider the map $\alpha : \mathfrak{R}_n^{\mathrm{uni}} \to \mathfrak{Q}_n$ defined by

$$(\rho_\Gamma)_\Gamma \mapsto \{\mathrm{part}(\Gamma) : \Gamma \text{ is a 1-layer tree and } \rho_\Gamma > 0\}.$$

Given a collection of partitions Q, we obtain an assignment of radii to radially aligned curves by assigning the radius ρ_r to Γ if r is the largest number such that $\mathrm{part}(\rho_r) \in Q$. This gives a map $\beta : \mathfrak{Q}_n \to \mathfrak{R}_n^{\mathrm{uni}}$.

Proposition 4.12. *The maps* $\alpha : \mathfrak{R}_n^{\mathrm{uni}} \to \mathfrak{Q}_n$ *and* $\beta : \mathfrak{Q}_n \to \mathfrak{R}_n^{\mathrm{uni}}$ *are well defined and are inverses.*

Proof. To show α is well defined, it suffices to show that its image is contained in $\mathfrak{Q}_n$. We do this by contradiction. Suppose that $Q = \alpha((\rho_\Gamma)_\Gamma)$ is not downward closed. We show that $(\rho_\Gamma)_\Gamma$ is not universal. We can find some $P \in Q$ such that a minimal coarsening of P is not in Q. Specifically, there will be a $P = \{p_1, \dots, p_k\} \in Q$ such that (up to reordering) $P' = \{p_1 \cup p_2, p_3, \dots, p_k\} \notin Q$, as otherwise Q

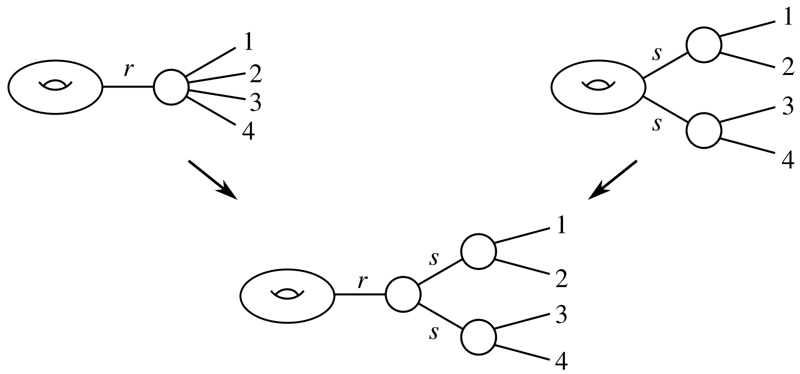


Figure 8. The tropical curves associated to the partition types P, P', and $P' \prec P$, where $P = \{\{1, 2\}, \{3, 4\}\}$ and $P' = \{\{1, 2, 3, 4\}\}$.

will be downward closed. Let the 1-layer trees Γ and Γ' correspond to the partition P and the coarsened partition P', respectively. Say Γ has edge length r and radius r and Γ' has edge length s and radius 0. There is a 2-layer curve, say $\tilde{\Gamma}$, that contracts to both Γ and Γ', and has edge lengths s and r and radius $s+r$ (see Figure 8). If the radius was universal, then we see that by contracting the edge of length r, Γ' must have a radius of s, not 0. Thus the radius is not universal, as claimed.

We now show that β is well defined. First, note that given a basic radially aligned curve Γ, ρ_Γ will be determined by the contractions to Γ_{δ_i} and Γ_{e_i}. To see this, note that the contraction $\Gamma_{\delta_i} \to \Gamma$ will send ρ_j to 0 for all j, and the contraction $\Gamma_{e_i} \to \Gamma$ will send ρ_j to e_i if $j \geq i$ and 0 if $j < i$. As these maps arise from projections from a product, ρ_Γ is uniquely determined by these contractions. Now pick $Q \in \mathfrak{Q}_n$. For any radially aligned Γ, Lemma 4.9 and the fact that ρ_Γ is determined by the contractions $\Gamma_{e_i} \to \Gamma$ imply that ρ_Γ is actually a distance to the core. To see that this is universal, we need only check that single edge contractions are compatible, i.e., if Γ has edge lengths $\{e_1, \ldots, e_n\}$, then the contraction $\Gamma' \to \Gamma$ sending e_j to 0, where Γ' has edge lengths $\{e_1, \ldots, \hat{e}_j, \ldots, e_n\}$, is compatible. This compatibility follows immediately from contracting both Γ and Γ' to each of the Γ_{e_i}'s.

Finally, note that α is injective because the assignment of a radius to a radially aligned curve Γ is uniquely determined by 1-layer trees. Furthermore, the discussion at the start of the previous paragraph shows that if Q is the collection of partitions corresponding to the 1-layer trees with nonzero radii in a universal radius, then ρ_r is the radius determined by contractions to 1-layer trees. This shows $\beta = \alpha^{-1}$. □

Theorem 4.13. *For each $Q \in \mathfrak{Q}_n$, there is a diagram of stacks*

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \\ \swarrow & & \searrow \\ \overline{\mathcal{M}}_{1,n} & & \overline{\mathcal{M}}_{1,n}(Q) \end{array}$$

such that both arrows are proper and restrict to an isomorphism on $\mathcal{M}_{1,n}$.

Proof. Theorem 3.18 gives us the arrow on the left; we only have to show that the arrow on the right has the claimed properties.

Let $(\rho_\Gamma) = \beta(Q)$ be the universal radius associated to Q. Note that for all Γ, ρ_Γ is a radius of Γ and $\mathrm{part}(\rho_\Gamma) \in Q$, by construction. By Lemma 4.5, the ρ_Γ's define a global section ρ of the characteristic sheaf of $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$. Let $\pi : C \to \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ be the universal curve. Theorem 4.1 constructs a Q-stable family of curves $\overline{\pi} : \overline{C} \to \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ associated to ρ, inducing the map $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n} \to \overline{\mathcal{M}}_{1,n}(Q)$. This map restricts to an isomorphism on $\mathcal{M}_{1,n}$ as the maps $C \leftarrow \widetilde{C} \to \overline{C}$ of Theorem 4.1 are isomorphisms where ρ restricts to 0. □

5. Construction of the Q-stable moduli spaces

Theorem 5.1. *For each $Q \in \mathfrak{Q}_n$, $\overline{\mathcal{M}}_{1,n}(Q)$ is a proper irreducible Deligne–Mumford stack over $\mathbb{Z}\left[\frac{1}{6}\right]$ containing $\mathcal{M}_{1,n}$.*

Our argument is brief since we may reuse much of the proof for the analogous result for m-stable spaces in [Smyth 2011a, Theorem 3.8]. In particular, to show that we have a Deligne–Mumford stack, it is enough to show that the moduli functor $\overline{\mathcal{M}}_{1,n}(Q)$ is deformation-open, bounded, and satisfies the valuative criterion for properness. Boundedness is immediate from [Smyth 2011a, Lemma 3.9], since every Q-stable curve is m-stable for some $m < n$. It is clear that $\overline{\mathcal{M}}_{1,n}(Q)$ contains $\mathcal{M}_{1,n}$. The resulting stack is therefore irreducible, since every Q-stable curve is m-stable for some m, and all m-stable curves are limits of curves in $\mathcal{M}_{1,n}$.

Theorem 5.2 (deformation-openness). *Let S be a noetherian scheme and let $\pi : C \to S$ be a flat, projective morphism with one-dimensional fibers and let $\sigma_1, \dots, \sigma_n$ be n sections. Then the set*

$$T = \left\{ s \in S \mid (\pi|_{\bar{s}} : C|_{\bar{s}} \to \bar{s}, \{\sigma_i(\bar{s})\}_{i=1}^n) \text{ is } Q\text{-stable} \right\}$$

is open.

Proof. As in [Smyth 2011a, Lemma 3.10], we may assume that the geometric fibers $C_{\bar{s}}$ of π are reduced, connected, of arithmetic genus one, with only Gorenstein singularities, and that $H^0(C_s, \Omega_C^\vee(-\Sigma)) = 0$ since these are open conditions.

It remains to show that the locus in S over which the level conditions hold is open. Since S is Noetherian, we may establish openness by showing that this locus is constructible and stable under generization. It is constructible since satisfaction of the level conditions is constant on combinatorial types (defined slightly ahead in Definition 6.1) and the curves with a given combinatorial type form a locally closed subset of S.

So assume S is the spectrum of a DVR with closed point $0 \in S$ and generic point $\eta \in S$. We must show that if $(C_{\bar{0}}, \sigma_1(\bar{0}), \dots, \sigma_n(\bar{0}))$ satisfies the level conditions, then so does $(C_{\overline{\eta}}, \sigma_1(\overline{\eta}), \dots, \sigma_n(\overline{\eta}))$. Replacing S by a finite base change if necessary, we may assume that the irreducible components of $C_{\overline{\eta}}$ are in bijection with the irreducible components of C.

Since level increases with the size of a subcurve, it is enough to check the subcurve level condition on minimal genus one subcurves. Let $E_{\overline{\eta}}$ be a minimal genus one subcurve of $C_{\overline{\eta}}$. Then the limit Z of $E_{\overline{\eta}}$ in $C_{\bar{0}}$ contains the minimal genus one subcurve E_0 of $C_{\bar{0}}$. Because $\mathrm{lev}(E_{\bar{0}}) \notin Q$ by hypothesis,

$\operatorname{lev}(E_{\bar{0}}) \preceq \operatorname{lev}(Z)$, and Q is downward closed, we have $\operatorname{lev}(Z) \notin Q$. Finally, taking limits of the components of $(C_{\bar{\eta}} - E_{\bar{\eta}}) \cup \Sigma$, we see that $\operatorname{lev}(E_{\bar{\eta}}) = \operatorname{lev}(Z)$, so $\operatorname{lev}(E_{\bar{\eta}}) \notin Q$ as required.

The level condition on singularities holds trivially if $C_{\bar{\eta}}$ has only nodes, so suppose that $C_{\bar{\eta}}$ has an elliptic l-fold singularity $q_{\bar{\eta}}$. Since elliptic l_1-fold singularities generize only to l_2-fold singularities with $l_2 \leq l_1$ or nodes, and nodes only generize to nodes and smooth points, the limit of $q_{\bar{\eta}}$ in $C_{\bar{0}}$ must be an elliptic m-fold singularity $q_{\bar{0}}$ with $m \geq l$. For each $i = 1, \ldots, l$ set $Z^i_{\bar{\eta}}$ to be the union of the i-th rational branch $B^i_{\bar{\eta}}$ of $q_{\bar{\eta}}$ with all rational tails attached to this branch. Next, set Z^i and B^i to be the closures of $Z^i_{\bar{\eta}}$ and $B^i_{\bar{\eta}}$ in C and write Z^i_{lim} and B^i_{lim} for their restrictions to $C_{\bar{0}}$. Similarly, let $B^j_{\bar{0}}$ and $Z^j_{\bar{0}}$ be respectively the j-th branch of $q_{\bar{0}}$ and the union of $B^j_{\bar{0}}$ with its rational tails for $j = 1, \ldots, m$. Observe that the partition of $\{1, \ldots, n\}$ induced by the markings in the Z^i_{lim} agrees with the level of $q_{\bar{\eta}}$, and the partition of $\{1, \ldots, n\}$ induced by the $Z^j_{\bar{0}}$ agrees with the level of $q_{\bar{0}}$.

We claim that each $Z^j_{\bar{0}}$ factors through exactly one of the Z^i_{lim}. Note that the limit of at most one branch $B^i_{\bar{\eta}}$ of $q_{\bar{\eta}}$ can contain $B^j_{\bar{0}}$ since $B^j_{\bar{0}}$ is irreducible and each B^i is an irreducible component of C. Observe that each irreducible component of $C_{\bar{0}}$ is connected via a nodal path to exactly one of the branches of $q_{\bar{0}}$. The remaining components of $Z^j_{\bar{0}}$ are connected to the $B^j_{\bar{0}}$ via a nodal path, so we conclude that these belong to Z^i_{lim} as well. The claim follows.

Therefore the level of $q_{\bar{\eta}}$ is a coarsening of the level of $q_{\bar{0}}$, that is, $\operatorname{lev}(q_{\bar{\eta}}) \prec \operatorname{lev}(q_{\bar{0}})$. Since $\operatorname{lev}(q_{\bar{0}}) \in Q$ and Q is downward closed, $q_{\bar{\eta}} \in Q$ too, and we are done. □

Theorem 5.3. *The stack $\overline{\mathcal{M}}_{1,n}(Q)$ is universally closed.*

Proof. Since $\mathcal{M}_{1,n}$ is dense in $\overline{\mathcal{M}}_{1,n}(Q)$, it is enough to show that limits of families of smooth n-pointed curves admit Q-stable limits. Suppose S is the spectrum of a discrete valuation ring with generic point η and $\pi_\eta : C_\eta \to \eta$ is a smooth and proper family of n-pointed curves. Since $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ is proper, after replacing S by a finite cover if necessary, we may find a limit basic radially aligned log curve $\pi^{\mathrm{rad}} : C^{\mathrm{rad}} \to S$ extending $C_\eta \to \eta$. The image of $C^{\mathrm{rad}} \to S$ under the map of Theorem 4.13 corresponding to Q gives the required limit Q-stable curve. □

We now recall the definition of balanced subcurve [Smyth 2011a, Definition 2.11].

Definition 5.4. Given a connected nodal curve E and connected subcurves F_1 and F_2, we say that the *nodal distance $l(F_1, F_2)$ from F_1 to F_2* is the least number of edges in a path from F_1 to F_2 in the dual graph of E. If $p \in E$ is a smooth point, then there is a unique irreducible component F of E containing p, and we write $l(-, p)$ instead of $l(-, F)$.

Definition 5.5. If $(E, \{p_i\}_{i=1}^m)$ is a semistable curve of arithmetic genus one, we say that $(E, \{p_i\}_{i=1}^m)$ is *balanced* if

$$l(Z, p_1) = l(Z, p_2) = \cdots = l(Z, p_m),$$

where $Z \subseteq E$ is the minimal elliptic subcurve of E.

We remark that the nodal distance is an integer, while the distances $\lambda(v)$ defined for genus one log curves take values in the characteristic monoid of the base. The two notions are related by the fact that if

(i) A is the spectrum of a DVR with uniformizer t,

(ii) $S = \operatorname{Spec} A$ is given the log structure associated to the chart $\mathbb{N}\delta \to A$ sending $\delta \mapsto t$,

(iii) $\pi : C \to S$ is a log curve of genus one with smooth generic fiber,

(iv) C has regular total space, and

(v) $Z \subseteq C$ is the minimal genus one subcurve,

then all smoothing parameters of the nodes of C_0 are equal to δ and $\lambda(F) = l(Z, F)\delta$ for all irreducible components F of C.

Theorem 5.6. *The stack $\overline{\mathcal{M}}_{1,n}(Q)$ is separated.*

Proof. We must show that given a pair of Q-stable families $\pi : C \to S$ and $\pi' : C' \to S$ over the spectrum of a discrete valuation ring with generic point η and special point x, an isomorphism $\psi : C|_\eta \to C'|_\eta$ of pointed curves extends to all of S. As in [Smyth 2011a, 3.3.2], we may assume that there is a flat and proper pointed semistable nodal curve $C^{\mathrm{ss}} \to S$ with regular total space and a diagram of pointed S-schemes

$$C \xleftarrow{\varphi} C^{\mathrm{ss}} \xrightarrow{\varphi'} C',$$

where φ and φ' are proper birational morphisms, and it will be enough to show that the exceptional loci of φ and φ' coincide.

If $C|_{\bar{x}}$ or $C'|_{\bar{x}}$ is nodal, then we may argue exactly as in [Smyth 2011a, 3.3.2] to conclude. Therefore we may assume that $C|_{\bar{x}}$ and $C'|_{\bar{x}}$ possess elliptic Gorenstein singularities p and p' respectively. Let $E = \varphi^{-1}(p)$ and $E' = \varphi^{-1}(p')$. As in [Smyth 2011a], we know that E and E' are balanced, with E (resp. E') consisting of all components of $C^{\mathrm{ss}}|_{\bar{x}}$ with nodal distance to the core of $C^{\mathrm{ss}}|_{\bar{x}}$ less than l (resp. less than l'). Without loss of generality we may assume $E \subseteq E'$. If the containment is proper, then $\varphi(E') \subseteq C|_{\bar{x}}$ is a subcurve of genus one containing p and all of its branches. Examining the dual graphs of the various curves over $\bar{x}$, we have $\operatorname{lev}(p') = \operatorname{lev}(\varphi(E'))$. Then, since C is Q-stable, $\operatorname{lev}(\varphi(E')) \notin Q$. On the other hand, $\operatorname{lev}(p') \in Q$, since C' is Q-stable. This is a contradiction, so we have $E = E'$.

The remainder of the argument follows as in [Smyth 2011a, 3.3.2]. □

6. Classification of semistable Gorenstein modular compactifications of $\mathcal{M}_{1,n}$

Our goal in this section is to prove Theorem 1.9, classifying the modular compactifications of $\mathcal{M}_{1,n}$. To aid our classification, we introduce the notion of the combinatorial type of a curve in $\mathcal{U}_{1,n}$. This is analogous to the dual graph of a nodal curve, with the difference that the combinatorial type also keeps track of elliptic m-fold singularities.

Definition 6.1. Let C be a connected, proper, reduced, 1-dimensional scheme over an algebraically closed field k with (at worst) nodes and elliptic Gorenstein singularities. The *combinatorial type of C* consists of the following data:

(i) A set V of *vertices*, equal to the set of components of C.

(ii) A set E of *singularities*, equal to the set of singular points of C.

(iii) A *genus function* $g : V \cup E \to \mathbb{N}$ taking a component of C to the genus of its normalization and taking each singularity of C to its genus as a singularity.

(iv) An *incidence function* $i : V \times E \to \{0, 1\}$ taking $(v, e) \mapsto 1$ if $e \in v$ and to 0 otherwise.

(v) A *marking function* $x : V \to 2^{\{1,\dots,n\}}$ taking a component v to the set of indices of the markings incident to v.

Two combinatorial types $\Gamma_1 = (V_1, E_1, g_1, i_1, x_1)$ and $\Gamma_2 = (V_2, E_2, g_2, i_2, x_2)$ are *isomorphic* if there is a bijection $f : V_1 \cup E_1 \to V_2 \cup E_2$ such that

(i) $f(V_1) \subseteq V_2$ and $f(E_1) \subseteq E_2$;

(ii) $x_1 = x_2 \circ f$;

(iii) $g_1 = g_2 \circ f$;

(iv) $i_1(v, e) = i_2(f(v), f(e))$ for all $(v, e) \in V_1 \times E_1$.

Let $\mathcal{Z}_\Gamma$ be the locus in $\mathcal{U}_{1,n}$ of curves with combinatorial type Γ. These loci have a natural structure of locally closed substack of $\mathcal{U}_{1,n}$. (This follows from the fact that the deformations of a curve preserving its singularities form a closed subspace of the full deformation space of the curve. See [Smyth 2011b, Lemma 2.1], for example.) Note that for each n, there is a finite set of isomorphism classes of combinatorial types of curves in $\mathcal{U}_{1,n}$ and altogether the $\mathcal{Z}_\Gamma$'s form a stratification of $\mathcal{U}_{1,n}$ into locally closed substacks.

Let $\mathcal{M}$ be a modular compactification in our sense. Since $\mathcal{M}$ is assumed to be an open substack of $\mathcal{U}_{1,n}$, it is uniquely determined by its points. Our strategy is to show that $\mathcal{M}$ must be a union of the $\mathcal{Z}_\Gamma$'s. Then, analyzing the possible limits of curves, we will find that the choices of combinatorial types making up $\mathcal{M}$ necessarily agree with a Q-stability condition.

Lemma 6.2. *$\mathcal{U}_{1,n}$ is the union of the $\overline{\mathcal{M}}_{1,n}(m)$'s.*

Proof. Suppose that $(\pi : C \to S, \sigma_1, \dots, \sigma_n)$ is a family in $\mathcal{U}_{1,n}$. We want to show that S possesses an open cover such that the restriction of C to each part of the cover factors through some $\overline{\mathcal{M}}_{1,n}(m)$. Let $\bar{s}$ be a geometric point of S. Then the fiber $C_{\bar{s}}$ of π over $\bar{s}$ is a Gorenstein curve of genus one with n distinct marked points and no infinitesimal automorphisms. Recall that the only Gorenstein singularities of genus less than or equal to one are the elliptic Gorenstein singularities and the node. If $C_{\bar{s}}$ has an elliptic m-fold singularity for some m, then since C has no infinitesimal automorphisms, the number of markings and nodes on the minimal genus one subcurve must be at least $m+1$. It follows that $C_{\bar{s}} \in \overline{\mathcal{M}}_{1,n}(m)$. If $C_{\bar{s}}$ does not have an elliptic Gorenstein singularity, then $C_{\bar{s}} \in \overline{\mathcal{M}}_{1,n} = \overline{\mathcal{M}}_{1,n}(0)$. Since the $\overline{\mathcal{M}}_{1,n}(m)$'s are deformation-open [Smyth 2011a], for each $\bar{s}$ there is an open neighborhood U_s of the image of $\bar{s}$ in S such that $C|_{U_s}$ factors through one of the stacks $\overline{\mathcal{M}}_{1,n}(m)$. ☐

Lemma 6.3. *Let Γ be a combinatorial type. Then $\mathcal{Z}_\Gamma$ is irreducible.*

Proof. If Γ possesses no m-fold points, then $\mathcal{Z}_\Gamma$ is a product of copies of $\mathcal{M}_{g,n}$'s coming from the vertices of Γ. Since each of the stacks $\mathcal{M}_{g,n}$ is geometrically irreducible, so is $\mathcal{Z}_\Gamma$.

Next, let Γ be a combinatorial type consisting of a single elliptic m-fold point with k rational branches $E_1, \dots, E_k$ with $n_1, \dots, n_k$ markings, respectively. Let $n = n_1 + \cdots + n_k$.

Let

$$A = \left\{ (f_i(t_i))_{i=1}^m \in \prod_{i=1}^m \mathbb{Z}\big[\tfrac{1}{6}, t_i\big] \;\middle|\; f_i(0) = f_j(0) \text{ for all } i \text{ and } \sum_{i=1}^m f_i'(0) = 0 \right\}.$$

This gives a standard affine model of the m-fold point with rational branches. Form a proper curve $D \to \operatorname{Spec} \mathbb{Z}\big[\frac{1}{6}\big]$ by gluing $\operatorname{Spec} \mathbb{Z}\big[\frac{1}{6}, t_i^{-1}\big]$ to the i-th branch of $\operatorname{Spec} A$ for each i. If C is a minimal unmarked Gorenstein curve of genus one over an algebraically closed field with an m-fold point, then C appears as a geometric fiber of D.

Now let

$$S = \prod_{i=1}^m \Big(\operatorname{Spec} \mathbb{Z}\big[\tfrac{1}{6}, s_{i,1}, \dots, s_{i,n_i}\big] - \Delta_{n_i} \Big),$$

where Δ_{n_i} is the locus where any pair of coordinates of $\mathbb{A}^{n_i}_{\mathbb{Z}[1/6]} = \operatorname{Spec} \mathbb{Z}\big[\frac{1}{6}, s_{i,1}, \dots, s_{i,n_i}\big]$ coincide. We construct a family of pointed curves $D \times S \to S$ by taking the j-th marking on the i-th branch of D to be located at $t_i^{-1} = s_{i,j}$.

Now, every pointed curve of type Γ appears as some geometric fiber of the family $D \times S \to S$. Therefore, the image of S in $\mathcal{U}_{1,n}$ under the map induced by $D \times S \to S$ is precisely $\mathcal{Z}_\Gamma$. Since S is irreducible, the result follows.

Finally, consider a general Γ. Let $\Gamma_{\min}$ be the combinatorial type of the minimal genus one subcombinatorial type of Γ with markings at the outgoing edges. Clearly, $\mathcal{Z}_\Gamma$ is a product of $\mathcal{M}_{0,n}$'s and $\mathcal{Z}_{\Gamma_{\min}}$, all of which are already known to be geometrically irreducible, so $\mathcal{Z}_\Gamma$ is irreducible too. □

The following lemma is the crucial one: it reduces the classification of Gorenstein compactifications to combinatorics.

Lemma 6.4. *$\mathcal{M}$ is a union of the $\mathcal{Z}_\Gamma$'s.*

Proof. It suffices to show that if $\mathcal{M}$ shares a geometric point with $\mathcal{Z}_\Gamma$ for some Γ, then $\mathcal{M}$ contains all points of $\mathcal{Z}_\Gamma$. By the previous lemma, for any pair of geometric points $C_p \in \mathcal{Z}_\Gamma(\operatorname{Spec} k(p))$, $C_q \in \mathcal{Z}_\Gamma(\operatorname{Spec} k(q))$, there are families of curves $C_S \in \mathcal{Z}_\Gamma(S)$, $C_T \in \mathcal{Z}_\Gamma(T)$, where S and T are spectra of discrete valuation rings, such that

(i) S and T have isomorphic geometric generic points,

(ii) C_S is isomorphic to C_T over this common geometric generic point,

(iii) there is a map $\operatorname{Spec} k(p) \to S$ onto the special point of S along which C_S pulls back to C_p,

(iv) there is a map $\operatorname{Spec} k(q) \to T$ onto the special point of T along which C_T pulls back to C_q.

We know $\mathcal{M}$ is closed under generization and has a specialization for any 1-dimensional family. It follows that if $\mathcal{M}$ contains C_p, then $\mathcal{M}(S)$ contains C_S. To conclude, we show that $\mathcal{M}(T)$ contains C_T too.

Let η be the generic point of T, and replacing T by a finite base change if necessary, let $C'_T \in \mathcal{M}(T)$ be the unique limit of $C_T|_\eta$ in $\mathcal{M}$. We have an isomorphism of η-schemes $C'_T|_\eta \cong C_T|_\eta$. Since $\mathcal{U}_{1,n}$ is a union of the open substacks $\overline{\mathcal{M}}_{1,n}(m)$, there is some m such that C'_T lives in $\overline{\mathcal{M}}_{1,n}(m)(T)$. Considering $C'_T|_{\bar{\eta}}$, we conclude that curves of combinatorial type Γ are m-stable. In particular, both C_T and C'_T are families in $\overline{\mathcal{M}}_{1,n}(m)$. Since $\overline{\mathcal{M}}_{1,n}(m)$ is separated, we conclude that $C'_T \cong C_T$ over T, completing the proof. □

Our strategy now is to produce families of curves witnessing enough of the relationships between the loci $\mathcal{Z}_\Gamma$ that $\mathcal{M}$ is forced to be Q-stable.

Definition 6.5. Let

$$\mathcal{P} : P_1 \prec P_2 \prec \cdots \prec P_k$$

be a strictly increasing chain of partitions of $\{1, \dots, n\}$ not including the partition $\{\{1\}, \dots, \{n\}\}$. We say that a family of radially aligned curves $\pi : C \to S$ is a *test curve of type $\mathcal{P}$ centered at a geometric point s of S* if

(i) $(\pi : C \to S, s)$ satisfies the conclusions of Theorem 3.10;

(ii) the tropicalization of the central fiber $\operatorname{trop}(C|_s)$ has a basic radially aligned log structure;

(iii) the log structure on S is divisorial, that is, it is the log structure associated to a normal crossings divisor [Kato 1989, (1.5)];

(iv) the tropicalization of the central fiber has partition type $\mathcal{P}$ (Definition 3.20).

Lemma 6.6. *For any strictly increasing chain of partitions of* $\{1, \dots, n\}$

$$\mathcal{P} : P_1 \prec P_2 \prec \cdots \prec P_k$$

not including the partition $\{\{1\}, \dots, \{n\}\}$, *there is a test curve of type* $\mathcal{P}$.

Proof. Choose an algebraically closed field κ. Pick a smooth genus one curve E over κ arbitrarily. Add a rational component $Z_A^{(1)}$ for each part A of P_1 and attach them nodally to E at distinct smooth points. Repeat this process for each $i = 2, \dots, k$, adding components $Z_A^{(i)}$ for each part $A \in P_i$, where $Z_A^{(i)}$ is nodally attached to the unique component $Z_B^{(i-1)}$ where $B \in P_{i-1}$ is the part containing A. Finally, mark points $p_1, \dots, p_n$, where each p_i is a smooth point of the component $Z_A^{(k)}$, where A is the part of P_k containing i. Call the whole pointed nodal curve we have constructed C_0. Give $S_0 = \operatorname{Spec} \kappa$ the log structure associated to the map $\bigoplus_{i=1}^k \mathbb{N} e_i \to \kappa$ sending everything to 0 except for the identity element. Choose a log structure on $\pi_0 : C_0 \to S_0$ compatible with the log structure on S_0 making $\pi_0 : C_0 \to S_0$ into a log curve so that

(i) the edges between the $Z_A^{(1)}$'s and E are all of length e_1;

(ii) for $i = 2, \dots, k$, the edges between the $Z_A^{(i)}$'s and $Z_A^{(i-1)}$'s are all of length e_i.

Stabilize to obtain a log curve $\pi_0^s : C_0^s \to S_0$. Since π_0^s is radially aligned with the basic radially aligned log structure, there is an associated map $f : S_0 \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Choose a factorization of it through an étale chart $U \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Now, set S to be a neighborhood of the image s of S_0 in U using Theorem 3.10.

We now claim that the pullback of the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to S gives the desired test curve. Properties (i) and (ii) hold by construction. Property (iii) holds since S was chosen to be an étale neighborhood of a point of the chart U, which has a divisorial log structure.

It remains to check property (iv). If i is an integer with $1 \leq i < k$, there is at least one $Z_A^{(i)}$ with at least two descendants, since $P_i \neq P_{i+1}$. If $i = k$, all of the $Z_A^{(i)}$'s possess at least three marked points. Either way, for each integer $i = 1, \ldots, k$, there is a component $Z_A^{(i)}$ that survives the stabilization step, and it lives at radius $\lambda(Z_A^{(i)}) = e_1 + \cdots + e_i$ by construction. Conversely, every radius is of this form. Now, if we subdivide $\mathrm{trop}(C|_s)$ where $\lambda = e_1 + \cdots + e_i$, the effect is to reintroduce the components $Z_A^{(i)}$ for $A \in P_i$. Deleting the locus where $\lambda < e_1 + \cdots + e_i$ leaves us with the trees rooted at the $Z_A^{(i)}$'s as A varies through the elements of P_i. By construction, for each $A \in P_i$ the tree rooted at $Z_A^{(i)}$ contains the markings indexed by A. Therefore, the partition type of $\mathrm{trop}(C|_s)$ is precisely $\mathcal{P}$. □

With test curves in hand, we may form several families of contracted curves. Exactly one of these will be Q-stable for any Q.

Lemma 6.7. *Let $\pi : C \to S$ be a test curve of type $\mathcal{P}$ centered at s. Let $\rho_0 = 0 < \rho_1 < \cdots < \rho_k$ be the distinct radii of the tropicalization of the central fiber. For each $i = 0, \ldots, k$, let $\overline{C}_i \to S$ be the contraction of $C \to S$ associated to the steep mesa with radius ρ_i and let Γ_i be the combinatorial type of the fiber of $\overline{C}_i$ over s.*

Let $Q \in \mathfrak{Q}_{1,n}$ be a downward closed set of partitions. Then there is exactly one index i such that Γ_i is Q-stable, namely the greatest index i for which $P_i \in Q$.

Proof. Observe that

(i) $\overline{C}_0|_s$ is nodal and its minimal elliptic subcurve has level P_1;

(ii) for $0 < i < k$, $\overline{C}_i|_s$ possesses an elliptic singularity of level P_i and its minimal elliptic subcurve has level P_{i+1};

(iii) for $i = k$, $\overline{C}_i|_s$ possesses an elliptic singularity of level P_k and its minimal elliptic subcurve (namely $\overline{C}_k|_s$) has level $\{\{1\}, \ldots, \{n\}\}$.

The result follows immediately. □

With some more work, we see that exactly one of the contracted curves belongs to our arbitrary modular compactification $\mathcal{M}$ as well.

Lemma 6.8. *Choose notation as in Lemma 6.7. Then $\mathcal{M}$ contains exactly one of the families $\overline{C}_i \to S$ and exactly one of the loci $\mathcal{Z}_{\Gamma_i}$.*

Proof. Choose t to be a point of S generizing s over which C is smooth. Let $T \to S$ be a map from the spectrum of a DVR with special point x mapping to s and generic point η mapping to t.

Replacing T by a finite base change if necessary, find the limit curve $C_{\mathcal{M}} \to T$ in $\mathcal{M}$ of the smooth curve $C|_\eta$. After a second base change if necessary, pick a regular family of semistable curves $C^{\mathrm{ss}} \to T$ dominating $C_{\mathcal{M}} \to T$ and each of the families $\overline{C}_i|_T \to T$, formed by subdividing and contracting the i-th radius of C. Note that $\overline{C}_0 = C|_T$. Let E be the exceptional locus of $\varphi : C^{\mathrm{ss}} \to C_{\mathcal{M}}$ and let E_i be the exceptional locus of $\varphi_i : C^{\mathrm{ss}} \to \overline{C}_i|_T$ for each i.

If $C_{\mathcal{M}}$ is stable then both $C|_T$ and $C_{\mathcal{M}}$ are stable limits of $C|_\eta$, so they must agree.

Otherwise, $C_{\mathcal{M}}$ possesses a unique elliptic m-fold point p. By [Smyth 2011a, Proposition 2.12], $\varphi^{-1}(p)$ is a balanced subcurve of $C^{\mathrm{ss}}|_x$, with $\varphi^{-1}(p)$ consisting of all components of $C^{\mathrm{ss}}|_x$ whose nodal distance from the core of $C^{\mathrm{ss}}|_x$ is less than some integer l. Since $C_{\mathcal{M}}$ possesses no infinitesimal automorphisms, at least one of the components of $C^{\mathrm{ss}}|_x$ of distance exactly l from the core of $C^{\mathrm{ss}}|_x$ has at least three special points. Then E is the union of $\varphi^{-1}(p)$ with the semistable components of $C^{\mathrm{ss}}|_x$ disjoint from $\varphi^{-1}(p)$.

We may repeat this argument for each of the exceptional loci E_i for $i > 0$. Therefore, for each i,

(i) there is a balanced subcurve F_i of $C^{\mathrm{ss}}|_x$ consisting of all components of nodal distance from the core of $C^{\mathrm{ss}}|_x$ less than some integer l_i,

(ii) there is a component of $C^{\mathrm{ss}}|_x$ of distance exactly l_i from the core with at least three markings, and

(iii) E_i is the union of F_i with the semistable components of $C^{\mathrm{ss}}|_x$ disjoint from F_i.

Moreover, we claim that the loci E_i exhaust the subcurves of $C^{\mathrm{ss}}|_x$ with these properties. To see this, suppose F' is a balanced subcurve of $C^{\mathrm{ss}}|_x$ whose components are those with nodal distance less than l', and so that there is a stable component G of $C^{\mathrm{ss}}|_x$ of distance l' from the core. Then $\varphi(G)$ maps to a stable component of $C|_x$. The component G lives at the distance ρ_i from the core of $C|_x$ for some i. Then, for this same i, we have that $l' = l_i$ and the rest follows.

Therefore $E = E_i$ for some $i > 0$. Since both $C_{\mathcal{M}}$ and $\overline{C}_i|_T$ are normal and obtained from contracting the same locus of C^{ss}, they are isomorphic.

This exhibits a curve, namely $\overline{C}_i|_x$, in $\mathcal{Z}_{\Gamma_i}$ in $\mathcal{M}$. By Lemma 6.4, $\mathcal{M}$ contains the whole locus. By separatedness, $\mathcal{M}$ cannot contain any of the other $\overline{C}_j|_T$ for $j \neq i$. Therefore $\mathcal{M}$ cannot contain $\mathcal{Z}_{\Gamma_j}$ for any $j \neq i$.

Finally, we wish to show that in fact $\mathcal{M}$ contains the entire family $\overline{C}_i \to S$. This will follow from the closure of $\mathcal{M}$ under generization. Write $\bigoplus_i e_i \oplus \bigoplus_j \delta_j$ for $\overline{M}_{S,s} \cong \Gamma(S, \overline{M}_S)$ so that the e_i's are the differences between consecutive radii, as in Definition 3.17. The base S of $C \to S$ possesses a stratification by locally closed subsets $\{W_I\}$ indexed by subsets $I \subseteq \{1, \ldots, k\}$, where

$$W_I = \bigcap_{j \in I} \mathrm{Supp}(e_j) \cap \bigcap_{j \in I^c} (S - \mathrm{Supp}(e_j)).$$

Since $C \to S$ satisfies the conclusions of Theorem 3.10, the tropicalization of the fibers of C is constant on the subsets W_I. It follows that the same holds of the combinatorial types of the fibers of $\overline{C}_i \to S$. Since the log structure of S is divisorial, for each $I \subseteq \{1, \ldots, k\}$, there is a generization $\eta_I \to s$, where $\eta_I \in W_I$. Since $\mathcal{M}$ is closed under generization and $\mathcal{M}$ contains $\overline{C}_i|_s$, $\mathcal{M}$ must also contain $\overline{C}_i|_{\eta_I}$. Then, since membership in $\mathcal{M}$ is determined by combinatorial type, $\mathcal{M}$ contains all of $\overline{C}_i|_{W_I}$. We conclude that the whole family $\overline{C}_i \to S$ belongs to $\mathcal{M}$. □

Remark 6.9. At first glance, the conclusion of the lemma looks different for test curves whose chains of partitions agree but whose cores have different combinatorial types, since their combinatorial types Γ_0 will be distinct. Such pairs of test curves share combinatorial types Γ_i for $i \geq 1$, so, comparing the conclusion of the lemma for the various test curves, either $\mathcal{M}$ chooses to contract the core of all test curves of type $\mathcal{P}$ or it contracts the core of none of them.

When the chain $\mathcal{P}$ consists of a single partition P, Lemma 6.8 tells us there are only two "choices" that $\mathcal{M}$ could make: either $\mathcal{M}$ contains the contracted curve $\overline{C}_1$ or $\mathcal{M}$ contains the uncontracted curve C. In the former case, say that $\mathcal{M}$ *contracts* P. Our next claim is that for any test curve C, the combinatorial type of the contraction of C included by $\mathcal{M}$ is determined solely by the partitions which $\mathcal{M}$ contracts.

Lemma 6.10. *Choose notation as in Lemma 6.8. Let i be the index of the family $\overline{C}_i \to S$ contained in $\mathcal{M}$. Let j be any integer with $1 \leq j \leq k$. Then $j \leq i$ if and only if $\mathcal{M}$ contracts P_j.*

Proof. Write $\bigoplus_i e_i \oplus \bigoplus_j \delta_j$ for $\overline{M}_{S,s} \cong \Gamma(S, \overline{M}_S)$ so that the e_i's are the differences between consecutive radii of trop(C), as in Definition 3.17. Let $C' \to S'$ be the restriction of $C \to S$ to the complement of the support of $e_1, \dots, \hat{e}_j, \dots, e_k$. Choose s' to be a point of the support of e_j inside S'. It is not difficult to check that the resulting family is a test family of type P_j (i.e., the chain of partitions of length one containing just the partition P_j) whose log structure is generated by e_j. The unique nonzero radius of this test family is e_j. Since the contraction of a family of mesa curves commutes with base change, $\overline{C}_i|_{S'}$ will be the contraction of C' with respect to $\rho|_{S'}$. Now, the restriction of ρ_i from S to S' is either e_j if $j \leq i$ or 0 if $i < j$. In the first case, $\mathcal{M}$ contracts P_j. In the latter $\mathcal{M}$ does not. □

Lemma 6.11. *Every combinatorial type Γ in $\mathcal{U}_{1,n}$ is the combinatorial type of some contraction of a test curve.*

Proof. Choose a representative curve C_0 for Γ. Find a smoothing $C \to T$ of C_0 over the spectrum of a discrete valuation ring with special point x and generic point η. Replacing T by a finite base change if necessary, let $C^{\text{rad}} \to T$ be the limit radially aligned curve of $C|_\eta \to \eta$ in $\overline{\mathcal{M}}^{\text{rad}}_{1,n}$ with the basic radially aligned log structure.

Arguing as in Lemma 6.6, we can find a family $C^{\text{test}} \to S$ centered at s with central fiber $C^{\text{rad}}|_x$ by choosing a sufficiently small étale neighborhood of $C^{\text{rad}}|_x$ in an étale chart for $\overline{\mathcal{M}}^{\text{rad}}_{1,n}$. This neighborhood may be assumed to satisfy the conditions of Theorem 3.10 and its log structure will be divisorial since it is pulled back from the log structure of $\overline{\mathcal{M}}^{\text{rad}}_{1,n}$ along an étale morphism. This family $C^{\text{test}} \to S$ is therefore a test curve of some type, with the type given by the partitions induced by the various radii of the tropicalization of its central fiber.

Moreover, replacing T by a finite base change if necessary, we may assume that $C^{\text{rad}} \to T$ is pulled back from $C^{\text{test}} \to S$. We are now in exactly the situation of the proof of Lemma 6.8, so we conclude that C_0 is one of the contractions of the central fiber of C^{test}, as required. □

Now that everything is in place, our classification result follows easily.

Proof of Theorem 1.9. Let $Q_{\mathcal{M}}$ be the set of nondiscrete partitions P of $\{1, \dots, n\}$ such that $\mathcal{M}$ contracts P. By Lemma 6.10, $Q_{\mathcal{M}}$ is downward closed.

By Lemmas 6.7 and 6.10, the combinatorial types that appear as contractions of test curves contained in $\mathcal{M}$ and $\overline{\mathcal{M}}_{1,n}(Q_{\mathcal{M}})$ are the same. On the other hand, by Lemma 6.11, every combinatorial type appears as Γ_i for some test curve C^{test}. It follows that $\mathcal{M}$ contains precisely the $Q_{\mathcal{M}}$-stable curves. □

7. Interpolation of Q-stable spaces

We learned in Section 4 that the choices of universal radii are in bijection with compatible choices of radii on 1-layer trees. Each 1-layer tree Γ has two possible radii: the zero radius and a unique nonzero radius; let's call it e_Γ. Thinking tropically, it is tempting to choose radii intermediate to 0 and e_Γ to contract. If we do this and follow the image of the universal curve under the resulting contraction, we find Artin stacks that interpolate between the Q-stable spaces. The moduli problems of these stacks admit an elegant description, so we will define them and verify their algebraicity directly prior to a tropical construction.

We therefore imagine choosing an intermediate radius for each 1-layer tree Γ in the form $c_\Gamma e_\Gamma$ with $c_\Gamma \in [0, 1]$. The possible choices of such radii form a cube complex, which we now describe directly.

Definition 7.1. Let $X_{1,n}$ be the subset of $[0, 1]^{\text{Part}(n)}$ of tuples $(c_P)_{P \in \text{Part}(n)}$ with the properties that

(i) whenever P_1 and P_2 are two partitions of $\{1, \dots, n\}$ with $P_1 \prec P_2$ and $c_{P_2} > 0$, then $c_{P_1} = 1$;

(ii) $c_{\{\{1\},\dots,\{n\}\}} = 0$.

We give $X_{1,n}$ the structure of a cube complex whose open cells are the intersections of the open cells of $[0, 1]^{\text{Part}(n)}$ with $X_{1,n}$.

Note that the vertices of $X_{1,n}$ are in bijection with $\mathfrak{Q}_n$: the correspondence is given by taking a subset Q of $\text{Part}(n)$ to its indicator function.

Example 7.2. $X_{1,3}$ consists of the tuples

$$(x, y, z, w) = (c_{\{\{1,2,3\}\}}, c_{\{\{1\},\{2,3\}\}}, c_{\{\{2\},\{1,3\}\}}, c_{\{\{3\},\{1,2\}\}})$$

in $[0, 1]^4$ where $y = z = w = 0$ unless $x = 1$. Then $X_{1,3}$ consists of the 1-dimensional cube

$$\{(x, 0, 0, 0) \mid x \in [0, 1]\}$$

attached to the 3-dimensional solid cube

$$\{(1, y, z, w) \mid y, z, w \in [0, 1]\}.$$

The latter cube fills in the interior of the cube visible in Figure 1.

The stacks associated to these imagined universal radii are as follows.

Definition 7.3. Let $(c_P)_{P \in \text{Part}(n)}$ be an element of $X_{1,n}$. Let

$$Q_{\text{sing}} = \{P \in \text{Part}(n) \mid c_P > 0\} \quad \text{and} \quad Q_{\text{curve}} = \{P \in \text{Part}(n) \mid c_P < 1\}.$$

An n-pointed family of Gorenstein curves $(\pi : C \to S, \sigma_1, \dots, \sigma_n)$ is $(c_P)_{P \in \mathrm{Part}(n)}$-*stable* if

> $(\pi : C \to S, \sigma_1, \dots, \sigma_n)$ is a flat and proper family of connected, reduced, Gorenstein curves of arithmetic genus one with n distinct marked points,

and for each geometric fiber C_s, the following conditions hold:

(i) If $p \in C_s$ is an elliptic Gorenstein singularity, then $\mathrm{lev}(p) \in Q_{\mathrm{sing}}$.

(ii) If $Z \subseteq C_s$ is a connected subcurve of genus one, then $\mathrm{lev}(Z) \in Q_{\mathrm{curve}}$.

(iii) If $Z_1 \subsetneq Z_2$ is a proper inclusion of connected subcurves of C_s of arithmetic genus one, then $\mathrm{lev}(Z_1) \prec \mathrm{lev}(Z_2)$.

(iv) If F is an irreducible component of the minimal genus one subcurve $Z_{\min}$ of C_s, then F meets $\overline{C_s - Z_{\min}} \cup \{\sigma_1(s), \dots, \sigma_n(s)\}$ in at least one point.

Let $\overline{\mathcal{M}}_{1,n}((c_P))$ be the stack whose S-points are the (c_P)-stable families of curves over S.

Remark 7.4. Note that Q_{sing} is downward closed, Q_{curve} is upward closed, and the two sets intersect in the partitions P where $0 < c_P < 1$.

The new feature of this definition is that a (c_P)-stable curve C may have a minimal genus one subcurve Z with an elliptic Gorenstein singularity p such that $\mathrm{lev}(p) = \mathrm{lev}(Z)$, so long as $\mathrm{lev}(p) \in Q_{\mathrm{sing}} \cap Q_{\mathrm{curve}}$. If this happens, then each of the branches of the singularity p will have only one special point other than p. By [Smyth 2011a, Corollary 2.4], C then has infinitesimal automorphisms. (In fact, it has a $\mathbb{G}_m$ of automorphisms.) It follows that whenever $Q_{\mathrm{sing}} \cap Q_{\mathrm{curve}}$ is nonempty, $\overline{\mathcal{M}}_{1,n}((c_P))$ is not a Deligne–Mumford stack.

Theorem 7.5. *If (c_P) consists only of 1's and 0's, then $\overline{\mathcal{M}}_{1,n}((c_P)) = \overline{\mathcal{M}}_{1,n}(Q)$, where*

$$Q = \{P \in \mathrm{Part}(n) \mid c_P = 1\}.$$

Proof. The only differently stated conditions for membership in $\overline{\mathcal{M}}_{1,n}((c_P))$ and $\overline{\mathcal{M}}_{1,n}(Q)$ are those on geometric fibers, so it suffices to show their geometric points coincide.

Suppose C is a Q-stable curve over some algebraically closed field. Then it is clear that conditions (i) and (ii) hold since $Q = Q_{\mathrm{sing}}$ and $\mathrm{Part}(n) - Q = Q_{\mathrm{curve}}$. Condition (iv) holds since C has no infinitesimal automorphisms.

To see that condition (iii) holds, suppose that $Z_1 \subsetneq Z_2$ is a proper inclusion of connected subcurves C of arithmetic genus one. Then there is a rational component F of Z_2 meeting Z_1 in a point. Since F has at least three special points, $\mathrm{lev}(Z_1) \prec \mathrm{lev}(Z_1 \cup F) \preceq \mathrm{lev}(Z_2)$.

Conversely, suppose that C is (c_P)-stable. The level conditions for Q on C hold by (i) and (ii). It remains to show that C has no infinitesimal automorphisms, that is, by [Smyth 2011a, Corollary 2.4],

(a) each irreducible component F of C with genus zero not meeting an elliptic Gorenstein singularity has at least three special points;

(b) if C has an elliptic Gorenstein singularity q, then each component B of the minimal elliptic subcurve containing q has at least one special point other than q and there is at least such component with two special points other than q.

To address (a), let F be an irreducible component of C with genus zero not meeting an elliptic Gorenstein singularity. Then either C is nodal and F belongs to the core of C, or F is not a component of the minimal subcurve of genus one. In the former case, we are done by (iv). In the latter, let Z_1 be the union of the minimal genus one subcurve E together with the components on the unique path from E to F, not including F. Let $Z_2 = Z_1 \cup F$. Then, since $\operatorname{lev}(Z_1) \prec \operatorname{lev}(Z_2)$, F must have at least three special points.

