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On unipotent radicals of motivic Galois groups

Payman Eskandari and V. Kumar Murty

Let $\boldsymbol{T}$ be a neutral Tannakian category over a field of characteristic zero with unit object $\mathbb{1}$, and equipped with a filtration $W_\bullet$ similar to the weight filtration on mixed motives. Let M be an object of $\boldsymbol{T}$, and $\underline{\mathfrak{u}}(M) \subset W_{-1}\underline{\mathrm{Hom}}(M, M)$ the Lie algebra of the kernel of the natural surjection from the fundamental group of M to the fundamental group of $\mathrm{Gr}^W M$. A result of Deligne gives a characterization of $\underline{\mathfrak{u}}(M)$ in terms of the extensions $0 \to W_pM \to M \to M/W_pM \to 0$: it states that $\underline{\mathfrak{u}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{Hom}}(M, M)$ such that the sum of the aforementioned extensions, considered as extensions of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{Hom}}(M, M)$, is the pushforward of an extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$. We study each of the above-mentioned extensions individually in relation to $\underline{\mathfrak{u}}(M)$. Among other things, we obtain a refinement of Deligne's result, where we give a sufficient condition for when an individual extension $0 \to W_pM \to M \to M/W_pM \to 0$ is the pushforward of an extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$. In the second half of the paper, we give an application to mixed motives whose unipotent radical of the motivic Galois group is as large as possible (i.e., with $\underline{\mathfrak{u}}(M) = W_{-1}\underline{\mathrm{Hom}}(M, M)$). Using Grothendieck's formalism of *extensions panachées* we prove a classification result for such motives. Specializing to the category of mixed Tate motives we obtain a classification result for 3-dimensional mixed Tate motives over $\mathbb{Q}$ with three weights and large unipotent radicals.

1. Introduction

1.1. ***About this paper.*** Let $\boldsymbol{T}$ be a neutral Tannakian category over a field K of characteristic zero, equipped with a weight filtration $W_\bullet$ similar to the weight filtration on mixed motives (functorial, increasing, finite on every object, exact, and respecting the tensor structure). For example, one might keep in mind the category of mixed Hodge structures. In fact, this is a concrete example that illustrates well the main results.

Let M be an object of $\boldsymbol{T}$, and $\underline{\mathfrak{u}}(M)$ the Lie algebra of the kernel of the natural map from the fundamental group of M to that of $\mathrm{Gr}^W M$. A result of Deligne describes $\underline{\mathfrak{u}}(M)$ in terms of extensions that arise naturally from the weight filtration of M. For each integer p, let $\mathscr{E}_p(M)$ be the extension

$$0 \to W_pM \to M \to M/W_pM \to 0, \tag{1}$$

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considered as an element in $\mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M))$ (where $\underline{\mathrm{End}}(M)$ means $\underline{\mathrm{Hom}}(M, M)$, the latter being the internal Hom). Deligne characterizes $\underline{\mathfrak{u}}(M)$ in terms of the sum

$$\mathscr{E}(M) := \sum_p \mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)).$$

The first half of this paper refines this by developing conditions under which the individual extensions $\mathscr{E}_p(M)$ can be related to $\underline{\mathfrak{u}}(M)$.

The second half of the paper specializes to the setting of mixed motives and gives an application of the first half to mixed motives whose unipotent radical of the motivic Galois group is as large as possible (i.e., with $\underline{\mathfrak{u}}(M) = W_{-1}\underline{\mathrm{End}}(M)$). These motives are in particular interesting for the transcendence properties of their periods: in view of Grothendieck's period conjecture the field generated by their periods should have the highest possible transcendence degree among all motives with the same associated graded.

A particularly striking implication of our result is that a suggestion of Euler about $\zeta(3)$ is incompatible with Grothendieck's period conjecture. Euler [1785] speculated that there may be rational numbers α and β and an expression of the form

$$\zeta(3) = \alpha(\log 2)^3 + \beta\pi^2(\log 2).$$

See the article of Dunham [2021] which gives a very readable account of this statement and Euler's remarkable work on evaluating the Riemann zeta function at integer arguments.

In Section 6.8, we construct a mixed Tate motive with periods (essentially) $\zeta(3)$, $\log 2$, π and a fourth period. Moreover, we use our results to show that the dimension of the Galois group in this case is 4. Thus, the period conjecture would predict that these four periods are algebraically independent, and this is incompatible with Euler's expectation stated above. A more detailed description of this mixed Tate motive is given below.

1.2. $\underline{\mathfrak{u}}(M)$ *and the extensions* $\mathscr{E}_p(M)$. To be more precise, $\underline{\mathfrak{u}}(M)$ is the subobject of $W_{-1}\underline{\mathrm{End}}(M)$ with the property that if ω is any fiber functor over K, then

$$\omega\underline{\mathfrak{u}}(M) \subset \omega W_{-1}\underline{\mathrm{End}}(M) = W_{-1}\,\mathrm{End}(\omega M)$$

is the Lie algebra of

$$\mathscr{U}(M, \omega) := \ker(\mathscr{G}(M, \omega) \xrightarrow{\text{restriction}} \mathscr{G}(\mathrm{Gr}^W M, \omega)),$$

where $\mathscr{G}(-, \omega)$ denotes the fundamental group of the indicated object with respect to ω. If $\mathrm{Gr}^W M$ is semisimple (which will be the case if $\boldsymbol{T}$ is a category of motives), then $\mathscr{U}(M, \omega)$ is the unipotent radical of $\mathscr{G}(M, \omega)$.

As stated above, Deligne (see [Jossen 2014, Appendix]) describes $\underline{\mathfrak{u}}(M)$ in terms of extensions that arise naturally from the weight filtration on M: For each integer p, let $\mathscr{E}_p(M)$ be the *p-th extension class of M* given by (1), considered as an extension of the unit object $\mathbb{1}$ by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$. Pushing this extension forward along the natural injection

$$\underline{\mathrm{Hom}}(M/W_pM,\, W_pM) \to W_{-1}\underline{\mathrm{End}}(M)$$

we get an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)$, which we also denote by $\mathscr{E}_p(M)$. The *total* extension class of M is then the extension

$$\mathscr{E}(M) := \sum_p \mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1},\, W_{-1}\underline{\mathrm{End}}(M)).$$

Deligne's result asserts that $\underline{\mathfrak{u}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{End}}(M)$ such that $\mathscr{E}(M)$ is in the image of the pushforward

$$\mathrm{Ext}^1(\mathbb{1},\, \underline{\mathfrak{u}}(M)) \to \mathrm{Ext}^1(\mathbb{1},\, W_{-1}\underline{\mathrm{End}}(M)) \tag{2}$$

under the inclusion $\underline{\mathfrak{u}}(M) \subset W_{-1}\underline{\mathrm{End}}(M)$. Deligne proves this in part by exploiting the weight filtration to construct an explicit extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$ which pushes forward to $\mathscr{E}(M)$.

The first half of this paper is dedicated to the study of the relation between $\underline{\mathfrak{u}}(M)$ and the individual extensions $\mathscr{E}_p(M)$, with a view to refining Deligne's result. In general, the individual extensions $\mathscr{E}_p(M)$ may not be in the image of the pushforward map (2); an example involving 1-motives can be given using the work of Jacquinot and Ribet [1987] on deficient points on semiabelian varieties; see Section 6.10 and the remarks at its end. The main result of the first half of the paper gives a sufficient condition for when the extension $\mathscr{E}_p(M)$ is in the image of (2); see Theorem 5.3.1 and its corollaries.

1.3. ***A more detailed overview.*** We continue this introduction by giving a more detailed overview of the contents of the paper, starting with the first half. Fix an integer p and an object M of $\boldsymbol{T}$. It is natural to expect $\mathscr{E}_p(M)$ to be related to the subobject

$$\underline{\mathfrak{u}}_p(M) := \underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_pM,\, W_pM)$$

of $\underline{\mathfrak{u}}(M)$, where we have considered $\underline{\mathrm{Hom}}(M/W_pM,\, W_pM)$ as a subobject of $W_{-1}\underline{\mathrm{End}}(M)$ via the natural injection. This is indeed the case: Write $\mathscr{E}_p(M)$ explicitly as

$$0 \to \underline{\mathrm{Hom}}(M/W_pM,\, W_pM) \to \underline{\mathrm{Hom}}(M/W_pM,\, M)^{\dagger} \to \mathbb{1} \to 0; \tag{3}$$

see Section 4.5 for the explicit description of the middle object. Then by Theorem 3.3.1 of [Eskandari and Murty 2021] (which is proved by a small modification of the proof of [Hardouin 2011, Theorem 2]), we have:

$$\begin{gathered}\underline{\mathfrak{u}}_p(M) \textit{ is the smallest subobject of } \underline{\mathrm{Hom}}(M/W_pM,\, W_pM) \textit{ such that} \\ \underline{\mathrm{Hom}}(M/W_pM,\, M)^{\dagger}/\underline{\mathfrak{u}}_p(M) \textit{ belongs to the subcategory } \langle W_pM,\, M/W_pM\rangle^{\otimes}.\end{gathered} \tag{$*$}$$

Here, as usual, the notation $\langle\ \rangle^{\otimes}$ means the smallest full Tannakian subcategory containing the indicated objects and closed under subobjects. The first contribution of the present article is to reformulate this statement in a more natural way, in the language of extensions originating from subcategories (discussed in Section 3). Given a full Tannakian subcategory $\boldsymbol{S}$ of $\boldsymbol{T}$ which is closed under subobjects, we say an extension $\mathscr{E}$ of $\mathbb{1}$ by an object A of $\boldsymbol{T}$ originates from $\boldsymbol{S}$ if there is an object A' of $\boldsymbol{S}$, an extension $\mathscr{E}'$ of $\mathbb{1}$ by A' in $\boldsymbol{S}$, and a morphism $A' \to A$ under which $\mathscr{E}'$ pushes forward to $\mathscr{E}$. While this is a very natural and simple generalization of the notion of splitting of sequences (as an extension splits if and only if it originates from a semisimple $\boldsymbol{S}$), it opens the door to refinements of $(*)$ and Deligne's theorem. The reformulated version of $(*)$ is given in Theorem 4.9.1. It asserts that $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that the pushforward $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ of $\mathscr{E}_p(M)$ under the quotient map originates from the subcategory

$$\langle W_pM, M/W_pM\rangle^{\otimes}. \tag{4}$$

Note that one advantage of formulating the statement in this language is that here we may think of $\mathscr{E}_p(M)$ as an extension of $\mathbb{1}$ by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ or by $W_{-1}\underline{\mathrm{End}}(M)$; see Section 4.

Our next goal is to find refinements of Theorem 4.9.1 in which the category (4) is replaced by smaller categories. Ideally, this category can be replaced by a semisimple category, in which case the pushforward $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ of $\mathscr{E}_p(M)$ along the quotient map will split. (By weight considerations and the long exact sequence for Ext groups this is equivalent to $\mathscr{E}_p(M)$ being in the image of (2).) But from the examples of 1-motives mentioned earlier we know that in general, this will not be the case.

Let $q \le p$. The second contribution of this paper is to show that if M satisfies certain "independence axioms", then in the statement of Theorem 4.9.1 the category (4) can be replaced by the smaller category $\langle W_qM, \mathrm{Gr}^W M\rangle^{\otimes}$ (smaller because $q \le p$); this is Theorem 5.3.1 in Section 5. The independence axioms are given in Section 5.2, and in fact, only depend on $\mathrm{Gr}^W M$. Roughly speaking, they require the subobject

$$\bigoplus_{\substack{i,j\\ j>q,i}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M) \tag{5}$$

of $W_{-1}\underline{\mathrm{End}}(\mathrm{Gr}^W M)$ to suitably decompose as a direct sum of two "independent" summands. In the weak sense, here the word "independent" means not having any nonzero isomorphic subobjects, and in the strong sense, it means having disjoint sets of weights; see the axioms $(IA1)_{\{p,q\}}$ and $(IA2)_{\{p,q\}}$ in Section 5.2.

An interesting consequence of Theorem 5.3.1 is the following refinement of Deligne's theorem (see Corollary 5.3.2): If $\mathrm{Gr}^W M$ is semisimple (e.g., if $\boldsymbol{T}$ is a category of motives) and the weak independence axioms hold for all $q \le p$, then $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. In particular, if $\mathrm{Gr}^W M$ is semisimple and $W_{-1}\underline{\mathrm{End}}(M)$ has $\binom{n}{2}$ distinct weights where n is the number of weights of M (e.g., if M has weights $0, -1, -3, -7$), then $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits for every p; see Corollary 5.3.3.

The proof of Theorem 5.3.1 is similar to the proof of $(*)$ (or rather, of Theorem 4.9.1), albeit with two added ingredients. Let $\underline{\mathfrak{u}}_{\ge q}(M)$ be the Lie algebra of the kernel of the restriction map from the fundamental group of M to that of $W_qM \oplus \mathrm{Gr}^W M$. The first new component is thanks to the independence axioms: they guarantee that $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\ge q}(M)$ (which is a subobject of (5)) decomposes according to the decomposition of (5) into our independent objects; see Lemma 5.5.1. This is the only place in the proof of Theorem 5.3.1 that the independence axioms play a part. Taking ω to be any fiber functor, this gives a decomposition of $\omega\,\mathrm{Gr}^W\,\underline{\mathfrak{u}}_{\geq q}(M)$. The second added ingredient is that we use the fundamental theorem of Tannakian categories with $\omega\circ\mathrm{Gr}^W$ as the fiber functor (rather than using ω itself). Notice the difference in the nature of this type of argument and Deligne's argument in [Jossen 2014, Appendix], which explicitly constructs an extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$ that pushes forward to the total class of M. We should point out that the idea of working with the associated graded fiber functor already appears in [Deligne 1994], and since then has featured frequently in the literature, especially in the setting of categories of mixed Tate motives; e.g., [Deligne and Goncharov 2005].

It would be interesting to give a more conceptual explanation (or geometric interpretation, in the case of motives) for the fact that the independence axioms force the individual extension classes $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ (or $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$) to split.

We now discuss the contents of the second half of the paper (Section 6). Let $\boldsymbol{T}$ be a Tannakian category of mixed motives over a field K of characteristic zero, e.g., the Tannakian categories of Nori or Ayoub of mixed motives over K, or Voevodsky's Tannakian category of mixed Tate motives over a number field, or categories of mixed motives defined in terms of realizations. We say $\underline{\mathfrak{u}}(M)$ is *large* (or that M has a large $\underline{\mathfrak{u}}$) if $\underline{\mathfrak{u}}(M)$ is equal to $W_{-1}\underline{\mathrm{End}}(M)$. As we pointed out earlier, such motives are interesting from the point of view of the transcendence properties of their periods. Our original motivation for this part of the paper was to study (or ideally, classify up to isomorphism) all objects M with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to

$$\mathbb{Q}(n)\oplus A\oplus\mathbb{1},$$

where A is a given pure object of weight p with $-2n<p<0$. We then realized that much of the discussion can be given in more generality, leading to the contents of this part of the paper as currently presented (and reviewed below).

Suppose tentatively that M is an extension of $\mathbb{1}$ by an object L of highest weight p with $p<0$. It is easy to see that if $\underline{\mathfrak{u}}(M)$ is large, then so are $\underline{\mathfrak{u}}(L)$ and $\underline{\mathfrak{u}}(M/W_{p-1}(L))$. The first main result of Section 6 (Theorem 6.3.1) gives a sufficient condition for the converse statement. The result asserts that if M satisfies a suitable independence axiom, and if $\underline{\mathfrak{u}}(L)$ and $\underline{\mathfrak{u}}(M/W_{p-1}(L))$ are large, then so is $\underline{\mathfrak{u}}(M)$. This is an application of Corollary 5.3.2. As in the case of the latter corollary, examples involving 1-motives show that the conclusion of Theorem 6.3.1 is in general false without the hypothesis about the independence axiom; see Section 6.10.

Theorem 6.3.1 suggests a way to obtain more complicated objects with large $\underline{\mathfrak{u}}$ by "patching together" smaller such objects. More precisely, given an object L of highest weight p with $p<0$ which has a large $\underline{\mathfrak{u}}$, and an object N which is an extension of $\mathbb{1}$ by $\mathrm{Gr}^W_p L$ and also has a large $\underline{\mathfrak{u}}$, we can look for objects M such that $W_pM\simeq L$ and $M/W_{p-1}M\simeq N$; assuming the relevant independence axiom (which only depends on $\mathrm{Gr}^W M\simeq\mathrm{Gr}^W L\oplus\mathbb{1}$) holds, any such M has a large $\underline{\mathfrak{u}}$. The answer to the question of existence of such M is given by Grothendieck's formalism of *extensions panachées* [SGA 7_{I} 1972]: The obstruction is an element of $\mathrm{Ext}^2(\mathbb{1}, W_{p-1}L)$. Moreover, the object M is unique up to isomorphism if $\mathrm{Ext}^1(\mathbb{1}, W_{p-1}L)=0$; see Lemma 6.4.1.

We consider the following classification problem in Sections 6.4–6.7: Given B of weights $< p$ and with a large $\underline{\mathfrak{u}}$, and a nonzero pure object A of negative weight p, classify up to isomorphism all M with large $\underline{\mathfrak{u}}$ satisfying

$$W_{p-1}M \simeq B, \quad \mathrm{Gr}_p^W M \simeq A \quad \text{and} \quad M/W_pM \simeq \mathbb{1}$$

(with the isomorphisms not part of the data). We manage to give a complete solution to this problem when $B \oplus A \oplus \mathbb{1}$ satisfies an independence axiom and $\mathrm{Ext}^1(\mathbb{1}, B) = 0$; the solution is summarized in Section 6.7, just before Corollary 6.7.1. To get there, in Sections 6.4–6.6 we study the *extensions panachées* problem in the setting of an abelian category with weights.[1] The main result is summarized in Proposition 6.6.1; see also Lemma 6.5.1. As a special case of these results, in Corollary 6.7.1 we give an answer to our original motivating classification problem about objects with associated graded isomorphic to $\mathbb{Q}(n) \oplus A \oplus \mathbb{1}$.

In Section 6.8 we specialize to the category $\mathbf{MT}(\mathbb{Q})$ of (say, Voevodsky) mixed Tate motives over $\mathbb{Q}$. The nice feature here is that the Ext groups are known. We use Corollary 6.7.1 to give a complete classification, up to isomorphism, of all 3-dimensional mixed Tate motives over $\mathbb{Q}$ with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(k) \oplus \mathbb{1}$ with $n > k > 0$ and $n \neq 2k$; the very last condition is the independence axiom in this situation.

Let us consider an example from Section 6.8 here. Let r be an integer > 1 and N the Kummer 1-motive $[\mathbb{Z} \xrightarrow{1 \mapsto r} \mathbb{G}_m]$, considered as an object of $\mathbf{MT}(\mathbb{Q})$. Let n be an even integer ≥ 4, and L an object which is a nontrivial extension of $\mathbb{1}$ by $\mathbb{Q}(n-1)$ (so with $(2\pi i)^{1-n}\zeta(n-1)$ as a period). Since Ext^2 groups vanish in $\mathbf{MT}(\mathbb{Q})$ and $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$, the two objects $L(1)$ and N can be patched together to form an object M of $\mathbf{MT}(\mathbb{Q})$, unique up to isomorphism, such that $W_{-2}M \simeq L(1)$ and $M/\mathbb{Q}(n) \simeq N$.[2] Moreover, M satisfies the required independence axiom (as $n \neq 2$), so that it follows from Theorem 6.3.1 that $\underline{\mathfrak{u}}(M)$ is large. According to Grothendieck's period conjecture, the field generated over $\mathbb{Q}$ by the periods of M should have transcendence degree equal to

$$\dim(\mathcal{G}(M, \omega_B)) = \dim(\omega_B \underline{\mathfrak{u}}(M)) + \dim(\mathcal{G}(\mathrm{Gr}^W M, \omega_B)) = 3 + 1 = 4$$

(where ω_B = Betti realization). The nonzero entries of the period matrix of M with respect to suitably chosen bases of de Rham and Betti realizations are $(2\pi i)^{-n}$, $(2\pi i)^{-n}\zeta(n-1)$, $(2\pi i)^{-1}$ (coming from $L(1)$), $(2\pi i)^{-1}\log(r)$, 1 (coming from N), and a "new period". So Grothendieck's period conjecture predicts that

$$2\pi i, \quad \zeta(n-1), \quad \log(r), \quad \text{and the new period of } M$$

must be algebraically independent over $\mathbb{Q}$.

