# Algebra & Number Theory

Volume 17
2023
No. 1

## Distinction inside L-packets of SL(n)

U. K. Anandavardhanan and Nadir Matringe

msp

ALGEBRA AND NUMBER THEORY 17:1 (2023)
https://doi.org/10.2140/ant.2023.17.45

# Distinction inside L-packets of SL(n)

U. K. Anandavardhanan and Nadir Matringe

If $E/F$ is a quadratic extension $p$-adic fields, we first prove that the $\mathrm{SL}_n(F)$-distinguished representations inside a distinguished unitary L-packet of $\mathrm{SL}_n(E)$ are precisely those admitting a degenerate Whittaker model with respect to a degenerate character of $N(E)/N(F)$. Then we establish a global analogue of this result. For this, let $E/F$ be a quadratic extension of number fields, and let $\pi$ be an $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$. Let $(\sigma, d)$ be the unique pair associated to $\pi$, where $\sigma$ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_E)$ with $n = dr$. Using an unfolding argument, we prove that an element of the L-packet of $\pi$ is distinguished with respect to $\mathrm{SL}_n(\mathbb{A}_F)$ if and only if it has a degenerate Whittaker model for a degenerate character $\psi$ of type $r^d := (r, \ldots, r)$ of $N_n(\mathbb{A}_E)$ which is trivial on $N_n(E + \mathbb{A}_F)$, where $N_n$ is the group of unipotent upper triangular matrices of $\mathrm{SL}_n$. As a first application, under the assumptions that $E/F$ splits at infinity and $r$ is odd, we establish a local–global principle for $\mathrm{SL}_n(\mathbb{A}_F)$-distinction inside the L-packet of $\pi$. As a second application we construct examples of distinguished cuspidal automorphic representations $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$ such that the period integral vanishes on some canonical realization of $\pi$, and of everywhere locally distinguished representations of $\mathrm{SL}_n(\mathbb{A}_E)$ such that their L-packets do not contain any distinguished representation.

## 1. Introduction

The present work fits in the study of local distinction and periods of automorphic forms, with respect to Galois pairs of reductive groups. It is motivated by earlier works, namely, [Anandavardhanan and Prasad 2003; 2018] in the local context and [Anandavardhanan and Prasad 2006; 2013] in the global context, which investigated distinction in the presence of L-packets.

In probing distinction inside an L-packet for SL(2), the key finding of [Anandavardhanan and Prasad 2003; 2006] was that distinction inside an L-packet that contains at least one distinguished representation can be characterized in terms of Whittaker models; i.e., distinguished representations in such "distinguished" L-packets are precisely the ones which admit a Whittaker model with respect to a nontrivial character of $E/F$ (resp. $\mathbb{A}_E/(E + \mathbb{A}_F)$) in the local (resp. global) case. A crucial role in the global papers on SL(2) [Anandavardhanan and Prasad 2006; 2013] is played by "multiplicity one for SL(2)"; i.e., a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_L)$ appears exactly once in the space of cusp forms on $\mathrm{SL}_2(\mathbb{A}_L)$ [Ramakrishnan 2000].

More recently, the results of [AP 2018] generalized [AP 2003] from $n = 2$ to any $n$. Thus, in [AP 2018], it is proved, amongst many other results, that if $\pi$ is a generic $\mathrm{SL}_n(F)$-distinguished representation

*MSC2020:* primary 11F70; secondary 22E50.
*Keywords:* Galois distinction, Galois periods, SL($n$), unitary representations, automorphic representations.

of $\mathrm{SL}_n(E)$, then the distinguished members of the L-packet of $\pi$ are the representations which are $\psi$-generic with respect to some nondegenerate character $\psi$ satisfying $\psi^\theta = \psi^{-1}$, where $\theta$ denotes the Galois involution. Such a relationship between distinction and genericity is expected more generally [Prasad 2015]; indeed, if $\psi$ is a nondegenerate character such that $\psi^\theta = \psi^{-1}$, then according to [Prasad 2015, Conjecture 13.3, (3)], for any quasisplit Galois pair, $\psi$-generic members of a distinguished L-packet are distinguished.

Somewhat surprisingly, even the finite field analogue of this characterization of distinction in a generic L-packet turned out to be nontrivial and was settled only fairly recently [Anandavardhanan and Matringe 2020, Theorem 5.1].

In this paper, we first prove a generalization of the above-mentioned local result of [AP 2003; 2018] for unitary L-packets of $\mathrm{SL}_n(E)$ and degenerate Whittaker models (see Theorem 3.9).

**Theorem 1.1.** *If $\tilde{\pi}$ is an irreducible unitary representation of $\mathrm{GL}_n(E)$ of type $(n_1, \dots, n_d)$, where $(n_1, \dots, n_d)$ is the partition of $n$ defined in Section 3A, and if the* L*-packet associated to $\tilde{\pi}$ contains a representation distinguished by $\mathrm{SL}_n(F)$, then its distinguished members are those which admit a $\psi$-degenerate Whittaker model for $\psi$ of type $(n_1, \dots, n_d)$ satisfying $\psi^\theta = \psi^{-1}$.*

Our proof of Theorem 1.1 builds on the work of Matringe [2014], which classified unitary representations of $\mathrm{GL}_n(E)$ which are distinguished with respect to $\mathrm{GL}_n(F)$, making use of which we can adapt the techniques of [AP 2003] and [AP 2018] to the unitary context. Such a result hints at the possibility of a generalization of the prediction of Dipendra Prasad [2015] relating distinction for Galois pairs inside distinguished generic L-packets to distinguished Whittaker models, to nongeneric L-packets. We feel that, thanks in particular to the work [Kemarsky 2015], the same result could be obtained in the Archimedean setting and we leave this question to experts. This would allow the removal of the assumption that the number field is split at infinity in some of our global results.

Now we come to the global results of this paper. The study of global representations of $\mathrm{SL}(n)$, already quite involved for $n = 2$ as can be seen from [AP 2006; 2013], is considerably more difficult for several reasons, one of which is that "multiplicity one" is not true for $\mathrm{SL}(n)$ for $n \geq 3$, as was first shown in the famous work of D. Blasius [Blasius 1994; Lapid 1999].

In this paper, we prove the most basic result about characterizing distinction inside a distinguished L-packet in terms of Whittaker models, thus generalizing [AP 2006, Theorem 4.2] from $n = 2$ to any $n$, and we cover not just cuspidal representations but the full residual spectrum (see Theorem 6.10). We emphasize that the L-packets that we consider in this work are defined by restriction of cusp forms, except in the abstract, where the results are formulated in terms of "the" L-packet consisting of automorphic members of the representation-theoretic L-packet.

**Theorem 1.2.** *Suppose $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ is an irreducible square-integrable automorphic representation of $\mathrm{GL}_{dr}(\mathbb{A}_E)$, where $\sigma$ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_E)$. Assume that the* L*-packet determined by $\tilde{\pi}$ contains an $\mathrm{SL}_{dr}(\mathbb{A}_F)$-distinguished representation. Then an irreducible square-integrable automorphic representation $\pi$ of this* L*-packet is $\mathrm{SL}_{dr}(\mathbb{A}_F)$-distinguished if and only if there exists a degenerate* *character $\psi$ of type $r^d := (r, \dots, r)$* (see Section 5B) *of $N_n(\mathbb{A}_E)$, trivial on $N_n(E + \mathbb{A}_F)$, such that $\pi$ has a degenerate $\psi$-Whittaker model.*

There are two main ideas in proving Theorem 1.2. First we settle the cuspidal case by creating an inductive setup based on an unfolding method, and make use of the base case for $n = 2$, which is known by [AP 2006, Theorem 4.2]. We mention here that the method that we follow to create this inductive setup is very parallel to that employed in [Dijols and Prasad 2019, Section 5] (see Remark 6.4). Having established the cuspidal case for all $r$, we do one more induction, this time in $d$, where $n = dr$, the case $d = 1$ being the cuspidal case. In order to work this out, the key ingredient is the work of Yamana [2015], which is the global counterpart of [Matringe 2014], and we need to do one more unfolding argument as well.

As an application of Theorem 1.2, we establish a local–global principle for square-integrable representations for $(\mathrm{SL}_n(\mathbb{A}_E), \mathrm{SL}_n(\mathbb{A}_F))$ (see Theorem 8.4).

**Theorem 1.3.** *Let $E/F$ be a quadratic extension of number fields split at the Archimedean places. Suppose $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ is a square-integrable automorphic representation of $\mathrm{GL}_{dr}(\mathbb{A}_E)$, where $\sigma$ is a cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_E)$, where we assume that $r$ is odd. Suppose that the L-packet determined by $\tilde{\pi}$ contains an $\mathrm{SL}_{dr}(\mathbb{A}_F)$-distinguished representation. Let $\pi$ be an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ which belongs to this L-packet. Write $\pi = \bigotimes_v' \pi_v$, for $v$ varying through the places of $F$. Then $\pi$ is distinguished with respect to $\mathrm{SL}_n(\mathbb{A}_F)$ if and only if each $\pi_v$ is $\mathrm{SL}_n(F_v)$-distinguished.*

**Remark 1.4.** Such a local–global principle was proved in [AP 2006] for cuspidal representations of $\mathrm{SL}_2(\mathbb{A}_E)$ by quite involved arguments. In contrast, our proof is reasonably elementary, making use of the assumption that $r$ is odd.

Another important objective of the present paper is to analyze distinction vis-à-vis the phenomenon of higher multiplicity for SL($n$). As mentioned earlier, unlike in the case of SL(2), a cuspidal representation may appear in the space of cusp forms with multiplicity more than 1 for SL($n$) for $n \geq 3$ [Blasius 1994; Lapid 1999].

In our first set of examples, we give a precise answer regarding the nonvanishing of the period integral on the canonical realizations of a cuspidal representation inside the L-packets obtained from restricting the cusp forms on $\mathrm{GL}_n(\mathbb{A}_E)$. We exhibit two types of examples of cuspidal representations of $\mathrm{SL}_n(\mathbb{A}_E)$ of multiplicity $m(\pi)$ more than 1 in the space of cusp forms which are $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished (see Sections 10B and 10C). In one set of examples, $F$ is any number field and $E/F$ is chosen so that the period integral vanishes on some of the $m(\pi)$ many canonical realizations but not on all the canonical realizations. In the second set of examples, $F$ is any number field and $E/F$ is chosen so that the period integral does not vanish in any of the $m(\pi)$ many canonical realizations inside the L-packets.

Then we tweak the method employed to construct the above examples to also show that the local–global principle fails at the level of nondistinguished L-packets for SL($n$) (see Section 10D). Namely, we give examples of cuspidal representations $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$ which are distinguished at every place, but such that the L-packet of $\pi$ contains no distinguished representation. Such a phenomenon was observed for SL(2) as well by an explicit construction in [AP 2006, Theorem 8.2]. The construction in [AP 2006] is somewhat involved, whereas our analogous examples in Section 10D are conceptually simpler; however, the methods here are tailor-made for $n$ odd.

All our examples in Section 10 of cuspidal representations of $\mathrm{SL}_n(\mathbb{A}_E)$ of high multiplicity that are $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished, which highlight a variety of different phenomena, owe a lot to the examples of Blasius [1994] of high cuspidal multiplicity. Blasius makes use of the representation theory of the Heisenberg group $H$ and, in particular, the fact that different Heisenberg representations are such that their value at any element of the group are conjugate in $\mathrm{PGL}_n(\mathbb{C})$, but they are projectively inequivalent [Blasius 1994, Section 1.1]. To give a rough idea, Blasius [Blasius 1994] produces high multiplicity examples on $\mathrm{SL}_n(\mathbb{A}_E)$ by transferring this representation-theoretic information about Heisenberg groups to Galois groups of $L/E$ for suitable number fields, via Shafarevich's theorem, and then to the automorphic side via the strong Artin conjecture, which is a theorem in the situation at hand, because $\mathrm{Gal}(L/E) \simeq H$ is nilpotent, by Arthur and Clozel [1989, Theorem 7.1]. For our examples, we start with an involution on $H$ and consider the corresponding semidirect product $H \rtimes \mathbb{Z}/2$, which cuts out extensions $L \supset E \supset F$, and play with these involutions to construct a variety of examples answering several natural questions about distinction for the pair $(\mathrm{SL}_n(\mathbb{A}_E), \mathrm{SL}_n(\mathbb{A}_F))$.

Last we mention that we give proofs of some elementary, and probably standard, facts on Archimedean and global L-packets of $\mathrm{SL}_n$ for which we could not find accessible sources in the literature. They follow from [Aizenbud et al. 2015] in the Archimedean setting, and from [Jiang and Liu 2013] in the global setting.

## 2. Notation

We denote by $\delta_G$ the character of a locally compact group $G$ such that $\delta_G\lambda$ is a right-invariant Haar measure on $G$ if $\lambda$ is a left-invariant Haar measure on $G$. We denote by $\mathcal{M}_{a,b}$ the algebraic group of $a \times b$ matrices. We denote by $G_n$ the algebraic group $\mathrm{GL}_n$, by $T_n$ its diagonal torus and by $N_n$ the group of upper triangular matrices in $G_n$. We set

$$U_n = \left\{ u_n(x) = \begin{pmatrix} I_{n-1} & x \\ \cdot & 1 \end{pmatrix} : x \in (\mathbb{A}^1)^{n-1} \right\} \subset N_n,$$

where $\mathbb{A}^1$ denotes the affine line. For $k \leq n$, we embed $G_k$ inside $G_n$ via $g \mapsto \mathrm{diag}(g, I_{n-k})$ and set $P_n = G_{n-1}U_n$, the mirabolic subgroup of $G_n$. We denote by $N_{n,r}$ the group of matrices

$$k(a, x, u) = \begin{pmatrix} a & x \\ \cdot & u \end{pmatrix}$$

with $a \in G_{n-r}$, $x \in \mathcal{M}_{n-r,r}$ and $u \in N_r$. We denote by $U_{n,r}$ the unipotent radical of $N_{n,r}$, which consists of the matrices $k(I_{n-r}, x, u)$. Note that $N_{n,n} = N_n$ and

$$U_{n,r} = U_n \cdots U_{r+1}.$$

For a subgroup $H$ of $G_n$, we denote by $H^\circ$ the intersection of $H$ with $\mathrm{SL}_n$.

## 3. Non-Archimedean theory

Let $E/F$ be a quadratic extension of $p$-adic fields with Galois involution $\theta$. We denote by $|\cdot|_E$ and $|\cdot|_F$ the respective normalized absolute values. In this section, by abuse of notation, we set $G = G(E)$ for any algebraic group defined over $E$. We denote by $\nu_E$ (or $\nu$), the character $|\cdot|_E \circ \det$ of $G_n$. We fix a nontrivial character $\psi_0$ of $E$ which is trivial on $F$.

**3A.** ***The type of an irreducible* GL*-representation via derivatives.*** If $\psi$ is a nondegenerate (smooth complex) character of $N_n$, we denote by $\psi^k$ its restriction to $U_k$ for $k \le n$. We denote by $\mathrm{Rep}(\bullet)$ the category of smooth complex representations of $\bullet$. Bernstein and Zelevinsky [1976; 1977] introduced the functors

$$\Phi^-_{\psi^n} : \mathrm{Rep}(P_n) \to \mathrm{Rep}(P_{n-1}) \quad \text{and} \quad \Psi^- : \mathrm{Rep}(P_n) \to \mathrm{Rep}(G_{n-1}).$$

For $(\tau, V) \in \mathrm{Rep}(P_n)$, one has

$$\Phi^-_{\psi^n}(V) = V/V(U_n, \psi^n),$$

where $V(U_n, \psi^n)$ is the space spanned by the differences $\tau(u)v - \psi^n(u)v$ for $u \in U_n$ and $v \in V$, but the action of $P_{n-1}$ on $\Phi^-_{\psi^n}(V)$ is normalized by twisting by $\delta_{P_n}^{-1/2}$. Similarly

$$\Psi^-(V) = V/V(U_n, 1),$$

where the action of $G_{n-1}$ on $\Psi^-(V)$ is normalized by twisting by $\delta_{P_n}^{-1/2}$ again.

The functor $\Phi^-_{\psi^n}$ does not in fact depend on $\psi$ in the sense that for $\tau \in \mathrm{Rep}(P_n)$ one has $\Phi^-_{\psi^n}(\tau) \simeq \Phi^-_{\psi'^n}(\tau)$ whenever $\psi$ and $\psi'$ are nondegenerate characters of $N_n$. Hence we simply write $\Phi^-(\tau)$ for it. For $\tau \in \mathrm{Rep}(P_n)$, we set

$$\tau_{(k)} = (\Phi^-)^{k-1}(\tau) \in \mathrm{Rep}(P_{n+1-k}),$$

and

$$\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau) \in \mathrm{Rep}(G_{n-k}),$$

which is called the $k$-th derivative of $\tau$. The $k$-th shifted derivative of $\tau$ is given by

$$\tau^{[k]} = \nu^{1/2}\tau^{(k)}.$$

Note that these definitions apply when $\tau$ is a representation of $G_n$ which we consider as a representation of $P_n$ by restriction.

Let $\tilde{\pi}$ be an irreducible smooth representation of $G_n$. We denote by $\tilde{\pi}^{[n_1]}$ its highest (nonzero) shifted derivative, by $\tilde{\pi}^{[n_1,n_2]} := (\tilde{\pi}^{[n_1]})^{[n_2]}$ the highest shifted derivative of $\tilde{\pi}^{[n_1]}$, and so on. All the representations $\tilde{\pi}^{[n_1,n_2,\dots,n_i]}$ are irreducible thanks to [Zelevinsky 1980, Theorem 8.1]. This defines a finite sequence of positive integers $(n_1, \dots, n_d)$ such than $n_1 + \cdots + n_d = n$. In fact, [Zelevinsky 1980, Theorem 8.1] implies that this sequence is a partition of $n$, i.e., $n_1 \ge n_2 \ge \cdots \ge n_d$. We call $(n_1, \dots, n_d)$ the *partition associated to* $\tilde{\pi}$. We will also say that $\tilde{\pi}$ is *of type* $(n_1, \dots, n_d)$. Note that by [Bernstein 1984, Section 7.4], if $\tilde{\pi}$ is unitary, then all the representations $\tilde{\pi}^{[n_1,n_2,\dots,n_i]}$ are unitary as well.

**Example 3.1.** Using the product notation for normalized parabolic induction, if $\delta$ is an essentially square-integrable representation of $G_r$ we set

$$\mathrm{Sp}(d,\delta)=\mathrm{LQ}(|\cdot|_E^{(d-1)/2}\delta\times\cdots\times|\cdot|_E^{(1-d)/2}\delta)$$

to be the Langlands quotient of the parabolically induced representation

$$|\cdot|_E^{(d-1)/2}\delta\times\cdots\times|\cdot|_E^{(1-d)/2}\delta.$$

More generally, if $\tau=\delta_1\times\cdots\times\delta_l$ is a generic unitary representation of $G_r$ written as a commutative product of essentially square-integrable representations [Zelevinsky 1980, Theorem 9.7], we set

$$\mathrm{Sp}(d,\tau)=\mathrm{Sp}(d,\delta_1)\times\cdots\times\mathrm{Sp}(d,\delta_l),$$

which is a commutative product by the results of Tadić [1986, Theorem D]. In this situation, [Bernstein and Zelevinsky 1977, 4.5, Lemma], together with the computation of the highest derivative of Speh representations [Offen and Sayag 2008, § 3.5 (3.3); Tadić 1987, § 6.1], implies that the partition of $n=rd$ associated to $\mathrm{Sp}(d,\tau)$ is $r^d:=(r,\dots,r)$. Conversely one can check using the same results that an irreducible unitary representation of $G_n$ of type $r^d$ is of the form $\mathrm{Sp}(d,\tau)$ for a unitary generic representation $\tau$ of $G_r$. We refer to Section 4B for the details in the Archimedean setting, which are the same as in the non-Archimedean setting.