To address (b), suppose that C has an elliptic Gorenstein singularity q. Let $Z_{\min}$ be the minimal genus one subcurve of C. If B is an irreducible component of $Z_{\min}$, then B has at least one other special point by condition (iv). Next, since $Q_{\text{sing}} \cap Q_{\text{curve}} = \varnothing$, we must have a proper refinement $\operatorname{lev}(q) \prec \operatorname{lev}(Z_{\min})$. This implies that there is at least one branch of q with two special points other than q. □

Theorem 7.6. *The stack of (c_P)-stable curves is deformation-open. That is, if S is a noetherian scheme and $\pi : C \to S$ is a flat, projective morphism with one-dimensional fibers and sections $\sigma_1, \dots, \sigma_n$, then the set*

$$T = \left\{ s \in S \mid (\pi_s : C_{\bar{s}} \to \bar{s}, \{\sigma_i(\bar{s})\}_{i=1}^n) \text{ is } (c_P)\text{-stable} \right\}$$

is open in S.

Proof. As in Theorem 5.2, we may assume that the geometric fibers $C_{\bar{s}}$ of π are reduced, connected, and of arithmetic genus one with only Gorenstein singularities, since these are open conditions.

Again as in Theorem 5.2, the locus T is constructible since satisfaction of the remaining conditions is constant on combinatorial types and the curves with a given combinatorial type form a locally closed subset of S. Therefore it suffices to check that the remaining conditions hold under generization.

So assume S is the spectrum of a DVR with closed point $0 \in S$ and generic point $\eta \in S$. We must show that if $(C_{\bar{0}}, \sigma_1(\bar{0}), \dots, \sigma_n(\bar{0}))$ satisfies the remaining conditions, then so does $(C_{\bar{\eta}}, \sigma_1(\bar{\eta}), \dots, \sigma_n(\bar{\eta}))$. Write Σ for the divisor of markings. Since T is characterized by geometric fibers, we may apply a finite base change to S so that restriction to the geometric generic fiber induces a bijection from the components of C to the components of $C_{\bar{\eta}}$. The level conditions on singularities and subcurves are stable under generization by an identical argument to Theorem 5.2.

Next, we consider condition (iii). Suppose that $Z_1^{\bar{\eta}} \subset Z_2^{\bar{\eta}}$ is a proper inclusion of genus one subcurves of $C_{\bar{\eta}}$. Let $Z_1^{\bar{0}}$ and $Z_2^{\bar{0}}$ be the respective limits of $Z_1^{\bar{\eta}}$ and $Z_2^{\bar{\eta}}$ in $C_{\bar{0}}$. Then we must have a proper inclusion $Z_1^{\bar{0}} \subset Z_2^{\bar{0}}$. Taking limits of the connected components of $\overline{(C_{\bar{\eta}} - Z_1^{\bar{\eta}}) \cup \Sigma|_{\bar{\eta}}}$, we see that $\operatorname{lev}(Z_1^{\bar{0}}) = \operatorname{lev}(Z_1^{\bar{\eta}})$. Similarly, $\operatorname{lev}(Z_2^{\bar{0}}) = \operatorname{lev}(Z_2^{\bar{\eta}})$. By (c_P)-stability of the central fiber, $\operatorname{lev}(Z_1^{\bar{0}}) \prec \operatorname{lev}(Z_2^{\bar{0}})$. Because $\operatorname{lev}(Z_1^{\bar{0}}) = \operatorname{lev}(Z_1^{\bar{\eta}})$ and $\operatorname{lev}(Z_2^{\bar{0}}) = \operatorname{lev}(Z_2^{\bar{\eta}})$, we have the desired refinement $\operatorname{lev}(Z_1^{\bar{\eta}}) \prec \operatorname{lev}(Z_2^{\bar{\eta}})$ in the generic fiber too.

Finally, we show condition (iv) is stable under generization. Let

$$Z_{\min}^{\bar{\eta}} = \text{minimal genus one subcurve of } C_{\bar{\eta}} \quad \text{and} \quad Z_{\min}^{\bar{0}} = \text{minimal genus one subcurve of } C_{\bar{0}}.$$

Given an irreducible component $F_{\bar{0}}$ of $Z^{\bar{0}}_{\mathrm{min}}$, there is a unique irreducible component $F_{\bar{\eta}}$ of $Z_{\bar{\eta}}$ to which $F_{\bar{0}}$ generizes. Moreover, as $F_{\bar{0}}$ varies over the components of $Z^{\bar{0}}_{\mathrm{min}}$, $F_{\bar{\eta}}$ varies over all the irreducible components of $Z^{\bar{\eta}}_{\mathrm{min}}$.

By axiom (iv), $F_{\bar{0}}$ either meets a marking or it meets a connected rational tail T contained in $\overline{C_{\bar{0}} - Z^{\bar{0}}_{\mathrm{min}}}$. If $F_{\bar{0}}$ itself contains a marking, we set $Y^1_{\bar{0}} = F_{\bar{0}}$ and let $k = 1$. If not, then we may find a path in the dual graph of $C_{\bar{0}}$ from $F_{\bar{0}}$ to a marked component of T, i.e., a sequence $Y^1_{\bar{0}}, \dots, Y^k_{\bar{0}}$ of irreducible components of $C_{\bar{0}}$, where $Y^1_{\bar{0}} = F_{\bar{0}}$, $Y^{i+1}_{\bar{0}}$ meets $Y^i_{\bar{0}}$ in a node for all $1 \le i < k$, $Y^i_{\bar{0}} \not\subseteq Z_{\bar{0}}$ for each $i > 0$, and $Y^k_{\bar{0}}$ contains a marking. For each i we let $Y^i_{\bar{\eta}}$ be the unique irreducible component of $C_{\bar{\eta}}$ generizing $Y^i_{\bar{0}}$. After reindexing to omit repeats, the resulting sequence of components $Y^1_{\bar{\eta}}, \dots, Y^l_{\bar{\eta}}$ of $C_{\bar{\eta}}$ is again a sequence of components connected by nodes, beginning with $F_{\bar{\eta}}$, and ending in a component with a marking.

If all of the nodes of the path $Y^1_{\bar{0}}, \dots, Y^k_{\bar{0}}$ smooth out in the geometric generic fiber, then $l = 1$, and $F_{\bar{\eta}}$ meets a marking, and we are done. If not, then let $p \in C_{\bar{0}}$ be the first node on the path that does not smooth out in the family. Then, after a finite base change if necessary, there is a section $P : S \to C$ through the singular locus of C going through p and meeting $F_{\bar{\eta}}$. Blowing up along P, we see that $Y^2_{\bar{\eta}}$ does not belong to $Z^{\bar{\eta}}_{\mathrm{min}}$, since P separates the limit of $Y^2_{\bar{\eta}}$ from $Z^{\bar{0}}_{\mathrm{min}}$ in the central fiber. Then F_η meets $\overline{C_{\bar{\eta}} - F_\eta}$ in the point $P|_{\bar{\eta}}$, and we deduce that axiom (iv) holds in the generic fiber. □

Corollary 7.7. *$\overline{\mathcal{M}}_{1,n}((c_P))$ is an Artin stack.*

Proof. The preceding theorem shows that $\overline{\mathcal{M}}_{1,n}((c_P))$ is an open substack of the Artin stack of all n-pointed curves. (See [Smyth 2013, Appendix B].) □

Our next result is that these various moduli spaces are related by containments induced by the containment of faces in the cube complex $X_{1,n}$. This strikes us as analogous to the containments of moduli spaces seen at critical values of the log minimal model program in [Alper et al. 2017, Theorem 1.1], except that the "larger" stacks here are associated with larger strata.

Theorem 7.8. *Suppose that (c_P), $(d_P) \in X_{1,n}$ and (d_P) lies in a face of the cube containing (c_P), i.e., $d_P = c_P$ whenever $c_P = 0$ or $c_P = 1$. Then there is a fully faithful inclusion functor*

$$\overline{\mathcal{M}}_{1,n}((d_P)) \hookrightarrow \overline{\mathcal{M}}_{1,n}((c_P)).$$

Proof. Note that the sets Q_{sing} and Q_{curve} associated to (d_P) are subsets of the respective sets associated to (c_P). Then the result is clear from the definition. □

Corollary 7.9. *The stacks $\overline{\mathcal{M}}_{1,n}((c_P))$ are universally closed.*

Proof. Let $Q = \{P \in \mathrm{part}(n) \mid c_P = 1\}$. By Theorems 7.5 and 7.8, $\overline{\mathcal{M}}_{1,n}(Q)$ is a substack of $\overline{\mathcal{M}}_{1,n}((c_P))$.

Note that $\overline{\mathcal{M}}_{1,n}((c_P))$ contains $\mathcal{M}_{1,n}$ as a dense open, since nodes and elliptic Gorenstein singularities are smoothable. To check universal closedness it therefore suffices to verify that if S is the spectrum of a DVR with generic point η, and $C_\eta \to \eta$ a family of smooth n-marked curves of genus one, then there is an extension of C_η to a family of (c_P)-stable curves over S. Such an extension already exists as a Q-stable limit. □

A tropical resolution of the birational map from $\overline{\mathcal{M}}_{1,n}$ to $\overline{\mathcal{M}}_{1,n}((c_P))$. We now give the tropical construction that motivated the definition of $\overline{\mathcal{M}}_{1,n}((c_P))$ above. For each $(c_P) \in X_{1,n}$ we will build a modification $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n} \to \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ and a contraction inducing a morphism $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n} \to \overline{\mathcal{M}}_{1,n}((c_P))$. The ideas are essentially the same as in Section 4, so we are relatively brief.

Fix an element $(c_P)_{P\in\mathrm{Part}(n)} \in X_{1,n}$. Let $Q_{\min} = \{P \in \mathrm{Part}(n) \mid c_P = 1\}$. Let $P_1, \ldots, P_k$ be the partitions for which $0 < c_{P_i} < 1$. Then let $Q_i = Q_{\min} \cup \{P_i\}$ for $1 \le i \le k$. Consider the associated global sections $\rho_{\min}, \rho_1, \ldots, \rho_k$ of the characteristic sheaf of $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$. Notice that $\rho_i \ge \rho_{\min}$ for each i, so the differences $\delta_i = \rho_i - \rho_{\min}$ are again well-defined sections of the characteristic sheaf of $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$.

We recall (see, for example, [Olsson 2003, Proposition 5.17] with $P = \mathbb{N}$) that the stack $[\mathbb{A}^1/\mathbb{G}_m]$ may be given a log structure so that $[\mathbb{A}^1/\mathbb{G}_m]$ represents the functor on log schemes

$$X \mapsto \Gamma(X, \overline{M}_X).$$

Since this is a functor valued in commutative monoids, there is a morphism $\mu : [\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ induced by the multiplication of $\Gamma(X, \overline{M}_X)$.

Take products and form the pullback square of fs log algebraic stacks

$$\begin{array}{ccc}
\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n} & \xrightarrow{\prod_i (\delta_i^{(1)}, \delta_i^{(2)})} & ([\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m])^k \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mu^k} \\
\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n} & \xrightarrow{\prod_i \delta_i} & [\mathbb{A}^1/\mathbb{G}_m]^k
\end{array}$$

Note that $p^*\delta_i$ factors into the sum $\delta_i^{(1)} + \delta_i^{(2)}$ in the characteristic sheaf for each i. Moreover, $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ is universal with respect to such factorizations in the sense that, given any $g : T \to \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ and an expression of each $g^*\delta_i$ as a sum $\alpha_i^{(1)} + \alpha_i^{(2)}$, there is a unique factorization $h : T \to \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ of g through $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ such that $\alpha_i^{(1)} = h^*\delta_i^{(1)}$ and $\alpha_i^{(2)} = h^*\delta_i^{(2)}$ for each i.

The morphism μ is integral and saturated, so the underlying algebraic stack of $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ is the fiber product of the underlying algebraic stacks of the rest of the diagram. The complement $D(\delta_1, \ldots, \delta_k)$ in $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ of the vanishing of the Cartier divisors associated to $\delta_1, \ldots, \delta_k$ is precisely the preimage of the non-stacky point of $[\mathbb{A}^1/\mathbb{G}_m]^k$. As μ^k restricts to an isomorphism over this point, it follows that p restricts to an isomorphism over $D(\delta_1, \ldots, \delta_k)$. In particular, since the δ_i's are nonvanishing on smooth curves, p restricts to an isomorphism on $\mathcal{M}_{1,n}$. We identify $\mathcal{M}_{1,n}$ with its image in $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$.

Lemma 7.10. *Let $T = \operatorname{Spec} B$ be the spectrum of a discrete valuation ring with uniformizer t and field of fractions K. Let $b_1, \ldots, b_k$ be nonzero elements of B, and let*

$$A = \frac{B[x_1^{(1)}, x_1^{(2)}, \ldots, x_k^{(1)}, x_k^{(2)}]}{(x_i^{(1)} x_i^{(2)} - b_i : i = 1, \ldots, k)}.$$

Then $S = \operatorname{Spec} A$ is an integral scheme and its generic point maps to the generic point of T under the natural map $S \to T$.

Proof. Let $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$. Let $\eta = D(t)$ be the generic point of T and U its preimage in S. Then $U \cong \mathbb{G}_{m,K}^k$ is an irreducible open of S. Recall that the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is flat, so its k-fold product $(\mathbb{A}^1 \times \mathbb{A}^1)^k \to (\mathbb{A}^1)^k$ is also flat. Observe that $S \to T$ is a pullback of this morphism, so also flat. If x is any point of S not in U, the going-down theorem for flat morphisms implies that x possesses a generization in U. Therefore, U is dense in S, and S is integral. Since U maps to the generic point of T, we also have the claim about generic points. □

Proposition 7.11. *The open substack $\mathcal{M}_{1,n}$ is dense in $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. In particular $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is irreducible and p is birational.*

Proof. Give $\mathbb{A}^1$ the toric log structure. Recall that $\mathbb{A}^1$ represents the functor

$$(X, \alpha : M_X \to \mathscr{O}_X) \mapsto \Gamma(X, M_X).$$

There is a commutative square of fs log algebraic stacks

$$\begin{array}{ccc} ([\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m])^k & \longleftarrow & (\mathbb{A}^1 \times \mathbb{A}^1)^k \\ \big\downarrow{\scriptstyle \mu^k} & & \big\downarrow \\ [\mathbb{A}^1/\mathbb{G}_m]^k & \longleftarrow & (\mathbb{A}^1)^k \end{array}$$

where each vertical map is induced by the monoid law, and the horizontal maps are induced by $M_X \to \overline{M}_X$. The map of schemes underlying the right vertical map consists of k-copies of the multiplication map. By [Olsson 2003, Proposition 2.1], a map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ étale locally admits a lift to $\mathbb{A}^1$.

Let $s = (\operatorname{Spec} k, \alpha)$ be a geometric fs log point, and let $f_s : s \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ be a log morphism. Our strategy is to find a smoothing of f_s in $\overline{\mathcal{M}}_{1,n}$, then to lift this smoothing back to $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ using the commutative square above.

Observe that since s is a geometric point, the composite $s \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to ([\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m])^k$ factors through $(\mathbb{A}^1 \times \mathbb{A}^1)^k$.

Let $t = (\operatorname{Spec} k, \beta)$ denote the fs log point with log structure pulled-back from $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ along the map $p \circ f_s : s \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Write g_t for the induced map $t \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Note that we have a commutative square of fs log algebraic stacks

$$\begin{array}{ccc} s & \xrightarrow{f_s} & \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ t & \xrightarrow{g_t} & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \end{array}$$

By construction, g_t is strict. So we may use that $\mathcal{M}_{1,n}$ is dense in the Noetherian algebraic stack $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to find a strict map $g : T \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$, where

(i) T is an fs log scheme with underlying scheme the spectrum of a discrete valuation ring;

(ii) g_t factors through the special point of T;

(iii) $\eta \subseteq T$ is the generic point;

(iv) $g|_\eta \in \mathcal{M}_{1,n}(\eta)$.

Replacing T by a finite base change if necessary, we may assume that the composite $T \to \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n} \to [\mathbb{A}^1/\mathbb{G}_m]^k$ factors through $(\mathbb{A}^1)^k$. A diagram chase shows that there is an induced map $f : S \to \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ and $f_s : s \to \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ factors through it. By Lemma 7.10, S is irreducible and its generic point θ maps to the generic point η of T. By construction, $g(\eta)$ lies in the smooth locus of $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$, so $f(\theta)$ lies in the smooth locus of $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$. It follows that the image of s in $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ has a generization to a point factoring through $\mathcal{M}_{1,n} \subseteq \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$, as desired. □

Now let $\rho = p^*\rho_{\min} + \delta_1^{(1)} + \cdots + \delta_k^{(1)}$ in the characteristic sheaf of $\widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$. Let $C_{1,n} \to \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ be the pullback of the universal family of $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ to $\widetilde{\mathcal{M}}_{1,n}$ Notice that ρ is comparable with the radii of $C_{1,n}$, so we may make a log modification $\widetilde{C}_{1,n} \to C_{1,n}$ subdividing tropicalizations at the locus where $\rho = \lambda$. Then, as in Theorem 4.1, we may form a section $\bar{\lambda} \in \Gamma(\widetilde{C}, \overline{\mathcal{M}}_{\widetilde{C}})$ by the formula

$$\bar{\lambda} = \max\{\rho - \lambda, 0\}.$$

Once again, it is easy to check that this is a mesa in the sense of [Bozlee 2020], so the main result there yields a contraction of families of curves

$$\begin{array}{ccc} \widetilde{C}_{1,n} & \xrightarrow{\tau} & \bar{C}_{1,n} \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\bar{\pi}} & \\ \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n} & & \end{array}$$

Similar reasoning to that of Section 4 yields that $\bar{C}_{1,n} \to \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n}$ is a family of curves in $\overline{\mathcal{M}}_{1,n}((c_P))$. We therefore have a diagram of birational morphisms of algebraic stacks

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}}^{\mathrm{rad}}_{1,n} & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \bar{C}_{1,n}} & \\ & \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n} & & & \overline{\mathcal{M}}_{1,n}((c_P)) \\ \swarrow & & & & \\ \overline{\mathcal{M}}_{1,n} & & & & \end{array}$$

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On unipotent radicals of motivic Galois groups

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Let $\boldsymbol{T}$ be a neutral Tannakian category over a field of characteristic zero with unit object $\mathbb{1}$, and equipped with a filtration $W_\bullet$ similar to the weight filtration on mixed motives. Let M be an object of $\boldsymbol{T}$, and $\underline{\mathfrak{u}}(M) \subset W_{-1}\underline{\mathrm{Hom}}(M, M)$ the Lie algebra of the kernel of the natural surjection from the fundamental group of M to the fundamental group of $\mathrm{Gr}^W M$. A result of Deligne gives a characterization of $\underline{\mathfrak{u}}(M)$ in terms of the extensions $0 \to W_p M \to M \to M/W_p M \to 0$: it states that $\underline{\mathfrak{u}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{Hom}}(M, M)$ such that the sum of the aforementioned extensions, considered as extensions of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{Hom}}(M, M)$, is the pushforward of an extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$. We study each of the above-mentioned extensions individually in relation to $\underline{\mathfrak{u}}(M)$. Among other things, we obtain a refinement of Deligne's result, where we give a sufficient condition for when an individual extension $0 \to W_p M \to M \to M/W_p M \to 0$ is the pushforward of an extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$. In the second half of the paper, we give an application to mixed motives whose unipotent radical of the motivic Galois group is as large as possible (i.e., with $\underline{\mathfrak{u}}(M) = W_{-1}\underline{\mathrm{Hom}}(M, M)$). Using Grothendieck's formalism of *extensions panachées* we prove a classification result for such motives. Specializing to the category of mixed Tate motives we obtain a classification result for 3-dimensional mixed Tate motives over $\mathbb{Q}$ with three weights and large unipotent radicals.

1. Introduction

1.1. ***About this paper.*** Let $\boldsymbol{T}$ be a neutral Tannakian category over a field K of characteristic zero, equipped with a weight filtration $W_\bullet$ similar to the weight filtration on mixed motives (functorial, increasing, finite on every object, exact, and respecting the tensor structure). For example, one might keep in mind the category of mixed Hodge structures. In fact, this is a concrete example that illustrates well the main results.

Let M be an object of $\boldsymbol{T}$, and $\underline{\mathfrak{u}}(M)$ the Lie algebra of the kernel of the natural map from the fundamental group of M to that of $\mathrm{Gr}^W M$. A result of Deligne describes $\underline{\mathfrak{u}}(M)$ in terms of extensions that arise naturally from the weight filtration of M. For each integer p, let $\mathscr{E}_p(M)$ be the extension

$$0 \to W_p M \to M \to M/W_p M \to 0, \tag{1}$$

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considered as an element in $\mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M))$ (where $\underline{\mathrm{End}}(M)$ means $\underline{\mathrm{Hom}}(M, M)$, the latter being the internal Hom). Deligne characterizes $\underline{\mathfrak{u}}(M)$ in terms of the sum

$$\mathscr{E}(M) := \sum_p \mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)).$$

The first half of this paper refines this by developing conditions under which the individual extensions $\mathscr{E}_p(M)$ can be related to $\underline{\mathfrak{u}}(M)$.

The second half of the paper specializes to the setting of mixed motives and gives an application of the first half to mixed motives whose unipotent radical of the motivic Galois group is as large as possible (i.e., with $\underline{\mathfrak{u}}(M) = W_{-1}\underline{\mathrm{End}}(M)$). These motives are in particular interesting for the transcendence properties of their periods: in view of Grothendieck's period conjecture the field generated by their periods should have the highest possible transcendence degree among all motives with the same associated graded.

A particularly striking implication of our result is that a suggestion of Euler about $\zeta(3)$ is incompatible with Grothendieck's period conjecture. Euler [1785] speculated that there may be rational numbers α and β and an expression of the form

$$\zeta(3) = \alpha(\log 2)^3 + \beta\pi^2(\log 2).$$

See the article of Dunham [2021] which gives a very readable account of this statement and Euler's remarkable work on evaluating the Riemann zeta function at integer arguments.

In Section 6.8, we construct a mixed Tate motive with periods (essentially) $\zeta(3)$, $\log 2$, π and a fourth period. Moreover, we use our results to show that the dimension of the Galois group in this case is 4. Thus, the period conjecture would predict that these four periods are algebraically independent, and this is incompatible with Euler's expectation stated above. A more detailed description of this mixed Tate motive is given below.

1.2. $\underline{\mathfrak{u}}(M)$ *and the extensions* $\mathscr{E}_p(M)$. To be more precise, $\underline{\mathfrak{u}}(M)$ is the subobject of $W_{-1}\underline{\mathrm{End}}(M)$ with the property that if ω is any fiber functor over K, then

$$\omega\underline{\mathfrak{u}}(M) \subset \omega W_{-1}\underline{\mathrm{End}}(M) = W_{-1}\,\mathrm{End}(\omega M)$$

is the Lie algebra of

$$\mathscr{U}(M, \omega) := \ker(\mathscr{G}(M, \omega) \xrightarrow{\text{restriction}} \mathscr{G}(\mathrm{Gr}^W M, \omega)),$$

where $\mathscr{G}(-, \omega)$ denotes the fundamental group of the indicated object with respect to ω. If $\mathrm{Gr}^W M$ is semisimple (which will be the case if $\boldsymbol{T}$ is a category of motives), then $\mathscr{U}(M, \omega)$ is the unipotent radical of $\mathscr{G}(M, \omega)$.

As stated above, Deligne (see [Jossen 2014, Appendix]) describes $\underline{\mathfrak{u}}(M)$ in terms of extensions that arise naturally from the weight filtration on M: For each integer p, let $\mathscr{E}_p(M)$ be the *p-th extension class of M* given by (1), considered as an extension of the unit object $\mathbb{1}$ by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$. Pushing this extension forward along the natural injection

$$\underline{\mathrm{Hom}}(M/W_pM,\, W_pM) \to W_{-1}\underline{\mathrm{End}}(M)$$

we get an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)$, which we also denote by $\mathscr{E}_p(M)$. The *total* extension class of M is then the extension

$$\mathscr{E}(M) := \sum_p \mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1},\, W_{-1}\underline{\mathrm{End}}(M)).$$

Deligne's result asserts that $\underline{\mathfrak{u}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{End}}(M)$ such that $\mathscr{E}(M)$ is in the image of the pushforward

$$\mathrm{Ext}^1(\mathbb{1},\, \underline{\mathfrak{u}}(M)) \to \mathrm{Ext}^1(\mathbb{1},\, W_{-1}\underline{\mathrm{End}}(M)) \tag{2}$$

under the inclusion $\underline{\mathfrak{u}}(M) \subset W_{-1}\underline{\mathrm{End}}(M)$. Deligne proves this in part by exploiting the weight filtration to construct an explicit extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$ which pushes forward to $\mathscr{E}(M)$.

The first half of this paper is dedicated to the study of the relation between $\underline{\mathfrak{u}}(M)$ and the individual extensions $\mathscr{E}_p(M)$, with a view to refining Deligne's result. In general, the individual extensions $\mathscr{E}_p(M)$ may not be in the image of the pushforward map (2); an example involving 1-motives can be given using the work of Jacquinot and Ribet [1987] on deficient points on semiabelian varieties; see Section 6.10 and the remarks at its end. The main result of the first half of the paper gives a sufficient condition for when the extension $\mathscr{E}_p(M)$ is in the image of (2); see Theorem 5.3.1 and its corollaries.

1.3. ***A more detailed overview.*** We continue this introduction by giving a more detailed overview of the contents of the paper, starting with the first half. Fix an integer p and an object M of $\boldsymbol{T}$. It is natural to expect $\mathscr{E}_p(M)$ to be related to the subobject

$$\underline{\mathfrak{u}}_p(M) := \underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_pM,\, W_pM)$$

of $\underline{\mathfrak{u}}(M)$, where we have considered $\underline{\mathrm{Hom}}(M/W_pM,\, W_pM)$ as a subobject of $W_{-1}\underline{\mathrm{End}}(M)$ via the natural injection. This is indeed the case: Write $\mathscr{E}_p(M)$ explicitly as

$$0 \to \underline{\mathrm{Hom}}(M/W_pM,\, W_pM) \to \underline{\mathrm{Hom}}(M/W_pM,\, M)^\dagger \to \mathbb{1} \to 0; \tag{3}$$

see Section 4.5 for the explicit description of the middle object. Then by Theorem 3.3.1 of [Eskandari and Murty 2021] (which is proved by a small modification of the proof of [Hardouin 2011, Theorem 2]), we have:

$$\begin{gathered}\underline{\mathfrak{u}}_p(M) \textit{ is the smallest subobject of } \underline{\mathrm{Hom}}(M/W_pM,\, W_pM) \textit{ such that}\\ \underline{\mathrm{Hom}}(M/W_pM,\, M)^\dagger/\underline{\mathfrak{u}}_p(M) \textit{ belongs to the subcategory } \langle W_pM,\, M/W_pM\rangle^{\otimes}.\end{gathered} \tag{$*$}$$

Here, as usual, the notation $\langle\ \rangle^{\otimes}$ means the smallest full Tannakian subcategory containing the indicated objects and closed under subobjects. The first contribution of the present article is to reformulate this statement in a more natural way, in the language of extensions originating from subcategories (discussed in Section 3). Given a full Tannakian subcategory $\boldsymbol{S}$ of $\boldsymbol{T}$ which is closed under subobjects, we say an extension $\mathscr{E}$ of $\mathbb{1}$ by an object A of $\boldsymbol{T}$ originates from $\boldsymbol{S}$ if there is an object A' of $\boldsymbol{S}$, an extension $\mathscr{E}'$ of $\mathbb{1}$ by A' in $\boldsymbol{S}$, and a morphism $A' \to A$ under which $\mathscr{E}'$ pushes forward to $\mathscr{E}$. While this is a very natural and simple generalization of the notion of splitting of sequences (as an extension splits if and only if it originates from a semisimple $\boldsymbol{S}$), it opens the door to refinements of $(*)$ and Deligne's theorem. The reformulated version of $(*)$ is given in Theorem 4.9.1. It asserts that $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that the pushforward $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ of $\mathscr{E}_p(M)$ under the quotient map originates from the subcategory

$$\langle W_pM, M/W_pM\rangle^{\otimes}. \tag{4}$$

Note that one advantage of formulating the statement in this language is that here we may think of $\mathscr{E}_p(M)$ as an extension of $\mathbb{1}$ by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ or by $W_{-1}\underline{\mathrm{End}}(M)$; see Section 4.

Our next goal is to find refinements of Theorem 4.9.1 in which the category (4) is replaced by smaller categories. Ideally, this category can be replaced by a semisimple category, in which case the pushforward $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ of $\mathscr{E}_p(M)$ along the quotient map will split. (By weight considerations and the long exact sequence for Ext groups this is equivalent to $\mathscr{E}_p(M)$ being in the image of (2).) But from the examples of 1-motives mentioned earlier we know that in general, this will not be the case.

Let $q \le p$. The second contribution of this paper is to show that if M satisfies certain "independence axioms", then in the statement of Theorem 4.9.1 the category (4) can be replaced by the smaller category $\langle W_qM, \mathrm{Gr}^W M\rangle^{\otimes}$ (smaller because $q \le p$); this is Theorem 5.3.1 in Section 5. The independence axioms are given in Section 5.2, and in fact, only depend on $\mathrm{Gr}^W M$. Roughly speaking, they require the subobject

$$\bigoplus_{\substack{i,j\\ j>q,i}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M) \tag{5}$$

of $W_{-1}\underline{\mathrm{End}}(\mathrm{Gr}^W M)$ to suitably decompose as a direct sum of two "independent" summands. In the weak sense, here the word "independent" means not having any nonzero isomorphic subobjects, and in the strong sense, it means having disjoint sets of weights; see the axioms $(IA1)_{\{p,q\}}$ and $(IA2)_{\{p,q\}}$ in Section 5.2.

An interesting consequence of Theorem 5.3.1 is the following refinement of Deligne's theorem (see Corollary 5.3.2): If $\mathrm{Gr}^W M$ is semisimple (e.g., if $\boldsymbol{T}$ is a category of motives) and the weak independence axioms hold for all $q \le p$, then $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. In particular, if $\mathrm{Gr}^W M$ is semisimple and $W_{-1}\underline{\mathrm{End}}(M)$ has $\binom{n}{2}$ distinct weights where n is the number of weights of M (e.g., if M has weights $0, -1, -3, -7$), then $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits for every p; see Corollary 5.3.3.

The proof of Theorem 5.3.1 is similar to the proof of $(*)$ (or rather, of Theorem 4.9.1), albeit with two added ingredients. Let $\underline{\mathfrak{u}}_{\ge q}(M)$ be the Lie algebra of the kernel of the restriction map from the fundamental group of M to that of $W_qM \oplus \mathrm{Gr}^W M$. The first new component is thanks to the independence axioms: they guarantee that $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\ge q}(M)$ (which is a subobject of (5)) decomposes according to the decomposition of (5) into our independent objects; see Lemma 5.5.1. This is the only place in the proof of Theorem 5.3.1 that the independence axioms play a part. Taking ω to be any fiber functor, this gives a decomposition of $\omega\,\mathrm{Gr}^W\,\underline{\mathfrak{u}}_{\geq q}(M)$. The second added ingredient is that we use the fundamental theorem of Tannakian categories with $\omega\circ\mathrm{Gr}^W$ as the fiber functor (rather than using ω itself). Notice the difference in the nature of this type of argument and Deligne's argument in [Jossen 2014, Appendix], which explicitly constructs an extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$ that pushes forward to the total class of M. We should point out that the idea of working with the associated graded fiber functor already appears in [Deligne 1994], and since then has featured frequently in the literature, especially in the setting of categories of mixed Tate motives; e.g., [Deligne and Goncharov 2005].

It would be interesting to give a more conceptual explanation (or geometric interpretation, in the case of motives) for the fact that the independence axioms force the individual extension classes $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ (or $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$) to split.

We now discuss the contents of the second half of the paper (Section 6). Let $\boldsymbol{T}$ be a Tannakian category of mixed motives over a field K of characteristic zero, e.g., the Tannakian categories of Nori or Ayoub of mixed motives over K, or Voevodsky's Tannakian category of mixed Tate motives over a number field, or categories of mixed motives defined in terms of realizations. We say $\underline{\mathfrak{u}}(M)$ is *large* (or that M has a large $\underline{\mathfrak{u}}$) if $\underline{\mathfrak{u}}(M)$ is equal to $W_{-1}\underline{\mathrm{End}}(M)$. As we pointed out earlier, such motives are interesting from the point of view of the transcendence properties of their periods. Our original motivation for this part of the paper was to study (or ideally, classify up to isomorphism) all objects M with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to

$$\mathbb{Q}(n)\oplus A\oplus\mathbb{1},$$

where A is a given pure object of weight p with $-2n<p<0$. We then realized that much of the discussion can be given in more generality, leading to the contents of this part of the paper as currently presented (and reviewed below).

Suppose tentatively that M is an extension of $\mathbb{1}$ by an object L of highest weight p with $p<0$. It is easy to see that if $\underline{\mathfrak{u}}(M)$ is large, then so are $\underline{\mathfrak{u}}(L)$ and $\underline{\mathfrak{u}}(M/W_{p-1}(L))$. The first main result of Section 6 (Theorem 6.3.1) gives a sufficient condition for the converse statement. The result asserts that if M satisfies a suitable independence axiom, and if $\underline{\mathfrak{u}}(L)$ and $\underline{\mathfrak{u}}(M/W_{p-1}(L))$ are large, then so is $\underline{\mathfrak{u}}(M)$. This is an application of Corollary 5.3.2. As in the case of the latter corollary, examples involving 1-motives show that the conclusion of Theorem 6.3.1 is in general false without the hypothesis about the independence axiom; see Section 6.10.

Theorem 6.3.1 suggests a way to obtain more complicated objects with large $\underline{\mathfrak{u}}$ by "patching together" smaller such objects. More precisely, given an object L of highest weight p with $p<0$ which has a large $\underline{\mathfrak{u}}$, and an object N which is an extension of $\mathbb{1}$ by $\mathrm{Gr}^W_p L$ and also has a large $\underline{\mathfrak{u}}$, we can look for objects M such that $W_pM\simeq L$ and $M/W_{p-1}M\simeq N$; assuming the relevant independence axiom (which only depends on $\mathrm{Gr}^W M\simeq\mathrm{Gr}^W L\oplus\mathbb{1}$) holds, any such M has a large $\underline{\mathfrak{u}}$. The answer to the question of existence of such M is given by Grothendieck's formalism of *extensions panachées* [SGA 7_{I} 1972]: The obstruction is an element of $\mathrm{Ext}^2(\mathbb{1}, W_{p-1}L)$. Moreover, the object M is unique up to isomorphism if $\mathrm{Ext}^1(\mathbb{1}, W_{p-1}L)=0$; see Lemma 6.4.1.

We consider the following classification problem in Sections 6.4–6.7: Given B of weights $< p$ and with a large $\underline{\mathfrak{u}}$, and a nonzero pure object A of negative weight p, classify up to isomorphism all M with large $\underline{\mathfrak{u}}$ satisfying

$$W_{p-1}M \simeq B, \quad \mathrm{Gr}^W_p M \simeq A \quad \text{and} \quad M/W_pM \simeq \mathbb{1}$$

(with the isomorphisms not part of the data). We manage to give a complete solution to this problem when $B \oplus A \oplus \mathbb{1}$ satisfies an independence axiom and $\mathrm{Ext}^1(\mathbb{1}, B) = 0$; the solution is summarized in Section 6.7, just before Corollary 6.7.1. To get there, in Sections 6.4–6.6 we study the *extensions panachées* problem in the setting of an abelian category with weights.[1] The main result is summarized in Proposition 6.6.1; see also Lemma 6.5.1. As a special case of these results, in Corollary 6.7.1 we give an answer to our original motivating classification problem about objects with associated graded isomorphic to $\mathbb{Q}(n) \oplus A \oplus \mathbb{1}$.

In Section 6.8 we specialize to the category $\mathbf{MT}(\mathbb{Q})$ of (say, Voevodsky) mixed Tate motives over $\mathbb{Q}$. The nice feature here is that the Ext groups are known. We use Corollary 6.7.1 to give a complete classification, up to isomorphism, of all 3-dimensional mixed Tate motives over $\mathbb{Q}$ with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(k) \oplus \mathbb{1}$ with $n > k > 0$ and $n \neq 2k$; the very last condition is the independence axiom in this situation.

Let us consider an example from Section 6.8 here. Let r be an integer > 1 and N the Kummer 1-motive $[\mathbb{Z} \xrightarrow{1 \mapsto r} \mathbb{G}_m]$, considered as an object of $\mathbf{MT}(\mathbb{Q})$. Let n be an even integer ≥ 4, and L an object which is a nontrivial extension of $\mathbb{1}$ by $\mathbb{Q}(n-1)$ (so with $(2\pi i)^{1-n}\zeta(n-1)$ as a period). Since Ext^2 groups vanish in $\mathbf{MT}(\mathbb{Q})$ and $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$, the two objects $L(1)$ and N can be patched together to form an object M of $\mathbf{MT}(\mathbb{Q})$, unique up to isomorphism, such that $W_{-2}M \simeq L(1)$ and $M/\mathbb{Q}(n) \simeq N$.[2] Moreover, M satisfies the required independence axiom (as $n \neq 2$), so that it follows from Theorem 6.3.1 that $\underline{\mathfrak{u}}(M)$ is large. According to Grothendieck's period conjecture, the field generated over $\mathbb{Q}$ by the periods of M should have transcendence degree equal to

$$\dim(\mathcal{G}(M, \omega_B)) = \dim(\omega_B \underline{\mathfrak{u}}(M)) + \dim(\mathcal{G}(\mathrm{Gr}^W M, \omega_B)) = 3 + 1 = 4$$

(where ω_B = Betti realization). The nonzero entries of the period matrix of M with respect to suitably chosen bases of de Rham and Betti realizations are $(2\pi i)^{-n}$, $(2\pi i)^{-n}\zeta(n-1)$, $(2\pi i)^{-1}$ (coming from $L(1)$), $(2\pi i)^{-1}\log(r)$, 1 (coming from N), and a "new period". So Grothendieck's period conjecture predicts that

$$2\pi i, \quad \zeta(n-1), \quad \log(r), \quad \text{and the new period of } M$$

must be algebraically independent over $\mathbb{Q}$.

The new period discussed above seems rather mysterious, and it would be very interesting to somehow calculate it.[3] When r is 2 (or a power of it), M is a mixed Tate motive over $\mathbb{Z}\left[\frac{1}{2}\right]$, and hence by Deligne's

[1] Actually, a slight variation of it; see the beginning of Section 6.4.

[2] This object is denoted by $M_{n,r}$ in Section 6.8.

[3] Ideally, one would like to do this by giving a geometric construction of M, but this may be too difficult especially when $n > 4$. In general, giving geometric constructions of mixed Tate motives with a few weights is a difficult problem; see [Brown 2016, Section 1.4].

work [2010] the new period will be a linear combination of alternating multiple zeta values, which one should be able to calculate using the formula of Goncharov [2005] and Brown [2012] for the motivic coaction on iterated integrals.[4] On the other hand, for general r, at least a priori, the new period may not be an iterated integral on the projective line $\mathbb{P}^1$ minus $\{0, \infty\} \cup \mu_r$. (This is related to the question of whether the category of mixed Tate motives over $\mathbb{Z}[1/r]$ is generated by the fundamental groupoid of $\mathbb{P}^1 \setminus (\{0, \infty\} \cup \mu_r)$, and for $r > 2$ one expects the answer to this question to be in general negative; see Section 3 of [Dan-Cohen and Wewers 2016] for a discussion of this question.)

After the discussion of 3-dimensional mixed Tate motives with large $\underline{\mathfrak{u}}$, in Section 6.9 we briefly consider some 4-dimensional examples; this leads again to some interesting questions about periods. One difference between the 4-dimensional and 3-dimensional examples is that in the former case (at least, a priori) one gets a family of motives with a large $\underline{\mathfrak{u}}$ when patching together a 3-dimensional L and a 2-dimensional N.

We end this introduction with some words on the organization of the paper. In Section 2 we review some basic material about Tannakian categories. The notion of extensions originating from subcategories of a Tannakian category is discussed in Section 3. Here we prove a few lemmas on this concept that will be useful throughout the paper. Starting from Section 4 we work in a Tannakian category with a weight filtration. In Section 4 we introduce the relevant objects and give the reformulation of $(*)$ (Theorem 4.9.1). The goal of Section 5 is to give the main results of the first part of the paper (Theorem 5.3.1 and its corollaries), in which we show that the independence axioms introduced in the same section result in refinements of Theorem 4.9.1 and Deligne's theorem. At the end of Section 5 we also prove a variant of Theorem 5.3.1 for $q > p$ case; see Theorem 5.7.1. Section 6 contains the application to motives with large unipotent radicals of motivic Galois groups, as discussed above. We should point out that prior to Section 6.8 we use the term "motive" only because we find it more suggestive: the discussion is valid in any Tannakian category with a weight filtration as long as the word "motive" is interpreted as "an object with a semisimple weight associated graded". In discussions where the Tate objects $\mathbb{Q}(n)$ play a role, we also need to assume that there is a pure object $\mathbb{Q}(1)$ of weight -2 such that the functor $-(1) := - \otimes \mathbb{Q}(1)$ is invertible. Sections 6.8 and 6.9 take place in the setting of a Tannakian category of mixed Tate motives over $\mathbb{Q}$ with the "correct" Ext groups. Finally, Section 6.10 uses 1-motives to give counterexamples to several statements in the paper, if the hypotheses regarding the independence axioms are omitted.

2. Preliminaries on Tannakian categories

The goal of this section is to review certain generalities about fundamental groups in Tannakian categories and fix some notation. None of the results in this section are new. The reader can refer to [Deligne and Milne 1982] for the basics of Tannakian categories, for instance. Throughout the paper, by a Tannakian subcategory we always mean a Tannakian subcategory that is closed under taking subobjects.

[4]This was told to us by Clément Dupont.

2.1. ***Notation.*** For any commutative ring R, we denote the category of R-modules (resp. commutative R-algebras) by $\mathbf{Mod}_R$ (resp. $\mathbf{Alg}_R$). We often denote the Hom and End groups in a category of modules simply by Hom and End, with the coefficient ring being understood from the context.

Throughout, K is a field of characteristic zero. If V is a vector space over K, we denote the general linear group of V by $\mathrm{GL}(V)$; it is an algebraic group over K. If G is an algebraic group over K, we denote the Lie algebra of G by $\mathrm{Lie}(G)$, and the category of finite-dimensional representations of G (over K) by $\mathbf{Rep}(G)$.

As usual, given a morphism $\alpha : \omega \to \omega'$ of functors, for any object M of the domain category the corresponding morphism $\omega M \to \omega' M$ in the target category is denoted by α_M.

Finally, in various contexts, we use the notation $f|_X$ for the restriction of f to X (whatever f and X are).

2.2. By a Tannakian category over K we mean a neutral Tannakian category over K, i.e., in the language of [Deligne and Milne 1982], a rigid abelian K-linear tensor category with K as the endomorphism algebra of the unit object, for which a fiber functor over K (= an exact faithful K-linear tensor functor from the category to $\mathbf{Mod}_K$) exists.[5]

If $\boldsymbol{T}$ is a Tannakian category over K and $\omega : \boldsymbol{T} \to \mathbf{Mod}_K$ is a fiber functor (over K), we denote the fundamental group of $\boldsymbol{T}$ with respect to ω by $\mathcal{G}(\boldsymbol{T}, \omega)$ (= $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ in the standard notation); thus (by the fundamental theorem of Tannakian categories) this is an affine group scheme over K with

$$\mathcal{G}(\boldsymbol{T}, \omega)(R) = \begin{cases} \text{the group of automorphisms of the functor} \\ \qquad \omega \otimes 1_R : \boldsymbol{T} \to \mathbf{Mod}_K \to \mathbf{Mod}_R \\ \qquad \text{respecting the tensor structures} \end{cases}$$

for any K-algebra R. For any object M of $\boldsymbol{T}$, we have a representation

$$\rho_M : \mathcal{G}(\boldsymbol{T}, \omega) \to \mathrm{GL}(\omega M), \quad \sigma \mapsto \sigma_M$$

and (again by the fundamental theorem) the functor

$$\boldsymbol{T} \to \mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega)), \quad M \mapsto (\omega M, \rho_M),$$

which with abuse of notation we also denote by ω, is an equivalence of categories.

2.3. Let $\boldsymbol{T}$ be a Tannakian category over K with unit object denoted by $\mathbb{1}$. Let

$$\omega : \boldsymbol{T} \to \mathbf{Mod}_K$$

be a fiber functor. For any full Tannakian subcategory $\boldsymbol{S}$ of $\boldsymbol{T}$, the inclusion $\boldsymbol{S} \subset \boldsymbol{T}$ gives a surjective restriction map

$$\mathcal{G}(T, \omega) \to \mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}})$$

(surjective because $\boldsymbol{S}$ is assumed to be closed under taking subobjects; see [Deligne and Milne 1982, Proposition 2.21]).

[5] Actually including faithfulness here is redundant, as it follows from the rest of the requirements; see [Deligne 1990, Sections 2.10 and 2.11].

2.4. Given any objects $M_1, \dots, M_n$ of $\boldsymbol{T}$, let $\langle M_1, \dots, M_n\rangle^{\otimes}$ be the Tannakian subcategory generated by $M_1, \dots M_n$; by definition, $\langle M_1, \dots, M_n\rangle^{\otimes}$ is the smallest full Tannakian subcategory of $\boldsymbol{T}$ which contains the M_i. Every object of $\langle M_1, \dots, M_n\rangle^{\otimes}$ is obtained from $M_1, \dots M_n$ and $\mathbb{1}$ by finitely many iterations of taking direct sums, tensor products, duals, and subobjects. We have

$$\langle M_1, \dots, M_n\rangle^{\otimes} = \big\langle \textstyle\bigoplus_{1\le i\le n} M_i\big\rangle^{\otimes}.$$

2.5. Let M be an object of $\boldsymbol{T}$. Given a fiber functor ω over K, we set

$$\mathcal{G}(M,\omega) := \mathcal{G}(\langle M\rangle^{\otimes}, \omega|_{\langle M\rangle^{\otimes}}) = \underline{\mathrm{Aut}}^{\otimes}(\omega|_{\langle M\rangle^{\otimes}});$$

we call this the fundamental group of M with respect to ω. Since every object of $\langle M\rangle^{\otimes}$ is obtained from M and $\mathbb{1}$ by finitely many iterations of taking direct sums, tensor products, duals and subobjects, the map

$$\rho_M : \mathcal{G}(M,\omega) \to \mathrm{GL}(\omega M)$$

(sending σ to σ_M) is injective. In particular, $\mathcal{G}(M,\omega)$ is an algebraic group over K.

Let $\mathfrak{g}(M,\omega)$ be the Lie algebra of $\mathcal{G}(M,\omega)$. In view of the equivalence of categories

$$\langle M\rangle^{\otimes} \to \mathbf{Rep}(\mathcal{G}(M,\omega))$$

given by ω, the adjoint representation of $\mathcal{G}(M,\omega)$ defines an object $\underline{\mathfrak{g}}(M,\omega)$ in $\langle M\rangle^{\otimes}$ such that

$$\omega\underline{\mathfrak{g}}(M,\omega) = \mathfrak{g}(M,\omega)$$

as representations of $\mathcal{G}(M,\omega)$, where the $\mathcal{G}(M,\omega)$-action on $\omega\underline{\mathfrak{g}}(M,\omega)$ corresponds to $\underline{\mathfrak{g}}(M,\omega)$ (i.e., is $\rho_{\underline{\mathfrak{g}}(M,\omega)}$) and the $\mathcal{G}(M,\omega)$-action on $\mathfrak{g}(M,\omega)$ is given by the adjoint representation.