The new period discussed above seems rather mysterious, and it would be very interesting to somehow calculate it.[3] When r is 2 (or a power of it), M is a mixed Tate motive over $\mathbb{Z}[\frac{1}{2}]$, and hence by Deligne's

[1] Actually, a slight variation of it; see the beginning of Section 6.4.

[2] This object is denoted by $M_{n,r}$ in Section 6.8.

[3] Ideally, one would like to do this by giving a geometric construction of M, but this may be too difficult especially when $n > 4$. In general, giving geometric constructions of mixed Tate motives with a few weights is a difficult problem; see [Brown 2016, Section 1.4].

work [2010] the new period will be a linear combination of alternating multiple zeta values, which one should be able to calculate using the formula of Goncharov [2005] and Brown [2012] for the motivic coaction on iterated integrals.[4] On the other hand, for general r, at least a priori, the new period may not be an iterated integral on the projective line $\mathbb{P}^1$ minus $\{0, \infty\} \cup \mu_r$. (This is related to the question of whether the category of mixed Tate motives over $\mathbb{Z}[1/r]$ is generated by the fundamental groupoid of $\mathbb{P}^1 \setminus (\{0, \infty\} \cup \mu_r)$, and for $r > 2$ one expects the answer to this question to be in general negative; see Section 3 of [Dan-Cohen and Wewers 2016] for a discussion of this question.)

After the discussion of 3-dimensional mixed Tate motives with large $\underline{\mathfrak{u}}$, in Section 6.9 we briefly consider some 4-dimensional examples; this leads again to some interesting questions about periods. One difference between the 4-dimensional and 3-dimensional examples is that in the former case (at least, a priori) one gets a family of motives with a large $\underline{\mathfrak{u}}$ when patching together a 3-dimensional L and a 2-dimensional N.

We end this introduction with some words on the organization of the paper. In Section 2 we review some basic material about Tannakian categories. The notion of extensions originating from subcategories of a Tannakian category is discussed in Section 3. Here we prove a few lemmas on this concept that will be useful throughout the paper. Starting from Section 4 we work in a Tannakian category with a weight filtration. In Section 4 we introduce the relevant objects and give the reformulation of (∗) (Theorem 4.9.1). The goal of Section 5 is to give the main results of the first part of the paper (Theorem 5.3.1 and its corollaries), in which we show that the independence axioms introduced in the same section result in refinements of Theorem 4.9.1 and Deligne's theorem. At the end of Section 5 we also prove a variant of Theorem 5.3.1 for $q > p$ case; see Theorem 5.7.1. Section 6 contains the application to motives with large unipotent radicals of motivic Galois groups, as discussed above. We should point out that prior to Section 6.8 we use the term "motive" only because we find it more suggestive: the discussion is valid in any Tannakian category with a weight filtration as long as the word "motive" is interpreted as "an object with a semisimple weight associated graded". In discussions where the Tate objects $\mathbb{Q}(n)$ play a role, we also need to assume that there is a pure object $\mathbb{Q}(1)$ of weight -2 such that the functor $-(1) := - \otimes \mathbb{Q}(1)$ is invertible. Sections 6.8 and 6.9 take place in the setting of a Tannakian category of mixed Tate motives over $\mathbb{Q}$ with the "correct" Ext groups. Finally, Section 6.10 uses 1-motives to give counterexamples to several statements in the paper, if the hypotheses regarding the independence axioms are omitted.

2. Preliminaries on Tannakian categories

The goal of this section is to review certain generalities about fundamental groups in Tannakian categories and fix some notation. None of the results in this section are new. The reader can refer to [Deligne and Milne 1982] for the basics of Tannakian categories, for instance. Throughout the paper, by a Tannakian subcategory we always mean a Tannakian subcategory that is closed under taking subobjects.

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[4] This was told to us by Clément Dupont.

2.1. ***Notation.*** For any commutative ring R, we denote the category of R-modules (resp. commutative R-algebras) by $\mathbf{Mod}_R$ (resp. $\mathbf{Alg}_R$). We often denote the Hom and End groups in a category of modules simply by Hom and End, with the coefficient ring being understood from the context.

Throughout, K is a field of characteristic zero. If V is a vector space over K, we denote the general linear group of V by $\mathrm{GL}(V)$; it is an algebraic group over K. If G is an algebraic group over K, we denote the Lie algebra of G by $\mathrm{Lie}(G)$, and the category of finite-dimensional representations of G (over K) by $\mathbf{Rep}(G)$.

As usual, given a morphism $\alpha : \omega \to \omega'$ of functors, for any object M of the domain category the corresponding morphism $\omega M \to \omega' M$ in the target category is denoted by α_M.

Finally, in various contexts, we use the notation $f|_X$ for the restriction of f to X (whatever f and X are).

2.2. By a Tannakian category over K we mean a neutral Tannakian category over K, i.e., in the language of [Deligne and Milne 1982], a rigid abelian K-linear tensor category with K as the endomorphism algebra of the unit object, for which a fiber functor over K (= an exact faithful K-linear tensor functor from the category to $\mathbf{Mod}_K$) exists.[5]

If $\boldsymbol{T}$ is a Tannakian category over K and $\omega : \boldsymbol{T} \to \mathbf{Mod}_K$ is a fiber functor (over K), we denote the fundamental group of $\boldsymbol{T}$ with respect to ω by $\mathcal{G}(\boldsymbol{T}, \omega)$ (= $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ in the standard notation); thus (by the fundamental theorem of Tannakian categories) this is an affine group scheme over K with

$$\mathcal{G}(\boldsymbol{T}, \omega)(R) = \begin{cases} \text{the group of automorphisms of the functor} \\ \qquad \omega \otimes 1_R : \boldsymbol{T} \to \mathbf{Mod}_K \to \mathbf{Mod}_R \\ \qquad \text{respecting the tensor structures} \end{cases}$$

for any K-algebra R. For any object M of $\boldsymbol{T}$, we have a representation

$$\rho_M : \mathcal{G}(\boldsymbol{T}, \omega) \to \mathrm{GL}(\omega M), \quad \sigma \mapsto \sigma_M$$

and (again by the fundamental theorem) the functor

$$\boldsymbol{T} \to \mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega)), \quad M \mapsto (\omega M, \rho_M),$$

which with abuse of notation we also denote by ω, is an equivalence of categories.

2.3. Let $\boldsymbol{T}$ be a Tannakian category over K with unit object denoted by $\mathbb{1}$. Let

$$\omega : \boldsymbol{T} \to \mathbf{Mod}_K$$

be a fiber functor. For any full Tannakian subcategory $\boldsymbol{S}$ of $\boldsymbol{T}$, the inclusion $\boldsymbol{S} \subset \boldsymbol{T}$ gives a surjective restriction map

$$\mathcal{G}(T, \omega) \to \mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}})$$

(surjective because $\boldsymbol{S}$ is assumed to be closed under taking subobjects; see [Deligne and Milne 1982, Proposition 2.21]).

[5] Actually including faithfulness here is redundant, as it follows from the rest of the requirements; see [Deligne 1990, Sections 2.10 and 2.11].

2.4. Given any objects $M_1, \dots, M_n$ of $\boldsymbol{T}$, let $\langle M_1, \dots, M_n\rangle^{\otimes}$ be the Tannakian subcategory generated by $M_1, \dots M_n$; by definition, $\langle M_1, \dots, M_n\rangle^{\otimes}$ is the smallest full Tannakian subcategory of $\boldsymbol{T}$ which contains the M_i. Every object of $\langle M_1, \dots, M_n\rangle^{\otimes}$ is obtained from $M_1, \dots M_n$ and $\mathbb{1}$ by finitely many iterations of taking direct sums, tensor products, duals, and subobjects. We have

$$\langle M_1, \dots, M_n\rangle^{\otimes} = \big\langle \textstyle\bigoplus_{1\le i\le n} M_i\big\rangle^{\otimes}.$$

2.5. Let M be an object of $\boldsymbol{T}$. Given a fiber functor ω over K, we set

$$\mathcal{G}(M, \omega) := \mathcal{G}(\langle M\rangle^{\otimes}, \omega|_{\langle M\rangle^{\otimes}}) = \underline{\mathrm{Aut}}^{\otimes}(\omega|_{\langle M\rangle^{\otimes}});$$

we call this the fundamental group of M with respect to ω. Since every object of $\langle M\rangle^{\otimes}$ is obtained from M and $\mathbb{1}$ by finitely many iterations of taking direct sums, tensor products, duals and subobjects, the map

$$\rho_M : \mathcal{G}(M, \omega) \to \mathrm{GL}(\omega M)$$

(sending σ to σ_M) is injective. In particular, $\mathcal{G}(M, \omega)$ is an algebraic group over K.

Let $\mathfrak{g}(M, \omega)$ be the Lie algebra of $\mathcal{G}(M, \omega)$. In view of the equivalence of categories

$$\langle M\rangle^{\otimes} \to \mathbf{Rep}(\mathcal{G}(M, \omega))$$

given by ω, the adjoint representation of $\mathcal{G}(M, \omega)$ defines an object $\underline{\mathfrak{g}}(M, \omega)$ in $\langle M\rangle^{\otimes}$ such that

$$\omega\underline{\mathfrak{g}}(M, \omega) = \mathfrak{g}(M, \omega)$$

as representations of $\mathcal{G}(M, \omega)$, where the $\mathcal{G}(M, \omega)$-action on $\omega\underline{\mathfrak{g}}(M, \omega)$ corresponds to $\underline{\mathfrak{g}}(M, \omega)$ (i.e., is $\rho_{\underline{\mathfrak{g}}(M,\omega)}$) and the $\mathcal{G}(M, \omega)$-action on $\mathfrak{g}(M, \omega)$ is given by the adjoint representation.

Identify $\mathcal{G}(M, \omega)$ as a subgroup of $\mathrm{GL}(\omega M)$ via ρ_M. This identifies

$$\mathfrak{g}(M, \omega) \subset \mathrm{Lie}(\mathrm{GL}(\omega M)) = \mathrm{End}(\omega M). \tag{6}$$

Denote $\underline{\mathrm{End}}(M) := \underline{\mathrm{Hom}}(M, M)$ (the internal Hom in $\boldsymbol{T}$). Then we can identify $\omega\underline{\mathrm{End}}(M) = \mathrm{End}(\omega M)$, with the action of $\mathcal{G}(M, \omega)$ on $\mathrm{End}(\omega M)$ corresponding to $\underline{\mathrm{End}}(M)$ being by conjugation. The inclusion (6) is compatible with the actions of $\mathcal{G}(M, \omega)$, making

$$\underline{\mathfrak{g}}(M, \omega) \subset \underline{\mathrm{End}}(M).$$

2.6. For any object N of $\langle M\rangle^{\otimes}$, let $\mathcal{G}(M, N, \omega)$ be the kernel of the surjection

$$\mathcal{G}(M, \omega) \to \mathcal{G}(N, \omega)$$

induced by the inclusion $\langle N\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$ (so for instance, $\mathcal{G}(M, \mathbb{1}, \omega) = \mathcal{G}(M, \omega)$). The Lie subalgebra

$$\mathfrak{g}(M, N, \omega) := \mathrm{Lie}(\mathcal{G}(M, N, \omega))$$

of $\mathfrak{g}(M, \omega)$ is invariant under the adjoint action of $\mathcal{G}(M, \omega)$, giving rise to a subobject

$$\underline{\mathfrak{g}}(M, N, \omega) \subset \underline{\mathfrak{g}}(M, \omega) \subset \underline{\mathrm{End}}(M).$$

2.7. The subobjects $\mathfrak{g}(M, N, \omega)$ of $\underline{\mathrm{End}}(M)$ do not depend on the choice of the fiber functor ω. More precisely, for every object N of $\langle M\rangle^{\otimes}$, there is a canonical subobject

$$\underline{\mathfrak{g}}(M, N) \subset \underline{\mathrm{End}}(M)$$

such that for every ω over K,

$$\omega\underline{\mathfrak{g}}(M, N) = \mathfrak{g}(M, N, \omega) \subset \mathrm{End}(\omega M).$$

This can be seen via the machinery of algebraic geometry over a Tannakian category [Deligne 1989, Section 5 and 6] and is well-known, but in the interest of keeping the paper more self-contained, we include a proof.

Proposition 2.7.1. *Suppose ω and ω' are two fiber functors $\boldsymbol{T} \to \mathbf{Mod}_K$. Then for any objects M of $\boldsymbol{T}$ and N of $\langle M\rangle^{\otimes}$,*

$$\underline{\mathfrak{g}}(M, N, \omega) = \underline{\mathfrak{g}}(M, N, \omega')$$

(*as subobjects of* $\underline{\mathrm{End}}(M)$).

Proof. By a theorem of Deligne [1990, Section 1.12 and 1.13], there exists a K-algebra R such that the two functors $\omega \otimes 1_R$ and $\omega' \otimes 1_R$ are isomorphic as $\otimes$-functors. Let

$$\alpha : \omega \otimes 1_R \to \omega' \otimes 1_R$$

be an isomorphism respecting the tensor structures. Then conjugation by $\alpha|_{\langle M\rangle^{\otimes}}$ gives an isomorphism

$$c_\alpha : \mathcal{G}(M, \omega)_R \to \mathcal{G}(M, \omega')_R.$$

On the other hand, conjugation by

$$\alpha_M : \omega M \otimes 1_R \to \omega' M \otimes 1_R$$

gives an isomorphism

$$c_{\alpha_M} : \mathrm{GL}(\omega M)_R \to \mathrm{GL}(\omega' M)_R.$$

The maps c_α and c_{α_M} are compatible with one another, i.e., we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}(M, \omega)_R & \xrightarrow{c_\alpha, \simeq} & \mathcal{G}(M, \omega')_R \\ \cap & & \cap \\ \mathrm{GL}(\omega M)_R & \xrightarrow{c_{\alpha_M}, \simeq} & \mathrm{GL}(\omega' M)_R, \end{array}$$

where the vertical inclusions are by the identifications via ρ_M for ω and ω' (i.e., are given by $\sigma \mapsto \sigma_M$). Going to the Lie algebras by taking derivatives we get a commutative diagram

$$\begin{array}{ccc} \omega\underline{\mathfrak{g}}(M,\omega)\otimes R = \mathfrak{g}(M,\omega)\otimes R & \xrightarrow{Dc_\alpha,\simeq} & \mathfrak{g}(M,\omega')\otimes R = \omega'\underline{\mathfrak{g}}(M,\omega')\otimes R \\ \cap & & \cap \\ \omega\underline{\mathrm{End}}(M)\otimes R = \mathrm{End}(\omega M)\otimes R & \xrightarrow{Dc_{\alpha_M},\simeq} & \mathrm{End}(\omega' M)\otimes R = \omega'\underline{\mathrm{End}}(M). \end{array}$$

The horizontal arrow in the second row is again just conjugation by α_M, so that

$$Dc_{\alpha_M} = \alpha_{\underline{\mathrm{End}}(M)}.$$

On recalling that $\underline{\mathfrak{g}}(M,\omega)$ is a subobject of $\underline{\mathrm{End}}(M)$ and by commutativity of the previous diagram, we get

$$\omega'\underline{\mathfrak{g}}(M,\omega)\otimes R = \alpha_{\underline{\mathrm{End}}(M)}(\omega\underline{\mathfrak{g}}(M,\omega)\otimes R) = \omega'\underline{\mathfrak{g}}(M,\omega')\otimes R \tag{7}$$

(as subspaces of $\mathrm{End}(\omega' M)\otimes R$). This shows that

$$\omega'\underline{\mathfrak{g}}(M,\omega) = \omega'\underline{\mathfrak{g}}(M,\omega')$$

and hence $\underline{\mathfrak{g}}(M,\omega) = \underline{\mathfrak{g}}(M,\omega')$.

If N is any object of $\langle M\rangle^{\otimes}$, by considering the analogous map to c_α for N one easily sees that c_α maps $\mathcal{G}(M,N,\omega)_R$ onto $\mathcal{G}(M,N,\omega')_R$. Thus

$$Dc_\alpha(\mathfrak{g}(M,N,\omega)\otimes R) = \mathfrak{g}(M,N,\omega')\otimes R,$$

and as in (7) we get

$$\omega'\underline{\mathfrak{g}}(M,N,\omega) = \omega'\underline{\mathfrak{g}}(M,N,\omega')$$

as subspaces of $\mathrm{End}(\omega' M)$. □

3. Extensions originating from a subcategory

The goal of this section is to introduce and prove a few lemmas about the basic but useful notion of extensions originating from subcategories of Tannakian categories. This concept will provide a natural language for the results of the paper. As in the previous section, K is a field of characteristic zero. Recall that by a Tannakian subcategory we mean one that is closed under taking subobjects.

3.1. Let G be an affine group scheme over K. Let H be a subgroup of G. Let V be an object of $\mathbf{Rep}(G)$. Denote by V^H the (K-) subspace of V which is fixed by H. More precisely,

$$V^H := \{v \in V : \forall R \in \mathbf{Alg}_K, \forall \sigma \in H(R), \sigma(v\otimes 1_R) = v\otimes 1_R\}.$$

Suppose H is normal in G. Then V^H is a G-subrepresentation of V (i.e., a subobject of V in $\mathbf{Rep}(G)$).

The restriction functor

$$\mathbf{Rep}(G/H) \to \mathbf{Rep}(G)$$

identifies $\mathbf{Rep}(G/H)$ as the full subcategory of $\mathbf{Rep}(G)$ consisting of those representation of G on which H acts trivially. It is evident that for every object V of $\mathbf{Rep}(G)$, the object V^H is the largest subobject of V which belongs to the subcategory $\mathbf{Rep}(G/H)$.

3.2. Let $\boldsymbol{T}$ be a Tannakian category over K, with ω a fiber functor $\boldsymbol{T} \to \mathbf{Mod}_K$. Let $\boldsymbol{S}$ be a full Tannakian subcategory of $\boldsymbol{T}$. The inclusion $\boldsymbol{S} \subset \boldsymbol{T}$ gives a surjection

$$\mathcal{G}(\boldsymbol{T}, \omega) \to \mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}}). \tag{8}$$

Denote the kernel of this map by $\mathcal{H}$.

Using the map (8) we may identify the category $\mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}}))$ as the full subcategory of $\mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega))$ consisting of all the objects on which $\mathcal{H}$ acts trivially. One has a commutative diagram

$$\begin{array}{ccc} \boldsymbol{S} & \xrightarrow{\omega|_{\boldsymbol{S}}, \simeq} & \mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}})) \\ \cap & & \cap \\ \boldsymbol{T} & \xrightarrow{\omega, \simeq} & \mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega)), \end{array} \tag{9}$$

where the horizontal arrows are the equivalences of categories given by the fundamental theorem of Tannakian categories. On recalling that $\boldsymbol{S}$ is closed under subobjects and hence in particular isomorphisms, it follows that any object A of $\boldsymbol{T}$ belongs to the subcategory $\boldsymbol{S}$ if and only if $\mathcal{H}$ acts trivially on ωA.

