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We use Macaulay's inverse system to study the Hilbert series for almost complete intersections generated by uniform powers of general linear forms. This allows us to give a classification of the weak Lefschetz property for these algebras, settling a conjecture by Migliore, Miró-Roig, and Nagel.

1. Introduction

Let \mathbb{k} be a field of characteristic zero and let $\ell_1, \ell_2, \dots, \ell_r \in \mathbb{k}[x_1, x_2, \dots, x_n]$ be general linear forms. For positive integers d_1, d_2, \dots, d_r , consider the quotient $\mathbb{k}[x_1, x_2, \dots, x_n]/\langle \ell_1^{d_1}, \ell_2^{d_2}, \dots, \ell_r^{d_r} \rangle$. This algebra ties together several areas of contemporary mathematics.

From the algebraic point of view, it is in a natural way linked to the long-standing conjectures by Fröberg [1985] and Iarrobino [1997] on the Hilbert series of generic forms.

Powers of general linear forms are also tightly connected to the study of fat-points schemes via Macaulay's inverse system, as was noticed by Emsalem and Iarrobino [1995]. This bridge to geometry relates the study of powers of general linear forms to the Alexander–Hirschowitz theorem [Alexander and Hirschowitz 1995; Chandler 2002] and the Segre–Gimigliano–Harbourne–Hirschowitz (SGHH) conjecture [Ciliberto 2001].

In this paper we consider the uniform almost complete intersection case, that is, algebras of the form $\mathbb{k}[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$, from the perspective of the weak Lefschetz property.

Recall that a graded algebra A satisfies the weak Lefschetz property (WLP) if there exists a linear form ℓ such that the multiplication map $\times \ell : A_i \rightarrow A_{i+1}$ has maximal rank for all degrees i , while A satisfies the strong Lefschetz property (SLP) if the multiplication map $\times \ell^j : A_i \rightarrow A_{i+j}$ has maximal rank for all i and all j . For an introduction to the Lefschetz properties, see, e.g., [Harima et al. 2013; Migliore and Nagel 2013].

The WLP for the class of algebras that we consider holds for $n = 1$ and $n = 2$, since all graded artinian quotients in one or two variables have the SLP, the argument being trivial for the univariate case, while the case of two variables, which requires characteristic zero, is attributed to Harima, Migliore, Nagel,

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and Watanabe [Harima et al. 2003]. For $n = 3$, Schenck and Seceleanu [2010] showed that the WLP holds for *any* quotient by an artinian ideal generated by powers of linear forms. Migliore, Miró-Roig, and Nagel [Migliore et al. 2012] showed that for even $n \geq 4$, the WLP fails for almost complete intersections generated by uniform powers, except in the case $(n, d) = (4, 2)$. They also gave results in the odd uniform case and in the mixed-degree case, and provided a conjecture for the unproven part of the odd uniform case.

To simplify the statement of the conjecture and the presentation in this paper in general, we let $R_{n,m,d}$ denote the ring $\mathbb{k}[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_m^d \rangle$, where $\ell_1, \ell_2, \dots, \ell_m$ are general linear forms and \mathbb{k} is a field of characteristic zero.

Conjecture 1 [Migliore et al. 2012, Conjecture 6.6]. *Let $n \geq 9$ be an odd integer. Then $R_{n,n+1,d} = \mathbb{k}[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$ fails the WLP if and only if $d > 1$. Furthermore, if $n = 7$, then $R_{n,n+1,d}$ fails the WLP when $d = 3$.*

Independently, Harbourne, Schenck, and Seceleanu [Harbourne et al. 2011] gave a more general but less precise conjecture.

Conjecture 2 [Harbourne et al. 2011, Conjecture 1.5]. *Let $r + 1 \geq n \geq 5$. Then the algebra $R_{n,r,d} = \mathbb{k}[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_r^d \rangle$ fails the WLP if $d \gg 0$.*

Since then, the main focus have been on [Conjecture 1](#). Miró-Roig [2016] has shown the failure of the WLP when $d = 2$, Nagel and Trok [2019] have shown the failure when both n and d are large enough, and also when $n \geq 9$, $d - 2 \gg 0$, and $d - 2$ is divisible by n . Ilardi and Vallès [2019] have settled the case $(n, d) = (7, 3)$, while Miró-Roig and Tran [2020] have shown the failure in the cases $9 \leq n = 2m + 1 \leq 17$ and $d \geq 4$, and in the cases $d = 2r$, $1 \leq r \leq 8$, and $9 \leq n \leq 4r(r + 2) - 1$.

We cover all the remaining cases, settling [Conjecture 1](#), and provide the following classification.

Theorem 1.1. *Let $d, n \geq 1$. Then $R_{n,n+1,d} = \mathbb{k}[x_1, x_2, \dots, x_n]/\langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$ fails the WLP except when $n \leq 3$, $d = 1$ or $(n, d) \in \{(4, 2), (5, 2), (5, 3), (7, 2)\}$, and in these cases, the WLP holds.*

In [Section 2](#), we introduce some notation. In [Section 3](#), we use the theory for inverse systems to determine the degree of the Hilbert series for $R_{n,n+2,d}$. In [Section 4](#), we give an upper bound for the degree of the Hilbert series for $R_{n,n+2,d}$ under the assumption that $R_{n+1,n+2,d}$ has the WLP. By comparing this upper bound with the actual degree, we can draw the conclusion that $R_{n+1,n+2,d}$ fails the WLP in all but a finite number of cases. The remaining cases are then dealt with separately in [Section 5](#).

2. Preliminaries

We begin by giving some background on Hilbert series, Fröberg's conjecture, and inverse systems.

The Hilbert series for a standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ is the power series $\sum \dim_{\mathbb{k}} A_i t^i$ and is denoted by $\text{HS}(A, t)$. The Hilbert function of A is the function $i \mapsto \dim_{\mathbb{k}} A_i$.

If A is an artinian graded algebra and f is a form in A of degree d such that the map $\times f : A_i \rightarrow A_{i+d}$ has maximal rank for all i , then it is an easy exercise to check that the Hilbert series for $A/(f)$ is equal to $[\text{HS}(A, t) \cdot (1 - t^d)]$. The bracket notation means that we truncate the series before the first nonpositive term.

Fröberg [1985] has conjectured that if A is the polynomial ring modulo an ideal generated by general forms, then the map induced by multiplication by a general form of degree d has maximal rank. Normally, this conjecture is expressed equivalently as follows: if f_1, \dots, f_r are general forms in $\mathbb{k}[x_1, x_2, \dots, x_n]$ of degrees d_1, d_2, \dots, d_r , then the Hilbert series for $\mathbb{k}[x_1, x_2, \dots, x_n]/\langle f_1, f_2, \dots, f_r \rangle$ equals $\left[\prod(1-t^{d_i})/(1-t)^n\right]$.

Fröberg [1985] also proves that $\left[\prod(1-t^{d_i})/(1-t)^n\right]$ is a lower bound for possible Hilbert series among forms of degrees d_1, d_2, \dots, d_r in the lexicographic sense, so for a fixed signature $(n, d_1, d_2, \dots, d_r)$, the conjecture can be verified with an example.