Identify $\mathcal{G}(M,\omega)$ as a subgroup of $\mathrm{GL}(\omega M)$ via ρ_M. This identifies

$$\mathfrak{g}(M,\omega) \subset \mathrm{Lie}(\mathrm{GL}(\omega M)) = \mathrm{End}(\omega M). \tag{6}$$

Denote $\underline{\mathrm{End}}(M) := \underline{\mathrm{Hom}}(M, M)$ (the internal Hom in $\boldsymbol{T}$). Then we can identify $\omega\underline{\mathrm{End}}(M) = \mathrm{End}(\omega M)$, with the action of $\mathcal{G}(M,\omega)$ on $\mathrm{End}(\omega M)$ corresponding to $\underline{\mathrm{End}}(M)$ being by conjugation. The inclusion (6) is compatible with the actions of $\mathcal{G}(M,\omega)$, making

$$\underline{\mathfrak{g}}(M,\omega) \subset \underline{\mathrm{End}}(M).$$

2.6. For any object N of $\langle M\rangle^{\otimes}$, let $\mathcal{G}(M,N,\omega)$ be the kernel of the surjection

$$\mathcal{G}(M,\omega) \to \mathcal{G}(N,\omega)$$

induced by the inclusion $\langle N\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$ (so for instance, $\mathcal{G}(M,\mathbb{1},\omega) = \mathcal{G}(M,\omega)$). The Lie subalgebra

$$\mathfrak{g}(M,N,\omega) := \mathrm{Lie}(\mathcal{G}(M,N,\omega))$$

of $\mathfrak{g}(M,\omega)$ is invariant under the adjoint action of $\mathcal{G}(M,\omega)$, giving rise to a subobject

$$\underline{\mathfrak{g}}(M,N,\omega) \subset \underline{\mathfrak{g}}(M,\omega) \subset \underline{\mathrm{End}}(M).$$

2.7. The subobjects $\underline{\mathfrak{g}}(M, N, \omega)$ of $\underline{\mathrm{End}}(M)$ do not depend on the choice of the fiber functor ω. More precisely, for every object N of $\langle M\rangle^{\otimes}$, there is a canonical subobject

$$\underline{\mathfrak{g}}(M, N) \subset \underline{\mathrm{End}}(M)$$

such that for every ω over K,

$$\omega\underline{\mathfrak{g}}(M, N) = \mathfrak{g}(M, N, \omega) \subset \mathrm{End}(\omega M).$$

This can be seen via the machinery of algebraic geometry over a Tannakian category [Deligne 1989, Section 5 and 6] and is well-known, but in the interest of keeping the paper more self-contained, we include a proof.

Proposition 2.7.1. *Suppose ω and ω' are two fiber functors $\boldsymbol{T} \to \mathbf{Mod}_K$. Then for any objects M of $\boldsymbol{T}$ and N of $\langle M\rangle^{\otimes}$,*

$$\underline{\mathfrak{g}}(M, N, \omega) = \underline{\mathfrak{g}}(M, N, \omega')$$

(*as subobjects of* $\underline{\mathrm{End}}(M)$).

Proof. By a theorem of Deligne [1990, Section 1.12 and 1.13], there exists a K-algebra R such that the two functors $\omega \otimes 1_R$ and $\omega' \otimes 1_R$ are isomorphic as $\otimes$-functors. Let

$$\alpha : \omega \otimes 1_R \to \omega' \otimes 1_R$$

be an isomorphism respecting the tensor structures. Then conjugation by $\alpha|_{\langle M\rangle^{\otimes}}$ gives an isomorphism

$$c_\alpha : \mathcal{G}(M, \omega)_R \to \mathcal{G}(M, \omega')_R.$$

On the other hand, conjugation by

$$\alpha_M : \omega M \otimes 1_R \to \omega' M \otimes 1_R$$

gives an isomorphism

$$c_{\alpha_M} : \mathrm{GL}(\omega M)_R \to \mathrm{GL}(\omega' M)_R.$$

The maps c_α and c_{α_M} are compatible with one another, i.e., we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}(M, \omega)_R & \xrightarrow{c_\alpha, \simeq} & \mathcal{G}(M, \omega')_R \\ \cap & & \cap \\ \mathrm{GL}(\omega M)_R & \xrightarrow{c_{\alpha_M}, \simeq} & \mathrm{GL}(\omega' M)_R, \end{array}$$

where the vertical inclusions are by the identifications via ρ_M for ω and ω' (i.e., are given by $\sigma \mapsto \sigma_M$). Going to the Lie algebras by taking derivatives we get a commutative diagram

$$\begin{array}{ccc} \omega\underline{\mathfrak{g}}(M,\omega)\otimes R = \mathfrak{g}(M,\omega)\otimes R & \xrightarrow{Dc_\alpha,\simeq} & \mathfrak{g}(M,\omega')\otimes R = \omega'\underline{\mathfrak{g}}(M,\omega')\otimes R \\ \cap & & \cap \\ \omega\underline{\mathrm{End}}(M)\otimes R = \mathrm{End}(\omega M)\otimes R & \xrightarrow{Dc_{\alpha_M},\simeq} & \mathrm{End}(\omega' M)\otimes R = \omega'\underline{\mathrm{End}}(M). \end{array}$$

The horizontal arrow in the second row is again just conjugation by α_M, so that

$$Dc_{\alpha_M} = \alpha_{\underline{\mathrm{End}}(M)}.$$

On recalling that $\underline{\mathfrak{g}}(M,\omega)$ is a subobject of $\underline{\mathrm{End}}(M)$ and by commutativity of the previous diagram, we get

$$\omega'\underline{\mathfrak{g}}(M,\omega)\otimes R = \alpha_{\underline{\mathrm{End}}(M)}(\omega\underline{\mathfrak{g}}(M,\omega)\otimes R) = \omega'\underline{\mathfrak{g}}(M,\omega')\otimes R \tag{7}$$

(as subspaces of $\mathrm{End}(\omega' M)\otimes R$). This shows that

$$\omega'\underline{\mathfrak{g}}(M,\omega) = \omega'\underline{\mathfrak{g}}(M,\omega')$$

and hence $\underline{\mathfrak{g}}(M,\omega) = \underline{\mathfrak{g}}(M,\omega')$.

If N is any object of $\langle M\rangle^{\otimes}$, by considering the analogous map to c_α for N one easily sees that c_α maps $\mathcal{G}(M,N,\omega)_R$ onto $\mathcal{G}(M,N,\omega')_R$. Thus

$$Dc_\alpha(\mathfrak{g}(M,N,\omega)\otimes R) = \mathfrak{g}(M,N,\omega')\otimes R,$$

and as in (7) we get

$$\omega'\underline{\mathfrak{g}}(M,N,\omega) = \omega'\underline{\mathfrak{g}}(M,N,\omega')$$

as subspaces of $\mathrm{End}(\omega' M)$. □

3. Extensions originating from a subcategory

The goal of this section is to introduce and prove a few lemmas about the basic but useful notion of extensions originating from subcategories of Tannakian categories. This concept will provide a natural language for the results of the paper. As in the previous section, K is a field of characteristic zero. Recall that by a Tannakian subcategory we mean one that is closed under taking subobjects.

3.1. Let G be an affine group scheme over K. Let H be a subgroup of G. Let V be an object of $\mathbf{Rep}(G)$. Denote by V^H the (K-) subspace of V which is fixed by H. More precisely,

$$V^H := \{v \in V : \forall R \in \mathbf{Alg}_K, \forall \sigma \in H(R), \sigma(v\otimes 1_R) = v\otimes 1_R\}.$$

Suppose H is normal in G. Then V^H is a G-subrepresentation of V (i.e., a subobject of V in $\mathbf{Rep}(G)$).

The restriction functor

$$\mathbf{Rep}(G/H) \to \mathbf{Rep}(G)$$

identifies $\mathbf{Rep}(G/H)$ as the full subcategory of $\mathbf{Rep}(G)$ consisting of those representation of G on which H acts trivially. It is evident that for every object V of $\mathbf{Rep}(G)$, the object V^H is the largest subobject of V which belongs to the subcategory $\mathbf{Rep}(G/H)$.

3.2. Let $\boldsymbol{T}$ be a Tannakian category over K, with ω a fiber functor $\boldsymbol{T} \to \mathbf{Mod}_K$. Let $\boldsymbol{S}$ be a full Tannakian subcategory of $\boldsymbol{T}$. The inclusion $\boldsymbol{S} \subset \boldsymbol{T}$ gives a surjection

$$\mathcal{G}(\boldsymbol{T}, \omega) \to \mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}}). \tag{8}$$

Denote the kernel of this map by $\mathcal{H}$.

Using the map (8) we may identify the category $\mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}}))$ as the full subcategory of $\mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega))$ consisting of all the objects on which $\mathcal{H}$ acts trivially. One has a commutative diagram

$$\begin{array}{ccc} \boldsymbol{S} & \xrightarrow{\omega|_{\boldsymbol{S}},\simeq} & \mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}})) \\ \cap & & \cap \\ \boldsymbol{T} & \xrightarrow{\omega,\simeq} & \mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega)), \end{array} \tag{9}$$

where the horizontal arrows are the equivalences of categories given by the fundamental theorem of Tannakian categories. On recalling that $\boldsymbol{S}$ is closed under subobjects and hence in particular isomorphisms, it follows that any object A of $\boldsymbol{T}$ belongs to the subcategory $\boldsymbol{S}$ if and only if $\mathcal{H}$ acts trivially on ωA.

3.3. Let A be an object of $\boldsymbol{T}$. Then $(\omega A)^{\mathcal{H}}$ is a $\mathcal{G}(\boldsymbol{T}, \omega)$-subrepresentation of ωA; hence there is a canonical subobject

$$A_{\boldsymbol{S}} \subset A$$

such that

$$\omega(A_{\boldsymbol{S}}) = (\omega A)^{\mathcal{H}}.$$

Since $(\omega A)^{\mathcal{H}}$ is the largest subobject of $\omega A \in \mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega))$ which belongs to the subcategory $\mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}}))$, it follows that $A_{\boldsymbol{S}}$ is the largest subobject of A which belongs to $\boldsymbol{S}$.

Taking $\mathcal{H}$-invariants gives a left exact functor

$$\mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega)) \to \mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}})).$$

Thus we have a left exact functor

$$-_{\boldsymbol{S}} : \boldsymbol{T} \to \boldsymbol{S}$$

which on objects acts like $A \mapsto A_{\boldsymbol{S}}$ (and on morphisms acts by restriction of domain and codomain).

3.4. Let A be an object of $\boldsymbol{T}$. Let $\mathscr{E}$ in $\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A)$ (= Yoneda Ext^1 group in $\boldsymbol{T}$) be the class of the short exact sequence

$$0 \to A \to E \to \mathbb{1} \to 0.$$

We say the extension $\mathscr{E}$ *originates from* or *comes from* $\boldsymbol{S}$ if there is a commutative diagram in $\boldsymbol{T}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \end{array} \tag{10}$$

where the rows are exact and the objects in the top row are in $\boldsymbol{S}$. In other words, we say $\mathscr{E}$ originates from $\boldsymbol{S}$ if there is an object A' of $\boldsymbol{S}$ and a morphism $A' \to A$ such that $\mathscr{E}$ is in the image of the pushforward map

$$\mathrm{Ext}^1_{\boldsymbol{S}}(\mathbb{1}, A') \to \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A).$$

We now give a few lemmas on the notion of extensions originating from subcategories which are useful in the later sections. The lemmas take place in the above setting (i.e., with $\mathscr{E}$, $\boldsymbol{S}$, and $\mathscr{H}$ as above). The first lemma highlights that the notion of extensions originating from subcategories is a generalization of the notion of splitting of sequences.

Lemma 3.4.1. *The following statements are equivalent*:

(i) *The extension $\mathscr{E}$ splits.*

(ii) *The extension $\mathscr{E}$ originates from some semisimple $\boldsymbol{S}$.*

(iii) *The extension $\mathscr{E}$ originates from every $\boldsymbol{S}$.*

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are trivial. As for (i) $\Longrightarrow$ (iii), note that if $\mathscr{E}$ splits, then it is the pushforward of the extension

$$0 \to 0 \to \mathbb{1} \to \mathbb{1} \to 0. \qquad \square$$

Lemma 3.4.2. *The following statements are equivalent*:

(i) *The extension $\mathscr{E}$ originates from $\boldsymbol{S}$.*

(ii) *The extension $\omega\mathscr{E}$*

$$0 \to \omega A \to \omega E \to K \to 0$$

splits in the category of representations of $\mathscr{H}$.

(iii) *The sequence*

$$0 \to (\omega A)^{\mathscr{H}} \to (\omega E)^{\mathscr{H}} \to K \to 0$$

(*obtained by taking $\mathscr{H}$-invariants of $\omega\mathscr{E}$*) *is exact.*

(iv) *The sequence in* $\boldsymbol{S}$

$$0 \to A_{\boldsymbol{S}} \to E_{\boldsymbol{S}} \to \mathbb{1} \to 0$$

obtained by applying $-_{\boldsymbol{S}}$ *to the defining sequence of* $\mathscr{E}$ *is exact.*

Proof. The equivalence of (iii) and (iv) is clear, as the sequence in (iii) is obtained by applying ω to the sequence in (iv). Note that since the functors $-^{\mathscr{H}}$ and $-_{\boldsymbol{S}}$ are left exact, the statements in (iii) and (iv) are really just statements about surjectivity of $(\omega E)^{\mathscr{H}} \to K$ and $E_{\boldsymbol{S}} \to \mathbb{1}$. The implication (iv) $\Longrightarrow$ (i) is also clear, as we can use the extension given in (iv) as the top row in (10).

(i) $\Longrightarrow$ (iv): Suppose $\mathscr{E}$ originates from $\boldsymbol{S}$, with a commutative diagram as in (10), with exact rows and the top row in $\boldsymbol{S}$. Since $\boldsymbol{S}$ is closed under taking subquotients, by replacing A' and E' if necessary by their images in A and E, we may assume without loss of generality that $A' \subset A$ and $E' \subset E$, with the vertical arrows being considered as inclusion maps. Since E' is in $\boldsymbol{S}$, we have $E' \subset E_{\boldsymbol{S}}$. This proves that the restriction of the surjection $E \to \mathbb{1}$ to $E_{\boldsymbol{S}}$ is still surjective, thus giving (iv).

(iii) $\Longrightarrow$ (ii): There is a commutative diagram of $\mathscr{G}(\boldsymbol{T}, \omega)$-representations

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\omega A)^{\mathscr{H}} & \longrightarrow & (\omega E)^{\mathscr{H}} & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \omega A & \longrightarrow & \omega E & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

where the bottom row is $\omega\mathscr{E}$, the vertical arrows are inclusion, and the rows are exact. Consider this diagram in the category of representations of $\mathscr{H}$. The top row splits, hence so does the bottom row (= the pushout of the top row).

(ii) $\Longrightarrow$ (iii): Suppose (ii) holds. Choose a section s of $\omega E \to K$ in $\mathbf{Rep}(\mathscr{H})$. Then $s(1)$ is fixed by $\mathscr{H}$ and thus belongs to $(\omega E)^{\mathscr{H}}$. It follows that $(\omega E)^{\mathscr{H}} \to K$ is surjective. □

Lemma 3.4.3. *Suppose* A *is an object of* $\boldsymbol{S}$*. Then* $\mathscr{E}$ *originates from* $\boldsymbol{S}$ *if and only if* E *is an object of* $\boldsymbol{S}$*.*

Proof. The "if" implication is trivial. As for the "only if" implication, suppose we have a diagram as in (10), with exact rows and the objects of the top row in $\boldsymbol{S}$. Then E is isomorphic to the fibered coproduct of A and E' over A'. Since A and E' are in $\boldsymbol{S}$, so is E. □

Lemma 3.4.4. *Let* A' *be a subobject of* A *such that the pushforward map*

$$\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' + A_{\boldsymbol{S}}) \to \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A)$$

(*along the inclusion* $A' + A_{\boldsymbol{S}} \to A$) *is injective. Suppose* $\mathscr{E}$ *is the pushforward of an extension*

$$\mathscr{E}' \in \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A')$$

along the inclusion map $A' \to A$*. Then* $\mathscr{E}$ *originates from* $\boldsymbol{S}$ *if and only if* $\mathscr{E}'$ *does.*

Proof. If $\mathscr{E}'$ originates from $\boldsymbol{S}$, then clearly so does $\mathscr{E}$. Suppose $\mathscr{E}$ originates from $\boldsymbol{S}$. Then $\mathscr{E}$ is the pushforward of the extension $\mathscr{E}_{\boldsymbol{S}}$ given in statement (iv) of Lemma 3.4.2 under the inclusion $A_{\boldsymbol{S}} \to A$.

Let $i : A_{\boldsymbol{S}} \to A' + A_{\boldsymbol{S}}$ and $i' : A' \to A' + A_{\boldsymbol{S}}$ be inclusion maps. Apply the δ-functor $\mathrm{Hom}_{\boldsymbol{T}}(\mathbb{1}, -)$ to the short exact sequence

$$0 \to A' \cap A_{\boldsymbol{S}} \to A' \oplus A_{\boldsymbol{S}} \xrightarrow{i-i'} A' + A_{\boldsymbol{S}} \to 0$$

(where the injective arrow is the diagonal embedding). We get exact

$$\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' \cap A_{\boldsymbol{S}}) \to \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A') \oplus \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A_{\boldsymbol{S}}) \xrightarrow{i_*-i'_*} \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' + A_{\boldsymbol{S}}),$$

where the lower stars denote pushforwards. The pushforward of the extension

$$i_*(\mathscr{E}_{\boldsymbol{S}}) - i'_*(\mathscr{E}') \in \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' + A_{\boldsymbol{S}})$$

in $\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A)$ is zero. By the injectivity hypothesis in the statement, $i_*(\mathscr{E}_{\boldsymbol{S}}) - i'_*(\mathscr{E}')$ is already zero. It follows that there is an extension $\mathscr{E}''$

$$0 \to A' \cap A_{\boldsymbol{S}} \to E'' \to \mathbb{1} \to 0$$

which pushes forward (under inclusion maps) to both $\mathscr{E}'$ and $\mathscr{E}_{\boldsymbol{S}}$. But then $A' \cap A_{\boldsymbol{S}}$ and E'', being subobjects of $A_{\boldsymbol{S}}$ and $E_{\boldsymbol{S}}$, belong to $\boldsymbol{S}$. Since $\mathscr{E}''$ pushes forward to $\mathscr{E}'$, the latter extension originates from $\boldsymbol{S}$. □

Remark. Note that the injectivity hypothesis in the statement of the previous lemma is guaranteed if

$$\mathrm{Hom}_{\boldsymbol{T}}(\mathbb{1}, A/(A' + A_{\boldsymbol{S}})) = 0$$

(and this will be the case whenever we use the result in the paper). This can be seen from the long exact sequence obtained by applying $\mathrm{Hom}_{\boldsymbol{T}}(\mathbb{1}, -)$ to

$$0 \to A' + A_{\boldsymbol{S}} \to A \to A/(A' + A_{\boldsymbol{S}}) \to 0.$$

4. Extension classes and subgroups of the fundamental group, part I

4.1. From this point on we suppose that $\boldsymbol{T}$ is a Tannakian category over a field K of characteristic zero, equipped with a functorial exact finite increasing filtration $W_\bullet$, compatible with the tensor structure. We refer to $W_\bullet$ as the weight filtration. Here, the expression "functorial exact finite increasing filtration $W_\bullet$" means that for every integer n, we have an exact functor $W_n : \boldsymbol{T} \to \boldsymbol{T}$, such that for every object M of $\boldsymbol{T}$, we have

$$W_{n-1}M \subset W_nM \quad (\forall n)$$
$$W_nM = 0 \quad (\forall n \ll 0)$$
$$W_nM = M \quad (\forall n \gg 0),$$

and such that the inclusions $W_nM \subset M$ for various M give a morphism of functors from W_n to the identity (and hence the W_n form an inductive system of functors). Compatibility with the tensor product means that for every objects M and N, we have

$$W_n(M \otimes N) = \sum_{\substack{p,q \\ p+q=n}} W_pM \otimes W_qN. \tag{11}$$

The associated graded functor Gr^W is the functor defined on objects by

$$\mathrm{Gr}^W M := \bigoplus_n \mathrm{Gr}^W_n M,$$

where $\mathrm{Gr}^W_n M := W_n M / W_{n-1} M$, and on morphisms in the obvious way using the fact that we have morphisms of functors $W_{n-1} \to W_n$. By the snake lemma, the associated graded functor (in fact, each Gr^W_n) is also exact. Also Gr^W is a graded tensor functor, in the sense that (via a canonical isomorphism) we have

$$\mathrm{Gr}^W(M \otimes N) = \mathrm{Gr}^W(M) \otimes \mathrm{Gr}^W(N),$$

with this identification being compatible with weights, i.e., being the direct sum of identifications

$$\mathrm{Gr}^W_n(M \otimes N) = \bigoplus_{\substack{p,q \\ p+q=n}} \mathrm{Gr}^W_p M \otimes \mathrm{Gr}^W_q M$$

induced by (11).

As it is customary, we call an object M with $W_{n-1}M = 0$ and $W_n M = M$ a pure object of weight n. Note that unless otherwise indicated, we do not assume that an object of the form $\mathrm{Gr}^W M$ (i.e., a direct sum of pure objects) is necessarily semisimple.

Given any fiber functor ω (over K) and any object M, set

$$W_\bullet \omega M := \omega(W_\bullet M).$$

This defines an exact $\otimes$-filtration on ω, in the language of Saavedra Rivano [1972, Chapter IV, Section 2], (note that Saavedra Rivano works with decreasing filtrations instead, and that his Condition FE 1) is guaranteed here because K is a field).

Given any objects M and N, we identify

$$\omega \underline{\mathrm{Hom}}(M, N) = \mathrm{Hom}(\omega M, \omega N).$$

One can then show that

$$\omega W_n \underline{\mathrm{Hom}}(M, N) = \{f \in \mathrm{Hom}(\omega M, \omega N) : f(W_\bullet \omega M) \subset W_{\bullet+n} \omega N\}.$$

4.2. Here and elsewhere in the paper, we shall use the notation and conventions of Section 2 for Tannakian fundamental groups and their Lie algebras.

Let M be an object of $\boldsymbol{T}$. Given any fiber functor ω, let $P(M, \omega)$ be the parabolic subgroup of $\mathrm{GL}(\omega M)$ which stabilizes the filtration $W_\bullet$. Then

$$\mathrm{Lie}(P(M, \omega)) = W_0 \,\mathrm{End}(\omega M).$$

The elements of $\mathcal{G}(M, \omega)$ (= the fundamental group of M with respect to ω) preserve subobjects of M, so that

$$\mathcal{G}(M, \omega) \subset P(M, \omega).$$

Going to the Lie algebras we have

$$\underline{\mathfrak{g}}(M) \subset W_0\underline{\mathrm{End}}(M).$$

Every element of $P(M, \omega)$ induces an automorphism of $\mathrm{Gr}^W \omega M$, giving rise to a homomorphism

$$P(M, \omega) \to \mathrm{GL}(\mathrm{Gr}^W \omega M).$$

Let $U(M, \omega)$ be the kernel of this map; then $U(M, \omega)$ is the unipotent radical of $P(M, \omega)$. It is easy to see that

$$\mathrm{Lie}(U(M, \omega)) = W_{-1}\,\mathrm{End}(\omega M).$$

Set

$$\mathcal{U}(M, \omega) := \mathcal{G}(M, \mathrm{Gr}^W M, \omega)$$

(= the kernel of the restriction map $\mathcal{G}(M, \omega) \to \mathcal{G}(\mathrm{Gr}^W M, \omega)$ induced by the inclusion $\langle \mathrm{Gr}^W M\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$). Then

$$\mathcal{U}(M, \omega) = \mathcal{G}(M, \omega) \cap U(M, \omega). \tag{12}$$

In particular, $\mathcal{U}(M, \omega)$ is a unipotent group. If $\mathcal{G}(\mathrm{Gr}^W M, \omega)$ happens to be reductive (i.e., if $\mathrm{Gr}^W M$ is semisimple), then $\mathcal{U}(M, \omega)$ will be the unipotent radical of $\mathcal{G}(M, \omega)$.

We set

$$\underline{\mathfrak{u}}(M) := \underline{\mathfrak{g}}(M, \mathrm{Gr}^W M) \quad \text{and} \quad \mathfrak{u}(M, \omega) := \mathrm{Lie}\,\mathcal{U}(M, \omega)$$

(= $\mathfrak{g}(M, \mathrm{Gr}^W M, \omega)$ in the notation of Section 2). Then (for every ω),

$$\omega\underline{\mathfrak{u}}(M) = \mathfrak{u}(M, \omega).$$

By (12), we have

$$\underline{\mathfrak{u}}(M) = \underline{\mathfrak{g}}(M) \cap W_{-1}\underline{\mathrm{End}}(M).$$

4.3. A result of Deligne (written by Jossen in the appendix of [Jossen 2014]) describes the subobject $\underline{\mathfrak{u}}(M)$ of $W_{-1}\underline{\mathrm{End}}(M)$ as follows.[6] From now on, if there is no ambiguity, we shall simply write Hom (resp. Ext^i) for the Hom groups $\mathrm{Hom}_{\boldsymbol{T}}$ (resp. the Yoneda $\mathrm{Ext}^i_{\boldsymbol{T}}$ groups) in $\boldsymbol{T}$.

Recall from the Introduction that for each integer p, the p-th extension class

$$\mathcal{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

of M is the extension corresponding to the sequence

$$0 \to W_pM \to M \to M/W_pM \to 0 \tag{13}$$

under the canonical isomorphism

$$\mathrm{Ext}^1(M/W_pM, W_pM) \cong \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)). \tag{14}$$

[6] We thank Peter Jossen for patiently explaining to us some parts of Deligne's argument from [Jossen 2014, Appendix].

Applying $\underline{\mathrm{Hom}}(M/W_pM, -)$ to the inclusion $W_pM \to M$ we get an injection

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM) \to \underline{\mathrm{Hom}}(M/W_pM, M).$$

On the other hand, applying $\underline{\mathrm{Hom}}(-, M)$ to the quotient map $M \to M/W_pM$ we get an injection

$$\underline{\mathrm{Hom}}(M/W_pM, M) \to \underline{\mathrm{End}}(M).$$

Composing the two injections, we get a map

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM) \to \underline{\mathrm{End}}(M). \tag{15}$$

After applying a fiber functor ω, this simply sends an element

$$f \in \mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM)$$

to the composition

$$\omega M \xrightarrow{\text{quotient}} \omega M/\omega W_pM \xrightarrow{f} \omega W_pM \xrightarrow{\text{inclusion}} \omega M. \tag{16}$$

From this it is clear that indeed, the image of the map (15) is contained in $W_{-1}\underline{\mathrm{End}}(M)$. We shall identify $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ as a subobject of $W_{-1}\underline{\mathrm{End}}(M)$ via the map (15). Note that $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ is an abelian Lie subalgebra of $W_{-1}\underline{\mathrm{End}}(M)$.

Pushing forward extensions along the inclusion map we get a map

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)), \tag{17}$$

which is injective, as (by weight considerations),

$$\mathrm{Hom}\left(\mathbb{1}, \frac{W_{-1}\underline{\mathrm{End}}(M)}{\underline{\mathrm{Hom}}(M/W_pM, W_pM)}\right) = 0.$$

To simplify the notation, we shall identify

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

with its image under (17).

Deligne defines the (total) extension class of M to be

$$\mathscr{E}(M) := \sum_p \mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M))$$

(this is denoted by $\mathrm{cl}(M)$ in [Jossen 2014]), and proves that the extension $\mathscr{E}(M)$ can be used to describe $\underline{\mathfrak{u}}(M)$. More precisely, he proves the following result:

Theorem 4.3.1 (Deligne, Appendix of [Jossen 2014]). *The subobject $\underline{\mathfrak{u}}(M) \subset W_{-1}\underline{\mathrm{End}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{End}}(M)$ such that the extension $\mathscr{E}(M)$ is the pushforward of an element of* $\mathrm{Ext}^1(\mathbb{1}, \underline{\mathfrak{u}}(M))$ *under the inclusion $\underline{\mathfrak{u}}(M) \to W_{-1}\underline{\mathrm{End}}(M)$.*

It is worth highlighting that the theorem asserts that $\underline{\mathfrak{u}}(M)$ is the smallest subobject with the stated property, not just the smallest Lie subobject with the property. Also note that by weight considerations, the pushforward map

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathfrak{u}}(M)) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)) \tag{18}$$

is injective, so that the element pushing forward to $\mathscr{E}(M)$ is indeed unique.

Remark. As we pointed out in the Introduction, in general, the individual extensions $\mathscr{E}_p(M)$ may not be in the image of the pushforward map (18). See Section 6.10 (and Remark (2) therein) for examples in the category of mixed Hodge structures using the Jacquinot–Ribet deficient points on semiabelian varieties.

4.4. We adopt the following notation for pushforwards of extensions along quotient maps. If $\mathscr{E}$ is an extension of an object A by B, then for any subobject B' of B we denote the pushforward of $\mathscr{E}$ along the quotient $B \to B/B'$ by $\mathscr{E}/B'$.

Given any subobject $A \subset W_{-1}\underline{\mathrm{End}}(M)$, applying the functor $\mathrm{Hom}(\mathbb{1}, -)$ to the short exact sequence

$$0 \to A \to W_{-1}\underline{\mathrm{End}}(M) \to W_{-1}\underline{\mathrm{End}}(M)/A \to 0$$

we get a long exact sequence. In particular, we have exact

$$\mathrm{Ext}^1(\mathbb{1}, A) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/A),$$

where the arrows are pushforwards along inclusion and quotient maps. Thus Deligne's result can be equivalently stated as that $\underline{\mathfrak{u}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{End}}(M)$ such that the pushforward

$$\mathscr{E}(M)/\underline{\mathfrak{u}}(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}(M))$$

of $\mathscr{E}(M)$ splits.

The formulation of Theorem 4.3.1 as given in the statement is more natural for Deligne's proof, as his argument goes by constructing an explicit extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$ which pushes forward to $\mathscr{E}(M)$. The formulation in terms of $\mathscr{E}(M)/\underline{\mathfrak{u}}(M)$ is however more natural when one wants to study the individual extensions $\mathscr{E}_p(M)$, as we shall see.

4.5. The canonical isomorphism (14) is given by first applying the functor

$$\underline{\mathrm{Hom}}(M/W_pM, -)$$

to an element of $\mathrm{Ext}^1(M/W_pM, W_pM)$, and then pulling back along the canonical map

$$\mathbb{1} \to \underline{\mathrm{End}}(M/W_pM)$$

(which after applying a fiber functor ω, sends 1 to the identity map on $\omega(M/W_pM)$). Going through this, we see that assuming $M/W_pM \neq 0$, the extension

$$\mathscr{E}_p \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

is the class of

$$0 \to \underline{\mathrm{Hom}}(M/W_pM, W_pM) \to \underline{\mathrm{Hom}}(M/W_pM, M)^\dagger \to \mathbb{1} \to 0, \tag{19}$$

where $\underline{\mathrm{Hom}}(M/W_pM, M)^\dagger$ is the subobject of $\underline{\mathrm{Hom}}(M/W_pM, M)$ characterized by

$$\begin{aligned}\omega\underline{\mathrm{Hom}}&(M/W_pM, M)^\dagger \\ &= \mathrm{Hom}(\omega M/\omega W_pM, \omega M)^\dagger \\ &:= \{f \in \mathrm{Hom}(\omega M/\omega W_pM, \omega M) : f \mod \omega W_pM = \lambda(f) Id_{\omega M/\omega W_pM} \text{ for some } \lambda(f) \in K\}\end{aligned}$$

for any fiber functor ω. The injective (resp. surjective) arrow in (19) is, after applying ω, the natural inclusion (resp. the map $f \mapsto \lambda(f)$, with $\lambda(f) \in K$ as in the definition of $\underline{\mathrm{Hom}}(M/W_pM, M)^\dagger$ above).

If $M/W_pM = 0$, set $\underline{\mathrm{Hom}}(M/W_pM, M)^\dagger := \mathbb{1}$; then $\mathscr{E}_p$ is again given by the sequence (19), with the surjective arrow being the identity map on $\mathbb{1}$.[7]

4.6. Fix an integer p. After applying a fiber functor ω to the identification

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM) \subset W_{-1}\underline{\mathrm{End}}(M)$$

we get an identification

$$\mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM) \subset W_{-1}\,\mathrm{End}(\omega M),$$

which thinks of $f : \omega M/W_pM \to \omega W_pM$ as the composition (16). This way,

$$\mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM) \tag{20}$$

becomes an abelian Lie subalgebra of $W_{-1}\,\mathrm{End}(\omega M)$. The exponential map

$$\exp : W_{-1}\,\mathrm{End}(\omega M) \to U(M, \omega)(K) \subset \mathrm{GL}(\omega M)(K)$$

is given by the usual exponential series. On the Lie subalgebra (20), it is simply given by

$$\exp(f) = I + f.$$

4.7. In this subsection we shall introduce certain Lie subalgebras of $\underline{\mathfrak{u}}(M)$ and subgroups of $\mathcal{U}(M, \omega)$ (for any ω) which play a crucial role in the paper. For any integer p, let

$$\underline{\mathfrak{u}}_p(M) := \underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_pM, W_pM)$$

---

[7] Equivalently, one can define $\underline{\mathrm{Hom}}(M/W_pM, M)^\dagger$ in the following way, which works in all cases: $\underline{\mathrm{Hom}}(M/W_pM, M)^\dagger$ is the subobject of $\underline{\mathrm{Hom}}(M/W_pM, M) \oplus \mathbb{1}$ whose image under any fiber functor ω is

$$\{(f, \lambda) \in \mathrm{Hom}(\omega M/\omega W_pM, \omega M) \oplus K : f \mod \omega W_pM = \lambda Id_{\omega M/\omega W_pM}\}.$$

That is, $\underline{\mathrm{Hom}}(M/W_pM, M)^\dagger$ is the kernel of the appropriate morphism $\underline{\mathrm{Hom}}(M/W_pM, M) \oplus \mathbb{1} \to \underline{\mathrm{End}}(M)$. The injective (resp. surjective) arrow in (19) is then induced by the inclusion (resp. projection) map into (resp. from) the direct sum. We shall however work with the first definition, as it will simplify the expressions in our proofs.

and for any ω,

$$\mathfrak{u}_p(M,\omega) := \omega\underline{\mathfrak{u}}_p(M) = \mathfrak{u}(M,\omega) \cap \mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M).$$

Then $\mathfrak{u}_p(M,\omega)$ is an abelian Lie subalgebra of $\mathfrak{u}(M,\omega)$.

For any Lie subalgebra $\mathfrak{l}$ of $W_{-1}\,\mathrm{End}(\omega M)$, we denote the subgroup of $U(M,\omega)$ whose Lie algebra is $\mathfrak{l}$ by $\mathcal{exp}(\mathfrak{l})$ (thus $\mathcal{exp}(\mathfrak{l})(K) = \exp(\mathfrak{l})$). Set

$$\begin{aligned}\mathcal{U}_p(M,\omega) &:= \mathcal{exp}(\mathfrak{u}_p(M,\omega))\\ &= \mathcal{U}(M,\omega) \cap \mathcal{exp}(\mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M))\\ &= \mathcal{G}(M,\omega) \cap \mathcal{exp}(\mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M)).\end{aligned}$$

This is an abelian unipotent subgroup of $\mathcal{U}(M,\omega)$.

Lemma 4.7.1. *$\mathcal{U}_p(M,\omega)$ is the kernel of the restriction homomorphism*

$$\mathcal{G}(M,\omega) \to \mathcal{G}(W_p M \oplus (M/W_p M), \omega)$$

(*induced by* $\langle W_p M \oplus (M/W_p M)\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$).

Proof. Tentatively, let us refer to the kernel of the homomorphism given in the statement of the lemma as U'. It is clear that U' is contained in $\mathcal{U}(M,\omega)$. In particular, U' is also unipotent and thus it is enough to show that U' and $\mathcal{U}_p(M,\omega)$ have the same K-valued points. We have

$$\mathcal{U}_p(M,\omega)(K) = \mathcal{G}(M,\omega)(K) \cap \exp(\mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M)).$$

Let $\sigma \in \mathcal{G}(M,\omega)(K)$. Then $\sigma \in U'(K)$ if and only if $\sigma_{W_p M} = I$ and $\sigma_{M/W_p M} = I$. Under the identification $\mathcal{G}(M,\omega) \subset P(M,\omega)$ (via $\sigma \mapsto \sigma_M$), $\sigma_{W_p M}$ is simply the restriction $\sigma|_{\omega W_p M}$ of σ to $\omega W_p M$, and $\sigma_{M/W_p M}$ is the map $\bar{\sigma}$ that σ, as an element of the parabolic subgroup $P(M,\omega)$, induces on $\omega M/\omega W_p M$ (given by $\bar{\sigma}(v + \omega W_p) = \sigma(v) + \omega W_p$, where $v \in \omega M$). On recalling that

$$\exp(\mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M)) = I + \mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M),$$

it is easy to see that the subgroup of $P(M,\omega)(K)$ which acts as identity on both $\omega W_p M$ and $\omega M/\omega W_p M$ is

$$\exp(\mathrm{Hom}(\omega M/\omega W_p M, \omega W_p M)).$$

The claim follows. □

Remark. Our examples in Section 6.10 (also see item (3) of the remark therein) show that in general, $\underline{\mathfrak{u}}(M)$ may not be generated by the $\underline{\mathfrak{u}}_p(M)$, even as a Lie algebra. It is however true that if $\mathcal{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits for every p, then $\underline{\mathfrak{u}}(M) = \sum_p \underline{\mathfrak{u}}_p(M)$. See item (2) of the remark at the end of Section 5.1. (Note that the sum $\sum_p \underline{\mathfrak{u}}_p(M)$ in general may not be a Lie subalgebra of $\underline{\mathfrak{u}}(M)$.)

4.8. Let us recall (∗) from the Introduction.

Proposition 4.8.1. *For any subobject A of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$, we have $\underline{\mathfrak{u}}_p(M) \subset A$ if and only if the quotient*

$$\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A$$

belongs to the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$.

This follows from Theorem 3.3.1 of [Eskandari and Murty 2021],[8] with L, N, and $\mathcal{U}(M)$ of [loc. cit.] being respectively W_pM, M/W_pM, and $\mathcal{U}_p(M, \omega)$ here. However, in the interest of keeping the paper more self-contained, let us recall the argument: The statement is trivial if $M/W_pM = 0$ so we may assume otherwise. To simplify the notation, let us tentatively denote the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ by $\boldsymbol{C}$. Let A be a subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ and ω a fiber functor. In view of Section 3.2 and Lemma 4.7.1, the quotient

$$\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A$$

belongs to $\boldsymbol{C}$ if and only if $\mathcal{U}_p(M, \omega)$ acts trivially on

$$\omega(\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A) = \omega\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/\omega A. \tag{21}$$

Choose a section of the natural surjection $\omega M \to \omega M/\omega W_pM$ to identify

$$\omega M = \omega W_pM \oplus \omega M/\omega W_pM$$

(as vector spaces). This also gives a decomposition of $\omega\underline{\mathrm{Hom}}(M/W_pM, M)$. In view of the sequence (19) and on noting that $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ belongs to $\boldsymbol{C}$, the group $\mathcal{U}_p(M, \omega)$ acts trivially on (21) if and only if it (or equivalently, $\mathcal{U}_p(M, \omega)(K)$) fixes the image of the element

$$\begin{aligned}(0, I) \in \mathrm{Hom}(\omega M/\omega W_pM, \omega M)^{\dagger} &\subset \mathrm{Hom}(\omega M/\omega W_pM, \omega M)\\ &= \mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM) \oplus \mathrm{End}(\omega M/\omega W_pM)\end{aligned}$$

in (21). Identifying $\mathrm{Hom}(\omega M/\omega W_pM, \omega M)$ as a subspace of $\mathrm{End}(\omega M)$ in the obvious way, given any $\sigma \in \mathcal{U}_p(M, \omega)(K)$, in view of the fact that σ fixes ωW_pM and $\omega M/\omega W_pM$, one calculates that

$$\sigma \cdot (0, I) - (0, I) = \log(\sigma).$$

Thus $\mathcal{U}_p(M, \omega)(K)$ fixes $(0, I)$ mod ωA if and only if ωA contains $\mathfrak{u}_p(M, \omega)$.

4.9. Proposition 4.8.1 can be reformulated in the language of extensions originating from subcategories of $\boldsymbol{T}$ (see Section 3.4) as follows:

Theorem 4.9.1. *Let A be a subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$. Then the extension $\mathcal{E}_p(M)/A$, viewed as an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/A$ or $\underline{\mathrm{Hom}}(M/W_pM, W_pM)/A$, originates from the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ if and only if A contains $\underline{\mathfrak{u}}_p(M)$.*

[8]Theorem 3.3.1 of [Eskandari and Murty 2021] is obtained by a slight modification of Hardouin's argument for Theorem 2 of [Hardouin 2011] and Théorème 2.1 of the unpublished article [Hardouin 2006].

In other words, $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$, viewed as an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}_p(M)$ or by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)/\underline{\mathfrak{u}}_p(M)$, originates from $\langle W_pM, M/W_pM\rangle^{\otimes}$.

Proof of Theorem 4.9.1. Let A be a subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$. By Lemma 3.4.4 (also see the remark after the same lemma), and in view of the facts (1) that the extension $\mathscr{E}_p(M)/A$ of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/A$ is the image of its namesake as an extension of $\mathbb{1}$ by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)/A$ under the obvious pushforward map

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)/A) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/A),$$

and (2) that (by weight considerations) there are no nonzero morphisms from $\mathbb{1}$ to objects of weight < 0, the following statements are equivalent for any full Tannakian subcategory $\boldsymbol{S}$ of $\boldsymbol{T}$:

(i) The extension $\mathscr{E}_p(M)/A$, viewed as an element of

$$\mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/A),$$

originates from $\boldsymbol{S}$.

(ii) The extension $\mathscr{E}_p(M)/A$, viewed as an element of

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)/A),$$

originates from $\boldsymbol{S}$.

In view of Lemma 3.4.3 and on recalling the explicit description of

$$\mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

from Section 4.5, Statement (ii) with $\boldsymbol{S}$ taken to be the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ is equivalent to the following statement:

(iii) The object

$$\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A$$

belongs to $\langle W_pM, M/W_pM\rangle^{\otimes}$.

Thus Theorem 4.9.1 is equivalent to Proposition 4.8.1 (or $(*)$ of the Introduction). □

5. Extension classes and subgroups of the fundamental group, part II

In the previous section we saw that $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from $\langle W_pM, M/W_pM\rangle^{\otimes}$. Our goal in this section is to give criteria under which the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ in this statement can be replaced by smaller subcategories. Of particular interest will be the case in which we can replace it with a semisimple category, as then $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ will split.

5.1. Let us first make an observation regarding the pushforwards of the extension $\mathscr{E}_p(M)$. Recall that we are using the same notation for

$$\mathscr{E}_p(M) \in \operatorname{Ext}^1(\mathbb{1}, \underline{\operatorname{Hom}}(M/W_pM, W_pM))$$

and its image in $\operatorname{Ext}^1(\mathbb{1}, W_{-1}\underline{\operatorname{End}}(M))$ under the pushforward map (17).

Lemma 5.1.1. *Let $\boldsymbol{S}$ be a full Tannakian subcategory of $\boldsymbol{T}$. Then the following statements are equivalent*:

(i) *The extension*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M) \in \operatorname{Ext}^1(\mathbb{1}, \underline{\operatorname{Hom}}(M/W_pM, W_pM)/\underline{\mathfrak{u}}_p(M))$$

originates from $\boldsymbol{S}$.

(ii) *The extension*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M) \in \operatorname{Ext}^1(\mathbb{1}, W_{-1}\underline{\operatorname{End}}(M)/\underline{\mathfrak{u}}_p(M))$$

originates from $\boldsymbol{S}$.

(iii) *The extension*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M) \in \operatorname{Ext}^1(\mathbb{1}, W_{-1}\underline{\operatorname{End}}(M)/\underline{\mathfrak{u}}(M))$$

originates from $\boldsymbol{S}$.

Proof. That (i) implies (ii) and (ii) implies (iii) is clear, as under the obvious maps the extension in (i) pushes forward to the extension in (ii) and then to the one in (iii) (in fact, we already observed the equivalence of (i) and (ii) in the proof of Theorem 4.9.1). That (iii) implies (i) follows similarly as in the proof of Theorem 4.9.1 from Lemma 3.4.4 on recalling that

$$\underline{\mathfrak{u}}(M) \cap \underline{\operatorname{Hom}}(M/W_pM, W_pM) = \underline{\mathfrak{u}}_p(M)$$

(so that the obvious map

$$\underline{\operatorname{Hom}}(M/W_pM, W_pM)/\underline{\mathfrak{u}}_p(M) \to W_{-1}\underline{\operatorname{End}}(M)/\underline{\mathfrak{u}}(M)$$

is injective). □

Remark. (1) In particular, by taking $\boldsymbol{S}$ to be the semisimple subcategory $\langle \mathbb{1} \rangle^{\otimes}$ we see that the three extensions in the lemma split at the same time.

(2) The lemma together with Deligne's Theorem 4.3.1 implies that if every $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits (i.e., if every $\mathscr{E}_p(M)$ is in the image of (18)), then $\underline{\mathfrak{u}}(M) = \sum_p \underline{\mathfrak{u}}_p(M)$. Indeed, let us tentatively set $\underline{\mathfrak{u}}' = \sum_p \underline{\mathfrak{u}}_p(M)$. If $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits for every p, then so does $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ and hence $\mathscr{E}_p(M)/\underline{\mathfrak{u}}'$ (the latter as an extension of $\mathbb{1}$ by $W_{-1}\underline{\operatorname{End}}(M)/\underline{\mathfrak{u}}'$). It follows that $\mathscr{E}(M)/\underline{\mathfrak{u}}'$ splits, so that by Deligne's theorem $\underline{\mathfrak{u}}(M) \subset \underline{\mathfrak{u}}'$.

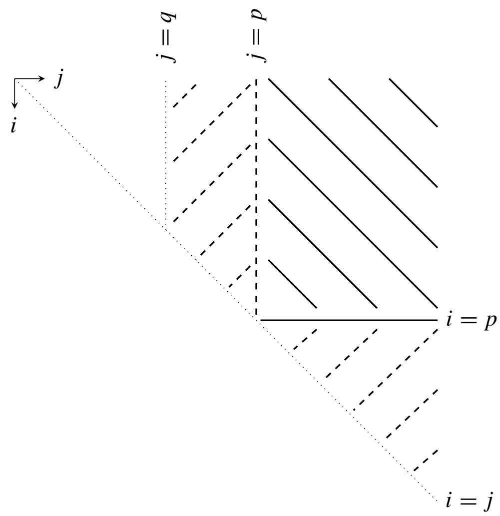

Figure 1. The set of lattice points in the region marked by solid (resp. thick dashed) lines is $J_1^{\{p,q\}}$ (resp. $J_2^{\{p,q\}}$).

5.2. For any integers p and q with $q \leq p$, define

$$J_1^{\{p,q\}} := \{(i, j) \in \mathbb{Z}^2 : i \leq p < j\},$$
$$J_2^{\{p,q\}} := \{(i, j) \in \mathbb{Z}^2 : i < j \text{ and } (q < j \leq p \text{ or } i > p)\}.$$

Figure 1 shows the two sets. In the figure, the axes are oriented according to the standard labeling of entries of a matrix (the pair (i, j) is placed where the entry ij of a matrix sits).

We consider the following *independence axioms* for an object M of $\boldsymbol{T}$:

- $(IA1)_{\{p,q\}}$: The two objects
$$\bigoplus_{(i,j)\in J_1^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M) \quad \text{and} \quad \bigoplus_{(i,j)\in J_2^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M)$$
have no nonzero isomorphic subobjects. Note that if $q' \leq q \leq p$, then $(IA1)_{\{p,q'\}}$ implies $(IA1)_{\{p,q\}}$.
- $(IA2)_{\{p,q\}}$: The two sets
$$J_1^{\{p,q\}}(M) := \{i - j : (i, j) \in J_1^{\{p,q\}}, \mathrm{Gr}_i^W M \neq 0, \mathrm{Gr}_j^W M \neq 0\}$$
and
$$J_2^{\{p,q\}}(M) := \{i - j : (i, j) \in J_2^{\{p,q\}}, \mathrm{Gr}_i^W M \neq 0, \mathrm{Gr}_j^W M \neq 0\}$$
are disjoint. (Note that $J_1^{\{p,q\}}(M)$ and $J_2^{\{p,q\}}(M)$ are respectively the set of weights of the two object in $(IA1)_{\{p,q\}}$ above.)
- $(IA3)$: The numbers
$$i - j \quad (i < j, \mathrm{Gr}_i^W M \neq 0, \mathrm{Gr}_j^W M \neq 0)$$
are all distinct. (Equivalently, if M has n distinct weights, then $W_{-1}\underline{\mathrm{End}}(M)$ has $\binom{n}{2}$ distinct weights.)

It is clear that $(IA2)_{\{p,q\}}$ implies $(IA1)_{\{p,q\}}$, and $(IA3)$ implies $(IA2)_{\{p,q\}}$ for every p and q. Also note that whether or not M satisfies any of these axioms only depends on $\mathrm{Gr}^W M$.