**3B.** ***Degenerate Whittaker models and* L*-packets.*** Let $\psi_{n_i}$ be a nondegenerate character of the group $N_{n_i}$. By [Zelevinsky 1980, Section 8], if the representation $\tilde{\pi}$ is of type $(n_1,\dots,n_d)$, then it has a unique degenerate Whittaker model with respect to

$$(\psi_{n_1}\otimes\cdots\otimes\psi_{n_d})\begin{pmatrix}u_d & \cdots & \cdot\\ & \ddots & \vdots\\ & & u_1\end{pmatrix}=\psi_{n_1}(u_1)\dots\psi_{n_d}(u_d)$$

for $u_i\in N_{n_i}$. We often use the notation

$$\psi_{n_1,\dots,n_d}:=\psi_{n_1}\otimes\cdots\otimes\psi_{n_d},$$

which has the advantage of being short but could mislead the reader, so we insist on the fact that $\psi_{n_1,\dots,n_d}$ depends on the characters $\psi_{n_i}$ and not only on the positive integers $n_i$. We will say that $\psi_{n_1,\dots,n_d}$ is *of type* $(n_1,\dots,n_d)$. If all the $n_i$ are equal then we set

$$\psi_{1,\dots,d}:=\psi_{n_1,\dots,n_d}.$$

The L-packet associated to $\tilde{\pi}$ is the finite set of irreducible representations of $G_n^\circ=\mathrm{SL}_n(E)$ appearing in the restriction of $\tilde{\pi}$, and is denoted by $\mathrm{L}(\tilde{\pi})$. We refer to [Hiraga and Saito 2012, Section 2] for its basic properties, which we now state (see also [Gelbart and Knapp 1982] or [Tadić 1992]). Any irreducible representation $\pi$ of $G_n^\circ$ arises in the restriction of an irreducible representation of $G_n$ and two irreducible representations of $G_n$ containing $\pi$ are twists of each other by a character. Hence it makes sense to set $\mathrm{L}(\pi)=\mathrm{L}(\tilde{\pi})$, and call this finite set the L-packet of $\pi$ (or the L-packet determined by $\tilde{\pi}$). *We define* *the type of* $\pi$ (*or the type of* $\mathrm{L}(\pi)$) *to be that of* $\tilde{\pi}$. Of course two irreducible representations of $G_n$ determining the same L-packet have the same type.

Clearly the group $\mathrm{diag}(E^\times, I_{n-1})$ acts transitively on $\mathrm{L}(\tilde{\pi})$ and the existence of a degenerate Whittaker model for irreducible representations of $G_n$ then has the following immediate consequence.

**Lemma 3.2.** *Suppose that* $\tilde{\pi}$ *is an irreducible representation of* $G_n$ *of type* $(n_1, \dots, n_d)$. *Then the group* $\mathrm{diag}(E^\times, I_{n-1})$ *acts transitively on* $\mathrm{L}(\tilde{\pi})$ *and every member of* $\mathrm{L}(\tilde{\pi})$ *has a* (*necessarily unique*) *degenerate* $\psi$*-Whittaker model for some* $\psi$ *of type* $(n_1, \dots, n_d)$.

Uniqueness of degenerate Whittaker models for $\tilde{\pi}$, together with Lemma 3.2, then has the following well-known consequence.

**Proposition 3.3.** *If* $\tilde{\pi}$ *is an irreducible representation of* $G_n$ *then the representations in* $\mathrm{L}(\tilde{\pi})$ *appear with multiplicity one in the restriction of* $\tilde{\pi}$ *to* $G_n^\circ$.

In fact we can be more precise. The following lemma follows from the fact that if $\tilde{\pi}$ is of type $(n_1, \dots, n_d)$ then $\tilde{\pi}^{[n_1,\dots,n_{k-1}]}$ is of type $(n_k, \dots, n_d)$ (see Section 3A).

**Lemma 3.4.** *If* $\tilde{\pi}$ *is an irreducible representation of* $G_n$ *of type* $(n_1, \dots, n_d)$, *then* $\mathrm{L}(\tilde{\pi}^{[n_1,\dots,n_{k-1}]})$ *contains a unique irreducible representation of* $G^\circ_{n_k+\dots+n_d}$ *with a degenerate Whittaker model with respect to* $\psi_{n_k,\dots,n_d}$.

Again $\mathrm{L}(\tilde{\pi}^{[n_1,\dots,n_{k-1}]})$ only depends on $\mathrm{L}(\tilde{\pi}) = \mathrm{L}(\pi)$ (because derivatives commute with character twists), and we set

$$\mathrm{L}(\pi)^{[n_1,\dots,n_{k-1}]} := \mathrm{L}(\tilde{\pi}^{[n_1,\dots,n_{k-1}]})$$

for any irreducible representation $\tilde{\pi}$ of $G_n$ such that $\pi \in \mathrm{L}(\tilde{\pi})$.

**Definition 3.5.** Let $\pi$ be an irreducible representation of $G_n^\circ$. Let $\pi^{[n_1,\dots,n_{k-1}]}(\psi_{n_k,\dots,n_d})$ denote the irreducible representation of $G^\circ_{n_k+\dots+n_d}$ isolated in Lemma 3.4, i.e., the unique representation in $\mathrm{L}(\pi)^{[n_1,\dots,n_{k-1}]}$ with a degenerate Whittaker model with respect to $\psi_{n_k,\dots,n_d}$. In particular, $\pi(\psi_{n_1,\dots,n_d})$ denotes the unique irreducible representation of $G_n^\circ$ in $\mathrm{L}(\pi)$ with a degenerate Whittaker model with respect to $\psi_{n_1,\dots,n_d}$.

**Remark 3.6.** We do not claim that if $\pi(\psi) = \pi(\psi')$, then $\psi$ and $\psi'$ are in the same $T_n^\circ$-conjugacy class.

**3C.** ***Distinguished representations inside a distinguished* L*-packet.*** Let $\tilde{\pi}$ be an irreducible representation of $G_n$ of type $(n_1, \dots, n_d)$. We start by making explicit the relation between the degenerate Whittaker models $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ and $\mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$.

**Lemma 3.7.** *The map*

$$W \mapsto W|_{G_{n-n_1}}$$

*is surjective from* $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ *to* $\mathcal{W}(\nu_E^{(n_1-1)/2}\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$.

*Proof.* By the same proof as in [Cogdell and Piatetski-Shapiro 2017, Proposition 1.2], the map

$$W \mapsto W|_{P_{n-n_1+1}}$$

is a surjection from $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ to $\mathcal{W}(\nu_E^{(n-n_1+1)/2}\tilde{\pi}_{(n_1-1)}, \psi_{n_2,\dots,n_d})$. But then, because $W(gu)=W(g)$ for $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$, $g \in G_{n-n_1}$ and $u \in U_{n-n_1+1}$, we deduce that $W|_{G_{n-n_1}} \in \mathcal{W}(\nu_E^{n_1/2}\tilde{\pi}^{(n_1)}, \psi_{n_2,\dots,n_d})$ and that

$$W \mapsto W|_{G_{n-n_1}}$$

is surjective from $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ to $\mathcal{W}(\nu_E^{n_1/2}\tilde{\pi}^{(n_1)}, \psi_{n_2,\dots,n_d})$. The result follows. □

We denote by $\mathcal{K}(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$ the generalized Kirillov model of $\tilde{\pi}$ (see [Zelevinsky 1980, Section 5]) with respect to $\tilde{\pi}^{[n_1]}$ and $\psi_{n_1}$. It is, by definition, the image of the unique embedding of $\tilde{\pi}|_{P_n}$ into the space of functions $K : P_n \to \pi^{[n_1]}$ which satisfy

$$K(k(a,x,u_1)p) = \nu(a)^{(n_1-1)/2}\psi_{n_1}(u_1)\pi^{[n_1]}(a)K(p)$$

for $k(a,x,u_1) \in N_{n,n_1}$.

Let $\tilde{\pi}$ be an irreducible representation of $G_n$ with degenerate Whittaker model $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$. Then, by Lemma 3.7, for any $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ and $g \in G_n$, the map

$$g_1 \mapsto \nu_E^{(n_1-1)/2}W(g_1 g)$$

belongs to $\mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$. We set

$$I(W) : G_n \to \mathcal{W}(\nu_E^{(n_1-1)/2}\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$$

to be the map defined by

$$I(W)(g) : g_1 \in G_{n-n_1} \mapsto W(g_1 g).$$

Hence $I$ realizes $\mathcal{W}(\tilde{\pi}, \psi_{n_1,n_2,\dots,n_d})$ inside the induced representation

$$\mathrm{Ind}_{N_{n,n_1}}^{G_n}(\mathcal{W}(\nu_E^{(n_1-1)/2}\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d}) \otimes \psi_{n_1}).$$

Then the map $W \mapsto I(W)|_{P_n}$ is a bijection

$$\mathcal{W}(\tilde{\pi}, \psi_{n_1,n_2,\dots,n_d}) \to \mathcal{K}(\tilde{\pi}, \mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d}), \psi_{n_1})).$$

The following is now a consequence of the results of [Matringe 2014].

**Proposition 3.8.** *Let $\tilde{\pi}$ be an irreducible unitary representation of $G_n$ of type $(n_1, \dots, n_d)$ which is distinguished with respect to $G_n^\theta$, with degenerate Whittaker model $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$, and suppose that $\psi_{n_1,\dots,n_d}$ is trivial on $N_n(F)$. Then the invariant linear form on $\tilde{\pi}$ is expressed as a local period on $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ by*

$$\lambda(W) = \int_{N_{n,n_1}^\theta \backslash P_n^\theta} \int_{N_{n-n_1,n_2}^\theta \backslash P_{n-n_1}^\theta} \cdots \int_{N^\theta_{n-\sum_{i=1}^{d-1} n_i, n_d} \backslash P^\theta_{n-\sum_{i=1}^{d-1} n_i}} W(p_d \cdots p_2 p_1)\, dp_d \cdots dp_2\, dp_1.$$

*For all $W \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$, the integral above is well-defined inductively in the sense that*

$$x \mapsto \int_{N^\theta_{n-\sum_{i=1}^{c} n_i, n_{c+1}} \backslash P^\theta_{n-\sum_{i=1}^{c} n_i}} \cdots \int_{N^\theta_{n-\sum_{i=1}^{d-1} n_i, n_d} \backslash P^\theta_{n-\sum_{i=1}^{d-1} n_i}} W(p_d \cdots p_{c+1} x)\, dp_d \cdots dp_{c+1}$$

*defines an absolutely convergent function on*

$$N^{\theta}_{n-\sum_{i=1}^{c-1} n_i, n_c} \setminus P^{\theta}_{n-\sum_{i=1}^{c-1} n_i}$$

*for $c$ descending from $d$ to* 1 *(for $c = d$ the first integral above is just $W$ by convention).*

*Proof.* The proof is by induction on $d$. For $d = 1$, the representation is unitary generic, and the fact that

$$W \mapsto \int_{N_n^{\theta} \setminus P_n^{\theta}} W(p)\, dp$$

is well defined is due to Flicker [1988, Section 4], and that it is $G_n^{\theta}$-invariant is a result due to Youngbin Ok (see [Matringe 2014, Proposition 2.5] for a more general statement in the unitary context). Then, for a general $d$, by [Matringe 2014, Proposition 2.4], if $\tilde{\pi}$ is distinguished, so is $\tilde{\pi}^{[n_1]}$, and we take $L \in \mathrm{Hom}_{G^{\theta}_{n-n_1}}(\tilde{\pi}^{[n_1]}, \mathbb{C}) \setminus \{0\}$. By [Matringe 2014, Propositions 2.2 and 2.5], the linear form

$$\lambda_K : K \mapsto \int_{N^{\theta}_{n,n_1} \setminus P_n^{\theta}} L(K(p_1))\, dp_1 \tag{1}$$

is, up to scaling, the unique $G_n^{\theta}$-invariant linear form on $\mathcal{K}(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$ and it is given by an absolutely convergent integral for all $K \in \mathcal{K}(\tilde{\pi}, \tilde{\pi}^{[n_1]}, \psi_{n_1})$. We realize $\tilde{\pi}$ as $\mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ and $\tilde{\pi}^{[n_1]}$ as $\mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$. Then by induction for all $W' \in \mathcal{W}(\tilde{\pi}^{[n_1]}, \psi_{n_2,\dots,n_d})$ we have

$$L(W') = \int_{N^{\theta}_{n-n_1,n_2} \setminus P^{\theta}_{n-n_1}} \cdots \int_{N^{\theta}_{n-\sum_{i=1}^{d-1} n_i, n_d} \setminus P^{\theta}_{n-\sum_{i=1}^{d-1} n_i}} W'(p_r \cdots p_2)\, dp_r \cdots dp_2,$$

which is well defined in the sense of the statement of the proposition because $\tilde{\pi}^{[n_1]}$ is unitary. Applying it to $W' = K(p_1) = I(W_K)(p_1)$ for the unique $W_K \in \mathcal{W}(\tilde{\pi}, \psi_{n_1,\dots,n_d})$ such that the previous equality holds, the result follows in view of the discussion preceding the proposition. □

**Theorem 3.9.** *Let $\pi$ be an irreducible unitary representation of $\mathrm{SL}_n(E)$ of type $(n_1, \dots, n_d)$ which is $\mathrm{SL}_n(F)$-distinguished. Then the $\mathrm{SL}_n(F)$-distinguished representations in $\mathrm{L}(\pi)$ are precisely the representations $\pi(\psi)$ for a character $\psi$ of $N_n$ of type $(n_1, \dots, n_d)$ such that $\psi|_{N_n^{\theta}} \equiv \mathbf{1}$.*

*Proof.* The proof follows exactly along the same lines of the generic case, as in [AP 2003, Section 3] and [AP 2018, Section 4], making use of Proposition 3.8 in lieu of Flicker's invariant linear form mentioned above. □

Theorem 3.9 has the following consequences.

**Proposition 3.10.** *Let $\pi$ be an irreducible unitary representation of $G_n^{\circ}$ of type $(n_1, \dots, n_d)$, and fix $\psi_{n_1,\dots,n_d}$, a character of $N_n$ of this type trivial on $N_n^{\theta}$. If $\pi(\psi_{n_1,\dots,n_d})$ is $\mathrm{SL}_n(F)$-distinguished, then the representation $\pi^{[n_1,\dots,n_{k-1}]}(\psi_{n_k,\dots,n_d})$ is $\mathrm{SL}_{\sum_{i=k}^{d} n_i}(F)$-distinguished for all $k = 1, \dots, d$.*

*Proof.* According to [AP 2018, Lemma 3.2], up to twisting $\tilde{\pi}$ by an appropriate character, we can suppose that it is $\mathrm{GL}_n(F)$-distinguished. Then $\tilde{\pi}^{[n_1,\dots,n_{k-1}]}$ is distinguished as we already saw (see proof of

Proposition 3.8). Now $\pi^{[n_1,\dots,n_{k-1}]}(\psi_{n_k,\dots,n_d})$ belongs to $\mathrm{L}(\pi^{[n_1,\dots,n_{k-1}]})$ and it has a degenerate Whittaker model with respect to the distinguished character $\psi_{n_k,\dots,n_d}$, so the result follows from Theorem 3.9. □

Proposition 3.10 can be strengthened for Speh representations.

**Theorem 3.11.** *Let $\tau$ be a generic representation of $G_r$ and let $\psi_i$ be a nondegenerate character of $N_r$ trivial on $N_r^\theta$ for $i = 1, \dots, d$. Fix $1 \le k \le d$, and then $\pi(\psi_{1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(d, \tau))$ is $\mathrm{SL}_n(F)$-distinguished if and only if $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(k, \tau))$ is $\mathrm{SL}_{kr}(F)$-distinguished.*

*Proof.* One direction follows from Proposition 3.10. Conversely suppose that

$$\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(k, \tau))$$

is $\mathrm{SL}_{kr}(F)$-distinguished. Then, thanks to [Matringe 2014, Theorem 2.13], up to a twist, $\mathrm{Sp}(k, \tau)$ is distinguished, so $\tau$ is, and hence $\mathrm{Sp}(d, \tau)$ is. But then because $\pi(\psi_{1,\dots,d}) \in \mathrm{L}(\mathrm{Sp}(d, \tau))$ has a $\psi_{1,\dots,d}$-degenerate Whittaker model and $\psi_{1,\dots,d}$ is trivial on $N_n^\theta$, we deduce that $\pi(\psi_{1,\dots,d})$ is $\mathrm{SL}_n(F)$-distinguished, thanks to Theorem 3.9. □

We will give the global analogue of this result in Theorem 6.16.

## 4. Archimedean prerequisites for the global theory

Here $E = \mathbb{C}$ or $\mathbb{R}$, and by abuse of notation we write $G = G(E)$ for any algebraic group defined over $E$. We set $|a+ib|_{\mathbb{C}} = a^2 + b^2$ and denote by $|\cdot|_{\mathbb{R}}$ the usual absolute value on $\mathbb{R}$. We then denote by $\nu_E$ the character of $G_n$ obtained by composing $|\cdot|_E$ with det. For $G$ a reductive subgroup of $G_n$ we write $\mathcal{SAF}(G)$ for the category of smooth admissible Fréchet representations of $G$ of moderate growth as in [Aizenbud et al. 2015], in which we work. We use the same product notation for parabolic induction in $\mathcal{SAF}(G_n)$ as in [Aizenbud et al. 2015].

We only consider unitary characters of $N_n$. The nondegenerate characters of $N_n$ are of the form

$$\psi_\lambda : \begin{pmatrix} 1 & z_1 & \cdots & \cdots & \cdot \\ & 1 & z_2 & \cdots & \cdot \\ & & \ddots & \ddots & \cdot \\ & & & 1 & z_{n-1} \\ & & & & 1 \end{pmatrix} \mapsto \exp\left( i \sum_{i=1}^{n-1} \Re(\lambda_i z_i) \right)$$

with $\lambda_i \in E^*$. Then for a partition $(n_1, \dots, n_r)$ of $n$ and nondegenerate characters $\psi_{n_i}$ of $N_{n_i}$ we define the degenerate character $\psi_{n_1,\dots,n_d}$ of $N_n$ as in Section 3B and we also write $\psi_{1,\dots,d} := \psi_{n_1,\dots,n_d}$ when all the $n_i$'s are equal. We again say $\psi_{n_1,\dots,n_d}$ is of type $(n_1, \dots, n_d)$, so that the set of characters of a given type forms a single $T_n$-conjugacy class. We call a member of this conjugacy class a degenerate character of type $(n_1, \dots, n_d)$. For a degenerate character $\psi$ of $N_n$ and an irreducible representation $\tilde{\pi}$ of $G_n$, by a $\psi$-Whittaker functional, we mean a nonzero continuous linear form $L$ from $\tilde{\pi}$ to $\mathbb{C}$ satisfying

$$L(\tilde{\pi}(n)v) = \psi(n)L(v)$$

for $n \in N_n$ and $v \in \tilde{\pi}$. We will say that $\tilde{\pi}$ has a unique $\psi$-Whittaker model if the space of $\psi$-Whittaker functionals on the space of $\tilde{\pi}$ is one-dimensional.

