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under specialization, II**

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Let G be a linear algebraic group over a field. We show that, under mild assumptions, in a family of primitive generically free G -varieties over a base variety B , the essential dimension of the geometric fibers may drop on a countable union of Zariski closed subsets of B and stays constant away from this countable union. We give several applications of this result.

1. Introduction

Let X be a complex algebraic variety (that is, a separated reduced \mathbb{C} -scheme of finite type) equipped with a faithful action of a finite group G . We will refer to X as a G -variety. Assume that the G -action on X is primitive, that is, G transitively permutes the irreducible components of X . In this paper, we will be interested in the essential dimension $\text{ed}_{\mathbb{C}}(X; G)$ and how it behaves in families. Essential dimension is an integer-valued birational invariant of the G -variety X . Its definition can be found in Section 2; for a more detailed discussion, see [Reichstein 2010] or [Merkurjev 2013]. When the group G is clear from the context, we will simply write $\text{ed}_{\mathbb{C}}(X)$ for $\text{ed}_{\mathbb{C}}(X; G)$. Clearly, $\text{ed}_{\mathbb{C}}(X) \leq \dim(X)$; if equality holds, we will say that the G -variety X is *incompressible*.

To date, the study of essential dimension has been primarily concerned with understanding versal G -varieties (once again, see Section 2 for the definition). A complete versal G -variety X has the following special property: X has an A -fixed rational point for every abelian subgroup $A \subset G$; see [Merkurjev 2013, Corollary 3.21].

At the other extreme are complete G -varieties X , where the action of G is free, i.e., no nontrivial element has a fixed point. Existing methods for proving lower bounds on $\text{ed}_{\mathbb{C}}(X)$ usually fail here. Until recently, there was only one family of interesting examples of complete incompressible G -varieties with a free G -action. These examples concern the action of $G = (\mathbb{Z}/p\mathbb{Z})^n$ on the product of elliptic curves $X = E_1 \times \cdots \times E_n$ over \mathbb{C} . Here, p is a prime; the generator of the i -th copy of $\mathbb{Z}/p\mathbb{Z}$ acts on E_i via translation by a point $x_i \in E_i(\mathbb{C})$ of order p , and trivially on E_j for $j \neq i$. Colliot-Thélène and Gabber [Colliot-Thélène 2002, Appendice] showed that for a very general choice of the elliptic curves E_i and torsion points $x_i \in E_i[p]$, a certain degree n cohomological invariant of G does not vanish on $\mathbb{C}(X)^G$.

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(Here, “very general” means “away from a countable union of proper subvarieties” in the moduli space of elliptic curves with a marked torsion point.) This implies that $\text{ed}_{\mathbb{C}}(X) = n = \text{ed}_{\mathbb{C}}(G)$. Additional examples of complete incompressible G -varieties X with a free G -action can be found in the recent work of Farb, Kisin and Wolfson [2021a; 2021b] and Fakhruddin and Saini [2022].

In this paper, we will give a systematic procedure for constructing such examples for an arbitrary finite group. In fact, we will work in a more general setting, where G is a linear algebraic group over an algebraically closed field k . Essential dimension and versality make sense in this more general setting, provided that we require our G -actions to be generically free and not just faithful; see Section 2.

If $\dim(G) > 0$, by Borel’s fixed point theorem, G cannot act freely on a complete variety. Nevertheless, the notion of a free action of a finite group on a projective variety can be generalized to the case of an arbitrary linear algebraic group G as follows: a generically free primitive G -variety X is said to be *strongly unramified* if X is G -equivariantly birationally isomorphic to the total space X' of a G -torsor $X' \rightarrow P$ over some smooth projective irreducible k -variety P . We are now ready to state our first main result.

Theorem 1.1. *Let G be a linear algebraic group over an algebraically closed field k of good characteristic (see Definition 2.1) and of infinite transcendence degree over its prime field, and let X be a generically free primitive G -variety. Then there exists a strongly unramified G -variety Y such that $\dim(Y) = \dim(X)$ and $\text{ed}_k(Y) = \text{ed}_k(X)$.*

Applying Theorem 1.1 to a versal G -variety X , we obtain a strongly unramified G -variety Y of maximal essential dimension, i.e., such that $\text{ed}_k(Y) = \text{ed}_k(G)$. When G is finite, Y is itself smooth and projective. Thus, by starting with an incompressible G -variety X , we obtain examples analogous to those of Colliot-Thélène and Gabber; Farb, Kisin and Wolfson; and Fakhruddin and Saini for an arbitrary finite group G . Note, however, that Farb, Kisin and Wolfson, as well as Fakhruddin and Saini, produce examples over $k = \overline{\mathbb{Q}}$, whereas Theorem 1.1 requires k to be of infinite transcendence degree over the prime field.

Our proof of Theorem 1.1 will rely on Theorems 1.2 and 1.4 below, which are of independent interest.

Theorem 1.2. *Let G be a linear algebraic group over a field k of good characteristic (see Definition 2.1). Let B be a noetherian k -scheme, and let $f : \mathcal{X} \rightarrow B$ be a flat G -equivariant morphism of finite type such that G acts trivially on B and the geometric fibers of f are generically free and primitive G -varieties (in particular, reduced). Then for any fixed integer $n \geq 0$ the subset of $b \in B$ such that $\text{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})}) \leq n$ for every (equivalently, some) geometric point \bar{b} above b is a countable union of closed subsets of B .*

Furthermore, assume that k is algebraically closed and of infinite transcendence degree over its prime field. (In particular, the latter condition is satisfied if k is uncountable.) Let $m \geq 0$ be the maximum of $\text{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})})$, where \bar{b} ranges over all geometric points of B . Then the set of those $b \in B(k)$ such that $\text{ed}_k(\mathcal{X}_b; G) = m$ is Zariski dense in B .

Informally, Theorem 1.2 can be restated as follows: in a family of G -varieties $\mathcal{X} \rightarrow B$, the essential dimension of the geometric fibers drops on a countable union of Zariski closed subsets of B , and stays constant away from this countable union. Several remarks concerning Theorem 1.2 are in order.

Remarks 1.3. (1) The assumption that the G -action on every geometric fiber \mathcal{X}_b of f is generically free and primitive ensures that $\text{ed}_{k(b)}(\mathcal{X}_b)$ is well defined.

(2) The countable union in the statement of Theorem 1.2 (a) cannot be replaced by a finite union, in general; see Propositions 5.2 and 5.3.

(3) The assumption that f is flat is necessary; see Remark 4.5. On the other hand, this assumption is rather mild. For example, when \mathcal{X} and B are smooth k -varieties, by “miracle flatness”, f is flat if and only if all of its fibers have the same dimension; see [Matsumura 1989, Theorem 23.1]. In the applications, one is usually interested in showing that the maximal value of $\text{ed}_k(\mathcal{X}_b)$ is attained at a very general point $b \in B(k)$. This can be done under a weaker flatness assumption on f ; see Theorem 9.1. (Here, once again, “very general” means “away from a countable union of proper subvarieties”.)

(4) If k is not algebraically closed, then the k -points $b \in B(k)$ such that $\text{ed}_k(\mathcal{X}_b) \leq n$ do not necessarily lie on a countable union of closed subvarieties of B ; see Section 6. In other words, Theorem 1.2 fails if we consider fibers of arbitrary closed points instead of just geometric fibers.

Our proof of Theorem 1.2 proceeds as follows: First we choose a subfield $k_0 \subset k$ finitely generated over the prime field, such that $G = G_0 \times_{\text{Spec}(k_0)} \text{Spec}(k)$, $f = f_0 \times_{\text{Spec}(k_0)} \text{Spec}(k)$, and the assumptions of Theorem 1.2 hold for k_0 , G_0 and $f_0: \mathcal{X}_0 \rightarrow B_0$. Then using arguments inspired by Gabber’s appendix [Colliot-Thélène 2002], we reduce Theorem 1.2 to the specialization property Proposition 3.1 and the rigidity property Lemma 4.2.

Note that the rigidity property may fail if k is not algebraically closed. This is the reason why in Theorem 1.2 we only consider the geometric fibers; see Remarks 1.3 (4).

Theorem 1.4. *Let k be an infinite field, G be a finite group, and let X_0 be an equidimensional generically free G -variety of dimension $e \geq 1$ (not necessarily primitive). Then there exist a smooth irreducible quasiprojective k -variety B , a smooth irreducible quasiprojective G -variety \mathcal{X} and a smooth G -equivariant morphism $f: \mathcal{X} \rightarrow B$ of constant relative dimension e defined over k such that*

- (i) G acts trivially on B and freely on \mathcal{X} ,
- (ii) there exists a dense open subscheme $U \subset B$ such that for every $b \in U$, the fiber \mathcal{X}_b is smooth, projective and geometrically irreducible,
- (iii) there exists $b_0 \in B(k)$ such that the fiber \mathcal{X}_{b_0} of f over b_0 is G -equivariantly birationally isomorphic to X_0 .

In particular, for any geometric point b of U , the G -action on the fiber \mathcal{X}_b is strongly unramified.

Our proof of Theorem 1.4 was motivated by Serre’s construction of a smooth projective n -dimensional complete intersection with a free G -action, for an arbitrary finite group G and an arbitrary positive integer n ; see [Serre 1958, Proposition 15].

The remainder of this paper is structured as follows. In Section 2, we set up notational conventions and recall some basic definitions concerning essential dimension and versality. Sections 3 and 4 are devoted

to the proof of Theorem 1.2. As we mentioned above, the proof is in two steps. The specialization property is proved in Section 3. In Section 4, we prove the rigidity property and complete the proof of Theorem 1.2. Sections 5 and 6 are devoted to examples, showing that various assumptions in the statement of Theorem 1.2 cannot be dropped. Section 7 collects elementary results concerning transversal intersection in projective space; these results are then used in the proof of Theorem 1.4 in Section 8. Theorem 1.1 is deduced from Theorems 1.2 and 1.4 in Section 9. Along the way we prove Theorem 9.1, a variant of Theorem 1.2, which is often easier to use in applications. In Section 10, we show that Proposition 3.1 and Theorems 1.1, 1.2 and 9.1 remain valid (even under slightly weaker assumptions) if essential dimension is replaced by essential dimension at a prime q . We also give an application of one of these results motivated by an open question from [Duncan and Reichstein 2014]; see Theorem 10.8. Another application can be found in Section 11.

This paper is a sequel to [Reichstein and Scavia 2022]. The main result of [Reichstein and Scavia 2022] is used in the proof of Proposition 3.1 (the specialization property of essential dimension). Other than that, this paper can be read independently of [Reichstein and Scavia 2022].

2. Notation and preliminaries

Group actions and essential dimension. Let k be a field, \bar{k} be an algebraic closure of k , G be a linear algebraic group over k and X be a G -variety, i.e., a separated geometrically reduced k -scheme of finite type endowed with a G -action over k . We will say that a G -variety X is primitive if $X \neq \emptyset$ and $G(\bar{k})$ transitively permutes the irreducible components of $X_{\bar{k}} := X \times_k \bar{k}$. We will say that a G -variety X is generically free if there exists a dense open subscheme $U \subset X$ such that for every $u \in U$ the scheme-theoretic stabilizer G_u of u is trivial.

By a G -compression of X , we will mean a dominant G -equivariant rational map $X \dashrightarrow Y$, where the G -action on Y is again generically free and primitive. The essential dimension of X , denoted by $\text{ed}_k(X; G)$, or $\text{ed}_k(X)$ if G is clear from the context, is defined as the minimal value of $\dim(Y)$, where the minimum is taken over all G -compressions $X \dashrightarrow Y$. The essential dimension $\text{ed}_k(G)$ of the group G is defined as the supremum of $\text{ed}_k(X)$, where X ranges over all faithful primitive G -varieties.

Good characteristic.

Definition 2.1. Let G be a linear algebraic group defined over a field k . We will say that G is in *good characteristic* if one of the following conditions holds:

- $\text{char } k = 0$, or
- $\text{char } k = p > 0$, the connected component G^0 is smooth reductive and there exists a finite subgroup $S \subset G(\bar{k})$ of order prime to p such that the induced map $H^1(K, S) \rightarrow H^1(K, G)$ is surjective for every field extension K/\bar{k} , or
- G is a finite discrete group, and if $\text{char } k = p > 0$, then the only normal p -subgroup of G is the trivial subgroup (that is, G is weakly tame in the sense of [Brosnan et al. 2018]).

Here are two large families of examples in prime characteristic:

Example 2.2. Suppose G is a smooth group over a field k of characteristic $p > 0$. Assume that the connected component G^0 of G is reductive. Let T be a maximal torus in G^0 , $r = \dim(T) \geq 0$, and let $W = N_G(T)/T$ be the Weyl group. If

- (a) G^0 is a split reductive group and p does not divide $2^r|W|$, or
- (b) G is connected and p does not divide $|W|$,

then G is in good characteristic. For a proof of (a), see [Reichstein and Scavia 2022, Proposition 5.1]. For a proof of (b), see [Chernousov et al. 2006, Theorem 1.1 (c)] and [Chernousov et al. 2008, Remark 4.1].

The following example shows that conditions (a) and (b) above can sometimes be relaxed.

Example 2.3. The split orthogonal group O_n , special orthogonal group SO_n and the spin group $Spin_n$ over a field k are in good characteristic as long as $\text{char } k \neq 2$. Indeed, let S be the group of diagonal $n \times n$ matrices of the form $\text{diag}(\epsilon_1, \dots, \epsilon_n)$, where each $\epsilon_i = \pm 1$, let $S_0 = S \cap SL_n$ and let \tilde{S} be the preimage of S_0 under the natural map $Spin_n \rightarrow SO_n$. Then $|S| = |\tilde{S}| = 2^n$ and $|S_0| = 2^{n-1}$. Moreover, if $\text{char } k \neq 2$, then the natural maps

$$H^1(K, S) \rightarrow H^1(K, O_n), \quad H^1(K, S_0) \rightarrow H^1(K, SO_n) \quad \text{and} \quad H^1(K, \tilde{S}) \rightarrow H^1(K, Spin_n)$$

are all surjective. The surjectivity of the first two maps follows from the fact that every quadratic form over a field of characteristic $\neq 2$ can be diagonalized. The surjectivity of the third map is proved in [Brosnan et al. 2007, Lemma 13.2].

Versality and essential dimension at q . A G -variety X is called *weakly versal* if every generically free primitive G -variety T admits a G -equivariant rational map $T \dashrightarrow X$. We will say that X is *versal* if every dense open G -invariant subvariety $U \subset X$ is weakly versal. In particular, if V is a vector space with a generically free action of G , then V is versal; see [Merkurjev 2013, Proposition 3.10]. If X is a generically free primitive versal G -variety, then X has maximal possible essential dimension,

$$\text{ed}_k(X; G) = \text{ed}_k(G); \tag{2.4}$$

see [Merkurjev 2013, Proposition 3.11].

Recall that a correspondence $X \rightsquigarrow Z$ between G -varieties X and Z of degree d is a diagram of G -equivariant rational maps of G -varieties having the form

$$\begin{array}{ccc} & X' & \\ & \downarrow & \searrow \\ \text{degree } d & X & Z \end{array} \tag{2.5}$$

where the vertical map is dominant of degree d . We say that $X \rightsquigarrow Z$ is *dominant* if the rational map $X' \dashrightarrow Z$ in the above diagram is dominant. Dominant correspondences may be composed in the obvious way. A correspondence of degree 1 is the same thing as a rational map.

The notions of essential dimension and versality have local versions at a prime q . These are obtained by replacing G -compressions by dominant correspondences of degree prime to q . Specifically, let G be a linear algebraic group defined over a field k and X be a generically free primitive G -variety. The essential dimension $\text{ed}_{k,q}(X; G)$ of X at q is defined as the minimal value of $\dim(Y)$, where the minimum is taken over all G -equivariant dominant correspondences $X \rightsquigarrow Y$ of degree prime to q . When the reference to G is clear from the context, we sometimes abbreviate $\text{ed}_{k,q}(X; G)$ to $\text{ed}_{k,q}(X)$. The maximal value of $\text{ed}_{k,q}(X; G)$ is called the essential dimension of G at q and is denoted by $\text{ed}_{k,q}(G)$. Here, the maximum is taken over all generically free primitive G -varieties X .

A G -variety X is called *weakly q -versal* if every generically free primitive G -variety T admits a G -equivariant rational map $T \rightsquigarrow X$ of degree prime to q . We will say that X is *q -versal* if every dense open G -invariant subvariety $U \subset X$ is weakly q -versal. If a generically free primitive G -variety X is q -versal, then

$$\text{ed}_{k,q}(X; G) = \text{ed}_{k,q}(G). \quad (2.6)$$

The proof of (2.6) is the same as the proof of (2.4), with rational maps replaced by correspondences of degree prime to q .

