A classification of modular compactifications of
the space of pointed elliptic curves by Gorenstein curves

Sebastian Bozlee, Bob Kuo and Adrian Neff

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We classify the Deligne–Mumford stacks $\mathcal{M}$ compactifying the moduli space $\mathcal{M}_{1,n}$ of smooth n-pointed curves of genus one under the condition that the points of $\mathcal{M}$ represent Gorenstein curves with distinct smooth markings. This classification uncovers new moduli spaces $\overline{\mathcal{M}}_{1,n}(Q)$, which we may think of as coming from an enrichment of the notion of level used to define Smyth's m-stable spaces. Finally, we construct a cube complex of Artin stacks interpolating between the $\overline{\mathcal{M}}_{1,n}(Q)$'s, a multidimensional analogue of the wall-and-chamber structure seen in the log minimal model program for $\overline{\mathcal{M}}_g$.

1. Introduction

The moduli stack $\mathcal{M}_{g,n}$ of smooth genus g algebraic curves with n marked points is not proper, so one searches for compactifications, that is, proper Deligne–Mumford stacks $\mathcal{M}$ such that $\mathcal{M}_{g,n}$ embeds as a dense open substack of $\mathcal{M}$. In this paper we construct a new family of modular compactifications of $\mathcal{M}_{1,n}$. We then show that these moduli spaces exhaust the semistable modular compactifications of $\mathcal{M}_{1,n}$ with Gorenstein singularities and distinct markings.

Let us now set up some notation in order to give the definition of these new moduli spaces.

Definition 1.1. Given a positive integer n, let $\mathrm{Part}(n)$ be the set of partitions of $\{1, \ldots, n\}$. Give $\mathrm{Part}(n)$ a partial order by $P_1 \preceq P_2$ if the partition P_1 is refined by the partition P_2.

Denote by $\mathfrak{Q}_n$ the collection of subsets $Q \subseteq \mathrm{Part}(n)$ such that

(i) Q is downward closed;

(ii) Q does not contain the discrete partition $\{\{1\}, \ldots, \{n\}\}$.

Definition 1.2. Let p be a closed point of an algebraic curve C over an algebraically closed field k, and let $\nu : \widetilde{C} \to C$ be the normalization. The *number of branches at* p is

$$m(p) = |\nu^{-1}(p)|.$$

The *delta invariant of* p is

$$\delta(p) = \dim_k(\nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C).$$

The *genus of* p is

$$g(p) = \delta(p) - m(p) - 1.$$

The genus of a singularity captures its contribution to the genus of C. In particular, if C is connected and proper, $p_1, \ldots, p_e$ are the singularities of C, and $\widetilde{C}_1, \ldots, \widetilde{C}_v$ are the irreducible components of $\widetilde{C}$, then the arithmetic genus of C is

$$g(C) = \sum_{i=1}^{e} g(p_i) + \sum_{j=1}^{v} g(\widetilde{C}_j) + b_1(\Delta_C),$$

where $b_1(\Delta_C)$ is the first Betti number of the simplicial complex Δ_C with vertices $\widetilde{C}_1, \ldots, \widetilde{C}_v$ and, for each p_i, an $(m(p_i)-1)$-simplex whose vertices are glued to the components meeting $\nu^{-1}(p_i)$.

Definition 1.3. A closed point p of an algebraic curve C over an algebraically closed field is an *elliptic Gorenstein singularity* if $\mathscr{O}_{C,p}$ is Gorenstein and $g(p) = 1$.

It is shown in [Smyth 2011a] that the elliptic Gorenstein singularities are classified by their number of branches, m. If $m = 1$, p is a cusp; for $m = 2$, p is a tacnode; for $m \geq 3$, p is the union of the coordinate axes of $\mathbb{A}^{m-1}$ with one more line transverse to each of the coordinate hyperplanes of $\mathbb{A}^{m-1}$. Given such a singularity, we will call the irreducible components to which p belongs the *branches* of p.

Definition 1.4. A *subcurve* Z of a proper algebraic curve C over an algebraically closed field is a connected reduced closed subscheme of C.

Let $(C, p_1, \ldots, p_n)$ be a curve of arithmetic genus one together with n marked closed points over an algebraically closed field. Let Z be a subcurve of C of genus one and let Σ be the divisor of markings. We define the *level of Z*, lev(Z), to be the partition of $\{1, \ldots, n\}$ where $a, b \in \{1, \ldots, n\}$ lie in the same subset if and only if the markings p_a and p_b lie in the same connected component of $(C - Z) \cup \Sigma$.

If $q \in C$ is an elliptic Gorenstein singularity, we say the *level of q*, lev(q) is the partition of $\{1, \ldots, n\}$ where $a, b \in \{1, \ldots, n\}$ lie in the same subset if and only if the markings p_a and p_b lie in the same connected component of the normalization of C at q (i.e., if the rational tails containing p_a and p_b are connected via a nodal path to the same branch of the singularity).

Remark 1.5. If Z_1 and Z_2 are two genus one subcurves of C and $Z_1 \subseteq Z_2$, then lev$(Z_1) \preceq$ lev(Z_2).

Remark 1.6. The level of C considered in [Smyth 2011a] is the cardinality $|\text{lev}(Z)|$, where Z is the minimal subcurve of C of genus one.

The level of an elliptic Gorenstein singularity q defined here was called the "combinatorial type of C" in [Smyth 2011b, Definition 2.15].

Definition 1.7. Let $Q \in \mathfrak{Q}_n$. A Q-stable curve over a scheme S consists of

  (i) $\pi : C \to S$, a flat and proper morphism of schemes, and

 (ii) $\sigma_1, \ldots, \sigma_n : S \to C$, sections of π with disjoint images,

such that, for each geometric fiber $(C_s, p_1, \ldots, p_n)$,

  (i) C_s is a connected reduced Gorenstein scheme of dimension 1 with arithmetic genus one;

 (ii) (level condition on subcurves) if $Z \subseteq C_s$ is a subcurve of genus one, then lev$(Z) \notin Q$;

(iii) (level condition on singularities) if $q \in Z$ is a genus one singularity, then $\mathrm{lev}(q) \in Q$;

(iv) $H^0(C, \Omega_C^\vee(-\Sigma)) = 0$.

We define the *moduli space* $\overline{\mathcal{M}}_{1,n}(Q)$ *of Q-stable n-marked curves of genus one* to be the stack over $\mathbb{Z}\left[\frac{1}{6}\right]$ whose S-points are the Q-stable curves over S.
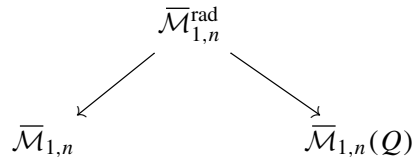
Our first main result is that this defines a modular compactification of $\mathcal{M}_{1,n}$.

Theorem 5.1. *For each $Q \in \mathfrak{Q}_n$, $\overline{\mathcal{M}}_{1,n}(Q)$ is a proper irreducible Deligne–Mumford stack over $\mathbb{Z}\left[\frac{1}{6}\right]$ containing $\mathcal{M}_{1,n}$.*

When $Q = \{S \in \mathrm{Part}(n) : |S| \leq m\}$ for some m, we recover the m-stable compactification $\overline{\mathcal{M}}_{1,n}(m)$ of Smyth [2011a]. We may regard the spaces $\overline{\mathcal{M}}_{1,n}(Q)$ as "combinatorial remixes" of the m-stable spaces, since each of the curves of $\overline{\mathcal{M}}_{1,n}(Q)$ for some Q belong to some $\overline{\mathcal{M}}_{1,n}(m)$ for various m. Despite this, the Q-stable spaces are surprisingly plentiful: for $n = 5$, there are only 5 m-stable spaces, but 79,814,831 Q-stable spaces.

All of the Q-stable spaces arise from compatible choices of how to contract the universal curve of the moduli space of radially aligned log curves (defined in [Ranganathan et al. 2019] and [Santos-Parker 2017]), analogously to the "extremal assignments" of [Smyth 2013]. It was systematic enumeration of such contractions using the log-geometric techniques of [Bozlee 2020] that led to the discovery of the Q-stable spaces. This leads to a resolution of the rational map between the Deligne–Mumford–Knudsen space and each Q-stable space.

Theorem 4.13. *For each $Q \in \mathfrak{Q}_n$, there is a diagram of stacks*

$$
\begin{array}{ccc}
 & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \\
 & \swarrow \qquad \searrow & \\
\overline{\mathcal{M}}_{1,n} & & \overline{\mathcal{M}}_{1,n}(Q)
\end{array}
$$

such that both arrows are proper and restrict to an isomorphism on $\mathcal{M}_{1,n}$.

We will also find the construction of contractions of families of curves to be helpful sporadically throughout the paper.

Our next main theorem is that the Q-stable spaces account for all sufficiently nice modular compactifications in genus one, taking us one step further in the classification of modular compactifications of the moduli space of pointed algebraic curves. To that end, we introduce some definitions.

Definition 1.8. Let $\mathcal{U}_{1,n}$ be the stack of Gorenstein, connected, reduced curves of genus one with n distinct smooth marked points and no infinitesimal automorphisms. For the purposes of this paper, a *modular compactification* is an open Deligne–Mumford substack $\mathcal{M}$ of $\mathcal{U}_{1,n}$, proper over $\mathrm{Spec}\,\mathbb{Z}\left[\frac{1}{6}\right]$.

In the language of [Smyth 2013], a modular compactification in our sense is a semistable modular compactification whose curves are Gorenstein with distinct smooth markings, except that the base is chosen as $\mathrm{Spec}\,\mathbb{Z}\left[\frac{1}{6}\right]$ instead of $\mathrm{Spec}\,\mathbb{Z}$.

Theorem 1.9. *If $\mathcal{M}$ is a modular compactification of $\mathcal{M}_{1,n}$, then $\mathcal{M} = \overline{\mathcal{M}}_{1,n}(Q)$ for some Q.*

We prove this classification theorem over the course of Section 6.

Finally, in Section 7, we construct a cube complex of mildly nonseparated Artin stacks interpolating between the $\overline{\mathcal{M}}_{1,n}(Q)$'s. This complex yields a multidimensional analogue of the wall-and-chamber structure seen in the log minimal model program for $\overline{\mathcal{M}}_g$.

This paper gives the first general classification of Gorenstein modular compactifications of $\mathcal{M}_{g,n}$ in genus greater than 0. In future work we hope to use similar ideas to construct and classify modular compactifications of $\mathcal{M}_{g,n}$. For instance, Battistella [2022] has constructed a sequence of modular compactifications of $\mathcal{M}_{2,n}$ parametrized by a level analogous to that of [Smyth 2011a], and our more flexible notion of level should also yield combinatorial variations of Battistella's moduli spaces.

It would also be natural to search for similar results on modular compactifications in which the marked points are permitted to come together, as in the spaces of weighted stable curves of [Hassett 2003]. The thesis of Andy Fry [2021] suggests that it is necessary to consider more general collisions of markings than those permitted by weights. This will be pursued in future work with Vance Blankers.

2. Examples

In this section we give some examples to illustrate the nature and variety of Q-stable spaces. We start by describing how to count Q-stability conditions.

Definition 2.1. An *antichain* in a partially ordered set P is a subset $A \subseteq P$ such that no distinct elements of A are comparable.

Proposition 2.2. *Let P be a finite partially ordered set. There is a bijection*

$$\{Q \subsetneq P : Q \text{ downward closed}\} \; \leftrightarrow \; \{A \subseteq P : A \text{ is a nonempty antichain}\}$$

given left-to-right by taking Q to the set of minimal elements of $P - Q$, and right-to-left by taking A to the complement of the upward closure of A.

Proof. Omitted. $\square$

The number of nonempty antichains of the lattice of partitions of n elements are counted in OEIS sequence A302251 [Machacek 2018]. We learn that there are

- 9 Q-stable compactifications of $\mathcal{M}_{1,3}$,

- 346 Q-stable compactifications of $\mathcal{M}_{1,4}$,

- 79,814,831 Q-stable compactifications of $\mathcal{M}_{1,5}$.

By contrast, for a given n, there are only n compactifications of $\mathcal{M}_{1,n}$ by m-stable spaces.

Since the properties of being downward closed and of being a proper subset are preserved by finite union and intersection, the set $\mathfrak{Q}_n$ forms a lattice under union and intersection.

Part(3)   downward closed proper subsets of Part(3)
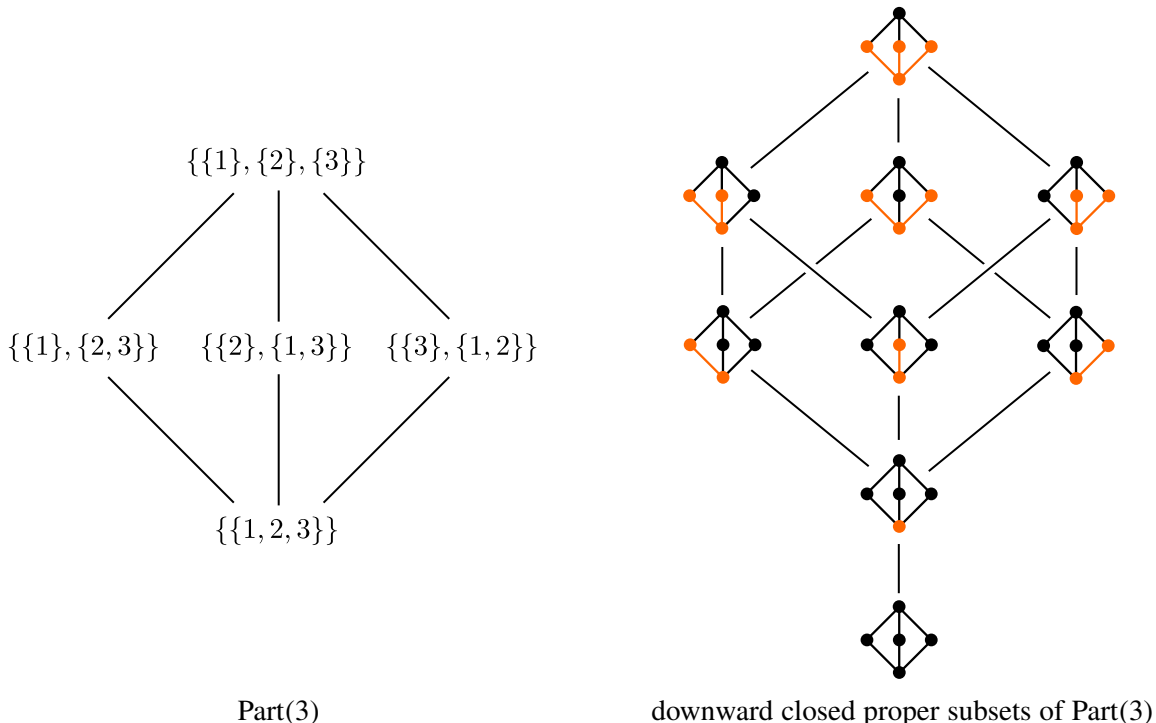
Figure 1. The partially ordered set of partitions of $\{1, 2, 3\}$ and the lattice of Q-stability conditions for $n = 3$. An orange dot indicates that the corresponding partition on the left is included in Q.

Example 2.3. Consider the case $n = 3$. The Hasse diagram of Part(3) and the corresponding lattice of Q-stability conditions for $n = 3$ are displayed in Figure 1. Visually, $\mathfrak{Q}_3$ consists of a cube and a whisker: we will show later that the lattice is always a "union of cubes" and consider a way to fill in the interior of the cube.

We see that there are 9 Q-stable spaces for $n = 3$, in agreement with the count just above. Three of those are m-stable spaces: $\overline{\mathcal{M}}_{1,3}$ corresponds to the subset at the bottom of the diagram, $\overline{\mathcal{M}}_{1,3}(1)$ to the subset just above, and $\overline{\mathcal{M}}_{1,3}(2)$ to the subset at the top of whole diagram.

In Figure 2 we give some examples of 3-pointed curves and the Q's for which they are considered stable.

Example 2.4. For $n = 3$ none of the new stability conditions—that is, the Q's such that $\overline{\mathcal{M}}_{1,n}(Q)$ is not an m-stable space—are symmetric with respect to the markings. This is a coincidence for low n.