**4A.** ***The Tadić classification of the unitary dual of $G_n$.*** We recall that irreducible square-integrable representations of $G_n$ for $n \geq 1$ exist only when $n = 1$ if $E = \mathbb{C}$ and when $n = 1$ or 2 if $E = \mathbb{R}$. When $n = 1$ these are just the unitary characters of $E^{\times}$. For $d \in \mathbb{N}$ and an irreducible square-integrable representation $\delta$ of $G_n$ ($n = 1$ or $n \in \{1, 2\}$ depending on whether $E$ is $\mathbb{C}$ or $\mathbb{R}$) we denote by

$$\mathrm{Sp}(d, \delta) = \mathrm{LQ}(\nu_E^{(d-1)/2}\delta \times \cdots \times \nu_E^{(1-d)/2}\delta)$$

the Langlands quotient of $\nu_E^{(d-1)/2}\delta \times \cdots \times \nu_E^{(1-d)/2}\delta$. In particular, $\mathrm{Sp}(d, \chi) = \chi \circ \det$ when $\chi$ is a unitary character of $G_1$. By [Tadić 2009], the representations

$$\pi(\mathrm{Sp}(d, \delta), \alpha) := \nu^{\alpha}\mathrm{Sp}(d, \delta) \times \nu^{-\alpha}\mathrm{Sp}(d, \delta)$$

are irreducible unitary when $\alpha \in \left(0, \frac{1}{2}\right)$, and any irreducible representation $\pi$ of $G_n$ can be written in a unique manner as a commutative product

$$\tilde{\pi} = \prod_{i=1}^{r} \mathrm{Sp}(d_i, \delta_i) \prod_{j=r+1}^{s} \pi(\mathrm{Sp}(d_j, \delta_j), \alpha_j).$$

When all the $d_i$ and $d_j$ are equal to one, the representation

$$\tau = \prod_{i=1}^{r} \delta_i \prod_{j=r+1}^{s} \pi(\delta_j, \alpha_j)$$

is generic unitary (it has a unique $\psi$-Whittaker model for any nondegenerate character $\psi$ of $N_n$), according to [Jacquet 2009, p. 4], and we set

$$\mathrm{Sp}(d, \tau) = \prod_{i=1}^{r} \mathrm{Sp}(d, \delta_i) \prod_{j=r+1}^{s} \pi(\mathrm{Sp}(d, \delta_j), \alpha_j),$$

which is thus an irreducible unitary representation.

We note that according to the proof of [Gourevitch and Sahi 2013, 4.1.1], which refers to [Vogan 1986] and [Sahi and Stein 1990], a Speh representation $\mathrm{Sp}(d, \delta)$ for $\delta$ an irreducible square-integrable representation of $G_2$ is the same thing as the Speh representations of Vogan's classification as presented in [Aizenbud et al. 2015, 4.1.2(c)]. Hence the Vogan classification as stated in [Aizenbud et al. 2015, 4.1.2] is immediately related to that of Tadić:

- The unitary characters of [Aizenbud et al. 2015, 4.1.2(a)] are the representations of the form $\mathrm{Sp}(d, \chi)$ for $\chi$ a unitary character of $G_1$.
- The Stein complementary series of [Aizenbud et al. 2015, 4.1.2(b)] are the representations of the form $\pi(\mathrm{Sp}(d, \chi), \alpha)$ for $\chi$ a unitary character of $G_1$.

- The Speh representations of [Aizenbud et al. 2015, 4.1.2(c)] are the representations of the form $\mathrm{Sp}(d,\delta)$ for $\delta$ an irreducible square-integrable representation of $G_2$.
- The Speh complementary series of [Aizenbud et al. 2015, 4.1.2(d)] are the representations of the form $\pi(\mathrm{Sp}(d,\delta),\alpha)$ for $\delta$ an irreducible square-integrable representation of $G_2$.

The third and fourth cases occur only when $E=\mathbb{R}$.

**4B.** ***Degenerate Whittaker models of irreducible unitary representations.*** In this section we recall the results of Aizenbud, Gourevitch, and Sahi on degenerate Whittaker models for $\mathrm{GL}_n(E)$ for $E=\mathbb{C}$ or $\mathbb{R}$. We believe that with the material developed by these authors, together with the real analogue of Ok's result due to Kemarsky [2015], the results obtained in [Matringe 2014] and Section 3 are in reach. However, being inexperienced in such matters, we leave this for experts, and simply recall immediate implications of the results in [Aizenbud et al. 2015] that we will need for our global applications.

To any irreducible representation $\tilde{\pi}$ of $G_n$, Sahi [1989] attached an irreducible representation $A(\tilde{\pi})$ of $G_{n-n_1}$ for some $0<n_1\le n$, the *adduced representation of* $\tilde{\pi}$, and proved that it satisfied

$$A\Big(\prod_{i=1}^{r}\mathrm{Sp}(d_i,\delta_i)\prod_{j=r+1}^{s}\pi(\mathrm{Sp}(d_j,\delta_j),\alpha_j)\Big)=\prod_{i=1}^{r}A(\mathrm{Sp}(d_i,\delta_i))\prod_{j=r+1}^{s}A\big(\pi(\mathrm{Sp}(d_j,\delta_j),\alpha_j)\big)$$

with respect to the Tadić classification. The adduced representation is the Archimedean highest shifted derivative, and from [Sahi 1990; Gourevitch and Sahi 2013; Aizenbud et al. 2015] (see [Aizenbud et al. 2015, Section 4]) one has

$$A\Big(\prod_{i=1}^{r}\mathrm{Sp}(d_i,\delta_i)\prod_{j=r+1}^{s}\pi(\mathrm{Sp}(d_j,\delta_j),\alpha_j)\Big)=\prod_{i=1}^{r}\mathrm{Sp}(d_i-1,\delta_i)\prod_{j=r+1}^{s}\pi(\mathrm{Sp}(d_j-1,\delta_j),\alpha_j). \tag{2}$$

One can then take the adduced of the adduced representation of the irreducible unitary representation $\tilde{\pi}$ and so on, and obtain the "depth sequence" $\bar{n}:=(n_1,\dots,n_d)$ attached to $\tilde{\pi}$, which forms a partition of $n$. We call this depth sequence the *type of* $\tilde{\pi}$. The combination of [Gourevitch and Sahi 2013, Theorem A] and [Aizenbud et al. 2015, Theorem 4.2.3] says:

**Theorem 4.1.** *Let $\tilde{\pi}$ be an irreducible unitary representation of $G_n$ of type $(n_1,\dots,n_d)$, and $\psi$ be any character of $N_n$ of type $(n_1,\dots,n_d)$. Then $\tilde{\pi}$ has a unique degenerate $\psi$-Whittaker model.*

For an irreducible representation $\pi$ of $\mathrm{SL}_n(E)$, the notion of a degenerate Whittaker model is defined similarly. This notion depends on the $T_n^\circ$-conjugacy class of the degenerate character $\psi$ and not just its type. The L-packet of $\pi$ is defined as in the $p$-adic case, and we refer to [Hiraga and Saito 2012, end of Section 2]. Note that [Hiraga and Saito 2012] deals with Harish-Chandra modules but their results remain valid in the context of $\mathcal{SAF}(G_n)$, thanks to the Casselman–Wallach equivalence of categories (see [Wallach 1988, Chapter 11]). If $\tilde{\pi}$ is an irreducible unitary representation of $G_n$, it follows from [Gourevitch and Sahi 2013, Theorem A] that the type of $\tilde{\pi}$ depends only on $\mathrm{L}(\tilde{\pi})$, and *we define the type of an irreducible unitary representation $\pi$ of $G_n^\circ$ to be that of any irreducible representation $\tilde{\pi}$ of $G_n$ such that $\pi\in\mathrm{L}(\tilde{\pi})$.*

**Remark 4.2.** If $\tilde{\pi}$ is an irreducible representation of $G_n^\circ$, then $\tilde{\pi}|_{G_n^\circ}$ contains an irreducible unitary representation if and only if it is unitary up to a character twist.

As in the $p$-adic case, Theorem 4.1 has the following consequence.

**Corollary 4.3.** *Let $\tilde{\pi} \in \mathcal{SAF}(G_n)$ be an irreducible unitary representation of type $(n_1, \dots, n_d)$. Then the group* $\mathrm{diag}(E^\times, I_{n-1})$ *acts transitively on* $\mathrm{L}(\tilde{\pi})$ *and every* $\pi \in \mathrm{L}(\tilde{\pi})$ *has a (necessarily unique) degenerate $\psi$-Whittaker model for some character $\psi$ of $N_n$ of type $(n_1, \dots, n_d)$. Moreover, $\tilde{\pi}|_{G_n^\circ}$ is multiplicity-free.*

We note that the computation of the adduced representation given in (2) implies:

**Theorem 4.4** (Aizenbud, Gourevitch, and Sahi). *Let $\tau$ be an irreducible generic representation of $G_r$. The Speh representation* $\mathrm{Sp}(d, \tau)$ *has type $r^d$, and conversely an irreducible unitary representation of $G_n$ of type $r^d$ is of the form* $\mathrm{Sp}(d, \tau)$ *for some unitary generic representation $\tau$ of $G_r$.*

We end by giving the Archimedean analogue of Definition 3.5 for Speh representations.

**Definition 4.5.** Let $\pi$ be an irreducible unitary representation of $G_n^\circ$ of type $r^d$, and let $\tau$ be an irreducible unitary generic representation of $G_r$ such that $\pi \in \mathrm{L}(\mathrm{Sp}(d, \tau))$. For $\psi_{1,\dots,d}$ a character of $N_n$ of type $r^d$, we denote by $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ the unique representation in $\mathrm{L}(\mathrm{Sp}(k, \tau))$ with a $\psi_{d-k+1,\dots,d}$-degenerate Whittaker model.

**Remark 4.6.** The representation $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ above depends only on $\mathrm{L}(\pi)$.

## 5. The global setting

In this section, $E/F$ is a quadratic extension of number fields with associated Galois involution $\theta$. We denote by $\mathbb{A}_E$ and $\mathbb{A}_F$ the rings of adeles of $E$ and $F$ respectively. We denote by $\mathrm{GL}_n(\mathbb{A}_E)^1$ the elements of $\mathrm{GL}_n(\mathbb{A}_E)$ which have determinant of adelic norm equal to 1, and for any subgroup $H$ of $\mathrm{GL}_n(\mathbb{A}_E)$, by $H^1$ we denote the intersection of $H$ with $\mathrm{GL}_n(\mathbb{A}_E)^1$. We recall that $\mathbb{A}_F^\times = \mathbb{A}_F^1 \times (\mathbb{A}_F)_{>0}$, where $(\mathbb{A}_F)_{>0}$ is $\mathbb{R}_{>0} \otimes_{\mathbb{Q}} 1 \subset \mathbb{R} \otimes_{\mathbb{Q}} F$ sitting inside $\mathbb{A}_F$. In particular, passing to the groups of unitary characters, we have $\widehat{\mathbb{A}_F^\times} = \widehat{\mathbb{A}_F^1} \times \widehat{(\mathbb{A}_F)_{>0}}$, and for $\lambda \in \mathbb{R}$ we denote by $\alpha_\lambda$ the unitary character of $\mathbb{A}_F^\times$ corresponding to $(\alpha, (|\cdot|_{\mathbb{A}_F}^{i\lambda})|_{\widehat{(\mathbb{A}_F)_{>0}}}) \in \widehat{\mathbb{A}_F^1} \times \widehat{(\mathbb{A}_F)_{>0}}$. Namely, extending $\alpha_0$ is the extension of $\alpha$ which is trivial on $\widehat{(\mathbb{A}_F)_{>0}}$ and $\alpha_\lambda = \alpha_0 |\cdot|_{\mathbb{A}_F}^{i\lambda}$. In particular $\alpha_\lambda$ is automorphic if and only if $\alpha \in \widehat{F^\times \backslash \mathbb{A}_F^1}$.

**5A.** ***Square-integrable automorphic representations and their* L*-packets.*** For $\omega \in \widehat{E^\times \backslash \mathbb{A}_E^\times}$, we denote by

$$L^2(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

the space of smooth $L^2$-automorphic forms on which the center $\mathbb{A}_E^\times$ of $\mathrm{GL}_n(\mathbb{A}_E)$ acts by $\omega$, and by

$$L_d^2(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

its discrete part. We then denote by $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ the dense $\mathrm{GL}_n(\mathbb{A}_E)$-submodule of $L_d^2(\mathbb{A}_E^\times \mathrm{GL}_n(E) \backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$ consisting of smooth automorphic forms (see [Cogdell 2004, Lecture 2]). We say that $\tilde{\pi}$ is a *square-integrable automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ if it is a closed (for the

Fréchet topology) irreducible $\mathrm{GL}_n(\mathbb{A}_E)$-submodule of $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E),\omega)$ for some Hecke character $\omega$. The space $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E),\omega)$ contains the space of smooth cusp forms

$$\mathcal{A}_0^\infty(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E),\omega)$$

as a $\mathrm{GL}_n(\mathbb{A}_E)$-invariant subspace. A *cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ is a closed irreducible $\mathrm{GL}_n(\mathbb{A}_E)$-submodule of $\mathcal{A}_0^\infty(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E),\omega)$, for some Hecke character $\omega$.

Let $\sigma$ be a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$, and

$$\tilde{\pi} = \mathrm{Sp}(d,\sigma) = \bigotimes_v{}' \, \mathrm{Sp}(d,\sigma_v)$$

be the restricted tensor product of the representations $\mathrm{Sp}(d,\sigma_v)$ for $v$ varying through the places of $E$. By [Jacquet 1984], this is a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, where $n = dr$. By [Mœglin and Waldspurger 1989], any irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ is of this form for a unique pair $(\sigma, d)$, and moreover $\mathrm{Sp}(d,\sigma)$ appears with multiplicity one in $L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E),\omega)$ (this of course was already known for $d=1$ by the pioneering independent results of Piatetski-Shapiro and Shalika).

We define the spaces $L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ and $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ in the same way that we defined their GL-analogues. Also, similarly, the notions of square-integrable and cuspidal automorphic representations of $\mathrm{SL}_n(\mathbb{A}_E)$ are defined. We set

$$L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c := \bigoplus_{\omega \in \widehat{E^\times\backslash \mathbb{A}_E^\times}} L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E),\omega),$$

which is well known to be multiplicity-free.

**Notation 5.1.** We denote by

$$\mathrm{Res}: L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c \to L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$$

the restriction of functions from $\mathrm{GL}_n(\mathbb{A}_E)$ to $\mathrm{SL}_n(\mathbb{A}_E)$.

We recall from [Hiraga and Saito 2012, Chapter 4] (see in particular [Hiraga and Saito 2012, Remark 4.23] for square-integrable representations) the following facts. If $\tilde{\pi} \subset L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c$ is an irreducible submodule, then by Corollary 5.5 of the next section the representation $\mathrm{Res}(\tilde{\pi})$ is multiplicity-free, and we denote by $\mathrm{L}(\tilde{\pi})$ the set of irreducible submodules of $\mathrm{Res}(\tilde{\pi})$, and call it the L-packet attached to $\tilde{\pi}$. Moreover if $\tilde{\pi}' \subset L^{2,\infty}(\mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E))_c$ is also an irreducible submodule, then $\mathrm{Res}(\tilde{\pi})$ and $\mathrm{Res}(\tilde{\pi}')$ are either in direct sum or equal, and they are equal if and only if $\tilde{\pi}$ and $\tilde{\pi}'$ are twists of each other by an automorphic character of $\mathbb{A}_E^\times$; i.e., $\mathrm{L}(\tilde{\pi}) \cap \mathrm{L}(\tilde{\pi}') \neq \varnothing$ if and only if they are equal if and only if $\tilde{\pi}$ and $\tilde{\pi}'$ are twists of each other by an automorphic character of $\mathbb{A}_E^\times$. For $\pi$ an irreducible submodule of $L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ we set

$$m(\pi) = \dim \mathrm{Hom}_{\mathrm{SL}_n(\mathbb{A}_E)}\big(\pi, L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))\big),$$

and call it the multiplicity of $\pi$ in $L^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$. This is known to be finite.

If $\pi$ is a square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$, then there are exactly $m(\pi)$ L-packets containing a representation isomorphic to $\pi$, and if $\pi_0$ is a representation isomorphic to $\pi$ contained in an L-packet, we call $\pi_0$ a *canonical realization* of $\pi$. In particular, if $\pi$ is such a canonical realization, the L-packet $\mathrm{L}(\pi)$ of $\pi$ is well defined (it is by definition equal to $\mathrm{L}(\tilde{\pi})$ for $\pi \subset \mathrm{Res}(\tilde{\pi})$).

**5B.** ***Degenerate Whittaker models and square-integrable* L*-packets.*** Let $n = dr$. Let $\sigma$ be a smooth unitary cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ and let $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ be the associated square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. We set $U_{r^d}$ to be the unipotent radical of the parabolic subgroup of type $r^d$ of $\mathrm{GL}(n)$, denoted by $P_{r^d}$. Let

$$\psi_{1,\dots,d}(\mathrm{diag}(n_1, \dots, n_d)u) = \prod_{i=1}^{d} \psi_i(n_i),$$

where $\psi_i$ is a nondegenerate character of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$ and $u \in U_{r^d}(\mathbb{A}_E)$. For $\varphi \in \pi$, we set

$$p_{\psi_{1,\dots,d}}(\varphi) = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi(n)\psi^{-1}_{1,\dots,d}(n)\, dn.$$

By [Jiang and Liu 2013, Corollary 3.4], there exists $\varphi \in \mathrm{Sp}(d, \sigma)$ such that $p_{\psi_{1,\dots,d}}(\varphi) \neq 0$: we will say that $\varphi$ *has a nonzero Fourier coefficient of type* $r^d$ or *a degenerate Whittaker model of type* $r^d$. Of course when $d = 1$ this result is due to the pioneering works of Piatetski-Shapiro and Shalika.

**Remark 5.2.** The result [Jiang and Liu 2013, Corollary 3.4] could also be deduced by the techniques used in Section 6, using the $E = F \times F$-analogue of Yamana's formula [2015, Theorem 1.1] (see Theorem 6.7). Also following Section 6 in the case where $E$ is split, one would conclude that any square-integrable representation of $\mathrm{SL}_n(\mathbb{A}_E)$ in the L-packet determined by $\mathrm{Sp}(d, \sigma)$ has a degenerate Whittaker model of type $r^d$. However for the sake of variety we offer a different proof of this fact here, using the results of [Jiang and Liu 2013] rather than those of [Yamana 2015] (or rather its split analogue).

**Definition.** We say that a square-integrable representation $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$ is *of type* $r^d$ if it belongs to $\mathrm{L}(\mathrm{Sp}(d, \sigma))$ for an irreducible (unitary) cuspidal automorphic representation $\sigma$ of $G_r(\mathbb{A}_E)$.

We say that $\tilde{\pi}$ (resp. $\pi$) *has a degenerate Whittaker model of type* $r^d$ if there is $\varphi \in \tilde{\pi}$ (resp. $\varphi \in \pi$) with a nonzero Fourier coefficient of type $r^d$. In particular $\mathrm{Sp}(d, \sigma)$ has a degenerate Whittaker model of type $r^d$.

We denote by $\psi$ a nondegenerate character of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$. We set

$$(\mathbf{1} \otimes \psi)\begin{pmatrix} I_{n-r} & x \\ \cdot & u_1 \end{pmatrix} = \psi(u_1) \quad \text{for} \quad \begin{pmatrix} I_{n-r} & x \\ \cdot & u_1 \end{pmatrix} \in U_{n,r}(\mathbb{A}_E).$$

For $\varphi \in \tilde{\pi}$, we set

$$\varphi_{U_{n,r},\psi}(g) = \int_{U_{n,r}(E)\backslash U_{n,r}(\mathbb{A}_E)} \varphi(u\, \mathrm{diag}(g, I_r))(\mathbf{1} \otimes \psi^{-1})(u)\, du$$

for $g \in \mathrm{GL}_{n-r}(\mathbb{A}_E)$.