5.3. We can now state the main result of this part of the paper:

Theorem 5.3.1. *Let $q \leq p$. Consider the following statements*:

(i) *M satisfies $(IA1)_{\{p,q\}}$ and $\mathrm{Gr}^W M$ is semisimple (= completely reducible).*

(ii) *M satisfies $(IA2)_{\{p,q\}}$.*

If either of the statements holds, then the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from the subcategory $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.

The proof of Theorem 5.3.1 shall be given in the Sections 5.4–5.6 below. Here we consider some consequences of the theorem:

(1) Since $q \leq p$, the subcategory $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$ is contained in the subcategory $\langle W_p M, M/W_p M\rangle^{\otimes}$. Thus combining Theorems 4.9.1 and 5.3.1 we get the following refinement of Theorem 4.9.1: If statements (i) or (ii) above hold for some $q \leq p$, then $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_p M, W_p M)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.

(2) Perhaps the most interesting application of Theorem 5.3.1 is in the following scenario: Fix p. Suppose $\mathrm{Gr}^W M$ is semisimple; for instance, this will be the case if $\boldsymbol{T}$ is a category of motives, or if $\boldsymbol{T}$ is the category of mixed Hodge structures and $\mathrm{Gr}^W M$ is polarizable. Suppose M satisfies $(IA1)_{\{p,q\}}$ for all $q \leq p$ (this holds for instance, if M satisfies $(IA3)$). Then $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_p M, W_p M)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from the semisimple subcategory $\langle \mathrm{Gr}^W M\rangle^{\otimes}$, i.e., splits. In particular, $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits. For future referencing, we record this as a corollary.

Corollary 5.3.2. *Fix p. Suppose $\mathrm{Gr}^W M$ is semisimple and that M satisfies $(IA1)_{\{p,q\}}$ for all $q \leq p$. Then $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_p M, W_p M)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. In particular,*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$$

splits.

As a special case, we obtain:

Corollary 5.3.3. *If $\mathrm{Gr}^W M$ is semisimple and $(IA3)$ holds, then for every p the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits.*

Remark. Recall that by Deligne's Theorem 4.3.1, the extension

$$\sum_p \mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$$

splits. As we pointed out earlier, in general, the individual extensions $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ may not split (see Section 6.10 and item (2) of the remark therein for examples). The above results give sufficient conditions for when an individual $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits.

5.4. From this point until the end of Section 5.6 our goal is to prove Theorem 5.3.1. Given any fiber functor ω, let $\mathcal{U}_{\geq q}(M,\omega)$ be the kernel of the surjection

$$\mathcal{G}(M,\omega) \to \mathcal{G}(W_q M \oplus \mathrm{Gr}^W M, \omega)$$

induced by the inclusion $\langle W_q M \oplus \mathrm{Gr}^W M\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$. Then $\mathcal{U}_{\geq q}(M,\omega)$ is the subgroup of $\mathcal{U}(M,\omega)$ which acts trivially on $\omega W_q M$. Let $U_{\geq q}(M,\omega)$ be the subgroup of $\mathrm{GL}(\omega M)$ consisting of the elements which fix the weight filtration, and act trivially on $\mathrm{Gr}^W \omega M$ and $\omega W_q M$:

$$U_{\geq q}(M,\omega) := \{\sigma \in U(M,\omega) : \sigma|_{\omega W_q M} = I\}.$$

Then

$$\mathcal{U}_{\geq q}(M,\omega) = \mathcal{U}(M,\omega) \cap U_{\geq q}(M,\omega).$$

We have

$$\mathrm{Lie}(U_{\geq q}(M,\omega)) = \mathrm{Hom}(\omega M/\omega W_q M, \omega M) \cap W_{-1}\,\mathrm{End}(\omega M),$$

where $\mathrm{Hom}(\omega M/\omega W_q M, \omega M)$ is identified as the subspace of $\mathrm{End}(\omega M)$ consisting of the elements which vanish on $\omega W_q M$. Then

$$\mathfrak{u}_{\geq q}(M,\omega) := \mathrm{Lie}(\mathcal{U}_{\geq q}(M,\omega)) = \mathfrak{u}(M,\omega) \cap \mathrm{Hom}(\omega M/\omega W_q M, \omega M).$$

Finally, set

$$\underline{\mathfrak{u}}_{\geq q}(M) := \underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_q M, M).$$

Here

$$\underline{\mathrm{Hom}}(M/W_q M, M)$$

is thought of as a subobject of $\underline{\mathrm{End}}(M)$ via the obvious injection induced by the quotient map $M \to M/W_q M$ (note that this is compatible with the previous identification of $\mathrm{Hom}(\omega M/\omega W_q M, \omega M)$ as a subspace of $\mathrm{End}(\omega M)$). We then have

$$\mathfrak{u}_{\geq q}(M,\omega) = \omega \underline{\mathfrak{u}}_{\geq q}(M).$$

5.5. Identifying

$$\mathrm{Gr}^W \underline{\mathrm{End}}(M) = \underline{\mathrm{End}}(\mathrm{Gr}^W M) = \bigoplus_{i,j} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M), \tag{22}$$

we have

$$\mathrm{Gr}^W W_{-1}\underline{\mathrm{End}}(M) = \bigoplus_{\substack{i,j\\ i<j}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M).$$

Then for every q,

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \subset \mathrm{Gr}^W \underline{\mathrm{Hom}}(M/W_q M, M) \cap \mathrm{Gr}^W W_{-1}\underline{\mathrm{End}}(M) = \bigoplus_{\substack{i,j\\ i,q<j}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M). \tag{23}$$

The following lemma is the only place in the proof of Theorem 5.3.1 that conditions (i) and (ii) of the theorem play a part.

Lemma 5.5.1. *Let $q \leq p$. Suppose statement* (*i*) *or* (*ii*) *of Theorem 5.3.1 holds. Then* $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M)$ *decomposes as the direct sum of*

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \cap \bigoplus_{(i,j)\in J_1^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M)$$

and

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \cap \bigoplus_{(i,j)\in J_2^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M).$$

Proof. The direct sum in (23) is over all pairs (i, j) in $J_1^{\{p,q\}} \sqcup J_2^{\{p,q\}}$, so that we can rewrite (23) as

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \subset \overbrace{\bigoplus_{(i,j)\in J_1^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M)}^{\text{(I)}} \oplus \overbrace{\bigoplus_{(i,j)\in J_2^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M)}^{\text{(II)}}.$$

First suppose $\mathrm{Gr}^W M$ is semisimple and M satisfies $(IA1)_{\{p,q\}}$. Then the object $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M)$ (living in the semisimple category $\langle \mathrm{Gr}^W M\rangle^{\otimes}$) is a direct sum of simple objects. By $(IA1)_{\{p,q\}}$, each simple direct factor either lives in (I) or (II).

On the other hand, if $(IA2)_{\{p,q\}}$ holds, then each nonzero graded component $Gr_n^W \underline{\mathfrak{u}}_{\geq q}(M)$ must live in (I) or (II) (whichever has a nonzero weight n part). □

5.6. We are ready to give the proof of Theorem 5.3.1. We may assume that M/W_pM is not zero. Consider $\mathscr{E}_p(M)$ as an extension of the unit object by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$, given by (19). In view of Section 5.4 and Lemma 3.4.2, it is enough to check right exactness of the sequence obtained by applying $\mathcal{U}_{\geq q}(M, \omega)$-invariance to $\omega(\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M))$ for a suitably chosen fiber functor ω. Let ω_0 be an arbitrary fiber functor. We shall take the composition

$$\omega^{\mathrm{gr}} : \boldsymbol{T} \xrightarrow{\mathrm{Gr}^W} \boldsymbol{T} \xrightarrow{\omega_0} \mathbf{Mod}_K$$

as our fiber functor ω.

Via the identification

$$\underline{\mathrm{Hom}}(M/W_pM, M) \subset \underline{\mathrm{End}}(M),$$

we think of the image under ω^{gr} of every subobject of $\underline{\mathrm{Hom}}(M/W_pM, M)$ as a subspace of $\omega^{\mathrm{gr}}\underline{\mathrm{End}}(M)$. Throughout, we shall write the elements of

$$\omega^{\mathrm{gr}}\underline{\mathrm{End}}(M) = \mathrm{End}(\omega^{\mathrm{gr}} M) = \mathrm{End}\Big(\bigoplus_n \omega_0 \,\mathrm{Gr}_n^W M\Big) = \bigoplus_{i,j} \mathrm{Hom}(\omega_0 \,\mathrm{Gr}_j^W M, \omega_0 \,\mathrm{Gr}_i^W M)$$

as 2 by 2 block matrices with rows (resp. columns) broken up as $\{i : i \le p\} \cup \{i : i > p\}$ (resp. the same with j replacing i). Then an element

$$f \in \omega^{\mathrm{gr}}\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger} = \mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}$$

looks like

$$\begin{pmatrix} 0 & * \\ 0 & \lambda(f)I \end{pmatrix}.$$

The surjective arrow

$$\mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger} \to K$$

in $\omega^{\mathrm{gr}}\mathscr{E}_p$ sends f to $\lambda(f)$.

Consider the element

$$f_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \in \mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}.$$

We will show that if conditions (i) or (ii) of Theorem 5.3.1 hold (and $q \le p$), then the element $f_0 + \omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)$ of

$$\frac{\mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}}{\omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)}$$

is fixed by $\mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})$; this proves surjectivity of

$$\left(\frac{\mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}}{\omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)}\right)^{\mathcal{U}_{\ge q}(M,\omega^{\mathrm{gr}})} \to K$$

and hence the theorem. Since $\mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})$ is unipotent, it is enough to verify that $f_0 + \omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)$ is fixed by every $\sigma \in \mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})(K) \subset \mathrm{GL}(\omega^{\mathrm{gr}}M)$. Given such a σ, we must show that

$$\sigma f_0 \sigma^{-1} - f_0 \in \omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M) = \mathfrak{u}_p(M, \omega^{\mathrm{gr}}). \tag{24}$$

Writing

$$\sigma = \begin{pmatrix} \sigma_1 & A \\ 0 & \sigma_2 \end{pmatrix},$$

we have

$$\log(\sigma) = \begin{pmatrix} \log\sigma_1 & * \\ 0 & \log\sigma_2 \end{pmatrix} \in \mathfrak{u}_{\ge q}(M, \omega^{\mathrm{gr}}) = \omega_0 \,\mathrm{Gr}^W \underline{\mathfrak{u}}_{\ge q}(M).$$

Applying ω_0 to the decomposition of $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\ge q}(M)$ given in Lemma 5.5.1, it follows that

$$\begin{pmatrix} \log\sigma_1 & 0 \\ 0 & \log\sigma_2 \end{pmatrix} \in \mathfrak{u}_{\ge q}(M, \omega^{\mathrm{gr}}),$$

so that

$$\delta := \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \in \mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})(K).$$

We thus have

$$\sigma f_0\sigma^{-1} - f_0 = \begin{pmatrix} 0 & A\sigma_2^{-1} \\ 0 & 0 \end{pmatrix} = \log(\sigma\delta^{-1}) \in \mathfrak{u}_{\geq q}(M, \omega^{\mathrm{gr}}).$$

We have shown that $\sigma f_0\sigma^{-1} - f_0$ is in $\mathfrak{u}(M, \omega^{\mathrm{gr}})$. Being an element of the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$, it will be actually in $\mathfrak{u}_p(M, \omega^{\mathrm{gr}})$, as desired.

5.7. We end this section with a variant of Theorem 5.3.1 for $q > p$, which again gives a sufficient condition to guarantee that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from the category $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.[9] two categories $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$ and $\langle W_p M, M/W_p M\rangle^{\otimes}$ necessarily contains the other.

For $q > p$, consider the following three sets:

$$J'^{\{p,q\}}_1 := \{(i, j) \in \mathbb{Z}^2 : i \leq p, j > q\}.$$
$$J'^{\{p,q\}}_2 := \{(i, j) \in \mathbb{Z}^2 : p < i \leq q < j\}.$$
$$J'^{\{p,q\}}_3 := \{(i, j) \in \mathbb{Z}^2 : q < i < j\}.$$

Say an object M of $\boldsymbol{T}$ satisfies $(IA1')_{\{p,q\}}$ if the objects

$$\bigoplus_{(i,j)\in J'^{\{p,q\}}_k} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M)$$

for $k = 1, 2, 3$ have no nonzero isomorphic subobjects. We say M satisfies $(IA2')_{\{p,q\}}$ if the sets of weights of these objects are disjoint. Then $(IA2')_{\{p,q\}}$ implies $(IA1')_{\{p,q\}}$, and $(IA3)$ implies $(IA2')_{\{p,q\}}$ for every p, q.

Theorem 5.7.1. *Let $q > p$. Suppose one of the following statements holds*:

(i) $\mathrm{Gr}^W M$ *is semisimple and M satisfies* $(IA1')_{\{p,q\}}$.

(ii) *M satisfies* $(IA2')_{\{p,q\}}$.

Then the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.

Proof. The proof is similar to the proof of Theorem 5.3.1. Note that the pairs (i, j) appearing in (23) are those in $J'^{\{p,q\}}_1 \cup J'^{\{p,q\}}_2 \cup J'^{\{p,q\}}_3$. Similar to Lemma 5.5.1, hypothesis (i) or (ii) above imply that $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M)$ is the direct sum of its intersections with the three objects

$$\bigoplus_{(i,j)\in J'^{\{p,q\}}_k} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M) \tag{25}$$

for $k = 1, 2, 3$. Taking ω^{gr} and f_0 as in the proof of Theorem 5.3.1, we shall show that for every $\sigma \in \mathscr{U}_{\geq q}(M, \omega^{\mathrm{gr}})(K)$,

$$\sigma f_0\sigma^{-1} - f_0 \in \mathfrak{u}(M, \omega^{\mathrm{gr}})$$

[9]The content of this subsection will not be used anywhere else in the paper. A reader mainly interested in the application to motives may skip to Section 6.

(it will then automatically be in $\mathfrak{u}_p(M, \omega^{\mathrm{gr}})$). Decompose

$$\log \sigma = \tau_1 + \tau_2 + \tau_3,$$

where τ_k is the component in (25); each τ_k is in $\mathfrak{u}_{\geq q}(M, \omega^{\mathrm{gr}})$, thanks to hypothesis (i) or (ii). Writing the elements of $\mathrm{End}(\omega^{\mathrm{gr}} M)$ as 3×3 block matrices with the rows (resp. columns) broken up as $\{i : i \leq p\} \cup \{i : p < i \leq q\} \cup \{i : i > q\}$ (resp. the same with j replacing i), we have

$$\log(\sigma) = \begin{pmatrix} 0 & & \tau_1 \\ & 0 & \tau_2 \\ & & \tau_3 \end{pmatrix}$$

(with zero missing entries), so that

$$\sigma = \begin{pmatrix} I & & \tau_1(\exp(\tau_3) - 1)/\tau_3 \\ & I & \tau_2(\exp(\tau_3) - 1)/\tau_3 \\ & & \exp(\tau_3) \end{pmatrix} \quad \text{and} \quad \sigma^{-1} = \begin{pmatrix} I & & \tau_1(\exp(-\tau_3) - 1)/\tau_3 \\ & I & \tau_2(\exp(-\tau_3) - 1)/\tau_3 \\ & & \exp(-\tau_3) \end{pmatrix},$$

where for brevity, for a nilpotent map N we have set

$$(\exp(N) - 1)/N := \sum_{n \geq 0} N^n/(n+1)!.$$

Then one calculates

$$\sigma f_0 \sigma^{-1} - f_0 = \begin{pmatrix} 0 & & \tau_1(1 - \exp(-\tau_3))/\tau_3 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

This belongs to $\mathfrak{u}(M, \omega^{\mathrm{gr}})$ because τ_1, τ_3 are in the Lie algebra $\mathfrak{u}(M, \omega^{\mathrm{gr}})$ and

$$[\tau_1, \tau_3] = \tau_1 \tau_3, \quad [[\tau_1, \tau_3], \tau_3] = \tau_1 \tau_3^2, \quad \ldots. \qquad \square$$

6. Motives with large unipotent radicals of motivic Galois groups

6.1. In this section, unless otherwise indicated, $\boldsymbol{T}$ is any reasonable Tannakian category of mixed motives in characteristic zero, or the category of mixed Hodge structures. Examples of the former include the (now known to be equivalent [Choudhury and Gallauer Alves de Souza 2017]) Tannakian categories of mixed motives over a subfield of $\mathbb{C}$ due to Nori [Huber and Müller-Stach 2017] and Ayoub [2014a; 2014b], Voevodsky's category of mixed Tate motives over $\mathbb{Q}$ (or those over $\mathbb{Z}$, etc.), and categories of mixed motives defined via realizations (see [Deligne 1989] or [Jannsen 1990]). See the remark at the end of this section for what we exactly need of $\boldsymbol{T}$. We shall use the word *motive* to refer to any object of $\boldsymbol{T}$ whose weight associated graded is semisimple. Of course, in the case that $\boldsymbol{T}$ is a reasonable category of mixed motives, this will simply mean an arbitrary object of $\boldsymbol{T}$. In the case of the category of mixed Hodge structures, this will include (graded-) polarizable objects, and in particular, the Hodge realizations of mixed motives.

Let M be a motive. We say $\underline{\mathfrak{u}}(M)$ is *large* (or that M has a large $\underline{\mathfrak{u}}$) if

$$\underline{\mathfrak{u}}(M) = W_{-1}\underline{\mathrm{End}}(M).$$

Similarly, we say $\underline{\mathfrak{u}}_p(M)$ is large if

$$\underline{\mathfrak{u}}_p(M) = \underline{\mathrm{Hom}}(M/W_pM, W_pM).$$

Then $\underline{\mathfrak{u}}(M)$ is large if and only if $\underline{\mathfrak{u}}_p(M)$ is large for every p. The interest in motives with large $\underline{\mathfrak{u}}$ is partly because of Grothendieck's period conjecture. If $\boldsymbol{T}$ is a good category of motives over a number field, among the motives with a fixed associated graded, the periods of a motive with large $\underline{\mathfrak{u}}$ should generate a field with the largest possible transcendence degree. We refer the reader to [André 2004] for a detailed discussion of Grothendieck's period conjecture.

Our main goal in this section is to use the earlier results of the paper to obtain motives with large $\underline{\mathfrak{u}}$ and three weights. We will be particularly interested in motives M with three weights $-2n < p < 0$, associated graded isomorphic to

$$\mathbb{Q}(n) \oplus A \oplus \mathbb{1}$$

where A is a given pure motive of weight p, and such that $\underline{\mathfrak{u}}(M)$ is large. We shall prove a precise classification result for such motives in terms of homological algebra, which completely classifies such motives up to isomorphism when $n \neq -p$ and $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$ (e.g., for even n if $\boldsymbol{T}$ is any reasonable category of motives over $\mathbb{Q}$). The condition $n \neq -p$ here is an independence axiom (referring to the language of the previous section). See Corollary 6.7.1 for the precise statement of the classification result. As an example, in Section 6.8 we shall consider the case where A is the simple Tate motive $\mathbb{Q}(k)$ and construct certain interesting mixed Tate motives over $\mathbb{Q}$.

It turns out that the machinery we shall need works in more generality with little extra effort. So we have decided to develop the results in more generality first and then apply them to the case of motives with three weights. We shall however start with the simplest case below, i.e., motives with only two weights; the observations made in this case will be useful when we deal with more than two weights.

Remark. Our restriction to the categories of motives and mixed Hodge structures here is for reasons to do with motivation and applications. Unless we explicitly say otherwise, the discussions can be assumed to take place in the following setting: Take $\boldsymbol{T}$ to be any Tannakian category over a field K of characteristic zero, equipped with a weight filtration (as in previous sections), and interpret the word "motive" as an object of $\boldsymbol{T}$ whose associate graded with respect to the weight filtration is semisimple. In discussions where the Tate objects $\mathbb{Q}(n)$ make an appearance, $\mathbb{Q}(n)$ may denote any object of weight $-2n$ and dimension 1 (even if $K \neq \mathbb{Q}$).

6.2. We shall use the following terminology: an extension of $\mathbb{1}$ by an object L is *totally nonsplit* if its pushforward to any nonzero quotient of L is nontrivial (= nonsplit); dually, we say an extension of an object L by $\mathbb{1}$ is totally nonsplit if its pullback to any nonzero subobject of L is nontrivial. Note that if L is simple, then "totally nonsplit" and "nonsplit" are equivalent.

Suppose M is an object with two weights, fitting in a short exact sequence

$$0 \to L \to M \to \mathbb{1} \to 0, \tag{26}$$

where L is a pure motive of weight $p < 0$.[10] Then

$$W_{-1}\underline{\mathrm{End}}(M) = \underline{\mathrm{Hom}}(\mathbb{1}, L) \cong L.$$

By Theorem 4.9.1 (or Deligne's Theorem 4.3.1 or [Hardouin 2011, Theorem 2], see also the latter's predecessors, [Bertrand 2001, Theorem 1.1] and [Hardouin 2006, Théorème 2.1]), $\underline{\mathfrak{u}}(M)$ (= $\underline{\mathfrak{u}}_p(M)$) is the smallest subobject of L such that the pushforward of the extension (26) to $\mathrm{Ext}^1(\mathbb{1}, L/\underline{\mathfrak{u}}(M))$ splits. (Indeed, note that via the identification of $\underline{\mathrm{Hom}}(\mathbb{1}, L)$ and L, the extension $\mathscr{E}_p(M)$ appearing in Theorem 4.9.1 is simply (26). Also note that the total class $\mathscr{E}(M)$ of M is a nonzero multiple of $\mathscr{E}_p(M)$.) Thus $\underline{\mathfrak{u}}(M)$ is large if and only if (26) is totally nonsplit. In particular, if L is simple, then

$$\underline{\mathfrak{u}}(M) = \begin{cases} L & \text{if } M \text{ is not semisimple,} \\ 0 & \text{if } M \text{ is semisimple.} \end{cases}$$

Remark. Let $\boldsymbol{T}$ be any Tannakian category over K with a weight filtration. Then for any object M of $\boldsymbol{T}$ with semisimple $\mathrm{Gr}^W M$ the following statements are equivalent:

(i) $\underline{\mathfrak{u}}(M)$ is zero.

(ii) M is semisimple.

(iii) M is isomorphic to $\mathrm{Gr}^W M$.

Indeed, choosing a fiber functor one easily sees (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) (note that among these the implication (i) $\Rightarrow$ (ii) is the only one that needs the assumption of semisimplicity of $\mathrm{Gr}^W M$). This gives another argument for the characterization of $\underline{\mathfrak{u}}(M)$ given above when L is simple.

6.3. In this section we will use the results of Sections 4 and 5 to give a criterion for a motive to have a large $\underline{\mathfrak{u}}$ in terms of its subobjects and subquotients.

Theorem 6.3.1. *Let $p < 0$ and M be a motive such that*

$$M/W_pM \simeq \mathbb{1}, \quad \mathrm{Gr}^W_p M \neq 0 \tag{27}$$

(*so that in particular*, 0 *and* p *are the highest two weights of* M). *Suppose moreover that*:

(i) $\underline{\mathfrak{u}}(W_pM)$ *is large.*

(ii) $\underline{\mathfrak{u}}(M/W_{p-1}M)$ *is large.*

(iii) M *satisfies* $(IA1)_{\{p,q\}}$ *for all* $q \leq p$.

Then $\underline{\mathfrak{u}}(M)$ *is large.*

[10] Note that this makes M also a motive (as $\mathrm{Gr}^W M \simeq L \oplus \mathbb{1}$ is semisimple).

Proof. Note that since M/W_pM is pure, for any choice of fiber functor ω, we have $\mathcal{U}_{\geq p}(M,\omega) = \mathcal{U}_p(M,\omega)$. Indeed, if σ is in $\mathcal{G}(M,\omega)$, then $\sigma_{\mathrm{Gr}^W M}$ and σ_{W_pM} are both identity if and only if $\sigma_{\mathrm{Gr}^W(M/W_pM)}$ and σ_{W_pM} are both identity, and by purity $\mathrm{Gr}^W(M/W_pM) \simeq M/W_pM$. Thus the kernel of the surjection

$$\underline{\mathfrak{u}}(M) \to \underline{\mathfrak{u}}(W_pM)$$

induced by the inclusion $\langle W_pM\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$ is $\underline{\mathfrak{u}}_p(M)$. In light of purity of M/W_pM, from this it follows that $\underline{\mathfrak{u}}(M)$ is large if and only if $\underline{\mathfrak{u}}(W_pM)$ and $\underline{\mathfrak{u}}_p(M)$ are both large.

In view of hypothesis (iii) and the fact that M is a motive, Corollary 5.3.2 tells us that $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that $\mathcal{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. Fix an isomorphism between M/W_pM and $\mathbb{1}$ to identify the two objects. Then

$$\underline{\mathfrak{u}}_p(M) \subset \underline{\mathrm{Hom}}(M/W_pM, W_pM) = \underline{\mathrm{Hom}}(\mathbb{1}, W_pM) \cong W_pM.$$

Via the latter identification, the extension

$$\mathcal{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(\mathbb{1}, W_pM)) = \mathrm{Ext}^1(\mathbb{1}, W_pM)$$

is simply the canonical extension

$$0 \to W_pM \to M \to \mathbb{1} \to 0, \tag{28}$$

where the surjective arrow is the quotient map $M \to M/W_pM = \mathbb{1}$. Let A be any subobject of W_pM such that $\mathcal{E}_p(M)/A$ splits. The goal is to show that $A = W_pM$.

Modding out by $W_{p-1}M$, the extension (28) pushes forward to

$$0 \to \mathrm{Gr}_p^W M \to M/W_{p-1}M \to \mathbb{1} \to 0. \tag{29}$$

By Section 6.2, $\underline{\mathfrak{u}}(M/W_{p-1}M)$ is large if and only if this extension is totally nonsplit. In view of hypothesis (ii), it follows that we must have

$$A + W_{p-1}M = W_pM. \tag{30}$$

Indeed, otherwise, by modding out (28) by $A + W_{p-1}M$ we see that the pushforward of (29) to a nonzero subquotient of $\mathrm{Gr}_p^W M$ splits, contradicting the fact that (29) is totally nonsplit.

Now consider the diagram:

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \cap & \searrow & & & \\ 0 & \longrightarrow & W_{p-1}M & \longrightarrow & W_pM & \longrightarrow & \mathrm{Gr}_p^W M & \longrightarrow & 0 \end{array}$$

We just saw that diagonal arrow is surjective. It follows that the extension in the diagram is the pushforward of an extension of $\mathrm{Gr}_p^W M$ by $A \cap W_{p-1}M$ (under inclusion map). Thus the extension

$$\mathcal{E}_{p-1}(W_pM) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, W_{p-1}M))$$

is the pushforward of an extension of $\mathbb{1}$ by

$$\underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M) \subset \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, W_{p-1}M),$$

i.e., that

$$\mathscr{E}_{p-1}(W_pM)/\underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M)$$

splits. By Theorem 4.9.1, we get

$$\underline{\mathfrak{u}}_{p-1}(W_pM) \subset \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M).$$

But since $\underline{\mathfrak{u}}(W_pM)$ is large, so is $\underline{\mathfrak{u}}_{p-1}(W_pM)$. Thus

$$\underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M) = \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, W_{p-1}M).$$

Since $\mathrm{Gr}_p^W M$ is nonzero, this implies that $W_{p-1}M \subset A$. Combining with (30) we get that $A = W_pM$, as desired.[11] □

Remark. (1) As pointed out in the proof, hypothesis (ii) of Theorem 6.3.1 is equivalent to the extension (29) being totally nonsplit. If we assume moreover that $\mathrm{Gr}_p M$ is simple, then this is equivalent to $M/W_{p-1}M$ not being semisimple.

(2) Let M be a motive which satisfies (27) (with $p < 0$). It is easy to see that if $\underline{\mathfrak{u}}_p(M)$ is large, then so is $\underline{\mathfrak{u}}(M/W_{p-1}M)$. Indeed, if the latter is not large, then the pushforward of (29) to a nonzero quotient of $\mathrm{Gr}_p^W M$ splits. The same split extension is then the pushforward of (28) to a nonzero quotient of W_pM, so that by Theorem 4.9.1 $\underline{\mathfrak{u}}_p(M)$ is not large.

Now suppose that $\underline{\mathfrak{u}}(M)$ is large. As we observed in the beginning of the proof of Theorem 6.3.1, this implies that both $\underline{\mathfrak{u}}(W_pM)$ and $\underline{\mathfrak{u}}_p(M)$ are large. We record the conclusion:

If M is a motive satisfying (27) (*with* $p < 0$) *and* $\underline{\mathfrak{u}}(M)$ *is large*, *then both* $\underline{\mathfrak{u}}(W_pM)$ *and* $\underline{\mathfrak{u}}(M/W_{p-1}M)$ *are large.*

Note that here we did not need to assume M satisfies any independence axiom. Theorem 6.3.1 asserts that if we further assume that M satisfies the independence axiom given in hypothesis (iii) of the theorem, then the converse to the statement above is also true.

(3) Hypothesis (iii) of the theorem (which was used in the proof to guarantee that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits) is actually important: the statement of the theorem is false if we remove Hypothesis (iii). See Section 6.10 for an example.

6.4. In view of Theorem 6.3.1 one may hope to form motives with large $\underline{\mathfrak{u}}$ by patching together suitable smaller such motives. The goal of the next few subsections is to try to classify, up to isomorphism, all motives M with large $\underline{\mathfrak{u}}$ which satisfy (27) and which, up to isomorphism, have a fixed $W_{p-1}M$ (with

[11] Note that the assumption that $\mathrm{Gr}_p^W M$ is nonzero is actually important for the proof. Thus when we want to apply Theorem 6.3.1 to show that a given motive M has a large $\underline{\mathfrak{u}}$, we do not have a choice about what to take as p; it is determined by the motive M.

large $\underline{u}$) and $\mathrm{Gr}_p^W M$ (with the isomorphisms not part of the data). To this end, let us first consider a related problem. For the discussion in this subsection, $\boldsymbol{T}$ can be any abelian category (we will eventually apply the discussion to our category of motives).

Throughout, we fix objects A, B and C in $\boldsymbol{T}$ (in our final application, these will be respectively (the fixed objects which are to be isomorphic to) $\mathrm{Gr}_p^W M$, $W_{p-1}M$, and $\mathbb{1}$). Grothendieck considers the following problem in [SGA 7_{I} 1972, Section 9.3 of Exposé 9]: Classify all tuples

$$(M;\,(M_i)_{-3\le i\le 0};\,\gamma_0,\,\gamma_{-1},\,\gamma_{-2})$$

where

$$M = M_0 \supset M_{-1} \supset M_{-2} \supset M_{-3} = 0$$

are objects of $\boldsymbol{T}$ and

$$M/M_{-1} = M_0/M_{-1} \xrightarrow{\gamma_0} C,\; M_{-1}/M_{-2} \xrightarrow{\gamma_{-1}} A,\; M_{-2}/M_{-3} = M_{-2} \xrightarrow{\gamma_{-2}} B$$

are isomorphisms. The classification is to be done up to isomorphisms of such tuples, defined in the obvious way. Here it is convenient for us to consider a slight variant of this problem, where we do not include the data of the isomorphisms γ_i in the tuple, but instead just require that the quotients M_0/M_{-1}, M_{-1}/M_{-2} and $M_{-2}/M_{-3} = M_{-2}$ are isomorphic to C, A and B, respectively.

We say that a pair of extension classes

$$(\mathscr{L}, \mathscr{N}) \in \mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(C, A)$$

is *compatible* if there is a commutative diagram in $\boldsymbol{T}$

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & L & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{31}$$

where the rows and columns are exact, the first (complete) row represents $\mathscr{L}$, and the second (complete) column represents $\mathscr{N}$. We say an object M is *attached to* the pair $(\mathscr{L}, \mathscr{N})$ if it fits in a diagram as above. Note that if we have a diagram as above, (by adjusting the maps when needed) we may replace the first row (resp. second column) by any other representative of $\mathscr{L}$ (resp. $\mathscr{N}$).

In [SGA 7_{I} 1972], a diagram as above is called an *extension panachée* of the second column sequence by the top row sequence.[12] Thus to say the pair $(\mathcal{L}, \mathcal{N})$ is compatible amounts to saying that an *extension panachée* of (an or every representative of) $\mathcal{N}$ by (an or every representative of) $\mathcal{L}$ exists, or that $(\mathcal{L}, \mathcal{N})$ is "*panachable*", in the language of [Bertrand 2013].

The theory of Yoneda extensions gives a simple characterization of compatible pairs. Let

$$\circ : \mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(C, A) \to \mathrm{Ext}^2(C, B)$$

be the Yoneda (composition) pairing; it sends the pair $(\mathcal{L}, \mathcal{N})$ with $\mathcal{L}$ given by

$$0 \to B \to L \xrightarrow{\pi} A \to 0 \tag{32}$$

and $\mathcal{N}$ given by

$$0 \to A \xrightarrow{\iota} N \to C \to 0$$

to the extension $\mathcal{L} \circ \mathcal{N}$ given by

$$0 \to B \to L \xrightarrow{\iota\circ\pi} N \to C \to 0.$$

Lemma 6.4.1. (a) *The pair* $(\mathcal{L}, \mathcal{N})$ *is compatible if and only if* $\mathcal{L} \circ \mathcal{N} = 0$.

(b) *Suppose* $\mathrm{Ext}^1(C, B) = 0$. *If* $(\mathcal{L}, \mathcal{N})$ *is compatible, then up to isomorphism there is a unique object attached to it.*

Proof. This is Lemma 9.3.8 of [SGA 7_{I} 1972]. Fix the extension (32) representing $\mathcal{L}$. If M is attached to the pair, fitting into a diagram as in (31), then the class

$$\mathcal{M} \in \mathrm{Ext}^1(C, L)$$

of the first column in the diagram pushes forward to $\mathcal{N}$ under π. Conversely, if $\mathcal{N}$ is in the image of the pushforward

$$\pi_* : \mathrm{Ext}^1(C, L) \to \mathrm{Ext}^1(C, A),$$

with $\mathcal{M}$ represented by

$$0 \to L \to M \to C \to 0$$

in the preimage of $\mathcal{N}$, then the object M is attached to our pair. Thus the pair $(\mathcal{L}, \mathcal{N})$ is compatible if and only if $\mathcal{N}$ is in the image of π_*. Now by the general theory of Yoneda extensions, applying the functor $\mathrm{Hom}(C, -)$ to (32) we get an exact sequence

$$\mathrm{Ext}^1(C, B) \to \mathrm{Ext}^1(C, L) \xrightarrow{\pi_*} \mathrm{Ext}^1(C, A) \xrightarrow{\delta = \mathcal{L}\circ -} \mathrm{Ext}^2(C, B);$$

see [Buchsbaum 1959, Section 3] or [Yoneda 1960, page 561]. This proves part (a).

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[12] Or as Bertrand translates in [Bertrand 1998], a *blended* extension.

As for the statement in part (b), if M and M' are attached to $(\mathscr{L}, \mathscr{N})$, fitting into diagrams as in (31) with the classes of the corresponding first columns denoted by $\mathscr{M}$ and $\mathscr{M}'$ (both in $\mathrm{Ext}^1(C, L)$) respectively, then it follows from the above long exact sequence that $\mathscr{M}$ and $\mathscr{M}'$ differ by an element in the image of $\mathrm{Ext}^1(C, B)$. If this Ext group is zero, then $\mathscr{M} = \mathscr{M}'$, and hence in particular M and M' are (noncanonically) isomorphic. □

Remark. In a reasonable Tannakian category of mixed motives over a number field it is expected that one should have $\mathrm{Ext}^2(X, Y) = 0$ for every objects X and Y. So in that context, every pair should be compatible. See the remark in the end of Section 6.7 for a more detailed discussion of the Ext groups in our particular categories of interest.

6.5. We shall continue in the setting of the previous subsection ($\boldsymbol{T}$ any abelian category, and B, A, C three fixed objects of $\boldsymbol{T}$). Our goal in this subsection is to see when the same object is attached to two compatible pairs of extensions.

We use the notation End( ) (resp. Aut( )) for the endomorphism algebra (resp. automorphism group) of an object in $\boldsymbol{T}$. The endomorphism algebra $\mathrm{End}(A)$ of A acts on both $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(C, A)$. Indeed, the action on $\mathrm{Ext}^1(A, B)$ is a right action given by pullback: if f is an endomorphism of A, set $\mathscr{L} \cdot f := f^*\mathscr{L}$ (f^* for pullback along f). The action on $\mathrm{Ext}^1(C, A)$ is a left action given by push forward: $f \cdot \mathscr{N} := f_*\mathscr{N}$ (to see the bilinearity properties of these actions, see [Buchsbaum 1959] or [Yoneda 1960]). If f is an automorphism of A, then $\mathscr{L} \cdot f$ and $f \cdot \mathscr{N}$ are simply obtained by twisting respectively the surjective and injective arrows of $\mathscr{L}$ and $\mathscr{N}$ by f^{-1}, i.e., $\mathscr{L} \cdot f$ (resp. $f \cdot \mathscr{N}$) is the class of the extension obtained by replacing the surjective (resp. injective) arrow π (resp. ι) in a representative of $\mathscr{L}$ (resp. $\mathscr{N}$) by $f^{-1} \circ \pi$ (resp. $\iota \circ f^{-1}$).

We restrict the two actions above on $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(C, A)$ to the actions of the group $\mathrm{Aut}(A)$. We also modify the action on $\mathrm{Ext}^1(C, A)$ so that it also becomes a right action, by setting $\mathscr{N} \cdot f := f_*^{-1}\mathscr{N}$. Thus $\mathscr{N} \cdot f$ is the class of the extension obtained by twisting the injective arrow of $\mathscr{N}$ by f. Similarly, we have right actions of $\mathrm{Aut}(B)$ (resp. $\mathrm{Aut}(C)$) on $\mathrm{Ext}^1(A, B)$ (resp. $\mathrm{Ext}^1(C, A)$).

We now equip the product

$$\mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(C, A) \tag{33}$$

with the following right actions of $\mathrm{Aut}(B)$, $\mathrm{Aut}(A)$, and $\mathrm{Aut}(C)$: the group $\mathrm{Aut}(B)$ (resp. $\mathrm{Aut}(C)$) acts by acting on the first (resp. second) factor, and $\mathrm{Aut}(A)$ acts diagonally, i.e., by the formula

$$(\mathscr{L}, \mathscr{N}) \cdot f := (\mathscr{L} \cdot f, \mathscr{N} \cdot f) = (f^*\mathscr{L}, f_*^{-1}\mathscr{N}).$$

The three actions commute with one another. Indeed, the actions of $\mathrm{Aut}(B)$ and $\mathrm{Aut}(C)$ trivially commute, and the commutativity of the actions of $\mathrm{Aut}(A)$ with each of $\mathrm{Aut}(B)$ and $\mathrm{Aut}(C)$ is clear from the description of the actions in terms of twisting the arrows, as different groups act by twisting different arrows. Thus we get an action of $\mathrm{Aut}(B) \times \mathrm{Aut}(A) \times \mathrm{Aut}(C)$ on the product (33). We say two pairs of extensions are *equivalent* if they are in the same orbit of this action.

Lemma 6.5.1. *Let* $(\mathscr{L}, \mathscr{N})$ *and* $(\mathscr{L}', \mathscr{N}')$ *be in* (33).

(a) *Suppose* $(\mathscr{L}, \mathscr{N})$ *and* $(\mathscr{L}', \mathscr{N}')$ *are equivalent. Then every object attached to the pair* $(\mathscr{L}, \mathscr{N})$ *is also attached to the pair* $(\mathscr{L}', \mathscr{N}')$. (*In particular*, $(\mathscr{L}, \mathscr{N})$ *is compatible if and only if* $(\mathscr{L}', \mathscr{N}')$ *is compatible.*)

(b) *Suppose every object of* $\boldsymbol{T}$ *is equipped with an exact functorial increasing filtration* $W_\bullet$ *which is finite on every object* (*we refer to this as the weight filtration*). *Suppose moreover that the highest weight of* B *is less than the lowest weight of* A, *and that the highest weight of* A *is less than the lowest weight of* C. *Then if there is an object* M *attached to both* $(\mathscr{L}, \mathscr{N})$ *and* $(\mathscr{L}', \mathscr{N}')$, *then the two pairs are equivalent.*

Proof. (a) Let $(\mathscr{L}', \mathscr{N}') = (\mathscr{L}, \mathscr{N}) \cdot (f_B, f_A, f_C)$ for some $f_B \in \mathrm{Aut}(B)$, $f_A \in \mathrm{Aut}(A)$, and $f_C \in \mathrm{Aut}(C)$. Suppose M is attached to $(\mathscr{L}, \mathscr{N})$. In a diagram as in (31) (with the first row and second column respectively representing $\mathscr{L}$ and $\mathscr{N}$), twist the arrows $B \to L$ and $B \to M$ by f_B, the arrow $L \to A$ by f_A^{-1}, the arrow $A \to N$ by f_A, and the arrows $M \to C$ and $N \to C$ by f_C, while keeping $L \to M$ and $M \to N$ unchanged. The diagram remains commutative and with exact rows and columns, and its first row (resp. second column) represents $\mathscr{L}'$ (resp. $\mathscr{N}'$).

(b) Suppose an object M is attached to both $(\mathscr{L}, \mathscr{N})$ and $(\mathscr{L}', \mathscr{N}')$. We consider two diagrams as in (31), one with objects L, N with the first row and second column representing $\mathscr{L}$ and $\mathscr{N}$, and the other with objects L', N' with the first row and second column representing $\mathscr{L}'$ and $\mathscr{N}'$. In the diagram for $(\mathscr{L}, \mathscr{N})$, we name the maps as follows: In the first row, (resp. second row, second column) the injective arrow is ι_L (resp. ι_M, ι_N) and the surjective arrow is π_L (resp. π_M, π_N). We refer to the maps $L \to M$ and $M \to C$ as α and β, respectively. Accordingly, denote the maps in the diagram for $(\mathscr{L}', \mathscr{N}')$ by $\iota_{L'}, \pi_{L'}, \iota'_M, \pi'_M, \iota_{N'}, \pi_{N'}, \alpha'$ and β' (each map being the analogue to its lookalike in the first diagram). (Note that the central object in both diagrams in M.)

Let b, a and c be respectively the highest weights of B, A and C. Focusing on the first diagram, using exactness of the weight filtration together with the hypothesis that every weight of B is less than every weight of A, which in turn is less than every weight of C, we see that

$$W_b L = \iota_L(B), \quad W_a L = L, \quad W_b N = 0, \quad W_a N = \iota_N(A), \quad W_c N = N$$

and

$$W_b M = \iota_M(B), \quad W_a M = \alpha(L), \quad W_c M = M.$$

We have similar equalities for the $'$-adorned analogues coming from the second diagram. In particular,

$$\iota_M(B) = \iota'_M(B) = W_b M, \qquad \alpha(L) = \alpha'(L') = W_a M.$$

Thus we get isomorphisms

$$\alpha^{-1}\alpha' : L' \to L \quad \text{and} \quad \iota_M^{-1}\iota'_M : B \to B$$

(uniquely) defined by the property that

$$\alpha(\alpha^{-1}\alpha') = \alpha' \quad \text{and} \quad \iota_M(\iota_M^{-1}\iota'_M) = \iota'_M.$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\iota_{L'}} & L' & \xrightarrow{\pi_{L'}} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_M^{-1}\iota'_M} & & \downarrow{\scriptstyle \alpha^{-1}\alpha'} & & \downarrow{\scriptstyle =:\gamma} & & \\ 0 & \longrightarrow & B & \xrightarrow{\iota_L} & L & \xrightarrow{\pi_L} & A & \longrightarrow & 0, \end{array}$$

where the rows are exact and the vertical arrows are isomorphisms (to see the commutativity of the first square further compose with α). Thus $\mathscr{L}'$ is obtained from $\mathscr{L}$ by twisting ι_L by $\iota_M^{-1}\iota'_M$ and twisting π_L by γ^{-1}.

On the other hand, since we have $\iota_M(B) = \iota'_M(B) = W_bM$, by exactness of the second rows in the diagrams of the two pairs, π_M and π'_M induce isomorphisms

$$\overline{\pi_M} : M/W_bM \to N, \quad \overline{\pi'_M} : M/W_bM \to N'.$$

Similarly, thanks to exactness of the first columns (and on recalling $\alpha(L) = \alpha'(L') = W_aM$), we have isomorphisms

$$\overline{\beta} : M/W_aM \to C, \quad \overline{\beta'} : M/W_aM \to C,$$

induced by β and β', respectively. We now have a commutative diagram

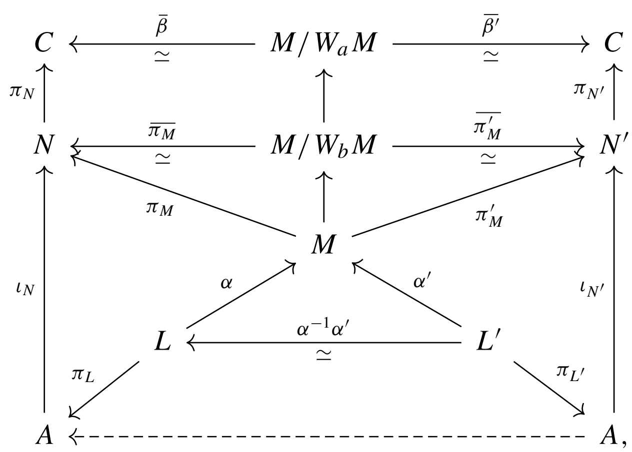
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota_{N'}} & N' & \xrightarrow{\pi_{N'}} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =:\lambda} & & \downarrow{\scriptstyle \overline{\pi_M}\,\overline{\pi'_M}^{-1}} & & \downarrow{\scriptstyle \overline{\beta}\,\overline{\beta'}^{-1}} & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota_N} & N & \xrightarrow{\pi_N} & C & \longrightarrow & 0, \end{array} \tag{34}$$

where the rows are exact and vertical arrows are isomorphisms (to see commutativity of the second square precompose with $\pi'_M : M \to N'$). It follows that $\mathscr{N}'$ is obtained from $\mathscr{N}$ by twisting ι_N by λ and twisting π_N by $\overline{\beta'}\,\overline{\beta}^{-1}$.

To complete the proof, it suffices to show that $\gamma = \lambda$, as then

$$(\mathscr{L}', \mathscr{N}') = (\mathscr{L}, \mathscr{N}) \cdot (\iota_M^{-1}\iota'_M, \gamma, \overline{\beta}\,\overline{\beta'}^{-1}).$$

Ignoring the dashed arrow, we have a commutative diagram



where the vertical arrows in the middle are the obvious maps. The map γ is the unique map that if it is included as the dashed arrow, it makes the bottom trapezoid of the diagram commute. But from the diagram we easily see that λ also does this job. Indeed, to check commutativity of the trapezoid with λ as the dashed arrow, it is enough to check commutativity after composing with ι_N. Now using commutativity of the rest of the diagram above and the left square in (34), we have

$$\iota_N \pi_L(\alpha^{-1}\alpha') = \pi_M \alpha' = \iota_N \lambda \pi_{L'}. \qquad \square$$

6.6. We now combine the results of the previous two subsections on compatible pairs. We shall assume that $\boldsymbol{T}$ is an abelian category equipped with a weight filtration (i.e., a functorial, exact, increasing filtration which is finite on every object). As in the previous two subsections, B, A, C are fixed objects of $\boldsymbol{T}$. The following result, which for future reference we record as a proposition, has been mostly already proved in the previous two subsections.

Proposition 6.6.1. *Suppose every weight of B is less than every weight of A, and that every weight of A is less than every weight of C. Let b, a, c be the highest weights of B, A, C, respectively.*

(a) *Any pair of extensions* $(\mathscr{L}, \mathscr{N})$ *in* (33) *is compatible if and only if*

$$\mathscr{L} \circ \mathscr{N} = 0 \quad \textit{in } \operatorname{Ext}^2(C, B).$$

(b) *If M is an object that is attached to some pair of extensions in* (33), *then we have*

$$B \simeq W_b M, \qquad A \simeq W_a M / W_b M, \qquad C \simeq M / W_a M. \tag{35}$$

(c) *Any object M satisfying* (35) *is attached to some pair* $(\mathscr{L}, \mathscr{N})$ *of extensions in* (33). *Moreover, M is attached to any other pair* $(\mathscr{L}', \mathscr{N}')$ *if and only if* $(\mathscr{L}', \mathscr{N}')$ *is equivalent to* $(\mathscr{L}, \mathscr{N})$. *We have a* (*well-defined*) *surjective map*

$$\left\{\begin{matrix}\textit{the collection of objects } M \textit{ satisfying} \\ (35), \textit{ up to isomorphism}\end{matrix}\right\} \to \left\{\begin{matrix}\textit{the collection of compatible pairs} \\ \textit{in } (33), \textit{ up to equivalence}\end{matrix}\right\}$$

which sends the isomorphism class of M to the equivalence class of any pair (*or all pairs*) $(\mathscr{L}, \mathscr{N})$ *to which M is attached.*

(d) *If* $\operatorname{Ext}^1(C, B) = 0$, *then the surjection above is a bijection.*

Proof. (a) This is Lemma 6.4.1(a).