3. A specialization property

The purpose of this section is to prove the following specialization property of essential dimension:

Proposition 3.1. *Let G be a linear algebraic group over a field k of good characteristic. Let R be a discrete valuation ring containing k having residue field k , and let l be the fraction field of R . Fix algebraic closures \bar{k} and \bar{l} of k and l , respectively. Let X be a flat R -scheme of finite type endowed with a G -action over R , whose fibers are generically free and primitive G -varieties. Then $\text{ed}_{\bar{l}}(X_{\bar{l}}) \geq \text{ed}_{\bar{k}}(X_{\bar{k}})$.*

For the proof, we will first reduce to the case where X is the total space of a G -torsor $X \rightarrow Y$, and then replace Y by $\text{Spec}(A)$, where A is a suitable discrete valuation ring containing R such that the inclusion $R \subset A$ is local. (Note that after the second reduction X is no longer of finite type over R .) In the latter case, the inequality $\text{ed}_{\bar{l}}(X_{\bar{l}}) \geq \text{ed}_{\bar{k}}(X_{\bar{k}})$ of Proposition 3.1 is established in [Reichstein and Scavia 2022, Theorem 6.4].

Proof of Proposition 3.1. By assumption, X_k (respectively, X_l) is a primitive generically free G_k -variety (respectively, G_l -variety). Our proof will be in several steps.

Claim 3.2. *There exists an integer $d \geq 0$ such that the irreducible components of $X_{\bar{k}}$ and of $X_{\bar{l}}$ are all of dimension d .*

Proof of Claim 3.2. For any finite field extensions $k' \supset k$ and $l' \supset l$, there exists a discrete valuation ring $R' \supset R$, finite and free over R , such that the residue field of R' contains k' and the fraction field of R' contains l' ; see [Serre 1979, I.4, Proposition 9 and Remark] and [Serre 1979, I.6, Proposition 15]. Thus, extending R if necessary, we may assume that the irreducible components of X_k (respectively, X_l) are geometrically irreducible and transitively permuted by $G(k)$ (respectively, $G(l)$).

After this reduction, the problem becomes to show that there exists an integer $d \geq 0$ such that the irreducible components of X_k and of X_l are all of dimension d . Since G acts transitively on the irreducible components of the fibers, it suffices to exhibit one irreducible component of X_k and one irreducible component of X_l of the same dimension.

Since X is flat over R , by [Liu 2002, Lemma 4.3.7] every irreducible component of X dominates $\text{Spec}(R)$. In other words, the open subscheme $X_l \subset X$ is dense. Therefore, each irreducible component of X is the closure of an irreducible component of X_l . Thus, since $X_k \neq \emptyset$, there exists an irreducible component $X' \subset X$ such that X'_k contains some irreducible component Z of X_k and such that X'_l is an irreducible component of X_l .

The composition $X' \hookrightarrow X \rightarrow \text{Spec}(R)$ is surjective, hence [Stacks 2005–, Tag 0B2J] implies that every irreducible component of X'_k has dimension $\dim(X'_l)$. In particular, $\dim(Z) = \dim(X'_l)$, as desired \square

Claim 3.3. *There exists a G -invariant R -fiberwise dense open subscheme $U \subset X$ such that G acts freely (i.e., with trivial stabilizers) on U .*

Proof of Claim 3.3. Since G_l acts generically freely on X_l , there exists a closed nowhere dense G_l -invariant subscheme $Z \subset X_l$ such that G_l acts freely on $X_l \setminus Z$. Let $W \subset Z$ be an irreducible component, and let \bar{W} be the closure of W in X . By [Stacks 2005–, Tag 0B2J], either $(\bar{W})_k$ is empty, or

$$\dim((\bar{W})_k) = \dim(W) \leq \dim(X_k) - 1.$$

It now follows from Claim 3.2 that \bar{W} does not contain any irreducible component of X_k . Therefore, the closure \bar{Z} of Z does not contain any irreducible component of X_k .

Since G_k acts generically freely on X_k , there exists a closed nowhere dense G_k -invariant subscheme $Z' \subset X_k$ such that G_k acts freely on $X_k \setminus Z'$. It follows that

$$U := X \setminus (\bar{Z} \cup Z')$$

is a fiberwise dense G -invariant open subscheme of X , such that G acts freely on U_l and U_k . To prove Claim 3.3, it remains to show that G acts freely on U , i.e., that the stabilizer U -group scheme

$$\mathcal{G} := U \times_{(U \times_R U)} (G \times_R U)$$

is trivial. Here, the fibered product is taken over the diagonal morphism $U \rightarrow U \times_R U$ and the action morphism $G \times_R U \rightarrow U \times_R U$. Since G acts freely on U_l and U_k , the U_l -group scheme \mathcal{G}_l and the U_k -group scheme \mathcal{G}_k are both trivial. In particular, for every $u \in U$, the fiber of the structure morphism $\pi : \mathcal{G} \rightarrow U$ at u is $k(u)$ -isomorphic to $\text{Spec}(k(u))$. Thus, [EGA IV₄ 1967, Proposition 17.2.6] implies that π is a monomorphism. Let $e : U \rightarrow \mathcal{G}$ be the identity section. We have $\pi \circ e = \text{id}_U$, hence $\pi \circ e \circ \pi = \pi$, which implies $e \circ \pi = \text{id}_{\mathcal{G}}$ because π is a monomorphism. Therefore, π is an isomorphism (with inverse e), that is, \mathcal{G} is trivial. \square

Claim 3.4. *For the purpose of proving Proposition 3.1, we may assume that X is the total space of a G -torsor $X \rightarrow Y$, where Y is a flat R -scheme of finite type, and that $\text{ed}_l(X_l) = \text{ed}_l(X_l)$.*

Proof of Claim 3.4. After replacing X by the open R -fiberwise dense subscheme U constructed in Claim 3.3, we may assume that G acts freely on X . We write $a : G \times_R X \rightarrow X \times_R X$ for the action

morphism, given by $(g, x) \mapsto (gx, x)$, and $\mathcal{R} := (G \times_R X \rightrightarrows X)$ for the action groupoid induced by a ; see [Stacks 2005–, Tag 03LJK]. By Claim 3.3, the map a is an isomorphism, and hence in particular it is quasi-finite. We apply [Anantharaman 1973, Appendice I, Corollaire 3] to the equivalence relation determined by \mathcal{R} and to a closed point x in X_k . We deduce that there exist a G -invariant dense open subscheme $V \subset X$ containing x , and a locally closed subscheme $Z \subset V$, such that the restriction \mathcal{R}_Z of \mathcal{R} to Z is flat, quasi-finite, finitely presented (equivalently, finite and locally free, see [Stacks 2005–, Tag 02KB]), and such that the natural morphism of fppf-sheaves $Z/\mathcal{R}_Z \rightarrow V/G$ is an isomorphism. Since V contains x and the G -variety X_k is primitive, V is R -fiberwise dense in X . We will prove that Z/\mathcal{R}_Z is a scheme by applying [Stacks 2005–, Tag 07S6]. Therefore, we need to verify that assumptions (1)-(3) of [Stacks 2005–, Tag 07S6] are satisfied by \mathcal{R}_Z . We have already checked (1) and (2), and so it remains to check (3): for a dense set of points of $z \in Z$, the \mathcal{R}_Z -equivalence class of z is contained in an affine open subscheme of Z . By [Berhuy and Favi 2003, Theorem 4.7], there exist an open subscheme $U' \subset V_l$ and a quotient map $U' \rightarrow U'/G$ in the category of schemes which is a G -torsor. Let $V' \subset U'$ be the inverse image of a dense affine open subscheme of U'/G . Then V' is an everywhere dense G -invariant affine open subscheme of V . Since the map $Z/\mathcal{R}_Z \rightarrow V/G$ is an isomorphism, it is in particular surjective, and hence every G -orbit intersects Z . Thus $V' \cap Z$ is a dense affine open subscheme of Z which is a union of \mathcal{R}_Z -equivalence classes. In other words, (3) is satisfied by all the points in the dense open subscheme $V' \cap Z \subset Z$. We may now apply [Stacks 2005–, Tag 07S6] to deduce that Z/\mathcal{R}_Z is represented by a scheme. Thus $V/G \simeq Z/\mathcal{R}_Z$ is also a scheme. Therefore, replacing X by V' , we may assume that X is the total space of an fppf G -torsor of schemes $X \rightarrow Y$. Since G is smooth, $X \rightarrow Y$ is in fact an étale G -torsor. The fact that Y is flat over R now follows from [Stacks 2005–, Tag 02JZ], since X is flat over R (this is one of the assumptions of Proposition 3.1) and over Y (because $X \rightarrow Y$ is a G -torsor). Since every $G_{\bar{l}}$ -equivariant compression of $X_{\bar{l}}$ over \bar{l} is defined over some finite extension of l , there is a finite subextension $l' \subset l' \subset \bar{l}$ such that $\text{ed}_{l'}(X_{l'}) = \text{ed}_{\bar{l}}(X_{\bar{l}})$. Let $R' \supset R$ be a discrete valuation ring with fraction field l' , and let $k' \supset k$ be the residue field of R' . The G_R -torsor $X \rightarrow Y$ over R lifts to a $G_{R'}$ -torsor on $X_{R'} \rightarrow Y_{R'}$, which is R' -fiberwise generically free and primitive. Since $\text{ed}_{k'}(X_{k'}) \geq \text{ed}_{\bar{k}}(X_{\bar{k}})$, we are allowed to replace R by R' and thus assume that $\text{ed}_l(X_l) = \text{ed}_{\bar{l}}(X_{\bar{l}})$. \square

We are now ready to complete the proof of Proposition 3.1. By Claim 3.4, we may assume that X is the total space of a G -torsor $X \rightarrow Y$, where Y is a flat R -scheme of finite type, and that $\text{ed}_l(X_l) = \text{ed}_{\bar{l}}(X_{\bar{l}})$.

Let $\eta \in Y$ be the generic point of Y_k , and let $A := \mathcal{O}_{Y, \eta}$. Then A is a discrete valuation ring with residue field $k(Y_k)$ and fraction field $l(Y_l)$. We have a Cartesian diagram

$$\begin{array}{ccc}
 P & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 \text{Spec}(A) & \longrightarrow & Y \\
 & & \downarrow \\
 & & \text{Spec}(R)
 \end{array}$$

where $P := X \times_{\text{Spec}(R)} \text{Spec}(A)$. By construction, the morphism $\text{Spec}(A) \rightarrow \text{Spec}(R)$ sends the closed point of $\text{Spec}(A)$ to the closed point of $\text{Spec}(R)$, and so it is local. Since G is in good characteristic, P satisfies the assumptions of [Reichstein and Scavia 2022, Theorem 6.4], hence $\text{ed}_l(P_l) \geq \text{ed}_k(P_k)$. Therefore,

$$\text{ed}_l(P_l) \geq \text{ed}_k(P_k) \geq \text{ed}_{\bar{k}}(P_{\bar{k}}).$$

The morphism $\text{Spec}(A) \rightarrow Y$ induces isomorphisms at the level of residue fields of the special and generic point. Thus $\text{ed}_l(X_l) = \text{ed}_l(P_l)$ and $\text{ed}_{\bar{k}}(X_{\bar{k}}) = \text{ed}_{\bar{k}}(P_{\bar{k}})$. Therefore,

$$\text{ed}_{\bar{l}}(X_{\bar{l}}) = \text{ed}_l(X_l) = \text{ed}_l(P_l) \geq \text{ed}_{\bar{k}}(P_{\bar{k}}) = \text{ed}_{\bar{k}}(X_{\bar{k}}). \quad \square$$

4. Proof of Theorem 1.2

Lemma 4.1. *Let k/k_0 be a field extension of infinite transcendence degree such that k is algebraically closed. Let B_0 be an irreducible k_0 -variety and $B := B_0 \times_{k_0} k$. Then the set of k -rational points of B mapping to the generic point of B_0 is dense in B .*

Proof. Let U be a nonempty open k -subscheme of B . It suffices to prove that U has a k -point which maps to the generic point of B_0 .

To prove this claim, note that the open embedding $U \hookrightarrow B$ is defined over some intermediate subfield $k_0 \subset k_1 \subset k$ such that the extension k_1/k_0 is finitely generated. In other words, $U \hookrightarrow B$ is obtained by base change from an affine open embedding $U_1 \hookrightarrow B_0 \times_{k_0} k_1$ defined over k_1 . In particular, the morphism $U_1 \rightarrow B_0$ is dominant. Let $\eta_1 : \text{Spec}(K_1) \rightarrow U_1$ be the generic point of U_1 .

Now consider a subfield $k_1 \subset L_1 \subset K_1$ such that L_1/k_1 is purely transcendental of finite transcendence degree and K_1/L_1 is finite. Since k/k_1 has infinite transcendence degree, there exists a field embedding $\iota : L_1 \hookrightarrow k$ compatible with the k_1 -algebra structures of L_1 and k . Since k is algebraically closed and K_1/L_1 is finite, we may extend ι to a field embedding $K_1 \hookrightarrow k$, again compatible with the k_1 -algebra structures of K_1 and k . This gives rise to a scheme morphism

$$u_1 : \text{Spec}(k) \rightarrow \text{Spec}(K_1) \xrightarrow{\eta_1} U_1.$$

Since $U = U_1 \times_{k_1} k$, the point u_1 uniquely lifts to a k -point u of U mapping to the generic point of U_1 . Since the morphism $U_1 \rightarrow B_0$ is dominant, the k -point u maps to the generic point of B_0 . This completes the proof of claim, and also completes the proof of Lemma 4.1. \square

We will make use of the following “rigidity property” of essential dimension. For a proof, see [Reichstein and Scavia 2022, Lemma 2.2].

Lemma 4.2. *Let k be an algebraically closed field, G be a k -group and X be a generically free primitive G -variety defined over k . Then $\text{ed}_k(X) = \text{ed}_l(X_l)$ for any field extension l/k .*

Now let $f : \mathcal{X} \rightarrow B$ be as in Theorem 1.2. For every integer n , we set

$$\Phi_f(n) := \{b \in B \mid \text{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}) \leq n \text{ for some geometric point } \bar{b} \text{ with image } b\}.$$

Lemma 4.3. (a) A point $b \in B$ belongs to $\Phi_f(n)$ if and only if $\text{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}) \leq n$ for every geometric point \bar{b} with image b .

(b) Let $\pi : B' \rightarrow B$ be a morphism of schemes, and let $f' : X \times_B B' \rightarrow B'$ be the base of change of f along π . Then $\Phi_{f'}(n) = \pi^{-1}(\Phi_f(n))$.

Proof. (a) Let \bar{b}_1 and \bar{b}_2 be two geometric points of B with image b . The ring $A := k(\bar{b}_1) \otimes_{k(b)} k(\bar{b}_2)$ is not zero. If \mathfrak{m} is a maximal ideal of A , the quotient A/\mathfrak{m} is a field containing $k(\bar{b}_1)$ and $k(\bar{b}_2)$. By considering an algebraic closure of A/\mathfrak{m} , we are thus reduced to the case when there is a field homomorphism $k(\bar{b}_1) \hookrightarrow k(\bar{b}_2)$. We may, thus, assume that $k(\bar{b}_1) \subset k(\bar{b}_2)$. In this case, (a) follows from Lemma 4.2.

(b) Let $b' \in B'$ and $b \in B$ be such that $\pi(b') = b$. Let \bar{b}' be a geometric point of B' with image b' , so that $\bar{b} := \pi \circ \bar{b}'$ is geometric point of B with image b . Then there is a natural isomorphism $\mathcal{X}_{\bar{b}'} \simeq \mathcal{X}_{\bar{b}} \times_{k(\bar{b})} k(\bar{b}')$ of $G_{\bar{b}'}$ -varieties, and

$$\text{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}) = \text{ed}_{k(\bar{b}')}(\mathcal{X}_{\bar{b}'} \times_{k(\bar{b})} k(\bar{b}')) = \text{ed}_{k(\bar{b}')}(\mathcal{X}_{\bar{b}'})$$

by Lemma 4.2. In particular, $b \in \Phi_f(n)$ if and only if $b' \in \Phi_{f'}(n)$, as desired. □

Proof of Theorem 1.2. We must show that $\Phi_f(n) \subset B$ is a union of countably many closed subsets of B . By noetherian approximation (see [EGA IV₃ 1966, IV, §8.10] or [Thomason and Trobaugh 1990, Appendix C]), the G -action on \mathcal{X} descends to a subfield k_0 of k which is finitely generated over its prime field. In other words, there exist

- a field $k_0 \subset k$ finitely generated over its prime field,
- a smooth group scheme G_0 of finite type over k_0 ,
- k_0 -schemes of finite type B_0 and \mathcal{X}_0 ,
- a G_0 -action on \mathcal{X}_0 over k_0 ,
- a flat G_0 -invariant morphism $f_0 : \mathcal{X}_0 \rightarrow B_0$ and
- a Cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}_0 \\ f \downarrow & & \downarrow f_0 \\ B & \xrightarrow{\pi} & B_0, \end{array} \tag{4.4}$$

such that $G = G_0 \times_{k_0} k$, and the base change of the G_0 -action on \mathcal{X}_0/B_0 along π is isomorphic to the G -action on \mathcal{X}/B .