Say that a proper downward closed subset Q of Part(n) is *symmetric* if Q is fixed by the natural S_n action. The orbits of partitions of $\{1, \ldots, n\}$ are in bijection with the integer partitions of n, so we may equivalently think of a symmetric Q as a proper downward closed subset of the partially ordered set of integer partitions of n ordered by refinement. Since the property of being symmetric is preserved under intersection and union, the set of symmetric Q-stable conditions for n also form a lattice under union and intersection.
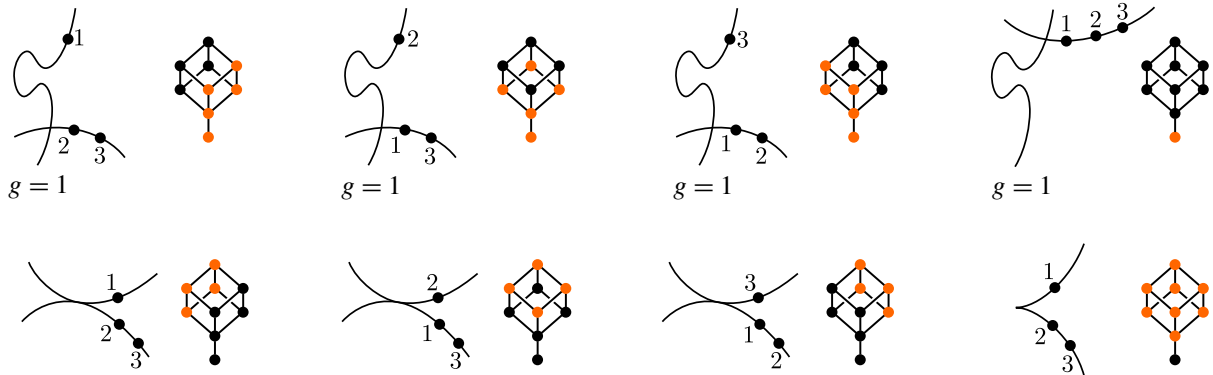
Figure 2. Some Gorenstein 3-pointed curves of genus one. Next to each curve we indicate for which choices of Q the curve is Q-stable: an orange dot means that the curve is stable with respect to the corresponding point in the diagram of downward closed subsets in Figure 1.

Consider the set of symmetric Q-stability conditions when $n = 5$. The Hasse diagram of the integer partitions of 5 and the lattice of symmetric Q-stable conditions, colored by the corresponding subset of the integer partitions of 5, is shown in Figure 3. The 5 m-stable spaces are given by the subsets down the middle of the diagram on the right; the 4 remaining subsets on the sides of that diagram yield new moduli spaces.
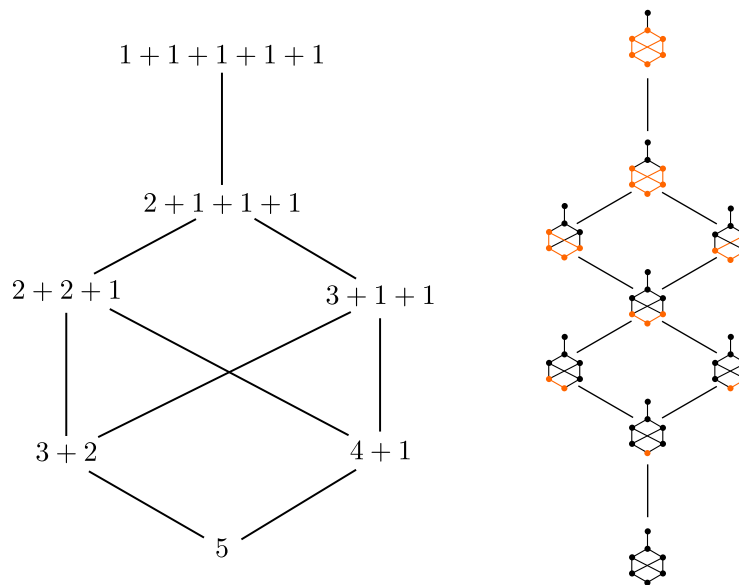
Figure 3. On the left, the integer partitions of 5. On the right, the lattice of symmetric Q-stability conditions when $n = 5$, thought of as subsets of the set of integer partitions. An orange dot indicates that the corresponding integer partition in the diagram on the left is included in Q.

3. Preliminaries

Tropical curves. Our main tool is the log-geometric approach to tropical geometry. We will use the framework of [Cavalieri et al. 2020]. All of our monoids will be commutative and we take $\mathbb{N}$ to include zero. We will prefer additive notation for the operation of P.

Recall that a monoid P is

(i) *sharp* if its only invertible element is the identity,

(ii) *integral* if $a + b = a + c$ implies $b = c$ for all $a, b, c \in P$,

(iii) *finitely generated* if there is a surjective monoid homomorphism $\mathbb{N}^r \to P$ for some integer r,

(iv) *saturated* if P is integral and for any $a \in P^{gp}$ and $n \in \mathbb{Z}_{>0}$, $n \cdot a \in P$ implies $a \in P$,

(v) *fs* if P is finitely generated, integral, and saturated.

We begin by recalling the definition of tropical curve, which is essentially a graph whose edges are labeled with "lengths" from an fs sharp monoid.

Definition 3.1. An n-marked *tropical curve* Γ *with edge lengths in an fs sharp monoid* P consists of:

(i) A finite set $X(\Gamma) = V(\Gamma) \sqcup F(\Gamma)$. The elements of $V(\Gamma)$ are called the *vertices* of Γ and the elements of $F(\Gamma)$ are called the *flags* of Γ.

(ii) A *root map* $r_\Gamma : X(\Gamma) \to X(\Gamma)$ which is idempotent with image $V(\Gamma)$.

(iii) An involution $\iota_\Gamma : X(\Gamma) \to X(\Gamma)$ that fixes $V(\Gamma)$. The subsets $\{f, \iota_\Gamma(f)\}$ of $F(\Gamma)$ of size two are called *edges*, and the set of all edges is denoted by $E(\Gamma)$. The subsets $\{f, \iota_\Gamma(f)\}$ of $F(\Gamma)$ of size one are called *legs*, and the set of all legs is denoted by $L(\Gamma)$.

(iv) A bijection $l : \{1, \ldots, n\} \to L(\Gamma)$.

(v) A function $g : V(\Gamma) \to \mathbb{N}$. Given a vertex v, $g(v)$ is called the *genus of* v.

(vi) A function $\delta : E(\Gamma) \to P$. Given an edge e, $\delta(e)$ is called the *length of* e.

We imagine that each flag f is half of an edge starting at the vertex $r_\Gamma(f)$. Given an edge $e = \{f, \iota_\Gamma(f)\}$, we say the vertices $r_\Gamma(f)$ and $r_\Gamma(\iota_\Gamma(f))$ are *incident* to e.

Definition 3.2. The *genus* of a tropical curve Γ is

$$g(\Gamma) = b_1(\Gamma) + \sum_{v \in V(\Gamma)} g(v),$$

where $b_1(\Gamma)$ is the first Betti number of Γ, that is, $|E(\Gamma)| - |V(\Gamma)| + n$, where n is the number of connected components of Γ.

Definition 3.3. A tropical curve is *stable* if it is connected and not an isolated vertex of genus one, and the valence of each of its vertices of genus 0 is at least 3.

Definition 3.4. A *piecewise linear function* f on a tropical curve Γ with edge lengths in P consists of

(i) a value $f(v) \in P$ for each vertex $v \in V(\Gamma)$,

(ii) a slope $m(l) \in \mathbb{N}$ for each leg $l \in L(\Gamma)$

such that whenever e is an edge with ends v and w, $f(v) - f(w)$ is an integer multiple of $\delta(e)$.

  The set of all piecewise linear functions on Γ is denoted by $\mathrm{PL}(\Gamma)$.

  Given a tropical curve Γ with edge lengths in P and a morphism of fs sharp monoids $\pi^\sharp : P \to P'$, we may apply $\pi^\sharp$ to the edge lengths of Γ and contract edges of length zero to arrive at a new tropical curve. Composing with an isomorphism gives us the notion of a weighted edge contraction, which we define below.

Definition 3.5. Let Γ and Γ' be tropical curves with edge lengths in P and P', respectively. A *weighted edge contraction* $\pi : \Gamma' \to \Gamma$ (note the variance!) consists of

(i) a function $\pi : X(\Gamma) \to X(\Gamma')$,

(ii) a morphism of monoids $\pi^\sharp : P \to P'$

such that

 (i) π preserves ends of flags, that is, $\pi \circ r_\Gamma = r_{\Gamma'} \circ \pi$;

 (ii) π preserves edges, that is, $\pi \circ \iota_\Gamma = \iota_{\Gamma'} \circ \pi$;

(iii) π sends legs of Γ bijectively to legs of Γ' and preserves their markings;

(iv) for each flag $f \in F(\Gamma')$, the preimage $\pi^{-1}(f)$ has exactly one element (automatically a flag);

 (v) for each vertex $v \in V(\Gamma')$, the preimage $\pi^{-1}(v)$ is a connected weighted graph of genus $g(v)$;

(vi) the flags of an edge $e \in E(\Gamma)$ are sent by π to a vertex of Γ' if and only if $\pi^\sharp(\delta(e)) = 0$;

(vii) for each edge $e \in E(\Gamma)$ with $\pi^\sharp(\delta(e)) \neq 0$, the image of e is an edge e' of length $\delta(e') = \pi^\sharp(\delta(e))$.

  We will call a weighted edge contraction a *face contraction* if there is a subset $S \subseteq P$ such that the map $\pi^\sharp$ is of the form

$$P \to S^{-1}P \to S^{-1}P/(S^{-1}P)^* \xrightarrow{\sim} P',$$

where the first arrow is localization, the second is the quotient by the submonoid of invertible elements, and the third is an isomorphism. (These are the edge contractions associated to face inclusions in the category of rational polyhedral cones [Cavalieri et al. 2020, Definition 2.25].) In the case that P is a finite free monoid $\mathbb{N}^r$, the face contractions are those induced by the projections of $\mathbb{N}^r$ onto subsets of its coordinates.

  Given a weighted edge contraction $\pi : \Gamma' \to \Gamma$ there is an induced map

$$\pi^* : \mathrm{PL}(\Gamma) \to \mathrm{PL}(\Gamma')$$

given by taking f with values $f(v)$ and slopes $m(l)$ to the piecewise linear function $\pi^* f$ with values $(\pi^* f)(v) = \pi^\sharp(f(v))$ for $v \in V(\Gamma)$ and the same slopes.

We take weighted edge contractions to be the morphisms in the category of tropical curves. In particular, an isomorphism of tropical curves is an invertible weighted edge contraction.

Log curves and their tropicalizations. The natural notion of family of curves in logarithmic geometry admits both an underlying family of pointed nodal curves and a tropicalization, connecting the tropical and algebrogeometric worlds. F. Kato [2000] introduced the notion of a family of log curves.

Definition 3.6. (cf. [Kato 2000, Definition 1.2]) Let S be an fs log scheme. A *log curve* over S is a log smooth and integral morphism $\pi : C \to S$ of fs log schemes such that every geometric fiber of π is a reduced and connected curve.

Kato [2000, Theorem 1.3] has shown that the underlying morphism of schemes of a log curve is a family of nodal curves, and the data in the log structure records some marked points. We borrow this statement of Kato's local structure theorem from [Ranganathan et al. 2019].

Theorem 3.7. *Let $\pi : C \to S$ be a family of proper log curves. If $x \in C$ is a geometric point with image $s \in S$, then there are étale neighborhoods V of x and U of s such that $V \to U$ has a strict morphism to an étale-local model $V' \to U'$, where $V' \to U'$ is one of the following:*

  (i) *The smooth germ: $V' = \mathbb{A}^1_{U'} \to U'$ and the log structure on V' is pulled back from the base.*

 (ii) *The germ of a marked point: $V' = \mathbb{A}^1_{U'} \to U'$ with the log structure pulled back from the toric log structure on $\mathbb{A}^1$.*

(iii) *The node: $V' = \mathrm{Spec}\,\mathscr{O}_{U'}[x, y]/(xy - t)$ for $t \in \mathscr{O}_{U'}$. The log structure on V is pulled back from the multiplication map $\mathbb{A}^2 \to \mathbb{A}^1$ of toric varieties along the morphism $U' \to \mathbb{A}^1$ of logarithmic schemes induced by t.*

The tropicalization of a log curve is the dual graph of its underlying nodal curve, enriched with the data of the smoothing parameters from its log structure.

Definition 3.8. Given an n-marked log curve $\pi : C \to S$, where S is a log point, the *tropicalization* $\mathrm{trop}(C)$ of C is the n-marked tropical curve with edge lengths in $\Gamma(S, \overline{M}_S)$ which has

  (i) a vertex for each component of C;

 (ii) an edge for each node of C, incident to the components of C which form the branches of the node;

(iii) a leg for each marked point of C, rooted at the component of C to which the marked point belongs;

and

  (a) for each vertex v, the genus $g(v)$ is the genus of the normalization of the corresponding component of C;

  (b) for each edge e, the length $\delta(e) \in \Gamma(S, \overline{M}_S)$ is the smoothing parameter of the node e.

See Figure 4 for an example of tropicalization. Note that the tropicalization may contain loops: consider the nodal cubic.

Sections of the characteristic sheaf of C are interpreted tropically as piecewise linear functions.
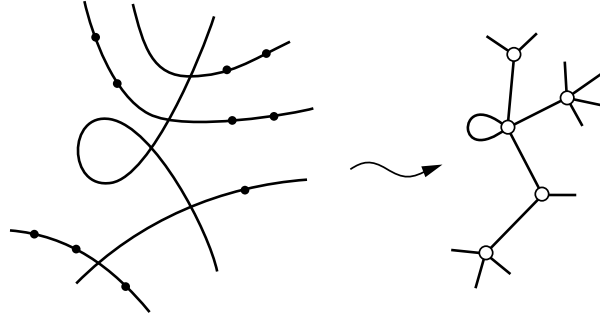
Figure 4. A typical log curve and its tropicalization.

Theorem 3.9. *Let* $\pi : C \to S$ *be a log curve over the spectrum of an algebraically closed field. Then there is a bijection*

$$\mathrm{PL} : \Gamma(C, \overline{M}_C) \xrightarrow{\sim} \mathrm{PL}(\mathrm{trop}(C)), \quad \sigma \mapsto \mathrm{PL}(\sigma),$$

where

(i) *the value of* $\mathrm{PL}(\sigma)$ *at a vertex* v *of* $\Gamma(C)$ *is the stalk of* σ *at the generic point of the corresponding component of* C;

(ii) *the slope of* $\mathrm{PL}(\sigma)$ *at a leg* l *of* $\Gamma(C)$ *is the image of* σ *in* $(\overline{M}_C/\pi^{-1}\overline{M}_S)_p \cong \mathbb{N}$, *where* p *is the marked point corresponding to* l.

Proof. See, for example, [Cavalieri et al. 2020, Remark 7.3]. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

For a general log curve, this interpretation extends nicely over an étale neighborhood of each point.

Theorem 3.10. *Let* $\pi : C \to S$ *be a log curve and let* s *be a geometric point of* S. *Then there is an étale neighborhood* U *of* s *in* S *such that*

(i) $\Gamma(U, \overline{M}_S) \to \overline{M}_{S,s}$ *and* $\Gamma(C|_U, \overline{M}_C) \to \Gamma(C|_s, \overline{M}_{C|_s})$ *are isomorphisms;*

(ii) *for each geometric point* t *of* U, *there is a canonical face contraction*

$$\mathrm{trop}(C_s) \to \mathrm{trop}(C_t)$$

induced by

$$\overline{M}_{S,s} \xleftarrow{\sim} \Gamma(U, \overline{M}_S) \to \overline{M}_{S,t}.$$

Moreover, this face contraction respects associated piecewise linear functions in the sense that

$$
\begin{array}{ccc}
\Gamma(C|_s, \overline{M}_{C|_s}) \xleftarrow{\sim} \Gamma(C|_U, \overline{M}_C) \longrightarrow \Gamma(C|_t, \overline{M}_{C|_t}) \\
{\scriptstyle \mathrm{PL}} \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathrm{PL}} \\
\mathrm{PL}(\mathrm{trop}(C|_s)) \longrightarrow \mathrm{PL}(\mathrm{trop}(C|_t))
\end{array}
$$

commutes.

Proof. This follows, for example, from the existence of "uniform sets of charts", constructed in [Bozlee 2020, Proposition 2.3.13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

It follows that to define a section of the characteristic sheaf of C it is equivalent to specify a piecewise linear function on each geometric fiber of C so that the resulting piecewise linear functions are compatible with generization.