The conjecture is, except for a few cases, open for $r-1 > n \geq 4$. For some recent results, see [Nenashev 2017]. The case $r = n+1$ is due to Stanley [1980] and is of particular importance for this paper.

Let A be a monomial complete intersection, i.e., $A = \mathbb{k}[x_1, \dots, x_n]/\langle x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n} \rangle$. Stanley showed that the multiplication map $\times(x_1 + x_2 + \dots + x_n)^d : A_i \rightarrow A_{i+d}$ has full rank for every i and d , not only settling the $n+1$ case of the Fröberg conjecture, but also opening up the area of the Lefschetz properties for graded algebras. If we perform a linear change of coordinates, Stanley's result is equivalent to the fact that complete intersections generated by powers of general linear forms have the SLP.

When restricted to the equigenerated case $d = d_1 = \dots = d_{n+1}$, this implies that

$$\text{HS}(R_{n,n+1,d}, t) = \left[\frac{(1-t^d)^{n+1}}{(1-t)^n} \right].$$

Suppose now that $R_{n,n+1,d}$ satisfies the WLP. Then the map induced by multiplication by a general linear form ℓ has maximal rank in every degree, so the Hilbert series for $R_{n,n+1,d}/(\ell)$ equals

$$\left[(1-t) \left[\frac{(1-t^d)^{n+1}}{(1-t)^n} \right] \right] = \left[(1-t) \frac{(1-t^d)^{n+1}}{(1-t)^n} \right] = \left[\frac{(1-t^d)^{n+1}}{(1-t)^{n-1}} \right],$$

where the first equality follows from [Fröberg 1985, Lemma 4].

Since $R_{n,n+1,d}/(\ell)$ is isomorphic to $R_{n-1,n+1,d}$, this gives that $R_{n,n+1,d}$ has the WLP if and only if the Hilbert series of $R_{n-1,n+1,d}$ is the one expected by Fröberg's conjecture, that is,

$$R_{n,n+1,d} \text{ has the WLP} \quad \text{if and only if} \quad \text{HS}(R_{n-1,n+1,d}, t) = \left[\frac{(1-t^d)^{n+1}}{(1-t)^{n-1}} \right]. \tag{1}$$

For an ideal I in $\mathbb{k}[x_1, x_2, \dots, x_n]$, we consider the dual polynomial ring $\mathbb{k}[X_1, X_2, \dots, X_n]$, where x_i acts like $\partial/\partial X_i$ for $i = 1, 2, \dots, n$, and the inverse system of I , denoted by I^{-1} , is the submodule annihilated by I under this action. We use the notation $f \circ F$ for the action of the form $f \in \mathbb{k}[x_1, x_2, \dots, x_n]$ on the form $F \in \mathbb{k}[X_1, X_2, \dots, X_n]$. By duality,

$$\dim_{\mathbb{k}}[I^{-1}]_d = \dim_{\mathbb{k}}[\mathbb{k}[x_1, x_2, \dots, x_n]/I]_d, \tag{2}$$

and this will enable us to use the inverse system in order to obtain lower bounds for the Hilbert series.

Finally, the action of the d -th power of a general linear form ℓ on a form F in $\mathbb{k}[X_1, X_2, \dots, X_n]$ will be of particular importance to us, and we therefore recall the general Leibniz rule,

$$\ell^d \circ F^n = \sum_{d_1+d_2+\dots+d_n=d} \frac{d!}{d_1!d_2!\dots d_n!} (\ell^{d_1} \circ F)(\ell^{d_2} \circ F) \dots (\ell^{d_n} \circ F).$$

3. The degree of the Hilbert series for $R_{n,n+2,d}$

By definition, the degree of the Hilbert series for an artinian graded algebra A is equal to $\max\{j \mid A_j \neq 0\}$. Since A is artinian and nonzero, this number also agrees with the Castelnuovo–Mumford regularity of A ; see [Eisenbud 2005].

Let

$$s(n, d) = \begin{cases} \frac{1}{2}(n+1)(d-1) & \text{if } n \text{ is odd,} \\ \lfloor \frac{1}{2}n(n+2)(d-1)/(n+1) \rfloor & \text{if } n \text{ is even.} \end{cases}$$

We will show that $\deg(\text{HS}(R_{n,n+2,d}, t)) = s(n, d)$ for all $n, d \geq 1$.

Sturmfels and Xu [2010] have shown that $\deg(\text{HS}(R_{n,n+2,2}, t)) = s(n, 2)$, and that the dimension of $R_{n,n+2,2}$ in degree $s(n, 2)$ is equal to $2^{n/2}$ if n is even, and equal to 1 if n is odd.

Nagel and Trok [2019] proved that $\deg(\text{HS}(R_{n,n+2,d}, t)) \leq s(n, d)$. They also proved that equality holds when n is odd, in which case the dimension of $R_{n,n+2,d}$ in degree $s(n, d)$ is equal to 1, and when n is even and $n+1$ divides $d-1$ or $d \geq n^2+n+2$, in which case the dimension of $R_{n,n+2,d}$ in degree $s(n, d)$ is equal to a binomial coefficient.

By (2), we have

$$\dim_{\mathbb{k}}[(\ell_1^d, \ell_2^d, \dots, \ell_{n+2}^d)^{-1}]_s \neq 0 \implies \deg(\text{HS}(R_{n,n+2,d}, t)) \geq s,$$

so in order to show that $s(n, d)$ is a lower bound, it is enough to show that the inverse system is nonzero in degree $s(n, d)$.

Although it is sufficient to show that $s(n, d)$ is a lower bound for $\deg(\text{HS}(R_{n,n+2,d}, t))$ in the unproven part of the case n even, we show, for completeness, that $s(n, d)$ is a lower bound for all n .

We begin with an alternative proof of the case n odd. The argument is short and also gives the main idea behind the more involved proof for the even case.

Proposition 3.1. *Let $n \geq 1$ be odd, and let $d \geq 1$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is nonzero in degree $s(n, d)$.*

Proof. When $d = 1$, we have $s(n, 1) = 0$, and the value of the Hilbert function in degree 0 is equal to 1. The case $d = 2$ follows from the result by Sturmfels and Xu; in particular, there is a form F of degree $s(n, d)$ such that $\ell_i^2 \circ F = 0$ for $i = 1, \dots, n+2$. For the case $d > 2$, it follows from the pigeonhole principle in conjunction with the general Leibniz rule that $\ell_i^d \circ F^{d-1} = 0$ for $i = 1, \dots, n+2$. Finally, the degree of F^{d-1} equals

$$(d-1)s(n, 2) = (d-1)\frac{n-1}{2} = s(n, d). \quad \square$$

We now turn to the even case. Also here we are able to reduce the argument to the result by Sturmfels and Xu in degree 2.

Lemma 3.2. *Let n be even and let $1 \leq d \leq 3$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is nonzero in degree $s(n, d)$.*

Proof. The cases $d = 1$ and $d = 2$ are dealt with similarly as in [Proposition 3.1](#).