By Lemma 4.3 (b), we have $\Phi_f(n) = \pi^{-1}(\Phi_{f_0}(n))$. Thus, since π is continuous, it suffices to prove that $\Phi_{f_0}(n)$ is a countable union of closed subsets of B_0 . In other words, we may assume that k is finitely generated over its prime field and that B is of finite type over k . In this case, the underlying topological space of B is countable, hence $\Phi_f(n)$ is countable. It remains to show that $\Phi_f(n)$ is a union of closed subsets of B . By elementary topology, it suffices to show that $\Phi_f(n)$ is closed under specialization;

see [Stacks 2005–, Tag 0EES]. In other words, if $b' \in B$ is a specialization of $b \in \Phi_f(n)$, i.e., $b' \in \overline{\{b\}}$, then we want to show that $b' \in \Phi_f(n)$.

By [EGA II 1961, Proposition 7.1.4], there exist a discrete valuation ring R with closed point s and generic point η , and a morphism $\text{Spec}(R) \rightarrow B$ sending s to b' and η to b . Precomposing with the completion map $\text{Spec}(\hat{R}) \rightarrow \text{Spec}(R)$, we may assume that R is complete. Since B is a k -scheme, the residue fields of b, b', s, η all have the same characteristic as k . Thus, R is complete and equicharacteristic and hence, by Cohen’s structure theorem we have an isomorphism $R \simeq k(s)[[t]]$. (In characteristic 0, this is an isomorphism of k -algebras, whereas in characteristic p , it is only an isomorphism of rings.) For the argument below, we only need an isomorphism of rings.) In particular, the residue field $k(s)$ is contained in R . By Proposition 3.1, letting $\bar{\eta}$ and \bar{s} be geometric points of $\text{Spec}(R)$ lying above η and s , respectively, we deduce that

$$\text{ed}_{k(\bar{\eta})}(X_{k(\bar{\eta})}) \geq \text{ed}_{k(\bar{s})}(X_{k(\bar{s})}).$$

Now, Lemma 4.3 (a) tells us that

$$n \geq \text{ed}_{k(\bar{b})}(X_{k(\bar{b})}) \geq \text{ed}_{k(\bar{b}')} (X_{k(\bar{b}')}),$$

where \bar{b} and \bar{b}' are geometric points of B lying above b and b' , respectively. This shows that $\Phi_f(n)$ is closed under specialization.

Assume now that k is algebraically closed and of infinite transcendence degree over its prime field, and let m be the maximum of $\text{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})})$, where \bar{b} ranges over all geometric points of B . Consider the diagram (4.4). Since $\Phi_{f_0}(m - 1)$ is a union of closed subsets of B_0 and it does not equal B_0 , it does not contain the generic point of B_0 . By Lemma 4.3 (b), we have $\Phi_f(m - 1) = \pi^{-1}(\Phi_{f_0}(m - 1))$, hence for every k -point b of B mapping to the generic point of B_0 , we have $\text{ed}_k(\mathcal{X}_b) = m$. By Lemma 4.1, the set of such k -points is Zariski dense in B . □

Remark 4.5. The following example shows that the flatness assumption in Theorem 1.2 is necessary.

Let n be a positive integer, and let k be an algebraically closed field of characteristic not dividing n . Consider the affine plane $\mathbb{A}_k^2 = \text{Spec}(k[x, y])$, with coordinates x, y . Let $\mathcal{X} \subset \mathbb{A}^2$ be defined by the equation $x(y^n - 1) = 0$, let $B = \mathbb{A}_k^1 = \text{Spec}(k[x])$ and let $f: \mathcal{X} \rightarrow B$ be the projection given by $(x, y) \mapsto x$. The group $\mu_n = \mathbb{Z}/n\mathbb{Z}$ acts on \mathcal{X} by $\zeta \cdot (x, y) \mapsto (x, \zeta y)$ and trivially on B . Clearly, f is μ_n -equivariant, and the μ_n -action on the fibers of f is generically free and primitive. We have $\text{ed}_k(X_a) = 0$ for every $a \in k^\times$, but $\text{ed}_k(X_0) = \text{ed}_k(\mu_n) = 1$.

Remark 4.6. To put Theorem 1.2 in perspective, we will conclude this section by recalling an analogous result for cohomological invariants from [Colliot-Thélène 2002, Appendix]. For an overview of the theory of cohomological invariants, see [Garibaldi et al. 2003].

Let k be a field, G be a linear algebraic k -group and X be a generically free primitive G -variety. After passing to a G -invariant open subvariety of X , we may assume that X is the total space of a G -torsor $\tau: X \rightarrow Y$, where Y is irreducible with function field $K = k(Y) = k(X)^G$; see [Berhuy and Favi 2003, Theorem 4.7]. Note that $k(X)^G$ is a field, since X is primitive. Pulling back to the generic point $\eta: \text{Spec}(K) \rightarrow Y$ of Y , we obtain a G -torsor $\tau_\eta: \mathcal{T}_\eta \rightarrow \text{Spec}(K)$. We denote the class of τ_η in $H^1(K, G)$ by $[X]$.

Now, suppose $f : \mathcal{X} \rightarrow B$ is a family of generically free primitive G -varieties, as in Theorem 1.2. Let i be a nonnegative integer, C be a finite $\text{Gal}(k_s/k)$ -module of order prime to the characteristic of k and $F \in \text{Inv}^i(G, C)$ be a cohomological invariant over k with values in the Galois cohomology ring $H^i(-, C)$. We will be interested in how $F([\mathcal{X}_b])$ behaves as b varies over B . First consider the generic point b_{gen} of B . Passing to a dense open subscheme of B if necessary, we may assume that $F([\mathcal{X}_{b_{\text{gen}}}]$ comes from a cohomology class $\alpha \in H_{\text{ét}}^i(B, C)$. In this case, by the compatibility of the specialization map in étale and Galois cohomology [Garibaldi et al. 2003, p. 15, Footnote], $\alpha_{\bar{b}} = F(k(X_{\bar{b}}))$ (up to sign) for every geometric point \bar{b} of B . From [Colliot-Thélène 2002, Proposition A7], we deduce that

$$B_0 := \{b \in B : F(k(X_{\bar{b}})) = 0 \text{ for some geometric point } \bar{b} \text{ above } b\}$$

is a countable union of closed subsets of B . Note that by the rigidity property for étale cohomology [Milne 1980, Corollary VI.2.6], one may replace “some” by “every” in the definition of B_0 , as in Lemma 4.3 (a).

5. Example: Elliptic curves with marked torsion points

In this section we will consider an example, showing that in Theorem 1.2 one may not replace “countable union” by “finite union”.

Let A be a commutative algebraic group over \mathbb{C} . Any choice of $v_1, \dots, v_r \in A[\ell]$ gives rise to a $(\mathbb{Z}/\ell\mathbb{Z})^r$ -action on A via

$$(n_1, \dots, n_r) : a \mapsto a + n_1 v_1 + \dots + n_r v_r.$$

This action is free if and only if v_1, v_2, \dots, v_r are linearly independent over $\mathbb{Z}/\ell\mathbb{Z}$. When we view A as a $(\mathbb{Z}/\ell\mathbb{Z})^r$ -variety via this action, we will denote it by $(A; v_1, \dots, v_r)$. We will focus on the case, where $r = 2$ and $A = E \times E$ is the direct product of two copies of a complex elliptic curve E . More specifically, we will investigate how $\text{ed}(E \times E; v_1, v_2)$ depends on the choice of E, v_1 and v_2 .

Recall that for every prime integer ℓ , there exists a complex curve B and a family of elliptic curves $\mathcal{E} \rightarrow B$, together with a nowhere zero ℓ -torsion section, such that every pair $(E; q)$ where E is a complex elliptic curve and $q \in E(\mathbb{C})[\ell] \setminus \{0\}$ arises as a fiber of $\mathcal{E} \rightarrow B$; see [Colliot-Thélène 2002, Proposition A4]. The group $\mathbb{Z}/\ell\mathbb{Z}$ acts freely on \mathcal{E} over B by translations by the ℓ -torsion section, and so $(\mathbb{Z}/\ell\mathbb{Z})^2$ acts freely on the fiber product

$$\Phi : \mathcal{X} = \mathcal{E} \times_B \mathcal{E} \rightarrow B \tag{5.1}$$

by translation. The fibers of Φ are $(\mathbb{Z}/\ell\mathbb{Z})^2$ -varieties having the form $(E \times E; (q, 0), (0, q))$, where $q \in E(\mathbb{C})[\ell] \setminus \{0\}$.

Proposition 5.2. *Let ℓ be an odd prime integer, E be an elliptic curve over \mathbb{C} , and $q \in E(\mathbb{C})[\ell] \setminus \{0\}$.*

- (i) *Suppose there exists an endomorphism $\phi : E \rightarrow E$ such that ϕ^2 is multiplication by an integer d , where d is not a square modulo ℓ . Then $\text{ed}(E \times E; (q, 0), (0, q)) = 1$.*
- (ii) *Assume $\text{End}(E) = \mathbb{Z}$. Then $\text{ed}(E \times E; (q, 0), (0, q)) = 2$.*

(iii) Consider the map $\Phi: \mathcal{X} \rightarrow B$ from (5.1). Then there are countably infinite subsets $B'_{\text{cm}} \subset B_{\text{cm}} \subsetneq B(\mathbb{C})$ such that

$$\text{ed}(\mathcal{X}_b) = \begin{cases} 1, & \text{if } b \in B'_{\text{cm}}, \\ 2, & \text{if } b \notin B_{\text{cm}}. \end{cases}$$

Proof. (i) It is obvious from the definition that $\text{ed}(E \times E : (q, 0), (0, q)) \geq 1$, so we only need to show that $\text{ed}(E \times E; (q, 0), (0, q)) \leq 1$. Since ϕ is an endomorphism, it restricts to a group homomorphism $E(\mathbb{C})[\ell] \rightarrow E(\mathbb{C})[\ell]$. Fixing a $(\mathbb{Z}/\ell\mathbb{Z})$ -basis of $E(\mathbb{C})[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$, ϕ gives rise to a matrix $A \in \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$. The matrix A does not have any eigenvalues in $\mathbb{Z}/\ell\mathbb{Z}$. Indeed, if $Av = \lambda v$ for some nonzero $v \in (\mathbb{Z}/\ell\mathbb{Z})^2$ and $\lambda \in \mathbb{Z}/\ell\mathbb{Z}$, then $d v = A^2 v = \lambda^2 v$, hence $d = \lambda^2$ in $\mathbb{Z}/\ell\mathbb{Z}$, which is impossible as d is not a square modulo ℓ . It follows that q and $\phi(q)$ are linearly independent, and so form a basis of $E(\mathbb{C})[\ell]$. Now, $\phi: (E; q) \rightarrow (E; \phi(q))$ is a $\mathbb{Z}/\ell\mathbb{Z}$ -equivariant morphism and the composition

$$(E \times E; (q, 0), (0, q)) \xrightarrow{(\text{id}, \phi)} (E \times E; (q, 0), (0, \phi(q))) \xrightarrow{+} (E; (q, \phi(q)))$$

is a $(\mathbb{Z}/\ell\mathbb{Z})^2$ -compression. Thus, $\text{ed}(E \times E; (q, 0), (0, q)) \leq 1$, as desired.

(ii) Assume the contrary. Then there exists a dominant $(\mathbb{Z}/\ell\mathbb{Z})^2$ -equivariant rational map

$$f: E \times E \dashrightarrow C,$$

where $E \times E$ stands for the $(\mathbb{Z}/\ell\mathbb{Z})^2$ -variety $(E \times E; (q, 0), (0, q))$ and C is some curve with a faithful action of $(\mathbb{Z}/\ell\mathbb{Z})^2$. We may assume that C is smooth and projective. Since ℓ is odd, $(\mathbb{Z}/\ell\mathbb{Z})^2$ cannot act faithfully on \mathbb{P}^1 . Thus, C is not isomorphic to \mathbb{P}^1 . For all but finitely many v , the map f restricts to a well-defined surjective morphism $E \simeq E \times \{v\} \rightarrow C$. We deduce from Hurwitz's formula that C has genus 1. After suitably choosing an origin for C , the map f becomes an everywhere defined homomorphism of abelian varieties. The restrictions of f to $E \times \{0\}$ and $\{0\} \times E$ give isogenies $f_1, f_2: E \rightarrow C$ such that the element $(1, 0)$ of $(\mathbb{Z}/\ell\mathbb{Z})^2$ acts on C via translation by $f_1(q)$, and the element $(0, 1)$ acts on C via translations by $f_2(q)$. Since the $(\mathbb{Z}/\ell\mathbb{Z})^2$ -action on C is faithful, we conclude that $f_1(q)$ and $f_2(q)$ form a basis of $C[\ell]$. On the other hand, recall from [Silverman 2009, Lemma III.4.2 (b)] that $\text{Hom}(E, C)$ is torsion-free \mathbb{Z} -module. Since

$$\text{Hom}(E, C) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \text{Hom}(E, E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q},$$

we conclude that $\text{Hom}(E, C) = \mathbb{Z}$. This implies that there exists homomorphism $h: E \rightarrow C$ such that f_1 and f_2 are multiples of h . In particular, $f_1(q)$ and $f_2(q)$ are linearly dependent, a contradiction. We conclude that C does not exist, and thus $\text{ed}(E \times E; (q, 0), (0, q)) = 2$, as claimed.

(iii) Recall that the endomorphism ring of an elliptic curve E over \mathbb{C} is either \mathbb{Z} (in which case we say that E has no complex multiplication) or the ring of integers in the field $\mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ and $d < 0$; see [Silverman 2009, Theorem III.9.3, Remark III.9.4.1].

Let B_{cm} be the set $b \in B(\mathbb{C})$ such that $\mathcal{E}_b = (E, q)$, where E is an elliptic curve with complex multiplication and q is an ℓ -torsion point. Let B'_{cm} be the set of $b \in B(\mathbb{C})$ such that $\mathcal{E}_b = (E, q)$, where $\text{End}(E)$ is the ring of integers in $\mathbb{Q}[\sqrt{d}]$, where $d < 0$ is a negative integer which is not a square modulo ℓ and $d \equiv 2$ or 3 modulo 4 . For d of this form, the ring of integers in $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\sqrt{d}]$.

By the Chinese remainder theorem, there exist infinitely many integers d of this form. Moreover, every ring of the form $\mathbb{Z}[\sqrt{d}]$, with d as above, arises as the endomorphism ring of a complex elliptic curve; see [Silverman 2009, Proposition VI.4.1]. Since there are only countably many complex elliptic curves with complex multiplication (see [Silverman 2009, Corollary 11.1.1 on p. 426]), we conclude that B_{cm} and B'_{cm} are both countably infinite subsets of $B(\mathbb{C})$. By part (i), we have $\text{ed}(\mathcal{X}_b) = 1$ for every $b \in B'_{\text{cm}}$ and by part (ii), we have $\text{ed}(\mathcal{X}_b) = 2$ for every $b \notin B_{\text{cm}}$. \square

The case where $\ell = 2$ is a bit more complicated but the end result is similar.

Proposition 5.3. *Consider the map $\Phi: X \rightarrow B$ from (5.1). Then:*

- (i) *There are countably infinitely many $b \in B(\mathbb{C})$ such that $\text{ed}(\mathcal{X}_b) = 1$.*
- (ii) *For a very general $b \in B(\mathbb{C})$, we have $\text{ed}(\mathcal{X}_b) = 2$.*

Proof. (i) Let E be an complex elliptic curve, corresponding to a lattice $\mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$, where $\mathbb{Q}(\tau)$ is an imaginary quadratic extension of \mathbb{Q} . Multiplication by τ respects $\mathbb{Z} \oplus \mathbb{Z}\tau$; hence, it induces an endomorphism $\phi: E \rightarrow E$. Since multiplication by τ sends $1/2$ to $\tau/2$, there exists a point $q \in E(\mathbb{C})[2] \setminus \{0\}$ such that $\tau(q) \neq q$. We may now conclude (as we did in the proof of Proposition 5.2 (i)) that the morphism $\phi: (E; q) \rightarrow (E; \phi(q))$ is $\mathbb{Z}/2\mathbb{Z}$ -equivariant and the composition

$$(E \times E; (q, 0), (0, q)) \xrightarrow{(\text{id}, \phi)} (E \times E; (q, 0), (0, \phi(q))) \xrightarrow{+} (E; (q, \phi(q)))$$

is a $(\mathbb{Z}/2\mathbb{Z})^2$ -compression. Thus, $\text{ed}(E \times E; (q, 0), (0, q)) = 1$. Since there are countably many such elliptic curves E , the proof of (i) is complete.