Definition 3.11. An n-marked log curve is a log curve $\pi : C \to S$ equipped with disjoint sections $\sigma_1, \ldots, \sigma_n : S \to C$ with image the marked points of C. An n-marked log curve is *stable* if its underlying family of marked nodal curves is Deligne–Mumford stable.

Theorem 3.12 [Kato 2000, Theorem 4.5]. *There is a log structure on $\overline{\mathcal{M}}_{g,n}$, called the **basic log structure**, such that $\overline{\mathcal{M}}_{g,n}$ represents the stack of stable n-marked log curves of genus g over the category of fs log schemes.*

We will generally regard $\overline{\mathcal{M}}_{g,n}$ as a stack with the basic log structure and will freely confuse $\overline{\mathcal{M}}_{g,n}$ with its underlying algebraic stack. A stable log curve $\pi : C \to S$ is said to have the *basic log structure* if its log structure is pulled back from that of the universal stable log curve of $\overline{\mathcal{M}}_{g,n}$. In the case that S is a geometric point, π has the basic log structure if and only if the characteristic monoid $\overline{M}_S$ is freely generated by the edge lengths of $\mathrm{trop}(C)$.

Radially aligned curves. We now build up the terminology to work with radially aligned curves. These were introduced by Santos-Parker [2017] under the name of ordered log curves and then popularized in [Ranganathan et al. 2019].

Definition 3.13. Let Γ be a tropical curve. A *path* W in Γ is a sequence $v_0 e_1 v_1 e_2 \cdots e_k v_k$ of vertices and edges in Γ such that the vertices v_i are distinct and v_{i-1} and v_i are the ends of the edge e_i for all i. Given subsets A and B of $V(\Gamma)$, we say that W is a path from A to B if $v_0 \in A$, $v_k \in B$, and $v_i \notin A \cup B$ for $i \neq 0, k$.

Definition 3.14. Given a proper curve C over the spectrum of an algebraically closed field, a *subcurve* of C is a union of irreducible components of C, possibly empty.

The *core* of C is the minimal connected subcurve of C with the same genus as C. Analogously the core of a tropical curve Γ is the minimal connected vertex-induced subgraph of the same genus as Γ.

Definition 3.15. Given a tropical curve Γ of genus one, we define a piecewise linear function λ on Γ measuring "distance from the core" as follows. If v is a vertex in the core of Γ, we set

$$\lambda(v) = 0.$$

If v is a vertex outside of the core of Γ, we let $W = v_0 e_1 v_1 e_2 \cdots e_k v_k$ be the unique path from the core of Γ to v and set

$$\lambda(v) = \sum_{i=1}^{k} \delta_{e_i}.$$

Finally, we set the slope of λ to be 1 at all marked points.

This is compatible with generization, so for any stable log curve $(\pi : C \to S; \sigma_1, \dots, \sigma_n)$ of genus one, we let $\lambda \in \Gamma(S, \overline{M}_S)$ be the unique section of the characteristic bundle whose restriction to geometric fibers has corresponding piecewise linear function as in the last paragraph.

Definition 3.16. If P is any fs sharp monoid, we give the elements of P a partial order by the rule $p \leq q$ if and only if there exists $r \in P$ with $q = p + r$.

Definition 3.17. A stable n-marked tropical curve of genus one with edge lengths in P is *radially aligned* if, for each pair of vertices v, w of Γ, $\lambda(v)$ is comparable to $\lambda(w)$ in P.

Given such a radially aligned curve, let

$$0 < \rho_1 < \cdots < \rho_k$$

be the distinct values of $\lambda(v)$ as v varies over the components of C, and let $\delta_1, \dots, \delta_l$ be the lengths of the edges of trop(C) internal to the core of Γ. Let $e_1 = \rho_1$, $e_2 = \rho_2 - \rho_1, \dots, e_k = \rho_k - \rho_{k-1}$. If P is freely generated by

$$\{e_1, \dots, e_k\} \cup \{\delta_1, \dots, \delta_l\},$$

then we say that Γ is a *basic radially aligned tropical curve*. An element of P is said to have *no contribution from the core* if it lies in the submonoid generated by $e_1, \dots, e_k$.

A stable log curve $(\pi : C \to S; \sigma_1, \dots, \sigma_n)$ of genus one with n markings is *radially aligned* or has a *basic radially aligned log structure* if the tropicalizations of its geometric fibers with their pulled back log structure are respectively radially aligned or basic radially aligned. An element $\rho \in \Gamma(S, \overline{M}_S)$ has *no contribution from the core* if the same holds of its stalks at the geometric points of S.

There is a moduli stack with log structure parametrizing radially aligned log curves.

Theorem 3.18 [Ranganathan et al. 2019, Proposition 3.3.4]. (i) *There is a Deligne–Mumford stack with locally free log structure $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ whose S-points for S an fs log scheme are the n-marked radially aligned curves $\pi : C \to S$ over S. We say its log structure is **the** basic radially aligned log structure.*

(ii) *There is a natural map $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}$ induced by a logarithmic blowup and it restricts to an isomorphism on $\mathcal{M}_{1,n}$.*

A stable log curve $(\pi : C \to S; \sigma_1, \dots, \sigma_n)$ has a basic radially aligned log structure precisely when the log structure on $\pi : C \to S$ is that pulled back from the universal curve $\mathcal{C}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ along the map $S \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. We remark that a fixed family of nodal curves may be enhanced to a family of basic radially aligned log curves in more than one way, which we illustrate with an example.

Example 3.19. Let $\pi : C \to S = \operatorname{Spec} k$ be a stable curve with the basic log structure over the spectrum of an algebraically closed field, and suppose that its tropicalization is
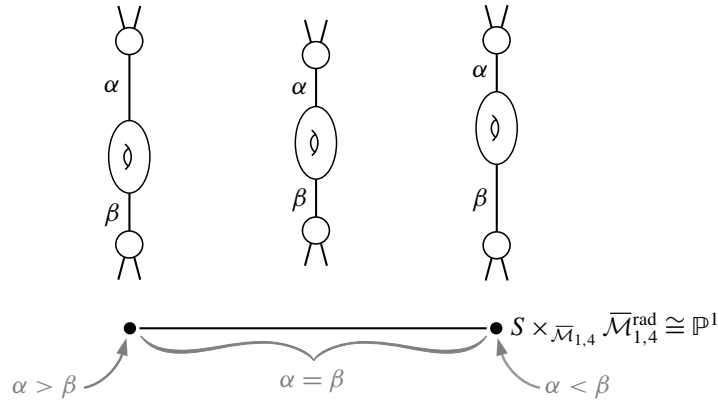
Figure 5. The $\mathbb{P}^1$ of radial alignments of a basic stable log curve with two edges.

There is an associated map $S \to \overline{\mathcal{M}}_{1,4}$. The basic log structure on S comes from the chart

$$\mathbb{N}\tilde{\alpha} \oplus \mathbb{N}\tilde{\beta} \to k$$

sending $\tilde{\alpha}, \tilde{\beta} \mapsto 0$. The edge lengths α and β are the respective images of $\tilde{\alpha}$ and $\tilde{\beta}$ in the characteristic sheaf. Locally in $\overline{\mathcal{M}}_{1,4}$ near the image of S, the map $\overline{\mathcal{M}}_{1,4}^{\text{rad}} \to \overline{\mathcal{M}}_{1,4}$ is given by the log blowup of the log ideal generated by α and β, as these are the distances that we wish to make comparable. We refer the interested reader to [Ogus 2018, Chapter III, Section 2.6] for details on log blowups.

We may compute all of the basic radially aligned log structures on C by computing the restriction of this blowup to S. We construct the blowup by first freely adjoining the element $\tilde{\alpha} - \tilde{\beta}$ to the log structure of S, adjoining an element to k for $\tilde{\alpha} - \tilde{\beta}$ to map to, doing likewise for $\tilde{\beta} - \tilde{\alpha}$, and finally gluing over the overlap. That is, $S \times_{\overline{\mathcal{M}}_{1,4}} \overline{\mathcal{M}}_{1,4}^{\text{rad}}$ possesses a cover by two open sets $U = \operatorname{Spec} k[t]$ (where $\alpha \geq \beta$) and $V = \operatorname{Spec} k[t^{-1}]$ (where $\beta \geq \alpha$) with log structure on U induced by

$$\mathbb{N}(\tilde{\alpha} - \tilde{\beta}) \oplus \mathbb{N}\tilde{\beta} \to k[t], \qquad \tilde{\alpha} - \tilde{\beta} \mapsto t \quad \text{and} \quad \tilde{\beta} \mapsto 0,$$

and log structure on V induced by

$$\mathbb{N}\tilde{\alpha} \oplus \mathbb{N}(\tilde{\beta} - \tilde{\alpha}) \to k[t^{-1}], \qquad \tilde{\alpha} \mapsto 0 \quad \text{and} \quad \tilde{\beta} - \tilde{\alpha} \mapsto t^{-1}.$$

The two charts are glued in the obvious way. Notice that on the intersection $U \cap V = \operatorname{Spec} k[t, t^{-1}]$, the sections $\tilde{\alpha} - \tilde{\beta}$ and $\tilde{\beta} - \tilde{\alpha}$ of the log structure restrict to units, so that their images in the characteristic sheaf are 0. It follows that α and β are equal over a $\mathbb{G}_m$'s worth of possible basic radially aligned enhancements of C. See Figure 5.

Definition 3.20. Let Γ be a radially aligned tropical curve with ordered radii $0 < \rho_1 < \cdots < \rho_k$.

Given a radius ρ, we may form a tropical curve $\widetilde{\Gamma}(\rho)$ by subdividing the edges and legs of Γ where $\lambda = \rho$, then deleting the locus where $\lambda < \rho$. We define the *partition associated to the radius ρ* to be the partition of $\{1, \ldots, n\}$ induced by the components of $\widetilde{\Gamma}(\rho)$, and we denote it by $\text{part}(\rho)$.
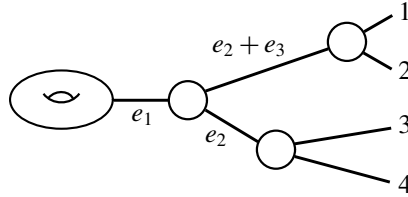
Figure 6. A basic radially aligned curve with partition type $\{\{1, 2, 3, 4\}\} \prec \{\{1, 2\}, \{3, 4\}\} \prec \{\{1, 2\}, \{3\}, \{4\}\}$. We draw a torus to indicate a vertex of genus one.

We say that the resulting strict chain of partitions

$$\mathrm{part}(\rho_1) \prec \mathrm{part}(\rho_2) \prec \cdots \prec \mathrm{part}(\rho_k)$$

is the *partition type* of Γ. See Figure 6 for an example.

It would be natural to include the partition $\mathrm{part}(0)$ in the partition type as well, but we choose not to for a few reasons. The first is that we always have $\mathrm{part}(0) = \{\{1, 2, \ldots, n\}\}$, since $\mathrm{part}(0)$ is the partition of the markings induced by deleting no components. So including $\mathrm{part}(0)$ in the list does not convey more information. For another, unlike the other comparisons, the comparison $\mathrm{part}(0) \preceq \mathrm{part}(\rho_1)$ need not be strict: it may be that both are the indiscrete partition. For example, see Figure 6.

4. Contractions of the universal radially aligned curve

Part of the utility of families of radially aligned curves is that they are easy to contract to families of curves with Gorenstein singularities, even at the level of a universal curve. By exploring the possible contractions of the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ we find regular birational maps $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}(Q)$ for each Q. It was this computation that identified the Q-stable moduli spaces.

The following theorem says that in order to contract a family of radially aligned curves, all we need is the data of a tropical radius for each curve in the family. This idea was key for the results of [Santos-Parker 2017] and [Ranganathan et al. 2019] and can be done using their language; see [Santos-Parker 2017, Section 5] and [Ranganathan et al. 2019, Section 3.7]. We give a proof using the language of [Bozlee 2020] for its convenience and generality.

Theorem 4.1. *Let $\pi : C \to S$ be a family of n-marked radially aligned log curves. Let $\rho \in \Gamma(S, \overline{M}_S)$ be a section of the characteristic monoid such that "ρ is a radius at all geometric points of s"; that is, for each geometric point s of S, there is a vertex v of $\mathrm{trop}(C|_s)$ such that $\lambda(v) = \rho|_s$.*

Then there is a diagram
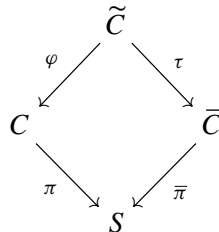
where

(i) φ *is a log blowup inducing the subdivision at the locus where* $\lambda = \rho$ *on tropicalizations*;

(ii) $\overline{\pi} : \overline{C} \to S$ *is a flat and proper family of Gorenstein curves of genus one*;

(iii) τ *is a surjective map whose restriction to geometric fibers contracts the locus (if nonempty) where* $\lambda < \rho$ *to an elliptic singularity of level* part$(\rho|_s)$ *and restricts further to an isomorphism in the complement of this locus.*

Moreover, formation of the diagram commutes with base change in S.

Proof. The construction of φ is standard. Using the language of [Bozlee 2020], we then define a mesa $\overline{\lambda} \in \Gamma(\widetilde{C}, \overline{M}_{\widetilde{C}})$ on the resulting family of log curves $\widetilde{C} \to S$ with the formula

$$\overline{\lambda} = \max\{\rho - \lambda, 0\}.$$

It is easy to check that $\overline{\lambda}$ defines a steep mesa with support on the locus where $\lambda < \rho$, so the main theorem of [Bozlee 2020] yields the claimed diagram with the required properties. To see that the elliptic singularities of $\overline{C}$ have the claimed level, we note that the branches of the singularity will be the images of the connected components of the locus in $\widetilde{C}$ where $\lambda \geq \rho$. These are precisely the connected components considered in the definition of part(ρ). $\qquad\square$

We want to apply this theorem to the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Our next task is to reduce the problem of enumerating the possible radii $\rho \in \Gamma(\overline{\mathcal{M}}_{1,n}, \overline{M}_{\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}})$ to something manageable.

Lemma 4.2. *Let Γ be a basic radially aligned curve. Then*:

(i) *If the core of Γ consists of a vertex with a self-loop, then the only nontrivial automorphism of Γ is the automorphism reversing the loop, but the identity on everything else.*

(ii) *If the core of Γ consists of a pair of vertices with two edges, the only nontrivial automorphism of Γ is the automorphism exchanging the edges of the core, and the identity on everything else.*

(iii) *Otherwise, Γ has no nonidentity automorphisms.*

Proof. Suppose Γ is a basic radially aligned tropical curve. Let $\varphi : \Gamma \to \Gamma$ be an invertible weighted edge contraction.

We argue that φ is the identity on the vertices of Γ. Let v be a vertex of Γ. Notice that the complement of the core of Γ consists of a forest of trees, each of which we can root at the vertex that attaches to the core. Furthermore, due to stability,

(i) if v is a vertex outside of the core of Γ, then v is uniquely identified by the markings that lie on v and the descendants of v;

(ii) if v is a vertex inside the core of Γ, there is at least one tree attached to v, and those trees are uniquely identified by their markings.

An automorphism of Γ must in particular preserve the markings. Therefore, φ fixes all of the vertices of Γ. This implies that φ fixes all edges except possibly those who share incident vertices. This implies the result. $\qquad\square$

Remark 4.3. For a fixed n, there are only finitely many isomorphism classes of n-marked tropical curves with the basic radially aligned log structure.

Lemma 4.4. *Let Γ be an n-marked basic radially aligned tropical curve with edge lengths in P. Write $\mathcal{M}_\Gamma^{\mathrm{rad}}$ for the substack of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ of curves whose tropicalizations are isomorphic to Γ. Then $\mathcal{M}_\Gamma^{\mathrm{rad}}$ is an irreducible locally closed substack of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$.*

Proof. Write $\mathcal{M}_\Gamma$ for the locally closed substack of stable n-marked curves of genus one whose dual graph is the underlying graph of Γ. Recall that $\mathcal{M}_\Gamma \cong \prod_{v \in V(\Gamma)} \mathcal{M}_{g(v),\mathrm{val}(v)}$, where the valence $\mathrm{val}(v)$ is the number of flags incident to v. Since the $\mathcal{M}_{g(v),\mathrm{val}(v)}$'s are geometrically irreducible, so is $\mathcal{M}_\Gamma$. There is a forgetful map $\mathcal{M}_\Gamma^{\mathrm{rad}} \to \mathcal{M}_\Gamma$ given by forgetting the log structure.