3.3. Let A be an object of $\boldsymbol{T}$. Then $(\omega A)^{\mathcal{H}}$ is a $\mathcal{G}(\boldsymbol{T}, \omega)$-subrepresentation of ωA; hence there is a canonical subobject

$$A_{\boldsymbol{S}} \subset A$$

such that

$$\omega(A_{\boldsymbol{S}}) = (\omega A)^{\mathcal{H}}.$$

Since $(\omega A)^{\mathcal{H}}$ is the largest subobject of $\omega A \in \mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega))$ which belongs to the subcategory $\mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}}))$, it follows that $A_{\boldsymbol{S}}$ is the largest subobject of A which belongs to $\boldsymbol{S}$.

Taking $\mathcal{H}$-invariants gives a left exact functor

$$\mathbf{Rep}(\mathcal{G}(\boldsymbol{T}, \omega)) \to \mathbf{Rep}(\mathcal{G}(\boldsymbol{S}, \omega|_{\boldsymbol{S}})).$$

Thus we have a left exact functor

$$-_{\boldsymbol{S}} : \boldsymbol{T} \to \boldsymbol{S}$$

which on objects acts like $A \mapsto A_{\boldsymbol{S}}$ (and on morphisms acts by restriction of domain and codomain).

3.4. Let A be an object of $\boldsymbol{T}$. Let $\mathscr{E}$ in $\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A)$ (= Yoneda Ext^1 group in $\boldsymbol{T}$) be the class of the short exact sequence

$$0 \to A \to E \to \mathbb{1} \to 0.$$

We say the extension $\mathscr{E}$ *originates from* or *comes from* $\boldsymbol{S}$ if there is a commutative diagram in $\boldsymbol{T}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \end{array} \tag{10}$$

where the rows are exact and the objects in the top row are in $\boldsymbol{S}$. In other words, we say $\mathscr{E}$ originates from $\boldsymbol{S}$ if there is an object A' of $\boldsymbol{S}$ and a morphism $A' \to A$ such that $\mathscr{E}$ is in the image of the pushforward map

$$\mathrm{Ext}^1_{\boldsymbol{S}}(\mathbb{1}, A') \to \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A).$$

We now give a few lemmas on the notion of extensions originating from subcategories which are useful in the later sections. The lemmas take place in the above setting (i.e., with $\mathscr{E}$, $\boldsymbol{S}$, and $\mathscr{H}$ as above). The first lemma highlights that the notion of extensions originating from subcategories is a generalization of the notion of splitting of sequences.

Lemma 3.4.1. *The following statements are equivalent*:

(i) *The extension $\mathscr{E}$ splits.*

(ii) *The extension $\mathscr{E}$ originates from some semisimple $\boldsymbol{S}$.*

(iii) *The extension $\mathscr{E}$ originates from every $\boldsymbol{S}$.*

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are trivial. As for (i) $\Longrightarrow$ (iii), note that if $\mathscr{E}$ splits, then it is the pushforward of the extension

$$0 \to 0 \to \mathbb{1} \to \mathbb{1} \to 0.$$

□

Lemma 3.4.2. *The following statements are equivalent*:

(i) *The extension $\mathscr{E}$ originates from $\boldsymbol{S}$.*

(ii) *The extension $\omega\mathscr{E}$*

$$0 \to \omega A \to \omega E \to K \to 0$$

splits in the category of representations of $\mathscr{H}$.

(iii) *The sequence*

$$0 \to (\omega A)^{\mathscr{H}} \to (\omega E)^{\mathscr{H}} \to K \to 0$$

(*obtained by taking $\mathscr{H}$-invariants of $\omega\mathscr{E}$*) *is exact.*

(iv) *The sequence in* $\boldsymbol{S}$

$$0 \to A_{\boldsymbol{S}} \to E_{\boldsymbol{S}} \to \mathbb{1} \to 0$$

obtained by applying $-_{\boldsymbol{S}}$ *to the defining sequence of* $\mathscr{E}$ *is exact.*

Proof. The equivalence of (iii) and (iv) is clear, as the sequence in (iii) is obtained by applying ω to the sequence in (iv). Note that since the functors $-^{\mathscr{H}}$ and $-_{\boldsymbol{S}}$ are left exact, the statements in (iii) and (iv) are really just statements about surjectivity of $(\omega E)^{\mathscr{H}} \to K$ and $E_{\boldsymbol{S}} \to \mathbb{1}$. The implication (iv) $\Longrightarrow$ (i) is also clear, as we can use the extension given in (iv) as the top row in (10).

(i) $\Longrightarrow$ (iv): Suppose $\mathscr{E}$ originates from $\boldsymbol{S}$, with a commutative diagram as in (10), with exact rows and the top row in $\boldsymbol{S}$. Since $\boldsymbol{S}$ is closed under taking subquotients, by replacing A' and E' if necessary by their images in A and E, we may assume without loss of generality that $A' \subset A$ and $E' \subset E$, with the vertical arrows being considered as inclusion maps. Since E' is in $\boldsymbol{S}$, we have $E' \subset E_{\boldsymbol{S}}$. This proves that the restriction of the surjection $E \to \mathbb{1}$ to $E_{\boldsymbol{S}}$ is still surjective, thus giving (iv).

(iii) $\Longrightarrow$ (ii): There is a commutative diagram of $\mathscr{G}(\boldsymbol{T}, \omega)$-representations

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\omega A)^{\mathscr{H}} & \longrightarrow & (\omega E)^{\mathscr{H}} & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \omega A & \longrightarrow & \omega E & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

where the bottom row is $\omega\mathscr{E}$, the vertical arrows are inclusion, and the rows are exact. Consider this diagram in the category of representations of $\mathscr{H}$. The top row splits, hence so does the bottom row (= the pushout of the top row).

(ii) $\Longrightarrow$ (iii): Suppose (ii) holds. Choose a section s of $\omega E \to K$ in $\mathbf{Rep}(\mathscr{H})$. Then $s(1)$ is fixed by $\mathscr{H}$ and thus belongs to $(\omega E)^{\mathscr{H}}$. It follows that $(\omega E)^{\mathscr{H}} \to K$ is surjective. □

Lemma 3.4.3. *Suppose* A *is an object of* $\boldsymbol{S}$. *Then* $\mathscr{E}$ *originates from* $\boldsymbol{S}$ *if and only if* E *is an object of* $\boldsymbol{S}$.

Proof. The "if" implication is trivial. As for the "only if" implication, suppose we have a diagram as in (10), with exact rows and the objects of the top row in $\boldsymbol{S}$. Then E is isomorphic to the fibered coproduct of A and E' over A'. Since A and E' are in $\boldsymbol{S}$, so is E. □

Lemma 3.4.4. *Let* A' *be a subobject of* A *such that the pushforward map*

$$\operatorname{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' + A_{\boldsymbol{S}}) \to \operatorname{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A)$$

(*along the inclusion* $A' + A_{\boldsymbol{S}} \to A$) *is injective. Suppose* $\mathscr{E}$ *is the pushforward of an extension*

$$\mathscr{E}' \in \operatorname{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A')$$

along the inclusion map $A' \to A$. *Then* $\mathscr{E}$ *originates from* $\boldsymbol{S}$ *if and only if* $\mathscr{E}'$ *does.*

Proof. If $\mathscr{E}'$ originates from $\boldsymbol{S}$, then clearly so does $\mathscr{E}$. Suppose $\mathscr{E}$ originates from $\boldsymbol{S}$. Then $\mathscr{E}$ is the pushforward of the extension $\mathscr{E}_{\boldsymbol{S}}$ given in statement (iv) of Lemma 3.4.2 under the inclusion $A_{\boldsymbol{S}} \to A$.

Let $i : A_S \to A' + A_S$ and $i' : A' \to A' + A_S$ be inclusion maps. Apply the δ-functor $\mathrm{Hom}_{\boldsymbol{T}}(\mathbb{1}, -)$ to the short exact sequence

$$0 \to A' \cap A_S \to A' \oplus A_S \xrightarrow{i-i'} A' + A_S \to 0$$

(where the injective arrow is the diagonal embedding). We get exact

$$\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' \cap A_S) \to \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A') \oplus \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A_S) \xrightarrow{i_*-i'_*} \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' + A_S),$$

where the lower stars denote pushforwards. The pushforward of the extension

$$i_*(\mathscr{E}_S) - i'_*(\mathscr{E}') \in \mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A' + A_S)$$

in $\mathrm{Ext}^1_{\boldsymbol{T}}(\mathbb{1}, A)$ is zero. By the injectivity hypothesis in the statement, $i_*(\mathscr{E}_S) - i'_*(\mathscr{E}')$ is already zero. It follows that there is an extension $\mathscr{E}''$

$$0 \to A' \cap A_S \to E'' \to \mathbb{1} \to 0$$

which pushes forward (under inclusion maps) to both $\mathscr{E}'$ and $\mathscr{E}_S$. But then $A' \cap A_S$ and E'', being subobjects of A_S and E_S, belong to $\boldsymbol{S}$. Since $\mathscr{E}''$ pushes forward to $\mathscr{E}'$, the latter extension originates from $\boldsymbol{S}$. □

Remark. Note that the injectivity hypothesis in the statement of the previous lemma is guaranteed if

$$\mathrm{Hom}_{\boldsymbol{T}}(\mathbb{1}, A/(A' + A_S)) = 0$$

(and this will be the case whenever we use the result in the paper). This can be seen from the long exact sequence obtained by applying $\mathrm{Hom}_{\boldsymbol{T}}(\mathbb{1}, -)$ to

$$0 \to A' + A_S \to A \to A/(A' + A_S) \to 0.$$

4. Extension classes and subgroups of the fundamental group, part I

4.1. From this point on we suppose that $\boldsymbol{T}$ is a Tannakian category over a field K of characteristic zero, equipped with a functorial exact finite increasing filtration $W_\bullet$, compatible with the tensor structure. We refer to $W_\bullet$ as the weight filtration. Here, the expression "functorial exact finite increasing filtration $W_\bullet$" means that for every integer n, we have an exact functor $W_n : \boldsymbol{T} \to \boldsymbol{T}$, such that for every object M of $\boldsymbol{T}$, we have

$$\begin{aligned} W_{n-1}M &\subset W_n M && (\forall n) \\ W_n M &= 0 && (\forall n \ll 0) \\ W_n M &= M && (\forall n \gg 0), \end{aligned}$$

and such that the inclusions $W_n M \subset M$ for various M give a morphism of functors from W_n to the identity (and hence the W_n form an inductive system of functors). Compatibility with the tensor product means that for every objects M and N, we have

$$W_n(M \otimes N) = \sum_{\substack{p,q \\ p+q=n}} W_p M \otimes W_q N. \tag{11}$$

The associated graded functor Gr^W is the functor defined on objects by

$$\mathrm{Gr}^W M := \bigoplus_n \mathrm{Gr}^W_n M,$$

where $\mathrm{Gr}^W_n M := W_n M / W_{n-1} M$, and on morphisms in the obvious way using the fact that we have morphisms of functors $W_{n-1} \to W_n$. By the snake lemma, the associated graded functor (in fact, each Gr^W_n) is also exact. Also Gr^W is a graded tensor functor, in the sense that (via a canonical isomorphism) we have

$$\mathrm{Gr}^W(M \otimes N) = \mathrm{Gr}^W(M) \otimes \mathrm{Gr}^W(N),$$

with this identification being compatible with weights, i.e., being the direct sum of identifications

$$\mathrm{Gr}^W_n(M \otimes N) = \bigoplus_{\substack{p,q \\ p+q=n}} \mathrm{Gr}^W_p M \otimes \mathrm{Gr}^W_q M$$

induced by (11).

As it is customary, we call an object M with $W_{n-1}M = 0$ and $W_n M = M$ a pure object of weight n. Note that unless otherwise indicated, we do not assume that an object of the form $\mathrm{Gr}^W M$ (i.e., a direct sum of pure objects) is necessarily semisimple.

Given any fiber functor ω (over K) and any object M, set

$$W_\bullet \omega M := \omega(W_\bullet M).$$

This defines an exact $\otimes$-filtration on ω, in the language of Saavedra Rivano [1972, Chapter IV, Section 2], (note that Saavedra Rivano works with decreasing filtrations instead, and that his Condition FE 1) is guaranteed here because K is a field).

Given any objects M and N, we identify

$$\omega \underline{\mathrm{Hom}}(M, N) = \mathrm{Hom}(\omega M, \omega N).$$

One can then show that

$$\omega W_n \underline{\mathrm{Hom}}(M, N) = \{ f \in \mathrm{Hom}(\omega M, \omega N) : f(W_\bullet \omega M) \subset W_{\bullet+n} \omega N \}.$$

4.2. Here and elsewhere in the paper, we shall use the notation and conventions of Section 2 for Tannakian fundamental groups and their Lie algebras.

Let M be an object of $\boldsymbol{T}$. Given any fiber functor ω, let $P(M, \omega)$ be the parabolic subgroup of $\mathrm{GL}(\omega M)$ which stabilizes the filtration $W_\bullet$. Then

$$\mathrm{Lie}(P(M, \omega)) = W_0 \,\mathrm{End}(\omega M).$$

The elements of $\mathcal{G}(M, \omega)$ (= the fundamental group of M with respect to ω) preserve subobjects of M, so that

$$\mathcal{G}(M, \omega) \subset P(M, \omega).$$

Going to the Lie algebras we have

$$\underline{\mathfrak{g}}(M) \subset W_0\underline{\mathrm{End}}(M).$$

Every element of $P(M,\omega)$ induces an automorphism of $\mathrm{Gr}^W\,\omega M$, giving rise to a homomorphism

$$P(M,\omega) \to \mathrm{GL}(\mathrm{Gr}^W\,\omega M).$$

Let $U(M,\omega)$ be the kernel of this map; then $U(M,\omega)$ is the unipotent radical of $P(M,\omega)$. It is easy to see that

$$\mathrm{Lie}(U(M,\omega)) = W_{-1}\,\mathrm{End}(\omega M).$$

Set

$$\mathcal{U}(M,\omega) := \mathcal{G}(M, \mathrm{Gr}^W M, \omega)$$

(= the kernel of the restriction map $\mathcal{G}(M,\omega) \to \mathcal{G}(\mathrm{Gr}^W M,\omega)$ induced by the inclusion $\langle \mathrm{Gr}^W M\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$). Then

$$\mathcal{U}(M,\omega) = \mathcal{G}(M,\omega) \cap U(M,\omega). \tag{12}$$

In particular, $\mathcal{U}(M,\omega)$ is a unipotent group. If $\mathcal{G}(\mathrm{Gr}^W M,\omega)$ happens to be reductive (i.e., if $\mathrm{Gr}^W M$ is semisimple), then $\mathcal{U}(M,\omega)$ will be the unipotent radical of $\mathcal{G}(M,\omega)$.

We set

$$\underline{\mathfrak{u}}(M) := \underline{\mathfrak{g}}(M, \mathrm{Gr}^W M) \quad \text{and} \quad \mathfrak{u}(M,\omega) := \mathrm{Lie}\,\mathcal{U}(M,\omega)$$

(= $\mathfrak{g}(M, \mathrm{Gr}^W M, \omega)$ in the notation of Section 2). Then (for every ω),

$$\omega\underline{\mathfrak{u}}(M) = \mathfrak{u}(M,\omega).$$

By (12), we have

$$\underline{\mathfrak{u}}(M) = \underline{\mathfrak{g}}(M) \cap W_{-1}\underline{\mathrm{End}}(M).$$

4.3. A result of Deligne (written by Jossen in the appendix of [Jossen 2014]) describes the subobject $\underline{\mathfrak{u}}(M)$ of $W_{-1}\underline{\mathrm{End}}(M)$ as follows.[6] From now on, if there is no ambiguity, we shall simply write Hom (resp. Ext^i) for the Hom groups $\mathrm{Hom}_{\boldsymbol{T}}$ (resp. the Yoneda $\mathrm{Ext}^i_{\boldsymbol{T}}$ groups) in $\boldsymbol{T}$.

Recall from the Introduction that for each integer p, the p-th extension class

$$\mathcal{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

of M is the extension corresponding to the sequence

$$0 \to W_pM \to M \to M/W_pM \to 0 \tag{13}$$

under the canonical isomorphism

$$\mathrm{Ext}^1(M/W_pM, W_pM) \cong \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)). \tag{14}$$

[6] We thank Peter Jossen for patiently explaining to us some parts of Deligne's argument from [Jossen 2014, Appendix].

Applying $\underline{\mathrm{Hom}}(M/W_pM, -)$ to the inclusion $W_pM \to M$ we get an injection

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM) \to \underline{\mathrm{Hom}}(M/W_pM, M).$$

On the other hand, applying $\underline{\mathrm{Hom}}(-, M)$ to the quotient map $M \to M/W_pM$ we get an injection

$$\underline{\mathrm{Hom}}(M/W_pM, M) \to \underline{\mathrm{End}}(M).$$

Composing the two injections, we get a map

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM) \to \underline{\mathrm{End}}(M). \tag{15}$$

After applying a fiber functor ω, this simply sends an element

$$f \in \mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM)$$

to the composition

$$\omega M \xrightarrow{\text{quotient}} \omega M/\omega W_pM \xrightarrow{f} \omega W_pM \xrightarrow{\text{inclusion}} \omega M. \tag{16}$$

From this it is clear that indeed, the image of the map (15) is contained in $W_{-1}\underline{\mathrm{End}}(M)$. We shall identify $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ as a subobject of $W_{-1}\underline{\mathrm{End}}(M)$ via the map (15). Note that $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ is an abelian Lie subalgebra of $W_{-1}\underline{\mathrm{End}}(M)$.

Pushing forward extensions along the inclusion map we get a map

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)), \tag{17}$$

which is injective, as (by weight considerations),

$$\mathrm{Hom}\left(\mathbb{1}, \frac{W_{-1}\underline{\mathrm{End}}(M)}{\underline{\mathrm{Hom}}(M/W_pM, W_pM)}\right) = 0.$$

To simplify the notation, we shall identify

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

with its image under (17).

Deligne defines the (total) extension class of M to be

$$\mathscr{E}(M) := \sum_p \mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M))$$

(this is denoted by $\mathrm{cl}(M)$ in [Jossen 2014]), and proves that the extension $\mathscr{E}(M)$ can be used to describe $\underline{\mathfrak{u}}(M)$. More precisely, he proves the following result:

Theorem 4.3.1 (Deligne, Appendix of [Jossen 2014]). *The subobject $\underline{\mathfrak{u}}(M) \subset W_{-1}\underline{\mathrm{End}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{End}}(M)$ such that the extension $\mathscr{E}(M)$ is the pushforward of an element of $\mathrm{Ext}^1(\mathbb{1}, \underline{\mathfrak{u}}(M))$ under the inclusion $\underline{\mathfrak{u}}(M) \to W_{-1}\underline{\mathrm{End}}(M)$.*

It is worth highlighting that the theorem asserts that $\underline{\mathfrak{u}}(M)$ is the smallest subobject with the stated property, not just the smallest Lie subobject with the property. Also note that by weight considerations, the pushforward map

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathfrak{u}}(M)) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)) \tag{18}$$

is injective, so that the element pushing forward to $\mathscr{E}(M)$ is indeed unique.

Remark. As we pointed out in the Introduction, in general, the individual extensions $\mathscr{E}_p(M)$ may not be in the image of the pushforward map (18). See Section 6.10 (and Remark (2) therein) for examples in the category of mixed Hodge structures using the Jacquinot–Ribet deficient points on semiabelian varieties.

4.4. We adopt the following notation for pushforwards of extensions along quotient maps. If $\mathscr{E}$ is an extension of an object A by B, then for any subobject B' of B we denote the pushforward of $\mathscr{E}$ along the quotient $B \to B/B'$ by $\mathscr{E}/B'$.