(ii) Suppose that there exists a dominant $(\mathbb{Z}/2\mathbb{Z})^2$ -equivariant rational map $f: E \times E \dashrightarrow C$, where C is a curve on which $(\mathbb{Z}/2\mathbb{Z})^2$ acts faithfully. In particular, there exists a surjective morphism $E \rightarrow C$. By the Hurwitz formula, C has genus 0 or 1. Since E is very general, we may suppose that $\text{End}(E) = \mathbb{Z}$. In the proof of Proposition 5.2 (ii), we showed that C cannot have genus 0, because $(\mathbb{Z}/p\mathbb{Z})^2$ cannot act faithfully on \mathbb{P}^1 for any odd prime p . We then used the condition that $\text{End}(E) = \mathbb{Z}$ to show that C cannot have genus 1. The latter argument goes through for $p = 2$ unchanged. Hence, we may assume that C has genus 0, i.e., $C = \mathbb{P}^1$. However, $(\mathbb{Z}/2\mathbb{Z})^2$ has a faithful action on \mathbb{P}^1 , so the case where $C = \mathbb{P}^1$ requires additional consideration. We cannot dismiss it quite as easily as we did in the proof of Proposition 5.2 (ii). To analyze the case where $C = \mathbb{P}^1$, we begin by recalling a construction due to Garcia-Armas [Garcia-Armas 2016]. Let G be a finite group, V be a finite-dimensional complex vector space, and $\rho: G \rightarrow \text{PGL}(V)$ be a faithful complex projective representation. Garcia-Armas [2016, §4] constructed a cohomological invariant

$$\Delta_\rho: H^1(K, G) \rightarrow H^2(K, \mathbb{G}_m) = \text{Br}(K),$$

where K/\mathbb{C} ranges over all field extensions. If X is a generically free primitive G -variety, we will denote by $[X] \in H^1(\mathbb{C}(X)^G, G)$ the class of the G -torsor corresponding to X , as in Remark 4.6. Suppose now that V is 2-dimensional, so that $\mathbb{P}(V) = \mathbb{P}^1$. Then by [Garcia-Armas 2016, Corollary 4.4], for any

generically free primitive G -variety X , we have $\Delta_\rho([X]) = 0$ if and only if there exists a G -equivariant compression $X \dashrightarrow \mathbb{P}(V)$. If the finite group G is abelian, then Δ_ρ is a group homomorphism and, in particular, it factors through $H^2(K, \mu_e) = \text{Br}(K)[e]$, where $e \geq 1$ is the exponent of G . Writing $C = \mathbb{P}^1$ as $C = \mathbb{P}(V)$, where V is a 2-dimensional complex vector space, we observe that there is only one possible faithful action of $(\mathbb{Z}/2\mathbb{Z})^2$ on \mathbb{P}^1 (up to an automorphism of \mathbb{P}^1), given by the unique faithful complex projective representation $\rho: (\mathbb{Z}/2\mathbb{Z})^2 \hookrightarrow \text{PGL}(V) = \text{PGL}_2(\mathbb{C})$. "

The family $\mathcal{E} \rightarrow B$ considered above contains a special fiber $(\mathbb{Z}/2\mathbb{Z})$ -equivariantly isomorphic to $(\mathbb{G}_m; -1)$. It follows that the family $\Phi: \mathcal{X} = \mathcal{E} \times_B \mathcal{E} \rightarrow B$ contains a special fiber \mathcal{X}_0 that is $(\mathbb{Z}/2\mathbb{Z})^2$ -equivariantly isomorphic to $(\mathbb{G}_m^2; (-1, 1), (1, -1))$. Replacing \mathbb{G}_m^2 by \mathbb{A}^2 , we see that \mathcal{X}_0 is $(\mathbb{Z}/2\mathbb{Z})^2$ -equivariantly birationally isomorphic to a linear action of $(\mathbb{Z}/2\mathbb{Z})^2$ on a 2-dimensional vector space. Hence, \mathcal{X}_0 is a versal $(\mathbb{Z}/2\mathbb{Z})^2$ -variety. Consequently, $\text{ed}_{\mathbb{C}}(\mathcal{X}_0) = \text{ed}_{\mathbb{C}}((\mathbb{Z}/2\mathbb{Z})^2) = 2$; see (2.4). In particular, there does not exist a $(\mathbb{Z}/2\mathbb{Z})^2$ -compression $\mathbb{G}_m^2 \dashrightarrow \mathbb{P}^1$. Now, [Garcia-Armas 2016, Corollary 4.4] implies that $\Delta_\rho([\mathcal{X}_0]) \neq 0$ in $H^2(K, \mu_2) = \text{Br}(K)[2]$, where $K = \mathbb{C}(\mathbb{G}_m^2)^{(\mathbb{Z}/2\mathbb{Z})^2}$. By Remark 4.6, it follows that $\Delta_\rho([\mathcal{X}_b]) \neq 0$ for all but countably many $b \in B(\mathbb{C})$. Applying [Garcia-Armas 2016, Corollary 4.4] one more time, we deduce that for any such b , a $(\mathbb{Z}/2\mathbb{Z})^2$ -equivariant rational map $f: \mathcal{X}_b \dashrightarrow \mathbb{P}^1$ does not exist. Consequently, $\text{ed}(\mathcal{X}_b) = 2$ for all but countably many $b \in B(\mathbb{C})$. \square

6. Example: Variety of quadratic forms

The following example shows that Theorem 1.2 fails if $\text{ed}_{k(\bar{b})}$ is replaced by $\text{ed}_{k(b)}$. In particular, if k is not algebraically closed, then the k -points $s \in B(k)$, where $\text{ed}_k(\mathcal{X}_s) \leq n$ do not necessarily lie on a countable union of closed subvarieties of B .

Let $k = \mathbb{R}$ be the field of real numbers and G be the orthogonal group O_2 defined over \mathbb{R} . Consider the action of $G = O_2$ on $\mathcal{X} = \text{GL}_2$ via multiplication on the right. Note that \mathcal{X} is the total space of a G -torsor $\tau: \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{Y} = \text{GL}_2 / O_2$ is naturally identified with the space of symmetric 2×2 matrices via $\tau: A \mapsto AA^T$.

Now consider the morphism

$$f: \mathcal{X} = \text{GL}_2 \longrightarrow B = \mathbb{A}^1 \setminus \{0\},$$

sending a matrix A to $\det(A)^2$. This morphism factors through τ as

$$f: \mathcal{X} \xrightarrow{\tau} \mathcal{Y} \xrightarrow{\det} B = \mathbb{A}^1 \setminus \{0\}.$$

Denote that fibers of \mathcal{X} and \mathcal{Y} over $s \in B$ by \mathcal{X}_s and \mathcal{Y}_s , respectively. Then \mathcal{X}_s is a G -torsor over \mathcal{Y}_s .

Proposition 6.1. *View a nonzero real number s as an \mathbb{R} -point of B . Then*

$$\text{ed}(\mathcal{X}_s) = \begin{cases} 0, & \text{if } s < 0, \\ 1, & \text{if } s > 0. \end{cases}$$

Proof. Note that \mathcal{Y}_s is the variety of symmetric matrices $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that $\det(B) = s$. Thus \mathcal{Y}_s is a rational surface over \mathbb{R} whose function field can be identified with $\mathbb{R}(a, b)$. Passing to the generic point of \mathcal{Y}_s ,

we see that $\text{ed}_{\mathbb{R}}(\mathcal{X}_s) = \text{ed}_{\mathbb{R}}(\tau_s)$, where $\tau_s \in H^1(\mathbb{R}(a, b), \mathcal{O}_2)$ is the \mathcal{O}_2 -torsor over $\mathbb{R}(a, b)$ obtained by pulling back τ to the generic point of \mathcal{Y}_s . Examining the long exact cohomology sequence associated to the exact sequence $1 \rightarrow \mathcal{O}_2 \rightarrow \text{GL}_2$ of algebraic groups and remembering that $H^1(\mathbb{R}(a, b), \text{GL}_2) = 1$ by Hilbert’s theorem 90, we see that $H^1(\mathbb{R}(a, b), \mathcal{O}_2)$ is in a natural bijective correspondence with the set of 2-dimensional nonsingular quadratic forms over $\mathbb{R}(a, b)$, up to equivalence, and the quadratic form q_s corresponding to τ_s is the form whose Gram matrix is $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where $c = (s + b^2)/a$. Note that, by definition, $\text{ed}_{\mathbb{R}}(\tau_s) = \text{ed}_{\mathbb{R}}(q_s)$ and the discriminant of q_s is s .

Since q assumes the value a and has discriminant s , the quadratic form q_s is isomorphic to $\langle a, as \rangle$. Here, $\langle a, as \rangle$ denote the 2-dimensional quadratic form $q_s(z, w) = az^2 + asw^2$ over $\mathbb{R}(a, b)$. If $s < 0$, then q is isotropic over $\mathbb{R}(a, b)$. Hence, q is hyperbolic over $\mathbb{R}(a, b)$, i.e., q_s is isomorphic to $\langle 1, -1 \rangle$; see [Lam 2005, Theorem I.3.2]. In particular, q_s descends to \mathbb{R} and hence, $\text{ed}_{\mathbb{R}}(q_s) = 0$.

On the other hand, suppose that $s > 0$. Then s is a complete square in $\mathbb{R}(a, b)$, so $q \simeq \langle a, a \rangle$. Clearly, q_s descends to $\mathbb{R}(a) \subset K$, so $\text{ed}_{\mathbb{R}}(q_s) \leq 1$. In order to complete the proof of Proposition 6.1, it remains to show that $\text{ed}_{\mathbb{R}}(q_s) \neq 0$. We argue by contradiction. Assume $\text{ed}_{\mathbb{R}}(q_s) = 0$, i.e., q_s descends to some intermediate extension $\mathbb{R} \subset K \subset \mathbb{R}(a, b)$, where $\text{trdeg}_{\mathbb{R}}(K) = 0$. In other words, K is algebraic over \mathbb{R} . Since \mathbb{R} is algebraically closed in $\mathbb{R}(a, b)$, this is only possible if $K = \mathbb{R}$, i.e., q descends to a 2-dimensional form q_0 defined over \mathbb{R} . Since $s > 0$, q is anisotropic over $\mathbb{R}(a, b)$, and hence, so is q_0 . Let $\nu_a: K^\times \rightarrow \mathbb{Z}$ be the valuation associated to the variable a . It is now easy to see that for any $(0, 0) \neq (f, g) \in (K^\times)^2$, $\nu_a(q_0(f, g))$ is even, whereas $\nu_a(q(f, g))$ is odd. This tells us that q and q_0 have no values in common, contradicting our assumption that q descends to q_0 . □

7. Transversal intersections in projective space

This section contains several preliminary results which will be used in the proof of Theorem 1.4. The common theme is transversal intersections of projective varieties with linear subspaces in projective space. Note that there are no algebraic groups or group actions here; they will come into play in the next section.

Recall that a commutative ring with identity is said to be *regular* if it is noetherian and all its localizations at prime ideals are regular local rings.

Lemma 7.1. *Let A be a regular semilocal noetherian ring and $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_r$ be the maximal ideals of A . For each $1 \leq i \leq r$, let $P_i \subset \mathfrak{m}_i$ be a prime ideal such that $P_i \not\subset \mathfrak{m}_j$ for any $j \neq i$ and such that each local ring A/P_i is regular. Assume that the prime ideals P_1, \dots, P_r have the same height, $\text{ht}(P_1) = \dots = \text{ht}(P_r) = c$. Then there exist $h_1, h_2, \dots, h_c \in A$ such that $P_i A_{\mathfrak{m}_i} = (h_1, \dots, h_c) A_{\mathfrak{m}_i}$ and h_1, \dots, h_c forms a regular sequence in $A_{\mathfrak{m}_i}$ for each i .*

Proof. We claim that, for all $1 \leq i \leq r$, there exist $f_{i,1}, \dots, f_{i,c} \in P_i$ whose images in the A/\mathfrak{m}_i -vector space $\mathfrak{m}_i/\mathfrak{m}_i^2$ are linearly independent. Indeed, letting $\bar{\mathfrak{m}}_i = \mathfrak{m}_i/P_i$ be the maximal ideal of A/P_i , we have a short exact sequence

$$0 \rightarrow P_i/(P_i \cap \mathfrak{m}_i^2) \rightarrow \mathfrak{m}_i/\mathfrak{m}_i^2 \rightarrow \bar{\mathfrak{m}}_i/\bar{\mathfrak{m}}_i^2 \rightarrow 0.$$

In view of the isomorphism $P_i/(P_i \cap \mathfrak{m}_i^2) \simeq (P_i + \mathfrak{m}_i^2)/\mathfrak{m}_i^2$ and the regularity of $A_{\mathfrak{m}_i}$ and A/P_i , this gives

$$\dim_k((P_i + \mathfrak{m}_i^2)/\mathfrak{m}_i^2) = \dim_k(\mathfrak{m}_i/\mathfrak{m}_i^2) - \dim_k(\bar{\mathfrak{m}}_i/\bar{\mathfrak{m}}_i^2) = \text{Kdim}(A_{\mathfrak{m}_i}) - \text{Kdim}(A/P_i) = c,$$

where Kdim denotes the Krull dimension. Therefore, we may choose $f_{i,1}, \dots, f_{i,c} \in P_i A_{\mathfrak{m}_i}$ whose images in $\mathfrak{m}_i/\mathfrak{m}_i^2$ are linearly independent. Multiplying each $f_{i,j}$ by a suitable unit in $A_{\mathfrak{m}_i}$, we may suppose that $f_{i,j} \in P_i$ for all j . This proves the claim.

For any $a \neq b$, the ideal $P_a + P_b$ is not contained in any \mathfrak{m}_i , hence $P_a + P_b = A$. By the Chinese remainder theorem the natural ring homomorphism

$$A/(P_1^2 \cdots P_r^2) \rightarrow (A/P_1^2) \times \cdots \times (A/P_r^2)$$

is an isomorphism; see [Stacks 2005–, Tag 00DT] or [Eisenbud 1995, Exercise 2.6]. In particular, the quotient map $A \rightarrow (A/P_1^2) \times \cdots \times (A/P_r^2)$ is surjective. Therefore, for all $1 \leq i \leq r$ and $1 \leq j \leq c$, we may find $h_{i,j} \in P_i$ whose reduction modulo P_i^2 coincides with the reduction of $f_{i,j}$, and whose reduction modulo P_s^2 is equal to 1 for all $s \neq i$. Then $h_{i,1}, \dots, h_{i,c} \in P_i$ satisfy the following conditions:

- (1) their images $\bar{h}_{i,1}, \dots, \bar{h}_{i,c}$ in the A/\mathfrak{m}_i -vector space $\mathfrak{m}_i/\mathfrak{m}_i^2$ form a basis of the subspace $(P_i + \mathfrak{m}_i^2)/\mathfrak{m}_i^2$,
- (2) $h_{i,j} - 1 \in P_s^2$ for all $s \neq i$.

For each $1 \leq j \leq c$, set $h_j := \prod_{i=1}^r h_{i,j}$. Then $h_j \in \cap_{i=1}^r P_i$ for all j . Moreover, for all $1 \leq i \leq r$, the images of h_1, \dots, h_c in $\mathfrak{m}_i/\mathfrak{m}_i^2$ are equal to $\bar{h}_{i,1}, \dots, \bar{h}_{i,c}$, and so form a basis of $(P_i + \mathfrak{m}_i^2)/\mathfrak{m}_i^2$. Thus, for each $i = 1, \dots, r$, the images of h_1, \dots, h_c in $\mathfrak{m}_i/\mathfrak{m}_i^2$ may be completed to a basis of $\mathfrak{m}_i/\mathfrak{m}_i^2$, hence h_1, \dots, h_c may be completed to a regular system of parameters for $\mathfrak{m}_i A_{\mathfrak{m}_i}$. Thus the elements h_1, \dots, h_c form a regular sequence in $\mathfrak{m}_i A_{\mathfrak{m}_i}$.