We now consider the case $d = 3$. Let F be a form such that $\ell_i^2 \circ F = 0$ for all i . Since F is in the inverse system of the ideal generated by $n + 2$ squares of general linear forms, we can choose F of degree $s(n, 2)$.

By the pigeonhole principle and the general Leibniz rule, we have $\ell_i^3 \circ F^2 = 0$. The degree of F^2 is $2s(n, 2)$. Since

$$s(n, 2) = \left\lfloor \frac{n(n+2)}{2(n+1)} \right\rfloor = \left\lfloor \frac{n(n+1)}{2(n+1)} + \frac{n}{2(n+1)} \right\rfloor = \left\lfloor \frac{n}{2} + \frac{n}{2(n+1)} \right\rfloor = \frac{n}{2}$$

and

$$s(n, 3) = \left\lfloor \frac{n(n+2)}{n+1} \right\rfloor = \left\lfloor n + \frac{n+1}{n+2} \right\rfloor = n,$$

we get that the degree of F^2 equals $s(n, 3)$, which shows that the value of the Hilbert function is nonzero in degree $s(n, 3)$. \square

Lemma 3.3. *Let n be even and let $4 \leq d \leq n + 1$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is nonzero in degree $s(n, d)$.*

Proof. Let F_i be such that $\ell_i \circ F_i = 0$ and $\ell_j^2 \circ F_i = 0$ for all j . Since F_i is in the inverse system of the ideal generated by one general linear form and $n + 1$ squares of general linear forms, we can choose F_i to be of degree $s(n - 1, 2)$.

Next, let G be such that $\ell_i \circ G = 0$ for $i \geq d$, and $\ell_i^2 \circ G = 0$ for all i . Now G is in the inverse system of the ideal generated by $n + 2 - d + 1$ general linear forms and $d - 1$ squares of general linear forms, so we can choose G of degree $s(n - (n + 2 - d + 1), 2) = s(d - 3, 2)$.

It follows by the pigeonhole principle and the general Leibniz rule that $\ell_i^d \circ GF_1 \cdots F_{d-1} = 0$ for $i = 1, \dots, n + 2$, and we are done if we can show that the degree of the form $GF_1 \cdots F_{d-1}$ is equal to $s(n, d)$, that is, that $s(d - 3, 2) + (d - 1)s(n - 1, 2) = s(n, d)$.

Suppose first that d is odd and write $d - 1 = 2c$. We get

$$s(d - 3, 2) = \left\lfloor \frac{(d-1)(d-3)}{2(d-2)} \right\rfloor = \left\lfloor \frac{c(2c-2)}{2c-1} \right\rfloor = \left\lfloor c - \frac{c}{2c-1} \right\rfloor = c - 1,$$

$$(d - 1) \cdot s(n - 1, 2) = cn,$$

$$s(n, d) = \left\lfloor \frac{(n+2)nc}{n+1} \right\rfloor = \left\lfloor nc + \frac{nc}{n+1} \right\rfloor = nc + \left\lfloor c - \frac{c}{n+1} \right\rfloor = nc + c - 1,$$

where we in the last step have used that $c < n + 1$. This proves the case d odd.

Suppose now that d is even. We get

$$\begin{aligned}
 s(d-3, 2) &= \frac{d-2}{2}, \\
 (d-1) \cdot s(n-1, 2) &= \frac{n(d-1)}{2}, \\
 s(n, d) &= \left\lfloor \frac{(n+2)n(d-1)}{2(n+1)} \right\rfloor = \left\lfloor \frac{n(d-1)}{2} + \frac{n(d-1)}{2(n+1)} \right\rfloor = \frac{n(d-1)}{2} + \left\lfloor \frac{n(d-1)}{2(n+1)} \right\rfloor \\
 &= \frac{n(d-1)}{2} + \left\lfloor \frac{d-1}{2} - \frac{d-1}{2(n+1)} \right\rfloor = \frac{n(d-1)}{2} + \left\lfloor \frac{d}{2} - \frac{1}{2} - \frac{d-1}{2(n+1)} \right\rfloor \\
 &= \frac{n(d-1)}{2} + \frac{d}{2} - 1,
 \end{aligned}$$

where we in the last step have used that $d-1 < n+1$. This finishes the proof. \square

Theorem 3.4. *The degree of the Hilbert series for $R_{n,n+2,d}$ equals $s(n, d)$.*

Proof. The case n odd was established by Nagel and Trok, so we only need to consider the case n even. Moreover, by [Nagel and Trok 2019, Theorem 4.4], the degree of the Hilbert series of $R_{n,n+2,d}$ is less than or equal to $s(n, d)$, so it is sufficient to prove that the $R_{n,n+2,d}$ is nonzero in degree $s(n, d)$.

Write $d = c + a(n+1)$, where $1 \leq c \leq n+1$. By Lemmas 3.2 and 3.3, there is a form F_c of degree $s(n, c)$ such that $\ell_i^c \circ F_c = 0$ for $i = 1, \dots, n+2$. Let $F = F_1 \cdots F_{n+2}$, where F_i is such that $\ell_i \circ F_i = 0$ and $\ell_j^2 \circ F_j = 0$ for all j . Then, by the pigeonhole principle and the general Leibniz rule, we have that $\ell^{(n+1)+1} \circ F = 0$, or more generalized, that $\ell^{a(n+1)+1} \circ F^a = 0$.

It follows that $\ell_i^{c+a(n+1)} \circ F_c F^a = 0$. Thus we are done if we can show that the degree of $F_c F^a$ is equal to $s(n, d)$, that is, that $s(n, c) + (n+2)a \cdot s(n-1, 2) = s(n, d)$, which we verify by the calculation

$$\begin{aligned}
 s(n, d) &= \left\lfloor \frac{(n+2)n(c+a(n+1)-1)}{2(n+1)} \right\rfloor = \left\lfloor \frac{(n+2)n(a(n+1))}{2(n+1)} + \frac{(n+2)n(c-1)}{2(n+1)} \right\rfloor \\
 &= \frac{(n+2)na}{2} + \left\lfloor \frac{(n+2)n(c-1)}{2(n+1)} \right\rfloor = (n+2)a \cdot s(n-1, 2) + s(n, c). \quad \square
 \end{aligned}$$

4. An upper bound for the smallest inflection point of the Hilbert function of a complete intersection

In order to use the results from the previous section to draw conclusions about the WLP, we need an upper bound for the degree of the expected Hilbert series

$$\left\lceil \frac{(1-t^d)^{n+2}}{(1-t)^n} \right\rceil \quad (3)$$

for $R_{n,n+2,d}$ given by Fröberg's conjecture. Since

$$\frac{(1-t^d)^{n+2}}{(1-t)^n} = (1-t)^2(1+t+\dots+t^{d-1})^{n+2},$$

we are interested in the lowest degree where the coefficients of the polynomial $(1-t)^2(1+t+\dots+t^{d-1})^{n+2}$

are nonpositive. We will provide the necessary bounds by induction on n . The induction step is [Lemma 4.1](#) below and the base of the induction is given in [Lemma 4.2](#).