Given any subobject $A \subset W_{-1}\underline{\mathrm{End}}(M)$, applying the functor $\mathrm{Hom}(\mathbb{1}, -)$ to the short exact sequence

$$0 \to A \to W_{-1}\underline{\mathrm{End}}(M) \to W_{-1}\underline{\mathrm{End}}(M)/A \to 0$$

we get a long exact sequence. In particular, we have exact

$$\mathrm{Ext}^1(\mathbb{1}, A) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/A),$$

where the arrows are pushforwards along inclusion and quotient maps. Thus Deligne's result can be equivalently stated as that $\underline{\mathfrak{u}}(M)$ is the smallest subobject of $W_{-1}\underline{\mathrm{End}}(M)$ such that the pushforward

$$\mathscr{E}(M)/\underline{\mathfrak{u}}(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}(M))$$

of $\mathscr{E}(M)$ splits.

The formulation of Theorem 4.3.1 as given in the statement is more natural for Deligne's proof, as his argument goes by constructing an explicit extension of $\mathbb{1}$ by $\underline{\mathfrak{u}}(M)$ which pushes forward to $\mathscr{E}(M)$. The formulation in terms of $\mathscr{E}(M)/\underline{\mathfrak{u}}(M)$ is however more natural when one wants to study the individual extensions $\mathscr{E}_p(M)$, as we shall see.

4.5. The canonical isomorphism (14) is given by first applying the functor

$$\underline{\mathrm{Hom}}(M/W_pM, -)$$

to an element of $\mathrm{Ext}^1(M/W_pM, W_pM)$, and then pulling back along the canonical map

$$\mathbb{1} \to \underline{\mathrm{End}}(M/W_pM)$$

(which after applying a fiber functor ω, sends 1 to the identity map on $\omega(M/W_pM)$). Going through this, we see that assuming $M/W_pM \neq 0$, the extension

$$\mathscr{E}_p \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

is the class of

$$0 \to \underline{\mathrm{Hom}}(M/W_pM, W_pM) \to \underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger} \to \mathbb{1} \to 0, \tag{19}$$

where $\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}$ is the subobject of $\underline{\mathrm{Hom}}(M/W_pM, M)$ characterized by

$$\begin{aligned}\omega\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger} &= \mathrm{Hom}(\omega M/\omega W_pM, \omega M)^{\dagger}\\ &:= \{f \in \mathrm{Hom}(\omega M/\omega W_pM, \omega M) : f \bmod \omega W_pM = \lambda(f) Id_{\omega M/\omega W_pM} \text{ for some } \lambda(f) \in K\}\end{aligned}$$

for any fiber functor ω. The injective (resp. surjective) arrow in (19) is, after applying ω, the natural inclusion (resp. the map $f \mapsto \lambda(f)$, with $\lambda(f) \in K$ as in the definition of $\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}$ above).

If $M/W_pM = 0$, set $\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger} := \mathbb{1}$; then $\mathscr{E}_p$ is again given by the sequence (19), with the surjective arrow being the identity map on $\mathbb{1}$.[7]

4.6. Fix an integer p. After applying a fiber functor ω to the identification

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM) \subset W_{-1}\underline{\mathrm{End}}(M)$$

we get an identification

$$\mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM) \subset W_{-1}\,\mathrm{End}(\omega M),$$

which thinks of $f : \omega M/W_pM \to \omega W_pM$ as the composition (16). This way,

$$\mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM) \tag{20}$$

becomes an abelian Lie subalgebra of $W_{-1}\,\mathrm{End}(\omega M)$. The exponential map

$$\exp : W_{-1}\,\mathrm{End}(\omega M) \to U(M, \omega)(K) \subset \mathrm{GL}(\omega M)(K)$$

is given by the usual exponential series. On the Lie subalgebra (20), it is simply given by

$$\exp(f) = I + f.$$

4.7. In this subsection we shall introduce certain Lie subalgebras of $\underline{\mathfrak{u}}(M)$ and subgroups of $\mathcal{U}(M, \omega)$ (for any ω) which play a crucial role in the paper. For any integer p, let

$$\underline{\mathfrak{u}}_p(M) := \underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_pM, W_pM)$$

[7] Equivalently, one can define $\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}$ in the following way, which works in all cases: $\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}$ is the subobject of $\underline{\mathrm{Hom}}(M/W_pM, M) \oplus \mathbb{1}$ whose image under any fiber functor ω is

$$\{(f, \lambda) \in \mathrm{Hom}(\omega M/\omega W_pM, \omega M) \oplus K : f \bmod \omega W_pM = \lambda Id_{\omega M/\omega W_pM}\}.$$

That is, $\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}$ is the kernel of the appropriate morphism $\underline{\mathrm{Hom}}(M/W_pM, M) \oplus \mathbb{1} \to \underline{\mathrm{End}}(M)$. The injective (resp. surjective) arrow in (19) is then induced by the inclusion (resp. projection) map into (resp. from) the direct sum. We shall however work with the first definition, as it will simplify the expressions in our proofs.

and for any ω,

$$\mathfrak{u}_p(M,\omega) := \omega\underline{\mathfrak{u}}_p(M) = \mathfrak{u}(M,\omega) \cap \operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M).$$

Then $\mathfrak{u}_p(M,\omega)$ is an abelian Lie subalgebra of $\mathfrak{u}(M,\omega)$.

For any Lie subalgebra $\mathfrak{l}$ of $W_{-1}\operatorname{End}(\omega M)$, we denote the subgroup of $U(M,\omega)$ whose Lie algebra is $\mathfrak{l}$ by $\mathcal{exp}(\mathfrak{l})$ (thus $\mathcal{exp}(\mathfrak{l})(K) = \exp(\mathfrak{l})$). Set

$$\begin{aligned}\mathcal{U}_p(M,\omega) &:= \mathcal{exp}(\mathfrak{u}_p(M,\omega))\\ &= \mathcal{U}(M,\omega) \cap \mathcal{exp}(\operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M))\\ &= \mathcal{G}(M,\omega) \cap \mathcal{exp}(\operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M)).\end{aligned}$$

This is an abelian unipotent subgroup of $\mathcal{U}(M,\omega)$.

Lemma 4.7.1. *$\mathcal{U}_p(M,\omega)$ is the kernel of the restriction homomorphism*

$$\mathcal{G}(M,\omega) \to \mathcal{G}(W_p M \oplus (M/W_p M), \omega)$$

(*induced by* $\langle W_p M \oplus (M/W_p M)\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$).

Proof. Tentatively, let us refer to the kernel of the homomorphism given in the statement of the lemma as U'. It is clear that U' is contained in $\mathcal{U}(M,\omega)$. In particular, U' is also unipotent and thus it is enough to show that U' and $\mathcal{U}_p(M,\omega)$ have the same K-valued points. We have

$$\mathcal{U}_p(M,\omega)(K) = \mathcal{G}(M,\omega)(K) \cap \exp(\operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M)).$$

Let $\sigma \in \mathcal{G}(M,\omega)(K)$. Then $\sigma \in U'(K)$ if and only if $\sigma_{W_p M} = I$ and $\sigma_{M/W_p M} = I$. Under the identification $\mathcal{G}(M,\omega) \subset P(M,\omega)$ (via $\sigma \mapsto \sigma_M$), $\sigma_{W_p M}$ is simply the restriction $\sigma|_{\omega W_p M}$ of σ to $\omega W_p M$, and $\sigma_{M/W_p M}$ is the map $\bar{\sigma}$ that σ, as an element of the parabolic subgroup $P(M,\omega)$, induces on $\omega M/\omega W_p M$ (given by $\bar{\sigma}(v + \omega W_p) = \sigma(v) + \omega W_p$, where $v \in \omega M$). On recalling that

$$\exp(\operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M)) = I + \operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M),$$

it is easy to see that the subgroup of $P(M,\omega)(K)$ which acts as identity on both $\omega W_p M$ and $\omega M/\omega W_p M$ is

$$\exp(\operatorname{Hom}(\omega M/\omega W_p M, \omega W_p M)).$$

The claim follows. □

Remark. Our examples in Section 6.10 (also see item (3) of the remark therein) show that in general, $\underline{\mathfrak{u}}(M)$ may not be generated by the $\underline{\mathfrak{u}}_p(M)$, even as a Lie algebra. It is however true that if $\mathcal{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits for every p, then $\underline{\mathfrak{u}}(M) = \sum_p \underline{\mathfrak{u}}_p(M)$. See item (2) of the remark at the end of Section 5.1. (Note that the sum $\sum_p \underline{\mathfrak{u}}_p(M)$ in general may not be a Lie subalgebra of $\underline{\mathfrak{u}}(M)$.)

4.8. Let us recall (∗) from the Introduction.

Proposition 4.8.1. *For any subobject A of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$, we have $\underline{\mathfrak{u}}_p(M) \subset A$ if and only if the quotient*

$$\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A$$

belongs to the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$.

This follows from Theorem 3.3.1 of [Eskandari and Murty 2021],[8] with L, N, and $\mathcal{U}(M)$ of [loc. cit.] being respectively W_pM, M/W_pM, and $\mathcal{U}_p(M,\omega)$ here. However, in the interest of keeping the paper more self-contained, let us recall the argument: The statement is trivial if $M/W_pM = 0$ so we may assume otherwise. To simplify the notation, let us tentatively denote the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ by $\boldsymbol{C}$. Let A be a subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ and ω a fiber functor. In view of Section 3.2 and Lemma 4.7.1, the quotient

$$\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A$$

belongs to $\boldsymbol{C}$ if and only if $\mathcal{U}_p(M,\omega)$ acts trivially on

$$\omega(\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A) = \omega\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/\omega A. \tag{21}$$

Choose a section of the natural surjection $\omega M \to \omega M/\omega W_pM$ to identify

$$\omega M = \omega W_pM \oplus \omega M/\omega W_pM$$

(as vector spaces). This also gives a decomposition of $\omega\underline{\mathrm{Hom}}(M/W_pM, M)$. In view of the sequence (19) and on noting that $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ belongs to $\boldsymbol{C}$, the group $\mathcal{U}_p(M,\omega)$ acts trivially on (21) if and only if it (or equivalently, $\mathcal{U}_p(M,\omega)(K)$) fixes the image of the element

$$\begin{aligned}(0, I) \in \mathrm{Hom}(\omega M/\omega W_pM, \omega M)^{\dagger} &\subset \mathrm{Hom}(\omega M/\omega W_pM, \omega M)\\ &= \mathrm{Hom}(\omega M/\omega W_pM, \omega W_pM) \oplus \mathrm{End}(\omega M/\omega W_pM)\end{aligned}$$

in (21). Identifying $\mathrm{Hom}(\omega M/\omega W_pM, \omega M)$ as a subspace of $\mathrm{End}(\omega M)$ in the obvious way, given any $\sigma \in \mathcal{U}_p(M,\omega)(K)$, in view of the fact that σ fixes ωW_pM and $\omega M/\omega W_pM$, one calculates that

$$\sigma\cdot(0, I) - (0, I) = \log(\sigma).$$

Thus $\mathcal{U}_p(M,\omega)(K)$ fixes $(0, I)$ mod ωA if and only if ωA contains $\mathfrak{u}_p(M,\omega)$.

4.9. Proposition 4.8.1 can be reformulated in the language of extensions originating from subcategories of $\boldsymbol{T}$ (see Section 3.4) as follows:

Theorem 4.9.1. *Let A be a subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$. Then the extension $\mathcal{E}_p(M)/A$, viewed as an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/A$ or $\underline{\mathrm{Hom}}(M/W_pM, W_pM)/A$, originates from the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ if and only if A contains $\underline{\mathfrak{u}}_p(M)$.*

[8] Theorem 3.3.1 of [Eskandari and Murty 2021] is obtained by a slight modification of Hardouin's argument for Theorem 2 of [Hardouin 2011] and Théorème 2.1 of the unpublished article [Hardouin 2006].

In other words, $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$, viewed as an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}_p(M)$ or by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)/\underline{\mathfrak{u}}_p(M)$, originates from $\langle W_pM, M/W_pM\rangle^{\otimes}$.

Proof of Theorem 4.9.1. Let A be a subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$. By Lemma 3.4.4 (also see the remark after the same lemma), and in view of the facts (1) that the extension $\mathscr{E}_p(M)/A$ of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/A$ is the image of its namesake as an extension of $\mathbb{1}$ by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)/A$ under the obvious pushforward map

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)/A) \to \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/A),$$

and (2) that (by weight considerations) there are no nonzero morphisms from $\mathbb{1}$ to objects of weight < 0, the following statements are equivalent for any full Tannakian subcategory $\boldsymbol{S}$ of $\boldsymbol{T}$:

(i) The extension $\mathscr{E}_p(M)/A$, viewed as an element of

$$\mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/A),$$

originates from $\boldsymbol{S}$.

(ii) The extension $\mathscr{E}_p(M)/A$, viewed as an element of

$$\mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)/A),$$

originates from $\boldsymbol{S}$.

In view of Lemma 3.4.3 and on recalling the explicit description of

$$\mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

from Section 4.5, Statement (ii) with $\boldsymbol{S}$ taken to be the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ is equivalent to the following statement:

(iii) The object

$$\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger}/A$$

belongs to $\langle W_pM, M/W_pM\rangle^{\otimes}$.

Thus Theorem 4.9.1 is equivalent to Proposition 4.8.1 (or $(*)$ of the Introduction). □

5. Extension classes and subgroups of the fundamental group, part II

In the previous section we saw that $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from $\langle W_pM, M/W_pM\rangle^{\otimes}$. Our goal in this section is to give criteria under which the subcategory $\langle W_pM, M/W_pM\rangle^{\otimes}$ in this statement can be replaced by smaller subcategories. Of particular interest will be the case in which we can replace it with a semisimple category, as then $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ will split.

5.1. Let us first make an observation regarding the pushforwards of the extension $\mathscr{E}_p(M)$. Recall that we are using the same notation for

$$\mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM))$$

and its image in $\mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M))$ under the pushforward map (17).

Lemma 5.1.1. *Let S be a full Tannakian subcategory of T. Then the following statements are equivalent*:

(i) *The extension*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_pM, W_pM)/\underline{\mathfrak{u}}_p(M))$$

originates from S.

(ii) *The extension*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}_p(M))$$

originates from S.

(iii) *The extension*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}(M))$$

originates from S.

Proof. That (i) implies (ii) and (ii) implies (iii) is clear, as under the obvious maps the extension in (i) pushes forward to the extension in (ii) and then to the one in (iii) (in fact, we already observed the equivalence of (i) and (ii) in the proof of Theorem 4.9.1). That (iii) implies (i) follows similarly as in the proof of Theorem 4.9.1 from Lemma 3.4.4 on recalling that

$$\underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_pM, W_pM) = \underline{\mathfrak{u}}_p(M)$$

(so that the obvious map

$$\underline{\mathrm{Hom}}(M/W_pM, W_pM)/\underline{\mathfrak{u}}_p(M) \to W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}(M)$$

is injective). □

Remark. (1) In particular, by taking S to be the semisimple subcategory $\langle\mathbb{1}\rangle^{\otimes}$ we see that the three extensions in the lemma split at the same time.

(2) The lemma together with Deligne's Theorem 4.3.1 implies that if every $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits (i.e., if every $\mathscr{E}_p(M)$ is in the image of (18)), then $\underline{\mathfrak{u}}(M) = \sum_p \underline{\mathfrak{u}}_p(M)$. Indeed, let us tentatively set $\underline{\mathfrak{u}}' = \sum_p \underline{\mathfrak{u}}_p(M)$. If $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits for every p, then so does $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ and hence $\mathscr{E}_p(M)/\underline{\mathfrak{u}}'$ (the latter as an extension of $\mathbb{1}$ by $W_{-1}\underline{\mathrm{End}}(M)/\underline{\mathfrak{u}}'$). It follows that $\mathscr{E}(M)/\underline{\mathfrak{u}}'$ splits, so that by Deligne's theorem $\underline{\mathfrak{u}}(M) \subset \underline{\mathfrak{u}}'$.

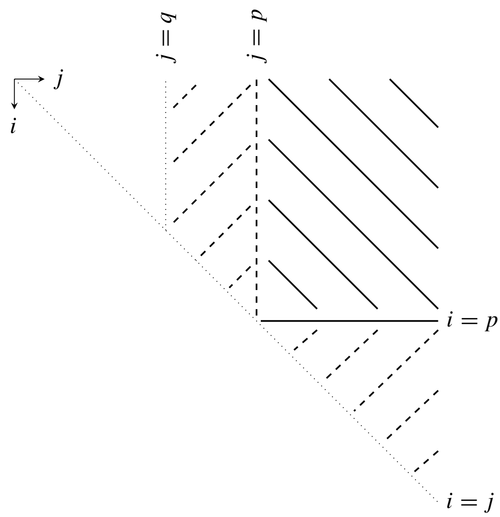


Figure 1. The set of lattice points in the region marked by solid (resp. thick dashed) lines is $J_1^{\{p,q\}}$ (resp. $J_2^{\{p,q\}}$).

5.2. For any integers p and q with $q \le p$, define

$$J_1^{\{p,q\}} := \{(i, j) \in \mathbb{Z}^2 : i \le p < j\},$$
$$J_2^{\{p,q\}} := \{(i, j) \in \mathbb{Z}^2 : i < j \text{ and } (q < j \le p \text{ or } i > p)\}.$$

Figure 1 shows the two sets. In the figure, the axes are oriented according to the standard labeling of entries of a matrix (the pair (i, j) is placed where the entry ij of a matrix sits).

We consider the following *independence axioms* for an object M of $\boldsymbol{T}$:

- $(IA1)_{\{p,q\}}$: The two objects
$$\bigoplus_{(i,j)\in J_1^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M) \quad \text{and} \quad \bigoplus_{(i,j)\in J_2^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}_j^W M, \mathrm{Gr}_i^W M)$$
have no nonzero isomorphic subobjects. Note that if $q' \le q \le p$, then $(IA1)_{\{p,q'\}}$ implies $(IA1)_{\{p,q\}}$.
- $(IA2)_{\{p,q\}}$: The two sets
$$J_1^{\{p,q\}}(M) := \{i - j : (i, j) \in J_1^{\{p,q\}}, \mathrm{Gr}_i^W M \ne 0, \mathrm{Gr}_j^W M \ne 0\}$$
and
$$J_2^{\{p,q\}}(M) := \{i - j : (i, j) \in J_2^{\{p,q\}}, \mathrm{Gr}_i^W M \ne 0, \mathrm{Gr}_j^W M \ne 0\}$$
are disjoint. (Note that $J_1^{\{p,q\}}(M)$ and $J_2^{\{p,q\}}(M)$ are respectively the set of weights of the two object in $(IA1)_{\{p,q\}}$ above.)
- $(IA3)$: The numbers
$$i - j \quad (i < j, \mathrm{Gr}_i^W M \ne 0, \mathrm{Gr}_j^W M \ne 0)$$
are all distinct. (Equivalently, if M has n distinct weights, then $W_{-1}\underline{\mathrm{End}}(M)$ has $\binom{n}{2}$ distinct weights.)

It is clear that $(IA2)_{\{p,q\}}$ implies $(IA1)_{\{p,q\}}$, and $(IA3)$ implies $(IA2)_{\{p,q\}}$ for every p and q. Also note that whether or not M satisfies any of these axioms only depends on $\mathrm{Gr}^W M$.

5.3. We can now state the main result of this part of the paper:

Theorem 5.3.1. *Let $q \leq p$. Consider the following statements*:

(i) *M satisfies $(IA1)_{\{p,q\}}$ and $\mathrm{Gr}^W M$ is semisimple* (= *completely reducible*).

(ii) *M satisfies $(IA2)_{\{p,q\}}$.*

If either of the statements holds, then the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from the subcategory $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.