Finally, we claim that $P_i A_{\mathfrak{m}_i} = (h_1, \dots, h_c) A_{\mathfrak{m}_i}$ for all $1 \leq i \leq r$. Since h_1, \dots, h_c are a regular sequence in $A_{\mathfrak{m}_i}$, by [Stacks 2005–, Tag 00NQ] the local ring $B_i := A_{\mathfrak{m}_i}/(f_{i,1}, \dots, f_{i,c})$ is a regular local ring of Krull dimension

$$\text{Kdim}(B_i) = \text{Kdim}(A_{\mathfrak{m}_i}) - c = \text{Kdim}(A/P_i).$$

Consider the surjection $\phi_i: B_i \rightarrow A/P_i$. We are going to show that ϕ_i is injective. Indeed, suppose that $0 \neq x \in B_i$ is in the kernel of ϕ_i . By [Stacks 2005–, Tag 00NP], the regular local ring B_i is a domain, hence by [Stacks 2005–, Tag 00KW], we have $\text{Kdim}(B_i/(x)) = \text{Kdim}(B_i) - 1 < \text{Kdim}(A/P_i)$. On the other hand, ϕ_i factors through a surjection $B_i/(x) \rightarrow A/P_i$, hence $\text{Kdim}(B_i/(x)) \geq \text{Kdim}(A/P_i)$, a contradiction. Therefore, ϕ_i is injective. This means that $P_i A_{\mathfrak{m}_i}$ is generated by h_1, \dots, h_c as an ideal of $A_{\mathfrak{m}_i}$, as claimed. \square

Let k be a field. For every $0 \leq d \leq n$, we denote by $\text{Gr}(n, n-d)$ the Grassmannian of codimension d hyperplanes of \mathbb{P}_k^n . If $W \subset \mathbb{P}_k^n$ is a k -subspace of codimension d , we will denote by $[W] \in \text{Gr}(n, n-d)(k)$ the k -point representing W in the Grassmannian. If $Z \subset \mathbb{P}_k^n$ is a closed subscheme, we will say that W intersects Z transversely at a smooth k -point $z \in Z$ if $z \in W$ and the tangent space $T_z(Z)$ intersects W transversely. Equivalently, W intersects Z transversely at z if W can be cut out by linear forms $h_1, \dots, h_d \in \Gamma(\mathbb{P}_k^n, \mathcal{O}(1))$ such that $h_1(z) = \cdots = h_d(z) = 0$ and $h_1/h, \dots, h_d/h$ form a regular sequence in the local ring $\mathcal{O}_{Z,z}$ for some (and, thus, any) $h \in \Gamma(\mathbb{P}_k^n, \mathcal{O}(1))$ with $h(z) \neq 0$; see [Eisenbud 1995, Section 10.3].

If $z \in Z$ is a smooth closed point, not necessarily k -rational, we say that W intersects Z transversely at z if $W_{\bar{k}}$ intersects $Z_{\bar{k}}$ transversely at every point \bar{k} -point of $Z \cap W$ lying above z .

Lemma 7.2. *Let Z be a closed subscheme of \mathbb{P}_k^n , $0 \leq c \leq n$ be an integer,*

$$I_{Z,c} \subset \text{Gr}(n, n - c) \times Z$$

be the incidence correspondence parametrizing pairs $([W], v)$ such that $v \in W \cap Z$, and

$$\phi: I_{Z,c} \rightarrow \text{Gr}(n, n - c), \quad ([W], v) \mapsto W$$

be the projection to the first component. Let z be a smooth closed point of Z , and let $W_0 \subset \mathbb{P}_k^n$ be a codimension c linear subspace such that W_0 and Z intersect transversely at z . Then ϕ is smooth at $([W_0], z)$.

Proof. Since smoothness may be verified after an fpqc base change [Stacks 2005–, Tag 02VL] and the morphism $\text{Spec}(\bar{k}) \rightarrow \text{Spec}(k)$ is fpqc, we may replace k by \bar{k} and assume that k is algebraically closed.

We claim that ϕ is flat at $([W_0], z)$. Note that the fiber $W \cap Z$ of ϕ over $[W]$ is smooth at z , so the lemma follows from this claim by [Stacks 2005–, Tag 01V8].

To prove the claim, we argue by induction on c . In the base case, $c = 0$, $\text{Gr}(n, n - c)$ is a point, ϕ is the identity map, and the claim is obvious.

For the induction step, assume that $c \geq 1$ and the claim holds when c is replaced by $c - 1$, for every $n \geq c$ and every closed subscheme Z of \mathbb{P}^n . Choose linear forms $h_1, \dots, h_c \in \Gamma(\mathbb{P}_k^n, \mathcal{O}(1))$ such that h_1, \dots, h_c cut out W_0 , and $h_1/h, \dots, h_c/h$ form a regular sequence in the local ring $\mathcal{O}_{Z,z}$ for some $h \in \Gamma(\mathbb{P}_k^n, \mathcal{O}(1))$ such that $h(z) \neq 0$.

Denote the zero locus of h_1 by \mathbb{P}^{n-1} , the intersection $Z \cap \mathbb{P}^{n-1}$ by Z' , the preimage of Z' under ϕ by $I'_{Z,c}$, and the restriction of ϕ to $I_{Z,c}$ by ϕ' . By [Eisenbud 1995, Corollary 6.9], it suffices to show that $\phi': I'_{Z,c} \rightarrow \text{Gr}(n, n - c)'$ is flat at $([W_0], z)$. Here, $\text{Gr}(n, n - c)'$ denotes the hypersurface in $\text{Gr}(n, n - c)$ consisting of $(n - c)$ -dimensional linear subspaces of \mathbb{P}^n which are contained in \mathbb{P}^{n-1} . We will view Z' as a closed subscheme of \mathbb{P}^{n-1} . Since h_1 cuts Z transversely at z , we see that z is a smooth point of Z' . Now observe that $\text{Gr}(n, n - c)'$ is naturally isomorphic to $\text{Gr}(n - 1, n - c)$ and $I'_{Z,c}$ is naturally isomorphic to $I_{Z',c-1}$ over Z' so that the following diagram commutes:

$$\begin{array}{ccccc} I_{Z',c-1} & \xrightarrow{\cong} & I'_{Z,c} & \hookrightarrow & I_{Z,c} \\ \downarrow & & \downarrow \phi' & & \downarrow \phi \\ \text{Gr}(n - 1, n - c) & \xrightarrow{\cong} & \text{Gr}(n, n - c)' & \hookrightarrow & \text{Gr}(n, n - c) \end{array}$$

Since $\text{Gr}(n - 1, n - c) = \text{Gr}(n - 1, (n - 1) - (c - 1))$, we can use the induction assumption to conclude that ϕ' is flat at $([W_0], z)$, as desired. □

Lemma 7.3. *Let Z be an irreducible quasiprojective k -variety, and let $Y \subset Z$ be a closed equidimensional subvariety, with irreducible components Y_1, \dots, Y_r . For every $1 \leq i \leq r$, let $z_i \in Y_i$ be a closed point such that Y and Z are smooth at z_i , and c be the codimension of Y_i in Z . Then there exist an integer $n \geq 0$, a closed embedding $Z \hookrightarrow \mathbb{P}_k^n$ and a codimension c subspace $W_0 \subset \mathbb{P}_k^n$ such that W_0 intersects Z transversely at z_i , and locally around z_i we have $Y = Y_i = Z \cap W_0$ (scheme-theoretically) for each $i = 1, \dots, r$.*

We emphasize that in the statement of Lemma 7.3 the closed points z_1, \dots, z_r are not assumed to be k -rational but the closed embedding of $Z \hookrightarrow \mathbb{P}_k^n$ and the codimension c subspace $W_0 \subset \mathbb{P}_k^n$ are defined over k .

Proof. Since Z is quasiprojective, there exists an affine open subset $\text{Spec}(B) \subset Z$ containing z_1, \dots, z_r . Let $\text{Spec}(A) \subset Z$ be the semilocalization of Z at $\{z_1, \dots, z_r\}$. By definition, the ring A is the semilocal ring obtained from B by localizing at the multiplicative subset consisting of elements which do not belong to the maximal ideal \mathfrak{m}_{z_i} for any $i = 1, \dots, r$. By Lemma 7.1, there exist $f_1, \dots, f_c \in A$, such that f_1, \dots, f_c form a regular sequence in \mathcal{O}_{Z, z_i} and generate the ideal of Y_i in \mathcal{O}_{Z, z_i} , for each $1 \leq i \leq r$.

Since Z is a quasiprojective, there exists a locally closed embedding $\iota: Z \hookrightarrow \mathbb{P}_k^n$. For every $0 \leq i \leq c$, f_i is the restriction of a rational function P_i/Q_i on \mathbb{P}_k^n , where P_i and Q_i are homogeneous polynomials of the same degree and Q_i does not vanish at z_1, \dots, z_r . We deduce that locally near each of the points z_1, \dots, z_r the variety Y is the scheme-theoretic intersection of Z and the closed subscheme defined by P_1, \dots, P_c . By a refinement of the graded prime avoidance lemma [Gabber et al. 2013, Lemma 4.11], there exist an integer $d \geq 1$ and homogeneous polynomials F_1, F_2 of degrees d and $d + 1$, respectively, such that F_1 and F_2 do not vanish at z_1, \dots, z_r . Since d and $d + 1$ are coprime, any sufficiently large positive integer is of the form $n_1d + n_2(d + 1)$ for some integers $n_1, n_2 \geq 0$. Therefore, multiplying the P_i and Q_i by suitable powers of F_1 and F_2 , we may assume that P_1, \dots, P_c have the same degree $D \geq 1$. After composing ι with the D -fold Veronese embedding of \mathbb{P}_k^n , we may further assume that each P_i has degree 1, that is, that each P_i cuts out a hyperplane in \mathbb{P}_k^n . Now the embedding $\iota: Z \hookrightarrow \mathbb{P}_k^n$ and the linear subspace W_0 of \mathbb{P}_k^n given by $P_1 = \dots = P_c = 0$ have the properties claimed in the lemma. \square

8. Proof of Theorem 1.4

Let $X_0 = X_0^{(1)} \cup \dots \cup X_0^{(r)}$ be the irreducible decomposition of X_0 . Then choose rational functions $\alpha_1, \dots, \alpha_d: X_0 \dashrightarrow \mathbb{A}_k^1$ such that the restriction of $\alpha_1, \dots, \alpha_d$ to $X_0^{(i)}$ generate the function field $k(X_0^{(i)})$ for every i . After adjoining all G -translates of $\alpha_1, \dots, \alpha_d$ to this set, we may assume that G permutes $\alpha_1, \dots, \alpha_d$. Consider the G -equivariant rational map $\alpha: X_0 \dashrightarrow \mathbb{P}(V)$, with $x \mapsto (1 : \alpha_1(x) : \dots : \alpha_d(x))$, for a suitable linear (permutation) representation of G on $V = k^{d+1}$. Note that since the G -action on X_0 is assumed to be faithful, the G -action on $\mathbb{P}(V)$ is faithful as well. Moreover, by our construction, α induces a G -equivariant birational isomorphism between X_0 and $\alpha(X_0)$. After replacing X_0 with the closure of $\alpha(X_0)$ in $\mathbb{P}(V)$, we may assume that X_0 is a closed subvariety of $\mathbb{P}(V)$.

Let $\mathbb{P}(V)_{\text{nonfree}}$ be the nonfree locus for the G -action on V , i.e., the union of the fixed point loci $\mathbb{P}(V)^g$ as g ranges over the nontrivial elements of G . Since G acts faithfully on $\mathbb{P}(V)$, we have

$$\dim \mathbb{P}(V) > \dim \mathbb{P}(V)_{\text{nonfree}}. \tag{8.1}$$

Let V^m be the direct sum of m copies of V (as a G -representation). We claim that

$$\text{the codimension of } \mathbb{P}(V^m)_{\text{nonfree}} \text{ in } \mathbb{P}(V^m) \text{ is } \geq m. \tag{8.2}$$

Indeed, for each $1 \neq g \in G$, let $\text{mult}(g, V)_\lambda$ be the dimension of the λ -eigenspace of g and $\text{mult}(g, V)$ be the maximal value of $\text{mult}(g, V)_\lambda$, where λ ranges over \bar{k} . Then $\dim \mathbb{P}(V)_{\text{nonfree}}$ is the maximal value

of $\text{mult}(g, V) - 1$, as g ranges over the nontrivial elements of G . Clearly, $\text{mult}(g, V^m) = \text{mult}(g, V) m$ for every g . Thus,

$$\dim \mathbb{P}(V^m)_{\text{nonfree}} = (\dim \mathbb{P}(V)_{\text{nonfree}} + 1) m - 1,$$

and the codimension of $\mathbb{P}(V^m)_{\text{nonfree}}$ in $\mathbb{P}(V^m)$ is

$$\begin{aligned} \dim \mathbb{P}(V^m) - \dim \mathbb{P}(V^m)_{\text{nonfree}} &= (\dim \mathbb{P}(V) + 1) m - 1 - ((\dim \mathbb{P}(V)_{\text{nonfree}} + 1) m - 1) \\ &= (\dim \mathbb{P}(V) - \dim \mathbb{P}(V)_{\text{nonfree}}) m \geq m, \end{aligned}$$

where the last inequality follows from (8.1). This completes the proof of (8.2).

The G -equivariant linear embedding $V \hookrightarrow V^m$, given by $v \mapsto (v, 0, \dots, 0)$, induces a G -equivariant closed embedding

$$X_0 \subset \mathbb{P}(V) \hookrightarrow \mathbb{P}(V^m).$$

This allows us to view X_0 as a G -invariant subvariety of $\mathbb{P}(V^m)$. By [Popov and Vinberg 1994, Theorem 4.14] or [SGA 1 1971, Exposé V, Propositions 1.8 and 3.1], there exists a geometric quotient map $\pi : \mathbb{P}(V^m) \rightarrow \mathbb{P}(V^m)/G$. Explicitly, write

$$\mathbb{P}(V^m) = \text{Proj}(k[V^m]),$$

where $k[V^m]$ is a polynomial ring in $\dim(V)m = (d + 1)m$ variables. Then $\mathbb{P}(V^m)/G \simeq \text{Proj}(A)$, where $A = k[V^m]^G$ is the ring of invariants and π is induced by the inclusion $A = k[V^m]^G \hookrightarrow k[V^m]$ of graded rings. Restricting π to $X_0 \subset \mathbb{P}(V^m)$, we obtain the geometric quotient map $X_0 \rightarrow X_0/G$. Note that

$$\dim \mathbb{P}(V^m)/G = \dim \mathbb{P}(V^m) = (d + 1)m - 1$$

and $\dim X_0/G = \dim X_0 = e$. Therefore, every irreducible component of X_0/G is of codimension $c = (d + 1)m - 1 - e$ in $\mathbb{P}(V^m)/G$.

By assumption (see Section 2), X_0 is geometrically reduced. By generic smoothness [Stacks 2005–, Tag 056V], there exists a dense open subscheme of X_0 which is smooth over k . Since the G -action on X_0 is assumed to be generically free, we may choose smooth closed points x_1, \dots, x_r , one on each irreducible component of X_0 , such that the (scheme-theoretic) stabilizer G_{x_i} is trivial for every i . We now apply Lemma 7.3 to $Z = \mathbb{P}(V^m)/G$, $Y = X_0/G$, z_1, \dots, z_r , where $z_i = \pi(x_i)$ for each i . Note that by our choice of x_1, \dots, x_r , both Y and Z are smooth at each z_i , for $i = 1, \dots, r$. We deduce that there exist a closed embedding $\mathbb{P}(V^m)/G \hookrightarrow \mathbb{P}^n$ defined over k and a subspace $W_0 \subset \mathbb{P}^n$ of codimension c such that $X_0/G = W_0 \cap (\mathbb{P}(V^m)/G)$ locally around z_i for each i . Consider the diagram

$$\begin{array}{ccc} T & \hookrightarrow & \text{Gr}(n, n - c) \times \mathbb{P}(V^m) \\ & \searrow & \downarrow \text{id} \times \pi \\ & & \text{Gr}(n, n - c) \times Z \\ & \searrow \phi & \downarrow \text{pr}_2 \\ & & \text{Gr}(n, n - c) \end{array}$$

$\bar{f} : T \rightarrow \text{Gr}(n, n - c)$

Here, $I_{Z,c}$ is the incidence correspondence parametrizing pairs $([W], q)$, where W is a linear subspace of \mathbb{P}^n_k having codimension c and $q \in W \cap Z$, as in Lemma 7.2, and T is the preimage of $I_{Z,e}$ in $\text{Gr}(n, n - c) \times \mathbb{P}(V^m)$.