**Remark 5.3.** Note that the function $\varphi_{U_{n,r},\psi}$ is nothing but the integral of the constant term of $\varphi$ along the $(n-r,r)$ parabolic against $\psi^{-1}$ on $N_r(E)\backslash N_r(\mathbb{A}_E)$. By [Yamana 2015, Lemma 6.1], there is a positive character $\delta$ of $\mathrm{GL}_{n-r}(\mathbb{A}_E)$ such that the function $\delta\otimes\varphi_{U_{n,r},\psi}$ belongs to $\mathrm{Sp}(d-1,\sigma)$; in particular, $(\varphi_{U_{n,r},\psi})|_H$ belongs to $\mathrm{Res}_H(\mathrm{Sp}(d-1,\sigma))$ (restriction of cusp forms) for any subgroup $H$ of $\mathrm{GL}_n(\mathbb{A}_E)^1$, for example $H=\mathrm{SL}_n(\mathbb{A}_E)$.

**Proposition 5.4.** *A square-integrable automorphic representation $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$ of type $r^d$ has a degenerate Whittaker model of type $r^d$.*

*Proof.* We will prove the stronger claim: for any $\varphi\in\tilde{\pi}$ such that $\varphi|_{\mathrm{SL}_n(\mathbb{A}_E)}\neq 0$, there is $h_0\in\mathrm{SL}_r(\mathbb{A}_E)$ (embedded in $\mathrm{SL}_n(\mathbb{A}_E)$ in the upper left block) such that $\rho(h_0)\varphi$ has a nonzero Fourier coefficient of type $r^d$. If $d=1$, we are in the cuspidal (and hence generic) case and the result follows from the same inductive procedure of Lemma 6.1 and Proposition 6.3, but applied to $E$ diagonally embedded inside $E\times E$ (instead of $F\subset E$ considered there). If $d\geq 2$, by [Jiang and Liu 2013, Proposition 3.1(1)] applied to $\varphi$, there is a nondegenerate character $\psi$ of $N_r(\mathbb{A}_E)$ trivial on $N_r(E)$ such that $\varphi_{U_{n,r},\psi}$ is nonzero on $\mathrm{SL}_{n-r}(\mathbb{A}_E)$ (because $N_{n,r}(E)\backslash P_n(E)=N^{\circ}_{n,r}(E)\backslash P^{\circ}_n(E)$ since $d\geq 2$). We conclude by induction, thanks to Remark 5.3. □

**Corollary 5.5.** *If $\tilde{\pi}$ is an irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ of type $r^d$, then $\mathrm{Res}(\tilde{\pi})$ is multiplicity-free. Moreover, for any automorphic character $\psi$ of $N_n(\mathbb{A}_E)$ of type $r^d$, the L-packet $\mathrm{L}(\tilde{\pi})$ contains a unique member $\pi(\psi)$ with a $\psi$-Whittaker model, and the group $\mathrm{diag}(E^\times, I_{n-1})$ acts transitively on $\mathrm{L}(\tilde{\pi})$.*

*Proof.* Thanks to multiplicity one inside local L-packets (see Proposition 3.3 and Corollary 4.3), it follows that the representations in $\mathrm{L}(\tilde{\pi})$ appear with multiplicity one in $\mathrm{Res}(\tilde{\pi})$. Moreover, we deduce that $T_n(E)$ acts transitively on $\mathrm{L}(\tilde{\pi})$: by Proposition 5.4 any representation in $\mathrm{L}(\tilde{\pi})$ has a degenerate Whittaker model of type $r^d$. Note that two automorphic characters of type $r^d$ of $N_n(\mathbb{A}_E)$ are conjugate to each other by $T_n(E)$ and this implies that for each automorphic character $\psi$ of type $r^d$ of $N_n(\mathbb{A}_E)$ there is a representation $\pi(\psi)$ in $\mathrm{L}(\tilde{\pi})$ with a $\psi$-Whittaker model. Moreover $\mathrm{L}(\tilde{\pi})$ has at most one representation with a $\psi$-Whittaker model by local multiplicity one of degenerate Whittaker models and this implies the uniqueness of $\pi(\psi)$ in the statement. Finally for $t\in T_n(E)$ and $t'=\mathrm{diag}(\det(t), I_{n-1})$, the representations $\pi^t$ and $\pi^{t'}$ in $\mathrm{L}(\tilde{\pi})$ are isomorphic, hence equal by multiplicity one inside $\mathrm{L}(\tilde{\pi})$. □

**5C. *Distinguished representations and distinguished* L*-packets.*** Take $\chi\in\widehat{F^\times\backslash\mathbb{A}_F^\times}$, and choose $\omega\in\widehat{E^\times\backslash\mathbb{A}_E^\times}$, a Hecke character such that $\omega|_{\mathbb{A}_F^\times}=\chi^n$. We denote by $\tilde{p}_{n,\chi}$ the linear form called the $\chi$-period integral on $L_d^{2,\infty}(\mathbb{A}_E^\times\mathrm{GL}_n(E)\backslash\mathrm{GL}_n(\mathbb{A}_E),\omega)$, given by

$$\tilde{p}_{n,\chi}(\phi)=\int_{\mathbb{A}_F^\times\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)}\phi(h)\chi^{-1}(\det(h))\,dh.$$

It is well defined on $\mathcal{A}_0^\infty(\mathbb{A}_E^\times\mathrm{GL}_n(E)\backslash\mathrm{GL}_n(\mathbb{A}_E),\omega)$ by [Ash et al. 1993, Proposition 1] and in general by [Yamana 2015, Lemma 3.1]. Indeed up to a positive constant, $\tilde{p}_{n,\chi}(\phi)$ is equal to

$$\int_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1}\phi(h)\chi^{-1}(\det(h))\,dh.$$

**Definition.** We say that a square-integrable automorphic representation

$$\tilde{\pi} \subset L_d^{2,\infty}(\mathbb{A}_E^\times \mathrm{GL}_n(E)\backslash \mathrm{GL}_n(\mathbb{A}_E), \omega)$$

is *$\chi$-distinguished* (or simply *distinguished* when $\chi \equiv 1$) if $\tilde{p}_{n,\chi}$ is nonvanishing on $\tilde{\pi}$.

We denote by $p_n$ the period integral on $L_d^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ given by

$$p_n(\phi) = \int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \phi(h)\, dh.$$

It is again well defined on $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ thanks to [Ash et al. 1993, Proposition 1] and on the space $L_d^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$ by the arguments in [Yamana 2015, Lemma 3.1].

**Definition.** We say that a square-integrable representation

$$\pi \subset L_d^{2,\infty}(\mathrm{SL}_n(E)\backslash \mathrm{SL}_n(\mathbb{A}_E))$$

is *distinguished* if $p_n$ does not vanish on $\pi$. We give another useful formula for the $\mathrm{SL}_n$-period integral following [AP 2006, Proposition 3.2].

**Proposition 5.6.** *Let $\tilde{\pi}$ be a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. The period integral*

$$\varphi \mapsto \int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h)\, dh$$

*is given by an absolutely convergent integral on $\mathrm{Res}(\tilde{\pi})$. Moreover, for any $\varphi \in \tilde{\pi}$, we have*

$$\int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h)\, dh = \sum_\alpha \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\alpha(\det(h))\, dh,$$

*where the sum is over all characters $\alpha$ of the compact abelian group $F^\times\backslash \mathbb{A}_F^1$.*

*Proof.* For the absolute convergence of the integrals, the arguments of [Yamana 2015, Lemma 3.1] adapt in a straightforward manner and we do not repeat them. The proof of the second point is now essentially that of [AP 2006, Proposition 3.2]. Indeed,

$$\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\, dh = \int_{F^\times\backslash \mathbb{A}_F^1} \left( \int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h \operatorname{diag}(x, I_{n-1}))\, dh \right) dx,$$

and one applies Fourier inversion on the compact abelian group $F^\times\backslash \mathbb{A}_F^1$. □

**Remark 5.7.** The sum of the $(\mathrm{GL}(n, \mathbb{A}_F)^1, \alpha)$-periods over all characters $\alpha$ of the group $F^\times\backslash \mathbb{A}_F^1$ is in fact a finite sum. We denote by $\omega_{\tilde{\pi}}$ the central character of $\tilde{\pi}$. The first observation is that we may assume that $\tilde{\pi}$ is distinguished with respect to $\mathrm{GL}_n(\mathbb{A}_F)$. Indeed if $\tilde{\pi}$ is $(\mathrm{GL}_n(\mathbb{A}_F)^1, \alpha)$-distinguished, then it is $(\mathrm{GL}_n(\mathbb{A}_F), \alpha')$-distinguished for $\alpha'$ the unique character of $\mathbb{A}_F^\times$ extending $\alpha$ and equal to $\omega_{\tilde{\pi}}^n$ on $(\mathbb{A}_F)_{>0}$, but then we take $\alpha'' \in \widehat{E^\times\backslash \mathbb{A}_E^\times}$ with $\alpha''|_{\mathbb{A}_F^\times} = \alpha'$ and replace $\tilde{\pi}$ by $\tilde{\pi} \otimes \alpha''^{-1}$. With this assumption $\tilde{\pi}$ is Galois conjugate self-dual by strong multiplicity one for the residual spectrum [Mœglin and Waldspurger 1989] and the fact that $\mathrm{Sp}(d, \sigma_v)$ is distinguished and hence Galois conjugate self-dual for any finite place $v$ [Flicker 1991]. Now, if the $(\mathrm{GL}(n,\mathbb{A}_F)^1,\alpha)$-period is also nonzero then we have $\tilde{\pi}\cong\tilde{\pi}\otimes\alpha'\circ N_{E/F}$ for the unique character $\alpha'\in\widehat{F^\times\backslash\mathbb{A}_F^\times}$ extending $\alpha$ and equal to $\omega_{\tilde{\pi}}^n$ on $(\mathbb{A}_F)_{>0}$, and writing $\tilde{\pi}=\mathrm{Sp}(d,\sigma)$, we see that $\sigma\cong\sigma\otimes\alpha'\circ N_{E/F}$. As $\sigma$ is a cuspidal representation and because $N_{E/F}(\mathbb{A}_E^\times)$ has finite index in $\mathbb{A}_F^\times$, the set of such characters $\alpha'$ (hence of that of the characters $\alpha$) follows from [Ramakrishnan 2000, Lemma 3.6.2] (which is [Hiraga and Saito 2012, Lemma 4.11]).

**Definition 5.8.** We say that the L-packet determined by a square-integrable representation of $\mathrm{GL}_n(\mathbb{A}_E)$ is distinguished if it contains a distinguished representation of $\mathrm{SL}_n(\mathbb{A}_E)$.

## 6. Distinction inside global L-packets

The aim of this section is to establish our main result, Theorem 6.10, which asserts that distinguished representations inside distinguished L-packets are those with a degenerate $\psi$-Whittaker model for some distinguished $\psi$, and to give a first application of it (Theorem 6.16). The proof is an induction based on the unfolding method, and has two steps, the first one being the cuspidal step (corresponding to $d=1$).

**6A.** ***The cuspidal case.*** Here we characterize members of distinguished L-packets of $\mathrm{SL}_n(\mathbb{A}_E)$ with nonvanishing $\mathrm{SL}_n(\mathbb{A}_F)$-period in terms of Whittaker periods. The following lemma is a generalization of [AP 2006, Lemma 4.3], but the proof there does not generalize to this case. We denote by $Q_n$ the proper parabolic subgroup of $\mathrm{SL}_n$ containing $P_n^\circ=\mathrm{SL}_{n-1}.U_n$. For $n\geq 3$, we set

$$R_n=\{\mathrm{diag}(x,I_{n-2},x^{-1}):x\in\mathbb{G}_m\},$$

so $Q_n$ is the semidirect product $P_n^1.R_n$.

**Lemma 6.1.** *Take $n\geq 3$. Let $\varphi$ be a cusp form on $\mathrm{SL}_n(\mathbb{A}_E)$ such that*

$$\int_{\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A}_F)}\varphi(h)\,dh\neq 0.$$

*Then there is $h_0\in\mathrm{SL}_n(\mathbb{A}_F)$ (and in fact in $R_n(\mathbb{A}_F)$) such that*

$$\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)}\varphi(hh_0)\,dh\neq 0,$$

*where this integral is absolutely convergent.*

*Proof.* By [Sakellaridis and Venkatesh 2017, Section 18.2], there is $s\in\mathbb{C}$ such that for $\Re(s)$ large enough, the integral $\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)}\varphi(p)\delta_{Q_n}^s(p)\,dp$ is absolutely convergent. Moreover, it has meromorphic continuation, and there is a meromorphic function $r(s)$ with $r(0)=0$ such that $r(s)\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)}\varphi(h)\delta_{Q_n}^s(h)\,dh$ tends to $\int_{\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A}_F)}\varphi(h)\,dh\neq 0$ when $s\to 0$. In particular there is an $s\in\mathbb{R}$ large enough in the realm of absolute convergence that

$$0\neq\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)}\varphi(p)\delta_{Q_n}^s(p)\,dp=\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)}\int_{R_n(F)\backslash R_n(\mathbb{A}_F)}\varphi(pa)\delta_{Q_n}^s(a)\,dp\,da,$$

and hence there is an $a\in R_n(\mathbb{A}_F)$ such that $\delta_{Q_n}^s(a)\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)}\varphi(pa)\,dp\neq 0$ and the result follows. $\square$

**Remark 6.2.** A result similar to Lemma 6.1 is [Dijols and Prasad 2019, Proposition 8], which is proved via unfolding an Eisenstein series $E(h,s)$ on $\mathrm{SL}_n(\mathbb{A}_F)$ and using that

$$\mathrm{Res}_{s=1}\biggl(\int_{\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}_F)} \varphi(h)E(h,s)\,dh\biggr) = \mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}(\varphi),$$

a trick that [DP 2019] attributes to [Ash et al. 1993]. A straightforward adaptation of the proof of [DP 2019, Proposition 8] can also be used to prove Lemma 6.1. Though our proof here looks much shorter where we appeal to [Sakellaridis and Venkatesh 2017, Section 18.2], the core of [SV 2017, Proposition 18.2.1] is, however, the equality (18.6) and what follows in [loc. cit.], and it relies on the exact same considerations on Eisenstein series as in [DP 2019, Proposition 8]. Hence the proof above is in fact essentially the same as that of [DP 2019, Proposition 8] but the main part of the argument is contained in the statement of [SV 2017, Section 18.2]. Note that [SV 2017, Section 18.2] is done in general for any semisimple group.

We recall that $U_{n,k} = U_n \cdots U_{k+1} < N_n = U_{n,1}$. For $\psi_{n,k}$ a character of $U_{n,k}(\mathbb{A}_E)$ and $\varphi$ a cusp form on $\mathrm{SL}_n(\mathbb{A}_E)$, we set

$$\varphi_{\psi_{n,k}}(x) = \int_{U_{n,k}(E)\backslash U_{n,k}(\mathbb{A}_E)} \varphi(nx)\psi_{n,k}^{-1}(n)\,dn$$

for $x \in \mathrm{SL}_n(\mathbb{A}_E)$. When $k=1$ and $\psi := \psi_{n,1}$ is nondegenerate, we write $\varphi_\psi = W_{\varphi,\psi}$. Note that the integrals defining $\varphi_{\psi_{n,k}}$ and $W_{\varphi,\psi}$ make sense for any smooth cuspidal function on $P_n^\circ(\mathbb{A}_F)$ and define smooth functions on $P_n^\circ(\mathbb{A}_E)$ which restrict to $P_{n-1}^\circ(\mathbb{A}_E)$ as smooth cuspidal functions again. This defines an appropriate setting for inductive proofs. The reader familiar with it will recognize what is often called the unfolding method in the following proof (see [Jacquet and Shalika 1990, Section 6] for a famous and difficult instance of this technique).

**Proposition 6.3.** *Let $\varphi$ be a smooth cuspidal function on $P_n^\circ(\mathbb{A}_E)$ such that*

$$\int_{P_n^\circ(F)\backslash P_n^\circ(\mathbb{A}_F)} \varphi(h)\,dh \neq 0.$$

*Then there is a nondegenerate character $\psi$ of $N_n(\mathbb{A}_E)/N_n(E+\mathbb{A}_F)$ such that $W_{\varphi,\psi}$ does not vanish on $\mathrm{SL}_{n-1}(\mathbb{A}_F)$. In particular, thanks to Lemma 6.1, if $\pi$ is an $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished cuspidal automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$, then it is $\psi$-generic for a nondegenerate character $\psi$ of $N_n(\mathbb{A}_E)/N_n(E+\mathbb{A}_F)$.*

*Proof.* We induct on $n$, and observe that the $n=2$ case is part of the proof of [AP 2006, Theorem 4.2]. Supposing that $n \geq 3$, we have, by hypothesis,

$$\int_{\mathrm{SL}_{n-1}(F)\backslash \mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \varphi(uh)\,du\,dh \neq 0.$$

Set

$$\varphi^{U_n,F}(x) = \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \varphi(ux)\,du$$

for $x \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$. By the Poisson formula for $(F\backslash\mathbb{A}_F)^{n-1} \subset (E\backslash\mathbb{A}_E)^{n-1}$, we have

$$\varphi^{U_n,F}(x) = \sum_{\psi_{n,n-1} \in \widehat{U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)}} \varphi_{\psi_{n,n-1}}(x),$$

which is in turn equal to

$$\sum_{\psi_{n,n-1}\in \widehat{U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)}\setminus\{1\}} \varphi_{\psi_{n,n-1}}(x)$$

by cuspidality of $\varphi$, where the convergence of the series is absolute. For fixed nondegenerate $\psi^0_{n,n-1}$ of $U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)$, one has

$$\varphi^{U_n,F}(x) = \sum_{\psi_{n,n-1}\in \widehat{U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)}} \varphi_{\psi_{n,n-1}}(x) = \sum_{\gamma\in P^\circ_{n-1}(F)\setminus \mathrm{SL}_{n-1}(F)} \varphi_{\psi^0_{n,n-1}}(\gamma x)$$

because, for $n\geq 3$, the group $\mathrm{SL}_{n-1}(F)$ acts transitively on the set of nontrivial characters of $U_n(\mathbb{A}_E)$ trivial on $U_n(E+\mathbb{A}_F)$, and the stabilizer of $\psi^0_{n,n-1}$ is $P^\circ_{n-1}(F)$. Hence

$$0\neq \int_{\mathrm{SL}_{n-1}(F)\setminus \mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{U_n(F)\setminus U_n(\mathbb{A}_F)} \varphi(uh)\,du\,dh = \int_{P^\circ_{n-1}(F)\setminus \mathrm{SL}_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(h)\,dh,$$

where the right-hand side is absolutely convergent (by Fubini). Now

$$\int_{P^\circ_{n-1}(F)\setminus \mathrm{SL}_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(h)\,dh = \int_{P^\circ_{n-1}(\mathbb{A}_F)\setminus \mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{P^\circ_{n-1}(F)\setminus P^\circ_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(hx)\,dh\,dx,$$

and this implies that

$$\int_{P^\circ_{n-1}(F)\setminus P^\circ_{n-1}(\mathbb{A}_F)} \varphi_{\psi^0_{n,n-1}}(hh_0)\,dh \neq 0$$

for some $h_0\in \mathrm{SL}_{n-1}(\mathbb{A}_F)$. The function $\varphi_0 = (\rho(h_0)\varphi)_{\psi^0_{n,n-1}} = \rho(h_0)\varphi_{\psi^0_{n,n-1}}$ restricts to a smooth cuspidal function on $P^\circ_{n-1}(\mathbb{A}_E)$, and we can apply our induction hypothesis to it, to conclude that $W_{\varphi_0,\psi'}$ is nonzero on $\mathrm{SL}_{n-1}(\mathbb{A}_F)$ for some nondegenerate character $\psi'$ of $N_{n-1}(\mathbb{A}_E)$ trivial on $N_{n-1}(\mathbb{A}_F+E)$. Setting

$$\psi := \psi'\otimes \psi^0_{n,n-1} : n'.u\mapsto \psi'(n')\psi^0_{n,n-1}(u),$$

one checks that, by definition,

$$W_{\varphi_0,\psi'}(x) = W_{\rho(h_0)\varphi,\psi}(x) = W_{\varphi,\psi}(xh_0)$$

for $x\in \mathrm{SL}_{n-1}(\mathbb{A}_E)$. The result follows. □

**Remark 6.4.** As mentioned in Section 1 our strategy in proving Proposition 6.3 is to have an inductive setup to reduce the proof to the case of $n=2$. In the finite field cuspidal case as well as in the $p$-adic field tempered case, such an inductive machinery can be set up via Clifford theory [DP 2019, Proposition 1], and this is carried out in [AP 2018, Proposition 4.2 and Remark 4]. A similar approach in the number field case can be carried out as well by making use of the global analogue of [DP 2019, Proposition 1], which is [DP 2019, Proposition 6]. This was brought to our attention by Prasad. In fact, [DP 2019, Proposition 6] is stated more generally and our inductive setup would follow by taking $H=\mathrm{SL}_{n-1}(\mathbb{A}_F)$ and $A = U_n(\mathbb{A}_E)/U_n(E+\mathbb{A}_F)$, in the notation of [DP 2019, Proposition 6].