(b) This follows from the observations made at the beginning of the proof of Lemma 6.5.1(b) about the weight filtration of M. (Note that the isomorphisms are noncanonical, as they depend on the particular choice of diagram (31).)

(c) Given M satisfying (35), we have a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_bM & \longrightarrow & W_aM & \longrightarrow & W_aM/W_bM & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_bM & \longrightarrow & M & \longrightarrow & M/W_bM & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M/W_aM & = & M/W_aM & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(with obvious maps, exact rows and columns). Now use some choice of isomorphisms (35) to replace W_bM, W_aM/W_bM, and M/W_aM respectively by B, A, and C. Take $\mathscr{L}$ (resp. $\mathscr{N}$) to be the extension class of the top row (resp. last column) in the new diagram. Then M is attached to the (compatible) pair $(\mathscr{L}, \mathscr{N})$. By Lemma 6.5.1(b), M is attached to another pair $(\mathscr{L}', \mathscr{N}')$ if and only if $(\mathscr{L}', \mathscr{N}')$ is equivalent to $(\mathscr{L}, \mathscr{N})$. On the other hand, if M' is isomorphic to M, then M' is clearly attached to the same pairs as M. Thus we have a well-defined map as in the statement. It is surjective by the definition of compatibility and Part (b).

(d) This follows from Lemmas 6.4.1(b) and 6.5.1(a). □

6.7. We now return to the discussion of motives with large $\underline{\mathfrak{u}}$ (with $\boldsymbol{T}$ again a Tannakian category of mixed motives or the category of rational mixed Hodge structures). Given any two motives A and B, let us say an extension class in $\operatorname{Ext}^1(A, B)$ has a large $\underline{\mathfrak{u}}$ if the object in the middle of a representing short exact sequence has a large $\underline{\mathfrak{u}}$. This is clearly well-defined, and moreover, the property of having a large $\underline{\mathfrak{u}}$ is invariant under the action of $\operatorname{Aut}(A) \times \operatorname{Aut}(B)$ (because the collection of the objects that can appear as the middle object for two extension classes in the same orbit are the same, as by twisting the arrows we can turn a representative of one extension class to a representative of another extension class in the same orbit). Note that if A is simple (resp. pure), then an extension class in $\operatorname{Ext}^1(\mathbb{1}, A)$ has a large $\underline{\mathfrak{u}}$ if and only if it is nonsplit (resp. totally nonsplit).

We say a pair of extensions $(\mathscr{L}, \mathscr{N})$ in (33) has a large $\underline{\mathfrak{u}}$ if both extensions $\mathscr{L}$ and $\mathscr{N}$ have a large $\underline{\mathfrak{u}}$. This property is invariant under our notion of equivalence of pairs.

We now fix an integer $p < 0$, and motives B and A with

$$B = W_{p-1}B, \quad A \cong \operatorname{Gr}^W_p A \neq 0.$$

(In other words, all weights of B are $< p$, and A is nonzero and pure of weight p; note that B may be mixed.) Proposition 6.6.1 gives a surjection (bijection if $\operatorname{Ext}^1(\mathbb{1}, B) = 0$) from the collection of motives

M satisfying

$$W_{p-1}M \simeq B, \quad \mathrm{Gr}^W_p M \simeq A, \quad M/W_pM \simeq \mathbb{1} \tag{36}$$

up to isomorphism to the collection of compatible pairs in

$$\mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(\mathbb{1}, A)$$

(= the kernel of the composition pairing into $\mathrm{Ext}^2(\mathbb{1}, B)$) up to equivalence (i.e., the action of $\mathrm{Aut}(B) \times \mathrm{Aut}(A) \times \mathrm{Aut}(\mathbb{1})$). By Theorem 6.3.1, if $B \oplus A \oplus \mathbb{1}$ satisfies the independence axiom $(IA1)_{\{p,q\}}$ for every $q \le p$, then given any compatible pair $(\mathcal{L}, \mathcal{N})$ with a large $\underline{\mathfrak{u}}$, any object M attached to the pair also has a large $\underline{\mathfrak{u}}$. Conversely, if an object M satisfying (36) has a large $\underline{\mathfrak{u}}$, then so does any pair $(\mathcal{L}, \mathcal{N})$ in the equivalence class of the extension pairs corresponding to M (see item (2) after Theorem 6.3.1; note that here no independence axiom needs to be satisfied).

We record the following special case as a corollary:

Corollary 6.7.1. *Let $-2n < p < 0$ and $p \neq -n$. Let A be a nonzero simple motive of weight p. Suppose moreover that* $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$. *Then there is a bijection*

$$\left\{\begin{matrix}\textit{the collection of objects } M \\ \textit{with } \mathrm{Gr}^W M \simeq \mathbb{Q}(n) \oplus A \oplus \mathbb{1} \\ \textit{and large } \underline{\mathfrak{u}}(M)\textit{, up to} \\ \textit{isomorphism}\end{matrix}\right\} \to \left\{\begin{matrix}\textit{the collection of compatible pairs} \\ \textit{of nonsplit extensions in} \\ \mathrm{Ext}^1(A, \mathbb{Q}(n)) \times \mathrm{Ext}^1(\mathbb{1}, A), \\ \textit{up to equivalence}\end{matrix}\right\}$$

which assigns to the isomorphism class of an object M the equivalence class of the compatible pairs to which M is attached. If we omit the condition $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$, *this map is well-defined and surjective.*

(Note that the condition $p \neq -n$ guarantees $(IA3)$.)

Remark. (1) In any reasonable Tannakian category of mixed motives over a number field, all the Ext^2 groups (and hence all the higher Ext group) are expected to vanish. The Ext^1 groups in such a category should be related to Chow groups and motivic cohomology (and algebraic K-theory). See for instance, the beautiful articles of Nekovar [1994] and Jannsen [1994]. The only case of a Tannakian category of motives where the Ext groups are actually known is the case of the category of mixed Tate motives. See part (2) for a discussion of this case.

(2) Let $\mathbf{MT}(F)$ be Voevodsky's category of mixed Tate motives over a number field F. The Ext^2 groups in $\mathbf{MT}(F)$ are zero, and the groups

$$\mathrm{Ext}^1_{\mathbf{MT}(F)}(\mathbb{1}, \mathbb{Q}(n))$$

are given by the K-theory of the field F modulo torsion, which in turn is described by theorems of Borel and Soulé (and Dirichlet in the case of K_1). In particular, if F is totally real and n is even, the Ext^1 group above vanishes. (See [Deligne and Goncharov 2005] for the precise description of the Ext groups in $\mathbf{MT}(F)$ and the subcategory of mixed Tate motives over the ring of integers of F. Note that if $\mathbf{MM}(F)$ is any category of mixed motives over F for which the full Tannakian subcategory generated by $\mathbb{Q}(1)$

and closed under extensions is equivalent to Voevodsky's $\mathbf{MT}(F)$, then the Ext^1 groups above are the same in $\mathbf{MM}(F)$ and $\mathbf{MT}(F)$.)

(3) In the category **MHS** of rational mixed Hodge structures, the Ext^2 groups vanish; see [Beĭlinson 1986]. The Ext^1 groups in this category are described by the results of Carlson [1980].

6.8. In this subsection, we shall take $\boldsymbol{T}$ to be Voevodsky's category $\mathbf{MT}(\mathbb{Q})$ of mixed Tate motives over $\mathbb{Q}$. As an application of the previous results, we shall classify (up to isomorphism) all 3-dimensional objects of $\mathbf{MT}(\mathbb{Q})$ with three distinct weights, large $\underline{\mathfrak{u}}$, and satisfying an independence axiom; see below for more details.[13] Note that for any 3-dimensional object M of $\mathbf{MT}(\mathbb{Q})$ with three distinct weights and large $\underline{\mathfrak{u}}$, the unipotent radical of the motivic Galois group $\mathcal{G}(M, \omega_B)$ (with ω_B the Betti realization functor) has dimension equal to 3 ($= \dim W_{-1} \operatorname{End}(\omega_B M)$). Since

$$\mathcal{G}(\mathrm{Gr}^W M, \omega_B) \simeq \mathbb{G}_m,$$

the motivic Galois group $\mathcal{G}(M, \omega_B)$ has dimension 4. Thus Grothendieck's period conjecture would predict that the transcendence degree of the field generated by the periods of M should be 4.

Let us first recall the description of the Ext groups between simple objects in $\mathbf{MT}(\mathbb{Q})$ (see [Deligne and Goncharov 2005], for instance)

$$\begin{gathered} \dim \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = \begin{cases} 0 & \text{if } n \text{ is even or } \leq 0 \\ 1 & \text{if } n \text{ is odd and } \geq 3 \end{cases} \\ \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1)) \cong \mathbb{Q}^\times \otimes \mathbb{Q} \end{gathered} \tag{37}$$

moreover, Ext^2 groups all vanish in $\mathbf{MT}(\mathbb{Q})$.

Back to our classification problem, we may assume that the highest weight of our motives is zero. We shall classify all motives with an associated graded of the form

$$\mathbb{Q}(n) \oplus \mathbb{Q}(k) \oplus \mathbb{1} \quad (n > k > 0, n \neq 2k)$$

which have a large $\underline{\mathfrak{u}}$. (The condition $n \neq 2k$ is an independence axiom. The case where $n = k$ is complicated, as then one can no longer use Theorem 6.3.1.) For any such motive, the pair $(\mathcal{L}, \mathcal{N})$ in

$$\mathrm{Ext}^1(\mathbb{Q}(k), \mathbb{Q}(n)) \times \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(k)) \tag{38}$$

associated to it by Corollary 6.7.1 (also see Proposition 6.6.1) has nonsplit entries. In view of the description of the Ext^1 groups in the category, we see that k must be odd and n must be even. We will then have a bijection as in Corollary 6.7.1.

Let us consider the action of $\mathrm{Aut}(\mathbb{Q}(n)) \times \mathrm{Aut}(\mathbb{Q}(k)) \times \mathrm{Aut}(\mathbb{1})$ on (38). Since the automorphism group of every $\mathbb{Q}(a)$ is $\mathbb{Q}^*$, it follows from bilinearity of the actions of $\mathrm{End}(A)$ on $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(B, A)$ (for any A, B in any K-linear category) that the action of $\mathrm{Aut}(\mathbb{Q}(k))$ can be absorbed into the actions of

[13] The classification is then valid in any Tannakian category $\mathbf{MM}(\mathbb{Q})$ of mixed motives over $\mathbb{Q}$ for which the smallest full Tannakian subcategory containing $\mathbb{Q}(1)$ and closed under extensions is equivalent to $\mathbf{MT}(\mathbb{Q})$.

the other two factors: $(\lambda, \gamma, \delta)$ acts the same as $(\lambda\gamma^{-1}, 1, \gamma\delta)$ (where $\lambda, \gamma, \delta \in \mathbb{Q}^*$). It follows that an orbit of the action of $\mathrm{Aut}(\mathbb{Q}(n)) \times \mathrm{Aut}(\mathbb{Q}(k)) \times \mathrm{Aut}(\mathbb{1})$ on (38) coincides with an element of

$$(\mathrm{Ext}^1(\mathbb{Q}(k), \mathbb{Q}(n)) / \mathrm{Aut}(\mathbb{Q}(n))) \times (\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(k)) / \mathrm{Aut}(\mathbb{1})) \tag{39}$$

(with both actions made right actions, as before).

Case I: $k = 1$. Then n is ≥ 4 (and even), and

$$\mathrm{Ext}^1(\mathbb{Q}(k), \mathbb{Q}(n)) \cong \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n-k))$$

is a 1-dimensional vector space over $\mathbb{Q}$, and all its nonzero elements are in the same $\mathrm{Aut}(\mathbb{Q}(n))$-orbit.

The extensions of $\mathbb{1}$ by $\mathbb{Q}(1)$ are the Kummer motives. For each positive rational number r, let

$$[r] \in \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$$

be the extension class arising from the weight filtration of the 1-motive (see [Deligne 1974])

$$K_r := [\mathbb{Z} \xrightarrow{1 \mapsto r} \mathbb{G}_m] \tag{40}$$

(considered as an object of $\mathbf{MT}(\mathbb{Q})$). Then $[r]$ is the element of $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$ corresponding to $r \otimes 1$ under the isomorphism (37). Thus $\{[p] : p \text{ prime} > 0\}$ is a basis of $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$ (over $\mathbb{Q}$). A complete inequivalent set of representatives for the nonzero orbits of the action of $\mathbb{Q}^* = \mathrm{Aut}(\mathbb{1})$ on $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$ is formed by the elements $[r]$, where r runs through all rational numbers > 1 which are not of the form s^a for any $s \in \mathbb{Q}$ and $a \in \mathbb{Z}$ with $a > 1$. In view of Corollary 6.7.1, each such $[r]$ gives a (unique, up to isomorphism) motive $M_{n,r}$ with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to

$$\mathbb{Q}(n) \oplus \mathbb{Q}(1) \oplus \mathbb{1}.$$

These motives are nonisomorphic, and are up to isomorphism, all the motives with associated graded as above and large $\underline{\mathfrak{u}}$.

A discussion of the periods of $M_{n,r}$ is in order. By construction, $W_{-2}M_{n,r}$ is a nontrivial extension of $\mathbb{Q}(1)$ by $\mathbb{Q}(n)$. Being a twist (by $\mathbb{Q}(1)$) of a nontrivial extension of $\mathbb{1}$ by $\mathbb{Q}(n-1)$, the motive $W_{-2}M_{n,r}$ has the period matrix

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n-1) \\ 0 & (2\pi i)^{-1} \end{pmatrix}$$

with respect to suitably chosen bases of Betti and de Rham realizations. (Note that $n-1$ is odd and ≥ 3. That a nontrivial extension of $\mathbb{1}$ by $\mathbb{Q}(n-1)$ has $\zeta(n-1)/(2\pi i)^{n-1}$ as a period follows from the work of Deligne [1989] in the setting of realizations, and later the work of Deligne and Goncharov [2005] in the setting of Voevodsky motives.) One the other hand, $M_{n,r}$ has the Kummer 1-motive K_r as a subquotient (by $W_{-2n}M_{n,r} = W_{-3}M_{n,r}$). With respect to suitably chosen bases of Betti and de Rham realizations, K_r has the period matrix

$$\begin{pmatrix} (2\pi i)^{-1} & (2\pi i)^{-1}\log r \\ 0 & 1 \end{pmatrix};$$

see [Deligne 1974] for the explicit realizations of 1-motives. With respect to suitably chosen bases, the matrix of periods of $M_{n,r}$ looks like

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n-1) & * \\ 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log r \\ 0 & 0 & 1 \end{pmatrix}.$$

As mentioned earlier, Grothendieck's period conjecture predicts the transcendence degree of the field generated over $\mathbb{Q}$ by the periods of $M_{n,r}$ to be 4. Thus assuming the period conjecture, the numbers

$$2\pi i, \quad \log r, \quad \zeta(n-1), \quad \text{and the entry denoted by } *$$

are algebraically independent over $\mathbb{Q}$.

It would be very interesting to somehow calculate the entry $*$ in a period matrix of $M_{n,r}$ as above. As we discussed in the Introduction, when $r \neq 2$, Deligne's work [2010] (and a fortiori Brown's [2012]) does not predict the nature of $*$.

<u>Case II</u>: $k > 1$ and $n \neq k+1$ (so $n \geq k+3$). Then both quotients in (39) are singletons. Thus up to isomorphism, there is a unique motive $Z_{n,k}$ with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(k) \oplus \mathbb{1}$. The subobject $W_{-2k}Z_{n,k}$ (resp. subquotient $Z_{n,k}/W_{-2k-1}Z_{n,k}$) of $Z_{n,k}$ is a nontrivial extension of $\mathbb{Q}(k)$ by $\mathbb{Q}(n)$ (resp. $\mathbb{1}$ by $\mathbb{Q}(k)$). The matrix of periods of $Z_{n,k}$ with respect to suitably chosen bases is of the form

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n-k) & * \\ 0 & (2\pi i)^{-k} & (2\pi i)^{-k}\zeta(k) \\ 0 & 0 & 1 \end{pmatrix}.$$

The period conjecture predicts that $2\pi i$, $\zeta(k)$, $\zeta(n-k)$ and the entry denoted by $*$ are algebraically independent over $\mathbb{Q}$. Again it would be interesting to find what the entry $*$ is. Note that the motive $Z_{n,k}$ is in the subcategory $\mathbf{MT}(\mathbb{Z})$, as from the beginning we may have done the entire discussion of this case in $\mathbf{MT}(\mathbb{Z})$ (as the relevant Ext groups in this case are the same in $\mathbf{MT}(\mathbb{Z})$ and $\mathbf{MT}(\mathbb{Q})$). Thus by Brown's work [2012], all periods of $Z_{n,k}$ will be in the algebra generated by $2\pi i$ and the multiple zeta values.

<u>Case III</u>: $k > 1$ and $n = k+1$. This case is the dual situation to Case I. Here the second factor of (39) is a singleton, and the motives under investigation are classified up to isomorphism by $\mathrm{Aut}(\mathbb{1})$-orbits of $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$. Consider the complete inequivalent set of representatives $\{[r]\}$ for these orbits as in Case I. Then for each r, we get an object $M'_{n,r}$ corresponding to the element of (39) with the orbit of $[r]$ as its first coordinate. The motives $M'_{n,r}$ are nonisomorphic and up to isomorphism, give all motives with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(n-1) \oplus \mathbb{1}$.

The motives obtained in this case are intimately related to the $M_{n,r}$ of Case I. Indeed, ${M'_{n,r}}^\vee(n)$ has a large $\underline{\mathfrak{u}}$ (as the property of having a large $\underline{\mathfrak{u}}$ is invariant under dualizing and tensoring by $\mathbb{Q}(1)$), and its associated graded is isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(1) \oplus \mathbb{1}$. Moreover, the quotient ${M'_{n,r}}^\vee(n)/W_{-2n}$ is isomorphic to the 1-motive K_r given in (40) (as by construction we have $W_{-2k}M'_{n,r} \simeq K_r(k)$, and K_r is isomorphic to its Cartier dual $K_r^\vee(1)$). It follows that $M'^{\vee}_{n,r}(n)$ is isomorphic to $M_{n,r}$ (as they both correspond to the same equivalence class of compatible pairs).

6.9. Let us continue to take $\boldsymbol{T} = \mathbf{MT}(\mathbb{Q})$. The motives of Section 6.8 together with the earlier results of the paper can be used to obtain 4-dimensional mixed Tate motives with 4 weights and a large $\underline{\mathfrak{u}}$.[14] We illustrate this with an example. Let M be the motive $M_{4,r}$ of the previous section, which has associated graded isomorphic to $\mathbb{Q}(4) \oplus \mathbb{Q}(1) \oplus \mathbb{1}$. The weight filtration of M gives an element $\mathscr{L}$ in $\mathrm{Ext}^1(\mathbb{1}, W_{-2}M)$. Let $\mathscr{N}$ a nonzero element of $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(5))$. Since Ext^2 groups vanish in $\mathbf{MT}(\mathbb{Q})$, there is an object in $\mathbf{MT}(\mathbb{Q})$ attached to the pair

$$(\mathscr{L}(5), \mathscr{N}) \in \mathrm{Ext}^1(\mathbb{Q}(5), (W_{-2}M)(5)) \times \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(5)).$$

Note that here, at least a priori, there might be nonisomorphic objects attached to the pair, as $\mathrm{Ext}^1(\mathbb{1}, (W_{-2}M)(5))$ is not zero. Any object $\widetilde{M}$ attached to the pair is 4-dimensional, with associated graded isomorphic to

$$\mathbb{Q}(9) \oplus \mathbb{Q}(6) \oplus \mathbb{Q}(5) \oplus \mathbb{1}.$$

Such $\widetilde{M}$ satisfies $(IA3)$, and hence by Theorem 6.3.1, $\underline{\mathfrak{u}}(\widetilde{M})$ is large (note that both M and $\mathscr{N}$ have a large $\underline{\mathfrak{u}}$). The field generated over $\mathbb{Q}$ by the periods of $\widetilde{M}$ contains $2\pi i$, $\zeta(3)$, $\log r$, the "new period" of M, and $\zeta(5)$. In fact, by the classification of Section 6.8, the quotient $\widetilde{M}/\mathbb{Q}(9)$ (which is easily seen to also have a large $\underline{\mathfrak{u}}$) must be isomorphic to the motive $M'_{6,r}$ (of Case III of Section 6.8), so that the new period of $M'_{6,r}$ will also be a period of $\widetilde{M}$. The period conjecture predicts that the field generated over $\mathbb{Q}$ by the periods of $\widetilde{M}$ should be of transcendence degree 7 $\big(= \binom{4}{2} + 1\big)$, so that $\widetilde{M}$ should have one more new period, which together with the aforementioned six numbers should form an algebraically independent set over $\mathbb{Q}$.

Remark. Note that $k = 5$ is the smallest positive integer such that

$$\mathrm{Gr}^W M(k) \oplus \mathbb{1}$$

satisfies the independence axiom required to be able to use Theorem 6.3.1.

6.10. Hypothesis (iii) of Theorem 6.3.1 was used in the proof to conclude that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. This hypothesis is actually important for the statement of the theorem to remain true. A counterexample to the statement without this condition can be given in the category **MHS** of rational mixed Hodge structures using the work of Jacquinot and Ribet [1987] on deficient (in the sense of [loc. cit.]) points on semiabelian varieties, as we shall discuss below. We shall freely use the basics of the theory of 1-motives (including the realizations of a 1-motive), as introduced by Deligne [1974].

[14] Inductively, one can obtain motives with more and more weights which have a large $\underline{\mathfrak{u}}$.

Consider a tuple (F, A, v, f), where

- F is a number field,
- A is a simple abelian variety over F with $\mathrm{rank}(A(F)) > 0$,
- $v \in A^t(F)$ (where A^t is the dual abelian variety),
- and $f : A^t \to A$ is an isogeny over F,

such that $f(v) - f^t(v) \in A(F)$ is a point of infinite order.[15] Let V be a semiabelian variety over F, an extension of A by $\mathbb{G}_m$, which under the canonical isomorphism

$$\mathrm{Ext}(A, \mathbb{G}_m) \cong A^t$$

corresponds to $v \in A^t(F)$. Denote the projection map $V \to A$ by π. In [Jacquinot and Ribet 1987, Section 4], a point $x_f \in V(F)$ is constructed such that

(i) $\pi(x_f) = f(v) - f^t(v)$, and

(ii) for every nonzero integer n the point x_f is divisible by n in $V(F_n)$, where F_n is the field obtained from F by adjoining the n-torsion subgroup of V (such a point is called a deficient point in [Jacquinot and Ribet 1987]).

Let M be the 1-motive $[\mathbb{Z} \xrightarrow{1 \mapsto x_f} V]$ over F. Fixing an embedding $F \subset \bar{F} \subset \mathbb{C}$, denote the Hodge realization of any 1-motive N over F by TN. Thus TM has weights -2, -1, 0 and

$$W_{-2}TM = H_1(\mathbb{G}_m) \simeq \mathbb{Q}(1), \quad W_{-1}TM = H_1(V), \quad \mathrm{Gr}_0^W TM = \mathbb{1}.$$

We shall see that (with $\boldsymbol{T} = \mathbf{MHS}$) $\underline{\mathfrak{u}}(TM)$ is not large, whereas both $\underline{\mathfrak{u}}(W_{-1}TM)$ and $\underline{\mathfrak{u}}(TM/W_{-2}TM)$ are large. This would provide a counterexample to the statement of Theorem 6.3.1 with hypothesis (iii) of the theorem omitted.

First, let us consider $W_{-1}TM$ and $TM/W_{-2}TM$. The former is a nonsplit extension of the simple Hodge structure $H_1(A)$ by $\mathbb{Q}(1)$ (because v has infinite order), and hence (by a similar argument as in Section 6.2) has a large $\underline{\mathfrak{u}}$. The latter is the Hodge realization of the 1-motive

$$[\mathbb{Z} \xrightarrow{1 \mapsto \pi(x_f)} A].$$

Since $\pi(x_f)$ is a point of infinite order, $TM/W_{-2}TM$ is a nonsplit extension of $\mathbb{1}$ by $H_1(A)$, and hence has a large $\underline{\mathfrak{u}}$.

To see that $\underline{\mathfrak{u}}(TM)$ is not large, let ℓ be a prime number. Given any 1-motive N over F, denote the ℓ-adic realization of N by $T_\ell N$, and let $\Pi_\ell(N)$ be the image of the natural map $\mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(T_\ell N)(\mathbb{Q}_\ell)$. Then Property (ii) above implies that the natural (restriction) map

$$\Pi_\ell(M) \to \Pi_\ell(W_{-1}M) \tag{41}$$

[15] For instance, take $A = A^t$ to be an elliptic curve with complex multiplication by $\mathbb{Z}[i]$, F large enough so that complex multiplication by i is defined over F and $A(F)$ has positive rank, v a point of infinite order in $A^t(F)$, and $f = i$ (so that $f^t = -i$).

(where $W_{-1}M = [0 \to V]$)) is injective (as well as surjective). By the Mumford–Tate conjecture for 1-motives on the unipotent parts (proved by Jossen [2014, Theorem 1]), the Hodge theoretic analogue of this map, i.e., the restriction map

$$\mathcal{G}(TM, \omega_B) \to \mathcal{G}(T(W_{-1}M), \omega_B) \quad (\omega_B = \text{the forgetful fiber functor})$$

is also injective (the two groups above are calculated in **MHS**). Thus $\underline{\mathfrak{u}}_{-1}(TM)$ is zero.

Remark. (1) Here we do not need the full power of the Mumford–Tate conjecture on the unipotent parts to go from the injectivity of (41) to the vanishing of $\underline{\mathfrak{u}}_{-1}(TM)$; just the more basic statement [Bertrand 1998, Theorem 1] is enough. Indeed, [Bertrand 1998, Theorem 1] and injectivity of (41) imply that $W_{-2}\underline{\mathfrak{u}}(TM)$ is zero. It follows that $\underline{\mathfrak{u}}(TM)$ and consequently $\underline{\mathfrak{u}}_{-1}(TM)$ is a pure object of weight -1. On the other hand,

$$\underline{\mathfrak{u}}_{-1}(TM) \subset \underline{\mathrm{Hom}}(TM/W_{-1}TM, W_{-1}TM) \cong W_{-1}TM.$$

It follows that $\underline{\mathfrak{u}}_{-1}(TM)$ is zero (as otherwise, in light of simplicity of $H_1(A)$ the extension $\mathcal{E}_{-2}(W_{-1}TM)$ would split).

(2) Note that the example given in this section shows that in general, without any independence axiom, the individual extensions $\mathcal{E}_p/\underline{\mathfrak{u}}$ need not split (see Corollaries 5.3.2 and 5.3.3 of Theorem 5.3.1, as well as Deligne's Theorem 4.3.1 and the remark after). Indeed, in the above example, $\mathcal{E}_{-1}(TM)/\underline{\mathfrak{u}}(TM)$ does not split: If it did, then by Lemma 5.1.1 so would $\mathcal{E}_{-1}(TM)/\underline{\mathfrak{u}}_{-1}(TM)$. But $\mathcal{E}_{-1}(TM)$ (= $\mathcal{E}_{-1}(TM)/\underline{\mathfrak{u}}_{-1}(TM)$) does not split as x_f is a point of infinite order.

(3) In fact, the example given in this section also shows that in general, $\underline{\mathfrak{u}}$ may not be generated as a Lie algebra by the subobjects $\underline{\mathfrak{u}}_p$. Indeed, with M as above, $\underline{\mathfrak{u}}(TM)$ is not zero (because TM is not semisimple), while both $\underline{\mathfrak{u}}_{-1}(TM)$ and $\underline{\mathfrak{u}}_{-2}(TM)$ are zero. (That the latter is zero can be seen by an argument similar to the one given in part (1): $\underline{\mathfrak{u}}_{-2}(TM)$ is pure of weight -1 and a subobject of $\underline{\mathrm{Hom}}(TM/W_{-2}TM, W_{-2}TM) \cong (TM/W_{-2}TM)^\vee(1)$; the latter object has no nonzero subobject of weight -1.)

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Support theory for Drinfeld doubles of some infinitesimal group schemes

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Consider a Frobenius kernel G in a split semisimple algebraic group, in very good characteristic. We provide an analysis of support for the Drinfeld center $Z(\text{rep}(G))$ of the representation category for G, or equivalently for the representation category of the Drinfeld double of kG. We show that thick ideals in the corresponding stable category are classified by cohomological support, and calculate the Balmer spectrum of the stable category of $Z(\text{rep}(G))$. We also construct a π-point style rank variety for the Drinfeld double, identify π-point support with cohomological support, and show that both support theories satisfy the tensor product property. Our results hold, more generally, for Drinfeld doubles of Frobenius kernels in any smooth algebraic group which admits a quasilogarithm, such as a Borel subgroup in a split semisimple group in very good characteristic.

In this paper we provide an in depth analysis of support theory for the Drinfeld double of a Frobenius kernel $G = \mathbb{G}_{(r)}$ in a sufficiently nice algebraic group $\mathbb{G}$. Equivalently, we study support for the Drinfeld center of the representation category $\text{rep}(G)$. As indicated in the abstract, we calculate the Balmer spectrum of thick prime ideals in the stable category of representations for the double, classify thick ideals in the stable category, and construct π-point style rank varieties for representations. Our rank variety construction is in line with that of Suslin, Friedlander and Bendel [Suslin et al. 1997b] and Friedlander and Pevtsova [2005; 2007].

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The present study occupies a somewhat unique position in the literature in that it is among the first semicomplete analyses of support for a class of “properly quantum” finite tensor categories; compare [Vashaw 2020, Section 3.1]. By properly quantum here we mean braided, but nonsymmetric. In our earlier papers [Friedlander and Negron 2018; Negron 2021], we verified the finite generation of cohomology for Drinfeld doubles of finite group schemes, a necessary foundational step for a theory of cohomological support varieties. We also made explicit computations of cohomology and briefly considered support varieties of irreducible representations. In contrast, our focus in this paper is the establishment of basic properties of support for Drinfeld doubles.

Support varieties have been employed to study various structural aspects of representations of groups and Hopf algebras. The stratification they provide for various stable module categories was presaged by Quillen’s stratification [1971a; 1971b] of the spectrum of the cohomology of finite groups. Indeed, cohomology (including Ext-groups) plays a central role in the formulation of support theories, revealing a surprising wealth of information about representations. Although the cohomology of a Hopf algebra A does not depend upon the coproduct of A, the tensor product certainly does and the behavior of tensor products is a fundamental underpinning of many applications of representation theory. Consequently, “the tensor product property” for a support theory $V \mapsto \mathrm{supp}(V)$ asserting that $\mathrm{supp}(V \otimes W) = \mathrm{supp}(V) \cap \mathrm{supp}(W)$ is of considerable interest.

As mentioned above, this text is dedicated to an analysis of support for the Drinfeld center $Z(\mathrm{rep}(G))$ of the representation category of an infinitesimal group scheme G. The center $Z(\mathrm{rep}(G))$ can be understood as the universal braided tensor category which admits a central tensor functor to $\mathrm{rep}(G)$, in the sense of [Bezrukavnikov 2004, Definition 2.1]. There are, however, a number of more explicit presentations of the center. For example, one can identify $Z(\mathrm{rep}(G))$ with the category $\mathrm{Coh}(G)^G$ of ad-equivariant sheaves on G. Or, even more concretely, $Z(\mathrm{rep}(G))$ is identified with the representation category of the smash product

$$D(G) := \mathscr{O}(G)\#_{\mathrm{ad}}kG$$

of the algebra of functions on G with the group ring of G. The algebra $D(G)$ is called the *Drinfeld double*, or *quantum double*, of the group ring kG. For more details one can see Section 2C below.

The Drinfeld center construction plays an essential role in studies of tensor categories and in related studies in mathematical physics. The important point here is that, unlike classical (symmetric) tensor categories, such as $\mathrm{rep}(G)$ itself, $Z(\mathrm{rep}(G)) = \mathrm{rep}(D(G))$ is highly *non*symmetric, and so behaves more like a *quantum* group than a classical group. In particular, the Drinfeld center is what is called a nonsemisimple modular tensor category. In order for the center $Z(\mathrm{rep}(G))$ to actually be a ribbon category some natural restrictions must be placed on G; see for example [Kauffman and Radford 1993; Humphreys 1978]. For applications of modular tensor categories to studies of conformal and topological field theories one can see for example [Reshetikhin and Turaev 1991; Fendley 2021; Brochier et al. 2021; Gannon and Negron 2021; Koshida and Kytölä 2022], and for some indications of the relevance of *cohomology* in such studies one can consult the texts [Lentner et al. 2020; Schweigert and Woike 2021; Costello et al. 2019; Creutzig et al. 2021].

Let us now turn to the specifics of this paper. For the remainder of the introduction we fix a field k of prime characteristic p, and consider the following:

- *Fix G to be the r-th Frobenius kernel in a split semisimple algebraic group $\mathbb{G}$, in very good characteristic.*
- *Fix $\mathfrak{D} = D(G)$ to be the corresponding Drinfeld double for kG.*

Here r is arbitrary, so that we are considering the family of normal subgroups $\mathbb{G}_{(r)}$ in $\mathbb{G}$.

For an explicit example, one could consider $\mathbb{G}$ to be $\mathrm{SL}_n(k)$ in odd characteristic p which does not divide n, or the symplectic group $\mathrm{Sp}_{2n}(k)$ in arbitrary odd characteristic. We note that all of the results listed below hold more generally when $\mathbb{G}$ is replaced by an arbitrary smooth algebraic group over k which admits a quasilogarithm; see Section 4 for a definition.

We recall the notion of cohomological support: For a finite-dimensional Hopf algebra A, and any A-representation V, we let $|A|$ denote the projective spectrum of cohomology, and $|A|_V$ denote the associated cohomological support space

$$|A| = \operatorname{Proj}\operatorname{Ext}^*_A(k,k), \quad |A|_V = \operatorname{Supp}_{|A|}\operatorname{Ext}^*_A(V,V)^{\sim}.$$

Here $\operatorname{Ext}^*_A(V,V)$ inherits a graded module structure over $\operatorname{Ext}^*_A(k,k)$ via the tensor structure on $\operatorname{rep}(A)$, and $\operatorname{Ext}^*_A(V,V)^{\sim}$ denotes the associated sheaf on the projective spectrum.

As a first point, we prove the following.

Theorem 6.11. *Consider G as above, with corresponding Drinfeld double $\mathfrak{D}$. Cohomological support for $\mathfrak{D}$ satisfies the tensor product property. That is to say, for finite-dimensional $\mathfrak{D}$-representations V and W we have*

$$|\mathfrak{D}|_{(V\otimes W)} = |\mathfrak{D}|_V \cap |\mathfrak{D}|_W. \tag{1}$$

From the perspective of tensor triangular geometry (e.g., [Balmer 2010b; Benson et al. 2011a]), Theorem 6.11 indicates that cohomological support may be used to “structure” both the derived and stable categories of representations for the double $\mathfrak{D}$. We elaborate on this point, and also on our findings in this direction.

Recall that the stable category $\mathrm{stab}(\mathfrak{D})$ for $\mathfrak{D}$ is the quotient of $\mathrm{rep}(\mathfrak{D})$ by the ideal of all morphisms which factor through a projective. This category inherits a triangulated structure from the abelian structure on $\mathrm{rep}(\mathfrak{D})$, and a monoidal structure from the monoidal structure on $\mathrm{rep}(\mathfrak{D})$. Also, by a thick ideal in $\mathrm{stab}(\mathfrak{D})$ we mean a thick subcategory — and in particular a full triangulated subcategory — which is stable under the tensor action of $\mathrm{stab}(\mathfrak{D})$ on itself. Finally, by a specialization closed subset in $|\mathfrak{D}|$, we mean a subset $\Theta \subset |\mathfrak{D}|$ which contains the closure $\bar{x} \subset \Theta$ of any point $x \in \Theta$. We prove the following.

Theorem 8.1. *Cohomological support provides an order preserving bijection*

$$\{\text{Specialization closed subsets in } |\mathfrak{D}|\} \xrightarrow{\sim} \{\text{thick ideals in } \mathrm{stab}(\mathfrak{D})\},$$
$$\Theta \longmapsto \mathscr{K}_\Theta,$$

where $\mathscr{K}_\Theta$ is the thick ideal of all objects V in $\mathrm{stab}(\mathfrak{D})$ *which are supported in the given set $|\mathfrak{D}|_V \subset \Theta$.*

One can compare with analogous classification results for finite groups [Rickard 1997], and finite group schemes [Friedlander and Pevtsova 2007]. By a thick *prime* ideal in $\mathrm{stab}(\mathfrak{D})$ we mean a thick ideal $\mathscr{P}$ in $\mathrm{stab}(\mathfrak{D})$ which satisfies the following: a product $V \otimes W$ is in $\mathscr{P}$ if and only if V or W is in $\mathscr{P}$. Balmer has shown that the collection of prime ideals in $\mathrm{stab}(\mathfrak{D})$ admits the structure of a locally ringed space, which he calls the spectrum of $\mathrm{stab}(\mathfrak{D})$.

Theorem 8.1 implies the following calculation of the Balmer spectrum $\mathrm{Spec}(\mathrm{stab}(\mathfrak{D}))$ for the Drinfeld double.

Theorem 8.2. *There is an isomorphism of locally ringed spaces*

$$f_{\mathrm{coh}} : |\mathfrak{D}| \xrightarrow{\cong} \mathrm{Spec}(\mathrm{stab}(\mathfrak{D})).$$

We note that the proofs of Theorems 8.1 and 8.2 rely on the construction of a certain "hybrid" Benson–Iyengar–Krause-type support theory [Benson et al. 2008] for infinite-dimensional $\mathfrak{D}$-representations. We discuss this support theory in Section 7 below.

Let us provide, in closing, an elaboration on the methods employed in our analysis of the center $Z(\mathrm{rep}(G)) = \mathrm{rep}(\mathfrak{D})$, and on a related π-point construction which appears in the appendix.

Elaborations on methods. Our proofs of the above results intertwine various approaches to support varieties in the literature. There are, however, some fundamental mechanisms which we leverage throughout the text.

Our basic approach to support for the double is as follows: We show in Section 5 that, for G a Frobenius kernel in a sufficiently nice algebraic group $\mathbb{G}$, the representation category of the Drinfeld double $\mathfrak{D} = D(G)$ admit an "effective comparison" with the representation category of an associated infinitesimal group scheme Σ. In particular, there is a linear abelian, *non*tensor, equivalence

$$\mathscr{L} : \mathrm{rep}(\mathfrak{D}) \xrightarrow{\sim} \mathrm{rep}(\Sigma) \tag{2}$$

which nonetheless transports support theoretic information back and forth. For example, we have an identification of cohomological supports $|\mathfrak{D}|_V = |\Sigma|_{\mathscr{L}(V)}$ for all V in $\mathrm{rep}(\mathfrak{D})$; see Lemma 6.9.

The fact that the equivalence $\mathscr{L}$ identifies support for $\mathfrak{D}$ with that of Σ is not a casual one, and requires one to "descend" the equivalence $\mathscr{L}$ to a family of local Hopf subalgebras $\mathfrak{D}_\psi \subset \mathfrak{D}$ which "covers" $\mathfrak{D}$. This family of subalgebras $\{\mathfrak{D}_\psi\}_{\psi \in V_r(G)}$ is parametrized by the scheme $V_r(G)$ of 1-parameter subgroups in G, and plays a fundamental role in our study. As a basic point, one can use the subalgebras $\mathfrak{D}_\psi$ to detect projectivity of $\mathfrak{D}$-representations. In particular, a given $\mathfrak{D}$-representation is projective if and only if its restriction to each $\mathfrak{D}_\psi$ is projective (Theorem 3.7). The ability of the $\mathfrak{D}_\psi$ to detect projectivity of $\mathfrak{D}$-representations is the covering property referred to above.

The effective comparison (2) is integral to our proofs of the tensor product property $|\mathfrak{D}|_{V \otimes W} = |\mathfrak{D}|_V \cap |\mathfrak{D}|_W$, and also to the classification results listed above. Additionally, the particular nature of our comparison indicates the existence of a π-point support theory for representations of the double, which we discuss in more detail below.

One might compare our approach with Avrunin and Scott's proof of Carlson's conjecture, where a certain change of coproduct result is used to relate supports for abelian restricted Lie algebras to those of elementary abelian groups [Avrunin and Scott 1982].

Conceptualizations via π-points. The introduction of π-points by Pevtsova and the first author [Friedlander and Pevtsova 2005; 2007] provide an alternate way to conceptualize our results. Our discussion of an analogous theory of π-points for the Drinfeld double $\mathfrak{D}$ is relegated to the appendix because they do not figure directly into the proofs of the results we have summarized. Instead, these results justify the intuition of π-points.

For us, a π-point for $\mathfrak{D}$ is a choice of field extension K/k, and a flat algebra map

$$\alpha : K[t]/(t^p) \to \mathfrak{D}_K$$

which admits an appropriate factorization through one of the local Hopf subalgebras $\mathfrak{D}_\psi \subset \mathfrak{D}_K$ (Definitions A.6 and A.8). We then construct the space $\Pi(\mathfrak{D})$ of equivalence classes of π-points, and a corresponding π-point support theory $V \mapsto \Pi(\mathfrak{D})_V$ for the double. The support spaces $\Pi(\mathfrak{D})_V$ are explicitly the locus of all π-points α at which the restriction $\mathrm{res}_\alpha(V_K)$ of V to $K[t]/(t^p)$ is nonprojective.

Two of our main results are that π-point support for the double $\mathfrak{D}$ satisfies the tensor product property

$$\Pi(\mathfrak{D})_{V\otimes W} = \Pi(\mathfrak{D})_V \cap \Pi(\mathfrak{D})_W \tag{3}$$

(Theorem A.14), and also agrees with cohomological support. In the statement of the following theorem we suppose that G is, as usual, a Frobenius kernel in a sufficiently nice algebraic group $\mathbb{G}$, i.e., one which admits a quasilogarithm.

Theorem A.15. *Consider G as above, and $\mathfrak{D} = D(G)$. There is a homeomorphism of topological spaces*

$$\Pi(\mathfrak{D}) \xrightarrow{\cong} |\mathfrak{D}|$$

which restricts to a homeomorphism of support spaces $\Pi(\mathfrak{D})_V \xrightarrow{\cong} |\mathfrak{D}|_V$ for each V in $\mathrm{rep}(\mathfrak{D})$.

We furthermore construct a "universal" π-point theory $\Pi_\otimes(\mathfrak{D})_\star$, and show that our specific π-point support theory $\Pi(\mathfrak{D})_\star$ agrees with this universal theory. One can see Theorem A.16 below.

In considering the π-point perspective for support, we open up the *possibility* of a deeper analysis of support for the double via explicit nilpotent operators. One can compare with the introduction of local Jordan types for group representations in [Carlson et al. 2008; Friedlander et al. 2007], and constructions of vector bundles on support spaces provided in [Friedlander and Pevtsova 2011; Benson and Pevtsova 2012]. Although we won't discuss the issue here, our methods also allow us to identify cohomological and hypersurface supports for Drinfeld doubles of first Frobenius kernels $\mathbb{G}_{(1)}$ in sufficiently nice algebraic groups; compare [Negron and Pevtsova 2020, Corollary 7.2, Section 13.3].

2. Preliminaries

We recall basic information about Hopf algebras, finite group schemes, and the Drinfeld double construction. We also recall the notion of cohomological support, and some basic results about Carlson modules. Throughout this text we work over a base field k which is of (finite) characteristic p.

2A. ***Hopf algebras and some generic notation.*** We set some global notations, and recall a strong form of the Larson–Sweedler theorem [Larson and Sweedler 1969]. We assume the reader has some familiarity with Hopf algebras, and our canonical reference for the topic is [Montgomery 1993].

For us, a representation of a finite-dimensional algebra A is the same thing as an A-module, and all representations/modules are *left* representations/modules. For a finite-dimensional Hopf algebra A we let

$$\mathrm{rep}(A) := \{\text{the tensor category of finite-dimensional } A\text{-representations}\}$$

and

$$\mathrm{Rep}(A) := \{\text{the monoidal category of all } A\text{-representation}\}.$$

To be clear, when we say $\mathrm{rep}(A)$ is a *tensor* category we recognize that all objects in $\mathrm{rep}(A)$ admit both left and right duals [Etingof et al. 2015, Section 2.10], whereas objects in $\mathrm{Rep}(A)$ are not dualizable in general. We let $\mathrm{Irrep}(A)$ denote the collection of all (isoclasses of) simple A-representations.

Throughout the text we denote finite-dimensional representations by the letters V and W, and reserve the letters M and N for possibly infinite-dimensional representations. This notation is employed throughout the text, without exception.

We recall the following basic result, which will be of use later.

Theorem 2.1 [Larson and Sweedler 1969]. *Any finite-dimensional Hopf algebra A is Frobenius. In particular*, *an A-representation M is projective if and only if M is injective.*

Proof. The algebra A is Frobenius by Larson and Sweedler [1969]. We note that if A is Frobenius then injectivity is the same as projectivity, even for infinite-dimensional modules, by [Faith and Walker 1967, Theorem 5.3]. □

2B. ***Finite group scheme.*** All group schemes in this text are affine. A group scheme G, over a base field k, is called finite if it is finite as a scheme over $\mathrm{Spec}(k)$. Rather, G is finite if it is affine and has finite-dimensional (Hopf) algebra of global functions $\mathscr{O}(G)$. For such finite G we let kG denote the associated group algebra $kG = \mathscr{O}(G)^*$. A finite group scheme is called *infinitesimal* if G is connected, i.e., if $\mathscr{O}(G)$ is local, and *unipotent* if the group algebra kG is local.

Following the framework of the previous section, we let $\mathrm{rep}(G)$ denote the category of finite-dimensional kG-modules, and $\mathrm{Rep}(G)$ denote the category of arbitrary kG-modules. Note that kG-modules are identified with $\mathscr{O}(G)$-comodules as in [Montgomery 1993, Lemma 1.6.4], so that finite-dimensional kG-modules are in fact identified with k-linear representations of the group scheme G.

2C. ***The Drinfeld double and the Drinfeld center.*** Let G be a finite group scheme. The adjoint action of G on itself induces an action of kG on $\mathscr{O}(G)$, and we can form the corresponding smash product, which is known as the *Drinfeld double*, or *quantum double* of kG, $D(G) = \mathscr{O}(G)\#kG$. We usually employ the generic notation $\mathfrak{D}$ for the Drinfeld double

$$\mathfrak{D} := D(G).$$

The algebra $\mathfrak{D}$ admits a unique Hopf algebra structure for which the two algebra inclusions $\mathscr{O}(G) \to \mathfrak{D}$ and $kG \to \mathfrak{D}$ are inclusions of Hopf algebras; see for example [Montgomery 1993, Corollary 10.3.10].

Remark 2.2. There is an analogous construction $A \rightsquigarrow D(A)$ of the Drinfeld double for an arbitrary finite-dimensional Hopf algebra A. So, we are only discussing a particular instance of a general construction.

Remark 2.3. If one compares directly with the presentation of [Montgomery 1993], then one finds an alternate description of the double as a smash product between the coopposite Hopf algebra $\mathscr{O}(G)^{\mathrm{cop}}$ and kG. However, by applying the antipode to the $\mathscr{O}(G)$ factor in $\mathfrak{D}$, one sees that the coopposite comultiplication on $\mathscr{O}(G)$ can be replaced with the usual one, up to Hopf isomorphism.

From a categorical perspective, we can consider the Drinfeld center of the representation category $\mathrm{rep}(G)$. This is the category of pairs

$$Z(\mathrm{rep}(G)) = \left\{ \begin{matrix} \text{pairs } (V, \gamma_V) \text{ of an object } V \text{ in } \mathrm{rep}(G), \text{ and} \\ \text{a choice of half braiding } \gamma_V : V \otimes - \to - \otimes V \end{matrix} \right\}$$

Such a half-braiding γ_V is required to be a natural isomorphism of endofunctors of $\mathrm{rep}(G)$, and we require that this natural isomorphism satisfies the expected compatibilities with the tensor structure on $\mathrm{rep}(G)$ [Kassel 1995, Definition XIII.4.1].