We recall from [Ranganathan et al. 2019, Proposition 3.3.4] that the map $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}$ is locally given as follows. Suppose given a map $S \to \overline{\mathcal{M}}_{1,n}$ such that S admits a global chart by a monoid Q. This induces a map $S \to V = \operatorname{Spec} \mathbb{Z}[Q]$. Let σ be the rational polyhedral cone dual to Q. Let Σ be the fan obtained by subdividing σ along the hyperplanes where $\lambda(v) = \lambda(w)$ as v and w range among the vertices of $\mathrm{trop}(C_s)$, and let W be the toric variety associated to Σ. Then

$$S \times_{\overline{\mathcal{M}}_{1,n}} \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \cong S \times_V W.$$

Suppose that $S \to \mathcal{M}_{1,n}^{\mathrm{rad}}$ factors through $\mathcal{M}_\Gamma^{\mathrm{rad}}$. Then Q is the free monoid on the edges of Γ. By construction, there is a cone of Σ associated to each possible choice of ordering of the distances $\lambda(v)$ as v varies over $V(\Gamma)$. Since Γ is basic radially aligned, these distances are ordered, and their ordering determines a cone τ of Σ. Let $W_\tau \subseteq W$ be the torus orbit associated to τ. One may check that the locus in $S \times_V W$ in which the tropicalization is isomorphic to Γ is precisely the locus $S \times_V W_\tau$: this is the locus in which the stalks of the characteristic sheaf agree with P. Letting S vary over a smooth cover of $\mathcal{M}_\Gamma^{\mathrm{rad}}$, we see that $\mathcal{M}_\Gamma^{\mathrm{rad}} \to \mathcal{M}_\Gamma$ is smooth-locally a W_τ-bundle. Since the target is irreducible and W_τ is irreducible, $\mathcal{M}_\Gamma^{\mathrm{rad}}$ is irreducible. Moreover it is a locally closed substack of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ as this is true of W_τ inside W. $\qquad\square$

Lemma 4.5. *Let I be the set of isomorphism classes of n-marked basic radially aligned tropical curves. Fix a representative Γ with edge lengths in P_Γ for each isomorphism class.*

Giving a section ρ of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ with no contribution from the core is equivalent to specifying for each isomorphism class of basic radially aligned tropical curves an element $\rho_\Gamma \in P_\Gamma$ with no contribution from the core such that whenever $\Gamma, \Gamma' \in I$ and $\pi : \Gamma \to \Gamma'$ is a face contraction with $\pi^\sharp : P_\Gamma \to P_{\Gamma'}$ a quotient map, then $\pi^* \rho_\Gamma = \rho_{\Gamma'}$

Proof. For brevity, write $\overline{M}$ for the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Since $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is a Deligne–Mumford stack, we may choose an étale cover $U \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ by a scheme U. For each point x of U, choose an algebraic

closure $\bar{k}(x)$ of the residue field $k(x)$, and write $\bar{x}^U : \operatorname{Spec} \bar{k}(x) \to U$ for the natural map to U and $\bar{x}$ for its composite with $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Write $\Gamma_{\bar{x}}$ for the tropicalization of the basic radially aligned log curve associated to $\bar{x}$ and write $P_{\bar{x}}$ for its associated monoid, i.e., $\overline{M}_{\bar{x}}$.

We may apply Theorem 3.10 to find an étale neighborhood $U_{\bar{x}}$ of $\bar{x}^U$ in U over which the pullback of the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ has the properties of the theorem. Since U is an étale cover of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ and the log structure on $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is divisorial, the maps $P_{\bar{x}} \xleftarrow{\sim} \Gamma(U_{\bar{x}}, \overline{M}) \to \overline{M}_t$ vary through *all* of the quotients of $P_{\bar{x}}$ by its generators as t varies through the geometric points of $U_{\bar{x}}$. Then, since the $U_{\bar{x}}$'s form a cover, we may identify the global sections of $\overline{M}$ with elements $(\rho_{\bar{x}})$ of $\prod_{\bar{x}} \Gamma(U_{\bar{x}}, \overline{M}) \cong \prod_{\bar{x}} P_{\bar{x}}$, suitably compatible on overlaps.

Now, for a pair of points $\bar{x}$ and $\bar{y}$, sections $\rho_{\bar{x}} \in \Gamma(U_{\bar{x}}, \overline{M})$ and $\rho_{\bar{y}} \in \Gamma(U_{\bar{y}}, \overline{M})$ agree on $U_{\bar{x},\bar{y}} := U_{\bar{x}} \times_{\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}} U_{\bar{y}}$ if and only if their stalks at geometric points $\bar{z}$ of $U_{\bar{x},\bar{y}}$ agree. This translates to the statement that whenever $\Gamma_{\bar{z}}$ is a face contraction of both $\Gamma_{\bar{x}}$ and $\Gamma_{\bar{y}}$, and $\varphi : \Gamma_{\bar{z}} \to \Gamma_{\bar{z}}$ is an automorphism, the stalk of $\rho_{\bar{x}}$ at $\bar{z}$ is $\varphi^{\sharp}$ applied to the stalk of $\rho_{\bar{y}}$ at $\bar{z}$.

Suppose $\bar{x}$ and $\bar{y}$ are points, $\Gamma \in I$ and $\Gamma_{\bar{x}} \cong \Gamma \cong \Gamma_{\bar{y}}$. Then, in the notation of the previous lemma, since $\mathcal{M}_{\Gamma}^{\mathrm{rad}}$ is irreducible, $U_{\bar{x},\bar{y}}$ must contain a point $\bar{z}$ that also maps into $\mathcal{M}_{\Gamma}^{\mathrm{rad}}$. Then the elements corresponding to $\rho_{\bar{x}}$ and $\rho_{\bar{y}}$ on P_{Γ} must differ by at most an automorphism of Γ. By Lemma 4.2, they are actually equal, so we have a well-defined element ρ_{Γ} of P_{Γ}. The agreement of stalks at other points implies that the ρ_{Γ}'s are compatible with edge contraction. We obtain the converse by reversing this construction. $\qquad\square$

In view of Lemma 4.5, we introduce the notion of *universal radius*.

Definition 4.6. An n-marked *universal radius* consists of the data of an element $\rho_{\Gamma} \in P_{\Gamma}$ for each n-marked basic radially aligned curve Γ so that

(i) for each Γ, $\rho_{\Gamma} = \lambda(v)$ for some vertex v of Γ;

(ii) if Γ and Γ' are two n-marked radially aligned curves and $\pi : \Gamma \to \Gamma'$ is a face contraction, then $\pi^* \rho_{\Gamma} = \rho_{\Gamma'}$.

We use the shorthand notation (ρ_{Γ}) for the tuple of radii making up a universal radius, and denote by $\mathfrak{R}_n^{\mathrm{uni}}$ the set of n-marked universal radii.

Remark 4.7. Condition (i) implies each ρ_{Γ} has no contribution from the core. Condition (ii) implies that we only need to keep track of the *finite* data of a choice of radius ρ_{Γ} for each isomorphism class of n-marked basic radially aligned curves. The maps $P_{\Gamma} \to P_{\Gamma'}$ induced by face contractions are just the coordinate projections. Therefore all we have to worry about to satisfy condition (ii) is what happens when we set various subsets of the generators of P_{Γ} (that is, $\delta_1, \ldots, \delta_l$ and $e_1, \ldots, e_k$ in the notation of Definition 3.17) to zero.

We have therefore reduced the problem of finding a section of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to giving the finite collection of tropical data that make up a universal radius. This is still a fair amount of data: see Figure 7 for an example when $n = 3$. We will see that we can reduce the data of a universal
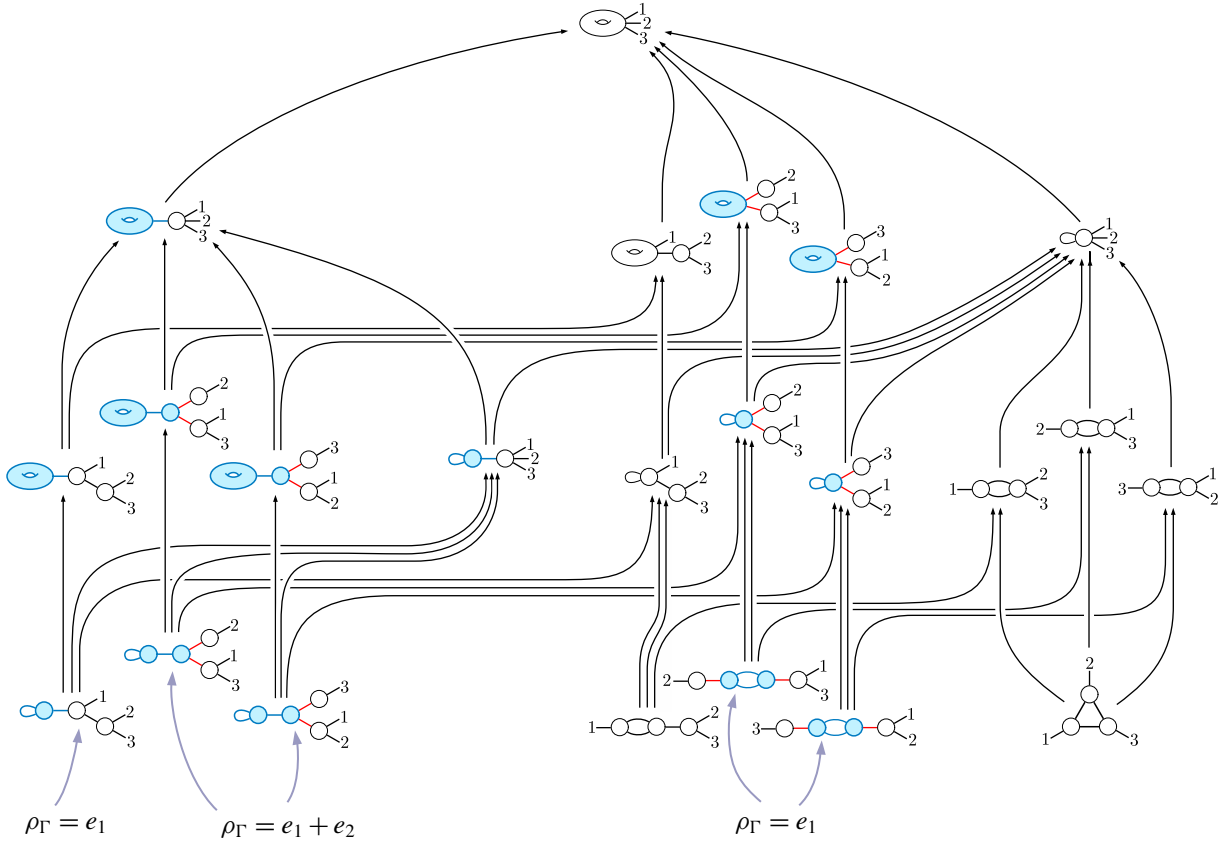
Figure 7. An example of the subdivided universal curve $\widetilde{C}$ on $\overline{\mathcal{M}}_{1,3}^{\mathrm{rad}}$ associated to a universal radius. A torus indicates a vertex of genus one. We have labeled the nonzero ρ_Γ's on the most degenerate tropical curves using the notation of Definition 3.17. The radii on other tropical curves can be deduced by following the indicated face contractions. The red edges of a particular curve have equal length: they come about by subdividing at the radius ρ_Γ. The blue components are those to be contracted.

radius further to that of a downward closed subset Q of partitions on $\{1, \ldots, n\}$. In Figure 7, for example, the corresponding downward closed subset will be

$$Q = \big\{\{\{1, 2, 3\}\},\ \{\{2\}, \{1, 3\}\},\ \{\{3\}, \{1, 2\}\}\big\}.$$

This can be read off from the second row of the figure.

Definition 4.8. We say that Γ is a *k-layer tree* if Γ is an n-marked basic radially aligned tropical curve with k nonzero radii $0 < \rho_1 < \cdots < \rho_k$ and smooth core.

Lemma 4.9. *Let Γ be a basic radially aligned tropical curve with k nonzero radii $0 < \rho_1 < \cdots < \rho_k$. Then, for each index $i < k$, there is a strict refinement* $\mathrm{part}(\rho_i) \prec \mathrm{part}(\rho_{i+1})$.

Proof. Notice that if Γ is replaced by the tropical curve in which all edges of the core are contracted, the sequence of partitions remains the same. Therefore we may assume that Γ is a k-layer tree.

Orient the tree Γ by taking its core as a root. Resuming the notation of Definition 3.20, let V_i be the set of roots of the forest $\widetilde{\Gamma}(\rho_i)$ for each i. Observe that for each i, there are bijections between V_i, the connected components of $\widetilde{\Gamma}(\rho_i)$, and the parts of the partition $\mathrm{part}(\rho_i)$. Notice that the connected components of $\widetilde{\Gamma}(\rho_{i+1})$ factor through the connected components of $\widetilde{\Gamma}(\rho_i)$, so the partition of $\{1, \ldots, n\}$ induced by the connected components of $\widetilde{\Gamma}(\rho_{i+1})$ refines that induced by the components of $\widetilde{\Gamma}(\rho_i)$.

To see that the refinement is strict, let v be a vertex of Γ such that $\lambda(v) = \rho_i$. By stability, there must be at least two flags leaving v in the direction of increasing λ. Then there are at least two vertices of V_{i+1} that belong to the component of $\Gamma(\rho_i)$ containing v. It follows that the refinement is strict. $\qquad\square$

Proposition 4.10. *1-layer trees are in bijection with the nondiscrete partitions of* $\{1, \ldots, n\}$.

Proof. Given a nondiscrete partition p of $\{1, \ldots, n\}$, we can construct a 1-layer tree $\Gamma(p)$ as follows. Let $p = \{p_1, \ldots, p_r\}$. Start with a genus 1 vertex v and then for each $1 \le i \le r$ attach a vertex v_i to v such that

 (i) v_i is distance ρ_1 from v, and

 (ii) the elements of p_i are precisely the legs attached to v_i.

After stabilizing, we obtain $\Gamma(p)$.

If Γ is a 1-layer tree, then $\Gamma \mapsto \mathrm{part}(\rho_1)$ gives a map from 1-layer trees to nondiscrete partitions. The maps $p \mapsto \Gamma(p)$ and $\Gamma \mapsto \mathrm{part}(\rho_1)$ are inverses. $\qquad\square$

Definition 4.11. Let Γ be a radially aligned tropical curve with monoid

$$\mathbb{N}e_1 \oplus \cdots \oplus \mathbb{N}e_k \oplus \mathbb{N}\delta_1 \oplus \cdots \oplus \mathbb{N}\delta_l,$$

and let Γ_{e_i} denote the tropical curve corresponding the monoid map $P \to \mathbb{N}$ taking $e_i \mapsto 1$ and $e_j \mapsto 0$ for all $j \ne i$, and $\delta_j \mapsto 0$ for all j. We define Γ_{δ_i} similarly; it is the tropical curve corresponding to the monoid map $P \to \mathbb{N}$ taking $\delta_i \mapsto 1$ and all other generators to 0.

Consider the map $\alpha : \mathfrak{R}_n^{\mathrm{uni}} \to \mathfrak{Q}_n$ defined by

$$(\rho_\Gamma)_\Gamma \mapsto \{\mathrm{part}(\Gamma) : \Gamma \text{ is a 1-layer tree and } \rho_\Gamma > 0\}.$$

Given a collection of partitions Q, we obtain an assignment of radii to radially aligned curves by assigning the radius ρ_r to Γ if r is the largest number such that $\mathrm{part}(\rho_r) \in Q$. This gives a map $\beta : \mathfrak{Q}_n \to \mathfrak{R}_n^{\mathrm{uni}}$.