**Remark 6.5.** Though not relevant to this paper, we remark here that the inductive strategy in the finite cuspidal and $p$-adic tempered cases mentioned in Remark 6.4 do not seem to generalize to cover all the generic representations. However, the final result, that distinction is characterized by genericity for a nondegenerate character of $N(E)/N(F)$, is established via other methods. In the $p$-adic case, this is done in [AP 2018], and this we have further generalized in Theorem 1.1 of the present paper. In the finite field case, the general result is established in [Anandavardhanan and Matringe 2020].

**Remark 6.6.** We seize the occasion to fill a small gap in the literature, using the ideas of this paper: namely, the unfolding of the Asai $L$-function. The proofs given in [Flicker 1988, p. 303] and [Zhang 2014, p. 558] are a bit quick. Here we add the details to the proof of [Flicker 1988, 2 Proposition, p. 303]. The transition between the second and third lines of the equality there relies on the following step: for $\varphi$ a cusp form on $\mathrm{GL}_n(\mathbb{A}_E)$,

$$\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \varphi(n)\,dn = \sum_{\gamma\in N_n(F)\backslash P_n(F)} W_{\varphi,\psi}(\gamma),$$

where both the "integrals" are absolutely convergent and $\psi$ is a nondegenerate character of $N_n(\mathbb{A}_E)$ trivial on $N_n(\mathbb{A}_F+E)$. We use the same notation as in Proposition 6.3, and denote by $\psi^0_{n,n-1}$ the restriction of $\psi$ to $U_n(\mathbb{A}_E)$.

Let us write

$$\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \varphi(n)\,dn = \int_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A}_F)} \varphi^{U_n,F}(n)\,dn.$$

By induction applied to the cusp form $\varphi^{U_n,F}$ on $\mathrm{GL}_{n-1}(\mathbb{A}_E)$, we have

$$\int_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A}_F)} \varphi^{U_n,F}(n)\,dn = \sum_{\gamma'\in N_{n-1}(F)\backslash P_{n-1}(F)} \varphi^{U_n,F}(\gamma').$$

Now replace $\varphi^{U_n,F}(\gamma')$ by $\sum_{\gamma\in P_{n-1}(F)U_n(F)\backslash P_n(F)} \varphi_{\psi^0_{n,n-1}}(\gamma\gamma')$ this time (still by the Poisson formula and because $P_n(F)$ also acts transitively on the set of nontrivial characters of $U_n(\mathbb{A}_E)$ trivial on $U_n(E+\mathbb{A}_F)$, the stabilizer of $\psi^0_{n,n-1}$ being $P_{n-1}(F)U_n(F)$). We get

$$\begin{aligned}\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \varphi(n)\,dn &= \sum_{\gamma'\in N_{n-1}(F)\backslash P_{n-1}(F)}\ \sum_{\gamma\in P_{n-1}(F)U_n(F)\backslash P_n(F)} W_{\varphi_{\psi^0_{n,n-1}},\psi|_{N_{n-1}(\mathbb{A}_E)}}(\gamma\gamma')\,dn\\ &= \sum_{\gamma'\in N_{n-1}(F)\backslash P_{n-1}(F)}\ \sum_{\gamma\in P_{n-1}(F)U_n(F)\backslash P_n(F)} W_{\varphi,\psi}(\gamma\gamma')\\ &= \sum_{\gamma\in N_{n-1}(F)U_n(F)\backslash P_n(F)} W_{\varphi,\psi}(\gamma),\end{aligned}$$

which is what we wanted.

**6B.** ***The square-integrable case.*** Our aim in this section is to show that if $\pi$ is distinguished then $\pi$ has a nonvanishing Fourier coefficient with respect to a character of type $r^d$ of $N_n(\mathbb{A}_E)$ which is trivial on $N_n(E+\mathbb{A}_F)$ (see Proposition 6.11). The key ingredient in achieving this is Proposition 6.8 below.

The following result is [Yamana 2015, Theorem 1.1] slightly reformulated for our purposes.

**Theorem 6.7.** *Let $n = rd$ with $r \geq 2$ and $d \geq 2$, and let $\psi$ be a nondegenerate unitary character of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$. Fix a character $\alpha$ of $F^\times\backslash\mathbb{A}_F^1$. Then, for $\varphi \in \tilde{\pi} = \mathrm{Sp}(d, \sigma)$, we have*

$$\int_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\alpha(\det h)\,dh = \int_{N^\circ_{n-1,r-1}(\mathbb{A}_F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{\mathrm{GL}_{n-r}(F)\backslash\mathrm{GL}_{n-r}(\mathbb{A}_F)^1} (\alpha\varphi)_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)\,\mathrm{diag}(h, 1))\,dm\,dh.$$

*Proof.* We denote by $\omega_\sigma$ the central character of $\sigma$. We extend $\alpha$ as $\alpha_0$ to $\mathbb{A}_F^\times$. We then extend $\alpha_0$ to an automorphic character of $\beta$ of $\mathbb{A}_E^\times$. Then we claim that the following equality holds:

$$\int_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)^1} \varphi(h)\alpha_0(\det h)\,dh = \int_{N_{n-1,r-1}(\mathbb{A}_F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \int_{\mathrm{GL}_{n-r}(F)\backslash\mathrm{GL}_{n-r}(\mathbb{A}_F)^1} (\alpha_0\varphi)_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)\,\mathrm{diag}(h, 1))\,dm\,dh.$$

Indeed, if $\alpha_0^r \cdot \omega_\sigma|_{\mathbb{A}_F^\times}$ is trivial, then this follows from the second part of [Yamana 2015, Theorem 1.1] applied to $\beta \otimes \pi$. If $\alpha_0^r \cdot \omega_\sigma|_{\mathbb{A}_F^\times} \not\equiv 1$, then it follows from the first part of [Yamana 2015, Theorem 1.1] applied to $\beta \otimes \pi$, with the extra observation that the right-hand side of the equality also vanishes, thanks to Remark 5.3 and the first part of [Yamana 2015, Theorem 1.1] again if $d \geq 3$, and for central character reasons when $d = 2$. We can now replace the quotient $N_{n-1,r-1}(\mathbb{A}_F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)$ by $N^\circ_{n-1,r-1}(\mathbb{A}_F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)$ and the statement follows. □

From Theorem 6.7, we deduce its $\mathrm{SL}(n)$ version by making use of Proposition 5.6.

**Proposition 6.8.** *With notation and assumptions $(r, d \geq 2)$ as in Theorem 6.7, for $\varphi \in \mathrm{Res}(\tilde{\pi})$ we have*

$$p_n(\varphi) = \int_{N^\circ_{n-1,r-1}(\mathbb{A}_F)\backslash\mathrm{SL}_{n-1}(\mathbb{A}_F)} \int_{\mathrm{SL}_{n-r}(F)\backslash\mathrm{SL}_{n-r}(\mathbb{A}_F)} \varphi_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)\,\mathrm{diag}(h, 1))\,dm\,dh.$$

*Proof.* We relate the $\mathrm{SL}(n, \mathbb{A}_F)$-period $p_n$ to the $(\mathrm{GL}(n, \mathbb{A}_F)^1, \alpha)$-periods via Proposition 5.6. Applying Theorem 6.7 to each summand of the sum over characters $\alpha$ of $F^\times\backslash\mathbb{A}_F^\times$ just selected, we once again apply Proposition 5.6 to the right-hand side sum to conclude the proof. □

Setting

$$(\rho(g)\varphi)_{n-r,\psi} := m \in \mathrm{GL}_{n-r}(\mathbb{A}_E) \mapsto \varphi_{U_{n,r},\psi}(\mathrm{diag}(m, I_r)g),$$

Proposition 6.8 implies the following observation, which we state as a lemma.

**Lemma 6.9.** *With notation and assumptions $(r, d \geq 2)$ as in Theorem 6.7, suppose that $\varphi \in \mathrm{Res}(\tilde{\pi})$ is such that $p_n(\varphi) \neq 0$. Then there is $h \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$ such that*

$$p_{n-r}\big((\rho(\mathrm{diag}(h, 1))\varphi)_{n-r,\psi}\big) \neq 0.$$

We now state the main theorem of this section, which holds without the previous assumptions on $r$ and $d$, as do all the results that we state from now on.

**Theorem 6.10.** *Let* $\mathrm{L}(\tilde{\pi})$ *be a distinguished square-integrable* L*-packet of* $\mathrm{SL}_n(\mathbb{A}_E)$ *of type* $r^d$*. Then the period integral* $p_n$ *does not vanish on* $\pi \in \mathrm{L}(\tilde{\pi})$ *if and only if there exists a degenerate character* $\psi_{1,\dots,d}$ *of type* $r^d$ *of* $N_n(\mathbb{A}_E)$ *trivial on* $N_n(E+\mathbb{A}_F)$ *such that* $p_{\psi_{1,\dots,d}}$ *does not vanish on* $\pi$*.*

The key direction of Theorem 6.10 is Proposition 6.11, which follows from Lemma 6.9 by an inductive argument (see also the proof of Proposition 6.3).

**Proposition 6.11.** *Let* $\pi$ *be an irreducible square-integrable automorphic representation of* $\mathrm{SL}_n(\mathbb{A}_E)$ *of type* $r^d$ *which is distinguished with respect to* $\mathrm{SL}_n(\mathbb{A}_F)$*, so that there exists* $\varphi \in \pi$ *such that* $p_n(\varphi) \neq 0$*. Then there exist* $d$ *nondegenerate characters* $\psi_i$ *of* $N_r(\mathbb{A}_E)$ *trivial on* $N_r(E+\mathbb{A}_F)$ *and* $\varphi' \in \pi$ *such that*

$$p_{\psi_{1,\dots,d}}(\varphi') = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi'(n)\psi^{-1}_{1,\dots,d}(n)\,dn \neq 0.$$

*Moreover,* $\varphi'$ *can be chosen to be a right* $\mathrm{SL}_{n-1}(\mathbb{A}_F)$*-translate of* $\varphi$*.*

*Proof.* The theorem is immediate from Lemma 6.9 by an inductive argument, but we have to treat the case $r = 1$ separately. If $r = 1$ then $\pi$ is the trivial character of $\mathrm{SL}_n(\mathbb{A}_E)$ and the claim is obvious. So we suppose that $r \geq 2$. If $d = 1$ the result is proved in Proposition 6.3, so we assume $d \geq 2$. Since $\varphi \in \pi$ is such that $p_n(\varphi) \neq 0$, by Lemma 6.9, we get $h \in \mathrm{SL}_{n-1}(\mathbb{A}_F)$ such that $p_{n-r}((\rho(h)\varphi)_{n-r,\psi}) \neq 0$. Therefore, by induction and thanks to Remark 5.3, we get $d-1$ nondegenerate characters $\psi_i$ for $i = 2, \dots, d$ of $N_r(\mathbb{A}_E)$, trivial on $N_r(E+\mathbb{A}_F)$, such that

$$p_{\psi_{2,\dots,d}}[\rho(x)(\rho(h)\varphi)_{n-r,\psi}] = \int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} (\rho(h)\varphi)_{n-r,\psi}(nx)\psi^{-1}_{2,\dots,d}(n)\,dn \neq 0,$$

for some $x = \mathrm{diag}(y, 1)$ for $y \in \mathrm{SL}_{n-r-1}(\mathbb{A}_F)$. But setting $\psi_1 := \psi$,

$$\begin{aligned}
&\int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} (\rho(h)\varphi)_{n-r,\psi_1}(nx)\psi^{-1}_{2,\dots,d}(n)\,dn \\
&\qquad = \int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} \varphi_{U_{n,r},\psi}(\mathrm{diag}(nx, I_r)h)\psi^{-1}_{1,\dots,d-1}(n)\,dn \\
&\qquad = \int_{N_{n-r}(E)\backslash N_{n-r}(\mathbb{A}_E)} \int_{U_{n,r}(E)\backslash U_{n,r}(\mathbb{A}_E)} \varphi(u\,\mathrm{diag}(nx, I_r)h)\psi^{-1}_{1,\dots,d-1}(n)(\mathbf{1}\otimes\psi^{-1})(u)\,dn\,du \\
&\qquad = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \varphi(n\,\mathrm{diag}(x, I_r)h)\psi^{-1}_{1,\dots,d}(n)\,dn,
\end{aligned}$$

and the result follows. □

To end the proof of Theorem 6.10, it now suffices to prove the following implication, which is part of the proof of [AP 2006, Theorem 4.2], which we repeat.

**Lemma 6.12.** *Let* $\mathrm{L}(\tilde{\pi})$ *be a distinguished* L*-packet of* $\mathrm{SL}_n(\mathbb{A}_E)$ *of type* $r^d$*. If* $\pi \in \mathrm{L}(\tilde{\pi})$ *is* $\psi$*-generic with respect to a degenerate character* $\psi$ *of type* $r^d$ *of* $N_n(\mathbb{A}_E)$ *trivial on* $N_n(E+\mathbb{A}_F)$*, then* $p_n$ *does not vanish on* $\pi$*.*

*Proof.* By definition there is $\pi' \in \mathrm{L}(\tilde{\pi})$ such that $p_n$ does not vanish on it. By Proposition 6.11, the representation $\pi'$ is $\psi'$-generic for a degenerate character $\psi'$ of type $r^d$ of $N_n(\mathbb{A}_E)$ trivial on $N_n(E+\mathbb{A}_F)$.

Now there is $t \in T_n(F)$ such that $\psi = \psi'^t$ where $\psi'^t(n) = \psi'(t^{-1}nt)$. And then the representation $\pi'^t$ given by $\pi'^t(g) = \pi'(t^{-1}gt)$ appears in $\mathrm{L}(\tilde{\pi})$ and is $\psi$-generic. We deduce that $\pi = \pi'^t$, by the local uniqueness of degenerate Whittaker models, and the result follows since $t \in \mathrm{GL}_n(F)$. □

Let us now state a simple but very useful consequence of Theorem 6.10, whose proof idea we have already employed in the proof of Lemma 6.12. We formulate this with an application in Section 9 in mind.

**Corollary 6.13.** *Let $\pi$ be a square-integrable automorphic $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished representation of $\mathrm{SL}_n(\mathbb{A}_E)$, and let $\mathrm{L}(\tilde{\pi}')$ be a distinguished* L*-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ containing an isomorphic copy of $\pi$. Then the period $p_n$ does not vanish on the unique representation in $\mathrm{L}(\tilde{\pi}')$ isomorphic to $\pi$.*

*Proof.* Call $\pi'$ the isomorphic copy of $\pi$ in $\mathrm{L}(\tilde{\pi}')$. Thanks to Theorem 6.10, $\pi$ is $\psi$-generic for $\psi$ a distinguished degenerate character of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$ of the correct type, and therefore $\pi'$ has a locally $\psi_v$-degenerate Whittaker model for every place $v$ of $F$. By Theorem 6.10 again, the $\psi$-generic representation $\pi''$ in $\mathrm{L}(\tilde{\pi}')$ is also $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished. But thanks to multiplicity one of local degenerate Whittaker models, two locally $\psi$-generic automorphic representations in the same L-packet are equal, so $\pi' = \pi''$, and we deduce that $p_n$ does not vanish on $\pi'$. □

As a corollary to Theorem 6.10, we state and prove one more variation of the above theme. This is applied in Section 8.

**Proposition 6.14.** *Let $\pi$ be a canonical realization of an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$. The group $\mathrm{diag}(F^\times, I_{n-1})$ acts transitively on the set of distinguished members of $\mathrm{L}(\pi)$.*

*Proof.* From Theorem 6.10 and the local uniqueness of degenerate Whittaker models, we easily deduce that $T_n(F)$ acts transitively on the set of distinguished members of $\mathrm{L}(\pi)$, and that the representations in $\mathrm{L}(\pi)$ appear with multiplicity one. However, for $t \in T_n(F)$ and $t' = \mathrm{diag}(\det(t), I_{n-1})$, the representations $\pi^t$ and $\pi^{t'}$ in $\mathrm{L}(\pi)$ are isomorphic, hence equal by multiplicity one inside $\mathrm{L}(\pi)$. □

**6C.** ***Automorphy and distinction of the highest derivative for $\mathrm{SL}_n(\mathbb{A}_E)$.*** As a first application of Theorem 6.10, we end this section with an analogue of [Yamana 2015, Theorem 1.2] in the context of $\mathrm{SL}_n(\mathbb{A}_E)$.

**Lemma 6.15.** *Let $\pi$ be a canonical realization of an irreducible square-integrable representation of $\mathrm{SL}_n(\mathbb{A}_E)$ of type $r^d$, and write*

$$\pi \simeq {\bigotimes_v}' \pi_v.$$

*Then, for any $k \in [1, d]$, the representation*

$$\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d}) := {\bigotimes_v}' \pi_v^{[r^{d-k}]}(\psi_{d-k+1,\dots,d,v})$$

(see Definitions 3.5 and 4.5) *is automorphic. If $\sigma$ is a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ such that a canonical realization of $\pi$ belongs to $\mathrm{L}(\mathrm{Sp}(d, \sigma))$, then $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ is in fact isomorphic to the unique element of $\mathrm{L}(\mathrm{Sp}(k, \sigma))$ with a $\psi_{d-k+1,\dots,d}$-Whittaker model.*

*Proof.* Let $\mu$ be the member of $\mathrm{L}(\mathrm{Sp}(k,\sigma))$ with a $\psi_{d-k+1,\dots,d}$-Whittaker model. Then for all places $v$, the representation $\mu_v$ is the member of $\mathrm{L}(\mathrm{Sp}(k,\sigma_v))$ with a $\psi_v$-Whittaker model, and therefore it must be $\pi_v^{[r^{d-d}]}(\psi_{d-k+1,\dots,d,v})$ and the result follows. □

Here is our SL-analogue of [Yamana 2015, Theorem 1.2].