The center $Z(\mathrm{rep}(G))$ inherits a tensor structure from that of $\mathrm{rep}(G)$, and admits a canonical braiding $c_{V,W} : V \otimes W \to W \otimes V$ induced by the given half-braidings on objects $\gamma_{V,W} : V \otimes W \to W \otimes V$. This braiding on $Z(\mathrm{rep}(G))$ is highly nonsymmetric, in any sense which one might consider [Shimizu 2019]. For example, any object V in $Z(\mathrm{rep}(G))$ for which the square braiding is trivial $c^2_{V,-} = \mathrm{id}_{V\otimes -}$ must itself be trivial, $V \cong \mathbf{1}^{\oplus \dim(V)}$.

We have the following categorical interpretation of the double.

Theorem 2.4 [Kassel 1995, Theorem XIII.5.1]. *For any finite group scheme G, there is an equivalence of tensor categories* $\mathrm{rep}(\mathfrak{D}) \cong Z(\mathrm{rep}(G))$.

As a corollary to this result, we see that the category $\mathrm{rep}(\mathfrak{D})$ of representations for the Drinfeld double is canonically braided. This point is relevant for many applications in mathematical physics, and is also relevant in studies of support and cohomology. Specifically, many support theoretic results which are stated in the context of symmetric tensor categories can be immediately extended to the braided setting.

Remark 2.5. As with the construction of the Drinfeld double, one can construct the Drinfeld center of an arbitrary finite tensor category. Furthermore, the obvious analog of Theorem 2.4 is valid when we replace $\mathrm{rep}(G)$ with the representation category of an arbitrary finite-dimensional Hopf algebra.

In addition to considering the double $\mathfrak{D}$ we also consider a certain class of Hopf subalgebras $\mathfrak{D}' \subset \mathfrak{D}$ which one associates to subgroups in G. The following lemma will prove useful for our analysis of the subalgebras $\mathfrak{D}'$.

Lemma 2.6. *Suppose that G is an infinitesimal group scheme, and let $H \subset G$ be a closed subgroup in G. Let H act on $\mathscr{O}(G)$ via the (restriction of the) adjoint action, and consider the smash product algebra $\mathscr{O}(G)\#kH$.*

Restriction along the surjective algebra map $\mathscr{O}(G)\#kH \to kH$, $f \otimes x \mapsto \epsilon(f)x$, provides a bijection

$$\operatorname{Irrep}(H) \xrightarrow{\cong} \operatorname{Irrep}(\mathscr{O}(G)\#kH).$$

Proof. Same as the proof of [Friedlander and Negron 2018, Proposition 5.5]. □

2D. *Cohomological support.*

Definition 2.7. We say a finite-dimensional Hopf algebra A (over k) has finite type cohomology (over k) if the following two conditions hold:

(a) The extensions $\operatorname{Ext}^*_A(k, k)$ form a finitely generated k-algebra.

(b) For any pair of finite-dimensional A-representations V and W, the extensions $\operatorname{Ext}^*_A(V, W)$ form a finitely generated module over $\operatorname{Ext}^*_A(k, k)$, via the tensor action

$$- \otimes - : \operatorname{Ext}^*_A(k, k) \otimes \operatorname{Ext}^*_A(V, W) \to \operatorname{Ext}^*_A(V, W).$$

Let A be a finite-dimensional Hopf algebra, and suppose that A has finite type cohomology. We take

$$|A| := \operatorname{Proj} \operatorname{Ext}^*_A(k, k).$$

Formally, $\operatorname{Proj} \operatorname{Ext}^*_A(k, k)$ is the topological space of homogeneous prime ideals in $\operatorname{Ext}^*_A(k, k)$, which we equip with the Zariski topology. Since $\operatorname{Ext}^*_A(k, k)$ is graded commutative and finitely generated, restriction along the inclusion $\operatorname{Ext}^{ev}_A(k, k) \to \operatorname{Ext}^*_A(k, k)$ provides a homeomorphism $\operatorname{Proj} \operatorname{Ext}^*_A(k, k) \cong \operatorname{Proj} \operatorname{Ext}^{ev}_A(k, k)$. The structure sheaf on $\operatorname{Proj} \operatorname{Ext}^*_A(k, k)$ is the expected one, whose sections over a basic open D_f, $f \in \operatorname{Ext}^n_A(k, k)$, are the degree 0 elements in the localization $\operatorname{Ext}^*_A(k, k)_f$.

For any finite-dimensional A-representation V, we can consider the self-extensions $\operatorname{Ext}^*_A(V, V)$ and the tensor action of $\operatorname{Ext}^*_A(k, k)$ on these extensions. Note that the extensions of V form a *graded* module over $\operatorname{Ext}^*_A(k, k)$, and we may consider the associated sheaf $\operatorname{Ext}^*_A(V, V)^\sim$ on $|A| = \operatorname{Proj} \operatorname{Ext}^*_A(k, k)$. We define the cohomological support of V as the support of its associated sheaf

$$|A|_V := \operatorname{Supp}_{|A|} \operatorname{Ext}^*_A(V, V)^\sim. \tag{4}$$

Remark 2.8. When we consider the support $|A|_V$ of a finite-dimensional A-representation V, we typically view this as a "subvariety" (i.e., a reduced subscheme) of the scheme $|A|$. However, at certain times we simply view $|A|_V$ as a subspace of the underlying topological space of $|A|$.

We have the following basic claim.

Lemma 2.9 [Pevtsova and Witherspoon 2009, Proposition 2]. *Suppose that A has finite type cohomology. A finite-dimensional A-representation V is projective if and only if its support vanishes*, $|A|_V = \varnothing$.

In considering the aforementioned collection of Hopf subalgebras $\mathfrak{D}' \subset \mathfrak{D}$ we also take account of the following.

Lemma 2.10. *Suppose that A has finite type cohomology, and that $B \to A$ is an inclusion of Hopf algebras. Then*:

(1) *B has finite type cohomology.*

(2) *The restriction map* $\mathrm{Ext}^*_A(k,k) \to \mathrm{Ext}^*_B(k,k)$ *is a finite algebra map, and the induced map on spectra*
$$\mathrm{res}^* : \mathrm{Spec}\,\mathrm{Ext}^*_B(k,k) \to \mathrm{Spec}\,\mathrm{Ext}^*_A(k,k)$$
is such that $(\mathrm{res}^*)^{-1}(0) = \{0\}$.

Proof. The algebra B has finite-type cohomology, and the algebra map of (2) is finite, by [Negron and Plavnik 2022, Proposition 3.3]. Since this map is finite, the fiber
$$k \otimes_{\mathrm{Ext}^*_A(k,k)} \mathrm{Ext}^*_B(k,k)$$
is a finite-dimensional nonnegatively graded algebra, and hence the irrelevant ideal is the unique prime ideal in this algebra. This implies that the preimage $(\mathrm{res}^*)^{-1}(0)$ is the singleton $\{0\}$. □

Lemma 2.10(2) tells us that restriction $\mathrm{res} : \mathrm{rep}(A) \to \mathrm{rep}(B)$ induces a *well-defined* map on projective spectra $|B| \to |A|$. This map is furthermore closed and has finite fibers.

2E. *Cohomological support for group schemes.* In considering a finite group scheme G (over k) we adopt the particular notation
$$|G| := |kG| = \mathrm{Proj}\,\mathrm{Ext}^*_G(k,k).$$
We may consider cohomological support for G-representations as described in Section 2D.

In addition to cohomological support, there are a number of additional support theories for $\mathrm{rep}(G)$ which one might employ in tandem. In particular, when G is an infinitesimal group scheme, one can consider the k-scheme $V_r(G)$ of 1-parameter subgroups in G and its associated support theory of [Suslin et al. 1997b]. Although we do not use this theory explicitly in the text, it does "run in the background" of our analysis. So we sketch a presentation of this support theory here.

At fixed $r \geq 0$, $V_r(G)$ is the moduli space of group scheme maps $\mathbb{G}_{a(r)} \to G$ [Suslin et al. 1997a], and for any finite-dimensional G-representation W one has an associated support space $V_r(G)_W$. The support space $V_r(G)_W$ is specifically a nonprojectivity locus of the representation W in $V_r(G)$. To elaborate, the group ring $k\mathbb{G}_{a(r)}$ is a truncated polynomial ring $k[t, t^{(1)}, \ldots, t^{(r-1)}]/(t^p, \ldots, t^{(r-1)p})$ generated by divided powers $t^{(i)}$, and $k\mathbb{G}_{a(r)}$ is a flat extension of the subalgebra $A_{\mathrm{top}} \subset \mathbb{G}_{a(r)}$ generated by the highest divided power $t^{(r-1)}$. A k-point $\alpha : \mathbb{G}_{a(r)} \to G$ is in the support $V_r(G)_W$, for example, precisely when the restriction $\mathrm{res}_\alpha(W)$ is nonprojective when restricted further to this highest power subalgebra $A_{\mathrm{top}} \subset \mathbb{G}_{a(r)}$. The moduli space $V_r(G)$ is a conical scheme, and the supports $V_r(G)_W$ are closed conical subschemes in $V_r(G)$.

By results of [Suslin et al. 1997b], we have a natural scheme map $\Psi : \mathbb{P}(V_r(G)) \to |G|$ from the projectivization of $V_r(G)$, and this map is a *homeomorphism* whenever G is of height $\leq r$. The map Ψ restricts to homeomorphisms $\Psi_W : \mathbb{P}(V_r(G)_W) \to |G|_W$ between support spaces at arbitrary $W \in \mathrm{rep}(G)$, again when G is of height $\leq r$. So the support theory $V_r(G)_\star$ provides a kind of group theoretic "realization" of cohomological support for infinitesimal group schemes.

Remark 2.11. Our notation $|G|$ conflicts slightly with the notation of [Suslin et al. 1997a; 1997b; Friedlander and Pevtsova 2007]. Namely, $|G|$ is used to denote the *affine* spectrum of $\mathrm{Ext}^*_G(k,k)$ in the aforementioned papers, while we use it to denote the projective spectrum.

Remark 2.12. By results of Friedlander and Pevtsova [2007], the support theory $W \mapsto V_r(G)_W$ for $\mathrm{rep}(G)$ has a reasonable extension to the category $\mathrm{Rep}(G)$ of arbitrary kG-representation.

2F. *Carlson modules and support.* Consider a finite-dimensional Hopf algebra A with finite type cohomology. Define the n-th syzygy $\Omega^n k$ of the trivial representation via any choice of projective resolution of k, $0 \to \Omega^n k \to P^{-(n-1)} \to \cdots \to P^0 \to k$. Given an extension $\zeta \in \mathrm{Ext}^n_A(k,k)$, we can represent ζ as a map $\tilde{\zeta} : \Omega^n k \to k$ and define

$$L_\zeta := \ker(\tilde{\zeta} : \Omega^n k \to k).$$

The object L_ζ is called a *Carlson module* associated to ζ.

The object L_ζ is clearly not uniquely defined by ζ, since the definition relies on a choice of representative for the map $\zeta : \Sigma^{-n} k \to k$ in the derived category $D^b(A)$. However, L_ζ is unique up to isomorphism in the stable category for A, and so is sufficiently unique for most support theoretic applications. Carlson modules have a number of exceedingly useful properties. We recall a few of these properties here.

Proposition 2.13 [Pevtsova and Witherspoon 2009, Proposition 3]. *Consider an arbitrary homogeneous extension $\zeta \in \mathrm{Ext}^{ev}_A(k,k)$. For any finite-dimensional A-representation V there is an equality of supports*

$$|A|_{(L_\zeta \otimes V)} = Z(\zeta) \cap |A|_V. \tag{5}$$

As a corollary to Proposition 2.13 we find

Corollary 2.14 [Pevtsova and Witherspoon 2009, Corollary 1]. *Any closed subset Θ in $|A|$ is realizable as the support of a product $L = L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_m}$ of Carlson modules, $\Theta = |A|_L$.*

Carlson modules also enjoy certain naturality properties with respect to exact tensor functors. We list a particular occurrence of such naturality here.

Lemma 2.15. *If $\iota : B \to A$ is an inclusion of Hopf algebras, and L_ζ is a Carlson module associated to an extension $\zeta \in \mathrm{Ext}^*_A(k,k)$ over A, then the restriction $\mathrm{res}_\iota(L_\zeta)$ is a Carlson module for the image of $\mathrm{res}_\iota(\zeta) \in \mathrm{Ext}^*_B(k,k)$ of this extension in $\mathrm{Ext}^*_B(k,k)$.*

Proof. By the Nichols–Zoeller theorem [Larson and Sweedler 1969], A is projective as a B-module. So the result just follows from the fact that a projective resolution $P \to k$ of the unit over A restricts to a projective resolution over B. □

3. The Hopf subalgebras $\mathfrak{D}_\psi$ and a projectivity test

Let G be an infinitesimal group scheme. We show that the Drinfeld double $\mathfrak{D} = D(G)$ admits a family of Hopf embeddings $\{\mathfrak{D}_\psi \to \mathfrak{D}\}_{\psi \in 1\text{-param}}$ which is parametrized by the space of 1-parameter subgroups in G. Each of the Hopf algebras $\mathfrak{D}_\psi$ is local, and so behaves like a "unipotent subgroup" in $\mathfrak{D}$.

We show that the family $\{\mathfrak{D}_\psi \to \mathfrak{D}\}_{\psi \in 1\text{-param}}$ can be used to check projectivity of arbitrary (possibly infinite-dimensional) $\mathfrak{D}$-representations. One can see Theorem 3.7 below for a specific statement. We furthermore show that the cohomological support $|\mathfrak{D}|_V$ of a finite-dimensional $\mathfrak{D}$-representation V can be reconstructed from the support spaces $|\mathfrak{D}_\psi|_{\mathrm{res}_\psi(V)}$ of the restrictions of V to the various $\mathfrak{D}_\psi$.

The family of embeddings $\{\mathfrak{D}_\psi \to \mathfrak{D}\}_{\psi \in 1\text{-param}}$ plays an integral role throughout our study, and is therefore a fundamental object of interest. As implied above, an analysis of support for the double $\mathfrak{D}$ will be shown to be reducible to an analysis of support for the local subalgebras $\mathfrak{D}_\psi$. One can compare with the group theoretic setting, where the support theory of a finite group scheme is similarly reducible to that of its unipotent subgroups; cf. [Friedlander and Pevtsova 2005; 2007].

3A. ***1-parameter subgroups.*** Let k be a field of characteristic $p > 0$, and G be an infinitesimal group scheme over k. We let G_K denote the base change along any given field extension $k \to K$.

Definition 3.1. An embedded 1-parameter subgroup for G is a pair (K, ψ) of a field extension $k \to K$ and a closed map of group schemes $\psi : \mathbb{G}_{a(s),K} \to G_K$. We call K the field of definition for such a 1-parameter subgroup ψ.

Of course, by $\mathbb{G}_{a(r),K}$ we mean the base change of the r-th Frobenius kernel in $\mathbb{G}_a$. Let us take a moment to compare with [Suslin et al. 1997a; 1997b].

In the texts [Suslin et al. 1997a; 1997b], by a 1-parameter subgroup the authors mean an *arbitrary* group map $\psi' : \mathbb{G}_{a(r),K} \to G_K$. Having fixed a preferred quotient $\mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$ for each $s \leq r$, such a group map specifies an integer $s \leq r$ and a unique factorization of ψ' as a composition of the quotient $\mathbb{G}_{a(r),K} \to \mathbb{G}_{a(s),K}$ followed by an embedding $\psi : \mathbb{G}_{a(s),K} \to G_K$. In this way, the moduli space of 1-parameter subgroups $V_r(G)$ employed in [Suslin et al. 1997b] is identified with the moduli space of embedded 1-parameter subgroups for G, provided G is of height $\leq r$. (One can define the moduli space of embedded 1-parameter subgroups in precise analogy with [Suslin et al. 1997a, Definition 1.1].) One thus translates freely between the language of [Suslin et al. 1997a; 1997b] and the language we employ in this text.

Having clarified with this point, we recall the following essential results of Suslin, Friedlander and Bendel [1997b, Proposition 7.6] and Pevtsova [2002; 2004, Theorem 2.2].

Theorem 3.2 [Suslin et al. 1997b; Pevtsova 2002]. *Consider an infinitesimal group scheme G. An arbitrary G-representation M is projective over G if and only if for every field extension $k \to K$, and embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$, the base change M_K is projective over $\mathbb{G}_{a(s),K}$.*

To be clear, when we say that M_K is projective over $\mathbb{G}_{a(s),K}$ we mean that M_K restricts to a projective $\mathbb{G}_{a(s),K}$-representation along the given map $\psi : \mathbb{G}_{a(s),K} \to G_K$.

When we consider a finite-dimensional representation V, and k is algebraically closed, it suffices to check projectivity of V after restricting to all 1-parameter subgroups which are defined over k.

Theorem 3.3 [Suslin et al. 1997b]. *Consider an infinitesimal group scheme G, and a finite-dimensional G-representation V. Suppose also that the base field k is algebraically closed. Then V is projective over G if and only if, for every embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s)} \to G$ which is defined over k, V is projective over $\mathbb{G}_{a(s)}$.*

Proof. Suppose that V is projective when restricted to all such ψ. Then [Suslin et al. 1997b, Corollary 6.8] tells us that V has no closed points in its support. Since the support $|G|_V$ is closed, we conclude that $|G|_V = \varnothing$, and hence that V is projective. □

Remark 3.4. Since the category $\mathrm{Rep}(G)$ is Frobenius, we can replace projectivity with injectivity, or even flatness, in the statements of Theorems 3.2 and 3.3.

3B. ***A family of local subalgebras, and projectivity.*** As we have just observed, 1-parameter subgroups play an essential role in studies of support for infinitesimal group schemes. We provide a corresponding family of Hopf subalgebras for the Drinfeld double.

Definition 3.5. Let G be an infinitesimal group scheme, and $\psi : \mathbb{G}_{a(s),K} \to G_K$ be an embedded 1-parameter subgroup. Let $\mathfrak{D} = D(G)$ denote the Drinfeld double for G. We define $\mathfrak{D}_\psi$ to be the Hopf algebra

$$\mathfrak{D}_\psi := \mathscr{O}(G_K)\#K\mathbb{G}_{a(s),K},$$

where $\mathbb{G}_{a(s),K}$ acts on $\mathscr{O}(G_K)$ by restricting the adjoint action of G_K along the given embedding ψ.

Note that each Hopf algebra $\mathfrak{D}_\psi$ embeds in the double $\mathfrak{D}_K$ via the map $\mathrm{id}_{\mathscr{O}} \otimes \psi : \mathfrak{D}_\psi \to \mathfrak{D}_K$. So we might speak of the $\mathfrak{D}_\psi$ as Hopf subalgebras in $\mathfrak{D}_K$, via a slight abuse of language.

Lemma 3.6. *Consider an infinitesimal group scheme G, with Drinfeld double $\mathfrak{D}$. For any embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s)} \to G$ the Hopf algebra $\mathfrak{D}_\psi$ is local.*

Proof. By changing base if necessary we may assume $K = k$. By Lemma 2.6 the restriction map provides an bijection $\mathrm{Irrep}(\mathbb{G}_{a(s)}) \to \mathrm{Irrep}(\mathfrak{D}_\psi)$. Now, since $\mathbb{G}_{a(s)}$ is unipotent, the trivial representation is the only simple object in $\mathrm{rep}(\mathbb{G}_{a(s)})$. So we observe that $\mathrm{rep}(\mathfrak{D}_\psi)$ has a unique simple object, and therefore that $\mathfrak{D}_\psi$ is local. □

We recall that, according to Theorem 3.2, 1-parameter subgroups in a given infinitesimal group scheme can be used to detect projectivity of G-representations. We observe an analogous detection property for the $\mathfrak{D}_\psi$.

Theorem 3.7. *Consider an arbitrary representation M over the Drinfeld double $\mathfrak{D}$ of an infinitesimal group scheme G. Then M is projective over $\mathfrak{D}$ if and only if for every field extension $k \to K$, and every embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$, the base change M_K is projective over $\mathfrak{D}_\psi$.*

When M is finite-dimensional, and k is algebraically closed, M is projective over $\mathfrak{D}$ if and only if, for all embedded 1-parameter subgroups $\psi : \mathbb{G}_{a(s)} \to G$ which are defined over k, M is projective over $\mathfrak{D}_\psi$.

Proof. Recall that $\mathfrak{D}$ is Frobenius, so that projectivity of M is equivalent to injectivity. It suffices to check projectivity/injectivity after changing base to the algebraic closure $\bar{k}$, so that we may assume $k = \bar{k}$. Furthermore, as with any finite dimensional algebra, injectivity of M is equivalent to vanishing of the extensions

$$\mathrm{Ext}^{>0}_{\mathfrak{D}}(S, M) = 0 \quad \text{from the sum } S \text{ of all simple } \mathfrak{D}\text{-reps.}$$

So we seek to establish the above vanishing of cohomology. In what follows we take $\mathscr{O} = \mathscr{O}(G)$.

If M is projective over $\mathfrak{D}$, then M is projective over the Hopf subalgebra $\mathscr{O} \subset \mathfrak{D}$ [Montgomery 1993, Theorem 3.1.5]. Thus M is injective over $\mathscr{O}$ in this case. Similarly, if M_K is projective over $\mathfrak{D}_\psi$, then M_K is projective over $\mathscr{O}_K$, and thus injective over $\mathscr{O}_K$ as well. It follows that M is injective over $\mathscr{O}$ itself. So it suffices to assume that M is injective over $\mathscr{O}$, and prove that in this case M is injective over $\mathfrak{D}$ if and only if M_K is injective over $\mathfrak{D}_\psi$ for all extensions $k \to K$ and embeddings $\psi : \mathbb{G}_{a(s),K} \to G_K$.

Let us assume that M is injective over $\mathscr{O}$. By Lemma 2.6, all simple $\mathfrak{D}$-representations are restrictions of simple G-representations along the projection $\mathfrak{D} \to kG$. It follows that we have a spectral sequence

$$\mathrm{Ext}^*_G(S, \mathrm{Ext}^*_{\mathscr{O}}(k, M)) \Rightarrow \mathrm{Ext}^*_{\mathfrak{D}}(S, M)$$

which reduces to an identification

$$\mathrm{Ext}^*_G(S, \mathrm{Hom}_{\mathscr{O}}(k, M)) = \mathrm{Ext}^*_{\mathfrak{D}}(S, M),$$

since M is injective over $\mathscr{O}$. Similarly, we have an identification

$$\mathrm{Ext}^*_{\mathbb{G}_{a(s),K}}(K, \mathrm{Hom}_{\mathscr{O}_K}(K, M_K)) = \mathrm{Ext}^*_{\mathfrak{D}_\psi}(K, M_K)$$

at any embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$. Hence M is injective over $\mathfrak{D}$ (resp. M_K is injective over $\mathfrak{D}_\psi$) if and only if the invariant subspace $\mathrm{Hom}_{\mathscr{O}}(k, M)$ is injective over G (resp. $\mathrm{Hom}_{\mathscr{O}_K}(K, M_K)$ is injective over $\mathbb{G}_{a(s),K}$).

Given the above information, we seek to establish the claim that

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{O}}(k, M) \text{ is injective over } G \\ &\quad \Longleftrightarrow \text{for each } \psi : \mathbb{G}_{a(s),K} \to G_K,\ \mathrm{Hom}_{\mathscr{O}}(k, M)_K = \mathrm{Hom}_{\mathscr{O}_K}(K, M_K) \text{ is injective over } \mathbb{G}_{a(s),K}. \end{aligned}$$

But this final claim follows by Theorem 3.2. Our specific claim about finite-dimensional M follows by Theorem 3.3. □

3C. ***(Re)constructing cohomological support.*** We consider cohomological support for *finite-dimensional* representations over the Drinfeld double. Fix an infinitesimal group scheme G and let $\mathfrak{D}$ denote its Drinfeld double $\mathfrak{D} = D(G)$. Recall our notation $|\mathfrak{D}|$ for the projective spectrum of cohomology, $|\mathfrak{D}| = \mathrm{Proj}\,\mathrm{Ext}^*_{\mathfrak{D}}(k, k)$. We have the following basic result of Friedlander and Negron [2018] and Negron [2021].

Theorem 3.8 [Friedlander and Negron 2018; Negron 2021]. *The Drinfeld double $\mathfrak{D}$ has finite type cohomology.*

We now apply Lemma 2.10 and Theorem 3.8 to find:

Corollary 3.9. *For any embedded* 1*-parameter subgroup* $\psi : \mathbb{G}_{a(s),K} \to G_K$, *the Hopf algebra* $\mathfrak{D}_\psi$ *has finite type cohomology, and the induced map on projective spectra* $\mathrm{res}^*_\psi : |\mathfrak{D}_\psi| \to |\mathfrak{D}_K|$ *is a finite map of schemes.*

Let us consider an arbitrary field extension $k \to K$. We note that the natural map $K \otimes \mathrm{Ext}^*_{\mathfrak{D}}(k, k) \xrightarrow{\cong} \mathrm{Ext}^*_{\mathfrak{D}_K}(K, K)$ is an isomorphism, and thus identifies the spectrum $|\mathfrak{D}_K|$ with the base change $|\mathfrak{D}|_K$. For any embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$, we therefore obtain a map of schemes

$$f_\psi : |\mathfrak{D}_\psi| \to |\mathfrak{D}| \tag{6}$$

given by composing the map $\mathrm{res}^*_\psi : |\mathfrak{D}_\psi| \to |\mathfrak{D}_K|$ induced by restriction with the projection $|\mathfrak{D}_K| = |\mathfrak{D}|_K \to |\mathfrak{D}|$.

We note that these f_ψ are not closed morphisms in general. This is simply because the projection $|\mathfrak{D}|_K \to |\mathfrak{D}|$ does not preserve closed points when the extension $k \to K$ is infinite. On the other hand, we see that any point x in $|\mathfrak{D}|$ is represented by — or rather lifts to — a closed point in the base change $|\mathfrak{D}|_{\bar{k}(x)}$. So, by employing base change one is able to treat arbitrary points in the spectrum $|\mathfrak{D}|$ as closed points, at least to a certain degree. We record a little lemma.

Lemma 3.10. *Consider any finite-dimensional* $\mathfrak{D}$*-representation* V.

(1) *For an arbitrary field extension* $k \to K$, *the support* $|\mathfrak{D}_K|_{V_K}$ *of* V_K *over* $\mathfrak{D}_K$ *is precisely the preimage of* $|\mathfrak{D}|_V$ *along the projection* $|\mathfrak{D}_K| \to |\mathfrak{D}|$. *In particular, the composition* $|\mathfrak{D}_K|_{V_K} \subset |\mathfrak{D}_K| \to |\mathfrak{D}|$ *is a surjection onto* $|\mathfrak{D}|_V$.

(2) *For any embedded* 1*-parameter subgroup* $\psi : \mathbb{G}_{a(s),K} \to G$ *the map* f_ψ *restricts to a morphism between support spaces* $|\mathfrak{D}_\psi|_{V_K} \to |\mathfrak{D}|_V$. *In particular, the image of* $|\mathfrak{D}_\psi|_{V_K}$ *under* f_ψ *is contained in* $|\mathfrak{D}|_V$.

Proof. Statement (1) follows from the fact that (a) For any scheme X, the projection $X_K \to X$ along a field extension $k \to K$ is surjective and (b) for any map of schemes $f : Y \to X$, and coherent sheaf $\mathscr{F}$ on X, we have $\mathrm{Supp}(f^*\mathscr{F}) = f^{-1}\,\mathrm{Supp}(\mathscr{F})$. For (2) it suffices to prove the result in the case $K = k$, by (1). We simply consider the diagram

$$\begin{array}{ccc} \mathrm{Ext}^*_{\mathfrak{D}}(k,k) & \xrightarrow{-\otimes V} & \mathrm{Ext}^*_{\mathfrak{D}}(V,V) \\ {\scriptstyle \mathrm{res}_\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_\psi} \\ \mathrm{Ext}^*_{\mathfrak{D}_\psi}(k,k) & \xrightarrow{-\otimes V} & \mathrm{Ext}^*_{\mathfrak{D}_\psi}(V,V) \end{array}$$

induced by the restriction functors, and note that the supports $|\mathfrak{D}|_V$ and $|\mathfrak{D}_\psi|_V$ are the subvarieties associated to the respective kernels of the algebra maps $-\otimes V$. □

We now observe that the support of V over $\mathfrak{D}$ can be reconstructed from the supports of V over the $\mathfrak{D}_\psi$, where we allow ψ to vary along all 1-parameter subgroups for G.

Proposition 3.11. *Let G be an infinitesimal group scheme and $\mathfrak{D} = D(G)$ be the associated Drinfeld double. For any finite-dimensional $\mathfrak{D}$-representation V there is an equality*

$$|\mathfrak{D}|_V = \bigcup_{\text{1-param subgroups}} f_\psi(|\mathfrak{D}_\psi|_{V_K}). \tag{7}$$

To be clear, the equality (7) is an equality of *sets*. Indeed, the support of a representation is itself simply a closed subset in the space $|\mathfrak{D}|$. Also, the union (7) is explicitly taken over the collection of all embedded 1-parameter subgroups in G, each of which consists of a pair of a field extension K/k and an embedding $\psi : \mathbb{G}_{a(s),K} \to G_K$.

Proof. If the support $|\mathfrak{D}|_V$ vanishes, i.e., if V is projective over $\mathfrak{D}$, then Theorem 3.7 tells us that all of the supports $|\mathfrak{D}_\psi|_{V_K}$ vanish as well. So the claimed equality holds when the support $|\mathfrak{D}|_V$ is empty.

Let us assume now that V is *not* projective over $\mathfrak{D}$, and hence that the support $|\mathfrak{D}|_V$ is nonvanishing. By considering base change, and Lemma 3.10, we see that the equality (7) can be obtained from the following claim:

Claim. *When k is algebraically closed, and x is a closed point in $|\mathfrak{D}|_V$, there is a 1-parameter subgroup $\psi : \mathbb{G}_{a(s)} \to G$ such that x is in the image $f_\psi(|\mathfrak{D}_\psi|_V)$.*

Let us verify this claim.

We suppose that $k = \bar{k}$ and consider a closed point x in $|\mathfrak{D}|_V$. Let L be a product of Carlson modules with $|\mathfrak{D}|_L = \{x\}$. Then $|\mathfrak{D}|_{L\otimes V} = \{x\}$ and for any 1-parameter subgroup $\psi : \mathbb{G}_{a(s)} \to G$ we have

$$f_\psi(|\mathfrak{D}_\psi|_{L\otimes V}) = \begin{cases} \{x\} & \text{if } x \in f_\psi(|\mathfrak{D}_\psi|_V) \\ \varnothing & \text{else.} \end{cases}$$

Indeed, the above formula follows from the fact that $|\mathfrak{D}_\psi|_L = f_\psi^{-1}(x)$, by Lemma 2.15, and the subsequent fact that

$$|\mathfrak{D}_\psi|_{L\otimes V} = f_\psi^{-1}(x) \cap |\mathfrak{D}_\psi|_V,$$

by Proposition 2.13.

Recall that, by the projectivity test of Theorem 3.7, projectivity of the restriction of $L \otimes V$ along each such ψ would imply that $L \otimes V$ is projective over $\mathfrak{D}$. Equivalently, vanishing of the supports $|\mathfrak{D}_\psi|_{L\otimes V}$ along all such ψ would imply vanishing of the support $|\mathfrak{D}|_{L\otimes V}$. Since we have chosen L so that the latter space explicitly *does not* vanish, we conclude that some support space $|\mathfrak{D}_\psi|_{L\otimes V}$ does not vanish. Rather, $x \in f_\psi(|\mathfrak{D}_\psi|_{L\otimes V})$ for some ψ, and thus $x \in f_\psi(|\mathfrak{D}_\psi|_V)$ for some ψ. So we have proved the above claim, and thus establish the identification (7). □

We remark, in closing, that one can prove analogs of the results of this section for arbitrary finite (rather than infinitesimal) group schemes. One simply replaces the "testing groups" $\mathbb{G}_{a(s)}$ with a larger class of unipotent group schemes; cf. [Friedlander and Pevtsova 2005].

4. Quasilogarithms for group schemes

In this short aside we recall the notion of a quasilogarithm for an affine group scheme. As we recall below, "most" familiar algebraic groups admit quasilogarithms. One can see Proposition 4.4 in particular. As our study of support for Drinfeld doubles becomes more focused, we employ quasilogarithms to gain some leverage on the *algebra* structure of the double $\mathfrak{D} = D(G)$.

4A. *Quasilogarithms.*

Definition 4.1 [Kazhdan and Varshavsky 2006]. Let G be an affine group scheme with Lie algebra $\mathfrak{g}$. We consider $\mathfrak{g}$ as an affine scheme $\mathfrak{g} = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^*))$. A quasilogarithm for G is a map of schemes $l : G \to \mathfrak{g}$ which

(a) is equivariant for the adjoint actions,

(b) sends $1 \in G$ to $\{0\} \in \mathfrak{g}$,

(c) induces the identity on tangent spaces $T_1 l = \mathrm{id}_{\mathfrak{g}}$.

Concretely, if we let $m \subset \mathscr{O}(G)$ denote the maximal ideal associated to the point $1 \in G$, then a quasilogarithm for G is a choice of ad-equivariant splitting $\mathfrak{g}^* \to m$ of the projection $m \to m/m^2 = \mathfrak{g}^*$. We note that, when G is smooth over the base field k, such a quasilogarithm induces an isomorphism on the respective formal neighborhoods $\hat{l} : \hat{G}_1 \xrightarrow{\cong} \hat{\mathfrak{g}}_0$. Also, when G is infinitesimal any quasilogarithm is a closed embedding.

The following lemma is straightforward.

Lemma 4.2. *Suppose a group scheme $\mathbb{G}$ admits a quasilogarithm $l : \mathbb{G} \to \mathfrak{g}$. Then for any positive integer r, the restriction $l|_{\mathbb{G}_{(r)}} : \mathbb{G}_{(r)} \to \mathfrak{g}$ provides a quasilogarithm for the Frobenius kernel $\mathbb{G}_{(r)}$.*

Through the remainder of the text we often adopt the following hypotheses: We assume $\mathbb{G}$ is a smooth algebraic group which admits a quasilogarithm, then consider the Frobenius kernels $G = \mathbb{G}_{(r)}$ at arbitrary $r > 0$. The previous lemma tells us that all such G naturally inherit quasilogarithms from any choice of quasilogarithm for the ambient group $\mathbb{G}$. So in this way we obtain various families of infinitesimal group schemes which admit quasilogarithms.

4B. *Appearances of quasilogs in nature.* We discuss the "generic" presence of quasilogarithms among reductive algebraic groups. Let $\boldsymbol{G}$ be an affine algebraic group which is defined over a localization $R = \mathbb{Z}[1/n]$ of the integers, and suppose that $\boldsymbol{G}$ is generically reductive. That is to say, suppose that the rational form $\boldsymbol{G}_{\mathbb{Q}}$ is reductive. Take $\mathscr{O} = \mathscr{O}(\boldsymbol{G})$.

Let $m \subset \mathscr{O}$ be the ideal associated to the identity $1 \in \boldsymbol{G}(R)$, and consider the coadjoint representation $\mathfrak{g}^* = m/m^2$. The surjection $m \to \mathfrak{g}^*$ admits an ad-equivariant splitting $\mathfrak{g}^*_{\mathbb{Q}} \to m_{\mathbb{Q}} \subset \mathscr{O}_{\mathbb{Q}}$ over the rationals, since $\boldsymbol{G}_{\mathbb{Q}}$ has semisimple representation theory [Milne 2017, Theorem 22.42]. This splitting is defined over a further localization $R' = \mathbb{Z}[1/N]$, so that we obtain a quasilogarithm $\boldsymbol{G}_{R'} \to \mathfrak{g}_{R'}$ defined over R'. It follows that for any field k of characteristic p which does not divide N, the group $\mathbb{G} = \boldsymbol{G}_k$ admits a quasilogarithm. We record this observation.

Proposition 4.3. *Let $\boldsymbol{G}$ be a algebraic group scheme which is defined over a localization $R = \mathbb{Z}[1/n]$ of the integers, and suppose that $\boldsymbol{G}$ is generically reductive. Then for any field k, in all but finitely many characteristics, the k-form $\mathbb{G} = \boldsymbol{G}_k$ admits a quasilogarithm.*

If we consider split semisimple algebraic groups, for example, we can be much more precise about the characteristics at which our group $\mathbb{G} = \boldsymbol{G}_k$ admits a quasilogarithm. We can also deduce quasilogarithms for various classes of algebraic groups which are related to such semisimple $\mathbb{G}$.

Proposition 4.4 [Friedlander and Negron 2018, Section 6.1]. *The following algebraic groups admit a quasilogarithm*:

- *The general linear group* GL_n, *over any field in any characteristic.*
- *Any split simple algebraic group in very good characteristic* (*relative to the corresponding Dynkin type*).
- *Any Borel subgroup inside a split simple algebraic group*, *in very good characteristic.*
- *The unipotent radical in such a Borel*, *in sufficiently large characteristic.*

5. The Drinfeld double $\mathfrak{D}$ via an infinitesimal group scheme

Let $\mathbb{G}$ be a smooth algebraic group over k which admits a quasilogarithm, and let G be a Frobenius kernel in $\mathbb{G}$. We consider the Drinfeld double $\mathfrak{D}$ for G. In this section we show that, for G as described, there is a *linear abelian* equivalence

$$\mathscr{L} : \mathrm{rep}(\mathfrak{D}) \xrightarrow{\sim} \mathrm{rep}(\Sigma)$$

between the representation category of the double and the representation category of an associated infinitesimal group scheme Σ. We show, furthermore, that this equivalence restricts to a corresponding abelian equivalence $\mathscr{L}_\psi : \mathrm{rep}(\mathfrak{D}_\psi) \xrightarrow{\sim} \mathrm{rep}(\Sigma_\psi)$ at all embedded 1-parameter subgroups in G.

Although these equivalences are not equivalences of tensor categories, they can be used in highly nontrivial ways in an analysis of support for the double, as we will see in Sections 6 and 7.

5A. ***The group schemes $\Sigma_V(G, r)$.*** Consider a finite group scheme G and any finite-dimensional G-representation V. To V we associate the algebra

$$S_r(V) := \mathrm{Sym}(V)/(v^{p^r} : v \in V).$$

This algebra has the natural structure of a cocommutative Hopf algebra in the symmetric tensor category $\mathrm{rep}(G)$, where the coproduct on $S_r(V)$ is defined by taking all of the generators $v \in V$ to be primitive $\Delta(v) = v \otimes 1 + 1 \otimes v$; cf. [Andruskiewitsch and Schneider 2001, Section 1.3]. Indeed, we may view V as an abelian Lie algebra in $\mathrm{rep}(G)$, and consider the universal enveloping algebra $U(V) = \mathrm{Sym}(V)$. We then obtain $S_r(V)$ as the quotient of $U(V)$ by the Hopf ideal generated by the primitive elements v^{p^r}, $v \in V$.

Now, since the forgetful functor $\mathrm{rep}(G) \to \mathrm{Vect}$ is a map of symmetric tensor categories, any Hopf algebra in $\mathrm{rep}(G)$ can be viewed immediately as a Hopf algebra in the classical sense, i.e., as a Hopf algebra in Vect. So we may view $S_r(V)$ as a Hopf algebra in rep(G) or as a Hopf algebra in Vect as needed. Furthermore, for any Hopf algebra S in rep(G) the smash product $S\#kG$ admits a unique Hopf algebra structure (in Vect) so that the two inclusions

$$S \to S\#kG \quad \text{and} \quad kG \to S\#kG$$

are maps of Hopf algebras (in Vect). Indeed, this is the standard bosonization procedure [Radford 2012, Theorem 1.6.9]. So, in the case discussed above, we obtain the following.

Lemma 5.1. *For any finite group scheme G and any finite-dimensional G-representation V, the smash product $S_r(V)\#kG$ admits a unique cocommutative Hopf algebra structure* (*in* Vect) *such that the following conditions hold*:

(a) *Each $v \in V$ is primitive.*

(b) *The inclusion $kG \to S_r(V)\#kG$ is a map of Hopf algebras.*

Proof. The existence of such a Hopf structure follows by the discussion above. Cocommutativity follows from the fact that the two Hopf subalgebras $S_r(V)$ and kG are cocommutative, and that the multiplication map

$$\text{mult} : S_r(V) \otimes kG \to S_r(V)\#kG$$

is a morphism, and hence an isomorphism, of coalgebras. □

The fact that $S_r(V)\#kG$ is cocommutative tells us that it serves as the group ring for an associated finite group scheme.

Definition 5.2. For any finite group scheme G, and any finite-dimensional G-representation V, we define the finite group scheme $\Sigma_V(G,r)$ to be the unique such group scheme with associated group algebra

$$k\Sigma_V(G,r) = S_r(V)\#kG.$$

Said another way, $\Sigma_V(G,r)$ is the spectrum of the dual Hopf algebra

$$\Sigma_V(G,r) = \text{Spec}((S_r(V)\#kG)^*).$$

Note that the group scheme $\Sigma_V(G,r)$ admits a normal subgroup $N_V(r) \subset \Sigma_V(G,r)$ which corresponds to the normal Hopf subalgebra $S_r(V) \subset k\Sigma_V(G,r)$, and that we have an exact sequence of group schemes

$$1 \to N_V(r) \to \Sigma_V(G,r) \to G \to 1. \tag{8}$$

Lemma 5.3. *Suppose that G is infinitesimal, and let V be an any finite-dimensional G-representation. Then $\Sigma_V(G,r)$ is infinitesimal. Furthermore, if G is unipotent then $\Sigma_V(G,r)$ is unipotent as well.*

Proof. Take $\Sigma = \Sigma_V(G,r)$. As a coalgebra $k\Sigma = S_r(V) \otimes kG$. So the algebra of functions $\mathscr{O}(\Sigma)$ is the tensor product $S_r(V)^* \otimes \mathscr{O}(G)$. Since $S_r(V)$ is a connected coalgebra, with primitive space $\text{Prim}(S_r(V)) = \{v^{p^s} : 0 \leq s < r\}$, it follows that the dual $S_r(V)^*$ is local. Since G is infinitesimal the algebra $\mathscr{O}(G)$ is also local. Now, since a tensor product of finite-dimensional local k-augmented algebras is also local, we see that $\mathscr{O}(\Sigma)$ is local. Hence Σ is infinitesimal.

For arbitrary G, the maximal ideal $m = (V) \subset S_r(V)$ is stable under the action of G, so that the ideal $m \otimes kG \subset k\Sigma$ is nilpotent. Hence the Jacobson radical of $k\Sigma$ is the preimage of the Jacobson radical in kG along the surjection $k\Sigma \to kG$. It follows that if kG is local then $k\Sigma$ is local. So we see that Σ is unipotent when G is unipotent. □

We note, finally, that the group scheme $\Sigma_V(G, r)$ can be defined entirely within the category of group schemes (rather than in the category of Hopf algebras). Indeed, the action of G on V induces an action on the r-th Frobenius kernel in the corresponding additive group scheme $V_a = (V, +)$, and hence on the Cartier dual $(N_V(r) =) V^\vee_{a(r)}$. We then recover $\Sigma_V(G, r)$ as the semidirect product $V^\vee_{a(r)} \rtimes G$. This construction is more in line with the standard perspective of, say, Jantzen [2003]. However, what is of interest to us is the algebra structure on $k\Sigma_V(G, r)$. So the above Hopf algebraic presentation is sufficiently informative for our purposes.

5B. ***Quasilogarithms and a system of linear equivalences.*** We consider the above construction $\Sigma_V(G, r)$ for the coadjoint representation of G.

Definition 5.4. For any finite group scheme G we define

$$\Sigma(G, r) := \Sigma_{\mathfrak{g}^*}(G, r),$$

where $\mathfrak{g}^*$ is the coadjoint representation. Additionally, for any embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$ we restrict the coadjoint representation of G_K along ψ to define

$$\Sigma(G, r)_\psi := \Sigma_{\mathfrak{g}^*_K}(\mathbb{G}_{a(s),K}, r).$$

When no confusion will arise we will be even more casual in our presentation, and write simply

$$\Sigma = \Sigma(G, r), \quad \Sigma_\psi = \Sigma(G, r)_\psi.$$

(We usually consider a Frobenius kernel $G = \mathbb{G}_{(r)}$ and the associated group scheme $\Sigma(G, r)$, so that the parameter r is already clear from the context.) Note that for any embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(r),K} \to G_K$ the product map $\mathrm{id}_{S_r} \otimes K\psi$ provides a natural inclusion of group schemes $\Sigma(G, r)_\psi \to \Sigma(G, r)_K$.

Lemma 5.5. *Let $\mathbb{G}$ be a smooth algebraic group which admits a quasilogarithm. Consider $G = \mathbb{G}_{(r)}$, $\mathfrak{D} = D(G)$, and $\Sigma = \Sigma(G, r)$ at arbitrary $r > 0$.*

Any choice of quasilogarithm l for G specifies an isomorphism of augmented k-algebras $a(l) : k\Sigma \to \mathfrak{D}$. Furthermore, for any 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$, we have a corresponding isomorphism of *augmented K-algebra $a(l)_\psi : K\Sigma_\psi \to \mathfrak{D}_\psi$. These isomorphisms fit into a diagram of K-algebra maps*:

$$\begin{array}{ccc} K\Sigma_K & \xrightarrow{a(L)_K} & \mathfrak{D}_K \\ \uparrow{\scriptstyle \text{incl}} & & \uparrow{\scriptstyle \text{incl}} \\ K\Sigma_\psi & \xrightarrow{a(L)_\psi} & \mathfrak{D}_\psi \end{array} \tag{9}$$

The augmentations considered above are, of course, the augmentations specified by the respective counits.

Proof. Take $S = S_r(\mathfrak{g}^*)$, with its G-action induced by the coadjoint action on $\mathfrak{g}^*$. Any quasilogarithm l specifies a G-equivariant map of algebras $a_0 : S \to \mathscr{O}(G)$ which is an isomorphism on cotangent spaces $m_0/m_0^2 \to m_1/m_1^2$. Indeed, a quasilogarithm for G is a choice of equivariant section $\mathfrak{g}^* \to m_1$ of the reduction map $m_1 \to m_1/m_1^2 = \mathfrak{g}^*$, and a_0 is the algebra map from the (truncated) symmetric algebra induced by this section. Since $\mathscr{O}(G)$ is local, such a map is necessarily surjective. Since furthermore $\dim(S) = \dim(\mathscr{O}(G)) = r^{\dim(\mathfrak{g})}$, it follows that a_0 is an isomorphism. Since both algebras in question are local, a_0 is an isomorphism of augmented algebras. (This point is also obvious from the construction of a_0.)

We obtain the desired isomorphism $a(l) : k\Sigma \to \mathfrak{D}$ as the product $a(l) = a_0 \otimes \mathrm{id}_{kG}$, and similarly $a(l)_\psi : K\Sigma_\psi \to \mathfrak{D}_\psi$ is the product $(a_0)_K \otimes \mathrm{id}_{k\mathbb{G}_{a(s)}}$. One sees directly that, since a_0 is an isomorphism of augmented algebras, $a(l)$ and $a(l)_\psi$ are also isomorphisms of augmented algebras. □

As a consequence of the above lemma, we see that any choice of quasilogarithm for the ambient group $\mathbb{G}$ specifies a "system of linear equivalences" for $\mathfrak{D}$, and its local family of Hopf subalgebras $\mathfrak{D}_\psi$.

Proposition 5.6. *For G as in Lemma 5.5, there is an equivalence of k-linear, abelian categories $\mathscr{L}$: $\mathrm{rep}(\mathfrak{D}) \xrightarrow{\sim} \mathrm{rep}(\Sigma)$ which preserves the trivial representation $\mathscr{L}(k) = k$. Furthermore, for any 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$ we have a corresponding equivalence of K-linear categories $\mathscr{L}_\psi : \mathrm{rep}(\mathfrak{D}_\psi) \xrightarrow{\sim} \mathrm{rep}(\Sigma_\psi)$ which preserves the trivial representation, and fits into a diagram of exact linear functors*:

$$\begin{array}{ccc} \mathrm{rep}(\mathfrak{D}_K) & \xrightarrow{\mathscr{L}_K} & \mathrm{rep}(\Sigma_K) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ \mathrm{rep}(\mathfrak{D}_\psi) & \xrightarrow{\mathscr{L}_\psi} & \mathrm{rep}(\Sigma_\psi). \end{array} \tag{10}$$

Proof. Define $\mathscr{L}$ and $\mathscr{L}_\psi$ as restriction along the algebra isomorphisms $a(l)$ and $a(l)_\psi$ of Lemma 5.5, respectively. □

For any 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$ we let

$$f'_\psi : |\Sigma_\psi| \to |\Sigma|$$

denote the corresponding map on projective spectra of cohomology. Specifically, we consider the composite

$$f'_\psi := \big(|\Sigma_\psi| \xrightarrow{\mathrm{res}^*} |\Sigma_K| = |\Sigma|_K \to |\Sigma|\big).$$

Proposition 5.6 tells us that, at any 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G$, the maps f'_ψ fit into a diagram

$$\begin{array}{ccc} |\mathfrak{D}_\psi| & \xrightarrow{\ f_\psi\ } & |\mathfrak{D}| \\ \cong \uparrow \mathscr{L}^*_\psi & & \cong \uparrow \mathscr{L}^* \\ |\Sigma_\psi| & \xrightarrow{\ f'_\psi\ } & |\Sigma| \end{array} \tag{11}$$

of maps of k-schemes, where f_ψ is as in (6).