The proof of Theorem 5.3.1 shall be given in the Sections 5.4–5.6 below. Here we consider some consequences of the theorem:

(1) Since $q \leq p$, the subcategory $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$ is contained in the subcategory $\langle W_p M, M/W_p M\rangle^{\otimes}$. Thus combining Theorems 4.9.1 and 5.3.1 we get the following refinement of Theorem 4.9.1: If statements (i) or (ii) above hold for some $q \leq p$, then $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_p M, W_p M)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.

(2) Perhaps the most interesting application of Theorem 5.3.1 is in the following scenario: Fix p. Suppose $\mathrm{Gr}^W M$ is semisimple; for instance, this will be the case if $\boldsymbol{T}$ is a category of motives, or if $\boldsymbol{T}$ is the category of mixed Hodge structures and $\mathrm{Gr}^W M$ is polarizable. Suppose M satisfies $(IA1)_{\{p,q\}}$ for all $q \leq p$ (this holds for instance, if M satisfies $(IA3)$). Then $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_p M, W_p M)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from the semisimple subcategory $\langle \mathrm{Gr}^W M\rangle^{\otimes}$, i.e., splits. In particular, $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits. For future referencing, we record this as a corollary.

Corollary 5.3.2. *Fix p. Suppose $\mathrm{Gr}^W M$ is semisimple and that M satisfies $(IA1)_{\{p,q\}}$ for all $q \leq p$. Then $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_p M, W_p M)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. In particular,*

$$\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$$

splits.

As a special case, we obtain:

Corollary 5.3.3. *If $\mathrm{Gr}^W M$ is semisimple and $(IA3)$ holds, then for every p the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits.*

Remark. Recall that by Deligne's Theorem 4.3.1, the extension

$$\sum_p \mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$$

splits. As we pointed out earlier, in general, the individual extensions $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ may not split (see Section 6.10 and item (2) of the remark therein for examples). The above results give sufficient conditions for when an individual $\mathscr{E}_p(M)/\underline{\mathfrak{u}}(M)$ splits.

5.4. From this point until the end of Section 5.6 our goal is to prove Theorem 5.3.1. Given any fiber functor ω, let $\mathcal{U}_{\geq q}(M,\omega)$ be the kernel of the surjection

$$\mathcal{G}(M,\omega) \to \mathcal{G}(W_q M \oplus \mathrm{Gr}^W M, \omega)$$

induced by the inclusion $\langle W_q M \oplus \mathrm{Gr}^W M\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$. Then $\mathcal{U}_{\geq q}(M,\omega)$ is the subgroup of $\mathcal{U}(M,\omega)$ which acts trivially on $\omega W_q M$. Let $U_{\geq q}(M,\omega)$ be the subgroup of $\mathrm{GL}(\omega M)$ consisting of the elements which fix the weight filtration, and act trivially on $\mathrm{Gr}^W \omega M$ and $\omega W_q M$:

$$U_{\geq q}(M,\omega) := \{\sigma \in U(M,\omega) : \sigma|_{\omega W_q M} = I\}.$$

Then

$$\mathcal{U}_{\geq q}(M,\omega) = \mathcal{U}(M,\omega) \cap U_{\geq q}(M,\omega).$$

We have

$$\mathrm{Lie}(U_{\geq q}(M,\omega)) = \mathrm{Hom}(\omega M/\omega W_q M, \omega M) \cap W_{-1}\,\mathrm{End}(\omega M),$$

where $\mathrm{Hom}(\omega M/\omega W_q M, \omega M)$ is identified as the subspace of $\mathrm{End}(\omega M)$ consisting of the elements which vanish on $\omega W_q M$. Then

$$\mathfrak{u}_{\geq q}(M,\omega) := \mathrm{Lie}(\mathcal{U}_{\geq q}(M,\omega)) = \mathfrak{u}(M,\omega) \cap \mathrm{Hom}(\omega M/\omega W_q M, \omega M).$$

Finally, set

$$\underline{\mathfrak{u}}_{\geq q}(M) := \underline{\mathfrak{u}}(M) \cap \underline{\mathrm{Hom}}(M/W_q M, M).$$

Here

$$\underline{\mathrm{Hom}}(M/W_q M, M)$$

is thought of as a subobject of $\underline{\mathrm{End}}(M)$ via the obvious injection induced by the quotient map $M \to M/W_q M$ (note that this is compatible with the previous identification of $\mathrm{Hom}(\omega M/\omega W_q M, \omega M)$ as a subspace of $\mathrm{End}(\omega M)$). We then have

$$\mathfrak{u}_{\geq q}(M,\omega) = \omega \underline{\mathfrak{u}}_{\geq q}(M).$$

5.5. Identifying

$$\mathrm{Gr}^W \underline{\mathrm{End}}(M) = \underline{\mathrm{End}}(\mathrm{Gr}^W M) = \bigoplus_{i,j} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M), \tag{22}$$

we have

$$\mathrm{Gr}^W W_{-1}\underline{\mathrm{End}}(M) = \bigoplus_{\substack{i,j\\ i<j}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M).$$

Then for every q,

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \subset \mathrm{Gr}^W \underline{\mathrm{Hom}}(M/W_q M, M) \cap \mathrm{Gr}^W W_{-1}\underline{\mathrm{End}}(M) = \bigoplus_{\substack{i,j\\ i,q<j}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M). \tag{23}$$

The following lemma is the only place in the proof of Theorem 5.3.1 that conditions (i) and (ii) of the theorem play a part.

Lemma 5.5.1. *Let $q \leq p$. Suppose statement (i) or (ii) of Theorem 5.3.1 holds. Then $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M)$ decomposes as the direct sum of*

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \cap \bigoplus_{(i,j)\in J_1^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M)$$

and

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \cap \bigoplus_{(i,j)\in J_2^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M).$$

Proof. The direct sum in (23) is over all pairs (i, j) in $J_1^{\{p,q\}} \sqcup J_2^{\{p,q\}}$, so that we can rewrite (23) as

$$\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M) \subset \overbrace{\bigoplus_{(i,j)\in J_1^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M)}^{\text{(I)}} \oplus \overbrace{\bigoplus_{(i,j)\in J_2^{\{p,q\}}} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M)}^{\text{(II)}}.$$

First suppose $\mathrm{Gr}^W M$ is semisimple and M satisfies $(IA1)_{\{p,q\}}$. Then the object $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M)$ (living in the semisimple category $\langle \mathrm{Gr}^W M\rangle^{\otimes}$) is a direct sum of simple objects. By $(IA1)_{\{p,q\}}$, each simple direct factor either lives in (I) or (II).

On the other hand, if $(IA2)_{\{p,q\}}$ holds, then each nonzero graded component $Gr_n^W \underline{\mathfrak{u}}_{\geq q}(M)$ must live in (I) or (II) (whichever has a nonzero weight n part). □

5.6. We are ready to give the proof of Theorem 5.3.1. We may assume that M/W_pM is not zero. Consider $\mathscr{E}_p(M)$ as an extension of the unit object by $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$, given by (19). In view of Section 5.4 and Lemma 3.4.2, it is enough to check right exactness of the sequence obtained by applying $\mathcal{U}_{\geq q}(M, \omega)$-invariance to $\omega(\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M))$ for a suitably chosen fiber functor ω. Let ω_0 be an arbitrary fiber functor. We shall take the composition

$$\omega^{\mathrm{gr}} : \boldsymbol{T} \xrightarrow{\mathrm{Gr}^W} \boldsymbol{T} \xrightarrow{\omega_0} \mathbf{Mod}_K$$

as our fiber functor ω.

Via the identification

$$\underline{\mathrm{Hom}}(M/W_pM, M) \subset \underline{\mathrm{End}}(M),$$

we think of the image under ω^{gr} of every subobject of $\underline{\mathrm{Hom}}(M/W_pM, M)$ as a subspace of $\omega^{\mathrm{gr}}\underline{\mathrm{End}}(M)$. Throughout, we shall write the elements of

$$\omega^{\mathrm{gr}}\underline{\mathrm{End}}(M) = \mathrm{End}(\omega^{\mathrm{gr}} M) = \mathrm{End}\Big(\bigoplus_n \omega_0 \,\mathrm{Gr}^W_n M\Big) = \bigoplus_{i,j} \mathrm{Hom}(\omega_0 \,\mathrm{Gr}^W_j M, \omega_0 \,\mathrm{Gr}^W_i M)$$

as 2 by 2 block matrices with rows (resp. columns) broken up as $\{i : i \le p\} \cup \{i : i > p\}$ (resp. the same with j replacing i). Then an element

$$f \in \omega^{\mathrm{gr}}\underline{\mathrm{Hom}}(M/W_pM, M)^{\dagger} = \mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}$$

looks like

$$\begin{pmatrix} 0 & * \\ 0 & \lambda(f)I \end{pmatrix}.$$

The surjective arrow

$$\mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger} \to K$$

in $\omega^{\mathrm{gr}}\mathscr{E}_p$ sends f to $\lambda(f)$.

Consider the element

$$f_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \in \mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}.$$

We will show that if conditions (i) or (ii) of Theorem 5.3.1 hold (and $q \le p$), then the element $f_0 + \omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)$ of

$$\frac{\mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}}{\omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)}$$

is fixed by $\mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})$; this proves surjectivity of

$$\left(\frac{\mathrm{Hom}(\omega^{\mathrm{gr}}(M/W_pM), \omega^{\mathrm{gr}}M)^{\dagger}}{\omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)}\right)^{\mathcal{U}_{\ge q}(M,\omega^{\mathrm{gr}})} \to K$$

and hence the theorem. Since $\mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})$ is unipotent, it is enough to verify that $f_0 + \omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M)$ is fixed by every $\sigma \in \mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})(K) \subset \mathrm{GL}(\omega^{\mathrm{gr}}M)$. Given such a σ, we must show that

$$\sigma f_0 \sigma^{-1} - f_0 \in \omega^{\mathrm{gr}}\underline{\mathfrak{u}}_p(M) = \mathfrak{u}_p(M, \omega^{\mathrm{gr}}). \tag{24}$$

Writing

$$\sigma = \begin{pmatrix} \sigma_1 & A \\ 0 & \sigma_2 \end{pmatrix},$$

we have

$$\log(\sigma) = \begin{pmatrix} \log\sigma_1 & * \\ 0 & \log\sigma_2 \end{pmatrix} \in \mathfrak{u}_{\ge q}(M, \omega^{\mathrm{gr}}) = \omega_0 \,\mathrm{Gr}^W\, \underline{\mathfrak{u}}_{\ge q}(M).$$

Applying ω_0 to the decomposition of $\mathrm{Gr}^W\, \underline{\mathfrak{u}}_{\ge q}(M)$ given in Lemma 5.5.1, it follows that

$$\begin{pmatrix} \log\sigma_1 & 0 \\ 0 & \log\sigma_2 \end{pmatrix} \in \mathfrak{u}_{\ge q}(M, \omega^{\mathrm{gr}}),$$

so that

$$\delta := \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \in \mathcal{U}_{\ge q}(M, \omega^{\mathrm{gr}})(K).$$

We thus have

$$\sigma f_0\sigma^{-1} - f_0 = \begin{pmatrix} 0 & A\sigma_2^{-1} \\ 0 & 0 \end{pmatrix} = \log(\sigma\delta^{-1}) \in \mathfrak{u}_{\geq q}(M, \omega^{\mathrm{gr}}).$$

We have shown that $\sigma f_0\sigma^{-1} - f_0$ is in $\mathfrak{u}(M, \omega^{\mathrm{gr}})$. Being an element of the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$, it will be actually in $\mathfrak{u}_p(M, \omega^{\mathrm{gr}})$, as desired.

5.7. We end this section with a variant of Theorem 5.3.1 for $q > p$, which again gives a sufficient condition to guarantee that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from the category $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.[9] two categories $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$ and $\langle W_p M, M/W_p M\rangle^{\otimes}$ necessarily contains the other.

For $q > p$, consider the following three sets:

$$J'^{\{p,q\}}_1 := \{(i, j) \in \mathbb{Z}^2 : i \leq p,\ j > q\}.$$
$$J'^{\{p,q\}}_2 := \{(i, j) \in \mathbb{Z}^2 : p < i \leq q < j\}.$$
$$J'^{\{p,q\}}_3 := \{(i, j) \in \mathbb{Z}^2 : q < i < j\}.$$

Say an object M of $\boldsymbol{T}$ satisfies $(IA1')_{\{p,q\}}$ if the objects

$$\bigoplus_{(i,j)\in J'^{\{p,q\}}_k} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M)$$

for $k = 1, 2, 3$ have no nonzero isomorphic subobjects. We say M satisfies $(IA2')_{\{p,q\}}$ if the sets of weights of these objects are disjoint. Then $(IA2')_{\{p,q\}}$ implies $(IA1')_{\{p,q\}}$, and $(IA3)$ implies $(IA2')_{\{p,q\}}$ for every p, q.

Theorem 5.7.1. *Let $q > p$. Suppose one of the following statements holds*:

(i) $\mathrm{Gr}^W M$ *is semisimple and M satisfies* $(IA1')_{\{p,q\}}$.

(ii) *M satisfies* $(IA2')_{\{p,q\}}$.

Then the extension $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ originates from $\langle W_q M, \mathrm{Gr}^W M\rangle^{\otimes}$.

Proof. The proof is similar to the proof of Theorem 5.3.1. Note that the pairs (i, j) appearing in (23) are those in $J'^{\{p,q\}}_1 \cup J'^{\{p,q\}}_2 \cup J'^{\{p,q\}}_3$. Similar to Lemma 5.5.1, hypothesis (i) or (ii) above imply that $\mathrm{Gr}^W \underline{\mathfrak{u}}_{\geq q}(M)$ is the direct sum of its intersections with the three objects

$$\bigoplus_{(i,j)\in J'^{\{p,q\}}_k} \underline{\mathrm{Hom}}(\mathrm{Gr}^W_j M, \mathrm{Gr}^W_i M) \tag{25}$$

for $k = 1, 2, 3$. Taking ω^{gr} and f_0 as in the proof of Theorem 5.3.1, we shall show that for every $\sigma \in \mathcal{U}_{\geq q}(M, \omega^{\mathrm{gr}})(K)$,

$$\sigma f_0\sigma^{-1} - f_0 \in \mathfrak{u}(M, \omega^{\mathrm{gr}})$$

[9]The content of this subsection will not be used anywhere else in the paper. A reader mainly interested in the application to motives may skip to Section 6.

(it will then automatically be in $\mathfrak{u}_p(M, \omega^{\mathrm{gr}})$). Decompose

$$\log \sigma = \tau_1 + \tau_2 + \tau_3,$$

where τ_k is the component in (25); each τ_k is in $\mathfrak{u}_{\geq q}(M, \omega^{\mathrm{gr}})$, thanks to hypothesis (i) or (ii). Writing the elements of $\mathrm{End}(\omega^{\mathrm{gr}} M)$ as 3×3 block matrices with the rows (resp. columns) broken up as $\{i : i \leq p\} \cup \{i : p < i \leq q\} \cup \{i : i > q\}$ (resp. the same with j replacing i), we have

$$\log(\sigma) = \begin{pmatrix} 0 & & \tau_1 \\ & 0 & \tau_2 \\ & & \tau_3 \end{pmatrix}$$

(with zero missing entries), so that

$$\sigma = \begin{pmatrix} I & & \tau_1(\exp(\tau_3) - 1)/\tau_3 \\ & I & \tau_2(\exp(\tau_3) - 1)/\tau_3 \\ & & \exp(\tau_3) \end{pmatrix} \quad \text{and} \quad \sigma^{-1} = \begin{pmatrix} I & & \tau_1(\exp(-\tau_3) - 1)/\tau_3 \\ & I & \tau_2(\exp(-\tau_3) - 1)/\tau_3 \\ & & \exp(-\tau_3) \end{pmatrix},$$

where for brevity, for a nilpotent map N we have set

$$(\exp(N) - 1)/N := \sum_{n \geq 0} N^n/(n+1)!.$$

Then one calculates

$$\sigma f_0 \sigma^{-1} - f_0 = \begin{pmatrix} 0 & & \tau_1(1 - \exp(-\tau_3))/\tau_3 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

This belongs to $\mathfrak{u}(M, \omega^{\mathrm{gr}})$ because τ_1, τ_3 are in the Lie algebra $\mathfrak{u}(M, \omega^{\mathrm{gr}})$ and

$$[\tau_1, \tau_3] = \tau_1\tau_3, \quad [[\tau_1, \tau_3], \tau_3] = \tau_1\tau_3^2, \quad \dots. \qquad \square$$

6. Motives with large unipotent radicals of motivic Galois groups

6.1. In this section, unless otherwise indicated, $\boldsymbol{T}$ is any reasonable Tannakian category of mixed motives in characteristic zero, or the category of mixed Hodge structures. Examples of the former include the (now known to be equivalent [Choudhury and Gallauer Alves de Souza 2017]) Tannakian categories of mixed motives over a subfield of $\mathbb{C}$ due to Nori [Huber and Müller-Stach 2017] and Ayoub [2014a; 2014b], Voevodsky's category of mixed Tate motives over $\mathbb{Q}$ (or those over $\mathbb{Z}$, etc.), and categories of mixed motives defined via realizations (see [Deligne 1989] or [Jannsen 1990]). See the remark at the end of this section for what we exactly need of $\boldsymbol{T}$. We shall use the word *motive* to refer to any object of $\boldsymbol{T}$ whose weight associated graded is semisimple. Of course, in the case that $\boldsymbol{T}$ is a reasonable category of mixed motives, this will simply mean an arbitrary object of $\boldsymbol{T}$. In the case of the category of mixed Hodge structures, this will include (graded-) polarizable objects, and in particular, the Hodge realizations of mixed motives.

Let M be a motive. We say $\underline{\mathfrak{u}}(M)$ is *large* (or that M has a large $\underline{\mathfrak{u}}$) if

$$\underline{\mathfrak{u}}(M) = W_{-1}\underline{\mathrm{End}}(M).$$

Similarly, we say $\underline{\mathfrak{u}}_p(M)$ is large if

$$\underline{\mathfrak{u}}_p(M) = \underline{\mathrm{Hom}}(M/W_pM, W_pM).$$

Then $\underline{\mathfrak{u}}(M)$ is large if and only if $\underline{\mathfrak{u}}_p(M)$ is large for every p. The interest in motives with large $\underline{\mathfrak{u}}$ is partly because of Grothendieck's period conjecture. If $\boldsymbol{T}$ is a good category of motives over a number field, among the motives with a fixed associated graded, the periods of a motive with large $\underline{\mathfrak{u}}$ should generate a field with the largest possible transcendence degree. We refer the reader to [André 2004] for a detailed discussion of Grothendieck's period conjecture.

Our main goal in this section is to use the earlier results of the paper to obtain motives with large $\underline{\mathfrak{u}}$ and three weights. We will be particularly interested in motives M with three weights $-2n < p < 0$, associated graded isomorphic to

$$\mathbb{Q}(n) \oplus A \oplus \mathbb{1}$$

where A is a given pure motive of weight p, and such that $\underline{\mathfrak{u}}(M)$ is large. We shall prove a precise classification result for such motives in terms of homological algebra, which completely classifies such motives up to isomorphism when $n \neq -p$ and $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$ (e.g., for even n if $\boldsymbol{T}$ is any reasonable category of motives over $\mathbb{Q}$). The condition $n \neq -p$ here is an independence axiom (referring to the language of the previous section). See Corollary 6.7.1 for the precise statement of the classification result. As an example, in Section 6.8 we shall consider the case where A is the simple Tate motive $\mathbb{Q}(k)$ and construct certain interesting mixed Tate motives over $\mathbb{Q}$.

It turns out that the machinery we shall need works in more generality with little extra effort. So we have decided to develop the results in more generality first and then apply them to the case of motives with three weights. We shall however start with the simplest case below, i.e., motives with only two weights; the observations made in this case will be useful when we deal with more than two weights.