**Theorem 6.16.** *Suppose that $\psi_{1,\dots,d}$ is a character of $N_n(\mathbb{A}_E)$ of type $r^d$ trivial on $N_n(E+\mathbb{A}_F)$. Let $\pi$ be a canonical realization of an irreducible square-integrable representation of $\mathrm{SL}_n(\mathbb{A}_E)$ of type $r^d$ and fix $k\in[1,d]$. Then $\pi(\psi_{1,\dots,d})$ is $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished if and only if $\pi^{[r^{d-k}]}(\psi_{d-k+1,\dots,d})$ is $\mathrm{SL}_{kr}(\mathbb{A}_F)$-distinguished.*

*Proof.* The proof is the same as that of Theorem 3.11, using [Yamana 2015, Theorem 1.2] in lieu of [Matringe 2014, Theorem 2.13]. □

## 7. Characterization of distinguished square-integrable global L-packets

Here we generalize the characterization of distinguished L-packets given in [AP 2006], which turns out to be convenient in the proof of our main applications, namely, the local–global principle inside distinguished L-packets of Section 8 and the study of the behavior of distinction with respect to higher multiplicity in Section 10. The proof is based on the following well-known theorem, which is a consequence of the work of Jacquet and Shalika [1981] on the one hand and Flicker and Zinoviev [1988; 1995] on the other.

**Theorem 7.1.** *Denote by $\omega_{E/F}$ the quadratic character attached to $E/F$ by global class field theory, and let $\tilde{\pi}$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then $\tilde{\pi}$ is conjugate self-dual, i.e., $\tilde{\pi}^\vee\simeq\tilde{\pi}^\theta$ if and only if $\pi$ is either distinguished or $\omega_{E/F}$-distinguished (and in fact not both together).*

*Proof.* Let $\pi_1$, $\pi_2$ and $\pi_3$ be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$. By the aforementioned references, the partial Rankin–Selberg $L^S(s,\pi_1,\pi_2)$ has a pole at $s=1$, which is necessarily simple, if and only if $\pi_2\simeq\pi_1^\vee$, whereas the partial Asai $L$-function $L^S_{\mathrm{As}}(s,\pi_3)$ has a pole (necessarily simple) at $s=1$ if and only if $\pi_3$ is $\mathrm{GL}_n(\mathbb{A}_F)$-distinguished. The result now follows from the equality

$$L^S(s,\pi_1,\pi_1^\theta)=L^S_{\mathrm{As}}(s,\pi_1)L^S_{\mathrm{As}}(s,\omega\otimes\pi_1),$$

where $\omega$ is any Hecke character of $\mathbb{A}_E^\times$ extending $\omega_{E/F}$. □

First it implies the following lemma.

**Lemma 7.2.** *Let $\alpha$ be a character of $F^\times\backslash\mathbb{A}_F^1$ and $\sigma$ be a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$ with central character $\omega$. The restriction of $\omega$ to $(\mathbb{A}_F)_{>0}$ coincides with the restriction of $|\cdot|_{\mathbb{A}_F}^{ir\lambda}$ for some $\lambda\in\mathbb{R}$, and we extend $\alpha$ to $\mathbb{A}_F^\times$ as the automorphic character $\alpha_{-\lambda}$. Suppose that the period integral*

$$\tilde{p}^1_{r,\alpha^{-1}}:\phi\mapsto\int_{\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F)^1}\phi(h)\alpha(\det(h))\,dh$$

*is nonzero on $\sigma$. Then $\alpha^r$ and $\omega^{-1}$ coincide on $\mathbb{A}_F^1$; i.e., $(\alpha_{-\lambda}\circ\det)^{-1}$ restricts as $\omega$ to $\mathbb{A}_F^\times$, and $\sigma$ is $\alpha_{-\lambda}^{-1}$-distinguished, and thus $\sigma^\vee\simeq(\alpha_{-\lambda}\circ N_{E/F})\otimes\sigma^\theta$.*

*Proof.* The fact that $\alpha^r$ and $\omega^{-1}$ must coincide on $\mathbb{A}_F^1$ if $\tilde{p}^1_{r,\alpha^{-1}}$ does not vanish on $\tilde{\pi}$ follows from central character considerations and the fact that $\tilde{p}^1_{r,\alpha^{-1}}$ is $\alpha^{-1}$-equivariant under $\mathrm{GL}_r(\mathbb{A}_F)^1$. But then for $\phi \in \tilde{\pi}$ the function $\alpha_{-\lambda} \otimes \phi : g \mapsto \alpha_{-\lambda}(\det(g))\phi(g)$ is $\mathbb{A}_F^\times$-invariant and we conclude that $\tilde{p}_{r,\alpha_{-\lambda}^{-1}}$ and $\tilde{p}^1_{r,\alpha^{-1}}$ agree up to a positive constant; in particular, $\sigma$ is $\alpha_{-\lambda}^{-1}$-distinguished. Therefore, for $\beta$ an automorphic character extending $\alpha_{-\lambda}$ to $\mathbb{A}_E^\times$, the representation $\beta \otimes \sigma$ is distinguished and we conclude that $\sigma^\vee \simeq (\alpha_{-\lambda} \circ N_{E/F}) \otimes \sigma^\theta$, thanks to Theorem 7.1. □

Now the characterization of square-integrable distinguished L-packets follows.

**Proposition 7.3.** *Let $\tilde{\pi} = \mathrm{Sp}(d, \sigma)$ an irreducible square-integrable representation of $\mathrm{GL}_n(\mathbb{A}_E)$, with $\sigma$ a unitary cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_E)$. Then $\mathrm{L}(\tilde{\pi})$ is distinguished if and only if there is an automorphic character $\alpha \in \widehat{F^\times \backslash \mathbb{A}_F^\times}$ such that $\tilde{\pi}^\vee \simeq (\alpha \circ N_{E/F}) \otimes \tilde{\pi}^\theta$ or, equivalently, $\sigma^\vee \simeq (\alpha \circ N_{E/F}) \otimes \sigma^\theta$.*

*Proof.* If $\tilde{\pi}^\vee \simeq (\alpha \circ N_{E/F}) \otimes \tilde{\pi}^\theta$, which is equivalent to $\sigma^\vee \simeq (\alpha \circ N_{E/F}) \otimes \sigma^\theta$, then $\alpha \otimes \sigma$ is conjugate self-dual hence an automorphic twist of $\sigma$ distinguished by $\mathrm{GL}_r(\mathbb{A}_F)$ thanks to Theorem 7.1. Hence by [Yamana 2015, Theorem 1.2] an automorphic twist of $\tilde{\pi}$ is distinguished by $\mathrm{GL}_n(\mathbb{A}_F)$, and $\mathrm{L}(\tilde{\pi})$ is distinguished thanks to Proposition 5.6 by a straightforward generalization of the second part of the proof of [AP 2006, Proposition 3.2]. Conversely if $\mathrm{L}(\tilde{\pi})$ is distinguished, then by Proposition 5.6 and Lemma 7.2, an automorphic twist of $\tilde{\pi}$ is distinguished and the result follows from Theorem 7.1. □

## 8. Local global principle for distinguished L-packets when $r$ is odd

This section establishes a local–global principle for distinction inside a square-integrable L-packet of type $r^d$ of $\mathrm{SL}_n(\mathbb{A}_E)$, when $r$ is odd.

Our proof makes use of the setup of [AP 2013, Section 7], where such a result is proved for a cuspidal L-packet of $\mathrm{SL}_2(\mathbb{A}_E)$. The proof there is somewhat intricate and relied crucially on an analysis of the fibers of the Asai lift (see [AP 2013, Remark in Section 7]). Here our arguments are more elementary due to the fact that $r$ is odd. This is consistent with the earlier works [Anandavardhanan 2005; AP 2018].

For the moment, however, $r$ is general. Let $\pi$ be a canonical realization of an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ and denote by $\tilde{\pi}$ a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi$ is realized in $\mathrm{Res}(\tilde{\pi})$.

We borrow the notation of [AP 2013, Section 7]. We consider $\mathbb{A}_E^\times$ as a subgroup of $\mathrm{GL}_n(\mathbb{A}_E)$ via the mapping $x \mapsto \mathrm{diag}(x, I_{n-1})$. This group acts by conjugation on isomorphism classes of an irreducible representation $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$. The orbit of $\pi$ under this action is the representation-theoretic L-packet of $\pi$, say $\mathrm{L}'(\pi)$. Let $G_\pi < \mathbb{A}_E^\times$ be the stabilizer of $\pi$. Then (see [Hiraga and Saito 2012, p. 23])

$$G_\pi = \bigcap_{\chi \in X(\tilde{\pi})} \operatorname{Ker} \chi,$$

where

$$X(\tilde{\pi}) = \{\chi \in \widehat{E^\times \backslash \mathbb{A}_E^\times} \mid \tilde{\pi} \otimes \chi \cong \tilde{\pi}\},$$

which is a finite abelian group (see Remark 5.7).

**Remark 8.1.** Note that $\mathrm{L}(\pi)$ identifies with the automorphic members of $\mathrm{L}'(\pi)$. Indeed $\mathrm{L}(\pi)$ clearly identifies with a subset of $\mathrm{L}'(\pi)$. On the other hand, if $\pi'$ is an automorphic member of $\mathrm{L}'(\pi)$, then any of its canonical realizations has a degenerate $\psi$-Whittaker model of type $r^d$ thanks to Proposition 5.4. However $\mathrm{L}(\pi)$ also contains a member $\pi''$ with a degenerate $\psi$-Whittaker model according to Corollary 5.5. We conclude that $\pi' \simeq \pi''$ by local uniqueness of degenerate Whittaker models.

We start with an elementary observation.

**Proposition 8.2.** *Suppose that $\tilde{\pi}$ is a square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ which is Galois conjugate self-dual, i.e., $\tilde{\pi}^\vee \cong \tilde{\pi}^\theta$, and that $\pi \in \mathrm{L}(\tilde{\pi})$. Then $G_\pi$ is stable under the action of $\theta$.*

*Proof.* As $\tilde{\pi}$ is Galois conjugate self-dual, it follows that the finite abelian group $X(\tilde{\pi})$ is stable under the Galois action, and thus $G_\pi$ is Galois stable. Alternatively, note that if $\pi_1$ and $\pi_2$ are in the same L-packet then $G_{\pi_1} = G_{\pi_2}$. Indeed, $\pi_2 = \pi_1^y$, for some $y \in \mathbb{A}_E^\times$, and by definition, $G_{\pi_2} = y^{-1} G_{\pi_1} y = G_{\pi_1}$ as the groups are abelian. In particular, $G_{\pi^\theta} = G_{\pi^\vee}$ as $\tilde{\pi}^\vee \cong \tilde{\pi}^\theta$. Observe also that $G_{\pi^\vee} = G_\pi$. Thus, if $x \in G_\pi$ then $x^\theta \in G_{\pi^\theta} = G_{\pi^\vee} = G_\pi$. □

**Assumption.** From now on, we assume that $E$ is split at the Archimedean places, so that the Archimedean analogue of Theorem 3.9 obviously holds.

As in [AP 2013, Section 7], we define the groups

$$H_0 = \mathbb{A}_E^\times, \quad H_1 = \mathbb{A}_F^\times G_\pi, \quad H_2 = E^\times G_\pi, \quad H_3 = F^\times G_\pi,$$

and we observe that:

(1) The set $H_0 \cdot \pi$ is the L-packet of representations of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by $\pi$ (see, for instance, [Hiraga and Saito 2012, Corollary 2.8]).

(2) The set $H_1 \cdot \pi$ is the set of locally distinguished representations in the L-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by $\pi$ (by Theorem 3.9 and its Archimedean analogue).

(3) The set $H_2 \cdot \pi$ is the set of automorphic representations in the L-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by $\pi$ (by Corollary 5.5).

(4) The set $H_3 \cdot \pi$ is the set of globally distinguished representations in the L-packet of $\mathrm{SL}_n(\mathbb{A}_E)$ determined by $\pi$ (by Proposition 6.14).

We also record the following observation as a lemma.

**Lemma 8.3.** *Let $\pi$ as above be of type $r^d$. Then, for an $x \in \mathbb{A}_E^\times$, we have $x^r \in G_\pi$.*

*Proof.* If $\pi$ has a $\psi_{1,\dots,d}$-Whittaker model with respect to the automorphic character $\psi_{1,\dots,d}$, then

$$\pi^{\mathrm{diag}(xI_r, I_{n-r})} \in \mathrm{L}'(\pi).$$

In particular, for finite places $v$, the local representation $\pi_v^{\mathrm{diag}(x_v I_r, I_{n-r})}$ has a $\psi_{1,\dots,d,v}$-Whittaker model because $\mathrm{diag}(x_v I_r, I_{n-r})$ fixes $\psi_{1,\dots,d,v}$ by conjugation, and hence both $\pi_v$ and $\pi_v^{\mathrm{diag}(xI_r, I_{n-r})}$ have a $\psi_{1,\dots,d,v}$-Whittaker model inside $\mathrm{L}(\pi_v)$, so they are equal, and the lemma follows. □

Next we state the local–global principle for $(\mathrm{SL}_n(\mathbb{A}_E), \mathrm{SL}_n(\mathbb{A}_F))$ for square-integrable automorphic representations (for $r$ odd).

**Theorem 8.4.** *Let $\pi$ be a canonical realization of an irreducible square-integrable automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ such that $\mathrm{L}(\pi)$ is distinguished. Assume that $r$ is odd and write $\pi = \bigotimes'_v \pi_v$, but this time for $v$ varying through the places of $F$ (hence here $\pi_v$ is $\pi_w$ for $w$ the place in $E$ lying over $v$ if $v$ does not split in $E$, and $\pi_v = \pi_{w_1} \otimes \pi_{w_2}$ if $v$ splits into $(w_1, w_2)$). Then $\pi$ is distinguished with respect to $\mathrm{SL}_n(\mathbb{A}_F)$ if and only if each $\pi_v$ is $\mathrm{SL}_n(F_v)$-distinguished.*

*Proof.* One direction is obvious, so we suppose that $\pi$ is locally distinguished. We can always suppose that $\tilde{\pi}$ is conjugate self-dual by Proposition 7.3.

The group $G_\pi$ is Galois stable by Proposition 8.2. As in [AP 2013, Theorem 7.1], we need to prove that the group

$$(H_1 \cap H_2)/H_3$$

is trivial. In order to show that $H_1 \cap H_2 \subseteq H_3$, we claim that $H_2 \cap \mathbb{A}_F^\times \subseteq H_3$.

So let $x \in E^\times G_\pi \cap \mathbb{A}_F^\times$. Note that $x^2 = xx^\theta$, as $x \in \mathbb{A}_F^\times$. Since $G_\pi$ is Galois stable, we see that $x^2 \in F^\times G_\pi = H_3$. Indeed, writing $x = hk$ for $h \in E^\times$ and $k \in G_\pi$, we get

$$x^2 = xx^\theta = hkh^\theta k^\theta = hh^\theta kk^\theta \in F^\times G_\pi.$$

Also $x^r \in G_\pi$ by Lemma 8.3. We have thus shown that both $x^2$ and $x^r$ are in $H_3$. It follows that $x \in H_3$, as $r$ is odd. □

**Remark 8.5.** The simplifying role played by the fact that $r$ is odd in the proof of Theorem 8.4 is quite analogous to its role in the proof of local multiplicity one, when $n$ is odd, for the pair $(\mathrm{SL}_n(E), \mathrm{SL}_n(F))$ (see [Anandavardhanan 2005, p. 183] or [AP 2018, p. 1703]).

## 9. Higher multiplicity for $\mathrm{SL}_n$

We now suppose $n \geq 3$ and recall consequences of the works of Blasius [1994], Lapid [1998; 1999], and Hiraga and Saito [2005; 2012]. This section contains no original result.

**9A.** ***Different notions of multiplicity.*** Let $\pi$ be a cuspidal automorphic representation $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$. There are several other notions of multiplicity for $\pi$, both on the automorphic side and on the Galois parameter side of the putative global Langlands correspondence. We shall need to pass from one to another, and we explain the process in this paragraph. We follow Lapid [1998, p. 293; 1999, p. 162]. First we consider the automorphic side. Thus, let $\tilde{\pi}$ and $\tilde{\pi}'$ be two cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_E)$. We write

(i) $\tilde{\pi} \sim_s \tilde{\pi}'$ if $\tilde{\pi} \simeq \tilde{\pi}' \otimes \eta$ for a Hecke character $\eta$ of $\mathbb{A}_E^\times$,

(ii) $\tilde{\pi} \sim_{ew} \tilde{\pi}'$ if $\tilde{\pi}_v \simeq \tilde{\pi}'_v \otimes \eta_v$ for a character $\eta_v$ of $E_v^\times$ at each place $v$ of $E$,

(iii) $\tilde{\pi} \sim_w \tilde{\pi}'$ if $\tilde{\pi}_v \simeq \tilde{\pi}'_v \otimes \eta_v$ for a character $\eta_v$ of $E_v^\times$ for almost places $v$ of $E$.

One denotes by $M(\mathrm{L}(\tilde{\pi}))$ the number of $\sim_s$ equivalence classes in the $\sim_{ew}$ equivalence class of $\tilde{\pi}$, and by $\mathcal{M}(\mathrm{L}(\tilde{\pi}))$ the number of $\sim_s$ equivalence classes in the $\sim_w$ equivalence class of $\tilde{\pi}$. It was expected by Labesse and Langlands [1979] that if $\pi$ is a cuspidal automorphic representation of $\mathrm{SL}_n(\mathbb{A}_E)$ contained in $\mathrm{L}(\tilde{\pi})$, then its multiplicity $m(\pi)$ inside the cuspidal automorphic spectrum is equal to $M(\mathrm{L}(\tilde{\pi}))$, so that in particular $M(\mathrm{L}(\tilde{\pi}))$ is finite. This was proved for $\mathrm{SL}_2(\mathbb{A}_E)$ in [Labesse and Langlands 1979] and in general for $\mathrm{SL}_n(\mathbb{A}_E)$ by Hiraga and Saito [2012, Theorem 1.6].

On the other hand, the multiplicity $\mathcal{M}(\mathrm{L}(\tilde{\pi}))$, which is conjectured to be finite and bounded by a function of $n$ in [Lapid 1999, Conjecture 1], is certainly at least equal to $M(\mathrm{L}(\tilde{\pi}))$ by definition, and related to a similar multiplicity on the "Galois parameter side". To this end we introduce equivalence relations $\sim_s$ and $\sim_w$ on the set of representations of a group $G$. Letting $\phi$ and $\phi'$ be two morphisms from $G$ to $\mathrm{GL}_n(\mathbb{C})$, we write

(i) $\phi \sim_s \phi'$ if there is $x \in \mathrm{PGL}_n(\mathbb{C})$ such that $\overline{\phi'(g)} = x^{-1}\overline{\phi(g)}x \in \mathrm{PGL}_n(\mathbb{C})$ for all $g \in G$, in which case we say that $\phi$ and $\phi'$ are strongly equivalent;

(ii) $\phi \sim_w \phi'$ if for all $g \in G$, there is $x_g \in \mathrm{PGL}_n(\mathbb{C})$ such that $\overline{\phi'(g)} = x_g^{-1}\overline{\phi(g)}x_g \in \mathrm{PGL}_n(\mathbb{C})$, in which case we say that $\phi$ and $\phi'$ are weakly equivalent.

We denote by $\mathcal{M}(\phi)$ the number of $\sim_s$ equivalence classes in the $\sim_w$ equivalence class of $\phi$. One of the main achievements of [Lapid 1998; 1999] is the following result (see [Lapid 1998, Theorems 6 and 2]).