Now, from [Suslin et al. 1997b, Corollary 5.4.1] we understand that any closed embedding $\Sigma_0 \to \Sigma_1$ of group schemes induces a map on projective spectra of cohomology $|\Sigma_0| \to |\Sigma_1|$ which is *universally injective*. The universal modifier here simply indicates that each base change $|\Sigma_0|_K \to |\Sigma|_K$ is also injective. So the above diagram (11) implies the following basic result.

Proposition 5.7. *Consider a smooth algebraic group* $\mathbb{G}$, *and take* $G = \mathbb{G}_{(r)}$. *Suppose that* $\mathbb{G}$ *admits a quasilogarithm. Let* $\psi : \mathbb{G}_{a(s)} \to G$ *be an embedded* 1*-parameter subgroup which is defined over* k. *Then the induced map on projective spectra of cohomology*

$$f_\psi : |\mathfrak{D}_\psi| \to |\mathfrak{D}|$$

is universally injective.

The system of equivalences (10), which we view as a family of equivalences parametrized by the space of 1-parameter subgroups in G, can be leveraged in quite substantive ways in an analysis of support for the double $\mathfrak{D}$. Indeed, the following two sections essentially argue this point in both the finite-dimensional and infinite-dimensional context.

6. Support and tensor products for finite-dimensional representations

As in the previous section, we consider a Frobenius kernel G in a smooth algebraic group $\mathbb{G}$ which admits a quasilogarithm. We prove that cohomological support for the Drinfeld double $\mathfrak{D} = D(G)$ satisfies the tensor product property

$$|\mathfrak{D}|_{(V\otimes W)} = |\mathfrak{D}|_V \cap |\mathfrak{D}|_W. \tag{12}$$

Here V and W are specifically *finite-dimensional* representations over $\mathfrak{D}$. This result appears in Theorem 6.11 below. Our proof of Theorem 6.11 relies on an analysis of cohomological support, and the tensor product property, for representations over the local family $\mathfrak{D}_\psi$.

For any given $\mathfrak{D}_\psi$ we argue that the behaviors of cohomological support are, essentially, independent of the choice of coproduct. We elaborate on this point in Sections 6B and 6C below.

In Section 7, we provide an extension of cohomological support, and of the identity (12), to the big representation category $\mathrm{Rep}(\mathfrak{D})$. Such an extension allows us to apply methods of Rickard [1997] to show that cohomological support can also be used to classify thick tensor ideals in the stable representation category for $\mathfrak{D}$.

6A. ***Comparison with the π-point support of the Appendix.*** Before we begin, let us make a few points of comparison between the material of this section and the material of the Appendix, for the π-point orientated reader. In the appendix we produce a π-point support theory for the double $\mathfrak{D}$, essentially by restricting to the local subalgebras $\mathfrak{D}_\psi$ and considering such a theory for $\mathfrak{D}_\psi$.

We note that the proof of the tensor product property for cohomological support is, arguably, more difficult than the proof for π-point support (Theorem A.14 below). However, the proof that π-support *agrees* with cohomological support uses precisely the same technology which is used in the proof of the tensor product property for cohomological support. So, depending on one's inclinations, one may view Theorem 6.11 below essentially *as* the claim that π-point support and cohomological support agree for Drinfeld doubles of the prescribed form.

6B. ***Supports and thick ideals for local Hopf algebras.*** Let A be a finite-dimensional, *local*, Hopf algebra. Suppose additionally that A has finite type cohomology.

For A as prescribed, the support (4) of a given finite-dimensional representation V can be computed as the support of the sheaf associated to the $\mathrm{Ext}^*_A(k,k)$-module $\mathrm{Ext}^*_A(k,V)$, where we act via the first coordinate

$$|A|_V = \mathrm{Supp}_{|A|}\,\mathrm{Ext}^*_A(k,V)^\sim; \tag{13}$$

see for example [Benson 1991, Proposition 5.7.1] or [Pevtsova and Witherspoon 2009, Proposition 2]. That is to say, the support spaces $|A|_V$ do not depend on the choice of Hopf structure on A.

Let us write $D^b(A)$ for the bounded derived category of finite-dimensional A-representations. Recall that a thick subcategory in $D^b(A)$ is a full triangulated subcategory which is closed under taking summands, and a thick ideal in $D^b(A)$ is a thick subcategory which is additionally closed under the (left and right) tensor actions of $D^b(A)$ on itself. The following lemma is strongly related to the above identification (13).

Lemma 6.1. *Consider a finite-dimensional local Hopf algebra A which has finite type cohomology. Any thick subcategory in $D^b(A)$ is stable under the tensor action of $D^b(A)$ on itself. That is to say, the collection of thick ideals in $D^b(A)$ is identified with the collection of thick subcategories in $D^b(A)$.*

Proof. Locality tells us that any complex V in $D^b(A)$ is obtainable from the trivial representation via a finite sequence of extensions. It follows that for any object W in $D^b(A)$, the product $V\otimes W$ is obtainable from $W = k\otimes W$ via a finite sequence of extensions. Hence $V\otimes W$ is contained in the thick subcategory generated by W, for arbitrary V and W in $D^b(A)$. Similarly, $W\otimes V$ is contained in the thick ideal generated by W.

Now, let $\mathscr{K} \subset D^b(A)$ be any thick subcategory. By the above discussion we have $V \otimes \mathscr{K} \subset \mathscr{K}$ and $\mathscr{K} \otimes V \subset \mathscr{K}$ for all V in $D^b(A)$. This shows that $\mathscr{K}$ is a thick ideal. Hence the inclusion

$$\{\text{thick ideals in } D^b(A)\} \to \{\text{thick subcategories in } D^b(A)\}$$

is an equality. □

We note that the definition of support (4) works perfectly well for arbitrary objects in the bounded derived category. Furthermore, when A is local the expression (13) remains valid for any V in $D^b(A)$.

For an exact triangle $V \to W \to V'$ in $D^b(A)$, the long exact sequence in cohomology provides an exact sequence of $\mathrm{Ext}^*_A(k, k)$-modules

$$\mathrm{Ext}^*_A(k, V) \to \mathrm{Ext}^*_A(k, W) \to \mathrm{Ext}^*_A(k, V').$$

So there is an inclusion of supports $|A|_W \subset (|A|_V \cup |A|_{V'})$ whenever we have such a triangle. Additionally, for any sum $V = V_1 \oplus V_2$ in $D^b(A)$ we have an equality $|A|_V = |A|_{V_1} \cup |A|_{V_2}$. From these observations we deduce an inclusion

$$|A|_W \subset |A|_V \quad \text{whenever } W \text{ is in the thick subcategory generated by } V.$$

Lemma 6.2. *Consider a finite-dimensional local Hopf algebra A. For any V and W in $D^b(A)$ there is an inclusion*

$$|A|_{(V\otimes W)} \subset (|A|_V \cap |A|_W).$$

Proof. The object $V \otimes W$ is in the thick ideal generated by V, and hence the thick subcategory generated by V by Lemma 6.1. So $|A|_{(V\otimes W)} \subset |A|_V$ by the above reasoning. We similarly find $|A|_{(V\otimes W)} \subset |A|_W$, which gives the claimed inclusion $|A|_{(V\otimes W)} \subset |A|_V \cap |A|_W$. □

We note that the inclusion of Lemma 6.2 does *not* hold for an arbitrary Hopf algebra A. One can see for example [Benson and Witherspoon 2014].

Remark 6.3. The familiar reader is free to replace the derived category $D^b(A)$ with the stable category $\mathrm{stab}(A)$ in the above discussion.

6C. *Classification of thick ideals for local algebras.*

Definition 6.4. Let A be a finite-dimensional Hopf algebra which has finite type cohomology. We say that cohomological support for A *classifies thick ideals* in $D^b(A)$ if an inclusion of supports $|A|_W \subset |A|_V$, for nonzero W and V in $D^b(A)$, implies that W is in the thick ideal generated by V in $D^b(A)$.

The supposition that W and V are nonzero (nonacyclic) is necessary to avoid issues with perfect complexes. Namely, any perfect complex has vanishing support, and yet the ideal of perfect complexes in $D^b(A)$ is not contained in the ideal of acyclic complexes. However, for nonzero V, we always have that $\mathrm{perf}(A)$ is contained in the thick ideal generated by V.

One can consider representation categories of finite group schemes, for example. In this case we understand [Friedlander and Pevtsova 2007] that cohomological support does in fact classify thick ideals in the associated derived category.

Theorem 6.5 [Friedlander and Pevtsova 2007, Theorem 6.3]. *For any finite group scheme G, cohomological support classifies thick ideals in $D^b(G)$.*

When G is furthermore *unipotent*, or rather when $\mathrm{rep}(G)$ is a local category, Theorem 6.5 and Lemma 6.1 combine to give the following.

Corollary 6.6. *Suppose that G is a finite unipotent group scheme. Then thick* ***subcategories*** *in $D^b(G)$ are classified by cohomological support.*

The following will prove quite useful in our analysis of support for the local Hopf algebras $\mathfrak{D}_\psi$.

Proposition 6.7. *Let A be a finite-dimensional local algebra. Suppose that A admits a Hopf algebra structure for which cohomological support classifies thick ideals in the derived category $D^b(A)$. Then under* ***any*** *choice of Hopf structure on A, and any choice of objects V and W in $D^b(A)$, we have an equality*

$$|A|_{(V\otimes W)} = |A|_V \cap |A|_W.$$

Proof. Let $\langle X\rangle$ denote the thick subcategory generated by a given object X in $D^b(A)$. For any object L in $\langle V\rangle$ the product $L\otimes W$ is in $\langle V\otimes W\rangle$, and hence $|A|_{(L\otimes W)}\subset |A|_{(V\otimes W)}$. Consider a product of Carlson modules L for which $|A|_L = |A|_V$. Since cohomological support classifies thick ideals, such equality of supports implies an equality $\langle L\rangle = \langle V\rangle$. Then by Proposition 2.13 we have

$$|A|_{(V\otimes W)} \supset |A|_{(L\otimes W)} = |A|_L \cap |A|_W = |A|_V \cap |A|_W.$$

The opposite inclusion is covered by Lemma 6.2, so that we obtain the desired equality. □

6D. ***Implications for $\mathfrak{D}_\psi$.*** Fix a smooth algebraic group $\mathbb{G}$ which admits a quasilogarithm and an arbitrary positive integer r. Let G be the r-th Frobenius kernel in $\mathbb{G}$. We consider the Drinfeld double $\mathfrak{D} = D(G)$.

For such G, we have the corresponding infinitesimal group scheme $\Sigma = \Sigma(G, r)$ of Definition 5.4, and for any 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$ we have an associated unipotent subgroup $\Sigma_\psi \subset \Sigma_K$. By Proposition 5.6, any choice of quasilogarithm for $\mathbb{G}$ determines a compatible collection of linear equivalences

$$\mathscr{L} : \mathrm{rep}(\mathfrak{D}) \xrightarrow{\sim} \mathrm{rep}(\Sigma) \quad \text{and} \quad \mathscr{L}_\psi : \mathrm{rep}(\mathfrak{D}_\psi) \xrightarrow{\sim} \mathrm{rep}(\Sigma_\psi), \tag{14}$$

which preserve the unit objects in the respective categories

Since cohomological support for a local Hopf algebra depends only on the abelian structure on the representation category, we see that the diagram of (11) restricts to a diagram

$$\begin{array}{ccc} |\mathfrak{D}_\psi|_V & \xrightarrow{f_\psi} & |\mathfrak{D}| \\ \cong \uparrow \mathscr{L}_\psi^* & & \cong \uparrow \mathscr{L}^* \\ |\Sigma_\psi|_{\mathscr{L}_\psi V} & \xrightarrow[f'_\psi]{} & |\Sigma| \end{array} \tag{15}$$

for any V in $D^b(\mathfrak{D}_\psi)$. Hence the discussions of Subsections 6B and 6C imply the following.

Proposition 6.8. *Let G be as above, and fix an embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$. Then the following hold:*

(1) *Thick ideals in $D^b(\mathfrak{D}_\psi)$ are classified by cohomological support.*

(2) *For any finite-dimensional $\mathfrak{D}_\psi$-representations V and W we have*

$$|\mathfrak{D}_\psi|_{(V\otimes W)} = |\mathfrak{D}_\psi|_V \cap |\mathfrak{D}_\psi|_W.$$

Proof. From the linear equivalence $\mathscr{L}_\psi$, Theorem 6.5, and Lemma 6.1, we understand that thick ideals in $D^b(\mathfrak{D}_\psi)$ are classified by cohomological support, establishing (1). A direct application of Proposition 6.7 now implies (2). □

6E. *Restrictions of support and the tensor product property.* As above, let G be the r-th Frobenius kernel in a smooth algebraic group $\mathbb{G}$, and suppose that $\mathbb{G}$ admits a quasilogarithm.

Lemma 6.9. *Let $\mathscr{L} : \mathrm{rep}(\mathfrak{D}) \to \mathrm{rep}(\Sigma)$ be the linear equivalence induced by a choice of quasilogarithm for $\mathbb{G}$. Then for any finite-dimensional $\mathfrak{D}$-representation V the isomorphism $\mathscr{L}^* : |\Sigma| \xrightarrow{\cong} |\mathfrak{D}|$ restricts to an isomorphism of supports $|\Sigma|_{\mathscr{L}V} \xrightarrow{\cong} |\mathfrak{D}|_V$.*

Proof. Via the diagram of equivalences of Proposition 5.6, and Theorem 3.7, we understand that a Σ-representation is projective if and only if its restriction to each of the Σ_ψ is projective. We can therefore repeat the proof of Proposition 3.11 to obtain a reconstruction of support

$$|\Sigma|_W = \bigcup_{\text{1-param subgroups}} f'_\psi(|\Sigma_\psi|_{W_K})$$

for any Σ-representation W, where the f'_ψ are the maps on projective spectra induced by restriction.

The above expression, and the analogous expression of Proposition 3.11, therefore imply the claimed equality. To argue this point more clearly, take a point $x \in |\Sigma|_{\mathscr{L}V}$. Then x is in the image of some map $f'_\psi : |\Sigma_\psi|_{\mathscr{L}_\psi V_K} \to |\Sigma|$. It follows by the diagram (15) that $\mathscr{L}^*(x) \in |\mathfrak{D}|$ is in the image of the corresponding map $f_\psi : |\mathfrak{D}_\psi|_{V_K} \to |\mathfrak{D}|$. Hence $\mathscr{L}^*(x) \in |\mathfrak{D}|_V$. This gives an inclusion $\mathscr{L}^*(|\Sigma|_{\mathscr{L}V}) \subset |\mathfrak{D}|_V$. Since this argument is completely symmetric, we obtain the opposite inclusion as well and find that we have an identification $\mathscr{L}^*(|\Sigma|_{\mathscr{L}V}) = |\mathfrak{D}|_V$. □

Recall from Proposition 5.7 that, for any embedded 1-parameter subgroup ψ which is defined over k, the map $f_\psi : |\mathfrak{D}_\psi| \to |\mathfrak{D}|$ is universally injective. Furthermore, in this case f_ψ is simply the map induced by restriction (i.e., it involves no base change).

Proposition 6.10. *Consider any embedded* 1*-parameter subgroup* $\psi : \mathbb{G}_{a(s)} \to G$ *which is defined over* k*, and identify* $|\mathfrak{D}_\psi|$ *with a closed subscheme in* $|\mathfrak{D}|$ *via the map induced by restriction* (*Proposition 5.7*)*. Then for any finite-dimensional* $\mathfrak{D}$*-representation* V *we have*

$$|\mathfrak{D}_\psi|_V = |\mathfrak{D}_\psi| \cap |\mathfrak{D}|_V.$$

Proof. By the diagram (15), and Lemma 6.9, it suffices to check that we have an equality

$$|\Sigma_\psi|_W = |\Sigma_\psi| \cap |\Sigma|_W$$

for any finite-dimensional Σ-representation W. However, the above equality follows from the analysis of support for infinitesimal group schemes given in [Suslin et al. 1997b] — in particular [loc. cit., Corollary 5.4.1, Proposition 7.4]. □

We can now prove that cohomological support for the Drinfeld double $\mathfrak{D}$ satisfies the tensor product property.

Theorem 6.11. *Consider a Frobenius kernel* $G = \mathbb{G}_{(r)}$ *in a smooth algebraic group* $\mathbb{G}$*. Suppose also that* $\mathbb{G}$ *admits a quasilogarithm. Then for any finite-dimensional* $\mathfrak{D}$*-representations* V *and* W *we have*

$$|\mathfrak{D}|_{(V\otimes W)} = |\mathfrak{D}|_V \cap |\mathfrak{D}|_W$$

Proof. Consider any point in the intersection $x \in |\mathfrak{D}|_V \cap |\mathfrak{D}|_W$, and let $\psi : \mathbb{G}_{a(s),K} \to G_K$ be any embedded 1-parameter subgroup for which x is in the image of the map $|\mathfrak{D}_\psi| \to |\mathfrak{D}|$. Let $x' \in |\mathfrak{D}_K|$ be any lift of x. Since the support of V_K (resp. W_K) over $\mathfrak{D}_K$ is simply the preimage of $|\mathfrak{D}|_V$ (resp. $|\mathfrak{D}|_W$) along the projection $|\mathfrak{D}_K| \to |\mathfrak{D}|$, by Lemma 3.10, we have $x' \in |\mathfrak{D}_K|_{V_K} \cap |\mathfrak{D}_K|_{W_K}$. So, by changing base, we may assume that x is in the image of $|\mathfrak{D}_\psi|$, where now $\psi : \mathbb{G}_{a(s)} \to G$ a 1-parameter subgroup which is defined over k.

Since x is in $|\mathfrak{D}|_V$, $|\mathfrak{D}|_W$, and $|\mathfrak{D}_\psi|$, Proposition 6.10 implies

$$x \in |\mathfrak{D}_\psi|_V \cap |\mathfrak{D}_\psi|_W.$$

By the tensor product property for $\mathfrak{D}_\psi$, Proposition 6.8, we then have $x \in |\mathfrak{D}_\psi|_{(V\otimes W)}$. From the inclusion $|\mathfrak{D}_\psi|_X \subset |\mathfrak{D}|_X$, for arbitrary X, we see that x is in $|\mathfrak{D}|_{(V\otimes W)}$ as well. We therefore have an inclusion $(|\mathfrak{D}|_V \cap |\mathfrak{D}|_W) \subset |\mathfrak{D}|_{(V\otimes W)}$.

For the opposite inclusion $|\mathfrak{D}|_{(V\otimes W)} \subset (|\mathfrak{D}|_V \cap |\mathfrak{D}|_W)$, one can restrict to some choice of $\mathfrak{D}_\psi$ and argue similarly. However, since the representation category rep($\mathfrak{D}$) is braided, this opposite inclusion actually comes for free; see for example [Bergh et al. 2021, Proposition 3.3]. □

7. Support and tensor products for infinite-dimensional representations

We consider support for infinite-dimensional representations over the Drinfeld double $\mathfrak{D} = D(G)$. The support theory which we employ is a kind of "hybrid theory", which we produce via the restriction functors $\mathrm{rep}(\mathfrak{D}) \to \mathrm{rep}(\mathfrak{D}_\psi)$ and the Benson–Iyengar–Krause (local cohomology) support theory for the $\mathfrak{D}_\psi$. We prove that this hybrid support theory detects projectivity of arbitrary $\mathfrak{D}$-representations, and admits a sufficiently strong tensor product property.

The results of this section provide the necessary foundations for our analysis of thick ideals in the (small) stable category $\mathrm{stab}(\mathfrak{D})$ in Section 8.

7A. ***Stable categories.*** Let A be a finite-dimensional Hopf algebra. We consider the stable categories $\mathrm{stab}(A)$ and $\mathrm{Stab}(A)$ for A. These are the quotient categories of $\mathrm{rep}(A)$ and $\mathrm{Rep}(A)$, respectively, by the tensor ideal consisting of all morphisms which factor through a projective.

In addition to the derived category $D^b(A)$ of finite-dimensional representations over A, we consider

$$D^b_{\mathrm{big}}(A) = \{\text{The bounded derived category of arbitrary } A\text{-representations}\}.$$

We have canonical equivalences to the Verdier quotients

$$\mathrm{stab}(A) \xrightarrow{\sim} D^b(A)/\langle \mathrm{proj}(A)\rangle, \quad \mathrm{Stab}(A) \xrightarrow{\sim} D^b_{\mathrm{big}}(A)/\langle \mathrm{Proj}(A)\rangle$$

[Rickard 1989; Friedlander 2021b, Theorem 4.2], which provide the stable categories with triangulated structures. These equivalences also provide actions of the extension algebra $\mathrm{Ext}^*_A(k,k)$ on the stable representation categories

$$-\otimes M : \mathrm{Ext}^*_A(k,k) \to \mathrm{Hom}^*_{\mathrm{Stab}}(M,M) \quad \forall M \in \mathrm{Stab}(A).$$

The inclusion $\mathrm{stab}(A) \to \mathrm{Stab}(A)$ is exact and fully faithful, and identifies the small stable category with the subcategory of compact objects in $\mathrm{Stab}(A)$.

7B. ***Local cohomology support.*** Let A be a finite-dimensional Hopf algebra with finite type cohomology. We suppose additionally that cohomological support for finite-dimensional A-representations satisfies the inclusion

$$|A|_{V\otimes W} \subset (|A|_V \cap |A|_W). \tag{16}$$

For example, we might consider A to be a local Hopf algebra with finite type cohomology (see Lemma 6.2).

Take $E_A := \mathrm{Ext}^*_A(k,k)$. As remarked above, we have natural actions of E_A on objects in the big stable category $\mathrm{Stab}(A)$, which collectively constitute a map to the graded center $E_A \to Z(\mathrm{Stab}(A)) = \mathrm{End}_{\mathrm{Fun}}(\mathrm{id}_{\mathrm{Stab}(A)})$. Given this situation, we can consider the local cohomology support of Benson, Iyengar, and Krause [Benson et al. 2008]. This support theory is defined via certain triangulated endofunctors $\Gamma_p : \mathrm{Stab}(A) \to \mathrm{Stab}(A)$ associated to (arbitrary) points in the *homogeneous* spectrum $|A| \cup \{m\} = \mathrm{Proj}(E_A) \cup \{m\}$. Here m is the maximal ideal of all positive degree elements in E_A, i.e., the irrelevant ideal, and the homogeneous spectrum is topologized in such a way that m becomes the unique closed point, and the complement $|A|$ to m is given its usual topology as the projective spectrum of cohomology. We have explicitly

$$\mathrm{supp}^{lc}_A(M) := \{p \in |A| \cup \{m\} : \Gamma_p(M) \neq 0\} \tag{17}$$

[Benson et al. 2008, Section 5.1]. We note that the points p appearing in the above formula are not necessarily closed, and that supports of objects in $\mathrm{Stab}(A)$ are not necessarily closed in the space $|A| \cup \{m\}$.

Since the support theory (17) is defined via the vanishing of certain triangulated endofunctors, it behaves appropriately under sums, shifts, and exact triangles. Specifically, the support of a sum $M \oplus M'$ is the union of the supports of M and M', support is invariant under the shift automorphism, and the support of an object N which fits into a triangle $M \to N \to M' \to \Sigma M$ is contained in the union $\mathrm{supp}^{lc}_A(M) \cup \mathrm{supp}^{lc}_A(M')$.

The following lemma is implicit in the literature, though we did not find a direct proof; cf. [Benson et al. 2008, Section 10]. So we give a proof here.

Lemma 7.1. *Let A be as above. The irrelevant ideal m is not contained in the local cohomology support $\mathrm{supp}^{lc}_A(M)$ of any object in $\mathrm{Stab}(A)$. Furthermore, for any finite-dimensional representation V there is an identification $\mathrm{supp}^{lc}_A(V) = |A|_V$.*

Proof. Let S be the sum of all simple A-representations, and consider any point p in the homogeneous spectrum $|A| \cup \{m\}$. The Koszul object $S/\!\!/p$ of [Benson et al. 2008] is, up to a shift, the tensor product $L_p \otimes S$ where L_p is a product of Carlson modules whose cohomological support is equal to the (projectivized) vanishing locus of p, $|A|_{L_p} = Z(p)$. In particular, L_m has vanishing cohomological support, and is thus projective over A. It follows that $L_m \otimes S$ vanishes in the stable category, as does $S/\!\!/m$.

We apply [Benson et al. 2008, Proposition 5.12] to see that vanishing of $S/\!\!/m$ implies vanishing of the stable morphisms $\mathrm{Hom}_{\mathrm{Stab}}(S, \Gamma_m(M))$, for any M in $\mathrm{Stab}(A)$. Since $\mathrm{Stab}(A)$ is generated by the simple A-representations, vanishing of $\mathrm{Hom}_{\mathrm{Stab}}(S, \Gamma_m(M))$ implies that $\Gamma_m(M) = 0$ in the stable category. Hence $m \notin \mathrm{supp}^{lc}_A(M)$ and we see that local cohomology support takes values in the projective spectrum $|A|$, as claimed.

We now consider the equality $\mathrm{supp}^{lc}_A(V) = |A|_V$ for finite-dimensional V. Let W be an arbitrary finite-dimensional representation. We have the natural map $f : \mathrm{Ext}^*_A(W, V) \to \mathrm{Hom}^*_{\mathrm{Stab}}(W, V)$ induced by the functor $D^b(A) \to \mathrm{Stab}(A)$. This map has m-torsion kernel and cokernel; see, e.g., [Benson and Krause 2002, Equation (2.3)]. It follows that f induces an isomorphism on all localizations $\mathrm{Ext}^*_A(W, V)_p \cong \mathrm{Hom}^*_{\mathrm{Stab}}(W, V)_p$ at points p in the projective spectrum $|A|$. Hence by [Benson et al. 2008, Lemma 2.2] the homogeneous supports of these two objects, defined as in [loc. cit., Section 2], agree modulo a consideration of the maximal ideal m. (That is to say, the homogeneous supports have the same intersection with $|A|$.) We consider the case where W is the sum of the simples, and note again that $m \notin \mathrm{supp}^{lc}_A(V)$, to observe finally that $\mathrm{supp}^{lc}_A(V) = |A|_V$ by [loc. cit., Theorem 5.13]. □

By Lemma 7.1 we can now consider local cohomology support supp^{lc}_A as a support theory which takes values in the projective, rather than homogeneous spectrum. Indeed, we can simply omit the extraneous point m from the definition and write simply

$$\mathrm{supp}^{lc}_A(M) = \{p \in |A| : \Gamma_p(M) \neq 0\} \subset |A|.$$

We understand furthermore that the support supp^{lc}_A provides an extension of cohomological support, which we have only defined for the small stable category, to all of $\mathrm{Stab}(A)$.

By pulling back along the quotient $D^b_{\mathrm{big}}(A) \to \mathrm{Stab}(A)$, we may consider local cohomology support supp^{lc}_A as a support theory which takes A-complexes as inputs as well.

Theorem 7.2 [Benson et al. 2008]. *For A as above, the following hold*:

(1) *M vanishes in* $\mathrm{Stab}(A)$ *if and only if* $\mathrm{supp}^{lc}_A(M) = \varnothing$.

(2) *For arbitrary M and N in $D^b_{\mathrm{big}}(A)$, local cohomology support satisfies*

$$\mathrm{supp}^{lc}_A(M \otimes N) \subset (\mathrm{supp}^{lc}_A(M) \cap \mathrm{supp}^{lc}_A(N)).$$

Proof. Statement (1) is covered in [Benson et al. 2008, Theorem 5.13]. For the claimed inclusion (2), we note that for any specialization closed subset $\Theta \subset |A|$ the containment (16) tells us that the subcategory

$$\mathscr{K}_\Theta := \{V \text{ in } \mathrm{stab}(A) : |A|_V \subset \Theta\}$$

is a thick ideal in $\mathrm{stab}(A)$. Thus one follows the proof of [loc. cit., Theorem 8.2] to see that

$$\Gamma_p(M \otimes N) = M \otimes \Gamma_p(N) = \Gamma_p(M) \otimes N.$$

From the above equation, and the definition of the support supp^{lc}_A, we deduce the inclusion of (2). □

7C. *$\boldsymbol{\psi}$-local support for $\mathfrak{D}$-representations.* Consider an infinitesimal group scheme G, with associated Drinfeld double $\mathfrak{D} = D(G)$. Let M be an object in the bounded derived category $D^b_{\mathrm{big}}(\mathfrak{D})$ of arbitrary $\mathfrak{D}$-representations, and recall the maps $f_\psi : |\mathfrak{D}_\psi| \to |\mathfrak{D}|$ induced by restriction (6). We define the support

$$\mathrm{supp}^{\boldsymbol{\psi}\text{-loc}}(M) := \bigcup_{\text{1-param subgroups}} f_\psi(\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(\mathrm{res}_\psi M_K)), \tag{18}$$

where the union runs over all embedded 1-parameter subgroups $\psi : \mathbb{G}_{a(s),K} \to G_K$, and $\mathrm{res}_\psi : \mathrm{rep}(G_K) \to \mathrm{rep}(\mathbb{G}_{a(s),K})$ denotes the restriction functor. As in Proposition 3.11, (18) defines the support $\mathrm{supp}^{\boldsymbol{\psi}\text{-loc}}(M)$ as a union of sub*sets* in the projective spectrum of cohomology $|\mathfrak{D}|$.

We refer to the support (18) as the $\boldsymbol{\psi}$-local support of M. Note that this support takes values in the projective spectrum of cohomology $|\mathfrak{D}|$. By pulling back along the quotient map

$$D^b_{\mathrm{big}}(\mathfrak{D}) \to \mathrm{Stab}(\mathfrak{D})$$

we freely consider the $\boldsymbol{\psi}$-local support as a support theory for the bounded derived category of arbitrary $\mathfrak{D}$-representations as well.

Remark 7.3. We have used a boldface $\boldsymbol{\psi}$ in our notation to indicate that $\boldsymbol{\psi}$ might be thought of as a coordinate which ranges over the space of 1-parameter subgroups.

We list some basic properties of ψ-local support.

Lemma 7.4. *For any infinitesimal group scheme G, ψ-local support satisfies the following*:

- $\mathrm{supp}^{\psi\text{-loc}}(M)=\varnothing$ *if and only if M vanishes in the stable category* $\mathrm{Stab}(\mathfrak{D})$.
- $\mathrm{supp}^{\psi\text{-loc}}(M\oplus N)=\mathrm{supp}^{\psi\text{-loc}}(M)\cup\mathrm{supp}^{\psi\text{-loc}}(N)$.
- *For any triangle $M\to N\to M'$*,
$$\mathrm{supp}^{\psi\text{-loc}}(N)\subset(\mathrm{supp}^{\psi\text{-loc}}(M)\cup\mathrm{supp}^{\psi\text{-loc}}(M')).$$
- $\mathrm{supp}^{\psi\text{-loc}}(M\otimes N)\subset(\mathrm{supp}^{\psi\text{-loc}}(M)\cap\mathrm{supp}^{\psi\text{-loc}}(N))$.
- $\mathrm{supp}^{\psi\text{-loc}}(\Sigma M)=\mathrm{supp}^{\psi\text{-loc}}(M)$.
- *For any V in $D^b(\mathfrak{D})$*, $\mathrm{supp}^{\psi\text{-loc}}(V)=|\mathfrak{D}|_V$.

In the above formulas M, M', and N are arbitrary objects in $D^b_{\mathrm{big}}(\mathfrak{D})$.

Proof. The first point follows by the projectivity test of Theorem 3.7, and the detection property for local cohomology support over $\mathfrak{D}_\psi$. The four subsequent points follow directly from the corresponding properties for the local cohomology supports $\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}$, and the fact that restriction is an exact tensor functor. The final point follows from the identification $\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(V_K)=|\mathfrak{D}_\psi|_{V_K}$ and the reconstruction formula of Proposition 3.11. □

7D. *ψ-local support and tensor products.*

Theorem 7.5. *Consider a Frobenius kernel G in a smooth algebraic group $\mathbb{G}$. Suppose that $\mathbb{G}$ admits a quasilogarithm. Then for any object V in $D^b(\mathfrak{D})$, and any M in $D^b_{\mathrm{big}}(\mathfrak{D})$, we have*
$$\mathrm{supp}^{\psi\text{-loc}}(V\otimes M)=\mathrm{supp}^{\psi\text{-loc}}(V)\cap\mathrm{supp}^{\psi\text{-loc}}(M). \tag{19}$$

Note that, since $\mathrm{Rep}(\mathfrak{D})$ is a braided monoidal category, an identification (19) implies the corresponding equality for the action of finite-dimensional representations (or complexes) on the right
$$\mathrm{supp}^{\psi\text{-loc}}(M\otimes V)=\mathrm{supp}^{\psi\text{-loc}}(M)\cap\mathrm{supp}^{\psi\text{-loc}}(V),$$
simply because $V\otimes M\cong M\otimes V$. In the language of [Negron and Pevtsova 2021, Definition 4.7], we are claiming that cohomological support for $\mathfrak{D}$ is a *lavish support theory* for the stable category $\mathrm{stab}(\mathfrak{D})$.

Before proving Theorem 7.5, we prove its local analog.

Proposition 7.6. *Let G be as in the statement of Theorem 7.5, and consider an embedded 1-parameter subgroup $\psi:\mathbb{G}_{a(s)}\to G$ which is defined over k. Then for W in $D^b(\mathfrak{D}_\psi)$, and N in $D^b_{\mathrm{big}}(\mathfrak{D}_\psi)$, local cohomology support satisfies*
$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W\otimes N)=\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W)\cap\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N).$$

Proof. It suffices to prove the inclusion

$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N)\subset \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W\otimes N),$$

since the opposite inclusion follows by Theorem 7.2. Since the local cohomology support is defined via the vanishing of the exact endomorphisms Γ_p, we understand that if Q' in $\mathrm{Stab}(\mathfrak{D}_\psi)$ is in the thick subcategory generated by Q then $\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(Q')\subset \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(Q)$. So it suffices to prove that there is an equality

$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N)= \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(L\otimes N)$$

for some L in the thick subcategory generated by W in $\mathrm{stab}(\mathfrak{D}_\psi)$.

Let L be a product of Carlson modules such that $\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(L)=\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W)$. By Lemma 6.1 and Proposition 6.8, the object L is in the thick subcategory generated by W in $\mathrm{stab}(\mathfrak{D}_\psi)$ and thus $L\otimes N$ is in the thick subcategory generated by $W\otimes N$ in $\mathrm{Stab}(\mathfrak{D}_\psi)$.

Recall that, in the stable category, the Carlson module L_ζ associated to an extension $\zeta : k\to\Sigma^n k$ is isomorphic to a shift of the mapping cone $\mathrm{cone}(\zeta)$. So by [Benson et al. 2011a, Lemma 2.6] we have

$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(L_\zeta\otimes N)=Z(\zeta)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N)=\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(L_\zeta)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N)$$

for any such L_ζ. It follows that, for our product of Carlson modules L, we have

$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(L\otimes N)=\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(L)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N)=\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(W)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(N),$$

as desired. □

We now prove our theorem.

Proof of Theorem 7.5. We have already observed one inclusion in Lemma 7.4. So we need only establish the inclusion

$$\mathrm{supp}^{\boldsymbol{\psi}\text{-loc}}(V)\cap \mathrm{supp}^{\boldsymbol{\psi}\text{-loc}}(M)\subset \mathrm{supp}^{\boldsymbol{\psi}\text{-loc}}(V\otimes M). \tag{20}$$

Consider any point x in the above intersection, and choose an embedded subgroup $\psi:\mathbb{G}_{a(s),K}\to G_K$ for which x is the image of a point $x'\in \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(M_K)$. The naturality property

$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(V_K)=|\mathfrak{D}_\psi|\cap|\mathfrak{D}_K|_{V_K}$$

of Proposition 6.10 implies that x' is in $\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(V_K)$ as well. (See also Lemma 3.10.) We apply the equality

$$\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(V_K\otimes M_K)=\mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(V_K)\cap \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(M_K)$$

of Proposition 7.6 to see that $x'\in \mathrm{supp}^{lc}_{\mathfrak{D}_\psi}(V_K\otimes M_K)$, and hence $x\in \mathrm{supp}^{\boldsymbol{\psi}\text{-loc}}(V\otimes M)$ by the definition of the $\boldsymbol{\psi}$-local support. We thus verify the inclusion (20), and obtain the proposed tensor product property. □

8. Thick ideals and the Balmer spectrum

We provide a classification of thick ideals in the stable category $\mathrm{stab}(\mathfrak{D})$, for $\mathfrak{D}$ the Drinfeld double of an appropriate Frobenius kernel. We then apply results of Balmer to calculate the spectrum of prime ideals in the stable category $\mathrm{stab}(\mathfrak{D})$. In particular, we show that thick ideals are classified by specialization closed subsets in the projective spectrum of cohomology $|\mathfrak{D}|$, and we show that the Balmer spectrum is isomorphic to the cohomological spectrum $|\mathfrak{D}|$ as a locally ringed space.

8A. ***Classification of thick ideals and prime ideal spectra.*** Let $\mathfrak{D}$ be the Drinfeld double of a finite group scheme. Recall that a specialization closed subset Θ in $|\mathfrak{D}| = \operatorname{Proj}\operatorname{Ext}^*_{\mathfrak{D}}(k,k)$ is a subset which contains the closures of all of its points. Equivalently, a specialization closed subset is an arbitrary union of closed subsets in $|\mathfrak{D}|$.

For any specialization closed subset Θ in $|\mathfrak{D}|$ we have the associated thick ideal

$$\mathscr{K}_\Theta := \{V \in \mathrm{stab}(\mathfrak{D}) : |\mathfrak{D}|_V \subset \Theta\}$$

in the stable category $\mathrm{stab}(\mathfrak{D})$. To see that $\mathscr{K}_\Theta$ is in fact closed under the tensor actions $\mathrm{stab}(\mathfrak{D})$ on the left and right, one simply consults the inclusion $|\mathfrak{D}|_{V\otimes W} \subset (|\mathfrak{D}|_V \cap |\mathfrak{D}|_W)$ provided by the braiding on $\mathrm{rep}(\mathfrak{D})$ [Bergh et al. 2021, Proposition 3.3]. Similarly, for any thick ideal $\mathscr{K} \subset \mathrm{stab}(\mathfrak{D})$ we have the associated support space

$$|\mathfrak{D}|_{\mathscr{K}} := \cup_{V\in\mathscr{K}} |\mathfrak{D}|_V,$$

which is a specialization closed subset in $|\mathfrak{D}|$. We note that the formal properties of cohomological support imply an equality $|\mathfrak{D}|_V = |\mathfrak{D}|_{\langle V\rangle_\otimes}$ between the support of a given object V, and the support of the thick ideal $\langle V\rangle_\otimes$ which it generates in $\mathrm{stab}(\mathfrak{D})$.

The two above operations define maps of sets

$$\{\text{thick ideals in } \mathrm{stab}(\mathfrak{D})\} \underset{|\mathfrak{D}|_?}{\overset{\mathscr{K}_?}{\rightleftarrows}} \{\text{specialization closed subsets in } |\mathfrak{D}|\} \tag{21}$$

which preserve the respective orderings by inclusion. In rephrasing Definition 6.4, we say cohomological support for $\mathfrak{D}$ *classifies thick ideals* in $\mathrm{stab}(\mathfrak{D})$ if the two maps in (21) are mutually inverse bijections.

At this point it is a formality to deduce a classification of thick ideals in the stable category $\mathrm{stab}(\mathfrak{D})$ from the support theoretic results of Lemma 7.4 and Theorem 7.5. One can see for example [Rickard 1997]. We follow the generic presentation of [Negron and Pevtsova 2021].

Theorem 8.1. *Consider a smooth algebraic group $\mathbb{G}$ which admits a quasilogarithm, and let G be a Frobenius kernel in $\mathbb{G}$. Then, for the Drinfeld double $\mathfrak{D} = D(G)$, cohomological support classifies thick ideals in the stable category* $\mathrm{stab}(\mathfrak{D})$. *That is to say, the two maps of* (21) *are mutually inverse bijections.*

Proof. Theorem 7.5 tells us that cohomological support is a lavish support theory for $\mathrm{stab}(\mathfrak{D})$, in the language of [Negron and Pevtsova 2021, Section 4.3]. So the claimed classification follows by [loc. cit., Proposition 5.2]. (Note that all of the centralizing hypotheses in [loc. cit.] are obviated by the existence of a braiding on $\mathrm{rep}(\mathfrak{D})$.) □

We note that, by pulling back along the projection $\pi : D^b(\mathfrak{D}) \to \mathrm{stab}(\mathfrak{D})$, we can similarly use cohomology to classify thick ideals in the bounded derived category for $\mathfrak{D}$. Namely, under the map π thick ideals in $\mathrm{stab}(\mathfrak{D})$ are identified with thick ideals in $D^b(\mathfrak{D})$ which contain the ideal $\mathrm{perf}(\mathfrak{D})$ of bounded complexes of projectives. This subcollection of ideals in $D^b(\mathfrak{D})$ is precisely the collection of nonvanishing ideals in $D^b(\mathfrak{D})$. So we obtain a classification

$$\{\text{thick ideals in } D^b(\mathfrak{D})\} \cong \{\text{specialization closed subsets in } |\mathfrak{D}|\} \cup \{0\}.$$

8B. ***Prime ideal spectra for Drinfeld doubles.*** Consider again the Drinfeld double $\mathfrak{D}$ of a finite group scheme G.

We recall that the sublattice of thick *prime* ideals in $\mathrm{stab}(\mathfrak{D})$ forms a locally ringed space, which is referred to as the Balmer spectrum

$$\mathrm{Spec}(\mathrm{stab}(\mathfrak{D})) := \left\{ \begin{matrix} \text{the collection of thick prime ideals in } \mathrm{stab}(\mathfrak{D}) \\ \text{with the topology and ringed structure described in [Balmer 2005]} \end{matrix} \right\}. \tag{22}$$

As one might expect, by a thick prime ideal in $\mathrm{stab}(\mathfrak{D})$ we mean a proper thick ideal $\mathscr{P}$ for which an inclusion $V \otimes W \in \mathscr{P}$ implies either $V \in \mathscr{P}$ or $W \in \mathscr{P}$. We do not recall the topology or the ringed structure on the spectrum here, and refer the reader instead to the highly readable text [Balmer 2005, Sections 1 and 6].

As explained in [Balmer 2005; 2010a], a classification of thick ideals in $\mathrm{stab}(\mathfrak{D})$ via cohomological support implies a corresponding calculation of the prime ideal spectrum.

Theorem 8.2. *For G as in Theorem 8.1, there is a homeomorphism*

$$f_{\mathrm{coh}} : |\mathfrak{D}| = \mathrm{Proj}\,\mathrm{Ext}^*_{\mathfrak{D}}(k,k) \xrightarrow{\cong} \mathrm{Spec}(\mathrm{stab}(\mathfrak{D}))$$

defined by taking $f_{coh}(x) = \{V \in \mathrm{stab}(\mathfrak{D}) : x \notin |\mathfrak{D}|_V\}$. Furthermore, f_{coh} can be upgraded to an isomorphism of locally ringed spaces.

Proof. Given Theorem 8.1, the fact that f_{coh} is a homeomorphism follows from [Balmer 2005, Theorem 5.2]. By [Balmer 2010a, Proposition 6.11], the homeomorphism f_{coh} furthermore enhances to an isomorphism of locally ringed spaces. To elaborate, in [loc. cit., Definitions 5.1, 6.10] a map of ringed spaces $\rho : \mathrm{Spec}(\mathrm{stab}(\mathfrak{D})) \to |\mathfrak{D}|$ is constructed. One sees directly that the composite $\rho \circ f_{\mathrm{coh}} : |\mathfrak{D}| \to |\mathfrak{D}|$ is the identity, as a map of topological spaces. Since f_{coh} is a homeomorphism, we see that ρ is a homeomorphism as well. It follows by [loc. cit., Proposition 6.11] that ρ is an isomorphism of (locally) ringed spaces, and so provides the homeomorphism $f_{\mathrm{coh}} = \rho^{-1}$ with ringed structure under which it is also an isomorphism of locally ringed spaces. □

Remark 8.3. In [Balmer 2005; 2010a] Balmer only considers symmetric tensor triangulated categories. However, all of the definitions, results, and proofs from [Balmer 2005; 2010a] apply verbatim in the braided context. So, implicitly, we use the fact that $\mathrm{rep}(\mathfrak{D}) = Z(\mathrm{rep}(G))$ admits a canonical (highly non-symmetric!) braided structure in the definition (22), and also in the proof of Theorem 8.2. One can alternatively refer to [Negron and Pevtsova 2021, Section 6] and in particular [loc. cit., Theorem 6.10].

Appendix: A π-point rank variety for the Drinfeld double

We introduce a π-point rank variety $\Pi(\mathfrak{D})$ for the Drinfeld double $\mathfrak{D}$, whose points consist of certain classes of flat algebra maps $K[t]/(t^p) \to \mathfrak{D}_K$. For any $\mathfrak{D}$-representation V we construct an associated support space $\Pi(\mathfrak{D})_V$ in $\Pi(\mathfrak{D})$. We show that the support theory $V \mapsto \Pi(\mathfrak{D})_V$ behaves in the expected manner when we consider the Drinfeld double of a Frobenius kernel $G = \mathbb{G}_{(r)}$ in a sufficiently nice algebraic group $\mathbb{G}$. In particular, the support space $\Pi(\mathfrak{D})_V$ vanishes if and only if the given representation V is projective, and the support spaces satisfy the tensor product property

$$\Pi(\mathfrak{D})_{V\otimes W} = \Pi(\mathfrak{D})_V \cap \Pi(\mathfrak{D})_W.$$

Furthermore, we establish an identification with cohomological support $\Pi(G)_\star \xrightarrow{\cong} |\mathfrak{D}|_\star$. We also show that our π-support can be identified with a certain "universal" π-point support, which we define in Section A5.

Since these results of this section are isolated from those of the body of the text, in a technical sense, we collect them here in an appendix.

A1. ***π-points and support for finite group schemes.*** Throughout this subsection G is a finite group scheme over our base field k. We recall some definitions and results from [Friedlander and Pevtsova 2007].

Definition A.1. A π-point for a finite group scheme G, over k, is a pair of a field extension $k \to K$ and a flat algebra map $\alpha : K[t]/(t^p) \to KG$ which factors through the group ring of an abelian, unipotent subgroup $U \subset G_K$.

We generally abuse notation and simply write α for the pair $(K/k, \alpha)$. Any π-point defines a corresponding point p_α in the projective spectrum of cohomology $|G|$, which is explicitly the homogeneous prime ideal

$$p_\alpha := \ker\big(\mathrm{Ext}^*_G(k,k) \xrightarrow{K\otimes -} \mathrm{Ext}^*_{G_K}(K,K) \xrightarrow{\mathrm{res}_\alpha} \mathrm{Ext}^*_{K[t]/t^p}(K,K)_{\mathrm{red}} = K[T]\big). \tag{23}$$

In the above formula T is a variable of cohomological degree 2 (or 1 in characteristic 2). Flatness of the extension α ensures that the ideal p_α is not all of $\mathrm{Ext}^{>0}_G(k,k)$, so that p_α does in fact define a point in the projective spectrum [Friedlander and Pevtsova 2005, Lemma 3.4]; compare to [Andruskiewitsch et al. 2022, Theorem 3.2.1].

Definition A.2. For a given finite group scheme G, we say two π-points $\alpha : K[t]/(t^p) \to KG$ and $\beta : L[t]/(t^p) \to LG$ are equivalent if any finite-dimensional G-representation V which restricts to a projective $K[t]/(t^p)$-representation $\mathrm{res}_\alpha(V_K)$ along α also restricts to a projective $L[t]/(t^p)$-representation $\mathrm{res}_\beta(V_L)$ along β, and vice versa.

We let $\Pi(G)$ denote the collection of equivalence classes of π-points

$$\Pi(G) = \{[\alpha] : \alpha : K[t]/(t^p) \to KG \text{ is a } \pi\text{-point for } G\}.$$

For any finite-dimensional G-representation V we define the π-support space $\Pi(G)_V$ as

$$\Pi(G)_V = \{[\alpha] : \mathrm{res}_\alpha(V_K) \text{ is nonprojective over } K[t]/(t^p)\}.$$

The collection of subsets $\{\Pi(G)_V : V \in \mathrm{rep}(G)\}$ in $\Pi(G)$ is closed under finite unions, since $\Pi(G)_V \cup \Pi(G)_W = \Pi(G)_{V \oplus W}$. Hence there is a uniquely defined topology on $\Pi(G)$ for which the supports of objects $\Pi(G)_V$ provide a basis of closed subsets.