Proposition 4.12. *The maps* $\alpha : \mathfrak{R}_n^{\mathrm{uni}} \to \mathfrak{Q}_n$ *and* $\beta : \mathfrak{Q}_n \to \mathfrak{R}_n^{\mathrm{uni}}$ *are well defined and are inverses.*

Proof. To show α is well defined, it suffices to show that its image is contained in $\mathfrak{Q}_n$. We do this by contradiction. Suppose that $Q = \alpha((\rho_\Gamma)_\Gamma)$ is not downward closed. We show that $(\rho_\Gamma)_\Gamma$ is not universal. We can find some $P \in Q$ such that a minimal coarsening of P is not in Q. Specifically, there will be a $P = \{p_1, \ldots, p_k\} \in Q$ such that (up to reordering) $P' = \{p_1 \cup p_2, p_3, \ldots, p_k\} \notin Q$, as otherwise Q
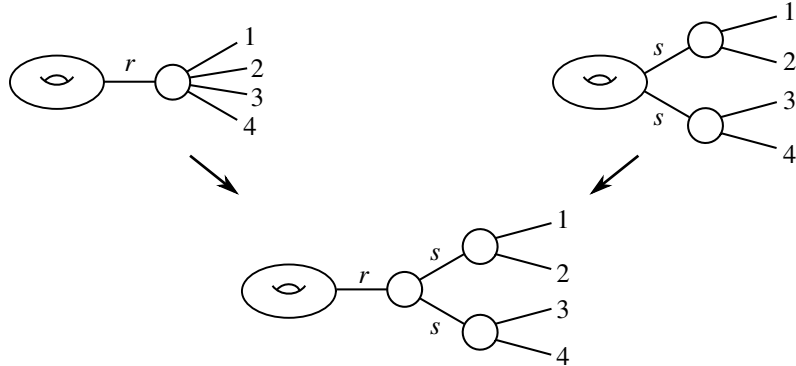
Figure 8. The tropical curves associated to the partition types P, P', and $P' \prec P$, where $P = \{\{1, 2\}, \{3, 4\}\}$ and $P' = \{\{1, 2, 3, 4\}\}$.

will be downward closed. Let the 1-layer trees Γ and Γ' correspond to the partition P and the coarsened partition P', respectively. Say Γ has edge length r and radius r and Γ' has edge length s and radius 0. There is a 2-layer curve, say $\tilde{\Gamma}$, that contracts to both Γ and Γ', and has edge lengths s and r and radius $s+r$ (see Figure 8). If the radius was universal, then we see that by contracting the edge of length r, Γ' must have a radius of s, not 0. Thus the radius is not universal, as claimed.

We now show that β is well defined. First, note that given a basic radially aligned curve Γ, ρ_Γ will be determined by the contractions to Γ_{δ_i} and Γ_{e_i}. To see this, note that the contraction $\Gamma_{\delta_i} \to \Gamma$ will send ρ_j to 0 for all j, and the contraction $\Gamma_{e_i} \to \Gamma$ will send ρ_j to e_i if $j \geq i$ and 0 if $j < i$. As these maps arise from projections from a product, ρ_Γ is uniquely determined by these contractions. Now pick $Q \in \mathfrak{Q}_n$. For any radially aligned Γ, Lemma 4.9 and the fact that ρ_Γ is determined by the contractions $\Gamma_{e_i} \to \Gamma$ imply that ρ_Γ is actually a distance to the core. To see that this is universal, we need only check that single edge contractions are compatible, i.e., if Γ has edge lengths $\{e_1, \ldots, e_n\}$, then the contraction $\Gamma' \to \Gamma$ sending e_j to 0, where Γ' has edge lengths $\{e_1, \ldots, \hat{e}_j, \ldots, e_n\}$, is compatible. This compatibility follows immediately from contracting both Γ and Γ' to each of the Γ_{e_i}'s.

Finally, note that α is injective because the assignment of a radius to a radially aligned curve Γ is uniquely determined by 1-layer trees. Furthermore, the discussion at the start of the previous paragraph shows that if Q is the collection of partitions corresponding to the 1-layer trees with nonzero radii in a universal radius, then ρ_r is the radius determined by contractions to 1-layer trees. This shows $\beta = \alpha^{-1}$. $\square$

Theorem 4.13. *For each $Q \in \mathfrak{Q}_n$, there is a diagram of stacks*

$$
\begin{array}{ccc}
 & \overline{\mathcal{M}}^{\mathrm{rad}}_{1,n} & \\
 & \swarrow \qquad \searrow & \\
\overline{\mathcal{M}}_{1,n} & & \overline{\mathcal{M}}_{1,n}(Q)
\end{array}
$$

such that both arrows are proper and restrict to an isomorphism on $\mathcal{M}_{1,n}$.

Proof. Theorem 3.18 gives us the arrow on the left; we only have to show that the arrow on the right has the claimed properties.

Let $(\rho_\Gamma) = \beta(Q)$ be the universal radius associated to Q. Note that for all Γ, ρ_Γ is a radius of Γ and part$(\rho_\Gamma) \in Q$, by construction. By Lemma 4.5, the ρ_Γ's define a global section ρ of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Let $\pi : C \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ be the universal curve. Theorem 4.1 constructs a Q-stable family of curves $\bar{\pi} : \bar{C} \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ associated to ρ, inducing the map $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}(Q)$. This map restricts to an isomorphism on $\mathcal{M}_{1,n}$ as the maps $C \leftarrow \widetilde{C} \to \bar{C}$ of Theorem 4.1 are isomorphisms where ρ restricts to 0. $\square$

5. Construction of the Q-stable moduli spaces

Theorem 5.1. *For each $Q \in \mathfrak{Q}_n$, $\overline{\mathcal{M}}_{1,n}(Q)$ is a proper irreducible Deligne–Mumford stack over $\mathbb{Z}\left[\frac{1}{6}\right]$ containing $\mathcal{M}_{1,n}$.*

Our argument is brief since we may reuse much of the proof for the analogous result for m-stable spaces in [Smyth 2011a, Theorem 3.8]. In particular, to show that we have a Deligne–Mumford stack, it is enough to show that the moduli functor $\overline{\mathcal{M}}_{1,n}(Q)$ is deformation-open, bounded, and satisfies the valuative criterion for properness. Boundedness is immediate from [Smyth 2011a, Lemma 3.9], since every Q-stable curve is m-stable for some $m < n$. It is clear that $\overline{\mathcal{M}}_{1,n}(Q)$ contains $\mathcal{M}_{1,n}$. The resulting stack is therefore irreducible, since every Q-stable curve is m-stable for some m, and all m-stable curves are limits of curves in $\mathcal{M}_{1,n}$.

Theorem 5.2 (deformation-openness). *Let S be a noetherian scheme and let $\pi : C \to S$ be a flat, projective morphism with one-dimensional fibers and let $\sigma_1, \ldots, \sigma_n$ be n sections. Then the set*

$$T = \left\{ s \in S \mid (\pi|_{\bar{s}} : C|_{\bar{s}} \to \bar{s}, \{\sigma_i(\bar{s})\}_{i=1}^n) \text{ is } Q\text{-stable} \right\}$$

is open.

Proof. As in [Smyth 2011a, Lemma 3.10], we may assume that the geometric fibers $C_{\bar{s}}$ of π are reduced, connected, of arithmetic genus one, with only Gorenstein singularities, and that $H^0(C_s, \Omega_C^\vee(-\Sigma)) = 0$ since these are open conditions.

It remains to show that the locus in S over which the level conditions hold is open. Since S is Noetherian, we may establish openness by showing that this locus is constructible and stable under generization. It is constructible since satisfaction of the level conditions is constant on combinatorial types (defined slightly ahead in Definition 6.1) and the curves with a given combinatorial type form a locally closed subset of S.

So assume S is the spectrum of a DVR with closed point $0 \in S$ and generic point $\eta \in S$. We must show that if $(C_{\bar{0}}, \sigma_1(\bar{0}), \ldots, \sigma_n(\bar{0}))$ satisfies the level conditions, then so does $(C_{\bar{\eta}}, \sigma_1(\bar{\eta}), \ldots, \sigma_n(\bar{\eta}))$. Replacing S by a finite base change if necessary, we may assume that the irreducible components of $C_{\bar{\eta}}$ are in bijection with the irreducible components of C.

Since level increases with the size of a subcurve, it is enough to check the subcurve level condition on minimal genus one subcurves. Let $E_{\bar{\eta}}$ be a minimal genus one subcurve of $C_{\bar{\eta}}$. Then the limit Z of $E_{\bar{\eta}}$ in $C_{\bar{0}}$ contains the minimal genus one subcurve E_0 of $C_{\bar{0}}$. Because $\mathrm{lev}(E_{\bar{0}}) \notin Q$ by hypothesis,

$\mathrm{lev}(E_{\bar{0}}) \preceq \mathrm{lev}(Z)$, and Q is downward closed, we have $\mathrm{lev}(Z) \notin Q$. Finally, taking limits of the components of $(C_{\bar{\eta}} - E_{\bar{\eta}}) \cup \Sigma$, we see that $\mathrm{lev}(E_{\bar{\eta}}) = \mathrm{lev}(Z)$, so $\mathrm{lev}(E_{\bar{\eta}}) \notin Q$ as required.

The level condition on singularities holds trivially if $C_{\bar{\eta}}$ has only nodes, so suppose that $C_{\bar{\eta}}$ has an elliptic l-fold singularity $q_{\bar{\eta}}$. Since elliptic l_1-fold singularities generize only to l_2-fold singularities with $l_2 \leq l_1$ or nodes, and nodes only generize to nodes and smooth points, the limit of $q_{\bar{\eta}}$ in $C_{\bar{0}}$ must be an elliptic m-fold singularity $q_{\bar{0}}$ with $m \geq l$. For each $i = 1, \ldots, l$ set $Z_{\bar{\eta}}^i$ to be the union of the i-th rational branch $B_{\bar{\eta}}^i$ of $q_{\bar{\eta}}$ with all rational tails attached to this branch. Next, set Z^i and B^i to be the closures of $Z_{\bar{\eta}}^i$ and $B_{\bar{\eta}}^i$ in C and write Z_{lim}^i and B_{lim}^i for their restrictions to $C_{\bar{0}}$. Similarly, let $B_{\bar{0}}^j$ and $Z_{\bar{0}}^j$ be respectively the j-th branch of $q_{\bar{0}}$ and the union of $B_{\bar{0}}^j$ with its rational tails for $j = 1, \ldots, m$. Observe that the partition of $\{1, \ldots, n\}$ induced by the markings in the Z_{lim}^i agrees with the level of $q_{\bar{\eta}}$, and the partition of $\{1, \ldots, n\}$ induced by the $Z_{\bar{0}}^j$ agrees with the level of $q_{\bar{0}}$.

We claim that each $Z_{\bar{0}}^j$ factors through exactly one of the Z_{lim}^i. Note that the limit of at most one branch $B_{\bar{\eta}}^i$ of $q_{\bar{\eta}}$ can contain $B_{\bar{0}}^j$ since $B_{\bar{0}}^j$ is irreducible and each B^i is an irreducible component of C. Observe that each irreducible component of $C_{\bar{0}}$ is connected via a nodal path to exactly one of the branches of $q_{\bar{0}}$. The remaining components of $Z_{\bar{0}}^j$ are connected to the $B_{\bar{0}}^j$ via a nodal path, so we conclude that these belong to Z_{lim}^i as well. The claim follows.

Therefore the level of $q_{\bar{\eta}}$ is a coarsening of the level of $q_{\bar{0}}$, that is, $\mathrm{lev}(q_{\bar{\eta}}) \prec \mathrm{lev}(q_{\bar{0}})$. Since $\mathrm{lev}(q_{\bar{0}}) \in Q$ and Q is downward closed, $q_{\bar{\eta}} \in Q$ too, and we are done. $\square$

Theorem 5.3. *The stack $\overline{\mathcal{M}}_{1,n}(Q)$ is universally closed.*

Proof. Since $\mathcal{M}_{1,n}$ is dense in $\overline{\mathcal{M}}_{1,n}(Q)$, it is enough to show that limits of families of smooth n-pointed curves admit Q-stable limits. Suppose S is the spectrum of a discrete valuation ring with generic point η and $\pi_\eta : C_\eta \to \eta$ is a smooth and proper family of n-pointed curves. Since $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is proper, after replacing S by a finite cover if necessary, we may find a limit basic radially aligned log curve $\pi^{\mathrm{rad}} : C^{\mathrm{rad}} \to S$ extending $C_\eta \to \eta$. The image of $C^{\mathrm{rad}} \to S$ under the map of Theorem 4.13 corresponding to Q gives the required limit Q-stable curve. $\square$

We now recall the definition of balanced subcurve [Smyth 2011a, Definition 2.11].

Definition 5.4. Given a connected nodal curve E and connected subcurves F_1 and F_2, we say that the *nodal distance $l(F_1, F_2)$ from F_1 to F_2* is the least number of edges in a path from F_1 to F_2 in the dual graph of E. If $p \in E$ is a smooth point, then there is a unique irreducible component F of E containing p, and we write $l(-, p)$ instead of $l(-, F)$.

Definition 5.5. If $(E, \{p_i\}_{i=1}^m)$ is a semistable curve of arithmetic genus one, we say that $(E, \{p_i\}_{i=1}^m)$ is *balanced* if

$$l(Z, p_1) = l(Z, p_2) = \cdots = l(Z, p_m),$$

where $Z \subseteq E$ is the minimal elliptic subcurve of E.

We remark that the nodal distance is an integer, while the distances $\lambda(v)$ defined for genus one log curves take values in the characteristic monoid of the base. The two notions are related by the fact that if

(i) A is the spectrum of a DVR with uniformizer t,

(ii) $S = \operatorname{Spec} A$ is given the log structure associated to the chart $\mathbb{N}\delta \to A$ sending $\delta \mapsto t$,

(iii) $\pi : C \to S$ is a log curve of genus one with smooth generic fiber,

(iv) C has regular total space, and

(v) $Z \subseteq C$ is the minimal genus one subcurve,

then all smoothing parameters of the nodes of C_0 are equal to δ and $\lambda(F) = l(Z, F)\delta$ for all irreducible components F of C.

Theorem 5.6. *The stack $\overline{\mathcal{M}}_{1,n}(Q)$ is separated.*

Proof. We must show that given a pair of Q-stable families $\pi : C \to S$ and $\pi' : C' \to S$ over the spectrum of a discrete valuation ring with generic point η and special point x, an isomorphism $\psi : C|_\eta \to C'|_\eta$ of pointed curves extends to all of S. As in [Smyth 2011a, 3.3.2], we may assume that there is a flat and proper pointed semistable nodal curve $C^{\mathrm{ss}} \to S$ with regular total space and a diagram of pointed S-schemes

$$C \xleftarrow{\varphi} C^{\mathrm{ss}} \xrightarrow{\varphi'} C',$$

where φ and φ' are proper birational morphisms, and it will be enough to show that the exceptional loci of φ and φ' coincide.

If $C|_{\bar{x}}$ or $C'|_{\bar{x}}$ is nodal, then we may argue exactly as in [Smyth 2011a, 3.3.2] to conclude. Therefore we may assume that $C|_{\bar{x}}$ and $C'|_{\bar{x}}$ possess elliptic Gorenstein singularities p and p' respectively. Let $E = \varphi^{-1}(p)$ and $E' = \varphi^{-1}(p')$. As in [Smyth 2011a], we know that E and E' are balanced, with E (resp. E') consisting of all components of $C^{\mathrm{ss}}|_{\bar{x}}$ with nodal distance to the core of $C^{\mathrm{ss}}|_{\bar{x}}$ less than l (resp. less than l'). Without loss of generality we may assume $E \subseteq E'$. If the containment is proper, then $\varphi(E') \subseteq C|_{\bar{x}}$ is a subcurve of genus one containing p and all of its branches. Examining the dual graphs of the various curves over $\bar{x}$, we have $\operatorname{lev}(p') = \operatorname{lev}(\varphi(E'))$. Then, since C is Q-stable, $\operatorname{lev}(\varphi(E')) \notin Q$. On the other hand, $\operatorname{lev}(p') \in Q$, since C' is Q-stable. This is a contradiction, so we have $E = E'$.

The remainder of the argument follows as in [Smyth 2011a, 3.3.2]. $\qquad\square$

6. Classification of semistable Gorenstein modular compactifications of $\mathcal{M}_{1,n}$

Our goal in this section is to prove Theorem 1.9, classifying the modular compactifications of $\mathcal{M}_{1,n}$. To aid our classification, we introduce the notion of the combinatorial type of a curve in $\mathcal{U}_{1,n}$. This is analogous to the dual graph of a nodal curve, with the difference that the combinatorial type also keeps track of elliptic m-fold singularities.

Definition 6.1. Let C be a connected, proper, reduced, 1-dimensional scheme over an algebraically closed field k with (at worst) nodes and elliptic Gorenstein singularities. The *combinatorial type of C* consists of the following data:

(i) A set V of *vertices*, equal to the set of components of C.

(ii) A set E of *singularities*, equal to the set of singular points of C.

(iii) A *genus function* $g : V \cup E \to \mathbb{N}$ taking a component of C to the genus of its normalization and taking each singularity of C to its genus as a singularity.

(iv) An *incidence function* $i : V \times E \to \{0, 1\}$ taking $(v, e) \mapsto 1$ if $e \in v$ and to 0 otherwise.

(v) A *marking function* $x : V \to 2^{\{1,\ldots,n\}}$ taking a component v to the set of indices of the markings incident to v.