For the statements of these lemmas, we introduce the following notation. For a sequence a_0, a_1, \dots, a_n of integers, let $\Delta(a)$ be the sequence of differences $\Delta(a) = a_0, a_1 - a_0, \dots, a_n - a_{n-1}, -a_n$. For simplicity we will assume that all sequences are zero outside the range of indices for which they are defined. The generating series of this sequence is $\sum_{i=0}^{n+1} \Delta(a)_i t^i = (1-t) \sum_{i=0}^n a_i t^i$. Instead of looking at the range of indices for which the coefficients of the polynomial in [\(3\)](#) are nonpositive, we will look at where the first difference of the coefficients of $(1+t+\dots+t^{d-1})^{n+2}$ are decreasing.

Lemma 4.1. *Let $n \geq 4$, let $d \geq 1$, and let*

$$\sum_{i=0}^{n(d-1)} a_i t^i = (1+t+\dots+t^{d-1})^n \quad \text{and} \quad \sum_{i=0}^{(n+1)(d-1)} b_i t^i = (1+t+\dots+t^{d-1})^{n+1}.$$

Suppose that for some $s \in \frac{1}{2}\mathbb{N}$ with $\frac{1}{2}n(d-1) - s \geq \frac{1}{2}(d-1)$ we have that

$$\Delta(a)_i \geq \Delta(a)_{i+1} \quad \text{for } s \leq i \leq (d-1)n - s \tag{4}$$

and

$$\Delta(a)_{s-j} \geq \Delta(a)_{s+j+1} \quad \text{for } 0 \leq j \leq s. \tag{5}$$

Then

$$\Delta(b)_i \geq \Delta(b)_{i+1} \quad \text{for } s + \frac{1}{2}(d-1) \leq i \leq (d-1)(n+1) - (s + \frac{1}{2}(d-1)) \tag{6}$$

and

$$\Delta(b)_{s+(d-1)/2-j} \geq \Delta(b)_{s+(d-1)/2+j+1} \quad \text{for } 0 \leq j \leq s + \frac{1}{2}(d-1). \tag{7}$$

In all cases, the index $j \in \frac{1}{2}\mathbb{N}$ takes only values that make the indices integers.

Proof. For simplicity, we will denote $\frac{1}{2}(d-1)$ by m , which is an integer when d is odd and a half integer for even d . Observe that the sequences $\Delta(a)_i$ and $\Delta(b)_i$ are antisymmetric around $mn + \frac{1}{2}$ and $m(n+1) + \frac{1}{2}$ respectively and that they are positive in the first half and negative in the second half. In particular, this gives that it is sufficient to prove [\(6\)](#) for $i < m(n+1)$.

We have that $(1-t)(1+t+\dots+t^{d-1})^{n+1} = (1-t^d)(1+t+\dots+t^{d-1})^n$, which shows that $\Delta(b)_i = a_i - a_{i-d}$ and

$$\Delta(b)_i - \Delta(b)_{i+1} = a_i - a_{i-d} - a_{i+1} + a_{i+1-d} = \Delta(a)_{i+1-d} - \Delta(a)_{i+1} = \Delta(a)_{i-2m} - \Delta(a)_{i+1}. \tag{8}$$

Hence we can use [\(4\)](#) to prove [\(6\)](#) when $i-2m \geq s$ and $i \leq 2mn - s$, i.e., in the range $s+2m \leq i \leq 2mn - s$.

Since, by the antisymmetry of $\Delta(b)_i$, we do not need to look at $i \geq m(n+1)$ and since we have that $2mn - s \geq m(n+1)$ by the assumption that $mn - s \geq m$, it only remains to prove [\(6\)](#) for i in the range $s+m \leq i < s+2m$. In order to do this, we use [\(5\)](#), and for $0 \leq j \leq m$, we have that

$$\Delta(a)_{s-m+j} \geq \Delta(a)_{s+m-j+1}.$$

Moreover, by [\(4\)](#), we have

$$\Delta(a)_{s+m-j+1} \geq \Delta(a)_{s+m+j+1},$$

which with $i = s + m + j$ in (8) now gives

$$\Delta(b)_{s+m+j} \geq \Delta(b)_{s+m+j+1} \quad \text{for } 0 \leq j \leq m,$$

where we only consider the j that make $j + m$ an integer. This finishes the proof of (6).

For (7) we have two cases, $j \geq m$ and $j \leq m$. In the first case we write

$$\Delta(b)_{s+m-j} \geq \Delta(b)_{s+m+j+1} \iff a_{s+m-j} - a_{s-m-j-1} \geq a_{s+m+j+1} - a_{s-m+j},$$

and the latter can be written

$$\Delta(a)_{s+m-j} + \cdots + \Delta(a)_{s-m-j} \geq \Delta(a)_{s+m+j+1} + \cdots + \Delta(a)_{s-m+j+1}.$$

This holds termwise because of (5) if $j \geq m$.

For the second case, we write

$$\Delta(b)_{s+m-j} \geq \Delta(b)_{s+m+j+1} \iff a_{s-m+j} - a_{s-m-j-1} \geq a_{s+m+j+1} - a_{s+m-j},$$

and the latter can be written as

$$\Delta(a)_{s-m-j} + \cdots + \Delta(a)_{s-m+j} \geq \Delta(a)_{s+m+j+1} + \cdots + \Delta(a)_{s+m-j+1},$$

and again this holds termwise because of (5) when $j \leq m$. This finishes the proof of (7). \square

Even though we do have the WLP for $R_{3,4,d}$, we start the induction with this as the base case. The following lemma gives us the value of s that can be used in the induction step of Lemma 4.1.

Lemma 4.2. For $d > 1$, let $a_0 + a_1 t + \cdots + a_{4(d-1)} t^{4(d-1)} = (1 + t + \cdots + t^{d-1})^4$. Then for $s = \lfloor \frac{4}{3}(d-1) \rfloor$,

$$\Delta(a)_j \geq \Delta(a)_{j+1} \quad \text{for } s \leq j \leq 4(d-1) - s \quad \text{and} \quad \Delta(a)_{s-j} \geq \Delta(a)_{s+j+1} \quad \text{for } 0 \leq j \leq s.$$

Proof. The coefficients of the polynomial $(1 - t^d)^3 / (1 - t)^3 = (1 + t + \cdots + t^{d-1})^3$ are unimodal and symmetric around degree $\frac{3}{2}(d-1)$. Hence the coefficients of the polynomial

$$(1 - t^d)^4 / (1 - t)^3 = \Delta(a)_0 + \Delta(a)_1 t + \cdots + \Delta(a)_{4d-3}$$

are antisymmetric around $\frac{1}{2}(4(d-2) + 1) = 2d - 2$, having positive coefficients up to degree $2d - 2$ and thereafter negative coefficients satisfying $\Delta(a)_{2d-2-i} = -\Delta(a)_{2d-1+i}$ for $0 \leq i \leq 2d - 2$.