By Lemma 7.2, ϕ is smooth at $([W_0], z_i)$ for each $i = 1, \dots, r$. On the other hand, by our choice of x_1, \dots, x_r , the map π is smooth at each of these points; hence, $\bar{f} = \phi \circ (\pi \times \text{id}): T \rightarrow \text{Gr}(n, n - c)$ is smooth at $([W], x_i)$ for each i .

From now on, we will assume that $m > e$. Note that e is given in the statement of Theorem 1.4, whereas m is a feature of our construction, which we are free to choose. We will construct the family $f: \mathcal{X} \rightarrow B$ by restricting \bar{f} to a dense open subset $\mathcal{X} = T \setminus C$, where $C = T_{\text{sing}} \cup T_{\text{nonfree}}$. Here, T_{sing} is the singular locus of f and

$$T_{\text{nonfree}} \subset \text{Gr}_{n,n-c} \times \mathbb{P}(V^m)_{\text{nonfree}}$$

is the nonfree locus for the G -action in T . Recall that $\mathbb{P}(V^m)_{\text{nonfree}}$ was defined at the beginning of this section. The base B of our family is obtained by removing from $\text{Gr}(n, n - c)$ the locus of points $b \in \text{Gr}(n, n - c)$ such that the entire fiber T_b lies in C . In particular, $\bar{f}(\mathcal{X}) \subset B$. Since C is closed in T , B is open in $\text{Gr}(n, n - c)$. Note also that since \bar{f} is a proper morphism, $\bar{f}(C)$ is closed in B .

Let $b_0 := [W_0] \in \text{Gr}(n, n - c)(k)$. By our choice of x_1, \dots, x_r , none of the points $([W_0], x_i)$ lie in C . Hence, $b_0 \in B(k)$ and the union of the irreducible components of \mathcal{X}_{b_0} passing through x_1, \dots, x_r remains birationally isomorphic to X_0 . This is close to condition (iii) but a little weaker; we will return to this point at the end of the proof.

Note that by our construction \mathcal{X} and B are irreducible, the G -action on \mathcal{X} is free, f is smooth of constant relative dimension $e = \dim(X_0) = \dim(\mathcal{X}_{b_0})$. In particular, condition (i) of Theorem 1.4 holds for f . To prove that condition (ii) also holds for f , it suffices to check that when $m > e$, we have

- (a) $\bar{f}(T_{\text{nonfree}}) \neq \text{Gr}(n, n - c)$,
- (b) $\bar{f}(T_{\text{sing}}) \neq \text{Gr}(n, n - c)$,
- (c) there exists a dense open subset $U \subset \text{Gr}(n, n - c)$ such that the fibers of f over U are projective and irreducible.

To prove (a), recall that by (8.2), the codimension of $P(V^m)_{\text{nonfree}}$ in $P(V^m)$ is $\geq m$. Remembering that $m > e$, $\dim(\mathbb{P}(V^m)) = (d + 1)m - 1$, and $c = (d + 1)m - 1 - e$, we obtain

$$\dim \pi(\mathbb{P}(V^m)_{\text{nonfree}}) = \dim \mathbb{P}(V^m)_{\text{nonfree}} \leq \dim(\mathbb{P}(V^m)) - m < \dim(\mathbb{P}(V^m)) - e = c.$$

Consequently, an $(n - c)$ -dimensional linear subspace W in \mathbb{P}^n in general position will intersect $\pi(\mathbb{P}(V^m)_{\text{nonfree}}) = \mathbb{P}(V^m)_{\text{nonfree}}/G$ trivially. We conclude that $\bar{f}^{-1}(W) \cap T_{\text{nonfree}} = \emptyset$. This proves (a).

To prove (b), recall that the fiber of the morphism $\phi: I_{Z,c} \rightarrow \text{Gr}(n, n - c)$ over $[W]$ is $W \cap Z$. By Bertini's theorem, there exists a dense open subset $U \subset \text{Gr}(n, n - c)$ consisting of $(n - c)$ -dimensional linear subspaces W of \mathbb{P}^n such that $W \cap Z$ is smooth. By generic flatness [Stacks 2005–, Tag 0529], after replacing U by a smaller open subset, we may assume that $\phi: \phi^{-1}(U) \rightarrow U$ is flat. Appealing

to [Stacks 2005–, Tag 01V8], as we did in Lemma 7.2, we see that since $\phi: \phi^{-1}(U) \rightarrow U$ is a flat map with smooth fibers, it is smooth. Finally, after intersecting U with the complement of $\bar{f}(T_{\text{nonfree}})$ (which we know is a dense open subset of $\text{Gr}(n, n - c)$ by (a)), we may assume that $\pi \times \text{id}$ is smooth over $\phi^{-1}(U)$. Hence, the map $\bar{f}: \bar{f}^{-1}(U) \rightarrow U$, being a composition of two smooth maps, $\pi \times \text{id}: \bar{f}^{-1}(U) \rightarrow \phi^{-1}(U)$ and $\phi: \phi^{-1}(U) \rightarrow U$, is smooth. This proves (b).

To prove (c), recall that by definition, for $b = [W] \in B(k)$, the fiber $\phi^{-1}(b) = W \cap Z = W \cap \mathbb{P}(V^m)/G$ is a complete intersection in $\mathbb{P}(V^m)/G$. The fiber $\mathcal{X}_{[W]} = f^{-1}(W')$ is cut out by the same homogeneous polynomials, now viewed as elements of $k[V^m]$ instead of $k[V^m]^G$. Thus, \mathcal{X}_b is a smooth complete intersection in $\mathbb{P}(V^m)$. Since $\dim \mathcal{X}_b = e \geq 1$, by [Serre 1955, n. 78], \mathcal{X}_b is connected, hence irreducible. This completes the proof of (c), and thus of condition (ii) of Theorem 1.4.

As we mentioned above, condition (iii) may not hold for the family $f: \mathcal{X} \rightarrow B$ we have constructed. Our construction only ensures that X_0 is G -equivariantly birationally isomorphic to a union of irreducible components of the fiber \mathcal{X}_{b_0} , and condition (iii) requires X_0 to be birationally isomorphic to the entire fiber \mathcal{X}_{b_0} . To bridge the gap between the two, we will slightly modify \mathcal{X} as follows. (Note that B will remain unchanged.) Let Y_0 denote the union of all other components of \mathcal{X}_{b_0} , the ones that do not pass through any of the points x_1, \dots, x_r . Then the open embedding $\mathcal{X} \setminus Y_0 \hookrightarrow \mathcal{X}$ is flat. After replacing \mathcal{X} by $\mathcal{X} \setminus Y_0$ and $f: \mathcal{X} \rightarrow B$ by its restriction to $\mathcal{X} \setminus Y_0$, we obtain a family satisfying (i), (ii) and (iii). This completes the proof of Theorem 1.4. □

Remark 8.3. The family $f: \mathcal{X} \rightarrow B$ constructed in the proof of Theorem 1.4 has the additional property that the fibers of f over the dense open subset $U \subset B$ (the open subset in part (ii) of the statement of the theorem) are complete intersections in the projective space \mathbb{P}^n . With a bit of extra effort one can ensure that every fiber of f over U is of general type. Since we will not need this assertion in this paper, we leave the proof as an exercise for the interested reader.

We conclude this section with the following consequence of Theorem 1.4, where the group G is not necessarily finite.

Corollary 8.4. *Let k be an infinite field, G be a linear algebraic group, $e \geq 1$ be an integer and X_0 be an equidimensional generically free G -variety of dimension $e + \dim(G)$ (not necessarily primitive). Suppose that there exist a finite subgroup $S \subset G(k)$, a generically free S -variety Y_0 and a G -equivariant birational isomorphism $Y_0 \times^S G \sim X_0$.*

Then there exist a smooth irreducible k -variety B , a smooth G -variety \mathcal{X} and a smooth G -equivariant morphism $f: \mathcal{X} \rightarrow B$ of constant relative dimension $e + \dim(G)$ defined over k such that

- (a) G acts trivially on B and freely on \mathcal{X} ,
- (b) there exists a dense open subscheme $U \subset B$ such that for every $b \in U$ the fiber \mathcal{X}_b is a primitive G -variety and the total space of a G -torsor $\mathcal{X}_b \rightarrow \mathcal{X}_b/G$, where \mathcal{X}_b/G is smooth projective,
- (c) there exists a $b_0 \in B(k)$ such that the fiber \mathcal{X}_{b_0} of f over b_0 is G -equivariantly birationally isomorphic to X_0 .

In particular, for any geometric point b of U , the G -action on the fiber \mathcal{X}_b is strongly unramified.

Proof. Note that Y_0 is equidimensional of dimension

$$\dim(Y_0) = \dim(X_0) - \dim(G) = e.$$

Let $h: \mathcal{Y} \rightarrow B$ be a family obtained by applying Theorem 1.4 to the group S and the S -variety Y_0 . This is possible because S is a finite group, Y_0 is equidimensional and generically free, and $e \geq 1$.

Set $\mathcal{X} := \mathcal{Y} \times^S G$. A priori, \mathcal{X} is an algebraic space. However, since the k -variety \mathcal{Y} is quasiprojective and S is finite, by [SGA 1 1971, Exposé V, Proposition 1.8 and Proposition 3.1], the quotient $\mathcal{X}/G \simeq \mathcal{Y}/S$ exists as a scheme. Since G is affine, the morphism of algebraic spaces $\mathcal{X} \rightarrow \mathcal{X}/G$ is affine, hence representable (see [Stacks 2005–, Tag 03WG]), and so \mathcal{X} is a scheme. Let $f: \mathcal{X} \rightarrow B$ be the natural projection induced by h . Since h satisfies properties (i), (ii) and (iii) of Theorem 1.4, f satisfies properties (i), (ii) and (iii) of Corollary 8.4. \square

9. Proof of Theorem 1.1

Let $d := \dim(X)$ and $e := \text{ed}_k(X)$. We must show that there exists an irreducible smooth projective variety Z and a G -torsor $Y \rightarrow Z$ such that $\dim(Y) = \dim(X)$ and $\text{ed}_k(Y) = \text{ed}_k(X)$. If $e = 0$, Theorem 1.1 is obvious: we can take $Y = G \times \mathbb{P}^d$, where G acts by translations on the first factor and trivially on the second. Thus, we may assume without loss of generality that $e \geq 1$ and in particular, $d \geq 1$. In this case, we use the following strategy: construct a family $\mathcal{X} \rightarrow B$ as in Theorem 1.4 with $\mathcal{X}_{b_0} = X$, then take $Y = \mathcal{X}_b$, where b is a k -point of B in very general position. Theorem 1.4 tells us that $\dim(Y) = d$ and the G -action on Y is strongly unramified. We would like to appeal to Theorem 1.2 to conclude that $\text{ed}_k(Y) = e$. One difficulty in implementing this strategy is that

- (i) Theorem 1.4 requires G to be a finite group, whereas in Theorem 1.1, G is an arbitrary linear algebraic group over a field of good characteristic.

Even if we assume that G is a finite group, there is another problem:

- (ii) Theorem 1.2 requires all fibers of $f: \mathcal{X} \rightarrow B$ to be primitive G -varieties, whereas if f is as in Theorem 1.4, we only have control over fibers \mathcal{X}_b when $b \in U$.

We will overcome (i) by using Corollary 8.4 in place of Theorem 1.4 and (ii) by using Theorem 9.1 below in place of Theorem 1.2.

Theorem 9.1. *Let G be a linear algebraic group over an algebraically closed field k of good characteristic (see Definition 2.1), $f: \mathcal{X} \rightarrow B$ be a G -equivariant morphism of k -varieties such that B is irreducible, G act trivially on B and the generic fiber of f is a primitive and generically free $G_{k(B)}$ -variety. Let $b_0 \in B(k)$, $x_0 \in \mathcal{X}_{b_0}(k)$ and X_0 be a G -invariant reduced open subscheme of \mathcal{X}_{b_0} containing x_0 such that*

- (1) X_0 is a generically free primitive G -variety and
- (2) f is flat at x_0 .

Then for a very general $b \in B(k)$, \mathcal{X}_b is generically free and primitive, and $\text{ed}_k(\mathcal{X}_b) \geq \text{ed}_k(\mathcal{X}_{b_0})$. Furthermore, if k is of infinite transcendence degree over its prime field (in particular, if k is uncountable), then the set of those $b \in B(k)$ such that $\text{ed}_k(\mathcal{X}_b) \geq \text{ed}_k(\mathcal{X}_0)$ is Zariski dense in B .

Here, as always, “very general” means “away from a countable union of proper subvarieties”.

Proof. The proof is in several steps.

Claim 9.2. *There exists a dense open subscheme $V \subset B$ such that for all points $v \in V$ the fiber \mathcal{X}_v is a generically free $G_{k(v)}$ -variety.*

Proof. By [EGA IV₃ 1966, Proposition 9.6.1 (iii), Théorème 9.7.7 (iii)], the locus of points $b \in B$ such that the $k(b)$ -scheme of finite type \mathcal{X}_b is separated and geometrically irreducible, that is, a $G_{k(b)}$ -variety, is a locally constructible subset of B . (By [EGA III₁ 1961, Chapitre 0, Proposition 9.1.12], a subset of B is locally constructible if and only if it is constructible.) Since $\mathcal{X}_{k(B)}$ is a $G_{k(B)}$ -variety by assumption, we deduce the existence of a dense open subscheme $V_1 \subset B$ such that for all $v \in V_1$ the fiber \mathcal{X}_v is a $G_{k(v)}$ -variety.

Consider the stabilizer \mathcal{X} -group scheme

$$\mathcal{G} := \mathcal{X} \times_{(\mathcal{X} \times_B \mathcal{X})} (G \times_B \mathcal{X}),$$

where the fibered product is taken over the diagonal morphism $\mathcal{X} \rightarrow \mathcal{X} \times_B \mathcal{X}$ and the action morphism $G \times_B \mathcal{X} \rightarrow \mathcal{X} \times_B \mathcal{X}$. Since the $G_{k(B)}$ -variety $\mathcal{X}_{k(B)}$ is generically free, the geometric fibers of $\mathcal{G} \rightarrow \mathcal{X}$ at the generic points of the irreducible components of $\mathcal{X}_{k(B)}$ are trivial. By [EGA IV₃ 1966, Proposition 9.6.1 (xi)], the locus of points $x \in \mathcal{X}$ such that $\mathcal{G}_x \rightarrow \text{Spec}(k(x))$ is an isomorphism is locally constructible. (Again by [EGA III₁ 1961, Chapitre 0, Proposition 9.1.12], a subset of \mathcal{X} is locally constructible if and only if it is constructible.) Therefore, there exists an open subscheme $\mathcal{Y} \subset \mathcal{X}$ such that $\mathcal{Y}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense and such that for all $y \in \mathcal{Y}$ the scheme-theoretic stabilizer G_y is trivial. Since $\mathcal{Y}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense, by [Stacks 2005–, Tag 054X], there exists a dense open subscheme $V_2 \subset B$ such that $\mathcal{Y}_v \subset \mathcal{X}_v$ is dense for all $v \in V_2$. Letting $V := V_1 \cap V_2$, we conclude that \mathcal{X}_v is a generically free $G_{k(v)}$ -variety for all $v \in V$. \square

By [Brion 2015, Theorem 1.1], there exists a finite subgroup $S \subset G(k)$ such that the projection $S \rightarrow G/G^0$ is surjective. By assumption, $G_{k(B)}$ acts primitively on $\mathcal{X}_{k(B)}$, hence so does S .

Claim 9.3. *There exists an S -invariant affine open subscheme $\mathcal{U} \subset \mathcal{X}$ such that $\mathcal{U}_{k(B)}$ is dense in $\mathcal{X}_{k(B)}$ and the quotient map $\mathcal{U} \rightarrow \mathcal{U}/S$ is an étale S -torsor.*

Proof. By [Stacks 2005–, Tag 03J1], there exists a separated dense open subscheme $\mathcal{X}_1 \subset \mathcal{X}$. The intersection \mathcal{X}_2 of all the S -translates of \mathcal{X}_1 is an S -invariant dense open subscheme of \mathcal{X} , and it is separated by [Stacks 2005–, Tag 01L8]. Let $\mathcal{X}_3 \subset \mathcal{X}_2$ be the S -free locus, which is a separated open subscheme of \mathcal{X} . By [EGA I 1971, Chapitre 0, (2.1.8)], the generic points of the irreducible components of $\mathcal{X}_{k(B)}$ are also generic points of irreducible components of \mathcal{X} . Recall that S acts generically freely on $\mathcal{X}_{k(B)}$. Therefore, while \mathcal{X}_3 is not necessarily dense in \mathcal{X} , $(\mathcal{X}_3)_{k(B)}$ is dense in $\mathcal{X}_{k(B)}$.