Remark. Our restriction to the categories of motives and mixed Hodge structures here is for reasons to do with motivation and applications. Unless we explicitly say otherwise, the discussions can be assumed to take place in the following setting: Take $\boldsymbol{T}$ to be any Tannakian category over a field K of characteristic zero, equipped with a weight filtration (as in previous sections), and interpret the word "motive" as an object of $\boldsymbol{T}$ whose associate graded with respect to the weight filtration is semisimple. In discussions where the Tate objects $\mathbb{Q}(n)$ make an appearance, $\mathbb{Q}(n)$ may denote any object of weight $-2n$ and dimension 1 (even if $K \neq \mathbb{Q}$).

6.2. We shall use the following terminology: an extension of $\mathbb{1}$ by an object L is *totally nonsplit* if its pushforward to any nonzero quotient of L is nontrivial (= nonsplit); dually, we say an extension of an object L by $\mathbb{1}$ is totally nonsplit if its pullback to any nonzero subobject of L is nontrivial. Note that if L is simple, then "totally nonsplit" and "nonsplit" are equivalent.

Suppose M is an object with two weights, fitting in a short exact sequence

$$0 \to L \to M \to \mathbb{1} \to 0, \tag{26}$$

where L is a pure motive of weight $p < 0$.[10] Then

$$W_{-1}\underline{\mathrm{End}}(M) = \underline{\mathrm{Hom}}(\mathbb{1}, L) \cong L.$$

By Theorem 4.9.1 (or Deligne's Theorem 4.3.1 or [Hardouin 2011, Theorem 2], see also the latter's predecessors, [Bertrand 2001, Theorem 1.1] and [Hardouin 2006, Théorème 2.1]), $\underline{\mathfrak{u}}(M)$ (= $\underline{\mathfrak{u}}_p(M)$) is the smallest subobject of L such that the pushforward of the extension (26) to $\mathrm{Ext}^1(\mathbb{1}, L/\underline{\mathfrak{u}}(M))$ splits. (Indeed, note that via the identification of $\underline{\mathrm{Hom}}(\mathbb{1}, L)$ and L, the extension $\mathscr{E}_p(M)$ appearing in Theorem 4.9.1 is simply (26). Also note that the total class $\mathscr{E}(M)$ of M is a nonzero multiple of $\mathscr{E}_p(M)$.) Thus $\underline{\mathfrak{u}}(M)$ is large if and only if (26) is totally nonsplit. In particular, if L is simple, then

$$\underline{\mathfrak{u}}(M) = \begin{cases} L & \text{if } M \text{ is not semisimple,} \\ 0 & \text{if } M \text{ is semisimple.} \end{cases}$$

Remark. Let $\boldsymbol{T}$ be any Tannakian category over K with a weight filtration. Then for any object M of $\boldsymbol{T}$ with semisimple $\mathrm{Gr}^W M$ the following statements are equivalent:

(i) $\underline{\mathfrak{u}}(M)$ is zero.

(ii) M is semisimple.

(iii) M is isomorphic to $\mathrm{Gr}^W M$.

Indeed, choosing a fiber functor one easily sees (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) (note that among these the implication (i) $\Rightarrow$ (ii) is the only one that needs the assumption of semisimplicity of $\mathrm{Gr}^W M$). This gives another argument for the characterization of $\underline{\mathfrak{u}}(M)$ given above when L is simple.

6.3. In this section we will use the results of Sections 4 and 5 to give a criterion for a motive to have a large $\underline{\mathfrak{u}}$ in terms of its subobjects and subquotients.

Theorem 6.3.1. *Let $p < 0$ and M be a motive such that*

$$M/W_pM \simeq \mathbb{1}, \quad \mathrm{Gr}_p^W M \neq 0 \tag{27}$$

(*so that in particular*, 0 *and* p *are the highest two weights of* M). *Suppose moreover that*:

(i) *$\underline{\mathfrak{u}}(W_pM)$ is large.*

(ii) *$\underline{\mathfrak{u}}(M/W_{p-1}M)$ is large.*

(iii) *M satisfies $(IA1)_{\{p,q\}}$ for all $q \leq p$.*

Then $\underline{\mathfrak{u}}(M)$ is large.

[10] Note that this makes M also a motive (as $\mathrm{Gr}^W M \simeq L \oplus \mathbb{1}$ is semisimple).

Proof. Note that since M/W_pM is pure, for any choice of fiber functor ω, we have $\mathcal{U}_{\geq p}(M,\omega) = \mathcal{U}_p(M,\omega)$. Indeed, if σ is in $\mathcal{G}(M,\omega)$, then $\sigma_{\mathrm{Gr}^W M}$ and σ_{W_pM} are both identity if and only if $\sigma_{\mathrm{Gr}^W (M/W_pM)}$ and σ_{W_pM} are both identity, and by purity $\mathrm{Gr}^W(M/W_pM) \simeq M/W_pM$. Thus the kernel of the surjection

$$\underline{\mathfrak{u}}(M) \to \underline{\mathfrak{u}}(W_pM)$$

induced by the inclusion $\langle W_pM\rangle^{\otimes} \subset \langle M\rangle^{\otimes}$ is $\underline{\mathfrak{u}}_p(M)$. In light of purity of M/W_pM, from this it follows that $\underline{\mathfrak{u}}(M)$ is large if and only if $\underline{\mathfrak{u}}(W_pM)$ and $\underline{\mathfrak{u}}_p(M)$ are both large.

In view of hypothesis (iii) and the fact that M is a motive, Corollary 5.3.2 tells us that $\underline{\mathfrak{u}}_p(M)$ is the smallest subobject of $\underline{\mathrm{Hom}}(M/W_pM, W_pM)$ such that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. Fix an isomorphism between M/W_pM and $\mathbb{1}$ to identify the two objects. Then

$$\underline{\mathfrak{u}}_p(M) \subset \underline{\mathrm{Hom}}(M/W_pM,\, W_pM) = \underline{\mathrm{Hom}}(\mathbb{1},\, W_pM) \cong W_pM.$$

Via the latter identification, the extension

$$\mathscr{E}_p(M) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(\mathbb{1},\, W_pM)) = \mathrm{Ext}^1(\mathbb{1},\, W_pM)$$

is simply the canonical extension

$$0 \to W_pM \to M \to \mathbb{1} \to 0, \tag{28}$$

where the surjective arrow is the quotient map $M \to M/W_pM = \mathbb{1}$. Let A be any subobject of W_pM such that $\mathscr{E}_p(M)/A$ splits. The goal is to show that $A = W_pM$.

Modding out by $W_{p-1}M$, the extension (28) pushes forward to

$$0 \to \mathrm{Gr}^W_p M \to M/W_{p-1}M \to \mathbb{1} \to 0. \tag{29}$$

By Section 6.2, $\underline{\mathfrak{u}}(M/W_{p-1}M)$ is large if and only if this extension is totally nonsplit. In view of hypothesis (ii), it follows that we must have

$$A + W_{p-1}M = W_pM. \tag{30}$$

Indeed, otherwise, by modding out (28) by $A + W_{p-1}M$ we see that the pushforward of (29) to a nonzero subquotient of $\mathrm{Gr}^W_p M$ splits, contradicting the fact that (29) is totally nonsplit.

Now consider the diagram:

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & & \cap & \searrow & & & \\ 0 & \longrightarrow & W_{p-1}M & \longrightarrow & W_pM & \longrightarrow & \mathrm{Gr}^W_p M & \longrightarrow & 0 \end{array}$$

We just saw that diagonal arrow is surjective. It follows that the extension in the diagram is the pushforward of an extension of $\mathrm{Gr}^W_p M$ by $A \cap W_{p-1}M$ (under inclusion map). Thus the extension

$$\mathscr{E}_{p-1}(W_pM) \in \mathrm{Ext}^1(\mathbb{1}, \underline{\mathrm{Hom}}(\mathrm{Gr}^W_p M,\, W_{p-1}M))$$

is the pushforward of an extension of $\mathbb{1}$ by

$$\underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M) \subset \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, W_{p-1}M),$$

i.e., that

$$\mathscr{E}_{p-1}(W_pM)/\underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M)$$

splits. By Theorem 4.9.1, we get

$$\underline{\mathfrak{u}}_{p-1}(W_pM) \subset \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M).$$

But since $\underline{\mathfrak{u}}(W_pM)$ is large, so is $\underline{\mathfrak{u}}_{p-1}(W_pM)$. Thus

$$\underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, A \cap W_{p-1}M) = \underline{\mathrm{Hom}}(\mathrm{Gr}_p^W M, W_{p-1}M).$$

Since $\mathrm{Gr}_p^W M$ is nonzero, this implies that $W_{p-1}M \subset A$. Combining with (30) we get that $A = W_pM$, as desired.[11] □

Remark. (1) As pointed out in the proof, hypothesis (ii) of Theorem 6.3.1 is equivalent to the extension (29) being totally nonsplit. If we assume moreover that $\mathrm{Gr}_p M$ is simple, then this is equivalent to $M/W_{p-1}M$ not being semisimple.

(2) Let M be a motive which satisfies (27) (with $p < 0$). It is easy to see that if $\underline{\mathfrak{u}}_p(M)$ is large, then so is $\underline{\mathfrak{u}}(M/W_{p-1}M)$. Indeed, if the latter is not large, then the pushforward of (29) to a nonzero quotient of $\mathrm{Gr}_p^W M$ splits. The same split extension is then the pushforward of (28) to a nonzero quotient of W_pM, so that by Theorem 4.9.1 $\underline{\mathfrak{u}}_p(M)$ is not large.

Now suppose that $\underline{\mathfrak{u}}(M)$ is large. As we observed in the beginning of the proof of Theorem 6.3.1, this implies that both $\underline{\mathfrak{u}}(W_pM)$ and $\underline{\mathfrak{u}}_p(M)$ are large. We record the conclusion:

If M is a motive satisfying (27) (*with* $p < 0$) *and* $\underline{\mathfrak{u}}(M)$ *is large*, *then both* $\underline{\mathfrak{u}}(W_pM)$ *and* $\underline{\mathfrak{u}}(M/W_{p-1}M)$ *are large.*

Note that here we did not need to assume M satisfies any independence axiom. Theorem 6.3.1 asserts that if we further assume that M satisfies the independence axiom given in hypothesis (iii) of the theorem, then the converse to the statement above is also true.

(3) Hypothesis (iii) of the theorem (which was used in the proof to guarantee that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits) is actually important: the statement of the theorem is false if we remove Hypothesis (iii). See Section 6.10 for an example.

6.4. In view of Theorem 6.3.1 one may hope to form motives with large $\underline{\mathfrak{u}}$ by patching together suitable smaller such motives. The goal of the next few subsections is to try to classify, up to isomorphism, all motives M with large $\underline{\mathfrak{u}}$ which satisfy (27) and which, up to isomorphism, have a fixed $W_{p-1}M$ (with

[11] Note that the assumption that $\mathrm{Gr}_p^W M$ is nonzero is actually important for the proof. Thus when we want to apply Theorem 6.3.1 to show that a given motive M has a large $\underline{\mathfrak{u}}$, we do not have a choice about what to take as p; it is determined by the motive M.

large $\underline{u}$) and $\mathrm{Gr}_p^W M$ (with the isomorphisms not part of the data). To this end, let us first consider a related problem. For the discussion in this subsection, $\boldsymbol{T}$ can be any abelian category (we will eventually apply the discussion to our category of motives).

Throughout, we fix objects A, B and C in $\boldsymbol{T}$ (in our final application, these will be respectively (the fixed objects which are to be isomorphic to) $\mathrm{Gr}_p^W M$, $W_{p-1}M$, and $\mathbb{1}$). Grothendieck considers the following problem in [SGA 7_{I} 1972, Section 9.3 of Exposé 9]: Classify all tuples

$$(M;\, (M_i)_{-3\le i\le 0};\, \gamma_0, \gamma_{-1}, \gamma_{-2})$$

where

$$M = M_0 \supset M_{-1} \supset M_{-2} \supset M_{-3} = 0$$

are objects of $\boldsymbol{T}$ and

$$M/M_{-1} = M_0/M_{-1} \xrightarrow{\gamma_0} C,\; M_{-1}/M_{-2} \xrightarrow{\gamma_{-1}} A,\; M_{-2}/M_{-3} = M_{-2} \xrightarrow{\gamma_{-2}} B$$

are isomorphisms. The classification is to be done up to isomorphisms of such tuples, defined in the obvious way. Here it is convenient for us to consider a slight variant of this problem, where we do not include the data of the isomorphisms γ_i in the tuple, but instead just require that the quotients M_0/M_{-1}, M_{-1}/M_{-2} and $M_{-2}/M_{-3} = M_{-2}$ are isomorphic to C, A and B, respectively.

We say that a pair of extension classes

$$(\mathscr{L}, \mathscr{N}) \in \mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(C, A)$$

is *compatible* if there is a commutative diagram in $\boldsymbol{T}$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & L & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{31}$$

where the rows and columns are exact, the first (complete) row represents $\mathscr{L}$, and the second (complete) column represents $\mathscr{N}$. We say an object M is *attached to* the pair $(\mathscr{L}, \mathscr{N})$ if it fits in a diagram as above. Note that if we have a diagram as above, (by adjusting the maps when needed) we may replace the first row (resp. second column) by any other representative of $\mathscr{L}$ (resp. $\mathscr{N}$).

In [SGA $7_{\rm I}$ 1972], a diagram as above is called an *extension panachée* of the second column sequence by the top row sequence.[12] Thus to say the pair $(\mathscr{L}, \mathscr{N})$ is compatible amounts to saying that an *extension panachée* of (an or every representative of) $\mathscr{N}$ by (an or every representative of) $\mathscr{L}$ exists, or that $(\mathscr{L}, \mathscr{N})$ is "*panachable*", in the language of [Bertrand 2013].

The theory of Yoneda extensions gives a simple characterization of compatible pairs. Let

$$\circ : \mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(C, A) \to \mathrm{Ext}^2(C, B)$$

be the Yoneda (composition) pairing; it sends the pair $(\mathscr{L}, \mathscr{N})$ with $\mathscr{L}$ given by

$$0 \to B \to L \xrightarrow{\pi} A \to 0 \tag{32}$$

and $\mathscr{N}$ given by

$$0 \to A \xrightarrow{\iota} N \to C \to 0$$

to the extension $\mathscr{L} \circ \mathscr{N}$ given by

$$0 \to B \to L \xrightarrow{\iota\circ\pi} N \to C \to 0.$$

Lemma 6.4.1. (a) *The pair $(\mathscr{L}, \mathscr{N})$ is compatible if and only if $\mathscr{L} \circ \mathscr{N} = 0$.*

(b) *Suppose* $\mathrm{Ext}^1(C, B) = 0$. *If $(\mathscr{L}, \mathscr{N})$ is compatible, then up to isomorphism there is a unique object attached to it.*

Proof. This is Lemma 9.3.8 of [SGA $7_{\rm I}$ 1972]. Fix the extension (32) representing $\mathscr{L}$. If M is attached to the pair, fitting into a diagram as in (31), then the class

$$\mathscr{M} \in \mathrm{Ext}^1(C, L)$$

of the first column in the diagram pushes forward to $\mathscr{N}$ under π. Conversely, if $\mathscr{N}$ is in the image of the pushforward

$$\pi_* : \mathrm{Ext}^1(C, L) \to \mathrm{Ext}^1(C, A),$$

with $\mathscr{M}$ represented by

$$0 \to L \to M \to C \to 0$$

in the preimage of $\mathscr{N}$, then the object M is attached to our pair. Thus the pair $(\mathscr{L}, \mathscr{N})$ is compatible if and only if $\mathscr{N}$ is in the image of π_*. Now by the general theory of Yoneda extensions, applying the functor $\mathrm{Hom}(C, -)$ to (32) we get an exact sequence

$$\mathrm{Ext}^1(C, B) \to \mathrm{Ext}^1(C, L) \xrightarrow{\pi_*} \mathrm{Ext}^1(C, A) \xrightarrow{\delta = \mathscr{L}\circ -} \mathrm{Ext}^2(C, B);$$

see [Buchsbaum 1959, Section 3] or [Yoneda 1960, page 561]. This proves part (a).

---

[12] Or as Bertrand translates in [Bertrand 1998], a *blended* extension.

As for the statement in part (b), if M and M' are attached to $(\mathscr{L}, \mathscr{N})$, fitting into diagrams as in (31) with the classes of the corresponding first columns denoted by $\mathscr{M}$ and $\mathscr{M}'$ (both in $\mathrm{Ext}^1(C, L)$) respectively, then it follows from the above long exact sequence that $\mathscr{M}$ and $\mathscr{M}'$ differ by an element in the image of $\mathrm{Ext}^1(C, B)$. If this Ext group is zero, then $\mathscr{M} = \mathscr{M}'$, and hence in particular M and M' are (noncanonically) isomorphic. □

Remark. In a reasonable Tannakian category of mixed motives over a number field it is expected that one should have $\mathrm{Ext}^2(X, Y) = 0$ for every objects X and Y. So in that context, every pair should be compatible. See the remark in the end of Section 6.7 for a more detailed discussion of the Ext groups in our particular categories of interest.

6.5. We shall continue in the setting of the previous subsection ($\boldsymbol{T}$ any abelian category, and B, A, C three fixed objects of $\boldsymbol{T}$). Our goal in this subsection is to see when the same object is attached to two compatible pairs of extensions.

We use the notation End( ) (resp. Aut( )) for the endomorphism algebra (resp. automorphism group) of an object in $\boldsymbol{T}$. The endomorphism algebra $\mathrm{End}(A)$ of A acts on both $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(C, A)$. Indeed, the action on $\mathrm{Ext}^1(A, B)$ is a right action given by pullback: if f is an endomorphism of A, set $\mathscr{L} \cdot f := f^*\mathscr{L}$ (f^* for pullback along f). The action on $\mathrm{Ext}^1(C, A)$ is a left action given by push forward: $f \cdot \mathscr{N} := f_*\mathscr{N}$ (to see the bilinearity properties of these actions, see [Buchsbaum 1959] or [Yoneda 1960]). If f is an automorphism of A, then $\mathscr{L} \cdot f$ and $f \cdot \mathscr{N}$ are simply obtained by twisting respectively the surjective and injective arrows of $\mathscr{L}$ and $\mathscr{N}$ by f^{-1}, i.e., $\mathscr{L} \cdot f$ (resp. $f \cdot \mathscr{N}$) is the class of the extension obtained by replacing the surjective (resp. injective) arrow π (resp. ι) in a representative of $\mathscr{L}$ (resp. $\mathscr{N}$) by $f^{-1} \circ \pi$ (resp. $\iota \circ f^{-1}$).

We restrict the two actions above on $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(C, A)$ to the actions of the group $\mathrm{Aut}(A)$. We also modify the action on $\mathrm{Ext}^1(C, A)$ so that it also becomes a right action, by setting $\mathscr{N} \cdot f := f_*^{-1}\mathscr{N}$. Thus $\mathscr{N} \cdot f$ is the class of the extension obtained by twisting the injective arrow of $\mathscr{N}$ by f. Similarly, we have right actions of $\mathrm{Aut}(B)$ (resp. $\mathrm{Aut}(C)$) on $\mathrm{Ext}^1(A, B)$ (resp. $\mathrm{Ext}^1(C, A)$).