We can write down an explicit formula for the positive coefficients as

$$\Delta(a)_j = \begin{cases} \binom{j+2}{2} & \text{for } 0 \leq j \leq d-1, \\ \binom{j+2}{2} - 4\binom{j+2-d}{2} & \text{for } d \leq j \leq 2d-2. \end{cases}$$

The second expression can be written as a quadratic polynomial in j as

$$g(j) = \binom{j+2}{2} - 4\binom{j+2-d}{2} = -\frac{3}{2} \left(j^2 - \left(\frac{8d}{3} - 3 \right) j + \frac{4d^2}{3} - 4d + 2 \right),$$

which is symmetric around $j = \frac{4}{3}d - \frac{3}{2}$. Thus we have that $\Delta(a)_j \geq \Delta(a)_{j+1}$ when

$$\frac{1}{2}(j + (j+1)) \geq \frac{4}{3}d - \frac{3}{2} \iff j \geq \frac{4}{3}d - 2 \iff j \geq s$$

for the positive coefficients. For the negative coefficients, we use the antisymmetry to conclude that $\Delta(a)_j \geq \Delta(a)_{j+1}$ when $j \leq s + 2 \cdot (2d - 2 - s) = 4(d - 1) + s$.

Moreover, for $0 \leq j \leq s$, we have $\Delta(a)_{s-j} \geq g(s - j)$ and $\Delta(a)_{s+j+1} = g(s + j + 1)$ when $\Delta(a)_{s+j+1}$ is positive. Thus it is enough to verify that $g(s - j) \geq g(s + j + 1)$. Indeed,

$$\frac{(s - j) + (s + j + 1)}{2} \geq \frac{4d}{3} - \frac{3}{2} \iff s \geq \frac{4d}{3} - 2 = \left\lfloor \frac{4(d - 1)}{3} \right\rfloor. \quad \square$$

We now use Lemmas 4.1 and 4.2 to give an upper bound on the degree of the expected Hilbert series for $n + 2$ general forms of degree d .

Proposition 4.3. *An upper bound for the smallest inflection point of the Hilbert function of a complete intersection generated by $n + 2$ forms of degree d in $n + 2 \geq 4$ variables is given by $\lfloor \frac{4}{3}(d - 1) \rfloor + \frac{1}{2}(n - 2)(d - 1)$; that is, for $n \geq 2$, we have*

$$\deg\left(\left[\frac{(1 - t^d)^{n+2}}{(1 - t)^n}\right]\right) \leq \left\lfloor \frac{4(d - 1)}{3} \right\rfloor + \frac{(n - 2)(d - 1)}{2}.$$

Proof. Induction on n with Lemma 4.2 as the base case and with Lemma 4.1 as the induction step proves the statement about the inflection point. As seen before, this inflection point corresponds to the degree of the expected Hilbert series, since

$$\left[\frac{(1 - t^d)^{n+2}}{(1 - t)^n}\right] = \left[(1 - t)^2 \frac{(1 - t^d)^{n+2}}{(1 - t)^{n+2}}\right]. \quad \square$$

The upper bound in Proposition 4.3 is far from being sharp, and in the proof of Theorem 4.5 we will need a better bound in some cases. For this purpose we will in Lemma 4.4 below give a more general version of Proposition 4.3 that could be used with a different base case than Lemma 4.2. Lemma 4.4 also gives the connection to the WLP that will be used in Theorem 4.5.

Lemma 4.4. *Let $n \geq 4$ and $d \geq 1$, and suppose that \tilde{s} is an integer such that the assumptions in Lemma 4.1 are satisfied. If $\tilde{s} < s(n - 2, d)$, then*

$$\deg\left(\left[\frac{(1 - t^d)^{n+k}}{(1 - t)^{n-2+k}}\right]\right) < s(n - 2 + k, d) \quad \text{for all even integers } k \geq 0,$$

and in particular, $R_{n-1+k, n+k, d}$ fails the WLP for all even integers $k \geq 0$.

Proof. If n is even, a computation reveals that $s(n + k, d) - s(n - 2 + k, d) \geq d - 1$, and if n is odd, then $s(n + k, d) - s(n - 2 + k, d) = d - 1$. From this we conclude that $s(n - 2, d) + \frac{1}{2}(k + 2)(d - 1) \leq s(n + k, d)$.

On the other hand, repeated use of Lemma 4.1 gives $\deg([(1 - t^d)^{n+k}/(1 - t)^{n-2+k}]) \leq \tilde{s} + k \cdot \frac{1}{2}(d - 1)$. Thus we have

$$\deg\left(\left[\frac{(1 - t^d)^{n+k}}{(1 - t)^{n-2+k}}\right]\right) \leq \tilde{s} + k \cdot \frac{d - 1}{2} < s(n - 2, d) + ((k - 2) + 2) \cdot \frac{d - 1}{2} \leq s(n + k - 2, d).$$

The second part of the proposition follows since by (1), the WLP for $R_{m+1, m+2, d}$ fails if the Hilbert series for $R_{m, m+2, d}$ does not equal $[(1 - t^d)^{m+2}/(1 - t)^m]$. \square

Theorem 4.5. *Let $n \geq 4$ and $d \geq 2$. Then $R_{n,n+1,d}$ fails the WLP except possibly for*

$$(n, d) \in \{(4, 2), (5, 2), (5, 3), (5, 5), (7, 2), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}.$$

Proof. According to [Proposition 4.3](#), we have

$$\deg\left(\left[\frac{(1-t^d)^{n+2}}{(1-t)^n}\right]\right) \leq \left\lfloor \frac{4(d-1)}{3} \right\rfloor + (n-2)\frac{d-1}{2},$$

and by [Lemma 4.4](#), the WLP fails for $R_{n+1,n+2,d}$ if

$$s(n, d) > \left\lfloor \frac{4(d-1)}{3} \right\rfloor + (n-2)\frac{d-1}{2}.$$

Thus for even n the WLP fails for $R_{n+1,n+2,d}$ if

$$\left\lfloor \frac{n(n+2)(d-1)}{2(n+1)} \right\rfloor - \left\lfloor \frac{4(d-1)}{3} \right\rfloor \geq (n-2)\frac{d-1}{2} + 1.$$

In order to show this, it is sufficient to show that

$$\frac{n(n+2)(d-1)}{2(n+1)} - \frac{4(d-1)}{3} \geq (n-2)\frac{d-1}{2} + 2,$$

which can be written as

$$d \geq 2 + 12 \cdot \frac{n+1}{n-2},$$

which for $n = 4$ gives $d \geq 32$.

In the same way, for odd $n > 2$, we want to show that

$$\frac{(n+1)(d-1)}{2} - \left\lfloor \frac{4(d-1)}{3} \right\rfloor \geq (n-2)\frac{d-1}{2} + 1,$$

and here it is sufficient to show that

$$\frac{(n+1)(d-1)}{2} - \frac{4(d-1)}{3} \geq (n-2)\frac{d-1}{2} + 1,$$

which is equivalent to $d \geq 7$.

Thus to this point we have by [Lemma 4.4](#) that $R_{n+1,n+2,d}$ fails for even $n \geq 4$ and $d \geq 32$, and for odd $n \geq 3$ and $d \geq 7$.