By [Stacks 2005–, Tag 01ZV], we can find an affine dense open subscheme $\mathcal{X}_4 \subset \mathcal{X}_3$. Since \mathcal{X}_3 is separated, the intersection of any two affine open subschemes of \mathcal{X}_3 is affine. Therefore, the intersection \mathcal{U}

of all the S -translates of \mathcal{X}_4 is an S -invariant affine open subscheme of \mathcal{X} on which S acts freely, and the open embedding $\mathcal{U}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense. In particular, the quotient scheme \mathcal{U}/S exists. Since S acts freely on \mathcal{U} , the quotient map $\mathcal{U} \rightarrow \mathcal{U}/S$ is étale, hence \mathcal{U} is the total space of an étale S -torsor. \square

Claim 9.4. *There exists a dense open subscheme $V \subset B$ such that for all points $v \in V$ the fiber \mathcal{X}_v is a generically free and primitive $G_{k(v)}$ -variety.*

Proof. Let $\mathcal{U} \subset \mathcal{X}$ be as in Claim 9.3. Since $\mathcal{U} \rightarrow \mathcal{U}/S$ is an S -torsor, the formation of the quotient of \mathcal{U} by the S -action commutes with arbitrary base change $B' \rightarrow B$, and in particular, for every $b \in B$ the canonical morphism $\mathcal{U}_b/S \rightarrow (\mathcal{U}/S)_b$ is an isomorphism. Therefore, for every $b \in B$ the S -variety \mathcal{U}_b is either empty or an S -torsor.

By our assumptions $\mathcal{U}_{k(B)}$ is dense open in $\mathcal{X}_{k(B)}$ and S acts transitively on the geometric irreducible components of $\mathcal{X}_{k(B)}$. Thus S acts transitively on the geometric irreducible components of $\mathcal{U}_{k(B)}$, that is, the $k(B)$ -variety $\mathcal{U}_{k(B)}/S$ is geometrically irreducible. By [Stacks 2005–, Tag 0559], this implies the existence of a dense open subscheme $V \subset B$ such that the restriction $(\mathcal{U}/S)_V \rightarrow V$ has geometrically irreducible fibers, that is, such that for every $v \in V$ the S -variety \mathcal{U}_v over $k(v)$ is nonempty and primitive. Since $\mathcal{U}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense, we deduce from [Stacks 2005–, Tag 054X] that, possibly after shrinking V , the open embedding $\mathcal{U}_v \subset \mathcal{X}_v$ is dense for all $v \in V$, and so \mathcal{X}_v is primitive for all $v \in V$. By Claim 9.2, after shrinking V one more time, we may suppose that \mathcal{X}_v is also a generically free $G_{k(v)}$ -variety for all $v \in V$. \square

We are now ready to complete the proof of Theorem 9.1. Let X'_0 be the union of irreducible components of X_{b_0} which are not contained in X_0 . By openness of the flat locus [Stacks 2005–, Tag 0398], possibly after replacing x_0 by another k -point of X_0 , we may assume that $x_0 \notin X'_0$ and f remains flat at x_0 .

Furthermore, after replacing \mathcal{X} by $\mathcal{Y} = \mathcal{X} \setminus X'_0$, we may assume without loss of generality that $\mathcal{X}_{b_0} = X_0$. Indeed, the new projection $f': \mathcal{Y} \rightarrow B$ is the composition of f (which is flat at x_0) with the open immersion $\mathcal{Y} \hookrightarrow \mathcal{X}$ (which is flat everywhere). Hence, f' is flat at x_0 .

Let $V \subset B$ be as in Claim 9.4. By generic flatness [Stacks 2005–, Tag 0529], there exists a dense open subscheme $U \subset V$ such that f is flat over U . If $b_0 \in U(k)$, the conclusion follows from Theorem 1.2. We may, thus, assume that b_0 does not lie in U , and let $u \in U$ be the generic point of U . Since U is dense in B , b_0 is a specialization of u . By [EGA II 1961, Proposition 7.1.9], there exist a discrete valuation ring R and a separated morphism $\text{Spec}(R) \rightarrow B$ mapping the closed point s of $\text{Spec}(R)$ to b_0 and the generic point η of $\text{Spec}(R)$ to u .

Since f is flat at x_0 and G acts primitively on \mathcal{X}_{b_0} , by the openness of the flat locus [Stacks 2005–, Tag 0399], the flat locus of the base change $f_R: \mathcal{X}_R \rightarrow \text{Spec}(R)$ is dense in the component of the special fiber of f_R containing (the preimage of) x_0 . Since we are assuming that the generic fiber is primitive, we conclude that the flat locus of f_R is dense in the generic fiber. Therefore, after removing the complement of the flat locus from \mathcal{X}_R , we may assume that f_R is flat. Let $\bar{\eta}$ and \bar{s} be geometric points lying above η and s , respectively. By Proposition 3.1, we have

$$\text{ed}_{k(\bar{\eta})}(\mathcal{X}_{\bar{\eta}}) \geq \text{ed}_{k(\bar{s})}((X_0)_{\bar{s}}).$$

On the other hand, by Lemma 4.2,

$$\text{ed}_{k(\bar{s})}((X_0)_{\bar{s}}) = \text{ed}_k(X_0).$$

Therefore, $\text{ed}_{k(\bar{\eta})}(X_{\bar{\eta}}) \geq \text{ed}_k(X_0)$. Since $U \subset V$, the restriction of f to $f^{-1}(U)$ satisfies the assumptions of Theorem 1.2. The conclusion of Theorem 9.1 now follows from an application of Theorem 1.2 to the restriction of f to $f^{-1}(U)$. \square

Conclusion to the proof of Theorem 1.1. As we mentioned at the beginning of this section, in the course of proving Theorem 1.1, we may assume that $e \geq 1$.

We will now reduce the theorem to the case, where X is incompressible, i.e., $d = e + \dim(G)$. Indeed, by the definition of essential dimension there exists a G -equivariant dominant rational map $X \dashrightarrow X'$ such that $\dim(X') = d' := e + \dim(G)$. Suppose we know that Theorem 1.1 holds for X' . In other words, there exists a G -variety Y' such that $\dim(Y') = d'$, $\text{ed}(Y') = e$, and Y' is the total space of a G -torsor $t: Y' \rightarrow P'$ over a smooth projective variety P' . Clearly $d \geq d'$. Let $Y = \mathbb{P}^{d-d'} \times Y'$, where G acts trivially on $\mathbb{P}^{d-d'}$. Then $\dim(Y) = d$ and Y is a G -torsor $\text{id} \times t: Y = \mathbb{P}^{d-d'} \times Y' \rightarrow \mathbb{P}^{d-d'} \times P'$ over the smooth projective variety $\mathbb{P}^{d-d'} \times P'$. Moreover, by [Reichstein and Scavia 2022, Corollary 8.5], $\text{ed}(Y) = \text{ed}(Y') = e$, as desired (see also [Reichstein 2000, Lemma 3.3 (d)]).

From now on we will assume that $d = e + \dim(G)$ and $e \geq 1$. Denote by $[X]$ the class in $H^1(k(X)^G, G)$ associated to the generically free primitive G -variety X , as in Remark 4.6.

Now observe that there exists a finite subgroup S of G , such that for every field extension K/k , the natural map $H^1(K, S) \rightarrow H^1(K, G)$ is surjective. This follows from the definition of good characteristic if $\text{char}(k) > 0$ and from [Chernousov et al. 2006, Theorem 1.1 (a)] if $\text{char}(k) = 0$. (Recall that we are assuming that k is algebraically closed.) Suppose the S -torsor $T \rightarrow \text{Spec}(K)$ represents a class in $H^1(K, S)$ in the preimage of $[X]$. Spreading out the S -torsor $T \rightarrow \text{Spec}(K)$, we obtain a generically free primitive S -variety X' such that $X' \times^S G$ is G -equivariantly birationally equivalent to X . Therefore, X satisfies the assumptions of Corollary 8.4.

We let $f: \mathcal{X} \rightarrow B$ be a family obtained by applying Corollary 8.4 with $X_0 = X$. Then f is a smooth morphism of constant relative dimension $d = e + \dim(G)$, G acts freely on \mathcal{X} , \mathcal{X}_b is a primitive G -variety for every $b \in U(k)$, the G -action on \mathcal{X}_b is strongly unramified and X is G -equivariantly birationally isomorphic to \mathcal{X}_{b_0} . Clearly,

$$\text{ed}(\mathcal{X}_b) \leq \dim(\mathcal{X}_b) - \dim(G) = \dim(X) - \dim(G) = d - \dim(G) = e.$$

On the other hand, since k is algebraically closed and of infinite transcendence degree over its prime field, by Theorem 9.1 there exists a k -rational point b of B such that $\text{ed}_k(\mathcal{X}_b) \geq \text{ed}_k(X) = e$. Setting $Y := \mathcal{X}_b$, we obtain a generically free primitive G -variety Y such that $\dim(Y) = d$, $\text{ed}_k(Y) = e$ and the G -action on Y is strongly unramified, as desired. \square

Corollary 9.5. *Let G be a linear algebraic group over an algebraically closed field k of good characteristic and of infinite transcendence degree over its prime field. Then there exists a strongly unramified generically free G -variety Y such that $\dim(Y) = \text{ed}_k(G) + \dim G$ and $\text{ed}_k(Y) = \text{ed}_k(G)$.*

Proof. Let V be a generically free G -variety over k of essential dimension $e = \text{ed}_k(G)$. Then there is a G -compression $V \dashrightarrow X$, where X is a generically free G -variety of dimension $e + \dim(G)$. Clearly, $\text{ed}_k(X) \leq e$; on the other hand, $\text{ed}_k(X, G) \leq \text{ed}_k(V, G) = e = \dim(X)$. We conclude that $\dim(X) = e + \dim(G)$ and $\text{ed}_k(X) = e$. By Theorem 1.1 there exists a strongly unramified generically free variety Y of dimension $e + \dim(G)$ and essential dimension e . \square

10. Essential dimension at a prime

In this section, we extend some of the main results of this paper to essential dimension at a prime q and give an application in this setting.

We will now show that Proposition 3.1 and Theorems 1.1, 1.2 and 9.1 continue to hold if we replace essential dimension by essential dimension at a prime q . The proofs are largely unchanged; see below.

Note that while Proposition 3.1 and Theorems 1.1, 1.2 and 9.1 all require that the base field k should be of good characteristic, their q -analogues, Proposition 10.1 and Theorems 10.2, 10.3 and 10.5, do not need this assumption. These results hold whenever $\text{char}(k) \neq q$.

Proposition 10.1. *Let k be an algebraically closed field, R be a discrete valuation ring containing k and with residue field k , and l be the fraction field of R . Let G be a linear algebraic group over k and q be a prime number invertible in k . Let X be a flat R -scheme of finite type endowed with a G -action over R , whose fibers are generically free and primitive G -varieties. Then $\text{ed}_{l,q}(X_l) \geq \text{ed}_{k,q}(X_k)$.*

Proof. The proof is the same as that of Proposition 3.1, except that instead of the [Reichstein and Scavia 2022, Theorem 1.2], one should use [Reichstein and Scavia 2022, Theorem 11.1] which gives an analogous assertion for essential dimension at q . \square

Theorem 10.2. *Let G be a linear algebraic group over an algebraically closed field k , and let q be a prime number invertible in k . Let B be a noetherian k -scheme, $f : \mathcal{X} \rightarrow B$ be a flat G -equivariant morphism of finite type such that G acts trivially on B and the geometric fibers of f are generically free and primitive G -varieties (in particular, reduced). Then for any fixed integer $n \geq 0$, the subset of $b \in B$ such that $\text{ed}_{k(\bar{b}),q}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})}) \leq n$ for every (equivalently, some) geometric point \bar{b} above b is a countable union of closed subsets of B .*

Proof. Analogous to that of Theorem 1.2, replacing Proposition 3.1 by Proposition 10.1. \square

Theorem 10.3. *Let G be a linear algebraic group over an algebraically closed field k of infinite transcendence degree over its prime field, q be a prime number invertible in k , and X be a generically free primitive G -variety. Then there exists an irreducible smooth projective variety Z and a G -torsor $Y \rightarrow Z$ such that $\dim(Y) = \dim(X)$ and $\text{ed}_{k,q}(Y) = \text{ed}_{k,q}(X)$.*

Proof. Analogous to that of Theorem 1.1, replacing Theorem 1.2 by Theorem 10.2. \square

The next lemma is the analogue of Lemma 4.2 for essential dimension at a prime.

Lemma 10.4. *Let k be an algebraically closed field, q be a prime number, G be a k -group and X be a generically free primitive G -variety defined over k . Then $\text{ed}_{k,q}(X) = \text{ed}_{l,q}(X_l)$ for any field extension l/k .*

Proof. The proof is analogous to that of [Reichstein and Scavia 2022, Lemma 2.2], replacing G -compressions by G -equivariant (rational) correspondences of degree prime to q . We sketch it for the sake of completeness.

By a base change, every G -equivariant correspondence $X \rightsquigarrow Y$ gives rise to a G_l -equivariant correspondence $X_l \rightsquigarrow Y_l$ of the same degree. Moreover, if the G -variety Y is generically free and primitive, so is the G_l -variety Y_l . It follows that $\text{ed}_{k,q}(X) \geq \text{ed}_{l,q}(X_l)$.

Conversely, let $f : X_l \rightsquigarrow Y$ be a G_l -equivariant correspondence

$$\begin{array}{ccc} Z & & \\ \downarrow & \searrow & \\ X_l & & Y \end{array}$$

of degree prime to q defined over l , where the G_l -variety Y is generically free and primitive of smallest dimension $\dim(G) + \text{ed}_{l,q}(X_l)$. Since Z, Y and f are defined over a finitely generated subextension of l/k , we may suppose that l is finitely generated over k . Then there exist a k -variety U with function field $k(U) \simeq l$, G -invariant morphisms $\mathcal{Y} \rightarrow U$ and $Z \rightarrow U$ whose generic fibers are Y and Z , respectively, and G -equivariant dominant rational maps

$$\begin{array}{ccc} Z & & \\ \downarrow & \searrow & \\ X \times_k U & & \mathcal{Y} \end{array}$$

over U , whose base change along the generic point $\text{Spec}(l) \rightarrow U$ is f . Since k is algebraically closed, $U(k)$ is dense in U . If $u \in U(k)$ is a k -point in general position in U , the G -variety \mathcal{Y}_u is generically free and primitive, $\dim(\mathcal{Y}_u) = \dim(Y)$ and base change along $\text{Spec}(k(u)) \rightarrow U$ yields a G -equivariant correspondence $X \rightsquigarrow \mathcal{Y}_u$ of the same degree as f . We conclude that

$$\text{ed}_{k,q}(X) \leq \dim(\mathcal{Y}_u) - \dim(G) = \dim(Y) - \dim G = \text{ed}_{l,q}(X_l). \quad \square$$

Theorem 10.5. *Let G be a linear algebraic group over an algebraically closed field and q be a prime number invertible in k . Let $f : \mathcal{X} \rightarrow B$ be a G -equivariant morphism of k -varieties such that B is irreducible, G acts trivially on B and the generic fiber of f is a primitive and generically free $G_{k(B)}$ -variety. Let $b_0 \in B(k)$, $x_0 \in \mathcal{X}_{b_0}(k)$, and X_0 be a G -invariant reduced open subscheme of \mathcal{X}_{b_0} containing x_0 such that*

- (1) X_0 is a generically free primitive G -variety and
- (2) f is flat at x_0 .

Then, for a very general $b \in B(k)$, \mathcal{X}_b is generically free and primitive, and $\text{ed}_{k,q}(\mathcal{X}_b) \geq \text{ed}_{k,q}(X_0)$. Furthermore, if k is of infinite transcendence degree over its prime field (in particular, if k is uncountable), then the set of those $b \in B(k)$ such that $\text{ed}_{k,q}(\mathcal{X}_b) \geq \text{ed}_{k,q}(X_0)$ is Zariski dense in B .