**Theorem 9.1.** *Let $L$ be a Galois extension of $E$ with respective Weil groups $W_L$ and $W_E$ such that* $\mathrm{Gal}(L/E)$ *is nilpotent, and let $\chi$ be a Hecke character of $\mathbb{A}_E^\times$ such that $\phi = \mathrm{Ind}_{W_L}^{W_E}(\chi)$ is irreducible. Denote by $\tilde{\pi} = \tilde{\pi}(\phi)$ the cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E)$ *associated to* $\mathrm{Ind}_{W_E}^{W_F}(\chi)$ *by* [Arthur and Clozel 1989]. *Then $\mathcal{M}(\phi) = \mathcal{M}(\mathrm{L}(\tilde{\pi}))$.*

**Remark 9.2.** In the proof of this result Lapid invokes the Chebotarev density theorem to argue that for such representations, the relations $\sim_s$ and $\sim_w$ are compatible on the Galois parameter side and the automorphic side, and shows that if $\tilde{\pi}' \sim_w \tilde{\pi}$ (i.e., $\tilde{\pi}'$ is almost everywhere a twist of $\tilde{\pi}$) for $\tilde{\pi}$ as in the statement of Theorem 9.1, then $\tilde{\pi}'$ is of Galois type, i.e., there exists a Galois representation $\phi'$, necessarily unique, of $W_E$ with Satake parameters equal to those of $\tilde{\pi}'$ at almost every place of $E$. We shall use these facts as well in what follows.

**Remark 9.3.** In particular suppose that $\tilde{\pi}$ and $\mathcal{M}(\phi)$ are as in the statement of Theorem 9.1, and suppose moreover that the weak equivalence class of $\tilde{\pi}$ (its $\sim_w$ class) is the same as its $\sim_{ew}$ class. Then, for any $\pi \in \mathrm{L}(\tilde{\pi})$, we have

$$m(\pi) = M(\mathrm{L}(\tilde{\pi})) = \mathcal{M}(\mathrm{L}(\tilde{\pi})) = \mathcal{M}(\phi).$$

Note that the middle equality can in general be a strict inequality; see for example [Blasius 1994, Proposition 2.5].

**9B.** ***Examples of higher cuspidal multiplicity due to Blasius.*** In this section we recall the first fundamental construction, due to Blasius [1994], of representations appearing with a multiplicity greater than one in the cuspidal spectrum of $\mathrm{SL}_n(\mathbb{A}_E)$. In view of the more recent results of Lapid and of Hiraga and Saito recalled in Section 9A, we give a slightly more modern treatment of the construction of Blasius, however following its exact same lines. For $p$ a fixed prime number, we denote by $H_p$ the Heisenberg subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ of upper triangular unipotent matrices with order $p^3$. Blasius considers finite products of Heisenberg groups

$$H_{p_i} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z}/p_i \right\},$$

where for our purpose we restrict a finite number of odd primes $p_i$ possibly equal for $i \neq j$. For each index $i$, we denote by $Z_i$ the center of $H_{p_i}$, and by $\mathcal{L}_i$ the Lagrangian subgroup of $H_{p_i}$ given by $a = 0$. We then set $H = \prod_i H_{p_i}$, $\mathcal{L} = \prod_i \mathcal{L}_i$ and $Z = \prod_i Z_i$.

Now let $E$ be our number field. Since $H$ is a product of $p$-groups it is solvable, and therefore by the well-known result of Shafarevich in inverse Galois theory, there is a Galois extension $L/E$ such that $\mathrm{Gal}(L/E) = H$. Now take for each $i$ a nontrivial character $\chi_i$ of $Z_i$ and extend $\chi_i$ to a character $\tilde{\chi}_i$ of $\mathcal{L}_i$ by

$$\tilde{\chi}_i \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \chi_i \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now set $\chi = \bigotimes_i \chi_i$ to be the corresponding character of $Z$, and call it a *regular* character of $Z$ (meaning all the $\chi_i$ are nontrivial) and $\tilde{\chi} = \bigotimes_i \tilde{\chi}_i$ to be the corresponding character of $\mathcal{L} = \mathrm{Gal}(L/L_{\mathcal{L}})$ (for $L_{\mathcal{L}}$ an extension of $E$). This character can be seen as a Hecke character of the Weil group $W_{L_{\mathcal{L}}}$ (which is trivial on $W_L$). The induced representation $I_\chi = \mathrm{Ind}_{W_{L_{\mathcal{L}}}}^{W_E}(\tilde{\chi})$ is an irreducible representation of $H$ of dimension $n = \prod_i p_i$, and when $\chi$ varies, the representations $I_\chi$ are nonisomorphic and describe all the irreducible representations of $H$, their number being equal to

$$m(n) = \prod_i (p_i - 1).$$

We then set $\tilde{\pi}_\chi$ to be the cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ attached to $I_\chi$ in [Arthur and Clozel 1989]. By Theorem 9.1 we obtain the following result from Section 1.1 of [Blasius 1994].

**Proposition 9.4.** *In the situation above, let $\pi \subset \mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash\mathrm{SL}_n(\mathbb{A}_E))$ be an irreducible summand of $\tilde{\pi}_\chi$. Then $\mathcal{M}(\mathrm{L}(\tilde{\pi}_\chi)) = m(n)$.*

*Proof.* According to Theorem 9.1, it is sufficient to check that the conjugacy class of $I_\chi(w)$ in $\mathrm{PGL}_n(\mathbb{C})$ is independent of $\chi$ for any $w \in W_E$ but that the $I_\chi$'s are inequivalent projective representations. This is done in [Blasius 1994, Section 1.1]. □

We are, however, looking for information on $m(\pi)$ rather than $\mathcal{M}(\mathrm{L}(\tilde{\pi}_\chi))$. Therefore we follow Blasius again to put us in a situation where $\mathcal{M}(\mathrm{L}(\tilde{\pi}_\chi)) = M(\mathrm{L}(\tilde{\pi}_\chi))$ in order to apply Remark 9.3. To this end we select $L$ as in the proof of [Blasius 1994, Proposition 2.1], such that at all the places in $L$ lying above $p$ for each $p$ dividing $|H|$, $L$ is unramified.

Then in such a situation, by [Blasius 1994, Proposition 2.1(2)], we deduce that two representations $\tilde{\pi}_\chi$ and $\tilde{\pi}_{\chi'}$, for regular characters $\chi$ and $\chi'$ of $Z$, are not only weakly equivalent (which we already know from [Blasius 1994, Section 1.1] and Section 9A), but they are in fact in the same $\sim_{ew}$-class, i.e., they are twists of each other at every place of $E$. Finally, by Remark 9.2, if $\tilde{\pi}$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ weakly equivalent to $\tilde{\pi}_\chi$, it is of Galois type with Galois parameter, say, $\phi$. Because for every $w \in W_E$, the conjugacy class of $I_\chi(w)$ in $\mathrm{GL}_n(\mathbb{C})$ is equal to that of $\phi(w)$, we deduce that $I_\chi$ and $\phi$ have the same kernel, and are thus in fact both irreducible representations of $H$. This implies that $\phi$ is itself of the form $I_{\chi'}$ for a regular character $\chi'$ of $Z$; in particular, the $\sim_w$ class of $\pi$ is equal to its $\sim_{ew}$ class. In view of Remark 9.3, the outcome of this discussion is the following result, which also follows from the proof of [Blasius 1994, Proposition 3.3].

**Proposition 9.5.** *Let $E$ be a number field and let $L$ be an extension of $E$ such that* $\mathrm{Gal}(L/E) \simeq H$ *and such that $L$ is unramified at every place of $L$ lying over a prime divisor of the cardinality $n = |H|$. Let $\chi$ be a regular character of $Z$ and let $\pi \in \mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash\mathrm{SL}_n(\mathbb{A}_E))$ be an irreducible summand of $\tilde{\pi}_\chi$. Then $m(\pi) = m(n)$, and the* L*-packets containing a copy of $\pi$ are those of the form* $\mathrm{L}(\tilde{\pi}_{\chi'})$ *for a regular character $\chi'$ of $Z$, and they are all different.*

**Remark 9.6.** Such extensions $L$ of $E$ exist in abundance by Shafarevich's theorem in inverse Galois theory.

**Remark 9.7.** Blasius [1994] had conjectured that two L-packets, say $\mathrm{L}(\tilde{\pi})$ and $\mathrm{L}(\tilde{\pi}')$, would be isomorphic if $\tilde{\pi}$ and $\tilde{\pi}'$ are locally isomorphic at every place up to a character twist [Blasius 1994, Conjecture on p. 239]. This conjecture was later proved by Hiraga and Saito [2005]. Lacking the truth of the conjecture at that point in time, [Blasius 1994] resorted to a trick using complex conjugation. Note that reading out the precise multiplicity $m(\pi)$ is an immediate consequence of this result.

## 10. Two questions

In this section we attempt to answer two natural and important questions. We thank Raphaël Beuzart-Plessis and Prasad for posing the first of these questions to us in the context of this paper. We then consider one more question, which in the case of SL(2) was answered by an explicit construction in [AP 2006, Theorem 8.2]. The key ingredient in all our constructions is the explicit nature of the examples of cuspidal representations of high multiplicity in [Blasius 1994; Lapid 1998; 1999]. In these examples, we also need to make a crucial use of the main result of this paper (see Theorem 6.10).

**10A.** ***Questions.*** We formulate two natural questions, for each of which we provide answers in the later subsections.

**Question 10.1.** Consider the natural decomposition of $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash\mathrm{SL}_n(\mathbb{A}_E))$ into L-packets. Let $\pi_1$ and $\pi_2$ be two canonical realizations of an irreducible submodule of $\mathcal{A}_0^\infty(\mathrm{SL}_n(E)\backslash\mathrm{SL}_n(\mathbb{A}_E))$ such that $\pi_1 \simeq \pi_2$ but which belong to two different L-packets $\mathrm{L}(\tilde{\pi}_1) \neq \mathrm{L}(\tilde{\pi}_2)$. If $p_n$ does not vanish on $\pi_1$, then is it true that it does not vanish on $\pi_2$?

**Remark 10.2.** We shall see in Section 10C that the answer is no in general. Thus, for $n \geq 3$, there are cuspidal automorphic representations of $\mathrm{SL}_n(\mathbb{A}_E)$ which are locally distinguished, but with at least one canonical realization in the space of smooth cusp forms on which $p_n$ vanishes.

The following question arises immediately after the above remark.

**Question 10.3.** For $n \geq 3$, are there cuspidal automorphic representations of $\mathrm{SL}_n(\mathbb{A}_E)$ which are locally distinguished at every place of $F$, but not globally? In fact is it even possible to construct such a representation with no canonical realization belonging to a distinguished L-packet?

We shall see in Section 10D that such representations do exist. Note that though Question 10.1 is not meaningful for $\mathrm{SL}_2(\mathbb{A}_E)$ according to Ramakrishnan's multiplicity one result [Ramakrishnan 2000], the issues addressed by Remark 10.2, as well as Question 10.3, make sense for $n = 2$. In this case both questions are answered in [AP 2006]. In fact it is sufficient to answer Question 10.3 for $n = 2$, and this is done by [AP 2006, Theorem 8.2], the proof of which is quite involved: there are indeed cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ which are locally distinguished at every place of $F$ but not globally. We shall provide easier examples of this type in Section 10C for $n \geq 3$.

**10B.** ***Distinguished cuspidal representations of higher multiplicity.*** Now we need to construct cuspidal representations $\pi$ of $\mathrm{SL}_n(\mathbb{A}_E)$ which are $\mathrm{SL}_n(\mathbb{A}_F)$-distinguished with $m(\pi) \geq 2$ for odd $n$.

Let us explain our general recipe for this, using the examples of Blasius in Section 9B. We take $n \geq 3$ odd and write it as $n = \prod_i p_i$. We set $H = \prod_i H_{p_i}$ as before and take an involution $\theta$ of the group $H$. Associated to this involution is the semidirect product

$$G = H \rtimes \mathbb{Z}/2,$$

where $\mathbb{Z}/2$ acts on $H$ via $\theta$. Now let $F$ be any number field and let $L$ be an extension of $F$ such that $\mathrm{Gal}(L/F) \simeq G$. In fact we choose $L$ in such a way that $L/F$ is unramified at each place of $F$ lying above any $p$ dividing $n$. Note that all these can be done by Shafarevich's theorem since $G$ is solvable. Let $E$ be the fixed field of $H$ so that

$$\mathrm{Gal}(L/E) \simeq H \quad \text{and} \quad \mathrm{Gal}(E/F) = \langle\theta\rangle.$$

Take an irreducible representation $\rho$ of $H$. It identifies with $I_{\chi_\rho}$ for $\chi_\rho$ a regular character of $Z$ and we set $\tilde{\pi}(\rho) = \tilde{\pi}_{\chi_\rho}$ (see Section 9B). In particular, because $L/E$ is unramified at places of $E$ lying above the prime divisors of $n$, if $\pi$ belongs to $\mathrm{L}(\tilde{\pi}(\rho))$, we obtain $m(\pi) = m(n)$ thanks to Proposition 9.5. In this situation, we have the following very useful result due to the rigidity of the representation theory of Heisenberg groups, which we will apply in order to produce examples answering Question 10.1.

**Proposition 10.4.** *In the situation described above, take an irreducible representation $\rho$ of $H$ and denote by $c_\rho$ its central character. The* L*-packet* $\mathrm{L}(\tilde{\pi}(\rho))$ *is distinguished if and only if $c_\rho(z^\theta) = c_\rho(z^{-1})$ for all $z \in Z$.*

*Proof.* By Proposition 7.3, $\mathrm{L}(\tilde{\pi}(\rho))$ is distinguished if and only if $(\tilde{\pi}(\rho)^\vee)^\theta \simeq \mu \otimes \tilde{\pi}(\rho)$ for a Hecke character $\mu$ factoring through $N_{E/F}$. This is equivalent to $\tilde{\pi}((\rho^\vee)^\theta) \simeq \mu \otimes \tilde{\pi}(\rho)$. However as the L-packets determined by different irreducible representations are different thanks to Proposition 9.5, we easily deduce that $\mathrm{L}(\tilde{\pi}(\rho))$ is distinguished if and only if $\rho$ is conjugate self-dual, i.e., $\rho^\vee \simeq \rho^\theta$. The result now follows from the fact that $\rho$ is determined by its central character. □

In view of Corollary 6.13, a consequence of Proposition 10.4 is the following.

**Corollary 10.5.** *In the situation of Proposition 10.4, let $\rho$ be an irreducible representation of $H$ such that $c_\rho^\theta = c_\rho^{-1}$, and $\pi \in \mathrm{L}(\tilde{\pi}(\rho))$ such that $\mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}$ does not vanish on $\pi$. Then the canonical copies of $\pi$ on which $\mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}$ does not vanish are those contained in the* L*-packets of the form* $\mathrm{L}(\tilde{\pi}(\rho'))$ *with $\rho'$ an irreducible representation of $H$ such that $c_{\rho'}^\theta = c_{\rho'}^{-1}$.*

**10C.** ***Examples for Question 10.1.*** We first give two examples for which we answer Question 10.1. In the first one, all the canonical copies of the considered distinguished representation have a nonvanishing period, whereas in the second example only some of the canonical copies of the considered distinguished representation have a nonvanishing period and some others do not have a nonvanishing period.

For the first set of examples, the group $H$ is as in Section 10B and the involution that we consider on it, for $a$, $b$ and $c$ in $\prod_i \mathbb{Z}/p_i$, is given by

$$\theta : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & -c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case because the associated involution acts as the inversion on $Z$, Proposition 10.4 tells us that all L-packets $\mathrm{L}(\tilde{\pi}(\rho))$ are distinguished when $\rho$ varies in the set of irreducible classes of representations of $H$, and that if one fixes a representation $\pi$ in one L-packet on which $\mathcal{P}_{\mathrm{SL}_n(\mathbb{A}_F)}$ does not vanish, then it does not vanish on any of the $m(n)$ canonical copies of $\pi$.

For the second set of examples, we consider $H$ as above (of odd cardinality $n$) and $H' = H \times H$ (which is in fact a special type of $H$) endowed with the switching involution

$$\theta : (x, y) \mapsto (y, x).$$

In this case Proposition 10.4 tells us that the distinguished L-packets of $\mathrm{SL}_{n^2}(\mathbb{A}_E)$ of the form $\mathrm{L}(\tilde{\pi}(\rho'))$ are the $m(n)$ ones such that $\chi_{\rho'}$ is of the form $\chi \otimes \chi^{-1}$ with $\chi$ regular, whereas the others are not. Then again by Corollary 10.5 we conclude that if $\pi$ is a fixed distinguished representation of $\mathrm{SL}_{n^2}(\mathbb{A}_E)$ appearing in one of the $m(n)^2$ many L-packets above, then the period $\mathcal{P}_{\mathrm{SL}_{n^2}(\mathbb{A}_F)}$ does not vanish on the $m(n)$ canonical copies inside the distinguished $m(n)$ many distinguished L-packets, and does vanish on the $m(n)^2 - m(n)$ remaining ones.

**10D.** ***Examples for Question 10.3.*** Now we give a set of examples answering Question 10.3, using again Proposition 10.4.

For simplicity we take $H = H_p$ for $p$ an odd prime (i.e., $n = p$), and we also take $L/F$, hence in particular $E/F$, split at Archimedean places (however we explain in Remark 10.6 how to get rid of this assumption). Let $\theta$ be an involution of $H$ such that $z^\theta = z$ for all $z \in Z$. Thus, we may take the trivial involution or the involution of $H$ given by

$$\theta : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -a & c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $z^\theta = z$ for all $z \in Z$, Proposition 10.4 implies that no L-packet of the form $\tilde{\pi}(\rho)$ for $\rho$ an irreducible representation of $H$ is distinguished because, as $|Z|$ is odd, the only character of $Z$ of order $\leq 2$ is trivial.

It remains to prove that if we fix $\rho$ as above, and set $\tilde{\pi} = \tilde{\pi}(\rho)$, then $\mathrm{L}(\tilde{\pi})$ contains an automorphic representation $\pi$ such that $\pi_v$ is $\mathrm{SL}_p(F_v)$-distinguished for every place $v$ of $F$. This is equivalent to showing that $\tilde{\pi}_v$ is $(\mathrm{GL}_p(F_v), \gamma_v)$-distinguished for some character $\gamma_v$ of $F_v^\times$, which is what we do. Recall that by [Blasius 1994, Proposition 2.1],

$$\tilde{\pi}_v^\theta \simeq \tilde{\pi}_v^\vee \otimes \eta_v$$

at each place $v$ for a character $\eta_v$ of $E_v^\times$.

If a place $v$ of $F$ splits in $E$ as $(v_1, v_2)$ then the above condition implies $\tilde{\pi}_v$ is of the form $(\tau, \tau^\vee \otimes \nu)$, which is distinguished for the character $\nu$ of $\mathbb{F}_v^\times$.

Now let $v$ be such that it does not split in $E$; in particular, $v$ is finite. We set $B_p(E_v)$ the upper triangular Borel subgroup of $\mathrm{GL}_p(E_v)$.