For the remaining cases we introduce the notation

$$\tilde{s}(n, d) = \min\{s : s \text{ satisfies the hypotheses in } \text{Lemma 4.1}\}.$$

For $n = 3$, we check with [\[Macaulay2\]](#) that, in the range $2 \leq d < 7$ except for $d = 2$, we have $s(3, d) > \tilde{s}(3, d)$, so by [Lemma 4.4](#), the WLP for $R_{n+1,n+2,d}$ fails for all odd $n \geq 3$ and $d > 2$. Since $s(5, 2) = 3 > \tilde{s}(5, 2) = 2$, we also get that the WLP for $R_{n+1,n+2,d}$ fails for all odd $n \geq 5$ and $d = 2$.

For $n = 4$, we check that $s(4, d) > \tilde{s}(4, d)$ in the range $2 \leq d < 32$ except for $d \in \{2, 3, 5\}$. Since $s(6, 5) = 13 > \tilde{s}(6, 5) = 12$, we have that the WLP for $R_{n+1,n+2,d}$ fails for all even $n \geq 6$ and $d > 3$ according

to [Lemma 4.4](#). For $d \in \{2, 3\}$ and even n , we need to go to $n = 12$ to get $s(12, 2) = 6 > \tilde{s}(12, 2) = 5$ and $s(12, 3) = 12 > \tilde{s}(12, 3) = 11$.

Thus we have shown that the WLP for $R_{n,n+1,d}$ fails for all $n \geq 4$ and $d \geq 2$ except possibly for

$$(n, d) \in \{(4, 2), (5, 2), (5, 3), (5, 5), (7, 2), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}. \quad \square$$

5. Explicit formulas, the remaining cases, and the proof of [Theorem 1.1](#)

As we saw from the previous section there are ten cases to consider in order to finish the proof of our main theorem. Four of them do satisfy the WLP, and now we have to deal with the remaining cases, which are

$$(n, d) \in \{(5, 5), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}.$$

The case $(5, 5)$ was handled by Migliore, Miró-Roig and Nagel in [\[Migliore et al. 2012\]](#), the case $(7, 3)$ by Ilardi and Vallès [\[2019\]](#) and the cases $(9, 2)$ and $(11, 2)$ by Sturmfels and Xu [\[2010\]](#). Thus there are two remaining cases: $(n, d) = (9, 3)$ and $(n, d) = (11, 3)$. We will now deal with these two cases, but we will also give new arguments for the other four since the method we use is the same.

We will for each case provide a set of elements of generators for the inverse system in the top degree that shows that the Hilbert function of $R_{n-1,n+1,d}$ is not the one expected from the Fröberg conjecture.

We start by establishing an explicit formula for the form of degree $\frac{1}{2}(n+1)$ in $\mathbb{k}[X_1, X_2, \dots, X_n]$ that is annihilated by the squares of $n+2$ general linear forms when n is odd. We will do this in two different ways with different sets of parameters. The two versions are useful in different situations. In the first version, we observe that for $n+2$ general forms we can, by a change of variables, assume that n are the variables and one is the sum of the variables. The last form will have general coefficients.

Observe that the references to the result by Sturmfels and Xu [\[2010\]](#) in [Section 3](#) can be replaced by the use of [Theorem 5.1](#) to get a completely self-contained proof of our main result.

We will use $V(y_1, y_2, \dots, y_m)$ to denote the Vandermonde determinant in variables y_1, y_2, \dots, y_m .

Theorem 5.1. *Let $n = 2k - 1$ for a positive integer k . The form*

$$F = \det \begin{bmatrix} X_1 & a_1 X_1 & a_1^2 X_1 & \cdots & a_1^{k-1} X_1 & a_1 & a_1^2 & \cdots & a_1^{k-1} \\ X_2 & a_2 X_2 & a_2^2 X_2 & \cdots & a_2^{k-1} X_2 & a_2 & a_2^2 & \cdots & a_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_n & a_n X_n & a_n^2 X_n & \cdots & a_n^{k-1} X_n & a_n & a_n^2 & \cdots & a_n^{k-1} \end{bmatrix}$$

$$= \frac{1}{k!(k-1)!} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) V(a_{\sigma_1}, a_{\sigma_2}, \dots, a_{\sigma_k}) V(a_{\sigma_{k+1}}, a_{\sigma_{k+2}}, \dots, a_{\sigma_n}) \prod_{j=k+1}^n a_{\sigma_j} \prod_{j=1}^k X_{\sigma_j}$$

is the unique form of degree k in $\mathbb{k}[X_1, X_2, \dots, X_n]$ that is annihilated by the squares of the linear forms $x_1, x_2, \dots, x_n, x_1 + x_2 + \dots + x_n$, and $a_1 x_1 + a_2 x_2 + \dots + a_n x_n$.

Proof. By work of Nagel and Trok [\[2019\]](#), there is a unique such form and it is sufficient for us to prove that this particular form is annihilated by the squares of the linear forms. The equality between the two

formulas follows from the generalized Laplace expansion over the first k columns. Since the form is square-free, it is annihilated by the squares of the variables and it remains for us to check that it is annihilated by the squares of the last two linear forms, $\ell_{n+1} = x_1 + x_2 + \cdots + x_n$ and $\ell_{n+2} = a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

We can start by computing $\ell_{n+1}^2 \circ F = (x_1 + x_2 + \cdots + x_n)^2 \circ F$, where we by the Leibniz rule for determinants get a sum over terms where we substitute $X_i = 1$ for $i = 1, 2, \dots, n$ in two of the first k columns. In all these terms, there will be a repeated column so all terms are zero.

In the same way we get that $\ell_{n+2}^2 \circ F = (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 \circ F = 0$. This time we substitute $X_i = a_i$ for $i = 1, 2, \dots, n$ in two of the first k columns, which again results in repeated columns. \square

For the second version of this formula, we observe that $n + 2$ general points in \mathbb{P}^{n-1} are on a rational normal curve and we can, by a change of coordinates, assume that this curve is the moment curve with parametrization $(1 : t : t^2 : \cdots : t^{n-1})$. Thus we can assume that the $n + 2$ linear forms are given by $\ell_i = \sum_{j=1}^n \alpha_i^{j-1} x_j$ for $i = 1, 2, \dots, n + 2$, where $\alpha_1, \alpha_2, \dots, \alpha_{n+2}$ are general elements of the field \mathbb{k} .

Theorem 5.2. *For $n = 2k - 1$, the form of degree $k = \frac{1}{2}(n + 1)$ that is annihilated by the squares of the linear forms $\ell_i = \sum_{j=1}^n \alpha_i^{j-1} x_j$ for $i = 1, 2, \dots, n + 2$ is given by*

$$F = \det \begin{bmatrix} x_1 & x_2 & \cdots & x_k \\ x_2 & x_3 & \cdots & x_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k+1} & \cdots & x_n \end{bmatrix}.$$

Proof. Again we use that Nagel and Trok [2019] have shown that there is a unique such form and it is sufficient for us to prove that this determinant is annihilated by the squares of the linear forms.