Theorem A.3 [Friedlander and Pevtsova 2007, Theorem 3.6]. *If two π-points α and β for G are equivalent, then the corresponding points p_α, $p_\beta \in |G|$ are equal. Furthermore, the resulting map*

$$\Pi(G) \to |G|, \quad [\alpha] \mapsto p_\alpha$$

is a homeomorphism, and for any finite-dimensional representation V this homeomorphism restricts to a homeomorphism $\Pi(G)_V \to |G|_V$.

Note that Theorem A.3 tells us that the topological space $\Pi(G)$ is Noetherian. Hence the basic closed sets $\{\Pi(G)_V\}_{V \in \mathrm{rep}(G)}$ in $\Pi(G)$ provide the collection of *all* closed sets in $\Pi(G)$ [Friedlander and Pevtsova 2007, Proposition 3.4].

Remark A.4. One of the main advancements of [Friedlander and Pevtsova 2007] is the observation that one can reasonably define support spaces $\Pi(G)_M$ for *infinite-dimensional* G-representation M. So, the above presentation omits some of the more significant aspects of [loc. cit.]. One can see Remark A.12 below for additional context.

Remark A.5. For infinitesimal G, a direct comparison between π-point support and the rank variety support theory of [Suslin et al. 1997b] can be found at [Friedlander 2021b, Theorem 1.6].

A2. *π-point support for $\mathfrak{D}_\psi$.* We consider an infinitesimal group scheme G, with corresponding Drinfeld double $\mathfrak{D} = D(G)$.

Definition A.6. Consider any infinitesimal group scheme G, and fix an embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s)} \to G$ which is defined over k. A π-point for $\mathfrak{D}_\psi$ is a pair of a field extension $k \to K$, and a flat algebra map $\alpha : K[t]/(t^p) \to (\mathfrak{D}_\psi)_K$ such that:

(a) There exists an *algebra* identification $\mathfrak{D}_\psi = kH$ between $\mathfrak{D}_\psi$ and the group algebra of a finite group scheme H over k.

(b) Under some identification as in (a), α corresponds to a π-point for the given group scheme H.

Statements (a) and (b) above can alternately be stated as follows: a π-point for $\mathfrak{D}_\psi$ is a flat algebra map $\alpha : K[t]/(t^p) \to (\mathfrak{D}_\psi)_K$ which is a π-point for $\mathfrak{D}_\psi$ relative to some *alternate* choice of *cocommutative* Hopf structure Δ' on $\mathfrak{D}_\psi$. We note that any group scheme H as in (a) is necessarily unipotent, since $\mathfrak{D}_\psi$ is local.

We say two π-points $\alpha : K[t]/(t^p) \to (\mathfrak{D}_\psi)_K$ and $\beta : L[t]/(t^p) \to (\mathfrak{D}_\psi)_L$ for $\mathfrak{D}_\psi$ are equivalent if any finite-dimensional $\mathfrak{D}_\psi$-representation V with projective restriction $\mathrm{res}_\alpha(V_K)$ also has projective restriction $\mathrm{res}_\beta(V_L)$, and vice versa. We define the π-point space in the expected manner

$$\Pi(\mathfrak{D}_\psi) = \{[\alpha] : \alpha : K[t]/(t^p) \to (\mathfrak{D}_\psi)_K \text{ is a } \pi\text{-point}\},$$

and for any finite-dimensional $\mathfrak{D}_\psi$-representation V we define the π-support space

$$\Pi(\mathfrak{D}_\psi)_V = \{[\alpha] : \mathrm{res}_\alpha(V_K) \text{ is nonprojective over } K[t]/(t^p)\}.$$

We note that if $\mathfrak{D}_\psi$ admits no such identification with a group algebra kH, as required in Definition A.6(a), then the space $\Pi(\mathfrak{D}_\psi)$ is necessarily empty.

We topologize the space $\Pi(\mathfrak{D}_\psi)$ via the basis of closed sets $\{\Pi(\mathfrak{D}_\psi)_V : V \text{ in } \mathrm{rep}(\mathfrak{D}_\psi)\}$. As in (23), one sees that each π-point α defines a corresponding point p_α in the cohomological support space $|\mathfrak{D}_\psi|$.

Lemma A.7. *If two π-points α and β for $\mathfrak{D}_\psi$ are equivalent, then their corresponding points p_α and p_β in $|\mathfrak{D}_\psi|$ are equal. Furthermore, whenever the π-point space $\Pi(\mathfrak{D}_\psi)$ is nonempty, the map*

$$\Pi(\mathfrak{D}_\psi) \to |\mathfrak{D}_\psi|, \quad [\alpha] \mapsto p_\alpha$$

is a homeomorphism and for any finite-dimensional $\mathfrak{D}_\psi$-representation V this homeomorphism restricts to a homeomorphism $\Pi(\mathfrak{D}_\psi)_V \to |\mathfrak{D}_\psi|_V$.

Proof. If $\mathfrak{D}_\psi$ admits no cocommutative Hopf structure then the space $\Pi(\mathfrak{D}_\psi)$ is empty, and there is nothing to prove. So let us suppose that $\mathfrak{D}_\psi$ admits the necessary alternate Hopf structure.

Consider any cocommutative Hopf structure Δ' on the underlying algebra $\mathfrak{D}_\psi$, and corresponding identification $\mathfrak{D}_\psi = kH$. Since H is necessarily unipotent, as $\mathfrak{D}_\psi$ is local, the cohomological support spaces agree $|H|_V = |\mathfrak{D}_\psi|_V$ for all V in $\mathrm{rep}(\mathfrak{D}_\psi) = \mathrm{rep}(H)$. (See Section 6B.)

Now, Theorem A.3 tells us that a H-representation V is nonprojective at a π-point α' for H if and only if $p_{\alpha'} \in |H|_V$. So by the above information we see that a $\mathfrak{D}_\psi$-representation V is nonprojective at a π-point α if and only if $p_\alpha \in |\mathfrak{D}_\psi|_V$. Hence two π-points α and β for $\mathfrak{D}_\psi$ are equivalent if and only if $p_\alpha = p_\beta$. This shows that the map $\Pi(\mathfrak{D}_\psi) \to |\mathfrak{D}_\psi|$ is well-defined and injective. The map is furthermore surjective since, if we consider our identification $\mathfrak{D}_\psi = kH$, the map $\Pi(H) \to |H| (= |\mathfrak{D}_\psi|)$ is surjective, meaning every point in the cohomological support space is represented by a π-point $\alpha : K[t]/(t^p) \to KH = (\mathfrak{D}_\psi)_K$. □

Based on the presentation of Section 5B, we understand that $\mathfrak{D}_\psi$ admits a cocommutative Hopf structure whenever G is a Frobenius kernel in a smooth algebraic group which admits a quasilogarithm. So Lemma A.7 tells us that we have an identification of support theories $\Pi(\mathfrak{D}_\psi)_\star \cong |\mathfrak{D}_\psi|_\star$ in this case. In particular, the above lemma is not vacuous.

A3. *π-point support for $\mathfrak{D}$.* Fix an infinitesimal group scheme G and $\mathfrak{D} = D(G)$.

Definition A.8. A π-point α for $\mathfrak{D}$ is a pair of an embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s),K} \to G_K$ and a π-point $\bar{\alpha} : K[t]/(t^p) \to \mathfrak{D}_\psi$, defined as in Definition A.6.

For any given π-point $(\psi, \bar{\alpha})$, we are particularly concerned with the composition $K[t]/(t^p) \to \mathfrak{D}_K$ of the map $\bar{\alpha} : K[t]/(t^p) \to \mathfrak{D}_\psi$ with the inclusion $\mathfrak{D}_\psi \to \mathfrak{D}$. So we generally identify a π-point with its associated flat map $K[t]/(t^p) \to \mathfrak{D}_K$, and simply write $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ by an abuse of notation.

Definition A.9. Two π-points $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ and $\beta : L[t]/(t^p) \to \mathfrak{D}_L$ are said to be equivalent if any finite-dimensional representation V which restricts to a projective $K[t]/(t^p)$-representation $\mathrm{res}_\alpha(V_K)$ along α also restricts to a projective $L[t]/(t^p)$-representation $\mathrm{res}_\beta(V_L)$ along β, and vice versa.

We define the space of equivalence classes of π-points

$$\Pi(\mathfrak{D}) = \{[\alpha] : \alpha : K[t]/(t^p) \to \mathfrak{D}_K \text{ is a } \pi\text{-point}\},$$

and for any finite-dimensional $\mathfrak{D}$-representation V we define the π-support

$$\Pi(\mathfrak{D})_V = \{[\alpha] : \mathrm{res}_\alpha(V_K) \text{ is nonprojective}\}.$$

The space $\Pi(\mathfrak{D})$ is topologized via the basis of closed sets provided by the supports $\Pi(\mathfrak{D})_V$ of all finite-dimensional $\mathfrak{D}$-representations.

As in (23), any π-point $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ defines an associated point $p_\alpha \in |\mathfrak{D}|$ in the cohomological support space. One employs Carlson modules exactly as in [Friedlander and Pevtsova 2007, Proposition 2.9] to see that the two points p_α and p_β agree whenever α and β are equivalent. So we find

Proposition A.10. *There is a well-defined continuous map*

$$w : \Pi(\mathfrak{D}) \to |\mathfrak{D}|, \quad \alpha \mapsto p_\alpha.$$

For any finite-dimensional $\mathfrak{D}$-representation V, the above map restricts to a map between support spaces $\Pi(\mathfrak{D})_V \to |\mathfrak{D}|_V$.

Proof. As stated above, well-definedness can be argued as in [Friedlander and Pevtsova 2007]. The fact that $\Pi(\mathfrak{D})_V$ is mapped to $|\mathfrak{D}|_V$ can be reduced to the corresponding claim for π-support over the $\mathfrak{D}_\psi$, which is covered in Lemma A.7.

All that is left is to establish continuity of w. For continuity, we note that any closed set in $|\mathfrak{D}|$ is the support $|\mathfrak{D}|_L$ of a product of Carlson modules. The naturality properties of Lemma 2.15 then gives $w^{-1}(|\mathfrak{D}|_L) = \Pi(\mathfrak{D})_L$. This shows that the preimage of any closed set in $|\mathfrak{D}|$ along w is closed in $\Pi(\mathfrak{D})$. □

One can see from Theorem 3.7, and the arguments used in the proof of Proposition A.10, that the map $\Pi(\mathfrak{D}) \to |\mathfrak{D}|$ is in fact *surjective* when G is a Frobenius kernel in a sufficiently nice algebraic group $\mathbb{G}$. We leave the details to the interested reader, as we will observe a stronger result in Theorem A.15 below. As a related finding, we have the following.

Theorem A.11. *Suppose that G is a Frobenius kernel in an algebraic group $\mathbb{G}$, and that $\mathbb{G}$ admits a quasilogarithm. Then a given finite-dimensional $\mathfrak{D}$-representation V is projective if and only if $\Pi(\mathfrak{D})_V = \varnothing$.*

Proof. By Theorem 3.7, V is projective if and only if its restrictions to all $\mathfrak{D}_\psi$ are projective. The hypothesis on G, and Lemma 5.5, ensure that at all 1-parameter subgroups ψ the algebra $\mathfrak{D}_\psi$ admits an (alternative) cocommutative Hopf structures. Hence, by Lemma A.7, V_K is projective over $\mathfrak{D}_\psi$ if and only it V_K is projective at all π-points for $\mathfrak{D}_\psi$. Taking this information together, we see that V is projective over $\mathfrak{D}$ if and only if V is projective at all π-points $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ for $\mathfrak{D}$. □

Remark A.12. There are ways to define the π-support $\Pi(\mathfrak{D})_M$ of an arbitrary (possibly infinite-dimensional) $\mathfrak{D}$-module M so that Theorem A.11 remains valid at arbitrary M. However, it is unclear whether or not the equivalence relation on π-points $K[t]/(t^p) \to \mathfrak{D}_K$ defined via finite-dimensional representations agrees with the analogous one defined via arbitrary modules; cf. [Friedlander and Pevtsova 2007, Theorem 4.6]. Rather, in the language of [loc. cit.], it is unclear whether equivalent π-points are in fact *strongly* equivalent. So we do not know if the support space $\Pi(\mathfrak{D})_M$ can be defined in such a way that depends only on the classes $[\alpha]$ of π-points, and not the π-points themselves. We therefore leave a discussion of π-point support for infinite-dimensional modules to some later investigation.

A4. ***Tensor product properties and comparison with cohomological support.*** As discussed in Section 6A, one can read the material of Section 6 through the alternate lens of π-point support. In particular, the arguments of Section 6 imply that π-point support behaves well with respect to tensor products, and also agrees with cohomological support; cf. [Friedlander and Pevtsova 2005; 2007].

We have the following.

Proposition A.13. *For any infinitesimal group scheme G, and embedded 1-parameter subgroup $\psi : \mathbb{G}_{a(s)} \to G$, π-point support for $\mathfrak{D}_\psi$ satisfies the tensor product property*

$$\Pi(\mathfrak{D}_\psi)_{V\otimes W} = \Pi(\mathfrak{D}_\psi)_V \cap \Pi(\mathfrak{D}_\psi)_W.$$

Proof. If $\Pi(\mathfrak{D}_\psi)$ is empty there is nothing to prove. If $\Pi(\mathfrak{D}_\psi)$ is nonempty, then π-point support for $\mathfrak{D}_\psi$ is identified with cohomological support, via Lemma A.7. So the result follows by the tensor product property for cohomological support provided in the (proof of) Proposition 6.8. □

An important reading of Proposition A.13 is the following: given a π-point $\alpha : K[t]/(t^p) \to \mathfrak{D}_\psi$, and $\mathfrak{D}_\psi$-representations V and W, the restriction $\mathrm{res}_\alpha(V \otimes W)$ is nonprojective if and only both $\mathrm{res}_\alpha(V)$ and $\mathrm{res}_\alpha(W)$ are nonprojective. Since π-point support for the global algebra $\mathfrak{D}$ is itself defined via π-points for the varying $\mathfrak{D}_\psi$, the following result is immediate.

Theorem A.14. *For any infinitesimal group scheme G, π-point support for $\mathfrak{D}$ satisfies the tensor product property*

$$\Pi(\mathfrak{D})_{V\otimes W} = \Pi(\mathfrak{D})_V \cap \Pi(\mathfrak{D})_W.$$

Finally, when G is a Frobenius kernel in a sufficiently nice algebraic group $\mathbb{G}$, we find that π-point support is identified with cohomological support.

Theorem A.15. *Suppose that G is a Frobenius kernel in an algebraic group $\mathbb{G}$, and that $\mathbb{G}$ admits a quasilogarithm. Then the map $w : \Pi(\mathfrak{D}) \to |\mathfrak{D}|$ of Proposition A.10 is a homeomorphism, and restricts to a homeomorphism $\Pi(\mathfrak{D})_V \to |\mathfrak{D}|_V$ for all finite-dimensional $\mathfrak{D}$-representations V.*

Proof. Let $w : \Pi(\mathfrak{D}) \to |\mathfrak{D}|$ denote the map $[\alpha] \mapsto p_\alpha$ of Proposition A.10. Under the above hypotheses Lemma 5.5 tells us that all $\mathfrak{D}_\psi$ have nonvanishing π-support spaces $\Pi(\mathfrak{D}_\psi)$. So Lemma A.7 tells us that π-supports and cohomological supports are identified for all $\mathfrak{D}_\psi$.

Suppose we have two π-points $\alpha, \beta \in \Pi(\mathfrak{D})$ for which $p_\alpha = p_\beta$. Let V be any representation which is nonprojective at α. Write explicitly $\alpha : K[t]/(t^p) \to \mathfrak{D}_\psi \to \mathfrak{D}_K$ and $\beta : K'[t]/(t^p) \to \mathfrak{D}_{\psi'} \to \mathfrak{D}_{K'}$. Since, at any embedded 1-parameter subgroups η, the composites

$$\Pi(\mathfrak{D}_\eta) \to \Pi(\mathfrak{D}) \to |\mathfrak{D}| \quad \text{and} \quad \Pi(\mathfrak{D}_\eta) \xrightarrow{\cong} |\mathfrak{D}_\eta| \to |\mathfrak{D}|$$

are both given by $[\eta] \mapsto p_\eta$, i.e., since the two composites agree, Proposition 6.10 ensures that $[\alpha] \in \Pi(\mathfrak{D}_\psi)_{V_K}$ and $[\beta] \in \Pi(\mathfrak{D}_{\psi'})_{V_{K'}}$. Rather, both $\mathrm{res}_\alpha(V_K)$ and $\mathrm{res}_\beta(V_{K'})$ are nonprojective. Since V was chosen arbitrarily, this shows α is equivalent to β. So we see that w is injective. Surjectivity follows from Proposition 3.11, applied to $V = k$.

We understand now that $w : \Pi(\mathfrak{D}) \to |\mathfrak{D}|$ is a bijection of sets. One argues similarly to see that each restriction $\Pi(\mathfrak{D})_V \to |\mathfrak{D}|_V$ is a bijection. Finally, since all basic closed subsets in $\Pi(\mathfrak{D})$ and $|\mathfrak{D}|$ are realized as supports of finite-dimensional representation, we see that w is in fact a homeomorphism. □

A5. ***Comparing with a universal π-point space.*** Consider the Drinfeld double $\mathfrak{D}$ of an arbitrary finite group scheme — or really any Hopf algebra. We have a universal definition of "π-points", from the perspective of classifying thick tensor ideals in the stable category. Namely, we consider all flat algebra maps $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ which satisfy the tensor product property:

$\mathrm{res}_\alpha(V_K \otimes W_K)$ is nonprojective if and only if both $\mathrm{res}_\alpha(V_K)$ and $\mathrm{res}_\alpha(W_K)$ are nonprojective. (TPP)

As with our other classes of π-points, we consider the space $\Pi_\otimes(\mathfrak{D})$ of equivalence classes of all such universal π-points, and topologize this space in the expected way. To be clear, our equivalence relation for universal π-points is defined exactly as in Definition A.9, where we simply replace "π-point" with "universal π-point" in the definition. We have the supports

$$\Pi_\otimes(\mathfrak{D})_V = \{[\alpha] : \mathrm{res}_\alpha(V_K) \text{ is nonprojective}\}$$

and corresponding support theory $V \mapsto \Pi_\otimes(\mathfrak{D})_V$.

One notes that the class of universal π-points is chosen in the coarsest possible way to ensure that the tensor product property

$$\Pi_\otimes(\mathfrak{D})_{V\otimes W} = \Pi_\otimes(\mathfrak{D})_V \cap \Pi_\otimes(\mathfrak{D})_W$$

holds, and to ensure that the support $\Pi_\otimes(\mathfrak{D})_V$ depends only on the class of V in the stable category.

Now, if we specifically consider the Drinfeld double of an infinitesimal group scheme, Theorem A.14 tells us that any π-point $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ as in Definition A.8 is a universal π-point. Furthermore, the equivalence relations on π-points and universal π-points are exactly the same. So we obtain a topological embedding $\iota : \Pi(\mathfrak{D}) \to \Pi_\otimes(\mathfrak{D})$ for which we have

$$\Pi(\mathfrak{D})_V = \Pi(\mathfrak{D}) \cap \Pi_\otimes(\mathfrak{D})_V, \tag{24}$$

simply by the definitions of these supports.

Theorem A.16. *Suppose that G is a Frobenius kernel in an smooth algebraic group $\mathbb{G}$, and that $\mathbb{G}$ admits a quasilogarithm. Then the inclusion $\iota : \Pi(\mathfrak{D}) \to \Pi_\otimes(\mathfrak{D})$ is a homeomorphism, and all of the restrictions $\iota_V : \Pi(\mathfrak{D})_V \to \Pi_\otimes(\mathfrak{D})_V$ are also homeomorphisms.*

Proof. Take $\mathscr{X} = \mathrm{stab}(\mathfrak{D})$, and recall the isomorphism $w : \Pi(\mathfrak{D}) \to |\mathfrak{D}|$ of Theorem A.15. By the universal property of the Balmer spectrum [2005, Theorem 3.2], and Theorem A.14, we have continuous maps to the Balmer spectrum $f_\pi : \Pi(\mathfrak{D}) \to \mathrm{Spec}(\mathscr{X})$ and $f_\otimes : \Pi_\otimes(\mathfrak{D}) \to \mathrm{Spec}(\mathscr{X})$ which are compatible, in the sense that $f_\otimes \circ \iota = f_\pi$. Similarly, the map $f_{\mathrm{coh}} : |\mathfrak{D}| \to \mathrm{Spec}(\mathscr{X})$ of Theorem 8.2 is such that $f_{\mathrm{coh}} \circ w = f_\pi$. Since w and f_{coh} are homeomorphisms, by Theorems 8.2 and A.15, we see that f_π is a homeomorphism. Since f_π factors through $f_\otimes$, we see that $f_\otimes : \Pi_\otimes(\mathfrak{D}) \to \mathrm{Spec}(\mathscr{X})$ is surjective. We claim that this surjection is in fact a bijection.

We have explicitly

$$f_\otimes(\alpha) = \{V \in \mathscr{X} : [\alpha] \notin \Pi_\otimes(\mathfrak{D})_V\} = \{V \in \mathscr{X} : \mathrm{res}_\alpha(V_K) \text{ is projective}\}$$

[Balmer 2005, Theorem 3.2]. Hence $f_\otimes(\alpha) = f_\otimes(\beta)$ implies that any $\mathfrak{D}$-representation with projective restriction along α also has projective restriction along β, and vice versa. So, by definition, the two classes agree $[\alpha] = [\beta]$. So we see that $f_\otimes$ is injective, and therefore a bijection. It follows that $\iota : \Pi(\mathfrak{D}) \to \Pi_\otimes(\mathfrak{D})$ is a bijection. Since ι is a topological embedding, this bijection is furthermore a homeomorphism. The fact that all of the restrictions $\Pi(\mathfrak{D})_V \to \Pi_\otimes(\mathfrak{D})_V$ are homeomorphisms as well follows by the intersection formula (24). □

We collect our results about the support theory $\Pi_\otimes(\mathfrak{D})_\star$ from above to find the following, somewhat remarkable, corollary.

Corollary A.17. *Fix G as in Theorem A.16, and $\mathfrak{D}$ the corresponding Drinfeld double. Then:*

(1) *$\mathfrak{D}$ admits enough universal π-points, in the sense that a $\mathfrak{D}$-representation V is projective if and only if its restriction $\mathrm{res}_\alpha(V_K)$ along each universal π-point $\alpha : K[t]/(t^p) \to \mathfrak{D}_K$ is projective.*

(2) *The natural map $w : \Pi_\otimes(\mathfrak{D}) \to |\mathfrak{D}|$, $[\alpha] \mapsto p_\alpha$, is a homeomorphism. In particular, the universal π-point space $\Pi_\otimes(\mathfrak{D})$ has the structure of a projective scheme.*

(3) *Any flat map $\alpha : K[t]/(t^p) \to \mathfrak{D}$ which satisfies the tensor product property* (TPP) *is equivalent to one of the form required in Definition A.8.*

Of course, the issue with the universal π-support $\Pi_\otimes(\mathfrak{D})_\star$, in general, is that it is difficult to understand the space $\Pi_\otimes(\mathfrak{D})$ explicitly, or even to understand when this space is nonempty. So, one *needs* a practical construction of π-points, as above, in order to populate $\Pi_{\otimes}(\mathfrak{D})$ with enough points, and in order to see that this theory carries significant amounts of information.

Remark A.18. For a general Hopf algebra A, we can define the universal π-point support theory $V \mapsto \Pi_{\otimes}(A)_V$ exactly as above. We make no claim that this theory is well-behaved, or even nonvacuous in general. However, it is interesting that there are even any examples in characteristic 0 where one has enough universal π-points. For example, the results of [Pevtsova and Witherspoon 2015] imply that the support theory $\Pi_{\otimes}(A)_{\star}$ satisfies the conclusions of Corollary A.17(1) and (2), for A a "quantum elementary abelian group" over $\mathbb{C}$. Similarly, for finite group schemes, one can argue as in the proof of Theorem A.16 to see that the standard π-point support theory $\Pi(G)_{\star}$ and universal theory $\Pi_{\otimes}(G)_{\star}$ agree.

A6. ***Remaining questions.*** At this point we have recorded a number of nontrivial results concerning π-points and support for Drinfeld doubles of (some) infinitesimal group schemes. We record a number of remaining questions which the reader may consider.

Question A.19. (1) Can one provide an intrinsic proof of the tensor product property of Theorem A.14, i.e., one which follows from a direct analysis of π-points, and does not reference an auxiliary support theory? (Compare with [Friedlander and Pevtsova 2005; Pevtsova and Witherspoon 2009; Friedlander 2021a].)

(2) Does the Drinfeld double of a general infinitesimal group scheme G admit enough (universal) π-points, in the sense of Corollary A.17(1)?

(3) Is there a reasonable extension of π-point support $\Pi(\mathfrak{D})_M$ to infinite-dimensional M? In particular, does there exist such a definition which reproduces the tensor product property

$$\Pi(\mathfrak{D})_{M\otimes N} = \Pi(\mathfrak{D})_M \cap \Pi(\mathfrak{D})_N$$

at arbitrary M and N?

Of course, question (3) has to do with one's (in)ability to use π-point support in certain tensor triangular investigations, as in Section 8 and [Benson et al. 2011b; 2018; Balmer et al. 2019; Balmer 2020] for example.

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Correction à l'article Sous-groupe de Brauer invariant et obstruction de descente itérée

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Le paragraphe 2 de l'article dans l'en-tête utilise un énoncé sur la formule de Künneth de degré 2 publié par Skorobogatov et Zarhin en 2014. Il a été remarqué que cet énoncé n'est pas correct. Nous corrigeons notre paragraphe 2, le seul affecté par cette erreur.

Section 2 of the article in the header uses a statement on Künneth's formula in degree 2 published by Skorobogatov and Zarhin in 2014. It has been pointed out that this statement is incorrect. We correct our Section 2, the only one affected by the error.

Tous les résultats de [Cao 2020] sont corrects sauf certains résultats du paragraphe 2. La proposition 2.2 ainsi que les corollaires 2.4 et 2.7 sont corrects, et ce sont les seuls résultats que nous utilisons dans le reste de l'article. Mais la formule de Künneth de degré 2 (proposition 2.6) n'est pas valide. Sa démonstration repose sur [Skorobogatov et Zarhin 2014, Proposition 2.2], qui n'est pas vraie : voir la remarque 1.2 du correctif de cet article pour un contre-exemple. Plus précisément, le morphisme noté par $-\circ-$ dans le premier diagramme commutatif de la preuve de [Cao 2020, corollaire 2.3] n'est pas un isomorphisme. De ce fait, il y a des erreurs dans le corollaire 2.3 et donc dans le lemme 2.5 et la proposition 2.6. Dans ce corrigendum, nous corrigeons les énoncés [Cao 2020, proposition 2.6] (remplacé par le théorème 2.1 ci-dessous) et aussi la démonstration de [Cao 2020, corollaire 2.7] (remplacé par le corollaire 2.2 ci-dessous). De plus, dans le livre récent de Colliot-Thélène et Skorobogatov [2021], les auteurs corrigent aussi la formule de Künneth de [Skorobogatov et Zarhin 2014] en utilisant la méthode de l'auteur tirée de ce corrigendum (voir [Colliot-Thélène et Skorobogatov 2021, §5.7.3]).

Dans toute cette note, k est un corps quelconque, k_s une clôture séparable de k et $\Gamma_k := \mathrm{Gal}(k_s/k)$. Sauf mention explicite du contraire, une variété est une k-variété. Fixons un entier $n \geq 2$ avec $\mathrm{char}(k) \nmid n$. Soient U, V deux k-variétés géométriquement intègres. On considère le diagramme commutatif

$$\begin{array}{ccc} U\times_k V & \xrightarrow{p_2} & V \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q_2} \\ U & \xrightarrow{q_1} & \mathrm{Spec}\, k \end{array} \tag{0-1}$$

MSC2020 : 14G12.
Mots-clefs : Hasse principle, Brauer–Manin obstruction.

1. Rappels

Nous rappelons la notion de torseur universel de n-torsion, la formule de Künneth générale et l'homomorphisme ε dans [Skorobogatov et Zarhin 2014, §5] et dans [Cao 2020, (2-9)].

Soit X une k-variété géométriquement intègre. Soit S_X un k-groupe fini commutatif tel que $n \cdot S_X = 0$ et que $S_X^* := \mathrm{Hom}_{k_s}(S_X, \mu_n) \cong H^1(X_{k_s}, \mu_n)$ comme Γ_k-modules. Pour la notion de *torseur universel de* n*-torsion pour* X, nous renvoyons à [Cao 2020, définition 2.1]. La propriété fondamentale d'un torseur universel de n-torsion est la proposition 1.1 ci-dessous.

Proposition 1.1 [Cao 2020, proposition 2.2]. *Soit $\mathcal{T}_X$ un torseur universel de n-torsion pour X. Soit S un k-groupe fini commutatif tel que $n \cdot S = 0$. Alors, pour tout S-torseur Y sur X, il existe un unique homomorphisme $\phi : S_X \to S$ tel que*

$$\phi_*([\mathcal{T}_X]) - [Y] \in \mathrm{Im}(H^1(k, S) \to H^1(X, S)).$$

L'homomorphisme ϕ dans la proposition 1.1 est appelé *le n-type de* $[Y]$. Il induit un isomorphisme de Γ_k-modules

$$\tau_{X,S} : \mathrm{Hom}_{k_s}(S_X, S) \to H^1(X_{k_s}, S), \quad \phi \mapsto \phi_*([\mathcal{T}_X]). \tag{1-1}$$

Par exemple, $\tau_{X,\mu_n} : \mathrm{Hom}_{k_s}(S_X, \mu_n) \xrightarrow{\sim} H^1(X_{k_s}, \mu_n)$ est un isomorphisme.

Rappelons que $H^1(X_{k_s}, S)$ est fini.

De plus, si $X(k) \neq \varnothing$, alors pour chaque $x \in X(k)$ il existe alors un unique torseur universel de n-torsion $\mathcal{T}_X$ pour X tel que $x^*[\mathcal{T}_X] = 0 \in H^1(k, S_X)$ (voir [Cao 2020, p. 2157]).

Considérons le diagramme commutatif (0-1).

Rappelons la formule de Künneth dans [SGA 4½ 1977]. Soit m un entier avec $m \mid n$. Soient $D^-(k)$ et $D^+(k)$ les catégories dérivées bornées respectivement à droite et à gauche de la catégorie des $\mathbb{Z}/m$-faisceaux étales sur le petit site de $\mathrm{Spec}\, k$. Par abus de notation, pour un objet M de $\mathrm{Sh}(k)$, on note M l'objet de $D^+(k)$ représenté par le complexe qui consiste en M en degré 0.

Soient M, N deux k-groupes finis commutatifs tels que $M(k_s)$, $N(k_s)$ soient des $\mathbb{Z}/m$-modules plats. Alors $M \otimes^L_{\mathbb{Z}/m} N = M \otimes_{\mathbb{Z}/m} N$ et le cup-produit donne un quasi-isomorphisme ([SGA 4½ 1977, Théorèmes de finitude, corollaire 1.11, p. 236] ou [Fu 2011, Corollary 9.3.5]), c'est-à-dire que

$$\cup : Rq_{1,*}M \otimes^L_{\mathbb{Z}/m} Rq_{2,*}N \cong R(q_1 \circ p_1)_*(M \otimes_{\mathbb{Z}/m} N)$$

dans $D^-(k)$. Ceci induit le cup-produit [Fu 2011, Proposition 6.4.12]

$$\cup_j : \bigoplus_{r+s=j} R^r q_{1,*}M \otimes_{\mathbb{Z}/m} R^s q_{2,*}N \to \mathcal{H}^j(Rq_{1,*}M \otimes^L_{\mathbb{Z}/m} Rq_{2,*}N) \sim R^j(q_1 \circ p_1)_*(M \otimes_{\mathbb{Z}/m} N).$$

Dans le cas où $m = p$ est un nombre premier, puisque $\mathbb{Z}/p$ est un corps, tout $\mathbb{Z}/p$-module est plat et $- \otimes^L_{\mathbb{Z}/p} - = - \otimes_{\mathbb{Z}/p} -$. Le cup-produit ci-dessus induit pour tout j un isomorphisme

$$\cup_j : \bigoplus_{r+s=j} R^r q_{1,*}M \otimes_{\mathbb{Z}/p} R^s q_{2,*}N \xrightarrow{\sim} R^j(q_1 \circ p_1)_*(M \otimes_{\mathbb{Z}/p} N). \tag{1-2}$$

Soit S un k-groupe fini commutatif avec $n \cdot S = 0$.

Pour le diagramme (0-1), la formation des $Rq_{2,*}(S)$ sur $\operatorname{Spec} k$ commute avec tout changement de base [SGA 4½ 1977, Théorèmes de finitude, théorème 1.9 (ii), p. 236], et donc on obtient

$$Rp_{1,*}p_2^*(S) \cong q_1^* Rq_{2,*}(S). \tag{1-3}$$

En fait, pour appliquer ce théorème, on choisit $X = V$, $Y = S = \operatorname{Spec} k$, $S' = U$ pour les X, Y, S, S' dans [SGA 4½ 1977, Théorèmes de finitude, théorème 1.9, p. 236].

Rappelons maintenant l'homomorphisme ε de Skorobogatov et Zarhin.

Supposons qu'il existe des torseurs universels de n-torsion $\mathcal{T}_U$ pour U (sous le groupe S_U) et $\mathcal{T}_V$ pour V (sous le groupe S_V). Skorobogatov et Zarhin [2014, §5] introduisent un homomorphisme

$$\varepsilon' : \operatorname{Hom}_k(S_U \otimes_{\mathbb{Z}/n} S_V, S) \to H^2(U \times V, S), \quad \phi \mapsto \phi_*([\mathcal{T}_U] \cup [\mathcal{T}_V]), \tag{1-4}$$

où $\cup$ est le cup-produit $H^1(U, S_U) \times H^1(V, S_V) \to H^2(U \times V, S_U \otimes_{\mathbb{Z}/n} S_V)$.

Dans le cas où $S := M \otimes_{\mathbb{Z}/n} N \cong M \otimes_{\mathbb{Z}/m} N$ (puisque $m \mid n$), on a le diagramme commutatif

$$\begin{array}{ccccc}
\operatorname{Hom}(S_U, M) \otimes_{\mathbb{Z}/n} \operatorname{Hom}(S_V, N) & \xrightarrow{\xi} & \operatorname{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, M \otimes_{\mathbb{Z}/n} N) & \xrightarrow{=} & \operatorname{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S) \\
\cong \downarrow \tau_{U,M} \otimes \tau_{V,N} & & & & \downarrow \varepsilon' \\
H^1(U, M) \otimes_{\mathbb{Z}/n} H^1(V, N) & \xrightarrow{\cong} & H^1(U, M) \otimes_{\mathbb{Z}/m} H^1(V, N) & \xrightarrow{\cup_2} & H^2(U \times V, S)
\end{array} \tag{1-5}$$

où $\tau_{U,M} \otimes \tau_{V,N}$ est induit par (1-1) qui est donc un isomorphisme, $\cup_2$ est induit par (1-2) et $\xi(\phi_1, \phi_2) = \phi_1 \otimes_{\mathbb{Z}/n} \phi_2$.

Ce diagramme est commutatif, car pour tout $\phi_1 \in \operatorname{Hom}(S_U, M)$, $\phi_2 \in \operatorname{Hom}(S_V, N)$, on a

$$\cup_2 \circ (\tau_{U,M} \otimes \tau_{V,N})(\phi_1, \phi_2) = \phi_{1,*}[\mathcal{T}_U] \cup \phi_{2,*}[\mathcal{T}_V] \overset{(1)}{=} (\phi_1 \otimes \phi_2)_*([\mathcal{T}_U] \cup [\mathcal{T}_V]) = \varepsilon'(\phi_1 \otimes \phi_2) = (\varepsilon' \circ \xi)(\phi_1, \phi_2),$$

où (1) découle du diagramme commutatif

$$\begin{array}{ccccc}
H^1(U, S_U) & \times & H^1(V, S_V) & \xrightarrow{\cup} & H^2(U \times V, S_U \otimes_{\mathbb{Z}/n} S_V) \\
\downarrow \phi_{1,*} & & \downarrow \phi_{2,*} & & \downarrow (\phi_1 \otimes \phi_2)_* \\
H^1(U, M) & \times & H^1(V, N) & \xrightarrow{\cup} & H^2(U \times V, S)
\end{array}$$

Remarque 1.2. Dans le cas où $S = \mu_n$, le ε' dans (1-4) est essentiellement le ε dans [Skorobogatov et Zarhin 2014, §5] et dans [Cao 2020, (2-9)]. Plus précisément, $S_V^* := \operatorname{Hom}_{k_s}(S_V, \mu_n)$ et

$$\varepsilon : \operatorname{Hom}_k(S_U, S_V^*) \to H^2(U \times V, \mu_n), \quad \phi \mapsto \eta_*(\phi_*[\mathcal{T}_U] \cup [\mathcal{T}_V]),$$

où $\eta : S_V^* \otimes_{\mathbb{Z}/n} S_V \to \mu_n$ est le morphisme d'évaluation. Dans ce cas, on a

$$\varepsilon \circ \varphi = \varepsilon',$$

où $\varphi : \mathrm{Hom}_k(S_U \otimes_{\mathbb{Z}/n} S_V, \mu_n) \xrightarrow{\sim} \mathrm{Hom}_k(S_U, \mathrm{Hom}_{k_s}(S_V, \mu_n)) = \mathrm{Hom}_k(S_U, S_V^*)$ est l'isomorphisme canonique induit par les foncteurs adjoints $- \otimes_{\mathbb{Z}/p} S_V$ et $\mathrm{Hom}_{k_s}(S_V, -)$.

Ceci est valide, car pour tout $\phi \in \mathrm{Hom}_k(S_U, S_V^*)$, on a $\varphi^{-1}(\phi) = \eta \circ (\phi \otimes \mathrm{id}_{S_V})$ et donc

$$\varepsilon(\phi) = \eta_*(\phi_*[\mathcal{T}_U] \cup [\mathcal{T}_V]) = \eta_*((\phi \otimes \mathrm{id}_V)_*([\mathcal{T}_U] \cup [\mathcal{T}_V])) = \varphi^{-1}(\phi)_*([\mathcal{T}_U] \cup [\mathcal{T}_V]) = (\varepsilon' \circ \varphi^{-1})(\phi).$$

2. Théorème principal

Voici la version correcte de la formule de Künneth de degré 2, qui remplace [Cao 2020, proposition 2.6].

Théorème 2.1. *Supposons que k est séparablement clos. Soient U, V deux k-variétés géométriquement intègres et S un $\mathbb{Z}/n$-module fini (vu comme un k-groupe commutatif). Considérons le diagramme* (0-1) *et l'homomorphisme ε' dans* (1-4) *ci-dessus. Alors on a des isomorphismes*

$$\psi^1(S) : H^1(U, S) \oplus H^1(V, S) \xrightarrow{(p_1^*, p_2^*)} H^1(U \times V, S),$$
$$\psi^2(S) : H^2(U, S) \oplus H^2(V, S) \oplus \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S) \xrightarrow{(p_1^*, p_2^*, \varepsilon')} H^2(U \times V, S),$$

et un homomorphisme injectif $\psi^3(S) : H^3(U, S) \oplus H^3(V, S) \xrightarrow{(p_1^*, p_2^*)} H^3(U \times V, S)$.

Démonstration. Puisque k est séparablement clos, $U(k) \neq \varnothing$ et $V(k) \neq \varnothing$. Fixons deux points $u \in U(k)$, $v \in V(k)$. L'existence des sections

$$l_v : U \times v \to U \times V, \quad l_u : u \times V \to U \times V$$

de p_1, p_2 impliquent que l'homomorphisme $(p_1^*, p_2^*) : H^i(U, S) \oplus H^i(V, S) \to H^i(U \times V, S)$ est injectif et scindé pour tout $i \geq 1$. Soit

$$H^2_{\mathrm{prim}}(S) := \mathrm{Ker}(H^2(U \times V, S) \xrightarrow{(l_v^*, l_u^*)} H^2(U, S) \oplus H^2(V, S)).$$

Puisque $[\mathcal{T}_U]|_u = 0$, $[\mathcal{T}_V]|_v = 0$, on a $l_u^* \circ \varepsilon' = 0$ et $l_v^* \circ \varepsilon' = 0$, et donc $\mathrm{Im}(\varepsilon') \subset H^2_{\mathrm{prim}}(S)$.

Considérons la suite spectrale $E_2^{i,j} = R^i q_{1,*} R^j p_{1,*}(S) \Rightarrow H^{i+j}(U \times V, S)$. On a :

(i) Par définition, l'homomorphisme

$$H^i(U, S) = R^i q_{1,*}(S) = E_2^{i,0} \to H^i(U \times V, S)$$

est exactement $p_1^* : H^i(U, S) \to H^i(U \times V, S)$, qui est injectif et scindé.

(ii) Puisque

$$Rp_{1,*}(S) = Rp_{1,*}p_2^*(S) \overset{(1\text{-}3)}{\cong} q_1^* Rq_{2,*}(S) = q_1^* R\Gamma(V, S), \tag{2-1}$$

tout $R^j p_{1,*}(S)$ est le faisceau constant $H^j(V, S)$. La composition

$$H^j(V, S) \to H^j(U \times V, S) \to E_2^{0,j} = q_{1,*}(H^j(V, S)) = H^j(V, S)$$

est l'identité, puisqu'elle est l'identité sur $u \times V$. Donc $H^j(U \times V, S) \to E_2^{0,j}$ est surjectif et scindé.

(iii) D'après (1-1), $E_2^{1,1} = H^1(U, H^1(V, S)) \cong \mathrm{Hom}(S_U, \mathrm{Hom}(S_V, S)) \cong \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S)$, qui est un groupe fini. Ainsi $|E_2^{1,1}| = |\mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S)|$.

D'après (i) et (ii),

$$H^1(U \times V, S) \cong E_2^{1,0} \oplus E_2^{0,1},\ H^2(U \times V, S) \cong E_2^{2,0} \oplus E_2^{0,2} \oplus E_2^{1,1},$$

et, dans ces sommes directes, (p_1^*, p_2^*) induit les isomorphismes $H^j(U, S) \oplus H^j(V, S) \cong E_2^{j,0} \oplus E_2^{0,j}$ pour $j = 1, 2$. Alors $H^2_{\mathrm{prim}}(S) \cong E_2^{1,1}$ et, d'après (iii), $|\mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S)| = |H^2_{\mathrm{prim}}(S)|$. Ainsi $\varepsilon' : \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S) \to H^2_{\mathrm{prim}}(S)$ est un isomorphisme si et seulement si ε' est injectif.

On a donc montré :

(iv) $\psi^1(S)$ est un isomorphisme et $\psi^3(S)$ est injectif.

(v) $\psi^2(S)$ est un isomorphisme si et seulement si $\psi^2(S)$ est injectif, si et seulement si ε' est injectif.

Dans le cas où $|S|$ est un nombre premier p, on a $S \cong \mathbb{Z}/p$ et on considère le diagramme (1-5) avec $M \cong N \cong \mathbb{Z}/p$. Dans ce diagramme, ξ est un isomorphisme et, d'après (1-2), $\cup_2$ est injectif. Alors ε' est injectif et, d'après (v), $\psi^2(S)$ est un isomorphisme.

Dans le cas général, par récurrence sur $|S|$, on peut supposer que $\psi^2(S')$ est un isomorphisme pour tout $\mathbb{Z}/n$-module fini S' avec $|S'| < |S|$. Alors, si $|S|$ n'est pas un nombre premier, il existe une suite exacte $0 \to S_1 \to S \to S_2 \to 0$ de $\mathbb{Z}/n$-modules finis avec $S_1, S_2 \neq 0$. Ceci induit une suite exacte

$$0 \to \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S_1) \to \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S) \to \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S_2)$$

et donc un diagramme commutatif de suites exactes :

$$\begin{array}{ccccccc}
H^1(U, S_2) \oplus H^1(V, S_2) & \xrightarrow{f_1} & F(S_1) & \longrightarrow & F(S) & \longrightarrow & F(S_2) \\
\downarrow{\scriptstyle \psi^1(S_2)} & & \downarrow{\scriptstyle \psi^2(S_1)} & & \downarrow{\scriptstyle \psi^2(S)} & & \downarrow{\scriptstyle \psi^2(S_2)} \\
H^1(U \times V, S_2) & \longrightarrow & H^2(U \times V, S_1) & \longrightarrow & H^2(U \times V, S) & \longrightarrow & H^2(U \times V, S_2)
\end{array}$$

où $F(S) := H^2(U, S) \oplus H^2(V, S) \oplus \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S)$ et f_1 est la composition

$$H^1(U, S_2) \oplus H^1(V, S_2) \xrightarrow{(\partial_U, \partial_V, 0)} H^2(U, S_1) \oplus H^2(V, S_1) \oplus \mathrm{Hom}(S_U \otimes_{\mathbb{Z}/n} S_V, S_1) = F(S_1).$$

Puisque $|S_1|, |S_2| < |S|$, par récurrence, $\psi^2(S_1)$, $\psi^2(S_2)$ sont des isomorphismes. D'après (iv), $\psi^1(S_2)$ est un isomorphisme. Par (v) et le lemme des cinq, $\psi^2(S)$ est injectif et donc un isomorphisme, ce qui établit le théorème. □

Soient U, V deux variétés géométriquement intègres sur k. On considère le diagramme commutatif (0-1) ci-dessus. Soit S un k-groupe fini commutatif avec $n \cdot S = 0$.

Si $U(k) \neq \varnothing$, alors il existe un torseur universel de n-torsion pour U. Un point $u \in U(k)$ induit $u^* : Rq_{1,*}S \to S$ et un morphisme surjectif $u^* : H^i(U, S) \to H^i(k, S)$ pour tout i. Notons

$$H^i_u(U, S) := \mathrm{Ker}(H^i(U, S) \xrightarrow{u^*} H^i(k, S)).$$

Corollaire 2.2. *Sous les notations et hypothèses ci-dessus, supposons que $U(k) \neq \varnothing$, et qu'il existe des torseurs universels de n-torsion $\mathcal{T}_U$, $\mathcal{T}_V$ comme ci-dessus. Soit $u \in U(k)$. Alors, pour tout k-groupe fini commutatif S avec $n \cdot S = 0$, on a un isomorphisme*

$$H^2_u(U, S) \oplus H^2(V, S) \oplus \mathrm{Hom}_k(S_U \otimes_{\mathbb{Z}/n} S_V, S) \xrightarrow{(p_1^*, p_2^*, \varepsilon')} H^2(U \times V, S).$$

Démonstration. Considérons le morphisme dans $D^+(k)$

$$\psi := (p_1^* + p_2^*, u^* + 0) : Rq_{1,*}S \oplus Rq_{2,*}S \to (R(q_1 \circ p_1)_* S) \oplus S.$$

D'après le théorème 2.1, le cône C_ψ de ψ est dans $D^{\geq 2}(k)$ et la composition

$$\mathrm{Hom}_{k_s}(S_U \otimes_{\mathbb{Z}/n} S_V, S) \xrightarrow{\varepsilon'|_{k_s}} H^2(U_{k_s} \times V_{k_s}, S) = \mathcal{H}^2(R(q_1 \circ p_1)_* S) \to \mathcal{H}^2(C_\psi) \qquad (2\text{-}2)$$

est un isomorphisme. Ceci induit une suite exacte

$$0 \to H^2_u(U, S) \oplus H^2(V, S) \xrightarrow{(p_1^*, p_2^*)} H^2(U \times V, S) \to H^0(k, \mathcal{H}^2(C_\psi)).$$

D'après (2-2), la composition

$$\mathrm{Hom}_k(S_U \otimes_{\mathbb{Z}/n} S_V, S) \xrightarrow{\varepsilon'} H^2(U \times V, \mu_n) \to H^0(k, \mathcal{H}^2(C_\psi))$$

est un isomorphisme, d'où le résultat. □

Le corollaire 2.2 et la remarque 1.2 donnent directement [Cao 2020, corollaire 2.7].

Remarque 2.3. Lv [2020] utilise [Cao 2020, proposition 2.6] pour établir son lemme 3.3. Dans sa démonstration, cette proposition 2.6 peut être remplacée par le théorème 2.1 ci-dessus. Donc tous les résultats de [Lv 2020] restent corrects (sauf le lemme 3.2).

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