We now equip the product

$$\mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(C, A) \tag{33}$$

with the following right actions of $\mathrm{Aut}(B)$, $\mathrm{Aut}(A)$, and $\mathrm{Aut}(C)$: the group $\mathrm{Aut}(B)$ (resp. $\mathrm{Aut}(C)$) acts by acting on the first (resp. second) factor, and $\mathrm{Aut}(A)$ acts diagonally, i.e., by the formula

$$(\mathscr{L}, \mathscr{N}) \cdot f := (\mathscr{L} \cdot f, \mathscr{N} \cdot f) = (f^*\mathscr{L}, f_*^{-1}\mathscr{N}).$$

The three actions commute with one another. Indeed, the actions of $\mathrm{Aut}(B)$ and $\mathrm{Aut}(C)$ trivially commute, and the commutativity of the actions of $\mathrm{Aut}(A)$ with each of $\mathrm{Aut}(B)$ and $\mathrm{Aut}(C)$ is clear from the description of the actions in terms of twisting the arrows, as different groups act by twisting different arrows. Thus we get an action of $\mathrm{Aut}(B) \times \mathrm{Aut}(A) \times \mathrm{Aut}(C)$ on the product (33). We say two pairs of extensions are *equivalent* if they are in the same orbit of this action.

Lemma 6.5.1. *Let* $(\mathscr{L}, \mathscr{N})$ *and* $(\mathscr{L}', \mathscr{N}')$ *be in* (33).

(a) *Suppose* $(\mathscr{L}, \mathscr{N})$ *and* $(\mathscr{L}', \mathscr{N}')$ *are equivalent. Then every object attached to the pair* $(\mathscr{L}, \mathscr{N})$ *is also attached to the pair* $(\mathscr{L}', \mathscr{N}')$. (*In particular,* $(\mathscr{L}, \mathscr{N})$ *is compatible if and only if* $(\mathscr{L}', \mathscr{N}')$ *is compatible.*)

(b) *Suppose every object of* $\boldsymbol{T}$ *is equipped with an exact functorial increasing filtration* $W_\bullet$ *which is finite on every object* (*we refer to this as the weight filtration*). *Suppose moreover that the highest weight of* B *is less than the lowest weight of* A, *and that the highest weight of* A *is less than the lowest weight of* C. *Then if there is an object* M *attached to both* $(\mathscr{L}, \mathscr{N})$ *and* $(\mathscr{L}', \mathscr{N}')$, *then the two pairs are equivalent.*

Proof. (a) Let $(\mathscr{L}', \mathscr{N}') = (\mathscr{L}, \mathscr{N}) \cdot (f_B, f_A, f_C)$ for some $f_B \in \mathrm{Aut}(B)$, $f_A \in \mathrm{Aut}(A)$, and $f_C \in \mathrm{Aut}(C)$. Suppose M is attached to $(\mathscr{L}, \mathscr{N})$. In a diagram as in (31) (with the first row and second column respectively representing $\mathscr{L}$ and $\mathscr{N}$), twist the arrows $B \to L$ and $B \to M$ by f_B, the arrow $L \to A$ by f_A^{-1}, the arrow $A \to N$ by f_A, and the arrows $M \to C$ and $N \to C$ by f_C, while keeping $L \to M$ and $M \to N$ unchanged. The diagram remains commutative and with exact rows and columns, and its first row (resp. second column) represents $\mathscr{L}'$ (resp. $\mathscr{N}'$).

(b) Suppose an object M is attached to both $(\mathscr{L}, \mathscr{N})$ and $(\mathscr{L}', \mathscr{N}')$. We consider two diagrams as in (31), one with objects L, N with the first row and second column representing $\mathscr{L}$ and $\mathscr{N}$, and the other with objects L', N' with the first row and second column representing $\mathscr{L}'$ and $\mathscr{N}'$. In the diagram for $(\mathscr{L}, \mathscr{N})$, we name the maps as follows: In the first row, (resp. second row, second column) the injective arrow is ι_L (resp. ι_M, ι_N) and the surjective arrow is π_L (resp. π_M, π_N). We refer to the maps $L \to M$ and $M \to C$ as α and β, respectively. Accordingly, denote the maps in the diagram for $(\mathscr{L}', \mathscr{N}')$ by $\iota_{L'}, \pi_{L'}, \iota'_M, \pi'_M, \iota_{N'}, \pi_{N'}, \alpha'$ and β' (each map being the analogue to its lookalike in the first diagram). (Note that the central object in both diagrams in M.)

Let b, a and c be respectively the highest weights of B, A and C. Focusing on the first diagram, using exactness of the weight filtration together with the hypothesis that every weight of B is less than every weight of A, which in turn is less than every weight of C, we see that

$$W_b L = \iota_L(B), \quad W_a L = L, \quad W_b N = 0, \quad W_a N = \iota_N(A), \quad W_c N = N$$

and

$$W_b M = \iota_M(B), \quad W_a M = \alpha(L), \quad W_c M = M.$$

We have similar equalities for the $'$-adorned analogues coming from the second diagram. In particular,

$$\iota_M(B) = \iota'_M(B) = W_b M, \qquad \alpha(L) = \alpha'(L') = W_a M.$$

Thus we get isomorphisms

$$\alpha^{-1}\alpha' : L' \to L \quad \text{and} \quad \iota_M^{-1}\iota'_M : B \to B$$

(uniquely) defined by the property that

$$\alpha(\alpha^{-1}\alpha') = \alpha' \quad \text{and} \quad \iota_M(\iota_M^{-1}\iota'_M) = \iota'_M.$$

We have a commutative diagram

$$\begin{CD}
0 @>>> B @>\iota_{L'}>> L' @>\pi_{L'}>> A @>>> 0\\
@. @V\iota_M^{-1}\iota'_MVV @V\alpha^{-1}\alpha'VV @V=:\gamma VV @.\\
0 @>>> B @>\iota_L>> L @>\pi_L>> A @>>> 0,
\end{CD}$$

where the rows are exact and the vertical arrows are isomorphisms (to see the commutativity of the first square further compose with α). Thus $\mathscr{L}'$ is obtained from $\mathscr{L}$ by twisting ι_L by $\iota_M^{-1}\iota'_M$ and twisting π_L by γ^{-1}.

On the other hand, since we have $\iota_M(B) = \iota'_M(B) = W_bM$, by exactness of the second rows in the diagrams of the two pairs, π_M and π'_M induce isomorphisms

$$\overline{\pi_M} : M/W_bM \to N, \quad \overline{\pi'_M} : M/W_bM \to N'.$$

Similarly, thanks to exactness of the first columns (and on recalling $\alpha(L) = \alpha'(L') = W_aM$), we have isomorphisms

$$\overline{\beta} : M/W_aM \to C, \quad \overline{\beta'} : M/W_aM \to C,$$

induced by β and β', respectively. We now have a commutative diagram

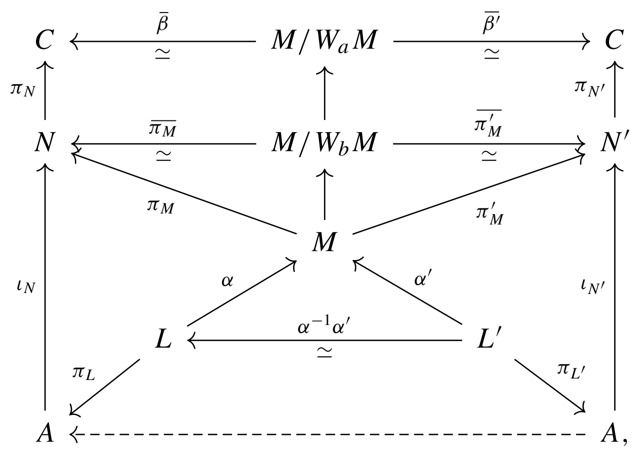
$$\begin{CD}
0 @>>> A @>\iota_{N'}>> N' @>\pi_{N'}>> C @>>> 0\\
@. @V=:\lambda VV @V\overline{\pi_M}\,\overline{\pi'_M}^{-1}VV @V\overline{\beta}\,\overline{\beta'}^{-1}VV @.\\
0 @>>> A @>\iota_N>> N @>\pi_N>> C @>>> 0,
\end{CD} \tag{34}$$

where the rows are exact and vertical arrows are isomorphisms (to see commutativity of the second square precompose with $\pi'_M : M \to N'$). It follows that $\mathscr{N}'$ is obtained from $\mathscr{N}$ by twisting ι_N by λ and twisting π_N by $\overline{\beta'}\,\overline{\beta}^{-1}$.

To complete the proof, it suffices to show that $\gamma = \lambda$, as then

$$(\mathscr{L}', \mathscr{N}') = (\mathscr{L}, \mathscr{N}) \cdot (\iota_M^{-1}\iota'_M, \gamma, \overline{\beta}\,\overline{\beta'}^{-1}).$$

Ignoring the dashed arrow, we have a commutative diagram



where the vertical arrows in the middle are the obvious maps. The map γ is the unique map that if it is included as the dashed arrow, it makes the bottom trapezoid of the diagram commute. But from the diagram we easily see that λ also does this job. Indeed, to check commutativity of the trapezoid with λ as the dashed arrow, it is enough to check commutativity after composing with ι_N. Now using commutativity of the rest of the diagram above and the left square in (34), we have

$$\iota_N \pi_L(\alpha^{-1}\alpha') = \pi_M \alpha' = \iota_N \lambda \pi_{L'}. \qquad \square$$

6.6. We now combine the results of the previous two subsections on compatible pairs. We shall assume that $\boldsymbol{T}$ is an abelian category equipped with a weight filtration (i.e., a functorial, exact, increasing filtration which is finite on every object). As in the previous two subsections, B, A, C are fixed objects of $\boldsymbol{T}$. The following result, which for future reference we record as a proposition, has been mostly already proved in the previous two subsections.

Proposition 6.6.1. *Suppose every weight of B is less than every weight of A, and that every weight of A is less than every weight of C. Let b, a, c be the highest weights of B, A, C, respectively.*

(a) *Any pair of extensions $(\mathscr{L}, \mathscr{N})$ in (33) is compatible if and only if*

$$\mathscr{L} \circ \mathscr{N} = 0 \quad \textit{in } \operatorname{Ext}^2(C, B).$$

(b) *If M is an object that is attached to some pair of extensions in (33), then we have*

$$B \simeq W_b M, \qquad A \simeq W_a M / W_b M, \qquad C \simeq M / W_a M. \tag{35}$$

(c) *Any object M satisfying (35) is attached to some pair $(\mathscr{L}, \mathscr{N})$ of extensions in (33). Moreover, M is attached to any other pair $(\mathscr{L}', \mathscr{N}')$ if and only if $(\mathscr{L}', \mathscr{N}')$ is equivalent to $(\mathscr{L}, \mathscr{N})$. We have a (well-defined) surjective map*

$$\left\{ \begin{matrix} \textit{the collection of objects } M \textit{ satisfying} \\ \textit{(35), up to isomorphism} \end{matrix} \right\} \to \left\{ \begin{matrix} \textit{the collection of compatible pairs} \\ \textit{in (33), up to equivalence} \end{matrix} \right\}$$

which sends the isomorphism class of M to the equivalence class of any pair (or all pairs) $(\mathscr{L}, \mathscr{N})$ to which M is attached.

(d) *If $\operatorname{Ext}^1(C, B) = 0$, then the surjection above is a bijection.*

Proof. (a) This is Lemma 6.4.1(a).

(b) This follows from the observations made at the beginning of the proof of Lemma 6.5.1(b) about the weight filtration of M. (Note that the isomorphisms are noncanonical, as they depend on the particular choice of diagram (31).)

(c) Given M satisfying (35), we have a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_b M & \longrightarrow & W_a M & \longrightarrow & W_a M / W_b M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_b M & \longrightarrow & M & \longrightarrow & M / W_b M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M / W_a M & = & M / W_a M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(with obvious maps, exact rows and columns). Now use some choice of isomorphisms (35) to replace $W_b M$, $W_a M / W_b M$, and $M / W_a M$ respectively by B, A, and C. Take $\mathscr{L}$ (resp. $\mathscr{N}$) to be the extension class of the top row (resp. last column) in the new diagram. Then M is attached to the (compatible) pair $(\mathscr{L}, \mathscr{N})$. By Lemma 6.5.1(b), M is attached to another pair $(\mathscr{L}', \mathscr{N}')$ if and only if $(\mathscr{L}', \mathscr{N}')$ is equivalent to $(\mathscr{L}, \mathscr{N})$. On the other hand, if M' is isomorphic to M, then M' is clearly attached to the same pairs as M. Thus we have a well-defined map as in the statement. It is surjective by the definition of compatibility and Part (b).

(d) This follows from Lemmas 6.4.1(b) and 6.5.1(a). □

6.7. We now return to the discussion of motives with large $\underline{\mathfrak{u}}$ (with $\boldsymbol{T}$ again a Tannakian category of mixed motives or the category of rational mixed Hodge structures). Given any two motives A and B, let us say an extension class in $\mathrm{Ext}^1(A, B)$ has a large $\underline{\mathfrak{u}}$ if the object in the middle of a representing short exact sequence has a large $\underline{\mathfrak{u}}$. This is clearly well-defined, and moreover, the property of having a large $\underline{\mathfrak{u}}$ is invariant under the action of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ (because the collection of the objects that can appear as the middle object for two extension classes in the same orbit are the same, as by twisting the arrows we can turn a representative of one extension class to a representative of another extension class in the same orbit). Note that if A is simple (resp. pure), then an extension class in $\mathrm{Ext}^1(\mathbb{1}, A)$ has a large $\underline{\mathfrak{u}}$ if and only if it is nonsplit (resp. totally nonsplit).

We say a pair of extensions $(\mathscr{L}, \mathscr{N})$ in (33) has a large $\underline{\mathfrak{u}}$ if both extensions $\mathscr{L}$ and $\mathscr{N}$ have a large $\underline{\mathfrak{u}}$. This property is invariant under our notion of equivalence of pairs.

We now fix an integer $p < 0$, and motives B and A with

$$B = W_{p-1} B, \qquad A \cong \mathrm{Gr}_p^W A \neq 0.$$

(In other words, all weights of B are $< p$, and A is nonzero and pure of weight p; note that B may be mixed.) Proposition 6.6.1 gives a surjection (bijection if $\mathrm{Ext}^1(\mathbb{1}, B) = 0$) from the collection of motives

M satisfying

$$W_{p-1}M \simeq B, \quad \mathrm{Gr}_p^W M \simeq A, \quad M/W_pM \simeq \mathbb{1} \tag{36}$$

up to isomorphism to the collection of compatible pairs in

$$\mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(\mathbb{1}, A)$$

(= the kernel of the composition pairing into $\mathrm{Ext}^2(\mathbb{1}, B)$) up to equivalence (i.e., the action of $\mathrm{Aut}(B) \times \mathrm{Aut}(A) \times \mathrm{Aut}(\mathbb{1})$). By Theorem 6.3.1, if $B \oplus A \oplus \mathbb{1}$ satisfies the independence axiom $(IA1)_{\{p,q\}}$ for every $q \le p$, then given any compatible pair $(\mathscr{L}, \mathscr{N})$ with a large $\underline{\mathfrak{u}}$, any object M attached to the pair also has a large $\underline{\mathfrak{u}}$. Conversely, if an object M satisfying (36) has a large $\underline{\mathfrak{u}}$, then so does any pair $(\mathscr{L}, \mathscr{N})$ in the equivalence class of the extension pairs corresponding to M (see item (2) after Theorem 6.3.1; note that here no independence axiom needs to be satisfied).

We record the following special case as a corollary:

Corollary 6.7.1. *Let $-2n < p < 0$ and $p \neq -n$. Let A be a nonzero simple motive of weight p. Suppose moreover that $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$. Then there is a bijection*

$$\left\{\begin{array}{c}\textit{the collection of objects } M\\ \textit{with } \mathrm{Gr}^W M \simeq \mathbb{Q}(n) \oplus A \oplus \mathbb{1}\\ \textit{and large } \underline{\mathfrak{u}}(M), \textit{ up to}\\ \textit{isomorphism}\end{array}\right\} \to \left\{\begin{array}{c}\textit{the collection of compatible pairs}\\ \textit{of nonsplit extensions in}\\ \mathrm{Ext}^1(A, \mathbb{Q}(n)) \times \mathrm{Ext}^1(\mathbb{1}, A),\\ \textit{up to equivalence}\end{array}\right\}$$

which assigns to the isomorphism class of an object M the equivalence class of the compatible pairs to which M is attached. If we omit the condition $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = 0$, this map is well-defined and surjective.

(Note that the condition $p \neq -n$ guarantees $(IA3)$.)

Remark. (1) In any reasonable Tannakian category of mixed motives over a number field, all the Ext^2 groups (and hence all the higher Ext group) are expected to vanish. The Ext^1 groups in such a category should be related to Chow groups and motivic cohomology (and algebraic K-theory). See for instance, the beautiful articles of Nekovar [1994] and Jannsen [1994]. The only case of a Tannakian category of motives where the Ext groups are actually known is the case of the category of mixed Tate motives. See part (2) for a discussion of this case.

(2) Let $\mathbf{MT}(F)$ be Voevodsky's category of mixed Tate motives over a number field F. The Ext^2 groups in $\mathbf{MT}(F)$ are zero, and the groups

$$\mathrm{Ext}^1_{\mathbf{MT}(F)}(\mathbb{1}, \mathbb{Q}(n))$$

are given by the K-theory of the field F modulo torsion, which in turn is described by theorems of Borel and Soulé (and Dirichlet in the case of K_1). In particular, if F is totally real and n is even, the Ext^1 group above vanishes. (See [Deligne and Goncharov 2005] for the precise description of the Ext groups in $\mathbf{MT}(F)$ and the subcategory of mixed Tate motives over the ring of integers of F. Note that if $\mathbf{MM}(F)$ is any category of mixed motives over F for which the full Tannakian subcategory generated by $\mathbb{Q}(1)$

and closed under extensions is equivalent to Voevodsky's $\mathbf{MT}(F)$, then the Ext^1 groups above are the same in $\mathbf{MM}(F)$ and $\mathbf{MT}(F)$.)

(3) In the category **MHS** of rational mixed Hodge structures, the Ext^2 groups vanish; see [Beĭlinson 1986]. The Ext^1 groups in this category are described by the results of Carlson [1980].

6.8. In this subsection, we shall take $\boldsymbol{T}$ to be Voevodsky's category $\mathbf{MT}(\mathbb{Q})$ of mixed Tate motives over $\mathbb{Q}$. As an application of the previous results, we shall classify (up to isomorphism) all 3-dimensional objects of $\mathbf{MT}(\mathbb{Q})$ with three distinct weights, large $\underline{\mathfrak{u}}$, and satisfying an independence axiom; see below for more details.[13] Note that for any 3-dimensional object M of $\mathbf{MT}(\mathbb{Q})$ with three distinct weights and large $\underline{\mathfrak{u}}$, the unipotent radical of the motivic Galois group $\mathcal{G}(M, \omega_B)$ (with ω_B the Betti realization functor) has dimension equal to 3 ($= \dim W_{-1}\operatorname{End}(\omega_B M)$). Since

$$\mathcal{G}(\mathrm{Gr}^W M, \omega_B) \simeq \mathbb{G}_m,$$

the motivic Galois group $\mathcal{G}(M, \omega_B)$ has dimension 4. Thus Grothendieck's period conjecture would predict that the transcendence degree of the field generated by the periods of M should be 4.

Let us first recall the description of the Ext groups between simple objects in $\mathbf{MT}(\mathbb{Q})$ (see [Deligne and Goncharov 2005], for instance)

$$\begin{gathered}\dim \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(n)) = \begin{cases} 0 & \text{if } n \text{ is even or } \le 0 \\ 1 & \text{if } n \text{ is odd and } \ge 3 \end{cases} \\ \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1)) \cong \mathbb{Q}^\times \otimes \mathbb{Q} \end{gathered} \tag{37}$$

moreover, Ext^2 groups all vanish in $\mathbf{MT}(\mathbb{Q})$.