We observe that $\ell_i^2 \circ F$ is given by a sum with signs over all ways of substituting x_j with α_i^{j-1} in two of the rows of the matrix. These two rows become linearly dependent and thus $\ell_i^2 \circ F = 0$ for $i = 1, 2, \dots, n + 2$. \square

The advantage of this second version is that the formula does not depend on the parameters. Moreover, we can also use it to find formulas for the unique forms that are annihilated by some of the linear forms and the squares of the remaining linear forms.

Theorem 5.3. *For $0 < k \leq \frac{1}{2}(n + 1)$, the unique form of degree k that is annihilated by the linear forms $\ell_i = \sum_{j=1}^n \alpha_i^{j-1} x_j$ for $i = 1, 2, \dots, n - 2k + 1$ and by the squares of the remaining $2k + 1$ linear forms is given by*

$$F = \det \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-k} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{n-2k+1} & \alpha_{n-2k+1}^2 & \cdots & \alpha_{n-2k+1}^{n-k} \\ x_1 & x_2 & x_3 & \cdots & x_{n+1-k} \\ x_2 & x_3 & x_4 & \cdots & x_{n+2-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_{k+1} & x_{k+2} & \cdots & x_n \end{bmatrix}.$$

Proof. The uniqueness is given by Nagel and Trok [2019] and it is enough for us to show the vanishing.

We have that $\ell^2 \circ F = 0$ for any linear form $\ell = \sum_{j=1}^n \alpha^j x_j$ for the same reason as in the previous theorem. Applying ℓ_i , where $i = 1, 2, \dots, n - 2k + 1$, to F gives a sum over the k determinants we get by replacing x_j with α_i^{j-1} in each of the k lowest rows. Hence $\ell_i \circ F = 0$ for $i = 1, 2, \dots, n - 2k + 1$. \square

We can now treat the sporadic cases not covered by Theorem 4.5 where the WLP fails.

Theorem 5.4. *The WLP fails for $R_{n,n+1,d} = \mathbb{k}[x_1, x_2, \dots, x_n] / \langle \ell_1^d, \ell_2^d, \dots, \ell_{n+1}^d \rangle$ in the cases $(n, d) \in \{(5, 5), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}$. In particular, the Hilbert series of $R_{4,6,5}$, $R_{6,8,3}$, $R_{8,10,2}$, $R_{8,10,3}$, $R_{10,12,2}$, and $R_{10,12,3}$ are:*

ring	Hilbert series
$R_{4,6,5}$	$1 + 4t + 10t^2 + 20t^3 + 35t^4 + 50t^5 + 60t^6 + 60t^7 + 45t^8 + 14t^9$
$R_{8,10,2}$	$1 + 6t + 21t^2 + 48t^3 + 78t^4 + 84t^5 + 43t^6$
$R_{8,10,2}$	$1 + 8t + 26t^2 + 40t^3 + 16t^4$
$R_{8,10,3}$	$1 + 8t + 36t^2 + 110t^3 + 250t^4 + 432t^5 + 561t^6 + 492t^7 + 171t^8$
$R_{10,12,2}$	$1 + 10t + 43t^2 + 100t^3 + 121t^4 + 32t^5$
$R_{10,12,3}$	$1 + 10t + 55t^2 + 208t^3 + 595t^4 + 1342t^5 + 2431t^6 + 3520t^7 + 3916t^8 + 2860t^9 + 683t^{10}$

These differ in the leading term from, respectively, the $10t^9$, $42t^6$, $15t^4$, $135t^8$, $22t^5$ and $88t^{10}$ that are expected by the Fröberg conjecture.

Proof. We consider the ring $R_{n,n+2,d}$ and denote our set of $n + 2$ general linear forms by

$$\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_{n+2}\} = \left\{ \sum_{i=1}^n a_1^{i-1} x_i, \sum_{i=1}^n a_2^{i-1} x_i, \dots, \sum_{i=1}^n a_{n+2}^{i-1} x_i \right\},$$

where a_1, a_2, \dots, a_{n+2} are general elements of \mathbb{k} .

Using Theorem 5.3, we can find a formula for the unique form of degree k that is annihilated by $n - 2k + 1$ of the $n + 2$ linear forms in \mathcal{L} and by the squares of the remaining $2k + 1$ linear forms in \mathcal{L} . For $\mathcal{S} \subseteq \mathcal{L}$ of size $n - 2k + 1$, we denote this unique form by $F_{\mathcal{S}}$. Observe that the coefficients of $F_{\mathcal{S}}$ are polynomials in the parameters a_1, a_2, \dots, a_{n+2} . In each of the six cases of the theorem, we produce a set of forms in the inverse system of $R_{n,n+2,d}$ in the top degree such that the dimension of the subspace they span agrees with the stated value of the Hilbert function in the socle degree. It will be enough to verify this for a specialization of the parameters, since a specialization can only lower the dimension. Thus this gives a lower bound for the Hilbert function in the socle degree. On the other hand, the computation of the Hilbert function of $R_{n,n+2,d}$ for a specialization provides an upper bound. Since they agree we can make the desired conclusion.

In all the cases, we use the specialization $a_i = i - 1$ for $i = 1, 2, \dots, n + 2$ to verify the dimension using Macaulay2.

For $R_{4,6,5}$ we need 14 linearly independent forms of degree 9. We use forms that can be written as $F = F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_3} F_{\mathcal{S}_4} F_{\mathcal{S}_5} F_{\mathcal{S}_6}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = 3$ and $|\mathcal{S}_4| = |\mathcal{S}_5| = |\mathcal{S}_6| = 1$, such that each linear

form in \mathcal{L} is contained in exactly two of the six subsets. Observe that the three first factors are linear and the three last are quadratic. For each $\ell \in \mathcal{L}$ we get $\ell^5 \circ F = 0$ by the pigeonhole principle and the general Leibniz rule since ℓ annihilates two of the factors and ℓ^2 annihilates the remaining four factors.

For $R_{6,8,3}$ we need 43 linearly independent forms of degree 6. We use forms that can be written as $F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_3} F_{\mathcal{S}_4}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = 5$ and $|\mathcal{S}_3| = |\mathcal{S}_4| = 3$, and each linear form in \mathcal{L} is contained in exactly two of the subsets. These forms are annihilated by the squares of all linear forms in \mathcal{L} since each linear form annihilates two of the factors and the square of the linear form annihilates the remaining two factors.

For $R_{8,10,2}$ we need 16 linearly independent forms of degree 4. These can be obtained as $F = F_{\mathcal{S}} F_{\mathcal{L} \setminus \mathcal{S}}$ for subsets \mathcal{S} of size five. These are products of two quadrics and they are annihilated by the squares of the linear forms in \mathcal{L} since for each ℓ in \mathcal{L} we have that ℓ annihilates one of the factors and ℓ^2 annihilates the other.

For $R_{8,10,3}$ we will produce a set of 171 linearly independent forms of degree 8 that are annihilated by the cubes of the linear forms. These forms are obtained as $F = F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_2} F_{\mathcal{S}_2}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3| = |\mathcal{S}_4| = 5$ and each linear form in \mathcal{L} is contained in two of the subsets. Thus we get that $\ell^3 \circ (F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_2} F_{\mathcal{S}_2}) = 0$ for all $\ell \in \mathcal{L}$ since ℓ annihilates two of the factors and ℓ^2 annihilates the remaining two.