Proof. Analogous to that of Theorem 9.1, replacing Proposition 3.1, Lemma 4.2 and Theorem 1.2 by Proposition 10.1, Lemma 10.4 and Theorem 10.2, respectively. □

We will now give an application of Theorem 10.5. Recall from Section 2 that

$$\mathrm{ed}_k(G) = \max \mathrm{ed}_k(X; G) = \mathrm{ed}_{k,q}(V; G)$$

and

$$\mathrm{ed}_{k,q}(G) = \max_q \mathrm{ed}_{k,q}(X; G) = \mathrm{ed}_{k,q}(W; G),$$

where X ranges over all generically free primitive G -varieties and V (respectively, W) is a versal (respectively, q -versal) generically free primitive G -variety. By definition,

$$\mathrm{ed}_k(G) \geq \max_q \mathrm{ed}_{k,q}(G), \quad (10.6)$$

where the maximum is taken over all primes q . One can think of $\mathrm{ed}_{k,q}(G)$ as a local version of $\mathrm{ed}_k(G)$. There is some tension between the two. On the one hand, the problem of computing $\mathrm{ed}_k(G)$ is more natural and, in some cases (e.g., for $G = S_n$ or PGL_n) is directly motivated by classical problems. On the other hand, $\mathrm{ed}_{k,q}(G)$ is usually more accessible: virtually all known techniques for bounding $\mathrm{ed}_k(G)$ from below actually yield a lower bound on $\mathrm{ed}_{k,q}(G)$ for some prime q . This means that when the obvious inequality (10.6) is not sharp, the exact value of $\mathrm{ed}_k(G)$ is difficult to establish.

In an attempt to probe the “gray area” between $\mathrm{ed}_k(G)$ and $\max_q \mathrm{ed}_{k,q}(G)$, Duncan and Reichstein [2014] defined poor man’s essential dimension, $\mathrm{pmed}_k(G)$, as the maximal value of $\mathrm{ed}_k(V; G)$, where V is a generically free primitive G -variety, which is q -versal for every prime q (but not necessarily versal). Clearly,

$$\mathrm{ed}_{k,q}(G) \leq \mathrm{pmed}_k(G) \leq \mathrm{ed}_k(G)$$

for every prime q . It is shown in [Duncan and Reichstein 2014, Theorem 1.4, Proposition 11.1] that

$$\mathrm{pmed}_k(G) = \max_q \mathrm{ed}_{k,q}(G) \quad (10.7)$$

for many finite groups G . Here the maximum is taken over all primes q , and $|G|$ is assumed to be invertible in k . Conjecturally, (10.7) holds for every finite group G ; see [Duncan and Reichstein 2014, Conjecture 11.5] (again, under the assumption that $|G|$ is invertible in k). We will now use Theorem 10.5 to prove the following result in a similar spirit:

Theorem 10.8. *Let G be a linear algebraic group defined over an uncountable algebraically closed field k . Suppose that at least one of the following conditions is satisfied:*

- (i) $\mathrm{char} k = 0$,
- (ii) $\mathrm{char} k = p > 0$, G^0 is reductive, and there exists a finite subgroup $S \subset G(k)$ of order prime to p such that for every $q \neq p$ and every q -closed field L containing k the natural map $H^1(L, S) \rightarrow H^1(L, G)$ is surjective.

Then there exists a strongly unramified generically free primitive G -variety X of dimension

$$\max_{q \neq \mathrm{char}(k)} \mathrm{ed}_{k,q}(G) + \dim(G)$$

such that $\mathrm{ed}_{k,q}(X; G) = \mathrm{ed}_{k,q}(G)$ for every prime q that is invertible in k .

To explain the relationship between Theorem 10.8 and the conjectural identity (10.7), assume for a moment that the G -variety X of Theorem 10.8 has been constructed. Then, on the one hand,

$$\text{ed}_k(X; G) \leq \dim(X) - \dim(G) = \max_{q \neq \text{char}(k)} \text{ed}_{k,q}(G)$$

and on the other hand,

$$\text{ed}_k(X; G) \geq \text{ed}_{k,q}(X; G) = \text{ed}_{k,q}(G)$$

for every prime q invertible in k . Thus,

$$\text{ed}_k(X; G) = \dim(X) - \dim(G) = \max_{q \neq \text{char}(k)} \text{ed}_{k,q}(G). \tag{10.9}$$

If we knew that X is q -versal for every prime q , then this equality would imply (10.7). Unfortunately, since X is strongly unramified, it will usually not be q -versal for every prime q different from $\text{char}(k)$, and the conjectural equality (10.7) remains open. On the other hand, one can think of the equality $\text{ed}_{k,q}(X; G) = \text{ed}_{k,q}(G)$ as a weaker substitute for q -versality. In this sense, one may view (10.9) (and thus, Theorem 10.8) as a weaker substitute for (10.7), valid for a wider class of algebraic groups G (not necessarily finite).

Our proof of Theorem 10.8 relies on the following well-known lemma. We include a proof for the sake of completeness.

Lemma 10.10. *Let G be a linear algebraic group defined over a field k . Then $\text{ed}_{k,q}(G) = 0$ for all but at most finitely many primes q .*

Proof. By [Merkurjev 2009, Proposition 4.4], $\text{ed}_{k,q}(G) > 0$ if and only if q is a torsion prime for G . It remains to show that every linear algebraic group G has only finitely many torsion primes.

Recall that q is a torsion prime for G if and only if q divides n_T for some G -torsor $T \rightarrow \text{Spec}(K)$, where K is a field containing k . Here, n_T denotes the index of $T \rightarrow \text{Spec}(K)$, i.e., the greatest common divisor of $[L : K]$, as L ranges over finite field extensions of K such that T splits over $\text{Spec}(L)$. For any G -torsor $T \rightarrow \text{Spec}(K)$ as above, the index n_T divides the so-called torsion index $t(G)$ of G ; see [Grothendieck 1958, Theorem 2]. Recall that the torsion index $t(G)$ is $\text{ind}(T_{\text{vers}})$, where $T_{\text{vers}} \rightarrow \text{Spec}(K_{\text{vers}})$ is a versal G -torsor. For an alternative definition and a discussion of the properties of the torsion index, see [Totaro 2005].

In summary, $\text{ed}_{k,q}(G) > 0$ only if q is a prime factor of $t(G)$, and the lemma follows. \square

Proof of Theorem 10.8. By Lemma 10.10, there are only finitely many primes q such that $\text{ed}_{k,q}(G) > 0$. Denote them by q_1, \dots, q_n . If $n = 0$, i.e., $\text{ed}_{k,q}(G) = 0$ for every prime $q \neq \text{char}(k)$, we can take $X = G$, viewed as a G -variety with respect to the translation action of G on itself. Then

$$\max_{q \neq \text{char}(k)} \text{ed}_{k,q}(G) = 0 = \dim(X) - \dim(G) \quad \text{and} \quad \text{ed}_{k,q}(X; G) = 0 = \text{ed}_{k,q}(G)$$

for every prime $q \neq \text{char}(k)$, as required.

We may thus assume without loss of generality that $n \geq 1$. Let $d_i = \text{ed}_{k,q_i}(G) + \dim(G)$, and let $d = \max(d_1, \dots, d_n)$. For each $i = 1, \dots, n$, let X_i be a q_i -versal G -variety of minimal possible dimension d_i . Let X_0 be the disjoint union of d -dimensional G -varieties $X_i \times \mathbb{P}^{d-d_i}$, where G acts

trivially on \mathbb{P}^{d-d_i} . The variety X_0 is equidimensional and the G -action on it is generically free. Note, however, that this action is not primitive unless $n = 1$.

There exists a subgroup $S \subset G(k)$ of order invertible in k such that for every $q \neq p$ and every q -closed field L containing k the natural map $H^1(L, S) \rightarrow H^1(L, G)$ is surjective. Indeed, if $\text{char } k = p > 0$, this follows from assumption (ii) of Theorem 10.8. If $\text{char}(k) = 0$, then S exist by [Chernousov et al. 2006, Theorem 1.1 (a)]. In other words, for every $i = 1, \dots, n$ there exists a G -equivariant dominant rational correspondence $X'_i \dashrightarrow X_i$ of degree prime to q and a generically free primitive S -variety Y_i such that $Y_i \times^S G$ is G -equivariantly birationally equivalent to X'_i . Now, let the S -variety Y_0 be the disjoint union of the $Y_i \times \mathbb{P}^{d-d_i}$, where S acts trivially on \mathbb{P}^{d-d_i} for each i . By our construction, $Y_0 \times^S G$ and X_0 are G -equivariantly birationally equivalent. Since $X'_i \dashrightarrow X_i$ has degree prime to q , we have $\text{ed}_{k,q}(X'_i; G) = \text{ed}_{k,q}(X_i; G)$ for all $i = 1, \dots, n$. Therefore, replacing X_i by X'_i , we may assume that X_0 admits reduction of structure to S , so that the assumptions of Corollary 8.4 apply to X_0 .

Let $f : \mathcal{X} \rightarrow B$ be a morphism constructed in Corollary 8.4, with respect to the G -variety X_0 we have just defined. We now want to take $X = \mathcal{X}_b$ to be the fiber of X over a very general point $b \in B(k)$. By Corollary 8.4, X is strongly unramified (again, for a very general $b \in B(k)$). By our construction,

$$\dim(\mathcal{X}_b) = d = \max(\text{ed}_{k,q_1}(G), \dots, \text{ed}_{k,q_n}(G)) + \dim(G) = \max_{q \neq \text{char}(k)} \text{ed}_{k,q}(G) + \dim(G),$$

as desired. Moreover, by Theorem 10.5,

$$\text{ed}_{k,q_i}(\mathcal{X}_b) \geq \text{ed}_{k,q_i}(X_i \times \mathbb{P}^{d-d_i}) = \text{ed}_{k,q_i}(X_i) = \text{ed}_{k,q_i}(G).$$

Here, the equality $\text{ed}_{k,q_i}(X_i \times \mathbb{P}^{d-d_i}) = \text{ed}_{k,q_i}(X_i)$ follows from [Reichstein and Scavia 2022, Corollary 10.3]. (It can also be deduced directly from the definition of essential dimension at q_i .) The opposite inequality, $\text{ed}_{k,q_i}(\mathcal{X}_b) \leq \text{ed}_{k,q_i}(G)$, follows from the definition of $\text{ed}_{k,q_i}(G)$. Thus,

$$\text{ed}_{k,q_i}(\mathcal{X}_b) = \text{ed}_{k,q_i}(G), \quad \text{for each } i = 1, \dots, n.$$

In summary, a very general fiber $X = \mathcal{X}_b$ of $f : \mathcal{X} \rightarrow B$ has the properties claimed in Theorem 10.8. \square

11. Example: Finite group actions on hypersurfaces

In this section, we give yet another application of Theorems 1.2 and 10.2.

Proposition 11.1. *Let k be an algebraically closed field of infinite transcendence degree over its prime field, W be a finite-dimensional k -vector space, G be a finite group and $G \hookrightarrow \text{GL}(W)$ be a faithful linear representation over k . Set $V := W \oplus k$, where G acts trivially on k . Let $f(x_1, \dots, x_n) \in k[W]$ be a G -invariant homogeneous polynomial of degree d and $f_0(x_{n+1})$ be an (inhomogeneous) polynomial of degree d with d distinct roots in k . Assume that the G -action on the affine hypersurface $Z(f) \subset \mathbb{A}(V) \simeq \mathbb{A}^{n+1}$ is primitive. Then*

- (a) *the G -action on the hypersurface $Z(f + f_0)$ is generically free and primitive,*
- (b) $\text{ed}_k(Z(f + f_0)) = \text{ed}_k(G)$,
- (c) $\text{ed}_{k,q}(Z(f + f_0)) = \text{ed}_{k,q}(G)$ for every prime integer q .

Proof. For $(0, 0) \neq (s, t) \in \mathbb{A}^2$, set

$$f_{(s,t)} = sf(x_1, \dots, x_n) + tf_0(x_{n+1})$$

and $Z_{[s:t]} = V(f_{(s,t)})$. Note that $Z_{[s:t]}$ depends only on the projective point $[s : t] \in \mathbb{P}^1$. The 2-dimensional torus \mathbb{G}_m^2 acts on $V = W \oplus k$ by scalar multiplication in W and k . This action commutes with the action of G on V . The hypersurfaces $Z_{[s:t]}$ with $s, t \neq 0$ all lie in the same orbit of this \mathbb{G}_m^2 -action; hence, they are isomorphic as G -varieties.

We now proceed with the proof of part (a). We claim that $Z_{[s:t]}$ is generically free and primitive whenever $s, t \neq 0$. As we saw above, the G -variety $Z_{[s:t]}$ is isomorphic to $Z_{[1:1]}$ whenever $s, t \neq 0$. Hence, the claim is equivalent to part (a). It will, however, be convenient for us to let $[s : t]$ vary over \mathbb{P}^1 , rather than focusing solely on $Z_{[1:1]}$.

To show that $Z_{[s,t]}$ is primitive, assume the contrary. Then $f_{(s,t)}$ can be written as a product of two polynomials, α and $\beta \in k[x_0, \dots, x_{n+1}]$ of degree $\leq d - 1$ such that G preserves α and β up to a scalar. Setting $x_{n+1} = 0$, we see that $f(x_1, \dots, x_n) = \alpha(x_1, \dots, x_n) \cdot \beta(x_1, \dots, x_n)$, which contradicts our assumption that $Z(f) = Z_{[1:0]}$ is primitive.

To prove that the G -action on $Z_{[s:t]}$ is generically free, assume the contrary. Then $Z_{[s:t]}$ is contained in $V_{\text{nonfree}} = W_{\text{nonfree}} \times k$, where V_{nonfree} the union of the fixed point loci V^g as g ranges over the nontrivial elements of G , as in Section 8, and similarly for W_{nonfree} . Consider the family $\mathcal{X} \rightarrow B = \mathbb{P}^1$ of G -varieties, where $\mathcal{X} \subset \mathbb{A}(V) \times \mathbb{P}^1$ consists of pairs $(v, [s : t])$ such that $v \in Z_{[s:t]}$. In other words, the fiber over $[s : t]$ is $Z_{[s:t]}$. The set of $[s : t]$ such that the entire fiber $\mathcal{X}_{[s:t]} = Z_{[s:t]}$ lies in V_{nonfree} is closed in \mathbb{P}^1 . On the other hand, note that $Z_{[0:1]} = Z(f_0)$ is a disjoint union of d copies of W . Since the G -action on W is faithful, hence generically free, we see that $Z_{[0:1]} = Z(f_0)$ is not contained in V_{nonfree} . Thus, the G -action on $Z_{[s:t]}$ can be nonfree for only finitely many $[s : t] \in \mathbb{P}^1$. Since $Z_{[s:t]} \simeq Z_{[1,1]}$ whenever $s, t \neq 0$, we conclude that the G -action on $Z_{[s:t]}$ is generically free whenever $s, t \neq 0$. This completes the proof of the claim and, hence, of part (a).

To prove part (b), let us examine the family $\mathcal{X} \rightarrow B = \mathbb{P}^1$ more closely. This family is obtained by a pullback from the universal family of affine hypersurfaces of degree $\leq d$ in V . In particular, it is flat. As we pointed out above, the fiber $\mathcal{X}_{[0:1]} = Z_{[0:1]}$ is a disjoint union of d copies of W (as a G -variety). Every copy of W has essential dimension equal to $\text{ed}_k(G)$; see (2.4). Let $\mathcal{X}' \subset \mathcal{X}$ be the Zariski open subvariety obtained by removing all irreducible components of $\mathcal{X}_{[0:1]}$ but one. Since the inclusion map $\mathcal{X}' \hookrightarrow \mathcal{X}$ is an open embedding, and open embeddings are flat, we conclude that the composition $\mathcal{X}' \hookrightarrow \mathcal{X} \rightarrow B$ is also flat. Theorem 1.2 now tells us that for a very general $[s : t]$ in \mathbb{P}^1 , we have

$$\text{ed}_k(Z_{[s:t]}) = \text{ed}_k(\mathcal{X}_{[s:t]}) = \text{ed}_k(\mathcal{X}'_{[s:t]}) \geq \text{ed}_k(\mathcal{X}'_{[0:1]}) = \text{ed}_k(G).$$

In particular, $\text{ed}_k(Z_{[s:t]}) \geq \text{ed}_k(G)$ for some $s, t \neq 0$. The opposite inequality, $\text{ed}_k(Z_{[s:t]}) \leq \text{ed}_k(G)$, is immediate from the definition of $\text{ed}_k(G)$. Thus, $\text{ed}_k(Z_{[s:t]}) = \text{ed}_k(G)$ for some $s, t \neq 0$. As we saw above, $Z_{[s,t]} \simeq Z_{[1:1]}$ for any $s, t \neq 0$. We conclude that $\text{ed}_k(Z_{[1:1]}) = \text{ed}_k(G)$. This proves part (b).

The proof of part (c) is the same, except that we use Theorem 10.2 in place of Theorem 1.2. □

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
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