Two combinatorial types $\Gamma_1 = (V_1, E_1, g_1, i_1, x_1)$ and $\Gamma_2 = (V_2, E_2, g_2, i_2, x_2)$ are *isomorphic* if there is a bijection $f : V_1 \cup E_1 \to V_2 \cup E_2$ such that

(i) $f(V_1) \subseteq V_2$ and $f(E_1) \subseteq E_2$;

(ii) $x_1 = x_2 \circ f$;

(iii) $g_1 = g_2 \circ f$;

(iv) $i_1(v, e) = i_2(f(v), f(e))$ for all $(v, e) \in V_1 \times E_1$.

Let $\mathcal{Z}_\Gamma$ be the locus in $\mathcal{U}_{1,n}$ of curves with combinatorial type Γ. These loci have a natural structure of locally closed substack of $\mathcal{U}_{1,n}$. (This follows from the fact that the deformations of a curve preserving its singularities form a closed subspace of the full deformation space of the curve. See [Smyth 2011b, Lemma 2.1], for example.) Note that for each n, there is a finite set of isomorphism classes of combinatorial types of curves in $\mathcal{U}_{1,n}$ and altogether the $\mathcal{Z}_\Gamma$'s form a stratification of $\mathcal{U}_{1,n}$ into locally closed substacks.

Let $\mathcal{M}$ be a modular compactification in our sense. Since $\mathcal{M}$ is assumed to be an open substack of $\mathcal{U}_{1,n}$, it is uniquely determined by its points. Our strategy is to show that $\mathcal{M}$ must be a union of the $\mathcal{Z}_\Gamma$'s. Then, analyzing the possible limits of curves, we will find that the choices of combinatorial types making up $\mathcal{M}$ necessarily agree with a Q-stability condition.

Lemma 6.2. *$\mathcal{U}_{1,n}$ is the union of the $\overline{\mathcal{M}}_{1,n}(m)$'s.*

Proof. Suppose that $(\pi : C \to S, \sigma_1, \ldots, \sigma_n)$ is a family in $\mathcal{U}_{1,n}$. We want to show that S possesses an open cover such that the restriction of C to each part of the cover factors through some $\overline{\mathcal{M}}_{1,n}(m)$. Let $\bar{s}$ be a geometric point of S. Then the fiber $C_{\bar{s}}$ of π over $\bar{s}$ is a Gorenstein curve of genus one with n distinct marked points and no infinitesimal automorphisms. Recall that the only Gorenstein singularities of genus less than or equal to one are the elliptic Gorenstein singularities and the node. If $C_{\bar{s}}$ has an elliptic m-fold singularity for some m, then since C has no infinitesimal automorphisms, the number of markings and nodes on the minimal genus one subcurve must be at least $m + 1$. It follows that $C_{\bar{s}} \in \overline{\mathcal{M}}_{1,n}(m)$. If $C_{\bar{s}}$ does not have an elliptic Gorenstein singularity, then $C_{\bar{s}} \in \overline{\mathcal{M}}_{1,n} = \overline{\mathcal{M}}_{1,n}(0)$. Since the $\overline{\mathcal{M}}_{1,n}(m)$'s are deformation-open [Smyth 2011a], for each $\bar{s}$ there is an open neighborhood U_s of the image of $\bar{s}$ in S such that $C|_{U_s}$ factors through one of the stacks $\overline{\mathcal{M}}_{1,n}(m)$. $\square$

Lemma 6.3. *Let Γ be a combinatorial type. Then $\mathcal{Z}_\Gamma$ is irreducible.*

Proof. If Γ possesses no m-fold points, then $\mathcal{Z}_\Gamma$ is a product of copies of $\mathcal{M}_{g,n}$'s coming from the vertices of Γ. Since each of the stacks $\mathcal{M}_{g,n}$ is geometrically irreducible, so is $\mathcal{Z}_\Gamma$.

Next, let Γ be a combinatorial type consisting of a single elliptic m-fold point with k rational branches $E_1, \ldots, E_k$ with $n_1, \ldots, n_k$ markings, respectively. Let $n = n_1 + \cdots + n_k$.

Let

$$A = \left\{ (f_i(t_i))_{i=1}^m \in \prod_{i=1}^m \mathbb{Z}\left[\tfrac{1}{6}, t_i\right] \; \middle| \; f_i(0) = f_j(0) \text{ for all } i \text{ and } \sum_{i=1}^m f_i'(0) = 0 \right\}.$$

This gives a standard affine model of the m-fold point with rational branches. Form a proper curve $D \to \operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{6}\right]$ by gluing $\operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{6}, t_i^{-1}\right]$ to the i-th branch of $\operatorname{Spec} A$ for each i. If C is a minimal unmarked Gorenstein curve of genus one over an algebraically closed field with an m-fold point, then C appears as a geometric fiber of D.

Now let

$$S = \prod_{i=1}^m \left(\operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{6}, s_{i,1}, \ldots, s_{i,n_i}\right] - \Delta_{n_i}\right),$$

where Δ_{n_i} is the locus where any pair of coordinates of $\mathbb{A}^{n_i}_{\mathbb{Z}[1/6]} = \operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{6}, s_{i,1}, \ldots, s_{i,n_i}\right]$ coincide. We construct a family of pointed curves $D \times S \to S$ by taking the j-th marking on the i-th branch of D to be located at $t_i^{-1} = s_{i,j}$.

Now, every pointed curve of type Γ appears as some geometric fiber of the family $D \times S \to S$. Therefore, the image of S in $\mathcal{U}_{1,n}$ under the map induced by $D \times S \to S$ is precisely $\mathcal{Z}_\Gamma$. Since S is irreducible, the result follows.

Finally, consider a general Γ. Let $\Gamma_{\min}$ be the combinatorial type of the minimal genus one subcombinatorial type of Γ with markings at the outgoing edges. Clearly, $\mathcal{Z}_\Gamma$ is a product of $\mathcal{M}_{0,n}$'s and $\mathcal{Z}_{\Gamma_{\min}}$, all of which are already known to be geometrically irreducible, so $\mathcal{Z}_\Gamma$ is irreducible too. $\qquad\square$

The following lemma is the crucial one: it reduces the classification of Gorenstein compactifications to combinatorics.

Lemma 6.4. $\mathcal{M}$ *is a union of the* $\mathcal{Z}_\Gamma$*'s.*

Proof. It suffices to show that if $\mathcal{M}$ shares a geometric point with $\mathcal{Z}_\Gamma$ for some Γ, then $\mathcal{M}$ contains all points of $\mathcal{Z}_\Gamma$. By the previous lemma, for any pair of geometric points $C_p \in \mathcal{Z}_\Gamma(\operatorname{Spec} k(p))$, $C_q \in \mathcal{Z}_\Gamma(\operatorname{Spec} k(q))$, there are families of curves $C_S \in \mathcal{Z}_\Gamma(S)$, $C_T \in \mathcal{Z}_\Gamma(T)$, where S and T are spectra of discrete valuation rings, such that

(i) S and T have isomorphic geometric generic points,

(ii) C_S is isomorphic to C_T over this common geometric generic point,

(iii) there is a map $\operatorname{Spec} k(p) \to S$ onto the special point of S along which C_S pulls back to C_p,

(iv) there is a map $\operatorname{Spec} k(q) \to T$ onto the special point of T along which C_T pulls back to C_q.

We know $\mathcal{M}$ is closed under generization and has a specialization for any 1-dimensional family. It follows that if $\mathcal{M}$ contains C_p, then $\mathcal{M}(S)$ contains C_S. To conclude, we show that $\mathcal{M}(T)$ contains C_T too.

Let η be the generic point of T, and replacing T by a finite base change if necessary, let $C_T' \in \mathcal{M}(T)$ be the unique limit of $C_T|_\eta$ in $\mathcal{M}$. We have an isomorphism of η-schemes $C_T'|_\eta \cong C_T|_\eta$. Since $\mathcal{U}_{1,n}$ is a union of the open substacks $\overline{\mathcal{M}}_{1,n}(m)$, there is some m such that C_T' lives in $\overline{\mathcal{M}}_{1,n}(m)(T)$. Considering $C_T'|_{\bar{\eta}}$, we conclude that curves of combinatorial type Γ are m-stable. In particular, both C_T and C_T' are families in $\overline{\mathcal{M}}_{1,n}(m)$. Since $\overline{\mathcal{M}}_{1,n}(m)$ is separated, we conclude that $C_T' \cong C_T$ over T, completing the proof. $\square$

Our strategy now is to produce families of curves witnessing enough of the relationships between the loci $\mathcal{Z}_\Gamma$ that $\mathcal{M}$ is forced to be Q-stable.

Definition 6.5. Let

$$\mathcal{P} : P_1 \prec P_2 \prec \cdots \prec P_k$$

be a strictly increasing chain of partitions of $\{1, \ldots, n\}$ not including the partition $\{\{1\}, \ldots, \{n\}\}$. We say that a family of radially aligned curves $\pi : C \to S$ is a *test curve of type $\mathcal{P}$ centered at a geometric point s of S* if

(i) $(\pi : C \to S, s)$ satisfies the conclusions of Theorem 3.10;

(ii) the tropicalization of the central fiber $\mathrm{trop}(C|_s)$ has a basic radially aligned log structure;

(iii) the log structure on S is divisorial, that is, it is the log structure associated to a normal crossings divisor [Kato 1989, (1.5)];

(iv) the tropicalization of the central fiber has partition type $\mathcal{P}$ (Definition 3.20).

Lemma 6.6. *For any strictly increasing chain of partitions of $\{1, \ldots, n\}$*

$$\mathcal{P} : P_1 \prec P_2 \prec \cdots \prec P_k$$

not including the partition $\{\{1\}, \ldots, \{n\}\}$, there is a test curve of type $\mathcal{P}$.

Proof. Choose an algebraically closed field κ. Pick a smooth genus one curve E over κ arbitrarily. Add a rational component $Z_A^{(1)}$ for each part A of P_1 and attach them nodally to E at distinct smooth points. Repeat this process for each $i = 2, \ldots, k$, adding components $Z_A^{(i)}$ for each part $A \in P_i$, where $Z_A^{(i)}$ is nodally attached to the unique component $Z_B^{(i-1)}$ where $B \in P_{i-1}$ is the part containing A. Finally, mark points $p_1, \ldots, p_n$, where each p_i is a smooth point of the component $Z_A^{(k)}$, where A is the part of P_k containing i. Call the whole pointed nodal curve we have constructed C_0. Give $S_0 = \mathrm{Spec}\,\kappa$ the log structure associated to the map $\bigoplus_{i=1}^k \mathbb{N}e_i \to \kappa$ sending everything to 0 except for the identity element. Choose a log structure on $\pi_0 : C_0 \to S_0$ compatible with the log structure on S_0 making $\pi_0 : C_0 \to S_0$ into a log curve so that

(i) the edges between the $Z_A^{(1)}$'s and E are all of length e_1;

(ii) for $i = 2, \ldots, k$, the edges between the $Z_A^{(i)}$'s and $Z_A^{(i-1)}$'s are all of length e_i.

Stabilize to obtain a log curve $\pi_0^s : C_0^s \to S_0$. Since π_0^s is radially aligned with the basic radially aligned log structure, there is an associated map $f : S_0 \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Choose a factorization of it through an étale chart $U \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Now, set S to be a neighborhood of the image s of S_0 in U using Theorem 3.10.

We now claim that the pullback of the universal curve of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to S gives the desired test curve. Properties (i) and (ii) hold by construction. Property (iii) holds since S was chosen to be an étale neighborhood of a point of the chart U, which has a divisorial log structure.

It remains to check property (iv). If i is an integer with $1 \le i < k$, there is at least one $Z_A^{(i)}$ with at least two descendants, since $P_i \ne P_{i+1}$. If $i = k$, all of the $Z_A^{(i)}$'s possess at least three marked points. Either way, for each integer $i = 1, \ldots, k$, there is a component $Z_A^{(i)}$ that survives the stabilization step, and it lives at radius $\lambda(Z_A^{(i)}) = e_1 + \cdots + e_i$ by construction. Conversely, every radius is of this form. Now, if we subdivide $\mathrm{trop}(C|_s)$ where $\lambda = e_1 + \cdots + e_i$, the effect is to reintroduce the components $Z_A^{(i)}$ for $A \in P_i$. Deleting the locus where $\lambda < e_1 + \cdots + e_i$ leaves us with the trees rooted at the $Z_A^{(i)}$'s as A varies through the elements of P_i. By construction, for each $A \in P_i$ the tree rooted at $Z_A^{(i)}$ contains the markings indexed by A. Therefore, the partition type of $\mathrm{trop}(C|_s)$ is precisely $\mathcal{P}$. $\square$

With test curves in hand, we may form several families of contracted curves. Exactly one of these will be Q-stable for any Q.

Lemma 6.7. *Let $\pi : C \to S$ be a test curve of type $\mathcal{P}$ centered at s. Let $\rho_0 = 0 < \rho_1 < \cdots < \rho_k$ be the distinct radii of the tropicalization of the central fiber. For each $i = 0, \ldots, k$, let $\overline{C}_i \to S$ be the contraction of $C \to S$ associated to the steep mesa with radius ρ_i and let Γ_i be the combinatorial type of the fiber of $\overline{C}_i$ over s.*

Let $Q \in \mathfrak{Q}_{1,n}$ be a downward closed set of partitions. Then there is exactly one index i such that Γ_i is Q-stable, namely the greatest index i for which $P_i \in Q$.

Proof. Observe that

(i) $\overline{C}_0|_s$ is nodal and its minimal elliptic subcurve has level P_1;

(ii) for $0 < i < k$, $\overline{C}_i|_s$ possesses an elliptic singularity of level P_i and its minimal elliptic subcurve has level P_{i+1};

(iii) for $i = k$, $\overline{C}_i|_s$ possesses an elliptic singularity of level P_k and its minimal elliptic subcurve (namely $\overline{C}_k|_s$) has level $\{\{1\}, \ldots, \{n\}\}$.

The result follows immediately. $\square$

With some more work, we see that exactly one of the contracted curves belongs to our arbitrary modular compactification $\mathcal{M}$ as well.

Lemma 6.8. *Choose notation as in Lemma 6.7. Then $\mathcal{M}$ contains exactly one of the families $\overline{C}_i \to S$ and exactly one of the loci $\mathcal{Z}_{\Gamma_i}$.*

Proof. Choose t to be a point of S generizing s over which C is smooth. Let $T \to S$ be a map from the spectrum of a DVR with special point x mapping to s and generic point η mapping to t.

Replacing T by a finite base change if necessary, find the limit curve $C_\mathcal{M} \to T$ in $\mathcal{M}$ of the smooth curve $C|_\eta$. After a second base change if necessary, pick a regular family of semistable curves $C^{\mathrm{ss}} \to T$ dominating $C_\mathcal{M} \to T$ and each of the families $\overline{C}_i|_T \to T$, formed by subdividing and contracting the i-th radius of C. Note that $\overline{C}_0 = C|_T$. Let E be the exceptional locus of $\varphi : C^{\mathrm{ss}} \to C_\mathcal{M}$ and let E_i be the exceptional locus of $\varphi_i : C^{\mathrm{ss}} \to \overline{C}_i|_T$ for each i.

If $C_\mathcal{M}$ is stable then both $C|_T$ and $C_\mathcal{M}$ are stable limits of $C|_\eta$, so they must agree.

Otherwise, $C_\mathcal{M}$ possesses a unique elliptic m-fold point p. By [Smyth 2011a, Proposition 2.12], $\varphi^{-1}(p)$ is a balanced subcurve of $C^{\mathrm{ss}}|_x$, with $\varphi^{-1}(p)$ consisting of all components of $C^{\mathrm{ss}}|_x$ whose nodal distance from the core of $C^{\mathrm{ss}}|_x$ is less than some integer l. Since $C_\mathcal{M}$ possesses no infinitesimal automorphisms, at least one of the components of $C^{\mathrm{ss}}|_x$ of distance exactly l from the core of $C^{\mathrm{ss}}|_x$ has at least three special points. Then E is the union of $\varphi^{-1}(p)$ with the semistable components of $C^{\mathrm{ss}}|_x$ disjoint from $\varphi^{-1}(p)$.

We may repeat this argument for each of the exceptional loci E_i for $i > 0$. Therefore, for each i,

  (i) there is a balanced subcurve F_i of $C^{\mathrm{ss}}|_x$ consisting of all components of nodal distance from the core of $C^{\mathrm{ss}}|_x$ less than some integer l_i,

 (ii) there is a component of $C^{\mathrm{ss}}|_x$ of distance exactly l_i from the core with at least three markings, and

(iii) E_i is the union of F_i with the semistable components of $C^{\mathrm{ss}}|_x$ disjoint from F_i.