Back to our classification problem, we may assume that the highest weight of our motives is zero. We shall classify all motives with an associated graded of the form

$$\mathbb{Q}(n) \oplus \mathbb{Q}(k) \oplus \mathbb{1} \quad (n > k > 0,\, n \neq 2k)$$

which have a large $\underline{\mathfrak{u}}$. (The condition $n \neq 2k$ is an independence axiom. The case where $n = k$ is complicated, as then one can no longer use Theorem 6.3.1.) For any such motive, the pair $(\mathcal{L}, \mathcal{N})$ in

$$\mathrm{Ext}^1(\mathbb{Q}(k), \mathbb{Q}(n)) \times \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(k)) \tag{38}$$

associated to it by Corollary 6.7.1 (also see Proposition 6.6.1) has nonsplit entries. In view of the description of the Ext^1 groups in the category, we see that k must be odd and n must be even. We will then have a bijection as in Corollary 6.7.1.

Let us consider the action of $\mathrm{Aut}(\mathbb{Q}(n)) \times \mathrm{Aut}(\mathbb{Q}(k)) \times \mathrm{Aut}(\mathbb{1})$ on (38). Since the automorphism group of every $\mathbb{Q}(a)$ is $\mathbb{Q}^*$, it follows from bilinearity of the actions of $\mathrm{End}(A)$ on $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(B, A)$ (for any A, B in any K-linear category) that the action of $\mathrm{Aut}(\mathbb{Q}(k))$ can be absorbed into the actions of

[13] The classification is then valid in any Tannakian category $\mathbf{MM}(\mathbb{Q})$ of mixed motives over $\mathbb{Q}$ for which the smallest full Tannakian subcategory containing $\mathbb{Q}(1)$ and closed under extensions is equivalent to $\mathbf{MT}(\mathbb{Q})$.

the other two factors: $(\lambda, \gamma, \delta)$ acts the same as $(\lambda\gamma^{-1}, 1, \gamma\delta)$ (where $\lambda, \gamma, \delta \in \mathbb{Q}^*$). It follows that an orbit of the action of $\operatorname{Aut}(\mathbb{Q}(n)) \times \operatorname{Aut}(\mathbb{Q}(k)) \times \operatorname{Aut}(\mathbb{1})$ on (38) coincides with an element of

$$(\operatorname{Ext}^1(\mathbb{Q}(k), \mathbb{Q}(n))/\operatorname{Aut}(\mathbb{Q}(n))) \times (\operatorname{Ext}^1(\mathbb{1}, \mathbb{Q}(k))/\operatorname{Aut}(\mathbb{1})) \tag{39}$$

(with both actions made right actions, as before).

Case I: $k = 1$. Then n is ≥ 4 (and even), and

$$\operatorname{Ext}^1(\mathbb{Q}(k), \mathbb{Q}(n)) \cong \operatorname{Ext}^1(\mathbb{1}, \mathbb{Q}(n-k))$$

is a 1-dimensional vector space over $\mathbb{Q}$, and all its nonzero elements are in the same $\operatorname{Aut}(\mathbb{Q}(n))$-orbit.

The extensions of $\mathbb{1}$ by $\mathbb{Q}(1)$ are the Kummer motives. For each positive rational number r, let

$$[r] \in \operatorname{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$$

be the extension class arising from the weight filtration of the 1-motive (see [Deligne 1974])

$$K_r := [\mathbb{Z} \xrightarrow{1 \mapsto r} \mathbb{G}_m] \tag{40}$$

(considered as an object of $\mathbf{MT}(\mathbb{Q})$). Then $[r]$ is the element of $\operatorname{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$ corresponding to $r \otimes 1$ under the isomorphism (37). Thus $\{[p] : p \text{ prime} > 0\}$ is a basis of $\operatorname{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$ (over $\mathbb{Q}$). A complete inequivalent set of representatives for the nonzero orbits of the action of $\mathbb{Q}^* = \operatorname{Aut}(\mathbb{1})$ on $\operatorname{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$ is formed by the elements $[r]$, where r runs through all rational numbers > 1 which are not of the form s^a for any $s \in \mathbb{Q}$ and $a \in \mathbb{Z}$ with $a > 1$. In view of Corollary 6.7.1, each such $[r]$ gives a (unique, up to isomorphism) motive $M_{n,r}$ with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to

$$\mathbb{Q}(n) \oplus \mathbb{Q}(1) \oplus \mathbb{1}.$$

These motives are nonisomorphic, and are up to isomorphism, all the motives with associated graded as above and large $\underline{\mathfrak{u}}$.

A discussion of the periods of $M_{n,r}$ is in order. By construction, $W_{-2}M_{n,r}$ is a nontrivial extension of $\mathbb{Q}(1)$ by $\mathbb{Q}(n)$. Being a twist (by $\mathbb{Q}(1)$) of a nontrivial extension of $\mathbb{1}$ by $\mathbb{Q}(n-1)$, the motive $W_{-2}M_{n,r}$ has the period matrix

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n-1) \\ 0 & (2\pi i)^{-1} \end{pmatrix}$$

with respect to suitably chosen bases of Betti and de Rham realizations. (Note that $n-1$ is odd and ≥ 3. That a nontrivial extension of $\mathbb{1}$ by $\mathbb{Q}(n-1)$ has $\zeta(n-1)/(2\pi i)^{n-1}$ as a period follows from the work of Deligne [1989] in the setting of realizations, and later the work of Deligne and Goncharov [2005] in the setting of Voevodsky motives.) One the other hand, $M_{n,r}$ has the Kummer 1-motive K_r as a subquotient (by $W_{-2n}M_{n,r} = W_{-3}M_{n,r}$). With respect to suitably chosen bases of Betti and de Rham realizations, K_r has the period matrix

$$\begin{pmatrix} (2\pi i)^{-1} & (2\pi i)^{-1}\log r \\ 0 & 1 \end{pmatrix};$$

see [Deligne 1974] for the explicit realizations of 1-motives. With respect to suitably chosen bases, the matrix of periods of $M_{n,r}$ looks like

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n-1) & * \\ 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log r \\ 0 & 0 & 1 \end{pmatrix}.$$

As mentioned earlier, Grothendieck's period conjecture predicts the transcendence degree of the field generated over $\mathbb{Q}$ by the periods of $M_{n,r}$ to be 4. Thus assuming the period conjecture, the numbers

$$2\pi i, \quad \log r, \quad \zeta(n-1), \quad \text{and the entry denoted by } *$$

are algebraically independent over $\mathbb{Q}$.

It would be very interesting to somehow calculate the entry $*$ in a period matrix of $M_{n,r}$ as above. As we discussed in the Introduction, when $r \neq 2$, Deligne's work [2010] (and a fortiori Brown's [2012]) does not predict the nature of $*$.

Case II: $k > 1$ and $n \neq k+1$ (so $n \geq k+3$). Then both quotients in (39) are singletons. Thus up to isomorphism, there is a unique motive $Z_{n,k}$ with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(k) \oplus \mathbb{1}$. The subobject $W_{-2k}Z_{n,k}$ (resp. subquotient $Z_{n,k}/W_{-2k-1}Z_{n,k}$) of $Z_{n,k}$ is a nontrivial extension of $\mathbb{Q}(k)$ by $\mathbb{Q}(n)$ (resp. $\mathbb{1}$ by $\mathbb{Q}(k)$). The matrix of periods of $Z_{n,k}$ with respect to suitably chosen bases is of the form

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n-k) & * \\ 0 & (2\pi i)^{-k} & (2\pi i)^{-k}\zeta(k) \\ 0 & 0 & 1 \end{pmatrix}.$$

The period conjecture predicts that $2\pi i$, $\zeta(k)$, $\zeta(n-k)$ and the entry denoted by $*$ are algebraically independent over $\mathbb{Q}$. Again it would be interesting to find what the entry $*$ is. Note that the motive $Z_{n,k}$ is in the subcategory $\mathbf{MT}(\mathbb{Z})$, as from the beginning we may have done the entire discussion of this case in $\mathbf{MT}(\mathbb{Z})$ (as the relevant Ext groups in this case are the same in $\mathbf{MT}(\mathbb{Z})$ and $\mathbf{MT}(\mathbb{Q})$). Thus by Brown's work [2012], all periods of $Z_{n,k}$ will be in the algebra generated by $2\pi i$ and the multiple zeta values.

Case III: $k > 1$ and $n = k+1$. This case is the dual situation to Case I. Here the second factor of (39) is a singleton, and the motives under investigation are classified up to isomorphism by $\mathrm{Aut}(\mathbb{1})$-orbits of $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(1))$. Consider the complete inequivalent set of representatives $\{[r]\}$ for these orbits as in Case I. Then for each r, we get an object $M'_{n,r}$ corresponding to the element of (39) with the orbit of $[r]$ as its first coordinate. The motives $M'_{n,r}$ are nonisomorphic and up to isomorphism, give all motives with large $\underline{\mathfrak{u}}$ and associated graded isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(n-1) \oplus \mathbb{1}$.

The motives obtained in this case are intimately related to the $M_{n,r}$ of Case I. Indeed, ${M'_{n,r}}^{\vee}(n)$ has a large $\underline{\mathfrak{u}}$ (as the property of having a large $\underline{\mathfrak{u}}$ is invariant under dualizing and tensoring by $\mathbb{Q}(1)$), and its associated graded is isomorphic to $\mathbb{Q}(n) \oplus \mathbb{Q}(1) \oplus \mathbb{1}$. Moreover, the quotient ${M'_{n,r}}^{\vee}(n)/W_{-2n}$ is isomorphic to the 1-motive K_r given in (40) (as by construction we have $W_{-2k}M'_{n,r} \simeq K_r(k)$, and K_r is isomorphic to its Cartier dual $K_r^{\vee}(1)$). It follows that $M'^{\vee}_{n,r}(n)$ is isomorphic to $M_{n,r}$ (as they both correspond to the same equivalence class of compatible pairs).

6.9. Let us continue to take $\boldsymbol{T} = \mathbf{MT}(\mathbb{Q})$. The motives of Section 6.8 together with the earlier results of the paper can be used to obtain 4-dimensional mixed Tate motives with 4 weights and a large $\underline{\mathfrak{u}}$.[14] We illustrate this with an example. Let M be the motive $M_{4,r}$ of the previous section, which has associated graded isomorphic to $\mathbb{Q}(4) \oplus \mathbb{Q}(1) \oplus \mathbb{1}$. The weight filtration of M gives an element $\mathscr{L}$ in $\mathrm{Ext}^1(\mathbb{1}, W_{-2}M)$. Let $\mathscr{N}$ a nonzero element of $\mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(5))$. Since Ext^2 groups vanish in $\mathbf{MT}(\mathbb{Q})$, there is an object in $\mathbf{MT}(\mathbb{Q})$ attached to the pair

$$(\mathscr{L}(5), \mathscr{N}) \in \mathrm{Ext}^1(\mathbb{Q}(5), (W_{-2}M)(5)) \times \mathrm{Ext}^1(\mathbb{1}, \mathbb{Q}(5)).$$

Note that here, at least a priori, there might be nonisomorphic objects attached to the pair, as $\mathrm{Ext}^1(\mathbb{1}, (W_{-2}M)(5))$ is not zero. Any object $\widetilde{M}$ attached to the pair is 4-dimensional, with associated graded isomorphic to

$$\mathbb{Q}(9) \oplus \mathbb{Q}(6) \oplus \mathbb{Q}(5) \oplus \mathbb{1}.$$

Such $\widetilde{M}$ satisfies $(IA3)$, and hence by Theorem 6.3.1, $\underline{\mathfrak{u}}(\widetilde{M})$ is large (note that both M and $\mathscr{N}$ have a large $\underline{\mathfrak{u}}$). The field generated over $\mathbb{Q}$ by the periods of $\widetilde{M}$ contains $2\pi i$, $\zeta(3)$, $\log r$, the "new period" of M, and $\zeta(5)$. In fact, by the classification of Section 6.8, the quotient $\widetilde{M}/\mathbb{Q}(9)$ (which is easily seen to also have a large $\underline{\mathfrak{u}}$) must be isomorphic to the motive $M'_{6,r}$ (of Case III of Section 6.8), so that the new period of $M'_{6,r}$ will also be a period of $\widetilde{M}$. The period conjecture predicts that the field generated over $\mathbb{Q}$ by the periods of $\widetilde{M}$ should be of transcendence degree 7 $\left(= \binom{4}{2} + 1\right)$, so that $\widetilde{M}$ should have one more new period, which together with the aforementioned six numbers should form an algebraically independent set over $\mathbb{Q}$.

Remark. Note that $k = 5$ is the smallest positive integer such that

$$\mathrm{Gr}^W M(k) \oplus \mathbb{1}$$

satisfies the independence axiom required to be able to use Theorem 6.3.1.

6.10. Hypothesis (iii) of Theorem 6.3.1 was used in the proof to conclude that $\mathscr{E}_p(M)/\underline{\mathfrak{u}}_p(M)$ splits. This hypothesis is actually important for the statement of the theorem to remain true. A counterexample to the statement without this condition can be given in the category **MHS** of rational mixed Hodge structures using the work of Jacquinot and Ribet [1987] on deficient (in the sense of [loc. cit.]) points on semiabelian varieties, as we shall discuss below. We shall freely use the basics of the theory of 1-motives (including the realizations of a 1-motive), as introduced by Deligne [1974].

[14] Inductively, one can obtain motives with more and more weights which have a large $\underline{\mathfrak{u}}$.

Consider a tuple (F, A, v, f), where

- F is a number field,
- A is a simple abelian variety over F with $\operatorname{rank}(A(F)) > 0$,
- $v \in A^t(F)$ (where A^t is the dual abelian variety),
- and $f : A^t \to A$ is an isogeny over F,

such that $f(v) - f^t(v) \in A(F)$ is a point of infinite order.[15] Let V be a semiabelian variety over F, an extension of A by $\mathbb{G}_m$, which under the canonical isomorphism

$$\operatorname{Ext}(A, \mathbb{G}_m) \cong A^t$$

corresponds to $v \in A^t(F)$. Denote the projection map $V \to A$ by π. In [Jacquinot and Ribet 1987, Section 4], a point $x_f \in V(F)$ is constructed such that

(i) $\pi(x_f) = f(v) - f^t(v)$, and

(ii) for every nonzero integer n the point x_f is divisible by n in $V(F_n)$, where F_n is the field obtained from F by adjoining the n-torsion subgroup of V (such a point is called a deficient point in [Jacquinot and Ribet 1987]).

Let M be the 1-motive $[\mathbb{Z} \xrightarrow{1 \mapsto x_f} V]$ over F. Fixing an embedding $F \subset \bar{F} \subset \mathbb{C}$, denote the Hodge realization of any 1-motive N over F by TN. Thus TM has weights -2, -1, 0 and

$$W_{-2}TM = H_1(\mathbb{G}_m) \simeq \mathbb{Q}(1), \quad W_{-1}TM = H_1(V), \quad \operatorname{Gr}_0^W TM = \mathbb{1}.$$

We shall see that (with $\boldsymbol{T} = \mathbf{MHS}$) $\underline{\mathfrak{u}}(TM)$ is not large, whereas both $\underline{\mathfrak{u}}(W_{-1}TM)$ and $\underline{\mathfrak{u}}(TM/W_{-2}TM)$ are large. This would provide a counterexample to the statement of Theorem 6.3.1 with hypothesis (iii) of the theorem omitted.

First, let us consider $W_{-1}TM$ and $TM/W_{-2}TM$. The former is a nonsplit extension of the simple Hodge structure $H_1(A)$ by $\mathbb{Q}(1)$ (because v has infinite order), and hence (by a similar argument as in Section 6.2) has a large $\underline{\mathfrak{u}}$. The latter is the Hodge realization of the 1-motive

$$[\mathbb{Z} \xrightarrow{1 \mapsto \pi(x_f)} A].$$

Since $\pi(x_f)$ is a point of infinite order, $TM/W_{-2}TM$ is a nonsplit extension of $\mathbb{1}$ by $H_1(A)$, and hence has a large $\underline{\mathfrak{u}}$.

To see that $\underline{\mathfrak{u}}(TM)$ is not large, let ℓ be a prime number. Given any 1-motive N over F, denote the ℓ-adic realization of N by $T_\ell N$, and let $\Pi_\ell(N)$ be the image of the natural map $\operatorname{Gal}(\bar{F}/F) \to \operatorname{GL}(T_\ell N)(\mathbb{Q}_\ell)$. Then Property (ii) above implies that the natural (restriction) map

$$\Pi_\ell(M) \to \Pi_\ell(W_{-1}M) \tag{41}$$

[15] For instance, take $A = A^t$ to be an elliptic curve with complex multiplication by $\mathbb{Z}[i]$, F large enough so that complex multiplication by i is defined over F and $A(F)$ has positive rank, v a point of infinite order in $A^t(F)$, and $f = i$ (so that $f^t = -i$).

(where $W_{-1}M = [0 \to V]$)) is injective (as well as surjective). By the Mumford–Tate conjecture for 1-motives on the unipotent parts (proved by Jossen [2014, Theorem 1]), the Hodge theoretic analogue of this map, i.e., the restriction map

$$\mathcal{G}(TM, \omega_B) \to \mathcal{G}(T(W_{-1}M), \omega_B) \quad (\omega_B = \text{the forgetful fiber functor})$$

is also injective (the two groups above are calculated in **MHS**). Thus $\underline{\mathfrak{u}}_{-1}(TM)$ is zero.

Remark. (1) Here we do not need the full power of the Mumford–Tate conjecture on the unipotent parts to go from the injectivity of (41) to the vanishing of $\underline{\mathfrak{u}}_{-1}(TM)$; just the more basic statement [Bertrand 1998, Theorem 1] is enough. Indeed, [Bertrand 1998, Theorem 1] and injectivity of (41) imply that $W_{-2}\underline{\mathfrak{u}}(TM)$ is zero. It follows that $\underline{\mathfrak{u}}(TM)$ and consequently $\underline{\mathfrak{u}}_{-1}(TM)$ is a pure object of weight -1. On the other hand,

$$\underline{\mathfrak{u}}_{-1}(TM) \subset \underline{\mathrm{Hom}}(TM/W_{-1}TM, W_{-1}TM) \cong W_{-1}TM.$$

It follows that $\underline{\mathfrak{u}}_{-1}(TM)$ is zero (as otherwise, in light of simplicity of $H_1(A)$ the extension $\mathcal{E}_{-2}(W_{-1}TM)$ would split).

(2) Note that the example given in this section shows that in general, without any independence axiom, the individual extensions $\mathcal{E}_p/\underline{\mathfrak{u}}$ need not split (see Corollaries 5.3.2 and 5.3.3 of Theorem 5.3.1, as well as Deligne's Theorem 4.3.1 and the remark after). Indeed, in the above example, $\mathcal{E}_{-1}(TM)/\underline{\mathfrak{u}}(TM)$ does not split: If it did, then by Lemma 5.1.1 so would $\mathcal{E}_{-1}(TM)/\underline{\mathfrak{u}}_{-1}(TM)$. But $\mathcal{E}_{-1}(TM)$ (= $\mathcal{E}_{-1}(TM)/\underline{\mathfrak{u}}_{-1}(TM)$) does not split as x_f is a point of infinite order.

(3) In fact, the example given in this section also shows that in general, $\underline{\mathfrak{u}}$ may not be generated as a Lie algebra by the subobjects $\underline{\mathfrak{u}}_p$. Indeed, with M as above, $\underline{\mathfrak{u}}(TM)$ is not zero (because TM is not semisimple), while both $\underline{\mathfrak{u}}_{-1}(TM)$ and $\underline{\mathfrak{u}}_{-2}(TM)$ are zero. (That the latter is zero can be seen by an argument similar to the one given in part (1): $\underline{\mathfrak{u}}_{-2}(TM)$ is pure of weight -1 and a subobject of $\underline{\mathrm{Hom}}(TM/W_{-2}TM, W_{-2}TM) \cong (TM/W_{-2}TM)^\vee(1)$; the latter object has no nonzero subobject of weight -1.)

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