For $R_{10,12,2}$ we provide a set of 32 linearly independent forms of degree 5 that are annihilated by the squares of the linear forms in \mathcal{L} . We do this by forms $F = F_{\mathcal{S}} F_{\mathcal{L} \setminus \mathcal{S}}$ for subsets \mathcal{S} of size five. Each linear form ℓ in \mathcal{L} annihilates one of the factors and ℓ^2 the other. Hence $\ell^2 \circ F_{\mathcal{S}} F_{\mathcal{L} \setminus \mathcal{S}} = 0$.

For $R_{10,12,3}$ we provide a set of 683 linearly independent forms of degree 10 that are annihilated by the cubes of the linear forms in \mathcal{L} . We do this by forms $F = F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_3} F_{\mathcal{S}_4}$, where $|\mathcal{S}_1| = |\mathcal{S}_2| = 5$, $|\mathcal{S}_3| = |\mathcal{S}_4| = 7$, and every $\ell \in \mathcal{L}$ is contained in two of the subsets. Now $\ell^3 \circ (F_{\mathcal{S}_1} F_{\mathcal{S}_2} F_{\mathcal{S}_3} F_{\mathcal{S}_4}) = 0$ for all $\ell \in \mathcal{L}$ since ℓ annihilates two of the factors and ℓ^2 annihilates the remaining two. \square

We can now prove [Theorem 1.1](#).

Proof of Theorem 1.1. By [Theorem 4.5](#) we have that the WLP for $R_{n,n+1,d}$ fails when $n \geq 4$ and $d \geq 2$ except possibly for the cases

$$(n, d) \in \{(4, 2), (5, 2), (5, 3), (5, 5), (7, 2), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}.$$

In the cases (4, 2), (5, 2), (5, 3) and (7, 2), we can verify by one example that they do satisfy the WLP and for the remaining cases [Theorem 5.4](#) shows that they fail to satisfy the WLP.

Finally, we refer to the introduction for references to the cases $n = 2$ and $n = 3$, and for the cases $n = 1$ and $d = 1$, the WLP is trivially satisfied. \square

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References

- [Alexander and Hirschowitz 1995] J. Alexander and A. Hirschowitz, “Polynomial interpolation in several variables”, *J. Algebraic Geom.* **4**:2 (1995), 201–222. [MR](#) [Zbl](#)
- [Chandler 2002] K. A. Chandler, “Linear systems of cubics singular at general points of projective space”, *Compositio Math.* **134**:3 (2002), 269–282. [MR](#) [Zbl](#)
- [Ciliberto 2001] C. Ciliberto, “Geometric aspects of polynomial interpolation in more variables and of Waring’s problem”, pp. 289–316 in *European Congress of Mathematics, I* (Barcelona, 2000), edited by C. Casacuberta et al., Progr. Math. **201**, Birkhäuser, Basel, 2001. [MR](#) [Zbl](#)
- [Eisenbud 2005] D. Eisenbud, *The geometry of syzygies: a second course in commutative algebra and algebraic geometry*, Graduate Texts in Mathematics **229**, Springer, 2005. [MR](#) [Zbl](#)
- [Emsalem and Iarrobino 1995] J. Emsalem and A. Iarrobino, “Inverse system of a symbolic power, I”, *J. Algebra* **174**:3 (1995), 1080–1090. [MR](#) [Zbl](#)
- [Fröberg 1985] R. Fröberg, “An inequality for Hilbert series of graded algebras”, *Math. Scand.* **56**:2 (1985), 117–144. [MR](#) [Zbl](#)
- [Harbourne et al. 2011] B. Harbourne, H. Schenck, and A. Seceleanu, “Inverse systems, Gelfand–Tsetlin patterns and the weak Lefschetz property”, *J. Lond. Math. Soc.* (2) **84**:3 (2011), 712–730. [MR](#) [Zbl](#)
- [Harima et al. 2003] T. Harima, J. C. Migliore, U. Nagel, and J. Watanabe, “The weak and strong Lefschetz properties for Artinian K -algebras”, *J. Algebra* **262**:1 (2003), 99–126. [MR](#) [Zbl](#)
- [Harima et al. 2013] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics **2080**, Springer, 2013. [MR](#) [Zbl](#)
- [Iarrobino 1997] A. Iarrobino, “Inverse system of a symbolic power, III: Thin algebras and fat points”, *Compositio Math.* **108**:3 (1997), 319–356. [MR](#) [Zbl](#)
- [Iardi and Vallès 2019] G. Iardi and J. Vallès, “Eight cubes of linear forms in \mathbb{P}^6 ”, preprint, 2019. [arXiv 1910.04035](#)
- [Macaulay2] D. R. Grayson and M. E. Stillman, “Macaulay2, a software system for research in algebraic geometry”, available at <https://faculty.math.illinois.edu/Macaulay2/>.
- [Migliore and Nagel 2013] J. Migliore and U. Nagel, “Survey article: a tour of the weak and strong Lefschetz properties”, *J. Commut. Algebra* **5**:3 (2013), 329–358. [MR](#) [Zbl](#)
- [Migliore et al. 2012] J. C. Migliore, R. M. Miró-Roig, and U. Nagel, “On the weak Lefschetz property for powers of linear forms”, *Algebra Number Theory* **6**:3 (2012), 487–526. [MR](#) [Zbl](#)
- [Miró-Roig 2016] R. M. Miró-Roig, “Harbourne, Schenck and Seceleanu’s conjecture”, *J. Algebra* **462** (2016), 54–66. [MR](#) [Zbl](#)
- [Miró-Roig and Tran 2020] R. M. Miró-Roig and Q. H. Tran, “On the weak Lefschetz property for almost complete intersections generated by uniform powers of general linear forms”, *J. Algebra* **551** (2020), 209–231. [MR](#) [Zbl](#)
- [Nagel and Trok 2019] U. Nagel and B. Trok, “Interpolation and the weak Lefschetz property”, *Trans. Amer. Math. Soc.* **372**:12 (2019), 8849–8870. [MR](#) [Zbl](#)
- [Nenashev 2017] G. Nenashev, “A note on Fröberg’s conjecture for forms of equal degrees”, *C. R. Math. Acad. Sci. Paris* **355**:3 (2017), 272–276. [MR](#) [Zbl](#)
- [Schenck and Seceleanu 2010] H. Schenck and A. Seceleanu, “The weak Lefschetz property and powers of linear forms in $\mathbb{K}[x, y, z]$ ”, *Proc. Amer. Math. Soc.* **138**:7 (2010), 2335–2339. [MR](#) [Zbl](#)
- [Stanley 1980] R. P. Stanley, “Weyl groups, the hard Lefschetz theorem, and the Sperner property”, *SIAM J. Algebraic Discrete Methods* **1**:2 (1980), 168–184. [MR](#) [Zbl](#)
- [Sturmfels and Xu 2010] B. Sturmfels and Z. Xu, “Sagbi bases of Cox–Nagata rings”, *J. Eur. Math. Soc. (JEMS)* **12**:2 (2010), 429–459. [MR](#) [Zbl](#)

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