Moreover, we claim that the loci E_i exhaust the subcurves of $C^{\mathrm{ss}}|_x$ with these properties. To see this, suppose F' is a balanced subcurve of $C^{\mathrm{ss}}|_x$ whose components are those with nodal distance less than l', and so that there is a stable component G of $C^{\mathrm{ss}}|_x$ of distance l' from the core. Then $\varphi(G)$ maps to a stable component of $C|_x$. The component G lives at the distance ρ_i from the core of $C|_x$ for some i. Then, for this same i, we have that $l' = l_i$ and the rest follows.

Therefore $E = E_i$ for some $i > 0$. Since both $C_\mathcal{M}$ and $\overline{C}_i|_T$ are normal and obtained from contracting the same locus of C^{ss}, they are isomorphic.

This exhibits a curve, namely $\overline{C}_i|_x$, in $\mathcal{Z}_{\Gamma_i}$ in $\mathcal{M}$. By Lemma 6.4, $\mathcal{M}$ contains the whole locus. By separatedness, $\mathcal{M}$ cannot contain any of the other $\overline{C}_j|_T$ for $j \neq i$. Therefore $\mathcal{M}$ cannot contain $\mathcal{Z}_{\Gamma_j}$ for any $j \neq i$.

Finally, we wish to show that in fact $\mathcal{M}$ contains the entire family $\overline{C}_i \to S$. This will follow from the closure of $\mathcal{M}$ under generization. Write $\bigoplus_i e_i \oplus \bigoplus_j \delta_j$ for $\overline{M}_{S,s} \cong \Gamma(S, \overline{M}_S)$ so that the e_i's are the differences between consecutive radii, as in Definition 3.17. The base S of $C \to S$ possesses a stratification by locally closed subsets $\{W_I\}$ indexed by subsets $I \subseteq \{1, \dots, k\}$, where

$$W_I = \bigcap_{j \in I} \mathrm{Supp}(e_j) \cap \bigcap_{j \in I^c} (S - \mathrm{Supp}(e_j)).$$

Since $C \to S$ satisfies the conclusions of Theorem 3.10, the tropicalization of the fibers of C is constant on the subsets W_I. It follows that the same holds of the combinatorial types of the fibers of $\overline{C}_i \to S$. Since the log structure of S is divisorial, for each $I \subseteq \{1, \dots, k\}$, there is a generization $\eta_I \to s$, where $\eta_I \in W_I$. Since $\mathcal{M}$ is closed under generization and $\mathcal{M}$ contains $\overline{C}_i|_s$, $\mathcal{M}$ must also contain $\overline{C}_i|_{\eta_I}$. Then, since

membership in $\mathcal{M}$ is determined by combinatorial type, $\mathcal{M}$ contains all of $\bar{C}_i|_{W_I}$. We conclude that the whole family $\bar{C}_i \to S$ belongs to $\mathcal{M}$. $\qquad\square$

Remark 6.9. At first glance, the conclusion of the lemma looks different for test curves whose chains of partitions agree but whose cores have different combinatorial types, since their combinatorial types Γ_0 will be distinct. Such pairs of test curves share combinatorial types Γ_i for $i \geq 1$, so, comparing the conclusion of the lemma for the various test curves, either $\mathcal{M}$ chooses to contract the core of all test curves of type $\mathcal{P}$ or it contracts the core of none of them.

When the chain $\mathcal{P}$ consists of a single partition P, Lemma 6.8 tells us there are only two "choices" that $\mathcal{M}$ could make: either $\mathcal{M}$ contains the contracted curve $\bar{C}_1$ or $\mathcal{M}$ contains the uncontracted curve C. In the former case, say that $\mathcal{M}$ *contracts* P. Our next claim is that for any test curve C, the combinatorial type of the contraction of C included by $\mathcal{M}$ is determined solely by the partitions which $\mathcal{M}$ contracts.

Lemma 6.10. *Choose notation as in Lemma 6.8. Let i be the index of the family $\bar{C}_i \to S$ contained in $\mathcal{M}$. Let j be any integer with $1 \leq j \leq k$. Then $j \leq i$ if and only if $\mathcal{M}$ contracts P_j.*

Proof. Write $\bigoplus_i e_i \oplus \bigoplus_j \delta_j$ for $\bar{M}_{S,s} \cong \Gamma(S, \bar{M}_S)$ so that the e_i's are the differences between consecutive radii of $\mathrm{trop}(C)$, as in Definition 3.17. Let $C' \to S'$ be the restriction of $C \to S$ to the complement of the support of $e_1, \dots, \hat{e}_j, \dots, e_k$. Choose s' to be a point of the support of e_j inside S'. It is not difficult to check that the resulting family is a test family of type P_j (i.e., the chain of partitions of length one containing just the partition P_j) whose log structure is generated by e_j. The unique nonzero radius of this test family is e_j. Since the contraction of a family of mesa curves commutes with base change, $\bar{C}_i|_{S'}$ will be the contraction of C' with respect to $\rho|_{S'}$. Now, the restriction of ρ_i from S to S' is either e_j if $j \leq i$ or 0 if $i < j$. In the first case, $\mathcal{M}$ contracts P_j. In the latter $\mathcal{M}$ does not. $\qquad\square$

Lemma 6.11. *Every combinatorial type Γ in $\mathcal{U}_{1,n}$ is the combinatorial type of some contraction of a test curve.*

Proof. Choose a representative curve C_0 for Γ. Find a smoothing $C \to T$ of C_0 over the spectrum of a discrete valuation ring with special point x and generic point η. Replacing T by a finite base change if necessary, let $C^{\mathrm{rad}} \to T$ be the limit radially aligned curve of $C|_\eta \to \eta$ in $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ with the basic radially aligned log structure.

Arguing as in Lemma 6.6, we can find a family $C^{\mathrm{test}} \to S$ centered at s with central fiber $C^{\mathrm{rad}}|_x$ by choosing a sufficiently small étale neighborhood of $C^{\mathrm{rad}}|_x$ in an étale chart for $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. This neighborhood may be assumed to satisfy the conditions of Theorem 3.10 and its log structure will be divisorial since it is pulled back from the log structure of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ along an étale morphism. This family $C^{\mathrm{test}} \to S$ is therefore a test curve of some type, with the type given by the partitions induced by the various radii of the tropicalization of its central fiber.

Moreover, replacing T by a finite base change if necessary, we may assume that $C^{\mathrm{rad}} \to T$ is pulled back from $C^{\mathrm{test}} \to S$. We are now in exactly the situation of the proof of Lemma 6.8, so we conclude that C_0 is one of the contractions of the central fiber of C^{test}, as required. $\qquad\square$

Now that everything is in place, our classification result follows easily.

Proof of Theorem 1.9. Let $Q_\mathcal{M}$ be the set of nondiscrete partitions P of $\{1, \ldots, n\}$ such that $\mathcal{M}$ contracts P. By Lemma 6.10, $Q_\mathcal{M}$ is downward closed.

By Lemmas 6.7 and 6.10, the combinatorial types that appear as contractions of test curves contained in $\mathcal{M}$ and $\overline{\mathcal{M}}_{1,n}(Q_\mathcal{M})$ are the same. On the other hand, by Lemma 6.11, every combinatorial type appears as Γ_i for some test curve C^{test}. It follows that $\mathcal{M}$ contains precisely the $Q_\mathcal{M}$-stable curves. $\square$

7. Interpolation of Q-stable spaces

We learned in Section 4 that the choices of universal radii are in bijection with compatible choices of radii on 1-layer trees. Each 1-layer tree Γ has two possible radii: the zero radius and a unique nonzero radius; let's call it e_Γ. Thinking tropically, it is tempting to choose radii intermediate to 0 and e_Γ to contract. If we do this and follow the image of the universal curve under the resulting contraction, we find Artin stacks that interpolate between the Q-stable spaces. The moduli problems of these stacks admit an elegant description, so we will define them and verify their algebraicity directly prior to a tropical construction.

We therefore imagine choosing an intermediate radius for each 1-layer tree Γ in the form $c_\Gamma e_\Gamma$ with $c_\Gamma \in [0, 1]$. The possible choices of such radii form a cube complex, which we now describe directly.

Definition 7.1. Let $X_{1,n}$ be the subset of $[0, 1]^{\text{Part}(n)}$ of tuples $(c_P)_{P \in \text{Part}(n)}$ with the properties that

 (i) whenever P_1 and P_2 are two partitions of $\{1, \ldots, n\}$ with $P_1 \prec P_2$ and $c_{P_2} > 0$, then $c_{P_1} = 1$;

(ii) $c_{\{\{1\}, \ldots, \{n\}\}} = 0$.

We give $X_{1,n}$ the structure of a cube complex whose open cells are the intersections of the open cells of $[0, 1]^{\text{Part}(n)}$ with $X_{1,n}$.

Note that the vertices of $X_{1,n}$ are in bijection with $\mathfrak{Q}_n$: the correspondence is given by taking a subset Q of $\text{Part}(n)$ to its indicator function.

Example 7.2. $X_{1,3}$ consists of the tuples

$$(x, y, z, w) = (c_{\{\{1,2,3\}\}}, c_{\{\{1\},\{2,3\}\}}, c_{\{\{2\},\{1,3\}\}}, c_{\{\{3\},\{1,2\}\}})$$

in $[0, 1]^4$ where $y = z = w = 0$ unless $x = 1$. Then $X_{1,3}$ consists of the 1-dimensional cube

$$\{(x, 0, 0, 0) \mid x \in [0, 1]\}$$

attached to the 3-dimensional solid cube

$$\{(1, y, z, w) \mid y, z, w \in [0, 1]\}.$$

The latter cube fills in the interior of the cube visible in Figure 1.

The stacks associated to these imagined universal radii are as follows.

Definition 7.3. Let $(c_P)_{P \in \text{Part}(n)}$ be an element of $X_{1,n}$. Let

$$Q_{\text{sing}} = \{P \in \text{Part}(n) \mid c_P > 0\} \quad \text{and} \quad Q_{\text{curve}} = \{P \in \text{Part}(n) \mid c_P < 1\}.$$

An n-pointed family of Gorenstein curves $(\pi : C \to S, \sigma_1, \ldots, \sigma_n)$ is $(c_P)_{P \in \mathrm{Part}(n)}$-*stable* if

> $(\pi : C \to S, \sigma_1, \ldots, \sigma_n)$ is a flat and proper family of connected, reduced, Gorenstein curves of arithmetic genus one with n distinct marked points,

and for each geometric fiber C_s, the following conditions hold:

(i) If $p \in C_s$ is an elliptic Gorenstein singularity, then $\mathrm{lev}(p) \in Q_{\mathrm{sing}}$.

(ii) If $Z \subseteq C_s$ is a connected subcurve of genus one, then $\mathrm{lev}(Z) \in Q_{\mathrm{curve}}$.

(iii) If $Z_1 \subsetneq Z_2$ is a proper inclusion of connected subcurves of C_s of arithmetic genus one, then $\mathrm{lev}(Z_1) \prec \mathrm{lev}(Z_2)$.

(iv) If F is an irreducible component of the minimal genus one subcurve $Z_{\min}$ of C_s, then F meets $\overline{C_s - Z_{\min}} \cup \{\sigma_1(s), \ldots, \sigma_n(s)\}$ in at least one point.

Let $\overline{\mathcal{M}}_{1,n}((c_P))$ be the stack whose S-points are the (c_P)-stable families of curves over S.

Remark 7.4. Note that Q_{sing} is downward closed, Q_{curve} is upward closed, and the two sets intersect in the partitions P where $0 < c_P < 1$.

The new feature of this definition is that a (c_P)-stable curve C may have a minimal genus one subcurve Z with an elliptic Gorenstein singularity p such that $\mathrm{lev}(p) = \mathrm{lev}(Z)$, so long as $\mathrm{lev}(p) \in Q_{\mathrm{sing}} \cap Q_{\mathrm{curve}}$. If this happens, then each of the branches of the singularity p will have only one special point other than p. By [Smyth 2011a, Corollary 2.4], C then has infinitesimal automorphisms. (In fact, it has a $\mathbb{G}_m$ of automorphisms.) It follows that whenever $Q_{\mathrm{sing}} \cap Q_{\mathrm{curve}}$ is nonempty, $\overline{\mathcal{M}}_{1,n}((c_P))$ is not a Deligne–Mumford stack.

Theorem 7.5. *If* (c_P) *consists only of* 1*'s and* 0*'s, then* $\overline{\mathcal{M}}_{1,n}((c_P)) = \overline{\mathcal{M}}_{1,n}(Q)$, *where*

$$Q = \{P \in \mathrm{Part}(n) \mid c_P = 1\}.$$

Proof. The only differently stated conditions for membership in $\overline{\mathcal{M}}_{1,n}((c_P))$ and $\overline{\mathcal{M}}_{1,n}(Q)$ are those on geometric fibers, so it suffices to show their geometric points coincide.

Suppose C is a Q-stable curve over some algebraically closed field. Then it is clear that conditions (i) and (ii) hold since $Q = Q_{\mathrm{sing}}$ and $\mathrm{Part}(n) - Q = Q_{\mathrm{curve}}$. Condition (iv) holds since C has no infinitesimal automorphisms.

To see that condition (iii) holds, suppose that $Z_1 \subsetneq Z_2$ is a proper inclusion of connected subcurves C of arithmetic genus one. Then there is a rational component F of Z_2 meeting Z_1 in a point. Since F has at least three special points, $\mathrm{lev}(Z_1) \prec \mathrm{lev}(Z_1 \cup F) \preceq \mathrm{lev}(Z_2)$.

Conversely, suppose that C is (c_P)-stable. The level conditions for Q on C hold by (i) and (ii). It remains to show that C has no infinitesimal automorphisms, that is, by [Smyth 2011a, Corollary 2.4],

(a) each irreducible component F of C with genus zero not meeting an elliptic Gorenstein singularity has at least three special points;

(b) if C has an elliptic Gorenstein singularity q, then each component B of the minimal elliptic subcurve containing q has at least one special point other than q and there is at least such component with two special points other than q.

To address (a), let F be an irreducible component of C with genus zero not meeting an elliptic Gorenstein singularity. Then either C is nodal and F belongs to the core of C, or F is not a component of the minimal subcurve of genus one. In the former case, we are done by (iv). In the latter, let Z_1 be the union of the minimal genus one subcurve E together with the components on the unique path from E to F, not including F. Let $Z_2 = Z_1 \cup F$. Then, since $\text{lev}(Z_1) \prec \text{lev}(Z_2)$, F must have at least three special points.

To address (b), suppose that C has an elliptic Gorenstein singularity q. Let $Z_{\min}$ be the minimal genus one subcurve of C. If B is an irreducible component of $Z_{\min}$, then B has at least one other special point by condition (iv). Next, since $Q_{\text{sing}} \cap Q_{\text{curve}} = \varnothing$, we must have a proper refinement $\text{lev}(q) \prec \text{lev}(Z_{\min})$. This implies that there is at least one branch of q with two special points other than q. □

Theorem 7.6. *The stack of (c_P)-stable curves is deformation-open. That is, if S is a noetherian scheme and $\pi : C \to S$ is a flat, projective morphism with one-dimensional fibers and sections $\sigma_1, \ldots, \sigma_n$, then the set*

$$T = \left\{ s \in S \mid (\pi_s : C_{\bar{s}} \to \bar{s}, \{\sigma_i(\bar{s})\}_{i=1}^n) \text{ is } (c_P)\text{-stable} \right\}$$

is open in S.

Proof. As in Theorem 5.2, we may assume that the geometric fibers $C_{\bar{s}}$ of π are reduced, connected, and of arithmetic genus one with only Gorenstein singularities, since these are open conditions.

Again as in Theorem 5.2, the locus T is constructible since satisfaction of the remaining conditions is constant on combinatorial types and the curves with a given combinatorial type form a locally closed subset of S. Therefore it suffices to check that the remaining conditions hold under generization.

So assume S is the spectrum of a DVR with closed point $0 \in S$ and generic point $\eta \in S$. We must show that if $(C_{\bar{0}}, \sigma_1(\bar{0}), \ldots, \sigma_n(\bar{0}))$ satisfies the remaining conditions, then so does $(C_{\bar{\eta}}, \sigma_1(\bar{\eta}), \ldots, \sigma_n(\bar{\eta}))$. Write Σ for the divisor of markings. Since T is characterized by geometric fibers, we may apply a finite base change to S so that restriction to the geometric generic fiber induces a bijection from the components of C to the components of $C_{\bar{\eta}}$. The level conditions on singularities and subcurves are stable under generization by an identical argument to Theorem 5.2.