We write as before $\tilde{\pi} = \tilde{\pi}(\rho)$ for $\rho$ an irreducible representation of $H$. We denote by $\mathcal{L}$ and $\mathcal{L}'$ the first and the second Lagrangian subgroups of $H$ given by $a = 0$ and $b = 0$ respectively (see Section 9B). By the proof of [Blasius 1994, Proposition 2.1] the local Galois group of $H_v$ is an abelian subgroup of $H$, hence either trivial or equal to $Z$, $\mathcal{L}$ or $\mathcal{L}'$. We recall that $\rho = \mathrm{Ind}_{\mathcal{L}}^{H}(\tilde{\chi})$, where

$$\tilde{\chi} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \chi(c)$$

for $\chi$ a nontrivial character of $\mathbb{Z}/p$. We fix $\mu$ a nontrivial character $\mathbb{Z}/p$ and set

$$\tilde{\mu} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \mu(b).$$

Similarly we set

$$\tilde{\chi}' \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \chi(c) \quad \text{and} \quad \tilde{\mu}' \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mu(a).$$

Clearly if $H_v$ is trivial or equal to $Z$, then $\rho|_{H_v}$ is a sum of copies of the same character, and hence $\tilde{\pi}_v$ is of the form

$$\mathrm{Ps}(\alpha, \dots, \alpha) = \mathrm{Ind}_{B_p(E_v)}^{\mathrm{GL}_p(E_v)}(\alpha \otimes \cdots \otimes \alpha),$$

where induction is normalized, hence $\alpha|_{F_v^\times}$-distinguished by [Matringe 2011, Theorem 5.2]. Now we consider the case $H_v = \mathcal{L}$. Then, by Mackey theory,

$$\rho|_{\mathcal{L}} = \tilde{\chi}.\Big(\bigoplus_{k=0}^{p-1} \tilde{\mu}^k\Big).$$

Thus the corresponding principal series is of the form

$$\tilde{\pi}_v = \mathrm{Ps}(\alpha, \alpha\beta, \alpha\beta^{-1}, \dots, \alpha\beta^{(p-1)/2}, \alpha\beta^{-(p-1)/2}). \tag{3}$$

If $\theta$ is the trivial involution we trivially have $\beta = \beta^\theta$ so (3) takes the form

$$\tilde{\pi}_v = \alpha \otimes \mathrm{Ps}(1, \beta, \beta^{-\theta}, \dots, \beta^{(p-1)/2}, (\beta^{(p-1)/2})^{-\theta}),$$

which is distinguished by [Matringe 2011, Theorem 5.2].

If $\theta$ is the nontrivial involution such that $z^\theta = z$ for $z \in Z$ then note that $\theta$ fixes $\tilde{\chi}$ whereas it sends $\tilde{\mu}$ to its inverse. We set $\mu_k = \alpha\beta^k$ for $k = 1, \dots, \frac{1}{2}(p-1)$, so that (3) takes the form

$$\tilde{\pi}_v = \mathrm{Ps}(\alpha, \mu_1, \mu_1^\theta, \dots, \mu_{(p-1)/2}, \mu_{(p-1)/2}^\theta).$$

Now because for $k = 1, \dots, \frac{1}{2}(p-1)$, one has $\alpha^2 = \mu_k\mu_k^\theta$ and hence $\alpha|_{F_v^\times}^2 = \mu_k|_{F_v^\times}^2$, but as both characters in this equality have odd order $p$ we deduce that $\alpha|_{F_v^\times} = \mu_k|_{F_v^\times}$. So

$$\tilde{\pi}_v = \alpha \otimes \mathrm{Ps}(1, \alpha^{-1}\mu_1, \alpha^{-1}\mu_1^\theta, \dots, \alpha^{-1}\mu_{(p-1)/2}, \alpha^{-1}\mu_{(p-1)/2}^\theta),$$

and all the characters appearing in the principal series have trivial restriction to $F_v^\times$, and thus we deduce again from [Matringe 2011, Theorem 5.2] that $\tilde{\pi}_v$ is $\alpha|_{F_v^\times}$-distinguished.

Finally, when $H_v = \mathcal{L}'$,

$$\rho|_{\mathcal{L}'} = \tilde{\chi}'.\Big(\bigoplus_{k=0}^{p-1} \tilde{\mu}'^k\Big),$$

and an analogous argument proves that $\tilde{\pi}_v$ is distinguished by a character.

Hence $\mathrm{L}(\tilde{\pi})$ does not contain any distinguished representation but it contains cuspidal representations which are everywhere locally distinguished.

**Remark 10.6.** In constructing examples in this section, we chose $L/F$ such that the Archimedean places split in order to have $E/F$ split at the Archimedean places. This assumption can be removed because the characterization of a generic distinguished principal series, as in [Matringe 2011, Theorem 5.2], is true also for $(\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{R}))$. Namely, a generic principal series $\mathrm{Ps}(\chi_1, \dots, \chi_n)$ of $\mathrm{GL}_n(\mathbb{C})$ is $\mathrm{GL}_n(\mathbb{R})$-distinguished if and only if there is an involution $\epsilon$ of in the symmetric group $S_n$ such that $\chi_{\epsilon(i)} = \chi_i^{-\theta}$ for any $i = 1, \dots, n$, and moreover, $(\chi_i)|_{\mathbb{R}^\times} = 1$ if $\epsilon(i) = i$. The direct implication is a special case of [Kemarsky 2015, Theorem 1.2], whereas the other implication can be obtained as follows. First up to reordering (which is possible as the principal series is generic by assumption) we can suppose that there is $1 \le s \le \lfloor \frac{1}{2}n \rfloor$ such that $\chi_{2i} = \chi_{2i-1}^{-\theta}$ for $i = 1, \dots, s$, and that $(\chi_i)|_{\mathbb{R}^\times} = 1$ for $i = 2s+1, \dots, n$. Now a principal series $\mathrm{Ps}(\chi, \chi^{-\theta})$ of $\mathrm{GL}_2(\mathbb{C})$ is $\mathrm{GL}_2(\mathbb{R})$-distinguished. Indeed by [Carmona and Delorme

1994, Théorème 3], for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough, there is a $\mathrm{GL}_2(\mathbb{R})$-invariant continuous linear form $L_s$ on $\mathrm{Ps}(\chi|\cdot|_{\mathbb{R}}^s, \chi^{-\theta}|\cdot|_{\mathbb{R}}^{-s})$, and a nonzero holomorphic function $h$ on $\mathbb{C}$ such that $h(s)L_s(f_s)$ extends to a holomorphic function on $\mathbb{C}$ for any flat section $f_s$ of $\mathrm{Ps}(\chi|\cdot|_{\mathbb{R}}^s, \chi^{-\theta}|\cdot|_{\mathbb{R}}^{-s})$. Moreover by [Carmona and Delorme 1994, Théorème 3] the meromorphic function $s \mapsto L_s(f_s)$ is nonzero for some choice of $f_s$, which by density we can suppose to be $U(2, \mathbb{C}/\mathbb{R})$-finite because $L_s$ is continuous for $\mathrm{Re}(s)$ large enough. A standard leading-term argument then allows to regularize $L_s$ at $s = 0$ to define a nonzero $\mathrm{GL}_2(\mathbb{R})$-invariant linear form $L$ on the dense subspace of $U(2, \mathbb{C}/\mathbb{R})$-finite vectors in $\mathrm{Ps}(\chi, \chi^{-\theta})$. Finally one extends $L$ to a necessarily nonzero element of $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{R})}(\mathrm{Ps}(\chi, \chi^{-\theta}), \mathbb{C})$ by [Brylinski and Delorme 1992, Théorème 1]. Once we have this result, the transitivity of parabolic induction together with a closed-orbit-contribution argument allows to define a nonzero $\mathrm{GL}_n(\mathbb{R})$-invariant linear form on $\mathrm{Ps}(\chi_1, \dots, \chi_n)$.

**Remark 10.7.** It is not hard to extend the examples obtained in this section in the cuspidal case, to the square-integrable case, using the results of this paper.


## Acknowledgements

The authors thank Raphaël Beuzart-Plessis and Yiannis Sakellaridis for useful comments and explanations. The content of Sections 10B and 10C grew out of a discussion with Beuzart-Plessis. The authors would like to especially thank Dipendra Prasad for his questions and comments over several e-mail conversations; his guidance in general has played a significant role in the writing of some parts of this paper. Finally, the authors warmly thank the referee for a meticulous reading of the manuscript and many helpful comments and suggestions which clarified several points of the paper including some of the proofs and statements.


## References


[Aizenbud et al. 2015] A. Aizenbud, D. Gourevitch, and S. Sahi, "Derivatives for smooth representations of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$", *Israel J. Math.* **206**:1 (2015), 1–38. MR Zbl

[Anandavardhanan 2005] U. K. Anandavardhanan, "Distinguished non-Archimedean representations", pp. 183–192 in *Algebra and number theory* (Hyderabad, India, 2003), edited by R. Tandon, Hindustan Book Agency, Delhi, 2005. MR Zbl

[Anandavardhanan and Matringe 2020] U. K. Anandavardhanan and N. Matringe, "Test vectors for finite periods and base change", *Adv. Math.* **360** (2020), art. id. 106915. MR Zbl

[Anandavardhanan and Prasad 2003] U. K. Anandavardhanan and D. Prasad, "Distinguished representations for SL(2)", *Math. Res. Lett.* **10**:5-6 (2003), 867–878. MR Zbl

[Anandavardhanan and Prasad 2006] U. K. Anandavardhanan and D. Prasad, "On the SL(2) period integral", *Amer. J. Math.* **128**:6 (2006), 1429–1453. MR Zbl

[Anandavardhanan and Prasad 2013] U. K. Anandavardhanan and D. Prasad, "A local-global question in automorphic forms", *Compos. Math.* **149**:6 (2013), 959–995. MR Zbl

[Anandavardhanan and Prasad 2018] U. K. Anandavardhanan and D. Prasad, "Distinguished representations for $\mathrm{SL}(n)$", *Math. Res. Lett.* **25**:6 (2018), 1695–1717. MR Zbl

[Arthur and Clozel 1989] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Ann. of Math. Stud. **120**, Princeton Univ. Press, 1989. MR Zbl

[Ash et al. 1993] A. Ash, D. Ginzburg, and S. Rallis, "Vanishing periods of cusp forms over modular symbols", *Math. Ann.* **296**:4 (1993), 709–723. MR Zbl

[Bernstein 1984] J. N. Bernstein, "$P$-invariant distributions on $\mathrm{GL}(N)$ and the classification of unitary representations of $\mathrm{GL}(N)$ (non-Archimedean case)", pp. 50–102 in *Lie group representations*, *II* (College Park, MD, 1982/1983), edited by R. Herb et al., Lecture Notes in Math. **1041**, Springer, 1984. MR Zbl

[Bernstein and Zelevinsky 1976] J. N. Bernstein and A. V. Zelevinskii, "Representations of the group $\mathrm{GL}(n, F)$, where $F$ is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3(189) (1976), 5–70. In Russian; translated in *Russ. Math. Surv.* **31**:3 (1976), 1–68. MR Zbl

[Bernstein and Zelevinsky 1977] I. N. Bernstein and A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, I", *Ann. Sci. École Norm. Sup.* (4) **10**:4 (1977), 441–472. MR Zbl

[Blasius 1994] D. Blasius, "On multiplicities for $\mathrm{SL}(n)$", *Israel J. Math.* **88**:1-3 (1994), 237–251. MR Zbl

[Brylinski and Delorme 1992] J.-L. Brylinski and P. Delorme, "Vecteurs distributions $H$-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégrales d'Eisenstein", *Invent. Math.* **109**:3 (1992), 619–664. MR Zbl

[Carmona and Delorme 1994] J. Carmona and P. Delorme, "Base méromorphe de vecteurs distributions $H$-invariants pour les séries principales généralisées d'espaces symétriques réductifs: equation fonctionnelle", *J. Funct. Anal.* **122**:1 (1994), 152–221. MR Zbl

[Cogdell 2004] J. W. Cogdell, "Lectures on $L$-functions, converse theorems, and functoriality for $\mathrm{GL}_n$", pp. 1–96 in *Lectures on automorphic $L$-functions*, Fields Inst. Monogr. **20**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[Cogdell and Piatetski-Shapiro 2017] J. W. Cogdell and I. I. Piatetski-Shapiro, "Derivatives and $L$-functions for $\mathrm{GL}_n$", pp. 115–173 in *Representation theory*, *number theory*, *and invariant theory* (New Haven, CT, 2015), edited by J. Cogdell et al., Progr. Math. **323**, Birkhäuser, Cham, 2017. MR Zbl

[Dijols and Prasad 2019] S. Dijols and D. Prasad, "Symplectic models for unitary groups", *Trans. Amer. Math. Soc.* **372**:3 (2019), 1833–1866. MR Zbl

[Flicker 1988] Y. Z. Flicker, "Twisted tensors and Euler products", *Bull. Soc. Math. France* **116**:3 (1988), 295–313. MR Zbl

[Flicker 1991] Y. Z. Flicker, "On distinguished representations", *J. Reine Angew. Math.* **418** (1991), 139–172. MR Zbl

[Flicker and Zinoviev 1995] Y. Z. Flicker and D. Zinoviev, "On poles of twisted tensor $L$-functions", *Proc. Japan Acad. Ser. A Math. Sci.* **71**:6 (1995), 114–116. MR Zbl

[Gelbart and Knapp 1982] S. S. Gelbart and A. W. Knapp, "$L$-indistinguishability and $R$ groups for the special linear group", *Adv. Math.* **43**:2 (1982), 101–121. MR Zbl

[Gourevitch and Sahi 2013] D. Gourevitch and S. Sahi, "Annihilator varieties, adduced representations, Whittaker functionals, and rank for unitary representations of $\mathrm{GL}(n)$", *Selecta Math.* (*N.S.*) **19**:1 (2013), 141–172. MR Zbl

[Hiraga and Saito 2005] K. Hiraga and H. Saito, "On restriction of admissible representations", pp. 299–326 in *Algebra and number theory* (Hyderabad, India, 2003), edited by R. Tandon, Hindustan Book Agency, Delhi, 2005. MR Zbl

[Hiraga and Saito 2012] K. Hiraga and H. Saito, *On $L$-packets for inner forms of* $\mathrm{SL}_n$, Mem. Amer. Math. Soc. **1013**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Jacquet 1984] H. Jacquet, "On the residual spectrum of $\mathrm{GL}(n)$", pp. 185–208 in *Lie group representations*, *II* (College Park, MD, 1982/1983), edited by R. Herb et al., Lecture Notes in Math. **1041**, Springer, 1984. MR Zbl

[Jacquet 2009] H. Jacquet, "Archimedean Rankin–Selberg integrals", pp. 57–172 in *Automorphic forms and $L$-functions*, *II*: *Local aspects* (Tel Aviv, 2006), edited by D. Ginzburg et al., Contemp. Math. **489**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[Jacquet and Shalika 1981] H. Jacquet and J. A. Shalika, "On Euler products and the classification of automorphic forms, II", *Amer. J. Math.* **103**:4 (1981), 777–815. MR Zbl

[Jacquet and Shalika 1990] H. Jacquet and J. Shalika, "Exterior square $L$-functions", pp. 143–226 in *Automorphic forms*, *Shimura varieties*, *and $L$-functions*, *II* (Ann Arbor, MI, 1988), edited by L. Clozel and J. S. Milne, Perspect. Math. **11**, Academic Press, Boston, 1990. MR Zbl

[Jiang and Liu 2013] D. Jiang and B. Liu, "On Fourier coefficients of automorphic forms of $\mathrm{GL}(n)$", *Int. Math. Res. Not.* **2013**:17 (2013), 4029–4071. MR Zbl

[Kemarsky 2015] A. Kemarsky, "Distinguished representations of $\mathrm{GL}_n(\mathbb{C})$", *Israel J. Math.* **207**:1 (2015), 435–448. MR Zbl

[Labesse and Langlands 1979] J.-P. Labesse and R. P. Langlands, "$L$-indistinguishability for SL(2)", *Canad. J. Math.* **31**:4 (1979), 726–785. MR Zbl

[Lapid 1998] E. M. Lapid, "A note on the global Langlands conjecture", *Doc. Math.* **3** (1998), 285–296. MR Zbl

[Lapid 1999] E. M. Lapid, "Some results on multiplicities for SL($n$)", *Israel J. Math.* **112** (1999), 157–186. MR Zbl

[Matringe 2011] N. Matringe, "Distinguished generic representations of GL($n$) over $p$-adic fields", *Int. Math. Res. Not.* **2011**:1 (2011), 74–95. MR Zbl

[Matringe 2014] N. Matringe, "Unitary representations of GL($n, K$) distinguished by a Galois involution for a $p$-adic field $K$", *Pacific J. Math.* **271**:2 (2014), 445–460. MR Zbl

[Mœglin and Waldspurger 1989] C. Mœglin and J.-L. Waldspurger, "Le spectre résiduel de GL($n$)", *Ann. Sci. École Norm. Sup.* (4) **22**:4 (1989), 605–674. MR Zbl

[Offen and Sayag 2008] O. Offen and E. Sayag, "Global mixed periods and local Klyachko models for the general linear group", *Int. Math. Res. Not.* **2008**:1 (2008), art. id. rnm136. MR Zbl

[Prasad 2015] D. Prasad, "A 'relative' local Langlands correspondence", preprint, 2015. arXiv 1512.04347

[Ramakrishnan 2000] D. Ramakrishnan, "Modularity of the Rankin–Selberg $L$-series, and multiplicity one for SL(2)", *Ann. of Math.* (2) **152**:1 (2000), 45–111. MR Zbl

[Sahi 1989] S. Sahi, "On Kirillov's conjecture for Archimedean fields", *Compos. Math.* **72**:1 (1989), 67–86. MR Zbl

[Sahi 1990] S. Sahi, "A simple construction of Stein's complementary series representations", *Proc. Amer. Math. Soc.* **108**:1 (1990), 257–266. MR Zbl

[Sahi and Stein 1990] S. Sahi and E. M. Stein, "Analysis in matrix space and Speh's representations", *Invent. Math.* **101**:2 (1990), 379–393. MR Zbl

[Sakellaridis and Venkatesh 2017] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, Astérisque **396**, Soc. Math. France, Paris, 2017. MR Zbl

[Tadić 1986] M. Tadić, "Spherical unitary dual of general linear group over non-Archimedean local field", *Ann. Inst. Fourier* (*Grenoble*) **36**:2 (1986), 47–55. MR Zbl

[Tadić 1987] M. Tadić, "Unitary representations of GL($n$), derivatives in the non-Archimedean case", Berichte 281 in *Mathematikertreffen Zagreb-Graz, V* (Graz, Austria, 1986), Forschungszentrum Graz, 1987. MR Zbl

[Tadić 1992] M. Tadić, "Notes on representations of non-Archimedean SL($n$)", *Pacific J. Math.* **152**:2 (1992), 375–396. MR Zbl

[Tadić 2009] M. Tadić, "GL($n, \mathbb{C}$)^ and GL($n, \mathbb{R}$)^", pp. 285–313 in *Automorphic forms and L-functions, II: Local aspects* (Tel Aviv, 2006), edited by D. Ginzburg et al., Contemp. Math. **489**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[Vogan 1986] D. A. Vogan, Jr., "The unitary dual of GL($n$) over an Archimedean field", *Invent. Math.* **83**:3 (1986), 449–505. MR Zbl

[Wallach 1988] N. R. Wallach, *Real reductive groups, I*, Pure Appl. Math. **132**, Academic Press, Boston, 1988. MR Zbl

[Yamana 2015] S. Yamana, "Periods of residual automorphic forms", *J. Funct. Anal.* **268**:5 (2015), 1078–1104. MR Zbl

[Zelevinsky 1980] A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, II: On irreducible representations of GL($n$)", *Ann. Sci. École Norm. Sup.* (4) **13**:2 (1980), 165–210. MR Zbl

[Zhang 2014] W. Zhang, "Automorphic period and the central value of Rankin–Selberg $L$-function", *J. Amer. Math. Soc.* **27**:2 (2014), 541–612. MR Zbl

Communicated by Wee Teck Gan
Received 2020-12-02 Revised 2021-08-02 Accepted 2022-01-03

anand@math.iitb.ac.in — *Department of Mathematics, Indian Institute of Technology Bombay, Mumbai, Maharashtra, India*

nadir.matringe@imj-prg.fr — *Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris Cité - Campus des Grands Moulins, Paris, France*

*Laboratoire Mathématiques et Applications, Université de Poitiers, Poitiers, France*

# Algebra & Number Theory

msp.org/ant





See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2023 is US $485/year for the electronic version, and $705/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

# Algebra & Number Theory

Volume 17 No. 1 2023