Next, we consider condition (iii). Suppose that $Z_1^{\bar{\eta}} \subset Z_2^{\bar{\eta}}$ is a proper inclusion of genus one subcurves of $C_{\bar{\eta}}$. Let $Z_1^{\bar{0}}$ and $Z_2^{\bar{0}}$ be the respective limits of $Z_1^{\bar{\eta}}$ and $Z_2^{\bar{\eta}}$ in $C_{\bar{0}}$. Then we must have a proper inclusion $Z_1^{\bar{0}} \subset Z_2^{\bar{0}}$. Taking limits of the connected components of $(C_{\bar{\eta}} - Z_1^{\bar{\eta}}) \cup \Sigma|_{\bar{\eta}}$, we see that $\text{lev}(Z_1^{\bar{0}}) = \text{lev}(Z_1^{\bar{\eta}})$. Similarly, $\text{lev}(Z_2^{\bar{0}}) = \text{lev}(Z_2^{\bar{\eta}})$. By (c_P)-stability of the central fiber, $\text{lev}(Z_1^{\bar{0}}) \prec \text{lev}(Z_2^{\bar{0}})$. Because $\text{lev}(Z_1^{\bar{0}}) = \text{lev}(Z_1^{\bar{\eta}})$ and $\text{lev}(Z_2^{\bar{0}}) = \text{lev}(Z_2^{\bar{\eta}})$, we have the desired refinement $\text{lev}(Z_1^{\bar{\eta}}) \prec \text{lev}(Z_2^{\bar{\eta}})$ in the generic fiber too.

Finally, we show condition (iv) is stable under generization. Let

$$Z_{\min}^{\bar{\eta}} = \text{minimal genus one subcurve of } C_{\bar{\eta}} \quad \text{and} \quad Z_{\min}^{\bar{0}} = \text{minimal genus one subcurve of } C_{\bar{0}}.$$

Given an irreducible component $F_{\bar{0}}$ of $Z^{\bar{0}}_{\min}$, there is a unique irreducible component $F_{\bar{\eta}}$ of $Z_{\bar{\eta}}$ to which $F_{\bar{0}}$ generizes. Moreover, as $F_{\bar{0}}$ varies over the components of $Z^{\bar{0}}_{\min}$, $F_{\bar{\eta}}$ varies over all the irreducible components of $Z^{\bar{\eta}}_{\min}$.

By axiom (iv), $F_{\bar{0}}$ either meets a marking or it meets a connected rational tail T contained in $\overline{C_{\bar{0}} - Z^{\bar{0}}_{\min}}$. If $F_{\bar{0}}$ itself contains a marking, we set $Y^1_{\bar{0}} = F_{\bar{0}}$ and let $k = 1$. If not, then we may find a path in the dual graph of $C_{\bar{0}}$ from $F_{\bar{0}}$ to a marked component of T, i.e., a sequence $Y^1_{\bar{0}}, \ldots, Y^k_{\bar{0}}$ of irreducible components of $C_{\bar{0}}$, where $Y^1_{\bar{0}} = F_{\bar{0}}$, $Y^{i+1}_{\bar{0}}$ meets $Y^i_{\bar{0}}$ in a node for all $1 \leq i < k$, $Y^i_{\bar{0}} \not\subseteq Z_{\bar{0}}$ for each $i > 0$, and $Y^k_{\bar{0}}$ contains a marking. For each i we let $Y^i_{\bar{\eta}}$ be the unique irreducible component of $C_{\bar{\eta}}$ generizing $Y^i_{\bar{0}}$. After reindexing to omit repeats, the resulting sequence of components $Y^1_{\bar{\eta}}, \ldots, Y^l_{\bar{\eta}}$ of $C_{\bar{\eta}}$ is again a sequence of components connected by nodes, beginning with $F_{\bar{\eta}}$, and ending in a component with a marking.

If all of the nodes of the path $Y^1_{\bar{0}}, \ldots, Y^k_{\bar{0}}$ smooth out in the geometric generic fiber, then $l = 1$, and $F_{\bar{\eta}}$ meets a marking, and we are done. If not, then let $p \in C_{\bar{0}}$ be the first node on the path that does not smooth out in the family. Then, after a finite base change if necessary, there is a section $P : S \to C$ through the singular locus of C going through p and meeting $F_{\bar{\eta}}$. Blowing up along P, we see that $Y^2_{\bar{\eta}}$ does not belong to $Z^{\bar{\eta}}_{\min}$, since P separates the limit of $Y^2_{\bar{\eta}}$ from $Z^{\bar{0}}_{\min}$ in the central fiber. Then F_η meets $\overline{C_{\bar{\eta}} - F_\eta}$ in the point $P|_{\bar{\eta}}$, and we deduce that axiom (iv) holds in the generic fiber. $\square$

Corollary 7.7. $\overline{\mathcal{M}}_{1,n}((c_P))$ *is an Artin stack.*

Proof. The preceding theorem shows that $\overline{\mathcal{M}}_{1,n}((c_P))$ is an open substack of the Artin stack of all n-pointed curves. (See [Smyth 2013, Appendix B].) $\square$

Our next result is that these various moduli spaces are related by containments induced by the containment of faces in the cube complex $X_{1,n}$. This strikes us as analogous to the containments of moduli spaces seen at critical values of the log minimal model program in [Alper et al. 2017, Theorem 1.1], except that the "larger" stacks here are associated with larger strata.

Theorem 7.8. *Suppose that* $(c_P), (d_P) \in X_{1,n}$ *and* (d_P) *lies in a face of the cube containing* (c_P), *i.e.,* $d_P = c_P$ *whenever* $c_P = 0$ *or* $c_P = 1$. *Then there is a fully faithful inclusion functor*

$$\overline{\mathcal{M}}_{1,n}((d_P)) \hookrightarrow \overline{\mathcal{M}}_{1,n}((c_P)).$$

Proof. Note that the sets Q_{sing} and Q_{curve} associated to (d_P) are subsets of the respective sets associated to (c_P). Then the result is clear from the definition. $\square$

Corollary 7.9. *The stacks* $\overline{\mathcal{M}}_{1,n}((c_P))$ *are universally closed.*

Proof. Let $Q = \{P \in \mathrm{part}(n) \mid c_P = 1\}$. By Theorems 7.5 and 7.8, $\overline{\mathcal{M}}_{1,n}(Q)$ is a substack of $\overline{\mathcal{M}}_{1,n}((c_P))$.

Note that $\overline{\mathcal{M}}_{1,n}((c_P))$ contains $\mathcal{M}_{1,n}$ as a dense open, since nodes and elliptic Gorenstein singularities are smoothable. To check universal closedness it therefore suffices to verify that if S is the spectrum of a DVR with generic point η, and $C_\eta \to \eta$ a family of smooth n-marked curves of genus one, then there is an extension of C_η to a family of (c_P)-stable curves over S. Such an extension already exists as a Q-stable limit. $\square$

A tropical resolution of the birational map from $\overline{\mathcal{M}}_{1,n}$ to $\overline{\mathcal{M}}_{1,n}((c_P))$. We now give the tropical construction that motivated the definition of $\overline{\mathcal{M}}_{1,n}((c_P))$ above. For each $(c_P) \in X_{1,n}$ we will build a modification $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ and a contraction inducing a morphism $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to \overline{\mathcal{M}}_{1,n}((c_P))$. The ideas are essentially the same as in Section 4, so we are relatively brief.

Fix an element $(c_P)_{P \in \mathrm{Part}(n)} \in X_{1,n}$. Let $Q_{\min} = \{P \in \mathrm{Part}(n) \mid c_P = 1\}$. Let $P_1, \ldots, P_k$ be the partitions for which $0 < c_{P_i} < 1$. Then let $Q_i = Q_{\min} \cup \{P_i\}$ for $1 \le i \le k$. Consider the associated global sections $\rho_{\min}, \rho_1, \ldots, \rho_k$ of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Notice that $\rho_i \ge \rho_{\min}$ for each i, so the differences $\delta_i = \rho_i - \rho_{\min}$ are again well-defined sections of the characteristic sheaf of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$.

We recall (see, for example, [Olsson 2003, Proposition 5.17] with $P = \mathbb{N}$) that the stack $[\mathbb{A}^1/\mathbb{G}_m]$ may be given a log structure so that $[\mathbb{A}^1/\mathbb{G}_m]$ represents the functor on log schemes

$$X \mapsto \Gamma(X, \overline{M}_X).$$

Since this is a functor valued in commutative monoids, there is a morphism $\mu : [\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ induced by the multiplication of $\Gamma(X, \overline{M}_X)$.

Take products and form the pullback square of fs log algebraic stacks

$$
\begin{array}{ccc}
\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \xrightarrow{\prod_i (\delta_i^{(1)}, \delta_i^{(2)})} & ([\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m])^k \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mu^k} \\
\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \xrightarrow{\prod_i \delta_i} & [\mathbb{A}^1/\mathbb{G}_m]^k
\end{array}
$$

Note that $p^* \delta_i$ factors into the sum $\delta_i^{(1)} + \delta_i^{(2)}$ in the characteristic sheaf for each i. Moreover, $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is universal with respect to such factorizations in the sense that, given any $g : T \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ and an expression of each $g^* \delta_i$ as a sum $\alpha_i^{(1)} + \alpha_i^{(2)}$, there is a unique factorization $h : T \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ of g through $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ such that $\alpha_i^{(1)} = h^* \delta_i^{(1)}$ and $\alpha_i^{(2)} = h^* \delta_i^{(2)}$ for each i.

The morphism μ is integral and saturated, so the underlying algebraic stack of $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is the fiber product of the underlying algebraic stacks of the rest of the diagram. The complement $D(\delta_1, \ldots, \delta_k)$ in $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ of the vanishing of the Cartier divisors associated to $\delta_1, \ldots, \delta_k$ is precisely the preimage of the non-stacky point of $[\mathbb{A}^1/\mathbb{G}_m]^k$. As μ^k restricts to an isomorphism over this point, it follows that p restricts to an isomorphism over $D(\delta_1, \ldots, \delta_k)$. In particular, since the δ_i's are nonvanishing on smooth curves, p restricts to an isomorphism on $\mathcal{M}_{1,n}$. We identify $\mathcal{M}_{1,n}$ with its image in $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$.

Lemma 7.10. *Let $T = \mathrm{Spec}\, B$ be the spectrum of a discrete valuation ring with uniformizer t and field of fractions K. Let $b_1, \ldots, b_k$ be nonzero elements of B, and let*

$$A = \frac{B[x_1^{(1)}, x_1^{(2)}, \ldots, x_k^{(1)}, x_k^{(2)}]}{(x_i^{(1)} x_i^{(2)} - b_i : i = 1, \ldots, k)}.$$

Then $S = \mathrm{Spec}\, A$ is an integral scheme and its generic point maps to the generic point of T under the natural map $S \to T$.

Proof. Let $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$. Let $\eta = D(t)$ be the generic point of T and U its preimage in S. Then $U \cong \mathbb{G}_{m,K}^k$ is an irreducible open of S. Recall that the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is flat, so its k-fold product $(\mathbb{A}^1 \times \mathbb{A}^1)^k \to (\mathbb{A}^1)^k$ is also flat. Observe that $S \to T$ is a pullback of this morphism, so also flat. If x is any point of S not in U, the going-down theorem for flat morphisms implies that x possesses a generization in U. Therefore, U is dense in S, and S is integral. Since U maps to the generic point of T, we also have the claim about generic points. $\square$

Proposition 7.11. *The open substack $\mathcal{M}_{1,n}$ is dense in $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. In particular $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is irreducible and p is birational.*

Proof. Give $\mathbb{A}^1$ the toric log structure. Recall that $\mathbb{A}^1$ represents the functor

$$(X, \alpha : M_X \to \mathcal{O}_X) \mapsto \Gamma(X, M_X).$$

There is a commutative square of fs log algebraic stacks

$$
\begin{array}{ccc}
([\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m])^k & \longleftarrow & (\mathbb{A}^1 \times \mathbb{A}^1)^k \\
\downarrow{\mu^k} & & \downarrow \\
[\mathbb{A}^1/\mathbb{G}_m]^k & \longleftarrow & (\mathbb{A}^1)^k
\end{array}
$$

where each vertical map is induced by the monoid law, and the horizontal maps are induced by $M_X \to \overline{M}_X$. The map of schemes underlying the right vertical map consists of k-copies of the multiplication map. By [Olsson 2003, Proposition 2.1], a map $X \to [\mathbb{A}^1/\mathbb{G}_m]$ étale locally admits a lift to $\mathbb{A}^1$.

Let $s = (\operatorname{Spec} k, \alpha)$ be a geometric fs log point, and let $f_s : s \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ be a log morphism. Our strategy is to find a smoothing of f_s in $\overline{\mathcal{M}}_{1,n}$, then to lift this smoothing back to $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ using the commutative square above.

Observe that since s is a geometric point, the composite $s \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to ([\mathbb{A}^1/\mathbb{G}_m] \times [\mathbb{A}^1/\mathbb{G}_m])^k$ factors through $(\mathbb{A}^1 \times \mathbb{A}^1)^k$.

Let $t = (\operatorname{Spec} k, \beta)$ denote the fs log point with log structure pulled-back from $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ along the map $p \circ f_s : s \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Write g_t for the induced map $t \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Note that we have a commutative square of fs log algebraic stacks

$$
\begin{array}{ccc}
s & \xrightarrow{f_s} & \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} \\
\downarrow & & \downarrow{p} \\
t & \xrightarrow{g_t} & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}
\end{array}
$$

By construction, g_t is strict. So we may use that $\mathcal{M}_{1,n}$ is dense in the Noetherian algebraic stack $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to find a strict map $g : T \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$, where

(i) T is an fs log scheme with underlying scheme the spectrum of a discrete valuation ring;

(ii) g_t factors through the special point of T;

(iii) $\eta \subseteq T$ is the generic point;

(iv) $g|_\eta \in \mathcal{M}_{1,n}(\eta)$.

Replacing T by a finite base change if necessary, we may assume that the composite $T \to \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \to [\mathbb{A}^1/\mathbb{G}_m]^k$ factors through $(\mathbb{A}^1)^k$. A diagram chase shows that there is an induced map $f : S \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ and $f_s : s \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ factors through it. By Lemma 7.10, S is irreducible and its generic point θ maps to the generic point η of T. By construction, $g(\eta)$ lies in the smooth locus of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$, so $f(\theta)$ lies in the smooth locus of $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. It follows that the image of s in $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ has a generization to a point factoring through $\mathcal{M}_{1,n} \subseteq \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$, as desired. $\square$

Now let $\rho = p^*\rho_{\min} + \delta_1^{(1)} + \cdots + \delta_k^{(1)}$ in the characteristic sheaf of $\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$. Let $C_{1,n} \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ be the pullback of the universal family of $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ to $\widetilde{\mathcal{M}}_{1,n}$ Notice that ρ is comparable with the radii of $C_{1,n}$, so we may make a log modification $\widetilde{C}_{1,n} \to C_{1,n}$ subdividing tropicalizations at the locus where $\rho = \lambda$. Then, as in Theorem 4.1, we may form a section $\bar{\lambda} \in \Gamma(\widetilde{C}, \overline{\mathcal{M}}_{\widetilde{C}})$ by the formula

$$\bar{\lambda} = \max\{\rho - \lambda, 0\}.$$

Once again, it is easy to check that this is a mesa in the sense of [Bozlee 2020], so the main result there yields a contraction of families of curves

$$
\begin{array}{ccc}
\widetilde{C}_{1,n} & \xrightarrow{\;\tau\;} & \overline{C}_{1,n} \\
{\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \overline{\pi}} & \\
\widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} & &
\end{array}
$$

Similar reasoning to that of Section 4 yields that $\overline{C}_{1,n} \to \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ is a family of curves in $\overline{\mathcal{M}}_{1,n}((c_P))$. We therefore have a diagram of birational morphisms of algebraic stacks

$$
\begin{array}{ccc}
& \widetilde{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \\
{\scriptstyle p}\swarrow & & \searrow {\scriptstyle \overline{C}_{1,n}} \\
\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} & & \overline{\mathcal{M}}_{1,n}((c_P)) \\
\searrow & & \\
& \overline{\mathcal{M}}_{1,n} &
\end{array}
$$

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